

# Math 120A (Differential Geometry)

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These are my lecture notes for Math 120A (Differential Geometry), which is taught by Fumiaki Suzuki. The textbook for this class is *Differential Geometry of Curves and Surfaces*, by Kristopher Tapp. Many of the figures I include in these notes are taken from Tapp's book.

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# 1 Jan 3, 2022

## 1.1 What is Differential Geometry?

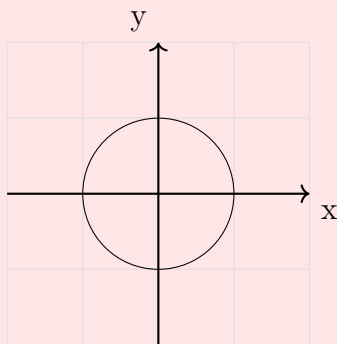
Differential geometry studies geometry via analysis and linear algebra.

| Geometry  | Analysis                             | Linear Algebra |
|-----------|--------------------------------------|----------------|
| Intuitive | Rigorous                             | Computable     |
| Curved    | $\xrightarrow{\text{tangent space}}$ | Linear         |
| Global    | Local                                |                |

## 1.2 Parametrized Curves

### Example 1.1

A unit circle  $S' = \{\vec{x} \text{ in } \mathbb{R}^2 \mid |\vec{x}| = 1\}$



$$\begin{aligned}\vec{\gamma} &: [0, 2\pi) \rightarrow \mathbb{R}^2 \\ t &\mapsto (\cos t, \sin t)\end{aligned}$$

$$\vec{\gamma}[0, 2\pi) = S'$$

### Definition 1.2 (Parametrized curve and Trace)

A (parametrized) curve is a smooth function  $\vec{\gamma}: I \rightarrow \mathbb{R}^n$ , where  $I$  is an interval in  $\mathbb{R}$ . The image

$$\vec{\gamma}(I) = \{\vec{\gamma}(t) \mid t \in I\}$$

is called the trace of  $\vec{\gamma}$ .

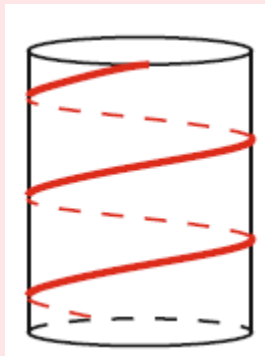
**Recall 1.3** An interval is a subset of  $\mathbb{R}$  that has one of the following forms:

$$(a, b), [a, b], (a, b], [a, b), (-\infty, b), (-\infty, b], (a, \infty), [a, \infty), (-\infty, \infty) = \mathbb{R}.$$

A function  $\vec{\gamma}: I \rightarrow \mathbb{R}^n$  is called smooth if  $\vec{\gamma}$  is infinitely differentiable, or equivalently, each of the component functions  $x_i: I \rightarrow \mathbb{R}$  is infinitely differentiable.

**Example 1.4**

$\vec{\gamma}(t) = (\cos t, \sin t, t)$ ,  $t \in (-\infty, \infty)$  is a curve, called a helix.

**Definition 1.5** (Derivative)

Let  $\vec{\gamma}: I \rightarrow \mathbb{R}^n$  be a curve. The derivative of  $\vec{\gamma}$  at  $t$  is defined as

$$\vec{\gamma}'(t) = \lim_{h \rightarrow 0} \frac{\vec{\gamma}(t+h) - \vec{\gamma}(t)}{h}$$

If  $t$  is on the boundaries of  $I$ , then use the left- or right-hand limit.

**Remarks 1.6**

- i. If  $\vec{\gamma}(t) = (x_1(t), x_2(t), \dots, x_n(t))$ , then  $\vec{\gamma}'(t) = (x_1'(t), x_2'(t), \dots, x_n'(t))$ .
- ii. The tangent line to the curve at  $\vec{\gamma}(t_0)$  is defined as

$$\vec{L}(t) = \vec{\gamma}(t_0) + t\vec{\gamma}'(t_0), \quad t \in (-\infty, \infty),$$

as soon as  $\vec{\gamma}'(t) \neq \vec{0}$ .

