Math 120A (Differential Geometry)
University of California, Los Angeles

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These are my lecture notes for Math 120A (Differential Geometry), which is taught by Fumiaki Suzuki. The textbook for this class is *Differential Geometry of Curves and Surfaces*, by Kristopher Tapp. Many of the figures I include in these notes are taken from Tapp's book.

Contents

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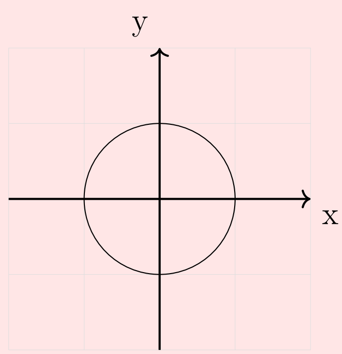
1.1 What is Differential Geometry?

Differential geometry studies geometry via analysis and linear algebra.

Geometry	Analysis	Linear Algebra
Intuitive	Rigorous	Computable
Curved	$\xrightarrow{\text{tangent space}}$	Linear
Global	Local	

1.2 Parametrized Curves

Example 1.1
A unit circle $S' = \{\vec{x} \text{ in } \mathbb{R}^2 \mid |\vec{x}| = 1\}$



$$\begin{aligned} \vec{\gamma} : [0, 2\pi) &\to \mathbb{R}^2 \\ t &\mapsto (\cos t, \sin t) \end{aligned}$$

$\vec{\gamma}[0, 2\pi) = S'$

Definition 1.2 (Parametrized curve and Trace)
A <u>(parametrized) curve</u> is a smooth function $\vec{\gamma}\colon I \to \mathbb{R}^n$, where I is an interval in $\mathbb{R}$. The image

$$\vec{\gamma}(I) = \{\vec{\gamma}(t) \mid t \in I\}$$

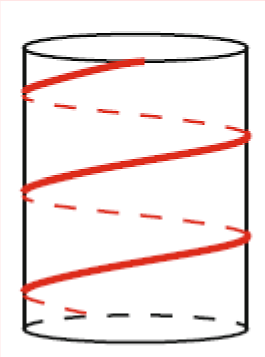
is called the <u>trace</u> of $\vec{\gamma}$.

Recall 1.3 An interval is a subset of $\mathbb{R}$ that has one of the following forms:

$$(a, b), [a, b], (a, b], [a, b), (-\infty, b), (-\infty, b], (a, \infty), [a, \infty), (-\infty, \infty) = \mathbb{R}.$$

A function $\vec{\gamma}\colon I \to \mathbb{R}^n$ is called <u>smooth</u> if $\vec{\gamma}$ is infinitely differentiable, or equivalently, each of the component functions $x_i\colon I \to \mathbb{R}$ is infinitely differentiable.

Example 1.4
$\vec{\gamma}(t) = (\cos t, \sin t, t)$, $t \in (-\infty, \infty)$ is a curve, called a helix.



Definition 1.5 (Derivative)
Let $\vec{\gamma}\colon I \to \mathbb{R}^n$ be a curve. The derivative of $\vec{\gamma}$ at t is defined as
$$\vec{\gamma}'(t) = \lim_{h \to 0} \frac{\vec{\gamma}(t+h) - \vec{\gamma}(t)}{h}$$
If t is on the boundaries of I, then use the left- or right-hand limit.

Remarks 1.6

i. If $\vec{\gamma}(t) = (x_1(t), x_2(t), \cdots, x_n(t))$, then $\vec{\gamma}'(t) = (x_1'(t), x_2'(t), \cdots, x_n'(t))$.

ii. The tangent line to the curve at $\vec{\gamma}'(t_0)$ is defined as
$$\vec{L}(t) = \vec{\gamma}(t_0) + t\vec{\gamma}'(t_0), \quad t \in (-\infty, \infty),$$
as soon as $\vec{\gamma}'(t) \neq \vec{0}$.

Definition 1.7 (Regular)
A curve $\vec{\gamma} : I \to \mathbb{R}^n$ is called regular if $\forall t \in I, \vec{\gamma}'(t) \neq \vec{0}$.

Remark 1.8 regular = tangent line is defined everywhere = the trace is smooth

Example 1.9
$$\vec{\gamma}(t) = (t^2, t^3), \quad t \in (-\infty, \infty)$$
Then $\vec{\gamma}$ is a curve that is not regular.
Indeed, $\vec{\gamma}'(t) = (2t, 3t^2)$, so $\vec{\gamma}'(0) = \vec{0}$.
Notice, $x(t) = t^2, y(t) = t^3$, so $x(t) = y(t)^{2/3}$. Hence, the trace is given by $x = y^{2/3}$ in $\mathbb{R}^2$.

