

Math 110B (Algebra)

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These are my lecture notes for Math 110B (Algebra), which is the second course in Algebra taught by Nicolle Gonzales. The textbook for this class is *Abstract Algebra: An Introduction, 3rd edition* by Hungerford.

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1.1 Groups

- Algebra \rightarrow study of mathematical structure.
- Rings \leftrightarrow “numbers” e.g. $\mathbb{R}, \mathbb{Z}, \mathbb{C}, \mathbb{Z}_p$
2 operations $(+, \cdot)$

Question 1.1: What happens if we have only 1 operation (either \cdot or $+$ but not both)?
What kind of structure is this more basic setup?

Answer: Groups! It turns out groups encode the mathematical structures of the symmetries in nature.

Definition 1.2 (Group)

A group $(G, *)$ is a nonempty set with a binary operation $*$: $G \times G \rightarrow G$ that satisfies

1. (Closure): $a * b \in G \quad \forall a, b \in G$
2. (Associativity): $(a * b) * c = a * (b * c) \quad \forall a, b, c \in G$
3. (Identity): $\exists e \in G$ such that $e * a = a = a * e \quad \forall a \in G$
4. (Inverse): $\forall a \in G, \exists d \in G$ such that $d * a = e = a * d$

Note:

- If $*$ is addition, we just divide $*$ by the usual $+$ sign. In this case

$$e = 0 \quad \text{and} \quad d = -a$$

- If the operation $*$ is multiplication, we just divide $*$ by the usual \cdot sign. In this case

$$e = 1 \quad \text{and} \quad d = a^{-1}$$

- Be aware that sometimes $*$ is neither.

Definition 1.3 (Abelian)

If the $*$ operation is commutative, i.e. $a * b = b * a$, then we say that G is abelian (named after the mathematician N.H. Abel)

Definition 1.4 (Order, Finite Group vs. Infinite Group)

The order of a group G , denoted $|G|$, is the number of elements it contains (as a set).
Thus, G is a finite group if $|G| < \infty$
and G is an infinite group if $|G| = \infty$

Examples 1.5 (Examples of a group)

1. Rings where you “forget” multiplication.
 $\rightarrow (\mathbb{Z}, +)$ integers with $*$ = $+$, $(\mathbb{R}[X], +)$, etc.
Note: $(\mathbb{Z}, *)$ with $*$ = \cdot is not a group. Why?

Theorem 1.6

Every ring is an abelian group under addition.

Proof. $e = 0$, inverse $= -a$ for each $a \in R$. □

Fact: If $R \neq 0$ then (R, \cdot) is never a group since 0 has no multiplicative inverse.

Examples 1.7 (More examples of a group)

2. Fields without zero.

Theorem 1.8

Let \mathbb{F}^* denote the nonzero elements of a field \mathbb{F} . Then (\mathbb{F}^*, \cdot) is an abelian group.

Recall: A unit in a ring R is an element $a \in R$ with a multiplicative inverse $a^{-1} \in R$ such that $aa^{-1} = 1 = a^{-1}a$.

Theorem 1.9

The set of units \mathcal{U} inside a ring R is a group under multiplication.

Examples 1.10 (More examples of a group cont.)

3. $\mathbb{Z}_n = \{m \mid (m, n) = 1\} \subseteq \mathbb{Z}_n$ is also a group, but under multiplication,

$n = 4$ $\mathbb{Z}_4 = \{0, 1, 2, 3\}$, $\mathcal{U}_4 = \{1, 3\}$

$(\mathbb{Z}_4, +)$ and (\mathcal{U}_4, \cdot) are groups with different binary operation!

$n = 6$ $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$, $\mathcal{U}_6 = \{1, 5\}$

(\mathcal{U}_6, \cdot) is a group

- $1 \cdot 5 = 5 \pmod{6} \in \mathcal{U}_6$ (closure)
- $1 = e$ (identity)
- $1 \cdot 1 = 1, \quad 5 \cdot 5 = 25 \equiv 1 \pmod{6}$ (inverse)

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