**Definition 1.7** (Regular)

A curve  $\vec{\gamma}: I \rightarrow \mathbb{R}^n$  is called regular if  $\forall t \in I, \vec{\gamma}'(t) \neq \vec{0}$ .

**Remark 1.8** regular = tangent line is defined everywhere = the trace is smooth

**Example 1.9**

$$\vec{\gamma}(t) = (t^2, t^3), \quad t \in (-\infty, \infty)$$

Then  $\vec{\gamma}$  is a curve that is not regular.

Indeed,  $\vec{\gamma}'(t) = (2t, 3t^2)$ , so  $\vec{\gamma}'(0) = \vec{0}$ .

Notice,  $x(t) = t^2, y(t) = t^3$ , so  $x(t) = y(t)^{2/3}$ . Hence, the trace is given by  $x = y^{2/3}$  in  $\mathbb{R}^2$ .

**Remark 1.10** The analogy with the physics is useful. If  $\vec{\gamma}: I \rightarrow \mathbb{R}^n$  is a curve, then  $\vec{\gamma}(t)$  is the position of a moving particle at time  $t$  in  $\mathbb{R}^2$ .

- $\vec{\gamma}'(t)$  velocity

- $\vec{\gamma}''(t)$  acceleration
- $|\vec{\gamma}'(t)|$  speed

In this analogy, regular = the speed is always nonzero = the particle never stops (hence no "corners" on the trace)

**Definition 1.11** (Arc length)

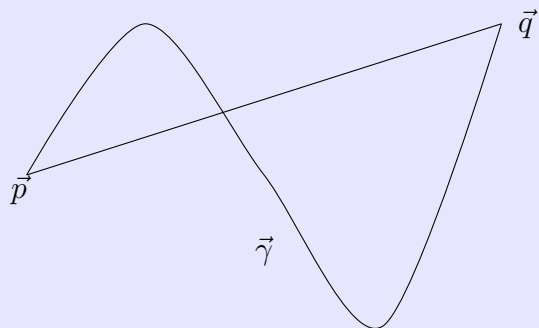
Let  $\vec{\gamma}(t): I \rightarrow \mathbb{R}^n$  be a regular curve. Then the arc length between times  $t_1, t_2$  is defined as

$$\int_{t_1}^{t_2} |\vec{\gamma}'(t)| dt$$

**Proposition 1.12**

Let  $\vec{\gamma}: [a, b] \rightarrow \mathbb{R}^n$  be a regular curve with the arc length  $L$ ,  $\vec{p} = \vec{\gamma}(a), \vec{q} = \vec{\gamma}(b)$ . Then  $L \geq |\vec{q} - \vec{p}|$ .

Moreover, the equality holds if and only if  $\vec{\gamma}$  parametrizes the line segment between  $\vec{p}, \vec{q}$ .



For the proof, we use the inner-product:

for  $\vec{x} = (x_1, x_2, \dots, x_n), \vec{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ ,

$$\langle x, y \rangle := x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

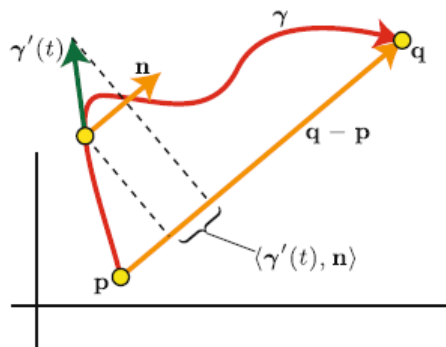
Basic properties:

- The inner product  $\langle -, - \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  is symmetric and bilinear.
- $\langle \vec{x}, \vec{y} \rangle = |\vec{x}| |\vec{y}| \cos \theta$ , where  $\theta$  is the angle between  $\vec{x}, \vec{y}$ . ( $\theta \in [0, 2\pi]$ )
- $\langle \vec{x}, \vec{y} \rangle = 0 \Leftrightarrow \vec{x}, \vec{y}$  are orthogonal to each other.
- $\langle \vec{x}, \vec{x} \rangle = |\vec{x}|^2$
- $\langle \vec{x}, \vec{y} \rangle \leq |\vec{x}| |\vec{y}|$  (Schwartz Inequality) and the equality holds if and only if  $\theta = 0$ .

