Math 136 (Partial Differential Equations)
University of California, Los Angeles

Aaron Chao

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These are my lecture notes for Math 136 (Partial Differential Equations) taught by Marcus Roper. The main textbook for this class is *Partial Differential Equations: An Introduction* by Walter Strauss.

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1 Mar 28, 2022

1.1 Motivation

Motivating example: Suppose we want to describe where the gas molecules are in a room.

Approach 1: Label every gas molecule and give x, y, z coordinates for each.

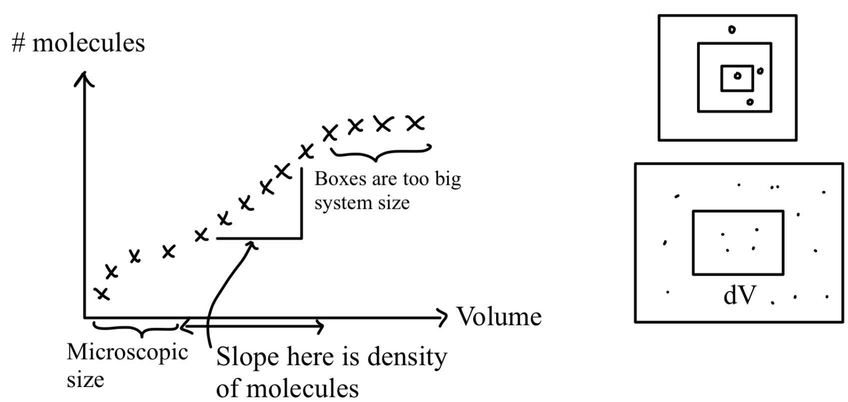
- Too much data.

Approach 2: Divide the room up into small volumes (boxes). Count the number of molecules in each volume.

- Number of molecules depends on the size of the box
- Take density/concentration:

$$\frac{\#\text{ molecules in the box}}{\text{volume of the box}}$$

We assume the distribution of molecules obeys the Continuum Hypothesis.



We assume our box sizes are in a region in which the number of molecules or volume, so density is well-defined. We defined. We define a field $u(\mathbf{x}, t)$ that describes the density of molecules.

At each point u counts molecules at $(\mathbf{x}, t)$; in the sense that if I make a box, volume dV, at $(\mathbf{x}, t)$; the number of molecules in box is: $u(\mathbf{x}, t)\mathrm{dV}$.

Note 1.1: u is dependent variable, and there are multiple independent variables; x, y, z, t

u (density) is one example of a field - a dependent variable that is defined at different points:

$$u\colon \mathbb{R}^3 \subset D \times \underbrace{[0, T]}_{\text{time interval}} \to \mathbb{R}$$
$$u\colon (\mathbf{x}, t) \mapsto u(\mathbf{x}, t)$$

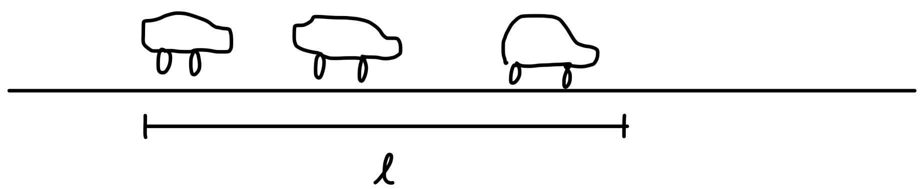
Other examples:

- Velocity (vector field); $\mathbf{v}\colon D \times [0,T] \to \mathbb{R}^3$
- temperature (scalar field); $\theta\colon D \times [0,T] \to \mathbb{R}$
- Electric field/magnetic field
- Distribution/density of cars
- Displacement of the ocean surface

We will derive (and solve) Partial Differential Equations (PDEs) as mathematical models for scalar and vector fields that depend on position (and in many cases, time).

1.2 Example of a PDE

We are modeling the density/distribution of cars on a freeway (looking at only one direction).



Count number of cars in some length ℓ of freeway.

$$\text{density of cars } = \frac{\#\text{ in length }\ell}{\ell}$$

e.g. if $\ell = 1$ km; and 1 count 30 cars $\implies u = 30/\text{km}$ given this density.

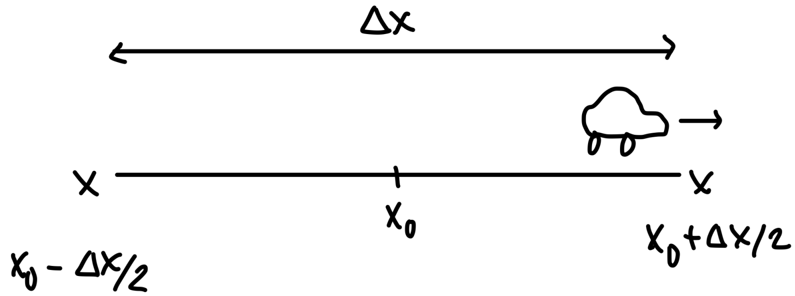
If $\ell = 500m$,

$$\begin{aligned}\# &= 30\text{ km } \times 500\text{ m}\\ &= 30/\text{km } \times 0.5\text{ km}\\ &= 15\text{ cars}\end{aligned}$$

$u \equiv u(x,t)$; u depends on x (distance along freeway) and t (time).

I will assume (Continuum-Hypothesis) that densities calculated at each poinn in the freeway give rise to a C^1 (continuously differentiable) field u. I want to derive an equation for u.

Calculus idea: If I know $u(x,t)$, I want to calculate the density shortly after; $u(x, t+\Delta t)$. If I can do this then I can calculate $u(x, t+2\Delta t), u(x, t+3\Delta t), \ldots$. Imagine that all cars drive at the same speed, c. Consider # cars in some interval.



At time t, there are $u(x_0, t)\Delta x$ cars in the interval.

$$\begin{aligned}\# \text{ cars at time } t + \Delta t = \# \text{ cars at time } t + \# \text{ entering at } x_0 - \Delta x/2 \\ - \# \text{ leaving at } x_0 + \Delta x/2\end{aligned}$$

This is the word statement of conservation of mass/cars.

2 Mar 30, 2022

2.1 Example of a PDE (Cont'd)

Recall 2.1

$$\begin{aligned}\#\text{ cars at time } t+\Delta t = \#\text{ cars at time } t + \#\text{ entering at } x_0 - \Delta x/2 \\ - \#\text{ leaving at } x_0 + \Delta x/2\end{aligned}$$

Therefore,

$$u(x_0, t+\Delta t)\Delta x = u(x_0, t)\Delta x + \underline{\hspace{4cm}}$$

Let's fill in the # cars entering or leaving.

Consider station at $x = x_0 + \Delta x/2$, how many cars pass this station in time Δt?

All of the cars to my left, that are within distance $c\Delta t$ of me, will pass in time Δt. In time Δt a car travels distance $c\Delta t$, so # cars in the interval is

$$\underbrace{u(x_0 + \Delta x/2, t)}_{\text{density}} \times \underbrace{c\Delta t}_{\text{length}}$$

We will show it doesn't change anything if we use $u(x_0 + \frac{\Delta x}{2} - \frac{1}{2}c\Delta t, t)$ instead.

Returning to conservation of cars:

$$u(x_0, t+\Delta t)\Delta x = u(x_0, t)\Delta x + \underbrace{u(x_0 - \Delta x/2, t)c\Delta t}_{\#\text{ entering}} - \underbrace{u(x_0 + \Delta x/2, t)c\Delta t}_{\#\text{ leaving}}$$

$$\Big(u(x_0, t+\Delta t) - u(x_0, t)\Big)\Delta x = -\Big(u(x_0 + \Delta x/2, t) - u(x_0 - \Delta x/2, t)\Big)c\Delta t \tag{2.1}$$

Recall

$$\frac{\partial u}{\partial t}(x, t) = \lim_{h\to 0}\left(\frac{u(x, t+h) - u(x, t)}{h}\right)$$

So,

$$\begin{aligned}\frac{\partial u}{\partial x}(x, t) &= \lim_{h\to 0}\left(\frac{u(x+h, t) - u(x, t)}{h}\right) \\ &= \lim_{h\to 0}\left(\frac{u(x+h/2, t) - u(x-h/2, t)}{h}\right)\end{aligned}$$

Now dividing (2.1) by $\Delta x\Delta t$

$$\frac{u(x_0, t+\Delta t) - u(x_0, t)}{\Delta t} = -c\left(\frac{u(x_0 + \Delta x/2, t) - u(x_0 - \Delta t/2, t)}{\Delta x}\right)$$

let $\Delta x \to 0, \Delta t \to 0$, then

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x}$$

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \tag{2.2}$$

u is a dependent variable, it depends on x and t as independent variables.

There is also a constant, parameter c.

Notation 2.2: Other notations are used for partial derivatives.

$$\underbrace{u_t + cu_x = 0}_{\text{Strauss}} \qquad \text{or} \qquad \underbrace{u_{,t} + cu_{,x}}_{\text{Roper}}$$

There are some solutions of (2.2).

$$u = 1 + \frac{1}{2}\sin(x - ct)$$
$$u = \frac{1}{2}(x - ct)^2$$
$$u = e^{-x+ct}$$

We can check these are solutions

$$u(x,t) = e^{-x+ct}$$
$$\frac{\partial u}{\partial t} = ce^{-x+ct}$$
$$\frac{\partial u}{\partial x} = -e^{-x+ct}$$

So

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = ce^{-x+ct} + c(-e^{-x+ct}) = 0$$

Compare with ODEs:

$$\frac{dy}{dx} = y$$

has solution $y(x) = Ce^x$ which has a constant of integration.

2.2 Linearity

In ODEs we use initial conditions to find our constants. To solve a PDE completely, we need both the PDE and an auxiliary or side condition. That is, we need either initial conditions or boundary conditions (or both) on u. (2.2) is an example of a PDE.

Definition 2.3 (Operator)
Most generally, a PDE takes the form:

$$\mathcal{L}[u] = 0$$

We call $\mathcal{L}$ an operator.

In this case:

$$\mathcal{L}[u] = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}$$

$\mathcal{L}$ includes derivatives of u, more derivatives are possible: e.g.:

$$\mathcal{L}[u] = \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}$$

Definition 2.4 (Linear operator)
We say an operator is linear if it has the following properties:

1. If $\mathcal{L}[u] = 0$ and a is constant, then $\mathcal{L}[au] = 0$.
2. If u_1, u_2 solve the PDE, $\mathcal{L}[u_1] = 0, \mathcal{L}[u_2] = 0$, then $v = u_1 + u_2$ also solves the PDE $\mathcal{L}[u_1 + u_2] = 0$.

For (2.2),

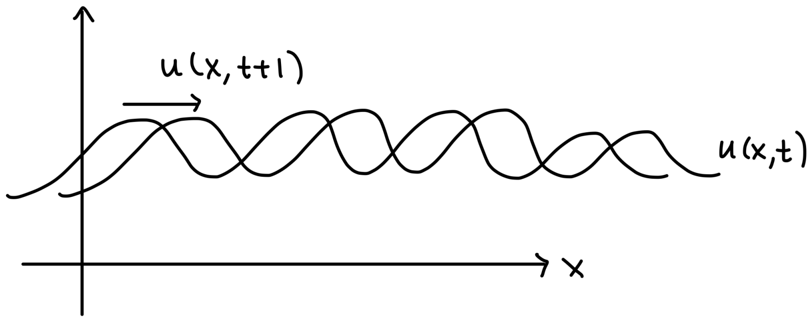
$$\begin{aligned}
\mathcal{L}[u_1 + u_2] &= \frac{\partial}{\partial t}(u_1 + u_2) + c\frac{\partial}{\partial x}(u_1 + u_2) \\
&= u_{1,t} + u_{2,t} + c(u_{1,x} + u_{2,x}) \\
&= (u_{1,t} + cu_{1,x}) + (u_{2,t} + cu_{2,x}) \\
&= \mathcal{L}[u_1] + \mathcal{L}[u_2]
\end{aligned}$$

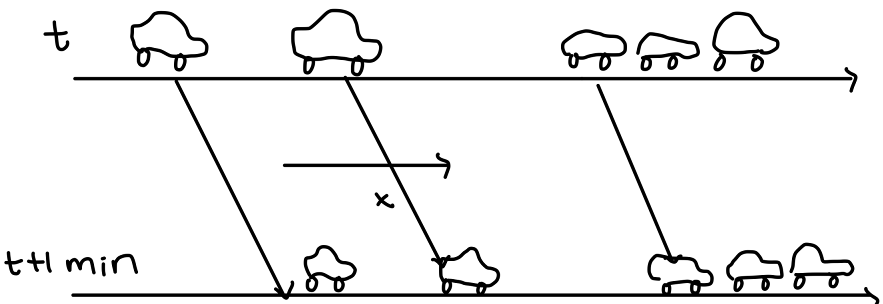
Example 2.5
Examples of linear versus non-linear PDEs

1. $\mathcal{L}[u] = u_{,x} + xu_{,y}$ (linear)
2. $\mathcal{L}[u] = u_{,t} + \underbrace{u_{,xxx}}_{\partial^3 u/\partial x^3}$ (linear)
3. $\mathcal{L}[u] = u_{,x} + \alpha u_{,y}, \alpha$ is a constant. (linear)
4. $\mathcal{L}[u] = u_{,x} + \sqrt{x^2 + y^2}e^{-x}u_{,y}$ (linear)
5. $\mathcal{L}[u] = u_{,x} + \sqrt{x^2 + u^2}u_{,xx}$ (non-linear)

1)

$$\begin{aligned}
\mathcal{L}[au] &= au_{,x} + xau_{,y} \\
&= a(u_{,x} + xu_{,y}) \\
&= a\mathcal{L}[u] \\
\mathcal{L}[u_1 + u_2] &= (u_1 + u_2)_{,x} + x(u_1 + u_2)_{,y} \\
&= (u_{1,x} + xu_{1,y}) + (u_{2,x} + xu_{2,y}) \\
&= \mathcal{L}[u_1] + \mathcal{L}[u_2]
\end{aligned}$$





The solution of the PDE is a traveling/shifting copy of $u(x,t)$; we call these solutions traveling waves.

3 Apr 1, 2022

3.1 Characteristics

We are studying the PDE:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \tag{3.1}$$

Based on our understanding of how cars move, if all travel at same speed c, then the solution will be a traveling wave. So, if at time $t = 0$, $u(x, 0) = g(x)$, we expect $u(x, t)$ to be the same graph shifted by ct units to the right.

> **Recall 3.1** If $y = f(x)$ has a certain graph $y = f(x - a)$ is the same graph shifted by a to the right.

So

$$u(x, t) = g(x - ct) \tag{3.2}$$

E.g. if $u(x, 0) = e^{-x}$ then at time t, $u(x, t) = e^{-(x-ct)} = e^{-x+ct}$. We can check that any function of the form (3.2) solves our PDE.

$$u_{,t} = -cg'(x - ct)$$

$$u_{,x} = (1)g'(x - ct)$$

Hence,

$$u_{,t} + cu_{,x} = -cg' + cg' = 0$$

To understand, mathematically, why (3.1) has traveling wave solutions, we need to study the advective derivative.

Given a field $\theta(x, t)$ derivatives give us information about the rate of change of θ. E.g.

$$\frac{\partial \theta}{\partial t} = \text{ time rate of change at fixed } x \text{ (Eulerian derivative)}$$

Another derivative comes from sampling θ at different points and times (time rate of change according to a moving observer).

Moving observer has a location $x(t)$, their rate of change is given by the advective derivative or Lagrangian derivative

$$\frac{d}{dt}\left(\theta(x(t), t)\right) = \underbrace{\frac{\partial \theta}{\partial x}\frac{dx}{dt}}_{\text{new term}} + \underbrace{\frac{\partial \theta}{\partial t}}_{\text{Eulerian derivative}} \tag{3.3}$$

Example 3.2
$\theta(x, t)$ is temperature, x is distance.
$\theta(x, t) = mx + b$ for some m and b constants.

Now imagine an observer enters room, and walks along x, at speed v so $\frac{dx}{dt} = v$. The time rate of change of temperature is

$$\frac{d\theta}{dt} = mv + 0 = mv$$

Compare (3.3)

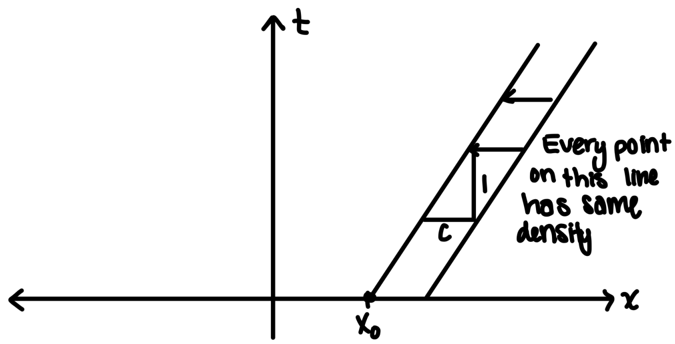
$$\frac{d\theta}{dt} = \frac{\partial\theta}{\partial t} + \left(\frac{dx}{dt}\right)\frac{\partial\theta}{\partial x}$$

with (3.1)

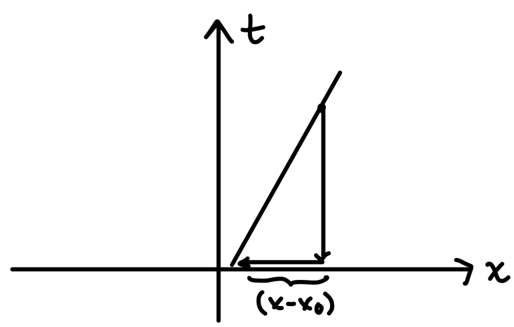
$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

(3.1) says that $\frac{du}{dt} = 0$ if $\frac{dx}{dt} = c$. In other words, according to an observer moving at speed $\frac{dx}{dt} = c$, the time rate of change is $\frac{du}{dt} = 0$. So u is constant according to this observer.

Suppose we know $u(x, 0)$, the initial density. I can draw a space-time diagram.



According to an observer moving at speed c, the density is constant. Similarly line starting at $x = x_1$. These lines called characteristics must be level curves / contours / isocontours of $u(x, t)$. Using characteristics, we can solve our equation.



Given (x, t) what is $u(x, t)$?

I need the PDE and some initial condition or boundary condition, suppose I know $u(x, 0) = g(x)$.

(x,t), where I want density, lies on a characteristic, every point on characteristic has the same density.

Follow the characteristic back to the x-axis $u(x,t) = u(x_0, 0)$ where x_0 is where the characteristic hits the x-axis $u(x,t) = g(x_0)$.

Given (x,t), I need to find x_0. From the picture,

$$\begin{aligned}(x - x_0) &= ct \\ x_0 &= x - ct\end{aligned}$$

hence

$$u(x,t) = g(x_0) = g(x - ct)$$

3.2 Using Characteristics to Solve More PDEs

Method of characteristics solves PDEs by tracing characteristics; lines or curves on which u is a constant. It can be used to solve PDEs of the form:

$$a(t,x)u_{,t} + b(t,x)u_{,x} = 0$$

where a and b are any functions. Or,

$$a(x,y)u_{,x} + b(x,y)u_{,y} = 0$$

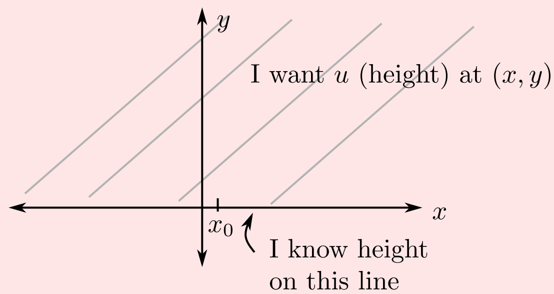
Note 3.3: Now we are using x and y for independent variables.

This is a linear PDE. it is also called first order (only one partial derivative in each term). To use it, start with example:

Example 3.4
We have

$$au_{,x} + bu_{,y} = 0$$

where a, b are both constants (a, b are not both 0). Solution is $u(x, y)$.



Assume I know $u(x, 0)$.

$$au_{,x} + bu_{,y} = 0 \iff (a, b)^T \cdot \nabla u = 0$$

i.e. the directional derivative of u, along $\begin{pmatrix} a \\ b \end{pmatrix}$ is equal to 0, i.e. u is constant in the direction $\begin{pmatrix} a \\ b \end{pmatrix}$.

Characteristics point in the direction of $\begin{pmatrix} a \\ b \end{pmatrix}$.

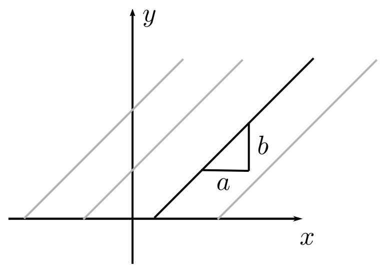
4 Apr 4, 2022

4.1 Using Characterizations to Solve More PDEs (Cont'd)

For the PDE:

$$au_{,x} + bu_{,y} = 0$$

Method #1: The lines parallel to $\begin{pmatrix} a \\ b \end{pmatrix}$ are characteristics.



Characteristics are lines $y = \frac{b}{a}x + c$ where c is a constant. Hence the characteristics are lines $ay - bx = ac = C$. Each value of C gives a different straight line, and $u \equiv f(C)$ or $u \equiv f(ay - bx)$. Any function $u \equiv f(ay - bx)$ is a solution of the PDE. E.g.

$$u(x, y) = \sin(ay - bx)$$

$$u(x, y) = (ay - bx)^2$$

Apply the auxiliary condition to find out what f is.

Example 4.1

Solve $2u_{,x} + 3u_{,y} = 0$ with an auxiliary condition $u(0, y) = y^3$ (u is known on the x-axis). Theory above says

$$u(x, y) = f(2y - 3x)$$

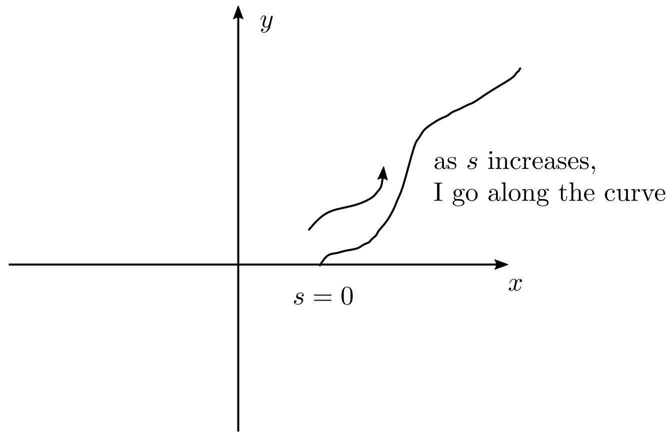
to satisfy the auxiliary condition:

$$\begin{aligned} u(0, y) &= y^3 \\ \implies f(2y) &= y^3 \\ \implies f(z) &= \left(\frac{z}{2}\right)^3 \\ \implies f(y) &= \left(\frac{y}{2}\right)^3 \end{aligned}$$

So

$$u(x, y) = f(2y - 3x) = \left(\frac{2y - 3x}{2}\right)^3$$

Method #2: If we didn't explicitly use the equation of a straight line, we know that the characteristics are parallel to $\begin{pmatrix} a \\ b \end{pmatrix}$. If a characteristic is a curve $(x(s), y(s))$.



Where,

$$\frac{dx}{ds} = a \qquad \frac{dy}{ds} = b$$

Chain rule says

$$\frac{dy}{dx} = \frac{\frac{dy}{ds}}{\frac{dy}{ds}} = \frac{b}{a}$$

Hence,

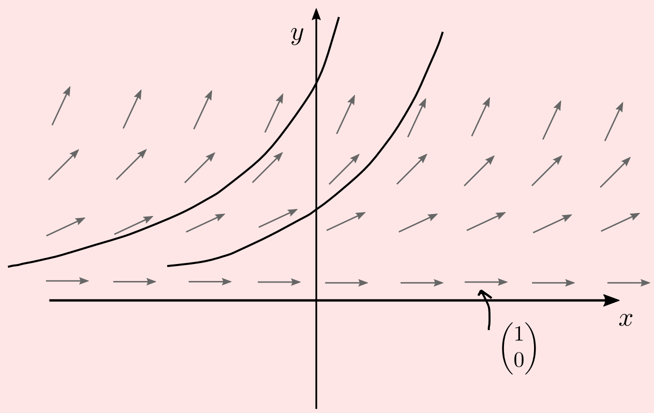
$$y = \left(\frac{b}{a}\right) x + c$$

We can follow Method #2, even when a and b are not constants.

Example 4.2
Solve $u_{,x} + yu_{,y} = 0$ with an auxiliary condition $u(0, y) = 5y$.
Our PDE gives:

$$\begin{pmatrix} 1 \\ y \end{pmatrix} \cdot \nabla u = 0$$

which implies ∇u is perpendicular to $\begin{pmatrix} 1 \\ y \end{pmatrix}$. In other words, I can define characteristics (on which u is constant) that are always parallel to $\begin{pmatrix} 1 \\ y \end{pmatrix}$.



The equation of any characteristic is:

$$\frac{dx}{ds} = 1 \qquad\qquad \frac{dy}{ds} = y$$

$$\frac{dy}{dx} = \frac{\frac{dy}{ds}}{\frac{dx}{ds}} = y$$

So the characteristics are lines $y(x) = Ce^x$. Different values of C give different characteristics, and therefore different values of u.

$$\begin{aligned} u &= f(C) \\ u &= f(ye^{-x}) \end{aligned}$$

$$\implies \begin{cases} u(x, y) = (ye^{-x})^2 \\ u(x, y) = \sinh(ye^{-x}) \end{cases}$$

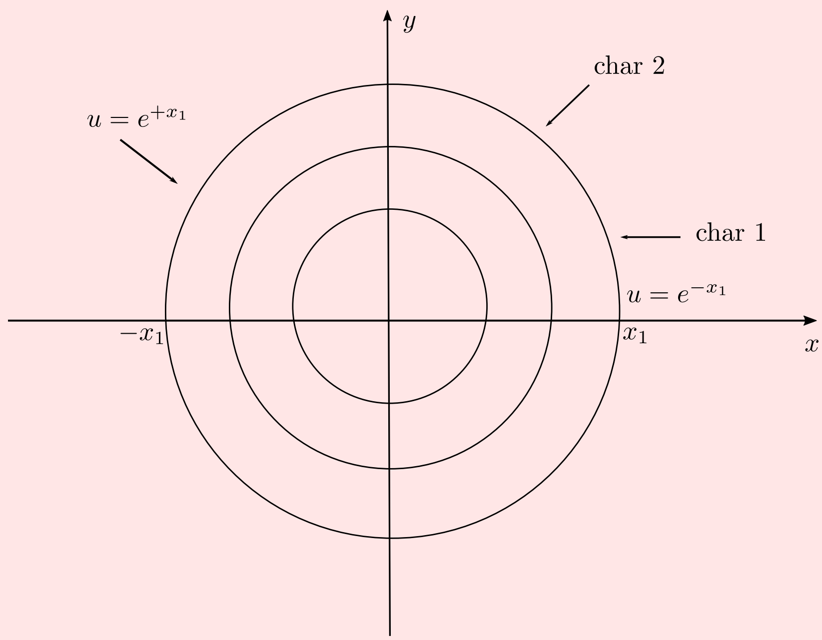
Appeal to the auxiliary condition

$$u(0, y) = 5y \implies \begin{cases} f(y) = 5y \\ f(z) = 5z \end{cases}$$

Hence, $u(x, y) = f(ye^{-x}) = 5ye^{-x}$.

