

# **Math 120A (Differential Geometry)**

## ***University of California, Los Angeles***

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These are my lecture notes for Math 120A (Differential Geometry), which is taught by Fumiaki Suzuki. The textbook for this class is *Differential Geometry of Curves and Surfaces*, by Kristopher Tapp. Many of the figures I include in these notes are taken from Tapp's book.

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# 1 Jan 3, 2022

## 1.1 What is Differential Geometry?

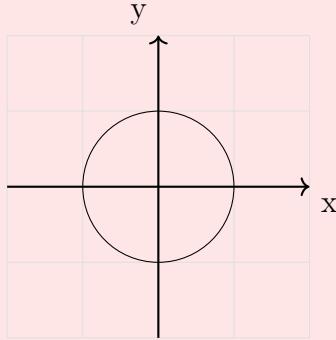
Differential geometry studies geometry via analysis and linear algebra.

Geometry	Analysis	Linear Algebra
Intuitive	Rigorous	Computable
Curved	<small>tangent space</small> $\xrightarrow{\quad}$	Linear
Global	Local	

## 1.2 Parametrized Curves

### Example 1.1

A unit circle  $S' = \{\vec{x} \text{ in } \mathbb{R}^2 \mid |\vec{x}| = 1\}$



$$\vec{\gamma}: [0, 2\pi) \rightarrow \mathbb{R}^2$$

$$t \mapsto (\cos t, \sin t)$$

$$\vec{\gamma}[0, 2\pi) = S'$$

### Definition 1.2 (Parametrized curve and Trace)

A parametrized curve is a smooth function  $\vec{\gamma}: I \rightarrow \mathbb{R}^n$ , where  $I$  is an interval in  $\mathbb{R}$ . The image

$$\vec{\gamma}(I) = \{\vec{\gamma}(t) \mid t \in I\}$$

is called the trace of  $\vec{\gamma}$ .

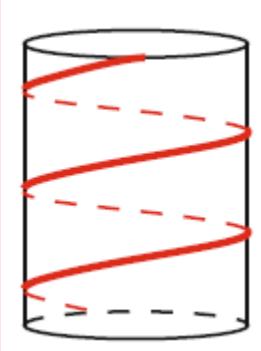
**Recall 1.3** An interval is a subset of  $\mathbb{R}$  that has one of the following forms:

$$(a, b), [a, b], (a, b], [a, b), (-\infty, b), (-\infty, b], (a, \infty), [a, \infty), (-\infty, \infty) = \mathbb{R}.$$

A function  $\vec{\gamma}: I \rightarrow \mathbb{R}^n$  is called smooth if  $\vec{\gamma}$  is infinitely differentiable, or equivalently, each of the component functions  $x_i: I \rightarrow \mathbb{R}$  is infinitely differentiable.

**Example 1.4**

$\vec{\gamma}(t) = (\cos t, \sin t, t)$ ,  $t \in (-\infty, \infty)$  is a curve, called a helix.

**Definition 1.5 (Derivative)**

Let  $\vec{\gamma}: I \rightarrow \mathbb{R}^n$  be a curve. The derivative of  $\vec{\gamma}$  at  $t$  is defined as

$$\vec{\gamma}'(t) = \lim_{h \rightarrow 0} \frac{\vec{\gamma}(t+h) - \vec{\gamma}(t)}{h}$$

If  $t$  is on the boundaries of  $I$ , then use the left- or right-hand limit.

**Remarks 1.6**

- i. If  $\vec{\gamma}(t) = (x_1(t), x_2(t), \dots, x_n(t))$ , then  $\vec{\gamma}'(t) = (x'_1(t), x'_2(t), \dots, x'_n(t))$ .
- ii. The tangent line to the curve at  $\vec{\gamma}'(t_0)$  is defined as

$$\vec{L}(t) = \vec{\gamma}(t_0) + t\vec{\gamma}'(t_0), \quad t \in (-\infty, \infty),$$

as soon as  $\vec{\gamma}'(t) \neq \vec{0}$ .

**Definition 1.7 (Regular)**

A curve  $\vec{\gamma}: I \rightarrow \mathbb{R}^n$  is called regular if  $\forall t \in I, \vec{\gamma}'(t) \neq \vec{0}$ .

**Remark 1.8** regular = tangent line is defined everywhere = the trace is smooth

**Example 1.9**

$$\vec{\gamma}(t) = (t^2, t^3), \quad t \in (-\infty, \infty)$$

Then  $\vec{\gamma}$  is a curve that is not regular.

Indeed,  $\vec{\gamma}'(t) = (2t, 3t^2)$ , so  $\vec{\gamma}'(0) = \vec{0}$ .

Notice,  $x(t) = t^2, y(t) = t^3$ , so  $x(t) = y(t)^{2/3}$ . Hence, the trace is given by  $x = y^{2/3}$  in  $\mathbb{R}^2$ .

**Remark 1.10** The analogy with the physics is useful. If  $\vec{\gamma}: I \rightarrow \mathbb{R}^n$  is a curve, then  $\vec{\gamma}(t)$  is the position of a moving particle at time  $t$  in  $\mathbb{R}^2$ .

- $\vec{\gamma}'(t)$  velocity
- $\vec{\gamma}''(t)$  acceleration
- $|\vec{\gamma}'(t)|$  speed

In this analogy, regular = the speed is always nonzero = the particle never stops (hence no "corners" on the trace)

### Definition 1.11 (Arc length)

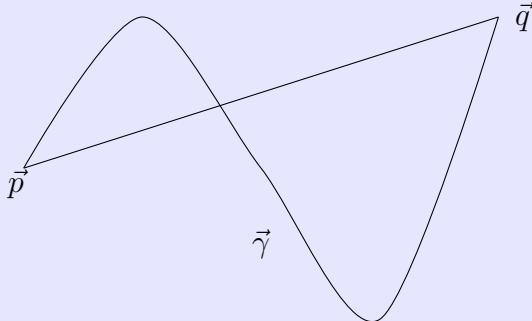
Let  $\vec{\gamma}(t): I \rightarrow \mathbb{R}^n$  be a regular curve. Then the arc length between times  $t_1, t_2$  is defined as

$$\int_{t_1}^{t_2} |\vec{\gamma}'(t)| dt$$

### Proposition 1.12

Let  $\vec{\gamma}: [a, b] \rightarrow \mathbb{R}^n$  be a regular curve with the arc length  $L$ ,  $\vec{p} = \vec{\gamma}(a), \vec{q} = \vec{\gamma}(b)$ . Then  $L \geq |\vec{q} - \vec{p}|$ .

Moreover, the equality holds if and only if  $\vec{\gamma}$  parametrizes the line segment between  $\vec{p}, \vec{q}$ .



For the proof, we use the inner-product:

for  $\vec{x} = (x_1, x_2, \dots, x_n), \vec{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ ,

$$\langle \vec{x}, \vec{y} \rangle := x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

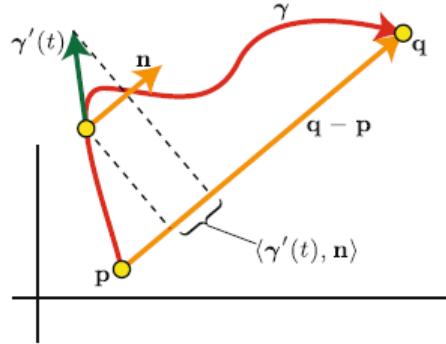
Basic properties:

- i. The inner product  $\langle -, - \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  is symmetric and bilinear.
- ii.  $\langle \vec{x}, \vec{y} \rangle = |\vec{x}| |\vec{y}| \cos \theta$ , where  $\theta$  is the angle between  $\vec{x}, \vec{y}$ . ( $\theta \in [0, 2\pi]$ )
- iii.  $\langle \vec{x}, \vec{y} \rangle = 0 \iff \vec{x}, \vec{y}$  are orthogonal to each other.
- iv.  $\langle \vec{x}, \vec{x} \rangle = |\vec{x}|^2$
- v.  $\langle \vec{x}, \vec{y} \rangle \leq |\vec{x}| |\vec{y}|$  (Schwartz Inequality) and the equality holds if and only if  $\theta = 0$ .

## 2 Jan 5, 2022

We start with the proof of Proposition 1.12.

**Proof.** Idea: Compare  $\vec{\gamma}'(t)$  and its projection onto  $\vec{q} - \vec{p}$ . Set  $\vec{n} = \frac{\vec{q} - \vec{p}}{|\vec{q} - \vec{p}|}$ ;  $\vec{n}$  is unit.



Tapp Pg.15

Then  $|\vec{\gamma}'(t)| \geq \langle \vec{\gamma}'(t), \vec{n} \rangle$  by Schwartz inequality.

Now,

$$\begin{aligned} L &= \int_a^b |\vec{\gamma}'(t)| dt \geq \int_a^b \langle \vec{\gamma}'(t), \vec{n} \rangle dt \\ &= [\langle \gamma(t), \vec{n} \rangle]_a^b = \langle \gamma(b), \vec{n} \rangle - \langle \gamma(a), \vec{n} \rangle \\ &= \left\langle \vec{q} - \vec{p}, \frac{\vec{q} - \vec{p}}{|\vec{q} - \vec{p}|} \right\rangle = |\vec{q} - \vec{p}| \end{aligned}$$

If the equality holds, then  $\forall t \in [a, b], \gamma'(t), \vec{n}$  are in the same direction. So,

$$\begin{aligned} \gamma'(t) &= \langle \gamma'(t), \vec{n} \rangle \vec{n}. \\ \gamma(t) &= \gamma(a) + \int_a^t \gamma'(u) du \\ &= \vec{p} + \left( \int_a^t \langle \gamma'(u), \vec{n} \rangle dt \right) \vec{n} \end{aligned}$$

parametrizes the line segment between  $\vec{p}, \vec{q}$ . □

### 2.1 Reparametrization

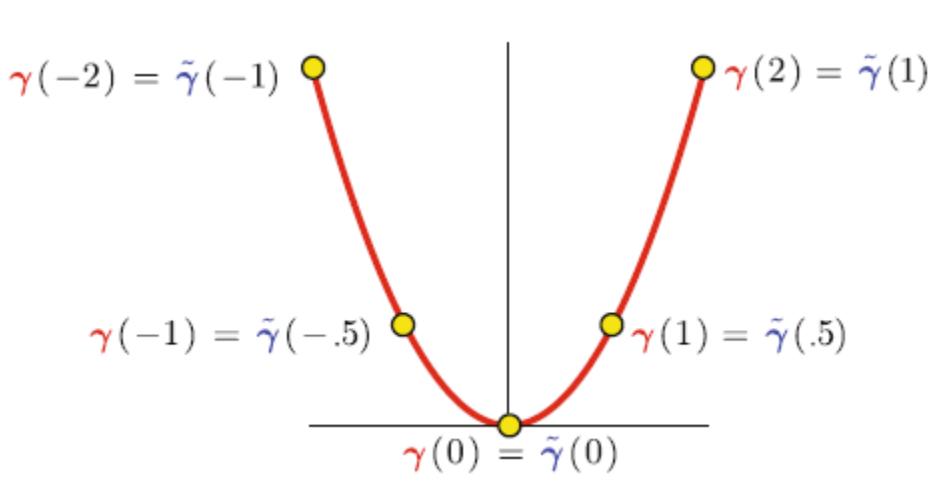
There are regular curves that share common properties. Which regular curves should we identify?

**Example 2.1**

$$\gamma(t) = (t, t^2), \quad t \in [-2, 2]$$

$$\tilde{\gamma}(t) = (-2t, (-2t)^2), t \in [-1, 1].$$

Then  $\gamma[-2, 2] = \tilde{\gamma}[-1, 1] =$



$\gamma, \tilde{\gamma}$  are the same, up to change in time:

Let  $\phi: [-1, 1] \rightarrow [-2, 2], \quad t \mapsto -2t.$

Then  $\tilde{\gamma} = \gamma \circ \phi$

**Definition 2.2 (Reparametrization)**

Let  $\gamma: I \rightarrow \mathbb{R}^n$  be a regular curve. A reparametrization of  $\gamma$  is a function of the form

$$\tilde{\gamma} = \gamma \circ \phi: \tilde{I} \rightarrow \mathbb{R}^n,$$

where  $\tilde{I}$  is an interval,  $\phi: \tilde{I} \rightarrow I$  is a smooth bijection such that  $\forall t \in \tilde{I}, \phi'(t) \neq 0$

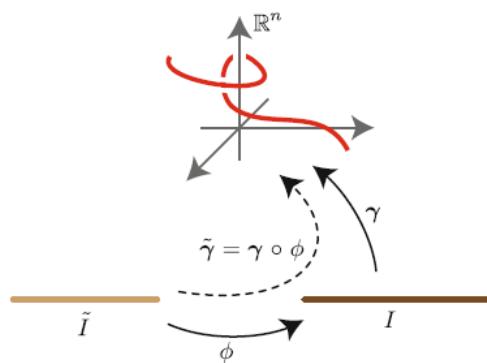


Figure 1: Kapp pg.19

**Proposition 2.3**

A reparametrization of a regular curve is a regular curve.

**Proof.** We use the same notations as the definition.

$\tilde{\gamma} = \gamma \circ \phi: \tilde{I} \rightarrow \mathbb{R}^n$  is the composition of smooth functions, so smooth.

Moreover,  $\forall t \in \tilde{I}, \tilde{\gamma}'(t) = \gamma'(\phi(t)) \cdot \phi'(t) \neq 0$

□

We will be interested in regular curves up to reparametrizations.

**Remarks 2.4**

1.  $\gamma, \tilde{\gamma}$  have the same trace.
2. There are regular curves with the same trace that cannot be reparametrized to each other. For instance,

$$\gamma_1(t) = (\cos(t), \sin(t)), t \in [0, 2\pi),$$

$$\gamma_2(t) = (\cos(t), \sin(t)), t \in [0, 4\pi),$$

**Question 2.5:** Is there a canonical reparametrization of a given regular curve?

**Definition 2.6 (Unit-speed)**

A regular curve  $\gamma: I \rightarrow \mathbb{R}^n$  is called unit-speed (or parametrized by arc length) if  $\forall t \in I, |\gamma'(t)| = 1$ .

**Remark 2.7** If  $\gamma: I \rightarrow \mathbb{R}^n$  is unit-speed, then,

$$\text{Arc length between } t_1, t_2 = \int_{t_1}^{t_2} |\gamma'(t)| dt = \int_{t_1}^{t_2} dt = t_2 - t_1$$

**Proposition 2.8**

A regular curve always has a unit-speed reparametrization.

**Proof.** Let  $\gamma: I \rightarrow \mathbb{R}^n$  be a regular curve. Fix  $t_0 \in I$ . Define  $s: I \rightarrow \mathbb{R}$  by

$$s(t) = \int_{t_0}^t |\gamma'(u)| du.$$

Let  $\tilde{I} = s(I) \subset \mathbb{R}$ . Then  $\tilde{I}$  is an interval by IVT.

Since  $s'(t) = |\gamma'(t)| > 0$  by FTC, regularity,  $s: I \rightarrow \tilde{I}$  is a smooth bijection. Then,  $\phi = s^{-1}: \tilde{I} \rightarrow I$  is a smooth bijection,

$$\phi'(t) = \frac{1}{s'(\phi(t))} = \frac{1}{|\gamma'(\phi(t))|} \neq 0.$$

Now  $\tilde{\gamma} = \gamma \circ \phi: \tilde{I} \rightarrow \mathbb{R}^n$  is a reparametrization of  $\gamma$ , that is unit-speed:

$$\begin{aligned} |\tilde{\gamma}'(t)| &= |\gamma'(\phi(t)) \cdot \phi'(t)| \\ &= |\gamma'(\phi(t))| \cdot 1/|\gamma'(\phi(t))| \\ &= 1 \end{aligned}$$

□

Note:

$$\begin{aligned}s^{-1} \cdot s(t) &= t \\(s^{-1})'(s(t)) \cdot s'(t) &= 1 \\(s^{-1})'(s(t)) &= 1/s'(t)\end{aligned}$$

# 3 Jan 7, 2022

## 3.1 Reparametrization (Cont'd)

### Example 3.1

$\gamma(t) = (\cos(t), \sin(t), t)$ ,  $t \in (-\infty, \infty)$  How can we find a unit-speed reparametrization of  $\gamma$ ? Compute the arc length function  $S: (-\infty, \infty) \rightarrow \mathbb{R}$ :

$$\begin{aligned} s(t) &= \int_0^t |\gamma'(u)| du = \int_0^t |(-\sin(u), \cos(u), 1)| du \\ &= \int_0^t \sqrt{2} du = \sqrt{2}t \end{aligned}$$

Set  $\phi = s^{-1}$ , then  $\phi(t) = t/\sqrt{2}$

$$\tilde{\gamma}(t) = \gamma(t) \circ \phi(t) = (\cos(t/\sqrt{2}), \sin(t/\sqrt{2}), t/\sqrt{2})$$

$t \in (-\infty, \infty)$ , is a unit speed reparametrization of  $\gamma$ .

We will be interested in invariants for a regular curve that are unchanged under any reparametrizations.

Examples include:

- trace
- arc-length
- curvature
- torsion

Non-examples include:

- position
- velocity
- speed
- acceleration

Sometimes we consider more specific reparametrization.

### Proposition 3.2

If  $\tilde{\gamma} = \gamma \cdot \phi: \tilde{I} \rightarrow \mathbb{R}^n$  is a reparametrization of a regular curve  $\gamma: I \rightarrow \mathbb{R}^n$ , then one of the following holds:

- i.  $\forall t \in \tilde{I}, \phi'(t) > 0$  i.e.  $\phi$  is strictly increasing
- ii.  $\forall t \in \tilde{I}, \phi'(t) < 0$  i.e.  $\phi$  is strictly decreasing

| **Proof.** Otherwise  $\exists t \in \tilde{I}, \phi'(t) = 0$  by IVT. This contradicts the assumption on  $\phi$ .  $\square$

**Definition 3.3** (Orientation-preserving vs. orientation-reversing)

Under the setting of the proposition, we say  $\tilde{\gamma}$  is orientation-preserving if (i) occurs, or orientation-reversing if (ii) occurs.

**Example 3.4** (Orientation-preserving)

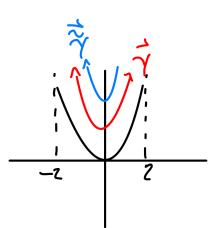
The arc length reparametrization of a regular curve  $\phi: I \rightarrow \tilde{I}$  is orientation-preserving, because  $\phi'(t) = 1/|\gamma'(\phi(t))| > 0 \quad \forall t \in I$

This shows an orientation-preserving unit-speed. Reparametrization always exists.

**Example 3.5** (Orientation-reversing)

$$\gamma(t) = (t, t^2), \quad t \in [-2, 2]$$

$$\tilde{\gamma}(t) = (-t, (-t)^2), \quad t \in [-2, 2]$$



$\tilde{\gamma}$  is an orientation-reversing reparametrization of  $\gamma$  by  $\phi: [-2, 2] \rightarrow [-2, 2], \quad t \mapsto -t$  (Indeed,  $\phi' = -1 < 0$ ).

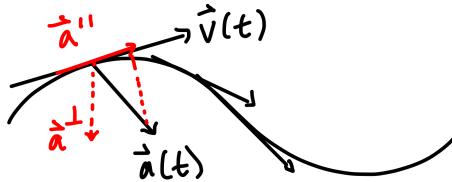
We will be interested in invariants that are unchanged under any orientation-preserving reparametrization.

- Signed curvature
- Rotation index

## 3.2 Curvature

The curvature measures how sharply the trace bends. What is a plausible definition of the curvature?

Let  $\gamma: I \rightarrow \mathbb{R}^n$  be a regular curve. Set  $\vec{v} = \gamma', \vec{a} = \gamma''$



$\vec{v}$  knows speed, direction of the motion

$\Rightarrow \vec{a}$  should know the change in speed, direction  $\rightarrow$  curvature.

We write

$$\vec{a} = \vec{a}^{\parallel} + \vec{a}^{\perp}$$

where

$$\begin{aligned}\vec{a}^{\parallel} &= \left\langle \vec{a}, \frac{\vec{v}}{|\vec{v}|} \right\rangle \frac{\vec{v}}{|\vec{v}|}: \text{ parallel to } \vec{v} \\ \vec{a}^{\perp} &= \vec{a} - \vec{a}^{\parallel}: \text{ orthogonal to } \vec{v}\end{aligned}$$

### Proposition 3.6

$$\begin{aligned}\frac{d}{dt} |\vec{v}(t)| &= \left\langle \vec{a}, \frac{\vec{v}}{|\vec{v}|} \right\rangle \\ &= \text{the parallel component of } \vec{a} \text{ with respect to } \vec{v}\end{aligned}$$

#### Proof.

$$\begin{aligned}\frac{d}{dt} |\vec{v}(t)| &= \frac{d}{dt} \langle \vec{v}(t), \vec{v}(t) \rangle^{1/2} \\ &= \frac{1}{2} \frac{1}{\langle \vec{v}(t), \vec{v}(t) \rangle^{1/2}} \cdot 2 \langle \vec{v}(t), \vec{v}'(t) \rangle \\ &= \left\langle \frac{\vec{v}(t)}{|\vec{v}(t)|}, \vec{a}(t) \right\rangle\end{aligned}$$

Note:  $\langle v, v' \rangle' = \langle v', v \rangle + \langle v, v' \rangle = 2\langle v', v \rangle$

□

So  $|\vec{a}^{\perp}(t)|$  would be a plausible definition of the curvature. However this depends on  $|t|$ . (Imagine a centripetal force for a car turning a corner.)

### Definition 3.7 (Curvature)

Let  $\gamma: I \rightarrow \mathbb{R}^n$  be a regular curve. The curvature function  $\kappa: I \rightarrow [0, \infty)$  is defined as

$$\kappa(t) = \frac{|\vec{a}^{\perp}(t)|}{|\vec{v}(t)|^2}$$

### Proposition 3.8

Curvature is independent of parametrizations.

**Proof.** Let  $\gamma$  be a regular curve.  $\tilde{\gamma} = \gamma \circ \phi$  is a reparametrization of  $\gamma$ .

Denote:

$\kappa$ : curvature function for  $\gamma$

$\tilde{\kappa}$ : curvature function for  $\tilde{\gamma}$

We need to show  $\tilde{\kappa} = \kappa \circ \phi$

Denote:

$v, a$ : velocity, acceleration of  $\gamma$

$\tilde{v}, \tilde{a}$ : velocity, acceleration of  $\tilde{\gamma}$ .