## 2 Jan 5, 2022

### 2.1 Proof of Proposition 1.12

**Proof.** Idea: Compare  $\vec{\gamma}'(t)$  and its projection onto  $\vec{q} - \vec{p}$ . Set  $\vec{n} = \frac{\vec{q} - \vec{p}}{|\vec{q} - \vec{p}|}$ ;  $\vec{n}$  is unit.



*Tapp Pg.15*

Then  $|\vec{\gamma}'(t)| \geq \langle \vec{\gamma}'(t), \vec{n} \rangle$  by Schwartz inequality.

Now,

$$\begin{aligned} L &= \int_a^b |\vec{\gamma}'(t)| dt \geq \int_a^b \langle \vec{\gamma}'(t), \vec{n} \rangle dt \\ &= [\langle \vec{\gamma}(t), \vec{n} \rangle]_a^b = \langle \vec{\gamma}(b), \vec{n} \rangle - \langle \vec{\gamma}(a), \vec{n} \rangle \\ &= \left\langle \vec{q} - \vec{p}, \frac{\vec{q} - \vec{p}}{|\vec{q} - \vec{p}|} \right\rangle = |\vec{q} - \vec{p}| \end{aligned}$$

If the equality holds, then  $\forall t \in [a, b]$ ,  $\vec{\gamma}'(t), \vec{n}$  are in the same direction. So,

$$\begin{aligned} \vec{\gamma}'(t) &= \langle \vec{\gamma}'(t), \vec{n} \rangle \vec{n}. \\ \vec{\gamma}(t) &= \vec{\gamma}(a) + \int_a^t \vec{\gamma}'(u) du \\ &= \vec{p} + \left( \int_a^t \langle \vec{\gamma}'(u), \vec{n} \rangle dt \right) \vec{n} \end{aligned}$$

parametrizes the line segment between  $\vec{p}, \vec{q}$ . □

### 2.2 Reparametrization

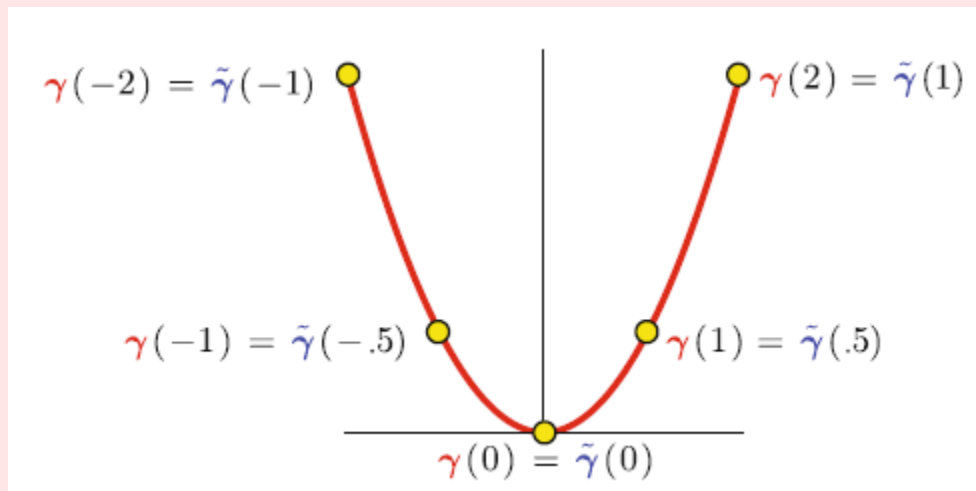
There are regular curves that share common properties. Which regular curves should we identify?