Example 4.3

Solve $yu_{,x} - xu_{,y} = 0$ with auxiliary condition $u(x,0) = e^{-x^2}$.



u is constant on characteristics

$$\left.\begin{aligned} \tfrac{dx}{ds} &= y \\ \tfrac{dy}{ds} &= -x \end{aligned}\right\} \implies \frac{dy}{dx} = \frac{-x}{y}$$

So

$$\begin{aligned} 2\int y\,dy &= 2\int -x\,dx \\ y^2 &= -x^2 + C \\ \implies x^2 + y^2 &= C \text{ on characteristics} \end{aligned}$$

So $u \equiv f(x^2 + y^2)$ and

$$\begin{aligned} u(x,0) = f(x^2) &= e^{-x^2} \\ f(z) &= e^{-z} \end{aligned}$$

hence $u(x,y) = e^{-x^2-y^2}$.

Note 4.4: Each point (x,y) lies on $\begin{cases} \text{a characteristic that meets the } x\text{-axis in 2 places} \\ \text{2 characteristics} \end{cases}$.

This can be a problem for some kinds of auxiliary conditions. E.g. if $u(x,0) = e^{-x}$.

4.2 PDE Models

We met

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

as an example of conservation of mass. This is an example of a transport model. Other examples start from the following idea:

$$u \text{ changes in an interval } \left(x_0 - \frac{\Delta x}{2}, x_0 + \frac{\Delta x}{2}\right)$$

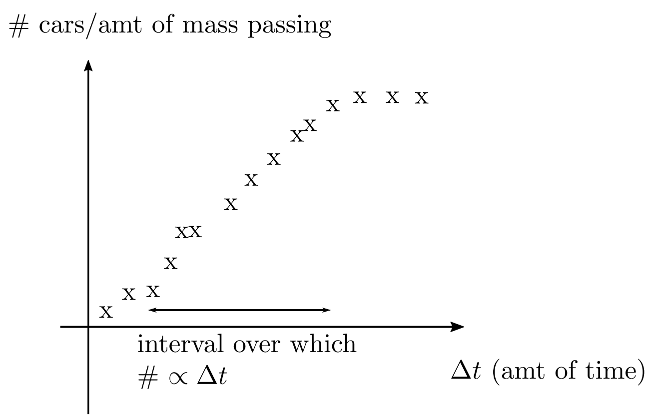
due to flaws at either end of the interval. We model these flows through a field q. $q(x, t)$ is the rate at which mass (e.g. cars) pass a station x.

$$q = \frac{\text{\# cars / amount of mass passing } x}{\text{time counted over}}$$

5 Apr 6, 2022

5.1 PDE Models (Cont'd)

We can think of $q(x,t)$ as being a new field, governed by the Continuum Hypothesis in the sense that.



Conservation of mass on an interval $(x_0 - \Delta x/2, x_0 + \Delta x/2)$ gives

$$u(x_0, t+\Delta t) = u(x_0,t) + \text{ amount entering at } x = x_0 - \Delta x/2 \\ - \text{amount leaving at } x = x_0 + \Delta x/2 \tag{5.1}$$

By default, I assume positive q means flow is from left to right.

$$q = \frac{\text{amount flowing left to right } - \text{ amount flowing right to left}}{\text{time observed over}}$$

Returning to (5.1):

$$u(x_0, t+\Delta t)\Delta x = u(x_0,t)\Delta x + q\left(x_0 - \Delta\frac{x}{2}, t\right)\Delta t - q\left(x_0 + \Delta\frac{x}{2}, t\right)\Delta t$$

On rearranging, dividing by $\Delta x \Delta t$ and letting $\Delta x, \Delta t \to 0$ we obtain the Conservation of mass law:

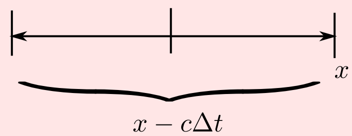
$$\frac{\partial u}{\partial t} = -\frac{\partial q}{\partial x} \tag{5.2}$$

We arrive at the PDEs of Math 136, typically by starting (5.2) and then doing more modeling to find how q is related to u.

Examples 5.1

1. Traffic flow with constant speed. Claim that $q = cu$.

   Amount passing station x in time Δt is



amt contained in interval of size $c\Delta t$

$$\begin{aligned} q(x,t)\Delta t &= u\left(x - \frac{1}{2}c\Delta t, t\right)\Delta x \\ &= u\left(x - \frac{1}{2}c\Delta t, t\right)c\Delta t \end{aligned}$$

   Now dividing by Δt, let $\Delta t \to 0$.

$$\begin{aligned} q(x,t) &= \lim_{\Delta t \to 0}\left[cu\left(x - \frac{1}{2}c\Delta t, t\right)\right] \\ &= cu(x,t) \end{aligned}$$

2. Lighthill - Whitham Richards model for traffic flow.

$$q(x,t) = c(u)u(x,t)$$

   that is, the velocity of cars depends on how dense the traffic is.

   (1) and (2) both may be written as a PDE

$$\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0$$

$$\implies \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\overbrace{c \cdot u}^{q} = 0$$

   In (1)

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

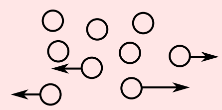
   In (2)

$$\frac{\partial u}{\partial t} + \frac{dq}{du} \cdot \frac{\partial u}{\partial x} = 0$$

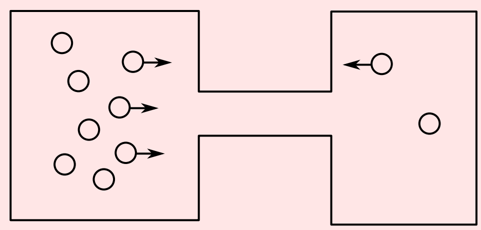
More examples,

Example 5.2 1. Random motion or diffusion.

A lot of matter moves around randomly. E.g. bacteria swimming around.



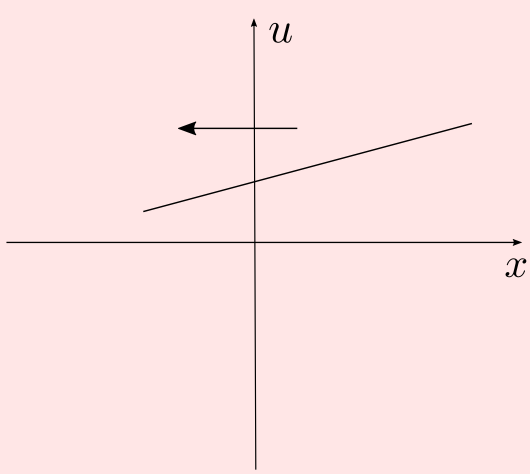
Follow random "run & tumble" paths. Consider a pair of linked boxes.



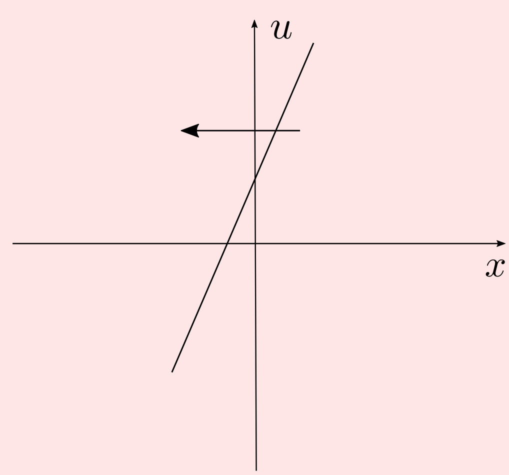
more flow left-to-right if
more bacteria in left box

We expect the flow to take bacteria on net out of regions where they are very dense/concentrated into regions where they are sparse/dilute. We posit that

$$q \propto -\frac{\partial u}{\partial x} \quad \text{(Fick's Law)}$$



flow is right to left
$q < 0$



flow is bigger when $\left|\frac{\partial u}{\partial x}\right|$ is bigger

$$q = -D\frac{\partial u}{\partial x}$$

D, the constant of proportionality is known as the diffusivity.

Consider units:

$$[u] = \#/m$$

$$[q] = \#/s$$

$$\left[\frac{\partial u}{\partial x}\right] = \frac{[u]}{[x]} = \#/m^2$$

$$[D] = \left[-\frac{q}{\frac{\partial u}{\partial x}}\right] = \frac{\#/s}{\#m^2} = m^2/s$$

Hence matter that is diffusing obeys a PDE

$$\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(-D\frac{\partial u}{\partial x}\right) = 0$$

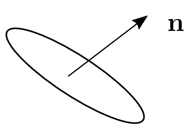
$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = 0 \tag{5.3}$$

assume D is a constant. This is a second order linear equation called the diffusion equation / heat equation.

We often are interested in quantities with multiple space-dimensions. Define u density per volume.

$$u(\mathbf{x}, t)dV = \text{ amount of matter in a small volume, } dV \text{ located at } (\mathbf{x}, t)$$

$$q(\mathbf{x}, t) \cdot \mathbf{n}dS = \text{ flow of matter across an element of surface, area } dS$$



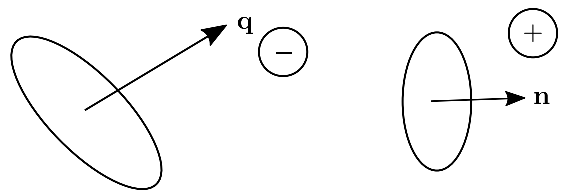
6 Apr 8, 2022

6.1 PDE Models (Cont'd)

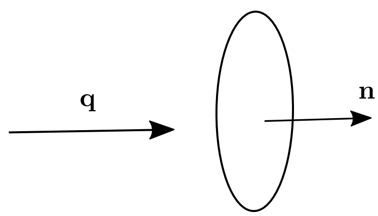
We introduce a vector field $q(\mathbf{x}, t)$. Vector q represents the amount of flow of matter at a point $\mathbf{x}$. Vector q gives how much flow crosses a surface $\mathbf{n}dS$ in time Δt as

$$q(\mathbf{x}, t) \cdot \mathbf{n}dS\Delta t.$$

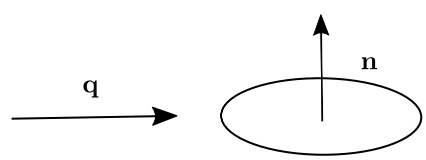
This is the net flow from the $\ominus$ side to the $\oplus$ side.



$\mathbf{q} \cdot \mathbf{n}$ projects $\mathbf{q}$
in the direction normal to the surface



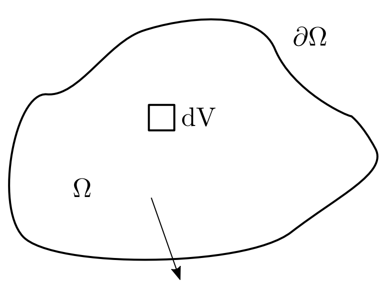
$\mathbf{n} \parallel \mathbf{q}$ maximizes flow through surface



$\mathbf{n} \perp \mathbf{q}$ means no flow

Amount of flow through surface is proportional to dS (area) and is proportional to Δt (time of observation).

To derive a PDE for conservation of mass, consider an arbitrary volume (called control volume) Ω, with piecewise differentiable and orientable boundary $\partial\Omega$, consider conservation of mass in Ω.

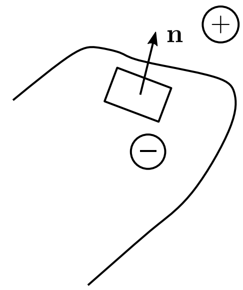


Total amount of matter in Ω is $\int u \cdot dV$ (divide Ω into boxes dV, amount in each box is $u(\mathbf{x}, t)dV$, sum up boxes and take dV $\to 0$). Change in matter in time $t \to \Delta t$ is

$$\int_\Omega u(\mathbf{x}, t + \Delta t)dV = \int_\Omega u(\mathbf{x}, t)dV + \text{ amt gained by flow through } \partial\Omega$$
$$(+\text{ amt of matter created } - \text{ amt of matter destroyed})$$

Divide boundary into surface elements $\mathbf{n}dS$. Flow through surface element in time Δt is

$$q(\mathbf{x}, t) \cdot \mathbf{n}dS\Delta t.$$



Total flow out of Ω (assuming $\mathbf{n}$ points out of Ω) through entire of $\partial\Omega$ is

$$\int_{\partial\Omega} q(\mathbf{x},t)\cdot\mathbf{n}dS\Delta t$$

summing all surface elements and letting dS $\to 0$. Amount gained in time Δt is:

$$-\int_{\partial\Omega} q(\mathbf{x},t)\cdot\mathbf{n}dS\Delta t$$

Hence,

$$\int_\Omega \Big(u(\mathbf{x},t+\Delta t)-u(\mathbf{x},t)\Big)dV = -\int_{\partial\Omega} q(\mathbf{x},t)\cdot\mathbf{n}dS\Delta t$$

Divide by Δt,

$$\int_\Omega \frac{\Big(u(\mathbf{x},t+\Delta t)-u(\mathbf{x},t)\Big)}{\Delta t}dV = -\int_{\partial\Omega} q(\mathbf{x},t)\cdot\mathbf{n}dS$$

let $\Delta t\to 0$,

$$\begin{aligned}\int_\Omega \frac{\partial u}{\partial t}dV &= -\int_{\partial\Omega} q(\mathbf{x},t)\mathbf{n}dS\\ &= -\int_\Omega \nabla\cdot q dV\end{aligned}$$

Use divergence theorem so that both sides are same type of integral. Now,

$$\int_\Omega\left(\frac{\partial u}{\partial t}+\nabla\cdot q\right)dV = 0$$

This integral vanishes for any choice of Ω, so the integrand must be 0. Hence

$$\frac{\partial u}{\partial t}+\nabla\cdot q = 0 \tag{6.1}$$

everywhere. In 1D,

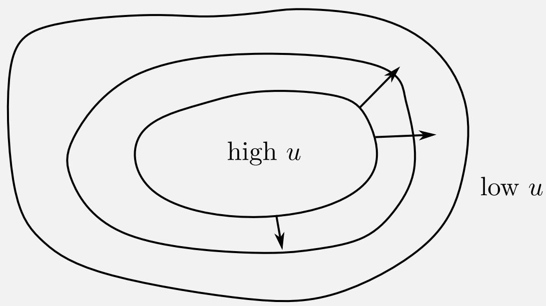
$$\frac{\partial u}{\partial t}+\frac{\partial q}{\partial x} = 0$$

To turn this into a PDE we can solve, we need to do additional modeling on $\mathbf{q}$.

Recall 6.1 The diffusion equation in 1D said

$$q = -D\frac{\partial u}{\partial x}$$

in n dimensions, visualize u through level surfaces. $q \propto -\nabla u$ (Fick's Law)



So

$$q = -D\nabla u$$
$$\implies \frac{\partial u}{\partial t} + \nabla \cdot (-D\nabla u) = 0$$

from (6.1)

$$\frac{\partial u}{\partial t} - D\nabla \cdot (\nabla u) \equiv \frac{\partial u}{\partial t} - D\nabla^2 u = 0 \tag{6.2}$$

which is the diffusion/heat equation.

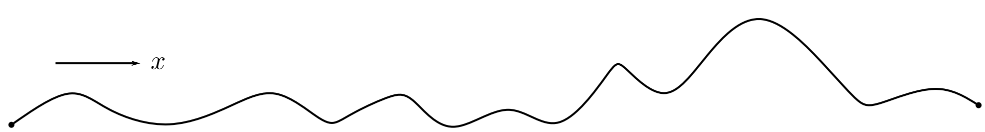
We are often interested in the steady state / equilibrium distribution of matter, in (6.2) we may consider what happens when $\frac{\partial u}{\partial t} = 0$, then

$$-D\nabla^2 u = 0 \quad \text{i.e.} \quad -\nabla^2 u = 0 \tag{6.3}$$

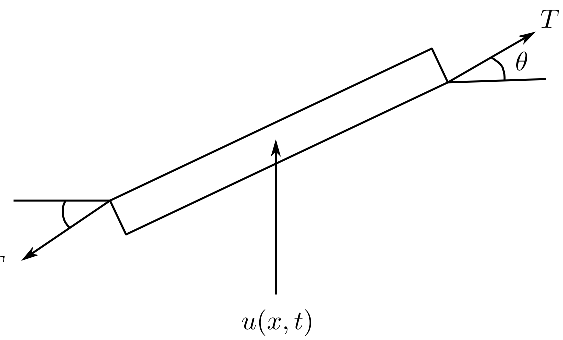
called Laplace's equation.

6.2 Wave Equation

Imagine a plucked string.



The string vibrates, define a field $u(x,t)$ that measures the displacement of the string (assumed planar). Consider a force balance on a piece of string of length Δx.

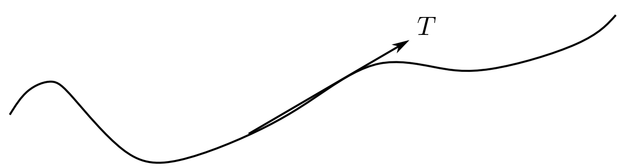


If the mass of unit length of string is ρ, then the mass of Δx length is $\rho \Delta x$.

$$\text{mass} \times \text{acceleration} = \text{net force}$$

$$\begin{aligned}\rho \Delta \frac{\partial^2 u}{\partial t^2} &= T \sin\theta(x + \Delta x/2, t) - T\sin\theta(x - \Delta x/2, t) \\ &= T\frac{\partial}{\partial x}\sin\theta \cdot \Delta x\end{aligned}$$

θ is the angle between the tension in the string and the x-axis, $\theta(x, t)$ is a field, so at each point T points along the tangent direction.



The string is at $(x, u(x, t))$ so the tangent vector is

$$\mathbf{t} = \frac{(1, \frac{\partial u}{\partial x})}{\sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2}}$$

Geometrically,

$$\sin\theta = \frac{\frac{\partial u}{\partial x}}{\sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2}}$$

which is approximately $\frac{\partial u}{\partial x}$ if $(\frac{\partial u}{\partial x})^2 \ll 1$ (small slope)

$$\rho \Delta x \frac{\partial^2 u}{\partial t^2} = T \frac{\partial}{\partial x}\sin\theta \Delta x$$

$$\frac{\partial^2 u}{\partial t^2} = \left(\frac{T}{\rho}\right)\frac{\partial^2 u}{\partial x^2} \tag{6.4}$$

Which is the wave equation.

7 Apr 11, 2022

7.1 Wave Equation (Cont'd)

Note 7.1: Notes on the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2}$$

for

$$\begin{aligned} u \colon \mathbb{R} \times \mathbb{R} &\to \mathbb{R} \\ (x,t) &\mapsto u(x,t) \end{aligned}$$

or equivalently,

$$u_{,tt} = c^2 u_{,xx} \quad \text{where} \quad c = \sqrt{\frac{T}{\rho}}$$

is the wave speed. Consider units:

$$\begin{aligned} [u_{,tt}] &= ms^{-2} \\ [u_{,xx}] &= \frac{m}{m^2} = m^{-1} \\ \implies [c^2] &= \left[\frac{u_{,tt}}{u_{,xx}}\right] = \frac{ms^{-2}}{m^{-1}} = (ms^{-1})^2 \end{aligned}$$

c has units of velocity.

Note 7.2: In $\mathbb{R}^n$, $u \equiv u(\mathbf{x}, t)$, the wave euqation is

$$u_{,tt} = c^2 \nabla^2 u (\equiv c^2 \Delta u)$$

where ∇^2 is the Laplacian. (see Math 272A)

Note 7.3: Compare Wave Equation and Diffusion Equation. $u(\mathbf{x}, t)$ solves

$$u_{,t} = D\nabla^2 u \tag{Diffusion}$$

$$u_{,tt} = c^2 \nabla^2 u \tag{Wave}$$

7.2 Summary

Transport PDE:

$$u_{,t} + \nabla \cdot (\mathbf{c}u) = 0$$

$$\text{in } \mathbb{R}^1 \quad u_{,t} + \frac{\partial}{\partial x}(cu) = 0$$

Diffusion equation:

$$u_{,t} = D\nabla^2 u$$

Wave equation:

$$u_{,tt} = c^2 \nabla^2 u$$

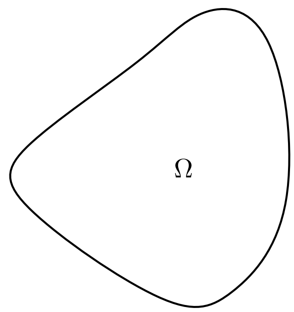
In both diffusion equation and heat equation, if the system reaches equilibrium, then:

$$-\nabla^2 u = 0 \tag{Laplace's equation}$$

Each equation can have forcing or reaction terms added to it. E.g. in the diffusion equation, we start with:

$$u_{,t} + \nabla \cdot \mathbf{q} = 0 \tag{conservation of mass}$$

(in homework # 3) we will add a reaction/source term.



$$u_{,t} + \nabla \cdot \mathbf{q} = s$$

s is the function representing creation of new matter. $s(\mathbf{x}, t)dV$ is the amount of new matter created in volume dV in one unit of time. Diffusion equation may become:

$$u_{,t} - D\nabla^2 u = s(\mathbf{x}, t)$$

which s is some function. This is the forced diffusion equation. In the equilibrium limit:

$$-\nabla^2 u = \frac{s(\mathbf{x}, t)}{D}$$

which is forced Laplace's equation or Poisson's equation.

7.3 Auxiliary Conditions

For first order PDEs, we need to know u on some lines in space time, i.e. we need some initial condition or we need some boundary conditions or both.

Example 7.4

If we are solving a model for traffic flow on $0 < x < L$ then we need

$$u(x, 0) \tag{IC}$$

and either

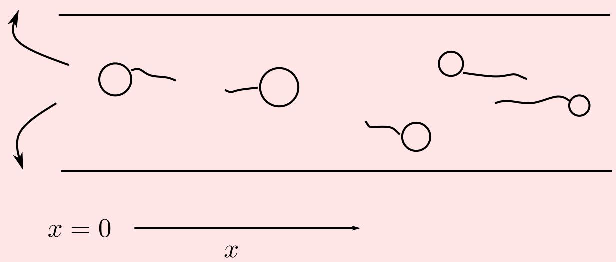
$$u(0, t) \quad \text{or} \quad u(L, t) \tag{BC}$$

Example 7.5
Given the diffusion equation:

$$u_{,t} = D\frac{\partial^2 u}{\partial x^2} \quad \text{on 0<x<L}$$

We need $u(x,0)$ and $\left.\begin{matrix} u(0,t) \\ \frac{\partial u}{\partial x}(0,t) \end{matrix}\right\}$ or and $\left.\begin{matrix} u(L,t) \\ \frac{\partial u}{\partial x}(L,t) \end{matrix}\right\}$ or.
We need either need the density at the boundary or the flux there.

$$q = -D\frac{\partial u}{\partial x}$$



What do bacteria do when they arrive at $x = 0$?
<u>Possibility # 1:</u> Bacteria leave; (become extremely dilute)

$$u(0,t) = 0$$

which is the Dirichlet boundary condition
<u>Possibility # 2:</u> End of tube is sealed.

Bacteria reaching end of tube reverse course. Since There is no net flow from right to left or conversely

$$q(0,t) = 0$$

$$-D\frac{\partial u}{\partial x}(0,t) = 0$$

which is the Neumann boundary condition.

7.4 Main Types of 2nd Order Linear PDE

There are 3 main types of 2nd order linear PDE:

1. Elliptic
2. Parabolic and
3. Hyperbolic

We restrict to PDEs in 2 variables

$$u(x,y)$$

solves

$$a_{11}(x,y)u_{,xx} \quad + \quad 2a_{12}u_{,xy} \quad + \quad a_{22}u_{,yy} \quad u_{,yy}$$

$$\begin{pmatrix} + \text{ first order and 0th order terms} \\ \text{e.g. } c_1 u_{,x} \quad c_2 u_{,y} \end{pmatrix}$$

We can rewrite our PDE as

$$\begin{aligned} 0 &= a_{11}\left(\frac{\partial}{\partial x} + \frac{a_{12}}{a_{11}}\frac{\partial}{\partial y}\right)^2 u + \left(a_{22} - \frac{a_{12}^2}{a_{11}}\right)\frac{\partial^2 u}{\partial y^2} \\ &= a_{11}\left(\frac{\partial^2}{\partial x^2} + \frac{2a_{12}}{a_{11}}\frac{\partial^2}{\partial x \partial y} + \frac{a_{12}^2}{a_{11}^2}\frac{\partial^2}{\partial y^2} + \text{ first \& 0th order terms}\right) \end{aligned}$$

If we do the right change of variables, we rewrite our PDE as:

$$0 = \frac{\partial^2}{\partial x^2}u + \left(a_{22} - \frac{a_{12}^2}{a_{11}}\right)\frac{\partial^2 u}{\partial y^2} + \text{ other terms}$$

Wave equation: in $\mathbb{R} \times \mathbb{R}$

$$u_{,xx} - \frac{u_{,tt}}{c^2} = 0$$

Laplace's equation: in $\mathbb{R}^2$

$$u_{,xx} + u_{,yy} = 0$$

So,

Definition 7.6 (Hyperbolic vs. elliptic vs. parabolic)
If

$$a_{22} - \frac{a_{12}^2}{a_{11}} < 0 \iff a_{12}^2 > a_{11}a_{22}$$

then our transformed equation looks like the wave equation. We call it hyperbolic.
If

$$a_{22} - \frac{a_{12}^2}{a_{11}} > 0 \iff a_{12}^2 < a_{11}a_{22}$$

then it looks like Laplace's equation. We call it elliptic.
If

$$a_{12}^2 = a_{11}a_{22}$$

then it looks like

$$D\frac{\partial^2 u}{\partial x^2} - u_{,t} = 0$$

the diffusion equation, we call it parabolic.