Remark 1.10 The analogy with the physics is useful. If $\vec{\gamma} : I \to \mathbb{R}^n$ is a curve, then $\vec{\gamma}(t)$ is the position of a moving particle at time t in $\mathbb{R}^2$.

- $\vec{\gamma}'(t)$ velocity

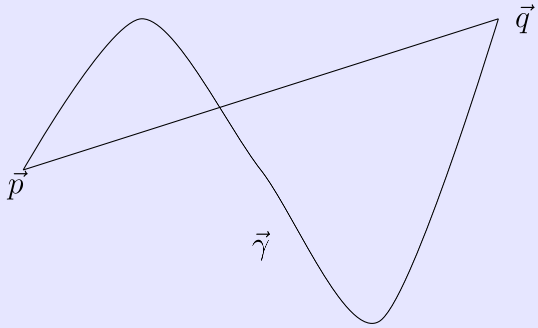
- $\vec{\gamma}''(t)$ acceleration
- $|\vec{\gamma}'(t)|$ speed

In this analogy, regular = the speed is always nonzero = the particle never stops (hence no "corners" on the trace)

Definition 1.11 (Arc length)
Let $\vec{\gamma}(t)\colon I \to \mathbb{R}^n$ be a regular curve. Then the arc length between times t_1, t_2 is defined as

$$\int_{t_1}^{t_2} |\vec{\gamma}'(t)|\, dt$$

Proposition 1.12
Let $\vec{\gamma}\colon [a,b] \to \mathbb{R}^n$ be a regular curve with the arc length L, $\vec{p} = \vec{\gamma}(a), \vec{q} = \vec{\gamma}(b)$. Then $L \geq |\vec{q} - \vec{p}|$.
Moreover, the equality holds if and only if $\vec{\gamma}$ parametrizes the line segment between $\vec{p}, \vec{q}$.



For the proof, we use the inner-product:
for $\vec{x} = (x_1, x_2, \cdots, x_n), \vec{y} = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$,

$$\langle x, y \rangle \coloneqq x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

Basic properties:

i. The inner product $\langle -, - \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is symmetric and bilinear.
ii. $\langle \vec{x}, \vec{y} \rangle = |\vec{x}||\vec{y}| \cos\theta$, where θ is the angle between $\vec{x}, \vec{y}$. ($\theta \in [0, 2\pi]$)
iii. $\langle \vec{x}, \vec{y} \rangle = 0 \Leftrightarrow \vec{x}, \vec{y}$ are orthogonal to each other.
iv. $\langle \vec{x}, \vec{x} \rangle = |\vec{x}|^2$
v. $\langle \vec{x}, \vec{y} \rangle \leq |\vec{x}||\vec{y}|$ (Schwartz Inequality) and the equality holds if and only if $\theta = 0$.

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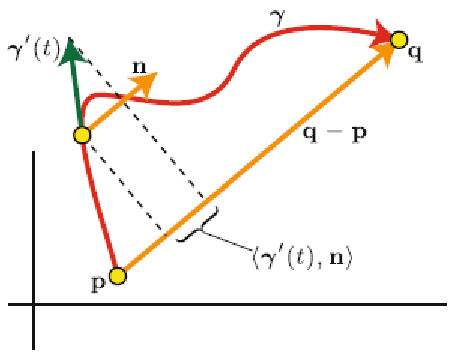
2.1 Proof of Proposition 1.12

Proof. Idea: Compare $\vec{\gamma}'(t)$ and its projection onto $\vec{q}-\vec{p}$. Set $\vec{n} = \frac{\vec{q}-\vec{p}}{|\vec{q}-\vec{p}|}$; $\vec{n}$ is unit.



