

Chapter 10

Solving time dependent heat problems

In this chapter, we revisit the time dependent heating and cooling problems introduced in Chapter 9. These resulted in ODEs which we now solve using standard analytic methods. Given such analytic solutions, we can answer some general questions about the problems. We also consider some extensions of time dependent heat transport, without conduction, in two case studies: a model describing house temperatures, and a model for predicting the spontaneous combustion of fish and chip crumble.

10.1 The cooling coffee problem revisited

In this section, we revisit the problem of determining the time it takes for a hot cup of coffee to cool. The exponential growth/decay differential equation which arises can be solved analytically, using different techniques, and this solution provides a more complete understanding of the cooling process than the particular numerical solution generated in Chapter 9.

Review of the model

In Section 9.1, we first introduced the problem of determining the temperature of a cup of coffee. In Section 9.2, we formulated the general differential equation for the temperature $U(t)$ as a function of time, which was

$$\frac{dU}{dt} = \frac{hS}{cm}(U - u_s), \quad (10.1)$$

where c is the specific heat of the coffee, m is the mass of the coffee, h is the Newton cooling coefficient, S is the surface area through which heat is lost and u_s is the constant temperature of the surroundings.

Analytic solution

We can write the differential equation as

$$\frac{dU}{dt} = -\lambda(U - u_s), \quad (10.2)$$

where λ is a constant of proportionality and equivalent to hS/cm . The solution to this differential equation can be found using the separable technique, as in the following example.

Example 10.1: Solve the differential equation (10.2) subject to the initial condition $U(0) = u_0$.

Solution: Separating the variables,

$$\frac{1}{U - u_s} \frac{dU}{dt} = -\lambda.$$

Integrating both sides with respect to t , assuming $U > u_s$, gives

$$\begin{aligned} \ln(U - u_s) &= -\lambda t + K \\ U &= e^{-\lambda t + K} + u_s. \end{aligned}$$

Applying the initial condition $U(0) = u_0$ gives $e^K = u_0 - u_s$ and so

$$U(t) = (u_0 - u_s)e^{-\lambda t} + u_s. \quad (10.3)$$

Applying the coffee cooling model

We return to our original problem (see Section 9.1) to find how long it takes a cup of coffee to cool to 40°C , and find the answer using the analytic solution we have just derived. A hot cup of coffee is at a temperature of 60°C and after 10 minutes the coffee has cooled to 50°C . The room temperature is 20°C . The following example shows how to find this time.

Example 10.2: Find how long it will take for the cup of coffee to cool to 40 °C.

Solution: We can find λ from the given information that $U = 50$ at time $t = 10$. Substituting this into the solution (10.3) we obtain

$$50 = (60 - 20)e^{-10\lambda} + 20, \quad \text{and } \lambda \approx 0.0288.$$

Now, if U is 20 °C then

$$40 = (60 - 20)e^{-0.0288t} + 20 \quad \text{and } t \approx 24 \text{ minutes.}$$

Thus the coffee will reach the temperature of 40 °C about 24 minutes after it was put into the room.

An alternative cooling law

The differential equation (10.2) applied Newton's law of cooling which assumes that air at room temperature is moving over and around the object. This is not necessarily the case in a room, or on a still day. A better model for cooling in still air is one in which the rate of the temperature increase/decrease is directly proportional to the $(5/4)$ th power of the difference in temperature between the object and its surroundings. This is the law of *natural cooling* and is given by

$$\left\{ \begin{array}{l} \text{rate of} \\ \text{change of} \\ \text{heat} \end{array} \right\} = \pm h_1 S(U - u_s)^{5/4}, \quad (10.4)$$

where the sign \pm depends on whether the object is cooling or heating, h_1 is a positive constant of proportionality and S is the surface area. Note that $h_1 \neq h$, the Newton cooling coefficient in (10.1).

For the problem of cooling a cup of coffee, the differential equation for the temperature is

$$\frac{dU}{dt} = -\lambda(U - u_s)^{5/4}, \quad \lambda = \frac{h_1 S}{cm}. \quad (10.5)$$

(Recall that m is the mass of the coffee and c its specific heat.) This differential equation is also separable and can be solved using the same technique as before. (See Exercises Question 1.) Using its solution we can show that the model predicts it will take approximately 26 minutes for the coffee to cool from 60 °C to 40 °C in a room of temperature 20 °C, given that it takes 10.3 minutes to cool to 50 °C. This estimate for the time is slightly longer than that obtained using Newton's law of cooling. A comparison between these two cooling models is illustrated in Figure 10.1. The difference is not large, for this problem.

The physical meaning of this is that the natural cooling law gives a slower rate of cooling than Newton's law. While the factor $(U - u_s)^{5/4}$ is larger than $(U - u_s)$, the associated values for λ (0.0115 and 0.0288 respectively) effectively render the rate for natural cooling less than that for Newton cooling. This makes sense because we expect cooling in an environment with a slight breeze (Newton's law of cooling) to be more effective than cooling in perfectly still air (natural law of cooling).

Summary of equations

Newton's law of cooling (cooling in a breeze) is

$$\frac{dU}{dt} = \pm \frac{hS}{cm}(U - u_s).$$

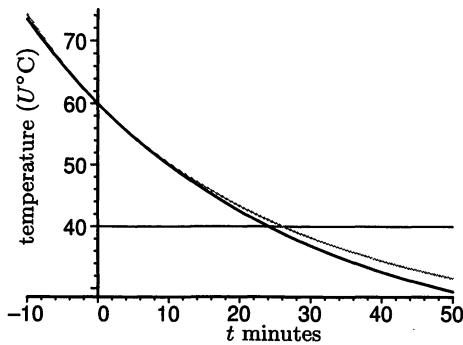


Figure 10.1: A comparison between Newton cooling (black line) and natural cooling (grey line). For comparison purposes, it has been assumed that the coffee takes 10.3 minutes to cool to 50 °C for natural cooling.

The natural law of cooling (applicable when there is no breeze) is

$$\frac{dU}{dt} = \pm \frac{h_1 S}{cm} (U - u_s)^{5/4}.$$

Summary of skills developed here:

- Solving a differential equation with a cooling law (e.g. Newton's law of cooling).
- Solving for a constant of proportionality and a constant of integration using a set of known results and an initial condition.
- Using a solution for the temperature to calculate the time of cooling (or time of heating).

10.2 The hot water heater problem revisited

We now return to the water heater problem introduced in Section 9.3. Again, our aim is to determine and interpret the analytic solution to indicate how such a solution provides a better general understanding of the system, compared with any particular numerical solution.

Review of the model

Suppose we have a domestic hot water heater. We wish to determine a formula for how long it takes to heat the water to, say 60 °C, and what physical quantities this depends on. One interesting question that we can pose is:

Does doubling the power output of the heating element halve the heating time?

We formulated the differential equation for the water temperature $U(t)$ in Section 9.3 as

$$cm \frac{dU}{dt} = q - hS(U - u_s), \quad (10.6)$$

which describes the temperature variation of the water with time. Here q is the rate of heat supplied, m the mass of water, S the surface area through which heat is lost, c the specific heat of the water, h the Newton cooling coefficient, u_s the temperature of the surroundings and u_0 the initial temperature of the water. The final temperature will be denoted by u_f , and here $u_f = 60^\circ\text{C}$.

Analytic solution

To solve the differential equation we first write it in the form

$$\frac{dU}{dt} = \beta - \alpha U, \quad U(0) = u_0, \quad (10.7)$$

where the constants α and β are defined as

$$\alpha = \frac{hS}{cm}, \quad \beta = \frac{q + hSu_s}{cm}.$$

In this form, it is easier to see that the differential equation is a first-order linear equation.

Example 10.3: Solve the differential equation (10.7) using the integrating factor technique.

Solution: Since the differential equation is first-order linear, we can find an integrating factor,

$$R(t) = e^{\int_0^t \alpha dt} = e^{\alpha t}.$$

Hence, we can write the differential equation as

$$\frac{d}{dt}(Ue^{\alpha t}) = \beta e^{\alpha t}.$$

Integrating both sides with respect to t

$$\begin{aligned} e^{\alpha t} U &= \frac{\beta}{\alpha} e^{\alpha t} + K, \\ \text{and } U &= \frac{\beta}{\alpha} + K e^{-\alpha t}, \end{aligned}$$

where K is the constant of integration. Now applying the initial condition $U(0) = u_0$ gives $u_0 = \beta/\alpha + K$ and so $K = u_0 - \beta/\alpha$. Hence the solution is

$$U = u_0 e^{-\alpha t} + \frac{\beta}{\alpha} (1 - e^{-\alpha t}).$$

Substituting for α and β this gives

$$U(t) = \left(u_0 - u_s - \frac{q}{hS}\right) e^{-(hS/cm)t} + u_s + \frac{q}{hS}. \quad (10.8)$$

Note that this equation is also separable and could have been solved using that technique. (See Question 3 of the Exercises.)

Applying the hot water heater model

We apply the above theory to a typical hot water cylinder used in an average Australian home (of four individuals) to provide hot water. Then, by considering that energy used to heat the system has a cost, which most of us would like to minimise, we examine how best to operate the system.

We suppose that we have a standard home hot water tank, which holds 250 litres of water, and is cylindrical with a height of 1.444 m and a diameter of 0.564 m. The water is heated by a heating element immersed in the water which supplies heat at a constant rate of 3600 Watts ($q = 3600 \text{ W} = 3600 \text{ J sec}^{-1}$). We assume the mass of the water to be 250 kg (as a litre of water has the mass of approximately 1 kg) with the surface area of the tank approximately 3.06 m^2 . We know the specific heat of water is $c = 4200 \text{ J kg}^{-1} \text{ }^\circ\text{C}^{-1}$ and the heat transfer coefficient is $h = 12 \text{ W m}^{-2} \text{ }^\circ\text{C}^{-1}$. Given that the temperature of the surroundings is $u_s = 15 \text{ }^\circ\text{C}$, and that initially the water in the tank is the same temperature as its environment ($u_s = u_0$), we are interested in how long it would take the water to reach a temperature of $60 \text{ }^\circ\text{C}$.

Example 10.4: Find the time it would take to heat the tank of water to $60 \text{ }^\circ\text{C}$.

Solution: Using solution equation (10.8) we need to solve for t . Again, we let

$$\alpha = \frac{hS}{cm}, \quad \beta = \frac{q + hSu_s}{cm},$$

and then the solution is

$$U = \left(u_0 - \frac{\beta}{\alpha} \right) e^{-\alpha t} + \frac{\beta}{\alpha}.$$

Solving for $e^{-\alpha t}$ gives

$$e^{-\alpha t} = \frac{U - \beta/\alpha}{u_0 - \beta/\alpha}.$$

Hence we obtain

$$t = \frac{1}{\alpha} \ln \left| \frac{\beta/\alpha - u_0}{\beta/\alpha - U} \right|.$$

With the above parameter values substituted into this equation, the time required to heat the water from $u_0 = u_s = 15 \text{ }^\circ\text{C}$ to $60 \text{ }^\circ\text{C}$ is $\approx 4 \text{ hours and 50 minutes}$.

This result can be compared with the numerical solution in the Maple produced diagram of Figure 9.4 in Section 9.3.

It is possible to improve on the efficiency of the tank by insulating it well, and the standard used in Australia is approximately 3 cm of compacted fibreglass. Then, assuming some cost of power per Watt, we can compare the efficiency of the above tank with that for a tank which is insulated. (See Question 5 of the Exercises.)

What is of major interest is how to run the most cost effective system. Clearly, the cost of heating the tank is proportional to the time it takes to heat the water, or the length of time for which the heater is on. Assuming all the constants as above, would it be cheaper to use a thermostat which switches the heater on each time the temperature drops below $50 \text{ }^\circ\text{C}$, or to switch the heater off for 8 hours a day relying on the thermostat at other times? Would this result change if the temperature of the environment were to fall to $10 \text{ }^\circ\text{C}$? (See Exercises, Question 4.)

We are also in a position to answer the question posed at the beginning of this section. This is done in the following example:

Example 10.5: Does doubling the power output halve the heating time?

Solution: We start with (10.6)

$$cm \frac{dU}{dt} = q - hS(U(t) - u_s)$$

and its solution as an expression for t is

$$t = \frac{cm}{hS} \ln \left| \frac{\frac{q}{hS} + u_s - u_0}{\frac{q}{hS} + u_s - U} \right|.$$

It is clear that if q is doubled, its effect on t is not to halve it. The effect of q is not quite so simple. While it will certainly reduce the natural logarithm function with the effect of decreasing the time, in general it will not halve it. Figure 10.2 illustrates the behaviour of t as q increases or decreases.

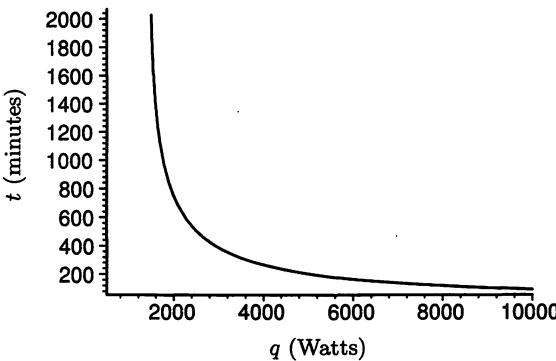


Figure 10.2: A plot of q in Watts as a function of t in minutes. U is fixed at $u_f = 60^\circ\text{C}$ and other parameters are as for Example 5.

The equilibrium temperature

Physically the differential equation (10.6) represents a balance between heat produced by the heating element and heat lost from the surface. This suggests the existence of an equilibrium temperature, or thermal equilibrium. To find the equilibrium temperature we set $dU/dt = 0$ and then solve for U . Thus

$$q - hS(U - u_s) = 0.$$

It follows that the equilibrium temperature U is

$$U = u_s + \frac{q}{hS}. \quad (10.9)$$

Note that, since q , h and S are all positive constants, the equilibrium temperature is always greater than the temperature of the surroundings. This makes sense intuitively.

Now consider the solution to differential equation (10.6) which is given by (10.8). If we take the limit of this solution as t becomes large, we see that

$$U \rightarrow u_s + \frac{q}{hS} \quad \text{as } t \rightarrow \infty.$$

We see that this limiting temperature, called the steady state temperature, is just the equilibrium temperature of (10.9).

The physical significance of this is that, as the steady state temperature is approached, heat is no longer used up to change the temperature. Instead a balance is attained where the heat input is equal to the heat escaping from the surface. Thus, in mathematical terms from the differential equation describing the heat balance, equation (10.6), the derivative term is zero, which is the condition for an equilibrium temperature.

Summary of skills developed here:

- Formulate a differential equation for a hot water heater with insulation of varying degrees of effectiveness.
- Formulate a model to describe a cooling device, such as a refrigerator.
- Be able to ascertain how each parameter in the model affects the heating/cooling times.

10.3 Case Study: It's hot and stuffy in the attic

We have considered some simple applications of heat transport, but how do these relate to larger systems? In the case study which follows, we adapt these ideas to consider the changing temperatures in a building. This is essential for good insulation characteristics. The following was based on (and substantially modified from) Sansgiry and Edwards (1996).

In different parts of the world dwellings and public buildings take on many different and imaginative forms. Mud, corrugated iron, fibro (asbestos) sheeting and glass are just some of a huge variety of materials used. One aspect considered in their construction is the climate. Certain materials used, or the aspect and design chosen, may effectively use the environment to enhance the heating/cooling mechanisms to suit a particular climate. Others, lacking this forethought, may be stuffy or cold.

To understand heat transfer in complicated structures, we need a model for the heat transfer between two regions at different temperatures. We assume

$$\left\{ \begin{array}{c} \text{rate} \\ \text{heat} \\ \text{transferred} \end{array} \right\} = \frac{S \times \Delta U}{R}$$

where S is the surface area between the two regions, ΔU is the temperature difference between them and R is the resistance to heat transport. For walls of houses R represents the total heat resistance due to the resistance of the material making up the wall, which might include insulation materials. The value of R for a given wall can be determined experimentally, or theoretically (as we see later in Chapter 11).

To understand how the materials used for heating/cooling processes work, we consider a model for the temperature of a simple brick house with a tiled roof, shown in the Figure 10.3. The building has two compartments: the interior at temperature $U_1(t)$, and the roof cavity (between the ceiling and the roof) at temperature $U_2(t)$. The temperature of the surrounding air is assumed to be $u_s = 35^\circ\text{C}$ corresponding to a house in a hot climate.

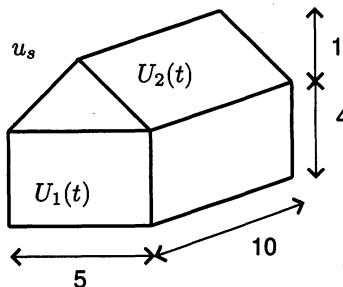


Figure 10.3: Schematic diagram for a model of a house with two compartments: an interior and a roof cavity. Here $V_1 = 200 \text{ m}^3$ and $V_2 = 25 \text{ m}^3$.

A system of differential equations, dependent on heat conduction, can be derived by considering the heat inputs and outputs in each of the two compartments. The resulting governing equations are

$$\begin{aligned} M_1 \frac{dU_1}{dt} &= \frac{S_{10}}{R_{10}}(u_s - U_1) + \frac{S_{12}}{R_{12}}(U_2 - U_1), \\ M_2 \frac{dU_2}{dt} &= \frac{S_{20}}{R_{20}}(u_s - U_2) + \frac{S_{12}}{R_{12}}(U_1 - U_2), \end{aligned} \quad (10.10)$$

where S is surface area, R total heat resistance due to the building material and M_1 and M_2 are parameters representing the overall thermal mass (air plus solid materials) of the two regions. For example, $M_1 = \rho c V_1 + \rho_b c_b V_b$, where ρ is the density and c the specific heat of air, V_1 the volume of region 1, ρ_b is the brick wall density, c_b the brick wall specific heat and V_b is the volume of the brick wall. (For the results generated here, these values are calculated as $M_1 = 0.6498018 \times 10^8$ and $M_2 = 0.5184672 \times 10^7$, both in SI units.) Also S_{10} , S_{12} and S_{20} denote the surface areas of the various boundaries of the regions (or compartments) and R_{10} , R_{12} and R_{20} are the corresponding thermal resistances. Typical values for these heat resistances are $R_{10} = 0.3$ for walls without insulation, $R_{12} = 0.1$ for ceilings (without insulation) and $R_{20} = 0.2$ for the roof. With wall and ceiling insulation these R values increase, with typically $R_{10} = 2$ and $R_{12} = 3$. Clearly, which building and insulation materials are chosen has a substantial impact on the resulting house temperatures.

Using Maple or MATLAB to find a numerical solution to the system provides the results illustrated in Figure 10.4.

A valid criticism of the above model is that over long periods of time the roof cavity temperature tends to the outside temperature. In reality, in warm climates the temperature of the roof cavity is often much hotter than the outside temperature, particularly on very hot days, as in the case under consideration. One reason for this is that the roof tiles heat up to a temperature substantially greater than that of the surroundings, due to their exposure to the sun, and then radiate this heat into the roof cavity. To incorporate this aspect into the model we include the transfer of heat by radiation.

Let us assume the temperature of the roof tiles to be 80°C . A model for the heat transfer by radiation is given by the Stefan-Boltzmann law,

$$\left\{ \begin{array}{c} \text{rate} \\ \text{heat} \\ \text{transferred} \end{array} \right\} = \epsilon \sigma S (T_r^4 - T_2^4)$$

where $\sigma = 5.6697 \times 10^{-8}$ is the Stefan-Boltzmann constant, the parameter ϵ is the emittance

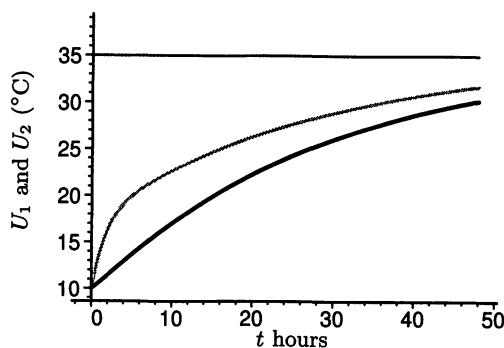


Figure 10.4: Plot of temperatures against time for the model without radiation from the roof tiles. The black line is U_1 , the interior temperature, the grey line is U_2 , the roof cavity temperature and the horizontal line is the temperature of the outside environment.

(typically $\epsilon \simeq 0.8$), T_r is the absolute temperature of the roof tiles (measured in degrees Kelvin) and T_2 is the absolute temperature of the roof cavity (also in degrees Kelvin). S is the surface area of the roof, as before. The quantities $u_s + 273$, $U_1(t) + 273$, $U_2(t) + 273$ and $u_r + 273$ now represent the absolute temperatures in degrees Kelvin (K), where u_r is the temperature of the roof tiles.

Incorporating this extra heat source into our model results in a modification of the second equation; note that the temperature is now expressed in degrees Kelvin. Setting $T_r = (u_r + 273)$ K and $T_2 = (U_2 + 273)$ K the resulting governing equations are now

$$\begin{aligned} M_1 \frac{dU_1}{dt} &= \frac{S_{10}}{R_{10}}(u_s - U_1) + \frac{S_{12}}{R_{12}}(U_2 - U_1), \\ M_2 \frac{dU_2}{dt} &= \frac{S_{20}}{R_{20}}(u_s - U_2) + \frac{S_{12}}{R_{12}}(U_1 - U_2) \\ &\quad + \epsilon\sigma S_{12} ((u_r + 273)^4 - (U_2 + 273)^4). \end{aligned} \quad (10.11)$$

Temperature predictions for this new system are given in Figure 10.5. Note that the temperature in the roof cavity $U_2(t)$ is now greater than the outside temperature. This has also caused the temperature of the interior $U_1(t)$ to increase compared with the first model, as would be expected.

Clearly, over a number of days, the outside temperature would change and also vary cyclically with a 24-hour period, and this time dependence could be incorporated into the model. However, it would still be appropriate to use this model to explore the effects of different insulation strategies. For example, would it be more cost efficient and effective to increase the insulation in the walls or in the ceiling? Or would different roofing materials improve, and possibly solve, the problem of a hot stuffy attic?

10.4 Spontaneous combustion

Spontaneous combustion may occur in a variety of situations when heat produced by a chemical reaction cannot escape from a system fast enough. Thus heat may build up and result in ignition. In this section, the basic mathematical theory describing this process is

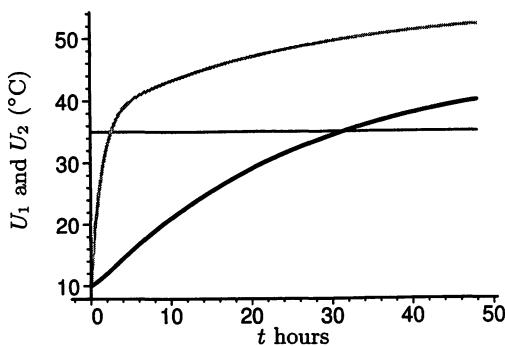


Figure 10.5: Inclusion of radiation from roof tiles. Plot of roof cavity temperature U_2 (grey) and interior temperature U_1 (black) against time in hours. The roof cavity temperature exceeds the temperature of the surroundings, $u_s = 35^\circ\text{C}$ (horizontal line).

developed, with both an analytical and numerical analysis. The complete theory (including conduction) is complex, beyond the scope of this book; thus, what is presented here is the simplest case, that of a *stirred chemical reactor* where there is no conduction of heat.

Background

Spontaneous combustion occurs when the temperature inside a body increases at a rate faster than heat is able to escape from its surface. The higher the temperature, the more heat is produced by the chemical reaction (usually oxidation), so it is important when designing systems which involve heat production to ensure that heat can escape sufficiently easily.