7.5 Wave Equation

Wave equation in 1D has the form:

$$u_{,tt} = c^2 u_{,xx}$$

In Homework 2, you showed a special case of the formula that:

$$u(x,t) = f(x - ct) + g(x + ct)$$

solves this PDE for any twice differentiable f and g. Solution is made up of two traveling waves, one going left at speed c and one going right at speed c.

8 Apr 13, 2022

8.1 Return to the Wave Equation and d'Alembert's Solution

Consider the wave equation on the real line

$$u_{,tt} = c^2 u_{,xx}, \quad -\infty < x < \infty$$

Typically, our auxilliary conditions take the form of initial conditions on u and $u_{,t}$ (starting position and starting velocity)

$$u(x,0) = \phi(x)$$

$$u_{,t}(x,0) = \psi(x)$$

Rewrite the wave equation as

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right) u = 0$$

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = 0$$

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = 0 \Longleftrightarrow u = f(x-ct)$$

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u = 0 \Longleftrightarrow u = g(x+ct)$$

Because the operators commute,

$$u = f(x-ct) + g(x+ct)$$

A second, slower but clearer, route to the same form of solution is to make a transformation of coordinates.

$$(x,t) \mapsto (\xi, \eta)$$

where

$$\begin{cases} \xi = x - ct \\ \eta = x + ct \end{cases} \quad \text{and} \quad \begin{cases} x = \frac{1}{2}(\xi + \eta) \\ t = \frac{1}{2c}(\eta - \xi) \end{cases}$$

I will rewrite the PDE in terms of ξ and η.

$$\begin{aligned} \left(\frac{\partial}{\partial t}\right)_x &= \left(\frac{\partial \xi}{\partial t}\right)_x \left(\frac{\partial}{\partial \xi}\right)_\eta + \left(\frac{\partial \eta}{\partial t}\right)_x \left(\frac{\partial}{\partial \eta}\right)_\xi \quad \text{by the Chain Rule} \\ &= -c\left(\frac{\partial}{\partial \xi}\right)_\eta + c\left(\frac{\partial}{\partial \eta}\right)_\xi \\ \left(\frac{\partial}{\partial x}\right)_t &= \left(\frac{\partial \xi}{\partial x}\right)_t \left(\frac{\partial}{\partial \xi}\right)_\eta + \left(\frac{\partial \eta}{\partial x}\right)_t \left(\frac{\partial}{\partial \eta}\right)_\xi \\ &= (1)\left(\frac{\partial}{\partial \xi}\right)_\eta + (1)\left(\frac{\partial}{\partial \eta}\right)_\xi \end{aligned}$$

So,

$$\begin{aligned}\frac{\partial^2 h}{\partial t^2} &= c^2\left(-\frac{\partial}{\partial \xi}+\frac{\partial}{\partial \eta}\right)^2 h\\ &= c^2\left(\frac{\partial^2 h}{\partial \xi^2}-\frac{2\partial^2 h}{\partial \xi\partial \eta}+\frac{\partial^2 h}{\partial \eta^2}\right)\\ \frac{\partial^2}{\partial x^2} &= \left(\frac{\partial}{\partial \xi}+\frac{\partial}{\partial \eta}\right)^2\\ &= \frac{\partial^2}{\partial \xi^2}+\frac{2\partial^2}{\partial \xi\partial \eta}+\frac{\partial^2}{\partial \eta^2}\end{aligned}$$

Hence

$$\frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial x^2} = -4\frac{c^2\partial^2 u}{\partial \xi\partial \eta} = 0$$

i.e.

$$\frac{\partial^2 u}{\partial \xi\partial \eta} = 0$$

which can be integrated

$$\begin{aligned}\frac{\partial u}{\partial \eta} &= A(\eta)\\ &= g(\eta)+f(\xi)\end{aligned}$$

where f and g are both arbitrary functions.

$$u(x,t) = f(x-ct)+g(x+ct) \tag{8.1}$$

is the only solutions, called d'Alembert's solution. Any solution of the wave equation is made up of two traveling waves, one with speed c, and one with speed $-c$.

We can choose f and g to satisfy the initial conditions.

$$u(x,0) = \phi(x)$$

From (8.1)

$$f(x)+g(x) = \phi(x) \tag{8.2}$$

$$u_{,t}(x,0) = \psi(x)$$

from (8.1),

$$-cf'(x)+cg'(x) = \psi(x) \tag{8.3}$$

Solve (8.2) and (8.3) for f and g. (8.2) implies

$$f'(x)+g'(x) = \psi'(x)$$

$$2cg'(x) = c\phi'(x)+\psi(x)$$

$$2cf'(x) = c\phi'(x)-\psi(x)$$

So

$$f(\xi) = \frac{1}{2}\phi(\xi) - \frac{1}{2c}\int_0^{\xi} \psi(y)dy + A$$
$$g(\eta) = \frac{1}{2}\phi(\eta) + \frac{1}{2c}\int_0^{\eta} \psi(y)dy + B$$

where A and B are constants of integration. Now what are A and B? Substitute into (8.2)

$$f(x) + g(x) = \phi(x)$$
$$\left(\frac{1}{2}\phi(x) - \frac{1}{2c}\int_0^{x} \psi(y)dy + A\right) + \left(\frac{1}{2}\phi(x) + \frac{1}{2c}\int_0^{x} \psi(y)dy + B\right) = \phi(x)$$

And so $A + B = 0$. So

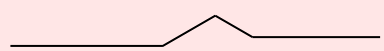
$$u(x,t) = f(\xi) + g(\eta)$$

so $A + B$ cancels. So

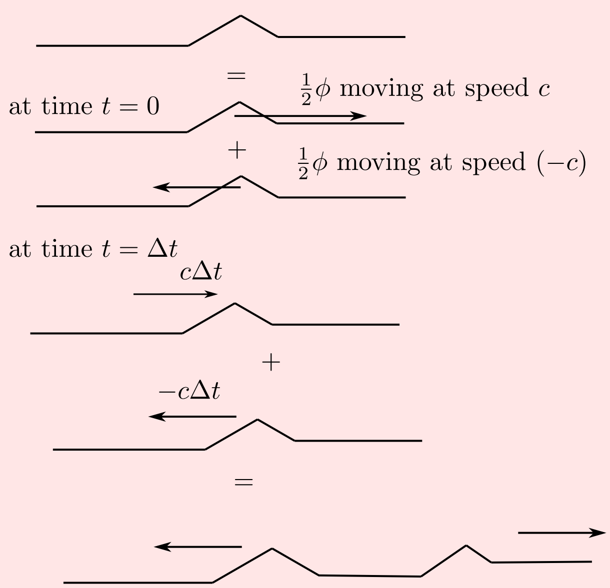
$$u(x,t) = \frac{1}{2}\phi(x-ct) - \frac{1}{2c}\int_0^{x-ct} \psi(y)dy + A + \frac{1}{2}\phi(x+ct) + \frac{1}{2c}\int_0^{x+ct} \psi(y)dy + B$$
$$u(x,t) = \frac{1}{2}\Big(\phi(x-ct) + \phi(x+ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy$$

which is d'Alembert's solution.

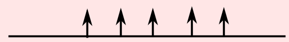
Example 8.1 (3 finger pluck)



infinitely long string has initial shape ϕ and $\psi = 0$, what is $u(x,t)$



Example 8.2 (Hammer punch)
Let $c = 1.$,



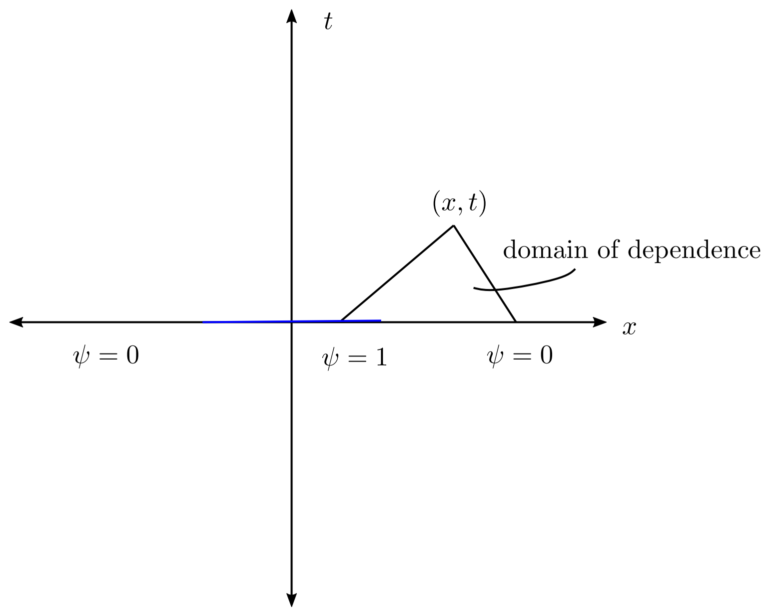
$u(x, 0) = 0$, $u_{,t}(x, 0) = \begin{cases} 1 & \text{if } |x| < 1. \\ 0 & \text{otherwise} \end{cases}$

9 Apr 15, 2022

9.1 D'Alembert's Formula

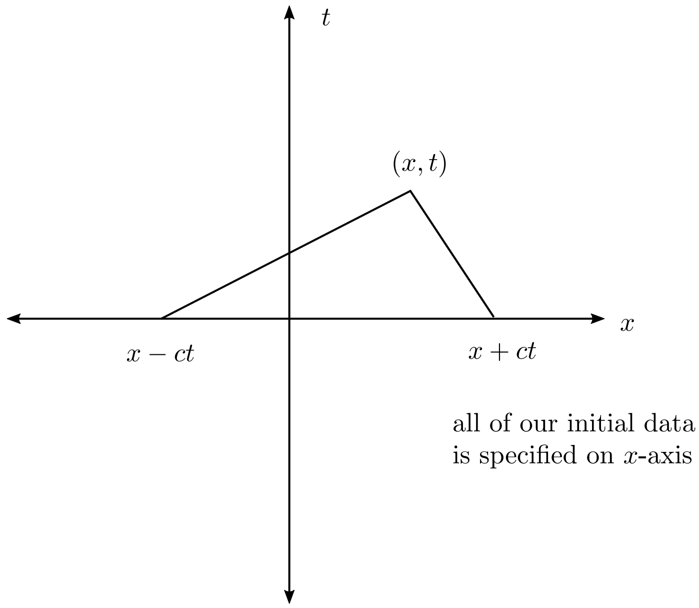
By d'Alembert's formula,

$$\begin{aligned} u &= \underbrace{0}_{\phi \text{ part}} + \frac{1}{2}\int_{x-ct}^{x+ct} \psi dy \\ &= \frac{1}{2} \text{ length } \{(x-ct, x+ct) \cap (-1,1)\} \end{aligned}$$



$$u = \frac{1}{2c} \times \text{ (length of interval above } \psi = 1 \text{ and that is contained in the domain of dependence)}$$

We will discuss the geometry in D'Alembert's formula, and then return to the hammer punch example.

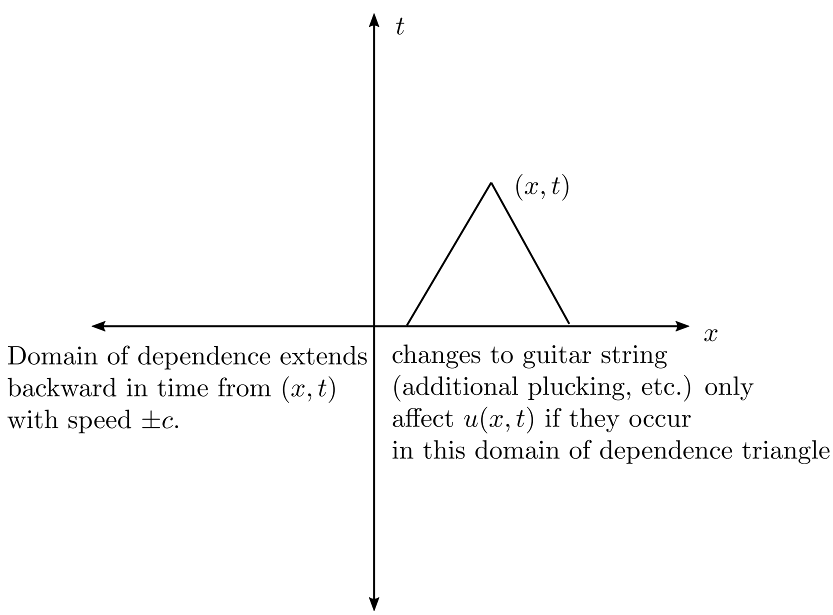


ϕ part of solution depends on values of $u(x-ct,,0)$ and $u(x+ct,0)$.

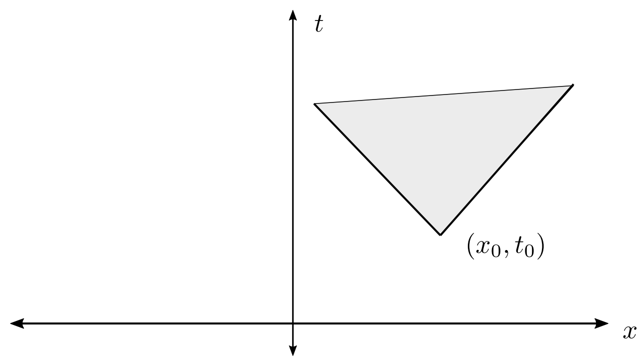
ψ part of solution depends on values of $u_{,t}(y,0)$ for $x-ct<y<x+ct$. In totality, $u(x,t)$ depends on initial condition information only between $x-ct$ and $x+ct$.

Definition 9.1 (Domain of dependence)
We call the region where the initial condition matters/ can affect (x,t) the domain of dependence.

As t increases, domain of dependence expands at speed $\pm c$.



The wave equations have a causality principle. Info about an initial condition or a change in u, can not travel faster than $\pm c$.



Suppose I pluck the string at (x_0, t_0), this can only affect $u(x,t)$ within a triangle of space-time that expands at speed $\pm c$.

Call this triangle the region of influence.

Recall 9.2 $c^2 = T/\rho$.

Causality is particularly insightful when our initial conditions on (ϕ and ψ) are both compactly supported, i.e. ϕ, ψ are both non-zero only within a finite interval/finite set of finite intervals (see homework).

9.2 Energy in the Wave Equation

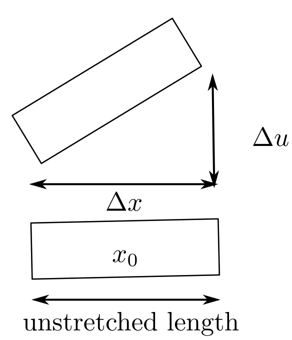
The wave equation was defined as a model for a stretched string that is vibrating.

We considered force balances, now we will consider energy. The element of string has two types of energy:

- Kinetic (movement)
- Potential (stored in stretching of string)

$$KE = \underbrace{\rho \Delta x}_{\text{mass}} \times \underbrace{\left(\frac{\partial u}{\partial t}\right)^2}_{\text{velocity sq'd}}$$

$$PE = \text{ force applied } \times \text{ amount of displacement (how far it stretches)}$$



$$\begin{aligned}
\Delta u &= u(x_0 + \Delta x/2, t) - u(x_0 - \Delta x/2, t) \\
&= \frac{\partial u}{\partial x}(x_0, t)\Delta x \quad \text{assuming small } \Delta x \\
\text{stretch } &= \sqrt{\Delta x^2 + \Delta u^2} - \Delta x \\
&= \Delta x \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2} - \Delta x
\end{aligned}$$

We assumed, to derive the wave equation, that $\left|\frac{\partial u}{\partial x}\right| \ll 1$. Hence,

$$\begin{aligned}
\text{stretch } &\approx \Delta x \left(1 + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 + \cdots\right) - \Delta x \\
&= \frac{1}{2}\Delta x \left(\frac{\partial u}{\partial x}\right)^2
\end{aligned}$$

The total energy is:

$$\frac{1}{2}\left(\rho u_{,t}^2 + T u_{,x}^2\right)\Delta x \quad \text{for an element of length } \Delta x$$

So the total energy of the entire string is:

$$E = \frac{1}{2}\int_{\text{length of string}} (\rho u_{,t}^2 + T u_{,x}^2) dx$$

summing the element contributions and letting $\Delta x \to 0$.

$$\frac{E}{\rho} = \frac{1}{2}\int_{\text{length of string}} (u_{,t}^2 - c^2 u_{,x}^2) dx$$

We believe that E (and E/ρ) are conserved i.e. stay constant over time.
Claim: If u solves the 1D wave equation:

$$u_{,tt} + c^2 u_{,xx}$$

and our initial conditions are such that

$$E = \frac{1}{2}\int_{-\infty}^{\infty} (u_{,t}^2 + c^2 u_{,x}^2) dx < \infty$$

then E is a constant.

Proof.

$$\begin{aligned}
\frac{dE}{dt} &= \frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} (u_{,t}^2 + c^2 u_{,x}^2) dx \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\partial}{\partial t}(u_{,t}^2 + c^2 u_{,x}^2) dx
\end{aligned}$$

by generalized Leibniz/FTC

$$\begin{aligned} &= \int_{-\infty}^{\infty} (u_{,t}u_{,tt} + c^2 u_{,x}u_{,xt})dx \\ &= \int_{-\infty}^{\infty} u_{,t}u_{,tt}dx + c^2[u_{,x}u_{,t}]_{-\infty}^{\infty} \end{aligned}$$

Use integration by parts

$$-c^2 \int_{-\infty}^{\infty} u_{,xx}u_{,t}dx$$

We continue this proof in the next lecture. □

10 Apr 18, 2022

10.1 Energy in the Wave Equation (Cont'd

Cont'd. We used integration by parts:

$$\tilde{u}v'dx = [\tilde{u}v] - \int \tilde{u}'vdx$$

let $\begin{cases} \tilde{u} = u_{,x} \\ v = u_{,xt} \end{cases}$ to apply this formula. For E to be a finite integral, we need its integrand

$$u_{,t}^2 + c^2 u_{,x}^2 \to 0 \quad \text{as } x \to \pm\infty$$

i.e.

$$u_{,t}^2 \to 0 \quad \text{and } u_{,x}^2 \to 0$$

So the boundary term

$$[u_{,x}u_{,t}]_{-\infty}^{\infty} = 0$$

Hence,

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} u_{,t} \underbrace{(u_{,tt} - c^2 u_{,xx})}_{\text{wave eq'n says } = 0} dx$$

Hence:

$$\frac{dE}{dt} = 0$$

So E is a constant. $\square$

Having a conserved energy gives us some insight into the wave equation and its solutions.

1. It is impossible if $u, u_{,t}$ are not both 0 at $t = 0$, for $u \to 0$ as $t \to \infty$.

   Because unless $E = 0$ at $t = 0$, it can not approach 0.

   A real guitar string loses energy over time.

   - Friction
   - Energy is radiated as sound waves

2. Energy arguments can be used to show the solutions of the wave equation are unique. In general, differential equations may not have unique solutions.

$$\frac{dy}{dx} = y^{1/2}, \quad y(0) = 0$$

has solution $y(x) = 0$ and

$$\begin{aligned} \int \frac{1}{y^{1/2}} dy &= \int dx \\ 2y^{1/2} &= x + C \\ y^{1/2} &= \frac{1}{4}(x + C) = \frac{1}{4}x^2 \end{aligned}$$

on applying the initial condition.

In models of the real world, we want solutions to be unique, i.e. the model predicts specifically only one possible "fate"/future set of behaviors. Energy arguments can show uniqueness.

Example 10.1
If

$$u_{,tt} = c^2 u_{,xx}$$
$$u(x,0) = \phi(x)$$
$$u_{,t}(x,0) = \psi(x)$$

are the initial conditions, there is at most one solution u, to this equation.
Proof: Suppose we have two solutions u_1, u_2.

$$u_{1,tt} = c^2 u_{1,xx} \quad \text{and} \quad u_1(x,0) = \phi, \quad u_{1,t}(x,0) = \psi$$

$$u_{2,tt} = c^2 u_{2,xx}$$

Let $w \equiv u_1 - u_2$.

$$w_{,tt} = u_{1,tt} - u_{2,tt}$$
$$c^2 w_{,xx} = c^2 u_{1,xx} - c^2 u_{2,xx}$$
$$\implies w_{,tt} = c^2 w_{,xx}$$

So w solves the wave equation.

$$\begin{aligned} w(x,0) &= u_1(x,0) - u_2(x,0) \\ &= \phi(x) - \phi(x) = 0 \end{aligned}$$

and similarly

$$w_{,t}(x,0) = 0$$

Define energy

$$E = \frac{1}{2}\int_{-\infty}^{\infty} (w_{,t}^2 + c^2 w_{,x}^2) dx$$

at $t = 0, E = 0$. In general, since E is conserved, $E = 0$ at all times. But the integrand is positive semi-definite (sum of two squares) so

$$w_{,t} \equiv 0 \quad \text{and} \quad w_{,x} \equiv 0 \quad \forall (x,t)$$

meaning w is a constant and because $w(x,0) = 0$, it follows $w \equiv 0$, so $u_1 \equiv u_2$ everywhere. □

10.2 Diffusion Equation

$$u_{,t} = D u_{,xx}$$

Book uses

$$u_{,t} = k u_{,xx}$$

D (or k) is a positive constant called the diffusivity. We will solve the diffusion equation on a finite interval $0 < x < L$ with boundary conditions

$$u(0,t) = a(t)$$

$$u(L,t) = b(t)$$

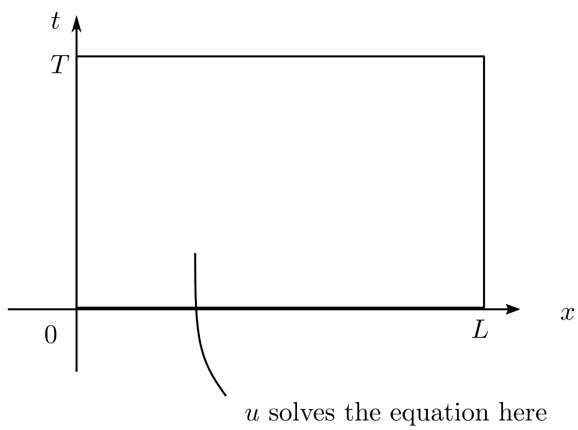
I know the value of u on the boundaries and with initial conditions

$$u(x,0) = g(x),$$

another known function. This equation has a maximum principle.
Claim: If u solves this diffusion equation, on a closed domain:

$$(x,t) \in [0,L] \times [0,T]$$

then

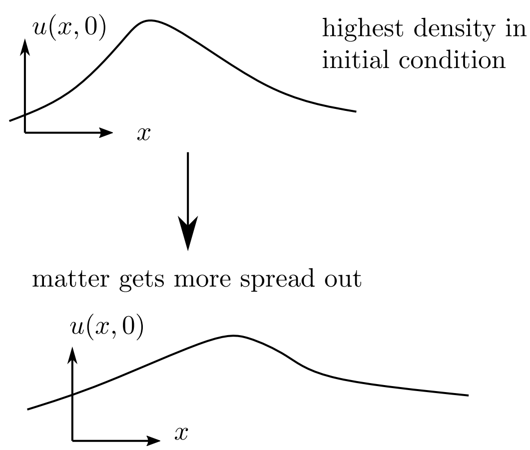


The maximum value of u is attained either at $x = 0$, $x = L$ or at $t = 0$

$$u \leq \max\{g(x), a(t), b(t)\}$$

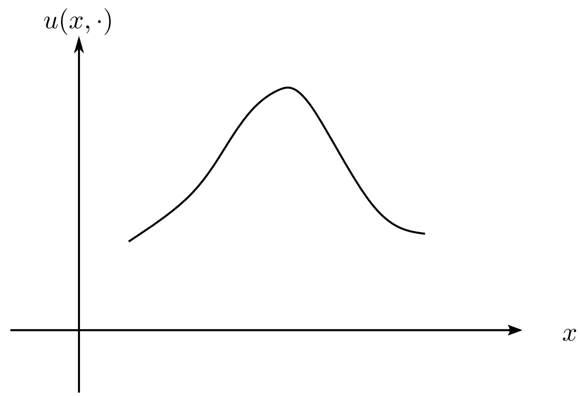
Say max value of u is attained on the parabolic boundary (either in initial data, or in boundary data)

Interpretation: If I start with matter unevenly distributed on a line segment.



If matter is introduced at end of interval, then highest density is at the end. Highest density is in boundary condition.

Proof of maximum principle. Assume that the max value is attained not at the parabolic boundary, i.e. at some (x_0, t_0), which is a local maximum. So $u_{,t} = 0$, $u_{,x} = 0$ at (x_0, t_0) and $u_{,xx} \leq 0$.



But

$$\begin{array}{rcl} u_{,t} & = & Du_{,xx} \\ 0 & \leq & 0 \end{array}$$

So we have a contradiction (almost). □

11 Apr 20, 2022

11.1 Diffusion Equation (Cont'd)

Proof of maximum principle (Cont'd). We want to choose w so that

$$w_{,t} - Dw_{,xx} < 0$$

then

$$\underbrace{u_{\varepsilon,t} - Du_{\varepsilon,xx}}_{\geq 0 \text{ at global max}} = \varepsilon \underbrace{(w_{,t} - Dw_{,xx})}_{<0}$$

leads to a contradiction at the global max. We need $w \geq 0$ and $w_{,t} - Dw_{,xx} < 0$. E.g.

$$w = x^2 \leftarrow \text{ the one we will use}$$

$$w = e^{-t}$$

Hence,

$$u_\varepsilon \equiv u + \varepsilon x^2$$

has its global max on the parabolic boundary.

$$\begin{aligned} u \leq u_\varepsilon \leq \max_{\substack{(x,t)\in \text{ parabolic} \\ \text{boundary}}} &= \max_{\substack{(x,t)\in \text{ parabolic} \\ \text{boundary}}} (u + \varepsilon x^2) \\ &\leq \max_{(x,t)\in \text{ p.b.}} (u + \varepsilon L^2) \end{aligned}$$

because $x^2 \leq L^2$

$$u \leq \max_{(x,t)\in \text{ p.b.}} + \varepsilon L^2 \quad \forall \varepsilon > 0$$

Let ε become arbitrary small (we can do this because our inequalities hold $\forall \varepsilon > 0$)

$$u \leq \max_{(x,t)\in \text{ p.b.}} u$$

which is the Maximum Principle. □

The max principle can be used to prove uniqueness of solutions because:

1. u also must obey a minimum principle. If u solves the diffusion equation, then so does $v = -u$. Hence,

$$\max_{(x,t)\in[0,L]\times[0,T]} = \max_{(x,t)\in \text{ p.b.}}$$

$$\max_{(x,t)\in[0,L]\times[0,T]} (-u) = \max_{(x,t)\in \text{ p.b.}} (-u)$$

$$\cancel{-} \min_{(x,t)\in[0,L]\times[0,T]} = \cancel{-} \min_{(x,t)\in \text{ p.b.}} (u)$$

2. Suppose we have two solutions u_1, u_2 of the diffusion equation with some boundary conditions and initial conditions.

$$u_{i,t} = Du_{i,xx} \quad i = 1, 2$$

and

$$\begin{cases} u_i = g(x) & \text{for } t = 0 \\ u_i = a(t) & \text{for } x = 0 \\ u_i = b(t) & \text{for } x = L \end{cases}$$

define $w \equiv u_1 - u_2$.

$$w_{,t} = Dw_{,xx}$$

$$\begin{cases} w = 0 & \text{for } t = 0 \\ w = 0 & \text{for } x = 0 \\ w = 0 & \text{for } x = L \end{cases}$$

$$\max_{(x,t)\in[0,L]\times[0,T]} w = \max_{\text{p.b.}} w = 0$$

$$w \leq 0 \quad \text{by max principle}$$

$$\min_{(x,t)\in[0,L]\times[0,T]} w = \min_{\text{p.b.}} w = 0$$

$$w \geq 0$$

So $w \equiv 0$ everywhere and $u_1 = u_2$ so solution is unique.