Then,

$$\begin{aligned}\tilde{\gamma} &= \gamma \circ \phi \\ \tilde{v} &= \gamma' \circ \phi \cdot \phi' = v \circ \phi \cdot \phi' \\ \tilde{a} &= \gamma'' \circ \phi \cdot (\phi')^2 + \gamma' \circ \phi \cdot \phi' \\ &= a \circ \phi \cdot (\phi')^2 + v \circ \phi \cdot \phi'\end{aligned}$$

So,  $\tilde{v}$  is parallel to  $v$ ,

$$\tilde{a}^\perp = a^\perp \circ \phi \cdot (\phi')^2$$

Therefore,

$$\begin{aligned}\tilde{\kappa} &= \frac{\tilde{a}^\perp}{|\tilde{v}|^2} = \frac{|a^\perp \circ \phi \cdot (\phi')^2|}{|v \circ \phi \cdot \phi'|^2} = \frac{|a^\perp \circ \phi|}{|v \circ \phi|^2} \\ &= \kappa \circ \phi\end{aligned}$$

□

# 4 Jan 10, 2022

Note: From now on, I will bold my vectors like this  $\mathbf{n}$  instead of  $\vec{n}$ .

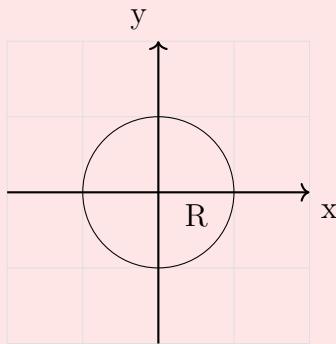
## 4.1 Curvature (Cont'd)

### Recall 4.1

$$\kappa(t) = \frac{|\mathbf{a}^\perp(t)|}{|\mathbf{v}(t)|^2}$$

### Example 4.2

$$\gamma(t) = (R \cos(t), R \sin(t)), \quad t \in (-\infty, \infty)$$



$$\mathbf{v}(t) = (-R \sin(t), R \cos(t))$$

$$\mathbf{a}(t) = (-R \cos(t), -R \sin(t))$$

Here,

$$\langle \mathbf{v}(t), \mathbf{a}(t) \rangle = -R^2 \sin(t) \cos(t) + R^2 \cos(t) \sin(t) = 0;$$

So,

$$\mathbf{v}(t) \perp \mathbf{a}(t) \implies \mathbf{a}(t) = \mathbf{a}^\perp(t).$$

Therefore,

$$\kappa(t) = \frac{|\mathbf{a}(t)|}{|\mathbf{v}(t)|^2} = \frac{R}{R^2} = \frac{1}{R} \xrightarrow{R \rightarrow +\infty} 0 \text{ (flat)}$$

Historically, the curvature of a regular curve was first defined by  $\kappa(t) = \frac{1}{R(t)}$ , where  $R(t)$  is the radius of the circle that best approximates the trace at  $t$  (The osculating circle; Read Tapp). Here we give another interpretation of the curvature using the osculating parabola.

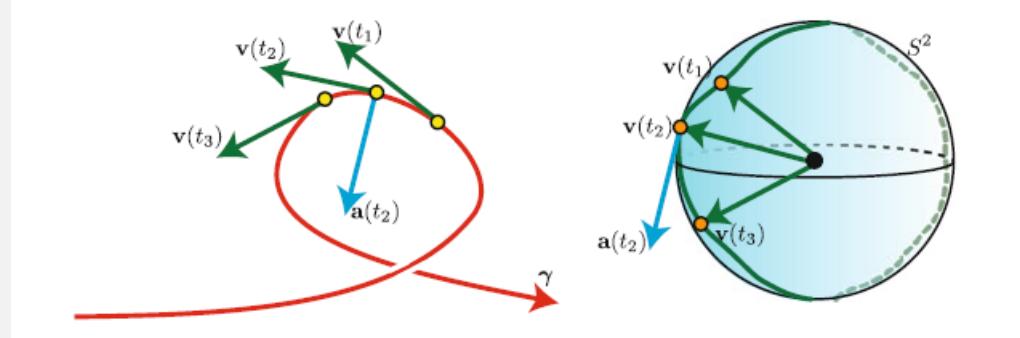
### Definition 4.3 (Unit tangent and normal vectors)

Let  $\gamma: I \rightarrow \mathbb{R}^n$  be a regular curve. Define the unit tangent and normal vectors as

$$\mathbf{t}(t_0) = \frac{\mathbf{v}(t_0)}{|\mathbf{v}(t_0)|}, \quad \mathbf{n}(t_0) = \underbrace{\frac{\mathbf{a}^\perp(t_0)}{|\mathbf{a}^\perp(t_0)|}}_{\text{defined only if } \kappa(t_0) \neq 0}$$

## Remarks 4.4

- i.  $\mathbf{t}(t_0), \mathbf{n}(t_0)$  are orthonormal, i.e. unit, orthogonal to each other



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- ii. The osculating plane at  $t_0$  is the plane through  $\mathbf{t}(t_0)$  spanned by  $\mathbf{t}(t_0), \mathbf{n}(t_0)$ . The osculating plane is the plane that  $\gamma$  is the closest to begin in, and contains the directions where the curve is heading and bending.

## **Proposition 4.5**

Let  $\gamma: I \rightarrow \mathbb{R}^n$  be a regular curve. Then  $|\mathbf{t}'| = \kappa |\mathbf{v}|^2$ , and  $\mathbf{t}' = \kappa |\mathbf{v}| \mathbf{n}$  if  $\mathbf{n}$  is defined. In particular, if  $\gamma$  is unit-speed, then

$|\mathbf{t}'| = \kappa$ , and  $\mathbf{t}' = \kappa \mathbf{n}$  if  $\mathbf{n}$  is defined.

## Proof.

$$\mathbf{t}' = \left( \frac{\mathbf{v}}{|\mathbf{v}|} \right)' = \frac{\mathbf{a}}{|\mathbf{v}|} - \mathbf{v} \frac{\langle \mathbf{a}, \mathbf{v} \rangle}{|\mathbf{v}|^3} = \frac{\mathbf{a} - \mathbf{a}^\parallel}{|\mathbf{v}|} = \frac{\mathbf{a}^\perp}{|\mathbf{v}|}$$

Hence  $|\mathbf{t}'| = \frac{|\mathbf{a}|^\perp}{|\mathbf{v}|^2} \cdot |\mathbf{v}| = \kappa |\mathbf{v}|$ , and

$$\mathbf{t}' = \frac{|\mathbf{a}^\perp|}{|\mathbf{v}|^2} |\mathbf{v}| \frac{\mathbf{a}^\perp}{|\mathbf{a}^\perp|} = \kappa |\mathbf{v}| \mathbf{n} \text{ if } \mathbf{n} \text{ is defined.}$$

A small, empty square box with a black border, likely a placeholder for a figure or diagram.

**Remark 4.6** Let  $\gamma: I \rightarrow \mathbb{R}^n$  be a unit-speed curve,  $t_0 \in I$  with  $\kappa(t_0) \neq 0$ .

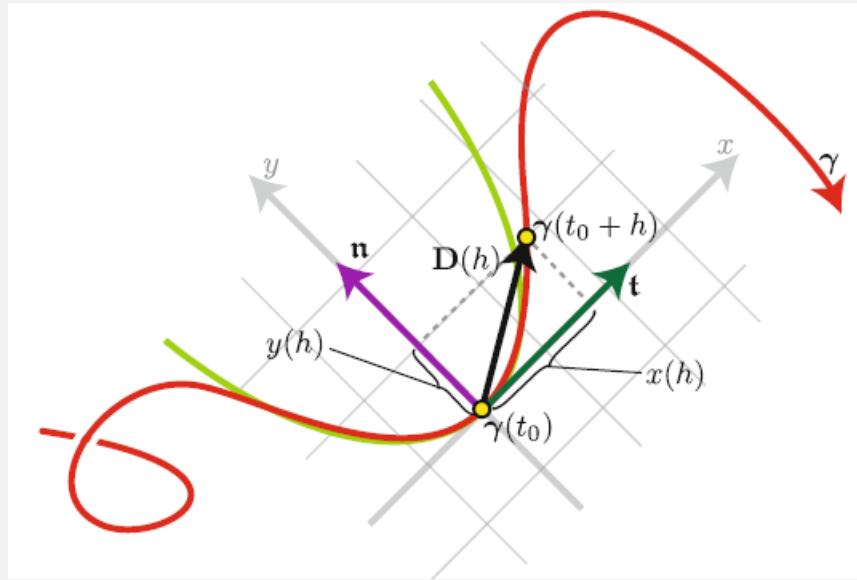
Then  $\gamma'(t_0) = \mathbf{t}$ ,  $\gamma''(t_0) = \mathbf{t}' = \kappa \mathbf{n}$ , and the 2nd order Taylor approximation at  $\gamma$  at  $t_0$  is

$$\begin{aligned}\gamma(t_0 + h) &\approx \gamma(t_0) + h\gamma'(t_0) + \frac{h^2}{2}\gamma''(t_0) \\ &= \gamma(t_0) + ht + \frac{\kappa h^2}{2} \mathbf{n}\end{aligned}$$

Set  $\mathbf{D}(h) = \gamma(t_0 + h) - \gamma(t_0) \approx h\mathbf{t} + \frac{\kappa h^2}{2}\mathbf{n}$ : displacement.

Then,

$$\left. \begin{array}{l} x(t) := \langle \mathbf{D}(h), \mathbf{t} \rangle \approx h \\ y(t) := \langle \mathbf{D}(h), \mathbf{n} \rangle \approx \frac{\kappa h^2}{2} \end{array} \right\} \text{the parabola } y = \frac{\kappa}{2}x^2 \text{ in the osculating plane}$$



Tapp Page 30

$\kappa(t_0)$  = the concavity of the parabola that best approximates the trace at  $t_0$

### Proposition 4.7

Let  $\gamma: I \rightarrow \mathbb{R}^n$  be a regular curve. If  $\forall t \in I, \kappa(t) = 0$ , then  $\gamma$  parametrizes a straight line.

**Proof.**

$$\begin{aligned} |\mathbf{t}'| = \kappa|\mathbf{v}| &= 0 \implies \mathbf{t}' = \mathbf{0} \\ &\implies \mathbf{t} = \mathbf{c} \text{ constant} \\ &\implies \mathbf{v} = |\mathbf{v}|\mathbf{c} \\ &\implies \text{fixing } t_0 \in I, \\ \gamma(t) &= \gamma(t_0) + \int_{t_0}^t \mathbf{v}(u) du \\ &= \gamma(t_0) + \left( \int_{t_0}^t |\mathbf{v}(u)| du \right) \mathbf{c} \end{aligned}$$

□

## 4.2 Plane Curves

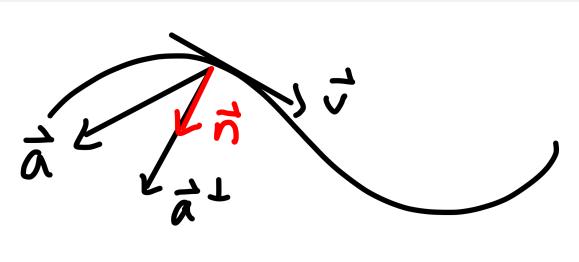
$\mathbb{R}^2$  is the only  $\mathbb{R}^n$  where the terms “clockwise” and “counter-clockwise” makes sense.

This allows us to define

“signed curvature” = curvature + turning direction with respect to  $\mathbf{v}$

**Recall 4.8**

$$\kappa = \frac{|\mathbf{a}^\perp|}{|\mathbf{v}|^2} = \frac{\langle \mathbf{a}, \mathbf{n} \rangle}{|\mathbf{v}|^2}$$


**Definition 4.9 (Signed curvature)**

Let  $\gamma: I \rightarrow \mathbb{R}^2$  be a regular plane curve. Then the signed curvature  $\kappa_s: I \rightarrow \mathbb{R}$  is defined as

$$\kappa_s = \frac{\langle \mathbf{a}, \mathbf{n}_s \rangle}{|\mathbf{v}|^2},$$

where,

$$\begin{aligned} \mathbf{n}_s &= R_{90}\mathbf{t} \\ &= \text{the counterclockwise } 90^\circ \text{ rotation of } \mathbf{t} \end{aligned}$$

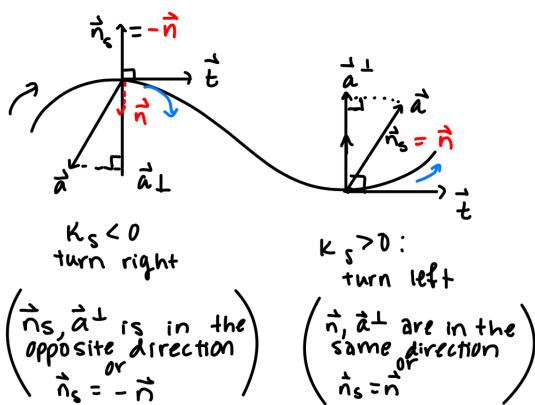
# 5 Jan 12, 2022

## 5.1 Plane Curves (Cont'd)

**Recall 5.1**

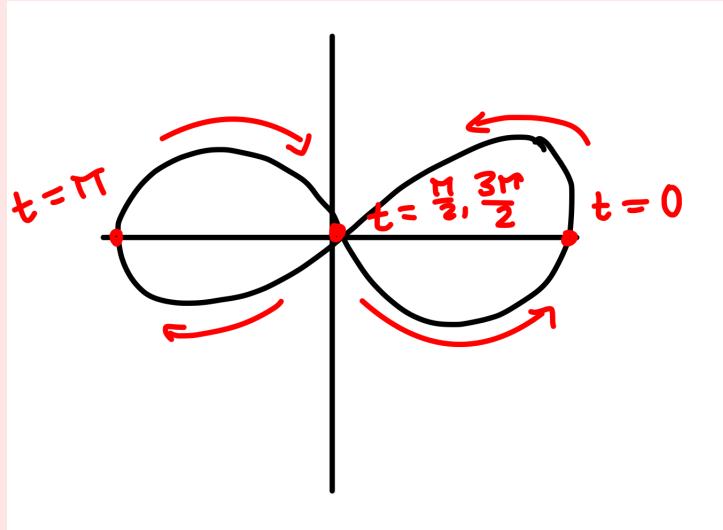
$$\kappa_s = \frac{\langle \mathbf{a}, \mathbf{n}_s \rangle}{|\mathbf{v}|^2}$$

where,  $\mathbf{n}_s = R_{90^\circ} \mathbf{t}$



**Example 5.2**

$$\gamma(g) = (\cos(t), \sin(2t)), \quad t \in [0, 2\pi]$$



Lissajous curve

$$\mathbf{v}(t) = (-\sin t, 2 \cos 2t)$$

$$\mathbf{a}(t) = (-\cos t, -4 \sin 2t)$$

$$|\mathbf{v}(t)| = \sqrt{\sin^2 t + 4 \cos^2 2t}$$

$$\mathbf{t}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = (-\sin t, 2 \cos 2t) \frac{1}{\sqrt{\sin^2 t + 4 \cos^2 2t}}$$

$$\mathbf{n}_s = R_{90}\mathbf{t} = (-2 \cos 2t, -\sin t) \frac{1}{\sqrt{\sin^2 t + 4 \cos^2 2t}}$$

$$\kappa_s = \frac{\langle \mathbf{a}, \mathbf{n}_s \rangle}{|\mathbf{v}|^2} = \frac{2 \cos t \cos 2t + 4 \sin t \sin 2t}{(\sin^3 t + 4 \cos^2 2t)^{3/2}}$$

$$\kappa_s(0) = \frac{2}{4^{3/2}} = \frac{2}{8} = \frac{1}{4} > 0$$

$$\kappa_s\left(\frac{\pi}{2}\right) = 0$$

$$\kappa_s(\pi) = \frac{-1}{4} < 0$$

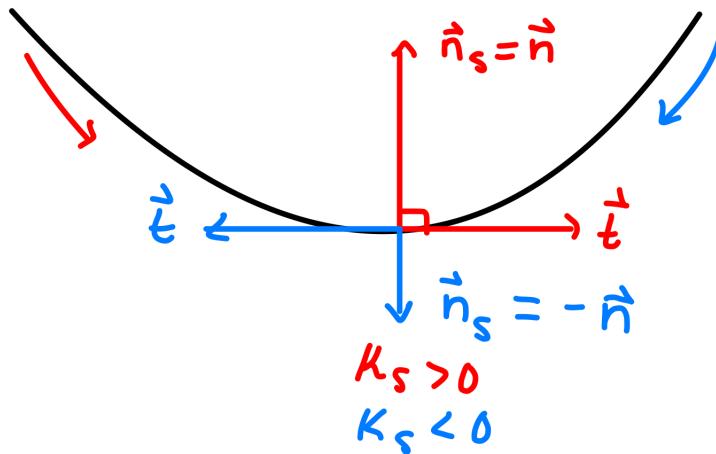
$$\kappa_s\left(\frac{3\pi}{2}\right) = 0$$

**Proposition 5.3**

Let  $\gamma: I \rightarrow \mathbb{R}^2$  be a plane curve. Then  $|\kappa_s| = \kappa$ .

**Proof.** Compare  $\kappa = \frac{\langle \mathbf{a}, \mathbf{n} \rangle}{|\mathbf{v}|^2}$ ,  $\kappa_s = \frac{\langle \mathbf{a}, \mathbf{n}_s \rangle}{|\mathbf{v}|^2}$

$\mathbf{n}_s = \pm \mathbf{n}$ , because they are both unit, orthogonal to  $\mathbf{t}$ . Hence  $\kappa_s$  coincides with  $\kappa$  up to signs.



□

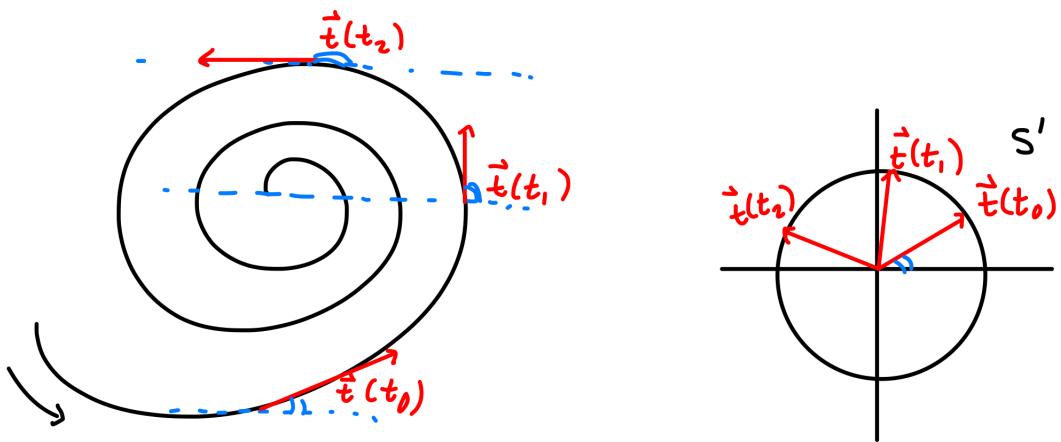
#### Proposition 5.4

Signed curvature is unchanged by any orientation-preserving reparametrizations.

| **Proof.** Exercise. □

#### Proposition 5.5

Let  $\gamma: I \rightarrow \mathbb{R}^2$  be a plane curve. Then there exists a smooth function  $\theta: I \rightarrow \mathbb{R}$  such that  $\forall t \in I, \mathbf{t}(t) = (\cos \theta(t), \sin \theta(t))$ .



What should  $\theta$  be?

$$\mathbf{t}' = \theta'(-\sin \theta, \cos \theta) = \theta' R_{90} \mathbf{t} = \theta' \mathbf{n}_s.$$

On the other hand,

$$\mathbf{t}' = \left( \frac{\mathbf{v}}{|\mathbf{v}|} \right)' = \frac{\mathbf{a}^\perp}{|\mathbf{v}|} = \frac{\langle \mathbf{a}, \mathbf{n}_s \rangle}{|\mathbf{v}|} \mathbf{n}_s = \kappa_s |\mathbf{v}| \mathbf{n}_s$$

By comparing the two formulas,  $\theta' = \kappa_s |\mathbf{v}|$ . In the proof, we solve this differential equation.

**Remark 5.6** If  $\gamma$  is unit-speed,  $\theta' = \kappa_s$ . This shows:

signed curvature =	the rate of change of the angle
curvature =	the rate of change of the angle

**Proof.** Fix  $t_0 \in I, \theta_0 \in \mathbb{R}$  such that  $\mathbf{t}(t_0) = (\cos \theta_0, \sin \theta_0)$ .

Define

$$\theta(t) = \theta_0 + \int_{t_0}^t \kappa_s(u) |\mathbf{v}(u)| du$$

We will show this  $\theta(t)$  works.

$\theta: I \rightarrow \mathbb{R}$  is a smooth function

$$\theta' = \kappa_s |\mathbf{v}|, \theta(t_0) = \theta_0.$$

Set  $\mathbf{t}_\theta = (\cos \theta, \sin \theta)$

We need to show  $\mathbf{t} = \mathbf{t}_\theta$ .

Observe  $\mathbf{t}, \mathbf{t}_\theta$  are unit.