The important mechanisms are the rate of heat production and the rate of heat conduction away from this heat source. The reaction causing the heat production may be one where some organic material intersects with the atmosphere or oxygen, such as woollen fibres in bales. It is characterised by its *exothermicity*, that is, the amount of heat produced by the reaction, and by the *rate of the reaction*, which determines the rate at which the heat is produced. The process by which heat is able to escape is typically conduction, and thus the surrounding materials and their ability to conduct heat also have a substantial effect on the system. Before formulating the differential equations governing this process, we need to introduce some further physical principles.

Arrhenius law for heat production in a reaction

The rate of the reaction (associated with the rate of heat gain) can be described by an exponential function $Ae^{-E/(RT)}$ where E is the *activation energy*, with R the universal gas constant¹, given by $R = 8.314 \text{ J K}^{-1} \text{ mol}^{-1}$, and where T is the absolute temperature, measured in degrees Kelvin (K) from absolute zero. T will be used here to distinguish between absolute temperature, measured in Kelvin, K, and the temperature measured in $^\circ\text{C}$, denoted by U , where $T = U + 273.15$. The parameter A is a further constant dependent on the type of the reaction. It is called the *pre-exponential factor* and has units of s^{-1} , where s is the SI unit of time, seconds. The rate of gain of heat (per unit volume) is given

¹The universal gas constant also occurs in the famous law $PV = nRT$ relating pressure, volume and temperature.

by

$$\left\{ \begin{array}{l} \text{rate} \\ \text{heat generated} \\ \text{by reaction} \\ \text{per unit volume} \end{array} \right\} = \rho Q A e^{-E/(RT)} \quad (10.12)$$

where ρ is the density (mass per unit volume) and Q is the 'heat of reaction', measured in Joules per unit mass.

Clearly, this function increases monotonically as the temperature rises, with

$$\lim_{T \rightarrow \infty} A^{-E/RT} = A.$$

From a graph of this reaction rate, we note that there is a domain of temperatures for which a small increase in T will result in a large increase in the reaction rate (see Figure 10.7 for the scaled equation, to come). This is associated with a substantial increase in the heat production.

Model assumptions and approach

We now make some simplifying assumptions on which to build the mathematical model.

- We neglect heat conduction by assuming a uniform temperature throughout the reacting material.
- The rate of heat created is given by the Arrhenius law.
- Heat is lost from the surface of the reacting material according to Newton's law of cooling.

General compartmental model

We can apply the balance law to this process, where the compartment is the container of material (drums, wool-bales, etc.). The heat input is via some exothermic reaction and the output is through the mechanism of heat conduction away from the surface of this compartment. (We have assumed above that the temperature inside the compartment is uniform.)

The process is illustrated in the input-output diagram of Figure 10.6.

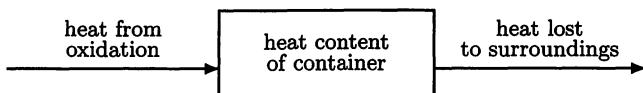


Figure 10.6: Input-output compartment diagram for the heat production and dissipation process.

In words, the heat balance equation is

$$\left\{ \begin{array}{l} \text{rate of} \\ \text{change of} \\ \text{heat content} \end{array} \right\} = \left\{ \begin{array}{l} \text{rate} \\ \text{heat generated} \\ \text{by reaction} \end{array} \right\} - \left\{ \begin{array}{l} \text{rate} \\ \text{heat lost} \\ \text{to surroundings} \end{array} \right\}. \quad (10.13)$$

Formulating the differential equation

The rate of heat loss is proportional to the surface area (S) of the body of material. It also depends on the type of material, for which there is an associated heat transfer coefficient h . Thus mathematically, the heat loss is proportional to the temperature difference between the material and its surroundings, and can be represented as

$$\left\{ \begin{array}{l} \text{rate of} \\ \text{heat} \\ \text{loss} \end{array} \right\} = hS(T - T_a), \quad (10.14)$$

where h is the Newton cooling coefficient, S is the surface area from which heat is lost, and T_a is the temperature of the surroundings, namely the ambient temperature (measured in degrees Kelvin).

Combining the expressions for heat loss (10.14) and gain (10.12) into an equation describing the rate of temperature change (10.13), and using the specific heat to evaluate the rate of change of heat in terms of the rate of change of temperature, we obtain the differential equation

$$\rho V c \frac{dT}{dt} = \rho V Q A e^{-E/(RT)} - hS(T - T_a), \quad (10.15)$$

where ρ is the reacting substance density (moles per unit volume), Q is a measure of heat gain associated with the reaction (Joules per mole) and c is the specific heat of the reacting substance. V and S are the volume and surface area of the material, respectively, and T_a is the ambient temperature (measured in degrees Kelvin from absolute zero). This model is known as the *Seminov* model of spontaneous combustion; see Jones (1993).

Numerical solution

In equation (10.15) there are numerous parameters, which make the analysis cumbersome. We therefore simplify by scaling the temperature. We define a dimensionless temperature θ (since E/R has the dimension of temperature)

$$\theta = \frac{T}{E/R}. \quad (10.16)$$

Then, with this change of variable, the system can be rewritten as

$$\sigma \frac{d\theta}{dt} = \lambda e^{-1/\theta} - (\theta - \theta_a) \quad (10.17)$$

with

$$\theta_a = \frac{T_a}{E/R}, \quad \lambda = \frac{\rho AVQ}{hS(E/R)}, \quad \sigma = \frac{\rho Vc}{hS}. \quad (10.18)$$

The constant θ_a denotes the dimensionless ambient temperature. The parameter λ is known as the reaction efficiency parameter, since high values of λ correspond to high levels of heat generation from the chemical reaction, compared with the heat loss through Newton cooling. Both λ and θ_a are dimensionless parameters. Parameter σ is a measure of the speed of the reaction. Equation (10.17) contains only three parameters² θ_a , λ , and σ .

Using MATLAB or Maple we can solve the differential equation (10.17) numerically, for different values of the reaction efficiency parameter λ . The results are illustrated in Figure 10.7. The MATLAB code used to obtain the graph is given in Listing 10.1. In each case the solution tends to a steady-state; however, as we vary λ from $\lambda = 2.84$ through to $\lambda = 2.85$ the amount by which the steady-state jumps increases dramatically.

²We could reduce this to only two parameters by writing $t = \sigma\tau$ and using the chain rule $\sigma d\theta/dt$ becomes $d\theta/d\tau$. The equation would then be completely dimensionless.

Listing 10.1: MATLAB code: c_ht_sponcombust.m

```

function c_ht_sponcomp
global lambda sigma theta_a;

sigma = 1.0;
theta_a = 0.2;
tend = 300;
y0 = theta_a;
lambda = 2.84;
[tsol, ysol] = ode45(@rhs, [0 tend] , y0);
plot(tsol, ysol, '-.');
hold on;
lambda = 2.85;
[tsol, ysol] = ode45(@rhs, [0 tend] , y0);
plot(tsol, ysol, ':');
lambda = 2.86;
[tsol, ysol] = ode45(@rhs, [0 tend] , y0);
plot(tsol, ysol, '-');
axis([0, tend, 0, 2])

function thetadot = rhs(t, theta)
global lambda sigma theta_a;
thetadot = lambda/sigma*exp(-1/theta) - 1/sigma*(theta - theta_a);

```

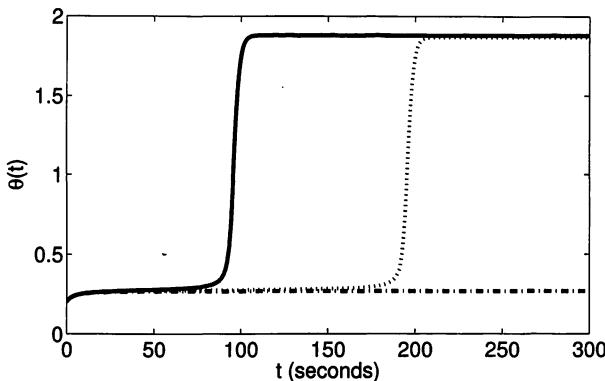


Figure 10.7: Numerical results for dimensionless temperature $\theta(t)$ against t , on solving equation (10.17). The parameter λ is gradually increased: $\lambda = 2.84$ (dot-dashed line), $\lambda = 2.85$ (solid line) and $\lambda = 2.86$ (dotted line).

With reference to equations (10.18), increasing λ can be considered as increasing the volume to surface area ratio. The system finds it more difficult to release heat to the surroundings, and so the reaction increases in intensity. The sudden jump in equilibrium temperature, apparent in the right hand diagram of Figure 10.7, corresponds to the spontaneous ignition of the material.

Analytic solution

Ideally, we would like to determine exactly which values of λ correspond to the sudden jump in equilibrium temperature. To find this value, we set $d\theta/dt = 0$, for equilibrium, and solve

the resulting equation graphically. Thus for equilibrium,

$$\lambda e^{-1/\theta} = \theta - \theta_a, \quad \lambda = \frac{\rho V A Q}{h S(E/R)} = \frac{\rho R V A Q}{h S E}. \quad (10.19)$$

Figure 10.8 illustrates the two curves for heat loss and heat gain given by

$$y = e^{-1/\theta}, \quad y = \frac{1}{\lambda}(\theta - \theta_a),$$

on the same system of axes. Clearly there can be at most three points of intersection where the heat loss equals the heat gain, that is $(d\theta/dt) = 0$ or $(dT/dt) = 0$. From (10.17) if the heat loss is greater than the heat gain, then $d\theta/dt < 0$, and if the heat loss is less than the heat gain then $d\theta/dt > 0$.

With reference to the solid straight line for heat loss in the figure (Figure 10.8), if the temperature is low so that $\theta < \theta_1$, then the heat gain is greater than the heat loss and the temperature increases with $\theta \rightarrow \theta_1$. If the temperature is such that $\theta_1 < \theta < \theta_2$, then the heat loss is greater than the heat gain and the temperature will decrease so that $\theta \rightarrow \theta_1$. Thus we consider the intersection at θ_1 as a stable equilibrium for the system. Arguing in a similar manner, we see that at θ_2 there is an unstable equilibrium, and at θ_3 another stable point.

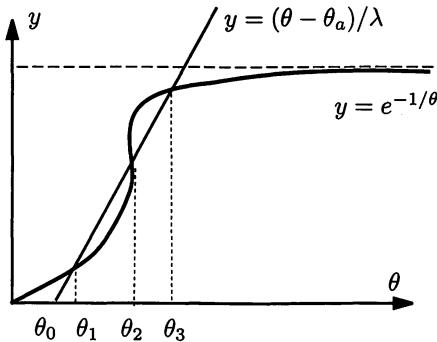


Figure 10.8: Sketch of the graphical solution of equation (10.17). There are, at most, 3 equilibrium solutions (θ_1 , θ_2 and θ_3), although for some values of λ there may be only one solution.

If the initial condition is such that the system approaches the equilibrium at θ_3 , we have the conditions for spontaneous ignition. However, if the initial conditions are different, ignition can be avoided. If we start with an initial temperature such that $\theta(0) = \theta_a$, then the temperature tends to the closest equilibrium. However, if λ is sufficiently large (i.e. the slope of the line, $1/\lambda$, is sufficiently small), for only one equilibrium point (corresponding to θ_3) to exist, then ignition will occur.

As we increase λ , which decreases the slope of the line $1/\lambda(\theta - \theta_a)$, we reach critical values where the number of equilibrium values change: from one through two to three, or from three through two to one. The values of λ for which this change occurs are called bifurcation points.

Example 10.6: Given $\theta_a = 0.2$ find the critical value of the parameter λ where ignition occurs.

Solution: The critical value occurs where the slope of the line in Figure 10.8 is tangential to the curve (which may occur in two places). Hence we solve the simultaneous equations

$$e^{-1/\theta_c} = \frac{1}{\lambda_c}(\theta_c - 0.2), \quad \frac{1}{\theta_c^2} e^{-1/\theta_c} = 1/\lambda_c,$$

where the first equation describes the point(s) of intersection, and the second equates the first derivatives of each, corresponding to the line and curve having the same slope. With a little algebra, we can deduce

$$\lambda_c = \theta_c^2 e^{1/\theta_c}, \quad \theta_c^2 - \theta_c + 0.2 = 0.$$

Solving the quadratic equation gives two roots

$$\begin{aligned}\theta_{c1} &= \frac{1}{2}(1 - \sqrt{1 - 4 \times 0.2}) \simeq 0.277, \\ \theta_{c2} &= \frac{1}{2}(1 + \sqrt{1 - 4 \times 0.2}) \simeq 0.724.\end{aligned}$$

After substituting these back into the expression for λ_c we get critical values of λ corresponding to a bifurcation of the equilibrium temperatures,

$$\lambda_c \simeq 2.847, \quad \lambda_c \simeq 2.085.$$

As we have seen above, it is the larger value $\lambda_c = 2.847$ which corresponds to spontaneous ignition. This value is consistent with the results of Figure 10.7.

Alternatively, we can hold λ fixed and think of varying θ_a . This corresponds to the slope of the line in Figure 10.8 remaining constant, but the intersection point it makes with the θ -axis moving to the right as θ_a increases. Again, we move from one equilibrium temperature (θ_1) to three equilibrium temperatures ($\theta_1, \theta_2, \theta_3$) and back to just one equilibrium temperature (θ_3), for the larger values of θ_a . It is the second bifurcation here that corresponds to thermal ignition, since, owing to the stability of the equilibrium points, the temperature to which the system evolves jumps suddenly from θ_1 to θ_3 . The value of θ_a where this occurs is the critical ambient temperature, θ_{ac} . (See Exercises, Question 10 for a numerical example of determining the critical ambient temperature.)

Further interpretation

Clearly the critical ambient temperature is crucial to any strategy for avoiding spontaneous combustion. The analysis above indicates that both the storage conditions and temperature of the material, as well as the ambient temperature, need to be carefully monitored in processes where spontaneous ignition may occur.

Spontaneous combustion can be avoided completely if the ambient temperature is sufficiently low so that there is only one stable equilibrium point on the lower branch of the heat gain curve. This occurs for low temperatures, below the ambient temperature θ_{ac} . However, such temperatures are not always practicable in the ‘real world’ and the regular monitoring of all conditions may be required.

Limitations and extensions

There are some important factors of which our model has not taken account.

- As the reaction proceeds fuel is used up. Here we have neglected the consumption of fuel.

- We have assumed the reactants are mixed (a stirred chemical reactor). In practice, perfect mixing will not occur, and the temperature is different at different points throughout the reacting material, which allows heat conduction to occur. This factor is especially important in solid materials.

Both these assumptions can be relaxed, but this leads to more complicated models. If we relax the first assumption, the model will comprise two simultaneous ODEs, one for the temperature and one for the concentration. Equilibrium values can be found by solving the two appropriate algebraic equations simultaneously, and bifurcations can be established as a parameter is varied. This is discussed briefly in Fulford and Broadbridge (2000).

If we relax the second assumption (no conduction), then our model results in a PDE for the temperature (see Section 12.1 for how partial derivatives arise). For further details of how to deal with this, see Fulford and Broadbridge (2000); Jones (1993).

In Section 10.5, we discuss a case study involving the spontaneous ignition of fish and chip crumble. Another practical example involving spontaneous ignition of moist milk powder is given in Rivers et al. (1996). In Lignola and Maio (1990), inclusion of reactant consumption in a stirred reactor shows that some interesting oscillatory behaviour can occur in these processes.

Summary of skills developed here:

- Use Maple or MATLAB to find a numerical solution for the temperature in a stirred chemical reactor.
- For a given dimensionless ambient temperature find the critical value of the reaction efficiency.
- For a given reaction efficiency parameter find the critical ambient temperature such that spontaneous ignition will occur.

10.5 Case Study: Fish and chips explode

There have been many recorded cases of spontaneous combustion, for example, the sudden combustion of a barn filled with hay, the spontaneous ignition of wool bales in transit or storage, and a case of spontaneous ignition in a fish and chips shop in Napier (New Zealand), which is the subject of the case study which follows. Such events are of great interest and importance to insurance companies, which are called upon to pay for the damage. The information for the following case derives from such a report, written for a New Zealand insurance company. This discussion is adapted from a report by Smedley and Wake (1987).

Fish and chips shops are not usually associated with explosives; however, in 1986 in Napier, New Zealand, one such shop was severely damaged when a drum of waste from the cooking scraps ignited spontaneously.

Fish and chips are cooked in large sieves submerged in vats of animal fat which are heated to about 180 °C. The hot crumble, or waste, which detaches during cooking, is allowed to cool and drain above the vat before being tipped into drums. The concern was that over a period of time the pile of crumble would (and the case in New Zealand proved that it could) spontaneously ignite through some heat producing mechanism within the waste.

The authors approached the problem from the point of view of constructing a mathematical model applicable to general problems of this nature, but which would be sufficiently simple for non-mathematicians to apply *in situ*. That is, they needed to provide adequate conditions to test for on site in order to prevent such 'explosive' events.

Spontaneous combustion is possible if the temperature inside the pile of crumble increases at a faster rate than heat is able to escape from the surface, where the crumble meets the air and drum wall. That is, if the exothermicity and the reaction rate are sufficiently great, such that the heat cannot dissipate, but instead builds up to a point of ignition. These processes need to be examined closely.

The exothermic reaction, which is the reaction producing the heat, is that of the organic material consisting of the fish and chip residue reacting with the atmosphere or oxygen. Since spontaneous combustion has occurred, this indicates that the exothermicity of this reaction is sufficient to cause ignition. In the case of oxidation for fish and chip residue, the activation energy was measured to be $E/R = 30\ 000$. (In comparison, for wool $E/R = 9\ 036$, or for cotton $E/R = 18\ 000$.) As mentioned above, heat is lost through the interface between the crumble and the air as well as through the crumble-drum wall interface. Thus a change in this area will affect this heat loss, and there is the possibility of changing the dimensions of the drums in order to exercise a degree of control with respect to the heat loss. For the fish and chips shop in question, the drums used were cylinders with a diameter and height of 38 cm.

In order to avoid spontaneous combustion, which is the aim, consider the relationship between these heat loss and heat gain processes, illustrated in Figure 10.8 of Section 10.4. The diagram illustrates the existence of a critical temperature (T_c) above which spontaneous combustion will always occur, and also a critical initial temperature (T_{0c}), dependent on the ambient temperature, below which combustion can be avoided. In order to predict the dynamics of such a system, particularly to avoid the event of spontaneous combustion, both the storage conditions and temperatures, as well as the ambient temperatures, need to be monitored closely in the waste disposal process.

We return to the formulation of strategies to adopt *in situ* which will avoid a repeated disaster in the fish and chips shop and keep the insurance company happy. The authors used a modified version of the model in Section 10.4. It was based on a heat balance, but was more complex, including details of the combustion process, and an approximation of the Arrhenius term (see Jones (1993)). Using the predictions of this more sophisticated model they were able to provide a simple look-up table of values from which the proprietor could read the critical ambient temperature and the associated critical initial temperature. Such figures could be derived for a variety of convenient drum sizes.

In practice the safety procedure is simple. A quick reference to the look-up table, and some intermittent monitoring of the temperatures will provide the required conditions for an uneventful evening of business down at the fish and chips shop.

10.6 Exercises for Chapter 10

10.1. Coffee cooling. Imagine you have a cup of coffee cooling according to the natural law of cooling, so that the temperature $U(t)$ satisfies

$$\frac{dU}{dt} = -\lambda(U - u_s)^{5/4},$$

where λ is a positive constant and $u_s = 20^\circ\text{C}$ is the temperature of the surroundings.

- (a) Assuming $U(0) = u_0$, separate the variables to obtain

$$\lambda t = 4 \left[(U - u_s)^{-1/4} - (u_0 - u_s)^{-1/4} \right].$$

- (b) If the cup of coffee cools from an initial temperature $u_0 = 60^\circ\text{C}$ to 50°C in 10.3 minutes, obtain λ and hence find the time it takes for the cup of coffee to cool to 40°C .
- (c) Using Maple or MATLAB compare the solution of this natural cooling model with the solution for Newton's law of cooling by plotting them on the same system of axes, as in Figure 10.1. (Assume that with Newton's law of cooling the coffee cools from $u_0 = 60^\circ\text{C}$ to 50°C in 10 minutes.)

10.2. Beer warming. A cold beer is at a temperature of 10°C . After 10 minutes the beer has warmed to a temperature of 15°C . If the room temperature is 30°C , how long will it take the beer to warm to 20°C , assuming that Newton's law of cooling applies?

10.3. Separating variables. Solve the equation developed to describe the hot water heater, (equation (10.7) from Section 10.2) using the separable technique, and show the solution to be equivalent to that provided in the text.

10.4. The hot water heater. Consider the problem of a water heater with parameters and model as in Section 10.2:

$$\frac{dU}{dt} = \beta - \alpha U, \quad U(0) = u_0, \quad (10.20)$$

where the constants α and β are defined as

$$\alpha = \frac{hS}{cm}, \quad \beta = \frac{q + hSu_s}{cm}.$$

(Parameters $q = 3600 \text{ W}$, $S = 2.7 \text{ m}^2$ and $u_s = 15^\circ\text{C}$ with constants $h = 12 \text{ W m}^{-2} \text{ }^\circ\text{C}^{-1}$ and $c = 4200 \text{ J kg}^{-1} \text{ }^\circ\text{C}^{-1}$.)

- (a) Suppose the temperature of the water in the hot water tank is 60°C , the heater is turned off so that no heat is supplied and the water is allowed to cool. What will the temperature of the water be after 8 hours?
- (b) With heat once again supplied ($q = 3600 \text{ W}$), how long would it take to heat the water back up to 60°C , after it has cooled for 8 hours in (a)?
- (c) Suppose again the water is 60°C , that the heater is turned off so that no heat is supplied and the water is allowed to cool. How long will it take for the water to reach 50°C ?
- (d) With the heater turned on again, how long will it take to heat the tank of water from 50°C to 60°C ?
- (e) What is of paramount interest is how to run the most cost efficient system and clearly the cost of heating the tank is directly related to the time for which the heater is on. Assuming all constants as above, would it be cheaper to use a thermostat which switches the heater on each time the temperature drops below 50°C and off each time the temperature reaches 60°C , or to switch the heater off for 8 hours each day relying on the thermostat at other times? Use the model and the results found above to substantiate your answer. Although actual costs are not needed to calculate the most economical solution, typical values in Australia are 8.22 c per kW hr.
- (f) Would your choice of the most economical system change if the temperature of the environment were to drop to 10°C ?

10.5. Insulating the hot water heater. Assume the model and parameters of Question 4. Suppose with good insulation no heat is lost to the environment. Adapt the model to accommodate this change and then, with a surrounding temperature of 15°C , and the water in the tank at this temperature, establish how long it would take to heat the tank to 60°C .

10.6. Horse temperatures. A vet is taking the temperature of a sick horse. Initially, the temperature of the thermometer is 27.8°C . Three minutes after insertion the reading is 32.2°C and three minutes later it is 34.4°C . The horse then has a violent convulsion which destroys the thermometer completely so that no final reading can be taken. You may assume that the rate of change of temperature of the thermometer is directly proportional to the difference between the temperature of the thermometer and that of the horse.

- (a) Let $U(t)$ be temperature at time t , with $U(0) = u_0$ the initial temperature of the thermometer and u_h the temperature of the horse. Model the system with a differential equation and show that its solution is

$$U(t) = u_h + (u_0 - u_h)e^{\lambda t},$$

where λ is the constant of proportionality.

- (b) Since $u_0 - u_h$ is a positive constant, show that

$$e^{3\lambda} = \frac{U(3) - u_h}{u_0 - u_h} = \frac{U(6) - u_h}{U(3) - u_h}.$$

- (c) Now find the temperature of the horse and also the value of the constant λ .

