Math 167 (Mathematical Game Theory)
University of California, Los Angeles

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Course description: Quantitative modeling of strategic interaction. Topics include extensive and normal form games, background probability, lotteries, mixed strategies, pure and mixed Nash equilibria and refinements, bargaining; emphasis on economic examples. Optional topics include repeated games and evolutionary game theory. More information can be found on Math UCLA website.

These are my lecture notes for Math 167 (Mathematical Game Theory) taught by Oleg Gleizer. The main textbook for this class is *Game Theory, Alive* by Anna Karlin and Yuval Peres and the supplementary textbook is *A Course in Game Theory* by Thomas Ferguson.

Contents

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1.1 Impartial Combinatorial Games

Definition 1.1 (Impartial combinatorial game)
In an impartial combinatorial game,

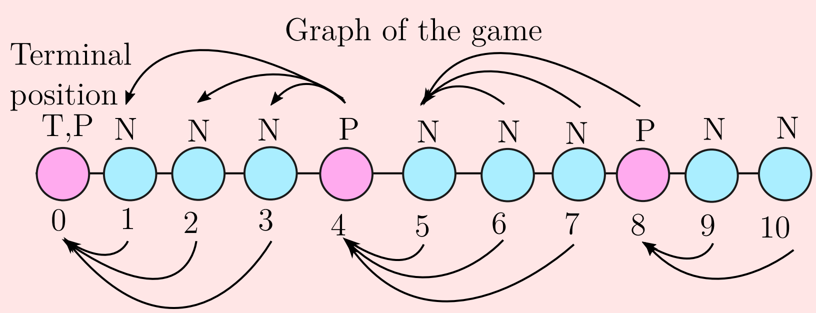
- Two-person
- Perfect information
- No chance moves
- Win-or-lose outcome

Example 1.2
Suppose

- A pile of n chips on the table
- Two players: P1 and P2
- A move consists of removing one, two, or three chips from the pile
- P1 makes the first move, players alternate then
- The player to remove the last chip wins (the last player to move wins. If a player can't move, they lose.)

Method to analyze: backward induction.
Positions:

- **N**, next player to take a move wins.
- **P**, previous (second) player to take a move wins.

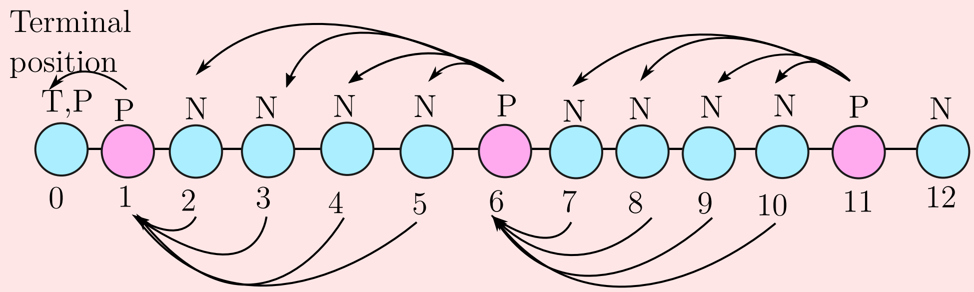


Any move from a **P** position leads to an **N** position. There always exists a move from an **N** position to a **P** position.
Ending condition: the game ends in a finite number of moves, no matter how played.
A **T** position is a **P** position.

Definition 1.3 (Normal play vs. misère play)
In a normal play, the last player to move wins. In a misère play, the last player to move loses.

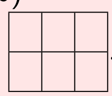
Example 1.4

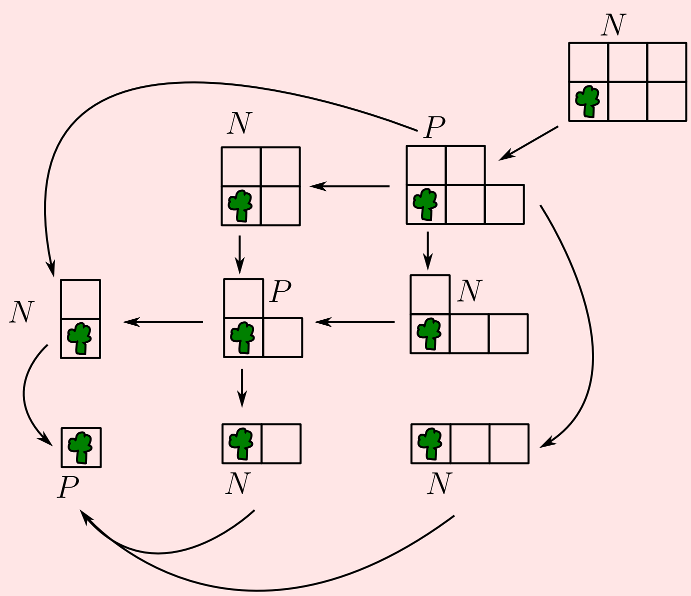
A misère game, a player can take 1-4 chips.



Note 1.5: Every position is either **N** or **P**, but not nothing or both.

Example 1.6 (The game of Chomp)

Let's play the game starting from .



Graph of the game:

- Positions correspond to vertices
- Moves correspond to oriented edges

Definition 1.7 (Strategy)

A function that assigns a move to each position, except for the terminal.

Definition 1.8 (Winning strategy from a position x)
A winning strategy from a position x is a sequence of moves, starting from x, that guarantees a win.

Consider a normal game. Let $\mathbf{N}_i/\mathbf{P}_i$ be the set of positions from which P1/P2 can win (reach the nearest terminal vertex of the same graph) in at most i moves.

$$\mathbf{P}_0 = \mathbf{P}_1 = \{\text{terminal positions}\}$$

$$\mathbf{N}_{i+1} = \{x\colon \text{ there is a move from } x \text{ to } \mathbf{P}_i\}$$
$$\mathbf{P}_{i+1} = \{y\colon \text{ each move leads to } \mathbf{N}_i\}$$

Note 1.9: $\mathbf{P}_0 = \mathbf{P}_1 \subseteq \mathbf{P}_2 \subseteq \mathbf{P}_3 \ldots$
$\mathbf{N}_1 \subseteq \mathbf{N}_2 \subseteq \mathbf{N}_3 \ldots$

$$\mathbf{N} = \bigcup_{i=1} \mathbf{N}_i, \quad \mathbf{P} = \bigcup_{i=0} \mathbf{P}_i$$

Definition 1.10 (Progressively bounded)
A game is called progressively bounded if for every position x there exists an upper bound $B(x)$ on the number of moves until the game terminates.

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2.1 Combinatorial Games (Cont'd)

Recall 2.1

- $\mathbf{P}_0 = \mathbf{P}_1 = \{\text{terminal positions}\}$
- $\mathbf{N}_{n+1} = \{x\colon \text{there is a move from } x \text{ to } \mathbf{P}_n\}$
- $\mathbf{P}_{n+1} = \{y\colon \text{each move from } y \text{ leads to } \mathbf{N}_n\}$
- $\mathbf{P}_0 = \mathbf{P}_1 \subseteq \mathbf{P}_2 \subseteq \dots$
- $\mathbf{N}_1 \subseteq \mathbf{N}_2 \subseteq \mathbf{N}_3 \subseteq$
- $\mathbf{P} = \bigcup\limits_{n=0} \mathbf{P}_n$
- $\mathbf{N} = \bigcup\limits_{n=1} \mathbf{N}_n$
- A game is called <u>progressively bounded</u> if for every position x there exists an upper bound $B(x)$ on the number of moves until the game stops.

Theorem 2.2
In a progressively bounded impartial full information combinatorial game, all positions are in $\mathbf{N} \cup \mathbf{P}$. Thus, for every position there exists a winning strategy.

Proof. Let $B(x) \leq n$. Let us prove by induction that $x \in \mathbf{N}_n \cup \mathbf{P}_n$.
<u>Base:</u> $n = 0$
x is a terminal vertex $\implies x \in \mathbf{P}_0 = \mathbf{P}_1$.
<u>Inductive hypothesis by $\mathbf{P}_0$:</u> $B(x) \leq n \implies x \in \mathbf{N}_n \cup \mathbf{P}_n$.
<u>Inductive step:</u> Show that $B(x) \leq n + 1 \implies x \in \mathbf{N}_{n+1} \cup \mathbf{P}_{n+1}$
Consider a move $x \to y$ and $B(y) \leq n$. Hence, $y \in \mathbf{N}_n \cup \mathbf{P}_n$. So either
<u>Case 1:</u> Each move from x leads to $y \in \mathbf{N}_n \implies x \in \mathbf{P}_{n+1}$.
<u>Case 2:</u> There exists a move from x to $y \notin \mathbf{N}_n$. Thanks to the inductive typo, $y \in \mathbf{N}_n \cup \mathbf{P}_n$ so $y \in \mathbf{P}_n \implies x \in \mathbf{N}_{n+1}$. □

2.2 The Game of Nim

- Several piles, each containing finitely many chips.
- A move: a player can remove any number of chips, from one to all from any pile
- P1 and P2 alternate taking moves
- The player to take the last chip wins

Example 2.3
We have two piles. The general case for k piles, we state: $(x_1, x_2, \ldots, x_k)$.
Nim-Sum:
Consider $x \oplus y$. We rewrite x and y as binary numbers and perform long addition of x_2 and y_2 without carry-over, i.e. mod 2.

Example 2.4

$$5 \oplus 7 = \begin{array}{cccc} & 1 & 0 & 1 \\ \oplus & 1 & 1 & 1 \\ \hline & 0 & 1 & 0 \end{array} = 2$$

Note 2.5: Nim is a progressively bounded game.

Theorem 2.6 (Bouton)
A position $x = (x_1, x_2, \ldots, x_k)$ is a $\mathbf{P}$ position $\Longleftrightarrow x_1 \oplus x_2 \oplus \cdots \oplus x_k = 0$.

3 April 1, 2022

3.1 The Game of Nim (Cont'd)

Recall 3.1 $x = (x_1, x_2, \ldots, x_k)$
Theorem (Bouton) says $x \in \mathbf{P} \iff x_1 \oplus x_2 \oplus \cdots \oplus x_k = 0$.

