Math 120A (Differential Geometry)
University of California, Los Angeles

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These are my lecture notes for Math 120A (Differential Geometry), which is taught by Fumiaki Suzuki. The textbook for this class is *Differential Geometry of Curves and Surfaces*, by Kristopher Tapp. Many of the figures I include in these notes are taken from Tapp's book.

Contents

1 Jan 3, 2022

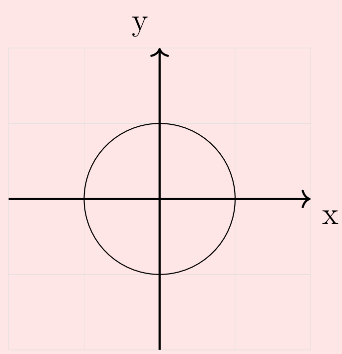
1.1 What is Differential Geometry?

Differential geometry studies geometry via analysis and linear algebra.

Geometry	Analysis	Linear Algebra
Intuitive	Rigorous	Computable
Curved	$\xrightarrow{\text{tangent space}}$	Linear
Global	Local	

1.2 Parametrized Curves

Example 1.1
A unit circle $S' = \{\vec{x} \text{ in } \mathbb{R}^2 \mid |\vec{x}| = 1\}$



$$\vec{\gamma} : [0, 2\pi) \to \mathbb{R}^2$$
$$t \mapsto (\cos t, \sin t)$$

$\vec{\gamma}[0, 2\pi) = S'$

Definition 1.2 (Parametrized curve and Trace)
A <u>(parametrized) curve</u> is a smooth function $\vec{\gamma} \colon I \to \mathbb{R}^n$, where I is an interval in $\mathbb{R}$. The image

$$\vec{\gamma}(I) = \{\vec{\gamma}(t) \mid t \in I\}$$

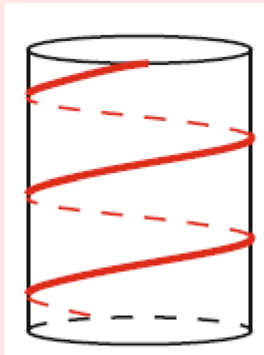
is called the <u>trace</u> of $\vec{\gamma}$.

Recall 1.3 An interval is a subset of $\mathbb{R}$ that has one of the following forms:

$$(a, b), [a, b], (a, b], [a, b), (-\infty, b), (-\infty, b], (a, \infty), [a, \infty), (-\infty, \infty) = \mathbb{R}.$$

A function $\vec{\gamma} \colon I \to \mathbb{R}^n$ is called <u>smooth</u> if $\vec{\gamma}$ is infinitely differentiable, or equivalently, each of the component functions $x_i \colon I \to \mathbb{R}$ is infinitely differentiable.

Example 1.4
$\vec{\gamma}(t) = (\cos t, \sin t, t)$, $t \in (-\infty, \infty)$ is a curve, called a helix.



Definition 1.5 (Derivative)
Let $\vec{\gamma}\colon I \to \mathbb{R}^n$ be a curve. The derivative of $\vec{\gamma}$ at t is defined as

$$\vec{\gamma}'(t) = \lim_{h \to 0} \frac{\vec{\gamma}(t+h) - \vec{\gamma}(t)}{h}$$

If t is on the boundaries of I, then use the left- or right-hand limit.

Remarks 1.6

i. If $\vec{\gamma}(t) = (x_1(t), x_2(t), \cdots, x_n(t))$, then $\vec{\gamma}'(t) = (x_1'(t), x_2'(t), \cdots, x_n'(t))$.

ii. The tangent line to the curve at $\vec{\gamma}'(t_0)$ is defined as

$$\vec{L}(t) = \vec{\gamma}(t_0) + t\vec{\gamma}'(t_0), \quad t \in (-\infty, \infty),$$

as soon as $\vec{\gamma}'(t) \neq \vec{0}$.

Definition 1.7 (Regular)
A curve $\vec{\gamma} : I \to \mathbb{R}^n$ is called regular if $\forall t \in I, \vec{\gamma}'(t) \neq \vec{0}$.

Remark 1.8 regular = the tangent line is defined everywhere = the trace is "smooth".