10.7. Modelling house temperatures. Develop a model for the temperature inside a building comprising two rooms with a flat ceiling (no roof cavity), which is described below.

One room has a heater which supplies heat at a constant rate q_0 . You may assume that heat is lost to the surroundings, from both rooms, according to Newton's law of cooling and that the heat transfer from one room to the other is proportional to the temperature difference between the two rooms. You may also assume the air inside the building is well mixed so that heat conduction can be ignored. Any radiation can also be ignored.

Give either a compartmental diagram or suitable word equations to help explain your formulation and define any symbols introduced. You are not required to solve any equations.

10.8. Stirred chemical reactor. For a stirred chemical reactor the heat balance equation may be given in terms of the scaled temperature (θ) as

$$\sigma \frac{d\theta}{dt} = \lambda e^{-1/\theta} - (\theta - \theta_a),$$

where θ_a , λ and σ are positive constants.

- (a) Refer to Figure 10.8 which illustrates the three equilibrium solutions. By interpreting each of the two curves as a heat gain or heat loss term in the differential equation determine the stability of each of the equilibrium solutions $\theta^{(1)}$, $\theta^{(2)}$, $\theta^{(3)}$.
- (b) Discuss what happens to $\theta(t)$ as $t \rightarrow \infty$ when θ_0 is small and when θ_0 large.

10.9. Spontaneous ignition and critical values. Consider the equation for the scaled absolute temperature in an exothermic oxidation reaction

$$\sigma \frac{d\theta}{dt} = \lambda e^{-1/\theta} - (\theta - \theta_a).$$

- (a) For $\theta_a = 0.3$, $\lambda = 2$, by plotting the curves, determine all the equilibrium solutions.
- (b) Repeat for $\theta_a = 0.1$ and $\lambda = 5$.

10.10. Critical ambient temperature. Consider the equation for the scaled absolute temperature in an exothermic oxidation reaction, with $\lambda = 2$,

$$\sigma \frac{d\theta}{dt} = 2e^{-1/\theta} - (\theta - \theta_a).$$

- (a) Deduce that the critical ambient temperature θ_{ac} where bifurcations occur satisfies the equations

$$\theta_{ac} = \theta_c - \theta_c^2, \quad \theta_c^2 = 2e^{-1/\theta_c},$$

where θ_c is the critical value of θ for which bifurcations occur.

- (b) By solving the second equation graphically, find the two values of θ_c and hence determine the approximate critical ambient temperature θ_{ac} above which spontaneous ignition occurs.

1920-1921 - 1921-1922 - 1922-1923 - 1923-1924

1924-1925 - 1925-1926 - 1926-1927 - 1927-1928

1928-1929 - 1929-1930 - 1930-1931 - 1931-1932

1932-1933 - 1933-1934 - 1934-1935 - 1935-1936

1936-1937 - 1937-1938 - 1938-1939 - 1939-1940

1940-1941 - 1941-1942 - 1942-1943 - 1943-1944

1944-1945 - 1945-1946 - 1946-1947 - 1947-1948

1948-1949 - 1949-1950 - 1950-1951 - 1951-1952

1952-1953 - 1953-1954 - 1954-1955 - 1955-1956

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2040-2041 - 2041-2042 - 2042-2043 - 2043-2044

2044-2045 - 2045-2046 - 2046-2047 - 2047-2048

2048-2049 - 2049-2050 - 2050-2051 - 2051-2052

2052-2053 - 2053-2054 - 2054-2055 - 2055-2056

2056-2057 - 2057-2058 - 2058-2059 - 2059-2060

2060-2061 - 2061-2062 - 2062-2063 - 2063-2064

2064-2065 - 2065-2066 - 2066-2067 - 2067-2068

2068-2069 - 2069-2070 - 2070-2071 - 2071-2072

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2076-2077 - 2077-2078 - 2078-2079 - 2079-2080

2080-2081 - 2081-2082 - 2082-2083 - 2083-2084

2084-2085 - 2085-2086 - 2086-2087 - 2087-2088

2088-2089 - 2089-2090 - 2090-2091 - 2091-2092

2092-2093 - 2093-2094 - 2094-2095 - 2095-2096

2096-2097 - 2097-2098 - 2098-2099 - 2099-20100

Chapter 11

Solving heat conduction problems

In the previous chapter we formulated differential equations for some heat conduction problems. We now see how to solve these differential equations, and apply suitable boundary conditions to obtain the temperature distribution.

11.1 Boundary value problems

In the previous chapter we discussed the formulation of differential equations. To solve these, we first need to solve the equations, and then evaluate the arbitrary constants using some known conditions. The problems we look at involve specifying conditions on the boundaries of a region. This is, technically, slightly different from applying initial conditions, although in practice the process is similar. We concentrate here on how to apply boundary conditions to the general solution of a differential equation. Such problems are known as boundary value problems.

Typical problems

The first problem we consider is the overall rate of heat loss through the wall of a building. We solve the appropriate differential equation to find the equilibrium temperature at any point inside the wall and then use Fourier's law to calculate the heat flux through the wall. The second problem analyses the rate of heat lost through an insulating jacket surrounding a water pipe. This problem is similar to the previous one, but involves radial heat flow rather than heat flow in one direction. In our final problem, we examine the cooling of a computer chip by mounted heat fins. A problem similar to this is the cooling of a motorcycle engine, also using heat fins.

Each of these problems involve first solving a differential equation for the equilibrium temperature. After finding the general solution, we then apply suitable boundary conditions to find the arbitrary constants, a process which requires the solution of simultaneous equations.

Boundary conditions

When we solve a second-order differential equation we obtain a solution with two arbitrary constants. In general, the number of arbitrary constants is the same as the order of the differential equation. (General principles used in solving second-order equations can be found in Appendix A.2 and Appendix A.5.)

For example, consider the differential equation

$$\frac{d^2U}{dx^2} - 4U = 0.$$

Since the differential equation is linear and its coefficients are constant we look for a solution of the form $U(x) = e^{\lambda x}$. Substitution of this into the differential equation gives

$$\lambda^2 e^{\lambda x} - 4e^{\lambda x} = 0$$

and hence we obtain $\lambda^2 - 4 = 0$. Thus we find two possible values for λ , $\lambda = 2$ or $\lambda = -2$. This yields two solutions e^{2x} and e^{-2x} .

The general solution of the differential equation is obtained by taking a linear combination of these two solutions

$$U(x) = C_1 e^{2x} + C_2 e^{-2x}. \quad (11.1)$$

This solution has two arbitrary constants C_1 and C_2 whose values are determined by applying boundary conditions.

Boundary conditions are conditions applied on either boundary of a region. Suppose the region is $0 \leq x \leq 1$. One pair of boundary conditions might be, for example,

$$U(0) = 0, \quad U(1) = 3.$$

Another example is

$$U(0) = 1, \quad U'(1) = 0.$$

This second example involves the derivative of U as one of the boundary conditions.

In solving differential equations where the independent variable is time, we have so far encountered initial conditions. The difference between initial and boundary conditions is the value of the independent variable they are applied to. Initial conditions are always applied at the same value (usually $t = 0$). Boundary conditions, however, are applied at two different values (e.g. $x = 0$ and $x = 1$).

Applying boundary conditions

The process of applying boundary conditions is exactly the same as applying initial conditions. Applying the conditions leads to, in general, two simultaneous equations for the two arbitrary constants.

As an example, consider the following differential equation with boundary conditions,

$$\frac{d^2U}{dx^2} - 4U = 0, \quad U(0) = 0, \quad U(1) = 3.$$

The solution to this differential equation is given by (11.1). Substituting $x = 0$ and then $x = 1$ into this solution, and then applying the boundary conditions, produces the pair of simultaneous equations

$$C_1 + C_2 = 0, \quad C_1 e^2 + C_2 e^{-2} = 3.$$

Solving simultaneously for C_1 and C_2 gives $C_1 = 3/(e^2 - e^{-2})$ and $C_2 = -3/(e^2 - e^{-2})$. We now substitute these values for C_1 and C_2 back into the general solution (11.1). Hence the solution U , which satisfies both boundary conditions, is

$$U(x) = 3 \frac{e^{2x} - e^{-2x}}{e^2 - e^{-2}}$$

for x between 0 and 1.

This can also be expressed in terms of the hyperbolic function $\sinh(2x) = (e^{2x} - e^{-2x})/2$ as

$$U(x) = 3 \frac{\sinh(2x)}{\sinh(2)}.$$

(For details on hyperbolic functions see Appendix B.4.)

In the following sections we consider some practical problems. These involve solving differential equations derived in the previous chapter, and then applying suitable boundary conditions.

Summary of skills developed here:

- Solve simultaneous equations to find the constants of integration.
- Differentiate between, and apply, initial conditions and boundary conditions.
- Understand the principles involved in solving simple second-order differential equations, with boundary conditions.

11.2 Heat loss through a wall

We solve a differential equation and apply boundary conditions to find the equilibrium temperature inside a brick wall. The boundary condition on the outside of the wall follows from Newton's law of cooling, while on the inside the wall is kept at a constant temperature.

Problem description

The design of energy efficient buildings has become more important as energy costs have risen, and there is a greater awareness of the responsible use of resources. As was seen in the earlier case study (Section 10.3), which examined the temperatures in a building, a crucial concern in such design is that of calculating the rate of heat loss from a building wall. We take a closer look at this heat loss process.

Consider a building with walls of thickness $\ell = 20$ cm. The inside of the wall is maintained at a comfortable temperature $u_i = 20^\circ\text{C}$. The temperature is maintained by a thermostat on the wall. The outside of the wall is exposed to air at a temperature $u_o = 5^\circ\text{C}$; however, the outside of the wall is not at 5°C , but loses heat to the atmosphere (see Figure 11.1). (Subscripts i and o denote variables on the inside and outside of the wall, respectively.)

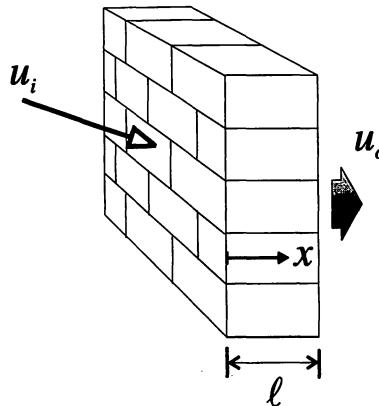


Figure 11.1: Heat flow through the wall of a house, $0 \leq x \leq \ell$.

We are interested in determining the rate of heat flow through the wall. One possible question we might ask is:

If the thickness of the wall is doubled what happens to the rate of flow of heat through the wall?

Model assumptions and approach

We let the x -axis measure distance from the inside of the wall, outwards. Closer to the inside the temperature will be closer to 20°C . Hence we may think of the temperature as a function of two variables x and t . For the equilibrium temperature, however, the temperature will be a function of x only. We make some assumptions and then build the model on these.

- We assume heat flows directly from the inside to the outside through the wall. Thus

we neglect any complicated heat flow at the edges, top and bottom. For most of the wall the heat flow will be in the x -direction.

- We also assume the wall to have no windows.
- Since the inside of the wall has a thermostat, once thermal equilibrium is established the temperature will be constant. We assume it stays constant at $u_i = 20^\circ\text{C}$.
- For thermal equilibrium the temperature on the outside of the wall will also be constant but it will not be the same as the temperature of the surroundings, $u_o = 5^\circ\text{C}$, or of the inside $u_i = 20^\circ\text{C}$. It will be somewhere between these temperatures. We assume the outside of the wall satisfies Newton's law of cooling. The temperature of the outside can be determined from the solution for the temperature inside the wall.

The differential equation

In the previous chapter we formulated a differential equation for the equilibrium temperature of some material through which heat was conducted in the x -direction. The differential equation (from Section 9.5, equation (9.18)) is

$$\frac{d^2U}{dx^2} = 0. \quad (11.2)$$

This is a second-order differential equation so we expect two arbitrary constants in the solution.

The equation arose from a heat balance in thermal equilibrium. Note that the differential equation does not depend on the conductivity of the material. The conductivity will be incorporated through the boundary conditions.

Analytic solution

It is a simple matter to solve this differential equation. We can integrate both sides twice. By integrating with respect to x once we obtain

$$\frac{dU}{dx} = C_1, \quad (11.3)$$

where C_1 is an arbitrary constant and then integrating again gives

$$U(x) = C_1x + C_2. \quad (11.4)$$

Here C_1 and C_2 are the two arbitrary constants. In order to solve for these constants we need two boundary conditions.

The boundary conditions

We require information at each boundary, $x = 0$ and $x = \ell$.

In modelling this problem we assumed that the temperature of the room is regulated by a thermostat so that the temperature of the inside of the wall is always at a given temperature u_i . Thus the boundary condition at $x = 0$ may be written as

$$U(0) = u_i. \quad (11.5)$$

We also assumed that the outside of the wall loses heat to the surroundings according to Newton's law of cooling, which states that the rate of heat loss is proportional to the surface area of the wall face and the temperature difference between the wall and the surroundings. We can equate this to the rate of heat conducted through the wall at $x = \ell$, given in terms

of the heat flux as $J(\ell)A$, where A is the area through which the heat passes. Then, dividing by the surface area,

$$J(\ell) = h(U(\ell) - u_o), \quad (11.6)$$

where h is the convective heat transfer coefficient.

Since the wall is hotter than the surroundings and the x -axis points towards the surroundings the sign is correct giving a positive heat flux for $U(\ell) > u_o$.

We now apply both boundary conditions (11.5) and (11.6) to determine the arbitrary constants C_1 and C_2 . This gives us two equations for the two unknowns C_1 and C_2 .

Example 11.1: Apply the boundary conditions (11.5) and (11.6) to determine the arbitrary constants C_1 and C_2 of equation (11.4). Hence obtain an expression for the temperature profile in the wall.

Solution: The boundary condition $U(0) = u_i$ states that $U = u_i$ at $x = 0$. To apply this we substitute $x = 0$ into the general solution (11.4) and obtain the equation

$$C_1 \times 0 + C_2 = u_i. \quad (11.7)$$

From this we can solve directly for C_2 giving

$$C_2 = u_i. \quad (11.8)$$

To apply the Newton cooling boundary condition (11.6) we first calculate $J(x)$ using Fourier's law $J = -k(dU/dx)$, where k is the conductivity. Using equation (11.4), we obtain

$$J(x) = -k \frac{dU(x)}{dx} = -kC_1.$$

At $x = \ell$ equation (11.6) becomes

$$-kC_1 = h(C_1\ell + C_2 - u_o). \quad (11.9)$$

Substituting (11.8) into (11.9) and dividing through by h we obtain

$$-\frac{k}{h}C_1 = C_1\ell + u_i - u_o.$$

Rearranging gives

$$-\left(\ell + \frac{k}{h}\right)C_1 = u_i - u_o,$$

which implies that

$$C_1 = \frac{-(u_i - u_o)}{\ell + \frac{k}{h}}. \quad (11.10)$$

Substituting the above expressions for the arbitrary constants (11.8) and (11.10) back into the general solution (11.4) gives the solution for the equilibrium temperature

$$U(x) = u_i - \left(\frac{u_i - u_o}{\ell + \frac{k}{h}}\right)x. \quad (11.11)$$

This is a simple linear function which is illustrated in Figure 11.2. The temperature $U(x)$ decreases with x . But note that the temperature of the outside of the wall is at a higher temperature to that of the surroundings, u_o .

As a quick check on the solution, note that as $h \rightarrow 0$, $U(\ell) \rightarrow u_i$. Physically, $h \rightarrow 0$ corresponds to not allowing any heat to escape so that the temperature on the outside would be the same as the temperature on the inside. Alternatively, consider the limit as $h \rightarrow \infty$. In this case $U(\ell) \rightarrow u_o$. This corresponds physically to the situation where the temperature of the outside wall is the same as the temperature of the outside surroundings.

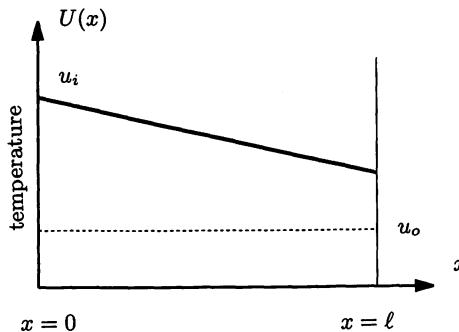


Figure 11.2: Equilibrium temperature in a wall where the inside of the wall is at temperature u_i and on the outside wall heat is lost to the surroundings, which are at temperature u_o .

Rate of heat loss from a wall

The rate of heat loss from the wall can be calculated by using Fourier's law. This gives the heat flux (the rate of flow of heat per unit time per unit area). By Fourier's law the heat flux at $x = l$, $J(l)$, can be evaluated from

$$J(x) = -k \frac{dU(x)}{dx}.$$

Example 11.2: Find an expression for the heat flux through the wall.

Solution: Using the solution for the temperature (11.11) we obtain

$$J(l) = -k \times -\frac{u_i - u_o}{\ell + \frac{k}{h}} = \frac{u_i - u_o}{\frac{\ell}{k} + \frac{1}{h}}. \quad (11.12)$$

As a quick check we see that this gives a positive value since $u_i > u_o$.

Let us carry out some calculations with typical numerical values. Suppose the inside temperature is $u_i = 20^\circ\text{C}$, whereas the outside temperature is 5°C . Consider a brick wall of thickness $\ell = 15\text{ cm} = 0.15\text{ m}$. The conductivity of brick is $k = 0.5\text{ SI units}$, and the convective heat transfer coefficient $h = 10\text{ SI units}$. With these values we calculate $\ell/k = 0.3$ and $1/h = 0.1$, so the the heat flux through the wall is

$$J = \frac{15}{0.3 + 0.1} = 37.5 \text{ W/m}^2. \quad (11.13)$$

Thus we can obtain an estimate for the total rate of heat loss from the walls of a brick house by multiplying this value by the total area of the walls.

Note that J is directly proportional to the temperature difference but not directly proportional to the conductivity k or to the thickness of the wall ℓ . Thus doubling the wall thickness will not halve the heat loss from the wall, and nor would doubling the conductivity (except in cases where $1/h$ is much smaller than ℓ/k). However, for decreased k and h there is a reduction in the heat flux J .

Heat resistance R-values

There is a neat interpretation to equation (11.12), which we can write as

$$J = \frac{u_i - u_o}{R}, \quad \text{where } R = \frac{\ell}{k} + \frac{1}{h}. \quad (11.14)$$

The parameter R is interpreted as a lumped thermal resistance. We can think of R as made up of two separate resistances:

- $R_1 = \ell/k$, the resistance to heat flow through the material, and
- $R_2 = 1/h$, the surface resistance to heat flow to the air.

Note that the resistance to heat flow through the material, ℓ/k , increases with the width of the material ℓ and decreases with the conductivity k , as we might expect. Companies which provide insulation batts often talk about the R -value of their batts. For example, R2.0 batts have a thermal resistance of $\ell/k = 2.0 \text{ W}^\circ\text{C}^{-1}$.

Using this notion of thermal resistances we can calculate the rate of heat loss for more complicated structures, such as an insulated wall of a house. Here, the total resistance is

$$R = \frac{\ell_1}{k_1} + \frac{\ell}{k} + \frac{1}{h},$$

where k_1 and ℓ_1 are the conductivity and width of the insulation batts, respectively, and k and ℓ are those values corresponding to brick. With R2.0 batts ($\ell_1/k_1 = 2.0$), a brick wall ($\ell/k = 0.3$) and $h = 10$ units this gives a total heat flux of

$$J = \frac{u_i - u_o}{R} = \frac{15}{2.0 + 0.3 + 0.1} = 6.25 \text{ W/m}^2.$$

This is much lower than the previously calculated flux through a single brick wall.

For a pane of glass, the total thermal resistance is made up of the resistance to heat passing through the glass ℓ/k together with the surface resistances for both the inside and outside surfaces (since there would not be a thermostat on a pane of glass). This gives

$$J = \frac{(u_i - u_o)}{\frac{1}{h} + \frac{\ell}{k} + \frac{1}{h}}.$$

For a 5-mm thickness of glass, $\ell = 5 \times 10^{-3} \text{ m}$. The conductivity of glass is $k = 0.8 \text{ SI units}$ and we take $h = 10 \text{ SI units}$. Substituting for these data values gives $\ell/k = 0.0062$ and $1/h = 0.1$. We obtain a value for the heat flux through the glass pane as

$$J = \frac{15}{0.1 + 0.0062 + 0.1} \simeq 73 \text{ W/m}^2.$$

For a window, the ratio ℓ/k measuring the resistance to heat flow through the material ($0.1/0.0062 \simeq 16$) is much greater than it is for a brick wall ($0.1/0.3 \simeq 0.33$). This indicates why heat loss through windows is generally more significant than through walls, especially when the walls are insulated. Note that if we had not included the two surface resistance terms (the two $1/h$ terms), then we would have calculated the heat flux erroneously, as $15/0.0062 \simeq 2.5 \times 10^3 \text{ W/m}^2$. An error of order 10^2 !

Summary of skills developed here:

- From a word description, write in mathematical form the boundary conditions corresponding to
 - prescribed temperature on a boundary,
 - prescribed heat flux on a boundary, and
 - Newton heat exchange with surroundings, on a boundary.
- Be able to apply the above boundary conditions to find the temperature for heat flow in one direction.
- Obtain the heat flux from the temperature.
- Be able to interpret the heat flux in terms of thermal resistances.

11.3 Case Study: Double glazing: What's it worth?

The following case study is adapted from Mesterton-Gibbons (1989) and information brochures on double glazing from manufacturing companies.

Is double glazing what it's cracked up to be? We saw in Section 11.2 that there is a simple relation between the heat flux and the temperature difference for heat flow through a slab of any material. It is

$$J = \frac{u_1 - u_2}{R},$$

where R is the heat resistance (commonly called the R -value). We consider here a case study which makes use of this result in an examination of the heat transfer process in windows. In particular we are interested in the extent to which double glazing reduces this heat loss and improves efficiency.

Advertising promotes double glazing as a very effective way of preventing heat loss from your home; however, how effective is it? The windows are expensive to buy and install so we need to know to what extent the benefits outweigh the cost.

A wall of any building transfers heat in one direction if the temperatures on either side are different. In the case of glass window panes there is a heat transfer outwards in winter and inwards in summer, neither of which may be desirable. Double glazing companies manufacture windows typically consisting of two 3 mm panes of glass separated by a 12 mm gap. Using the theory of heat flux we can get a very good idea of just how much heat loss is prevented by installing these double glazed windows.

Let u_1 be the temperature inside a room and u_2 the temperature outside. We are concerned with winter heat loss so $u_1 > u_2$. Let ℓ_g be the thickness of the glass in each pane and ℓ_a the width of the air cavity between the sheets of glass. We take u_a and u_b to be the temperatures on either side of the air cavity, as illustrated in Figure 11.3.

Let J (the heat flux) be the amount of heat transferred through a unit area (per unit time) of a plane perpendicular to the direction of the x -axis, at x . We are not interested here in the initial heat transfer to the glass or to the cavity, but rather in the behaviour of the system after it has settled down to a state of thermal equilibrium, where J is a constant.

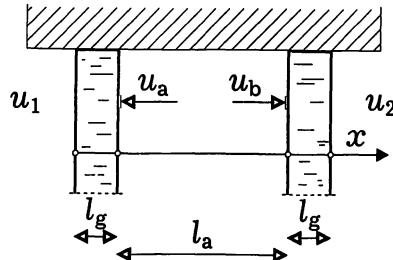


Figure 11.3: Schematic diagram of double glazing.