Proof of Theorem 2.6. We have
Terminal position: $x = (0, 0, \ldots 0) \in \mathbf{P}$ Let $x \in \mathbf{N}$. Then there exists a move $x \to y \in \mathbf{P}$.

$$x_1 \oplus x_2 \oplus \cdots \oplus x_k = \begin{array}{rcccccccc} & & 1 & * & * & \ldots & \ldots & * & * \\ \bigoplus & & & & 1 & * & \ldots & * & * \\ & & & & \vdots & \vdots & & \vdots & \vdots \\ & 1 & * & * & \ldots & \ldots & \ldots & * & * \\ \hline \end{array}$$

Find the left-most (most significant) column with an odd number of 1's. Change any number that has a 1 in the column so that there is an even number of 1's in every column. The 1 in the most significant position becomes a 0 which implies the number becomes smaller. So this is a legal move.
We have $x \in \mathbf{P} \implies$ any move $x \to y \in \mathbf{N}$ where

$$x = (x_1, x_2, \ldots, x_k) \mapsto y = (x_1', x_2, \ldots, x_k)$$

such that

$$x_1' < x_1 \text{ and } x_1 \oplus x_2 \oplus \cdots \oplus x_k = 0.$$

If

$$x_1' \oplus x_2 \oplus \cdots \oplus x_k = 0$$

then

$$x_1' \oplus x_2 \oplus \cdots \oplus x_k = 0$$

then $x_1' = x_0$, a contradiction. Hence

$$x_1' \oplus x_2 \oplus \cdots \oplus x_k \neq 0 \implies y \in \mathbf{N}.$$

□

Example 3.2

$$x_1 = 7$$
$$x_2 = 10$$
$$x_3 = 15$$

$$\begin{array}{cc|c|c} & 0 & 1 & 1 & 1 \\ \oplus & 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 & 1 \\ \hline & 0 & 0 & 1 & 0 \end{array} \implies \begin{array}{ccccc} & 0 & 1 & 0 & 1 \\ \oplus & 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 & 1 \\ \hline & 0 & 0 & 0 & 0 \end{array}$$

So we have that $(7, 10, 15) \mapsto (5, 10, 15)$

3.2 Subtraction Nim

Extra condition: A player can remove at most n chips.
We find pile sizes mod $n + 1$, i.e.

$$(x_1, x_2, \ldots, x_k) \mapsto (x_1 \bmod n + 1, x_2 \bmod n + 1, \ldots, x_k \bmod n + 1)$$

Now we find the Nim-sum and make a move.

$$x \bmod n + 1 = \underbrace{(x_1 \bmod n + 1, x_2 \bmod n + 1, \ldots, x_k \bmod n + 1)}_{(x_1 \bmod n+1)_2 \oplus (x_2 \bmod n+1)_2 \oplus \cdots \oplus (x_k \bmod n+1)_2} \implies \begin{cases} = 0 \iff \mathbf{P} \\ \neq 0 \iff \mathbf{N} \end{cases}$$

Example 3.3
We have $x = (12, 13, 14)$ and $n = 3$. So,

$$(12 \bmod 4, 13 \bmod 4, 14 \bmod 4) \equiv (0, 1, 2) = (0_2, 1_2, 10_2)$$

So

$$\begin{array}{ccc} & 0 & 0 \\ \oplus & 0 & 1 \\ & 1 & 0 \\ \hline & 1 & 1 \end{array} \neq 0$$

so we take away one chip from the third pile

$$\begin{array}{ccc} & 0 & 0 \\ \oplus & 0 & 1 \\ & 0 & 1 \\ \hline & 0 & 0 \end{array}$$

So we have that $(12, 13, 14) \mapsto (12, 13, 13)$.

Note 3.4: You can always make a legal move $\mathbf{N} \to \mathbf{P}$ by removing $i \leq n$ chips from a pile.

Note 3.5: To move from $\mathbf{P}$ to $\mathbf{P}$, you need to remove $n + 1$ chips from a pile. Not allowed! Hence, any move from $\mathbf{P}$ is to $\mathbf{N}$.

Example 3.6
We have $x = (12, 13, 13)$, with $n = 3$. So

$$x \mod 4 = (0, 1, 1)$$

therefore

$$\begin{array}{r} 0 \\ \bigoplus \quad 1 \\ 1 \\ \hline 0 \end{array}$$

3.3 Two-Person Zero Sum Games (Strategic Form)

We have

- P1: a non-empty set of strategies S1
- P2: a non-empty set of strategies S2
- A: S1 $\times$ S2 $\to \mathbb{R}$, the min function for P1 (payoff matrix)

Note 3.7: Since the game is zero-sum, a win for P1 is a loss for P2. $A(i, j)$ can be ≤ 0, so works both ways.

Pure strategies:

$$\begin{array}{cc|c|c|c|c|} & & \multicolumn{4}{c}{\text{P2}} \\ & & \text{S21} & \text{S22} & \cdots & \text{S2}n \\ \cline{3-6} & \text{S11} & a_{11} & a_{12} & \cdots & a_{1n} \\ \cline{3-6} \text{P1} & \text{S12} & a_{21} & a_{22} & \cdots & a_{2n} \\ \cline{3-6} & \vdots & \vdots & \vdots & \ddots & \vdots \\ \cline{3-6} & \text{S1}m & a_{m1} & a_{m2} & \cdots & a_{mn} \\ \cline{3-6} \end{array}$$

A game. P1 chooses the strategy S1i. Simultaneously, P2 chooses the strategy S2j. P1 wins a_{ij}.

Lemma 3.8
$\min_j \max_i a_{ij} \geq \max_i \min_j a_{ij}$

We will continue in the next lecture.

4 Apr 4, 2022

4.1 Two-Person Zero Sum Games in Strategic Form (Cont'd)

Recall 4.1 Recall that

$$
\begin{array}{ccc|c|c|c|c|}
 & & \multicolumn{5}{c}{\text{P2}} \\
 & & \text{S21} & \text{S22} & \cdots & \text{S2}n \\
\cline{3-6}
 & \text{S11} & a_{11} & a_{12} & \cdots & a_{1n} \\
\cline{3-6}
\text{P1} & \text{S12} & a_{21} & a_{22} & \cdots & a_{2n} \\
\cline{3-6}
 & \vdots & \vdots & \vdots & \ddots & \vdots \\
\cline{3-6}
 & \text{S1}m & a_{m1} & a_{m2} & \cdots & a_{mn} \\
\cline{3-6}
\end{array}
$$

P1 has a non-empty set of pure strategies

$$S1 = \{S11, S12, \dots, S1m\}$$

P2 has a non-empty set of pure strategies

$$S2 = \{S21, S22, \dots, S2n\}$$

$A\colon S1 \times S2 \to \mathbb{R}$, payoff matrix
P1, $S1i\colon a_{i1}, a_{i2}, \dots, a_{in}$

Betting on the worst possible outcome, P1 bets on $\min_{1 \le j \le n} a_{ij}$. Being intelligent, P1 chooses

$$\max_{1 \le i \le m} \min_{1 \le j \le m} a_{ij}.$$

Betting on the worst possible loss, P2 bets on $\max_{1 \le i \le m} a_{ij}$. Being intelligent, P2 chooses

$$\min_{1 \le j \le n} \max_{1 \le i \le m} a_{ij}$$

.

Lemma 4.2
$\max\limits_{1 \le i \le m} \min\limits_{1 \le j \le n} a_{ij} \le \min\limits_{1 \le i \le m} \max\limits_{1 \le j \le n} a_{ij}$

Proof. Let

$$\max_i \min_j a_{ij} = a_{pq}$$

$$\min_j \max_i a_{ij} = a_{rs}$$

$$
\begin{array}{c|c|c|c|c|c|}
 & & q & & s & \\
\hline
 & & & & & \\
\hline
p & & a_{pq} & \leq & a_{ps} & \\
\hline
 & & & & \rotatebox{-90}{$\leq$} & \\
\hline
r & & a_{rq} & & a_{rs} & \\
\hline
 & & & & & \\
\hline
\end{array}
$$

□

Example 4.3

Chooser (P1), Hider (P2). Hider hides behind their back

- Either left hand with one coin
- or right hand with two coins

Chooser chooses L or R,

		P2	
		L1	R2
P1	L	1	0
	R	0	2

$$P1\colon \max_j \min_i a_{ij} = 0$$

$$P2\colon \min_j \max_i a_{ij} = 1$$

<u>Mixed strategies</u>

		P2	
		L1, q	R2, $1-q$
P1	L, p	1	0
	R, $1-p$	0	2

P1: if P2 chooses the strategy L1, the expected gain is

$$1 \cdot p + 0 \cdot (1-p) = p$$

If P2 chooses R2, the expected gain is

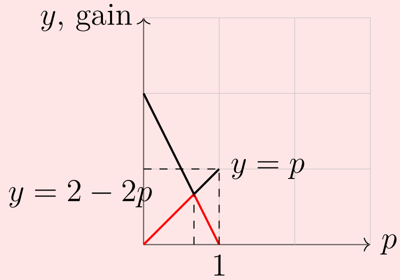
$$0 \cdot p + 2(1-p) = 2 - 2p.$$

If P1 is out of luck, then the expected gain is

$$\min\{p, 2-2p\}$$

Since P1 is intelligent, they choose p s.t. the gain is

$$\max_{0 \leq p \leq 1} \min\{p, 2-2p\}$$



$$2 - 2p = p$$

$$2 = 3p$$

$$p = \frac{2}{3}$$

The optimal strategy is

$$\frac{2}{3}L + \frac{1}{3}R$$

With expected gain $\geqslant \frac{2}{3}$.
P2 is thinking. If P1 chooses L, my expected loss is

$$1 \cdot q + 0 \cdot (1 - q) = q$$

If P1 chooses R, my expected loss is

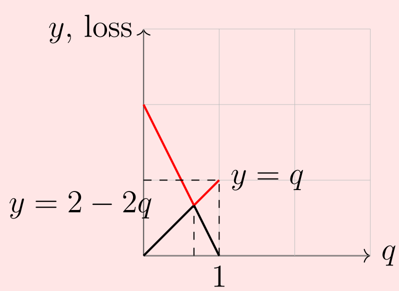
$$0 \cdot q + 2(1 - q) = 2 - 2q$$

Suppose I'm out of luck. Then my expected loss is

$$\max\{q, 2 - 2q\}$$

Being my very smart self,

$$\min_{0 \leq q \leq 1} \max\{q, 2 - 2q\}$$



The optimal strategy is

$$\frac{2}{3}L1 + \frac{1}{3}R2$$

With expected loss $\leq \frac{2}{3} = V$, the value of the game.

Let us generalize $A \in \mathbb{R}^{n\times m}$, an $n \times m$ payoff matrix.

$$\Delta_m = \left\{ \mathbf{p} \in \mathbb{R}^m \colon p_1 \geq 0, p_2 \geq 0, \dots, p_m \geq 0, \sum_{i=1}^{m} p_i = 1 \right\}$$

$$\Delta_n = \left\{ \mathbf{q} \in \mathbb{R}^n \colon q_1 \geq, q_2 \geq 0, \dots, q_n \geq 0, \sum_{j=1}^{n} q_j = 1 \right\}$$

A mixed strategy for P1 is determined by

$$\mathbf{p} \in \Delta_m$$

A mixed strategy for P2 is determined by

$$\mathbf{q} \in \Delta_n$$

Expected gain for P1 (expected loss for P2) = $(\mathbf{p})^T A\mathbf{q}$

$$\begin{array}{cc|c|c|c|c|}
 & & \multicolumn{4}{c}{\text{P2}} \\
 & & \multicolumn{1}{c}{q_1} & \multicolumn{1}{c}{q_2} & \multicolumn{1}{c}{\cdots} & \multicolumn{1}{c}{q_n} \\
\cline{3-6}
 & p_1 & a_{11} & a_{12} & \cdots & a_{1n} \\
\cline{3-6}
\text{P1} & p_2 & a_{21} & a_{22} & \cdots & a_{2n} \\
\cline{3-6}
 & \vdots & \vdots & \vdots & \ddots & \vdots \\
\cline{3-6}
 & p_m & a_{m1} & a_{m2} & \cdots & a_{mn} \\
\cline{3-6}
\end{array}$$

