Math 120A (Differential Geometry)
University of California, Los Angeles

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These are my lecture notes for Math 120A (Differential Geometry), which is taught by Fumiaki Suzuki. The textbook for this class is *Differential Geometry of Curves and Surfaces*, by Kristopher Tapp. Many of the figures I include in these notes are taken from Tapp's book.

Contents

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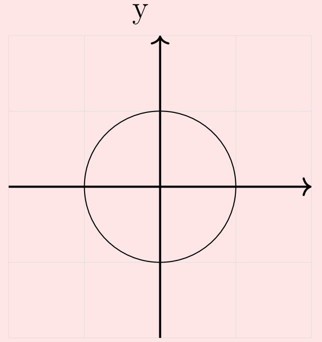
1.1 What is Differential Geometry?

Differential geometry studies geometry via analysis and linear algebra.

Geometry	Analysis	Linear Algebra
Intuitive	Rigorous	Computable
Curved	$\xrightarrow{\text{tangent space}}$	Linear
Global	Local	

1.2 Parametrized Curves

Example 1.1
A unit circle $S' = \{\vec{x} \text{ in } \mathbb{R}^2 \mid |\vec{x}| = 1\}$



$$\vec{\gamma} : [0, 2\pi) \to \mathbb{R}^2$$
$$t \mapsto (\cos t, \sin t)$$

$\vec{\gamma}[0, 2\pi) = S'$

Definition 1.2 (Parametrized curve and Trace)
A <u>(parametrized) curve</u> is a smooth function $\vec{\gamma} : I \to \mathbb{R}^n$, where I is an interval in $\mathbb{R}$. The image

$$\vec{\gamma}(I) = \{\vec{\gamma}(t) \mid t \in I\}$$

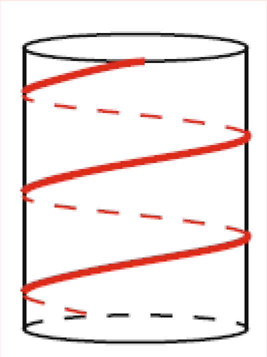
is called the <u>trace</u> of $\vec{\gamma}$.

Recall 1.3 An interval is a subset of $\mathbb{R}$ that has one of the following forms:

$$(a, b), [a, b], (a, b], [a, b), (-\infty, b), (-\infty, b], (a, \infty), [a, \infty), (-\infty, \infty) = \mathbb{R}.$$

A function $\vec{\gamma} : I \to \mathbb{R}^n$ is called <u>smooth</u> if $\vec{\gamma}$ is infinitely differentiable, or equivalently, each of the component functions $x_i : I \to \mathbb{R}$ is infinitely differentiable.

Example 1.4
$\vec{\gamma}(t) = (\cos t, \sin t, t)$, $t \in (-\infty, \infty)$ is a curve, called a helix.



Definition 1.5 (Derivative)
Let $\vec{\gamma} : I \to \mathbb{R}^n$ be a curve. The derivative of $\vec{\gamma}$ at t is defined as

$$\vec{\gamma}'(t) = \lim_{h \to 0} \frac{\vec{\gamma}(t+h) - \vec{\gamma}(t)}{h}$$

If t is on the boundaries of I, then use the left- or right-hand limit.

Remarks 1.6

i. If $\vec{\gamma}(t) = (x_1(t), x_2(t), \cdots, x_n(t))$, then $\vec{\gamma}'(t) = (x_1'(t), x_2'(t), \cdots, x_n'(t))$.

ii. The tangent line to the curve at $\vec{\gamma}'(t_0)$ is defined as

$$\vec{L}(t) = \vec{\gamma}(t_0) + t\vec{\gamma}'(t_0), \quad t \in (-\infty, \infty),$$

as soon as $\vec{\gamma}'(t) \neq \vec{0}$.

Definition 1.7 (Regular)
A curve $\vec{\gamma} : I \to \mathbb{R}^n$ is called regular if $\forall t \in I, \vec{\gamma}'(t) \neq \vec{0}$.

Remark 1.8 regular = the tangent line is defined everywhere = the trace is "smooth".

Example 1.9

$$\vec{\gamma}(t) = (t^2, t^3), \quad t \in (-\infty, \infty)$$

Then $\vec{\gamma}$ is a curve that is not regular.
Indeed, $\vec{\gamma}'(t) = (2t, 3t^2)$, so $\vec{\gamma}'(0) = \vec{0}$.
Notice, $x(t) = t^2, y(t) = t^3$, so $x(t) = y(t)^{2/3}$. Hence, the trace is given by $x = y^{2/3}$ in $\mathbb{R}^2$.

Remark 1.10 The analogy with the physics is useful. If $\vec{\gamma} : I \to \mathbb{R}^n$ is a curve, then $\vec{\gamma}(t)$ is the position of a moving particle at time t in $\mathbb{R}^2$.

- $\vec{\gamma}'(t)$ velocity

- $\vec{\gamma}''(t)$ acceleration
- $|\vec{\gamma}'(t)|$ speed

In this analogy, regular = the speed is always nonzero = the particle never stops (hence no "corners" on the trace)

Definition 1.11 (Arc length)
Let $\vec{\gamma}(t) : I \to \mathbb{R}^n$ be a regular curve. Then the arc length between times t_1, t_2 is defined as

$$\int_{t_1}^{t_2} |\vec{\gamma}'(t)|\, dt$$

Proposition 1.12
Let $\vec{\gamma} : [a,b] \to \mathbb{R}^n$ be a regular curve with the arc length L, $\vec{p} = \vec{\gamma}(a), \vec{q} = \vec{\gamma}(b)$. Then $L \geq |\vec{q} - \vec{p}|$.
Moreover, the equality holds if and only if $\vec{\gamma}$ parametrizes the line segment between $\vec{p}, \vec{q}$.

For the proof, we use the inner-product:
for $\vec{x} = (x_1, x_2, \cdots, x_n), \vec{y} = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$,

$$\langle x, y \rangle \coloneqq x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

Basic properties:

i. The inner product $\langle -, - \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is symmetric and bilinear.
ii. $\langle \vec{x}, \vec{y} \rangle = |\vec{x}||\vec{y}| \cos\theta$, where θ is the angle between $\vec{x}, \vec{y}$. ($\theta \in [0, 2\pi]$)
iii. $\langle \vec{x}, \vec{y} \rangle = 0 \Leftrightarrow \vec{x}, \vec{y}$ are orthogonal to each other.
iv. $\langle \vec{x}, \vec{x} \rangle = |\vec{x}|^2$
v. $\langle \vec{x}, \vec{y} \rangle \leq |\vec{x}||\vec{y}|$ (Schwartz Inequality) and the equality holds if and only if $\theta = 0$.