Global Derivatives, Trading & Risk Management

Amsterdam 18-22 May 2015

A Practical Robust Long Term Yield Curve Model

M A H Dempster

Centre for Financial Research, Statistical Laboratory
University of Cambridge
& Cambridge Systems Associates Limited

mahd2@cam.ac.uk www.cfr.statslab.cam.ac.uk

Co-workers: Elena Medova, Igor Osmolovskiy & Philipp Ustinov

Outline

- Introduction
- Multifactor yield curve models
- Difficulties with Gaussian affine models
- Black correction for negative rates
- HPC approaches to calibrating Black models
- Unscented Kalman filter implementation
- Conclusions

Introduction

- In all the world's major economies low interest rates have prevailed since the 2007-2008 financial crisis which were presaged by more than a decade in Japan

- This has posed a problem for the widespread use of diffusion based yield curve models for derivative and other structured product pricing and for forward rate simulation for systematic investment and asset liability management

- Sufficiently accurate for pricing and discounting in relatively high rate environments Gaussian models tend to produce an unacceptable proportion of negative forward rates at all maturities with Monte Carlo scenario simulation from initial conditions in low rate economies

- The implications for this question of negative nominal rates in deflationary regimes and the currently fashionable multi-curve models remain to be seen

Variety of Approaches to Yield Curve Modelling

- Investment bank pricing and hedging of fixed income products
  - Short term current market data calibration
  - Updated for re-hedging
  - Evaluated by realized hedging P&L
- Central bank forecasts for monetary policy making
  - Long term historical estimation for medium term forecasting
  - Updated for next forecast
  - Mainly evaluated by in-sample fit to historical data
- Consultants and fund managers advice for product pricing, investment advice and asset liability management over long horizons
  - Long term historical calibration to market data often using filtering techniques
  - Updated for decision points
  - Evaluated by consistency with out– of –sample market data: e. g. prices, returns

High Performance Computing Requirement

- Beginning with work in the Bank of Japan in the early 2000s there is currently considerable research in universities, central banks and financial services firms to develop yield curve models whose simulation produces nonnegative rate scenarios

- All this work is based on a suggestion of Fisher Black (1995) published posthumously to apply a call option payoff with zero strike to the model instantaneous short rate which leads to a piecewise nonlinearity in standard Gaussian affine yield curve model formulae for zero coupon (discount) bond prices and the corresponding yields and precludes their explicit closed form solution

- As a result most of the published solutions to Black-corrected yield curve models are approximations and even these require high performance computing techniques for numerical solution but we shall study here an obvious approximation which works extremely well as we shall see and is amenable to cloud computing for speed up

Multifactor Yield Curve Models

Yield Curve Model Applications

- Scenario simulation for predominantly long term asset liability management (ALM) problems in multiple currencies
- Valuation of complex structured derivatives and other products and portfolios with embedded derivatives in multiple currencies
- Risk assessment of portfolios and structured products

Model Requirements

- Continuous time
- Mean reversion
- Dynamic evolution under both pricing (risk neutral) and market (real world) measures
- Wide range of yield curve shapes and dynamics reproduced (LIBOR)
- Realistic zero lower bound (ZLB) modelling
- Feasible and efficient discount bond price or yield calculation
- Parameter estimation by efficient model calibration to market data to multiple yield curves and currency exchange rates
- Parsimony in parameter specification
- Time homogeneity

Dempster *et al.* (2010, 2014, 2015)

Multi-factor Yield Curve Models

- Three broad overlapping classes
  - Short rate models
  - Heath-Jarrow-Morton models
  - Market models
- Most rate variability captured by 3 stochastic factors
  Litterman & Scheinkman (1991)
- The 2 factor affine or quadratic short rate models are insufficient to reproduce the correlation structure of market rate changes but 3 to 5 factors suffice
  Rebonato & Cooper (1995) Nawalka & Rebonato (2011)
- The Nelson-Siegel (1987) 3-factor short rate model widely used by central banks has time inhomogeneous parameters and is neither parsimonious nor arbitrage free
- The Diebold-Rudebusch (2011, 2013) version of this model corrects both these faults
  Rebonato (2015)

3 Factor Affine Short Rate Models

- The 3 factors under the pricing (risk-neutral) measure Q satisfy the $A_0(3)$ SDE

$$d\boldsymbol{Y}_t = \Lambda(\Theta - Y_t)dt + \Sigma\sqrt{S_t}\,d\boldsymbol{W}_t$$

$$[S_t]_{ii} = \alpha_i + \beta_i' Y_t \qquad i = 1, 2, 3$$

- Discount bond prices are given in affine form as $P_t(\tau) = e^{A(\tau)+B(\tau)'Y_t}$
  and the instantaneous short rate similarly as $r_t = \phi_0 + \phi_x' Y_t$

- Then bond prices and yields are given respectively by $P_t(\tau) = E^Q\left[\exp\left(-\int_t^{t+\tau} \boldsymbol{r}_s ds\right)\right]$
  and $y_t = -\ln P_t(\tau)/\tau$

- A 3 rate vector satisfies the Ricatti equation $\dfrac{\partial R_t(\tau)}{\partial \tau} = \Lambda R_t(\tau) - \dfrac{1}{2} R_t(\tau)\Sigma S \Sigma' R_t(\tau)' + r_t 1$

Duffie & Kan (1996) Dai & Singleton (2000) Dempster *et al.* (2014)

3 Factor Gaussian Extended Vasicek Model

- Specified under P by

$$\Lambda := \begin{pmatrix} \lambda_{11} & 0 & 0 \\ \lambda_{21} & \lambda_{22} & 0 \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix}$$

$$\Theta := \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}$$

$$\Sigma := \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

$$S := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$r(t) := \delta_0 + \delta_1 y_1(t) + \delta_2 y_2(t) + \delta_3 y_3(t)$$

- This Dai & Singleton $A_0(3)$ model with **16 parameters** is not identified under P unless $\Theta := 0$ which is only appropriate to Q and has other difficulties

Joslin-Singleton-Zhu (JSZ) 3 Factor Affine Gaussian Yield Curve Model

- The underlying stochastic differential equation (SDE) for the continuous time evolution of the 3 factors $\boldsymbol{Y}$ is given by

$$d\boldsymbol{Y}(t) = \Lambda(\theta - Y(t) + \Pi(t))dt + \Sigma\sqrt{S(t)}d\boldsymbol{W}(t),$$

where now

$$\boldsymbol{\Pi}(\mathrm{t}) = k_0 + K_1\boldsymbol{Y}(\mathrm{t})$$

is the affine state dependent market price of risk vector

- JSZ estimate the discrete time version with 3 observed yield curve points – rates – fit exactly and extra rates least squares fit approximately as two standard econometric vector autoregression (VAR) models given respectively under the pricing and market (real world) measures Q and P

JSZ (2011)

Economic Factor Model

- A 3 factor extended Vasicek Gaussian model specified under the market measure P by

$$d\boldsymbol{X}(t) = (\mu_X - \lambda_X X(t) + \gamma_X \sigma_X)dt + \sum_{j=1}^{3} \sigma_{1j} d\boldsymbol{W}_j(t)$$

Long rate

$$d\boldsymbol{Y}(t) = (\mu_Y - \lambda_Y Y(t) + \gamma_Y \sigma_Y)dt + \sum_{j=1}^{3} \sigma_{2j} d\boldsymbol{W}_j(t)$$

Minus Slope

$$d\boldsymbol{R}(t) = \{k[\underline{X(t) + Y(t)} - R(t)] + \gamma_R \sigma_R\}dt + \sum_{j=1}^{3} \sigma_{3j} d\boldsymbol{W}_j(t)$$

Unobservable instantaneous short rate

- Its discretization is estimated from CMS swap data with many observed yield curve points – rates – from 1 day (Libor) to 30 years (Treasury) using the EM algorithm which iterates Kalman filtering and maximum likelihood estimation to convergence
- Specifying the constant market prices of risk in terms of volatility units solves the ***X*** & ***Y*** identification problem and setting them to zero generates the factor pricing process
- This workhorse model has been used for pricing complex products and ALM using daily to quarterly frequency data in US, UK , EU, Swiss and Japanese jurisdictions

State Space Model Formulation

Transition Equation

$$\boldsymbol{Y}_t = d + \Phi Y_{t-1} + \boldsymbol{\eta}_t,$$

$$E[\boldsymbol{Y}_t | Y_{t-1}] = d + \Phi Y_{t-1}$$

$$var(\boldsymbol{\eta}_t) = var(\boldsymbol{Y}_t | Y_{t-1}) = \Omega(Y_{t-1}) := \Omega_t$$

Measurement Equation

Substantial number of observed yields (e.g. 14)

$$y_t = \mathcal{A} + \mathcal{B} Y_t + \varepsilon_t$$

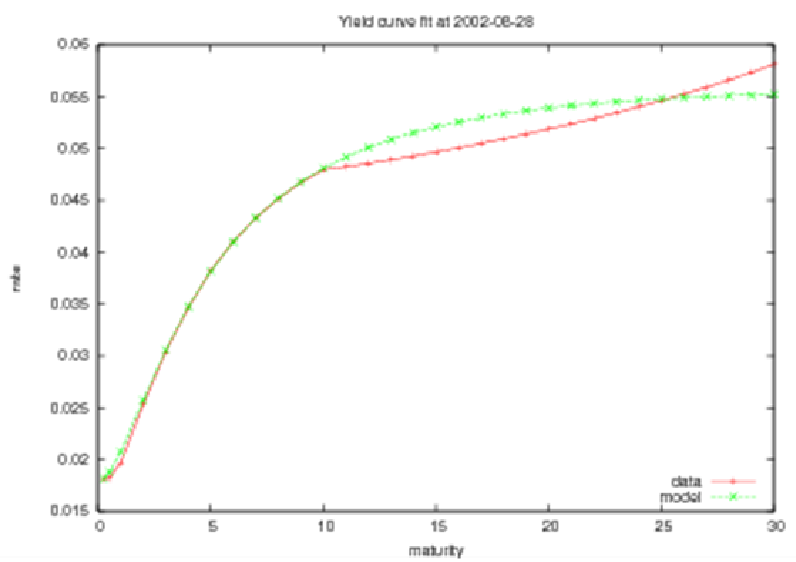
Calibrating the EFM Model

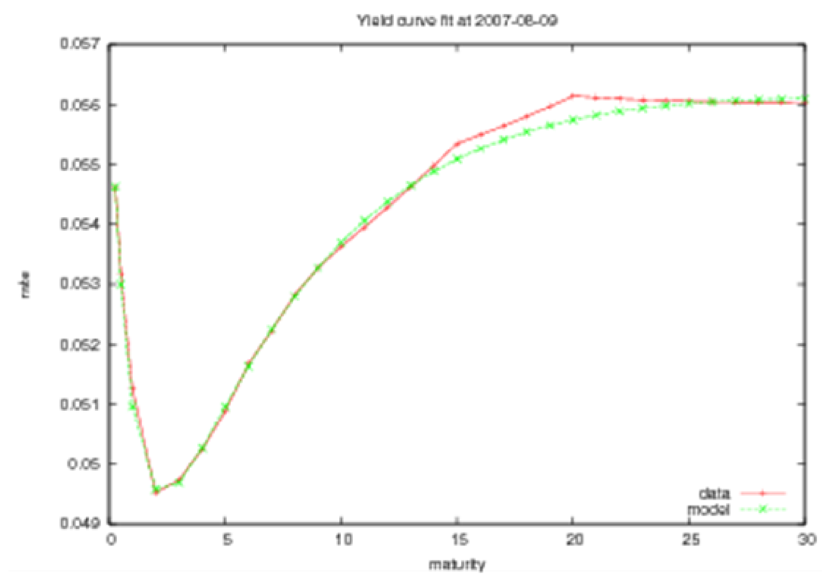
- Given the vector of parameters θ this Gaussian extended Vasicek model has rates (zero coupon bond yields) for maturity $\tau := T - t$ of the form

$$y(t,T) = \tau^{-1}[A(\tau,\theta)R_t + B(\tau,\theta)X_t + C(\tau,\theta)Y_t + D(\tau,\theta)]$$