So

$$(\mathbf{p})^t A\mathbf{q} = p_i(a_i q_1 + a_{i2}q_2 + \cdots + a_{in}q_n)$$

If P1 employs the strategy $\mathbf{P}$, then in the worst case their payoff is

$$\min_{\mathbf{q}\in\Delta_n} (\mathbf{p})^t A\mathbf{q} = \min_{1\leq j\leq n} \sum_{i=1}^{m} a_{ij}p_i$$

Hence, P1's winning strategy is

$$\max_{\mathbf{p}\in\Delta_m} \min_{\mathbf{q}\in\Delta_n} \mathbf{p}^T A\mathbf{q}$$

5 Apr 6, 2022

5.1 General Two-Person Zero-Sum Games in Strategic Form

Recall 5.1 Recall

$$\begin{array}{cc|c|c|c|c|}
 & & \multicolumn{4}{c}{\text{P2}} \\
 & & q_1 & q_2 & \cdots & q_n \\
\hline
 & p_1 & a_{11} & a_{12} & \cdots & a_{1n} \\
\hline
\text{P1} & p_2 & a_{21} & a_{22} & \cdots & a_{2n} \\
\hline
 & \vdots & \vdots & \vdots & \ddots & \vdots \\
\hline
 & p_m & a_{m1} & a_{m2} & \cdots & a_{mn} \\
\hline
\end{array}$$

With set of mixed strategies given by,

$$\Delta_m = \left\{ \mathbf{p} \in \mathbb{R}^m \colon \mathbf{p} \geq 0, \sum_{i=1}^{m} p_i = 1 \right\}$$

$$\Delta_n = \left\{ \mathbf{q} \in \mathbb{R}^n \colon \mathbf{q} \geq 0, \sum_{j=1}^{n} q_j = 1 \right\}$$

where $p_1 \geq 0, p_2 \geq 0, \ldots, p_m \geq 0$.

We have

$$\text{Expected gain of P1 } = (\mathbf{p})^t A\mathbf{q}$$

with $\mathbf{p} \in \Delta_m$ and $\mathbf{q} \in \Delta_n$.

The winning strategy for P1:

- Worst case: $\min\limits_{\mathbf{q} \in \Delta_n} (\mathbf{p})^t A\mathbf{q}$
- Smart choice: $\max\limits_{\mathbf{p} \in \Delta_m} \min\limits_{\mathbf{q} \in \Delta_n} (\mathbf{p})^t A\mathbf{q}$

$$\begin{aligned}
\min_{\mathbf{q} \in \Delta_n} (\mathbf{p})^t A\mathbf{q} &= \min_{\mathbf{q} \in \Delta_m} \sum_{j=1}^{n} q_j \sum_{i=1}^{m} a_{ij} p_i \\
&= \min_{1 \leq j \leq n} \sum_{i=1}^{m} a_{ij} p_i
\end{aligned}$$

The winning strategy for P2:

- Worst case: $\max\limits_{\mathbf{p} \in \Delta_m} (\mathbf{p})^t A\mathbf{q}$
- Smart choice: $\min\limits_{\mathbf{q} \in \Delta_n} \max\limits_{\mathbf{p} \in \Delta_m} (\mathbf{p})^t A\mathbf{q}$

$$\begin{aligned}\max_{\mathbf{p}\in\Delta_m}(\mathbf{p})^t A\mathbf{q} &= \max_{\mathbf{p}\in\Delta_m}\sum_{i=1}^{m} p_i \sum_{j=1}^{n} a_{ij}q_j \\ &= \max_{1\le i\le m}\sum_{j=1}^{n} a_{ij}q_j\end{aligned}$$

Definition 5.2 (Safety value for P1 vs. P2)
The value $\hat{\mathbf{p}}$ at which

$$\max_{\mathbf{p}\in\Delta_m}\min_{\mathbf{q}\in\Delta_n}(\mathbf{p})^t A\mathbf{q}$$

is attained is called the safety value for P1. The value $\hat{\mathbf{q}}$ at which

$$\min_{\mathbf{q}\in\Delta_n}\max_{\mathbf{p}\in\Delta_m}(\mathbf{p})^t A\mathbf{q}$$

is attained is called the safety value for P2.

Theorem 5.3 (Von Neumann Minimax Theorem)
For any two-person zero-sum game with $m \times n$ payoff matrix A, there is a number V, called the value of the game, satisfying

$$\min_{\mathbf{q}\in\Delta_n}\max_{\mathbf{p}\in\Delta_m}(\mathbf{p})^t A\mathbf{q} = \max_{\mathbf{p}\in\Delta_m}\min_{\mathbf{q}\in\Delta_n}(\mathbf{p})^t A\mathbf{q} = V$$

Let $\hat{\mathbf{p}}$ be an optimal solution for P1. Let $\hat{\mathbf{q}}$ be an optimal solution for P2. Then

$$\min_{\mathbf{q}\in\Delta_n}(\hat{\mathbf{p}})^t A\mathbf{q} = \max_{\mathbf{p}\in\Delta_m}(\hat{\mathbf{p}})^t A\hat{\mathbf{q}}$$

Proof. Proof seen here. □

Definition 5.4 (Value of the game)
Given conditions from Von Neumann Minimax Theorem, V is the value of the game.

Example 5.5 (Odd or Even)

- P1 and P2 simultaneously call out one of the numbers, 1 or 2.
- If the sum is odd, P1 wins and gets the sum of the numbers in \$
- If the sum is even, P2 wins and gets the sum of the numbers in \$

		P2: $1, q$	P2: $2, 1-q$
P1	$1, p$	-2	3
P1	$2, 1-p$	3	-4

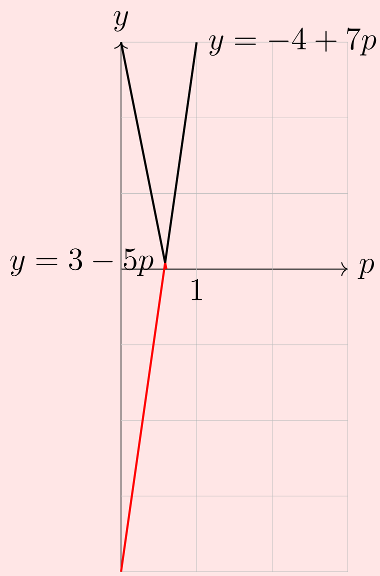
So P1's expected win (P2's expected loss) is

$$(\mathbf{p})^t A\mathbf{q} = \begin{bmatrix} p & 1-p \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} q \\ 1-q \end{bmatrix}$$
$$= -12pq + 7p + 7q - 4$$

P1's worst possible case:

$$f(p) = \min_{0 \le q \le 1} \{-12pq + 7p + 7q - 4\}$$

$$\begin{aligned} q, S21\colon &\quad -2p + 3(1-p) = 3 - 5p \\ 1-q, S22\colon &\quad 3p - 4(1-p) = -4 + 7p \end{aligned}$$



- If $3 - 5p \ge -4 + 7p$, then $q = 0$.
- If $3 - 5p < -4 + 7p$, then $q = 1$.

Hence,

$$f(p) = \min\{3 - 5p, 4 - 7p\}$$

Note that

$$(-12pq + 7p + 7q - 4)\Big|_{q=0} = -4 + 7p$$

$$(-12pq + 7p + 7q - 4)\Big|_{q=1} = 3 - 5p$$

P1: $\max\limits_{0 \le p \le 1} \min\limits_{0 \le q \le 1} q\Big(-2p + 3(1-p)\Big) + (1-q)\Big(3p - 4(1-p)\Big) = \max\limits_{0 \le p \le 1} \min\{3 - 5p, -4 + 7p\}$

$$3 - 5p = -4 + 7p$$
$$7 = 12p$$
$$p = \frac{7}{12}, \quad q = \frac{5}{12}$$

Now from P2:

P2: $\min\limits_{0 \le q \le 1} \max\limits_{0 \le p \le 1} p\Big(-2q + 3(1-q)\Big) + (1-p)\Big(3q - 4(1-q)\Big) = \min\limits_{0 \le q \le 1} \max\{3 - 5q, -4 + 7q\}$

6 Apr 8, 2022

6.1 Solving Small-Dimensional Two-Person Zero-Sum Games Pen-and-Paper

Definition 6.1 (Saddle point)
An element of A, a_{ij} is called a saddle point if

- a_{ij} is the min of the i-th row
- a_{ij} is the max of the j-th column

Then $p_i = 1$, $q_j = 1$, $V = a_{ij}$

Example 6.2
Given

$$\begin{array}{cc} & \text{min} \\ \begin{array}{c} \\ \\ \\ \text{max} \end{array} \begin{array}{c} \begin{bmatrix} 4 & 1 & -3 \\ 3 & 2 & 5 \\ 0 & 1 & 6 \end{bmatrix} \\ \begin{array}{ccc} 4 & 2 & 6 \end{array} \end{array} & \begin{array}{c} -3 \\ 2 \\ 0 \\ \\ \end{array} \end{array}$$

So $p_2 = q_2 = 1$ and $V = 2$.

Lemma 6.3
Let a_{pq} and a_{rs} be saddle points of a payoff matrix A. Then $a_{pq} = a_{rs}$.

		q		s	
p		a_{pq}	$\leq$	a_{ps}	
		$\geq$		$\leq$	
r		a_{rq}	$\geq$	a_{rs}	

6.2 Domination

Rows:

$$\begin{array}{ccccc} i\text{-th row} & a_{i1} & a_{12} & \cdots & a_{in} \\ & \geq & \geq & & \geq \\ k\text{-th row} & a_{k1} & a_{k2} & \cdots & a_{kn} \end{array}$$

So $p_k = 0$ so k-th row can be removed from A.

Strict domination: for $j = 1, 2, \ldots, n$,

$$a_{ij} > a_{kj}$$

Columns: the k-th column dominates the j-th column

$$\begin{array}{ccc} a_{1j} & \geqslant & a_{1k} \\ a_{2j} & \geqslant & a_{2k} \\ \vdots & \vdots & \vdots \\ a_{mj} & \geqslant & a_{mk} \end{array}$$

Strict domination: for $i = 1, 2, \ldots, m$,

$$a_{ij} > a_{ik}$$

where a_{ik} is dominant.

- Removing a dominant row or column does not change the value of the game, but may remove an optimal strategy.
- Removing a strictly dominant row or column does not change the set of optimal strategies.