The heat flux J is given by

$$J = \frac{u_1 - u_2}{R}, \quad (11.15)$$

where R is the total heat resistance and $R = 2R_g + R_a$. This can be derived from the relation

$$J = \frac{u_1 - u_a}{R_g} = \frac{u_a - u_b}{R_a} = \frac{u_b - u_2}{R_g},$$

which expresses the equality of heat flux through each part of the system resulting from thermal equilibrium, or constant J . R above is the sum of the three individual heat resistances, due to the two panes of glass R_g and the air cavity R_a . We later relate the heat resistances R_g and R_a to the glass and air conductivities and thicknesses.

We need to compare this flux with that in the case of a single paned window, that is, one which is not double glazed ($\ell_a = 0$), but where the pane now has thickness $2\ell_g$. The heat flux for the single paned window, J_s , is

$$J_s = \frac{(u_1 - u_2)}{2R_g}.$$

(We have taken the single pane to have a thickness of $2\ell_g$ so that the only difference between J and J_s is due to the air gap.) We define a relative heat loss Δ by

$$\Delta = \frac{J_s - J}{J_s} = \frac{2R_g + R_a}{2R_g}. \quad (11.16)$$

If we assume the temperature of the window is exactly the temperature of the air, then $R_g = \ell_g/k_g$ where ℓ_g is the width of a single pane of glass and k_g is its conductivity. Similarly, $R_a = \ell_a/k_a$ where ℓ_a is the distance between the panes and k_a is the conductivity of the air. This essentially follows the calculations of Mesterton-Gibbons (1989). Although this assumption is not very accurate, we shall see where it leads, and then extend the model with this assumption relaxed.

Using approximate values of the conductivities

$$k_g = 0.8 \text{ J m}^{-1} \text{s}^{-1} \text{ }^\circ\text{C}^{-1}, \quad k_a = 0.05 \text{ J m}^{-1} \text{s}^{-1} \text{ }^\circ\text{C}^{-1},$$

we have $k_g/k_a \simeq 16$. Also, we set $\ell_a/\ell_g = r$. Substituting these values into (11.16) we obtain the relative heat loss Δ_0 as

$$\Delta_0 = \frac{16r}{16r + 2} = \frac{r}{r + 1/8}.$$

The graph of this function can be seen in Figure 11.4. It is a monotonically increasing function with $\Delta = 0$ when $r = 0$ and $\lim_{r \rightarrow \infty}(r/(r + 1/8)) = 1$.

Thus for $r = 1$ the reduction in heat loss is 88%. For $r = 4$ the reduction has increased substantially to 97% but for $r = 8$ the reduction is not much better at 98%. To conclude then, if $\ell_a \approx 4\ell_g$, the heat loss can be reduced by 97% and not much is gained by increasing the gap (ℓ_a) between the glass panes. This is indeed a substantial reduction in heat loss, and these specifications are exactly those manufactured as standard by the companies considered.

However, the above calculations assumed the surface temperature of the glass to be the same as the air temperature. This is a unrealistic assumption for a typical pane of glass and can result in a huge overestimate of the heat flux. (Possibly by a factor of 3000% for a 15°C temperature difference!) In an improved model, we do not take the temperature of the glass surface to be the same as that of the air, but instead replace this with a Newton cooling condition. This results in heat resistances being increased by an amount $1/h$ for each glass-air contact, where h is the Newton cooling coefficient. If we now include the Newton cooling terms for only the external glass-air contacts (but neglect the ones for the internal air cavity) our expression for R_g becomes $R_g = \ell_g/k_g + 1/h$ and, with $k_g/k_a = 16$, $\ell_a/\ell_g = r$, the relative heat loss Δ_1 is

$$\Delta_1 = \frac{16r}{2\sigma + 16r + 2}, \quad \text{where } \sigma = \frac{k_g}{h\ell_g}.$$

If we also allow for Newton cooling from the glass to the air cavity and from the air cavity to the glass then R_a becomes $R_a = \ell_a/k_a + 2/h$ and we obtain, from (11.16), Δ_2

$$\Delta_2 = \frac{2\sigma + 16r}{4\sigma + 16r + 2}, \quad \text{where } \sigma = \frac{k_g}{h\ell_g}.$$

A comparison of the results from these three models, Δ_0 , Δ_1 and Δ_2 , is illustrated in Figure 11.4.

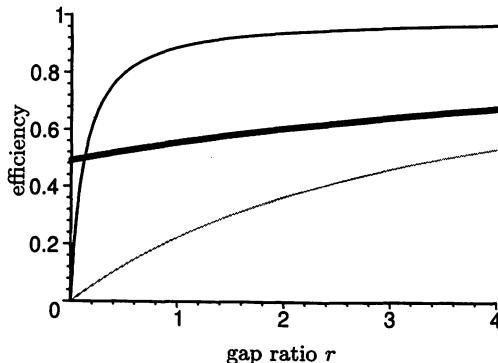


Figure 11.4: Relative heat loss efficiency calculations Δ_0 (thin black), Δ_1 (grey) and Δ_2 (thick black) plotted against r , the ratio of air gap to glass width. We have set $h = 10 \text{ W m}^{-2} \text{ }^\circ\text{C}^{-1}$ and $\ell_g = 0.003 \text{ m} = 3 \text{ mm}$.

The reduction in heat loss here is not quite the 97% of the first model; however, there is still a substantial improvement, particularly as window panes are a major cause of heat loss from homes during winter. Furthermore, in summer, the transmission of heat inwards is reduced to a significant degree, and double glazing provides a substantial reduction in sound transmission as well.

It should be mentioned here that k_a applies to dry, still air, and so this model is not applicable where air may contain moisture (which increases k_a and also the heat loss). For this reason double glazing requires a fully sealed cavity without moisture. Wall cavities, floors and ceilings use other materials for insulation, as it is not feasible to seal the cavities involved. In these cases, where there is moving air, heat can be convected (transport by a moving medium) whereas in the above double glazing example only the conduction of heat was considered (transport through a medium at rest).

Clearly, from the results of the above models, a substantial reduction in heat loss can be achieved through windows which are double glazed. Perhaps they are nearly as good as the manufacturers claim and worth the cost. Those without the economic resources, or in rental accommodation, have been known to use plastic-wrap (transparent sheet of plastic) attached against the inside of single window panes to create a crude type of double glazing in the colder months — with remarkably rewarding effects.

11.4 Insulating a water pipe

In this section we adapt the theory developed in Section 9.6, concerning the radial conduction of heat, to consider the insulation of a cylindrical pipe. After examining the physical processes we arrive at an unusual result — that sometimes we can be worse off with insulation than with no insulation at all!

Problem description

In cold climates a significant heat loss can occur from water pipes exposed to cold air. In extreme conditions, the heat loss may be such that the temperature of the water in the pipes drops below 0 °C and freezes. As the water freezes it expands and can burst the pipes. To restrict this heat loss and prevent such problems, a jacket of insulating material is placed around the pipe, or the pipe is lagged.

It is therefore useful to calculate the rate of heat loss from a pipe of given length. One of the things we should investigate is:

How does this rate of heat loss change as the thickness of the insulating material increases?

Model assumptions and approach

For a mathematical model we examine radial heat conduction inside the insulation jacket. We let the inner radius of the jacket be a and the outer radius be b . Hence the thickness is $(b - a)$ and the temperatures u_a and u_b are those on the inside and outside of the insulated pipe respectively. We denote by r the radial distance from the centre of the pipe. The geometry is shown in Figure 11.5.

We make some simplifying assumptions and then build our model on these.

- Since metal is such a good conductor of heat we assume the pipe metal to be at the same temperature as the insulation jacket, and to have good contact with it. We assume the inside of the insulation jacket is always at that temperature.
- On the outside of the insulation jacket the air temperature is u_b . We assume that heat is lost from this surface according to Newton's law of cooling.

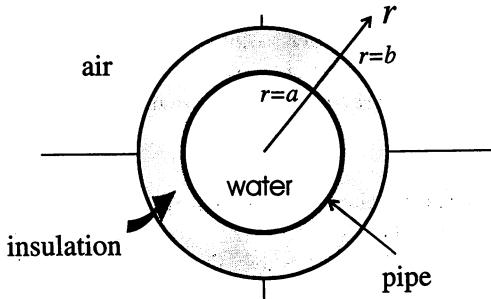


Figure 11.5: Geometry of the pipe insulation problem.

The differential equation

In a previous chapter we formulated the differential equation for radial heat conduction in a cylinder or cylindrical shell. The equilibrium temperature satisfied the differential equation (Section 9.6, equation (9.26))

$$\frac{d}{dr} \left(r \frac{dU}{dr} \right) = 0. \quad (11.17)$$

This differential equation is of second-order, but is not a constant coefficient differential equation. Using the product rule we can write this differential equation in the form

$$r \frac{d^2U}{dr^2} + \frac{dU}{dr} = 0.$$

This shows that the differential equation has coefficients which are not constant with respect to r . However, the original form is more convenient here for finding its solution.

Analytic solution

To obtain the solution to this differential equation we integrate both sides twice with respect to r .

Example 11.3: Solve the differential equation (11.17).

Solution: Integrating once gives

$$r \frac{dU}{dr} = C_1,$$

where C_1 is an arbitrary constant of integration. Hence

$$\frac{dU}{dr} = \frac{C_1}{r}.$$

Integrating again with respect to r ($r > 0$) gives

$$U(r) = C_1 \ln(r) + C_2, \quad (11.18)$$

where C_2 is the second arbitrary constant of integration.

We have two arbitrary constants which are required for the general solution of a second-order differential equation. The equilibrium temperature depends on the logarithmic of the radial distance from the centreline, rather than the linear distance as for the problem solved in Section 11.2.

The boundary conditions

We need two boundary conditions in order to solve for the two arbitrary constants. For this problem the boundary conditions are at the edges of the insulation at $r = a$ and $r = b$.

At $r = a$ we specified the temperature as u_a . In mathematical form this is written as

$$U(a) = u_a. \quad (11.19)$$

At $r = b$ the surface of the insulation loses heat to the surroundings. Newton's law of cooling states the rate of loss of heat to the surroundings is proportional to the surface area and the temperature difference. Since the heat flux is the rate of heat loss divided by the surface area, we can write

$$J(b) = h(U(b) - u_b), \quad (11.20)$$

where u_b is the temperature of the surrounding air. Since the temperature of the insulation is greater than the temperature of the air this gives a positive heat flux for $U(b) > u_b$.

Example 11.4: Apply the boundary conditions (11.19) and (11.20) and hence find an expression for the equilibrium temperature $U(x)$.

Solution: Applying the first boundary condition (11.19) at $r = a$, we obtain the equation

$$C_1 \ln(a) + C_2 = u_a. \quad (11.21)$$

To apply Newton's law of cooling at the boundary (11.20) we first calculate the heat flux. Using the general solution (11.18) and Fourier's law

$$J(r) = -k \frac{dU(r)}{dr} = -k \frac{C_1}{r} \quad (11.22)$$

and hence

$$J(b) = -k \frac{C_1}{b}. \quad (11.23)$$

The Newton cooling condition at $r = b$ now reduces to the equation

$$-k \frac{C_1}{b} = h(C_1 \ln(b) + C_2 - u_b). \quad (11.24)$$

Equations (11.21) and (11.24) give two simultaneous equations for C_1 and C_2 :

$$\ln(a)C_1 + C_2 = u_a, \quad (11.25)$$

$$\left(\ln(b) + \frac{k}{bh} \right) C_1 + C_2 = u_b. \quad (11.26)$$

Solving these equations we get

$$C_1 = \frac{-(u_a - u_b)}{\ln(b/a) + \frac{k}{bh}} \quad \text{and} \quad C_2 = u_a + \frac{(u_a - u_b)}{\ln(b/a) + \frac{k}{bh}} \ln(a),$$

and substituting these values for C_1 and C_2 back into the general solution (11.18) we have

$$U(r) = u_a - \frac{u_a - u_b}{\ln(b/a) + \frac{k}{bh}} \ln(r/a). \quad (11.27)$$

Rate of heat loss from a pipe

We calculate the total rate of heat loss from the pipe. For this we use Fourier's law of heat conduction to find the heat flux at $r = b$.

Example 11.5: Calculate the total rate of heat loss at $r = b$ of an insulated pipe of length ℓ .

Solution: Let Q denote the total rate of heat loss from a pipe of length ℓ . We can calculate this quantity by multiplying the heat flux at $r = b$ by the surface area of the insulation jacket (at $r = b$). The total surface area of the insulation jacket is $2\pi b\ell$. Hence

$$Q = 2\pi b\ell J(b).$$

Using Fourier's law, $J = -k dU/dr$, the rate of heat loss per unit length becomes

$$Q = -2k\pi b\ell \frac{dU(b)}{dr}$$

where we evaluate dU/dr at $r = b$.

Using the solution obtained for $U(r)$ from equation (11.27), we obtain

$$\begin{aligned} Q &= -2k\pi b\ell \left. \frac{d}{dr} \left(u_a - \frac{u_a - u_b}{\ln(b/a) + \frac{k}{bh}} \ln\left(\frac{r}{a}\right) \right) \right|_{r=b} \\ &= 2k\pi b\ell \frac{u_a - u_b}{\ln(b/a) + \frac{k}{bh}} \left. \frac{d}{dr} \ln\left(\frac{r}{a}\right) \right|_{r=b}. \end{aligned}$$

Evaluating the derivative and then substituting for $r = b$ gives

$$Q = 2k\pi b\ell \frac{u_a - u_b}{\ln(b/a) + \frac{k}{bh}} \times \frac{1}{b}. \quad (11.28)$$

Hence

$$Q = \frac{2k\pi\ell(u_a - u_b)}{\ln(b/a) + \frac{k}{bh}}. \quad (11.29)$$

As a quick check note that $u_a > u_b$ and $b/a > 1$, so we get a positive rate of heat loss per unit length, as would be expected.

Results and interpretation

In order to predict how the rate of heat loss varies as the insulation thickness is increased, we consider the above model with some appropriate parameter values.

Take $\ell = 1\text{ m}$, $a = 15\text{ mm}$, $u_a = 60^\circ\text{C}$, $u_b = 15^\circ\text{C}$ and $h = 5\text{ W m}^{-2}\text{ }^\circ\text{C}^{-1}$. We vary the outer radius b . The computed values of Q are given in Table 11.1 where we look at two different values of conductivity $k = 0.05$ and $k = 0.17$ SI units. The first value $k = 0.05$ corresponds to polyurethane and the value $k = 0.17$ corresponds to asbestos felt.

Note from Table 11.1 that the rate of heat loss for conductivity $k = 0.05$ decreases as the thickness of the insulation increases, as we might expect. However, for the larger conductivity of $k = 0.17$, we see that the rate of heat loss increases as the thickness of the insulation is increased. In this case it appears that we are better off not insulating the pipe at all!

Figure 11.6 gives a graph of the rate of heat loss from the insulated water pipe as a function of the outer radius of the insulation b . Note that there is a maximum rate of heat loss at the value $b = k/h$.

This suggests that there is a critical value for the outer radius b . For the case cited above, where the conductivity is $k = 0.05$, the critical value of the outer radius is 10 mm, but for $k = 0.17$ the critical radius is 34 mm. In this latter case one would need an outer radius of greater than 34 mm before the insulation would have the effect of reducing the heat loss

Table 11.1: Table of values of heat loss from an insulated water pipe for different thicknesses (in mm) of insulation and for two different conductivities (in SI units).

Insulation thickness $(b - a)$	Rate of heat loss $k = 0.05$	Rate of heat loss $k = 0.17$
0	21.2	21.2
1	20.5	22.0
2	19.8	22.6
5	18.0	24.2
10	15.5	25.7

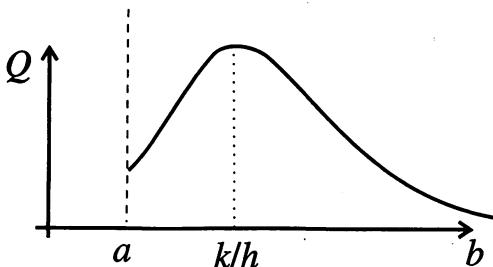


Figure 11.6: Diagram of the rate of heat loss from an insulated water pipe versus the outer radius of the insulation. There is a maximum at $b = k/h$. It illustrates the general behaviour for any values of k , h and b .

from the pipe. If this critical value is less than the inner radius, then placing insulation around the pipe has the desired effect of reducing the heat loss from it. If the critical value is greater than the inner radius, however, then it may be better not to insulate the pipe at all.

The physical explanation for this is that there is a balance between two competing effects. Increasing the thickness of the insulation means that there is more insulation through which the heat need flow. However, increasing the thickness of the insulation also increases the surface area through which heat is lost.

Solid cylinders and spheres

A different problem is that of a solid cylinder (or sphere). For this type of problem we usually do not have a boundary condition given at $r = 0$. Note that the general solution for a cylinder (11.18) cannot be evaluated at $r = 0$ due to the $\ln(r)$ term, which tends to $-\infty$. And for a solid sphere, there is a term r^{-1} which becomes infinite at $r = 0$.

For these problems a ‘hidden’, or implicit, boundary condition is applied. Physically we require the equilibrium temperature inside the cylinder to be finite. The only way this can be true is if the arbitrary constant coefficient in the $\ln(r)$ term is identically zero. This condition sets the value of one of the arbitrary constants and hence acts like a boundary condition, but in an implicit way.

Summary of skills developed here:

- *Solve for the temperature in a cylindrical shell, with any combination of prescribed temperature, heat flux or Newton cooling boundary conditions.*
- *Solve for temperature in a spherical shell with any combination of prescribed temperature, heat flux or Newton cooling boundary conditions.*
- *Extend to a cylinder or sphere with internal heat production.*
- *Solve for a solid cylinder or sphere with an implicit boundary condition, using that the temperature is finite at the centre ($r = 0$) to set one of the arbitrary constants to zero.*

11.5 Cooling a computer chip

We now consider a single heat fin from a set of heat fins attached to a computer chip. We solve the appropriate differential equation to find an analytic solution, and then examine this to understand what it tells us about the general behaviour of the temperature in the chip.

Problem description

In a computer, modern processing chips can generate a lot of heat. If the heat is not dissipated adequately the temperature of the chip can increase significantly, possibly causing the chip to cease functioning properly or to become permanently damaged. The Cyrix 6x86-P166 chip generates heat at a rate of approximately 24 W and Cyrix advises that the chip should not reach a temperature of over 70 °C (see Cyrix (1998)).

One method of dissipating heat is to mount heat fins on the chip with a cooling fan to blow air between the fins. For the Cyrix chip, 12 heat fins are used with a fan mounted above them. Each heat fin has dimensions of width $w = 5.1$ cm, thickness $b = 1$ mm and length $\ell = 1.5$ cm. A schematic diagram of the heat fin assembly and fan for a Cyrix 6x86 CPU chip is shown in Figure 11.7.

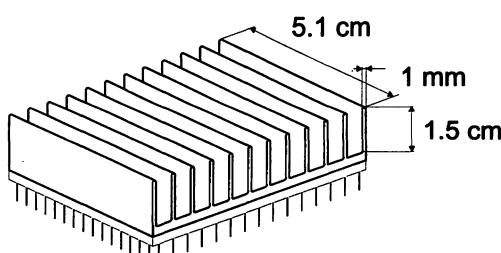


Figure 11.7: Heat dissipation assembly for a Cyrix CPU computer chip. It consists of a cooling fan and 12 heat fins attached to the chip.

The Cyrix 6x86 chip, for example, generates heat at a rate of $q = 24$ Watts, Cyrix (1998). If the only way the heat can dissipate is from the surface of the chip, then we can model the temperature of the chip by using Newton's law of cooling.

Example 11.6: Use Newton's law of cooling to calculate the equilibrium temperature of the chip if there are no cooling fins and the dimensions of the chip are $5.1\text{ cm} \times 5.3\text{ cm}$.

Solution: Newton's law of cooling states that the rate of heat loss Q is

$$Q = hS\Delta U, \quad (11.30)$$

where ΔU is the temperature difference between the chip and the ambient temperature. For the Cyrix chip, with surface area $S = 5.1\text{ cm} \times 5.3\text{ cm} = 2.7 \times 10^{-3}\text{ m}^2$ and using a value of $h = 5\text{ W m}^{-2}\text{ C}^{-1}$ we obtain

$$\Delta U \simeq \frac{24}{5 \times 2.7 \times 10^{-3}} \simeq 1800^\circ\text{C} \quad (11.31)$$

for the temperature difference between the chip face and the surrounding air.

Of course, the chip would cease to function long before this temperature were reached. Clearly we need to dissipate the heat more efficiently.

We are interested in calculating the temperature of the chip with the heat fins attached. We are also interested in answering the question:

How does this temperature vary if we increase or decrease the length of the heat fins?

Model assumptions and approach

We develop a model for a single heat fin where the geometry of the model is shown in Figure 11.8.

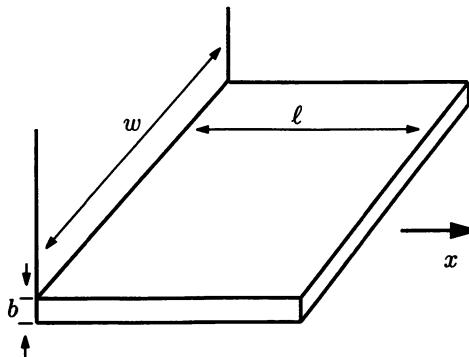


Figure 11.8: Diagram of model used for a single heat fin. Heat enters at $x = 0$ (left end) with flux q generated from the computer chip. We assume the heat flux from $x = l$ (right end) is zero.

We make some simplifying assumptions on which to build the mathematical model.

- We assume the heat loss is distributed evenly over the 12 heat fins.
- We assume the heat loss from the sides of the fin to be negligible compared with the heat loss from the top and bottom.
- We also neglect any heat loss from the end of the heat fin.

The differential equation

We can calculate the heat flux entering a single heat fin. This heat flux, denoted by q , is obtained by taking the rate of heat loss for a single fin $24/12 = 2 \text{ W}$ and then dividing by the area of the cross-section, bw . This gives a heat flux of $q = 3.92 \times 10^4 \text{ W/m}^2$.

In Chapter 9 (Section 9.7), we formulated an appropriate differential equation for the equilibrium temperature distribution, $U(x)$, along a single heat fin. This differential equation is

$$\frac{d^2U}{dx^2} = \beta(U - u_s), \quad \text{where } \beta = \frac{2h}{kb}. \quad (11.32)$$

Here $U(x)$ is the temperature, x the distance along the heat fin, k the conductivity, h the convective heat transfer coefficient and u_s the temperature of the surrounding air. This differential equation is an inhomogeneous, constant coefficient, second-order differential equation.

For simplicity we set the temperature of the heat fin relative to the temperature of the surrounding air. On this temperature scale, $u_s = 0$ and the differential equation (11.32) becomes

$$\frac{d^2U}{dx^2} = \beta U \quad \text{where } \beta = \frac{2h}{kb} \quad (11.33)$$

which is now homogeneous. This is not a major restriction as it is simple, with a change of variable, to convert a problem with non-zero u_s to one with $u_s = 0$. Equation(11.33) represents a *heat balance*. The LHS term comes from heat conduction and the RHS comes from heat lost to the surrounding air.

Analytic solution

First we derive a general solution to the differential equation. Then we find the associated constants and hence the particular solution, using the boundary conditions.

Example 11.7: Find the general solution to the differential equation (11.33).

Solution: The general solution can be found by substituting a trial solution of the form $U(x) = e^{\lambda x}$ into the differential equation (11.33). This gives rise to the characteristic equation $\lambda^2 = \beta$. Since β is a positive constant the roots of the characteristic equation are $\pm\sqrt{\beta}$. A general solution of the differential equation (11.33) is then

$$U(x) = C_1 e^{\sqrt{\beta}x} + C_2 e^{-\sqrt{\beta}x} \quad (11.34)$$

where C_1 and C_2 are arbitrary constants.

