# Math 110B (Algebra)
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These are my lecture notes for Math 110B (Algebra), which is the second course in Algebra taught by Nicolle Gonzales. The textbook for this class is *Abstract Algebra: An Introduction, 3rd edition* by Hungerford.

# Contents

# 1 Jan 3, 2022

## 1.1 Groups

- Algebra $\to$ study of mathematical structure.
- Rings $\leftrightarrow$ "numbers" e.g. $\mathbb{R}, \mathbb{Z}, \mathbb{C}, \mathbb{Z}_p$
  2 operations $(+, \cdot)$

**Question 1.1:** What happens if we have only 1 operation (either $\cdot$ or $+$ but not both)? What kind of structure is this more basic setup?

Answer: Groups! It turns out groups encode the mathematical structures of the symmetries in nature.

**Definition 1.2** (Group)
A group $(G, *)$ is a nonempty set with a binary operation $* : G \times G \to G$ that satisfies

1. (Closure): $a * b \in G \quad \forall a, b \in G$
2. (Associativity): $(a * b) * c = a * (b * c) \quad \forall a, b, c \in G$
3. (Identity): $\exists e \in G$ such that $e * a = a = a * e \quad \forall a \in G$
4. (Inverse): $\forall a \in G, \exists d \in G$ such that $d * a = e = a * d$

Note:

- If * is addition, we just divide * by the usual + sign. In this case

$$e = 0 \quad \text{and} \quad d = -a$$

- If the operation * is multiplication, we just divide * by the usual $\cdot$ sign. In this case

$$e = 1 \quad \text{and} \quad d = a^{-1}$$

- Be aware that sometimes * is neither.

**Definition 1.3** (Abelian)
If the * operation is commutative, i.e. $a * b = b * a$, then we say that $G$ is abelian (named after the mathematician N.H. Abel)

**Definition 1.4** (Order, Finite Group vs. Infinite Group)
The order of a group $G$, denoted $|G|$, is the number of elements it contains (as a set). Thus, $G$ is a finite group if $|G| < \infty$
and $G$ is an infinite group if $|G| = \infty$

**Examples 1.5** (Examples of a group)

1. Rings where you "forget" multiplication.
   $\to (\mathbb{Z}, +)$ integers with $* = +$, $(\mathbb{R}[X], +)$, etc.
   Note: $(\mathbb{Z}, *)$ with $* = \cdot$ is not a group. Why?

**Theorem 1.6**
Every ring is an abelian group under addition.

**Proof.** $e = 0$, inverse $= -a$ for each $a \in R$. $\square$

Fact: If $R \neq 0$ then $(R, \cdot)$ is never a group since 0 has no multiplicative inverse.

**Examples 1.7** (More examples of a group)

2. Fields without zero.

**Theorem 1.8**
Let $\mathbb{F}^*$ denote the nonzero elements of a field $\mathbb{F}$. Then $(\mathbb{F}^*, \cdot)$ is an abelian group.

Recall: A unit in a ring $R$ is an element $a \in R$ with a multiplicative inverse $a^{-1} \in R$ such that $aa^{-1} = 1 = a^{-1}a$.

**Theorem 1.9**
The set of units $\mathcal{U}$ inside a ring $R$ is a group under multiplication.

**Examples 1.10** (More examples of a group cont.)

3. $\mathcal{U}_n = \{m | (m, n) = 1\} \subseteq \mathbb{Z}_n$ is also a group, but under multiplication,
   $\underline{n = 4}$ $\quad \mathbb{Z}_4 = \{0, 1, 2, 3\}, \quad \mathcal{U}_4 = \{1, 3\}$
   $(\mathbb{Z}_4, +)$ and $(\mathcal{U}_4, \cdot)$ are groups with different binary operation!