Example 6.4

$$A_1 = \begin{bmatrix} 2 & 0 & 4 \\ 1 & 2 & 3 \\ 4 & 1 & 2 \end{bmatrix}$$

Note: $\left.\begin{array}{c} 0 < 4 \\ 2 < 3 \\ 1 < 2 \end{array}\right\}$, strict domination.

$$A_2 = \begin{bmatrix} 2 & 0 \\ 1 & 2 \\ 4 & 1 \end{bmatrix}$$

Note: $\begin{array}{cc} 2 & 0 \\ \wedge & \wedge \\ 4 & 1 \end{array}$, strict domination

$$\begin{array}{cc} & \text{min} \\ \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix} & \begin{array}{c} 1 \\ 1 \end{array} \\ \text{max} \;\; \begin{array}{cc} 4 & 2 \end{array} & \end{array}$$

Note: No saddle point

Remark 6.5 A row/column can be dominated by a weighted sum of rows columns. For example,

$$\begin{matrix} a_{i1} & a_{12} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{matrix}$$

For $\ell = 1, 2, \ldots, n$, we have

$$\alpha a_{i\ell} + (1-\alpha) a_{j\ell} \geqslant a_{k\ell}$$

Example 6.6

$$\begin{bmatrix} 0 & 4 & 6 \\ 5 & 7 & 4 \\ 9 & 6 & 3 \end{bmatrix}$$

$$4 > 3$$

$$7 > 4.5$$

$$5 \geq 6$$

7 Apr 11, 2022

7.1 Principle of Indifference

$$
\begin{array}{cc|c|c|c|c|}
 & & \multicolumn{4}{c}{\text{P2}} \\
 & & \multicolumn{1}{c}{q_1} & \multicolumn{1}{c}{q_2} & \multicolumn{1}{c}{\cdots} & \multicolumn{1}{c}{q_n} \\
\cline{3-6}
 & p_1 & a_{11} & a_{12} & \cdots & a_{1n} \\
\cline{3-6}
\text{P1} & p_2 & a_{21} & a_{22} & \cdots & a_{2n} \\
\cline{3-6}
 & \vdots & \vdots & \vdots & \ddots & \vdots \\
\cline{3-6}
 & p_m & a_{m1} & a_{m2} & \cdots & a_{mn} \\
\cline{3-6}
\end{array}
$$

Let $\hat{\mathbf{p}} = (p_1, p_2, \ldots, p_m)^t$ be an optimal strategy for P1 and let $q_j = 1$ be a pure strategy for P2.

$$\sum_{i=1}^{m} a_{ij} p_i \geqslant V \tag{1}$$

Let $\hat{\mathbf{q}} = (q_1, q_2, \ldots, q_n)^t$ be an optimal strategy for P2 and let $p_i = 1$ be a pure strategy for P1. Then

$$\sum_{j=1}^{n} a_{ij} q_j \leqslant V \tag{2}$$

Note 7.1: If both players use optimal strategies, then

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j = V$$

Proof. We have

$$
\begin{aligned}
V \leqslant \sum_{i=1}^{m} a_{ij} p_i = 1 \cdot \sum_{i=1}^{m} a_{ij} p_i &= \left(\sum_{j=1}^{m} q_j \right) \sum_{i=1}^{m} a_{ij} p_i \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j \\
&= \underbrace{\sum_{i=1}^{m} p_i}_{=1} \sum_{j=1}^{m} a_{ij} q_j \leqslant V \\
&= V
\end{aligned}
$$

□

Theorem 7.2 (The Equilibrium Theorem)
Let $\hat{\mathbf{p}} = (p_1, p_2, \ldots, p_m)$ and $\hat{\mathbf{q}} = (q_1, q_2, \ldots, q_n)$ be optimal strategies for P1 and P2 respectively. Then

$$\sum_{j=1}^{n} a_{ij} q_j = V \quad \forall i \text{ s.t. } p_i > 0$$

$$\sum_{i=1}^{m} a_{ij} p_i = V \quad \forall j \text{ s.t. } q_j > 0$$

Proof. Let $p_k > 0$ and let $\sum_{j=1}^{n} a_{kj} q_j \neq V \implies \sum_{j=1}^{n} a_{kj} q_j < V$. We have

$$V \leqslant \sum_{i=1}^{m} p_i \sum_{j=1}^{n} a_{ij} q_j < V$$

a contradiction. □

Example 7.3 (The game of Odd-and-Even)
Played with three numbers: 0, 1, and 2.

		P2, even 0	1	2
P1, Odd	$0, p_1$	0	1	-2
	$1, p_2$	1	-2	3
	$2, p_3$	-2	3	-4

$p_1 \geqslant 0, p_2 \geqslant 0, p_3 \geqslant 0$, and $p_1 + p_2 + p_3 = 1$. Then

$$\begin{cases} p_2 - 2p_3 - V = 0 \\ p_1 - 2p_2 + 3p_3 - V = 0 \\ -2p_1 + 3p_2 - 4p_3 - V = 0 \\ p_1 + p_2 + p_3 = 1 \end{cases}$$

7.2 Symmetric Games

Definition 7.4 (Symmetric Game)
The rules are the same for P1 and P2. So $A^t = -A$.

Theorem 7.5
The value of a finite size symmetric game is zero.

Proof. Note $V^t = V$. And

$$V = (\hat{\mathbf{p}})^t A \hat{\mathbf{p}} = \left[(\hat{\mathbf{p}})^t A \hat{\mathbf{p}}\right]^t = -\hat{\mathbf{p}} A \hat{\mathbf{p}} = -V$$

So

$$V = -V \implies V = 0$$

$\square$

Example 7.6 (Rock, Paper, Scissors)
We have

		P2		
		Rock	Paper	Scissors
P1	Rock	0	−1	1
	Paper	1	0	−1
	Scissors	−1	1	0

So

$$\begin{cases} \quad\ \ p_2 - p_3 = 0 \\ -p_1 \quad\ + p_3 = 0 \\ p_1 - p_2 \quad\ = 0 \\ p_1 + p_2 + p_3 = 1 \end{cases}$$

Therefore,

$$p_1 = p_2 = p_3 = \frac{1}{3}$$

8 Apr 13, 2022

8.1 The Equilibrium Theorem and Symmetric Games (Cont'd)

Recall 8.1 The Equilibrium Theorem:
Let $\hat{\mathbf{p}}$, $\hat{\mathbf{q}}$ be optimal solutions for P1 and P2 respectively. Then

$$\sum_{j=1}^{n} a_{ij} q_j = V \quad \forall i \text{ s.t. } p_i > 0$$

$$\sum_{i=1}^{m} a_{ij} p_i = V \quad \forall j \text{ s.t } q_j > 0$$

Recall 8.2 A game is called a symmetric game if $A^t = -A$.

Example 8.3 (A Mendelson Game)

	1	2	3	4	5	6		99	100
1	0	−1	2	2	2	2	⋯	2	2
2	1	0	−1	2	2	2	⋯	2	2
3	−2	1	0	−1	2	2	⋯	2	2
4	−2	−2	1	0	−1	2	⋯	2	2
5	−2	−2	−2	1	0	−1	⋯	2	2
6	−2	−2	−2	−2	1	0	⋯	2	2
⋮	−2	−2	−2	−2	−2	1	⋯		
99	−2	−2	−2	−2	−2	−2	⋯	0	−1
100	−2	−2	−2	−2	−2	−2	⋯		0

So

$$\begin{cases} \phantom{-p_1 + {}} p_2 - 2p_3 = 0 \\ -p_1 + \ p_3 = 0 \\ 2p_1 - p_2 = 0 \\ p_1 + p_2 + \ p_3 = 1 \end{cases}$$

$$\implies \begin{cases} p_3 = p_1 \\ p_2 = 2p_1 \end{cases} \implies p_1 + 2p_1 + p_1 = 1$$

$$\implies p_1 = \frac{1}{4} = p_3, \quad p_2 = \frac{1}{2}$$

So

$$\hat{\mathbf{p}} = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, 0, 0, \ldots, 0\right)$$

8.2 Invariance or Games with Symmetries

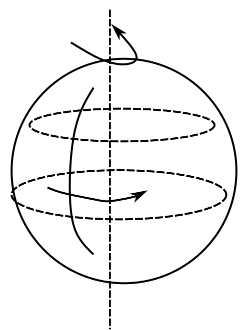
P2

		Rock	Paper	Scissors
	Rock	0	-1	1
P1	Paper	1	0	-1
	Scissors	-1	1	0

We shift to

P2

		Paper	Rock	Scissors
	Paper	0	1	1
P1	Rock	-1	0	-1
	Scissors	1	1	0



A group $= G$

$$G = \{\sigma \colon \sigma_1 \circ \sigma_2 \in G, e \in G, \sigma^{-1} \circ \sigma = \sigma \circ \sigma^{-1} = e\}$$

Let S1 $= (1, 2, \ldots, m)$ and S2 $= (1, 2, \ldots, n)$ be the sets of pure strategies for P1 and P2. Let σ be a permutation of S1.

$$\begin{matrix} 1 & 2 & \ldots & m \\ \sigma(1) & \sigma(2) & \ldots & \sigma(m) \end{matrix}$$

Definition 8.4 (Invariant)
A finite two-person zero-sum game is invariant under a permutation $\sigma \in S_m$ if $\forall i \in S1$, and $\forall j \in S2$, there exists a unique $j' \in S2$ such that

$$A(i, j) = A\Big(\sigma(i), j'\Big)$$

9 Apr 15, 2022

9.1 Invariance (Cont'd)

Recall 9.1 $S1 = \{1, 2, \dots, m\}, S2 = \{1, 2, \dots, n\}$.

Recall 9.2 σ, a permutation of S1, a bijection S1 $\to$ S1:

$$\begin{pmatrix} 1 & 2 & \cdots & m \\ \sigma(1) & \sigma(2) & \cdots & \sigma(m) \end{pmatrix}$$

Definition 9.3 (Invariant under a permutation σ)
A finite two-person zero-sum game is <u>invariant</u> under a permutation σ of S1 if $\forall i \in S1, j \in S2$, there exists a unique $j' \in S2$ such that

$$A(i, j) = A\Big(\sigma(i), j'\Big)$$

The uniqueness requirement: if $\forall i \in S1, \forall j \in S2$, $\exists j', j'' \in S2$ such that $j' \neq j''$ and

$$A(i, j) = A(\sigma(i), j') = A(\sigma(i), j'')$$

then the strategies j' and j'' have identical payoffs. Remembering the rows, we have

		j'	$\cdots$	j''
$\sigma(1)$		$a_{\sigma(1)}j'$	$\cdots$	$a_{\sigma(1)}j''$
$\sigma(2)$		$a_{\sigma(2)}j'$	$\cdots$	$a_{\sigma(2)}j''$
$\vdots$		$\vdots$		$\vdots$
$\sigma(m)$		$a_{\sigma(m)}j'$	$\cdots$	$a_{\sigma(m)}j''$

A duplicate strategy can be removed without loss of generality.

Notation 9.4: $\overline{\sigma}(j) = j'$

Lemma 9.5
$\overline{\sigma}$ is a permutation of S2.

Proof.

- $\overline{\sigma}$ is defined for any $j \in S2$ by definition.
- To show that $\overline{\sigma} \in S_n$, let us show that $\overline{\sigma}$ is one-to-one.

Assume the opposite: $j, k \in S2$, $j \neq k$, $\overline{\sigma}(j) = \overline{\sigma}(k)$. Then

$$A(i, j) = A\Big(\sigma(i), \overline{\sigma}(j)\Big) = A\Big(\sigma(i), \overline{\sigma}(k)\Big) = A(i, j) \quad \forall i \in S1$$

Hence, the strategies $S2j$ and $S2k$ are duplicates which implies assumption is incorrect. $\square$

Lemma 9.6
A game invariant under σ is invariant under σ^{-1}

Proof.
$$A(i,j) = A\Big(\sigma(i), \overline{\sigma}(j)\Big) \quad \forall i \in S1, j \in S2$$
Then
$$A\Big(\sigma(i), \overline{\sigma}(j)\Big) = A(i,j)$$
Let $i' = \sigma(i), j' = \sigma(j)$. Then
$$A(i',j') = A\Big(\sigma^{-1}(i'), \overline{\sigma}^{-1}(j')\Big) \quad \forall i' \in S1, j' \in S2$$
□

9.2 Multiplication of Permutation

Note 9.7:
$$\sigma_2 \circ \sigma_1(*) = \sigma_2(\sigma_1(*))$$

Example 9.8
Suppose
$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$$
Then,
$$\sigma_2 \circ \sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 2 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$$

Lemma 9.9
A game invariant under σ_1 and σ_2 is invariant under $\sigma_2 \circ \sigma_1$.

