Math 167 (Mathematical Game Theory) *University of California, Los Angeles*

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These are my lecture notes for Math 167 (Mathematical Game Theory) taught by Oleg Gleizer. The main textbook for this class is *Game Theory, Alive* by Anna Karlin and Yuval Peres and the supplementary textbook is *A Course in Game Theory* by Thomas Ferguson.

Contents

1 Mar 28, 2022

1.1 Impartial Combinatorial Games

Definition 1.1 (Impartial combinatorial game)
In an impartial combinatorial game,

- Two-person
- Perfect information
- No chance moves
- Win-or-lose outcome

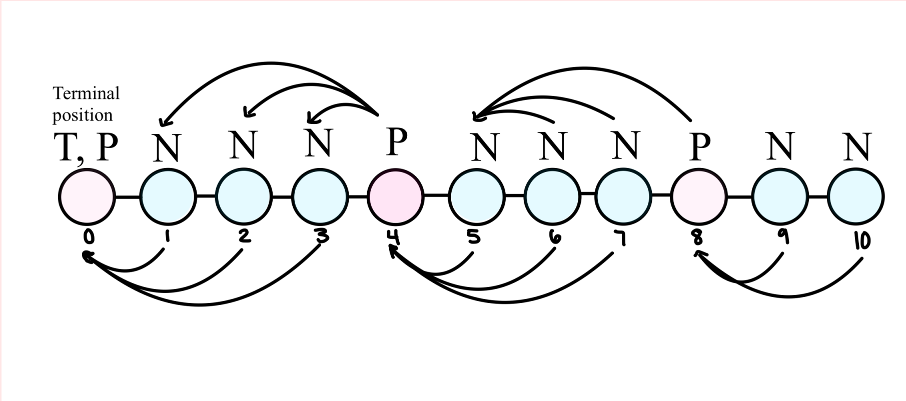
Example 1.2
Suppose

- A pile of n chips on the table
- Two players: P1 and P2
- A move consists of removing one, two, or three chips from the pile
- P1 makes the first move, players alternate then
- The player to remove the last chip wins (the last player to move wins. If a player can't move, they lose.)

Method to analyze: backward induction.
Positions:

- **N**, next player to take a move wins.
- **P**, previous (second) player to take a move wins.

Graph of the game



Any move from a **P** position leads to an **N** position. There always exists a move from an **N** position to a **P** position.
Ending condition: the game ends in a finite number of moves, no matter how played.
A **T** position is a **P** position.

Definition 1.3 (Normal play vs. misère play)
In a normal play, the last player to move wins. In a misère play, the last player to move loses.

Example 1.4
A misère game, a player can take 1-4 chips.
Every position is either **N** or **P**, but not nothing or both.

Example 1.5 (The game of Chomp)
Graph of the game:

- Positions correspond to vertices
- Moves correspond to oriented edges

Definition 1.6 (Strategy)
A function that assigns a move to each position, except for the terminal.

Definition 1.7 (Winning strategy from a position x)
A winning strategy from a position x is a sequence of moves, starting from x, that guarantees a win.

Consider a normal game. Let $\mathbf{N}_i/\mathbf{P}_i$ be the set of positions from which P1/P2 can win (reach the nearest terminal vertex of the same graph) in at most i moves.

$$\mathbf{P}_0 = \mathbf{P}_1 = \{\text{terminal positions}\}$$

$$\mathbf{N}_{i+1} = \{x\colon \text{ there is a move from } x \text{ to } \mathbf{P}_i\}$$

$$\mathbf{P}_{i+1} = \{y\colon \text{ each move leads to } \mathbf{N}_i\}$$

Note 1.8: $\mathbf{P}_0 = \mathbf{P}_1 \subseteq \mathbf{P}_2 \subseteq \mathbf{P}_3 \dots$
$\mathbf{N}_1 \subseteq \mathbf{N}_2 \subseteq \mathbf{N}_3 \dots$

$$\mathbf{N} = \bigcup_{i=1} \mathbf{N}_i, \quad \mathbf{P} = \bigcup_{i=0} \mathbf{P}_i$$

Definition 1.9 (Progressively bounded)
A game is called progressively bounded if for every position x there exists an upper bound $B(x)$ on the number of moves until the game terminates.