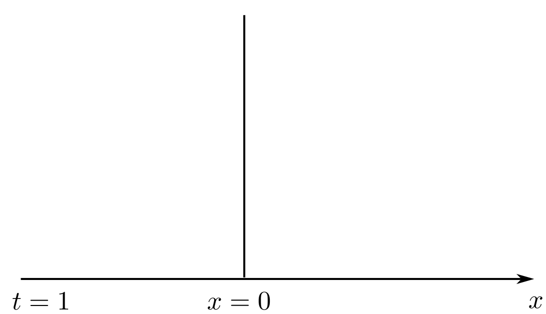
11.2 The Source or Fundamental Solution or Green's Function Solution of the Diffusion Equation

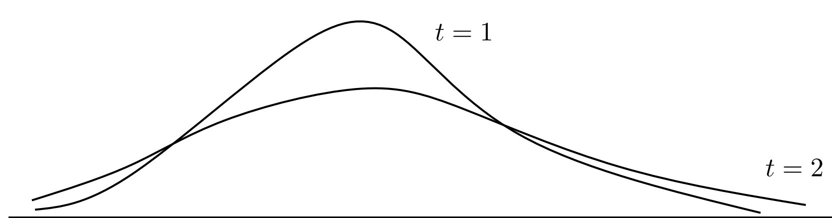
We have

$$u_{,t} = Du_{,xx} \quad x \in \mathbb{R}$$

with initial condition

$$u(x, 0) = \phi(x)$$





Ansatz: The density $u(x,t)$ will have the same shape at all times, but is transformed by being stretched in the x-direction and in the y-direction.

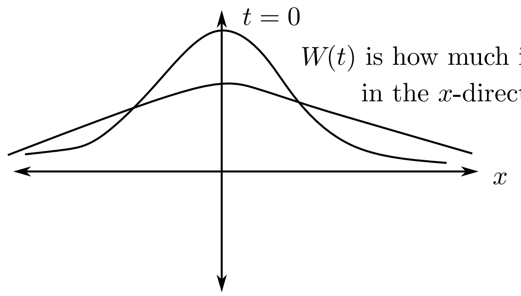
12 Apr 22, 2022

12.1 The Source or Fundamental Solution or Green's Function Solution of the Diffusion Equation (Cont'd)

We have

$$u_{,t} = Du_{,xx} \quad x \in \mathbb{R}, t > 0$$
$$u(x,0) = \phi(x) \quad t > 0$$

We started by asking what happens if all mass is initially at $x = 0$.



Hypothesize that $u(x,t)$ has the same shape at different times, but that its height and width change.

$$u(x,t) = \mathcal{H}(t) f\left(\frac{x}{W(t)}\right)$$

Where $\mathcal{H}(t)$ is how much function is dilated in the y-direction, and $x/W(t)$ is the same function, dilated by different amounts W, at different times.

Define a new variable $\eta \equiv x/W(t)$.

$$u(x,t) = \mathcal{H}(t) f(\eta)$$

I want to rewrite the diffusion equation in terms of η, t.

$$(x,t) \mapsto (\eta, t)$$

$$\begin{aligned}
\left(\frac{\partial}{\partial x}\right)_t &= \left(\frac{\partial t}{\partial x}\right)_t \left(\frac{\partial}{\partial t}\right)_\eta + \left(\frac{\partial \eta}{\partial x}\right)_t \left(\frac{\partial}{\partial \eta}\right)_t \\
&= 0 \cdot \left(\frac{\partial}{\partial t}\right)_\eta + \frac{1}{W}\left(\frac{\partial}{\partial \eta}\right)_t \\
\left(\frac{\partial}{\partial t}\right)_x &= \left(\frac{\partial t}{\partial t}\right)_x \left(\frac{\partial}{\partial t}\right)_\eta + \left(\frac{\partial \eta}{\partial t}\right)_x \left(\frac{\partial}{\partial \eta}\right) \\
&= (1)\left(\frac{\partial}{\partial t}\right)_\eta - \frac{\eta \dot{W}}{W}\left(\frac{\partial}{\partial \eta}\right)_t
\end{aligned}$$

$$\eta = x/W \implies \left(\frac{\partial \eta}{\partial t}\right) = -\frac{x}{W^2}\dot{W}$$
$$= -\eta\frac{\dot{W}}{W}$$

where $\dot{W} \equiv \frac{dW}{dt}$. For $u = Hf(n)$,

$$u_{,t} \equiv \dot{H}f - \frac{\eta\dot{W}}{W}\cdot Hf'$$

where $\dot{H} = \frac{dH}{dt}$.

$$Du_{,xx} = \frac{D}{W^2}Hf''$$

diffusion equation implies

$$\dot{H}f - \frac{\eta\dot{W}}{W}Hf' = \frac{DHf''}{W^2} \tag{12.1}$$

I went from a PDE in (x,t) to 3 unknown functions $H(t), W(t), f(n)$. We can turn the diffusion PDE into an ODE, by choosing H and W carefully. Compare terms:

$$-\eta\frac{\dot{W}H}{W}f' = -\underbrace{nf'}_{\text{function of }\eta}\frac{\dot{W}H}{W}$$

$$\underbrace{\frac{DH}{W^2}}_{\text{function of }t}\underbrace{f''}_{\text{function of }\eta}$$

Choose W so that:

$$\begin{aligned}\frac{DH}{W^2} &= \frac{\dot{W}H}{W}\\ D &= \dot{W}W\\ &= \frac{d}{dt}\left(\frac{1}{2}W^2\right)\\ Dt &= \frac{1}{2}W^2\\ W(t) &= \sqrt{2Dt}\end{aligned}$$

Choose H so that

$$\dot{H} = \frac{DH}{w^2} = \frac{H}{2t}$$

given $W(t) = \sqrt{2Dt}$.

$$\begin{cases}\frac{\dot{H}}{H} = \frac{1}{2t}\\ \log(H) = \frac{1}{2}\log(t)\end{cases} \implies H = t^{1/2}$$

A hand wavy way to find $H(t)$ and $W(t)$.

$$u_{,t} = Du_{,xx}$$

We want to estimate the size of the terms in the equation.

$$u_{,t} \sim H$$

where $\sim$ means ball-park estimate.

$$u_{,x} \sim \frac{H}{W}, \quad u_{,xx} \sim \frac{H}{W^2}$$

for equality we need:

$$u_{,t} = Du_{,xx}$$

$$\frac{H}{t} \sim \frac{DH}{W^2} \implies \begin{cases} W^2 \sim Dt \\ W \sim \sqrt{Dt} \end{cases}$$

but H can not be estimated from the equation. We also need an initial condition. Recall that the equation represents conservation of mass.

$$\begin{aligned} \underbrace{\frac{d}{dt}\int_{-\infty}^{\infty} u\,dx}_{\text{time rate of change of total mass}} &= \int_{-\infty}^{\infty} \frac{\partial u}{\partial t} dx = D\int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} dx \\ &= -\left[D\frac{\partial u}{\partial x}\right]_{-\infty}^{\infty} = 0 \end{aligned}$$

The total mass is constant, let

$$\int_{-\infty}^{\infty} u\,dx = 1$$

<u>Handwavy</u>: $HW \sim 1$. Which is height $\times$ width of function

$$H \sim \frac{1}{W}$$

$$H \sim \frac{1}{\sqrt{Dt}}$$

<u>Careful route</u>:

$$\begin{aligned} \int_{-\infty}^{\infty} u\,dx &= 1 \\ \int_{-\infty}^{\infty} H(t)f(\eta)dx &= 1 \\ &= HW\int_{-\infty}^{\infty} f(\eta)\eta \\ \left(\frac{\partial \eta}{\partial x}\right)_t &= \frac{1}{W} \\ W\,d\eta &= dx \\ \implies HW &= \frac{1}{\int_{-\infty}^{\infty} f(\eta)d\eta} \end{aligned}$$

$H \propto \frac{1}{W}$ because the integral is just a number.

Now assume

$$H(t) = \frac{1}{\sqrt{Dt}}, \quad W(t) = \sqrt{Dt}$$

Look for a solution of the equation with

$$u(x,t) = \underbrace{\frac{1}{\sqrt{Dt}}}_{H} f\left(\underbrace{\frac{x}{\sqrt{Dt}}}_{\eta = x/W}\right)$$

for some function f. Substitute into Equation 12.1:

$$\dot{H} = \frac{d}{dt}\left(\frac{1}{\sqrt{Dt}}\right) = -\frac{1}{2\sqrt{Dt^3}}$$

$$-\frac{1}{2\sqrt{Dt^3}} f - \frac{1}{2t\sqrt{Dt}} \eta f' =$$

$$-\frac{\eta \dot{W}}{W} H f' = (-nf')\left(\frac{\dot{W}H}{W} = \frac{1}{2t}\frac{1}{\sqrt{Dt}}\right)$$

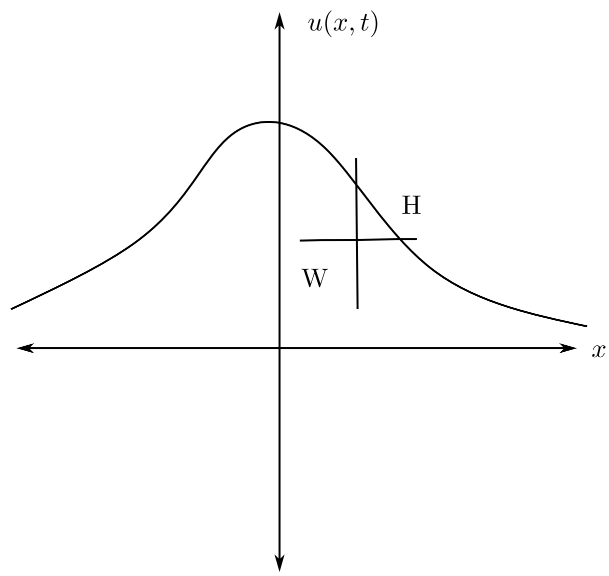
13 Apr 25, 2022

13.1 Continuuing the Fundamental Solution of Diffusion Equation

We solve:

$$u_{,t} = Du_{,xx}$$

assume all mass is initially at $x = 0$ and that the solution is self-similar; it has the same but scaled in height and width at all times.



$$u(x,t) = H(t) f\left(\frac{x}{W(t)}\right)$$

H is height, W is width.

$$u_{,t} = Du_{,xx}$$

1. $\frac{H}{t} \sim \frac{DH}{W^2} \implies W \sim \sqrt{Dt}$
2. The total mass is conserved:

   $$HW \sim 1 \implies H \sim \frac{1}{\sqrt{Dt}}$$

   Hence I look for a solution of form:

   $$u(x,t) = \frac{1}{\sqrt{Dt}} f \underbrace{\left(\frac{x}{\sqrt{Dt}}\right)}_{\eta}$$

   see books by G.I. Barenblatt.

From last class:

$$\left(\frac{\partial}{\partial t}\right)_x \to \left(\frac{\partial}{\partial t}\right)_\eta + \left(\frac{\partial \eta}{\partial t}\right)_x \left(\frac{\partial}{\partial \eta}\right)_t$$
$$= \left(\frac{\partial}{\partial t}\right)_\eta - \frac{\eta}{2t}\left(\frac{\partial}{\partial \eta}\right)_t$$
$$\left(\frac{\partial}{\partial x}\right)_t \to \frac{1}{\sqrt{Dt}}\frac{\partial}{\partial \eta}$$
$$u \equiv \frac{1}{\sqrt{Dt}} f(\eta)$$

LHS:

$$u_{,t} = -\frac{1}{2\sqrt{Dt^3}} f - \frac{\eta}{2\sqrt{Dt^3}} f'$$
$$Du_{,xx} = \frac{D}{\sqrt{Dt}(Dt)} f''$$
$$\implies -\frac{1}{2\sqrt{Dt^3}} f - \frac{\eta}{2\sqrt{Dt^3}} f' = \frac{1}{\sqrt{Dt^3}} f''$$
$$-\frac{1}{2} f - \frac{\eta}{2} f' = f''$$

thus is an ODE in η. Boundary conditions are $u \to 0$ as $x \to +\infty$, or equivalently, $f \to 0$ as $\eta \to \pm\infty$. Notice u is even in x, f is an even function of η.

$$-\frac{1}{2}(\eta f)' = f'' \tag{13.1}$$

$$c - \frac{1}{2}\eta f' = f'$$

since f is even, $f'(0) = 0$.

$$\implies c - 0 = 0 \implies c = 0$$

Hence

$$-\frac{1}{2}\eta f = f'$$
$$-\frac{1}{2}\eta = \frac{f'}{f}$$
$$c - \frac{1}{4}\eta^2 = \log|f|$$
$$\implies f = Ce^{-1/4\eta^2}$$

C a constant of integration. $f \to 0$ as $\eta \to \pm\infty$. We require:

$$\int_{-\infty}^{\infty} u(x,t)dx = 1 \quad \forall t$$

Notice

$$\eta = \frac{x}{\sqrt{Dt}} \implies d\eta = \frac{dx}{\sqrt{Dt}}$$

So

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{Dt}} f(\eta) \cdot \sqrt{Dt} d\eta = 1$$

$$\int_{-\infty}^{\infty} C_1 e^{-1/4\eta^2} d\eta = 1$$

$$C_1 = \frac{1}{\int_{-\infty}^{\infty} e^{-1/4\eta^2} d\eta}$$

Recall 13.1 (A fact about the normal distribution)

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2\sigma^2}} dx = 1$$

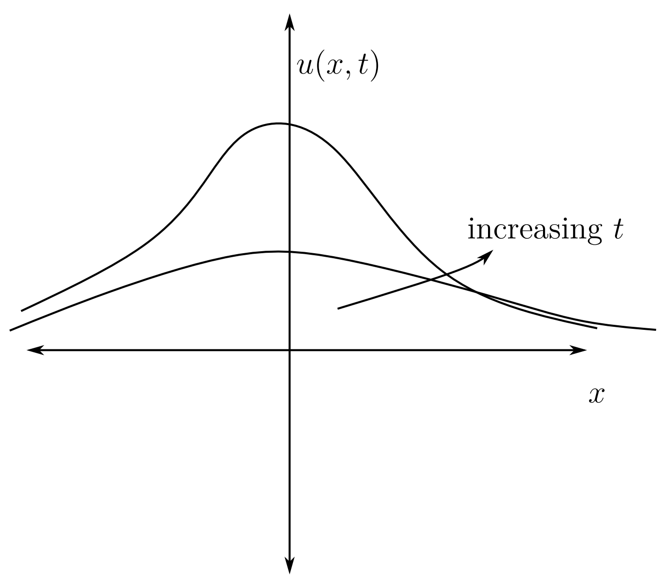
$$\implies \int_{-\infty}^{\infty} e^{\frac{-x^2}{2\sigma^2}} dx = \sqrt{2\pi\sigma^2}$$

Hence

$$C_1 = \frac{1}{\sqrt{4\pi}}$$

$$u(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$$

is a solution of the diffusion equation.



at $t = 0$,

$$u(x,0) = \begin{cases} 0 & \text{except at } x = 0 \\ \infty & \text{at } x = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} u(x,0) dx = 1$$

We call $u(x,0), \delta(x)$ the Dirac δ-function.

13.2 Solving the Diffusion Equation for Arbitrary Initial Conditions

Idea # 1: Suppose we start with all of our mass at $x = 1$.

We expect solution to be translated by $+1$,

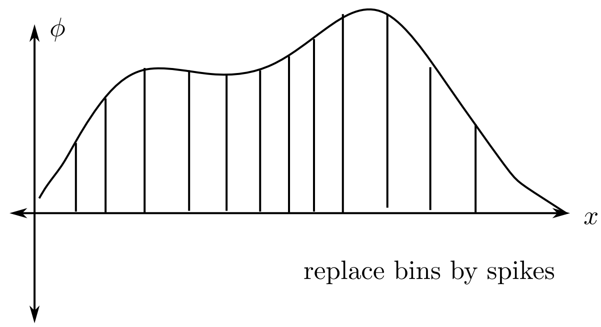
$$u(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{(x-1)^2}{4Dt}}$$

Solution for any starting location y, is:

$$S(x-y,t) = S(x,t;y) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{(x-y)^2}{4Dt}}$$

Idea # 2:

Given $u(x,0) = \phi(x)$, find $u(x,t)$ by approximating $\phi(x)$ by a set of spikes/δ-functions.



replace bins by spikes

Replace $\phi(x)$ by a set of spikes, spacing Δx. Spike at $x = x_i$, has mass $\phi(x_i)\Delta x$. Solution from this spike is

$$S(x - x_i, t)\phi(x_i)\Delta x$$

add all spikes together;

$$u(x,t) = \sum_i S(x - x_i, t)\phi(x_i)\Delta x$$

let $\Delta x \to 0$;

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy$$

14 Apr 27, 2022

14.1 Solving the Diffusion Equation for Arbitrary Initial Conditions (Cont'd)

Recall from last class, we derived a solution

$$\left.\begin{matrix} S(x,t;y) \\ S(x-y,t) \end{matrix}\right\} = \frac{1}{\sqrt{4\pi Dt}} e^{-(x-y)^2/4Dt}$$

Total mass 1, with all mass initially concentrated at y. We claim that if the initial condition is $u(x,0) = \phi(x)$, the solution is:

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy \tag{14.1}$$

Let's prove this is truly the solution.

$$\begin{aligned} u_{,t} - Du_{,xx} &= \left(\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2}\right)\int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy \\ &= \int_{-\infty}^{\infty} \underbrace{\left(\frac{\partial}{\partial t}S(x-y,t) - D\frac{\partial^2}{\partial x^2}S(x-y,t)\right)\phi(y)\,dy}_{\text{0 because } S \text{ solves PDE}} \end{aligned}$$

We also need to show the initial condition is satisfied. I.e. we need $u(x,0) = \phi(x)$. Want to substitute into equation 14.1:

$$\int_{-\infty}^{\infty} \underbrace{S(x-y,0)}_{-\frac{\partial}{\partial y}\mathcal{Q}(x-y)}\phi(y)dy = \left[-\mathcal{Q}(x-y)\phi(y)\right]\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\mathcal{Q}(x-y)\phi'(y)dy$$

We will use integration by parts to evaluate the integral.

> **Recall 14.1** $S(x,0) = 0$ except at $x = 0$.
> $\int_{-\infty}^{\infty} S(x,0)dx = 1$ because mass is conserved.

This implies

$$\int_I S(x,0)dx = 1$$

if I is any interval containing $x = 0$. Define

$$\mathcal{Q} = \int_{-\infty}^{x} S(\tilde{x},0)d\tilde{x} = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

this is called the Heaviside function.

$$\text{integral } = [0 + \phi(-\infty)] + \int_{-\infty}^{x}\phi'(y)dy$$

because

$$\mathcal{Q}(x-y) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{if } x < y \end{cases}$$

$$u(x,0) = \text{integral } = \phi(-\infty) + \phi(x) - \phi(-\infty) = \phi(x)$$

So Equation 14.1 satisfies the initial condition.

14.2 Midterm Review Topics

Need to be able:

- Calculate partial derivatives, e.g.

$$\frac{\partial}{\partial y} \sin(x-y)\cos(x+y)$$

- Linear first order PDEs
  - Recognize them
  - Geometric interpretation of characteristics for PDEs of form:

$$a(x,y)u_{,x} + b(x,y)u_{,y} = 0$$

with equation of characteristics

$$\frac{dy}{dx} = \frac{b}{a}$$

E.g.

$$xu_{,x} + yu_{,y} = 0$$

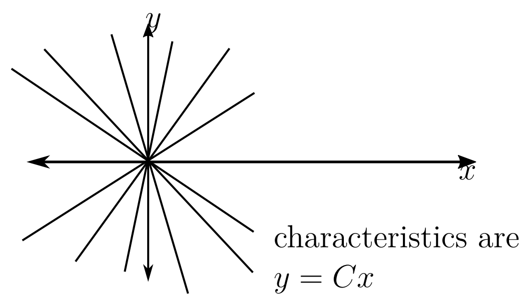
$$\frac{dy}{dx} = \frac{y}{x}$$

$$\int \frac{1}{y} dy = \int \frac{1}{x} dx$$

$$\log|y| = \log|x| + C$$

$$\implies \frac{y}{x} = \pm e^C \implies f\left(\frac{y}{x}\right) = u(x,y)$$

is a solution.



If I am given $u(x,1) = e^x$, how much of the x-y plane can I find the solution on?
I can find $u(x,y)$ for all $y > 0$ (over whole upper half plane)
I can not fill in the solution for $y \leq 0$.
For $y > 0$

$$u(x,y) = f\left(\frac{y}{x}\right)$$

$$f\left(\frac{1}{x}\right) = e^x$$

from auxiliary condition.

$$f(z) = e^{1/z}$$

$$u(x,y) = f\left(\frac{y}{x}\right) = e^{x/y}$$

- Interpreting PDEs:

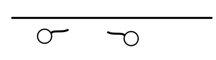
$$\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0$$

what is u, what is q, what does this PDE represent?

u: amount of shift at (x,t) (density)

q: (flow rate) amount of shift passing x in unit time.

  - $q = cu$ transport equation
  - $q = -D\frac{\partial u}{\partial x}$ diffusion equation



$-D\frac{\partial u}{\partial x} = 0$? at $(0,t)$ what does this mean? closed end
$u = 0?, u(0,t)$ no bacteria at $x = 0$

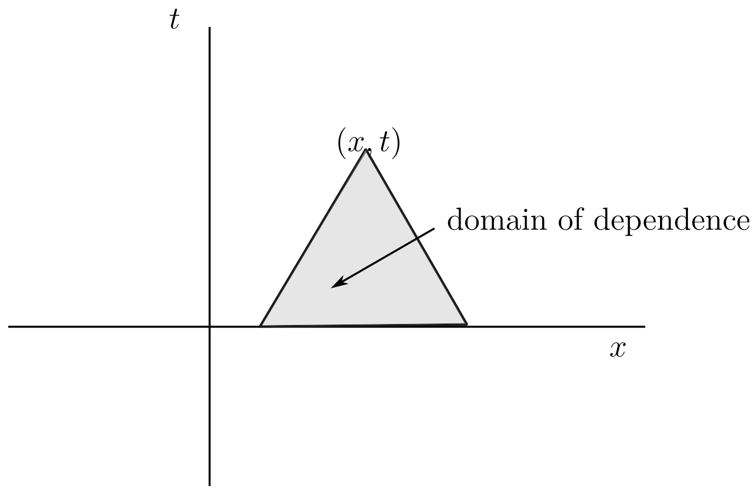
- Classifying PDEs as elliptic, hyperbolic, parabolic

$$a_{11}u_{,xx} + 2a_{12}u_{,xy} + a_{22}u_{,yy} + \ldots$$

- D'Alembert's Solution

$$u(x,t) = \begin{cases} \frac{1}{2}\Big(\phi(x+ct) + \phi(x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy \\ f(x+ct) + g(x-ct) \end{cases}$$

- Domain of dependence and region of influence (e.g. Q3 of homework 4)


t
(x, t)
domain of dependence
x

15 Apr 29, 2022

15.1 Midterm

16 May 2, 2022

16.1 Solution of the Heat Equation with General Initial Conditions

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy$$

where

$$S(x-y,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-(x-y)^2/4Dt}$$

is called the source/Green's function. $S(x,t)$ is infinitely differentiable with respect x and t except at $t=0$. We say S is smooth or $\mathcal{C}^\infty$.

If I differentiate

$$u; u_{,x} = \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial x} S(x-y,t) \right) \phi(y)dy$$

In fact all derivatives with respect to x or t are applied only to the S part of the integrand.

Solution is e^∞ for all $t > 0$, even if the initial conditions are bad. In practice, $u(x, 0+)$ (tiny time) is a smooth approximation of $\phi(x)$. e.g. it finds use in image analysis and in statistics.

16.2 Comparing the Diffusion Equation with the Wave Equation

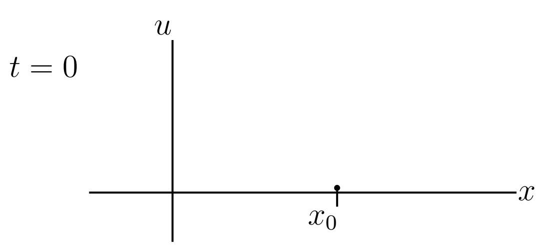
Information

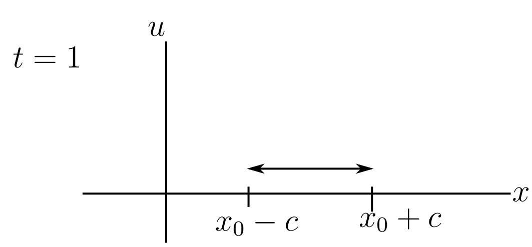
Question 16.1: How far away is a change in the initial conditions felt at time t?

First order PDE

At each time, the effect of the initial conditions is felt at exactly one point in space. Region of influence is a set of points in space time/curve/characteristic.

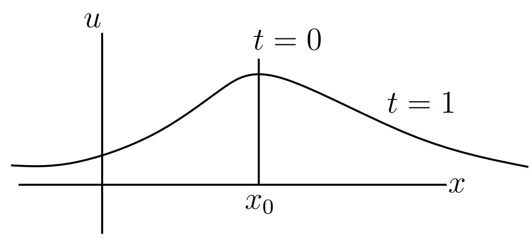
Wave equation:





Region of influence is a triangle in space-time, and expands at speed c. Information propagates at speed c.

