Math 110B (Algebra)
University of California, Los Angeles

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These are my lecture notes for Math 110B (Algebra), which is the second course in Algebra taught by Nicolle Gonzales. The textbook for this class is *Abstract Algebra: An Introduction, 3rd edition* by Hungerford.

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1.1 Groups

- Algebra $\rightarrow$ study of mathematical structure.
- Rings $\leftrightarrow$ "numbers" e.g. $\mathbb{R}, \mathbb{Z}, \mathbb{C}, \mathbb{Z}_p$
  2 operations $(+, \cdot)$

Question 1.1: What happens if we have only 1 operation (either $\cdot$ or $+$ but not both)? What kind of structure is this more basic setup?

Answer: Groups! It turns out groups encode the mathematical structures of the symmetries in nature.

Definition 1.2 (Group)
A group $(G, *)$ is a nonempty set with a binary operation $* : G \times G \to G$ that satisfies

1. (Closure): $a * b \in G \quad \forall a, b \in G$
2. (Associativity): $(a * b) * c = a * (b * c) \quad \forall a, b, c \in G$
3. (Identity): $\exists e \in G$ such that $e * a = a = a * e \quad \forall a \in G$
4. (Inverse): $\forall a \in G, \exists d \in G$ such that $d * a = e = a * d$

Note:

- If * is addition, we just divide * by the usual $+$ sign. In this case
$$e = 0 \quad \text{and} \quad d = -a$$
- If the operation * is multiplication, we just divide * by the usual $\cdot$ sign. In this case
$$e = 1 \quad \text{and} \quad d = a^{-1}$$
- Be aware that sometimes * is neither.

Definition 1.3 (Abelian)
If the * operation is commutative, i.e. $a * b = b * a$, then we say that G is abelian (named after the mathematician N.H. Abel)

Definition 1.4 (Order, Finite Group vs. Infinite Group)
The order of a group G, denoted $|G|$, is the number of elements it contains (as a set). Thus, G is a finite group if $|G| < \infty$
and G is an infinite group if $|G| = \infty$

Examples 1.5 (Examples of a group)

1. Rings where you "forget" multiplication.
   $\rightarrow (\mathbb{Z}, +)$ integers with $* = +, (\mathbb{R}[X], +)$, etc.
   Note: $(\mathbb{Z}, *)$ with $* = \cdot$ is not a group. Why?

Theorem 1.6
Every ring is an abelian group under addition.

Proof. $e = 0$, inverse $= -a$ for each $a \in R$. □

Fact: If $R \neq 0$ then $(R, \cdot)$ is never a group since 0 has no multiplicative inverse.

Examples 1.7 (More examples of a group)

2. Fields without zero.

Theorem 1.8
Let $\mathbb{F}^*$ denote the nonzero elements of a field $\mathbb{F}$. Then $(\mathbb{F}^*, \cdot)$ is an abelian group.

Recall: A unit in a ring R is an element $a \in R$ with a multiplicative inverse $a^{-1} \in R$ such that $aa^{-1} = 1 = a^{-1}a$.

Theorem 1.9
The set of units $\mathcal{U}$ inside a ring R is a group under multiplication.

Examples 1.10 (More examples of a group cont.)

3. $\mathcal{U}_n = \{m | (m, n) = 1\} \subseteq \mathbb{Z}_n$ is also a group, but under multiplication,
   $n = 4$ $\mathbb{Z}_4 = \{0, 1, 2, 3\}$, $\mathcal{U}_4 = \{1, 3\}$
   $(\mathbb{Z}_4, +)$ and $(\mathcal{U}_4, \cdot)$ are groups with different binary operation!

   $n = 6$ $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$, $\mathcal{U}_6 = \{1, 5\}$
   $(\mathcal{U}_6, \cdot)$ is a group
   - $1 \cdot 5 = 5 \pmod 6 \in \mathcal{U}_6$ (closure)
   - $1 = e$ (identity)
   - $1 \cdot 1 = 1$, $5 \cdot 5 = 25 \equiv 1 \pmod 6$ (inverse)
   - Associativity is clear

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2.1 Groups (Cont'd)

Examples 2.1

4. $(M_{n\times m}(\mathbb{F}), +) = m \times n$ matrices over $\mathbb{F}$ under addition
   $e =$ zero matrix, inverse of a matrix $-M$

Definition 2.2 (General linear group)
Denote by $GL_n(\mathbb{F})$ the set of nxn invertible matrices under multiplication. $(\det(A) \neq 0 \quad \forall A \in GL_n)$

- Closed: $\det(A \cdot B) = \det(A) \cdot \det(B) \neq 0 \implies AB \in GL_n \quad \forall A, B \in GL_N$
- Associativity: Obvious.
- Identity: $\det(I) = 1 \neq 0 \implies I \in GL_n(\mathbb{F})$
- Inverse: $A \in GL_n; \det(A^{-1}) = \frac{1}{\det(A)} \neq 0 \implies A^{-1} \in GL_n(\mathbb{F})$