Tapp Pg.15

Then $|\vec{\gamma}'(t)| \geq \langle \vec{\gamma}'(t), \vec{n}\rangle$ by Schwartz inequality.
Now,

$$\begin{aligned} L &= \int_a^b |\vec{\gamma}'(t)|\,dt \geq \int_a^b \langle \vec{\gamma}'(t), \vec{n}\rangle\,dt \\ &= [\langle \vec{\gamma}(t), \vec{n}\rangle]_a^b = \langle \vec{\gamma}(b), \vec{n}\rangle - \langle \vec{\gamma}(a), \vec{h}\rangle \\ &= \left\langle \vec{q}-\vec{p}, \frac{\vec{q}-\vec{p}}{|\vec{q}-\vec{p}|}\right\rangle = |\vec{q}-\vec{p}| \end{aligned}$$

If the equality holds, then $\forall t \in [a,b], \vec{\gamma}'(t), \vec{n}$ are in the same direction. So,

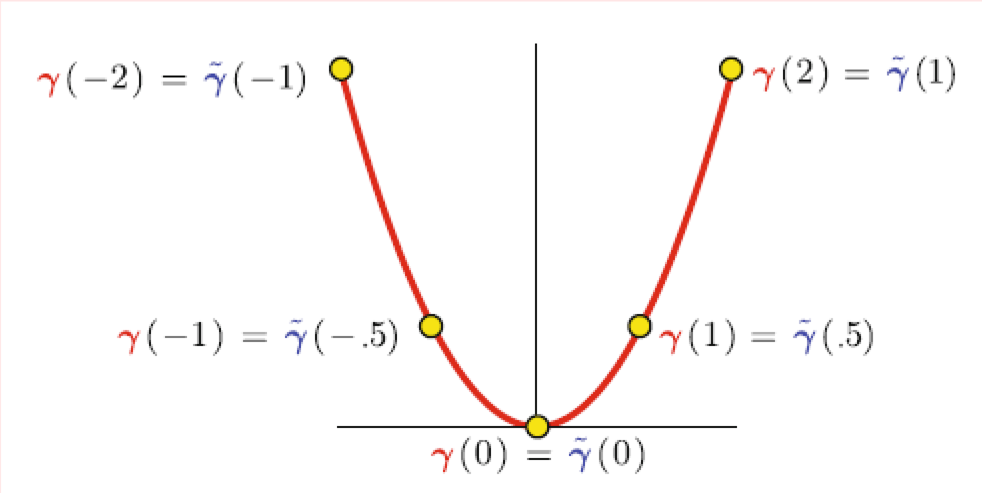
$$\begin{aligned} \vec{\gamma}'(t) &= \langle \vec{\gamma}'(t), \vec{n}\rangle \vec{n}. \\ \vec{\gamma}(t) &= \vec{\gamma}(a) + \int_a^t \vec{\gamma}'(u)du \\ &= \vec{p} + \left(\int_a^t \langle \vec{\gamma}'(u), \vec{n}\rangle dt\right)\vec{n} \end{aligned}$$

parametrizes the line segment between $\vec{p}, \vec{q}$. $\square$

2.2 Reparametrization

There are regular curves that share common properties. Which regular curves should we identify?

Example 2.1
$\vec{\gamma}(t) = (t, t^2), \quad t \in [-2, 2]$
$\tilde{\vec{\gamma}}(t) = (-2t, (-2t)^2), t \in [-1, 1].$
Then $\vec{\gamma}[-2, 2] = \tilde{\vec{\gamma}}[-1, 1] =$



$\vec{\gamma}, \tilde{\vec{\gamma}}$ are the same, up to change in time:
Let $\phi \colon [-1, 1] \to [-2, 2], \quad t \mapsto -2t.$
Then $\tilde{\vec{\gamma}} = \vec{\gamma} \circ \phi$

Definition 2.2 (Reparametrization)
Let $\vec{\gamma} \colon I \to \mathbb{R}^n$ be a regular curve. A reparametrization of $\vec{\gamma}$ is a function of the form

$$\tilde{\vec{\gamma}} = \vec{\gamma} \circ \phi : \tilde{I} \to \mathbb{R}^n,$$

where $\tilde{I}$ is an interval, $\phi \colon \tilde{I} \to I$ is a smooth bijection such that $\forall t \in \tilde{I}$, $\phi'(t) \neq 0$

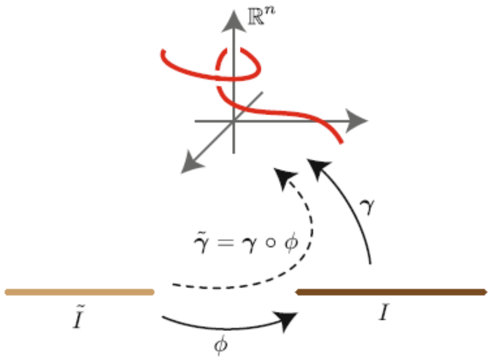


Figure 1: Kapp pg.19

Proposition 2.3
A reparametrization of a regular curve is a regular curve.