Example 1.9

$$\vec{\gamma}(t) = (t^2, t^3), \quad t \in (-\infty, \infty)$$

Then $\vec{\gamma}$ is a curve that is not regular.
Indeed, $\vec{\gamma}'(t) = (2t, 3t^2)$, so $\vec{\gamma}'(0) = \vec{0}$.
Notice, $x(t) = t^2, y(t) = t^3$, so $x(t) = y(t)^{2/3}$. Hence, the trace is given by $x = y^{2/3}$ in $\mathbb{R}^2$.

Remark 1.10 The analogy with the physics is useful. If $\vec{\gamma} : I \to \mathbb{R}^n$ is a curve, then $\vec{\gamma}(t)$ is the position of a moving particle at time t in $\mathbb{R}^2$.

- $\vec{\gamma}'(t)$ velocity

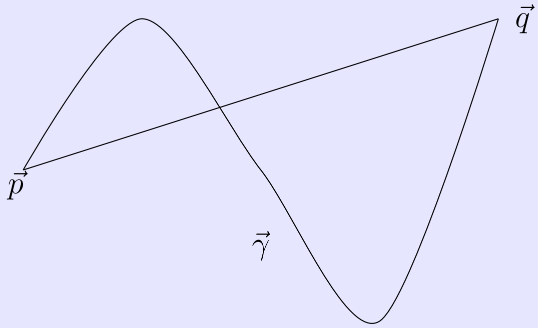
- $\vec{\gamma}''(t)$ acceleration
- $|\vec{\gamma}'(t)|$ speed

In this analogy, regular = the speed is always nonzero = the particle never stops (hence no "corners" on the trace)

Definition 1.11 (Arc length)
Let $\vec{\gamma}(t)\colon I \to \mathbb{R}^n$ be a regular curve. Then the arc length between times t_1, t_2 is defined as

$$\int_{t_1}^{t_2} |\vec{\gamma}'(t)|\, dt$$

Proposition 1.12
Let $\vec{\gamma}\colon [a,b] \to \mathbb{R}^n$ be a regular curve with the arc length L, $\vec{p} = \vec{\gamma}(a), \vec{q} = \vec{\gamma}(b)$. Then $L \geq |\vec{q} - \vec{p}|$.
Moreover, the equality holds if and only if $\vec{\gamma}$ parametrizes the line segment between $\vec{p}, \vec{q}$.



For the proof, we use the inner-product:
for $\vec{x} = (x_1, x_2, \cdots, x_n), \vec{y} = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$,

$$\langle x, y\rangle \coloneqq x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

Basic properties:

i. The inner product $\langle -, - \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is symmetric and bilinear.
ii. $\langle \vec{x}, \vec{y}\rangle = |\vec{x}||\vec{y}|\cos\theta$, where θ is the angle between $\vec{x}, \vec{y}$. ($\theta \in [0, 2\pi]$)
iii. $\langle \vec{x}, \vec{y}\rangle = 0 \Leftrightarrow \vec{x}, \vec{y}$ are orthogonal to each other.
iv. $\langle \vec{x}, \vec{x}\rangle = |\vec{x}|^2$
v. $\langle \vec{x}, \vec{y}\rangle \leq |\vec{x}||\vec{y}|$ (Schwartz Inequality) and the equality holds if and only if $\theta = 0$.

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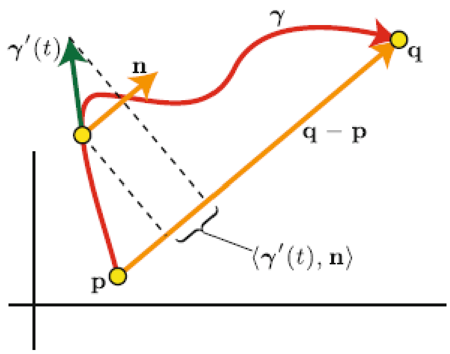
2.1 Proof of Proposition 1.12

Proof. Idea: Compare $\vec{\gamma}'(t)$ and its projection onto $\vec{q}-\vec{p}$. Set $\vec{n} = \frac{\vec{q}-\vec{p}}{|\vec{q}-\vec{p}|}$; $\vec{n}$ is unit.