   $\underline{n = 6}$ $\quad \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}, \quad \mathcal{U}_6 = \{1, 5\}$
   $(\mathcal{U}_6, \cdot)$ is a group

   - $1 \cdot 5 = 5 \pmod 6 \in \mathcal{U}_6$ (closure)
   - $1 = e$ (identity)
   - $1 \cdot 1 = 1, \quad 5 \cdot 5 = 25 \equiv 1 \pmod 6$ (inverse)
   - Associativity is clear

# 2 Jan 5, 2022

## 2.1 Groups (Cont'd)

**Examples 2.1**

4. $(M_{n\times m}(\mathbb{F}), +) = m \times n$ matrices over $\mathbb{F}$ under addition

   $e =$ zero matrix, inverse of a matrix $-M$

**Definition 2.2** (General linear group)
Denote by $GL_n(\mathbb{F})$ the set of $n \times n$ invertible matrices under multiplication. $(\det(A) \neq 0 \quad \forall A \in GL_n)$

- Closed: $\det(A \cdot B) = \det(A) \cdot \det(B) \neq 0 \implies AB \in GL_n \quad \forall A, B \in GL_N$
- Associativity: Obvious.
- Identity: $\det(I) = 1 \neq 0 \implies I \in GL_n(\mathbb{F})$
- Inverse: $A \in GL_n; \det(A^{-1}) = \frac{1}{\det(A)} \neq 0 \implies A^{-1} \in GL_n(\mathbb{F})$

Fact: $GL_n(\mathbb{F})$ is a group for any field $\mathbb{F}$.
Comment:

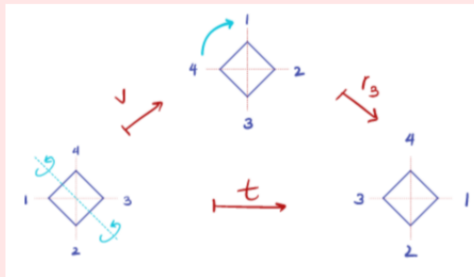
- $\det(A + B) \neq \det(A) + \det(B)$
- $\det(AB) = \det(A) \cdot \det(B)$

**Definition 2.3** (Special linear group)
Let $SL_n(\mathbb{F})$ denote the set of invertible matrices over $\mathbb{F}$ with $\det = 1$

**Exercise.** Show that $SL_n(\mathbb{F})$ is a group.

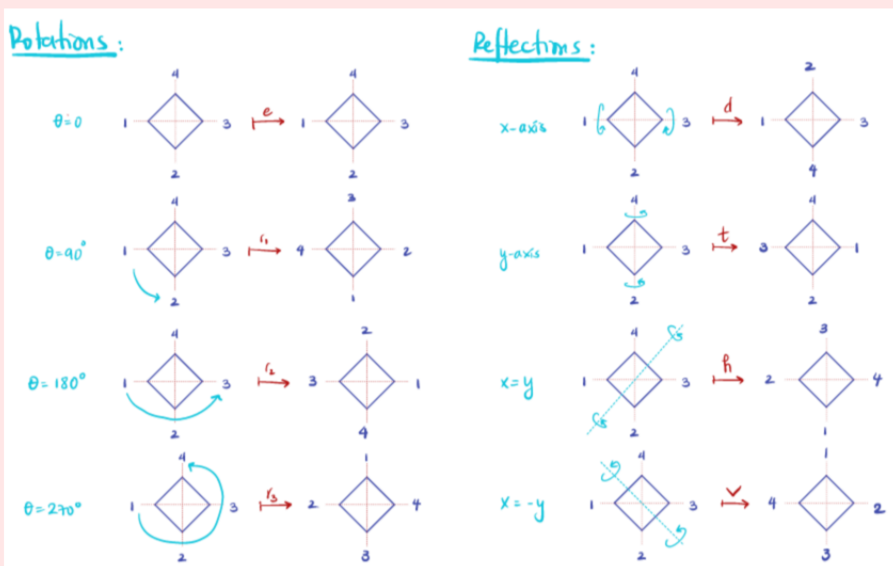
## 2.2 Symmetries

**Example 2.4** (Symmetries over a square)
Rotations and reflection These operations (maps) form a group under composition. So $* = 0$. For instance, suppose
$r_3 \circ t = h$



The group of rotations/reflections of a square is called <u>Dihedral Group of degree 4</u>, denoted $D_4$.

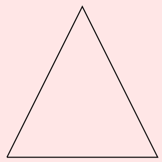
$$D_4 = \{r_1, r_2, r_3, r_4, d, t, h, v \mid \text{under } \circ\}$$



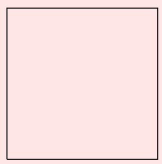

These are Professor Gonzales's lovely drawings.

**Example 2.5** (Symmetries of a regular polygon with $n$ sides)
Called the dihedral groups of degree $n$, $D_n$.

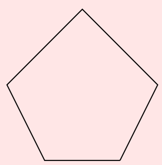
- n=3



- n=4



- n=5



- n=6

  etc...