Proof. We use the same notations as the definition.
$\tilde{\vec{\gamma}} = \vec{\gamma} \circ \phi \colon \tilde{I} \to \mathbb{R}^n$ is the composition of smooth functions, so smooth.
Moreover, $\forall t \in \tilde{I}, \tilde{\vec{\gamma}}'(t) = \vec{\gamma}'(\phi(t)) \cdot \phi'(t) \neq 0$ □

We will be interested in regular curves up to reparametrizations.

Remarks 2.4

1. $\vec{\gamma}, \tilde{\vec{\gamma}}$ have the same trace.
2. There are regular curves with the same trace that cannot be reparametrized to each other. For instance,
$$\vec{\gamma}_1(t) = (\cos(t), \sin(t)), t \in [0, 2\pi),$$
$$\vec{\gamma}_2(t) = (\cos(t), \sin(t)), t \in [0, 4\pi),$$

Question 2.5: Is there a canonical reparametrization of a given regular curve?

Definition 2.6 (Unit-speed)
A regular curve $\vec{\gamma} \colon I \to \mathbb{R}^n$ is called unit-speed (or parametrized by arc length) if $\forall t \in I$, $|\vec{\gamma}'(t)| = 1$.

Remark 2.7 If $\vec{\gamma} \colon I \to \mathbb{R}^n$ is unit-speed, then,
$$\text{Arc length between } t_1, t_2 = \int_{t_1}^{t_2} |\vec{\gamma}'(t)| dt = \int_{t_1}^{t_2} dt = t_2 - t_1$$

Proposition 2.8
A regular curve always has a unit-speed reparametrization.

Proof. Let $\vec{\gamma} \colon I \to \mathbb{R}^n$ be a regular curve. Fix $t_0 \in I$. Define $s \colon I \to \mathbb{R}$ by
$$s(t) = \int_{t_0}^{t} |\vec{\gamma}'(u)|\, du.$$
Let $\tilde{I} = s(I) \subset \mathbb{R}$. Then $\tilde{I}$ is an interval by IVT.
Since $s'(t) = |\vec{\gamma}'(t)| > 0$ by FTC, regularity, $s \colon I \to \tilde{I}$ is a smooth bijection. Then, $\phi = s^{-1} \colon \tilde{I} \to I$ is a smooth bijection,
$$\phi'(t) = \frac{1}{s'(\phi(t))} = \frac{1}{|\vec{\gamma}'(\phi(t))|} \neq 0.$$
Now $\tilde{\vec{\gamma}} = \vec{\gamma} \circ \phi \colon \tilde{I} \to \mathbb{R}^n$ is a reparametrization of $\vec{\gamma}$, that is unit-speed:
$$\begin{aligned} |\tilde{\vec{\gamma}}'(t)| &= |\vec{\gamma}'(\phi(t)) \cdot \phi'(t)| \\ &= |\vec{\gamma}'(\phi(t))| \cdot 1/|\vec{\gamma}'(\phi(t))| \\ &= 1 \end{aligned}$$
□

Note:

$$s^{-1} \cdot s(t) = t$$
$$(s^{-1})'(s(t)) \cdot s'(t) = 1$$
$$(s^{-1})'(s(t)) = 1/s'(t)$$

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3.1 Reparametrization (Cont'd)

Example 3.1
$\vec{\gamma}(t) = (\cos(t), \sin(t), t), \quad t \in (-\infty, \infty)$ How can we find a unit-speed reparametrization of $\vec{\gamma}$? Compute the arc length function $S\colon (-\infty, \infty) \to \mathbb{R}$:

$$s(t) = \int_0^t |\vec{\gamma}'(u)|\, du = \int_0^t |(-\sin(u), \cos(u), 1)|\, du$$
$$= \int_0^t \sqrt{2}\, du = \sqrt{2}t$$

Set $\phi = s^{-1}$, then $\phi(t) = t/\sqrt{2}$

$$\tilde{\vec{\gamma}}(t) = \vec{\gamma}(t) \circ \phi(t) = \Big(\cos\big(t/\sqrt{2}\big), \sin\big(t/\sqrt{2}\big), t/\sqrt{2}\Big)$$

$t \in (-\infty, \infty)$, is a unit speed reparametrization of $\vec{\gamma}$.

We will be interested in invariants for a regular curve that are unchanged under any reparametrizations.

Examples include:

- trace
- arc-length
- curvature
- torsion

Non-examples include:

- position
- velocity
- speed
- acceleration

Sometimes we consider more specific reparametrization.