Enough to show  $\langle \mathbf{t}, \mathbf{t}_\theta \rangle = 1$

On the other hand,

$$\begin{aligned} \mathbf{t}_\theta(t_0) &= (\cos \theta(t_0), \sin \theta(t_0)) \\ &= (\cos \theta_0, \sin \theta_0) \\ &= \mathbf{t}(t_0) \end{aligned}$$

So,

$$\langle \mathbf{t}(t_0), \mathbf{t}_\theta(t_0) \rangle = 1$$

Enough to show  $\langle \mathbf{t}, \mathbf{t}_\theta \rangle' = 0$

$$\begin{aligned} \mathbf{t}' &= \kappa_s |\mathbf{v}| \mathbf{n}_s = \kappa_s |\mathbf{v}| R_{90} \mathbf{t} \\ \mathbf{t}'_\theta &= \theta'(-\sin \theta, \cos \theta) = \kappa_s |\mathbf{v}| R_{90} \mathbf{t}_\theta \end{aligned}$$

Therefore,

$$\begin{aligned} \langle \mathbf{t}, \mathbf{t}_\theta \rangle' &= \langle \mathbf{t}', \mathbf{t}_\theta \rangle + \langle \mathbf{t}, \mathbf{t}'_\theta \rangle \\ &= \kappa_s |\mathbf{v}| (\langle R_{90} \mathbf{t}, \mathbf{t}_\theta \rangle + \langle \mathbf{t}, R_{90} \mathbf{t}_\theta \rangle) \\ &= \kappa_s |\mathbf{v}| (\langle R_{90} \mathbf{t}, \mathbf{t}_\theta \rangle + \langle R_{90} \mathbf{t}, R_{90}(\mathbf{t}_\theta) \rangle) && R_{90} \text{ is orthogonal} \\ &= \kappa_s |\mathbf{v}| (\langle R_{90} \mathbf{t}, \mathbf{t}_\theta \rangle - \langle R_{90} \mathbf{t}, \mathbf{t}_\theta \rangle) && R_{90} \circ R_{90} = R_{180} = -1 \\ &= 0 \end{aligned}$$

□

**Remark 5.7** The angle function  $\theta$  is unique up to an integer multiple of  $2\pi$ .

Indeed if  $\Theta: I \rightarrow \mathbb{R}$  is a smooth function such that  $\forall t \in I, \gamma = (\cos \Theta, \sin \Theta)$ , then,

$$\begin{aligned}\Theta' &= \theta' = \kappa_s |\mathbf{v}| \\ \implies |\Theta - \theta'| &= 0 \\ \implies \Theta - \theta &= \text{constant}\end{aligned}$$

On the other hand,

$$(\cos \theta, \sin \theta) = (\cos \Theta, \sin \Theta) = \mathbf{t}$$

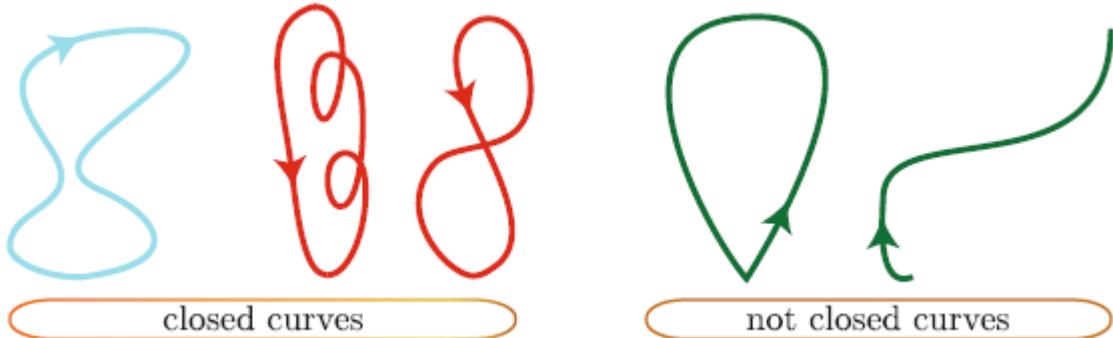
So  $\Theta - \theta \in 2\pi \cdot \mathbb{Z}$

# 6 Jan 14, 2022

## 6.1 Plane Curves(Cont'd)

**Definition 6.1** (Closed curve)

A regular curve  $\gamma: [a, b] \rightarrow \mathbb{R}^n$  is called closed if  $\gamma(a) = \gamma(b)$ , and  $\forall n \in \mathbb{N}, \gamma^{(n)}(a) = \gamma^{(n)}(b)$



**Definition 6.2** (Rotation index)

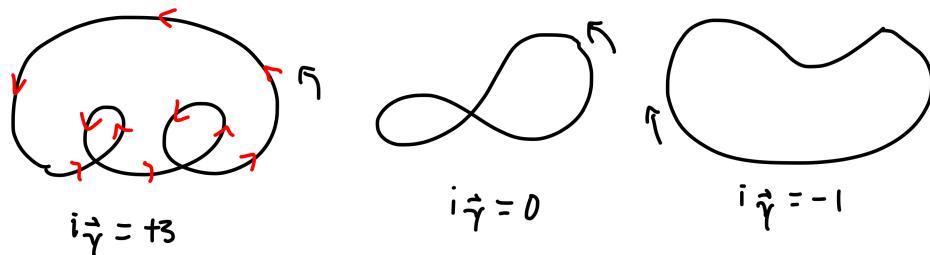
Let  $\gamma: [a, b] \rightarrow \mathbb{R}^2$  be a closed plane curve. The rotation index of  $\gamma$  is defined as

$$i_\gamma = \frac{1}{2\pi}(\theta(b) - \theta(a)),$$

where  $\theta$  is the angle function from proposition 5.5.

### Remarks 6.3

- i.  $i_\gamma \in \mathbb{Z}$ , because  $\mathbf{t}(a) = \mathbf{t}(b)$ , so  $\theta(b) - \theta(a) \in 2\pi\mathbb{Z}$
- ii. Later on, we will show  $i_\gamma = \pm 1$  if  $\gamma$  has no self-intersection.



### Proposition 6.4

Let  $\gamma: [a, b] \rightarrow \mathbb{R}^2$  be a closed plane curve. Then

$$i_\gamma = \frac{1}{2\pi} \int_a^b \kappa_s(t) |\mathbf{v}(t)| dt$$

| **Proof.** This follows from the construction of the angle function.  $\square$

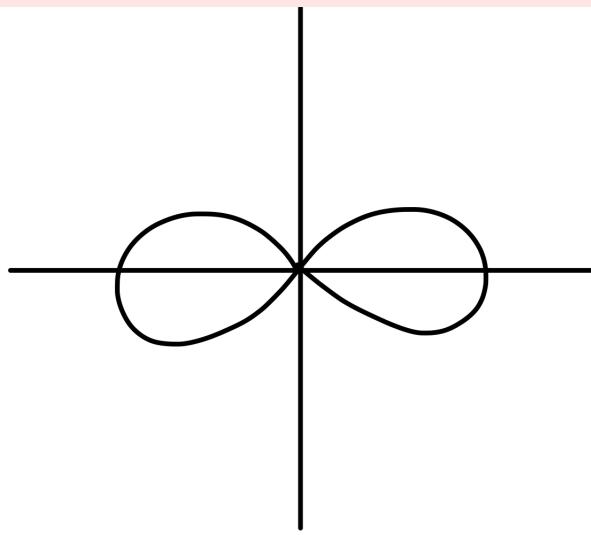
**Proposition 6.5**

Rotation index is unchanged under any orientation-preserving reparametrizations.

| **Proof.** Exercise.  $\square$

**Example 6.6**

$$\gamma(t) = (\cos t, \sin 2t), t \in [0, 2\pi]$$



Recall:

$$\kappa_s(t) = \frac{2 \cos t \cos 2t + 4 \sin t \sin 2t}{(\sin^2 t + 4 \cos^2 2t)^{3/2}}$$

$$|\mathbf{v}| = (\sin^2 t + 4 \cos^2 2t)^{1/2}$$

Therefore,

$$\begin{aligned} i_\gamma &= \frac{1}{2\pi} \int_0^{2\pi} \frac{2 \cos t \cos 2t + 4 \sin t \sin 2t}{\sin^2 t + 4 \cos^2 2t} dt \\ &= \frac{1}{2\pi} \left( \underbrace{\int_0^\pi \dots dt}_{t=s+\pi, \text{ then the integrand is multiplied by } -1} + \underbrace{\int_\pi^{2\pi} \dots dt}_{t=s+\pi, \text{ then the integrand is multiplied by } -1} \right) \\ &= 0 \end{aligned}$$

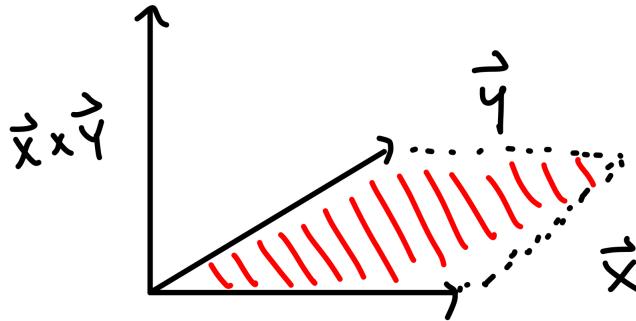
## 6.2 Space Curves

What's special about  $\mathbb{R}^3$ ?  
 $\mathbb{R}^3$  has the cross product.

**Recall 6.7**  $\mathbf{x} = (x_1, x_2, x_3), \mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$ ,  
 $\mathbf{x} \times \mathbf{y} = (x_2 y_3 - x_3 y_2, -(x_1 y_3 - x_3 y_1), x_1 y_2 - x_2 y_1) \in \mathbb{R}^3$

Basic properties:

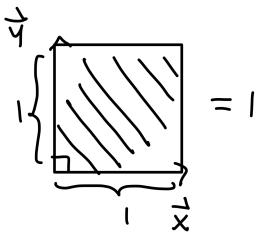
- i.  $\times: \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is bilinear, and antisymmetric.  
 (i.e.  $\mathbf{y} \times \mathbf{x} = -\mathbf{x} \times \mathbf{y}$ )
- ii.  $|\mathbf{x} \times \mathbf{y}| = |\mathbf{x}| |\mathbf{y}| \sin(\theta)$ , where  $\theta$  is the angle between  $\mathbf{x}, \mathbf{y}$   
 = the area of the parallelogram spanned by  $\mathbf{x}, \mathbf{y}$
- iii.  $\mathbf{x} \times \mathbf{y}$  is orthogonal to  $\mathbf{x}, \mathbf{y}$ ;  
 $\{\mathbf{x}, \mathbf{y}, \mathbf{x} \times \mathbf{y}\}$  is a right-handed system.



### Example 6.8

If  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$  are orthonormal, then  $\{\mathbf{x}, \mathbf{y}, \mathbf{x} \times \mathbf{y}\}$  is an orthonormal basis for  $\mathbb{R}^3$ :

- $\mathbf{x} \times \mathbf{y}$  is orthogonal to  $\mathbf{x}, \mathbf{y}$ , and
- $|\mathbf{x} \times \mathbf{y}| =$



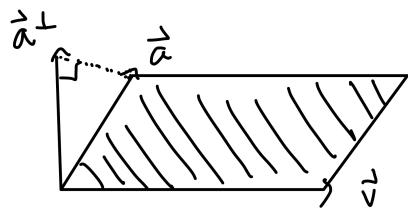
$$= 1$$

### Proposition 6.9

Let  $\gamma: I \rightarrow \mathbb{R}^3$  be a space curve, then

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$$

| **Proof.**  $|\mathbf{v} \times \mathbf{a}| =$

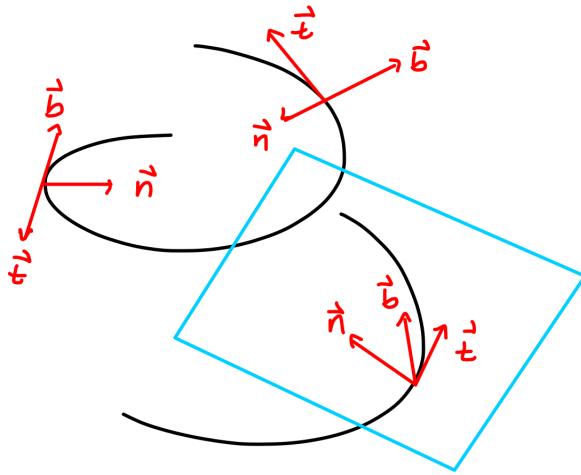


$$= |\mathbf{v}| |\mathbf{a}^\perp| \\ \implies \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{|\mathbf{a}^\perp|}{|\mathbf{v}|^2} = \kappa$$

□

**Definition 6.10** (Unit binormal vector and Frenet frame)

Let  $\gamma: I \rightarrow \mathbb{R}^3$  be a space curve. The unit binormal vector for  $\gamma$  at  $t \in I$  is defined as  $\mathbf{b}(t) = \mathbf{t}(t) \times \mathbf{n}(t)$  (only if  $\kappa(t) \neq 0$ ). The orthonormal basis  $\{\mathbf{t}(t), \mathbf{n}(t), \mathbf{b}(t)\}$  for  $\mathbb{R}^3$  is called the Frenet frame for  $\gamma$  at  $t$ .

**Remark 6.11**  $\mathbf{b}(t)$  is a unit normal vector to the osculating plane of  $\gamma$  at  $t$ .

$\implies \mathbf{b}$  encodes the tilt of the osculating plane of  $\gamma$ .

We want to define the “torsion” as the measurement of the change of the tilt of the osculating plane.

**Definition 6.12** (Torsion)

Let

$$\gamma: I \rightarrow \mathbb{R}^3 \text{ be a space curve,}$$

$$t \in I \text{ s.t. } \kappa(t) \neq 0$$

The torsion of  $\gamma$  at  $t$  is defined as

$$\tau(t) = -\frac{\langle \mathbf{b}'(t), \mathbf{n}(t) \rangle}{|\mathbf{v}(t)|}$$

**Remark 6.13** Why is this definition plausible?

- i.  $\mathbf{b}'(t)$  is parallel to  $\mathbf{n}(t)$  (later).  
So  $\langle \mathbf{b}'(t), \mathbf{n}(t) \rangle = \pm |\mathbf{b}'(t)|$
- ii.  $\langle \mathbf{b}'(t), \mathbf{n}(t) \rangle$  depends on parametrizations.

**Proposition 6.14**

Torsion is independent of parametrizations.

**Proof.** Read Tapp for the details.

Sketch:

$\varphi$  is orientation-preserving.

$$\tilde{t} = t \circ \varphi, \tilde{n} = n \circ \varphi$$

$$\implies \tilde{b} = b \circ \varphi$$

$$\implies \tilde{b}' = b' \circ \varphi \cdot \varphi'$$

□

# 7 Jan 19, 2022

## 7.1 Space Curves (Cont'd)

**Recall 7.1**  $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ ,  $\tau = -\frac{\langle \mathbf{b}', \mathbf{n} \rangle}{|\mathbf{v}|}$

Note:  $\mathbf{b}' = -\tau |\mathbf{v}| \mathbf{n}$

### Proposition 7.2

Let  $\gamma: I \rightarrow \mathbb{R}^3$  be a space curve such that  $\forall t \in I, \kappa(t) \neq 0$ . Then the following conditions are equivalent:

- i. The trace of  $\gamma$  is contained in a plane in  $\mathbb{R}^3$ .
- ii.  $\forall t \in I, \tau(t) = 0$ .

**Remark 7.3** The torsion measures the failure of a space curve to remain in a plane in  $\mathbb{R}^3$ .

**Proof.** (i.) is equivalent to:

(i.)'  $\exists \mathbf{w} \neq \mathbf{0} \in \mathbb{R}^3, c \in \mathbb{R}, \forall t \in I, \langle \gamma, \mathbf{w} \rangle = c$

We show (i.)'  $\iff$  (ii.).

$$(\iff) \mathbf{b}' = -\tau |\mathbf{v}| \mathbf{n} = 0, \text{ so}$$

$$\mathbf{b} = \text{constant} =: \mathbf{w} \neq 0$$

$$\langle \gamma(t), \mathbf{w} \rangle' = \langle \mathbf{v}(t), \mathbf{w} \rangle = \langle |\mathbf{v}(t)| \mathbf{t}(t), \mathbf{b}(t) \rangle = 0, \text{ so}$$

$$\langle \gamma(t), \mathbf{w} \rangle = \text{constant}.$$

$$(\implies) \langle \gamma(t), \mathbf{w} \rangle = \text{constant}, \text{ so}$$

$$\langle \gamma(t), \mathbf{w} \rangle = \langle \mathbf{a}(t), \mathbf{w} \rangle = 0$$

$$\mathbf{t}(t), \mathbf{n}(t) \in \text{span}(\mathbf{v}(t), \mathbf{a}(t)), \text{ so}$$

$$\langle \mathbf{t}(t), \mathbf{w} \rangle = \langle \mathbf{n}(t), \mathbf{w} \rangle = 0.$$

This shows that  $\mathbf{w}$  is normal to the osculating plane spanned by  $\mathbf{t}(t), \mathbf{n}(t)$ , so

$$\mathbf{b}(t) = \pm \frac{\mathbf{w}}{|\mathbf{w}|} = \text{constant}, \text{ so}$$

$$\mathbf{b}'(t) = \mathbf{0}, \text{ so}$$

$$\tau(t) = -\frac{\langle \mathbf{b}'(t), \mathbf{n}(t) \rangle}{|\mathbf{v}(t)|} = 0$$

□

There are differential equations for  $\mathbf{t}, \mathbf{n}, \mathbf{b}$  determined by  $\kappa, \tau$ .

**Proposition 7.4** (Frenet equations)

Let  $\gamma: I \rightarrow \mathbb{R}^3$  be a space curve such that  $\forall t \in I, \kappa(t) \neq 0$ .

Then,

$$\begin{aligned}\mathbf{t}' &= \kappa |\mathbf{v}| \mathbf{n} \\ \mathbf{n}' &= -\kappa |\mathbf{v}| \mathbf{t} + \tau |\mathbf{v}| \mathbf{b} \\ \mathbf{b}' &= -\tau |\mathbf{v}| \mathbf{n}\end{aligned}$$

In particular, if  $\gamma$  is unit-speed, then

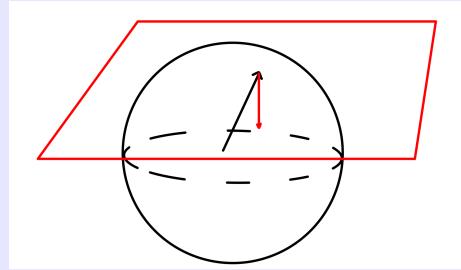
$$\begin{aligned}\mathbf{t}' &= \kappa \mathbf{n} \\ \mathbf{n}' &= -\kappa \mathbf{t} + \tau \mathbf{b} \\ \mathbf{b}' &= -\tau \mathbf{n}\end{aligned}$$

**Remark 7.5** This suggests that a space curve is completely determined by the functions  $\kappa, \tau$  up to initial conditions. (Fundamental Theorem of Space Curves)

**Lemma 7.6**

Let  $\gamma, \delta: I \rightarrow \mathbb{R}^n$  be curves (not necessarily regular).

- i. If  $\exists c \in \mathbb{R}, \forall t \in I, |\gamma(t)| = c$ , then  $\forall t \in I, \gamma'(t)$  is orthogonal to  $\gamma(t)$ .



- ii. If  $\exists D \in \mathbb{R}, \forall t \in I, \langle \gamma(t), \delta(t) \rangle = D$ , then  $\forall t \in I, \langle \gamma'(t), \delta(t) \rangle = -\langle \gamma(t), \delta'(t) \rangle$ .

**Remark 7.7** Both the assumptions are satisfied if  $\forall t \in I, \gamma(t), \delta(t)$  are orthogonal.

**Proof of Lemma.**

- i.  $c^2 = |\gamma(t)|^2 = \langle \gamma(t), \gamma(t) \rangle$ .  
 $\implies 0 = 2 \langle \gamma(t), \gamma'(t) \rangle$   
 $\implies \langle \gamma(t), \gamma'(t) \rangle = 0$
- ii.  $\langle \gamma(t), \delta(t) \rangle = D$   
 $\implies \langle \gamma'(t), \delta(t) \rangle + \langle \gamma(t), \delta'(t) \rangle = 0$   
 $\implies \langle \gamma'(t), \delta(t) \rangle = -\langle \gamma(t), \delta'(t) \rangle$

□

**Proof of Proposition 7.4.** We have proved  $\mathbf{t}' = \kappa |\mathbf{v}| \mathbf{n}$ . As for  $\mathbf{n}', \mathbf{b}'$ , it is enough to compute their components with respect to the Frenet frame  $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ .  $\langle \mathbf{n}', \mathbf{t} \rangle = -\langle \mathbf{n}, \mathbf{t}' \rangle = -\langle \mathbf{n}, \kappa |\mathbf{v}| \mathbf{n} \rangle = -\kappa |\mathbf{v}|$

$$\begin{aligned}\langle \mathbf{n}', \mathbf{n} \rangle &= 0 \\ \langle \mathbf{n}', \mathbf{b} \rangle &= -\langle \mathbf{n}, \mathbf{b}' \rangle = \tau |\mathbf{v}|\end{aligned}$$

Therefore,

$$\mathbf{n}' = -\kappa |\mathbf{v}| \mathbf{t} + \tau |\mathbf{v}| \mathbf{b}.$$

$$\langle \mathbf{b}', \mathbf{t} \rangle = -\langle \mathbf{b}, \mathbf{t}' \rangle = -\langle \mathbf{b}, -\kappa |\mathbf{v}| \mathbf{n} \rangle = 0$$

$$\langle \mathbf{b}', \mathbf{n} \rangle = -\tau |\mathbf{v}|$$

$$\langle \mathbf{b}', \mathbf{b} \rangle = 0$$

Therefore,

$$\mathbf{b}' = -\tau |\mathbf{v}| \mathbf{n}$$

□

**Remark 7.8** Another interpretation of the torsion can be given by the Frenet equations.

Let  $\gamma: I \rightarrow \mathbb{R}^3$  be a unit-speed space curve.

Then,

$$\gamma' = \mathbf{t}, \gamma'' = \mathbf{t}' = \kappa \mathbf{n},$$

$$\gamma''' = (\kappa \mathbf{n})' = \kappa' \mathbf{n} + \kappa \mathbf{n}' = -\kappa^2 \mathbf{t} + \kappa' \mathbf{n} + \kappa \tau \mathbf{b}$$

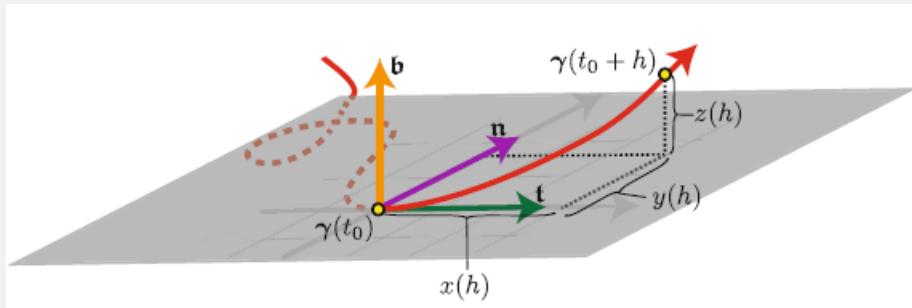
So the 3rd order Taylor approximation at  $t_0 \in I, \kappa(t_0) > 0$  is as follows:

$$\begin{aligned}\mathbf{D}(h) &= \gamma(t_0 + h) - \gamma(t_0) \\ &\approx h \gamma'(t_0) + \frac{h^2}{2} \gamma''(t_0) + \frac{h^3}{6} \gamma'''(t_0) \\ &= \left(h - \frac{\kappa^2 h^3}{6}\right) \mathbf{t} + \left(\frac{\kappa h^2}{2} + \frac{\kappa' h^3}{6}\right) \mathbf{n} + \frac{\kappa \tau h^3}{6} \mathbf{b}\end{aligned}$$

Therefore,

$$\begin{aligned}x(h) &= \langle \mathbf{D}(h), \mathbf{t} \rangle \approx h - \frac{\kappa^2 h^3}{6} \\ y(h) &= \langle \mathbf{D}(h), \mathbf{n} \rangle \approx \frac{\kappa h^2}{2} + \frac{\kappa' h^3}{6} \\ z(h) &= \langle \mathbf{D}(h), \mathbf{b} \rangle \approx \frac{\kappa \tau h^3}{6}\end{aligned}$$

If  $\tau(t_0) > 0$ , then the curve passes through the osculating plane from below.

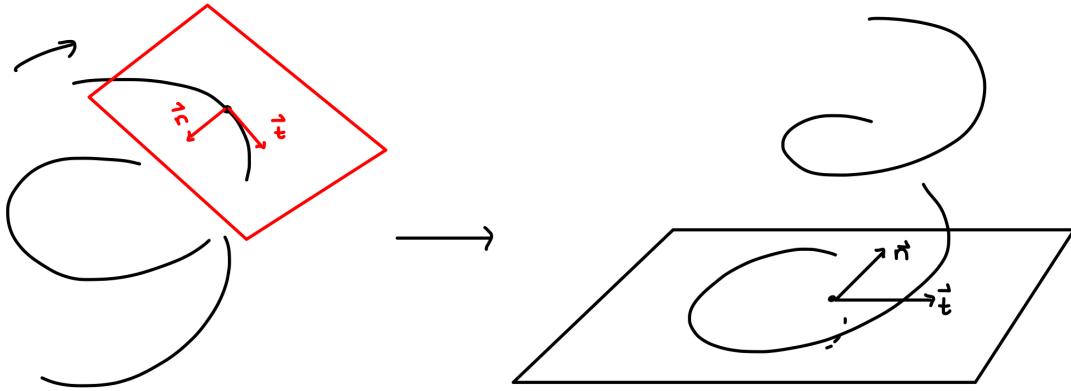


If  $\tau(t_0) < 0$ , then the curve passes through the osculating plane from above.