Tapp Pg.15

Then $|\vec{\gamma}'(t)| \geq \langle \vec{\gamma}'(t), \vec{n} \rangle$ by Schwartz inequality.
Now,

$$\begin{aligned} L &= \int_a^b |\vec{\gamma}'(t)|\, dt \geq \int_a^b \langle \vec{\gamma}'(t), \vec{n} \rangle\, dt \\ &= [\langle \vec{\gamma}(t), \vec{n} \rangle]_a^b = \langle \vec{\gamma}(b), \vec{n} \rangle - \langle \vec{\gamma}(a), \vec{h} \rangle \\ &= \left\langle \vec{q}-\vec{p}, \frac{\vec{q}-\vec{p}}{|\vec{q}-\vec{p}|} \right\rangle = |\vec{q}-\vec{p}| \end{aligned}$$

If the equality holds, then $\forall t \in [a,b], \vec{\gamma}'(t), \vec{n}$ are in the same direction. So,

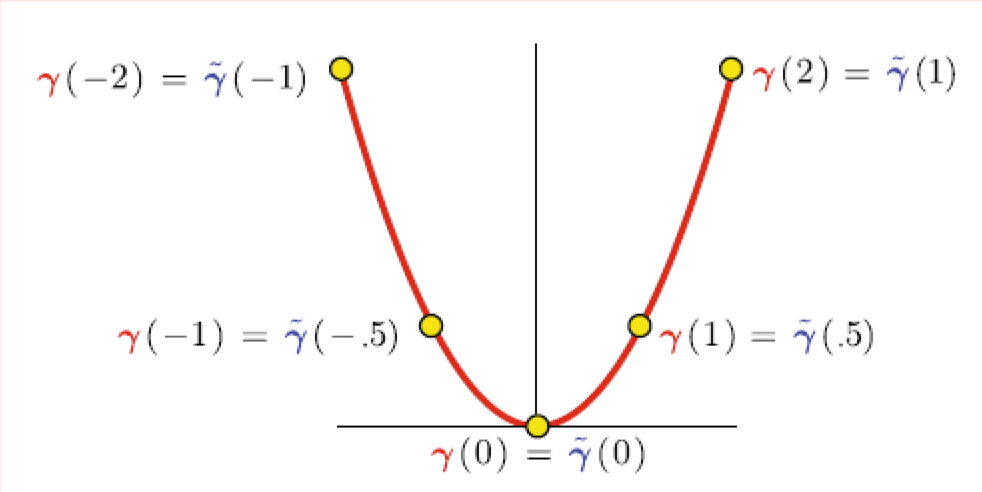
$$\begin{aligned} \vec{\gamma}'(t) &= \langle \vec{\gamma}'(t), \vec{n} \rangle \vec{n}. \\ \vec{\gamma}(t) &= \vec{\gamma}(a) + \int_a^t \vec{\gamma}'(u) du \\ &= \vec{p} + \left(\int_a^t \langle \vec{\gamma}'(u), \vec{n} \rangle dt \right) \vec{n} \end{aligned}$$

parametrizes the line segment between $\vec{p}, \vec{q}$. $\square$

2.2 Reparametrization

There are regular curves that share common properties. Which regular curves should we identify?

Example 2.1
$\vec{\gamma}(t) = (t, t^2), \quad t \in [-2, 2]$
$\tilde{\vec{\gamma}}(t) = (-2t, (-2t)^2), t \in [-1, 1]$.
Then $\vec{\gamma}[-2, 2] = \tilde{\vec{\gamma}}[-1, 1] =$



$\vec{\gamma}, \tilde{\vec{\gamma}}$ are the same, up to change in time:
Let $\phi \colon [-1, 1] \to [-2, 2], \quad t \mapsto -2t$.
Then $\tilde{\vec{\gamma}} = \vec{\gamma} \circ \phi$

Definition 2.2 (Reparametrization)
Let $\vec{\gamma} \colon I \to \mathbb{R}^n$ be a regular curve. A <u>reparametrization</u> of $\vec{\gamma}$ is a function of the form

$$\tilde{\vec{\gamma}} = \vec{\gamma} \circ \phi : \tilde{I} \to \mathbb{R}^n,$$

where $\tilde{I}$ is an interval, $\phi \colon \tilde{I} \to I$ is a smooth bijection such that $\forall t \in \tilde{I}$, $\phi'(t) \neq 0$