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2.1 Combinatorial Games (Cont'd)

Recall 2.1
- $\mathbf{P}_0 = \mathbf{P}_1 = \{\text{terminal positions}\}$
- $\mathbf{N}_{n+1} = \{x\colon \text{there is a move from } x \text{ to } \mathbf{P}_n\}$
- $\mathbf{P}_{n+1} = \{y\colon \text{each move from } y \text{ leads to } \mathbf{N}_n\}$
- $\mathbf{P}_0 = \mathbf{P}_1 \subseteq \mathbf{P}_2 \subseteq \dots$
- $\mathbf{N}_1 \subseteq \mathbf{N}_2 \subseteq \mathbf{N}_3 \subseteq$
- $\mathbf{P} = \bigcup\limits_{n=0} \mathbf{P}_n$
- $\mathbf{N} = \bigcup\limits_{n=1} \mathbf{N}_n$
- A game is called progressively bounded if for every position x there exists an upper bound $B(x)$ on the number of moves until the game stops.

Theorem 2.2
In a progressively bounded impartial full information combinatorial game, all positions are in $\mathbf{N} \cup \mathbf{P}$. Thus, for every position there exists a winning strategy.

Proof. Let $B(x) \leq n$. Let us prove by induction that $x \in \mathbf{N}_n \cup \mathbf{P}_n$.
Base: $n = 0$
x is a terminal vertex $\implies x \in \mathbf{P}_0 = \mathbf{P}_1$.
Inductive hypothesis by $\mathbf{P}_0$: $B(x) \leq n \implies x \in \mathbf{N}_n \cup \mathbf{P}_n$.
Inductive step: Show that $B(x) \leq n+1 \implies x \in \mathbf{N}_{n+1} \cup \mathbf{P}_{n+1}$
Consider a move $x \to y$ and $B(y) \leq n$. Hence, $y \in \mathbf{N}_n \cup \mathbf{P}_n$. So either
Case 1: Each move from x leads to $y \in \mathbf{N}_n \implies x \in \mathbf{P}_{n+1}$.
Case 2: There exists a move from x to $y \notin \mathbf{N}_n$. Thanks to the inductive typo, $y \in \mathbf{N}_n \cup \mathbf{P}_n$ so $y \in \mathbf{P}_n \implies x \in \mathbf{N}_{n+1}$. $\square$

2.2 The Game of Nim

- Several piles, each containing finitely many chips.
- A move: a player can remove any number of chips, from one to all from any pile
- P1 and P2 alternate taking moves
- The player to take the last chip wins

Example 2.3
We have two piles. The general case for k piles, we state: $(x_1, x_2, \ldots, x_k)$.
Nim-Sum:
Consider $x \oplus y$. We rewrite x and y as binary numbers and perform long addition of x_2 and y_2 without carry-over, i.e. mod 2.

Example 2.4

$$5 \oplus 7 = \begin{array}{cccc} & 1 & 0 & 1 \\ \oplus & 1 & 1 & 1 \\ \hline & 0 & 1 & 0 \end{array} = 2$$

Note 2.5: Nim is a progressively bounded game.

Theorem 2.6 (Bouton)
A position $x = (x_1, x_2, \ldots, x_k)$ is a $\mathbf{P}$ position $\iff x_1 \oplus x_2 \oplus \cdots \oplus x_k = 0$.

3 April 1, 2022

3.1 The Game of Nim (Cont'd)

Recall 3.1 $x = (x_1, x_2, \dots, x_k)$
Theorem (Bouton) says $x \in \mathbf{P} \iff x_1 \oplus x_2 \oplus \cdots \oplus x_k = 0$.