Proof. $\forall i \in$ S1, $j \in$ S2,
$$\begin{aligned} A(i,j) = A\Big(\sigma(i), \overline{\sigma}(j)\Big) = A(i',j') &= A\Big(\sigma_2(i'), \overline{\sigma}_2(j')\Big) \\ &= A\Big(\sigma_2 \circ \sigma_1(i), \overline{\sigma}_2 \circ \overline{\sigma}_1(j)\Big) \end{aligned}$$
□

<u>**Conclusion**</u>: Invariant permutations of pure strategies form a group.

Definition 9.10 (Invariant under a group σ)
A game (S1, S2, A) is <u>invariant under a group σ</u> if it is invariant under any $\sigma \in G$

10 Apr 18, 2022

10.1 Midterm 1

11 Apr 20, 2022

11.1 Invariance (Cont'd)

Recall 11.1 A group G is a set such that

1. $\forall \sigma_1, \sigma_2 \in G, \sigma_2 \circ \sigma_1 \in G$, i.e. there exists an operation of group multiplication.
2. $\exists e \in G$ such that $e \circ \sigma = \sigma \circ e = \sigma$, $\forall \sigma \in G$.
3. $\forall \sigma \in G, \exists \sigma^{-1} \in G$ such that $\sigma^{-1} \circ \sigma = \sigma \circ \sigma^{-1} = e$

Let $\mathrm{S1} = \{1, \ldots, m\}$, $\mathrm{S2} = \{1, \ldots, n\}$. Let G be a subgroup of S_m.

Definition 11.2 (Homomorphism)
A homomorphism of groups G and $\overline{G}$ is a map $h\colon G \to \overline{G}$ preserving the group structure, i.e.

1. $h(\sigma_2 \circ \sigma_1) = \overline{\sigma_2} \circ \overline{\sigma_1}, \forall \sigma_1, \sigma_2 \in G$
2. $h(e) = \overline{e}$
3. $h(\sigma^{-1}) = \overline{\sigma}^{-1}$, $\forall \sigma \in G$

Definition 11.3 (Invariant under a game)
Let G be a subgroup of S_m. A game (S1, S2, A) is invariant under G if it is invariant $\forall \sigma \in G$.

Recall 11.4 $\forall i \in \mathrm{S1}$, $j \in \mathrm{S2}$, there exists a unique $\bar{\sigma} \in \mathrm{S}_n$ such that $A(i, j) = A\big(\sigma(i), \bar{\sigma}(j)\big)$. In other words, there exists a homomorphism

$$\begin{aligned} h\colon G &\to S_n \\ \sigma &\mapsto \bar{\sigma} \end{aligned}$$

such that

$$a_{ij} = a_{\sigma(i), \bar{\sigma}(j)} \quad \forall i \in S1, j \in S2$$

Definition 11.5 (Orbit of a group action)
The orbit of a group action is the set

$$O_i = \{\sigma(i)\colon i \in S1, \sigma \in G\}$$

Note 11.6: The relation "being in the same orbit" is an equivalence relation.

Action of G on mixed strategies

$$p_1 S11 + p_2 S12 + \cdots + p_m S1m$$

$$\big\downarrow \sigma$$

$$p1S1\sigma(1) + p_2S1\sigma(2) + \cdots + p_mS1\sigma(m)$$

so some $\sigma(k)$ in the sum above equals 1

$$\sigma(k) = 1 \quad \sim \quad k = \sigma^{-1}(1)$$

$$p_kS1\sigma(k) = p_{\sigma^{-1}(1)}S11$$

Similarly for $2, 3, \ldots, m$,

$$p_{\sigma^{-1}(1)}S11 + p_{\sigma^{-1}(2)}S12 + \cdots + P_{\sigma^{-1}(m)}S1m$$

Definition 11.7 (G-invariant)
A mixed strategy $\mathbf{p}$ is <u>G-invariant</u> if $\sigma(\mathbf{p}) = \mathbf{p}$ for all $\sigma \in G$.

Theorem 11.8
If a game (S1, S2, A) is invariant under G, then each player has a G-invariant optimal strategy.

Proof. Let $\hat{\mathbf{p}} = (p_1, p_2, \ldots, p_m)^t$ be an optimal strategy for P1. Then

$$\sum_{i=1}^{m} p_iA(i,j) \geqslant V \quad \forall j \in S2$$

Let

$$\hat{\mathbf{p}}_G = \frac{1}{|G|}\sum_{\sigma\in G}\sigma(\hat{\mathbf{p}}),$$

i.e. $\forall i = 1, 2, \ldots, m$,

$$(\hat{\mathbf{p}}_G)_i = \frac{1}{|G|}\sum_{\sigma\in G} p_{\sigma^{-1}(i)},$$

the average of the ith coordinate over the orbit O.
Invariance of $\hat{\mathbf{p}}_G$:

$$\forall \alpha \in G, \quad \alpha(\hat{\mathbf{p}}_G) = \frac{1}{|G|}\sum_{\sigma\in G}\alpha\circ\sigma(\hat{\mathbf{p}})$$

- $\alpha \circ \sigma$ is defined $\forall \sigma \in G$
- Assume $\sigma_1 \neq \sigma_2$, $\alpha \circ \sigma_1 = \alpha \circ \sigma_2$. Then

$$\alpha^{-1}\circ\alpha\circ\sigma_1 = \alpha^{-1}\circ\alpha\circ\sigma_2 \sim \sigma_1 = \sigma_2$$

Contradiction implies multiplication by α is a one-to-one map $G \to G$.

Hence, multiplication by α is a bijection $G \to G$.

$$\frac{1}{|G|}\sum_{\sigma\in G}\sigma(\hat{\mathbf{p}}) = \hat{\mathbf{p}}_G$$

We shall continue this proof in the next lecture. □

12 Apr 22, 2022

12.1 Invariance (Cont'd)

Recall 12.1

$$\sigma\Big(\mathbf{p} = (p_1, p_2, \ldots, p_m)^t\Big) = (p_{\sigma^{-1}(1)}, p_{\sigma^{-1}(2)}, \ldots, p_{\sigma^{-1}(m)})$$

A mixed strategy $\mathbf{p}$ is G-invariant if $\sigma(\mathbf{p}) = \mathbf{p}$ for all $\sigma \in G$.

Proof of Theorem 11.8. $\alpha\colon G \to G$, $\alpha(\sigma) = \alpha \circ \sigma$ is a bijection of G. Thus,

$$\hat{\mathbf{p}}_G = \frac{1}{|G|}\sum_{\sigma \in G}\sigma(\hat{\mathbf{p}}) = \frac{1}{|G|}\sum_{\sigma \in G}\alpha \circ \sigma(\hat{\mathbf{p}}) = \alpha(\hat{\mathbf{p}}_G)$$

<u>Optimality of $\hat{\mathbf{p}}_G$:</u>

$$\begin{aligned}
\sum_{i=1}^{m}(\hat{\mathbf{p}}_G)_i A(i,j) &= \sum_{i=1}^{m}\frac{1}{|G|}\sum_{\sigma \in G}(\hat{\mathbf{p}})_{\sigma^{-1}(i)}A(i,j) \\
&= \frac{1}{|G|}\sum_{\sigma \in G}\sum_{i=1}^{m}(\hat{\mathbf{p}})_{\sigma^{-1}(i)}A(i,j) \\
&= \frac{1}{|G|}\sum_{\sigma \in G}\sum_{i=1}^{m}p_i A\Big(\sigma(i), \bar{\sigma}(j)\Big) \\
&= \frac{1}{|G|}\sum_{\sigma \in G}\sum_{i=1}^{m}p_i A(i,j) \geqslant V
\end{aligned}$$

because

$$\sum_{i=1}^{m}p_i A(i,j) \geqslant V \quad \forall j \in S2$$

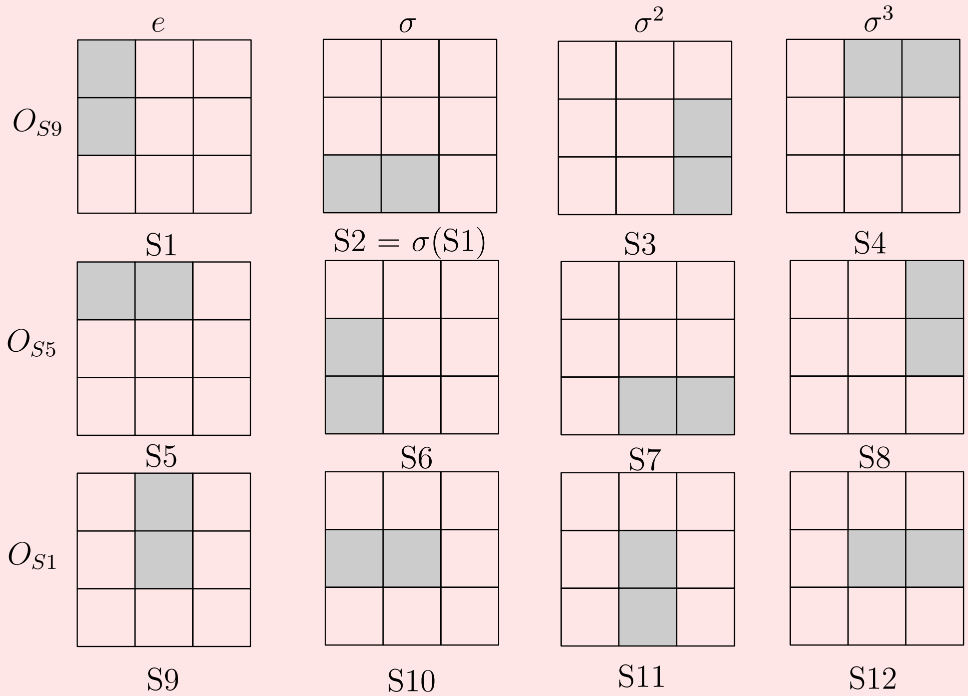
□

Example 12.2 (Battleships)
We have a 3×3 grid, a submarine, P1 is the bomber, G-counter-clockwise rotations of the grid.
G is cyclic: $\{e, \sigma, \sigma^2, \sigma^3\}$ and note $\sigma^4 = e$.