**Example 2.1**

$$\vec{\gamma}(t) = (t, t^2), \quad t \in [-2, 2]$$

$$\tilde{\gamma}(t) = (-2t, (-2t)^2), \quad t \in [-1, 1].$$

Then  $\vec{\gamma}[-2, 2] = \tilde{\gamma}[-1, 1] =$



$\vec{\gamma}, \tilde{\gamma}$  are the same, up to change in time:

Let  $\phi: [-1, 1] \rightarrow [-2, 2], \quad t \mapsto -2t.$

Then  $\tilde{\gamma} = \vec{\gamma} \circ \phi$

**Definition 2.2** (Reparametrization)

Let  $\vec{\gamma}: I \rightarrow \mathbb{R}^n$  be a regular curve. A reparametrization of  $\vec{\gamma}$  is a function of the form

$$\tilde{\gamma} = \vec{\gamma} \circ \phi: \tilde{I} \rightarrow \mathbb{R}^n,$$

where  $\tilde{I}$  is an interval,  $\phi: \tilde{I} \rightarrow I$  is a smooth bijection such that  $\forall t \in \tilde{I}, \phi'(t) \neq 0$

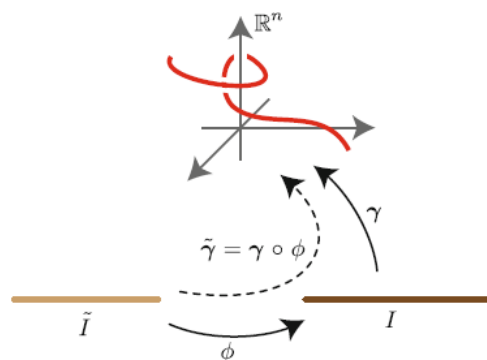


Figure 1: Kapp pg.19

**Proposition 2.3**

A reparametrization of a regular curve is a regular curve.

**Proof.** We use the same notations as the definition.

$\tilde{\gamma} = \gamma \circ \phi: \tilde{I} \rightarrow \mathbb{R}^n$  is the composition of smooth functions, so smooth.

Moreover,  $\forall t \in \tilde{I}, \tilde{\gamma}'(t) = \gamma'(\phi(t)) \cdot \phi'(t) \neq 0$  □

We will be interested in regular curves up to reparametrizations.

**Remarks 2.4**

1.  $\gamma, \tilde{\gamma}$  have the same trace.
2. There are regular curves with the same trace that cannot be reparametrized to each other. For instance,

$$\begin{aligned}\gamma_1(t) &= (\cos(t), \sin(t)), t \in [0, 2\pi), \\ \gamma_2(t) &= (\cos(t), \sin(t)), t \in [0, 4\pi),\end{aligned}$$

**Question 2.5:** Is there a canonical reparametrization of a given regular curve?

**Definition 2.6 (Unit-speed)**

A regular curve  $\gamma: I \rightarrow \mathbb{R}^n$  is called unit-speed (or parametrized by arc length) if  $\forall t \in I, |\gamma'(t)| = 1$ .

**Remark 2.7** If  $\gamma: I \rightarrow \mathbb{R}^n$  is unit-speed, then,

$$\text{Arc length between } t_1, t_2 = \int_{t_1}^{t_2} |\gamma'(t)| dt = \int_{t_1}^{t_2} dt = t_2 - t_1$$

**Proposition 2.8**

A regular curve always has a unit-speed reparametrization.

**Proof.** Let  $\gamma: I \rightarrow \mathbb{R}^n$  be a regular curve. Fix  $t_0 \in I$ . Define  $s: I \rightarrow \mathbb{R}$  by  $s(t) = \int_{t_0}^t \gamma'(u) du$ .

Let  $\tilde{I} = s(I) \subset \mathbb{R}$ . Then  $\tilde{I}$  is an interval by IVT.

Since  $s'(t) = |\gamma'(t)| > 0$  by FTC, regularity,  $s: I \rightarrow \tilde{I}$  is a smooth bijection. Then,  $\phi = s^{-1}: \tilde{I} \rightarrow I$  is a smooth bijection,

$$\phi'(t) = \frac{1}{s'(\phi(t))} = \frac{1}{|\gamma'(\phi(t))|} \neq 0.$$

Now  $\tilde{\gamma} = \gamma \circ \phi: \tilde{I} \rightarrow \mathbb{R}^n$  is a reparametrization of  $\gamma$ , that is unit-speed:

$$\begin{aligned}|\tilde{\gamma}'(t)| &= |\gamma'(\phi(t)) \cdot \phi'(t)| \\ &= |\gamma'(\phi(t))| \cdot 1/|\gamma'(\phi(t))| \\ &= 1\end{aligned}$$