Proposition 3.2
If $\tilde{\vec{\gamma}} = \vec{\gamma} \cdot \phi\colon \tilde{I} \to \mathbb{R}^n$ is a reparametrization of a regular curve $\vec{\gamma}\colon I \to \mathbb{R}^n$, then one of the following holds:

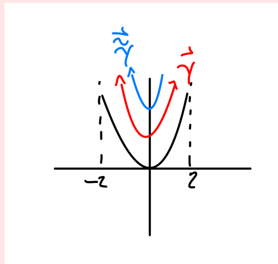
i. $\forall t \in \tilde{I}, \phi'(t) > 0$ i.e. ϕ is strictly increasing
ii. $\forall t \in \tilde{I}, \phi'(t) < 0$ i.e. ϕ is strictly decreasing

Proof. Otherwise $\exists t \in \tilde{I}, \phi'(t) = 0$ by IVT. This contradicts the assumption on ϕ. $\square$

Definition 3.3 (Orientation-preserving vs. orientation-reserving)
Under the setting of the proposition, we say $\tilde{\vec{\gamma}}$ is orientation-preserving if (i) occurs, or orientation-reversing if (ii) occurs.

Example 3.4 (Orientation-preserving)
The arc length reparametrization of a regular curve $\phi\colon I \to \tilde{I}$ is orientation-preserving, because $\phi'(t) = 1/|\vec{\gamma}'(\phi(t))| > 0 \quad \forall t \in I$
This shows an orientation=preserving unit-speed. Reparametrization always exists.

Example 3.5 (Orientation-reversing)
$\vec{\gamma}(t) = (t, t^2), \quad t \in [-2, 2]$
$\tilde{\vec{\gamma}}(t) = (-t, (-t)^2), \quad t \in [2, 2]$



$\tilde{\vec{\gamma}}$ is an orientation-reserving reparametrization of $\vec{\gamma}$ by $\phi\colon [-2, 2] \to [-2, 2], \quad t \mapsto -t$
(Indeed, $\phi' = -1 < 0$).

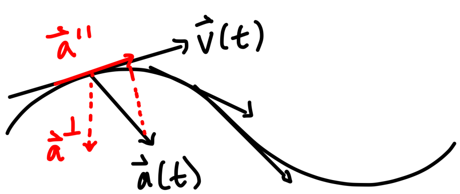
We will be interested in invariants that are unchanged under any orientation-preserving reparametrization.

Example 3.6
signed curvature, rotation index

3.2 Curvature

The curvature measures how sharply the trace bends. What is a plausible definition of the curvature?

Let $\vec{\gamma}\colon I \to R^n$ be a regular curve. Set $\vec{v} = \vec{\gamma}', \vec{a} = \vec{\gamma}''$



$\vec{v}$ knows speed, direction of the motion
$\implies$ $\vec{a}$ should know the change in speed, direction $\to$ curvature.
We write

$$\vec{a} = \vec{a}'' + \vec{a}^\perp$$

where

$$\begin{aligned} \vec{a}'' &= \left\langle \vec{a}, \frac{\vec{v}}{|\vec{v}|} \right\rangle : \quad \text{parallel to } \vec{v} \\ \vec{a}^{\perp} &= \vec{a} - \vec{a}'' : \quad \text{orthogonal to } \vec{v} \end{aligned}$$

Proposition 3.7
$\frac{d}{dt}|\vec{v}(t)| = \left\langle \vec{a}, \frac{\vec{v}}{|\vec{v}|} \right\rangle =$ the parallel component of $\vec{a}$ with respect to $\vec{v}$

Proof.

$$\begin{aligned} \frac{d}{dt}|\vec{v}(t)| &= \frac{d}{dt}\langle \vec{v}(t), \vec{v}(t) \rangle^{1/2} \\ &= \frac{1}{2}\frac{1}{\langle \vec{v}(t), \vec{v}(t) \rangle^{1/2}} \cdot 2\langle \vec{v}(t), \vec{v}'(t) \rangle \\ &= \left\langle \frac{\vec{v}(t)}{|\vec{v}(t)|}, \vec{a}(t) \right\rangle \end{aligned}$$

Note: $\langle v, v \rangle' = \langle v', v \rangle + \langle v, v' \rangle = 2\langle v', v \rangle$ $\square$

So $|\vec{a}^{\perp}(t)|$ would be a plausible definition of the curvature. however this depends on $|\vec{t}|$. (Imagine a centripetal force for a car turning a corner.)