Observe: $|D_n| = 2n$ because you have $n$-axes of reflection and $n$-angles of notation.

**Example 2.6** (The symmetric group)
Let $n \in N$, and $S_n$ be the set of all permutations of the numbers $\{1, ..., n\}$.
Note: any permutation of $\{1, ..., n\}$ can be thought of as a bijection $\{1, ..., n\} \to \{1, ..., n\}$.
$\implies$ This allows us to compose permutations just like functions.
$\implies$ $S_n$ is a group!

**Definition 2.7** (Symmetric group)
The symmetric group $S_n$ is the group of permutations of the integers of the integers $\{1, ..., n\}$.
Given any permutation $\sigma \in S_n$,

$$\begin{aligned} \sigma : & \{1, ..., n\} \to \{1, ..., n\}, \\ & i \mapsto \sigma_i \end{aligned}$$

$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_{n-1} & \sigma_n \end{pmatrix} \to e = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}$
$\sigma^{-1} = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma_1^{-1} & \sigma_2^{-1} & \cdots & \sigma_n^{-1} \end{pmatrix}$

Group operation: function composition.

**Example 2.8**
n=2:

$$e = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \tau = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$\tau \circ \tau = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = e$$

$\tau \circ e = \tau$
$e \circ \tau = \tau$
$e \circ \tau = e$
$\implies S_2 = \{e, \tau\}$ is a group
$e^{-1} = e$
$\tau^{-1} = \tau$
Associativity: obvious because of function composition

# 3 Jan 7, 2022

## 3.1 Symmetries (Cont'd)

**Example 3.1**
n=3 $S_3$: permutations of $\{1,2,3\}$

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \tau_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \tau_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$\tau_{21} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \tau_{12} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \tau_{121} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

So,

$$\begin{aligned} \tau_1 \circ \tau_2 \circ \tau_1 &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \tau_{121} \end{aligned}$$

Note: $\tau_{21} = \tau_2 \circ \tau_1$, $\tau_{12} = \tau_1 \circ \tau_2$
$\tau_{21} \neq \tau_{12} \implies S_3$ is not abelian!

**Exercise.** $\tau_{212}$?

## 3.2 Direct Product of Groups

**Definition 3.2** (Direct product)
Given $(G, *), (H, \star)$ both groups define the binary operation:

$$\begin{aligned} \Box \colon (G \times H) \times (G \times H) &\to G \times H \\ (g, h) \,\Box\, (g', h') &\mapsto (g * g', h \star h') \end{aligned}$$

Side note: $(S, ☺)$
$☺ \colon S \times S \to S \implies S$ group

**Example 3.3**
$S_2 \times D_4$ :
$(\tau_1, r_{270^\circ}) \,\Box\, (\tau_1, v) = (\tau_1 \circ \tau_1, r_{270^\circ} v) = (e, t)$

**Example 3.4**
$(\mathbb{R}, +) \times (\mathbb{R}^*, \cdot)$
$(5, 2) \,\Box\, (-5, \pi) = (0, 2\pi)$

**Example 3.5**
$\mathbb{Z}_n \times \mathbb{Z}_m \quad n, m \in \mathbb{N}$.
$(a,b) \,\square\, (a',b') = (\underbrace{a+a'}_{\text{mod } n}, \underbrace{b+b'}_{\text{mod } m})$

$$\begin{aligned}(5,5) \overset{\mathbb{Z}_8\times\mathbb{Z}_6}{\square} (2,2) &= (5+2, 5+2)\\ &= (7,1)\end{aligned}$$

## 3.3 Properties of Groups

Notation: Going forward, we omit $*$ in the notation: $(G,*) \to G$. Use multiplicative notation for abstract groups. Instead $a*b \to ab$.

$$\underbrace{a*a*a*a\cdots*a}_{n \text{ times}} \to a^n$$

However, for very explicit groups like
$(\mathbb{Z},+), (\mathbb{R},+), (\mathbb{Z}_n,+)$, etc, we use additive notation. $(* = +)$

$$a*b \to a+b$$

$$\underbrace{a*\cdots*a}_{n \text{ times}} \to n\cdot a$$

(Review notation on page 198 of book)

**Theorem 3.6**
$G$ group, $a,b,c \in G$. Then

1. $e \in G$ is unique
2. if $ab = ac$ or $ba = ca \implies b = c$
3. $\forall a \in G : a^{-1}$ is unique.