# 8 Jan 21, 2022

## 8.1 Rigid Motions

In geometry, it is often useful to “tilt your head”, or choose an orthonormal set of vectors at a point, adapted to the problem at hand:



This is achieved by rigid motions.

**Definition 8.1** (Rigid motion)

A rigid motion in  $\mathbb{R}^n$  is a function  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  that preserves the distances:

$$\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n, |f(\mathbf{x}) - f(\mathbf{y})| = |\mathbf{x} - \mathbf{y}|$$

**Example 8.2**

The translation by  $\mathbf{p} \in \mathbb{R}^n$

$$T_{\mathbf{p}}: \mathbb{R}^n \rightarrow \mathbb{R}^n, \mathbf{x} \mapsto \mathbf{x} + \mathbf{p}$$

is a rigid motion. Indeed,

$$\begin{aligned} |T_{\mathbf{p}}(\mathbf{x}) - T_{\mathbf{p}}(\mathbf{y})| &= |\mathbf{x} + \mathbf{p} - (\mathbf{y} + \mathbf{p})| \\ &= |\mathbf{x} - \mathbf{y}| \end{aligned}$$

Note:  $T_{\mathbf{p}}$  is never linear if  $\mathbf{p} \neq \mathbf{0}$ , because  $T_{\mathbf{p}}(\mathbf{0}) = \mathbf{p} \neq \mathbf{0}$ .

**Theorem 8.3**

Let  $L_A: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear transformation represented by an  $n \times n$  matrix  $A$ . The following conditions are equivalent:

1.  $L_A$  is a rigid motion.
2.  $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n, \langle L_A(\mathbf{x}), L_A(\mathbf{y}) \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$ .
3. If  $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$  is an orthonormal basis of  $\mathbb{R}^n$ , so is  $\{L_A \mathbf{x}_1, \dots, L_A \mathbf{x}_n\}$ .
4. The column vectors of  $A$  form an orthonormal basis of  $\mathbb{R}^n$ .
5.  $A^T A = I_n$

**Definition 8.4**

A linear rigid motion and its matrix are called orthogonal.

$$O(n) := \text{the set of all } n \times n \text{ orthogonal matrices}$$

**Proposition 8.5**

Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a rigid motion. Then,

$$\exists! \mathbf{p} \in \mathbb{R}^n, \exists! A \in O(n), f = T_{\mathbf{p}} \circ L_A$$

**Sketch of proof.** Step 1:  $(f(\mathbf{0}) = \mathbf{0}): \exists! A \in O(n), f = L_A$

Step 2: (General Case): Set  $\mathbf{p} = f(\mathbf{0})$ . Then apply Step 1 to  $(T_{\mathbf{p}})^{-1} \circ f = T_{-\mathbf{p}} \circ f$   
Indeed,

$$(T_{\mathbf{p}})^{-1} \circ f(\mathbf{0}) = T_{-\mathbf{p}} \circ f(\mathbf{0}) = T_{-\mathbf{p}}(\mathbf{p}) = \mathbf{0},$$

So,

$$\exists! A \in O(n), (T_{\mathbf{p}})^{-1} \circ f = L_A$$

$$\implies f = T_{\mathbf{p}} \circ L_A \text{ Read Tapp for the details.} \quad \square$$

We can classify rigid motions:

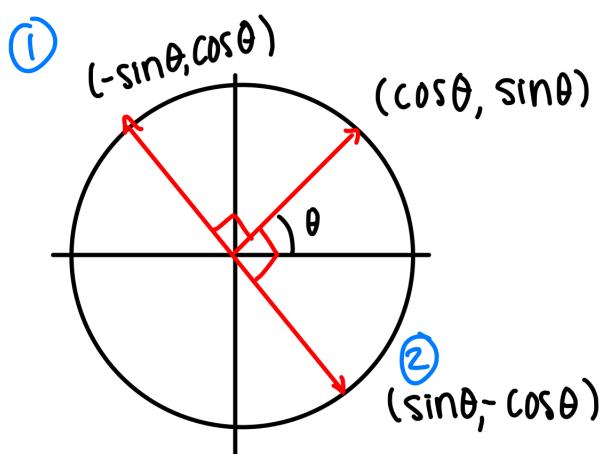
**Lemma 8.6**

$$A \in O(n) \implies \det(A) = \pm 1$$

**Proof.**  $A^T A = \mathbb{I}_n$ , so  $1 = \det(A^T A) = \det(A^T) \det(A) = \det(A)^2$   $\square$

**Example 8.7**

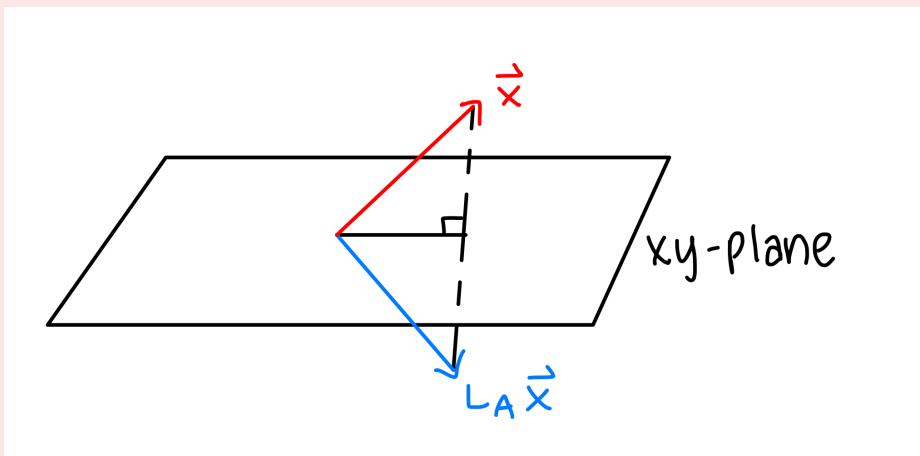
Let  $A \in O(2)$ . The column vectors of  $A$  are orthonormal:



$$A = \underbrace{\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}}_{\substack{\text{rotation,} \\ \text{det}=1 \\ \text{proper}}} \text{ or } \underbrace{\begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}}_{\substack{\text{reflection} \\ \text{det}=-1 \\ \text{improper}}}$$

**Example 8.8**

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in O(3)$  represents the reflection about the  $xy$  plane:



$\det(A) = -1$ , so  $L_A$  is improper.

**Remark 8.9** proper = physically performable (e.g. rotations)

improper = physically unperformable (e.g. reflections)

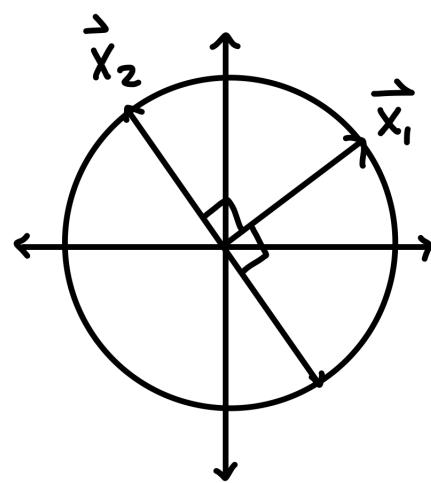
Another interpretation of proper (improper) rigid motions is given in terms of the orientation of orthonormal basis.

**Definition 8.10** (Ordered orthonormal basis and Positively oriented vs. Negatively oriented)

An ordered orthonormal basis (o.o.b.)  $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$  of  $\mathbb{R}^n$  is called positively oriented (p.o.) if the orthogonal matrix whose column vectors are  $\mathbf{x}_1, \dots, \mathbf{x}_n$  has  $\det = 1$ , and negatively oriented (n.o.) if it has  $\det = -1$ .

**Example 8.11**

$\{\mathbf{x}_1, \mathbf{x}_2\}$  are o.o.b of  $\mathbb{R}^2$ .



$$\text{p.o.} \iff \mathbf{x}_2 = R_{90}\mathbf{x}_1$$

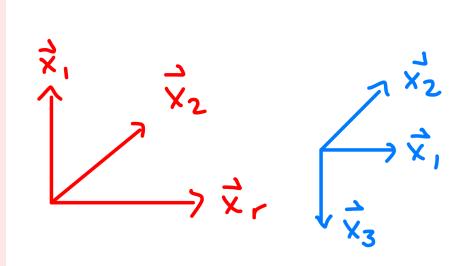
$$\text{n.o.} \iff \mathbf{x}_2 = R_{-90}\mathbf{x}_1$$

**Example 8.12**

$\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$  o.o.b. of  $\mathbb{R}^3$ .

p.o.  $\iff \mathbf{x}_3 = \mathbf{x}_1 \times \mathbf{x}_2 \iff$  right-hand

n.o.  $\iff \mathbf{x}_3 = -\mathbf{x}_1 \times \mathbf{x}_2 \iff$  left-hand

**Proposition 8.13**

Let  $A \in O(n)$ . Then  $A$  preserves the orientation of any o.o.b.  $\iff \det(A) = +1$

$A$  reserves the orientation of any o.o.b.  $\iff \det(A) = -1$ .

**Proof.** Let  $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$  be an o.o.b. of  $\mathbb{R}^n$ . Set

$$B := (\mathbf{x}_1 \cdots \mathbf{x}_n) \in O(n).$$

Then,

$$AB = (L_A \mathbf{x}_1 \cdots L_A \mathbf{x}_n)$$

Note,  $\det(AB) = \det(A) \det(B)$ .

Therefore,

$$\det(AB) = \begin{cases} \det(B) & \text{if } \det(A) = 1 \\ -\det(B) & \text{if } \det(A) = -1 \end{cases}$$

□

**Proposition 8.14**

The following functions are unchanged by proper rigid motions:

- i. Curvature for a regular curve
- ii. Torsion for a space curve
- iii. Signed curvature for a plane curve.

By improper rigid motions, (i) is unchanged, (ii) and (iii) are multiplied by  $-1$ .

# 9 Jan 24, 2022

## 9.1 Rigid Motions (Cont'd)

**Proof of Proposition 8.14.** Let  $\gamma: I \rightarrow \mathbb{R}^n$  be a regular curve,  $f = T_p \circ L_A: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a rigid motion. Set  $\hat{\gamma} = f \circ \gamma: I \rightarrow \mathbb{R}^n$ . Then,

$$\begin{aligned}\hat{\gamma} &= A\gamma(t) + p \\ \hat{v}(t) &= (A\gamma(t) + p)' = Av(t), \\ \hat{a}(t) &= (Av(t))' = Aa(t).\end{aligned}$$

Note:  $\hat{\gamma}: I \rightarrow \mathbb{R}^n$  is a regular curve, because  $\hat{\gamma}$  is smooth, and

$$\forall t \in I, \quad |\hat{v}(t)| = |Av(t)| = |v(t)| \neq 0.$$

Moreover,

$$\begin{aligned}\hat{t}(t) &= \frac{\hat{v}(t)}{|\hat{v}(t)|} = \frac{Av(t)}{|Av(t)|} = A \frac{v(t)}{|v(t)|} = At(t), \\ \hat{t}'(t) &= At'(t), \\ \hat{n}(t) &= \frac{\hat{t}'(t)}{|\hat{t}'(t)|} = \frac{At'(t)}{|At'(t)|} = A \frac{t'(t)}{|t'(t)|} = An(t)\end{aligned}$$

$$\text{i. } \hat{\kappa} = \frac{|\hat{t}'|}{|\hat{v}|} = \frac{|At'|}{|Av|} = \frac{|t'|}{|v|} = \kappa$$

ii.  $\hat{b} \stackrel{?}{\leftrightarrow} Ab$ . Compare  $\{\hat{t}, \hat{n}, \hat{b}\}, \{At, An, Ab\}$ :

- (1)  $\forall t \in I, \{\hat{t}(t), \hat{n}(t), \hat{b}(t)\}, \{At(t), An(t), Ab(t)\}$  are o.o.b.
- (2)  $\hat{t} = At, \hat{n} = An$
- (3)  $\{\hat{t}(t), \hat{n}(t), \hat{b}(t)\}$  is p.o.,

$$\{At, An, Ab\} \text{ is } \begin{cases} \text{p.o. if } \det(A) = 1, \text{ proper} \\ \text{n.o. if } \det(A) = -1, \text{ improper} \end{cases}$$

Therefore,

$$\hat{b} = \pm Ab, \text{ where } \begin{cases} + & \text{if } \det(A) = 1 \\ - & \text{if } \det(A) = -1 \end{cases}$$

$$\begin{aligned}\implies \hat{\tau} &= -\frac{\langle \hat{b}', \hat{n} \rangle}{|\hat{v}|^2} = -\frac{\langle \pm Ab', An \rangle}{|Av|^2} \\ &= \pm \left( -\frac{\langle b', n \rangle}{|v|^2} \right) = \pm \tau\end{aligned}$$

iii. Similar. □

**Theorem 9.1 (Fundamental Theorems for Plane and Space Curves)**

- If  $\kappa_s: I \rightarrow \mathbb{R}$  is a smooth function, then there exists a unit-speed plane curve  $\gamma: I \rightarrow \mathbb{R}^2$  whose signed curvature =  $\kappa_s$ . If  $\gamma, \hat{\gamma}: I \rightarrow \mathbb{R}^2$  are two such curves, then there exists a proper rigid motion  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $\hat{\gamma} = f \circ \gamma$ .
- If  $\kappa, \tau: I \rightarrow \mathbb{R}$  are smooth functions with  $\kappa > 0$ , then there exists a unit-speed space curve  $\gamma: I \rightarrow \mathbb{R}^3$  whose curvature =  $\kappa$ , torsion =  $\tau$ . If  $\gamma, \hat{\gamma}: I \rightarrow \mathbb{R}^3$  are two such curves, then there exists a proper rigid motion  $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  such that  $\hat{\gamma} = f \circ \gamma$ .

### Proof.

- Read the proof in Tapp.
- (Sketch, full proof uploaded on Canvas):

Fix  $t_0 \in I$ . We will show that, given the initial Frenet frame  $\{t_0, n_0, b_0\}$ , position  $\gamma_0$ , there exists a unique unit-speed space curve  $\gamma: I \rightarrow \mathbb{R}^3$  such that  $\gamma(t_0) = \gamma_0$ ,  $\{t(t_0), n(t_0), b(t_0)\} = \{t_0, n_0, b_0\}$ .

Step 1: Solve the Frenet equations for  $t, n, b$ :

$$\begin{cases} \mathbf{t}' = \kappa \mathbf{n} \\ \mathbf{n}' = -\kappa \mathbf{t} + \tau \mathbf{b} \\ \mathbf{b}' = -\tau \mathbf{n} \end{cases} \quad \text{the system of } 3 \times 3 = 9 \text{ ODEs}$$

The Picard theorem from the theory of ODEs implies there is a unique solution  $\{t, n, b\}$  such that  $\{t(t_0), n(t_0), b(t_0)\} = \{t_0, n_0, b_0\}$ . (Read textbooks for ODEs)

Step 2: Show  $\forall t \in I$ ,  $\{t(t), n(t), b(t)\}$  is orthonormal. It is important that

$$\begin{pmatrix} 0 & \kappa(t) & 0 \\ -\kappa(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{pmatrix}$$

is skew-symmetric i.e.  $c(t)^T = -c(t)$ .

Step 3:  $\gamma' = t$  has a unique solution such that  $\gamma(t_0) = \gamma_0$ , namely

$$\gamma(t) = \gamma_0 + \int_{t_0}^t t(u) du$$

Show  $\gamma: I \rightarrow \mathbb{R}^3$  is a unit-speed space curve whose

$$\begin{aligned} \text{Frenet-frame} &= \{t, n, b\} \\ \text{curvature} &= \kappa \\ \text{torsion} &= \tau \end{aligned}$$

The result follows. Finally, suppose  $\gamma, \hat{\gamma}: I \rightarrow \mathbb{R}^3$  are unit-speed space curves whose curvature =  $\kappa$ , torsion =  $\tau$ . Want to find a proper rigid motion  $f = T_p \circ L_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  such that  $\hat{\gamma} = f \circ \gamma$ . Fix  $t_0 \in I$ . Set

$$A := (\hat{t}(t_0) \ \hat{n}(t_0) \ \hat{b}(t_0))^{-1} (t(t_0) \ n(t_0) \ b(t_0)).$$

Note  $A \in O(3)$ ,  $\det(A) = 1$  because  $(t(t_0) \ n(t_0) \ b(t_0))$ ,  $(\hat{t}(t_0) \ \hat{n}(t_0) \ \hat{b}(t_0))$  have the same property. Set

$$\begin{aligned} p &:= \hat{\gamma}(t_0) - \gamma(t_0) \\ f &:= T_p \circ L_A: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \text{ proper rigid motion} \end{aligned}$$

We want to show  $\hat{\gamma} = f \circ \gamma$ . Enough to show their initial positions and Frenet frames are the same:

$$\begin{aligned} \hat{\gamma}(t_0) &= f \circ \gamma(t_0), & \hat{t}(t_0) &= At(t_0), \\ \hat{n}(t_0) &= An(t_0), & \hat{b}(t_0) &= Ab(t_0). \end{aligned}$$

These are true by the choice of  $A, p$ .

□

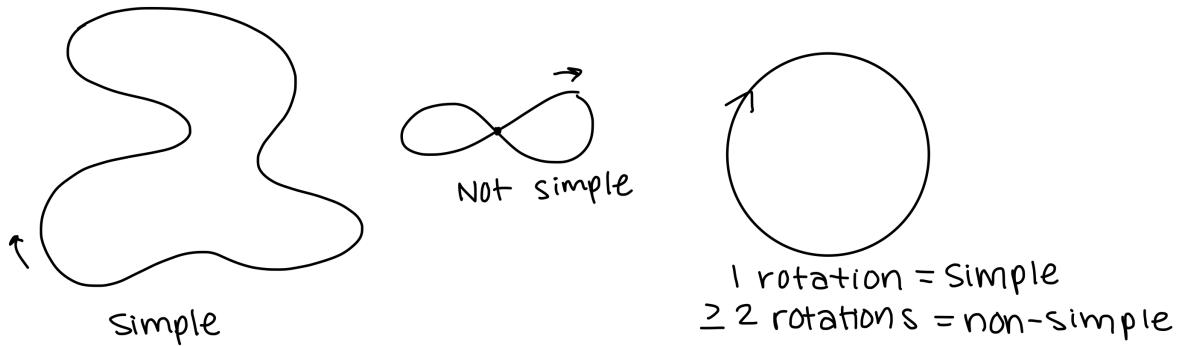
# 10 Jan 26, 2022

## 10.1 Hopf's Theorem

**Definition 10.1** (Simple)

A closed regular curve  $\gamma: [a, b] \rightarrow \mathbb{R}^n$  is called simple if  $\gamma$  is one-to-one on  $[a, b]$ .

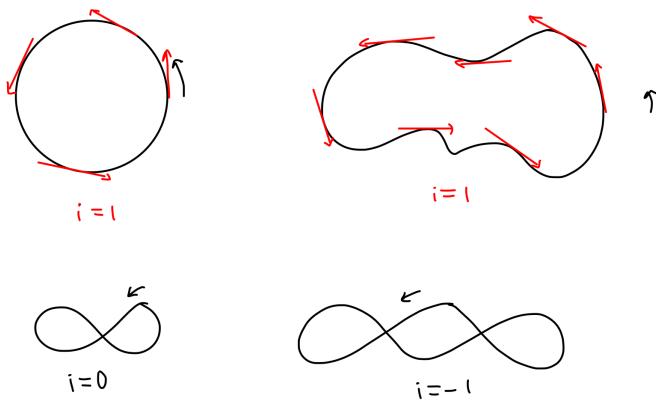
**Remark 10.2** simple = no self-intersection + 1 full rotation



**Theorem 10.3** (Hopf's Umlaufsatz)

Let  $\gamma: [a, b] \rightarrow \mathbb{R}^2$  be a simple closed plane curve. Then  $i_\gamma = \pm 1$ .

**Recall 10.4**  $i_\gamma = \frac{1}{2\pi}(\theta(b) - \theta(a))$  = “degree” for  $t$ , where  $\theta$  is a smooth angle function from  $[a, b]$  to  $\mathbb{R}$  such that  $\forall t \in [a, b], t(t) = (\cos \theta(t), \sin \theta(t))$ .



Idea: Deform the unit tangent to another function, while the “degree” is constant in a family of continuous functions.

**Proposition 10.5**

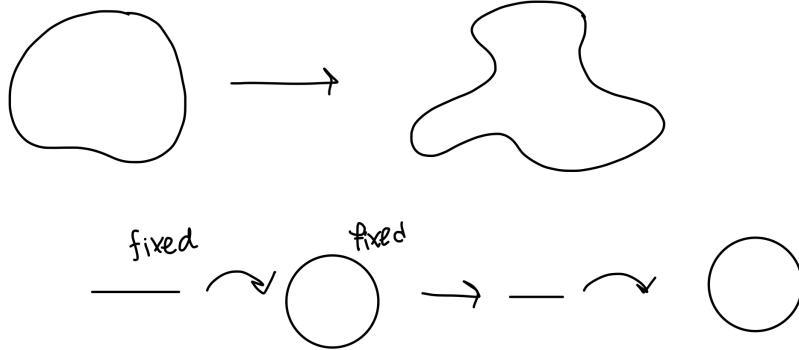
Let  $f: [a, b] \rightarrow S^1 = \{x \in \mathbb{R}^2 \mid |x| = 1\}$  be a continuous function. Then there exists a continuous angle function  $\theta: [a, b] \rightarrow \mathbb{R}$  such that  $\forall t \in [a, b], f(t) = (\cos \theta(t), \sin \theta(t))$ . The angle function  $\theta$  is unique up to adding a multiple of  $2\pi$ . If  $f(a) = f(b)$ , then  $\frac{1}{2\pi}(\theta(b) - \theta(a)) \in \mathbb{Z}$  and the integer is called the degree of  $f$ , denoted by  $\deg(f)$ .

**Remark 10.6** If  $\gamma: [a, b] \rightarrow \mathbb{R}^2$  is a closed plane curve, then the unit tangent gives  $t: [a, b] \rightarrow S^1$ , and  $\deg(t) = i_\gamma$ .

**Proof of Proposition (Sketch).** Using  $\cos^{-1}, \sin^{-1}$ , define  $\theta$  locally, then patch them to define  $\theta$  globally so that  $\theta$  is continuous on entire  $[a, b]$ .  $\square$

**Proposition 10.7**

$\deg(f)$  is locally constant under deformation (continuous change of shapes) of  $f: [a, b] \rightarrow S^1$ . Loosely speaking, the proposition says that  $\deg(f)$  does not change by small continuous change in  $f$ .