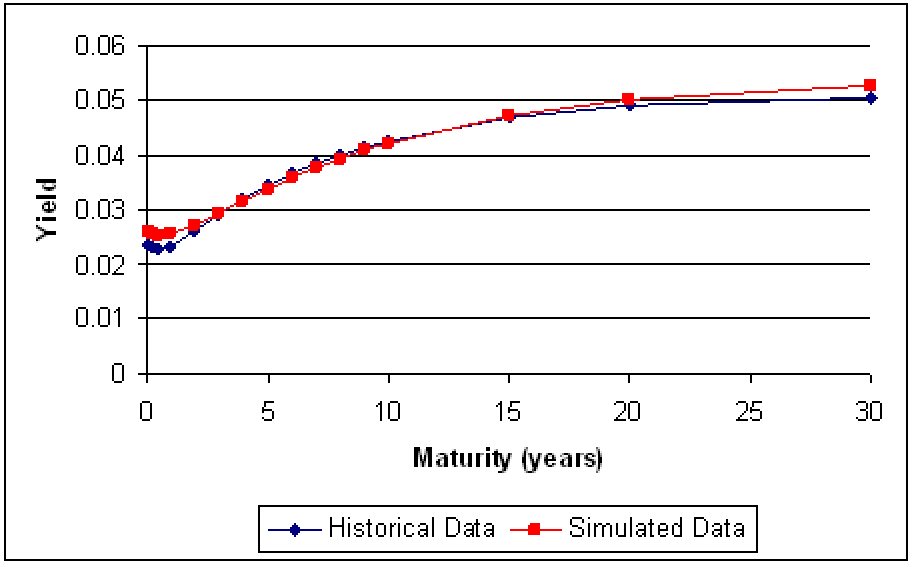
- We interpolate the appropriate swap curve linearly to obtain swap rates at all maturities and then use 1, 3 and 6 month LIBOR rates and the swap curve to recursively back out a zero coupon bond yield curve for each day from the basic swap pricing equation Ron (2000)
- This gives the input data for model calibration to give the parameter estimates $\hat{\theta}$
- Calibration is accomplished using the EM algorithm which iterates successively the Kalman filter (KF) and maximum likelihood estimation from an initial estimate θ_0
- At each iteration multi-extremal likelihood optimization in θ is accomplished using a global optimization technique followed by an approximate conjugate direction search
- The procedure is run on a Dell 32 Intel core system using parallelization techniques and we have also investigated the use of cloud computing for these calculations

Goodness of Fit to Historical Yield Curves

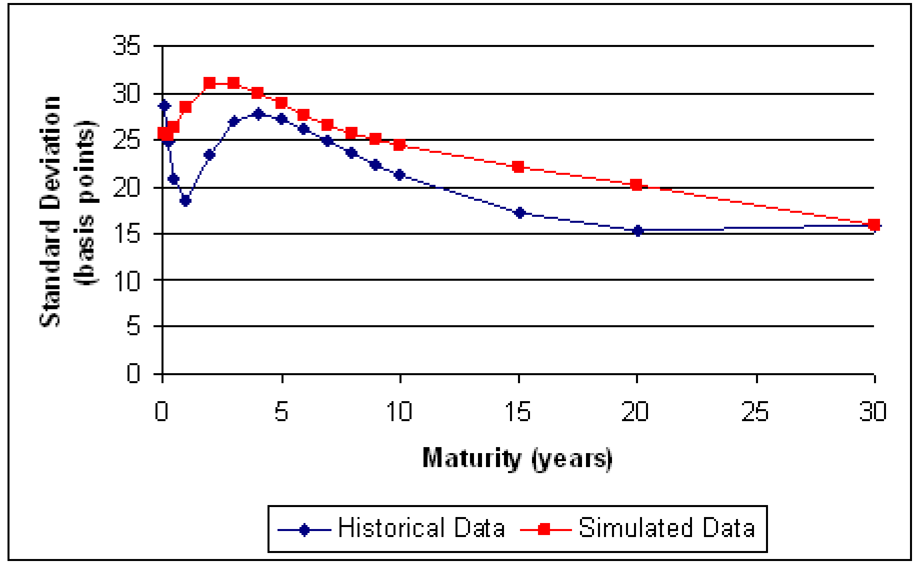




EFM Model EU Yield Curve Prediction 2003



Mean level of yields over 2003 for historical and simulated weekly data



Weekly standard deviation of yields over 2003 for historical and simulated data

Dempster, Medova & Villaverde (2010)

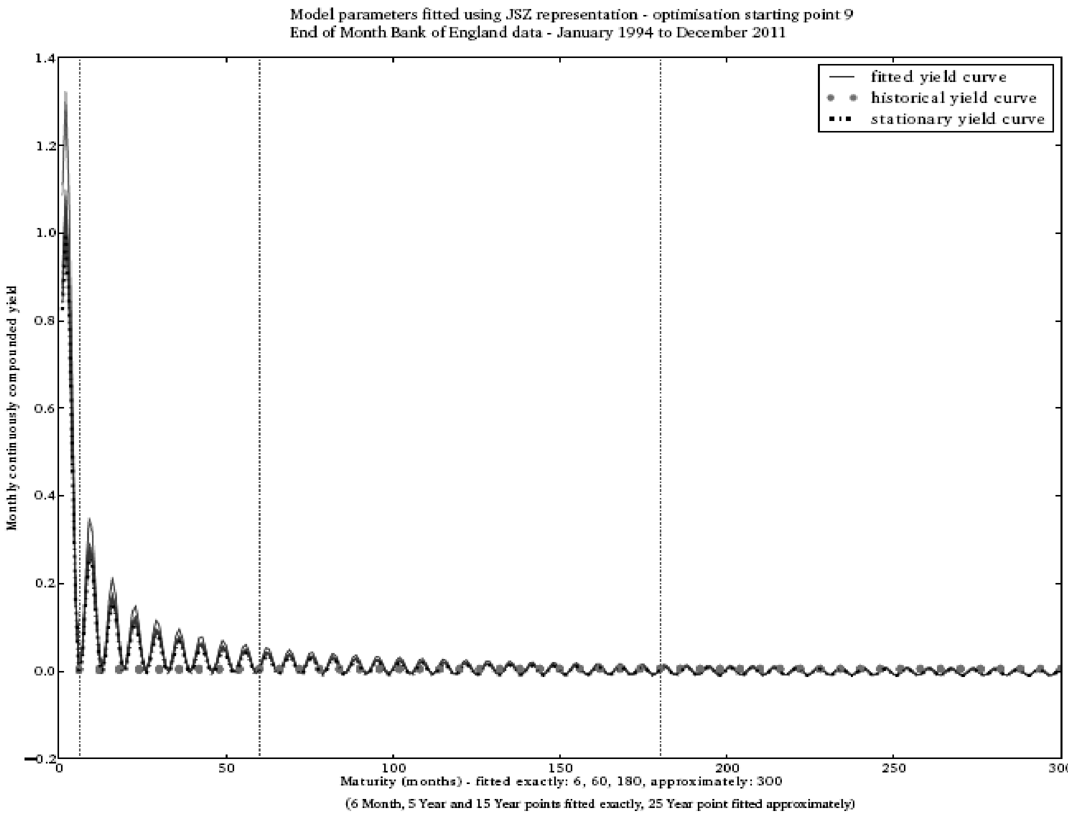
- Longer term out-of-sample yield curve prediction has recently been independently found to be superior to the arbitrage-free Nelson-Siegel model of Christensen, Diebold & Rudebusch (2011) widely used by central banks

Monte Carlo Structured Deal Valuation

- OTC deal valuation may require *several* yield curve estimates together with CMS swap rates and cross currency rates which are all assumed correlated with *fixed* values
- The estimated factor dynamics of (**X,Y,R**) are simulated forward under the Q measure for pricing with the fixed market prices of risk set to 0
- The corresponding curves and FX rates are simulated to maturity together with a daily time step from respectively the valuation day yield curve estimates and FX data using 10,000 paths
- For OTC client deals optionality is typically in the form of bank cancellation rights (without compensation) at prescribed dates – usually at all reset dates after some initial period from inception
- We use an augmented version of a sub-optimal cancellation rule due to Andersen (1999) which relies on a score function $s_t(x, y, r)$ and cancels if $s_t \prec s_t^*$
- The exercise thresholds s_t^* are determined by a separate set of 10,000 paths for (**X,Y,R**) as the discounted value of all the remaining net payouts to the bank along the average factor path

Difficulties With Gaussian Affine Models

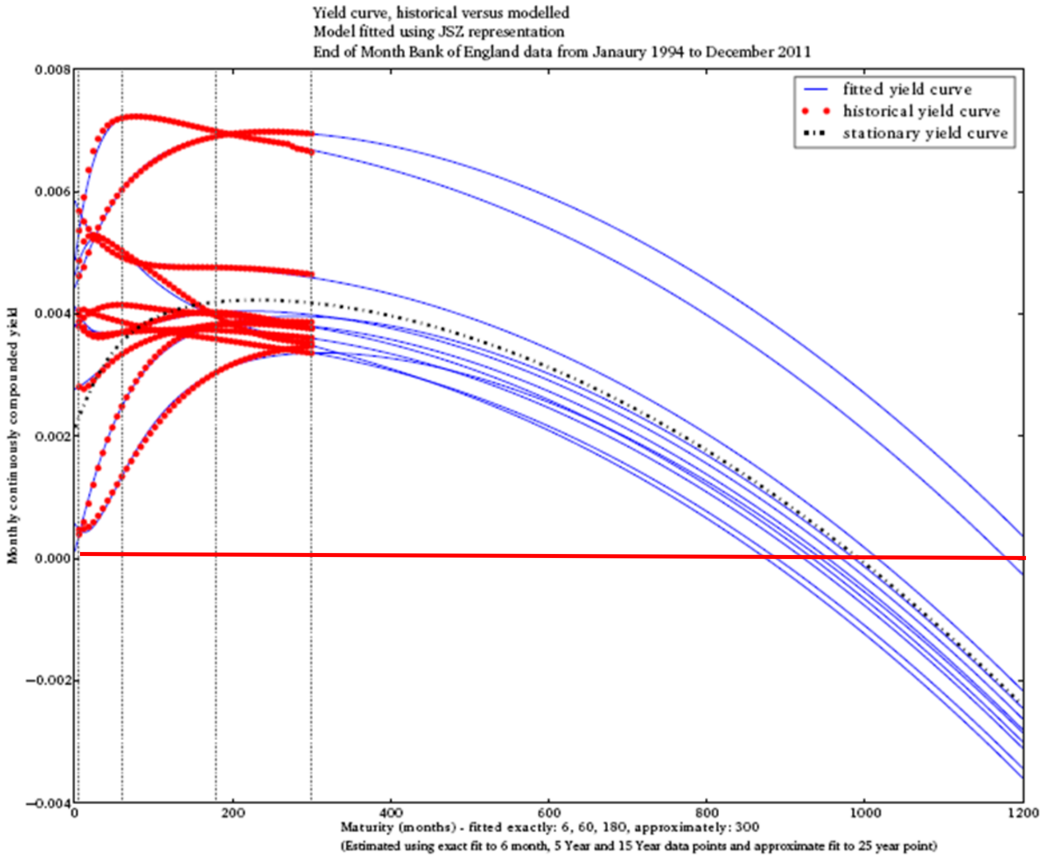
JSZ Affine Model Fit Numerical Instability