P1:

S1	S2	S3
S4	S5	S6
S7	S8	S9

P2: sub



$$A, \text{ a } 9 \times 12 \text{ matrix}$$

Let $\hat{\mathbf{p}}_G = (p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9)$ be a G-invariant optimal strategy for P1. Then

$$p_1 = p_3 = p_7 = p_9 = \tilde{p}_1$$
$$p_2 = p_4 = p_6 = p_8 = \tilde{p}_2$$
$$p_5 = \tilde{p}_5$$

Similarly,

$$q_1 = q_2 = q_3 = q_4 = \tilde{q}_1$$
$$q_5 = q_6 = q_7 = q_8 = \tilde{q}_2$$
$$q_9 = q_{10} = q_{11} = q_{12} = \tilde{q}_3$$

		$O_1, \tilde{q}_1$	$O_5, \tilde{q}_2$	$O_9, \tilde{q}_3$	min
P1	$O_1, \tilde{p}_1$	$\frac{1}{4}$	$\frac{1}{4}$	0	0
	$O_2, \tilde{p}_2$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
	$O_5, \tilde{p}_3$	0	0	1	0
	max	$\frac{1}{4}$	$\frac{1}{4}$	1	

<u>Note</u>: $A_{21} = A_{22} = \frac{1}{4}$ are saddle points.

$$C1 = C2 \implies \tilde{q}_1 = \tilde{q}_2$$

Solution of the game: $V = \frac{1}{4}$,

$$\tilde{p_2} = 1, \quad \tilde{q_1} = \tilde{q_2} = \frac{1}{2}$$

i.e,

$$p_2 = p_4 = p_6 = p_8 = \frac{1}{4}$$

$$q_1 = q_2 = q_3 = q_4 = q_5 = q_6 = q_7 = q_8 = \frac{1}{8}$$

13 Apr 25, 2022

13.1 Colonel Blotto Games

Example 13.1

Colonel Blotto has 4 regiments with which to occupy two posts. Lieutenant Kije has 3 regiments for the task.
Payoffs:

- The side sending more units to a post captures it as well as the enemy units sent to the post. The payoff is 1 point for the post and 1 point for each captured enemy unit.
- If both sides send the same number of units, the forces withdraw with no payoff.

	$(3,0), q_1$	$(2,1), q_2$	$(1,2), q_2$	$(0,3), q_1$
$p_1, (4,0)$	4	2	1	0
$p_2, (3,1)$	1	3	0	-1
$p_3, (2,2)$	-2	2	2	-2
$p_2, (1,3)$	-1	0	3	1
$p_1, (0,4)$	0	1	2	4

So $G = \mathbb{Z}_2 = \{e, \sigma \colon \sigma^2 = e\}$, acts swapping posts.

	$O_1\colon (3,0) \sim (0,3)$, $\tilde{q}_1 = 2q_1$	$O_2\colon (2,1) \sim (1,2)$, $\tilde{q}_2 = 2q_1$
$O_1\colon (4,0) \sim (0,4)$, $\tilde{p}_1 = 2p_1$	$\dfrac{4+0+4+0}{4} = 2$	$\dfrac{2+1+1+2}{4} = \dfrac{3}{2}$
$O_2\colon (3,1) \sim (1,3)$, $\tilde{p}_2 = 2p_2$	0	$\dfrac{3}{2}$
$O_3\colon (2,2)$, $\tilde{p}_3 = p_3$	$\dfrac{-2-2}{2} = -2$	$\dfrac{2+2}{2} = 2$

$$\begin{array}{cc} 2 & \frac{3}{2} \\ \vee\!\!\vee & \vee\!\!\vee \\ 0 & \frac{3}{2} \end{array} \implies \tilde{p}_2 = 0 \implies p_2 = 0$$

	$\tilde{q}_1$	$1-\tilde{q}_1 = \tilde{q}_2$
$\tilde{p}_1$	2	$\frac{3}{2}$
$\tilde{p}_3 = 1-\tilde{p}_1$	-2	2

$$\text{P1: } \max_{0\le p\le 1} \min_{0\le q\le 1} q\Big(2p - 2(1-p)\Big) + \Big(1-q\Big)\left(\frac{3}{2}p + 2(1-p)\right) = \max_{0\le p\le 1} \min\left\{4p-2, -\frac{p}{2}+2\right\}$$

$$\implies 4p - 2 = -\frac{p}{2} + 2 \implies \frac{9p}{2} = 4 \implies 9p = 8 \implies \boxed{\tilde{p}_1 = \frac{8}{9}}$$

$$V = 2 - \frac{\tilde{p}_1}{2} = 2 - \frac{4}{9} = \boxed{\frac{14}{9}}$$

Since $0 < \tilde{p}_1 < 1$, we can use the indifference principle to find $\tilde{q}_1$.

$$\begin{aligned} 2q + \frac{3}{2}(1-q) &= \frac{14}{9} \\ \frac{q}{2} + \frac{3}{2} &= \frac{14}{9} \\ q + 3 &= \frac{28}{9} \\ q &= \frac{28}{9} - 3 = \frac{1}{9} \end{aligned}$$

$$\implies \boxed{\tilde{q}_1 = \frac{1}{9}}$$

$$(\hat{\mathbf{p}})^t = \left(\frac{4}{9}, 0, \frac{1}{9}, 0, \frac{4}{9}\right)$$

$$(\hat{\mathbf{q}})^t = \left(\frac{1}{18}, \frac{4}{9}, \frac{4}{9}, \frac{1}{18}\right)$$

14 Apr 27, 2022

14.1 Linear Programming

Example 14.1
We want to maximize

$$x + 5y$$

such that

$$5x + 6y \leq 30$$

$$3x + 2y \leq 12$$

$$x \geq 0, y \geq 0$$

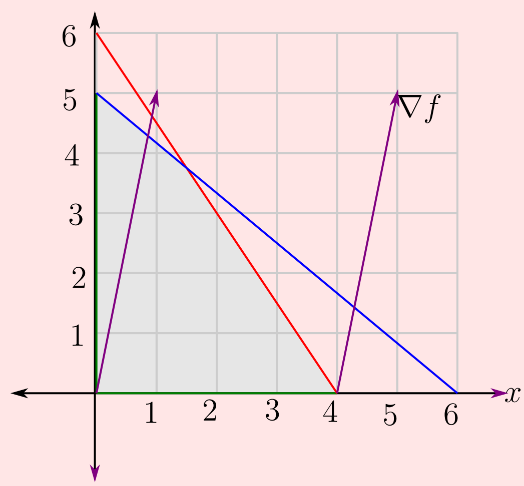
So

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

And

$$6y = 30 - 5x$$

$$\implies \boxed{y = 6 - \frac{3}{2}x} \quad \boxed{y = 5 - \frac{5}{6}x}$$



Standard form of a LP

We want to minimize

$$v = c^t x = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

such that

$$Ax \geq b$$

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \geq b_1$$

$$a_{12}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \geq b_2$$

$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \geq b_m$$
$$x \geq 0$$

i.e.

$$x_1 \geq 0, \ldots, x_n \geq 0$$

Example 14.2 (Example 14.1 in the standard form)
We want to minimize

$$-x - 5y$$

such that

$$-5x - 6y \geq -30$$
$$-3x - 2y \geq 0$$
$$x \geq 0$$
$$y \geq 0$$

14.2 Two-Person Zero-Sum Games as LP Programs in the Standard Form

We have

$$\begin{aligned}\text{P1:}\ \max_{\mathbf{p}\in\Delta_m} V(p_1, p_2, \ldots, p_m) &= \max_{\mathbf{p}\in\Delta_m} \min_{\mathbf{q}\in\Delta_n} (\mathbf{p})^t A\mathbf{q} \\ &= \max_{\mathbf{p}\in\Delta_n} \left(\min_{\mathbf{q}\in\Delta_n} q_1 \sum_{i=1}^{m} p_1 a_{i1} + q_2 \sum_{i=1}^{m} p_i a_{i2} + \cdots + q_n \sum_{i=1}^{n} p_i a_{in} \right) \\ &= \max_{\mathbf{p}\in\Delta_m} \min \left\{ \sum_{i=1}^{n} p_i a_{ij}, j = 1, \ldots, n \right\}\end{aligned}$$

Example 14.3
P1 is solving the following problem:
Choose $p_1, p_2, \dots, p_m, V$, to maximize V such that

$$V \leq \sum_{i=1}^{m} p_i a_{i1}, V \leq \sum_{i=1}^{m} p_i a_{i2}, \dots, V \leq \sum_{i=1}^{m} p_i a_{in}$$

$$\implies 0 \leq \sum_{i=1}^{m} p_i a_{i1} - V, 0 \leq \sum_{i=1}^{m} p_i a_{i2} - V, \dots, 0 \leq \sum_{i=1}^{m} p_i a_{in} - V$$

$$\mathbf{p} \in \Delta_m \quad \boxed{p_1 \geq 0, p_2 \geq 0, \dots, p_m \geq 0, p_1 + p_2 + \cdots + p_m = 1}$$

So

$$V = V_1 - V_2, V_1 \geq 0, V_2 \geq 0$$

$$p_1 + p_2 + \cdots + p_m \geq 1$$

$$-p_1 - p_2 - \cdots - p_m \geq -1$$

Example 14.4 (Colonel Blotto)
We have

	$(3,0), q$	$(2,1), q_2$	$(1,2), q_2$	$(0,3), q_4$
$(4,0), p_1$	4	2	1	0
$(3,1), p_2$	1	3	0	-1
$(2,2), p_3$	-2	2	2	-2
$(1,3), p_4$	-1	0	3	1
$(0,4), p_5$	0	1	2	4

So

$$p_1 + p_2 + p_3 + p_4 + p_5 = 1$$

$$p_1 \geq 0, p_2 \geq 0, p_3 \geq 0, p_4 \geq 0, p_5 \geq 0$$

LP in the standard form

$$V = V_1 - V_2, V_1 \geq 0, V_2 \geq 0$$

$$-V \to \min \sim V_2 - V_1 \to \min \text{ s.t.}$$

$$\begin{cases} 4p_1 + p_2 - 2p_3 - p_4 - V_1 + V_2 \geq 0 \\ 2p_1 + 3p_2 + 2p_3 + p_5 - V_1 + V_2 \geq 0 \\ p_1 + 2p_3 + 3p_4 + 2p_5 - V_1 + V_2 \geq 0 \\ - p_2 - 2p_3 + p_4 + 4p_5 - V_1 + V_2 \geq 0 \\ p_1 + p_2 + p_3 + p_4 + p_5 = 1 \end{cases}$$

So

$$\left\{p_1 = \frac{4}{9}, p_2 = 0, p_3 = \frac{1}{9}, p_4 = 0, p_5 = \frac{4}{9}, V_1 = \frac{14}{9}, V_2 = 0\right\}$$

15 Apr 29, 2022

15.1 LP Duality

We have the problem

$$V = c^t x \to \min \text{ s.t.}$$
$$Ax \geq b$$
$$x \geq 0$$

And the Dual problem is

$$w = b^t y \to \max \text{ s.t.}$$
$$A^t y \leq c$$
$$y \geq 0$$

Lemma 15.1
Dual to dual = primary

Proof.