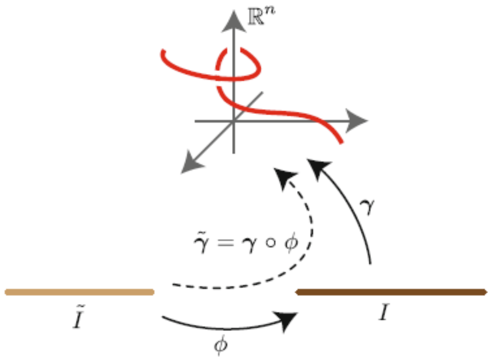


Figure 1: Kapp pg.19

Proposition 2.3
A reparametrization of a regular curve is a regular curve.

Proof. We use the same notations as the definition.
$\tilde{\vec{\gamma}} = \vec{\gamma} \circ \phi \colon \tilde{I} \to \mathbb{R}^n$ is the composition of smooth functions, so smooth.
Moreover, $\forall t \in \tilde{I}, \tilde{\vec{\gamma}}'(t) = \vec{\gamma}'(\phi(t)) \cdot \phi'(t) \neq 0$ □

We will be interested in regular curves up to reparametrizations.

Remarks 2.4

1. $\vec{\gamma}, \tilde{\vec{\gamma}}$ have the same trace.
2. There are regular curves with the same trace that cannot be reparametrized to each other. For instance,
$$\vec{\gamma}_1(t) = (\cos(t), \sin(t)), t \in [0, 2\pi),$$
$$\vec{\gamma}_2(t) = (\cos(t), \sin(t)), t \in [0, 4\pi),$$

Question 2.5: Is there a canonical reparametrization of a given regular curve?

Definition 2.6 (Unit-speed)
A regular curve $\vec{\gamma} \colon I \to \mathbb{R}^n$ is called unit-speed (or parametrized by arc length) if $\forall t \in I$, $|\vec{\gamma}'(t)| = 1$.

Remark 2.7 If $\vec{\gamma} \colon I \to \mathbb{R}^n$ is unit-speed, then,
$$\text{Arc length between } t_1, t_2 = \int_{t_1}^{t_2} |\vec{\gamma}'(t)| dt = \int_{t_1}^{t_2} dt = t_2 - t_1$$

Proposition 2.8
A regular curve always has a unit-speed reparametrization.

Proof. Let $\vec{\gamma} \colon I \to \mathbb{R}^n$ be a regular curve. Fix $t_0 \in I$. Define $s \colon I \to \mathbb{R}$ by $s(t) = \int_{t_0}^t \vec{\gamma}'(u) du$.
Let $\tilde{I} = s(I) \subset \mathbb{R}$. Then $\tilde{I}$ is an interval by IVT.
Since $s'(t) = |\vec{\gamma}'(t)| > 0$ by FTC, regularity, $s \colon I \to \tilde{I}$ is a smooth bijection. Then, $\phi = s^{-1} \colon \tilde{I} \to I$ is a smooth bijection,
$$\phi'(t) = \frac{1}{s'(\phi(t))} = \frac{1}{|\vec{\gamma}'(\phi(t))|} \neq 0.$$
Now $\tilde{\vec{\gamma}} = \vec{\gamma} \circ \phi \colon \tilde{I} \to \mathbb{R}^n$ is a reparametrization of $\vec{\gamma}$, that is unit-speed:
$$\begin{aligned} |\tilde{\vec{\gamma}}'(t)| &= |\vec{\gamma}'(\phi(t)) \cdot \phi'(t)| \\ &= |\vec{\gamma}'(\phi(t))| \cdot 1/|\vec{\gamma}'(\phi(t))| \\ &= 1 \end{aligned}$$
□

Note:
$$s^{-1} \cdot s(t) = t$$

$$(s^{-1})'(s(t)) \cdot s'(t) = 1$$
$$(s^{-1})'(s(t)) = 1/s'(t)$$