Fact: $GL_n(\mathbb{F})$ is a group for any field $\mathbb{F}$.
Comment:

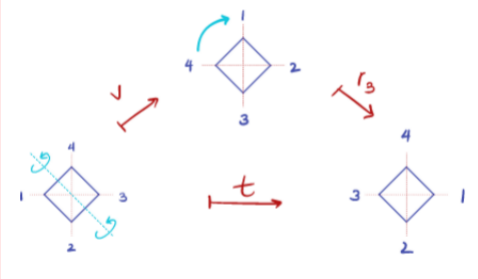
- $\det(A + B) \neq \det(A) + \det(B)$
- $\det(AB) = \det(A) \cdot \det(B)$

Definition 2.3 (Special linear group)
Let $SL_n(\mathbb{F})$ denote the set of invertible matrices over $\mathbb{F}$ with $\det = 1$

Exercise. Show that $SL_n(\mathbb{F})$ is a group.

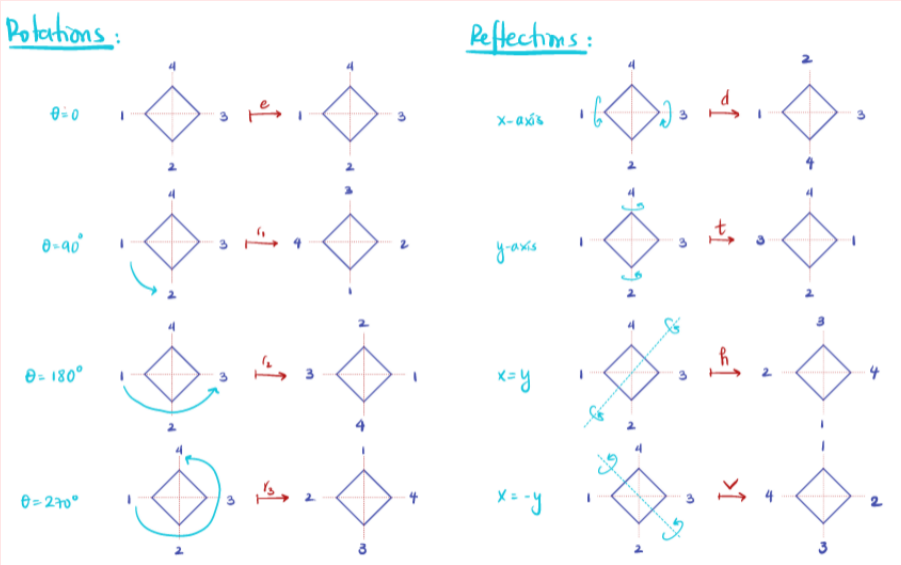
2.2 Symmetries

Example 2.4 (Symmetries over a square)
Rotations and reflection These operations (maps) form a group under composition. So $* = 0$. For instance, suppose
$r_3 \circ t = h$



The group of rotations/reflections of a square is called Dihedral Group of degree 4, denoted D_4.

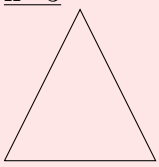
$$D_4 = \{r_1, r_2, r_3, r_4, d, t, h, v \mid \text{under } \circ\}$$



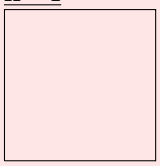
These are Professor Gonzales's lovely drawings.

Example 2.5 (Symmetries of a regular polygon with n sides)
Called the dihedral groups of degree n, D_n.

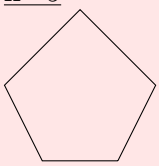
- n=3



- n=4



- n=5



- n=6

  etc...

Observe: $|D_n| = 2n$ because you have n-axes of reflection and n-angles of notation.

Example 2.6 (The symmetric group)
Let $n \in N$, and S_n be the set of all permutations of the numbers $\{1, ..., n\}$.
Note: any permutation of $\{1, ..., n\}$ can be thought of as a bijection $\{1, ..., n\} \to \{1, ..., n\}$.
$\implies$ This allows us to compose permutations just like functions.
$\implies$ S_n is a group!

Definition 2.7 (Symmetric group)
The symmetric group S_n is the group of permutations of the integers of the integers $\{1, ..., n\}$.
Given any permutation $\sigma \in S_n$,

$$\sigma : \{1, ..., n\} \to \{1, ..., n\},$$
$$i \mapsto \sigma_i$$

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_{n-1} & \sigma_n \end{pmatrix} \to e = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}$$

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma_1^{-1} & \sigma_2^{-1} & \cdots & \sigma_n^{-1} \end{pmatrix}$$

Group operation: function composition.

Example 2.8

n=2:

$e = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ $\tau = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$\tau \circ \tau = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = e$

$\tau \circ e = \tau$

$e \circ \tau = \tau$

$e \circ \tau = e$

$\implies S_2 = \{e, \tau\}$ is a group

$e^{-1} = e$

$\tau^{-1} = \tau$

Associativity: obvious because of function composition