**Proof.**

1. Suppose $\exists e' \in G$ s.t $e \neq e'$ but $e'a = a = ae'$ $\forall a \in G$. $\implies$ let $a = e \implies e'e = e = ee'$

   On the other hand $e\cdot e' = e' = e'e$

   $\implies e = e'$

2. $ab = ac, \quad a,b,c \in G$.

   Since $a^{-1} \in G$

$$\begin{aligned}\implies \underbrace{a^{-1}a}_{e}b &= \underbrace{a^{-1}a}_{e}c\\ \implies e\cdot b &= e\cdot c\\ \implies b &= c\end{aligned}$$

3. Suppose $a \in G$ $\exists$ two distinct inverses.

   $d_1, d_2 \in G$.

   $d_1 a = e = a d_1$

   $d_2 a = e = a d_2$

   $\implies d_1 = d_1 e = d_1 a d_2 = e \cdot d_2 = d_2$

$\square$

**Corollary 3.7**
$G$ group, $a, b \in G$. Then

1. $(ab)^{-1} = b^{-1}a^{-1}$
2. $(a^{-1})^{-1} = a$

**Proof.** Exercise. $\square$

Note: $ab = ba$ ($G$ is abelian)

$$\implies (ab)^{-1} = a^{-1}b^{-1} = b^{-1}a^{-1}$$

Generally: $ab \neq ba \implies a^{-1}b^{-1} \neq b^{-1}a^{-1}$

## 3.4 Order of an Element

**Definition 3.8** (Order (of an element) and Finite vs. Infinite order)
The order of an element $a \in G$ is the smallest $k \in \mathbb{N}$ such that $a^k = e$. We denote this by $|a|$.
If $k$ is finite $\implies$ $a$ has finite order.
If $k$ is infinite $\implies$ $a$ has infinite order.

**Example 3.9**
$S_2$; $e, \tau_1 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$
Notice,

$$|e| = 1; e^1 = e$$

$$|\tau_1| = 2 \quad \tau_1^2 = \tau_1 \circ \tau_1 = e$$

$$\tau_1^4 = \tau_1^2 \circ \tau_1^2 = e \circ e = e$$

**Example 3.10**
$\mathbb{Z} \leftarrow e = 0$.
$|1| = ?$
$1 \cdot n = 0$ for which $n$?
None, so $\implies |1| = \infty$

# 4 Jan 10, 2022

## 4.1 Order of an Element (Cont'd)

**Theorem 4.1**
$G$-group, $a \in G$

1. If $|a| = \infty$, then $a^i \neq a^j$ for any $i, j \in \mathbb{Z}$ with $i \neq j$.
2. If $\exists i \neq j$ such that $a^i = a^j \implies |a| < \infty$.

**Proof.** We prove (2) (because $1 \iff 2$).
WLOG suppose $i > j$, then if $a^i = a^j \implies a^{i-j} = a^i a^{-j} \implies = a^j a^{-j} = a^0 = e$
$\implies |a| \leq i - j < \infty$ □

**Theorem 4.2**
$G$ group, $a \in G \quad |a| = n$

1. $a^k = e \iff n \mid k \quad (n \leq k)$
2. $a^i = a^j \iff i \equiv j \pmod{n}$
3. if $n = td \quad d \geq 1 \implies |a^t| = d$.

**Proof.**

1. If $a^k = e$ and since $a^n = e$ with $n$-smallest such integer, then $k > n$, and so $k = nd + r$ with $0 \leq r < n$
$$a^k = a^{nd+r} = (a^n)^d a^r = e^d a^r = a^r$$
If $0 < r < n \implies a^r \neq e \implies a^k \neq e$
$\implies r = 0 \implies k = nd \implies n \mid k$.
2. If $a^i = a^j \implies a^{i-j} = e$
$\implies n \mid i - j$ by (1).
$\implies i - j \equiv 0 \pmod{n}$
$\implies i \equiv j \pmod{n}$
3. If $n = td$ $(d \geq 1) \overset{?}{\implies} |a^t| = d$
Since $a^n = e \implies (a^t)^d = e \implies |a^t| \leq d$.
If $|a^t| = k < d \implies (a^t)^k = a^{tk} = e$
But $tk < td = n \implies a^{tk} = e$ for $tk < n \implies \neq$ because $n$ is the smallest positive integer such that $a^n = e$.
$\implies k = d \implies |a^t| = d$.