It is also possible to express this solution in terms of the hyperbolic functions¹ $\cosh(\sqrt{\beta}x)$ and $\sinh(\sqrt{\beta}x)$. Often this form can lead to simpler algebra when applying certain types of boundary conditions. (Further details of hyperbolic functions can be found in Appendix B.4.)

¹The hyperbolic functions \cosh and \sinh are defined by

$$\cosh(z) = \frac{1}{2}(e^z + e^{-z}), \quad \sinh(z) = \frac{1}{2}(e^z - e^{-z}).$$

The boundary conditions

We require two boundary conditions, one at the end $x = 0$, the origin, and one at the end $x = \ell$, to give us two simultaneous equations for the two arbitrary constants C_1 and C_2 . At $x = 0$ we know the value of the heat flux to be q . We write this boundary condition as

$$J(0) = q. \quad (11.35)$$

At the end $x = \ell$ we have assumed we can neglect the heat loss. This is expressed mathematically as

$$J(\ell) = 0. \quad (11.36)$$

Example 11.8: Apply the boundary conditions to obtain an expression for the temperature along the heat fin.

Solution: Both boundary conditions involve the heat flux J . From Fourier's law ($J(x) = -k dU/dx$) and the general solution (11.34), we obtain

$$J(x) = -k\sqrt{\beta} \left(C_1 e^{\sqrt{\beta}x} - C_2 e^{-\sqrt{\beta}x} \right). \quad (11.37)$$

Applying the first boundary condition $J(0) = q$ and evaluating this at $x = 0$ gives the equation

$$C_1 - C_2 = \frac{-q}{k\sqrt{\beta}}. \quad (11.38)$$

To apply the second boundary condition, $J(\ell) = 0$, we substitute $x = \ell$ into the expression for $J(x)$ to get

$$C_1 e^{\sqrt{\beta}\ell} - C_2 e^{-\sqrt{\beta}\ell} = 0. \quad (11.39)$$

We solve equations (11.38) and (11.39) simultaneously giving an expression for C_2 . Substituting this into (11.38) gives

$$\left(1 - e^{2\sqrt{\beta}\ell} \right) C_1 = \frac{-q}{k\sqrt{\beta}},$$

which allows us to solve for C_1 as

$$C_1 = \frac{q}{k\sqrt{\beta}} \left(\frac{1}{e^{2\sqrt{\beta}\ell} - 1} \right). \quad (11.40)$$

Substituting this back into our expression for C_2 we obtain

$$C_2 = \frac{q}{k\sqrt{\beta}} \left(\frac{e^{2\sqrt{\beta}\ell}}{e^{2\sqrt{\beta}\ell} - 1} \right). \quad (11.41)$$

With these expressions for C_1 and C_2 we have, from the general solution (11.34), an expression for the temperature along the heat fin, namely

$$U(x) = \frac{q}{k\sqrt{\beta}} \left(\frac{e^{\sqrt{\beta}x} + e^{2\sqrt{\beta}\ell} e^{-\sqrt{\beta}x}}{e^{2\sqrt{\beta}\ell} - 1} \right). \quad (11.42)$$

From this general solution for the temperature in equation (11.42), we can immediately write down an expression for the temperature of the computer CPU chip, which is the quantity of interest. This is the temperature at $x = 0$ and is given by

$$U(0) = \frac{q}{k\sqrt{\beta}} \left(\frac{e^{2\sqrt{\beta}\ell} + 1}{e^{2\sqrt{\beta}\ell} - 1} \right). \quad (11.43)$$

Results and interpretation

Using some appropriate parameter values for copper heat fins we take $k = 386$ and $h = 32$ SI units. The value of the convective heat transfer coefficient h is higher than the one we used in the calculation in equation (11.31) to incorporate the effect of the fan.² We also have $b = 1 \text{ mm} = 10^{-3} \text{ m}$, $w = 5.1 \times 10^{-2} \text{ m}$ and $\ell = 1.5 \text{ cm} = 1.5 \times 10^{-2} \text{ m}$. We then calculate $\beta = 174$ (in SI units). We have previously calculated $q = 3.92 \times 10^4 \text{ W/m}^2$ for each heat fin. Substituting these values into (11.43) we obtain

$$U(0) = 41.4^\circ\text{C}.$$

This is the temperature rise above the ambient temperature, or the surrounding temperature, as our model assumed $u_s = 0$.

In Figure 11.9, we plot the temperature along the length of the heat fin. The temperature is highest at the chip surface and decreases along the length of the heat fin, as expected.

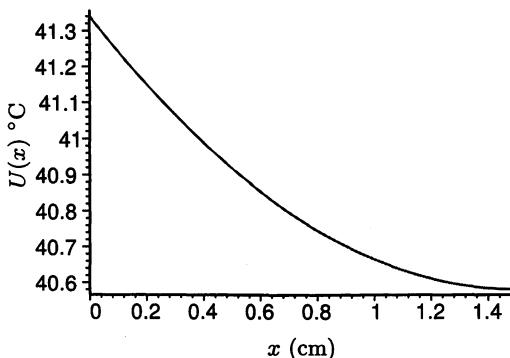


Figure 11.9: Graph of temperature in °C along the length of the heat fin.

It is more interesting to sketch a graph of the temperature of the computer chip for different lengths of the heat fin. This graph, illustrated in Figure 11.10, provides some insight into how long to make the heat fin for the best results. The temperature of the chip decreases as we increase the length of the heat fin. Physically this makes sense, since increasing the length of the heat fin increases the area through which heat is lost. However, $U(0)$ appears to approach a limiting value as ℓ (the length of the fin) increases.

From (11.43) we can calculate the observed limiting temperature as we let ℓ become very large. This will provide a lower bound for the coolest temperature which the chip can reach. To find this limit we let $\ell \rightarrow \infty$ in (11.43). Dividing both top and bottom by $e^{2\sqrt{\beta}\ell}$, to avoid infinite limits in the numerator and denominator, yields

$$U(0) = \frac{q}{k\sqrt{\beta}} \left(\frac{1 + e^{-2\sqrt{\beta}\ell}}{1 - e^{-2\sqrt{\beta}\ell}} \right). \quad (11.44)$$

Hence, the limiting value as $\ell \rightarrow \infty$ is

$$U_{\min} = \frac{q}{k\sqrt{\beta}}. \quad (11.45)$$

²To calculate h a formula from Holman (1981) was used, assuming an air speed of 1 ms^{-1} .

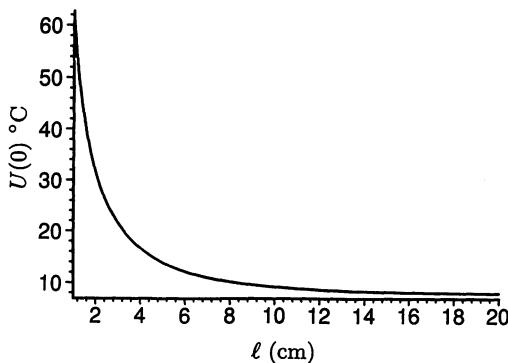


Figure 11.10: The temperature $U(0)$ in °C of the computer chip as a function of the length ℓ of the heat fin.

For the same parameter values as above, this gives a minimum temperature for U of $U_{\min} = 8.1^\circ\text{C}$. Clearly, from Figure 11.10, the temperature is close to 8.1°C for a fin length of much less than $\ell = \infty$, and thus not much is gained by increasing the fin length beyond a certain point (say 10 cm). However, there is a substantial gain by increasing the length from 2 to 4 cm.

Summary of skills developed here:

- Obtaining the general solution to the homogeneous heat fin equations.
- Applying boundary conditions to this general solution, including other combinations of boundary conditions, such as one or both boundaries at prescribed temperatures.
- Interpreting the solution for the temperature of a computer chip to find a theoretical minimum temperature.

11.6 Exercises for Chapter 11

11.1. Solving a boundary value problem. Consider the differential equation

$$\frac{d^2U}{dx^2} = 1$$

which satisfies the boundary conditions

$$U(0) = 1, \quad U(2) = 0.$$

- (a) Find the general solution.
- (b) Apply the boundary conditions to find the solution.

- (c) Suppose the two boundary conditions are replaced with

$$U(0) = 1, \quad \frac{dU}{dx}(2) = 0.$$

Apply these boundary conditions and find the solution.

- (d) Suppose the two boundary conditions are replaced with

$$\frac{dU}{dx}(0) = 1, \quad \frac{dU}{dx}(2) = U(2).$$

Apply these boundary conditions and find the solution.

Find the solution if the boundary condition at $x = 1$ is replaced with the boundary condition (at $x = 1$)

$$\frac{dU}{dx} = 0.$$

11.2. Formulating a boundary value problem. Write down, in mathematical form, boundary conditions for the following. Express them in terms of temperature U or heat flux J .

- (a) A wall of a furnace gains heat from the end $x = 0$ at a fixed rate of 300 W, while the other end is maintained at temperature 30 °C.
- (b) The outside of a wall of a house loses heat according to Newton's law of cooling, to the surrounding air at temperature 10 °C, while the inside gains heat according to Newton's law of cooling from the inside of the house which is at a temperature of 25 °C.
- (c) A slab of material has its right end held at temperature 80 °C and the left end gaining heat according to Newton's law of cooling.

11.3. Heat flux through a wall. In an internal wall of thickness d , the temperature at $x = 0$ is maintained (by a thermostat) at temperature u_1 and the other side is maintained at temperature u_2 .

- (a) Find an expression for the temperature at any point inside the wall.
- (b) Hence find an expression for the heat flux through the wall.

11.4. Heat flux through a window. Consider the flow of heat through a window pane of width ℓ . Assume that heat is gained from the inside and lost from the outside of the window according to Newton's law of cooling. The temperature of the air on the inside is u_i and the temperature on the outside is u_o . The Newton cooling coefficients are h_i and h_o for the inside surface and outside surface of the window, respectively. You are given that the equilibrium temperature, inside the window, satisfies

$$\frac{d^2U}{dx^2} = 0.$$

- (a) What is the general solution for the equilibrium temperature?
- (b) What are the boundary conditions for this problem?
- (c) Find the equilibrium temperature as a function of x .
- (d) Hence deduce the heat flux from the outer surface $x = \ell$ is given by the expression

$$\frac{k(u_i - u_o)}{\ell + \frac{k}{h_i} + \frac{k}{h_o}}.$$

- (e) Interpret this formula in terms of heat resistances.

11.5. Heat flux and thermal resistance. Suppose we have two different materials with widths d_1 and d_2 , conductivities k_1 and k_2 , and which are joined together at $x = 0$. The equilibrium temperatures $U_1(x)$ and $U_2(x)$ both satisfy the basic equilibrium heat equations

$$\frac{d^2U_1}{dx^2} = 0, \quad \frac{d^2U_2}{dx^2} = 0.$$

The temperature on the inside ($x = -d_1$) is held (by a thermostat) at a temperature u_i and the temperature on the outside ($x = d_2$) is held (by a thermostat) at a temperature u_o .

- (a) Find expressions for the two temperatures $U_1(x)$ and $U_2(x)$. You may assume the temperature and heat flux are continuous where the materials join at $x = 0$.
- (b) Show the heat flux is given by

$$J = \frac{u_i - u_o}{\frac{d_1}{k_1} + \frac{d_2}{k_2}}.$$

- (c) Interpret this in terms of thermal resistances.

11.6. Internal floor heating. Houses are often heated via a heating source located within a concrete slab floor (e.g. by attaching an electrical resistance wire to the reinforcing bars embedded within the concrete). Suppose such a heat source is located in a slab of thickness d and generates heat at a constant rate q (per unit volume). The equilibrium temperature $U(x)$ satisfies the differential equation

$$k \frac{d^2U}{dx^2} + q = 0$$

where k is the thermal conductivity. If each side of the slab is maintained at the same temperature u_0 find an expression for $U(x)$, the temperature inside the slab.

11.7. Nuclear reactor. In the shielding wall of a nuclear reactor the rate of heat generated by gamma rays interacting with the wall, the internal heat generation (per unit volume) $q(x)$, may be modelled by

$$q(x) = q_0 e^{-ax}$$

where q_0 and a are positive constants. The equilibrium temperature inside the wall satisfies the differential equation

$$k \frac{d^2U}{dx^2} + q(x) = 0.$$

- (a) Sketch $q(x)$ as a function of x .
- (b) Determine the general solution for the equilibrium temperature.
- (c) Given that the inside of the wall $x = 0$ is maintained at a constant temperature of u_1 and the outside of the wall is maintained at a constant temperature of u_2 , find an expression for the equilibrium temperature.

11.8. Combining heat resistances. To prove that the individual heat resistances are additive, consider

$$J = \frac{u_1 - u_a}{R_g} = \frac{u_a - u_b}{R_a} = \frac{u_b - u_2}{R_g}.$$

Find expressions for u_a and u_b and substitute them back into one of the equations for J . This will establish equation (11.15) of the double glazing case study (Section 11.3).

11.9. Double glazing. Following the case study on double glazing (Section 11.3), assume that the inside room temperature is 20°C , the outside environment temperature is 0°C , the thickness of each of the two panes of glass is $d = 4\text{ mm}$, the conductivity of glass is $k_g = 0.8 \text{ J m}^{-1} \text{s}^{-1} \text{ }^\circ\text{C}^{-1}$ and the conductivity of air is $k_a = 0.05 \text{ J m}^{-1} \text{s}^{-1} \text{ }^\circ\text{C}^{-1}$. Take Newton cooling coefficient $h = 10 \text{ W m}^{-2} \text{ }^\circ\text{C}^{-1}$.

Calculate the heat flux for a gap width of twice the thickness of a single pane of glass, using each of the models corresponding to each of the Δ_0 , Δ_1 and Δ_2 models defined in the Case Study. Compare this to the heat flux through a single pane of glass with width $2d$.

11.10. Comparing glazing models. Using Maple or MATLAB, draw graphs of the three delta functions in the case study on double glazing (Section 11.3). Look at limits over time, and with the standard specifications of manufacturers how these three compare.

11.11. Cooling lizards. Lizards at rest are known to generate heat internally at a rate of $q = 0.5 \text{ W/kg}$. We might model the lizard as a cylinder of radius a and length ℓ . The equilibrium temperature satisfies the differential equation

$$\frac{k}{r} \frac{d}{dr} \left(r \frac{dU}{dr} \right) + \rho q = 0,$$

where ρ is the (average) density of lizard tissue, k is the conductivity, and r measures radial distance from the centre of the lizard's body (a cylinder).

- (a) Starting from a suitable word equation, outline a derivation of this differential equation.
- (b) Find the general solution.
- (c) For a solid cylinder (our lizard) we require the equilibrium temperature to be finite at $r = 0$. What does this tell you about the value of one of the arbitrary constants?
- (d) Suppose heat is lost to the surroundings according to Newton's law of cooling, where the surroundings are at temperature u_0 . Find an expression for the equilibrium temperature inside the lizard (cylinder). Where is the equilibrium temperature a maximum?

11.12. Equilibrium temperature in a spherical shell. The equilibrium temperature inside a spherical shell, of inner radius a and outer radius b , satisfies the differential equation

$$\frac{d}{dr} \left(r^2 \frac{dU}{dr} \right) = 0.$$

- (a) Find the general solution of the differential equation.
- (b) Find the equilibrium temperature if the inner surface $r = a$ is maintained at temperature u_1 and the outer surface $r = b$ is maintained at temperature u_2 .

11.13. Hyperbolic functions. Verify that

$$U(x) = D_2 \cosh(\sqrt{\beta}x) + D_1 \sinh(\sqrt{\beta}x)$$

is a general solution of the differential equation

$$\frac{d^2U}{dx^2} = \beta U.$$

Now find the solution which satisfies the boundary conditions applied at $x = 0$,

$$U(0) = 1, \quad \frac{dU(x)}{dx} = 0 \quad \text{at } x = 0.$$

For details on hyperbolic functions see Appendix B.4.

11.14. Cooling an engine. Heat fins are often attached to engines to cool them when in operation. Find an expression for the temperature inside a heat fin where one end of the heat fin is attached to a metal engine, and is maintained at temperature u_1 . You may neglect any heat loss from the other end of the heat fin.

The equilibrium temperature satisfies the differential equation

$$\frac{d^2U}{dx^2} = \beta U,$$

where β is a constant and the temperature of the surroundings is zero.

11.15. Rectangular heat fin. In Section 11.5 it was shown that the relative temperature in a rectangular heat fin satisfies the differential equation

$$\frac{d^2U}{dx^2} = a^2U, \quad a^2 = \frac{2h}{kb},$$

where b is the thickness of the heat fin, k is the conductivity, h the Newton cooling coefficient, and we have taken the temperature of the surrounding air to be zero (see Section 11.5, equation (11.32)).

- (a) Write down the boundary conditions assuming that the temperature at the end attached to the heat source is u_1 and the heat lost from the other end is given by Newton's law of cooling.
- (b) Hence obtain an expression for the equilibrium temperature. (Hint: you will find the algebra easier if you use the hyperbolic form of the general solution

$$U(x) = C_1 \cosh(ax) + C_2 \sinh(ax),$$

where C_1 and C_2 are arbitrary constants.)

11.16. Formulating boundary conditions. In the following, write down boundary conditions involving the temperature U or the heat flux J . (It may be useful to draw a diagram).

- (a) In a spherical shell defined by $r = a$ to $r = b$, heat flows radially. The temperature on the outside is u_2 and the inside of the shell receives heat at a rate of 10 W.
- (b) In a wall of length L the temperature at one end $x = L$ is u_1 and the temperature at the other end is u_2 .
- (c) In a cylindrical heat fin, of small radius a , heat is conducted along the length of the wire. The heat fin is attached to a surface whose temperature is maintained at 100 °C. The other end of the heat fin is assumed to lose no heat.
- (d) A wall located between $x = 0$ and $x = \ell$ has temperature at $x = \ell$ fixed at u_1 . The other side of the wall gains heat from the surroundings, at temperature u_2 , according to Newton's law of heat exchange.
- (e) The material inside a cylindrical insulating layer, of inner radius 1 and outer radius 2, gains heat from the surroundings at temperature $3u_1$, according to Newton's law of heat exchange. The temperature at the inner radius is maintained at u_1 .

Chapter 12

Introduction to partial differential equations

This chapter provides a brief introduction to the derivation of partial differential equations (PDEs). It is a natural extension of the previous models of heat and lake pollution, and indicates how they might be improved and generalised. A case study is included to illustrate how such equations are applied in practice.

12.1 The heat conduction equation

In previous chapters we have mentioned the dependence of certain functions on two independent variables: time and position. We now see how to derive a PDE for the temperature when the system is not in thermal equilibrium, that is when there are changes in both of these independent variables.

Introduction

So far we have assumed thermal equilibrium so that, for all calculations, the heat flux is dependent on x (position) alone. Let us now consider what happens when the material is not in thermal equilibrium. We have to take account of the heat used in raising (or lowering) the temperature. The temperature will now depend on both position x and time t . Hence we let $U(x, t)$ be the temperature, and denote the heat flux as $J(x, t)$, at position x and time t .

Formulating a PDE

We return to the problem of heat flow through a wall, where the wall is now no longer in thermal equilibrium. From Section 9.5, the word equation describing the rate of change of heat content in a small section x to $x + \Delta x$ of the wall is

$$\left\{ \begin{array}{l} \text{rate of} \\ \text{change of} \\ \text{heat in section} \end{array} \right\} = \left\{ \begin{array}{l} \text{rate heat} \\ \text{conducted} \\ \text{in at } x \end{array} \right\} - \left\{ \begin{array}{l} \text{rate heat} \\ \text{conducted} \\ \text{out at } x + \Delta x \end{array} \right\}. \quad (12.1)$$

Since the temperature is no longer in thermal equilibrium we must regard the temperature and heat flux as functions of two variables, $U(x, t)$ and $J(x, t)$, where x measures distance and t measures time.

The rate of change of heat is calculated from the rate at which heat is ‘converted’ into an increase in temperature. For the LHS of this equation we use equation (9.1), from Section 9.2, which relates heat to temperature as

$$\left\{ \begin{array}{l} \text{rate of} \\ \text{change of} \\ \text{heat in section} \end{array} \right\} = c \times \{ \text{mass} \} \times \frac{\partial U}{\partial t}, \quad (12.2)$$

with c the specific heat. Note the partial derivative with respect to time, since U depends on both x and t . (For more details on partial derivatives see Appendix B.2.) The mass of the section can be expressed in terms of the wall density ρ , as $\rho A \Delta x$, where A is the usual surface area.

The heat conduction terms can be expressed in terms of the heat flux

$$\left\{ \begin{array}{l} \text{rate heat} \\ \text{conducted} \\ \text{in at } x \end{array} \right\} = AJ(x, t), \quad \left\{ \begin{array}{l} \text{rate heat} \\ \text{conducted} \\ \text{out at } x + \Delta x \end{array} \right\} = AJ(x + \Delta x, t). \quad (12.3)$$

Then, from equations (12.1) and (12.2),

$$c\rho \frac{\partial U}{\partial t} \Delta x = -J(x + \Delta x, t) + J(x, t). \quad (12.4)$$

Example 12.1: For equation (12.4) take the limit as $\Delta x \rightarrow 0$ and then use Fourier's law to obtain a differential equation for the temperature.

Solution: We first divide through by Δx to obtain

$$c\rho \frac{\partial U}{\partial t} = - \left[\frac{J(x + \Delta x, t) - J(x, t)}{\Delta x} \right].$$

Recall the definition of a partial derivative,

$$\frac{\partial J}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{J(x + \Delta x, t) - J(x, t)}{\Delta x},$$

where we are looking at the change in J with respect to the x variable, while the t variable remains constant. If we now take the limit $\Delta x \rightarrow 0$, we obtain

$$c\rho \frac{\partial U}{\partial t} = - \frac{\partial J}{\partial x}. \quad (12.5)$$

Finally, we apply Fourier's law to express the equation in terms of temperature. For quantities which vary with both space x and time t , Fourier's law must be written as

$$J(x, t) = -k \frac{\partial U}{\partial x}. \quad (12.6)$$

Substituting Fourier's law (12.6) back into (12.5) gives

$$c\rho \frac{\partial U}{\partial t} = - \frac{\partial}{\partial x} \left(-k \frac{\partial U}{\partial x} \right).$$

For constant k this simplifies to

$$\frac{\partial U}{\partial t} = \alpha \frac{\partial^2 U}{\partial x^2}, \quad \text{where } \alpha = \frac{k}{c\rho}. \quad (12.7)$$

This second-order partial differential equation is called the *heat equation*, and the parameter α is known as the thermal diffusivity. Note that for equilibrium temperatures, that is for $\partial U / \partial t = 0$, (12.7) reduces to

$$\frac{d^2 U}{dx^2} = 0,$$

which was the ordinary differential equation obtained in Section 9.5.

Other applications of the heat/diffusion equation

This heat equation (12.7) is widely known, arising in many other applications. In such contexts, where it models the changes in concentration due to the movement of mass by random motion, it is called the *diffusion equation*. One other area of application is finance, where another similar partial differential equation, called the Black-Scholes equation, is used to compute for share option prices. The Black-Scholes equation can be transformed into the heat equation by a change of variables. See Wilmott (1998). Variants of this equation are also used in population dynamics with spatial dependence, as in Murray (1990), where the random movement of animals can be modelled by a diffusion term.

One application of mass diffusion is the diffusion of chemicals. Partial differential equations would provide a valuable extension to the model we developed in Chapter 2 on lake pollution. There we assumed that pollution concentration within the lake was uniformly distributed, while in reality it is a function of both time and distance from its source. We see how to do this later in this chapter.