Definition 3.8 (Curvature)
Let $\vec{\gamma} \colon I \to \mathbb{R}^n$ be a regular curve. The curvature function $\kappa \colon I \to [0, \infty)$ is defined as

$$\kappa(t) = \frac{|\vec{a}^{\perp}(t)|}{|\vec{v}(t)|^2}$$

Proposition 3.9
Curvature is independent of parametrizations.

Proof. Let γ be a regular curve. $\tilde{\gamma} = \gamma \cdot \phi$ is a reparametrization of γ.
Denote:

$$\begin{aligned} \kappa &\colon \text{curvature function for } \gamma \\ \tilde{\kappa} &\colon \text{curvature function for } \tilde{\gamma} \end{aligned}$$

We need to show $\tilde{\kappa} = \kappa \circ \phi$
Denote:

$$\begin{aligned} v, a &: \ \text{velocity, acceleration of } \gamma \\ \tilde{v}, \tilde{a} &: \ \text{velocity, acceleration of } \tilde{\gamma}. \end{aligned}$$

Then,

$$\begin{aligned} \tilde{\gamma} &= \gamma \cdot \phi \\ \tilde{v} &= \gamma' \cdot \phi \cdot \phi' = v \circ \phi \cdot \phi' \\ \tilde{a} &= \gamma'' \circ \phi \cdot (\phi')^2 + \gamma' \circ \phi \cdot \phi' \\ &= a \circ \phi \cdot (\phi')^2 + v \circ \phi \cdot \phi' \end{aligned}$$

So, $\tilde{v}$ is parallel to v,

$$\tilde{a}^{\perp} = a^{\perp} \circ \phi \cdot (\phi')^2$$

Therefore,

$$\tilde{\kappa} = \frac{\tilde{a}^\perp}{|\tilde{v}|^2} = \frac{|a^\perp \circ \phi \cdot (\phi')^2|}{|v \circ \phi \cdot \phi|^2} = \frac{|a^\perp \cdot \phi|}{|v \cdot \phi|^2}$$
$$= \kappa \circ \phi$$

□

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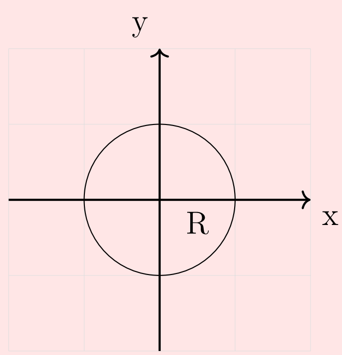
Note: From now on, I will bold my vectors like this $\mathbf{n}$ instead of $\vec{n}$.

4.1 Curvature (Cont'd)

Recall 4.1

$$\kappa(t) = \frac{|\mathbf{a}^\perp(t)|}{|\mathbf{v}(t)|^2}$$

Example 4.2
$\gamma(t) = (R\cos(t), R\sin(t)), \quad t \in (-\infty, \infty)$



$\mathbf{v}(t) = (-R\sin(t), R\cos(t))$
$\mathbf{a}(t) = (-R\cos(t), -R\sin(t))$
Here $\langle \mathbf{v}(t), \mathbf{a}(t)\rangle = -R^2\sin(t)\cos(t) + R^2\cos(t)\sin(t) = 0$;
So $\mathbf{v}(t) \perp \mathbf{a}(t) \implies \mathbf{a}(t) = \mathbf{a}^\perp(t)$.
Therefore,

$$\kappa(t) = \frac{|\mathbf{a}(t)|}{|\mathbf{v}(t)|^2} = \frac{R}{R^2} = \frac{1}{R} \overset{R\to+\infty}{\to} 0 \text{ (flat)}$$

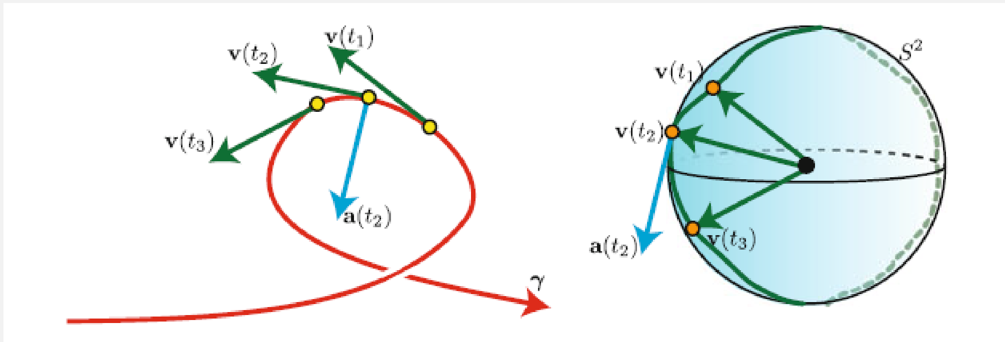
Historically, the curvature of a regular curve was first defined by $\kappa(t) = \frac{1}{R(t)}$, where $R(t)$ is the radius of the circle that best approximates the trace at t (The osculating circle; Read Tapp). Here we give another interpretation of the curvature using the osculating parabola.