Dempster, Evans & Medova (2014)

JSZ Model 25 Year Yield Curve Projections

In-sample



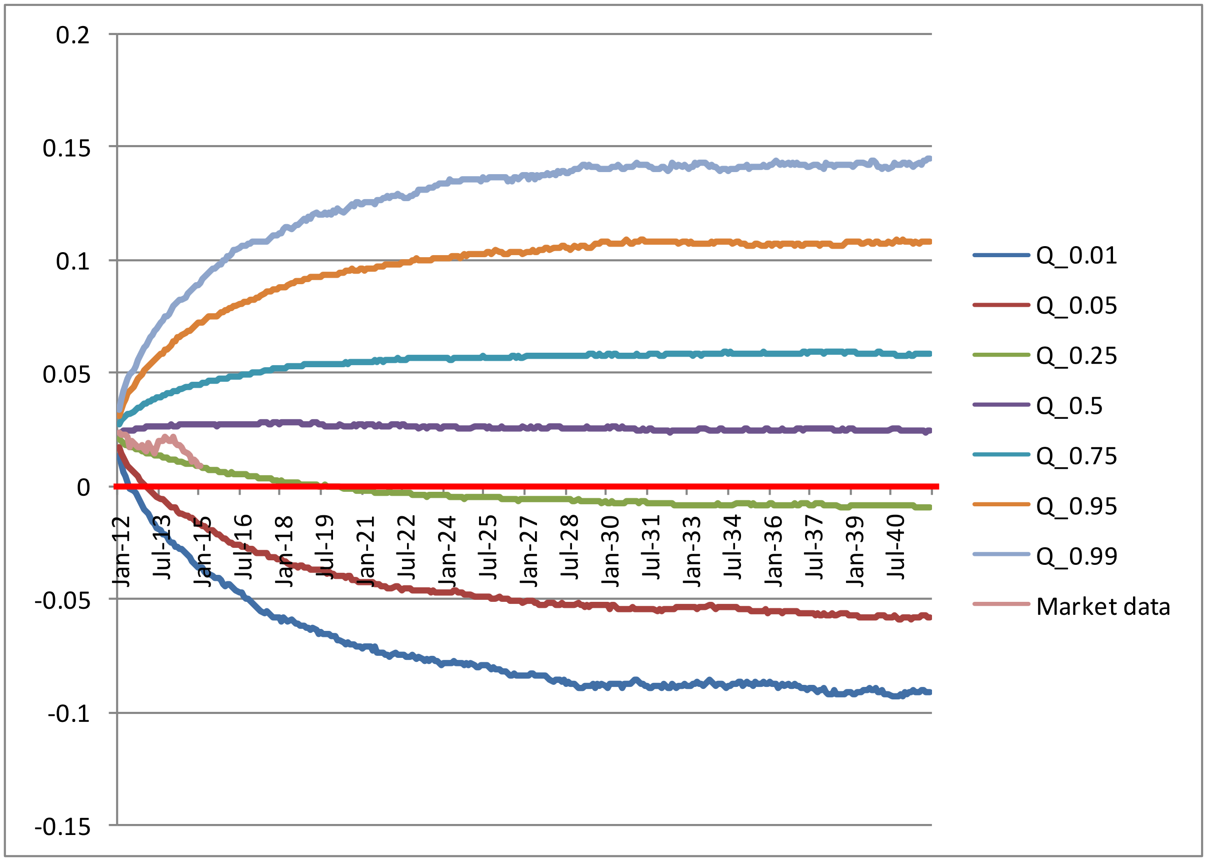
Out-of-sample

Mean

Dempster, Evans & Medova (2014)

EFM Model Euro 10 Year Rate for 30 Years

Quantiles based on 100,000 scenarios



Black Correction for Negative Rates

Nonlinear 3-Factor Black Model

- In a posthumously published paper Fisher Black (1995) suggested correcting *a priori* a Gaussian short rate model for a shadow short rate $\boldsymbol{r}$ to give the actual short rate as

$$\boldsymbol{r}_{actual,t} := \max[0, \boldsymbol{r}_{shadow,t}] := 0 \vee \boldsymbol{r}_{shadow,t}$$

- Applied to an affine 3-factor Gaussian yield curve model such as that of JSZ or our EFM model this yields a hard nonlinear estimation problem posed by the bond price

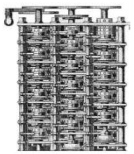
Joslin, Singleton & Zhu (2011)

$$P_t(\tau) = E^Q \exp[-\int_t^{t+\tau} 0 \vee \boldsymbol{r}_{shadow,s}\, ds]$$

- Such models have been studied in the 2-factor case by the Bank of Japan and at Stanford but their discount bond pricing (rate) PDE methods do not easily extend to 3 factors

Ichuie & Ueno (2007) Kim & Singleton (2011)

Christensen & Rudebusch (2013) Kim & Priebsch (2013)



3-Factor Black Model Stylized Properties

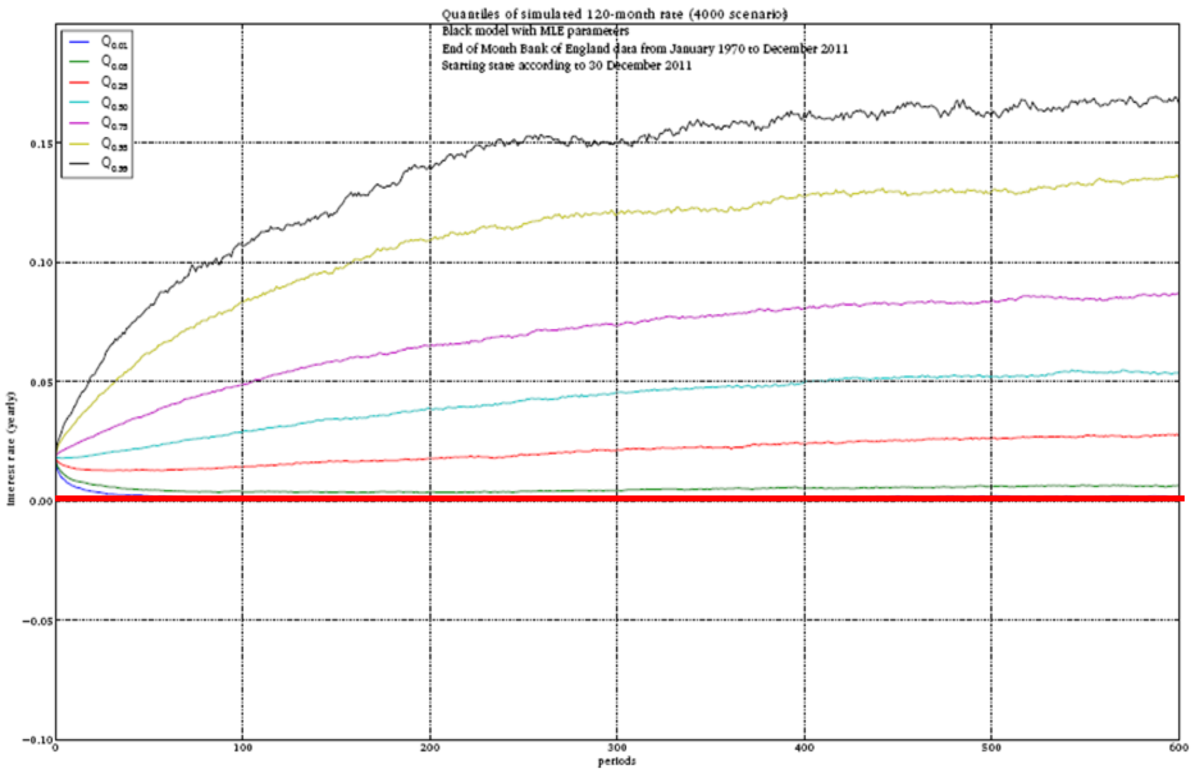
Stylized Fact Properties	Yield Curve Model					
	CIR	BDFS	Vasicek	JSZ/HW	JSZ/HW/BRW	Black
	$A_3(3)$	$A_3(3)$	$A_3(3)$	$A_1(3)$	$A_0(3)$	$A_0(3)$
Mean Reverting Rates	Yes	Yes	Yes	Yes	No	Yes
Nonnegative Rates	Yes	No	No	No	No	Yes
Stochastic Rate Volatility	Yes	Yes	No	No	No	Yes*
Closed Form Bond Prices	Yes	Yes	Yes	Yes	Yes	No
Replicates All Observed Curves	No	Yes	Yes	Yes	Y es	Yes
State Dependent Risk Premia	No	No	No	Yes	Yes	Yes
Good for Long Term Simulations	No	No	No	No	No	Yes
Slow Mean Reversion Under *Q*	No	No	No	No	No	Yes
+ve Rate/Volatility Correlation	No	No	No	No	No	Yes
Effective in Low Rate Regimes	No	No	No	No	No	Yes

Table 1. Properties of evaluated yield curve models with regard to stylized facts

*Rate volatilities are piecewise constant punctuated by random jumps to 0 at rate 0 boundary hitting points.

Black Model 10 Year Gilt Rate 50 Year Predicted Distribution 2011-2061

Quantiles based on 10,000 scenarios



HPC Approaches to Calibrating Black Models

3 Factor Black Model Approaches

- The differences between current approaches to Black models based on 3 factor affine shadow rate models may be categorized in terms of handling the three steps crucial to the solution process
- Method of inferring (3 factor) states from observed market rates
  - inverse mapping or least squares
  - extended or iterated extended Kalman filter (EKF or IEKF) with piecewise linearization
  - unscented Kalman filter (UKF) with averaged multiple displaced KF paths
- Method of parameter estimation
  - method of moments
  - maximum likelihood (MLE) or quasi maximum likelihood (QMLE)
- Method of calculating bond prices or yields
  - Monte Carlo simulation
  - PDE solution
  - approximation

Monte Carlo Bond Pricing

- Calibration of the nonlinear Black model with any underlying 3 factor Gaussian shadow rate model is **more** computationally intensive than for the underlying affine model
- Dempster, Evans & Medova (2014) use cloud facilities and Monte Carlo simulation with a JSZ 4 yield curve point model
- In more detail:
  - For short rates the closed form numerical rate calculations of Kim & Singleton (2011) are used
  - For long rates the averages of Monte Carlo forward simulated paths -- which automatically take account of the convexity adjustment otherwise required for this model – are used
- With this approach filtering a multi-curve EFM model for OTC structured derivative valuation becomes **very** computationally intensive

PDE Bond Pricing

- A possible key to calibration of both the JSZ and EFM models is the efficient solution for discount bond prices $P(\tau)$ of all maturities τ at each time t of a 3-dimensional parabolic partial differential equation (PDE) of the form

$$\partial P_t(\tau)/\partial\tau = \sum_{i,j=1}^{3} a_{ij}\partial^2 P_t(\tau)/\partial y_i \partial y_j + \sum_{i=1}^{3} b_i \partial P_i(\tau)/\partial y_i + cP_t(\tau)$$

- Kim and Singleton's 2-dimensional alternating direction implicit (ADI) solution method will not cope with the 3-D case
  Kim & Singleton (2011)   Lipton (2015)
- Having evaluated simulation-based techniques we intend to investigate applying a fast robust 3D PDE solver based on an interpolating wavelet-specified irregular mesh implicit method that we have developed for complex derivative valuation and is expected to form a part of the NAG library   Jameson (1998)   Carton de Wiart & Dempster (2011)