□

Note:

$$s^{-1} \cdot s(t) = t$$

$$(s^{-1})'(s(t)) \cdot s'(t) = 1$$

$$(s^{-1})'(s(t)) = 1/s'(t)$$



## 3 Jan 7, 2022

### 3.1 Reparametrization (Cont'd)

#### Example 3.1

$\vec{\gamma}(t) = (\cos(t), \sin(t), t)$ ,  $t \in (-\infty, \infty)$  How can we find a unit-speed reparametrization of  $\vec{\gamma}$ ? Compute the arc length function  $S: (-\infty, \infty) \rightarrow \mathbb{R}$ :

$$\begin{aligned} s(t) &= \int_0^t |\vec{\gamma}'(u)| \, du = \int_0^t |(-\sin(u), \cos(u), 1)| \, du \\ &= \int_0^t \sqrt{2} \, du = \sqrt{2}t \end{aligned}$$

Set  $\phi = s^{-1}$ , then  $\phi(t) = t/\sqrt{2}$

$$\tilde{\gamma}(t) = \vec{\gamma}(\phi(t)) = (\cos(t/\sqrt{2}), \sin(t/\sqrt{2}), t/\sqrt{2})$$

$t \in (-\infty, \infty)$ , is a unit speed reparametrization of  $\vec{\gamma}$ .

We will be interested in invariants for a regular curve that are unchanged under any reparametrizations.

Examples include:

- trace
- arc-length
- curvature
- torsion

Non-examples include:

- position
- velocity
- speed
- acceleration

Sometimes we consider more specific reparametrization.

#### Proposition 3.2

If  $\tilde{\gamma} = \vec{\gamma} \cdot \phi: \tilde{I} \rightarrow \mathbb{R}^n$  is a reparametrization of a regular curve  $\vec{\gamma}: I \rightarrow \mathbb{R}^n$ , then one of the following holds:

- i.  $\forall t \in \tilde{I}, \phi'(t) > 0$  i.e.  $\phi$  is strictly increasing
- ii.  $\forall t \in \tilde{I}, \phi'(t) < 0$  i.e.  $\phi$  is strictly decreasing

**Proof.** Otherwise  $\exists t \in \tilde{I}, \phi'(t) = 0$  by IVT. This contradicts the assumption on  $\phi$ .  $\square$

#### Definition 3.3 (Orientation-preserving vs. orientation-reversing)

Under the setting of the proposition, we say  $\tilde{\gamma}$  is orientation-preserving if (i) occurs, or orientation-reversing if (ii) occurs.

**Example 3.4** (Orientation-preserving)

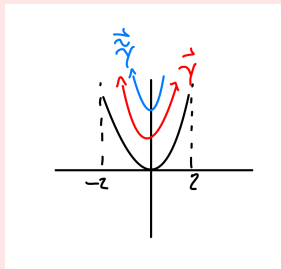
The arc length reparametrization of a regular curve  $\phi: I \rightarrow \tilde{I}$  is orientation-preserving, because  $\phi'(t) = 1/|\tilde{\gamma}'(\phi(t))| > 0 \quad \forall t \in I$

This shows an orientation=preserving unit-speed. Reparametrization always exists.

**Example 3.5** (Orientation-reversing)

$$\tilde{\gamma}(t) = (t, t^2), \quad t \in [-2, 2]$$

$$\tilde{\gamma}(t) = (-t, (-t)^2), \quad t \in [2, 2]$$



$\tilde{\gamma}$  is an orientation-reversing reparametrization of  $\tilde{\gamma}$  by  $\phi: [-2, 2] \rightarrow [-2, 2], \quad t \mapsto -t$  (Indeed,  $\phi' = -1 < 0$ ).

We will be interested in invariants that are unchanged under any orientation-preserving reparametrization.

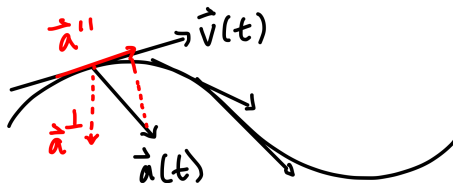
**Example 3.6**

signed curvature, rotation index

### 3.2 Curvature

The curvature measures how sharply the trace bends. What is a plausible definition of the curvature?