Diffusion Equation



u changes at arbitrarily early times. $S \neq 0$ for all $t > 0$, no matter how far we are from source. Information propagates at ∞ speed.

	Wave equation	Diffusion equation
Uniqueness	✓(energy)	✓(max principle)
Max principle	× (hammer punch)	✓
Speed of propogation	c	∞
Behavior as $t \to \infty$	Continues to vibrate forever (conserved energy)	$u \to 0$

Behavior of the solution of the diffusion equation as $t \to \infty$. Define an energy for u, satisfying the diffusion equation,

$$E \equiv \int_{-\infty}^{\infty} u^2 \, dx$$

$$\begin{aligned}
\frac{dE}{dt} &= 2 \int_{-\infty}^{\infty} u u_{,t} dx \\
&= 2D \int_{-\infty}^{\infty} u u_{,xx} dx \\
&= 2D \underbrace{[u u_{,x}] \Big|_{-\infty}^{\infty}}_{\text{assume 0 because of BCs}} - 2D \int_{-\infty}^{\infty} u_{,x}^2 dx
\end{aligned}$$

$$\implies \frac{dE}{dt} = -2D \int_{-\infty}^{\infty} u_{,x}^2 dx \leq 0$$

with equality only if u is a constant. So E is monotonic decreasing (in fact it can be shown that $E \to 0$).

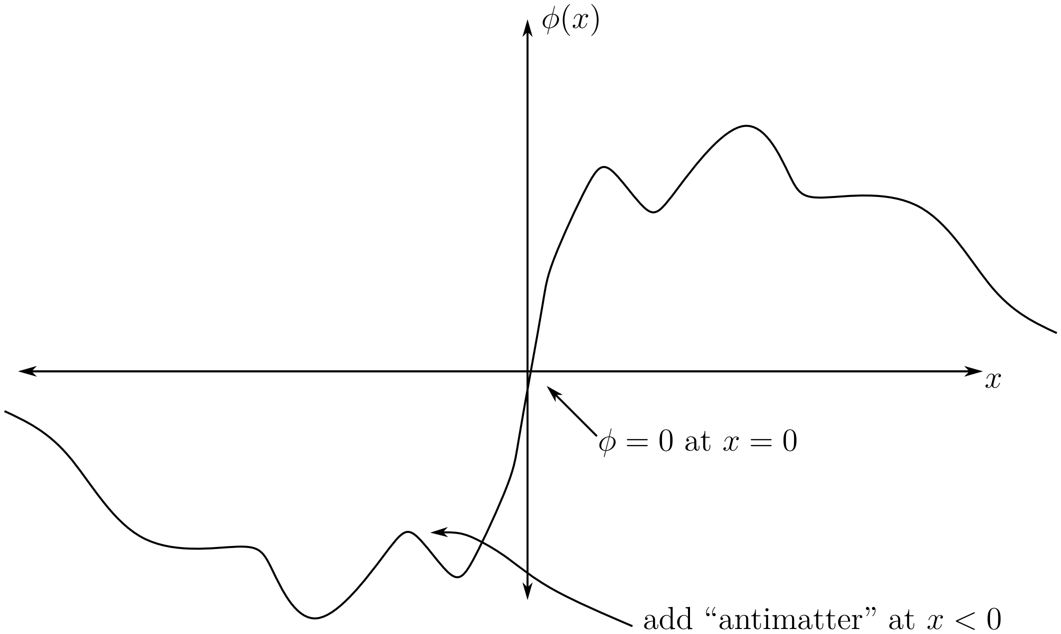
16.3 Diffusion Equation on a Half Infinite Line

Source solution is only valid for $x \in \mathbb{R}$ (domain is whole real line), in real life domains are usually finite. For a paradigmatic example, consider a half-infinite line:

$$u_{,t} = D u_{,xx}; \quad x > 0$$

$$u(x, 0) = \phi(x)$$

We need to know what happens to diffusing matter at $x = 0$.
Case 1: $u = 0$ at $x = 0$; Dirichlet boundary condition
Case 2: $-Du_{,x} = 0 \iff u_{,x} = 0$; Neumann boundary condition
Case 1: Method of reflections



Construct an odd extension of $\phi(x)$

$$\phi_{\text{odd}}(x) = \begin{cases} \phi(x) & \text{when } x > 0 \\ -\phi(-x) & \text{when } x < 0 \end{cases}$$

Construct solution:

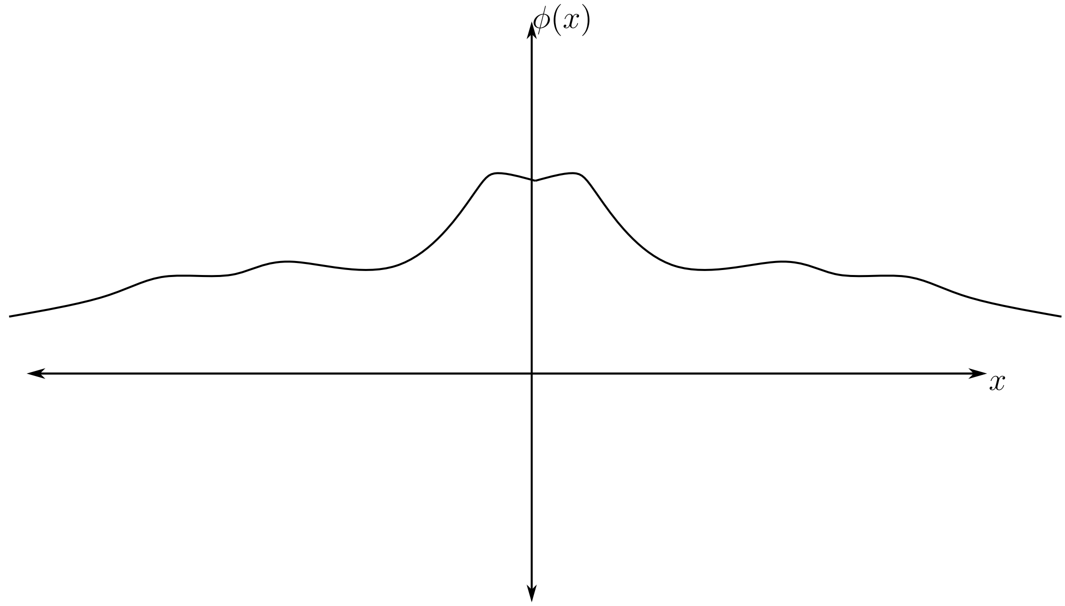
$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi_{\text{odd}}(y)dy$$

We can check that u satisfies the PDE.

We can check it satisfies the initial conditions. And

$$u(0,t) = \int_{-\infty}^{\infty} S(-y,t)\phi_{\text{odd}}(y)dy = 0$$

Case 2:



Construct

$$\phi_{\text{even}}(x) = \begin{cases} \phi(x) & \text{if } x > 0 \\ \phi(-x) & \text{if } x < 0 \end{cases}$$
$$= \phi(|x|)$$

and let

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi_{\text{even}}(y)dy$$

17 May 4, 2022

17.1 Diffusion Equation on Half Infinite Line

Last class we discussed diffusion equation with Dirichlet boundary conditions:

$$\begin{aligned} u(x,t) &= \int S(x-y,t)\phi_{\text{odd}}(y)dy \\ &= \underbrace{\int_0^\infty S(x-y,t)\phi_{\text{odd}}(y)dy}_{1} + \underbrace{\int_{-\infty}^0 S(x-y,t)\phi_{\text{odd}}(y)dy}_{2} \end{aligned}$$

in which we broken integration into $y \geq 0, y < 0$

$$= \underbrace{\int_0^\infty S(x-y,t)\phi(y)dy}_{1}$$

to evaluate 2, let $Y = -y$, $-\infty < y < 0$, $\infty > y > 0$

$$+ \int_\infty^0 \underbrace{(-dY)S(x+Y,t)}_{2}\underbrace{\phi_{\text{odd}}(-Y)}_{-\phi(Y)}$$

also $\begin{cases} dY = -dy \\ -dY = dy \end{cases}$.

$$\begin{aligned} u(x,t) &= \int_0^\infty S(x-y,t)\phi(y)dy + \int_0^\infty dY S(x+Y,t)(-\phi(Y)) && (17.1) \\ &= \int_0^\infty \Big(S(x-y,t) - S(x+y,t)\Big)\phi(y)dy && (17.2) \end{aligned}$$

For Neumann boundary conditions, we claim that:

$$u(x,t) = \int_{-\infty}^\infty S(x-y,t)\phi_{\text{even}}(y)dy$$

where $\phi_{\text{even}}(y)$ is the even extension of ϕ to $\mathbb{R}$.

Note 17.1: This certainly satisfies the PDE, and initial conditions (if we evaluate $u(x,0)$ we get $\phi_{\text{even}}(x) = \phi(x)$ for $x \geq 0$)

It also satisfies the boundary conditions $-D\frac{\partial u}{\partial x} = 0$ at $x = 0$,

$$\begin{aligned} \frac{\partial u}{\partial x} &= \int_{-\infty}^\infty \left(\frac{\partial}{\partial x}S(x-y,t)\right)\phi_{\text{even}}(y)dy \\ &= -\int_{-\infty}^\infty \left(\frac{\partial}{\partial y}S(x-y,t)\right)\phi_{\text{even}}(y)dy \end{aligned}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial y} S(-y,t)\right) \phi_{\text{even}}(y)dy$$

$S(y,t)$ is an even function of y

$$= \int_{-\infty}^{\infty} \underbrace{\left(\frac{\partial}{\partial y} S(y,t)\right)}_{\text{odd function}} \phi_{\text{even}}(y)dy$$
$$= 0$$

We can rewrite the solution as

$$u(x,t) = \int_0^{\infty} \Big(S(x-y,t) + S(x+y,t)\Big)\phi(y)dy \tag{17.3}$$

by following the same steps as for the Dirichlet Boundary condition.

17.2 Wave Equation on the Half Infinite Line

We have

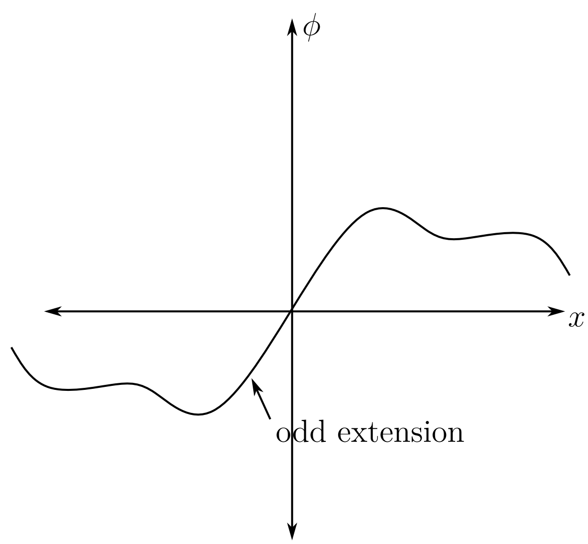
$$u_{,tt} = cu_{,xx} \quad x \geq 0$$

with initial conditions

$$u(x,0) = \phi(x)$$
$$u_{,t}(x,0) = \psi(x)$$

Assume Dirichlet boundary conditions at $x = 0$, $u(0,t) = 0$. Let's make an odd extension of ϕ and ψ

$$\phi_{\text{odd}}(x) = \begin{cases} \phi(x) & \text{when } x \geq 0 \\ -\phi(-x) & \text{when } x < 0 \end{cases}$$



and similarly for ψ

D'Alembert's solution gives:

$$u(x,t) = \frac{1}{2}\Big(\phi_{\text{odd}}(x+ct) + \phi_{\text{odd}}(x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{odd}}(y)dy \tag{17.4}$$

- This satisfies the PDE and its initial conditions

$$u(x,0) = \phi_{\text{odd}}(x) = \phi(x)$$

  when $x \geq 0$

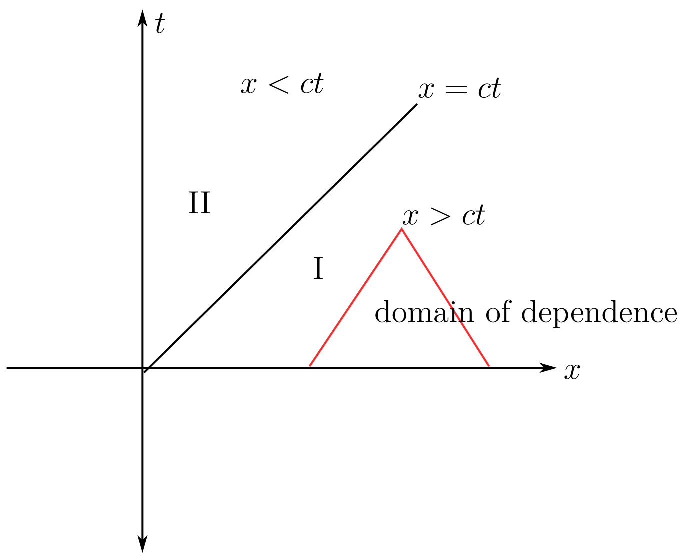
- It satisfies the boundary condition because

$$u(0,t) = \frac{1}{2}\Big(\phi_{\text{odd}}(ct) + \phi_{\text{odd}}(-ct)\Big) + \frac{1}{2c}\int_{-ct}^{ct} \psi_{\text{odd}}(y)dy$$

We want to rewrite Equation (17.4) in terms of ϕ and ψ only. We are only interested in the solution for $x \geq 0$, $t \geq 0$, $x + ct \geq 0$ for all such (x,t). So

$$\phi_{\text{odd}}(x+ct) = \phi(x+ct)$$

$$\phi_{\text{odd}}(x-ct) = \begin{cases} \phi(x-ct) & \text{if } x \geq ct \\ -\phi\Big(-(x-ct)\Big) = -\phi(ct-x) & x < ct \end{cases}$$



We divide space-time into two regions:

$$I : x \geq ct$$

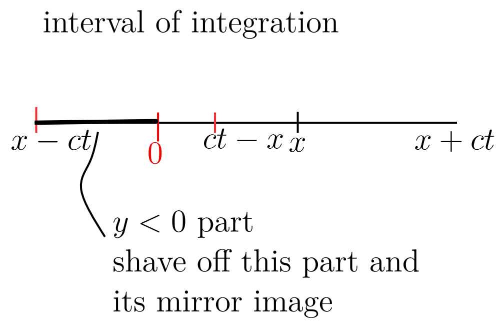
$$II : x - ct$$

In I: $x - ct \geq 0$.

$$u(x,t) = \left.\frac{1}{2}\Big(\phi(x+ct) + \phi(x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy\right\}\text{usual d'Alembert's formula.}$$

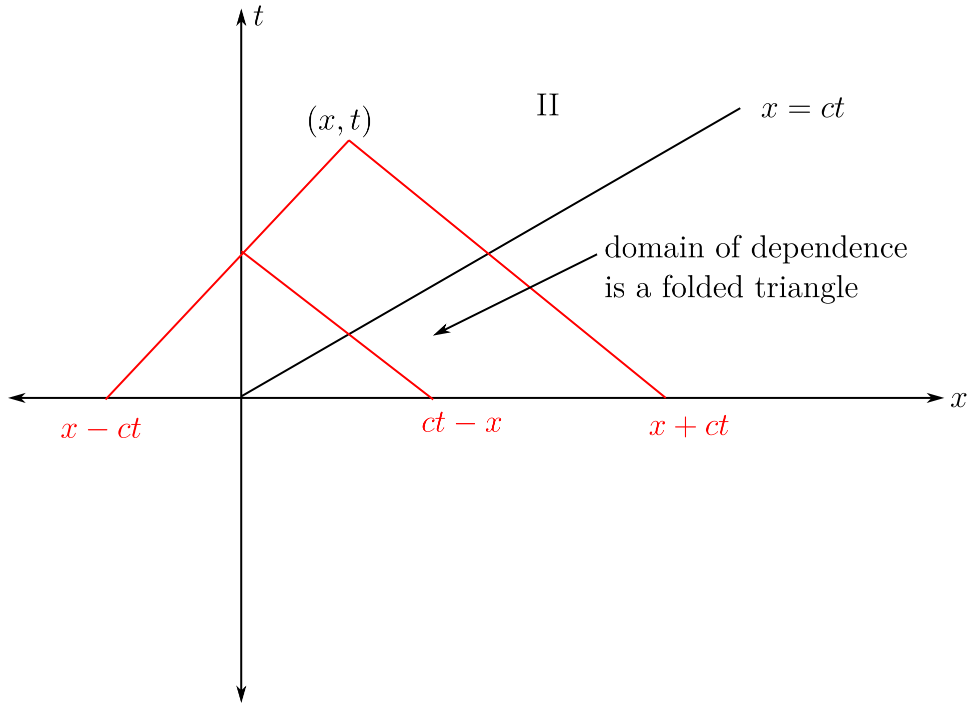
If $x > ct$, the boundary condition doesn't affect our solution at (x,t). $x = 0$ is not in the domain of dependence of such points.
In II: $x - ct < 0$

$$u(x,t) = \frac{1}{2}\Big(\phi(ct+x) - \phi(ct-x)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{odd}}(y)dy$$



$$u(x,t) = \frac{1}{2}\Big(\phi(ct+x) - \phi(ct-x)\Big) + \frac{1}{2c}\int_{ct-x}^{ct+x} \psi(y)dy$$

for $x < ct$.



18 May 6, 2022

18.1 Solving PDEs with Source Terms

Hitherto, all of the PDEs we have studied have been homogenous, meaning we can write them as $\mathcal{L}[y] = 0$, where $\mathcal{L}$ is a linear PDE operator. E.g.,

$$u_{,t} = Du_{,xx}$$

has

$$\mathcal{L}[u] = \left(\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2}\right)u.$$

Now we will consider inhomogenous PDEs, meaning:

$$\underbrace{\mathcal{L}[u]}_{\text{linear PDE operator}} = f(\mathbf{x}, t)$$

where the RHS is a known function, defined on the same domain as u.

Such PDEs arise as models for

1. Conservation laws, where matter is being created or destroyed: (q represents flows of matter)
$$\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = \begin{cases} 0 & u \text{ is conserved} \\ s(x,t) & \end{cases}$$

2. Force balance equation (e.g. wave equation).
$$\underbrace{\rho\frac{\partial^2 u}{\partial t^2}}_{\text{mass} \times \text{acceleration of unit length of string}} = \underbrace{T\frac{\partial^2 u}{\partial x^2}}_{\text{resultant tension}} + \underbrace{f(x,t)}_{\text{the force applied on unit length of string. (e.g. due to bowing)}}$$

We want to expand our knowledge of PDEs to include solutions of the inhomogeneous version of each PDE. Throughout we will find that our PDEs obey Duhamel's principle, which states that the solution of the inhomgeous PDE is built of a linear combination of homogenous solutions.

18.2 Inhomogeneous Solutions of ODEs

Think about ODEs:

$$\frac{du}{dt} = f(t)$$

a inhomogeonous ODE. Apply initial condition

$$u(0) = u_0$$

Because this is a linear PDE, I can break the problem into two parts.

$$(1)\frac{du}{dt} = 0 \qquad\qquad (2)\frac{du}{dt} = f(t)$$
$$u(0) = u_0 \qquad\qquad u(0) = 0$$
$$u(t) = u_0 \qquad\qquad u(t) = \int_0^t f(s)ds$$

add the two solutions together

$$u(t) = u_0 + \int_0^t f(s)ds$$

Now let's solve

$$\frac{du}{dt} = \delta(t-s) \quad \text{with } s > 0 \text{ for } t > 0$$

δ is the Dirac δ-function, it is defined by

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

or over any interval including the origin. Hence

$$\delta(t-s) = \begin{cases} 0 & \text{if } t \neq s \\ \infty & \text{if } t = s \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t-s)dt = 1$$

Physically, $\delta(t-s)$ represents a point force/source that is applied or occurs only when $t = s$.

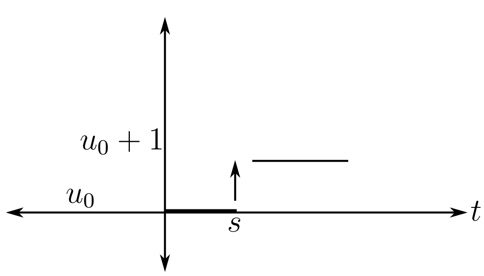
Solution of the equation is

$$u(t) = u_0 + \int_0^t \delta(\tilde{t}-s)d\tilde{t}$$

where we change dummy variable from s to $\tilde{t}$

$$= \begin{cases} u_0 + 0 & \text{if } t < s \text{ because } \delta(\tilde{t}-s) = 0, \forall 0 \leq \tilde{t} \leq t \\ u_0 + 1 & t > s \end{cases}$$

$u(t)$ jumps by 1 when t crosses s.



δ-function forcing/source causes our solution $u(t)$ to jump by 1.

Consider a more complicated first order ODE:

$$\frac{du}{dt} = g(u) + \delta(t-s)$$

for example $g(u) = -u$, ODE is:

$$\frac{du}{dt} = -u + \delta(t-s)$$

If we want to solve this equation for $t > 0$, $s > 0$. If $t < s$, then

$$\frac{du}{dt} = g(u)$$

we can solve this equation. If $g(u) = -u$, then $u(t) = u_0 e^{-t}$. e.g. if $g(u) = -u$, then $u(t) = Ce^{-t}$.

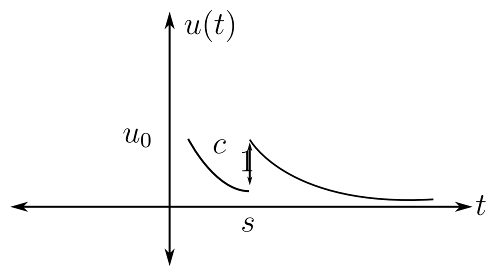


Figure 1: C does not have to be u_0, so our solution need not be continuous

What happens to the solution when $t = s$? I claim $u(t)$ will jump by 1 when t crosses s.

$$[u(t)]_{s-}^{s+} = 1$$

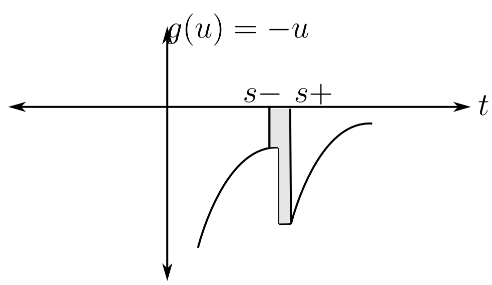
To see why, integrate the equation:

$$\int_{s-}^{s+} \frac{du}{dt} dt = \int_{s-}^{s+} g(u) dt + \int_{s-}^{s+} \delta(t-s) dt$$

$$u(t)]_{s-}^{s+} = 0 + 1$$

by reckless FTC application.

If u is discontinuous, then so is $g(u)$.



Let the interval become vanishingly small, then the area of the strip $\to 0$.

Importantly, apart from at $t = s$, our solution is made up of the solution of the homogeneous ODE, we need to match these solutions at $t = s$.

$$u(t) = \begin{cases} u_0 e^{-t} & \text{for } t < s \\ (u_0 e^{-s} + 1)e^{-(t-s)} & t > s \end{cases}$$

If instead we solve a second order ODE:

$$\frac{d^2u}{dt^2} = g\left(u, \frac{du}{dt}\right) + \delta(t - s)$$

Claim is that the solutions for $\begin{cases} t > s \\ t < s \end{cases}$ are both solutions of the homogeneous ODE, and there will be a discontinuity in $\frac{du}{dt}$ at $t = s$. For $t < s, t > s$, I am solving the homogenous ODE. For $t = s$, I integrate the ODE:

$$\int_{s-}^{s+} \frac{d^2u}{dt^2} dt = \int_{s-}^{s+} g\left(u, \frac{du}{dt}\right) + \int_{s-}^{s+} \delta(t - s) dt$$

$$\left[\frac{du}{dt}\right]_{s-}^{s+} = 0 + 1$$

u is continuous at s, du/dt has a jump discontinuity, thus $\to 0$ as interval $\to 0$.

19 May 9, 2022

19.1 Solving ODEs using δ-functions/ Green's Function/ Fundamental Solution for Inhomogeneous ODEs

Recall 19.1 Recall that for a first order ODE:

$$\frac{du}{dt} = g(u) + \delta(t-s) \quad t, s > 0$$

$u()$ is a homogeneous solution for $t > s, t < s$.

$$[u]_{s-}^{s+} = 1$$

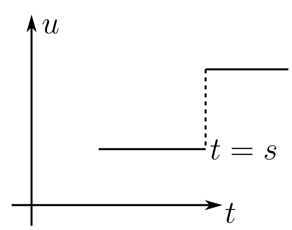
Whereas for a 2nd order ODE:

$$\frac{d^2u}{dt^2} = g\left(u, \frac{du}{dt}\right) + \delta(t-s)$$

$u(t)$ is a homogeneous solution for $t < s, t > s$,

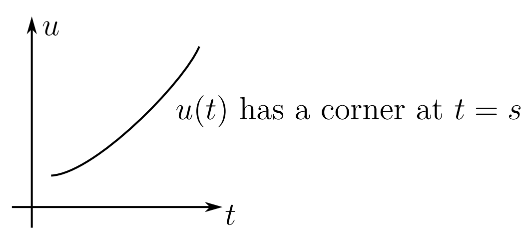
$$[u]_{s-}^{s+} = 0 \text{ while } \left[\frac{du}{dt}\right]_{s-}^{s+} = 1$$

We expect u to be continuous, because if u has a discontinuity,



then $\frac{du}{dt}$ will look $\delta(t-s$.

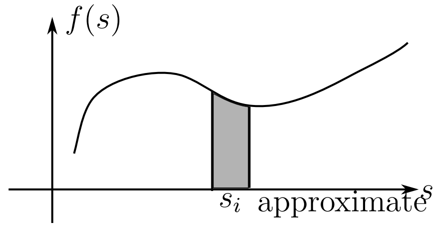
For a second ODE,



We used δ-functions to approximate continuous initial conditions when we solved the diffusion equation. We can do the same thing to solve inhomogenous ODEs.