$$(-b)^t y \to \min \text{ s.t.}$$
$$(-A)^t y \geq -c$$
$$y \geq 0$$

Next,

$$(-c)^t x \to \max \text{ s.t.}$$
$$-Ax \leq -b$$
$$x \geq 0$$

Finally,

$$c^t x \to \min \text{ s.t.}$$
$$Ax \geq b$$
$$x \geq 0$$

□

Lemma 15.2
Dual to LP for P1 = LP for P2

Proof. We have

$$\begin{aligned} a_{11}p_1 + a_{21}p_2 + \cdots + a_{m1}p_m - V_1 + V_2 &\geq 0 \\ a_{12}p_1 + a_{22}p_2 + \cdots + a_{m2}p_m - V_1 + V_2 &\geq 0 \\ &\vdots \\ a_{1n}p_1 + a_{2n}p_2 + \cdots + a_{mn}p_m - V_1 + V_2 &\geq 0 \\ -p_1 - p_2 + \cdots - p_m &\geq -1 \\ p_1 + p_2 + \cdots + p_m &\geq 1 \end{aligned}$$

With

$$p_1 \geq 0$$
$$p_2 \geq 0$$
$$\vdots$$
$$p_m \geq 0$$
$$v_1 \geq 0$$
$$v_2 \geq 0$$
$$p_1 + p_2 + \cdots + p_m = 1$$

So

$$V = V_1 - V_2$$
$$\mathbf{x}^t = (p_1, p_2, \ldots, p_m, V_1, V_2)$$
$$V = V(p_1, p_2, \ldots, p_m) \to \max$$
$$V_2 - V_1 \to \min$$

So

$$Ax = \begin{bmatrix} a_{11} & a_{21} & \ldots & a_{m1} & -1 & 1 \\ a_{12} & a_{22} & \ldots & a_{m2} & -1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \ldots & a_{mn} & -1 & 1 \\ -1 & -1 & \ldots & -1 & 0 & 0 \\ 1 & 1 & \ldots & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ v_1 \\ v_2 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

and

$$c^t = (\underbrace{0, \ldots, 0}_{m}, -1, 1) \quad x \geq 0$$

So

$$(\mathbf{x})^t = (q_1, q_2, \ldots, q_n, W_1, W_2)$$

So the objective function is

$$W_2 - W_1 \to \max \text{ s.t.}$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & -1 & 1 \\ a_{21} & a_{22} & \dots & a_{2n} & -1 & 1 \\ \vdots & \vdots & & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} & -1 & 1 \\ -1 & -1 & \dots & -1 & 0 & 0 \\ 1 & 1 & \dots & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \\ W_1 \\ W_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

with

$$q_1 \geq 0, W_1 \geq 0, W_2 \geq 0$$

Then

$$\begin{aligned} -q_1 - q_2 - \dots + q_n &\leq -1 \\ q_1 + q_2 + \dots + q_n &\leq 1 \\ \underline{q_1 + q_2 + \dots + q_n \geq 1} \\ q_1 + q_2 + \dots + q_n &= 1 \end{aligned}$$

So

$$a_{11}q_1 + a_{12}q_2 + \dots + a_m q_n \leq W = W_1 - W_2$$

$$\vdots$$

$$a_{m1}q_1 + a_{12}q_2 + \dots + a_{1n}q_n \leq W \to \min$$

Furthermore, we are maximizing

$$V(p_1, p_2, \dots, p_m)$$

and minimizing

$$W(q_1, q_2, \dots, q_n)$$

□

Theorem 15.3 (Weak Duality)
Let x be a feasible point for the primary problem:

$$V = c^t x \to \min \text{ subject to}$$

$$Ax \geq b, x \geq 0$$

Let y be a feasible point for the dual primary problem

$$W = b^t y \to \max \text{ subject to}$$

$$A^t y \leq c, y \geq 0.$$

Then

$$c^t x \geq b^t y$$

Proof. Write

$$c^t \geq y^t A$$

So

$$-V = c^t x \geq y^t A x \geq v^t b = -W$$

$$-V \geq -W \implies W \geq V$$

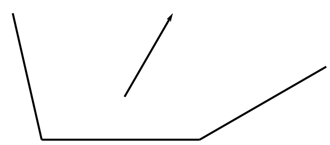
So

$$\boxed{\overbrace{\max_{p \in \Delta_m} \min_{q \in \Delta_n} p^t A q}^{V} \leq \overbrace{\min_{q \in \Delta_n} \max_{p \in \Delta_m} p^t A q}^{W}}$$

□

Corollary 15.4
- If the primary problem is unbounded, then the dual problem is infeasible.
- If the dual problem is unbounded, then the problem is infeasible.

Proof. $c^t x \geq b^t y$



□

16 May 2, 2022

16.1 Using LP Duality to Prove Von Neumann's Theorem

Recall 16.1 (Weak Duality Theorem) Let x be a feasible point for a linear program.

$$-V = c^t x \to \min \text{ subject to}$$

$$Ax \geq b, x \geq 0$$

where

$$x^t = (p_1, p_2, \dots, p_m, V_1, V_2)$$
$$V = V_1 - V_2$$

Let y be a feasible point for the dual LP

$$-W = b^t y \to \max \text{ subject to}$$

$$A^t y \leq c, y \geq 0$$

where

$$y^t = (q_1, q_2, \dots, q_n, W_1, W_2)$$
$$W = W_1 - W_2$$

Then

$$c^t x \geq b^t y$$
$$-V \geq -W$$

Corollary 16.2
If x is a feasible solution to the primary problem and y is a feasible solution to the dual problem $c^t x = b^t y$, then x and y are optimal.

Lemma 16.3
$\max\limits_{p \in \Delta_m} \min\limits_{q \in \Delta_n} p^t A q$ exists.

Proof.

$$\begin{aligned}
\max_{p \in \Delta_m} \min_{q \in \Delta_n} p^t A q &= \max_{p \in \Delta_m} \min \left\{ \sum_{i=1}^m a_{ij} p_i, j = 1, \dots, n \right\} \\
&= \underbrace{\max_{p \in \Delta_m}}_{\text{compact set}} \underbrace{V(p_1, p_2, \dots, p_m)}_{\substack{\text{piecewise linear, hence} \\ \text{continuous function}}}
\end{aligned}$$

□

> **Recall 16.4** (Weierstrass Extreme Value Theorem) A real-valued function continuous on a compact set attains its min and max on the set.

> **Theorem 16.5** (LP: Strong Duality Theorem)
> If one of the two dual linear programs has an optimal solution, then so does the other and the values of the objective functions at the optimal points are equal.

16.2 General Sum Games

Prisoner's Dilemma

Two members of a criminal organization are arrested and put into solitary confinement. They have no means to communicate with each other. Prosecution lacks evidence to convict each on the principal charge. Prosecution has enough evidence to convict each on a lesser charge. each prisoner is given a possibility to cooperate (Rat) or stay silent (Mum). The payoff bimatrix is given by

		S21	S2
	P2 \ P1	R	M
S11	R	-8, -8	0, -10
S12	M	-10, 0	-1, 1

$$A = \begin{bmatrix} -8 & 0 \\ -10 & -1 \end{bmatrix}, \quad B^t = \begin{bmatrix} -8 & -10 \\ 0 & -1 \end{bmatrix}$$

Dominant strategy: Whatever P2 chooses, P1 is better off

Definition 16.6 (Maxmin strategy)
Let A and B be $m \times n$ payoff matrices for P1 and P2.

$$V_1 = \max_{p \in \Delta_m} \min_{q \in \Delta_n} p^t A q = \max_{p \in \Delta_m} \min_{j=1,\ldots,n} \sum_{i=1}^{m} p_i a_{ij} = \mathrm{Val}(A)$$

where $\mathrm{Val}(A) =$ safety level for P1. And

$$V_2 = \max_{q \in \Delta_n} \min_{p \in \Delta_m} p^t B q = \max_{q \in \Delta_n} \min_{i=1,\ldots,m} \sum_{j=1}^{n} b_{ij} q_j = \mathrm{Val}(B^t)$$

Noting that

$$p^t B q = q^t B^t p \quad V_2^t = V_2$$

Also note that safety level strategies disregard payoff for the other player. A strategy p that achieves $\mathrm{Val}(A)$ for P1 is called a maxmin strategy.

Note 16.7: This game is symmetric if $A = B^t$.

Example 16.8

Recall our payoff matrix

P1/P2	R	M
R	$-8, -8$	$0, -10$
M	$-10, 0$	$-1, -1$

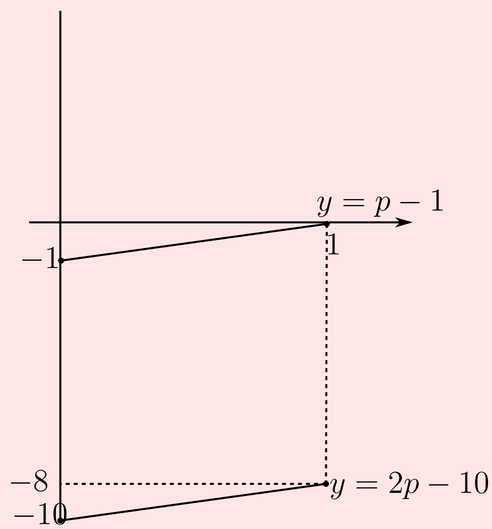
$\rightarrow$

	P1/P2	R	M
p	R	-8	0
$1-p$	M	-10	-1

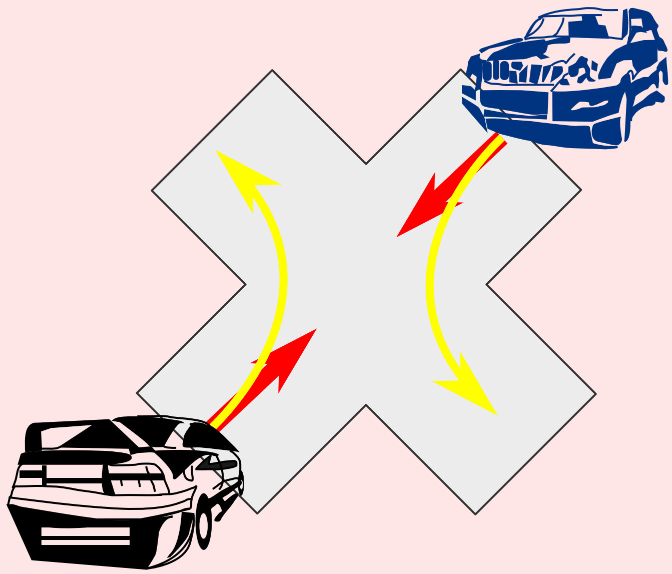
The safety strategy is given by

$$\begin{aligned}\mathrm{Val}(A) = \max_{0\le p\le 1} \min\{-8p - 10(1-p), p-1\} &= \max_{0\le p\le 1} \min\{2p-10, p-1\} \\ &= \max_{0\le p\le 1}(2p-10) = -8\end{aligned}$$

$$\boxed{p = 1, 1-p = 0}$$



Example 16.9 (The Game of Chicken)
Two drivers speed head-on towards each other at crossroads.