Let  $\tilde{\gamma}: I \rightarrow R^n$  be a regular curve. Set  $\vec{v} = \tilde{\gamma}', \vec{a} = \tilde{\gamma}''$



$\vec{v}$  knows speed, direction of the motion

$\implies \vec{a}$  should know the change in speed, direction  $\rightarrow$  curvature.

We write

$$\vec{a} = \vec{a}'' + \vec{a}^{\perp}$$

where

$$\begin{aligned}\vec{a}'' &= \left\langle \vec{a}, \frac{\vec{v}}{|\vec{v}|} \right\rangle : \text{parallel to } \vec{v} \\ \vec{a}^\perp &= \vec{a} - \vec{a}'' : \text{orthogonal to } \vec{v}\end{aligned}$$

**Proposition 3.7**

$\frac{d}{dt}|\vec{v}(t)| = \left\langle \vec{a}, \frac{\vec{v}}{|\vec{v}|} \right\rangle$  = the parallel component of  $\vec{a}$  with respect to  $\vec{v}$

**Proof.**

$$\begin{aligned}\frac{d}{dt}|\vec{v}(t)| &= \frac{d}{dt} \langle \vec{v}(t), \vec{v}(t) \rangle^{1/2} \\ &= \frac{1}{2} \frac{1}{\langle \vec{v}(t), \vec{v}(t) \rangle^{1/2}} \cdot 2 \langle \vec{v}(t), \vec{v}(t) \rangle' \\ &= \left\langle \frac{\vec{v}(t)}{|\vec{v}(t)|}, \vec{a}(t) \right\rangle\end{aligned}$$

Note:  $\langle v, v \rangle' = \langle v', v \rangle + \langle v, v' \rangle = 2\langle v', v \rangle$  □

So  $|\vec{a}^\perp(t)|$  would be a plausible definition of the curvature. however this depends on  $|\vec{t}|$ . (Imagine a centripetal force for a car turning a corner.)

**Definition 3.8 (Curvature)**

Let  $\tilde{\gamma}: I \rightarrow \mathbb{R}^n$  be a regular curve. The curvature function  $\kappa: I \rightarrow [0, \infty)$  is defined as

$$\kappa(t) = \frac{|\vec{a}^\perp(t)|}{|\vec{v}(t)|^2}$$

**Proposition 3.9**

Curvature is independent of parametrizations.

**Proof.** Let  $\gamma$  be a regular curve.  $\tilde{\gamma} = \gamma \cdot \phi$  is a reparametrization of  $\gamma$ .

Denote:

$\kappa$ : curvature function for  $\gamma$

$\tilde{\kappa}$ : curvature function for  $\tilde{\gamma}$

We need to show  $\tilde{\kappa} = \kappa \circ \phi$

Denote:

$v, a$ : velocity, acceleration of  $\gamma$

$\tilde{v}, \tilde{a}$ : velocity, acceleration of  $\tilde{\gamma}$ .

Then,

$$\begin{aligned}\tilde{\gamma} &= \gamma \cdot \phi \\ \tilde{v} &= \gamma' \cdot \phi \cdot \phi' = v \cdot \phi \cdot \phi' \\ \tilde{a} &= \gamma'' \circ \phi \cdot (\phi')^2 + \gamma' \circ \phi \cdot \phi'' \\ &= a \circ \phi \cdot (\phi')^2 + v \circ \phi \cdot \phi''\end{aligned}$$

So,  $\tilde{v}$  is parallel to  $v$ ,

$$\tilde{a}^\perp = a^\perp \circ \phi \cdot (\phi')^2$$

Therefore,

$$\begin{aligned}\tilde{\kappa} &= \frac{\tilde{a}^\perp}{|\tilde{v}|^2} = \frac{|a^\perp \circ \phi \cdot (\phi')^2|}{|v \circ \phi \cdot \phi|^2} = \frac{|a^\perp \cdot \phi|}{|v \cdot \phi|^2} \\ &= \kappa \circ \phi\end{aligned}$$

□