Suppose we want to solve:

$$\frac{du}{dt} = f(t), \quad u(0) = u_0, \text{ for } t > 0$$



approximate the total forcing
$f(s_i)\Delta s$, by a single spike
at $s = s_i$: $f\underbrace{(s_i)\Delta s}\Delta s(s - s_i)$
has same total area as
the shaded region

Break Equation (7.1.1) into a set of ODES.

$$\text{Homogeneous problem } : \frac{du}{dt} = 0; u(0) = u_0$$

$$\text{solution} : u(t) = u_0$$

$$\text{Inhomogenous problems} : \frac{du}{dt} = f(s_i)\Delta s\delta(t - s_i)$$

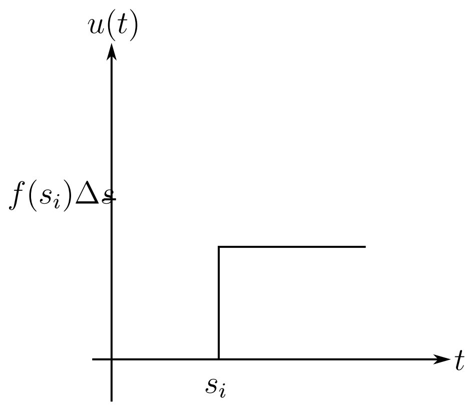
$$u(0) = 0$$

Solution of (7.1.1) and its initial condition is the sum of all of these functions.

$$\frac{du}{dt} = f(s_i)\Delta s\delta(t - s_i)$$

is a homogeneous solution when $t > s_i, t < s_i$, and it has a discontinuity of size $[u]_{s_i-}^{s_i+} = f(s_i)\Delta s$.

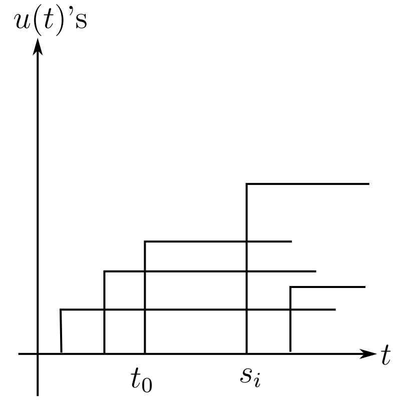
$$u = \begin{cases} C_1 & \text{for } t < s_i \\ C_2 & \text{for } t > s_i \end{cases} u(0) = 0 \implies C_1 = 0$$



$$u(t) = \begin{cases} 0 & \text{for } t < s_i \\ f(s_i)\Delta s & t > s_i \end{cases} \tag{19.1}$$

My solution of the inhomogenous ODE is given by the sum of all the above solutions.

$$u(t) = u_0 + \sum \text{ solutions 19.1}$$



Only the solutions (19.1) with $s_i < t_0$ when contribute to $u(t_0)$

$$\begin{aligned} u(t) &= u_0 + \sum_{s_i<t} \text{ solutions (19.1)} \\ &= u_0 + \sum_{s_i<t} f(s_i)\Delta s \to u_0 + \underbrace{\int_0^t f(s)ds}_{\text{limits arise because we needed } s_i>0, \text{ and } s_i<t} \end{aligned}$$

Homework: We will derive a solution of:

$$\frac{du}{dt} = u + f(t)$$

which can be solved by integrating.

19.2 Using δ-functions to Solve the Diffusion Equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} \quad \begin{cases} -\infty < x < \infty \\ t > 0 \\ u(x,0) = \phi(x) \end{cases}$$

We can find homogeneous solutions using the source solution method;

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy$$

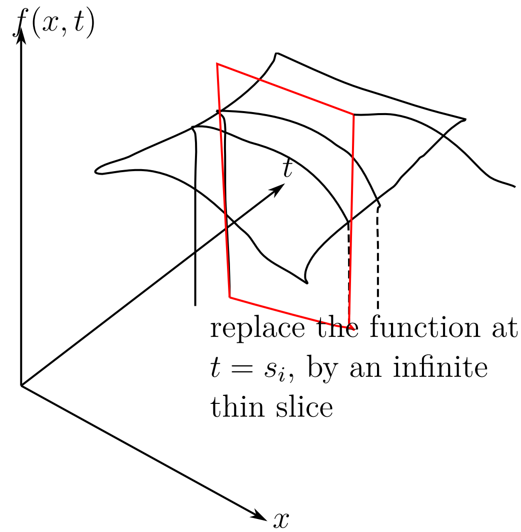
Now let's consider the inhomogenous version;

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(x,t) \quad \begin{cases} x \in R \\ t > 0 \\ u(x,0) = \phi(x) \end{cases}$$

Approach by decomposing into a set of problems:

1.
$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}; \quad u(x,0) = \phi(x)$$

2.
$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(x,s_i)\delta(t-s_i)\Delta s$$



Solution of (1):
$$u(x,t) = \int_{-\infty}^{\infty} \phi(y)S(x-y,t)dy$$

Solution of (2): $u(x,t)$ solve the homogeneous PDE for $t < s_i$, for $t > s_i$

Discontinuity at $t = s_i$ is given by integrating the PDE across $t = s_i$

$$\int_{s_i-}^{s_i+} \frac{\partial u}{\partial t}dt = D\int_{s_i-}^{s_i+} \frac{\partial^2 u}{\partial x^2}dt + \int_{s_i-}^{s_i+} f(x,s_i)\delta(t-s_i)\Delta s dt$$

$$[u]_{s_i-}^{s_i+} = 0(*) + f(x,s_i)\Delta s$$

$(*)$ even if u is discontinuous, then integral $\to 0$

$$u(x,s_i+) - u(x,s_i-) = f(x,s_i)\Delta s \tag{A}$$

$$u(x,0) = 0 \tag{B}$$

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} \text{ for } t < s_i, t > s_i \tag{C}$$

for $0 < t < s_i$, $u(x,t) = 0$ from (B) and (C).

For $t > s_i$, $\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$, and $u(x,s_i) = f(x,s_i)\Delta s$.

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y, \underbrace{t-s_i}_{\text{because sol'n is sought at time } t\text{, starting IC at } s_i})f(y,s_i)\Delta s dy$$

Putting solutions (1) and (2) together, we obtain:

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)dy + \int_0^t ds\int_{-\infty}^{\infty} dyS(x-y,t-s)f(y,s) \tag{19.2}$$

20 May 11, 2022

20.1 Solving ODEs using δ-functions (Cont'd)

We gave an argument to derive Equation (19.2) last class. We now want to verify that it truly solves the PDE and its initial conditions. At $t = 0$,

$$u(x,0) = \int_{-\infty}^{\infty} \phi(y)S(x-y,0)dy + 0 = \phi(x),$$

so initial conditions is satisfied. To check that our solution satisfies the PDE, we need to calculate its derivatives.

Recall 20.1 If

$$w(t) = \int_0^{g(t)} h(s,t)ds$$

then

$$\frac{dw}{dt} = \int_0^{g(t)} \frac{\partial h}{\partial t} ds + h(g(t),t)g'(t)$$

e.g. if $g(t) = t$,

$$\frac{dw}{dt} = \int_0^t \frac{\partial h}{\partial t} ds + h(t,t)$$

which is the Generalized Fundamental Theorem of Calculus.

To verify that u satisfies the PDE, ignore the first term (which we already know satisfies the homogeneous PDE)

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \int_0^t ds \int_{-\infty}^{\infty} dy \frac{\partial S}{\partial t} f(y,s) + \lim_{S\to t} \int_{-\infty}^{\infty} dy S(x-y,t-s)f(y,s) \\
&= \int_0^t ds \int_{-\infty}^{\infty} dy D\frac{\partial^2 S}{\partial x^2} f(y,s) + \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \underbrace{S(x-y,\epsilon)f(y,s)}_{f(x,t)} dy
\end{aligned}$$

because $\frac{\partial S}{\partial t} = D\frac{\partial^2 S}{\partial x^2}$.

$$\begin{aligned}
&= D\frac{\partial^2}{\partial x^2} \underbrace{\int_0^t ds \int_{-\infty}^{\infty} dy S(x-y,t-s)f(y,s)}_{u} + f(x,t) \\
&= D\frac{\partial^2 u}{\partial x^2} + f(x,t) \\
\frac{\partial u}{\partial t} &= D\frac{\partial^2 u}{\partial x^2} + f(x,t)
\end{aligned}$$

So u satisfies the PDE.

Note 20.2:

$$\int_{-\infty}^{\infty}\int_0^2 \sqrt{x^2+y^2}dydx = \int_0^2\int_{-\infty}^{\infty}\sqrt{x^2+y^2}dxdy = \int_0^2 dy\int_{-\infty}^{\infty} dx\sqrt{x^2+y^2}$$

20.2 Inhomogeneous Wave Equation

$$u_{tt} = c^2 u_{xx} + \underbrace{f(x,t)}_{\substack{\text{inhomogeneous term} \\ \text{represents an extra force}}} \qquad t > 0, x \in \mathbb{R}$$

with initial conditions

$$u(x,0) = \phi(x)$$
$$u_t(x,0) = \psi(x)$$

Problem (1):

$$u_{tt} = c^2 u_{xx}$$
$$u(x,0) = \phi(x)$$
$$u_t(x,0) = \psi(x)$$

Problem (2):

$$u_{tt} = c^2 u_{xx} + f(x,t)$$
$$u(x,0) = 0$$
$$u_t(x,0) = 0$$

Duhamel's principle states that the solution can be made up of solutions of the homogeneous problem.

1. Has solution (d'Alembert)

$$u(x,t) = \frac{1}{2}\Big(\phi(x-ct) + \phi(x+ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy$$

2. Break $f(x,t)$ up by time slices into functions like

$$f(x,s_i)\Delta s\delta(t-s_i)$$

and solve:

$$u_{tt} = c^2 u_{xx} + f(x,s_i)\Delta s\delta(t-s_i)$$
$$u(x,0) = 0$$
$$u_t(x,0) = 0$$

So if $t \lesseqgtr s_i$, then:

$$u_{tt} = c^2 u_{xx} \tag{A}$$
$$u(x,0) = 0, \quad u_t(x,0) = 0 \quad + \text{ jump conditions} \tag{B}$$
$$\left.\begin{aligned} u(x,s_i+) - u(x,s_i-) &= 0 \\ u_t(x,s_i+) - u_t(x,s_i-) &= f(x,s_i)\Delta s \end{aligned}\right\} \tag{C}$$

Solve for $t < s_i$, from (A) and (B)

$$u(x,t) = 0$$

for $t > s_i$, from (A) and (C),

$$u_{tt} = c^2 u_{xx} \tag{A}$$

$$\left.\begin{aligned} u(x, s_i) &= 0 \leftarrow \phi(x) \\ u_t(x, s_i) &= f(x, s_i)\Delta s \end{aligned}\right\} \tag{C}$$

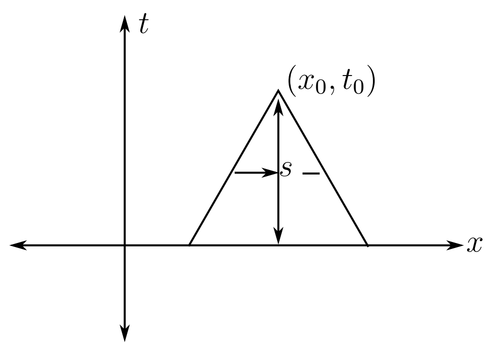
D'Alembert's solution implies

$$\begin{aligned} u(x,t) &= 0 + \frac{1}{2c}\int_{x-c(t-s_i)}^{x+c(t-s_i)} \psi(y)dy \\ &= \frac{1}{2c}\int_{x-c(t-s_i)}^{x+c(t-s_i)} f(y, s_i)\Delta s dy \end{aligned} \tag{20.1}$$

To solve problem (2), we will add together all of our solutions of form Equation (20.1).

$$u(x,t) = \frac{1}{2}\Big(\phi(x-ct) + \phi(x+ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy + \underbrace{\sum_{s_i : s_i < t} \frac{1}{2c}\int_{x-c(t-s_i)}^{x+c(t-s_i)} f(y, s_i)\Delta s_i dy}_{\to \frac{1}{2c}\int_0^t ds \int_{x-c(t-s)}^{x+c(t-s)} dy f(y,s)}$$

Now term is a double integral. To calculate $u(x_0, t_0)$,



Find domain of dependence of (x_0, t_0) (a triangle in (x,t) space) integrate f over the triangle by $2x$

21 May 13, 2022

21.1 Inhomogeneous Wave Equation (Cont'd)

From last class, we derived a formula for the solution of the inhomogeneous wave equation:

$$u(x,t) = \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds \tag{21.1}$$

solves $u_{tt} = c^2 u_{xx} + f$ with initial conditions $u(x,0) = u_t(x,0) = 0$.
We check that Equation (21.1) satisfies the PDE and its initial conditions (This proof is non-examinable).

Proof.

$$\frac{\partial u}{\partial t} = \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dy\Big|_{s=t} + \frac{1}{2c}\int_0^t \Big(cf(x+c(t-s),s\Big) + cf\Big(x-c(t-s),s)\Big)ds$$

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= \frac{1}{2}\Big(f(x+c(t-s),s\Big) + f(x-c(t-s),s)\Big)\Big|_{s=t} \\ &\quad + \frac{1}{2}\int_0^t \frac{\partial}{\partial t}\bigg(f\Big(x+c(t-s),s\Big) + f\Big(x-c(t-s),s\Big)\bigg)ds \\ &= f(x,t) + \frac{c}{2}\int_0^t \frac{\partial}{\partial x}\bigg(f\Big(x+c(t-s),s\Big) - f\Big(x-c(t-s),s\Big)\bigg)ds\end{aligned}$$

$$\frac{\partial u}{\partial x} = \frac{1}{2c}\int_0^t \bigg(f\Big(x+c(t-s),s\Big) - f\Big(x-c(t-s),s\Big)\bigg)ds$$

$$c^2\frac{\partial^2 u}{\partial x^2} = \frac{c}{2}\int_0^t \frac{\partial}{\partial x}\bigg(f\Big(x+c(t-s),s\Big) - f\Big(x-c(t-s),s\Big)\bigg)ds$$

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = f(x,t)$$

as required. And

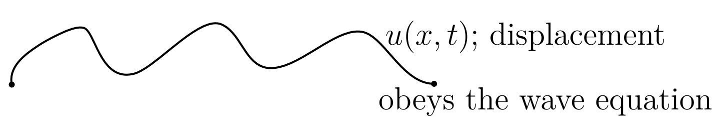
$$u(x,0) = 0$$

$$u_t(x,0) = 0$$

because both involve integrals form $\int_0^0 ds = 0$. $\square$

21.2 Spectral Expansions and Boundary Value Problems

Suppose we want to model a vibrating string



We have studied ∞ strings (d'Alembert's solution) and half ∞ strings (method of reflections). We now want to solve wave equation

$$u_{tt} = c^2 u_{xx} \quad 0 < x < L$$

with a finite domain. And assume for now Dirichlet boundary conditions

$$u(0,t) = u(L,t) = 0$$

Wouldn't it be great if the solutions were separable,

$$u(x,t) = X(x)T(t)$$

for some functions X, T and hope that this ansatz leads to ODEs for X, T

$$u_{tt} = X\ddot{T} \leftarrow X\frac{d^2T}{dt^2}$$

$$u_{xx} = X''T \leftarrow \frac{d^2X}{dx^2}T$$

For $u = XT$ to solve the PDE, I need:

$$X\ddot{T} = c^2 X''T$$
$$\underbrace{\frac{\ddot{T}}{c^2T}}_{\text{function of } t \text{ only}} = \underbrace{\frac{X''}{X}}_{\text{function of } x \text{ only}} = -\beta^2$$

Both sides must be equal to a constant. Assume for now the constant is < 0, hence we can write it as $-\beta^2$. But don't forget the boundary conditions,

$$u(0,t) = u(L,t) = 0$$

$$\iff X(0)T(t) = X(L)T(L) = 0 \quad \forall t$$

so $X(0) = X(L) = 0$ for the solution to be non-trivial.

Please solve

$$X'' + \beta^2 X = 0$$

with $X(0) = X(L) = 0$. $X = 0$ is a trivial solution or

$$X(x) = A\cos\beta x + B\sin\beta x$$

with A, B chosen to satisfy the initial conditions.

$$X(0) = 0 \implies A(1) + B(0) = 0 \implies A = 0$$

$$X(L) = 0 \implies B\sin\beta L = \begin{cases} 0 \text{ trivial} \\ \beta L = n\pi \text{ for some } n \in \mathbb{N} \end{cases}$$

Hence if $\beta = \frac{n\pi}{L}$ for some $n \in \mathbb{N}$, we can solve the X equation. Now

$$\frac{\ddot{T}}{c^2T} = -\beta^2 \implies \ddot{T} + \beta^2c^2T = 0$$

has solutions

$$T(t) = A\cos(\beta ct) + B\sin(\beta ct)$$

Any function of the form:

$$u(x,t) = \left(A_n\cos\left(\frac{n\pi ct}{L}\right) + B_n\sin\left(\frac{n\pi ct}{L}\right)\right) \times \sin\left(\frac{n\pi x}{L}\right)$$

solves the PDE and its boundary conditions. We hope that given any initial conditions $\phi(x), \psi(x)$, we can choose A_n, B_n so that:

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n\cos\left(\frac{n\pi ct}{L}\right) + B_n\sin\left(\frac{n\pi ct}{L}\right)\right)\sin\frac{n\pi x}{L}$$

satisfies the initial conditions.

We can solve the Diffusion equation in the same way.

$$u_t = Du_{xx}$$

$$u(0,t) = u(L,t) = 0$$

with ansatz $u(x,t) = X(x)T(t)$

$$X\dot{T} = DX''T$$
$$\frac{\dot{T}}{DT} = \frac{X''}{X} = -\beta^2$$
$$\implies X(x) = \sin\frac{n\pi x}{L}$$

with $\beta = \frac{n\pi}{L}$ for some $n \in \mathbb{N}$ and

$$\dot{T} + \beta^2Dt = 0$$
$$\implies T(t) = A\exp\Big(-\beta^2Dt\Big)$$

Hence

$$u(x,t) = \sum_{n=1}^{\infty}A_n\exp\left(-\frac{n^2\pi^2Dt}{L^2}\right)\sin\frac{n\pi x}{L}$$

solves the PDE, and we hope that we can choose $\{A_n\}$ to realize any initial conditions

$$u(x,0) = \phi(x)$$

We still need to show that there are no solutions with the constant $(-\beta^2)$ not being negative.

22 May 16, 2022

22.1 Spectral Expansions and Boundary Value Problems (Cont'd)

Last class we studied separation of variables for wave equation and diffusion equation on a finite interval $0 < x < L$, with Dirichlet boundary conditions.

We can construct solutions like:

e.g. Wave Equation:

$$u = XT$$

where

$$X'' + \beta^2 X = 0 \implies X = \sin \beta x$$
$$B = \frac{n\pi}{L} \quad \text{for some } n \in \mathbb{Z}$$

similarly

$$\ddot{T} = -c^2\beta^2 T$$
$$\implies T = \left\{ \begin{matrix} \cos \\ \sin \end{matrix} \right\} \left(\frac{cn\pi t}{L} \right)$$

Note: check dimensions

- Any linear combination of these solutions is also a valid solution. How to choose this linear combination?
- Recall that our separation of variables produced.

$$\frac{\ddot{T}}{c^2 T} = \frac{X''}{X} = -\beta^2$$

We assumed that the RHS is non-positive.

Why could we do this: Let's suppose the RHS were positive:

$$\frac{X''}{X} = +\alpha^2$$
$$\implies X'' - \alpha^2 X = 0$$
$$X(x) = e^{\alpha x} \text{ or } e^{-\alpha x}$$

or

$$X(x) = \underbrace{\sinh \alpha x}_{\frac{1}{2}(e^{\alpha x} - e^{-\alpha x})} \quad \text{or} \quad \underbrace{\cosh \alpha x}_{\frac{1}{2}(e^{\alpha x} + e^{-\alpha x})}$$

Recall 22.1

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$\sinh(0) = 0, \cosh(0) = 1$

- Boundary conditions are often easier to apply when we write our solutions using sinh + cosh rather than exponentials

Recall 22.2

$$\sinh x = \frac{1}{2}(e^x - e^{-x}); \quad \cosh x = \frac{1}{2}(e^x + e^{-x})$$

so

$$\frac{d}{dx}(\sinh x) = \frac{1}{2}(e^x + e^{-x}) = \cosh x$$
$$\frac{d}{dx}(\cosh x) = \frac{1}{2}(e^x - e^{-x}) = \sinh x$$

Solution of $X'' - \alpha^2 X = 0$ is:

$$X(x) = A \cosh \alpha x + B \sinh \alpha x$$

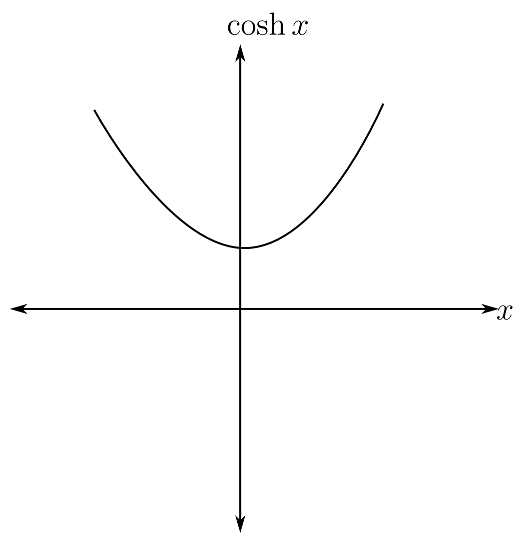
Now apply boundary conditions:

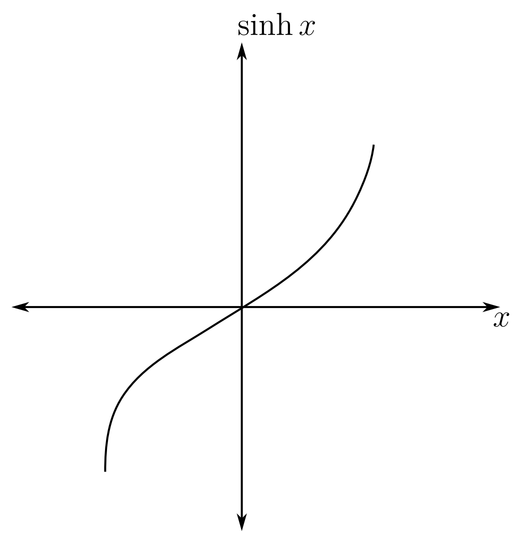
$$X(0) = 0 \implies A = 0$$

$$\implies X(x) = B \sinh \alpha x$$

then

$$X(L) = 0 \implies B \sinh \alpha L = 0$$





Since $\sinh x = 0$ only when $x = 0$, we infer either $\alpha = 0$ or $B = 0$. i.e. only trivial solutions are possible.

In general, we will need to choose the coefficients in our series to match the given initial conditions, $u(x, 0) = \phi(x)$ (and $u_t(x, 0) = \psi(x)$ for wave equation).

22.2 Now Consider Neumann Boundary Conditions

Either:

$$\left.\begin{aligned} u_{tt} &= c^2 u_{xx} \\ u_t &= D u_{xx} \end{aligned}\right\} 0 < x < L$$

$$u_x = 0 \text{ at } x = 0, L$$

Seek a separable solution:

$$u = XT$$

wave equation	diffusion equation
$\dfrac{\ddot{T}}{c^2T} = \dfrac{X''}{X} = -\beta^2$	$\dfrac{\dot{T}}{DT} = \dfrac{X''}{X} = -\beta^2$
so solutions are sines and cosines rather than sinh and cosh	Ditto
$X'' + \beta^2 X = 0$	ditto
$X(x) = A\cos\beta x + B\sin\beta x$	
$u_x = X'T$	
$X'(0) = X'(L) = 0$	ditto

$$X'(x) = -\beta A\sin\beta x + \beta B\cos\beta x$$

$$X'(0) = 0 \implies \beta B(1) = 0$$

so either $\beta = 0$ or $B = 0$.

1. If $\beta = 0$ then $X(x) = A$. (+ thus satisfies 2nd Boundary condition)
2. If $B = 0$, then $X(x) = A\cos\beta x$.

$$X'(L) = -\beta A\sin\beta L = 0$$

$\implies$ for nontrivial solutions $\beta = \frac{n\pi}{L}$ for some $n \in \mathbb{N}$.

In both wave + diffusion equation cases, we have

$$X = (A)\cos\frac{n\pi x}{L} \quad \text{or} \quad 1 \qquad \text{(ignoring coefficients)}$$

The solution for T is identical to the previous problem of Dirichlet Boundary conditions.

wave equation	diffusion equation
$T = \cos\dfrac{n\pi ct}{L}$ or $\sin\frac{n\pi ct}{L}$	$T = \exp\left(-\dfrac{n^2\pi^2 Dt}{L^2}\right)$

22.3 Cosine + Sine Sum Formulae

Recall 22.3

$$\cos(a+b) = \cos a\cos b - \sin a\sin b \tag{22.1}$$

$$\sin(a+b) = \sin a\cos b + \cos a\sin b \tag{22.2}$$

$$\begin{aligned}
e^{i\theta} &= \cos\theta + i\sin\theta \\
\cos\theta &= \text{Re}(e^{i\theta}) \\
&= \text{Re}(e^{ia}\cdot e^{ib}) \\
&= \text{Re}\Big((\cos a + i\sin a)(\cos b + i\sin b)\Big) \\
&= \text{Re}(\cos a\cos b - \sin a\sin b + i \times \text{ something}) \\
&= \cos a\cos b - \sin a\sin b
\end{aligned}$$

These formulae are very useful when calculating integrals of products of sines or cosines.