Black Model Calibration Progress

- Bonfim (2003) estimated his 2 factor model only on yields safely above 0 where the underlying shadow rate affine model rates and the Black rates agree, used the standard KF in the EM algorithm and solved the 2D parabolic quasilinear bond price PDE with finite differences
- Bauer & Rudebusch (2014) took the same approach to the 3 factor model employing the EKF in the EM algorithm and evaluated bond prices using 500 path Monte Carlo simulation as do Lemke & Vladu (2014) with more paths
- Dempster *et al.* (2014) used least squares with 4 observed yields, QMLE and analytical approximation for short yields and 10,000 path Monte Carlo for longer maturity yields as noted above

Further Progress

- Krippner (2013) noted that Black model bond prices for short maturities τ at time t are the underlying shadow rate model closed form bond price less the value of an American call option struck at t with strike 1 and maturity τ which he approximates with the corresponding Black Scholes European value but although this is accurate under P it is not under Q and hence is not arbitrage free. These prices can however be used as control variates with Monte Carlo bond price evaluation
- Richard (2013) solves the 3D bond price PDE for the Black model using finite differences in 2 to 4 weeks on a supercomputer!
- Priebsch (2013) notes that the Black log bond price is the value at -1 of the conditional cumulant generating function of the random integral term involved under Q which can be expanded as $$\ln P_t(\tau) = \ln E_t^Q\left[\exp\left(\int_t^{t+\tau} 0 \vee \boldsymbol{r}_s \; ds\right)\right] = \sum_{j=1}^{\infty} (-1)^j \frac{\kappa_j^Q}{j!}$$ and then 1 or 2 terms used
- We apply the Black correction to the measurement equation for yields within the unscented Kalman filter together with QMLE in the EM algorithm and EFM bond prices

Unscented Kalman Filter Bond Pricing

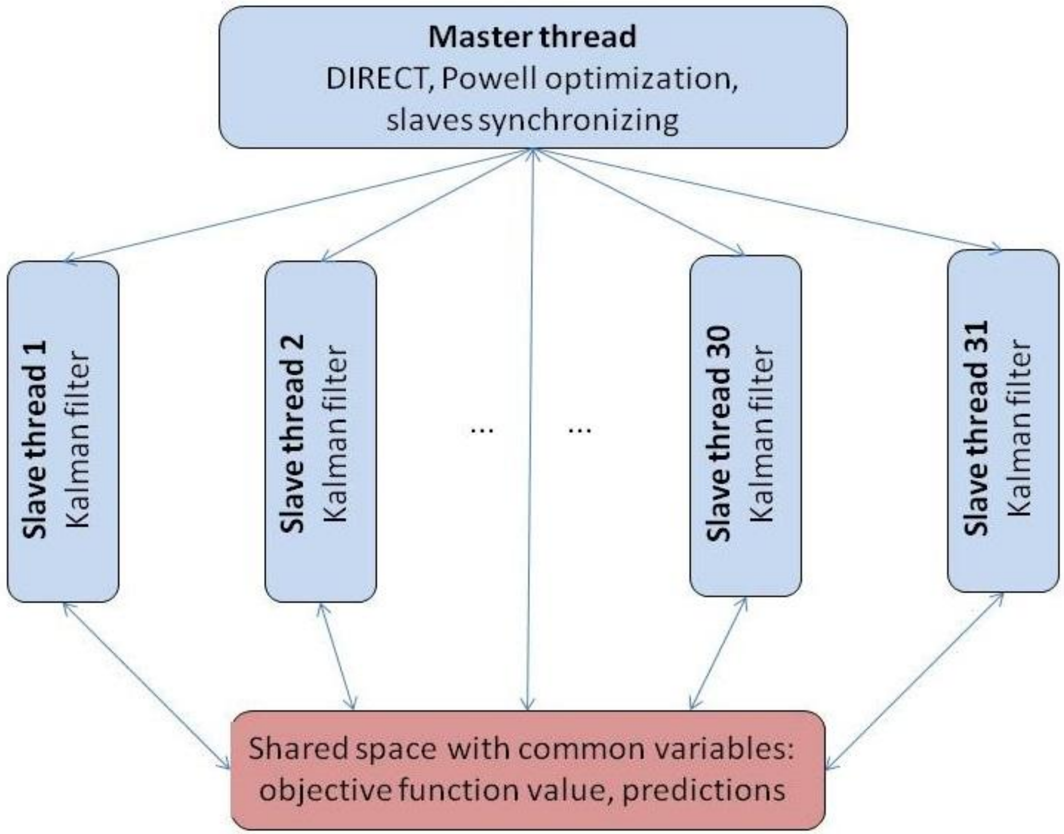
- Here we calibrate the Black EFM model with our current EM algorithm approach using the (NAG) unscented Kalman filter to handle the "hockey stick" nonlinearity

  Julier & Uhlmann (1997)

- Working with yields directly as we do rather than bond prices computed or approximated numerically from integrals of the instantaneous short rate as in the references to Black model calibration previously cited significantly accelerates computation
- Putting the EFM 3-factor yield curve dynamics in state-space form shows that the factor state dynamics remain linear Gaussian while the Black nonlinearity may be directly applied to each observed maturity market rate in the shadow rate affine measurement equation – longer maturity yields typically need no correction
- With this approach the 35 (34 sigma points plus original) duplicate KF calculations of the unscented Kalman filter averaged at each daily time step can be mindlessly parallelized to handle the Black nonlinearity in essentially the same running time as the calibration of the underlying EFM model using basic linear Kalman filtering

Unscented Kalman Filter Implementation

Parallelization Schema with MPI



Data

- Combination of LIBOR data and fixed interest rate swap rates (the ISDA fix) for each of 4 currency areas (EUR, GBP, USD, JPY) to bootstrap the yield curve daily for 14 maturities:

  3 month, 6 month, 1 year, 2 years, 3 years, 4 years, 5 years,

  6 years, 7 years, 8 years, 9 years, 10 years, 20 years, 30 years

- In the case of the Swiss franc (CHF), only 12 maturities are available:

  3 month, 6 month, 1 year, 2 years, 3 years, 4 years, 5 years,
  6 years, 7 years, 8 years, 9 years, 10 years

- Calibration periods used for these 5 currencies are the following:

  EUR: 02.01.2001 to 02.01.2012

  CHF: 02.01.2001 to 31.05.2013

  GBP: 07.10.2008 to 31.05.2013

  USD: 02.01.2001 to 31.05.2013

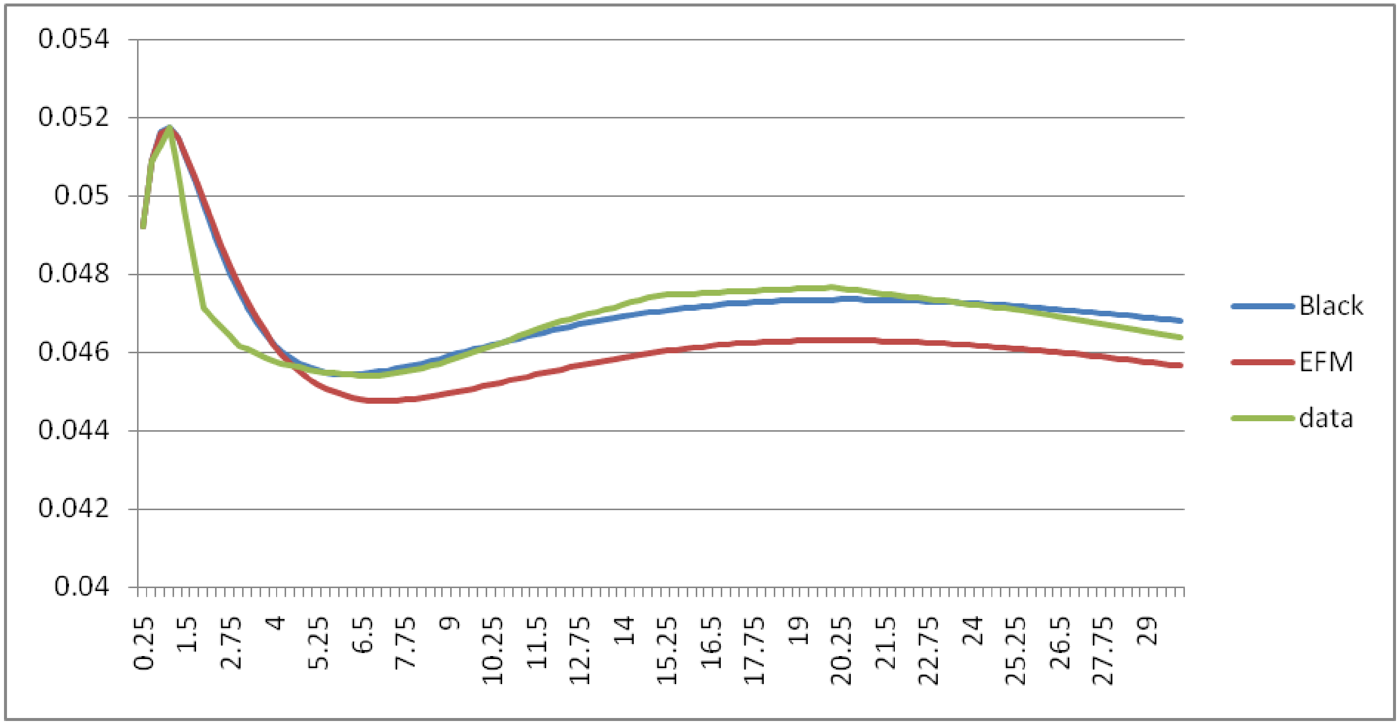
  JPY: 30.03.2009 to 31.05.2013

- The data was obtained from Bloomberg

In-sample Yield Curve Goodness-of-fit

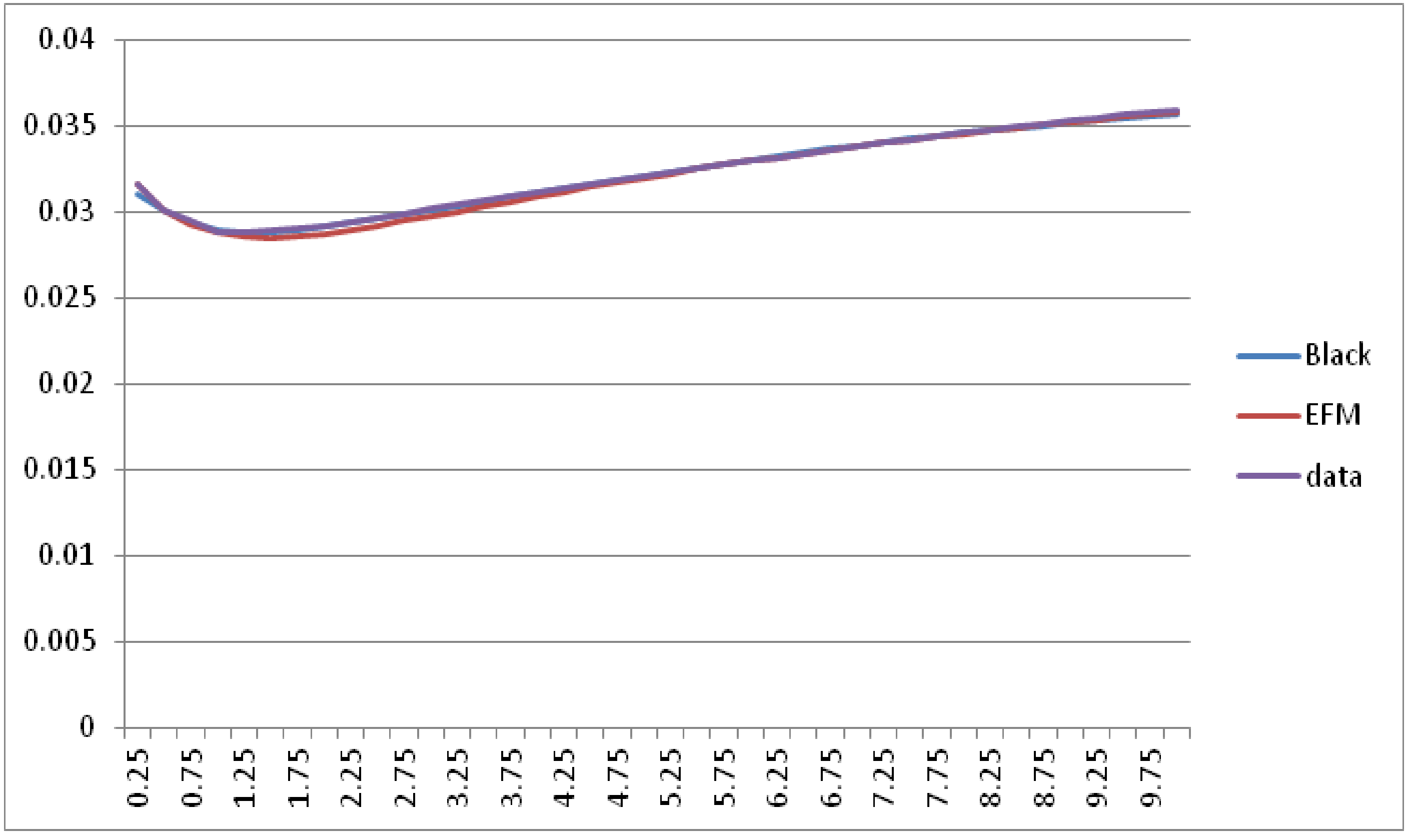
EUR **Date: 22 Aug 2008**

Model	RMSE
EFM	11 bp
Black EFM	6 bp



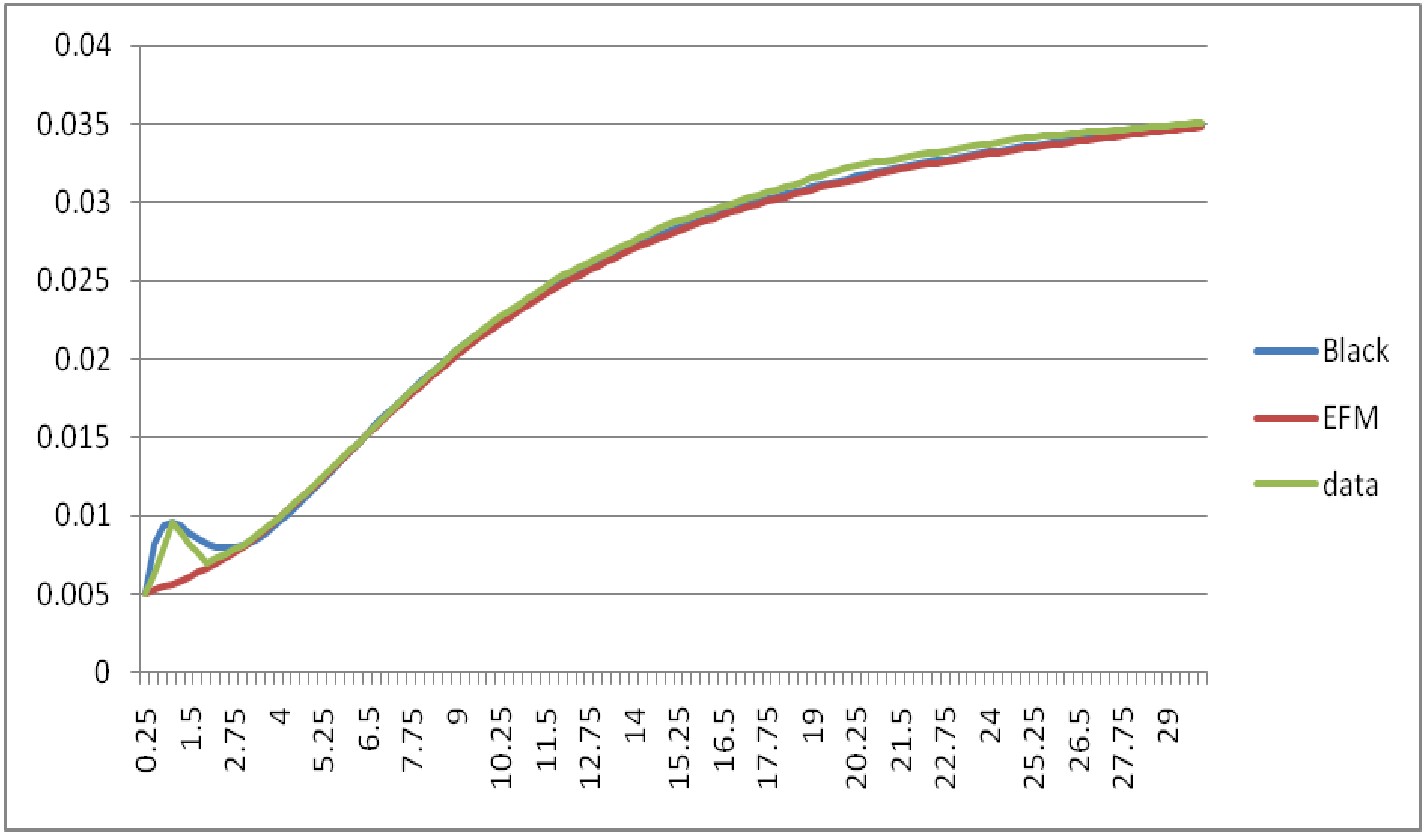
CHF Date: 20 Aug 2001

Model	RMSE
EFM	2 bp
Black EFM	2 bp



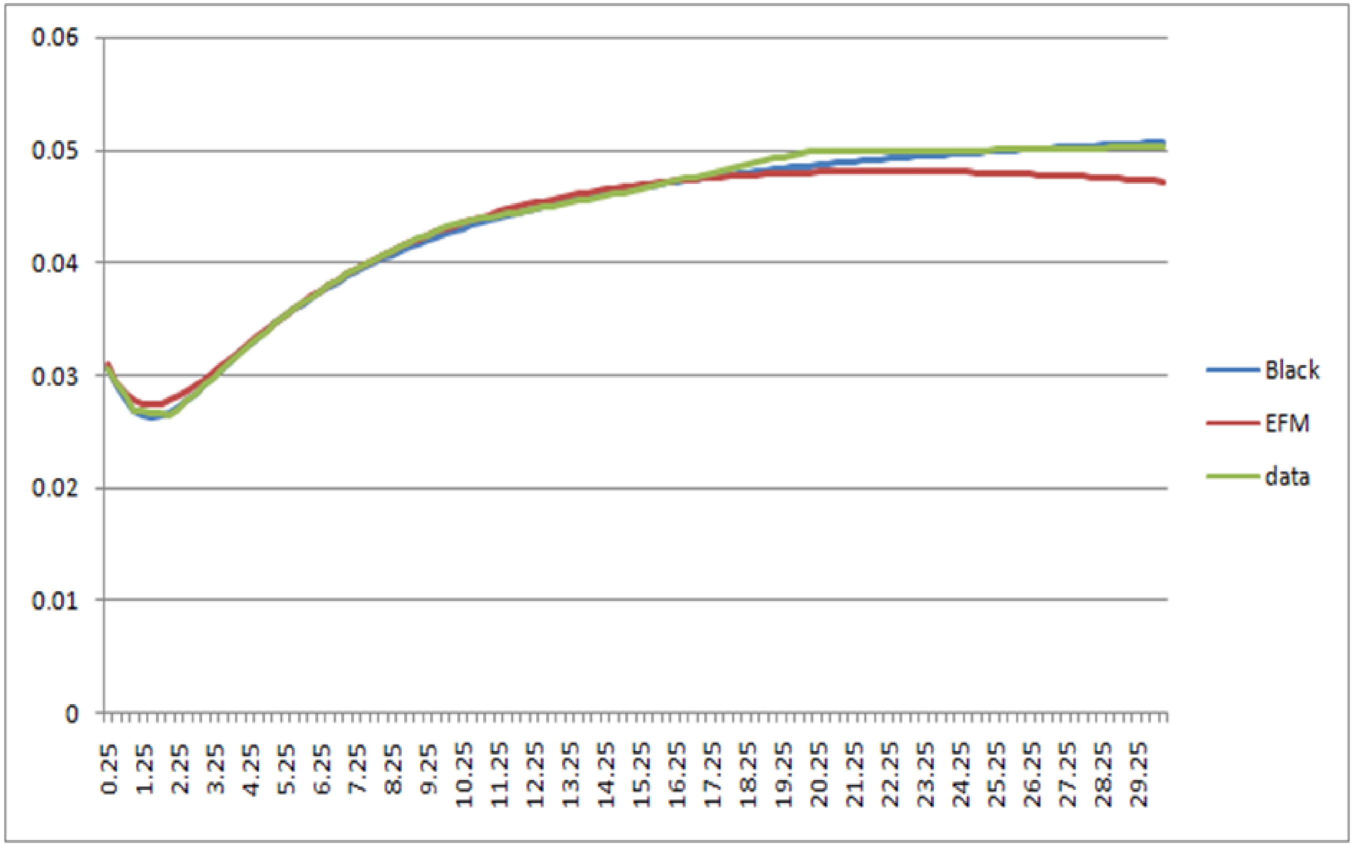
GBP Date: 18 Feb 2013

Model	RMSE
EFM	8 bp
Black EFM	5 bp



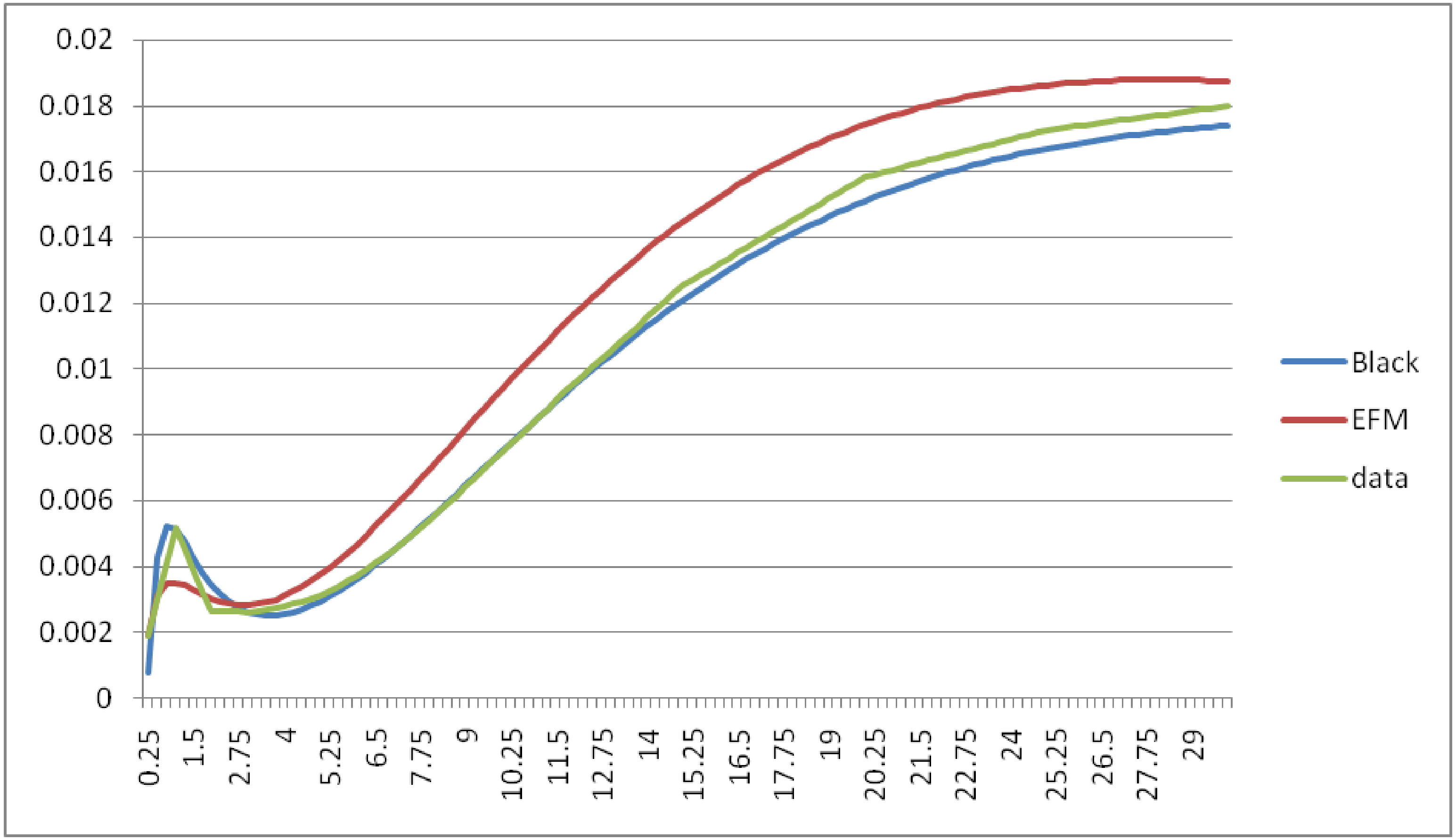
Model	RMSE
EFM	14 bp
Black EFM	5 bp