Proof of Theorem 2.6. We have
Terminal position: $x = (0, 0, \dots 0) \in \mathbf{P}$ Let $x \in \mathbf{N}$. Then there exists a move $x \to y \in \mathbf{P}$.

$$x_1 \oplus x_2 \oplus \cdots \oplus x_k = \bigoplus \begin{array}{cccccccccc} & & 1 & * & * & \dots & \dots & * & * \\ & & & & 1 & * & \dots & * & * \\ & & & & \vdots & \vdots & & \vdots & \vdots \\ & 1 & * & * & \dots & \dots & \dots & * & * \\ \hline \end{array}$$

Find the left-most (most significant) column with an odd number of 1's. Change any number that has a 1 in the column so that there is an even number of 1's in every column. The 1 in the most significant position becomes a 0 which implies the number becomes smaller. So this is a legal move.
We have $x \in \mathbf{P} \implies$ any move $x \to y \in \mathbf{N}$ where

$$x = (x_1, x_2, \dots, x_k) \mapsto y = (x_1', x_2, \dots, x_k)$$

such that

$$x_1' < x_1 \text{ and } x_1 \oplus x_2 \oplus \cdots \oplus x_k = 0.$$

If

$$x_1' \oplus x_2 \oplus \cdots \oplus x_k = 0$$

then

$$x_1' \oplus x_2 \oplus \cdots \oplus x_k = 0$$

then $x_1' = x_0$, a contradiction. Hence

$$x_1' \oplus x_2 \oplus \cdots \oplus x_k \neq 0 \implies y \in \mathbf{N}.$$

□

Example 3.2

$$x_1 = 7$$
$$x_2 = 10$$
$$x_3 = 15$$

$$\begin{array}{c|cc|c|c} & 0 & 1 & 1 & 1 \\ \oplus & 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 & 1 \\ \hline & 0 & 0 & 1 & 0 \end{array} \implies \begin{array}{ccccc} & 0 & 1 & 0 & 1 \\ \oplus & 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 & 1 \\ \hline & 0 & 0 & 0 & 0 \end{array}$$

So we have that $(7, 10, 15) \mapsto (5, 10, 15)$

3.2 Subtraction Nim

Extra condition: A player can remove at most n chips.
We find pile sizes mod $n+1$, i.e.

$$(x_1, x_2, \ldots, x_k) \mapsto (x_1 \bmod n+1, x_2 \bmod n+1, \ldots, x_k \bmod n+1)$$

Now we find the Nim-sum and make a move.

$$x \bmod n+1 = \underbrace{(x_1 \bmod n+1, x_2 \bmod n+1, \ldots, x_k \bmod n+1)}_{(x_1 \bmod n+1)_2 \oplus (x_2 \bmod n+1)_2 \oplus \cdots \oplus (x_k \bmod n+1)_2} \implies \begin{cases} = 0 \iff \mathbf{P} \\ \neq 0 \iff \mathbf{N} \end{cases}$$

Example 3.3
We have $x = (12, 13, 14)$ and $n = 3$. So,

$$(12 \bmod 4, 13 \bmod 4, 14 \bmod 4) \equiv (0, 1, 2) = (0_2, 1_2, 10_2)$$

So

$$\begin{array}{ccc} & 0 & 0 \\ \oplus & 0 & 1 \\ & 1 & 0 \\ \hline & 1 & 1 \end{array} \neq 0$$

so we take away one chip from the third pile

$$\begin{array}{ccc} & 0 & 0 \\ \oplus & 0 & 1 \\ & 0 & 1 \\ \hline & 0 & 0 \end{array}$$

So we have that $(12, 13, 14) \mapsto (12, 13, 13)$.

Note 3.4: You can always make a legal move $\mathbf{N} \to \mathbf{P}$ by removing $i \leq n$ chips from a pile.

Note 3.5: To move from $\mathbf{P}$ to $\mathbf{P}$, you need to remove $n+1$ chips from a pile. Not allowed! Hence, any move from $\mathbf{P}$ is to $\mathbf{N}$.

