ST2334 Cheatsheet 19/20 S1 Midterms

Definitions
- Sample Space: S is the set of all possible outcomes
  - eg. For rolling 2 dice: $S = \{(1,1),(1,2)...(6,5),(6,6)\}$
- Sample Point: Any element/outcome in the sample space S
- Event: Any subset E of the sample space
- Sure Event: the sample space itself
- Null Event: empty set $\emptyset$

Counting

Choose k from n	Order Matters	Not Matter
With Replacement	n^k	$\binom{n+k-1}{k}$
Without Replacement	$\frac{n!}{(n-k)!}$	$\binom{n}{k}$

- In a circle: $(n-1)!$

Probability
Inclusion-Exclusion Principle
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A \cup B \cup C) = P(A) + P(B) + P(C) - [P(A\cap B) + P(A\cap C) + P(B \cap C)] + P(A \cap B \cap C)$

Independent Events
- $P(A \cap B) = P(A) \times P(B)$
- $P(A|B) = P(A)$
- $P(A) = P(A \cap B) + P(A \cap B^c)$

Mutually Exclusive Events
- $P(A \cap B) = 0$ (B cannot happen if A happens)
- $P(A|B) = 0$
- $P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$

Two non-trivial ($P > 0$) events can only be independent, or mutually exclusive, or neither, but **never both** at the same time

Conditional Probability
- $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}$
- $P(A|B\cap C) = \frac{P(A\cap B \cap C)}{P(B\cap C)} = \frac{P(B|A\cap C)P(A|C)}{P(B|C)} = \frac{P(B\cap C|A)P(A)}{P(B\cap C)}$
- $P(A \cap B) = P(A)P(B|A) = P(B)(A|B)$

De Morgan's Law
- $(A \cup B)^c = A^c \cap B^c$
- $(A \cap B)^c = A^c \cup B^c$

Partition
- If $B_1, B_2, ..., B_n$ are mutually exclusive and exhaustive (they are disjoint and their union $= S$), then $B_1, B_2, ..., B_n$ is a partition of S

Law of Total Probability (Bayes' Formula 1)

Let $B_1, ..., B_n$ be a partition of S:
- $P(A) = \sum_{i=1}^n P(B_i \cap A) = \sum_{i=1}^n P(B_i)P(A|B_i)$

With extra conditioning:
- $P(A|C) = \sum_{i=1}^n P(A|B_i \cap C)P(B_i|C) = \sum_{i=1}^n P(A \cap B_i|C)$

Special case when B and B^c are the partitions:
- $P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$
- $P(A) = P(A \cap B) + P(A \cap B^c)$

Bayes' Theorem

Let $B_1, \ldots, B_n$ be a partition of S. $\forall k \in 1, \ldots, n$,
- $P(B_k|A) = \frac{P(B_k)P(A|B_k)}{\sum_{i=1}^n P(B_i)(PA|B_i)}$

Discrete Random Variables
Probability Mass Function, $f_X(x)$ (PMF)
- Probability that a discrete random variable $= x$
- Given by $f_X(x) = P(X = x)$
- When asked to find PMF: find $\forall x$, $P(X = x)$

Properties:
1. $0 \le f_X(x) \le 1$
2. $\sum_x f_X(x) = 1$
3. $P(X \in E) = \sum_{x\in E} f_X(x)$

Cumulative Distribution Function, $F_X(x)$ (CDF)
- Probability that a discrete random variable is $\le x$
- $F_X(x) = P(X \le x) = \sum_{t \le x} P(X = t)$

Properties:
1. $F_X(x)$ is a non-decreasing function: $x_1 < x_2 \Rightarrow F_X(x_1) \le F_X(x_2)$
2. $0 \le F_X(x) \le 1$

Continuous Random Variables
- $P(X = x) = 0$
- $P(a < X \le b) = \int_a^b f_X(x)\,dx = F_X(b) - F_X(a)$

Probability Density Function, $f_X(x)$ (PDF)

The function f_X is the PDF of the continuous random variable X $\iff$
1. $\forall x$, $f_X(x) \ge 0$
2. $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

Cumulative Distribution Function, $F_X(x)$ (CDF)

CDF of a continuous random variable X with PDF $f_X(x)$ is given by
- $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$

Properties:
1. $F_X(x)$ is a non-decreasing function of x
2. $\lim_{x\to-\infty} F_X(x) = 0$ **AND** $\lim_{x\to\infty} F_X(x) = 1$

Mean & Variance
Mean, μ (1^{st} Moment)
- Discrete: $E(X) = \sum_x xP(X = x)$ **OR** $X \in \mathbb{N} \Rightarrow \sum_{k=1}^{\infty} P(X \ge k)$
- Random: $E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$
- LOTUS: $E[g(X)] = \sum_x g(x)f_X(x)$ **OR** $\int_{-\infty}^{\infty} g(x)f_X(x)\,dx$
  - eg. to find k^{th} moment: $E(X^k) = \sum_x (x)^k P(X = x)$