Summary of skills developed here:

- Formulate a PDE to describe the time dependent conduction of heat.
- Distinguish cases where the use of an ODE is appropriate, and where a PDE is appropriate.

12.2 Oscillating soil temperatures

We consider an interesting and fundamental property of heat processes, namely the oscillatory nature of temperatures of large masses, such as the ocean and the soil. It will provide a theoretical basis for the case study concerning the detection of land mines.

Problem description

It is well known that soil acts as an excellent thermal insulator for variations in temperature. Since wine demands an environment with as close to a constant temperature as possible, wine cellars are constructed underground making use of this thermal property. The thermal mass of the ocean, which has a substantial impact on the milder climatic temperatures experienced in coastal regions, is another well-known case of this phenomenon.

The theory we develop concerns the soil mass and its temperatures over time. We intend to develop a model which reflects the oscillatory nature of this thermal mass.

Model assumptions and approach

Let us consider the problem of determining the temperature in the soil due to periodic heating by day and cooling by night. We consider an infinite soil depth, with the surface $x = 0$ corresponding to the soil surface and the x -axis pointing into the ground.

We let $U(x, t)$ denote the relative temperature variation in the ground, at some time t , and at a distance x into the ground. We measure the temperature relative to some mean temperature, in this case the mean surface temperature. From Section 12.1 the governing differential equation is the standard non-equilibrium heat conduction equation,

$$\frac{\partial U}{\partial t} = \alpha \frac{\partial^2 U}{\partial x^2}, \quad (12.8)$$

where α is the thermal diffusivity for soil. (Recall that $\alpha = k/c\rho$, with k the conductivity, ρ the density and c the specific heat.)

Boundary conditions for the PDE

On the surface of the ground, $x = 0$, and we have to specify the heat flux $Q(t)$ (for which we develop a model shortly). Due to heating by the sun during the day and then cooling at night, we require a suitable function which oscillates with time.

The change in temperature is driven by a diurnal cycle. Thus we model the surface heat flux $Q(t)$ with a simple trigonometric function, $\cos(\omega t)$, where ω is determined from the period being (1 day), that is,

$$Q(t) = q \cos(\omega t), \quad \omega = \frac{2\pi}{60 \times 60 \times 24} \simeq 7.3 \times 10^{-5} \text{ seconds.} \quad (12.9)$$

Hence, using Fourier's law of heat conduction, we can write the boundary condition on the ground surface ($x = 0$) as

$$-k \frac{\partial U}{\partial x} = q \cos(\omega t) \quad \text{at } x = 0. \quad (12.10)$$

(We could alternatively have considered a more complicated boundary condition by allowing Newton cooling/heating from the ground surface, but we shall not do so here.)

The second boundary condition states that the temperature variation goes to zero at a long distance from the surface, that is,

$$-k \frac{\partial U}{\partial x} \rightarrow 0 \quad \text{as } x \rightarrow \infty. \quad (12.11)$$

Complex valued temperatures

To solve the PDE (12.8), with the two boundary conditions (12.10) and (12.11), we assume a suitable form of the solution. Bearing in mind the PDE (12.8) has constant coefficients, we would assume exponential forms for both the x -dependence and the t -dependence. However, this would not match with the cosine term in the boundary condition (12.10). Similarly, assuming a form $U(x, t) = F(x) \cos(\omega t)$, while matching in the boundary condition (12.10) will not help to solve the PDE (12.8).

A useful trick is to use complex numbers. This technique allows us to simplify the problem we have to solve. We write the problem in terms of complex exponentials creating a problem which is easier to solve. Since our original problem is the real part of the complex problem, the solution to our problem is the real part of the complex problem solution. We represent the trigonometric functions as complex exponentials, using Euler's identity

$$e^{iz} = \cos(z) + i \sin(z), \quad \text{where} \quad i = \sqrt{-1}.$$

(See Appendix B.3 for an introduction to complex numbers.) We define a complex valued temperature \hat{U} , where U is the real part of \hat{U} ,

$$U(x, t) = \text{Real}(\hat{U}(x, t)). \quad (12.12)$$

The problem we now solve is

$$\frac{\partial \hat{U}}{\partial t} = \alpha \frac{\partial^2 \hat{U}}{\partial x^2}, \quad -k \frac{\partial \hat{U}}{\partial x} = q e^{i\omega t} \quad \text{at } x = 0, \quad -k \frac{\partial \hat{U}}{\partial x} = 0 \quad \text{as } x \rightarrow \infty. \quad (12.13)$$

We then take the real part of the solution, which corresponds to the solution of the original problem (12.8).

Analytic solution

Assume the solution for the complex valued temperature can be written in the form

$$\hat{U}(x, t) = F(x) e^{i\omega t}. \quad (12.14)$$

Now substituting (12.14) into equation (12.13) we obtain

$$i\omega F(x) e^{i\omega t} = \alpha \frac{d^2 F(x)}{dx^2} e^{i\omega t}.$$

Note that the derivative is an ordinary derivative, which is the case since F is a function of x alone. Dividing each equation by $e^{i\omega t}$, we obtain an ODE for $F(x)$,

$$\alpha \frac{d^2 F}{dx^2} = i\omega F. \quad (12.15)$$

We now find its general solution.

Example 12.2: Find the general solution to the differential equation (12.15).

Solution: The equation is a constant coefficient differential equation. Hence let $F(x) = e^{\lambda x}$. Substituting this solution form into equation (12.15),

$$\lambda^2 = \frac{i\omega}{\alpha}, \quad \text{so} \quad \lambda = \pm \sqrt{\frac{\omega}{\alpha}} \times i^{1/2}.$$

From Appendix B.3 we see that

$$i^{1/2} = \pm \frac{(1+i)}{\sqrt{2}}.$$

So the solution is (substituting back into $F(x) = e^{\lambda x}$ and taking a linear combination of the solutions)

$$F(x) = C_1 e^{\sqrt{\frac{\omega}{2\alpha}}(1+i)x} + C_2 e^{-\sqrt{\frac{\omega}{2\alpha}}(1+i)x},$$

where C_1 and C_2 are arbitrary complex valued constants. Defining

$$b = \sqrt{\frac{\omega}{2\alpha}}(1+i) \quad (12.16)$$

we can write

$$F(x) = C_1 e^{bx} + C_2 e^{-bx}. \quad (12.17)$$

Using (12.14) and (12.17) the complex valued temperature is given by

$$\hat{U}(x, t) = (C_1 e^{bx} + C_2 e^{-bx}) e^{i\omega t}, \quad \text{where } b = \sqrt{\frac{\omega}{2\alpha}}(1+i). \quad (12.18)$$

The arbitrary constants C_1 and C_2 are found by applying the boundary conditions in (12.13), as shown in the following example.

Example 12.3: Apply the boundary conditions to find values for C_1 and C_2 and hence obtain the complex valued temperature.

Solution: Applying the boundary condition at $x = 0$, $\hat{U} = qe^{i\omega t}$ gives

$$-kb(C_1 - C_2)e^{i\omega t} = qe^{i\omega t}, \quad (12.19)$$

and hence

$$C_1 - C_2 = \frac{-q}{kb}.$$

Applying the other boundary condition, as $x \rightarrow \infty$, noting that $e^{-bx} \rightarrow 0$ and $e^{bx} \rightarrow \infty$ (see Exercises, Question 4), the condition can only be satisfied if

$$C_1 = 0.$$

Hence we obtain $C_2 = q/kb$.

Substituting for C_1 and C_2 back into (12.18), the complex valued temperature $\hat{U}(x, t)$ is now given by

$$\hat{U}(x, t) = \frac{q}{kb} e^{-bx} e^{i\omega t}. \quad (12.20)$$

The actual temperature is obtained by taking the real part of this complex valued solution. Using the expression for b from (12.16), and Euler's identity $e^{i\theta} = \cos(\theta) + i \sin(\theta)$ (whence $(1+i)/\sqrt{2} = e^{i\pi/4}$), allows us to write (12.20) in the form

$$\begin{aligned}\hat{U}(x, t) &= \left(\frac{q}{k} \sqrt{\frac{\alpha}{\omega}} \right) e^{-i\pi/4} e^{-\sqrt{\omega/2\alpha}(1+i)x} e^{i\omega t} \\ &= \left(\frac{q}{k} \sqrt{\frac{\alpha}{\omega}} e^{-\sqrt{\omega/2\alpha}x} \right) e^{i(\omega t - \sqrt{\omega/2\alpha}x - \pi/4)}.\end{aligned}$$

Taking the real part of $\hat{U}(x, t)$ (using Euler's identity) yields the expression for the real-valued temperature

$$U(x, t) = \left(\frac{q}{k} \sqrt{\frac{\alpha}{\omega}} e^{-\sqrt{\omega/\alpha}x} \right) \cos \left(\omega t - \sqrt{\frac{\omega}{\alpha}}x - \pi/4 \right). \quad (12.21)$$

This can also be done using **Maple**, with the built-in functions **Re** and **evalc**. This is illustrated in the **Maple** code Listing 12.1.

Listing 12.1: **Maple** code: c_hp_soiltemp.txt

```
> restart;
> interface(imaginaryunit=i); I:='I':
> assume(omega>0,alpha>0,k>0);
> b:=(omega/alpha)^(1/2)(1+_i)/2^(1/2);
> U:=-q/(k*b)*exp(-b*x)*exp(_i*omega*t);
> evalc(Re(U));
```

We can summarise the solution technique described above for oscillatory problems involving linear PDEs:

- Introduce a complex valued function $\hat{U}(x, t) = U(x, t) + iV(t)$ where $U(x, t)$ is the solution we seek, and assume its form to be $\hat{U} = F(x)e^{i\omega t}$.
- Substitute this into the PDE, where all the $e^{i\omega t}$ terms cancel leaving a constant coefficient, ODE for $F(x)$.
- Substitute for $F(x) = e^{\lambda x}$ and hence obtain the general solution for \hat{U} .
- Apply the boundary conditions to find the arbitrary constants.
- Take the real part of $\hat{U}(x, t)$ to obtain $U(x, t)$ (using Euler's identity).

Interpreting the solution

A general oscillation can be expressed in the form

$$A \cos(\omega(t - \phi)),$$

where A is the *amplitude* of the oscillation and ϕ is the *phase-shift*. The amplitude measures the maximum extent of the oscillation and the phase-shift measures the time difference between the oscillatory behaviour of the above function and that of $A \cos(\omega t)$. The first term in brackets in (12.21), which is the coefficient of the cosine term, represents the amplitude of the oscillation. This term decreases exponentially with depth x . The phase-shift is proportional to the depth. This means the oscillation lags behind what is happening at the ground surface, by an amount which increases with depth. Furthermore, the temperature on the surface also lags behind the heat flux applied to the surface by an amount $\pi/(4\omega)$.

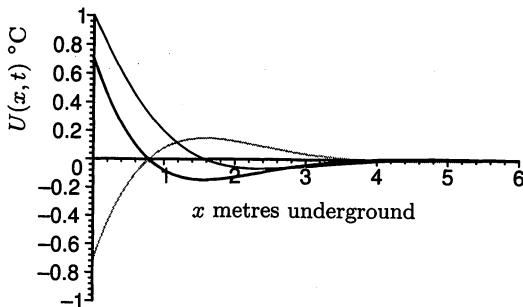


Figure 12.1: Maple graph of the ground temperature at different times resulting from an oscillating heat flux applied at the surface. (Solution to equation (12.21) with parameters $k = 1$, $\alpha = 1$ and $q = 1$.) The temperature is plotted at times $\omega t = 0$ (black), $\omega t = \pi/4$ (thin black) and $\omega t = \pi$ (grey) radians, with $\omega = 1$.

(which is equivalent to 1/8 of a day). Figure 12.1 provides a typical sketch of the temperature profiles at different times, which illustrates this decrease in amplitude, and an increase in phase-shift, with depth.

In practice, for soils, growth in the exponential term is sufficiently large for the amplitude to die off quickly. This implies that soil is a very good insulator from temperature variations caused by daily fluctuations. As was mentioned at the beginning of the section, wine requires a close to constant temperature for storage. Thus wine is stored in cellars; which do not have to be very deep to reduce the temperature variation to less than 1 °C.

Summary of skills developed here:

- Write down appropriate governing equations for a heat conduction problem with an oscillatory driving temperature.
- For a problem with oscillating temperatures, convert to a complex valued problem to simplify the analysis.
- Convert back from the complex valued temperatures to real valued temperatures.

12.3 Case Study: Detecting land mines

In the following case study we use the above theory of oscillatory temperatures of a soil mass to detect non-soil objects buried beneath the earth's surface. The main idea is that such objects change the oscillatory pattern sufficiently to be detected and, furthermore, different buried objects such as land mines or rocks produce changes sufficiently different to allow us to distinguish between them. This case study is based on Miller (1995), Davies (1994) and discussions with Tony Richings.

Land mines are a serious problem. Decades after a war, land mines still maim and cripple civilians with an enormous impact on their communities. Large areas on the Falkland islands are fenced off and unsafe for use as a result of a war fought more than 15 years ago. In November 1998, Hurricane Mitch together with floods and mud slides in Nicaragua and Honduras exposed long buried land mines, carrying them to previously clear regions and creating further distress. Many countries around the world are affected by mines: El Salvador, Ethiopia, Peru, Argentina, Afghanistan, Angola, Mozambique, Iraq, Iran, Kuwait, Korea and Cambodia, to name some. Cambodia is probably one of the worst affected countries in that the mines are widespread and regularly moved by monsoon floods. Although the war there ended some six years ago, there were in 1999 an average of 350 accidents a month due to land mines, which represents a substantial proportion of the total Cambodian population of six million. The safe detection and removal of land mines thus assumes crucial importance.

A common method for spotting land mines is to use metal detectors; however, land mines are now often made of plastic to avoid detection. Another approach, being investigated by the Defence Science and Technology Organisation in Australia, is to use thermal infrared imaging. This works because the temperature of the ground just above a buried land mine is different from the temperature of the surrounding ground. The mine produces a 'hot spot' on the surface during the day and a 'cold spot' at night.

To gain a better understanding of this process, mathematical models were developed by CSIRO (Miller (1995)), which could be used to investigate such effects as the depth and size of the land mine, the thermal properties of land mine and soil, as well as the weather conditions. In fact two models were produced: a full three dimensional model used for quantitative predictions and a one dimensional model for initial investigations. We consider a slightly simplified version of the one dimensional model.

Consider a land mine buried at a depth d of $d = 5 \text{ mm} = 0.005 \text{ m}$. The ground is heated by the sun during the day and cools at night, and we make use of this oscillatory heat flux $Q(t)$ to model the heating and cooling processes.

We neglect any horizontal heat flow in the ground above the land mine and thus consider a one dimensional model where the heat flow is only in the vertical direction. This is illustrated in Figure 12.2 below. We also let $U_1(x, t)$ denote the temperature of the soil and $U_2(x, t)$ denote the temperature of the land mine, where t is time and x denotes distance into the ground from the surface. The surface of the soil is taken at $x = -d$ and the interface between the soil and mine at $x = 0$.

The temperature at any point will be oscillating with time due to the diurnal cycle of heating and cooling of the sun, therefore the temperatures will not reach equilibrium but depend continuously on time and depth. The heat equation is used to model these temperatures of the soil above the land mine, as well as the land mine (with the configuration illustrated in Figure 12.2) resulting in the equation system

$$\frac{\partial U_1}{\partial t} = \alpha_1 \frac{\partial^2 U_1}{\partial x^2}, \quad -d < x < 0, \quad (12.22)$$

$$\frac{\partial U_2}{\partial t} = \alpha_2 \frac{\partial^2 U_2}{\partial x^2}, \quad 0 < x < \infty, \quad (12.23)$$

where

$$\alpha_1 = \frac{k_1}{\rho_1 c_1}, \quad \alpha_2 = \frac{k_2}{\rho_2 c_2}. \quad (12.24)$$

The parameter values k_1 and k_2 are the conductivities of the soil and land mine respectively, ρ_1 and ρ_2 the densities of the soil and land mine and c_1 and c_2 the specific heats of the soil and mine.

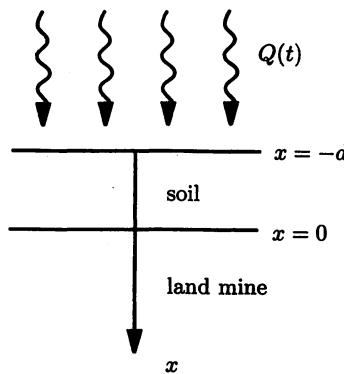


Figure 12.2: The 1-D model for thermal detection of a land mine. The soil surface is heated by day and cools at night; this is represented by an oscillatory heat flux $Q(t)$ applied to the soil surface.

The associated boundary conditions are as follows. On the surface of the soil ($x = -d$) the heat flux has a specified value $Q(t) = q \cos(\omega t)$, where q is the amplitude of the oscillatory heat flux. By Fourier's law this gives the boundary condition at $x = -d$

$$-k_1 \frac{\partial U_1}{\partial x} = q \cos(\omega t) \quad \text{at } x = -d. \quad (12.25)$$

Where the soil and the top of the land mine join ($x = 0$) we impose a condition for the continuity of temperature and heat flux,

$$U_1(0, t) = U_2(0, t), \quad -k_1 \frac{\partial U_1}{\partial x} = -k_2 \frac{\partial U_2}{\partial x} \quad \text{at } x = 0. \quad (12.26)$$

A further boundary condition, specified for $x \rightarrow \infty$, states that the rate of flow of heat, or heat flux, is zero. Hence

$$-k_2 \frac{\partial U_2}{\partial x} = 0, \quad \text{at } x = \infty. \quad (12.27)$$

The governing equations can be solved to obtain a solution for the soil surface temperature (at $x = -d$), given by the real part of the following complex function:

$$\hat{U}_1(-d, t) = \frac{q}{k_1 b_1} \left(\frac{(1+r) + (1-r)e^{-2b_1 d}}{(1+r) - (1-r)e^{-2b_1 d}} \right) e^{i\omega t}, \quad (12.28)$$

where

$$r = \frac{k_2 b_2}{k_1 b_1}, \quad b_1 = \sqrt{\frac{\omega}{\alpha_1}} \frac{(1+i)}{\sqrt{2}}, \quad b_2 = \sqrt{\frac{\omega}{\alpha_2}} \frac{(1+i)}{\sqrt{2}}.$$

Note that, in the limit as $d \rightarrow \infty$, this expression for the surface temperature reduces to the (complex valued) temperature for the surface temperature for a semi-infinite soil. (Semi-infinite soil here refers to deep soil with no buried land mine, as would be found in the vicinity of the land mine.) This is the same as setting $r = 1$, so that the thermal parameters of the land mine are the same as those of the surrounding soil.

One thing we notice about the solution (12.28) is that the temperature is proportional to q , the amplitude of the periodic heat flux term. In practice this quantity might be difficult to measure directly. However, it can be determined by measuring the temperature in soil where there are no buried objects.

We might then be interested in comparing the relative temperature of the soil surface above the land mine with the temperature in an infinite homogeneous soil. This is shown in Figure 12.3 for a MK5 land mine, buried under 5 mm of soil. We have taken

$$\alpha_1 = 0.6 \times 10^{-6}, \quad \alpha_2 = 0.1 \times 10^{-6}, \quad k_1 = 1.2, \quad k_2 = 0.26.$$

The temperature on the soil surface above the land mine (solid line) at approximately noon, $t = 0$ days, is greater than the surface temperature for soil without any buried mine (dotted line). Similarly, at night ($t = 0.5$ days) the temperature is lower than that for the soil without a land mine. (Actually, there is a small phase-lag in the soil temperatures without the land mine which can be observed by examining the graph carefully.) Thus, where the land mine is buried, we see a ‘hot spot’ on the soil surface during the day and a ‘cold spot’ at night.

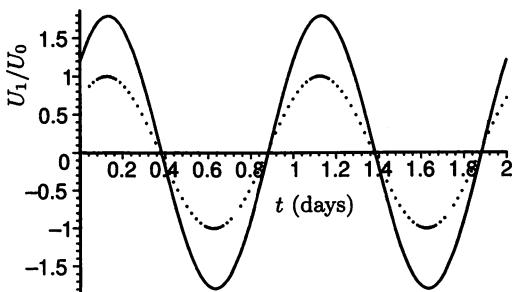


Figure 12.3: Normalised temperature of soil surface versus time in days t . The solid line is the normalised surface temperature for an MK5 land mine buried 5 mm below the surface. The dotted line is the normalised temperature of the soil surface when there is no buried object. Both temperatures have been expressed as multiples of the maximum surface temperature, $U_0 = q/(kb)$, for a soil with no buried land mine.

A similar calculation for a buried rock ($\alpha_2 = 1.1 \times 10^{-6}$, $k_2 = 2.5$) shows the opposite, that is, a ‘cold spot’ during the day and a ‘hot spot’ at night. This makes thermal detection attractive from the point of view of discriminating land mines from other buried objects.

Our model predictions are not as accurate as we might have hoped: the predicted temperature variations are far too large. This is because the model is only one-dimensional and fails to account for heat conduction in the horizontal direction. When the model is extended to include this conduction the temperature variations become far smaller.

Hopefully, such models will be relevant in the clearing of land mines in affected countries. There is a current political drive to ban land mines completely and most countries, including Australia, have signed such an agreement at the Ottawa Convention in 1997. However, the USA, China and Vietnam refused. It is an interesting (and possibly unrelated) fact that while land mines are manufactured in several countries, the USA, China and Vietnam are the principal manufacturers of land mines today.

12.4 Lake pollution revisited

We return to the problem of modelling the levels of pollution in a lake from Section 2.5. There are a number of processes by which a mass of pollutant may travel through a lake.

Previously we considered a lake in which the pollution was well mixed and was transported with the water flow. This means of transport is advection, which is equivalent to the convection of heat. Below, we relax the ‘well mixed’ assumption, allowing the pollution levels to be different at different locations within the lake, although again advection is responsible for the movement of pollution. Modelling this process leads to a PDE for the pollution concentration, which we derive and solve analytically using a technique known as the method of characteristics. We also indicate how **Maple** can be used to solve the equation, and generate solutions for given boundary conditions. In the final section, the much slower diffusive process of pollution transport is introduced, and a PDE to model this developed.

Model assumptions and approach

We assume pollution to travel solely by water flow through the lake, that is, by advection. We assume a unidirectional flow of water from its entrance into, to its exit from, the lake. Thus if x is in the direction of water flow, and y and z are the other axes in 3D space, then, for any fixed x , there is no change in pollution concentration in the y and z directions. Once again, we consider the volume V to be constant, as well as the flow of water into, and out of, the lake, and the pollution source will be from a feeder river at one end. However, unlike in our previous lake pollution model from Section 2.5, in this case pollution will not be assumed to be of uniform concentration throughout the lake. Alternatively, we model pollution levels at any location x within the lake, and at any time t . To do this we make an assumption about the shape of the lake: it is taken as rectangular! As justification for this simplification, in many lakes there is a main channel of flow from entrance to exit, where advection is the dominant means of pollution movement through the lake. This channel is often, roughly, rectangular in shape. As the formulation progresses below, you will see how the non-rectangular shape of real lakes might be dealt with, as well as changes in pollution concentration in the y and z directions.

The concentration C of pollutant depends on both position x (horizontal distance from its entry point) in the lake, and time t . Hence we let $C(x, t)$ be the concentration, and examine how it varies throughout the lake with time. The pollution mass flux (across a unit of area per unit of time) will be denoted $J(x, t)$, at position x and time t . We consider the movement of pollution in one direction only: the direction of changing x .