This follows from another lemma:

**Lemma 10.8**

Let  $f_1, f_2: [a, b] \rightarrow S^1$  be continuous functions. If  $\deg(f_1) \neq \deg(f_2)$ , then  $\exists t_0 \in [a, b], f_1(t_0) = -f_2(t_0)$ .

**Remark 10.9** If  $f_1, f_2$  never point in the opposite directions, then  $\deg(f_1) = \deg(f_2)$ .

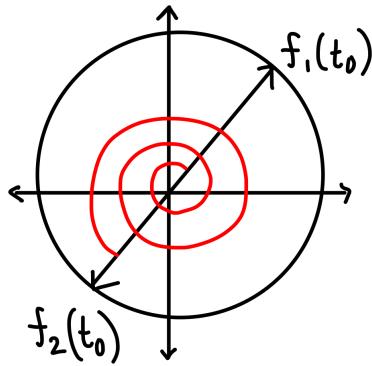
**Proof (sketch).**  $\theta_1, \theta_2$ : angle functions for  $f_1, f_2, \theta := \theta_1 - \theta_2$ . Then

$$\begin{aligned} |\theta(a) - \theta(b)| &= \left| \underbrace{(\theta_1(a) - \theta_1(b))}_{2\pi \deg(f_1)} - \underbrace{(\theta_2(a) - \theta_2(b))}_{2\pi \deg(f_2)} \right| \\ &\geq 2\pi \end{aligned}$$

$\implies \exists$  odd multiple of  $\pi$  between  $\theta(a), \theta(b), (2n - 1)\pi$ .

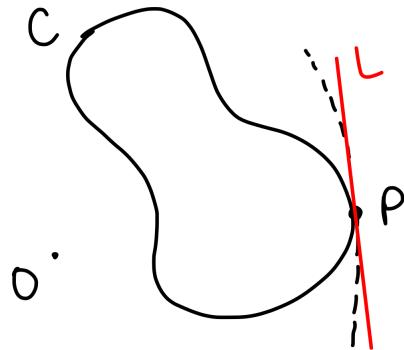
$$\underset{\text{IVT}}{\implies} \exists t_0 \in [a, b], \quad \theta(t_0) = (2n - 1)\pi$$

$$\theta_1(t_0) = \theta_2(t_0) + (2n - 1)\pi$$

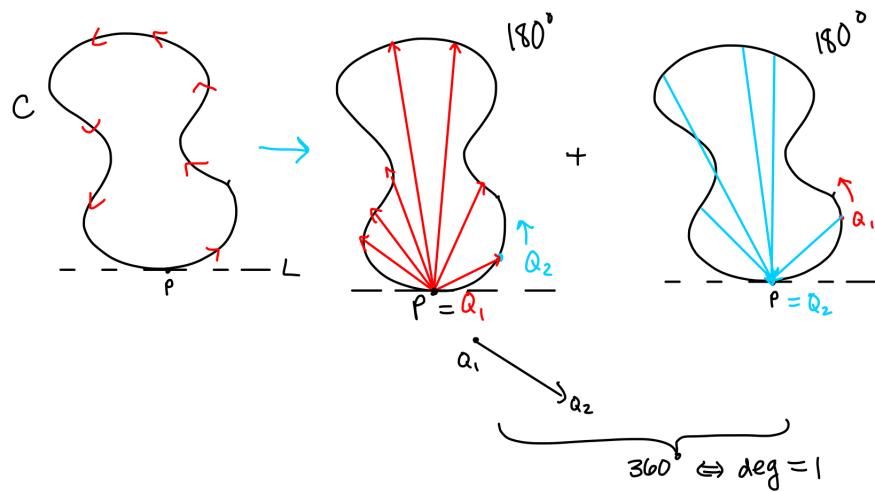


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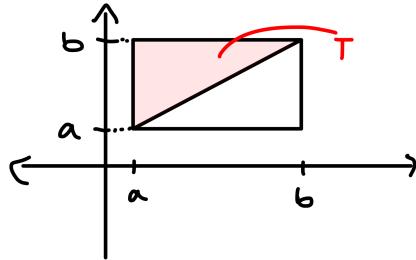
**Proof of Hopf's Umlaufsatz.** Let  $\gamma: [a, b] \rightarrow \mathbb{R}^2$  be a simple closed curve,  $c :=$  the trace of  $\gamma$ . We need to show  $i_\gamma = \pm 1$ . Let  $p \in C$  such that  $|\gamma|$  has the maximum at  $p$ .



Then  $C$  is entirely on one side of the tangent line  $L$  to  $C$  at  $p$ . We may assume  $\gamma$  is unit-speed,  $p = \gamma(a)$ .



Set  $T := \{(t_1, t_2) \in \mathbb{R}^2 \mid a \leq t_1 \leq t_2 \leq b\}$



Define  $\psi: T \rightarrow S^1$  as follows:

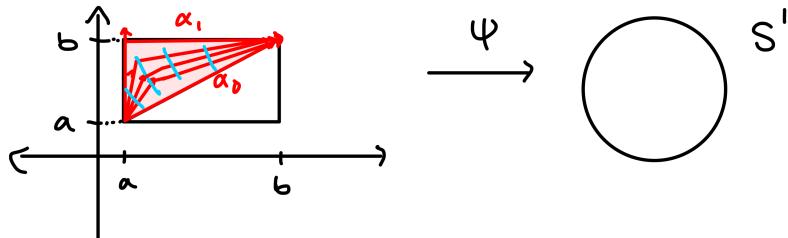
$$\psi(t_1, t_2) := \begin{cases} \gamma'(t_1) = t(t_1) & \text{if } t_1 = t_2 \\ \frac{\gamma(t_2) - \gamma(t_1)}{|\gamma(t_2) - \gamma(t_1)|} & \text{if } t_1 \neq t_2 \cap (t_1, t_2) \neq (a, b) \\ -\gamma'(a) & \text{if } (t_1, t_2) = (a, b) \end{cases}$$

$\psi$  is well-defined because  $\gamma$  is simple, and  $\psi$  is continuous. For instance,

$$\psi(t_1, t_2) = \gamma'(t_1) = \lim_{t_2 \rightarrow t_1} \frac{\gamma(t_2) - \gamma(t_1)}{|\gamma(t_2) - \gamma(t_1)|} = \lim_{t_2 \rightarrow t_1} \psi(t_1, t_2)$$

Consider paths:

$$\begin{aligned} \alpha_0: [0, 1] &\rightarrow T \quad (a, a) \rightarrow (b, b) \\ \alpha_1: [0, 1] &\rightarrow T \quad (a, a) \rightarrow (a, b) \rightarrow (b, b) \end{aligned}$$

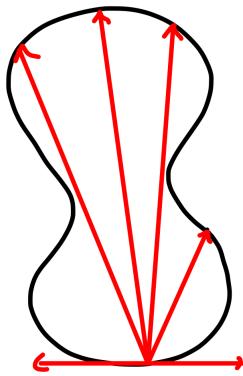


$\alpha_0$  deforms to  $\alpha_1$  in a family of continuous functions

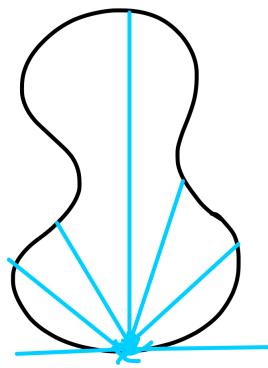
$$\begin{aligned}\alpha_s &= (1-s)\alpha_0 + s\alpha_1, \quad s \in [0, 1] \\ \implies \psi \circ \alpha_0 &\text{ deforms to } \psi \circ \alpha_1: [0, 1] \rightarrow S^1 \\ \implies \deg(\psi \circ \alpha_0) &= \deg(\psi \circ \alpha_1) \\ \deg(\psi \circ \alpha_0) &= \deg(t) = i_\gamma\end{aligned}$$

Enough to show  $\deg(\psi \circ \alpha_1) = \pm 1$ .

$$(a, a) \rightarrow (a, b): \psi(a, t) = \frac{\gamma(t) - \gamma(a)}{|\gamma(t) - \gamma(a)|}$$



$$(a, b) \rightarrow (b, b): \psi(t, b) = \frac{\gamma(b) - \gamma(t)}{|\gamma(b) - \gamma(t)|}$$



□

# 11 Jan 28, 2022

## 11.1 Midterm 1

# 12 Jan 31, 2022

## 12.1 Jordan's Theorem

**Definition 12.1** (Path-connected)

A subset  $S \subseteq \mathbb{R}^n$  is called path-connected if any two points in  $S$  are connected by a continuous path in  $S$ .

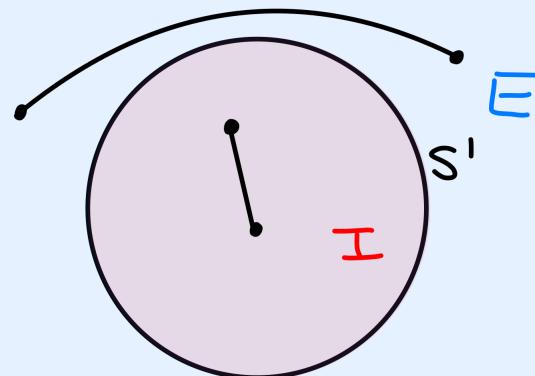
**Example 12.2**

We have

$$I := \{\vec{x} \in \mathbb{R}^2 \mid |\vec{x}| < 1\}$$

$$E := \{\vec{x} \in \mathbb{R}^2 \mid |\vec{x}| > 1\}$$

$I, E$  are both path-connected.



**Definition 12.3** (Path-connected component)

A path-connected component of a subset  $S \subseteq \mathbb{R}^n$  is a maximal path-connected subset of  $S$ .

**Example 12.4**

$\mathbb{R}^2 - S^1$  has exactly two connected components, namely  $I, E$ .

**Theorem 12.5** (Jordan's Theorem)

Let  $\gamma: [a, b] \rightarrow \mathbb{R}^2$  be a simple closed plane curve, and  $C$  the trace of  $\gamma$ . Then  $\mathbb{R}^2 - C$  has exactly two path-connected components. One is bounded (called the interior), and the other is unbounded (called the exterior).

**Remark 12.6** Intuitively clear, but a rigorous proof is not easy.

**Recall 12.7**  $f: [a, b] \rightarrow S^1$  continuous,  $f(a) = f(b), \forall t \in [a, b]$ ,

$$f(t) = (\cos \theta(t), \sin \theta(t))$$

where  $\theta: [a, b] \rightarrow \mathbb{R}$  continuous

$$\deg f := \frac{1}{2\pi}(\theta(b) - \theta(a)) \in \mathbb{Z}$$

**Proposition 12.8**

Let  $D \subseteq \mathbb{R}^n$  be a subset. Let  $\{f_s\}_{s \in D}$  be a continuous family of continuous functions  $f_s: [a, b] \rightarrow S^1$ , i.e.

$$\begin{aligned} [a, b] \times D &\rightarrow S^1 \text{ is continuous} \\ (t, s) &\mapsto f_s(t) \end{aligned}$$

Then,

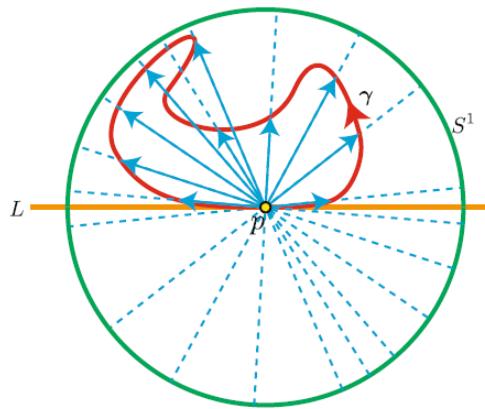
$$\begin{aligned} \deg: D &\rightarrow \mathbb{Z}, \\ s &\mapsto \deg f_s \end{aligned}$$

is constant on every path-connected component of  $D$ .

**Proof of Jordan's Theorem (Sketch).** Let  $\gamma: [a, b] \rightarrow \mathbb{R}^2$  be simple closed,  $C = \text{im } \gamma$ . For  $p \in \mathbb{R}^2 - C$ ,

$$f_p(t) := \frac{\gamma(t) - p}{|\gamma(t) - p|},$$

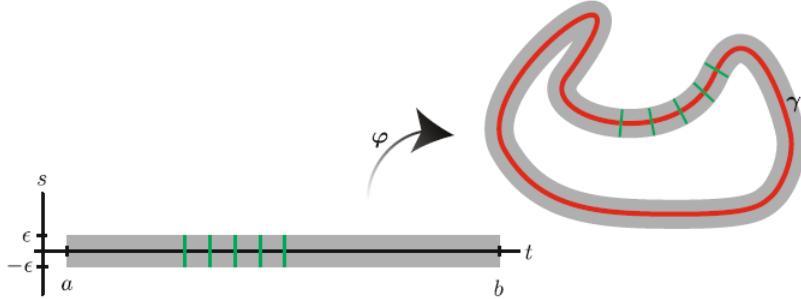
$f_p: [a, b] \rightarrow S^1$  continuous



$\{f_p\}_{p \in \mathbb{R}^2 - C}$  continuous family.

We show:  $\mathbb{R}^2 - C$  has exactly two path-component components, one on which  $\underbrace{\deg f_p = 0}_{\text{unbounded}}$ , the other on which  $\underbrace{\deg f_p = 1 \text{ or } -1}_{\text{bounded}}$ .

Idea: Consider a tubular neighborhood of  $C$ . A tubular neighborhood being a thickening of  $C$  by  $\pm \varepsilon$  in the normal direction of  $C$ .



The tubular neighborhood has no self-intersection, as  $C$  is simple. Take  $P, Q$  in the zoom window that are very close to each other, but on the opposites of  $C$ .

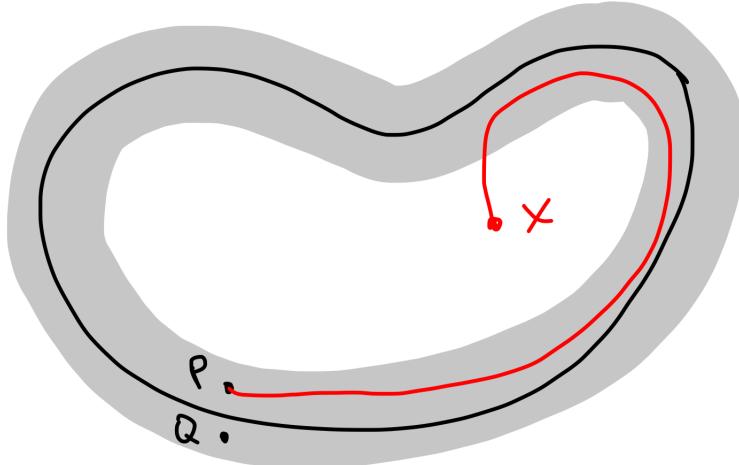
Step 1: Show  $|\deg f_P - \deg f_Q| = 1 \implies \mathbb{R}^2 - C$  has at least two components.

$f_P - f_Q$  almost makes  $\pm 1$  rotation on  $[a, c]$ , while  $f_P - f_Q$  makes only a very small change in  $[c, b]$ , too small to contribute to the change of the degree.

Step 2: Show  $\mathbb{R}^2 - C$  has exactly two path-connected components.

Let  $x \in \mathbb{R}^2 - C$ . Then  $x$  is connected to either  $P$  or  $Q$  by a continuous path in  $\mathbb{R}^2 - C$  as follows:

- Choose a shortest path from  $x$  to  $C$ .
- Before reaching  $C$ , the path reaches the tubular neighborhood of  $C$ .
- Then, inside the tubular neighborhood, the path can be connected to either  $P$  or  $Q$ .



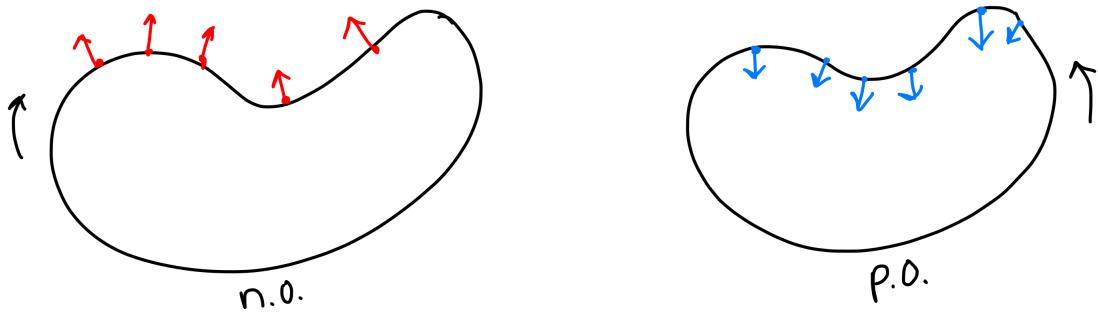
□

**Definition 12.9** (Positively oriented vs. negatively oriented)

A simple closed plane curve  $\gamma: [a, b] \rightarrow \mathbb{R}^2$  is positively oriented if the interior is always on one's left as one traverses  $\gamma$ :

$$\forall \varepsilon > 0, \forall t \in [a, b], \forall S \in (0, \varepsilon),$$

$\gamma(t) + Sn_s(t)$  is in the interior and negatively oriented if the exterior is always on one's left and  $\gamma(t) + Sn_s(t)$  is in the exterior.

**Remarks 12.10**

- i.  $\gamma$  is either positively oriented or negatively oriented, as  $\deg f_{\gamma(t)+Sn_s(t)}, t(s) \in \underbrace{[a, b] \times (0, \varepsilon)}_{\text{path-connected}}$  is constant, hence 0 (n.o.) or  $\pm 1$  (p.o.)
- ii.
  - p.o.  $\iff i = 1$
  - n.o.  $\iff i = -1$

# 13 Feb 2, 2022

## 13.1 Jordan's Theorem (Cont'd)

**Definition 13.1** (Piecewise regular curve, closed, simple, positively vs. negatively oriented)

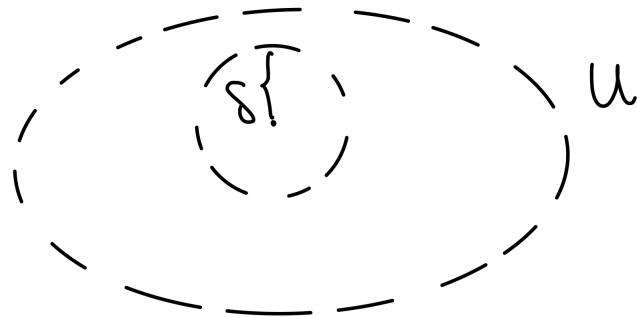
A piecewise regular curve is a continuous function  $\gamma: [a, b] \rightarrow \mathbb{R}^n$  with partition  $a = t_0 < t_1 < \dots < t_m = b$  such that  $\gamma_i := \gamma|_{[t_{i-1}, t_i]}: [t_{i-1}, t_i] \rightarrow \mathbb{R}^n$  is a regular curve for each  $i = 1, \dots, m$ . Such a curve is called closed if  $\gamma(a) = \gamma(b)$ , simple if  $\gamma$  is one-to-one on  $[a, b]$ .

When  $n = 2$ , such a curve is called positively oriented if  $\vec{n}_s(t)$  points toward the interior of  $C$  for all  $t \in [a, b]$  corresponding to smooth points, and negatively oriented if  $\vec{n}_s(t)$  points toward the exterior of  $C$ .

**Remark 13.2** Jordan's theorem is true for piecewise regular simple closed plane curves.

## 13.2 Green's Theorem

**Recall 13.3**  $U \subset \mathbb{R}^n$  is open  $\iff \forall x \in U, \exists \delta > 0, \forall y \in \mathbb{R}^n, |y - x| < \delta \implies y \in U$ .

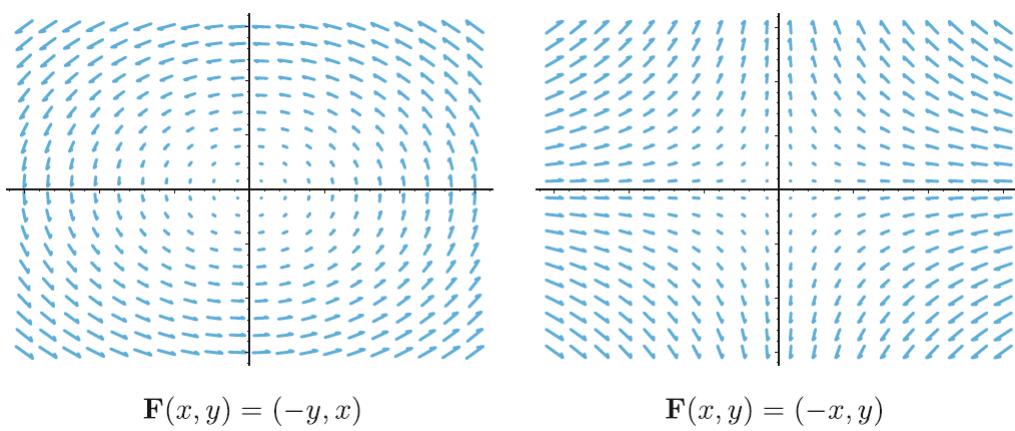


**Definition 13.4** (Vector field)

A vector field on an open subset  $U \subset \mathbb{R}^n$  is a smooth function  $\vec{F}: U \rightarrow \mathbb{R}^n$ , where smooth means

$$\vec{F} = (F_1, \dots, F_n), F_1, \dots, F_n \text{ smooth}$$

**Remark 13.5** A vector field assigns to each point in  $U$  a vector in  $\mathbb{R}^n$ .

**Example 13.6**

An “oriented curve” will mean a regular curve  $\gamma: I \rightarrow \mathbb{R}^n$  with its trace.

**Definition 13.7** (line integral)

Let  $C$  be an oriented curve parametrized as  $\gamma: [a, b] \rightarrow \mathbb{R}^n$ ,  $\mathbf{F}$  be a vector field whose domain contains  $C$ . The line integral of  $\mathbf{F}$  along  $C$  is defined as:

$$\int_C \mathbf{F} \cdot d\gamma := \int_a^b \langle \mathbf{F}(\gamma(t)), \gamma'(t) \rangle dt$$

when  $C$  is simple closed, then the line integral is also denoted by

$$\oint_C \mathbf{F} \cdot d\gamma$$

**Remark 13.8**  $\mathbf{F}$  force field  $\implies \int_C \mathbf{F} \cdot d\gamma$  total work along  $C$ .

**Proposition 13.9**

The line integral is unchanged by any orientation-preserving reparametrization, and multiplied by  $-1$  by any orientation-reversing reparametrization.

| **Proof.** Homework. □

**Remark 13.10** This shows that the line integral is well-defined for an equivalence class of oriented curves modulo orientation-preserving reparametrization. We will work with such a class, instead of an oriented curve itself.

**Example 13.11**

$$\mathbf{F}(x, y) = (-y, x).$$

$C_1 :=$  the counterclockwise circle of radius 3 centered at the origin.