Properties:
1. $E(aX + bY + c) = aE(X) + bE(Y) + c$

Variance, σ^2
- Discrete: $V(X) = E[(X - \mu_X)^2] = \sum_x (x - \mu_X)^2 f_X(x)$
- Random: $V(X) = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x)\,dx$
- $SD(X) = \sqrt{V(X)}$

Properties:
1. $V(X) \ge 0$
2. $V(X) = E(X^2) - [E(X)]^2$
3. $V(X) = 0 \Rightarrow P(X = \mu_X) = 1$
4. $V(a + bX) = b^2 V(X)$

Chebyshev's Inequality

If a random variable X has mean, μ, and SD, σ, the probability of getting a value which deviates from μ by at least $k\sigma$ is at most $\frac{1}{k^2}$
- $P(|X - \mu| > k\sigma) \le \frac{1}{k^2}$ **OR** $P(|X - \mu| \le k\sigma) \ge 1 - \frac{1}{k^2}$
- Applying $k = 2$, we conclude that for any random variable X, there is at most $\frac{1}{4}$ chance that it is 2 SD or further away from its mean

Joint Distribution
Joint Probability Mass Function
1. $f_{(X,Y)}(x, y) \ge 0$, $\forall (x, y) \in R_{X,Y}$
2. $\sum_x \sum_y f_{X,Y}(x, y) = \sum_x \sum_y P(X = x, Y = y) = 1$
3. $P((X, Y) \in A) = \sum_{(x,y)\in A} f_{X,Y}(x, y)$

Joint Probability Density Function
1. $f_{X,Y}(x, y) \ge 0$, $\forall (x, y) \in R_{X,Y}$
2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x, y)\,dydx = 1$

Examples
Suppose it is given that

$$F(0) = 0.58,\ F(1) = 0.72,\ F(2) = 0.76,\ F(3) = 0.81,\ F(4) = 0.88,\ F(5) = 0.94.$$

Compute the follow probabilities.

(a) $P(2 \le X \le 5)$

(b) $P(X = 3)$

Solution:

(a)
$$P(2 \le X \le 5) = F(5) - F(2^-) = F(5) - F(1) = 0.22.$$

(b)
$$P(X = 3) = F(3) - F(3^-) = F(3) - F(2) = 0.05.$$

Suppose that the random variable X is continuous with the following probability mass function:

$$f(x) = \begin{cases} \dfrac{x}{225}, & 0 < x < 15 \\ \dfrac{30-x}{225}, & 15 \le x \le 30 \\ 0, & \text{otherwise} \end{cases}.$$

Find $E(X)$ and $V(X)$.

Solution:

We first obtain

$$\begin{aligned} E(X) &= \int_0^{15} x \cdot \frac{x}{225}\,\mathrm{dx} + \int_{15}^{30} x \cdot \frac{30-x}{225}\,\mathrm{dx} \\ &= \frac{1}{225}\left(\left[\frac{x^3}{3}\right]_0^{15} + \left[15x^2 - \frac{x^3}{3}\right]_{15}^{30}\right) \\ &= 15. \end{aligned}$$

Now,

$$\begin{aligned} E(X^2) &= \int_0^{15} x^2 \cdot \frac{x}{225}\,\mathrm{dx} + \int_{15}^{30} x^2 \cdot \frac{30-x}{225}\,\mathrm{dx} \\ &= \frac{1}{225}\left(\left[\frac{x^4}{4}\right]_0^{15} + \left[10x^3 - \frac{x^4}{4}\right]_{15}^{30}\right) \\ &= \frac{525}{2}. \end{aligned}$$

Thus

$$V(X) = E(X^2) - [E(X)]^2 = \frac{525}{2} - 15^2 = 37.5.$$

Let X denote the amount of time for which a book on 2-hour reserve at the Science Library is checked out by a randomly selected student and suppose X has the probability density function

$$f(x) = \begin{cases} \frac{x}{2}, & 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}.$$

(a) Find $E(X)$, $V(X)$ and σ_X.

(b) If the borrower is charged an amount $h(X) = X^2$ when checkout duration is X, compute the expected charge $E[h(X)]$.

Solution:

(a)

$$\begin{aligned} E(X) &= \int_0^2 x \cdot \frac{x}{2} \, \mathrm{d}x = \frac{4}{3}. \\ E(X^2) &= \int_0^2 x^2 \cdot \frac{x}{2} \, \mathrm{d}x = 2. \\ V(X) &= 2 - (4/3)^2 = 2/9. \\ \sigma_X &= \sqrt{2/9}. \end{aligned}$$

(b) $E[h(X)] = E(X^2) = 2$.

The probability density function of a continuous random variable X, the total number of hours, in units of 100 hours, that a family runs a vacuum cleaner over a period of one year, is given as follows.

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}.$$

Find the average number of hours per year that families run their vacuum cleaners.

Solution:

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f(x) \, \mathrm{d}x \\ &= \int_0^1 x^2 \, \mathrm{d}x + \int_1^2 x(2-x) \, \mathrm{d}x \\ &= \left[\frac{x^3}{3}\right]_0^1 + \left[x^2 - \frac{x^3}{3}\right]_1^2 = 1. \end{aligned}$$

Families run their vacuum cleaners 100 hours per year on average.