Example 3.6
We have $x = (12, 13, 13)$, with $n = 3$. So

$$x \mod 4 = (0, 1, 1)$$

therefore

$$\begin{array}{r} 0 \\ \oplus \quad 1 \\ 1 \\ \hline 0 \end{array}$$

3.3 Two-Person Zero Sum Games (Strategic Form)

We have

- P1: a non-empty set of strategies S1
- P2: a non-empty set of strategies S2
- A: S1 $\times$ S2 $\to \mathbb{R}$, the min function for P1 (payoff matrix)

Note 3.7: Since the game is zero-sum, a win for P1 is a loss for P2. $A(i, j)$ can be ≤ 0, so works both ways.

Pure strategies:

$$\begin{array}{cc|c|c|c|c|} & & \multicolumn{4}{c}{\text{P2}} \\ & & \text{S21} & \text{S22} & \cdots & \text{S2}n \\ \cline{3-6} & \text{S11} & a_{11} & a_{12} & \cdots & a_{1n} \\ \cline{3-6} \text{P1} & \text{S12} & a_{21} & a_{22} & \cdots & a_{2n} \\ \cline{3-6} & \vdots & \vdots & \vdots & \ddots & \vdots \\ \cline{3-6} & \text{S1}m & a_{m1} & a_{m2} & \cdots & a_{mn} \\ \cline{3-6} \end{array}$$

A game. P1 chooses the strategy S1i. Simultaneously, P2 chooses the strategy S2j. P1 wins a_{ij}.

Lemma 3.8
$\min_j \max_i a_{ij} \geq \max_i \min_j a_{ij}$

We will continue in the next lecture.

4 Apr 4, 2022

4.1 Two-Person Zero Sum Games (Cont'd)

Lemma 4.1
$\max_{1\leq i\leq m} \min_{1\leq j\leq n} a_{ij} \leq \min_{1\leq i\leq m} \max_{1\leq j\leq n} a_{ij}$

Example 4.2
Chooser (P1), Hider (P2).

	$L1$	$R2$
L	1	0
R	0	2

We have

$$\max_{1\leq p\leq 0} \min\{p, 2-2p\}$$

$$\begin{aligned} p &= 2-2p \\ 3p &= 2 \\ p &= \frac{2}{3} \end{aligned}$$

Now $\min_{0\leq q\leq 1} \max\{q, 2-2q\}$

Let us generalize $A \in \mathbb{R}^{n\times m}$, an $n \times m$ matrix (the payoffs).

$$\Delta_m = \left\{ \mathbf{p} \in \mathbb{R}^m \colon p_1 \geq 0, p_2 \geq 0, \ldots, p_m \geq 0, \sum_{i=1}^{m} p_i = 1 \right\}$$

$$\Delta_n = \left\{ \mathbf{q} \in \mathbb{R}^n \colon q_1 \geq, q_2 \geq 0, \ldots, q_n \geq 0, \sum_{j=1}^{n} q_j = 1 \right\}$$

Expected gain for P1: $(\mathbf{p})^T A\mathbf{q}$

	q_1	q_2	$\ldots$	q_n
p_1	a_{11}	a_{12}	$\ldots$	a_{1n}
p_2	a_{21}	a_{22}	$\ldots$	a_{2n}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
p_m	a_{m1}	a_{m2}	$\ldots$	a_{mn}

So

$$(\mathbf{p})^t A\mathbf{q} = p_i(a_i q_1 + a_{i2} q_2 + \cdots + a_{in} q_n) =$$

A mixed strategy for P1 is a point $\mathbf{p} \in \Delta_m$. A mixed strategy for P2 is a point $\mathbf{q} \in \Delta_n$.

Expected gain for P1:
If P1 employs the strategy $\mathbf{P}$, then the worst case payoff is

$$\min_{q\in\Delta_n} \mathbf{p}^T A\mathbf{q} = \min \sum_{i=1}^{m} a_{ij}p_i$$

$$\max_{\mathbf{p}\in\Delta_m} \min_{\mathbf{q}\in\Delta_n} \mathbf{p}^T A\mathbf{q}$$