$C_1$  can be parametrized by  $\gamma_1(t) = (3 \cos t, 3 \sin t), t \in [0, 2\pi]$

$$\begin{aligned} \oint_{C_1} \mathbf{F} \cdot d\gamma_1 &= \int_0^{2\pi} \langle \mathbf{F}(\gamma_1(t)), \gamma'(t) \rangle dt \\ &= \int_0^{2\pi} \langle (-3 \sin t, 3 \cos t), (-3 \sin t, 3 \cos t) \rangle dt \\ &= \int_0^{2\pi} 9 \sin^2 t + 9 \cos^2 t dt \\ &= 9 \int_0^{2\pi} dt = 18\pi \end{aligned}$$

$C_2 :=$  the graph of the parabola  $y = x^2$  from  $(-1, 1)$  to  $(1, 1)$ .

$$\gamma_2(t) = (t, t^2), \quad t \in [-1, 1].$$

$$\begin{aligned} \int_{C_2} \mathbf{F}(t) \cdot d\gamma_2 &= \int_{-1}^1 \langle \mathbf{F}(\gamma_2(t)), \gamma'(t) \rangle dt \\ &= \int_{-1}^1 \langle (-t^2, t), (1, 2t) \rangle dt \\ &= \int_{-1}^1 (-t^2 + 2t^2) dt \\ &= \left[ \frac{t^3}{3} \right]_{-1}^1 = \frac{2}{3} \end{aligned}$$

**Remark 13.12** The line integral can be defined for a piecewise-regular curve  $(C, \gamma)$  with smooth pieces  $(C_i, \gamma_i)$ :

$$\int_C \mathbf{F} \cdot d\gamma := \sum_i \int_{C_i} \mathbf{F} \cdot d\gamma_i$$

**Theorem 13.13 (Green's Theorem)**

Let  $C$  be a positively oriented piecewise-regular simple closed plane curve parametrized by  $\gamma: [a, b] \rightarrow \mathbb{R}^2$ ,  $D$  be the interior of  $C$ . Let  $\mathbf{F}$  be a vector field whose domain contains  $C \cup D$ . Then,

$$\oint_C \mathbf{F} \cdot d\gamma = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy,$$

where  $\mathbf{F} = (P, Q)$ .

| **Proof.** Read Tapp for the proof. □

**Remark 13.14** Green's Theorem is a special case of the generalized Stokes Theorem:

$$\int_{\partial D} F = \int_D dF,$$

where  $D$  is a “region” in  $\mathbb{R}^n$  with boundary  $\partial D$ ,  $F$  is a “function” on  $D$ ,  $d$  is a “derivative”.

**Corollary 13.15**

Let  $(C, \gamma), \mathbf{F}$  as in Green's Theorem. Write  $\gamma(t) = (x(t), y(t))$ . Then,

$$\begin{aligned} \text{Area}(D) &= \int_a^b x(t)y'(t) dt \\ &= - \int_a^b x'(t)y(t) dt \end{aligned}$$

**Proof.** Apply Green's Theorem to:

$$\mathbf{F}_1(x, y) = (0, x), \quad \mathbf{F}_2(x, y) = (-y, 0).$$

For instance,

$$\int_C \mathbf{F}_1 \cdot d\gamma = \iint_D \left( \frac{\partial x}{\partial x} - \frac{\partial 0}{\partial y} \right) dx dy$$

$$\begin{aligned} \text{L.H.S.} &= \int_a^b \langle (0, x(t)), (x'(t), y'(t)) \rangle dt \\ &= \int_a^b x(t)y'(t) dt \\ \text{R.H.S.} &= \iint_D dx dy = \text{Area}(D) \end{aligned}$$

□

# 14 Feb 4, 2022

## 14.1 Isoperimetric Inequality

**Theorem 14.1** (Isoperimetric Inequality)

Let  $C$  be a simple closed plane curve,  $\ell$  be the arc length of  $C$ ,  $A$  be the area of the interior of  $C$ . Then

$$\ell^2 \geq 4\pi A$$

Moreover,

$$\text{“} = \text{”} \iff C \text{ is a circle.}$$

**Remark 14.2** Theorem says among all simple closed plane curves with fixed perimeter, the circle bounds the largest area.

3 main ingredients for the proof

i. (Corollary of) Green's Theorem:

Let  $C$  be a positively oriented piecewise-regular simple closed plane curve, parametrized by  $\gamma(t) = (x(t), y(t))$ ,  $t \in [a, b]$ ,  $D$  be the interior of  $C$ . Then

$$\text{Area}(D) = \int_a^b x(t)y'(t) dt = - \int_a^b x'(t)y(t) dt$$

ii. Schwartz inequality:

$\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ , then

$$\langle \mathbf{x}, \mathbf{y} \rangle \leq |\mathbf{x}| \cdot |\mathbf{y}|$$

and,

“ $=$ ”  $\iff \mathbf{x}, \mathbf{y}$  point toward the same direction

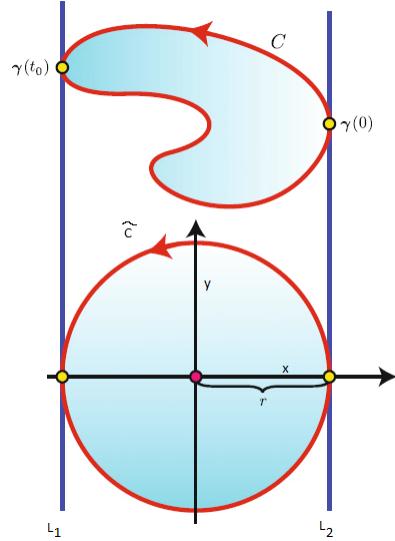
$$\iff_{\text{if } \mathbf{x} \neq \mathbf{0}} \exists c \geq 0, \mathbf{y} = c\mathbf{x}$$

iii. AM-GM inequality:

$a, b \geq 0$ , then

$$\sqrt{ab} \leq \frac{a+b}{2}, \text{ and “} = \text{”} \iff a = b$$

**Proof of Isoperimetric Inequality.** Let  $C$  be a simple closed plane curve. Let  $L_1, L_2$  be two parallel tangent lines to  $C$  so that  $C$  is between  $L_1, L_2$ :



Set  $r := (\text{the distance between } L_1, L_2) \times \frac{1}{2}$

Let  $\tilde{C}$  be a circle tangent to  $L_1, L_2$ . Then  $\tilde{C}$  has radius  $r$ .

Choose a coordinate system  $\{x, y\}$  for  $\mathbb{R}^2$  so that  $\tilde{C}$  has center  $(0, 0)$ ,  $L_1 = \{x = -r\}$ ,  $L_2 = \{x = r\}$ .

Let  $\gamma: [0, \ell] \rightarrow \mathbb{R}^2$  be a positively oriented unit-speed parametrization of  $C$ . May assume  $\gamma(0) =$  the tangent point with  $L_2$ . Let  $t_0 \in [0, \ell]$  such that  $\gamma(t_0) =$  the tangent point with  $L_1$ .

Let:  $\mathbf{B}: [0, \ell] \rightarrow \mathbb{R}^2$  be the parametrization of  $\tilde{C}$  given by  $\mathbf{B}(t) = (x(t), \tilde{y}(t))$ , where

$$\tilde{y}(t) := \begin{cases} \sqrt{r^2 - x(t)^2} & \text{if } t \in [0, t_0] \\ -\sqrt{r^2 - x(t)^2} & \text{if } t \in [t_0, \ell] \end{cases}$$

Note:  $\mathbf{B}$  might not be regular, nor simple, but no issue when computing the area. By Green's Theorem,

$$\begin{aligned} A &= \int_0^\ell x(t)y'(t) dt \\ \pi r^2 &= - \int_0^\ell x'(t)\tilde{y}(t) dt \end{aligned}$$

Then,

$$\begin{aligned} A + \pi r^2 &= \int_0^\ell (x(t)y'(t) - x'(t)\tilde{y}(t)) dt \\ &= \int_0^\ell \langle (x'(t), y'(t)), (-\tilde{y}(t), x(t)) \rangle dt \\ &\leq \int_0^\ell \underbrace{|(x'(t), y'(t))|}_{\substack{=1 \text{ unit-speed} \\ \text{by Schwartz Inequality}}} \cdot \underbrace{|(-\tilde{y}(t), x(t))|}_r dt \\ &= \int_0^\ell r dt = \ell r \end{aligned}$$

By AM-GM inequality,

$$\begin{aligned}\sqrt{A \cdot \pi r^2} &\leq \frac{A + \pi r^2}{2} \leq \frac{\ell r}{2} \\ A \cdot \pi r^2 &\leq \frac{\ell^2 r^2}{4} \\ 4\pi A &\leq \ell^2\end{aligned}$$

The first statement is proved!

Suppose  $4\pi A = \ell^2$ .

Then the Schwartz, AM-GM inequalities are equalities:

1.  $\exists c \geq 0, (-\tilde{y}(t), x(t)) = c(x'(t), y'(t))$
2.  $A = \pi r^2$ .

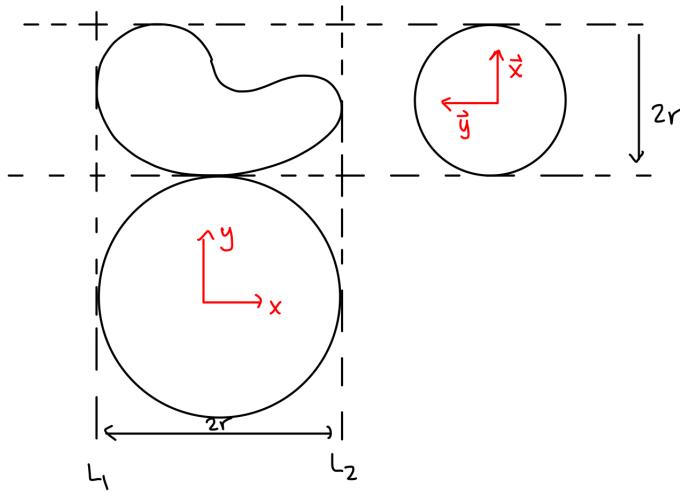
From (i),

$$\begin{aligned}\underbrace{|\tilde{y}(t), x(t)|}_{=r} &= c \cdot \underbrace{|(x'(t), y'(t))|}_{=1} \\ \implies c &= r \\ \implies (-\tilde{y}(t), x(t)) &= r(x'(t), y'(t)) \\ \implies x(t) &= ry'(t)\end{aligned}\tag{1}$$

From (ii),

$$r = \sqrt{\frac{A}{\pi}}$$

This shows  $r$  does not depend on the directions of  $L_1, L_2$ .



Now we repeat the process for two parallel tangent lines perpendicular to  $L_1, L_2$ .

Let  $\{\bar{x}, \bar{y}\}$  be the corresponding coordinate system. Then  $\bar{x}(t) = r\bar{y}'(t)$ . On the other hand,

$$\begin{cases} \bar{x}(t) = y(t) + d \\ \bar{y}(t) = -x(t) = \ell \end{cases} \quad \text{for } d, \ell \in \mathbb{R}$$

Then

$$y(t) + d = -rx'(t) \quad (2)$$

Then

$$\begin{aligned} & x(t)^2 + (y(t) + d)^2 \\ &= (ry'(t))^2 + (-rx'(t))^2 \quad (\text{by 1 and 2}) \\ &= r^2(x'(t)^2 + y'(t)^2) = r^2 \quad (\text{unit-speed}) \end{aligned}$$

Therefore  $C$  is a circle of radius  $r$ .  $\square$

## 14.2 The Derivative of Functions from $\mathbb{R}^m$ to $\mathbb{R}^n$

**Definition 14.3** (Partial derivatives,  $C^r$  on  $U$ , smooth on  $U$ )

Let  $f: U \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$  be a function, when  $U \subset \mathbb{R}^m$  is an open subset.

The partial derivative of  $f$  with respect to  $x$  at  $p \in U$  is defined as

$$\frac{\partial f}{\partial x_i}(p) = f_{x_i}(p) := \lim_{h \rightarrow 0} \frac{f(p + he_i) - f(p)}{h},$$

where  $e_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^n$ .

A second order partial derivative is a partial derivative of a partial derivative, and so on.

For instance,

$$\frac{\partial^2 f}{\partial x_j \partial x_i} = f_{x_i, x_j} := \frac{\partial}{\partial x_j} \left( \frac{\partial f}{\partial x_i} \right)$$

$f$  is called  $C^r$  on  $U$  if all  $r$ -th partial derivatives exist and are continuous on  $U$ .  $f$  is smooth on  $U$  if  $\forall r \in \mathbb{Z} > 0$ ,  $f$  is  $C^r$  on  $U$ .

# 15 Feb 7, 2022

## 15.1 The Derivative of Functions from $\mathbb{R}^m$ to $\mathbb{R}^n$ (Cont'd)

**Recall 15.1** Given  $f: U \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$ ,

$f$  is  $C^r$  on  $U \iff$  All  $r$ -th partial derivatives exist and are continuous on  $U$ .

$f$  is smooth on  $U \iff \forall r \in \mathbb{Z}_{>0}, f$  is  $C^r$  on  $U$ .

### Remarks 15.2

- i. If  $f$  is  $C_2$ , then  $\forall i, j, f_{x_i, x_j} = f_{x_j, x_i}$ .  
If  $f$  is smooth, then mixed partials of any order commute.
- ii.  $f$  is smooth  $\iff \forall i, f_i$  is smooth, where  $f = (f_1, \dots, f_n)$ .

### Example 15.3

Let  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be the smooth function defined by

$$f(x, y) = (\sin(xy), e^{x+5y}, x^2y^3)$$

Then,

$$f_x = (y \cos(xy), e^{x+5y}, 3xy^3)$$

$$f_y = (x \cos(xy), 5e^{x+5y}, 3x^2y^2)$$

and

$$f_{xx} = (-y^2 \sin(xy), e^{x+5y}, 2y^3)$$

$$f_{xy} = (f_x)_y = (\cos(xy) - xy \sin(xy), 5e^{x+5y}, 6xy^2) = f_{yx}$$

$$f_{yy} = (-x^2 \sin(xy), 25e^{x+5y}, 6x^2y)$$

### Proposition 15.4

If  $g: V \subset \mathbb{R}^\ell \rightarrow \mathbb{R}^m, f: U \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$  are smooth and  $g(V) \subset U$ , then  $f \circ g: V \subset \mathbb{R}^\ell \rightarrow \mathbb{R}^n$  is smooth.

### Example 15.5

We have

$$f(x, y) = (\sin(xy), e^{x+5y}, x^2y^3)$$

so  $f$  is the composition of the smooth functions:

$$(x, y) \mapsto (xy, x + 5y, x^2y^3)$$

and

$$(u, v, w) \mapsto (\sin(u), e^v, w)$$

**Definition 15.6** (Differentiable)

Let  $f: U \subset \mathbb{R}^m \rightarrow \mathbb{R}^n, p \in U$ .  $f$  is differentiable at  $p \in U$  if there exists an  $n \times m$  matrix  $M$  such that

$$f(p + v) = f(p) + M \cdot v + E(v),$$

where

$$\lim_{v \rightarrow 0} \frac{E(v)}{|v|} = 0.$$

In this case,  $M$  is called the derivative of  $f$  at  $p$ , and denoted by  $f'(p)$ .

**Remarks 15.7**

- i.  $f'(p)$  is uniquely determined if it exists.
- ii.  $f(p + v) = f(p) + f'(p) \cdot v + E(v)$  is the first order Taylor approximation of  $f$  at  $p$ .
- iii. Smooth  $\Rightarrow$  differentiable  $\Rightarrow f_{x_1}, \dots, f_{x_m}$  all exist.

**Proposition 15.8**

Suppose  $f: U \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$  is differentiable at  $p \in U$ . Then

$$f'(p) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_1}{\partial x_m}(p) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(p) & \cdots & \frac{\partial f_n}{\partial x_m}(p) \end{pmatrix}$$

where  $f'(p)$  is the Jacobian matrix of  $f$  at  $p$ .

**Example 15.9**

Let  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be defined by

$$f(x, y) = (x^2 y^3, x + y^4, y + 1)$$

Then,

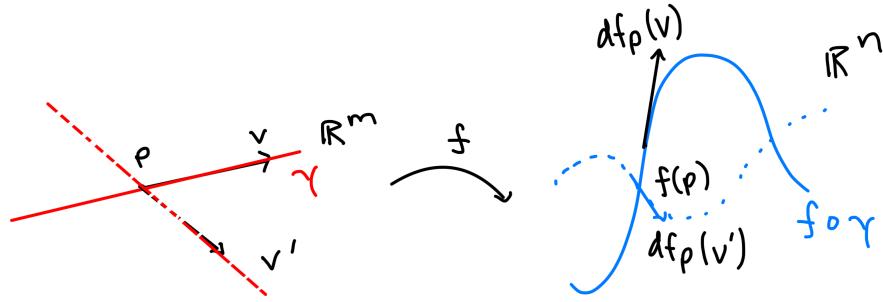
$$f' = \begin{pmatrix} \frac{\partial}{\partial x}(x^2 y^3) & \frac{\partial}{\partial y}(x^2 y^3) \\ \frac{\partial}{\partial x}(x + y^4) & \frac{\partial}{\partial y}(x + y^4) \\ \frac{\partial}{\partial x}(y + 1) & \frac{\partial}{\partial y}(y + 1) \end{pmatrix} = \begin{pmatrix} 2xy^3 & 3x^2 y^2 \\ 1 & 4y^3 \\ 0 & 1 \end{pmatrix}$$

**Definition 15.10** (Directional derivative)

Let  $f: U \subset \mathbb{R}^m \rightarrow \mathbb{R}^n, p \in U, v \in \mathbb{R}^m$ . The directional derivative of  $f$  in the direction of  $v$  at  $p$  is defined as

$$df_p(v) := \lim_{t \rightarrow 0} \frac{f(p + tv) - f(p)}{t} = (f \circ \gamma)'(0),$$

where  $\gamma(t) = p + tv$ .

**Remarks 15.11**

- i.  $\frac{\partial f}{\partial x_i}(p) = df_p(e_i)$
- ii. differentiable at  $p \implies \forall v \in \mathbb{R}^m, df_p(v)$  exists

**Proposition 15.12**

Suppose  $f: U \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$  is differentiable at  $p \in U$ . Then  $\forall v \in \mathbb{R}^m, df_p(v) = f'(p) \cdot v$ .

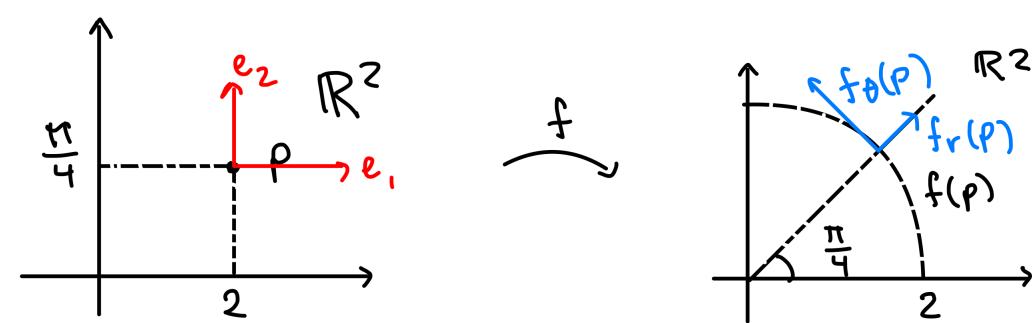
**Remark 15.13** Proposition says:

- $df_p: \mathbb{R}^m \rightarrow \mathbb{R}^n$  is a linear transformation represented by  $f'(p)$ .
- $df_p$  is also called the derivative of  $f$  at  $p$  and will be identified with the matrix:  
 $df_p = f'(p)$

**Example 15.14**

Let  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2, (r, \theta) \mapsto (r \cos \theta, r \sin \theta)$ ,  $p = (2, \frac{\pi}{4}) \in \mathbb{R}^2$ . Then,

$$\begin{aligned} df_p &= \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} \Big|_{(r, \theta) = (2, \frac{\pi}{4})} \\ &= \begin{pmatrix} \frac{\sqrt{2}}{2} & -\sqrt{2} \\ \frac{\sqrt{2}}{2} & \sqrt{2} \end{pmatrix} \\ &\quad \underbrace{f_r(p)}_{\sqrt{2}} \quad \underbrace{f_\theta(p)}_{\sqrt{2}} \end{aligned}$$

**Proposition 15.15** (Chain Rule)

If  $g: V \subset \mathbb{R}^\ell \rightarrow \mathbb{R}^m, f: U \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$  are differentiable and  $g(V) \subset U$ , then  $\forall q \in V$ ,

$$(f \circ g)'(q) = \underbrace{f'(g(q))}_{n \times m} \cdot \underbrace{g'(q)}_{m \times \ell}$$

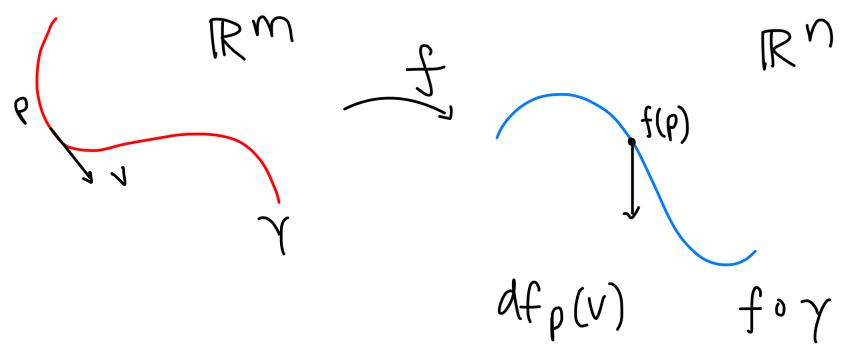
or,

$$d(f \circ g)_q = (df)_{g(q)} \circ (dg)_q$$

**Corollary 15.16**

Let  $f: U \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$  be a differentiable function,  $\gamma: I \rightarrow U \subset \mathbb{R}^m$  be a regular curve such that  $0 \in I, \gamma(0) = p \in U, \gamma'(0) = v \in \mathbb{R}^m$ . Then

$$df_p(v) = (f \circ \gamma)'(0).$$



| **Proof.** Apply the Chain rule to  $f$  and  $g = \gamma$ . □

# 16 Feb 9, 2022

## 16.1 The Derivative of Functions from $\mathbb{R}^m$ to $\mathbb{R}^n$ (Cont'd)

**Recall 16.1**  $f: U \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$  smooth,  $P \in U$ ,  $df_p: \mathbb{R}^m \rightarrow \mathbb{R}^n$  is represented by

$$f'(p) = \left( \frac{\partial f_i}{\partial x_j}(p) \right)$$

Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be smooth. Suppose  $f$  is a smooth bijection such that  $f^{-1}: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is smooth. Then  $\forall p \in \mathbb{R}^n$ ,  $df_p$  is an invertible linear transformation. Indeed, by Chain rule,

$$I_n = d(f \circ f^{-1})_{f(p)} = (df)_p \circ (df^{-1})_{f(p)}$$

$$I_n = d(f^{-1} \circ f)_p = (df^{-1})_{f(p)} \circ (df)_p$$

Therefore,  $(df)_p$  is invertible.