Example 22.4

$$\int \sin\alpha x\sin\beta x dx = \frac{1}{2}\int\Big(\cos(\alpha-\beta)x - \cos(\alpha+\beta)x\Big)dx$$

We will use (22.1) +

$$\begin{aligned}
\cos(a-b) &= \cos a\cos b + \sin a\sin b \\
\sin a\sin b &= \frac{1}{2}\Big(\cos(a-b) - \cos(a+b)\Big) \qquad (22.3)
\end{aligned}$$

Let's assume $a \neq \pm\beta$, then

$$\int \sin\alpha x\sin\beta x dx = \frac{1}{2}\cdot\left(\frac{\sin(\alpha-\beta)x}{(\alpha-\beta)} - \frac{\sin(\alpha+\beta)x}{(\alpha+\beta)}\right) + C$$

Idea: We know solutions of the diffusion equation with Dirichlet Boundary Conditions are of the form:

$$u(x,t) = \sum_{n=1}^{\infty} A_n \exp\left(-\frac{n^2\pi^2 Dt}{L^2}\right)\sin\left(\frac{n\pi x}{L}\right)$$

If we have an initial condition:

$$u(x,0) = \phi(x)$$

we need to choose A_n such that

$$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) = \phi(x)$$

23 May 18, 2022

23.1 Expressing Functions as Sums of Sines or Cosines

Given a function $\phi(x)$, we'd like to choose coefficients $\{A_n\}$ such that

$$\phi(x) = \sum_{n=0}^{\infty} A_n \sin \frac{n\pi x}{L}$$

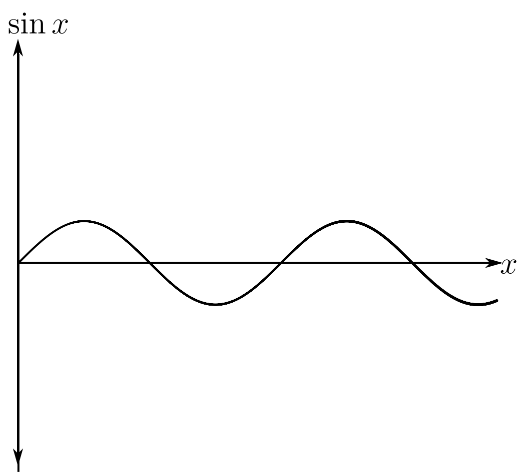
Returning to the formula from last class

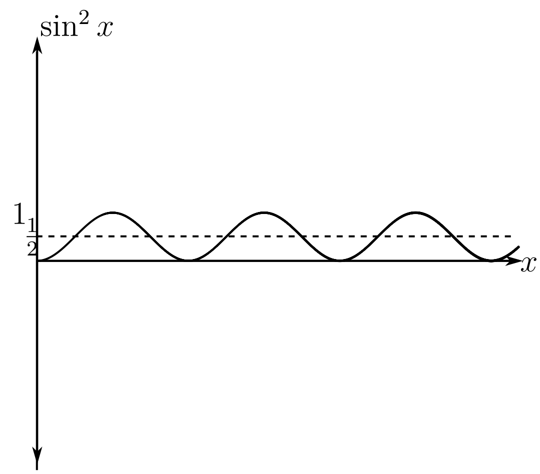
$$\int_0^L \underbrace{\sin \frac{n\pi x}{L}}_{\alpha = n\pi/L} \sin \underbrace{\frac{m\pi x}{L}}_{\beta = m\pi/L}\ dx = \frac{1}{2}\left[\frac{\sin((n-m)\pi x/L)}{((n-m)\pi/L) - \frac{\sin((n+m)\pi x/L)}{((n+m)\pi/L)}}\right]_0^L \tag{23.1}$$

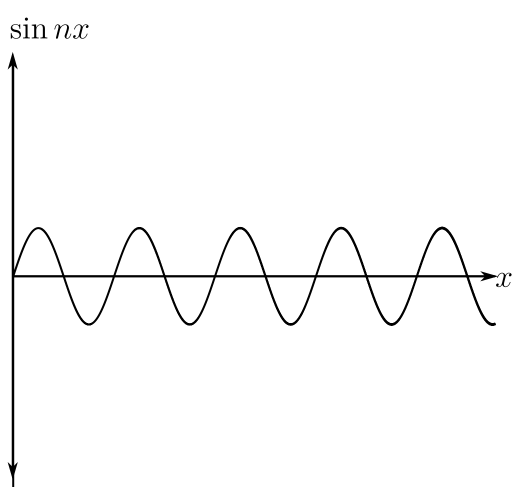
assuming $n \neq m$, (23.1) $= 0$ because n, m are integers.
Conversely if $n = m$, integral is

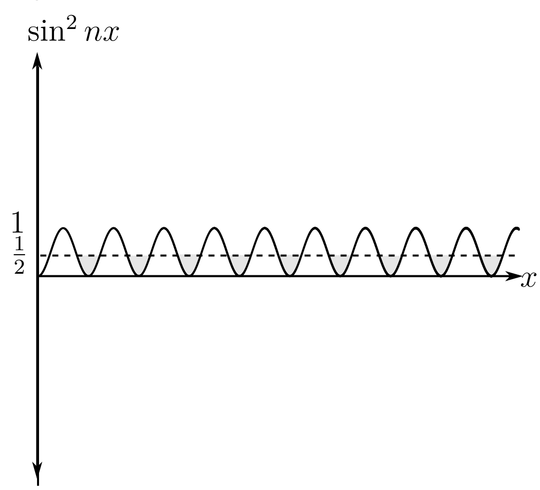
$$\int_0^L \sin^2 \frac{n\pi x}{L} dx = \frac{1}{2} L \tag{23.2}$$

evaluate by graph.









$$\sin^2 nx \text{ has as much area above } y = \tfrac{1}{2} \text{ as below.}$$

Hence, if ℓ is any multiple of its period, the average value of function

$$\frac{1}{\ell}\int_0^\ell \sin^2 nx dx = \frac{1}{2}$$

$$\implies \int_0^\ell \sin^2 nx dx = \frac{1}{2}\ell$$

For a more careful evaluation, use the sine-sum formula.

$$\begin{aligned} \int \sin^2 \alpha x dx &= \frac{1}{2}\int (\cos 0 - \cos 2\alpha x) dx \\ &= \frac{1}{2}\int (1 - \cos 2\alpha x) dx \\ &= \frac{1}{2}\left(x - \frac{1}{2\alpha}\sin 2\alpha x\right) + C \\ \implies \int_0^L \sin^2 \frac{n\pi x}{L} dx &= \frac{1}{2}L \end{aligned}$$

Assume $\phi(x)$ can be written as a sum of sines, and this sum is absolutely and uniformly convergent. Then

$$\phi(x) = \sum_{n=0}^{\infty} A_n \sin \frac{n\pi x}{L}$$

- We can multiply functions into the series on the RHS, by multiplying each term in the series
- We can integrate the series on the RHS term by term

Our philosophy is to be reckless + cheerful.

$$\phi(x)\sin\frac{m\pi x}{L} = \sum_{n=0}^{\infty} A_n \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}$$

And integrate,

$$\int_0^L \phi(x)\sin\frac{m\pi x}{L} dx = \sum_{n=1}^{\infty} A_n \int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L} dx \tag{23.3}$$

Our results on integrals of products of sines can be expressed as

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L} dx = \frac{1}{2}L\cdot\delta_{mn}$$

And the Kronecker delta is defined by

$$\delta_{mn} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$$

Hence (23.3) implies

$$\int_0^L \phi(x)\sin\frac{m\pi x}{L}dx = \sum_{n=1}^{\infty} A_n \underbrace{\frac{1}{2}L\cdot\delta_{mn}}_{\text{isolates } n=m \text{ term}} = \frac{1}{2}A_m$$

In other words,

$$A_m = \frac{2}{L}\int_0^L \phi(x)\sin\frac{m\pi x}{L}dx$$

Example 23.1

If $\phi(x) = x$.

$$A_m = \frac{2}{L}\int_0^L x\sin\frac{m\pi x}{L}dx$$

Let $u = x$ and $v' = \sin\frac{m\pi x}{L}$, $u' = 1$, and $v = -\frac{L}{m\pi}\cos\left(\frac{m\pi x}{L}\right)$

$$= \frac{2}{L}\left[-\frac{Lx}{m\pi}\cos\left(\frac{m\pi x}{L}\right)\right]_0^L + \frac{L}{m\pi}\underbrace{\int_0^L \cos\frac{m\pi x}{L}dx}_{\to 0}$$

$$= \frac{2}{m\pi}(-1)\cos(m\pi)$$

Consider a table

m	$\cos m\pi$
0	1
1	-1
2	1
m	$(-1)^m$

$$\implies A_m = \frac{2L}{m\pi}(-1)^{m+1}$$

No way, there can be convergence at $x = L$, because $\phi(L) = L$, whereas all sines vanish at $x = L$.

Example 23.2

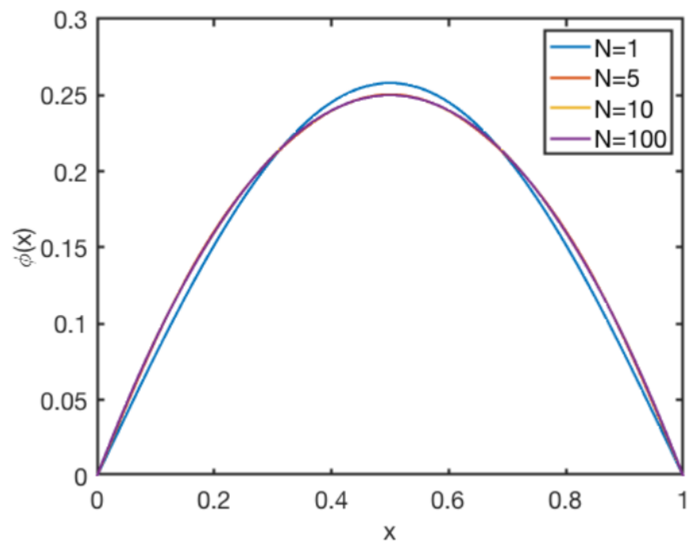
$$\phi(x) = x(1-x) \quad 0 < x < 1$$

actually satisfies boundary conditions.

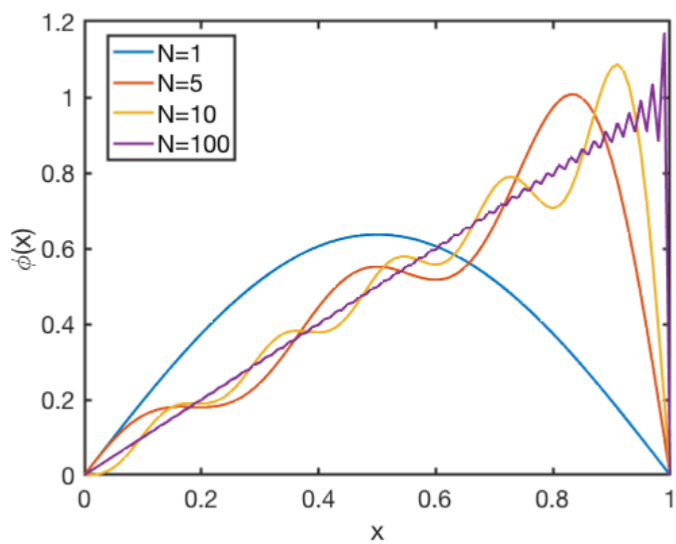
$$A_m = 2\int_0^1 x(1-x)\sin m\pi x dx$$
$$A_m = \frac{4-4(-1)^m}{m^3\pi^3}$$

We get a good approximation with ~ 2 terms.

phi(x) = x(1-x)



phi(x) = x



24 May 20, 2022

24.1 Self-Adjoint Boundary Value Problems

When we solved the wave equation or the diffusion equation by separation of variables, we encountered an ODE for X;

$$-X'' = \lambda X \qquad (= \beta^2 X) \tag{24.1}$$

This problem has non-trivial solutions that represent boundary conditions only for discrete values of λ; ($\lambda = n^2\pi^2/L^2$ for the Neumann + Dirichlet BVPs)
We noticed that these solutions are orthogonal in the sense that

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx = 0$$

unless $n = m$. These properties arise because we are solving a self-adjoint boundary value problem.

Definition 24.1 (Inner product)
Define an inner product of two real functions f, g as follows

$$(f, g) \equiv \int_0^L f(x)g(x)dx$$

Note:

$$(f, f) = \int_0^L f(x)^2 dx \equiv \|f\|^2$$

Definition 24.2 (Eigenvalue and Eigenfunction)
Problems like (24.1) are examples of ODE BVPs (boundary value problems). There is an $\mathcal{L}(\equiv -d^2/dx^2)$ that can act on functions. We look for solutions of

$$\mathcal{L}[X] = \lambda X$$

for some constant λ, such that X satisfies given boundary conditions, e.g.

$$X(0) = X(L) = 0 \tag{Dirichlet}$$

or

$$X'(0) = X'(L) = 0 \tag{Neumann}$$

($A\mathbf{x} = \lambda\mathbf{x}$ where A is a matrix from 33A)
We call λ an eigenvalue and X the corresponding eigenfunction.

Example 24.3
If $\mathcal{L}[x] \equiv -\frac{d^2X}{dx^2}$ and BCs are $X'(0) = X'(L) = 0$, then

$$\lambda = \frac{n^2\pi^2}{L^2} \quad \text{and} \quad X = \cos\frac{n\pi x}{L}$$

are the eigenvalues and eigenfunctions.

Definition 24.4 (Self-adjoint)
We say that the operator (+ its BCs) are <u>self-adjoint</u> if given any f, g in the domain of operator, that satisfy the boundary conditions, the following fact holds,

$$(f, \mathcal{L}[g]) = (\mathcal{L}[f], g)$$

Example 24.5
$\mathcal{L}[x] = -d^2X/dx^2$ with $X'(0) = X'(L) = 0$ is self-adjoint because if f, g are any two $\mathcal{C}^2$ functions that satisfy the boundary conditions, then

$$\begin{aligned}
(f, \mathcal{L}[g]) &= -\int_0^L \underbrace{f(x)}_{u}\underbrace{g''(x)}_{v'}\,dx \\
&= -[fg']_0^L + \int_0^L \underbrace{f'}_{u}\,\underbrace{g'}_{v'}\,dx \\
&= [f'g - fg']_0^L - \underbrace{\int_0^L f''g dx}_{(\mathcal{L}[f],g)}
\end{aligned}$$

But $f'(0) = f'(L) = 0$ and $g'(0) = g'(L) = 0$

$$(f, \mathcal{L}[g]) = 0 + (\mathcal{L}[f], g)$$

which is what we needed to show.

Example 24.6
Similarly $\mathcal{L}[X] = -X''$ with $X(0) = X(L) = 0$ is self-adjoint. In fact, any boundary conditions that make

$$[f'g - fg']_0^L = 0$$

will make the problem self-adjoint.

Example 24.7
$X(0) = X'(L) = 0$ Dirichlet + Neumann on the other.

Example 24.8
$X(0) = X(L), X'(0) = X'(L)$ (periodic boundary conditions)

Self-adjoint boundary value problems have delightful eigenfunctions and eigenvalues (for a full description see 266A, maybe 135).

24.2 Properties of Eigenfunctions of Self-Adjoint (Hermitian) BVPs

Proposition 24.9
If X_1, X_2 are eigenfunctions with different eigenvalues, then X_1, X_2 are orthogonal, i.e.:

$$(X_1, X_2) = 0$$

Proof. Consider:

$$(\mathcal{L}[X_1], X_2) = (X_1, \mathcal{L}[X_2])$$

since X_1, X_2 satisfy the boundary conditions and $\mathcal{L}$ is self-adjoint. Since X_i is an eigenfunction,

$$\mathcal{L}[X_i] = \lambda_i X_i$$

for some λ_i.

$$\implies (\lambda_1 X_1, X_2) = (X_1, \lambda_2 X_2)$$

$$\lambda_1 (X_1, X_2) = \lambda_2 (X_1, X_2)$$

(Look at definition of inner product)

$$\implies (\lambda_1 - \lambda_2)(X_1, X_2) = 0$$

Since $\lambda_1 \neq \lambda_2$, it follows $(X_1, X_2) = 0$. $\square$

Proposition 24.10
The eigenvalues are real.

Proof. Requires redefining the inner product (f, g) for complex f, g $\square$

Proposition 24.11
The eigenfunctions are complete; meaning that given any f that satisfies the boundary conditions, we can write:

$$f(x) = \sum_{n-1}^{\infty} A_n X_n(x)$$

i.e. f can be written as a sum of eigenfunctions, and this sum converges absolutely and uniformly. Given any $\varepsilon > 0$, I can define N, such that:

$$\left| f(x) - \sum_{n=1}^{N} A_n X_n(x) \right| < \varepsilon \quad \forall x \in [0, L]$$

And we choose the coefficients A_n, by applying orthogonality:

$$\begin{aligned} (f(x), X_m(x)) &= \sum A_n \Big(X_n(x), X_m(x) \Big) \\ &= A_m(X_m, X_m) \\ \implies A_m &= \frac{(f, X_m)}{(X_m, X_m)} \end{aligned}$$

We can use the orthogonality for any set of eigenfunctions of a self-adjoint BVP.

Example 24.12
Expand $f(x)$ as a sum of $X_n = \cos \frac{n\pi x}{L}$ (eigenfunctions of $-X'' = \lambda X$ with $X'(0) = X'(L) = 0$).
Because they are eigenfunctions:

$$(X_n, X_m) = 0 \quad \text{if } m \neq n$$

and

$$(X_m, X_m) = \int_0^L \cos^2 \frac{m\pi x}{L} dx = \frac{1}{2} L$$

(by graphing, we find average value is $\frac{1}{2}$)

$$= \int_0^L \frac{1}{2} \left(1 + \cos \frac{2m\pi x}{L} \right) dx = \frac{1}{2} L$$

Because:

$$\begin{aligned} \cos 2\alpha &= \cos^2 \alpha - \sin^2 \alpha \\ &= 2\cos^2 \alpha - 1 \\ \implies \cos^2 \alpha &= \frac{1}{2}(\cos 2\alpha + 1) \end{aligned}$$

25 May 23, 2022

25.1 Cosine Series for Neumann Boundary Conditions

Example 25.1
$\phi(x) = \frac{1}{2}x^2 = \frac{1}{3}x^3, 0 < x < 1$
$\phi'(x) = x(1-x)$ vanishes at $x = 0, 1$.
Expressed as a cosine series ;

$$\begin{aligned}\phi(x) &= \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \\ &= \sum_{n=0}^{\infty} A_n \cos(n\pi x)\end{aligned}$$

Note: Book: $\frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}$
By orthogonality:

$$A_n = \frac{(\phi, \cos n\pi x)}{(\cos n\pi x, \cos n\pi x)}$$

$$A_0 = \frac{(\phi, 1)}{(1, 1)} = \frac{\int_0^1 \left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right) dx}{\int_0^1 1 dx} = \frac{1}{12}$$

$$A_n = \frac{\int_0^1 \left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right) \cos n\pi x dx}{\int_0^1 \cos^2 n\pi x dx} = \frac{\frac{2((-1)^n - 1)}{n^4\pi^4}}{\frac{1}{2}} = \frac{4((-1)^n - 1)}{n^4\pi^4} \qquad (*)$$

$(*)$ Picks out odd values of n.
It turns out 2 terms are enough to approximate the function $\phi(x)$.

25.2 More Properties of the Eigenfunctions of a Self-Adjoint BVP

Proposition 25.2
The eigenfunctions are complete, in the sense that if $\phi(x)$ is any $\mathcal{C}[0, L]$ that satisfies the boundary conditions, then:

$$\phi(x) = \sum_{n=0}^{\infty} A_n X_n(x)$$

and the series converges uniformly to $\phi(x)$.

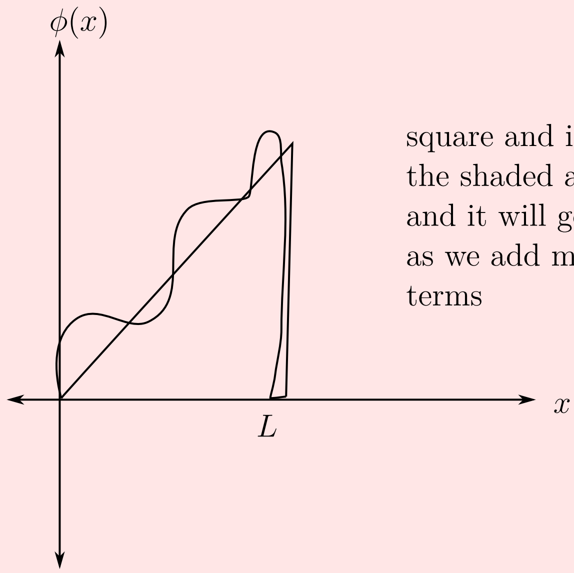
Proposition 25.3
Even if $\phi(x)$ does not satisfy the boundary conditions, then we can write:

$$\phi(x) = \sum_{n=0}^{\infty} A_n X_n(x)$$

where the series converges in L^2 to $\phi(x)$.

$$\int_0^L \left(\phi(x) - \sum_{n=0}^{N} A_n X_n(x)\right)^2 dx \to 0 \text{ as } N \to \infty$$

Example 25.4
$\phi(x) = x$ using sine series



square and integrate the shaded area, and it will go $\to 0$ as we add more sine terms

For more (non-examinable) details, consult the book, or an ODE ref, like: Coddington and Lennson Theory of Ordinary Differential Equations.

25.3 Solving PDEs using Separation of Variables

Idea: Express the initial conditions on the PDE in terms of separable solutions of the PDE.

Example 25.5

Solve

$$u_t = u_{xx} \quad 0 < x < 1$$

with Neumann boundary conditions $-u_x = 0$ at $x = 0, 1$ and initial conditions

$$u(x) = \phi(x) = \frac{1}{2}x^2 - \frac{1}{3}x^3$$

Idea: Write the solution as a sum of separable solutions:

$$\begin{aligned} u(x,t) &= \sum_n X_n(x)T_n(t) \\ &= \sum_{n=0}^{\infty} A_n \cos n\pi x e^{-n^2\pi^2 t} \end{aligned}$$

Eigenfunctions are dictated by the BCs. Choose A_n to satisfy the initial conditions,

$$\begin{aligned} u(x,0) &= \sum_{n=0}^{\infty} A_n \cos n\pi x \\ &= \frac{1}{2}x^2 - \frac{1}{3}x^3 \end{aligned}$$

And

$$A_0 = \frac{1}{12}, \quad A_n = \frac{4((-1)^n - 1)}{n^4\pi^4} \text{ for } n > 0$$

Solution is:

$$u(x,t) = \frac{1}{12} + \sum_{n=1}^{\infty} \frac{4((-1)^n - 1)}{n^4\pi^4} e^{-n^2\pi^2 t} \cos n\pi x \tag{25.1}$$

This series converges rapidly, (first term gets us practically close to the real solution). We can sum the series to find $u(x,t)$ numerically. We notice $u(x,t) \to \frac{1}{12}$ as $t \to \infty$; $\forall x$ physically this is because diffusion evens out gradients. Mathematically, every summand in (25.1) except for $\frac{1}{12}$ is exponentially decaying.

Example 25.6
We can do the same thing for wave equation:

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right) \times \sin\left(\frac{n\pi x}{L}\right)$$

using

$$u(x,0) = \phi(x)$$
$$u_t(x,0) = \psi(x)$$

So

$$\sum_n A_n \sin\left(\frac{n\pi x}{L}\right) = \phi(x)$$
$$\sum_n \left(\frac{n\pi c}{L}\right) B_n \sin\frac{n\pi x}{L} = \psi(x)$$

25.4 What to do if the BCs are not simple Neumann/Dirichlet

So far we have assumed $u = 0$ or $u_x = 0$ at boundaries. If we have boundary conditions where u is non-zero, e.g. $u = 1$ at $x = 0$, what do we do?

Example 25.7
Solve

$$u_t = u_{xx}$$
$$u(0) = 1$$
$$u(1) = 0$$
$$u(x,t) = 1 - x + \sum_{n=1}^{\infty} A_n \sin n\pi x e^{-n^2\pi^2 Dt}$$

where $1 - x$ is a simple function that satisfies the PDE and the non-zero boundary conditions. Choose $\{A_n\}$ to satisfy the initial condition.

26 May 25, 2022

26.1 Solving Diffusion Equation with Inhomogeneous Boundary Conditions

Hitherto we have solved diffusion + wave equation homogeneous boundary conditions; e.g.

$$\left.\begin{array}{l}\text{homogeneous Dirichlet } u = 0 \\ \text{or homogeneous Neumann } u_x = 0\end{array}\right\} \text{ at either boundary}$$

Often, we are interested in problems with inhomogeneous boundary conditions; e.g. $u = a \leftarrow$ end of bar maintained at some temperature

$$-D\frac{\partial u}{\partial x} = b \leftarrow \text{ flux of morphogens into embryo}$$

Strategy is to write:

$$u(x,t) = w(x,t) + v(x,t)$$

require $w(x,t)$ satisfy the BCs + the PDE but not necessarily the IC, while
$v(x,t)$ satisfy the PDE + the homogeneous version of the BCs + fixes the IC. (we have been finding these functions for the last week)

Example 26.1
Solve

$$u_t = Du_{xx} \quad 0 < x < L$$
$$u = 1 \text{ at } x = 0$$
$$u = 2 \text{ at } x = L$$

are BCs. And

$$u(x,0) = \phi(x)$$

is IC.
First, we find $w(x,t)$ that satisfies

$$w_t = Dw_{xx}$$

$$\left.\begin{aligned} w &= 1 \\ w &= 2 \end{aligned}\right\} \text{ at } x = 0, L \text{ resp.}$$

To get w, we will assume $w_t = 0$ i.e. $w \equiv w(x)$.

$$0 = Dw_{xx} \quad w = 1, 2 \text{ at } x = 0, L \text{ resp.}$$

$$\implies w = 1 + x/L$$

Then:

$$u(x,t) = 1 + \frac{x}{L} + v(x,t)$$

where $v(x,t)$ satisfies;

$$v_t = Dv_{xx}$$
$$v = 0 \text{ at } x = 0, L$$

and

$$\begin{aligned} v(x,0) &= u(x,0) - 1 - \frac{x}{L} \\ &= \phi(x) - 1 - \frac{x}{L} \end{aligned}$$

which can all be solved by separation of variables.

26.2 Laplace's Equation + Poisson's Equation

We have studied (a lot) the diffusion, which comes from conservation of mass,

$$\underbrace{u_t + \nabla \cdot q = 0}_{\text{stuff is conserved}}$$

q = flow rate / flux = $-D\nabla u$ (Fick's law). Stuff moves around randomly.

In many examples, we find that u converges to some time-independent function:

$$u(\mathbf{x}, t) \to u(\mathbf{x})$$

u reaches some equilibrium distribution. We expect the equilibrium distribution to satisfy the diffusion equation:

$$\nabla \cdot (-D\nabla u) = 0$$

since $u_t = 0$ at equilibrium.

$$\iff -\nabla^2 u = 0$$

(some people write $-\Delta u = 0$) i.e. Laplace's equation. We may also be interested in equilibria that arise when there is a source/force;

$$u_t + \nabla \cdot q = s(\mathbf{x}, t)$$

where s is source strength. If we assume s is steady; $s \equiv s(\mathbf{x})$, if there is an equilibrium distribution, it satisfies;

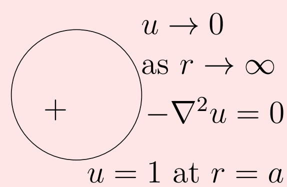
$$-D\nabla^2 u = s$$
$$\iff -\nabla^2 u = f(\mathbf{x})$$

where $f(\mathbf{x})$ is a function of $\mathbf{x}$ only, i.e. Poisson's equation.