USD Date: 14 Oct 2008



JPY **Date: 12 Nov 2012**

Model	RMSE
EFM	16 bp
Black EFM	4 bp



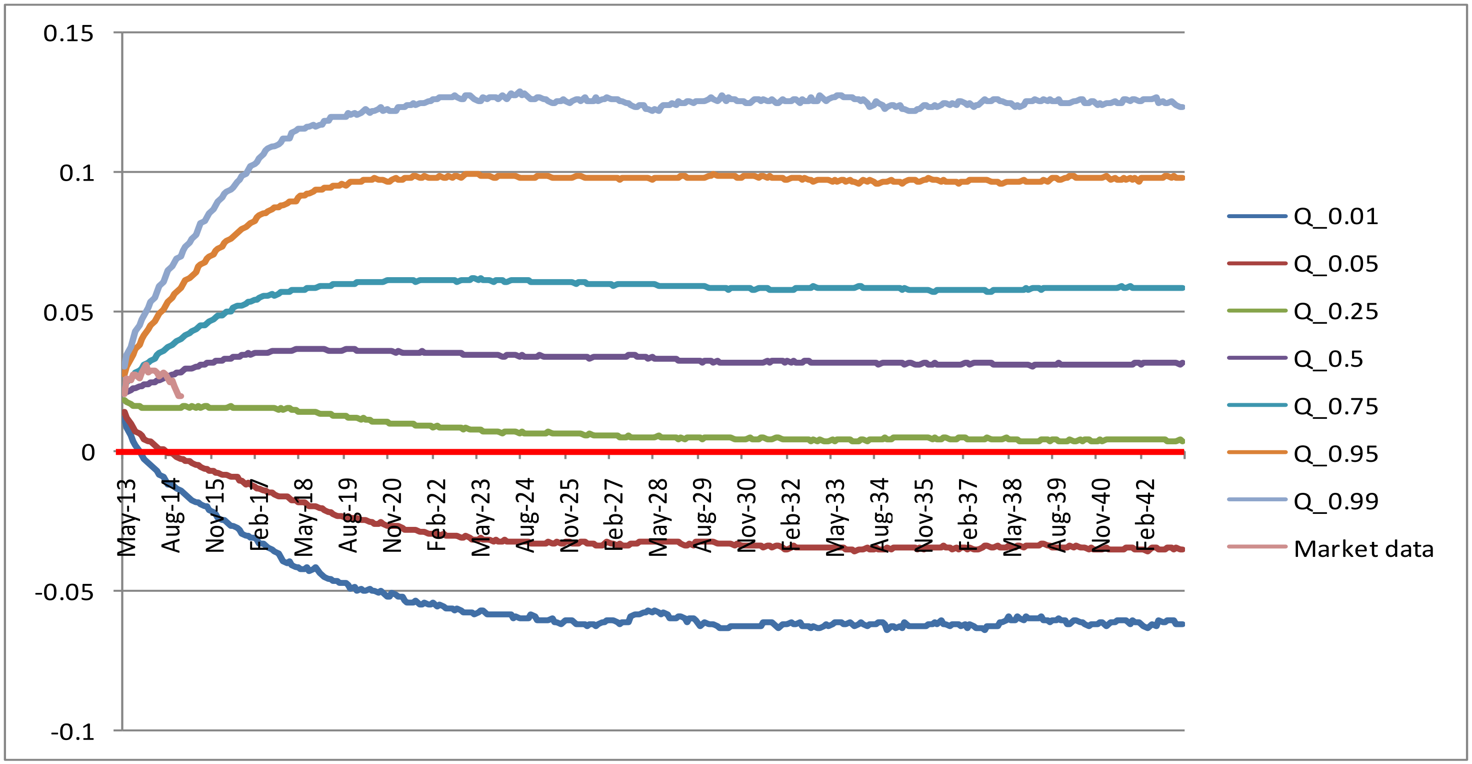
Overall In-sample Goodness of Fit

Currency	Observations	Calibration	log likelihood	Sample fit MSE (vol)
EUR	2817	EFM	232,652	15 bp
		EFM UKF	252,500	17 bp
		Black EFM α:=0.0025	259,436	15 bp
CHF	3100	EFM	232,100	8 bp
		EFM UKF	250,391	10 bp
		Black EFM α:=1.0	253,095	8 bp
GBP	1171	EFM	98,021	16 bp
		EFM UKF	103,529	15 bp
		Black EFM α:=0.0001	105,368	14 bp
USD	3093	EFM	279,114	15 bp
		EFM UKF	280,745	25 bp
		Black EFM α:=0.0001	288,422	16 bp
JPY	950	EFM	91,014	6 bp
		EFM UKF	84,564	28 bp
		Black EFM α:=0.006	102,544	6 bp