**Recall 16.2** A neighborhood of  $p \in \mathbb{R}^n$  is an open subset  $U \subset \mathbb{R}^n$  such that  $p \in U$ .

### Theorem 16.3 (Inverse Function Theorem)

Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be smooth on a neighborhood of  $p \in \mathbb{R}^n$ . Suppose  $df_p$  is an invertible linear transformation. Then

$$\exists U \subset \mathbb{R}^n: \text{neighborhood of } p$$

$$\exists V \subset \mathbb{R}^n: \text{neighborhood of } f(p)$$

such that  $f: U \rightarrow V$  is a smooth bijection with  $f^{-1}: V \rightarrow U$  smooth.

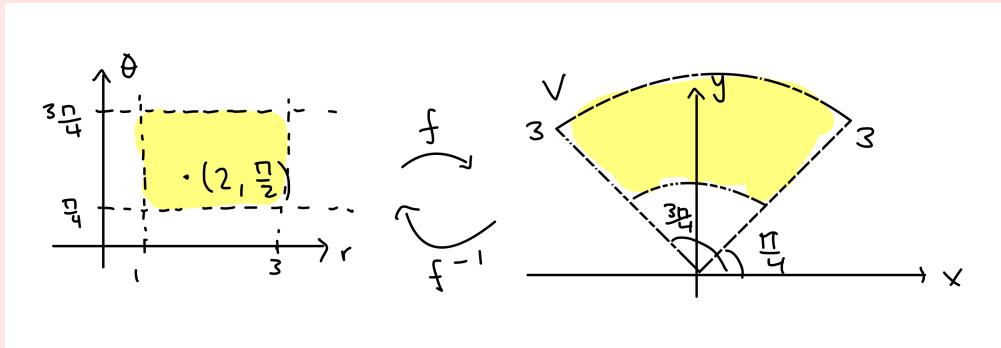
**Example 16.4**

Suppose  $f(r, \theta) = (r \cos \theta, r \sin \theta)$ ,  $(r, \theta) \in \mathbb{R}^2$ . Then,

$$df_{(r, \theta)} = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix}$$

So,

$$\det df_{(r, \theta)} = r \cos^2 \theta + r \sin^2 \theta = r \neq 0 \iff df_{(r, \theta)} \text{ is invertible.}$$



where

$$f^{-1}(x, y) = \left( \sqrt{x^2 + y^2}, \cos^{-1} \frac{x}{\sqrt{x^2 + y^2}} \right).$$

$f$  is not a bijection on  $\{(r, \theta) \in \mathbb{R}^2 \mid r \neq 0\}$ , and

$$f(r, \theta) = f(r, \theta + 2\pi m), \quad m \in \mathbb{Z}$$

**Definition 16.5 (Smooth (on a set))**

Let  $X \subset \mathbb{R}^m$  be a subset (not necessarily open).  $f: X \rightarrow \mathbb{R}^n$  is called smooth if  $\forall p \in X, \exists U$ : neighborhood of  $p$  in  $\mathbb{R}^m$ ,  $\exists \tilde{f}: U \rightarrow \mathbb{R}^n$  smooth such that

$$f|_{U \cap X} = \tilde{f}|_{U \cap X}.$$

**Example 16.6**

Consider  $S^2 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ .

Define

$$f(x, y, z) = x^2 + y^2 + z^2,$$

$$g(x, y, z) = \frac{1}{x^2 + y^2 + z^2}.$$

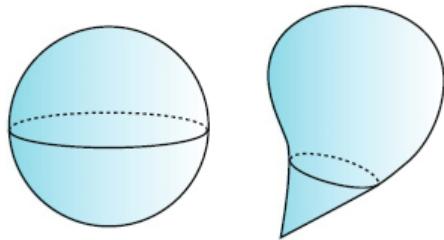
Then  $f, g: S^2 \rightarrow \mathbb{R}$  are both smooth. As for  $g$ ,  $g$  is smooth on  $\mathbb{R}^3 - \{(0, 0, 0)\}$ , and  $S^2 \subseteq \mathbb{R}^3 - \{(0, 0, 0)\}$ .

## 16.2 Diffeomorphisms

**Definition 16.7** (Diffeomorphic/Diffeomorphism)

$X \subset \mathbb{R}^m, Y \subset \mathbb{R}^n$  are called diffeomorphic if  $\exists f: X \rightarrow Y$  smooth bijection such that  $f^{-1}: Y \rightarrow X$  is smooth. In this case,  $f$  is called a diffeomorphism between  $X, Y$ .

**Remark 16.8** If we require “continuous” instead of “smooth” in the above definition, then we get the definition of homeomorphism.



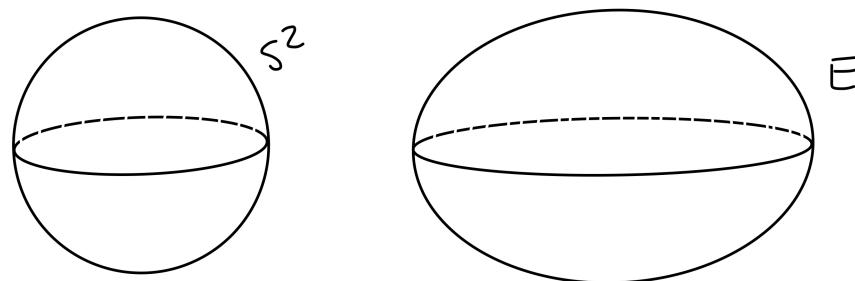
Homeomorphic  
but not diffeomorphic

**Example 16.9**

Consider  $S^2$ , and

$$E = \left\{ (x, y, z) \in \mathbb{R}^3 \mid \left( \frac{x}{a} \right)^2 + \left( \frac{y}{b} \right)^2 + \left( \frac{z}{c} \right)^2 = c \right\}$$

where  $a, b, c > 0$  (fixed).

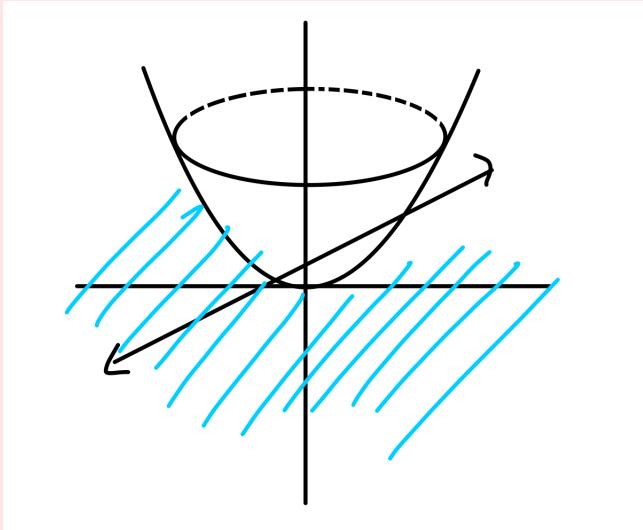


$S^2, E$  are diffeomorphic:

$$\begin{aligned} f: S^2 &\rightarrow E, (x, y, z) \mapsto (ax, by, cz), \\ f^{-1}: E &\rightarrow S^2, (x, y, z) \mapsto \left( \frac{x}{a}, \frac{y}{b}, \frac{z}{c} \right), \end{aligned}$$

**Example 16.10**

Consider  $P := \{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 + y^2\}$



Show  $P, \mathbb{R}^2$  are diffeomorphic.

Define

$$\begin{aligned} f: P &\rightarrow \mathbb{R}^2, (x, y, z) \mapsto (x, y) \\ g: \mathbb{R}^2 &\rightarrow P, (x, y) \mapsto (x, y, x^2 + y^2) \end{aligned}$$

Now  $f, g$  are both smooth. Also  $f$  is a bijection with  $f = g^{-1}$ . Indeed,

$$g \circ f(x, y, z) = g(x, y) = (x, y, x^2 + y^2) = (x, y, z)$$

So  $g \circ f = id_p$ . Similarly,  $f \circ g = id_{\mathbb{R}^2}$ . Therefore,  $f: P \rightarrow \mathbb{R}^2$  is a smooth bijection with smooth inverse  $g$ . So  $f$  is a smooth diffeomorphism.

# 17 Feb 11, 2022

## 17.1 Diffeomorphisms (Cont'd)

**Recall 17.1**  $X \subset \mathbb{R}^m, Y \subset \mathbb{R}^n$

$$\begin{aligned} f: X \rightarrow \mathbb{R}^n \text{ is smooth} &\iff \forall p \in X, \exists U \subseteq \mathbb{R}^m \text{ nbd of } P, \\ &\quad \exists \tilde{f}: U \rightarrow \mathbb{R}^n \\ &\quad \text{s.t. } f = \tilde{f} \text{ on } X \cap U. \end{aligned}$$

$f: X \rightarrow Y$  is a diffeomorphism  $\iff f$  is a smooth bijection with  $f^{-1}$  smooth.

### Lemma 17.2

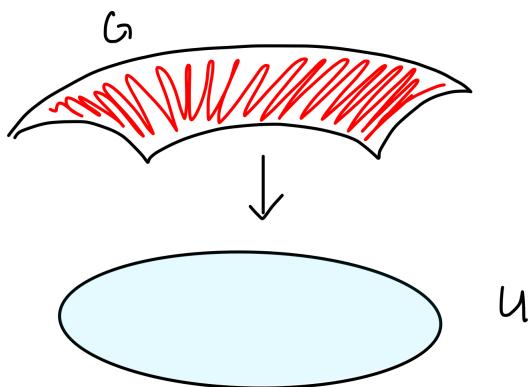
Let  $f: U \subset \mathbb{R}^2 \rightarrow \mathbb{R}$  be smooth,

$$G := \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in U, z = f(x, y)\}$$

Then  $G$  is diffeomorphic to  $U$  by

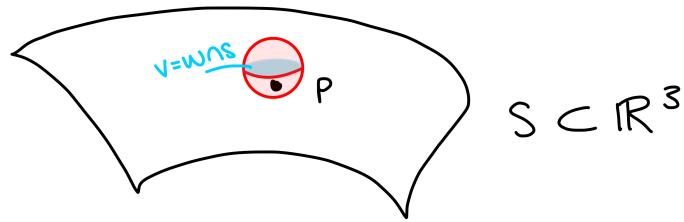
$$\sigma: G \rightarrow U, (x, y, z) \mapsto (x, y),$$

$$\sigma^{-1}: U \rightarrow G, (x, y) \mapsto (x, y, f(x, y))$$



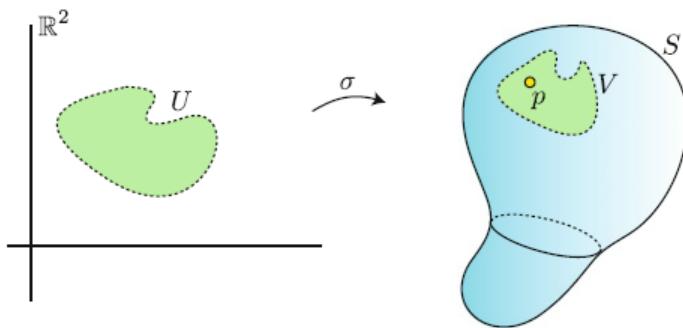
## 17.2 Regular Surfaces

**Recall 17.3** Consider  $S \subset \mathbb{R}^n$ .  $V \subset S$  is open if  $\exists W \in \mathbb{R}^n$  open,  $V = S \cap W$ . A neighborhood (or nbd) of  $p \in S$  in  $S$  is an open subset of  $S$  that contains  $p$ .


**Definition 17.4** (Regular surface, surface patch)

$S \subset \mathbb{R}^3$  is called a regular surface if  $\forall p \in S, \exists V \subseteq S$  neighborhood of  $p, \exists U \subseteq \mathbb{R}^2$  open,  $\exists \sigma: U \rightarrow V$  diffeomorphism.

$\sigma$  is called a surface patch or coordinate chart. A collection of surface patches that cover every point of  $S$  is called an atlas for  $S$ .



A surface patch  $\sigma: U \rightarrow V$  that covers the point  $p \in S$

**Remark 17.5** An analogue of a regular curve is a parametrized surface, which will be defined later.

**Example 17.6**

Let  $U \subset \mathbb{R}^2 \subset \mathbb{R}^3$  be an open subset. Then  $U$  is a regular surface. Indeed,  $\text{id}: U \rightarrow U$  is a surface patch that covers  $U$ .

**Example 17.7**

Let  $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$

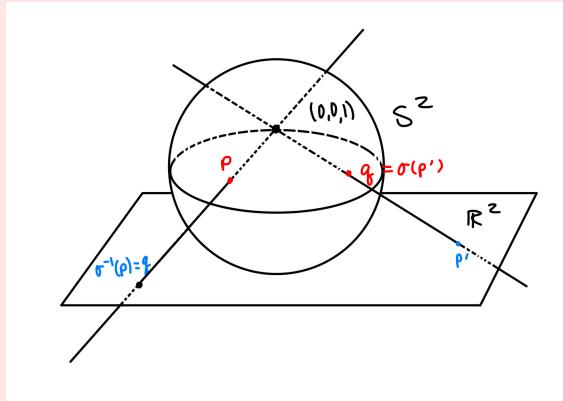
Show  $S^2$  is a regular surface.

We will define 2 surface patches that cover  $S^2$ :

$$\sigma: \mathbb{R}^2 \rightarrow S^2 - \{(0, 0, 1)\},$$

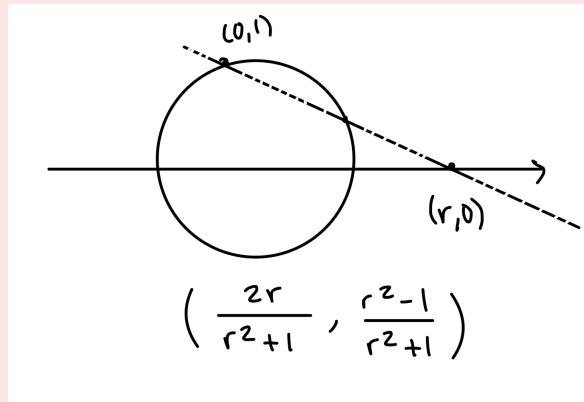
$$\tau: \mathbb{R}^2 \rightarrow S^2 - \{(0, 0, -1)\}$$

We use the stereographic projection:



Where,

$$\sigma(x, y) = \left( \frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} \right)$$



Then,  $\sigma^{-1}: S^2 - \{(0, 0, 1)\} \rightarrow \mathbb{R}^2$  is defined by

$$\sigma^{-1}(x, y, z) = \left( \frac{2x}{1-z}, \frac{2y}{1-z} \right)$$

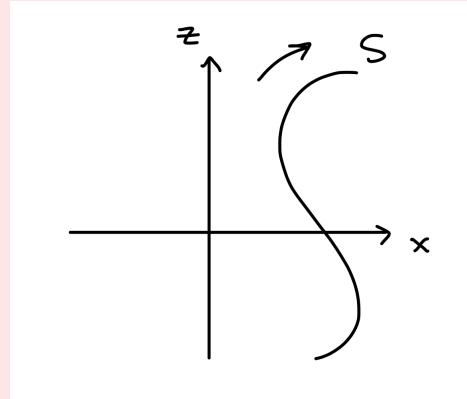
$\sigma, \sigma^{-1}$  are both smooth, so  $\sigma$  is a diffeomorphism. Therefore,  $\sigma$  is a surface patch. Similar for  $\tau$ .

**Example 17.8**

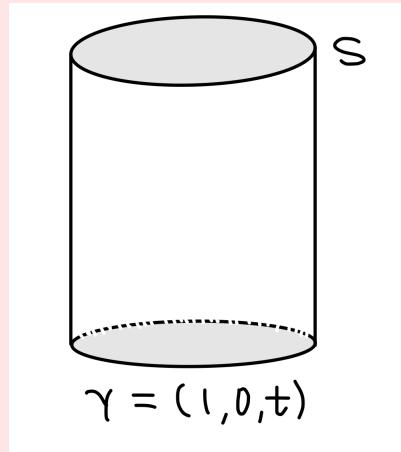
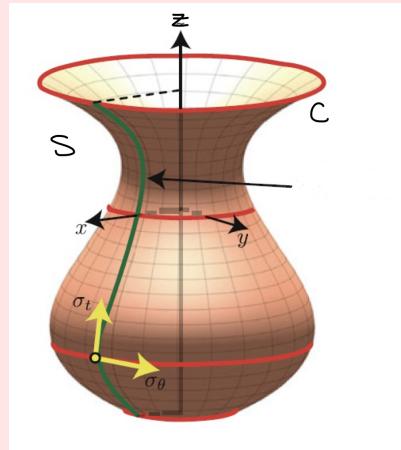
Let  $\gamma(t) = (x(t), 0, z(t)), t \in (a, b)$  be a regular curve in the  $xz$  planes. Define  $c :=$  the trace of  $\gamma$ . Suppose:

- i.  $\forall t \in (a, b), x(t) > 0$
- ii.  $\forall t \in (a, b), z'(t) > 0$

(i)  $\implies c$  does not intersect the  $z$ -axis  
(ii)  $\implies c$  has no self-intersection

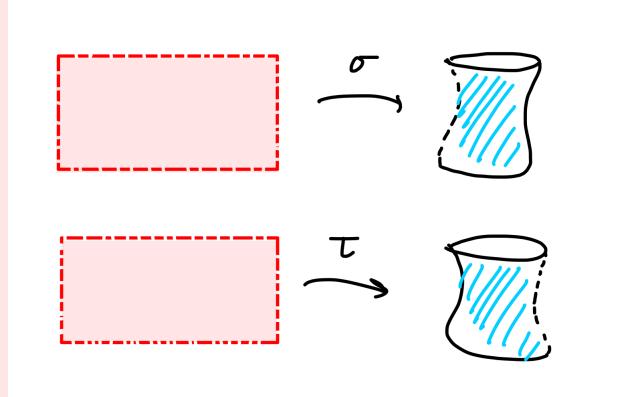


Let  $S$  be the surface of revolution resulting from revolving  $C$  about the  $z$ -axis.



We show  $S$  is a regular surface.

Idea:



Define  $\sigma: (a, b) \times (-\pi, \pi) \rightarrow S$  by

$$\begin{aligned}\sigma(t, \theta) &= (x(t) \cos \theta, x(t) \sin \theta, z(t)) \\ &= R_\theta(\gamma(t)),\end{aligned}$$

where,

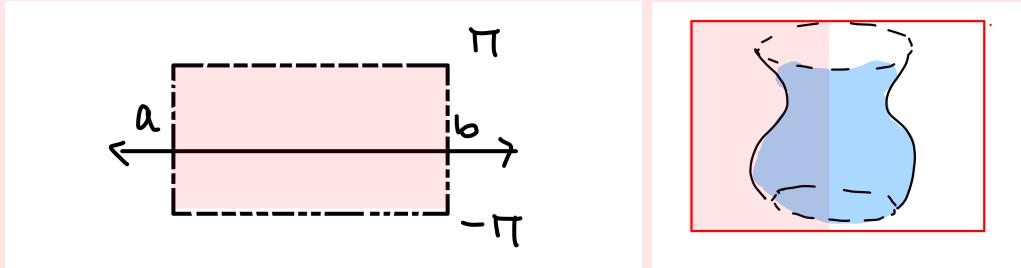
$$R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is the rotation by  $\theta$  about the  $z$ -axis.

We need to show  $\sigma$  is a surface patch.

$$U := (a, b) \times (-\pi, \pi) \subset \mathbb{R}^2$$

$$V := \sigma(U) = S - \underbrace{\{(x, y, z) \mid y = 0, x \leq 0\}}_{\text{closed}} \subset S \text{ open}$$



Enough to show  $\sigma: U \rightarrow V$  is a diffeomorphism.

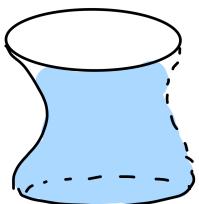
$\sigma$  is a smooth bijection. As for  $\sigma^{-1}$ , the smoothness can be checked locally, for instance,

$$\{(x, y, z) \in V \mid x > 0\},$$

then

$$\sigma^{-1}(x, y, z) = (t(z), \tan^{-1}(y/x)),$$

where  $t(z)$  is the inverse of  $z(t)$ . Therefore  $\sigma$  is a surface patch. Similarly, we define  $\tau: (a, b) \times (0, 2\pi) \rightarrow S$  by  $\tau(t, \theta) = (x(t) \cos \theta, x(t) \sin \theta, z(t))$ .  $\tau$  covers



Therefore,  $S$  is a regular surface covered by  $\sigma, \tau$ .

# 18 Feb 14, 2022

## 18.1 Regular Surfaces (Cont'd)

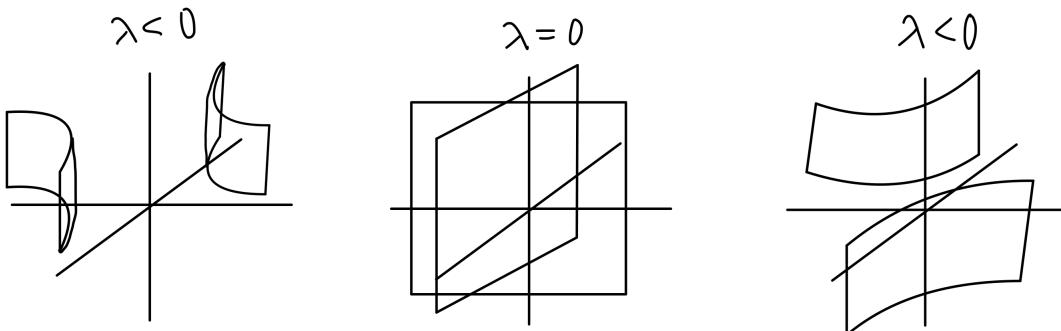
**Recall 18.1**  $f: U \subset \mathbb{R}^3 \rightarrow \mathbb{R}, \lambda \in \mathbb{R}$

$$f^{-1} = \{p \in U \mid f(p) = \lambda\}$$

### Example 18.2

Suppose  $f(x, y, z) = xy, \lambda \in \mathbb{R}$ . Then,

$$f^{-1}(\lambda) = \{(x, y, z) \in \mathbb{R}^3 \mid xy = \lambda\} \subset \mathbb{R}^3$$



For which  $\lambda$  is  $f^{-1}(\lambda)$  a regular surface?

### Definition 18.3 (Regular value)

Let  $f: U \subset \mathbb{R}^3 \rightarrow \mathbb{R}$  be smooth,  $\lambda \in \mathbb{R}$ .  $\lambda$  is called a regular value of  $f$  if

$$\forall p \in f^{-1}(\lambda), \quad df_p = \left( \frac{\partial f}{\partial x}(p) \quad \frac{\partial f}{\partial y}(p) \quad \frac{\partial f}{\partial z}(p) \right) \neq (0 \quad 0 \quad 0)$$

That is, one of

$$\frac{\partial f}{\partial x}(p), \frac{\partial f}{\partial y}(p), \frac{\partial f}{\partial z}(p) \neq 0$$

### Theorem 18.4 (Regular Value Theorem)

Under the same setting, if  $\lambda$  is a regular value, then  $f^{-1}$  is a regular surface.