Definition 4.3 (Unit tangent and normal vectors)
Let $\gamma\colon I \to \mathbb{R}^n$ be a regular curve. Define the unit tangent and normal vectors as

$$\mathbf{t}(t_0) = \frac{\mathbf{v}(t_0)}{|\mathbf{v}(t_0)|}, \quad \underbrace{\mathbf{n}(t_0) = \frac{\mathbf{a}^\perp(t_0)}{|\mathbf{a}^\perp(t_0)|}}_{\text{defined only if } \kappa(t_0)\neq 0}$$

Remarks 4.4

i. $\mathbf{t}(t_0), \mathbf{n}(t_0)$ are orthonormal, i.e. unit, orthogonal to each other



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ii. The osculating plane at t_0 is the plane through $\mathbf{t_0}$ spanned by $\mathbf{t}(t_0), \mathbf{n}(t_0)$. The osculating plane is the plane that γ is the closest to begin in, and contains the directions where the curve is $\underset{(\mathbf{t})}{\text{heading}}$ and $\underset{(\mathbf{n})}{\text{bending}}$.

Proposition 4.5
Let $\gamma\colon I \to \mathbb{R}^n$ be a regular curve. Then $|\mathbf{t}'| = \kappa|\mathbf{v}|^2$, and $\mathbf{t}' = \kappa|\mathbf{v}|\mathbf{n}$ if $\mathbf{n}$ is defined. In particular, if γ is unit-speed, then

$$|\mathbf{t}'| = \kappa, \quad \text{and } \mathbf{t}' = \kappa\mathbf{n} \text{ if } \mathbf{n} \text{ is defined.}$$

Proof.

$$\mathbf{t}' = \left(\frac{\mathbf{v}}{|\mathbf{v}|}\right)' = \frac{\mathbf{a}}{|\mathbf{v}|} - \mathbf{v}\frac{\langle \mathbf{a}, \mathbf{v}\rangle}{|\mathbf{v}|^3} = \frac{\mathbf{a} - \mathbf{a}''}{|\mathbf{v}|} = \frac{\mathbf{a}^\perp}{|\mathbf{v}|}$$

Hence $|\mathbf{t}'| = \frac{|\mathbf{a}|^\perp}{|\mathbf{v}|^2} \cdot |\mathbf{v}| = \kappa|\mathbf{v}|$, and

$$\mathbf{t}' = \frac{|\mathbf{a}^\perp|}{|\mathbf{v}|^2}|\mathbf{v}|\frac{\mathbf{a}^\perp}{|\mathbf{a}^\perp|} = \kappa|\mathbf{v}|\mathbf{n} \text{ if } \mathbf{n} \text{ is defined.}$$

