Math 132 (Complex Analysis for Applications)
University of California, Los Angeles

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These are my lecture notes for Math 132 (Complex Analysis for Applications), which is taught by Tyler James Arant. The textbook for this class is *Complex Analysis*, by Theodore W. Gamelin.

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1.1 What are the Complex Numbers?

We first recall the basic algebraic properties of the real numbers, $\mathbb{R}$. For all $a, b, c \in \mathbb{R}$,

1. (Commutative law of addition): $a + b = b + a$
2. (Commutative law of multiplication): $a \cdot b = b \cdot a$
3. (Associative law of addition): $(a + b) + c = a + (b + c)$
4. (Associative law of multiplication): $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
5. (Distributive law): $a(b + c) = a \cdot b + a \cdot c$

The system of real numbers $\mathbb{R}$ has many more (non-algebraic) properties which make it suitable for calculus. However, it lacks a particular desirable property: $\mathbb{R}$ does not contain roots for all of its polynomial equations, e.g., there is not a solution to the equation

$$x^2 + 1 = 0 \quad \text{in } \mathbb{R}.$$

It turns out (by the non-trivial fundamental theorem of algebra) that we can get a number system for which every polynomial equation has a root by "appending" $i = \sqrt{-1}$ to $\mathbb{R}$.

Definition 1.1 (Complex number)
A <u>complex number</u> is an expression of the form

$$x + iy \quad \text{where } x, y \in \mathbb{R},$$

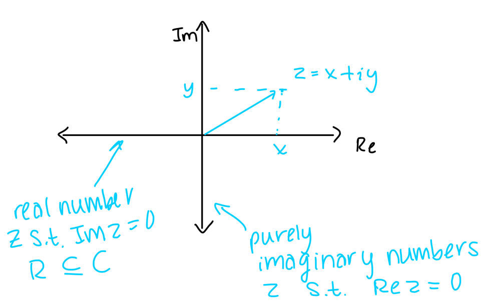
Two complex numbers $a + ib$ and $c + id$ are equal if and only if $a = c$ and $b = d$
We denote by $\mathbb{C}$ the set of all complex numbers.
For a complex number $z = x + iy$, we define its real and imaginary parts as follows:

$$\operatorname{Re} z = x$$
$$\operatorname{Im} z = y$$

There is a one-to-one correspondence between $\mathbb{C}$ and $\mathbb{R}^2$:

$$z \mapsto (\operatorname{Re} z, \operatorname{Im} z)$$

This can be visualized as the *complex plane*, where we can identify the real numbers and the *purely imaginary numbers*.



Example 1.2 (Addition and multiplication on $\mathbb{C}$)
We can define operations of addition and multiplication on $\mathbb{C}$ as follows:

$$z = x + iy, \quad w = a + ib$$

$z + w = (x + iy) + (a + ib) = (x + a) + i(y + b)$
$zw = (x + iy)(a + ib) = xa + ixb + iya + i^2yb$
$= (xa - yb) + i(xb + ya)$

Example 1.3 (Multiplicative inverse in $\mathbb{C}$)
Every nonzero complex number $z = x + iy \neq 0$ has a multiplicative inverse,

$$\frac{1}{z} = \frac{x - iy}{x^2 + y^2}.$$

Need to check $z \cdot \frac{1}{z} = 1$

$$z \cdot \frac{1}{z} = (x + iy)\left(\frac{x - iy}{x^2 + y^2}\right) = \left(\frac{x^2 - ixy + ixy - i^2y^2}{x^2 + y^2}\right) = \left(\frac{x^2 + y^2}{x^2 + y^2}\right) = 1$$

In addition to having additive and multiplicative inverses, the complex numbers also have the following algebraic properties:
For all $z_1, z_2, z_3 \in \mathbb{C}$,

1. (Commutative law of addition): $z_1 + z_2 = z_2 + z_1$
2. (Commutative law of multiplication): $z_1 \cdot z_2 = z_2 \cdot z_1$
3. (Associative law of addition): $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
4. (Associative law of multiplication): $z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$
5. (Distributive law): $z_1(z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$

1.2 Complex Conjugates and the Modulus

Definition 1.4 (Complex conjugate)
The complex conjugate of the number $z = x + iy$ is the number

$$\overline{z} = x - iy.$$

Some basic facts about complex conjugation. All are simple to prove, so we only discuss the proof of a few.

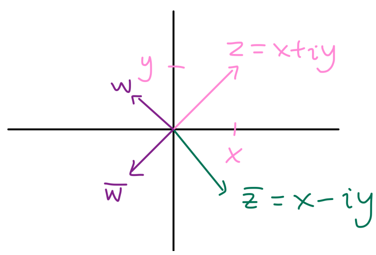
- $\overline{\overline{z}} = z$
- $z = \overline{z}$ if and only if z is a real number
- $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$
- $\overline{z_1 z_2} = \overline{z}_1 \cdot \overline{z}_2$
- $\overline{\left(\frac{1}{z}\right)} = 1/\overline{z}$

Proof. We want to show that $\overline{\left(\frac{1}{z}\right)} = 1/\overline{z}$.

$$\frac{1}{\overline{z}} = \frac{1}{x-iy} = \frac{x-(-iy)}{x^2+y^2} = \frac{x+iy}{x^2+y^2} = \overline{\left(\frac{x-iy}{x^2+y^2}\right)} = \overline{\left(\frac{1}{z}\right)}$$

□

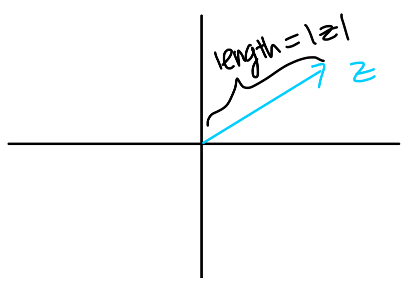
Geometrically, conjugation reflects z across the real axis:



Definition 1.5 (Absolute value/modulus)
The absolute value or modulus of $z = x + iy$ is

$$|z| = \sqrt{x^2+y^2}$$

Geometrically, $|z|$ is the length of z as a vector in the complex plane:



Some properties relating complex conjugation and absolute value:

- $|z|^2 = z\overline{z}$

$$\begin{aligned} z\overline{z} = (x+iy)(x-iy) &= x^2 - ixy + ixy - i^2y^2 \\ &= x^2 + y^2 \\ &= |z|^2 \end{aligned}$$

- $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$

$$\frac{1}{z} = \frac{x-iy}{x^2+y^2} = \frac{\overline{z}}{|z|^2}$$

- We have

$$\operatorname{Re} z = \frac{z+\overline{z}}{2}, \quad \operatorname{Im} z = \frac{z-\overline{z}}{2i}$$

$$\frac{z+\overline{z}}{2} = \frac{x+iy+x-iy}{2} = \frac{2x}{2} = x$$

Note: $\frac{1}{i} = -i$

- For $z, w \in \mathbb{C}, |zw| = |z| \cdot |w|$.

$$\begin{aligned} |z|^2 \cdot |w|^2 = z\overline{z} \cdot w\overline{w} &= (zw)(\overline{z} \cdot \overline{w}) \\ &= (zw)\overline{(zw)} = |zw|^2 \end{aligned}$$

Then take a square root.