**Remark 18.5** Theorem shows the existence of some surface patches that cover  $f^{-1}(\lambda)$ , but does not give an explicit construction of such patches.

**Example 18.6**

$$f(x, y, z) = x^2 + y^2 - z^2 \quad f: \mathbb{R}^3 \rightarrow \mathbb{R} \text{ is smooth}$$

Therefore,

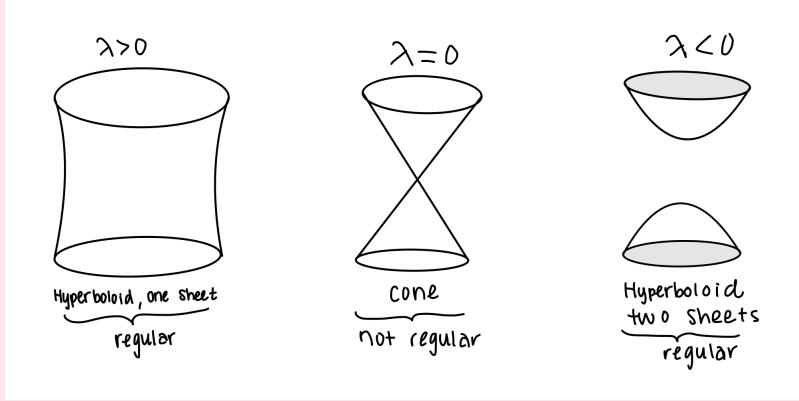
$$df_{(x,y,z)} = (2x \quad 2y \quad 2z),$$

so,

$$df_{(x,y,z)} = 0 \iff (x, y, z) = (0, 0, 0)$$

so,

$$f^{-1}(\lambda) = \{(x, y, z) \in \mathbb{R}^3 \mid \underbrace{x^2 + y^2 - z^2}_{x^2 + y^2 = z^2 + \lambda} = \lambda\}$$



**Proof of Regular Value Theorem (Sketch).**  $f: U \subset \mathbb{R}^3 \rightarrow \mathbb{R}$  is smooth,  $\lambda \in \mathbb{R}$  a regular value.

$$S := f^{-1}(\lambda), p \in S.$$

We know

$$df_p = \left( \frac{\partial f}{\partial x}(p) \quad \frac{\partial f}{\partial y}(p) \quad \frac{\partial f}{\partial z}(p) \right) \neq (0 \quad 0 \quad 0)$$

We may assume

$$\frac{\partial f}{\partial z}(p) \neq 0.$$

Define

$$g: U \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3 \text{ by } g(x, y, z) = (x, y, f(x, y, z))$$

Then  $g$  is smooth, and

$$dg_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial f}{\partial x}(p) & \frac{\partial f}{\partial y}(p) & \frac{\partial f}{\partial z}(p) \end{pmatrix}$$

So  $\det dg_p = \frac{\partial f}{\partial z}(p) \neq 0$ . So  $dg_p$  is invertible.

By the inverse function theorem,

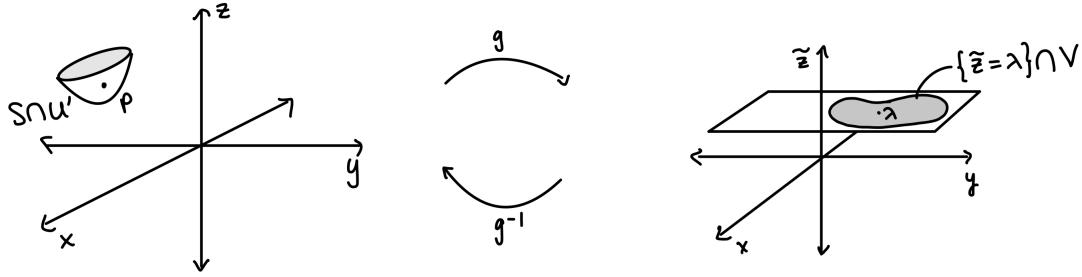
$$p \in \exists U' \subset U \text{ nbd}, \quad g(p) \in \exists V \subset \mathbb{R}^3 \text{ nbd}$$

such that

$$g: U' \rightarrow V \text{ is a diffeomorphism}$$

Then  $g$  restricts to a diffeomorphism:

$$S \cap U' = \{f(x, y, z) = \lambda\} \cap U' \mapsto \{\tilde{z} = \lambda\} \cap V$$



$\{\tilde{z} = \lambda\} \cap V$  can be identified with an open subset of  $\mathbb{R}^2$ , and  $y^{-1}: \{\tilde{z} = \lambda\} \cap V \rightarrow S \cap U'$  is a surface patch containing  $p$ .  $\square$

## 18.2 Parametrized Surfaces

### Definition 18.7 (Parametrized surface)

A parametrized surface is a smooth function  $f: U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$  such that  $\forall p \in U, \text{rank } df_p = 2$ .

### Remarks 18.8

- i. A parametrized surface is a function, but a regular surface is a set.
- ii. A regular curve is a smooth function  $\gamma: I \subset \mathbb{R} \rightarrow \mathbb{R}^n$  such that  $\forall p \in I, \text{rank } d\gamma_p = 1 \implies \text{rank } \gamma'(p) \iff \gamma'(p) \neq 0$
- iii.  $\sigma(U)$  is not a regular surface in general, as it might have a self-intersection.

**Example 18.9**

$$\sigma(x, y) = (\cos x, \sin 2x, y)$$

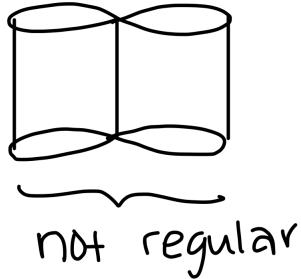
Then  $\sigma: \mathbb{R}^2 \rightarrow \mathbb{R}^3$  is smooth. So

$$d\sigma_{(x,y)} = \underbrace{\begin{pmatrix} -\sin x & 0 \\ 2\cos 2x & 0 \\ 0 & 1 \end{pmatrix}}_{\text{first column can never be 0}}$$

$$\implies \text{rank } d\sigma_{(x,y)} = 2$$

$\implies \sigma$  is a parametrized surface

So,  $\sigma(\mathbb{R}^2)$  is



which is not regular.

**Lemma 18.10**

A surface patch  $\sigma: U \subset \mathbb{R}^2 \rightarrow V \subset S \subset \mathbb{R}^3$  is a parametrized surface.

**Proof.** Let  $p \in U, q := \sigma(p)$ .

$\sigma^{-1}$  locally extends to a smooth function on  $\mathbb{R}^3$ .

$$\begin{aligned} I_2 &= d(\sigma^{-1} \circ \sigma)_p = d\sigma_q^{-1} \circ d\sigma_p \\ &\implies \ker(d\sigma_p) = 0 \\ &\stackrel{\text{rank-nullity}}{\implies} \text{rank}(d\sigma_p) = 0 \end{aligned}$$

□

Roughly speaking, a surface patch is equivalent to the image of a parametrized surface that is one-to-one on the domain. Strictly speaking, this is not true. In addition,  $\sigma: U \rightarrow \sigma(U)$  has to be a homeomorphism. Read §3.10 in Tapp for the details.

# 19 Feb 16, 2022

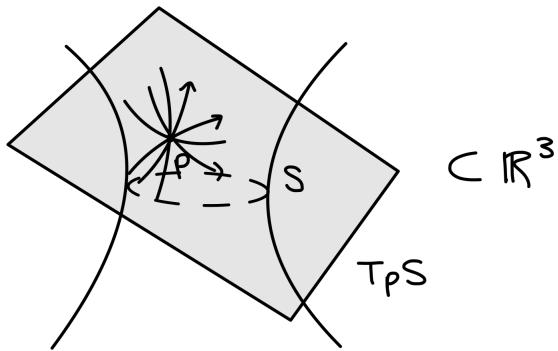
## 19.1 Tangent Planes

**Definition 19.1** (Tangent space, tangent vector)

Let  $S \subset \mathbb{R}^n$  be a subset,  $p \in S$ . The tangent space to  $S$  at  $p$  is defined as follows:

$$T_p S := \{\gamma'(0) \mid \gamma: (-\varepsilon, \varepsilon) \rightarrow S \subset \mathbb{R}^n \text{ curve s.t. } \gamma(0) = p\}$$

A vector in  $T_p S$  is called a tangent vector to  $S$  at  $p$ .

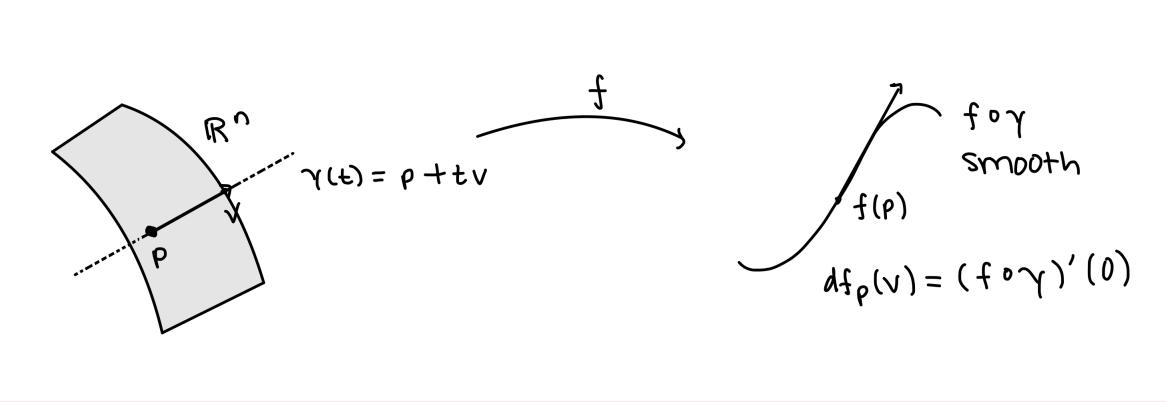


### Remarks 19.2

- i.  $0 \in T_p S$ , as one can consider  $\gamma(t) = p$  (constant)
- ii. Another possible definition of the tangent space is:  $p + T_p S$  (tangent space placed at  $p$ ).
- iii. If  $p \in U \subset S$  neighborhood, then  $T_p U = T_p S$  as one can shrink the domain of  $\gamma$ .

**Example 19.3**

Let  $p \in \mathbb{R}^n$ . Then  $T_p \mathbb{R}^n = \mathbb{R}^n$ .



If  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is smooth, then  $df_p: T_p \mathbb{R}^n \rightarrow T_{f(p)} \mathbb{R}^m$ .

If  $p \in S \subset \mathbb{R}^n$ , then  $T_p S \subset T_p \mathbb{R}^n = \mathbb{R}^n$ .

**Proposition 19.4**

Let  $S \subset \mathbb{R}^3$  be a regular surface,  $p \in S$ . If

$$\begin{aligned}\sigma: U \subset \mathbb{R}^2 &\rightarrow V \subset S \\ q &\mapsto p\end{aligned}$$

is a surface patch, then

$$T_p S = \text{Im } d\sigma_p = \text{span}(\sigma_x(p), \sigma_y(p)) \subset T_p \mathbb{R}^3$$

If  $p \in S \subset \mathbb{R}^n$ , then

$$T_p S \subset T_p \mathbb{R}^n = \mathbb{R}^n.$$

In particular,  $T_p S \subset T_p \mathbb{R}^3 = \mathbb{R}^3$  is a 2-dimensional linear subspace.

**Remark 19.5**  $d\sigma_p: T_q \mathbb{R}^2 \rightarrow T_p S \subset T_p \mathbb{R}^3$  is a linear isomorphism.

**Proof.**

$$\begin{aligned}T_p S &= \{\gamma'(0) \mid \gamma \text{ is a curve on } V \text{ s.t. } \gamma(0) = p\} \\ &= \{(\sigma \circ B)'(0) \mid B \text{ curve on } U \text{ s.t. } B(0) = q\} \\ &\rightarrow \sigma \text{ is a diffeomorphism, so it identifies regular curves} \\ &\text{on } U \text{ and those on } V \\ &= \text{Im } d\sigma_q\end{aligned}$$

□

**Example 19.6**

$$S := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$$

So,

$$p = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1 \right) \in S$$

Compute  $T_p S$ .

Recall:

$$\sigma(\theta, t) = (\cos \theta, \sin \theta, t)$$

gives a surface patch for  $S$ .

$$\sigma\left(\frac{\pi}{4}, 1\right) = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1 \right) = p$$

Set  $q = \left(\frac{\pi}{4}, 1\right)$ . So,

$$\begin{aligned} d\sigma_q &= \begin{pmatrix} -\sin \theta & 0 \\ \cos \theta & 0 \\ 0 & 1 \end{pmatrix}_q \\ &= \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

$$\implies T_p S = \text{Im } d\sigma_q = \text{Span} \left( \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

Then,

$$n = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

So,

$$T_p S = \{(x, y, z) \in \mathbb{R}^3 \mid x + y = 0\}$$

**Example 19.7**

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2\}, p = (0, 0, 0) \in S.$$

Then,  $T_p S = S$  (homework) is not a subspace. Hence,  $S$  is not a regular surface.

**Definition 19.8** (Derivative (domain is a surface) )

Let  $S \subset \mathbb{R}^n$  be a subset,  $f: S \rightarrow \mathbb{R}^m$  be smooth,  $p \in S$ . The derivative of  $f$  at  $p$  is defined as

$$df_p: T_p S \rightarrow T_{f(p)} \mathbb{R}^m, \quad v \mapsto (f \circ \gamma)'(0),$$

where  $\gamma$  is any curve on  $S$  such that  $\gamma(0) = p, \gamma'(0) = v$ .

**Lemma 19.9**

$df_p$  is well-defined.

**Proof.** We need to check:

- i.  $f \circ \gamma(-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^m$  is smooth
- ii.  $df_p(v)$  is independent of  $\gamma$ .

$f$  locally extends to a smooth function  $\tilde{f}$  on a neighborhood of  $p$  in  $\mathbb{R}^n$ .

i.)  $f \circ \gamma = \tilde{f} \circ \gamma$  is the composition of smooth functions whose domains are open in some vector space, so smooth.

ii.)  $df_p(v) = (f \circ \gamma)'(0) = (\tilde{f} \circ \gamma)'(0) = \tilde{f}' \circ \gamma(0) \cdot \gamma'(0) = \tilde{f}'(p) \cdot v$

$\implies$  This does not depend on  $\gamma$ . □

# 20 Feb 18, 2022

## 20.1 Tangent Planes (Cont'd)

**Recall 20.1**  $S \subset \mathbb{R}^n$  a subset,  $p \in S$ ,

$$T_p S = \{\gamma'(0) \mid \gamma \text{ is a curve on } S \text{ s.t. } \gamma(0) = p\}$$

$f: S \rightarrow \mathbb{R}^m$  smooth,

$$\begin{aligned} df_p: T_p S &\rightarrow T_{f(p)} \mathbb{R}^m \\ v = \gamma'(0) &\mapsto (f \circ \gamma)'(0) \end{aligned}$$

### Proposition 20.2

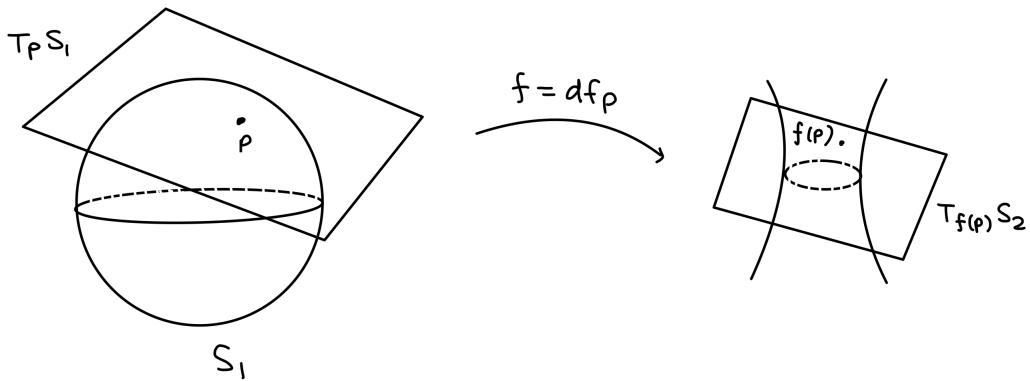
Let  $S \subset \mathbb{R}^3$  be a regular surface,  $f: S \rightarrow \mathbb{R}^m$  be smooth,  $p \in S$ . Then  $df_p: T_p S \rightarrow T_{f(p)} \mathbb{R}^m$  is a linear transformation.

**Proof.**  $f$  extends locally to a smooth function  $\tilde{f}$  on a neighborhood of  $p$  in  $\mathbb{R}^3$ .

$\forall v \in T_p S$ ,

$$df + p(v) = (f \circ \gamma)'(0) = (\tilde{f} \circ \gamma)'(0) = d\tilde{f}_p(v)$$

$\Rightarrow df_p$  is the restriction of the linear transformation  $d\tilde{f}_p: T_p \mathbb{R}^3 \rightarrow T_{f(p)} \mathbb{R}^m$  to the subspace  $T_p S \subset T_p \mathbb{R}^3 = \mathbb{R}^3$ .  $\square$



### Theorem 20.3 (Inverse Function Theorem for Regular Surfaces)

Let  $S_1, S_2 \subset \mathbb{R}^3$  be regular surfaces,  $f: S_1 \rightarrow S_2$  be smooth. Suppose  $\exists p \in S_1, df_p: T_p S_1 \rightarrow T_{f(p)} S_2$  is invertible, then  $f$  is locally a diffeomorphism at  $p$ :

$$p \in \exists V_1 \subset S_1 \text{ nbd}$$

$$f(p) \in \exists V_2 \subset S_2 \text{ nbd}$$

such that  $f: V_1 \rightarrow V_2$  is a diffeomorphism.

**Proof.** We take surface patches at  $p \in S_1, f(p) \in S_2$ ,

$$\sigma_1: U_1 \rightarrow V_1, \quad \sigma_2: U_2 \rightarrow V_2$$

We may assume  $f(V_1) \subset V_2$  by shrinking  $U_1, V_1$  if necessary.

$$\begin{array}{ccc} V_1 \subset S_1 & \xrightarrow{f} & V_2 \subset S_2 \\ \sigma_1 \uparrow & & \uparrow \sigma_2 \\ U_1 \subset \mathbb{R}^2 & \xrightarrow{\psi} & U_2 \subset \mathbb{R}^2 \end{array}$$

$$\text{Define } \psi := \sigma_2^{-1} \circ f \circ \sigma_1$$

Then  $\psi$  is smooth, as  $\sigma_1, f, \sigma_2^{-1}$  are all smooth. Set  $q := \sigma^{-1}(p) \in U_1$ . Then

$$\begin{array}{ccc} T_p S_1 & \xrightarrow{df_p} & T_{f(p)} S_2 \\ d(\sigma_1)_q \uparrow & & \uparrow d(\sigma_2)_{\psi(q)} \text{ by the chain rule} \\ T_q \mathbb{R}^2 & \xrightarrow{d\psi_q} & T_{\psi(q)} \mathbb{R}^2 \end{array}$$

Here,  $(d\sigma_1)_{q_1}, (d\sigma_2)_{\psi(q)}$  are invertible, and  $(df)_p$  is invertible by assumption.

$$\implies d\psi_q: T_q \mathbb{R}^2 \rightarrow T_{\psi(q)} \mathbb{R}^2 \text{ is invertible}$$

and by Inverse Function theorem,

$$\implies \psi: U_1 \rightarrow U_2 \text{ is a diffeomorphism}$$

by shrinking  $U_1$  if necessary.

$$\implies f = \sigma_2 \circ \psi \circ (\sigma_1)^{-1}$$

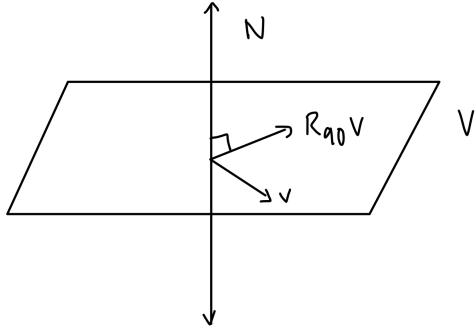
is the composition of diffeomorphisms  $\sigma_2, \psi, \sigma_1^{-1}$ , hence  $f$  is a diffeomorphism.  $\square$

## 20.2 Orientable Surfaces

An orientation for a regular surface allows us to define the notions of “clockwise” and “counterclockwise” in a constant manner. First, we define an orientation for a 2-dimensional vector space  $V \subset \mathbb{R}^3$ .

**Definition 20.4** (External definition for orientation)

An orientation for  $\mathcal{V}$  is a choice of a unit normal vector  $N$  to  $\mathcal{V}$ .

**Remarks 20.5**

- i. There are exactly 2 orientations for  $\mathcal{V}$ .
- ii. There is no normal orientation for  $\mathcal{V}$ .
- iii. Given an orientation  $N$  for  $\mathcal{V}$ ,  $\forall v \in \mathcal{V}, R_{90}v := N \times v$ .

**Definition 20.6 (Internal definition for orientation)**

An equivalence relation on the set of ordered basis for  $\mathcal{V}$  is defined as follows:

$$\{v_1, v_2\} \sim \{v'_1, v'_2\}$$

$$\iff \begin{cases} v'_1 = a_1 v_1 + a_2 v_2 \\ v'_2 = b_1 v_1 + b_2 v_2 \end{cases} \text{ then } \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} > 0$$

An orientation for  $\mathcal{V}$  is an equivalence class of ordered basis for  $\mathcal{V}$ .

**Lemma 20.7**

External def  $\iff$  internal def

More precisely,

$$\{\text{eq. classes}\} \rightarrow \{\text{unit normals}\}$$

$$\{v_1, v_2\} \mapsto \frac{v_1 \times v_2}{|v_1 \times v_2|}$$

**Proof.** Let  $\{v_1, v_2\}, \{v'_1, v'_2\}$  be ordered basis for  $\mathbb{R}^2$ . Then, we can write

$$\begin{cases} v'_1 = a_1 v_1 + a_2 v_2 \\ v'_2 = b_1 v_1 + b_2 v_2 \end{cases}$$

Then,

$$\begin{aligned} v'_1 \times v'_2 &= (a_1 v_1 + a_2 v_2) \times (b_1 v_1 + b_2 v_2) \\ &= (a_1 b_2 - a_2 b_1) v_1 \times v_2 \\ &= \det(A) v_1 \times v_2 \end{aligned}$$

where  $A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$ . Then,

$$\frac{v'_1 \times v'_2}{|v'_1 \times v'_2|} = \begin{cases} \frac{v_1 \times v_2}{|v_1 \times v_2|} & \text{if } \det(A) > 0 \\ -\frac{v_1 \times v_2}{|v_1 \times v_2|} & \text{if } \det(A) < 0. \end{cases}$$

□