Monte Carlo Out-of-sample 30 Year Projection

Quantiles based on 100,000 scenarios

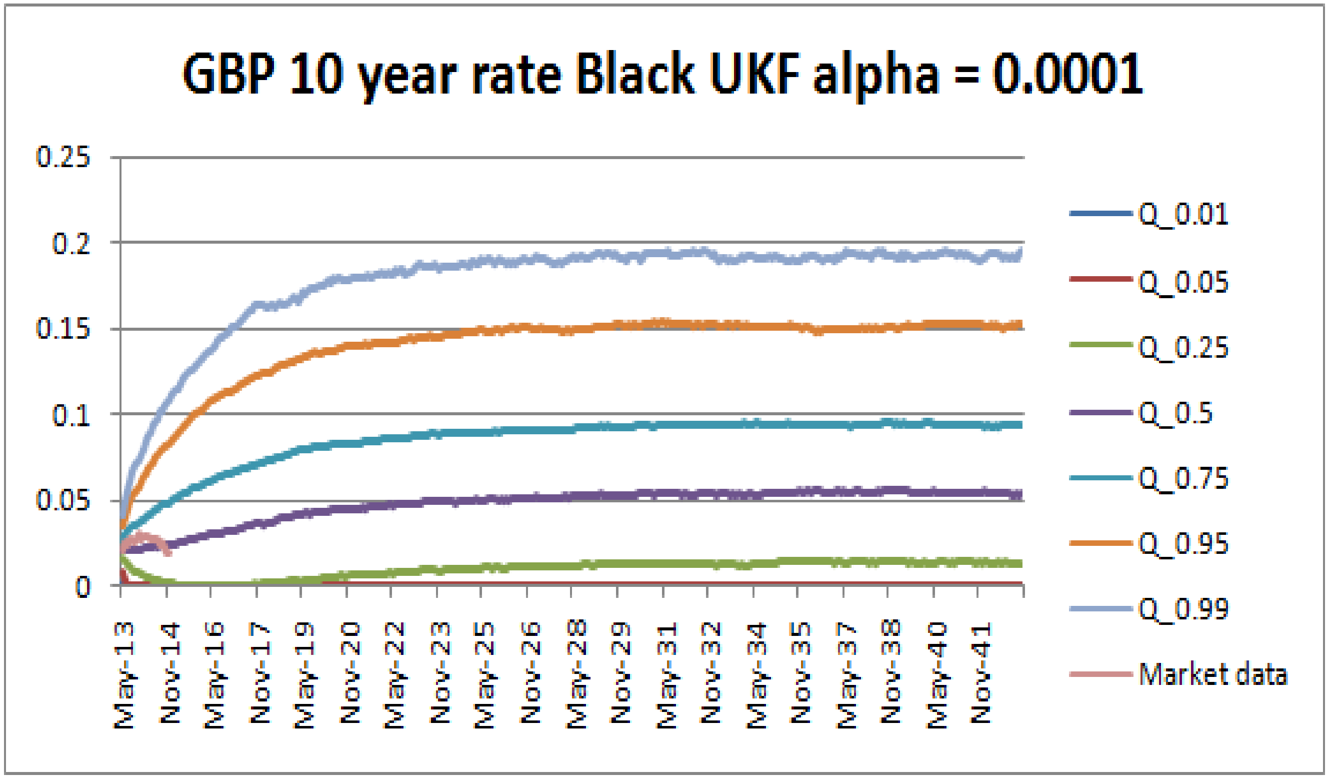
30 Year EFM GPB 10 Year Rate



Monte Carlo Out-of-sample Projection

Quantiles based on 100,000 scenarios

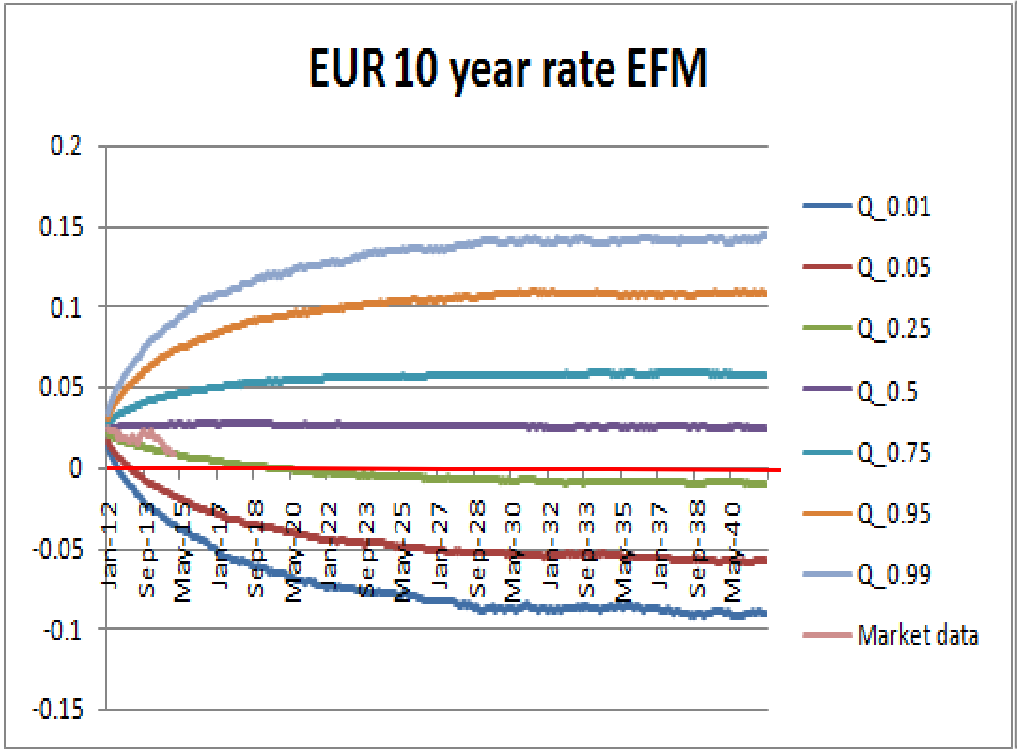
30 Year Black EFM GPB 10 Year Rate

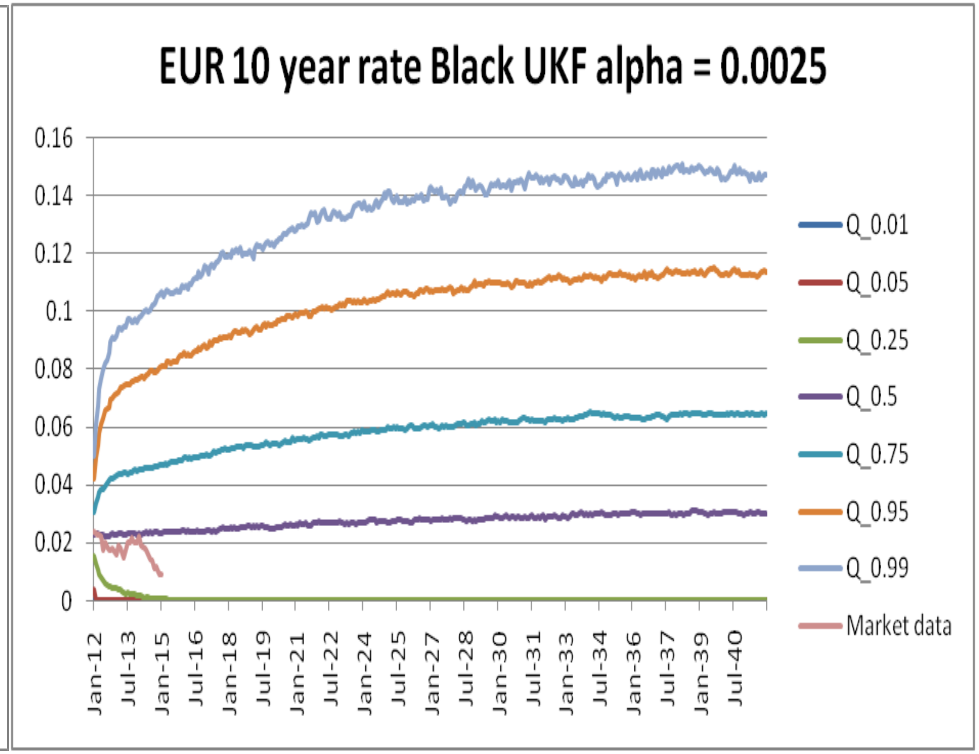


GBP 10 year rate forecast RMSE over 20 months

Black median	0.48%
EFM median	0.45%

EUR 10 Year Rate Out-of-sample Projections

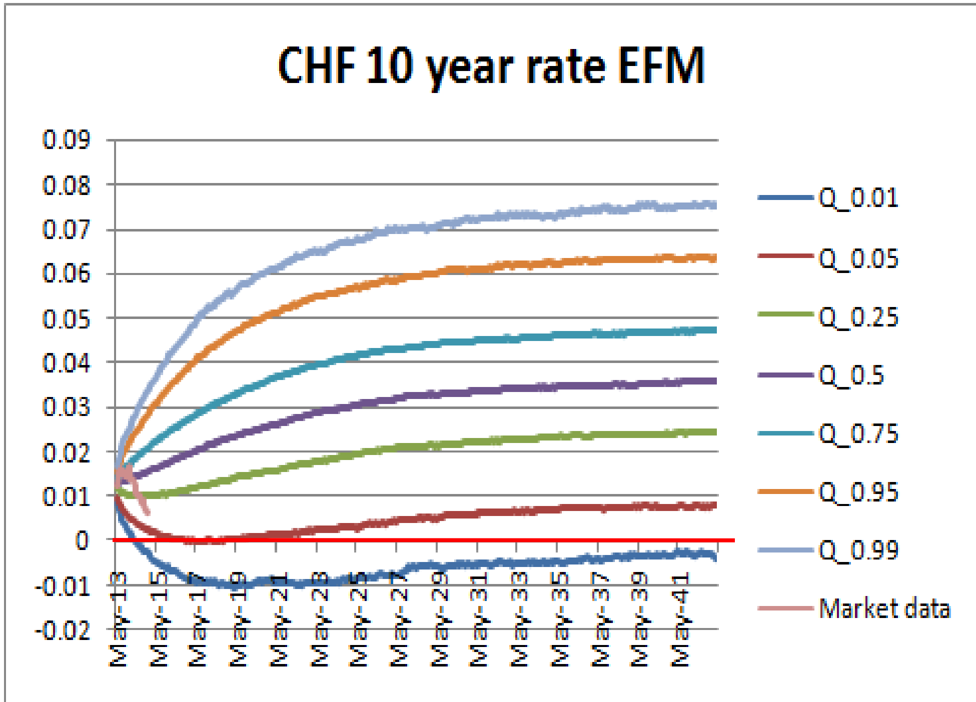


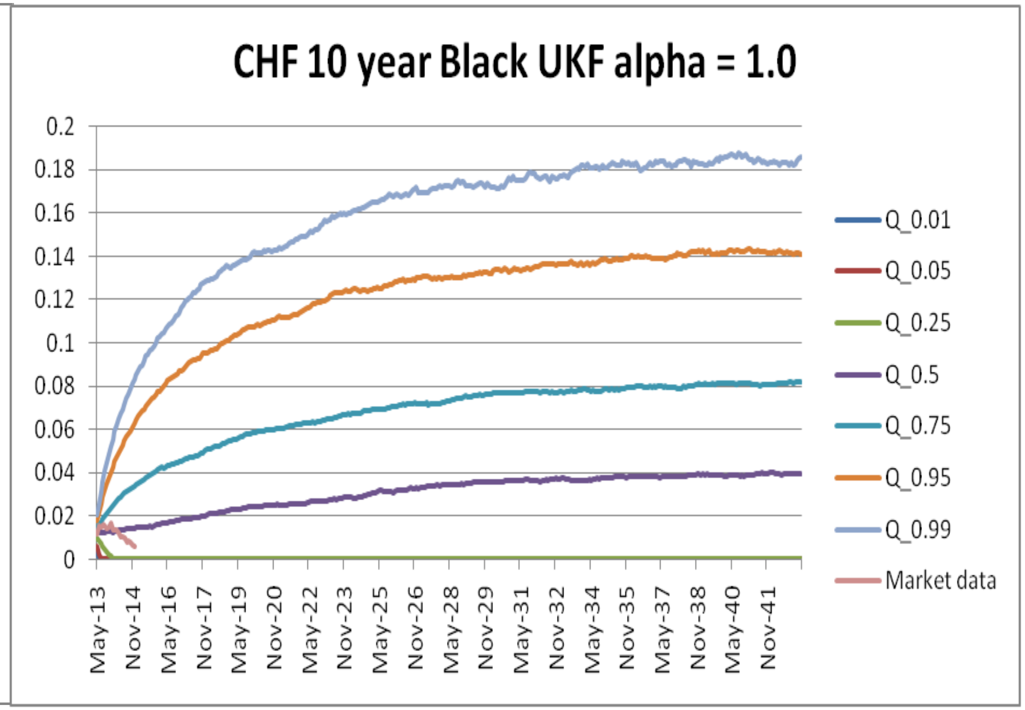


EUR 10 year rate forecast RMSE over 37 months

Black median	1.76%
EFM median	1.37%

CHF 10 Year Rate Out-of-sample Projections

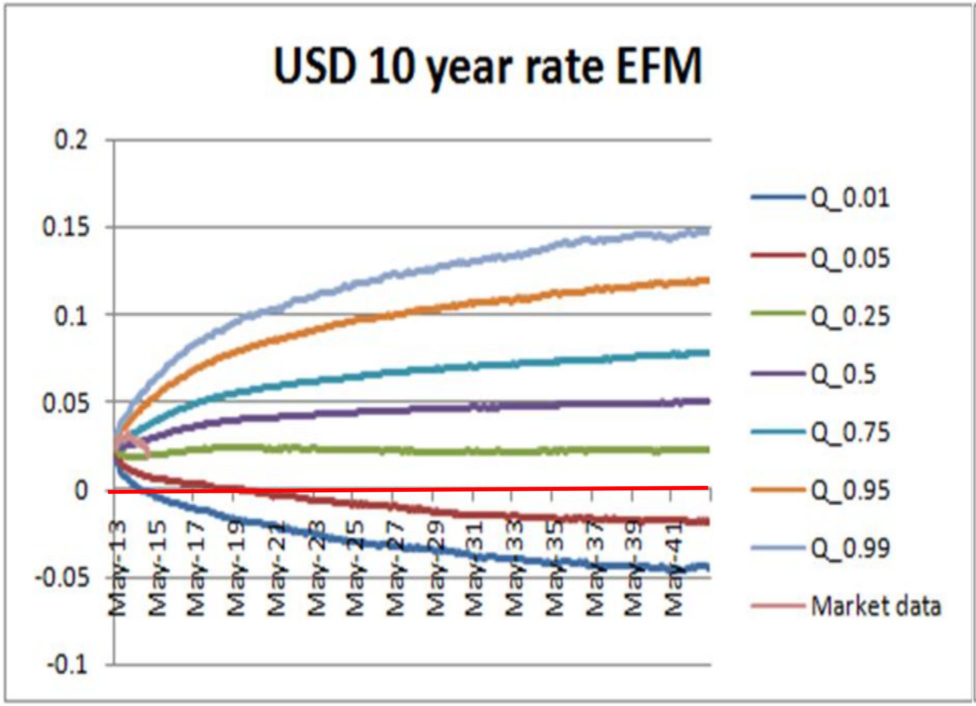


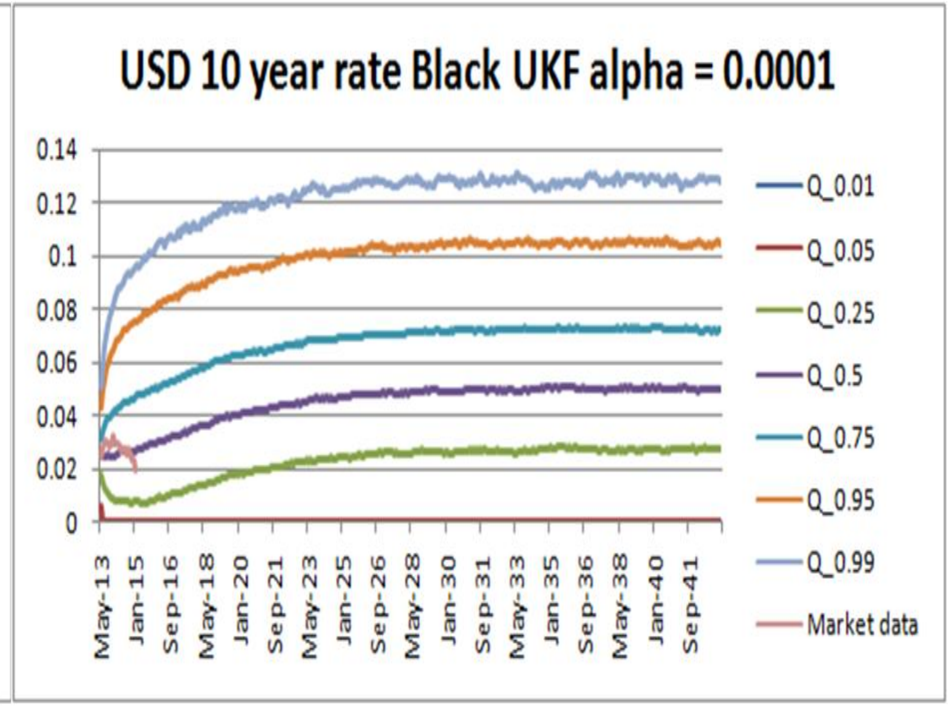


CHF 10 year rate forecast RMSE over 20 months

Black median	**0.40%**
EFM median	**0.45%**

USD 10 Year Rate Out-of-sample Projections

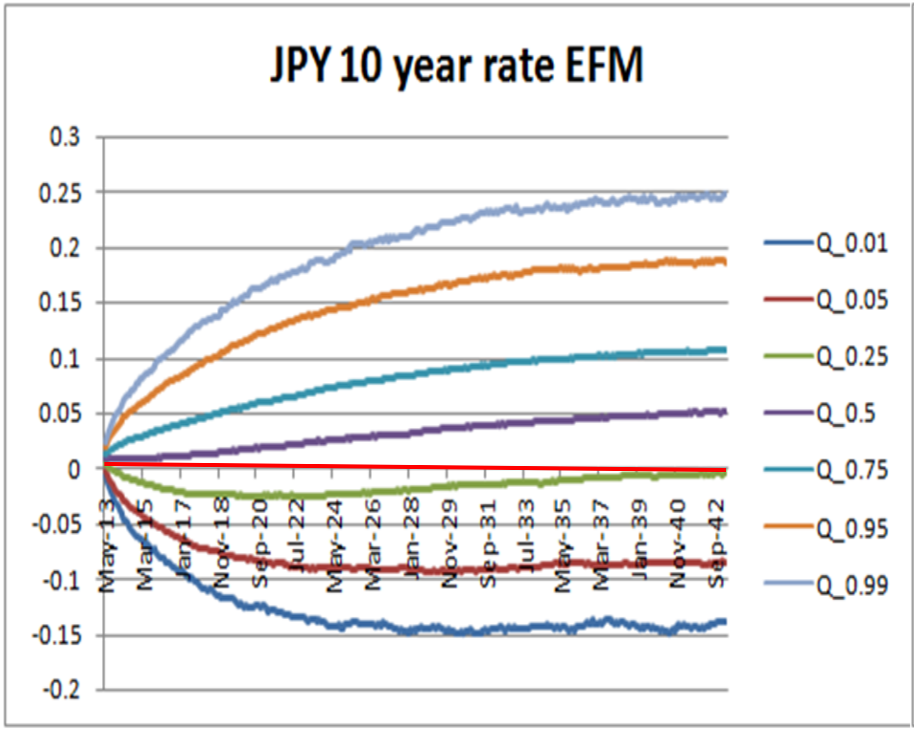


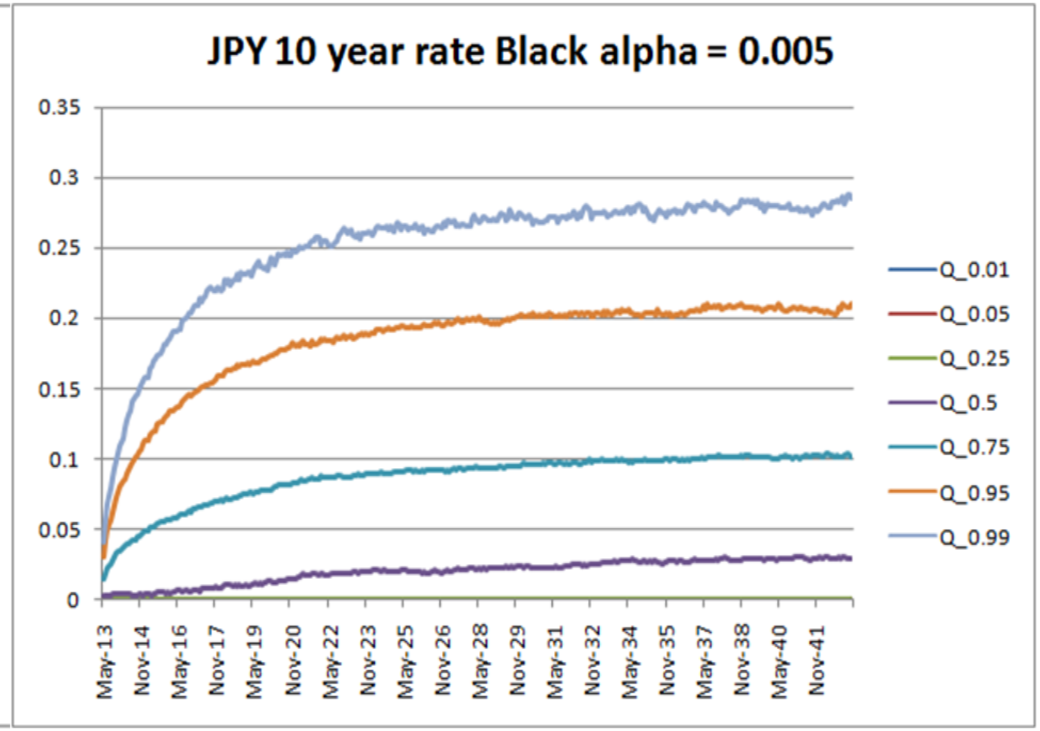


USD 10 year rate forecast RMSE over 21 months

Black median	0.39%
EFM median	0.43%

JPY 10 Year Rate Out-of-sample Projections





Conclusion

Conclusions

- We have developed a Black-corrected version of our workhorse 3 factor affine Gaussian yield curve Economic Factor Model implemented using the unscented Kalman filter to handle the Black nonlinearity and HPC techniques
- Although this method generates an approximation to the full Black model its accuracy is comparable to and computing run time only about twice that of the basic EFM model – unlike all the alternatives published to date which are very heavily computationally intensive
- Using the unreleased NAG UKF algorithm (g13 ejc) with tuned α parameter setting both the in- and out-of-sample accuracy of the method exceeds that of the affine EFM model and it possesses much better dynamics
- Using the cloud we can reduce calibration times on big samples to minutes

References

- L Andersen (1998). A simple approach to pricing Bermudan swaptions in the multifactor LIBOR market model. Working Paper, General Re Financial Products
- F Black (1995). Interest rates as options. *Journal of Finance* **50** 1371-1376
- A N Bomfim (2003) "Interest rates as options:" Assessing the markets' view of the liquidity trap. Working Paper 2003-45, Finance and Economics Discussion Series, Federal Reserve Board, Washington, DC
- M D Bauer & G D Rudebusch (2014), Monetary Policy Expectations at the Zero Lower Bound. Working Paper, Federal Reserve Bank of San Francisco
- B Carton de Wiart & M A H Dempster (2011). Wavelet optimized valuation of financial derivatives. *International Journal of Theoretical and Applied Finance* **14.7** 1113-1137
- J H E Christensen, F X Diebold & G D Rudebusch (2011). The affine arbitrage free class of Nelson-Siegel term structure models. *Journal of Econometrics* **164** 4-20
- J H E Christensen & G D Rudebusch (2013). Modeling yields at the zero lower bound: Are shadow rates the solution? Working Paper 2013-39, Federal Reserve Bank of San Francisco
- J H E Christensen & G D Rudebusch (2014). Estimating shadow rate term structure models with near-zero yields. *Journal of Financial Econometrics*, to appear

- M A H Dempster, J Evans & E A Medova (2014). Developing a practical yield curve model: An odyssey. In: *New Developments in Macro-Finance Yield Curves*, J Chadha, A Durre, M Joyce & L Sarnio, eds., Cambridge University Press (2014) 251-290.
- M A H Dempster, E A Medova, I Osmolovskiy & P Ustinov (2015). A practical robust long term yield curve model. *In preparation*