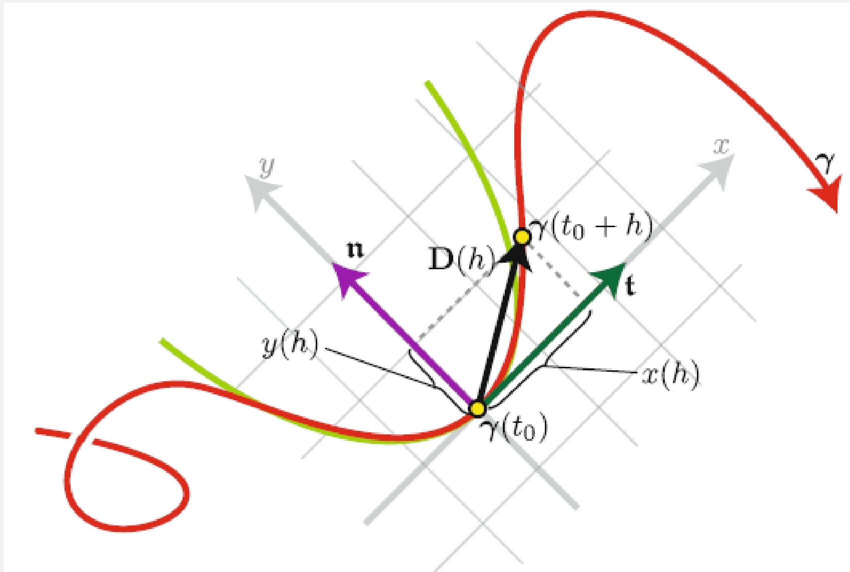
$\square$

Remark 4.6 Let $\gamma\colon I \to \mathbb{R}^n$ be a unit-speed curve, $t_0 \in I$ with $\kappa(t_0) \neq 0$.
Then $\gamma'(t_0) = \mathbf{t}, \gamma''(t_0) = \mathbf{t}' = \kappa\mathbf{n}$, and the 2nd order Taylor approximation at γ at t_0 is

$$\begin{aligned}\gamma(t_0 + h) &\approx \gamma(t_0) + h\gamma'(t_0) + \frac{h^2}{2}\gamma''(t_0)\\ &= \gamma(t_0) + h\mathbf{t} + \frac{\kappa h^2}{2}\mathbf{n}\end{aligned}$$

Set $\mathbf{D}(h) = \gamma(t_0 + h) - \gamma(t_0) \approx h\mathbf{t} + \frac{\kappa h^2}{2}\mathbf{n}$: displacement.
Then,

$$\left.\begin{aligned} x(t) &:= \langle \mathbf{D}(h), \mathbf{t}\rangle \approx h \\ y(t) &:= \langle \mathbf{D}(h), \mathbf{n}\rangle \approx \tfrac{\kappa h^2}{2}\end{aligned}\right\} \text{ the parabola } y = \frac{\kappa}{2}x^2 \text{ in the osculating plane}$$



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$\kappa(t_0)$ = the concavity of the parabola that best approximates the trace at t_0

Proposition 4.7
Let $\gamma\colon I \to \mathbb{R}^n$ be a regular curve. If $\forall t \in I, \kappa(t) = 0$, then γ parametrizes a straight line.

Proof.

$$\begin{aligned}
|\mathbf{t}'| = \kappa|\mathbf{v}| = 0 &\implies \mathbf{t}' = \mathbf{0} \\
&\implies \mathbf{t} = \mathbf{0} \text{ constant} \\
&\implies \mathbf{v} = |\mathbf{v}|\mathbf{c} \\
&\implies \text{ fixing } t_0 \in I, \\
\gamma(t) &= \gamma(t_0) + \int_{t_0}^{t} \mathbf{v}(u)\, du \\
&= \gamma(t_0) + \left(\int_{t_0}^{t} |\mathbf{v}(u)|\, du\right)\mathbf{c}
\end{aligned}$$

□

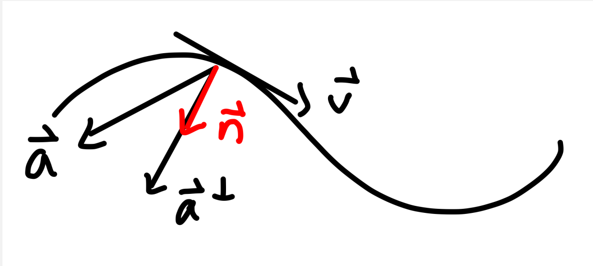
4.2 Plane Curves

$\mathbb{R}^2$ is the only $\mathbb{R}^n$ where the terms “clockwise” and “counter-clockwise” makes sense. This allows us to define

“signed curvature” = curvature + turning direction with respect to $\mathbf{v}$

Recall 4.8

$$\kappa = \frac{|\mathbf{a}^{\perp}|}{|\mathbf{v}|^2} = \frac{\langle \mathbf{a}, \mathbf{n}\rangle}{|\mathbf{v}|^2}$$



Definition 4.9 (Signed curvature)
Let $\gamma: I \to \mathbb{R}^2$ be a regular plane curve. Then the signed curvature $\kappa_s: I \to \mathbb{R}$ is defined as

$$\kappa_s = \frac{\langle \mathbf{a}, \mathbf{n}_s \rangle}{|\mathbf{v}|^2},$$

where,

$$\begin{aligned} \mathfrak{n}_s &= R_{90}\mathbf{t} \\ &= \text{ the counterclockwise } 90^\circ \text{ rotation of } \mathbf{t} \end{aligned}$$