The probability density function $f(x)$ of a random variable X is

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}.$$

Find the cumulative distribution function of X.

Solution:

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t) \, \mathrm{d}t \\ &= \begin{cases} \int_{-\infty}^{x} 0 \, \mathrm{d}t, & \text{for } x < 0 \\ \int_{-\infty}^{0} 0 \, \mathrm{d}t + \int_0^x 2t \, \mathrm{d}t, & \text{for } 0 < x < 1 \\ \int_{-\infty}^{0} 0 \, \mathrm{d}t + \int_0^1 2t \, \mathrm{d}t + \int_1^x 0 \, \mathrm{d}t, & \text{for } x \ge 1 \end{cases} \\ &= \begin{cases} 0, & \text{for } x < 0 \\ x^2, & \text{for } 0 < x < 1 \\ 1, & \text{for } x \ge 1 \end{cases}. \end{aligned}$$

The daily production of electric motors at a certain factory averaged 120 with a standard deviation of 10. Use the Chebyshev's Inequality to find an interval that contains at least 90% of the daily production levels.

Solution:

Let X be the daily production of electric motors at that factory. Chebyshev's Inequality gives

$$P(\mu - k\sigma < X < \mu + k\sigma) \ge 1 - \frac{1}{k^2}.$$

Setting $1 - \frac{1}{k^2} = 0.90$ we obtain $k = \sqrt{10}$, so

$$P\left(\mu - \sqrt{10}\sigma < X < \mu + \sqrt{10}\sigma\right) \ge 0.90.$$

The interval we seek is

$$\left(120 - \sqrt{10} \times 10, 120 + \sqrt{10} \times 10\right) = (88.4, 151.6).$$

A company has 2 production lines, A and B, which produces at most 5 and 3 machines respectively. Assume that the number of machines produced is a random variable.

Let (X,Y) represent the 2-dimensional random variable yielding the numbers of machines produced by Line A and Line B respectively on a given day.

The joint probability function, $f_{X,Y}(x,y)$ of (X,Y) is given as follow

$f_{X,Y}(x,y)$

y \ x	0	1	2	3	4	5	Row Total
0	0	0.01	0.02	0.05	0.06	0.08	**0.22**
1	0.01	0.03	0.04	0.05	0.05	0.07	**0.25**
2	0.02	0.03	0.05	0.06	0.06	0.07	**0.29**
3	0.02	0.04	0.03	0.04	0.06	0.05	**0.24**
Column Total	**0.05**	**0.11**	**0.14**	**0.20**	**0.23**	**0.27**	1

What is the probability that more chips are produced by Line A than by Line B on a given day?

Solution:

Let $B = \{X > Y\}$. Then

$$\begin{aligned} P(B) &= P(X > Y) \\ &= P((X,Y) = (1,0)) + P((X,Y) = (2,0)) \\ &\quad + P((X,Y) = (2,1)) + \cdots + P((X,Y) = (5,3)) \\ &= f_{X,Y}(1,0) + f_{X,Y}(2,0) + f_{X,Y}(2,1) + \cdots f_{X,Y}(5,3) \\ &= 0.01 + 0.02 + 0.04 + \cdots + 0.06 + 0.05 = 0.73 \end{aligned}$$

Suppose the two-dimensional continuous random variable (X,Y) has the joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} x^2 + \frac{xy}{3}, & \text{for } 0 \le x \le 1, 0 \le y \le 2 \\ 0, & \text{otherwise} \end{cases}.$$

Verify that $f_{X,Y}$ is a joint probability density function and compute $P(X + Y \ge 1)$

Solution:

It is clear that $f_{X,Y}(x,y) \ge 0$ for all $0 \le x \le 1, 0 \le y \le 2$.

$$\begin{aligned} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y) \, \mathrm{d}x \, \mathrm{d}y &= \int_0^2 \int_0^1 \left(x^2 + \frac{xy}{3}\right) \mathrm{d}x \, \mathrm{d}y \\ &= \int_0^2 \left[\frac{x^3}{3} + \frac{x^2 y}{6}\right]_0^1 \mathrm{d}y \\ &= \int_0^2 \left(\frac{1}{3} + \frac{y}{6}\right) \mathrm{d}y = \left[\frac{y}{3} + \frac{y^2}{12}\right]_0^2 = 1. \end{aligned}$$

Thus $f_{X,Y}$ is a joint probability density function.

$$\begin{aligned} P(X + Y \ge 1) &= \int_0^1 \int_{1-x}^2 \left(x^2 + \frac{xy}{3}\right) \mathrm{d}y \, \mathrm{d}x \\ &= \int_0^1 \left[x^2 y + \frac{xy^2}{6}\right]_{1-x}^2 \mathrm{d}x \\ &= \int_0^1 \frac{1}{6}(5x^3 + 8x^2 + 3x) \, \mathrm{d}x \\ &= \frac{1}{6}\left[5x^4/4 + 8x^3/3 + 3x^2/2\right]_0^1 = \frac{65}{72}. \end{aligned} \tag{3.1}$$