- M A H Dempster, E A Medova & M Villaverde (2010). Long term interest rates and consol bond valuation. *Journal of Asset Management* **11.2-3** 113-135
- Q Dai & K J Singleton (2000). Specification analysis of affine term structure models. *Journal of Finance* **50** 1943-1978
- D Davydov & V Linetsky (2003). Pricing options on scalar diffusions: An eigenfunction expansion approach. *Operations Research* **51** 185–209
- F X Diebold & G D Rudebusch (2013). *Yield Curve Modelling and Forecasting - The Dynamic Nelson-Siegel Approach.* Princeton University Press
- J D Duffie & R Kan (1996). A yield-factor model of interest rates. *Mathematical Finance* **6** 379-406
- H Ichiue & Y Ueno (2007). Equilibrium interest rate and the yield curve in a low interest rate environment. Working Paper 07-E-18, Bank of Japan
- L Jameson (1998). A wavelet-optimized, very high order adaptive grid and order numerical method. *SIAM Journal on Scientific Computing* **19** 1980-2013

- Joslin, S, K J Singleton & H Zhu (2011). A new perspective on Gaussian dynamic term structure models. *Review of Financial Studies* **24** 926-970.
- S J Julier & J K Uhlmann (1997). A new extension of the Kalman filter to nonlinear systems. *International Symposium on Aerospace/Defense Sensing, Simulation and Control* **3**
- D H Kim & M Priebsch (2013). Estimation of multi-factor shadow-rate term structure models. Working Paper, Research Division, Board of Governors of the Federal Reserve System, Washington, DC, October 2013
- D H Kim & K J Singleton (2011). Term structure models and the zero bound: An empirical investigation of Japanese yields. Working Paper, Stanford Graduate School of Business, June 2011
- L Krippner (2013a). A tractable framework for zero lower bound Gaussian term structure models. CAMA Working Paper No. 49/2013, Australian National University
- L Krippner (2013b). Faster solutions for Black zero lower bound term structure models. CAMA Working Paper 66/2013, Australian National University
- W Lemke & A L Vladu (2014). A shadow-rate term structure for the Euro Area. http://www.ecb.europa.eu/events/pdf/conferences/140908/lemke_vladu.pdf?68b4093d2247e1d0ff5159d1b7b0347b
- NAG (2015). NAG Library, Mark 25. Numerical Algorithms Group, Oxford UK, May 2015. http://www.nag.co.uk/numeric/CL/nagdoc_c125/html/FRONTMATTER/mancont s.html

- V Linetsky (2004): The spectral decomposition of the option value. *International Journal of Theoretical and Applied Finance* **7.3** 337-384
- R Litterman & J A Scheinkman (1991). Common factors affecting bond returns. *Journal of Fixed Income* **1** 54-61
- S K Nawalkha & R Rebonato(2011). What interest rate models to use? Buy side versus sell side. SSRN electronic journal 01/2011
- C R Nelson & A F Siegel (1987). Parsimonious modeling of yield curves. *Journal of Business* **60** 473-489
- M A Priebsch (2013). Computing arbitrage-free yields in multi-factor Gaussian shadow-rate term structure models. Working Paper 2013 - 63, Finance and Economics Discussion Series, Federal Reserve Board
- R Rebonato (2015). Review of *Yield Curve Modelling and Forecasting - The Dynamic Nelson-Siegel Approach* by F X Diebold & G D Rudebusch. *Quantitative Finance*, to appear
- R Rebonato & I Cooper (1995). The limitations of simple two-factor interest rate models. *Journal of Financial Engineering* **5** 1-16
- S F Richard (2013). A non-linear macroeconomic term structure model. Working Paper, University of Pennsylvania
- U Ron (2000) A practical guide to swap curve construction. Working Paper 2000-17, Bank of Canada, August 2000