We are typically interested in solving these equations in $\mathbb{R}^n$ where $n > 1$, ($n = 2$ or 3). We solve these equations initially in high symmetry situations.

Example 26.2

$n = 2$



Because the problem is axisymmetric, we expect $u \equiv u(r)$, where r is distance from center of ring.

(r, θ) are plane polar coordinates.

$$\nabla^2 \equiv \nabla \cdot (\nabla)$$

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \frac{\mathbf{e}_\theta}{r}\frac{\partial}{\partial \theta}$$

where $\mathbf{e}_r, \mathbf{e}_\theta$ are unit basis vectors. In Cartesians,

$$\mathbf{e}_r = (\cos\theta, \sin\theta)$$
$$\mathbf{e}_\theta = (-\sin\theta, \cos\theta)$$

Hence,

$$\begin{aligned} \nabla^2 u(r) &= \left(\mathbf{e}_r \frac{\partial}{\partial r} + \frac{\mathbf{e}_\theta}{r}\frac{\partial}{\partial \theta}\right) \cdot \left(\mathbf{e}_r \frac{du}{dr}\right) \\ &= \frac{d^2u}{dr^2 2} + \frac{1}{r}\frac{du}{dr} = \frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) \\ \frac{\partial \mathbf{e}_r}{\partial \theta} &= (-\sin\theta, \cos\theta) = \mathbf{e}_\theta \end{aligned}$$

So

$$\mathbf{e}_\theta \cdot \frac{\partial \mathbf{e}_r}{\partial \theta} = 1$$

Hence our Laplace equation becomes:

$$-\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = 0 \text{ for } r > a$$

with BCs

$$u = 1 \text{ when } r = a$$

$$u \to 0 \text{ when } r \to \infty$$

$$\implies \frac{d}{dr}\left(r\frac{du}{dr}\right) = 0$$

$$\implies r\frac{du}{dr} = A \implies \frac{du}{dr} = \frac{A}{r}$$

$$\implies u = A \log r$$

Try to apply the BCs;

$$u(a) = 1 \implies A\log a + B = 1$$

$$u(\infty) = 0 \implies A = 0, B = 0$$

We can't find A, B satisfying both BCs, "Stokes' paradox". Unbounded problems lead to divergence in 2D.

In general in plane polar coordinates:

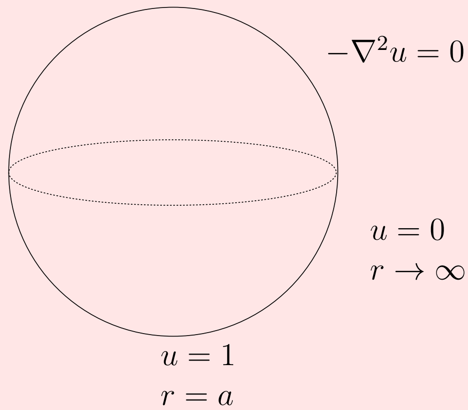
$$\nabla^2 u \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

if $u \equiv u(r,\theta)$. In spherical polar coordinates; (r, θ, φ)

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}$$

Example 26.3

Consider Laplace's equation outside of a sphere



$u \equiv u(r)$ (problem is spherically symmetric)

$$-\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{du}{dr}\right) = 0,$$

$u(a) = 1, u(\infty) = 0$

$$\implies \frac{d}{dr}\left(r^2\frac{du}{dr}\right) = 0$$

$$\implies r^2\frac{du}{dr} = A$$

integrate w.r.t r.

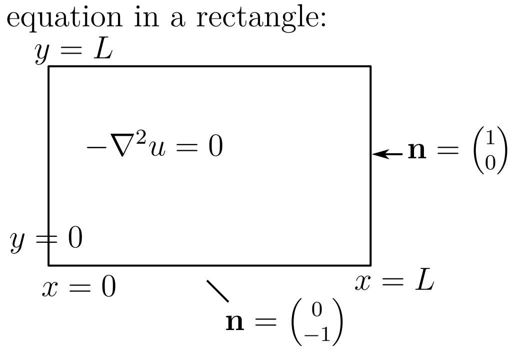
$$\implies \frac{du}{dr} = \frac{A}{r^2} \implies u = -\frac{A}{r} + B$$

apply BCs;

$$u = \frac{a}{r}$$

26.3 Separation of Variables in Laplace's Equation

Consider Laplace's equation in a rectangle:



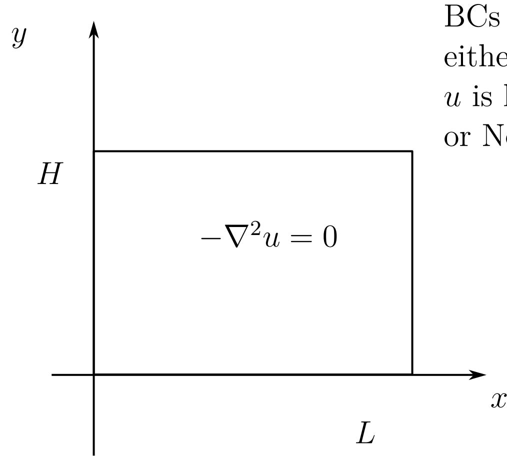
BCs may be Dirichlet; $u = h$, a known function

or Neumann; $-n \cdot \nabla u = h$

(Flux across a surface is $q \cdot \mathbf{n}$, $q = -(D)\nabla u$)

27 May 27, 2022

27.1 Separation of Variables in a Rectangle



BCs will be either Dirichlet; u is known. or Neumann

Neumann BCs are statements about flux;

$$\mathbf{q} = -D\nabla u$$

for diffusion equation. Flux through an element of boundary is

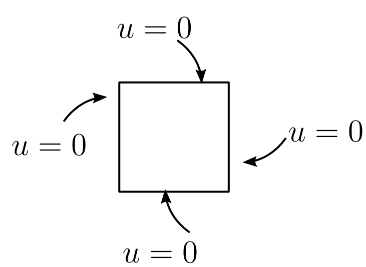
$$\mathbf{n} \cdot \mathbf{q} dS$$

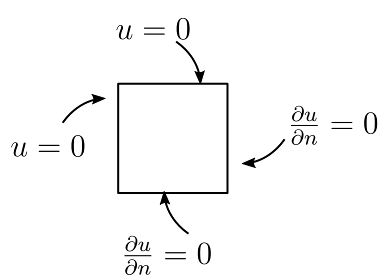
We constrain $\mathbf{n} \cdot \mathbf{q}$ on boundaries where $\mathbf{n}$ (by convention) is the unit outward normal, at a Neumann boundary; we know

$$\mathbf{n} \cdot \mathbf{q} = -D\mathbf{n} \cdot \nabla u$$

for Laplace's equation we specify $\mathbf{n} \cdot \nabla u \equiv \frac{\partial u}{\partial n}$.

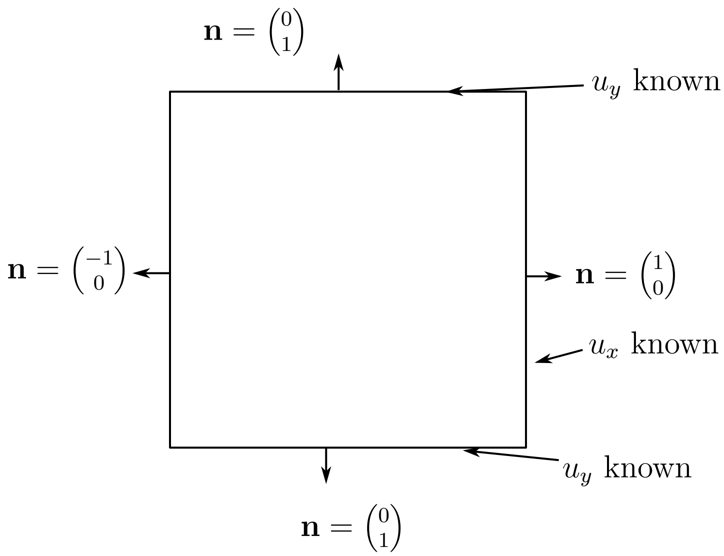
Note 27.1: If all BCs are homogeneous, then





Replacing some of the BCs by Neumann, we still get only trivial solutions.

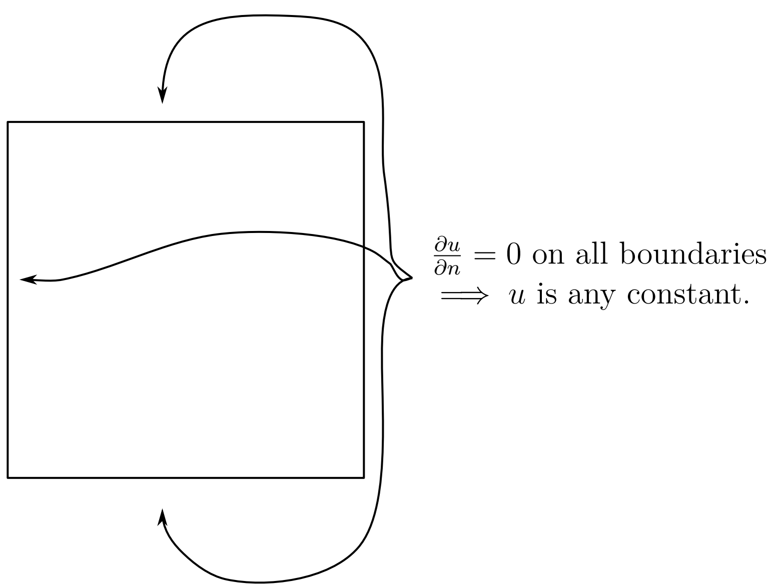
Let's be more explicit about what Neumann BCs represent.



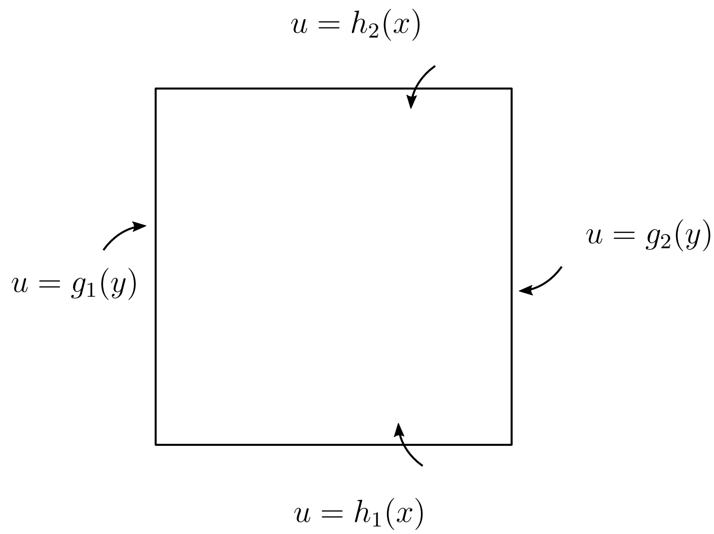
If $\mathbf{n} = \binom{0}{-1}$, then:

$$\begin{aligned} \frac{\partial u}{\partial n} &\equiv \mathbf{n} \cdot \nabla u \\ &= \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} u_x \\ u_y \end{pmatrix} \\ &= -u_y \end{aligned}$$

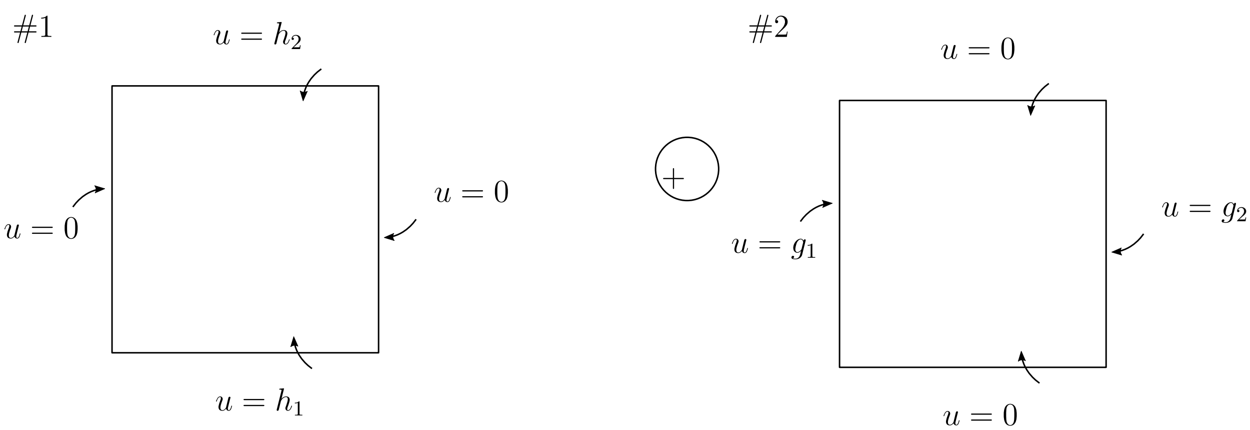
On a Neumann boundary either u_x or u_y is known.



Notice solutions are not unique here.
For interesting solutions, we need inhomogeneous BCs.



To solve this problem appeal to linearity:



Solve both problems by separation of variables. Look for solutions of the form:

$$u(x, y) = X(x)Y(y)$$

$$-X''Y - XY'' = 0$$

Divide by XY and separate x and y functions;

$$\implies -\frac{X''}{X} = \frac{Y''}{Y} = \lambda$$

We will impose homogeneous boundary conditions on each separable solution in problem #1,

$$X(0) = X(L) = 0$$

So $\lambda > 0$ for sines and cosines to be solutions. We get solutions

$$X_n = \sin \frac{n\pi x}{L}$$

eigenvalues λ are $\lambda_n = \frac{n^2\pi^2}{L^2}$ with

$$\frac{Y_n''}{Y_n} = \frac{n^2\pi^2}{L^2}$$

the corresponding functions Y.

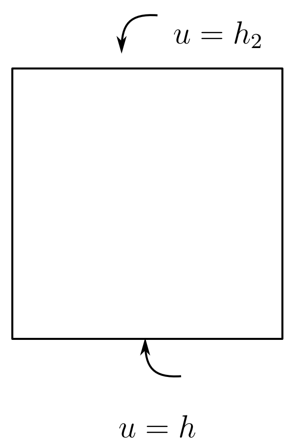
$$\implies Y_n = A_n \cosh \frac{n\pi y}{L} + B_n \sinh \frac{n\pi y}{L}$$

My separable solutions are:

$$u(x,y) = \sum_{n=1}^{\infty} \left(A_n \cosh \frac{n\pi y}{L} + B_n \sinh \frac{n\pi y}{L} \right) \sin \frac{n\pi x}{L}$$

where I must find coefficients $\{A_n\}$, $\{B_n\}$

- u satisfies the PDE ✓
- u satisfies the BCs at $x = 0, L$ ✓
- It does not satisfy the BCs at $y = 0, H$, yet. Choose A_n, B_n to satisfy the other BCs.



$$\sum_n A_n \sin \frac{n\pi x}{L} = h_1(x)$$

So

$$\begin{aligned} A_m &= \frac{(h_1(x), \sin m\pi x/L)}{(\sin m\pi x/L, \sin m\pi x/L)} \\ &= \frac{2}{L} \int_0^L h_1(x) \sin \frac{m\pi x}{L} dx \end{aligned}$$

$u(x, H) = h_2(x)$

$$\implies \sum_{n=1}^{\infty} \left(A_n \cosh \frac{n\pi H}{L} + B_n \sinh \frac{n\pi H}{L} \right) \sin \frac{n\pi x}{L} = h_2(x)$$

<u>Trick</u>:

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

has an analogue for cosh + sinh:

$$\cosh(A+B) = \frac{e^{A+B} + e^{-(A+B)}}{2}$$

we will use a formula to rewrite our 2 functions. $\cosh n\pi y/L, \sinh n\pi y/L$ so that one vanishes on each boundary.
Claim: any function of the form:

$$A_n \cosh \frac{n\pi y}{L} + B_n \sinh \frac{n\pi y}{L}$$

can be written as:

$$\underbrace{C_n \sinh \frac{n\pi(y-H)}{L}}_{=0 \text{ when } y=H} + D_n \underbrace{\sinh \frac{n\pi Y}{L}}_{=0 \text{ when } y=0}$$

because:

$$\begin{gathered}
\sin(A+B) = \frac{e^{A+B} - e^{-A-B}}{2} \\
\sinh A = \frac{e^A - e^{-A}}{2}, \quad \cosh A = \frac{e^A + e^{-A}}{2} \\
\sinh B = \frac{e^B - e^{-B}}{2}, \quad \cosh B = \frac{e^B + e^{-B}}{2} \\
\sinh A \cosh B = \frac{e^{A+B} + e^{A-B} - e^{-A+B} - e^{-A-B}}{4} \\
\sinh B \cosh A = \frac{e^{A+B} + e^{B-A} - e^{A-B} - e^{-A-B}}{4} \\
\sinh(A+B) = \sinh A \cosh B + \sinh B \cosh A
\end{gathered}$$

Hence,

$$\sinh \frac{n\pi(y-H)}{L} = \sinh \frac{n\pi y}{L} \cosh \frac{n\pi H}{L} - \cosh \frac{n\pi y}{L} \sinh \frac{n\pi H}{L}$$

So

$$\begin{aligned}
C_n \sinh \frac{n\pi(y-H)}{L} + D_n \sinh \frac{n\pi y}{L} = -C_n \sinh \frac{n\pi H}{L} \cosh \frac{n\pi y}{L} \\
+ \left(D_n + C_n \cosh \frac{n\pi H}{L}\right) \sinh \frac{n\pi y}{L}
\end{aligned}$$

Hence

$$u(x,y) = \sum_n \left(C_n \sinh \frac{n\pi(y-H)}{L} + D_n \sinh \frac{n\pi y}{L}\right) \times \sin \frac{n\pi x}{L}$$

$u(x,0) = h_1$

$$\implies C_n \sinh \frac{n\pi(-H)}{L} \sin \frac{n\pi x}{L} = h_1(x)$$

$$\implies C_n = -\frac{2}{L \sinh\left(\frac{n\pi H}{L}\right)} \int_0^L h_1(x) \sin \frac{n\pi x}{L} dx$$

and

$$D_n = \frac{2}{L \sinh\left(\frac{n\pi H}{L}\right)} \int_0^L h_2(x) \sin \frac{n\pi x}{L} dx$$

I can do the same thing for problem #2.

$$u(x, y) = \sum_{n=1}^{\infty} \left(A_n \sinh \frac{n\pi x}{H} + B_n \sinh \frac{n\pi(L - x)}{H} \right) \sin \frac{n\pi y}{H}$$

In practice in 136, we will not meet problems that have 4 on homogeneous BCs to satisfy.

28 June 1, 2022

28.1 Example of Solution By Separation of Variables

Example 28.1

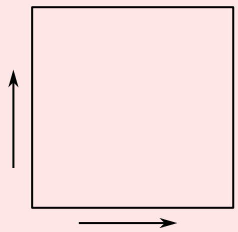
Solve $-\nabla^2 u = xy$ on

$$0 < x < 1$$

$$0 < y < 1$$

BCs

$$u = 0 \text{ on } x = 0, 1, y = 0, 1$$



Choose a function (particular integral) so that

$$u = v + w$$

where $-\nabla^2 v = xy$; and $-\nabla^2 w = 0$ and w fixes the BCs. Guess

$$v = f(x)y$$

$$-\nabla^2 v = -f''(x)y + 0 = xy$$

$$\implies f'' = -x$$

$$\implies f = -\frac{1}{6}x^3 + Ax + B$$

where A, B are constants.

$$v = \left(-\frac{1}{6}x^3 + Ax + B\right)y$$

$$v(0, y) = By$$

$$v(1, y) = \left(A + B - \frac{1}{6}\right)y$$

$$v(x, 0) = 0$$

$$v(x, 1) = -\frac{1}{6}x^3 + Ax + B$$

I can choose A, B so that 3 boundary conditions are satisfied; $B = 0, A = 1/6$,

$$v(x, y) = \frac{1}{6}(x - x^3)y$$

Then, solve for w:

$$-\nabla^2 w = 0$$

with BCs: $v + w = 0$ on all 4 boundaries,

$$\implies w = 0 \text{ on } x = 0, 1$$

$$w = 0 \text{ on } y = 0$$

$$w = \frac{1}{6}(x^3 - x) \text{ on } y = 1$$

Look for solutions of this form $w = XY$ where

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda, \text{ a constant}$$

Since $X = 0$ at $x = 0, 1$, we need sines or cosines for solutions, $\implies \lambda \leq 0$. Applying BCs

$$X(x) = \sin n\pi x, \quad \lambda = -n^2\pi^2$$

$$\implies \frac{Y''}{y} = n^2\pi^2$$

$$\implies Y(y) = A_n \sinh n\pi y + B_n \sinh n\pi(y-1)$$

Separable solutions are

$$w = \sum_{n=1}^{\infty} (A_n \sinh n\pi y + B_n \sinh n\pi(y-1)) \sin n\pi x$$

Apply BCs on y:

$$w(x, 0) = 0; \implies \sum_{n=1}^{\infty} (A_n(0) + B_n \sinh(n\pi(-1)))$$

$$\implies B_n = 0 \quad \forall n$$

$$w(x, 1) = \frac{1}{6}(x^3 - x)$$

$$\implies \sum_{n=1}^{\infty} A_n \sinh n\pi \sin n\pi x = \frac{1}{6}(x^3 - x)$$

$$\int_0^1 \sum_{n=1}^{\infty} A_n \sinh n\pi \sin n\pi x \sin m\pi x dx = \int_0^1 \frac{1}{6}(x^3 - x) \sin m\pi x dx$$

$$\frac{1}{2} A_m \sinh m\pi = \frac{(-1)^m}{m^3\pi^3}$$

$$\implies A_m = \frac{2(-1)^m}{(\sinh m\pi) m^3\pi^3}$$

$$\implies u(x, y) = -\frac{1}{6}(x^3 - x)y + \sum_{n=1}^{\infty} \frac{2(-1)^n}{\sinh n\pi n^3\pi^3} \sinh n\pi y$$

28.2 Some More Properties of Laplace's Equation

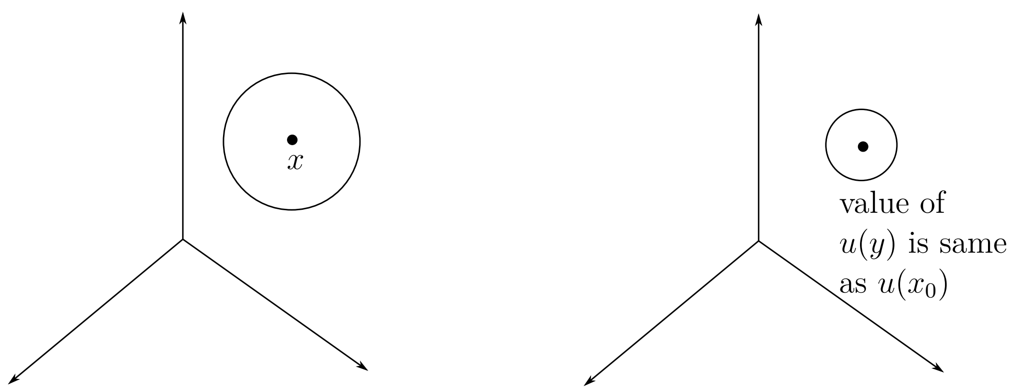
Definition 28.2 (Harmonic function)
We say a function is harmonic if it satisfies Laplace's equation.

Theorem 28.3 (Mean Value Property)
If u is harmonic, then it satisfies the mean value property, which means, if I make any ball $B(x_0, R)$, a ball of radius R, centered at x_0, that is contained in u's domain, then

$$u(x_0) = \frac{1}{4\pi R^2} \int_{\partial B(x_0,R)} u(y)dS$$

(True in any number of dimensions, but we will work in 3D for definiteness).

Proof. The result is true for any continuous function u, in the limit as $R \to 0$.



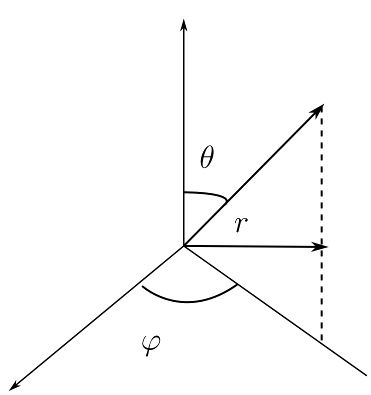
2) For a harmonic function, we will show the value of the integral is independent of R, let

$$I(R) = \frac{1}{4\pi R^2} \int_{\partial B(x_0,R)} u(y)dS$$

and I will show $I(R)$ is a constant, so

$$I(R) = \lim_{R\to 0} I(R) = u(x_0)$$

Use spherical polar coordinates to write out the integral.



$$I(R) = \frac{1}{4\pi R^2}\int_0^\pi \int_0^{2\pi} u(r,\theta,\varphi) r^2 \sin\theta d\varphi d\theta$$
$$= \frac{1}{4\pi}\int_0^\pi \int_0^{2\pi} u(R,\theta,\varphi)\sin\theta d\varphi d\theta$$

Want to show I is independent of R,

$$I'(R) = \frac{1}{4\pi}\int_0^\pi \int_0^{2\pi} \frac{\partial u}{\partial R}\sin\theta d\varphi d\theta$$
$$= \frac{1}{4\pi R^2}\int_{\partial B(x_0,R)} \frac{\partial u}{\partial R} dS$$
$$= \frac{1}{4\pi R^2}\int_{\partial B(x_0,R)} \mathbf{n}\cdot\nabla u dS$$

because on the sphere, $\mathbf{n}$ is in the radial direction, so $\mathbf{n}\cdot\nabla \equiv \frac{\partial}{\partial R}$

$$= \frac{1}{4\pi R^2}\int_{B(x_0,R)} \nabla^2 u dV = 0$$

as required.

□

Mean Value Property immediately gives us the maximum property theorem.

Theorem 28.4 (Max Theorem)
If u satisfies Laplace's equation on D then u takes its largest value on ∂D (the boundary of D).

Let's suppose u takes its max value at x_0, because of mean value theorem, if I put any sphere $B(x_0, R)$ around $x_0 \implies u$ is constant on $\partial B(x_0, R)$. It can't be larger than $u(x_0)$ on the ball, and average is $u(x_0) \implies u$ is constant in a neighborhood of x_0. Extend neighborhood until it reaches ∂D, then the value $u(x_0)$ must also be attained somewhere on ∂D.

Maximum principle can be used to prove uniqueness of solutions.