Consider a lake, estimated by a rectangular shape of length $(b - a)$, and constant cross-sectional area A , as illustrated in Figure 12.4. Thus $a \leq x \leq b$. Here $F(x, t)$ is taken as the

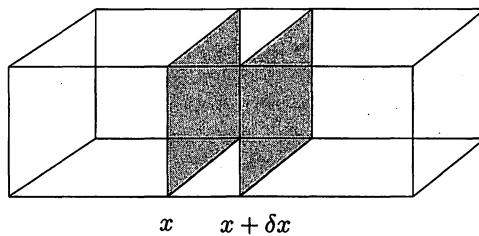


Figure 12.4: Diagram of a ‘rectangular’ lake, from $x = 0$ to $x = b$.

horizontal water flow (across area A per unit of time), so that $F(a, t)$ is the horizontal water flow across area A at the entrance to the lake, and $F(b, t)$ the horizontal water-pollution flow across area A leaving the lake.

Formulating the PDE

The input-output principle, or balance law, provides us with a basis on which to build the equations. Applied to the total mass of pollutant in a section of width δx ,

$$\left\{ \begin{array}{l} \text{rate of change} \\ \text{of pollutant} \\ \text{mass} \end{array} \right\} = \left\{ \begin{array}{l} \text{rate pollutant} \\ \text{mass enters} \\ \text{at } x \end{array} \right\} - \left\{ \begin{array}{l} \text{rate pollutant} \\ \text{mass leaves} \\ \text{at } x + \delta x \end{array} \right\}. \quad (12.29)$$

If we let $C(x, t)$ denote the pollutant concentration, and let $J(x, t)$ denote the pollutant mass flux at position x and time t , and if A is the area of the cross-section (assumed constant), then equation (12.29) becomes

$$\frac{\partial}{\partial t}(CA\delta x) = J(x, t)A - J(x + \delta x, t)A. \quad (12.30)$$

Note that each of the three terms in (12.30) has units of mass divided by time. Furthermore, the rate of change with time is given by a partial derivative since the quantities involved depend on both time t and position x . We divide throughout by $A\delta x$ to obtain

$$\frac{\partial C}{\partial t} = - \left(\frac{J(x + \delta x, t) - J(x, t)}{\delta x} \right).$$

Letting $\delta x \rightarrow 0$ yields the equation

$$\frac{\partial C}{\partial t} = - \frac{\partial J}{\partial x}. \quad (12.31)$$

We have assumed that the pollutant is transported solely by the motion of the fluid: that is, via advection. We need an expression for the pollutant mass flux due to the motion of the water-pollutant mixture. F was defined as the flow rate (litres per second) of the mixture, which we assumed to be constant. So the rate of flow of pollutant mass (at the point x) is FC . The pollutant mass flux (rate of flow of mass per unit area per unit time) is thus given by

$$J(x, t) = \frac{FC(x, t)}{A}. \quad (12.32)$$

Substituting this into equation (12.31) yields a PDE for the pollutant concentration,

$$\frac{\partial C}{\partial t} + \frac{F}{A} \frac{\partial C}{\partial x} = 0. \quad (12.33)$$

On the boundaries of the lake we place conditions, which allow the constants of integration to be established, just as initial conditions were required for the solution of differential equations in previous sections. (For further details on boundary conditions see Section 11.1.) We assume that pollution is entering the lake at $x = a$, such that

$$C(a, t) = \frac{g(t)}{F}, \quad (12.34)$$

where $g(t)$ is a time dependent function describing the total amount of pollution entering the lake at a . A second condition is placed on the pollution concentration at location x , when $t = 0$: that is, the initial concentration at each location x within the lake.

$$C(x, 0) = P(x). \quad (12.35)$$

Analytical solution

We now need to solve the PDE (12.33), together with its boundary conditions, equations (12.34) and (12.35). We examine the simple case where the system has settled to some steady state, or equilibrium, in the exercises (see Question 9), and deal with the general case below.

We are interested in establishing the changing level of pollution over time, at any location in the lake, before any steady-state solution is reached. For this we introduce the *method of characteristics*, in its simplest form. It is a method commonly used for the solution of PDEs, but the idea is not extended in this text. (The reader is referred to any text on PDEs for further details on this, and other, solution methods.)

Example 12.4: Suppose pollution is entering a lake from a river at one end. Modelling the spread of pollution with (12.33), establish the level of pollution at any time and position (horizontal distance from the pollution source) in the lake.

Solution: We need to solve

$$\begin{aligned} \frac{\partial C}{\partial t} + \frac{F}{A} \frac{\partial C}{\partial x} &= 0 \\ C(a, t) &= \frac{g(t)}{F} \quad (\text{possibly } 0) \\ C(x, 0) &= P(x). \end{aligned} \tag{12.36}$$

Rearranging the PDE above

$$\frac{\partial C}{\partial t} = -\frac{F}{A} \frac{\partial C}{\partial x}. \tag{12.37}$$

This implies that the rate of change of C with respect to t is a constant ($-F/A$) times the rate of change of C with respect to x . Set constant $(F/A) = v$. Then,

$$\frac{\partial C / \partial x}{\partial C / \partial t} = -\frac{A}{F} = -\frac{1}{v} = -\frac{dt}{dx}.$$

So v is a velocity, and solving the simple differential equation on the RHS, $t = x/v - K_1$ (with K_1 an arbitrary constant), or $x = vt + K$ where $K = vK_1$. What we now show is that pollution concentration C is constant along these lines (curves) defined by x and t .

From the definition for the derivative of a function of two variables $C(x, t)$, we have that

$$\begin{aligned} \frac{dC}{dt} &= \frac{\partial C}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial C}{\partial t} \frac{\partial t}{\partial t} \\ &= \frac{\partial C}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial C}{\partial t}. \end{aligned}$$

Let us return now to equation system (12.36) and apply this definition. With $x = vt + K$,

$$\frac{\partial C}{\partial t} + \frac{F}{A} \frac{\partial C}{\partial x} = 0 = \frac{dC}{dt}.$$

Thus along the lines $x = vt + K$, the pollutant concentration C is constant. This is a very powerful result (as will be seen below), and these lines are known as characteristics.

Next we establish the value of C on each characteristic (and thus at any specific time and location) using the boundary conditions. Consider the (x, t) -plane: The particular line passing through $(a, 0)$ can be determined by setting $x = a$ and $t = 0$ in $x = vt + K$, and solving for K . The associated characteristic is

$$t = \frac{A}{F}(x - a).$$

Similarly a series of characteristics can be established. These are illustrated in Figure 12.5 in the positive (x, t) -plane.

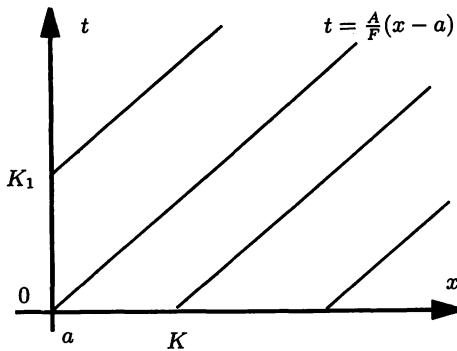


Figure 12.5: Diagram of characteristics.

Notice that when $t < A(x - a)/F$ then the characteristics cut the x -axis at $(K, 0)$, which implies that the value of C (which is constant along line $x = vt + K$) is $P(x) = P(K)$, from the second boundary condition in system (12.36). These values for $P(x)$ are the initial values of pollution at each x -location in the lake.

If $t > A(x - a)/F$ then $t = A(x - a)/F + K_1$ for some positive constant K_1 . The point of intersection between this characteristic and the boundary $x = a$ is at $t = K_1$. On substitution into the first boundary condition of system (12.36),

$$C(a, t) = \frac{g(t - A(x - a)/F)}{F} = \frac{g(K_1)}{F}.$$

Thus we have as a solution for any location x , $a \leq x \leq b$, and with $v = F/A$,

$$C(x, t) = \begin{cases} g\left(t - \frac{A(x-a)}{F}\right)/F & \text{if } t > \frac{A(x-a)}{F} \\ P(x - \frac{Ft}{A}) & \text{if } t < \frac{A(x-a)}{F}. \end{cases} \quad (12.38)$$

We can now look at the cleaning time this model of a lake predicts, in the case where no further pollution enters the lake, and compare it with the cleaning time in our previous model from Section 2.5 where the automatic mixing of a pollutant was assumed. This is a particular case of the example above, when no pollution enters the lake, and thus the boundary condition at $x = a$ and $t > 0$ is $g(t) = 0$, so that $C(a, t) = 0$. The resulting solution is

$$C(x, t) = \begin{cases} 0 & \text{if } t > \frac{x-a}{v} \\ P(x - vt) & \text{if } t < \frac{x-a}{v}. \end{cases} \quad (12.39)$$

Note that for $t > (b - a)/v$ (or $x > b$) there are no positive values for the pollution concentration in the lake, whence it is clean. The complete cleaning time is given by $(b - a)/v = A(b - a)/F$. Thus v is the velocity, in the x -direction, of the pollution through the lake.

Equation (12.39) describes a very important phenomenon: that of *wave motion*. Over time, the wave (of pollution) propagates in the x -direction, with velocity v . This notion can be understood by examining the equation solution as t increases. Notice that $P(a) = P(x - vt)$ for some x and t (from (12.39)), which implies that the pollution level at location $x - vt$ is the same as that at a . Alternatively stated, wherever $a = x - vt$ or $x = a + vt$. This latter equation implies that the pollution level at location a will have moved in the positive x -direction with velocity v in time t : the wave shifts to the right with velocity v . This moving wave phenomenon via advection is clarified in the sequence of diagrams of Figure 12.6.

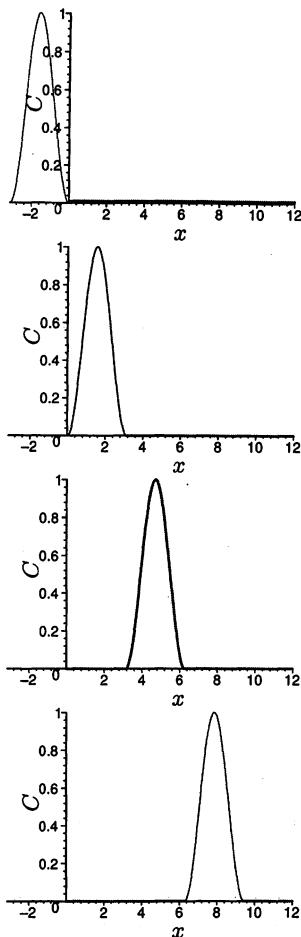


Figure 12.6: Illustration of solution (12.39), with a wave of pollution $P(x) = \sin^2(x - vt)$ arriving at an unpolluted lake. For simplicity $v = 1$, and the ‘rectangular’ lake stretches from $a = 0$ to $b = 12$.

The system is initialised with a ‘wave’ of pollution, $g(t)$, at the boundary $a = 0$, as in the first graph. For illustration the lake is assumed pollution free at this initial time. Each subsequent plot is the level of pollution in the lake for $a \leq x \leq b$, at a later time, with $t = t + nh$, where $n = 1, 2, \dots$ and $h = \pi$ for the illustrated figure.

In order to calculate the time required for a polluted lake, with fresh water flowing into it, to have its pollution level reduced to 5% of its current level, we apply (12.39), with $g(t) = 0$ and $P(x, 0) \neq 0$ for some x , where $a \leq x \leq b$. To establish the time required for such a reduction in the amount of pollution in the entire lake, we convert concentration to mass $M(x, t)$, using $C = M/V(x)$, where $V = \Delta x \times \text{width} \times \text{depth}$ is the volume of water at location x . Since width and depth are assumed constant, and $C\Delta x$ is a measure of mass, the solution of t in

$$\int_a^b C(x, t) dx = 0.05 \int_a^b P(x) dx$$

is the answer sought. This can be simplified to

$$\int_a^b P(x - vt) dx = 0.05 \int_a^b P(x) dx$$

$$\int_{a-vt}^{b-vt} P(s) ds = 0.05 \int_a^b P(x) dx.$$

Numerical solution

Maple can be used to solve the PDE (12.33), and obtain a general solution. Then, with specified boundary conditions, we can examine how the system changes with time and space. The code for this is given in Listing 12.2. The inclusion of the orientation here allows you to plot t or x against pollution concentration. Alternatively, excluding orientation keyword will result in a 3D plot, with a time axis, a spatial axis, and a pollution concentration axis.

Listing 12.2: Maple code: c_hp_lakepde.txt

```
> restart:with(PDEtools):
> pdelake := diff(C(x,t),t) + F/A*diff(C(x,t),x) = 0;
> generalSolution:=pdssolve(pdelake);

#Plotting the solution for given conditions
> PDEplot(pdelake,[y,5*(y-2),1+sin(5*(y-2))],
           y=0..50,orientation=[1,90]);
> PDEplot(pdelake,[y,5*(y-2),1+sin(5*(y-2))],
           y=0..50,orientation=[-90,90]);
```

Extensions and diffusion

Although advection of pollution in the main central channel of a lake is the main contributor to the movement of pollution through it, there are many other processes at work; chemical processes, for example. Acting at a molecular level, certain nutrients react with chemicals to create further pollutants.

Another process operating at the microscopic level is diffusion. Diffusion may be the main process for reducing pollution in sheltered bays, away from the main advection channel, and as such is essential to understand in order to ascertain safety levels for food sources from, or recreational sports in, these regions. Diffusion operates in all directions. Thus the diffusion of chemicals is a slow process, compared with the advection modelled above, and occurring in all directions puts it beyond the scope of this book. However, we will develop the diffusion equation in a single direction to provide an understanding of how the process works.

For simplicity we will return to Figure 12.4, and imagine that diffusion may occur in the y direction only, at right angles to the direction of the advection in direction x . Using the same thin rectangular slice (in Figure 12.4) as before, let the face of the slice facing the y direction have area B . Thus $B = \Delta x \times \text{depth}$ and is comparable with A of the advection model. Let $J(y, t)$ be the pollution flux through a diffusive process across a unit of area per unit of time. Then, in the same way as PDE (12.33) was developed for $C(x, t)$, with pollution flux $J(x, t)$ via advection in the x -direction across area A , we can develop a PDE for diffusion in the y direction across area B , to give

$$\frac{\partial C}{\partial t} + \frac{\partial J}{\partial y} = 0. \quad (12.40)$$

Since pollution moves from regions of high concentration to regions of low concentration, the flux J is proportional to the negative gradient of pollution concentration C . Thus we

write

$$J = -D \frac{\partial C}{\partial y},$$

where D is the constant of proportionality (called the diffusivity). Thus we have the diffusion equation in one direction,

$$\frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial y^2}. \quad (12.41)$$

We will not solve this equation here, nor develop it to 3D space, but the interested reader will find details in any text on PDEs. However, the solution to the diffusion equation (12.41) will follow from the solution to the heat equation (see Section 12.2), which takes the same form, although with different boundary conditions.

Summary of skills developed here:

- Formulate a PDE to describe the advection of a pollutant from some source in a lake, such as a river.
- Understand the differences between modelling pollution spread via advection and diffusion.
- Distinguish cases where the use of an ODE is appropriate, and where a partial differential equation is appropriate.

12.5 Exercises for Chapter 12

12.1. Non-equilibrium temperature model. For Question 2, from Chapter 9, show the non-equilibrium temperature satisfies a PDE of the form

$$\frac{\partial U}{\partial t} = \alpha \frac{\partial^2 U}{\partial x^2} + \beta$$

where α and β are constants you must determine.

12.2. Non-equilibrium temperature for a cylinder. Extend the formulation in Question 6 of Chapter 9 to obtain a PDE for the non-equilibrium temperature $U(r, t)$.

12.3. Non-equilibrium temperature for a cylindrical heat fin. In Question 8 of Chapter 9 you found an ODE for the equilibrium temperature. Modify the derivation to obtain the following PDE for time-dependent temperature

$$c\rho \frac{\partial U}{\partial t} = k \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial U}{\partial r} \right) - 2h(U - u_s),$$

where c is the specific heat, ρ is the density, k is the conductivity and h is the convective heat transfer coefficient.

12.4. Using Euler's identity. In Section 12.2 limits were required to establish values for the constants of integration, which led to equation (12.20). Given $b = a(1 + i)$ where a is a positive constant, what are

$$\lim_{x \rightarrow \infty} e^{bx} \quad \text{and} \quad \lim_{x \rightarrow -\infty} e^{-bx}?$$

What if $a < 0$? Show that your results lead to the solution given in (12.20). (Hint: Use Euler's identity.)

12.5. Oscillating surface temperature. Consider the PDE

$$\frac{\partial U}{\partial t} = \alpha \frac{\partial^2 U}{\partial x^2}$$

with the boundary conditions

$$U(0, t) = u_1 \cos(\omega t), \quad -k \frac{\partial U}{\partial x} = 0, \quad \text{at } x = \infty.$$

These are the equations for the temperature in the ground due to a given oscillating temperature on the surface.

Solve these equations, leaving your answer as the real part of a complex valued temperature.

12.6. Amplitude of oscillating temperature. Find the amplitude of the complex valued temperature,

$$\hat{U}(x, t) = \frac{q}{kb} e^{-bx} e^{i\omega t}, \quad b = \sqrt{\frac{\omega}{2\alpha}}(1 + i), \quad i = \sqrt{-1}.$$

(Hint: Take the modulus of the complex valued temperature.)

12.7. Land mines. Solve the heat equation system from the land mine case study in the same way as the infinite soil problem from Section 12.2. The following differential equations and boundary conditions (equations (12.22) to (12.24) of Section 12.3 with boundary conditions (12.25) to (12.27)) are a model for the temperature of a land mine, $U_2(x, t)$, and the temperature of the soil, $U_1(x, t)$, above the land mine:

$$\begin{aligned} \frac{\partial U_1}{\partial t} &= \alpha_1 \frac{\partial^2 U_1}{\partial x^2}, & -d < x < 0, \\ \frac{\partial U_2}{\partial t} &= \alpha_2 \frac{\partial^2 U_2}{\partial x^2}, & x > 0, \\ -k \frac{\partial U_1}{\partial x} &= q \cos(\omega t), & x = -d \\ U_1 &= U_2, & x = 0 \\ -k \frac{\partial U_1}{\partial x} &= -k_1 \frac{\partial U_2}{\partial x}, & x = 0 \\ -k_1 \frac{\partial U_2}{\partial x} &= 0, & x = \infty. \end{aligned}$$

- (a) Solve a modified problem, introducing complex valued functions by replacing $\cos(\omega t)$ with $e^{i\omega t}$.

Assume a form of the solution $\hat{U}_1(x, t) = F_1(x)e^{i\omega t}$, and $\hat{U}_2(x, t) = F_2(x)e^{i\omega t}$. Substitute into the PDEs (partial differential equations) and obtain ODEs (ordinary differential equations) by using the fact that F_i is a function of x alone.

Solve the ODEs to get a general complex valued solution, by assuming $F_1(x) = e^{\lambda_1 x}$ and $F_2(x) = e^{\lambda_2 x}$.

Apply the boundary conditions to find the arbitrary constants.

Hence obtain a solution for the soil surface temperature (at $x = -d$), given by

$$U_1 = \frac{q}{k_1 b_1} \left(\frac{(1+r) + (1-r)e^{-2b_1 d}}{(1+r) - (1-r)e^{-2b_1 d}} \right) e^{i\omega t}, \quad x = -d$$

$$r = \frac{k_2 b_2}{k_1 b_1}, \quad b_1 = \sqrt{\frac{\omega}{\alpha_1}} \frac{(1+i)}{\sqrt{2}}, \quad b_2 = \sqrt{\frac{\omega}{\alpha_2}} \frac{(1+i)}{\sqrt{2}}.$$

- (b) We finally need to extract the real part of this complex valued expression to get an appropriate surface temperature. Use Maple to do this.
- (c) Show that, in the limit as $d \rightarrow \infty$, this expression for the surface temperature reduces to the (complex valued) temperature for the surface temperature of a semi-infinite soil which was obtained directly in Section 12.2.

12.8. Depth of buried land mine. For the land mine case study, Section 12.3, use Maple to produce a graph of the amplitude of the soil surface temperature as a function of the depth the land mine is buried. Discuss the physical reasons for the behaviour in the graph.

12.9. Lake pollution in steady state. Suppose pollution has been entering a lake from a river at one end over a period of time, such that a stable steady state situation has been reached. Modelling the spread of pollution with (12.33), establish the level of pollution at any time and position (horizontal distance from the pollution source) in the lake.

12.10. Including a variable flow rate into the lake pollution model. In the case of a non-constant F , with $F(x,t)/A = v(x,t)$, velocity, show that equation (12.33) becomes

$$\frac{\partial C}{\partial t} + \frac{\partial}{\partial x} (C(x,t) v(x,t)) = 0.$$

Appendix A

Differential equations

A.1 Properties of differential equations

A *differential equation* is an equation which involves the derivative(s) of some unknown function. Such equations occur in many forms and thus specific terminology, outlined below, is used to classify them. Solving a differential equation involves finding an expression for the unknown function, for which there are a variety of analytical techniques available. Since not all equations can be solved analytically, numerical methods have been developed and are widely used. Some analytical techniques relevant to this text are introduced below, with a brief introduction to numerical methods covered in Chapter 4.

First some terminology relevant to the discussion of differential equations is introduced:

- The *order* of a differential equation is the highest derivative occurring in it. Note that the order of the differential equation is the same as the number of arbitrary constants occurring in the general solution. For example, the differential equation

$$\left\{ \begin{array}{l} \frac{dy}{dt} - y = 0 \end{array} \right.$$

is first-order and has a general solution $y = C_1 e^t$, which contains only one arbitrary constant C_1 . The differential equation

$$\left\{ \begin{array}{l} \frac{d^2y}{dt^2} - y = 0 \end{array} \right.$$

is second-order. Its general solution is $y = \underbrace{C_1 \cosh(t)}_{\text{arbitrary constants}} + \underbrace{C_2 \sinh(t)}$, which contains two arbitrary constants.

- A differential equation is *linear* if it can be expressed in the form

$$a_n(t) \frac{d^n y}{dt^n} + a_{n-1}(t) \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1(t) \frac{dy}{dt} + a_0(t)y = F(t),$$

where the a_i and F are functions of t alone. Otherwise it is *nonlinear*. For example

$$\frac{dy}{dt} - y = 0 \quad \text{and} \quad t \frac{dy}{dt} - y = \sin t$$

are linear, whereas

$$y^3 \frac{dy}{dt} - y = 0 \quad \text{and} \quad \frac{dy}{dt} - \sin y = 0$$

are nonlinear.

- An equation is said to be *homogeneous* when it involves only terms which are functions of the dependent variable and/or its derivatives. That is, no term must be a constant term or a function of the independent variable alone. The equation is *inhomogeneous* if a term involving the independent variable is included. Thus, for example,

$$\frac{dy}{dt} - y = 0$$

is homogeneous while

$$\frac{dy}{dt} - y = 2t$$

is inhomogeneous.

- A *system of differential equations* is a group of differential equations (more than one) which together describe some phenomenon. The equations are *coupled* if they depend on each other, such as in a predator-prey system

$$\frac{dx}{dt} = \beta x - c_1 xy, \quad \frac{dy}{dt} = c_2 yx - \alpha y.$$

They are *uncoupled* otherwise.

- The system is called a *dynamical system* when the independent variable is time and the solution used to predict future states. That is, the system describes how the modelled phenomenon changes with time. The coupled system above is thus a dynamical system.
- Differential equations are *ordinary differential equations* (ODEs) when there is only one independent variable, such as space or time. In the case of two independent variables, such as dependence on both space and time together, the equations are called *partial differential equations* (PDEs) in which case the derivatives are written with 'curly' d's, as for $y = y(x, t)$ an example PDE is

$$\frac{\partial y}{\partial t} = 2xyt + \sin t.$$

A.2 Solution by inspection

Some differential equations are sufficiently simple for us to spot the general solution by inspection. This requires a good knowledge of the derivatives of the elementary functions such as polynomials, exponential functions, hyperbolic functions and trigonometric functions.