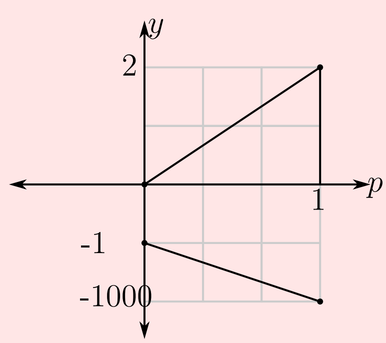
		P2: Drive	P2: Chicken
P1	Drive, p	$-1000, 1000$	$2, -1$
	Chicken, $1-p$	$-1, 2$	$0, 0$

$$\begin{aligned}\mathrm{Val}(A) &= \max_{0\le p\le 1} \min\{-1000p-(1-p), 2p\} \\ &= \max_{0\le p\le 1} \min\{-999p-1, 2p\}\end{aligned}$$

So

$$p=0, \quad 1-p=1$$

$$\mathrm{Val}(A) = -1$$



17 May 4, 2022

17.1 Nash Equilibrium in General Sum Games

Recall 17.1 For an $m \times n$ matrix A,

$$\mathrm{Val}(A) = \max_{p \in \Delta_m} \min_{q \in \Delta_n} p^t A q$$

which is the safety level of P1.

Definition 17.2 (Nash Equilibrium (NE))
A pair $(\hat{p}, \hat{q})$ is called a Nash Equilibrium, if

$*$ $(\hat{p})^t A \hat{q} \geqslant p A \hat{q} \quad \forall p \in \Delta_m$. i.e. $\hat{p}$ is the best response to $\hat{q}$

$**$ $(\hat{p})^t B \hat{q} \geqslant \hat{p} B q \quad \forall q \in \Delta_n$. i.e. $\hat{q}$ is the best response to $\hat{p}$.

Definition 17.3 (Pure Nash Equilibrium (PNE))
A Pure Nash Equilibrium is a Nash Equilibrium such that $\hat{p} = e_i \in \mathbb{R}^m$ and $\hat{q} = e_j \in \mathbb{R}^n$, where

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{th position}$$

Note 17.4: Pure Nash Equlibriums are easy to find.

Example 17.5
We have

	S21	S22	S23	S24	S25	S26
S11	2,1	4,3	$7^*, 2$	$7^*, 4$	0,5*	3,2
S12	$4^*, 0$	$5^*, 4$	$1, 6^*$	0,4	0,3	$5^*, 1$
S13	1,3*	$5^*, 3^*$	3,2	4,1	$1^*, 0$	4,3*
S14	4*,3	2,5*	4,0	1,0	$1^*, 5^*$	2,1

So

$$\mathrm{PNE} = \{(S13, S22), (S14, S25)\}$$

Lemma 17.6
Nash equilibria are "individually rational", i.e. delivery at least the safety value.

Proof. Let p^* be a solution to the problem

$$\max_{p\in\Delta_n}\min_{q\in\Delta_n} p^t Aq = \mathrm{Val}(A)$$

i.e.

$$(p^*)^t Aq \geqslant \mathrm{Val}(A)_q \quad \forall q\in\Delta_n$$

So,

$$(\hat{p})^t A\hat{q} \geqslant (p^*)^t Aq \geqslant \mathrm{Val}(A)$$
$$(\hat{p})^t A\hat{q} \geqslant pA\hat{q}$$

□

Lemma 17.7
Nash Equilibria survive elimination of strongly dominated strategies.

Proof.

$$\left[\begin{array}{cc|c|c} a_{11} & a_{12} & a_{1j} & a_{1n} \\ a_{21} & a_{22} & a_{2j} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \hline a_{i1} & a_{i2} & a_{ij} & a_{in} \\ \hline \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{mj} & a_{mn} \end{array}\right]$$

Suppose the ith row is strictly domination by other rows, then

$$\exists \alpha_1 \geq 0, \alpha_2 \geq 0, \dots, \alpha_{i-1} \geq 0$$

$$\alpha_{i+1} \geq 0, \dots, \alpha_m \geq 0$$
$$\sum_{\substack{k=1\\k\neq i}}^{m} \alpha_i = 1$$

and

$$a_{ij} < \sum_{\substack{k=1\\k\neq i}}^{\alpha_k a_{kj}} \quad \text{for } j = 1, 2, \dots, n$$

Suppose that $(\hat{p}, \hat{q})$ is a Nash Equlibrium such that $\hat{p}_i \neq 0$. So

$$\begin{aligned}
(\hat{p})^t A\hat{q} = \sum_{i,j} a_{ij}\hat{p}_i\hat{q}_j &= \hat{p}_1 \sum_{j=1}^{n} a_{ij}\hat{q}_j + \cdots + \hat{p}_i \sum_{j=1}^{n} a_{ij}\hat{q}_j + \cdots + \hat{p}_m \sum_{j=1}^{n} a_{mj}\hat{q}_j \\
&< \hat{p}_1 \sum_{j=1}^{n} a_{ij}\hat{q}_j + \cdots + \hat{p}_i \sum_{j=1}^{n} \sum_{\substack{k=1 \\ k\neq i}}^{m} \alpha_k a_{kj}\hat{q}_j + \cdots + \hat{p}_m \sum_{j=1}^{n} a_{mj}\hat{q}_j \\
&= \hat{p}_1 \sum_{j=1}^{n} a_{1j}\hat{q}_j + \cdots + \sum_{\substack{k=1 \\ k\neq i}}^{n} \hat{p}_i\alpha_k \sum_{j=1}^{n} a_{kj}\hat{q}_j + \cdots + \hat{p}_n \sum_{j=1}^{n} a_{mj}\hat{q}_j \\
&= (\hat{p}_1 + \hat{p}_i\alpha_1) \sum_{j=1}^{n} a_{ij}\hat{q}_j + \cdots + 0 + \cdots + (\hat{p}_m + \hat{p}_1\alpha_n) \sum_{j=1}^{n} a_{mj}\hat{q}_j
\end{aligned}$$

This way, the vector,

$$(p^*)^t = (\hat{p}_1 + \hat{p}_i\alpha, \hat{p}_2 + \hat{p}_i\alpha_2, \ldots, 0, \ldots \hat{p}_m + p_i\alpha_m)$$

is a better response to $\hat{q}$ than

$$(\hat{p})^t = (\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_i, \ldots, \hat{p}_m)$$

Thus, contrary to the assumption, $\hat{p}_i = 0$. $\square$

18 May 6, 2022

18.1 Equalizing Strategies

P1 plays a strategy equalizing payoff of P2. P2 has no incentive to leave whatever strategy they are playing. A pair of equalizing strategies forms a Nash Equilibrium. We return back to the game of Chicken.

Example 18.1 (Game of Chicken (Cont'd))
We have two Pure Nash Equilibriums

P2

P1		D	C
	D, p	$-1000, -1000$	$2^*, -1^*$
	C, $1-p$	$-1^*, 2^*$	$0, 0$

So

$$\text{PNE} = \{(D, C), (C, D)\}$$
$$-1000p + 2(1-p) = -p$$
$$\implies 2 = 1001p$$
$$\implies \begin{cases} p = \frac{2}{1001} \\ 1 - p = \frac{999}{1001} \end{cases}$$

So,

P2

P1		D, $\frac{2}{1001}$	C, $\frac{999}{1001}$
	D, $\frac{2}{1001}$	$-1000, -1000$	$2^*, -1^*$
	C, $\frac{999}{1001}$	$-1^*, 2^*$	$0, 0$

The expected payoff is $\frac{2}{1001}$.

Example 18.2 (Cheetahs and Antelopes)

- Two cheetahs chasing two antelopes, L and S
- Each cheetah can catch either antelope
- If catch same, have to share

P2

P1		L	S
	L	$\frac{\ell}{2}, \frac{\ell}{2}$	ℓ, s
	S	s, ℓ	$\frac{s}{2}, \frac{s}{2}$

and $\ell > s$.
Case 1: $\frac{\ell}{2} > s \implies \ell > 2s$ (Strict domination)

P2

P1		L	S
	L	$\frac{\ell}{2}^*, \frac{\ell}{2}^*$	ℓ^*, s
	S	s, ℓ^*	$\frac{s}{2}, \frac{s}{2}$

PNE: (L, L)

<u>Case 2</u>: $\frac{\ell}{2} = s \sim \ell = 2s$

P2

P1		L	S
	L	$\frac{\ell}{2}^*, \frac{\ell}{2}^*$	ℓ^*, s^*
	S	s^*, ℓ^*	$\frac{s}{2}, \frac{s}{2}$

$$\text{PNE: } \{(L, L), (L, S), (S, L)\}$$

<u>Case 3</u>: $s < \ell < 2s$

P2

P1		L	S
	L	$\frac{\ell}{2}, \frac{\ell}{2}$	ℓ^*, s^*
	S	s^*, ℓ^*	$\frac{s}{2}, \frac{s}{2}$

$$\text{PNE: } \{(L, S), (S, L)\}$$

Equalizing strategy for Cases 2 and 3:

$$s < \ell \leq 2s$$

P2

P1		L	S
	L, p	$\frac{\ell}{2}, \frac{\ell}{2}$	ℓ, s
	S, $1-p$	s, ℓ	$\frac{s}{2}, \frac{s}{2}$

$$p\frac{\ell}{2} + (1-p)\ell = ps + (1-p)\frac{s}{2}$$
$$\frac{\ell}{2}p + \ell - \ell p = sp + \frac{s}{2} - \frac{s}{2}p$$
$$-\frac{\ell}{2}p + \ell = \frac{s}{2}p + \frac{s}{2}$$
$$\ell - \frac{s}{2} = \frac{\ell + s}{2}p$$
$$2\ell - s = (\ell + s)p$$

$$\boxed{\hat{p} = \frac{2\ell - s}{\ell + s}}, \quad \boxed{1 - \hat{p} = \frac{2s - \ell}{\ell + s}}$$

$$\begin{aligned}\text{payoff} &= \frac{s}{2}(\hat{p}+1) = \frac{s}{2}\left(\frac{2\ell - s}{\ell + s} + 1\right) \\ &= \frac{s}{2}\frac{2\ell - s + \ell + s}{\ell + s} \\ &= \frac{3\ell s}{2(\ell + s)}\end{aligned}$$

So we have the payoff for the L strategy:

$$L(p) = \ell - \frac{\ell}{1}p; \quad S(p) = \frac{s}{2} + \frac{s}{2}p$$

$$\hat{p} = \frac{2\ell - s}{\ell + s}$$

$$p < \hat{p}$$

Evolutionary stability:
Let $0 \le p \le 1$ be the part of greedy cheetahs (always choose L) in the population.

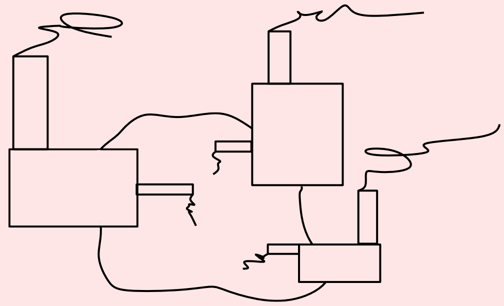
- For $0 \le p < \hat{p}, L(p) > S(p)$, so greedy cheetahs have an advantage
- For $\hat{p} < p \le 1, L(p) < S(p)$, so non-greedy cheetahs have an advantage.

So, evolution pushes p to $\hat{p}$.

18.2 General Sum Games with More Than Two Players

Example 18.3

We have



P3: Purify

		P2	
		S21: purify	S22: pollute
P1	S11	1,1,1	1,0,1
	S12	0,1,1	3,3,4

P3: pollute

		P2	
		S21	S22
P1	S11	1,1,0	4,3,3
	S12	3,4,3	3,3,3