Example 1.1: Find a solution to the differential equation

$$\frac{d^2y}{dt^2} = -b^2 y,$$

where b is a non-zero constant (hence $-b^2$ is negative).

Solution: We look for functions y with the property that their second derivative is a constant squared (b^2), times the function itself (y), and is negative. Functions with this property are the trigonometric functions $\cos(bt)$ and $\sin(bt)$. Hence the general solution is

$$y = C_1 \cos(bt) + C_2 \sin(bt),$$

where C_1 and C_2 are arbitrary constants.

A few other common examples of differential equations whose solutions may be obtained by inspection include:

- $\frac{dy}{dt} = ay$, with solution $y = Ce^{at}$.
- $\frac{dy}{dt} = at$, with solution $y = \frac{a}{2}t^2 + C$.
- $\frac{d^2y}{dt^2} = b^2 y$, with solution $y = C_1 e^{bt} + C_2 e^{-bt}$ or, equivalently, with solution

$$y = c_1 \cosh(bt) + c_2 \sinh(bt).$$

A.3 First-order separable equations

These are first-order differential equations which can be put in the form

$$\frac{dy}{dt} = f(y)g(t),$$

where $f(y)$ is a function of y only and $g(t)$ is a function of t only. We can solve these equations by ‘separating the variables’, that is, by dividing the differential equation through by $f(y)$ and then integrating both sides with respect to the independent variable t .

Example 1.2: Solve the differential equation $\frac{dy}{dt} = 2ty + 2t$ using indefinite integrals.

Solution: We identify this differential equation as a separable differential equation by writing it in the form

$$\frac{dy}{dt} = 2(y + 1)t.$$

Thus, separating the variables, we obtain

$$\frac{1}{y+1} \frac{dy}{dt} = 2t.$$

Integrating both sides with respect to t we obtain

$$\int \frac{1}{y+1} \frac{dy}{dt} dt = \int 2t dt.$$

On the LHS the change of variables rule for integration allows us to change the integration variable from t to y , giving

$$\int \frac{1}{y+1} dy = \int 2t dt.$$

Carrying out the integrations we obtain

$$\ln|y+1| = t^2 + C$$

where C is an arbitrary constant. Hence, solving for y ,

$$|y+1| = e^{t^2+C},$$

or

$$y(t) = Ae^{t^2} - 1,$$

where $A = \pm e^C$ is a convenient relabelling of the arbitrary constant.

Alternatively, the above example could be solved on some fixed interval using definite integrals, and thus not requiring the arbitrary constants, as follows.

Example 1.3: Solve the differential equation $\frac{dy}{dt} = 2ty + 2t$ on the interval $[t_0, t]$.

Solution: As above, this differential equation is a separable differential equation and can be written in the form

$$\frac{dy}{dt} = 2(y + 1)t.$$

Thus, separating the variables, we obtain

$$\frac{1}{y+1} \frac{dy}{dt} = 2t.$$

Integrating both sides with respect to t

$$\int_{t_0}^t \frac{1}{y+1} \frac{dy}{dt} dt = \int_{t_0}^t 2t dt.$$

On the LHS the change of variables rule for integration allows us to change the integration variable from t to y , giving

$$\int_{y_0}^y \frac{1}{y+1} dy = \int_{t_0}^t 2t dt$$

where $y(t_0) = y_0$. Carrying out the integrations we obtain

$$\ln|y+1| - \ln|y_0+1| = t^2 - t_0^2,$$

and then

$$\ln \left| \frac{y+1}{y_0+1} \right| = t^2 - t_0^2.$$

From here y can be solved for as before

$$\left(\frac{y+1}{y_0+1} \right) = \pm e^{t^2 - t_0^2}.$$

This can be written as

$$y(t) = Ae^{t^2} - 1,$$

where $A = \pm(y_0 + 1)e^{-t_0^2}$ is a convenient relabelling of the constant.

A.4 First-order linear equations and integrating factors

These are differential equations which can be put into the form

$$\frac{dy}{dt} + p(t)y = q(t)$$

where functions p and q are functions of t only. To solve these we find an *integrating factor* which helps us to integrate the equation directly. The integrating factor $R(t)$ is given by the formula

$$R(t) = e^{\int_a^t p(t) dt}.$$

If we then multiply throughout by this integrating factor, the differential equation can be converted into the form

$$\frac{d}{dt} (R(t)y(t)) = R(t)q(t),$$

and then the integral (with respect to t) of the LHS is simple to find, with the function on the right a function of t alone.

Example 1.4: Solve the differential equation $\frac{dy}{dt} = 2t(y + 1)$.

Solution: First we see that the differential equation can be written in the appropriate form:

$$\frac{dy}{dt} - 2ty = 2t.$$

Thus $p(t) = -2t$, $q(t) = 2t$ and the integrating factor $R(t)$ is given by

$$R(t) = e^{\int_a^t p(t) dt} = e^{\int_a^t -2t dt} = e^{-t^2+a^2} = Ae^{-t^2},$$

where $A = e^{a^2}$ is constant. Hence, multiplying throughout by this factor, the differential equation becomes

$$\frac{d}{dt} (e^{-t^2} y) = 2te^{-t^2}.$$

Integrating both sides with respect to t we obtain

$$e^{-t^2} y = -e^{-t^2} + C$$

(where C is the arbitrary constant of integration) and so

$$y = -1 + Ce^{t^2}.$$

A.5 Homogeneous equations

We consider here *constant coefficient equations*. These are linear differential equations where the coefficients of each derivative, including that of zero order, are constants and the coefficient of y^0 is 0. This last condition defines the equation as homogeneous. Thus they are a linear combination of derivatives of the dependent variable and include no functions of the independent variable alone.

For first-order equations they are of the general form

$$\frac{dy}{dt} + a_0 y = 0,$$

and for second-order equations they are of the general form

$$\frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = 0,$$

where a_1 and a_0 are constants.

The characteristic equation

To solve such equations we use prior knowledge about their solutions: we look for a solution of the form $y(t) = e^{\lambda t}$, where the constant λ is to be determined. This always leads to an auxiliary equation, or characteristic equation, for λ . The general solution is then a linear combination of the solutions obtained.

Example 1.5: Solve $\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0$.

Solution: Substitute $y = e^{\lambda t}$ into the differential equation, to obtain

$$\lambda^2 e^{\lambda t} - \lambda e^{\lambda t} - 2e^{\lambda t} = 0 \quad \text{so} \quad (\lambda^2 - \lambda - 2)e^{\lambda t} = 0.$$

Hence we obtain the characteristic (auxiliary) equation

$$\lambda^2 - \lambda - 2 = (\lambda + 1)(\lambda - 2) = 0.$$

This quadratic equation has roots $\lambda = -1$ and $\lambda = 2$ and thus the solutions are e^{-t} and e^{2t} . The general solution is a linear combination of e^{-t} and e^{2t} ,

$$y(t) = C_1 e^{-t} + C_2 e^{2t}.$$

Classification of roots

For second-order equations the characteristic equation is a quadratic equation and thus we do not always obtain distinct real roots. The form of the solution depends on the nature of the roots, for which there are three possible cases:

- Real roots: $\lambda = m_1$ and $\lambda = m_2$. Then

$$y = C_1 e^{m_1 t} + C_2 e^{m_2 t}.$$

- Complex roots $\lambda = \alpha \pm \beta i$. Then

$$y = e^{\alpha t} (C_1 \cos(\beta t) + C_2 \sin(\beta t)).$$

- Equal roots: $\lambda = m_1$. A second independent solution can be obtained by multiplying by t . Then

$$y = e^{m_1 t} (C_1 + C_2 t).$$

Two special cases of the above are worth noting:

- Real perfect squares: $\lambda = \pm m_1$. Then

$$y = C_1 e^{m_1 t} + C_2 e^{-m_1 t},$$

or an equivalent form is

$$y = c_1 \cosh(m_1 t) + c_2 \sinh(m_1 t),$$

where the arbitrary constants c_1 and c_2 are linearly related to the arbitrary constants C_1 and C_2 .

- Purely imaginary roots: $\lambda = \pm m_1 i$. Then

$$y = C_1 \cos(m_1 t) + C_2 \sin(m_1 t).$$

A.6 Inhomogeneous equations

We consider again *constant coefficient equations*. This time we include a forcing term, that is, include a function dependent on t alone, or independent of the dependent variable. The inclusion of forcing defines the equation as inhomogeneous. For first-order equations the general form is

$$\frac{dy}{dt} + a_0 y = F(t),$$

and for second-order equations

$$\frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = F(t),$$

where a_1 and a_0 are constants and F is a non-zero function called the forcing term. (When $F(t) = 0$ the equation is homogeneous. Alternatively, when $F(t) \neq 0$ the equation is inhomogeneous.)

Splitting the solution

We solve this equation by splitting the solution into a homogeneous solution $y_h(t)$ and a *particular solution* $y_p(t)$, whence

$$y(t) = y_h(t) + y_p(t).$$

Here y_h is the *general solution* of the homogeneous equation (with $F(t) = 0$) and y_p is any particular solution satisfying the original equation.

Finding particular solutions

A particular solution of a differential equation is any function which satisfies the differential equation. If the RHS is a simple elementary function then it is usually possible to find a particular solution by making an educated guess based on the form of the RHS and its derivatives.

Example 1.6: Find a particular solution of $\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 3$.

Solution: Since the RHS is constant, we seek a solution of the form $y_p = A$ where A is a constant. Substituting into the differential equation we obtain $0 + 0 - 2A = 3$. Hence $A = 2/3$ and $y_p = 2/3$.

Example 1.7: Find a particular solution of $\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = t^2$.

Solution: We look for a solution of the form

$$y_p = A + Bt + Ct^2,$$

where A , B and C are constants. Substituting this into the differential equation we get $2C - (2Ct + B) - 2(Ct^2 + Bt + A) = t^2$. Equating coefficients of the powers of t (noting that the RHS may be expressed as $t^2 + 0t + 0$) we obtain the set of simultaneous equations

$$\begin{aligned} -2C &= 1, \\ -2C - 2B &= 0, \\ 2C - B - 2A &= 0. \end{aligned}$$

Solving these $C = -1/2$, $B = 1/2$ and $A = -3/4$. Hence

$$y_p = -\frac{1}{2}t^2 + \frac{1}{2}t - \frac{3}{4}.$$

Example 1.8: Find a particular solution of $\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 2e^{5t}$.

Solution: We look for a solution of the form

$$y_p = Ae^{5t}.$$

Substituting into the differential equation we obtain

$$25Ae^{5t} - 5Ae^{5t} - 2Ae^{5t} = 2e^{5t}.$$

Hence

$$(25 - 5 - 2)A = 2$$

and $A = 1/9$. Thus

$$y_p = \frac{1}{9}e^{5t}.$$

Example 1.9: Find a particular solution of $\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 2\sin(2t)$.

Solution: We try a solution in the form

$$y_p = A\cos(2t) + B\sin(2t).$$

Substituting this form into the differential equation we obtain

$$\begin{aligned} -4A\cos(2t) - 4B\sin(2t) - [-2A\sin(2t) + 2B\cos(2t)] - 2[A\cos(2t) + B\sin(2t)] \\ = 2\sin(2t) + 0\cos(2t). \end{aligned}$$

Equating the coefficients of $\cos(2t)$ and $\sin(2t)$ we obtain

$$\begin{aligned} -6A - 2B &= 0, \\ -6B + 2A &= 2. \end{aligned}$$

Solving for A and B gives $A = 1/10$ and $B = -3/10$ and hence

$$y_p = -\frac{1}{10}\cos(2t) + \frac{3}{10}\sin(2t).$$

Thus we have some general rules we can apply:

- For constant RHS try $y_p = A$, where A is a constant.
- For polynomial RHS try a general polynomial up to the order of the RHS polynomial.
- For exponential RHS try a general exponential with the same argument.
- For trigonometric RHS try a linear combination of \sin and \cos .

There are some exceptions which should be noted. When the particular solution is the same as the homogeneous solution then the usual approach for finding the general solution will fail. In the context of oscillations, this is called the resonance condition. In this case try the appropriate choice for a particular solution multiplied by t , as illustrated in the following example.

Example 1.10: Find a particular solution of $\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 3e^{2t}$.

Solution: Note that the homogeneous solution is $C_1e^{2t} + C_2e^{-t}$. For the particular solution, rather than try $y_p = Ae^{2t}$, try $y_p = Ate^{2t}$.

Substituting this into the differential equation yields

$$(4Ate^{2t} + 4Ae^{2t}) - (2Ate^{2t} + Ae^{2t}) - 2(Ate^{2t}) = 3e^{2t}.$$

Equating coefficients of te^{2t} we gain no information ($0 = 0$). However, from the coefficients of e^{2t} we obtain $2A = 3$. Hence a particular solution is

$$y_p(t) = \frac{3}{2}te^{2t}$$

and the general solution is

$$y(t) = C_1e^{2t} + C_2e^{-t} + \frac{3}{2}te^{2t}.$$

Appendix B

Further mathematics

B.1 Linear algebra

This appendix on linear algebra presents only the theory and concepts required to support the text. A general text on linear algebra, such as Lay (1994), is recommended for further background and a complete presentation of the theory.

Matrix notation and algebra

Square matrices, that is, matrices with the same number of rows and columns (in our case 2×2), can be added (subtracted), multiplied, or multiplied by a constant as follows:

- (a) $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} + \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 + c_1 & b_1 + d_1 \\ a_2 + c_2 & b_2 + d_2 \end{bmatrix}$
- (b) $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \times \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} (a_1c_1 + b_1c_2) & (a_1d_1 + b_1d_2) \\ (a_2c_1 + b_2c_2) & (a_2d_1 + b_2d_2) \end{bmatrix}$
- (c) $k \times \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \begin{bmatrix} ka_1 & kb_1 \\ ka_2 & kb_2 \end{bmatrix}$.

They can also be post-multiplied by a vector with an equal number of rows (in our case 2):

$$(d) \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1x + b_1y \\ a_2x + b_2y \end{bmatrix}.$$

For this last case we use the form

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Matrices and vectors provide a very useful way of expressing systems of equations. With this above notation, the system

$$\begin{aligned} a_1x + b_1y &= 0, \\ a_2x + b_2y &= 0, \end{aligned}$$

can be given as

$$\mathbf{A}\mathbf{x} = \mathbf{0},$$

where

$$\mathbf{A} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

It should be noted that while the above notation is given for 2×2 matrices, the concepts can be extended to $n \times n$ matrices. There is a wealth of linear algebra theory which can then be applied to these systems, some of which is given below. (For a full understanding of the concepts a text on linear algebra, such as Lay (1994), is recommended.)

Determinants

Suppose \mathbf{A} is the 2×2 matrix $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ is a vector.

Then $\mathbf{Ax} = \mathbf{0}$ is

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (\text{B.1})$$

or

$$\begin{aligned} a_1x + b_1y &= 0, \\ a_2x + b_2y &= 0. \end{aligned}$$

The *trivial solution* to this is $x = y = 0$. The *nontrivial solution* excludes this above possibility, and thus

$$x = -\frac{b_1}{a_1}y \quad \text{if } a_1 \neq 0.$$

Then

$$-\frac{a_2b_1}{a_1}y + b_2y = 0$$

so that

$$y \left(b_2 - \frac{a_2b_1}{a_1} \right) = 0$$

and then, since $y \neq 0$, for a nontrivial solution to the vector equation (B.1) we must have

$$a_1b_2 - a_2b_1 = 0.$$

This value $(a_1b_2 - a_2b_1)$ is called the *determinant* of A , and is denoted $\det A$ or $|A|$. It is the difference between the diagonal products of A in the case of a 2×2 matrix. The important conclusion is that for a nontrivial solution to a vector equation $\mathbf{Ax} = \mathbf{0}$, we require $|A| = 0$. (Such notions can be extended to $n \times n$ matrices.)

Eigenvectors and eigenvalues

An *eigenvector* of a 2×2 matrix \mathbf{A} is a non-zero vector \mathbf{x} such that $\mathbf{Ax} = \lambda\mathbf{x}$ for some scalar λ . A scalar λ is an *eigenvalue* of A if there is a nontrivial solution \mathbf{x} to $\mathbf{Ax} = \lambda\mathbf{x}$ such that \mathbf{x} is the eigenvector corresponding to the eigenvalue λ .

Note that, using the results of the previous section, the existence of a nontrivial (i.e. non-zero) solution \mathbf{x} to

$$\mathbf{Ax} = \lambda\mathbf{x} \quad \text{or} \quad \mathbf{Ax} - \lambda\mathbf{x} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

means that the determinant of the matrix $(\mathbf{A} - \lambda\mathbf{I})$ is zero. That is; $|\mathbf{A} - \lambda\mathbf{I}| = 0$. (Here I is the identity matrix, and

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

for 2×2 matrices.) Recall,

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} a_1 - \lambda & b_1 \\ a_2 & b_2 - \lambda \end{bmatrix}$$

and then

$$\begin{aligned} |\mathbf{A} - \lambda\mathbf{I}| &= (a_1 - \lambda)(b_2 - \lambda) - a_2b_1 \\ &= a_1b_2 - \lambda b_2 - \lambda a_1 + \lambda^2 - a_2b_1 \\ &= \lambda^2 - \lambda(a_1 + b_2) + (a_1b_2 - a_2b_1). \end{aligned}$$

Notice that this is $\lambda^2 - \lambda(\text{trace}\mathbf{A}) + (\det \mathbf{A})$, where $\text{trace}\mathbf{A}$ (or $\text{tr } \mathbf{A}$) is the sum of the diagonal terms of A , and $\det \mathbf{A} = |\mathbf{A}| = a_1b_2 - a_2b_1$.

We call

$$|\mathbf{A} - \lambda\mathbf{I}| = \lambda^2 - \lambda(a_1 + b_2) + (a_1b_2 - a_2b_1)$$

the *characteristic polynomial* of \mathbf{A} , and $|\mathbf{A} - \lambda\mathbf{I}| = 0$ is called the *characteristic equation* of \mathbf{A} . This characteristic equation can be solved to find the eigenvalues. Clearly, in this case, there can be at most two distinct eigenvalues, with the possibility of complex valued eigenvalues.

Linear dependence and independence

Any point in the (x, y) -plane can be expressed as a linear combination of x and y , which have direction vectors $[1, 0]^T$ and $[0, 1]^T$ respectively.¹ We say that $[1, 0]^T$ and $[0, 1]^T$ span \mathbb{R}^2 , or that they form a *basis* for \mathbb{R}^2 : that is, every point in the plane can be expressed in terms of them. Further,

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq c \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq c \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

for any constant c . This property is called *linear independence* of the vectors. Thus we say that two vectors are linearly independent if neither can be expressed as a linear combination of the other. Another way to say this (the typical mathematical definition) is:

A set of vectors is linearly independent if the vector equation

$$c_1 v_1 + c_2 v_2 + \dots + c_k v_k = 0$$

has only the trivial solution $c_1 = c_2 = \dots = c_k = 0$.

Theorem 4 (Linear independence of eigenvectors.) Suppose v_1, \dots, v_k are eigenvectors that correspond to k distinct eigenvalues $\lambda_1, \dots, \lambda_k$ of some matrix A . Then the set of eigenvectors $\{v_1, \dots, v_k\}$ is linearly independent.

Proof Suppose $\{v_1, \dots, v_k\}$ is not linearly independent (i.e. linearly dependent), then there exists some p such that v_p is a linear combination of the eigenvectors $\{v_1, \dots, v_{p-1}\}$, with constants c_i ,

$$v_p = c_1 v_1 + \dots + c_{p-1} v_{p-1}. \quad (\text{B.2})$$

Now, multiplying by A ,

$$A v_p = c_1 A v_1 + \dots + c_{p-1} A v_{p-1}$$

and then

$$\lambda_p v_p = c_1 \lambda_1 v_1 + \dots + c_{p-1} \lambda_{p-1} v_{p-1}, \quad (\text{B.3})$$

since $A v_i = \lambda_i v_i$ by the definition of eigenvalues and eigenvectors. Also, multiplying (B.2) by λ_p gives

$$\lambda_p v_p = c_1 \lambda_p v_1 + \dots + c_{p-1} \lambda_p v_{p-1} \quad (\text{B.4})$$

and then subtracting (B.4) from (B.3)

$$0 = c_1 (\lambda_1 - \lambda_p) v_1 + \dots + c_{p-1} (\lambda_{p-1} - \lambda_p) v_{p-1}.$$

Since $\{v_1, \dots, v_{p-1}\}$ are linearly independent and non-zero, the expressions $(\lambda_i - \lambda_p)$ for $i = 1, \dots, p-1$ are all zero. However, this is a contradiction because the eigenvalues are all distinct. This means that $\{v_1, \dots, v_k\}$ are linearly independent.

What we have shown is that, in the case of A a 2×2 matrix with two distinct eigenvalues and two associated eigenvectors (u and v), the eigenvectors are linearly independent. And if we have two linearly independent vectors in \mathbb{R}^2 , they must necessarily span \mathbb{R}^2 , that is, form a basis for \mathbb{R}^2 . Thus x and y can be expressed as a linear combination of the eigenvectors

$$x = z_1 u + z_2 v.$$

¹The superscript T denotes the transpose. Thus

$$[1, 0]^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Matrix invertibility

An $n \times n$ matrix \mathbf{A} is *invertible* if there exists a matrix \mathbf{A}^{-1} such that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} = \mathbf{A}\mathbf{A}^{-1}$, where \mathbf{I} is the identity matrix.

Theorem 5 (Matrix invertibility.) Let \mathbf{A} be an $n \times n$ matrix (a square matrix). Then the following are equivalent, that is, if any one of them is true then so are the rest.

- (a) \mathbf{A} is invertible
- (b) the columns of \mathbf{A} are linearly independent
- (c) the columns form a basis for \mathbb{R}^n
- (d) $|\mathbf{A}| \neq 0$ (i.e. $\det \mathbf{A} \neq 0$)
- (e) $\mathbf{Ax} = \mathbf{0}$ has no solutions other than the trivial one
- (f) the rows of \mathbf{A} are linearly independent
- (g) the rows of \mathbf{A} form a basis for \mathbb{R}^n .

Proof. The details are omitted, but for those who are interested, see any text on linear algebra, for example Lay (1994).

B.2 Partial derivatives and Taylor expansions

Partial derivatives

Suppose z is dependent on two variables x and y . We say $z = f(x, y)$. If we keep y constant ($y = y_1$ say), then z can be considered as a function of x alone, and if the derivative of $f(x, y_1)$ exists then it is called the partial derivative of f with respect to x . Mathematically, it is described with ‘curly’ d’s as

$$\frac{\partial f}{\partial x} = f_x \quad \text{— a first-order partial derivative.}$$

Similarly, with x fixed ($x = x_1$ say) we have (if the derivative exists)

$$\frac{\partial f}{\partial y} = f_y \quad \text{— the other first-order partial derivative.}$$

Example 2.1: $z = f(x, y) = x^2y + x \sin y$

$$\frac{\partial f}{\partial x} = 2xy + \sin y, \quad \frac{\partial f}{\partial y} = x^2 + x \cos y,$$

and

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = 2y, & \frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = -x \sin y, \\ \frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = 2x + \cos y, & \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = 2x + \cos y. \end{aligned}$$

These are the four second-order partial derivatives.

The partial derivatives represent slopes in the x - and y -directions respectively (see Figure B.1).