□

**Corollary 4.3**
$G$- abelian group with $|a| < \infty \quad \forall a \in G$. Suppose $c \in G$ such that $|a| \leq |c| \quad \forall a \in G$. Then $|a| \mid |c|$.

**Proof.** Suppose not. $\exists$ some $a \in G$ such that $|a| \nmid |c|$. Consider prime factorizations of $|a|$ and $|c|$.
$\implies$ Then $\exists$ some prime $p$ such that $|a| = p^r m \quad |c| = p^s n$ where $r > s$ ($s$ might be zero) and $(p_1 m) = 1 = (p_1 n)$.
Then by (3) of Theorem 4.2,

$$|a^m| = p^r \quad \text{and} \quad |c^{p^s}| = n$$

$$\underset{\text{because } (p^r,n)=1}{\implies} |\underbrace{a^m \cdot c^{p^s}}_{\in G}| = p^r \cdot n$$

<u>Note:</u> $|a| = n, |b| = m, |a \cdot b| \neq n \cdot m$ unless $(n, m) = 1$
<u>Recall:</u> $|c| = p^s \cdot n$ where $s < r$
$\implies p^r > p^s$
$\implies p^r n > p^s n$
$\implies |a^m \cdot c^{p^s}| > |c|$
$\implies \neq$ because $c$ is the element in $G$ with maximal order! So $a^m c^{p^s} \in G$ cannot have order larger than $c$. $\square$

## 4.2 Subgroups

**Definition 4.4** (Subgroup)
A subset $H \subseteq G$ is a <u>subgroup</u> of $(G, *)$ if it is also a group under $*$.
<u>Note:</u>
$G \subseteq G \implies G$ is always a subgroup of itself (Improper subgroup)
$\{e\} \subseteq G \implies \{e\}$ is always a subgroup of G (Trivial subgroup of $G$)
$\implies$ Any subgroup $e \neq H \neq G$ is called a <u>nontrivial proper subgroup</u>.

**Examples 4.5**

- $(\mathbb{Z}, +) \subseteq (\mathbb{Q}, +)$
- $\{e, r_{90}, r_{180}, r_{270}\} \subseteq D_4$
- $SL_n(\mathbb{F}) \subseteq GL_n(\mathbb{F})$

<u>Note:</u> any subgroup always contains $e$.

**Theorem 4.6**
A nonempty subset $H$ of $G$ is a subgroup if:

1. $ab \in H \quad \forall a, b \in H$
2. $a^{-1} \in H \quad \forall a \in H$

**Proof.** Since $H \neq \emptyset \quad \exists a \in H$. By (2), $\exists a^{-1} \in H$. $\implies$ By (1) $aa^{-1} = e \in H \implies e \in H$. $\square$

**Theorem 4.7**
Any closed nonempty finite subset $H$ of $G$ is a subgroup.

**Proof.** By Theorem 4.6, we need only show that H contains inverses.
If $a \in H \quad a^k \in H \quad \forall k \in \mathbb{Z}$.
Since $H$ is finite, not all $a^k$ can be distinct.
$\implies |a| = n < \infty$ for some $n \in \mathbb{N}$.
$\implies a^n = e$
$\implies a^{n-1} \cdot a = e = a \cdot a^{n-1}$
$\implies a^{n-1} = a^{-1}$
If $n > 1 \implies a^{-1} \in H$
If $n = 1 \implies a^{-1} = e \implies a = e \implies a^{-1} = e \in H$. $\square$

# 5 Jan 12, 2022

## 5.1 Subgroups (Cont'd)

**Example 5.1**
$\mathbb{Z}_5 \leftarrow$ group under addition $= \{0, 1, 2, 3, 4\}$
Units of $\mathbb{Z}_5$: $\mathcal{U}_5 = \{1, 2, 3, 4\}$
Clearly, $\mathcal{U}_5 \subseteq \mathbb{Z}_5$
Question: Is $\mathcal{U}_5$ a subgroup of $\mathbb{Z}_5$
No, because $\mathcal{U}_5$ is a group under multiplication.

**Example 5.2**
$S_3$: set of permutations that fix 1.
$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ $\quad \tau_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$

$$\left.\begin{aligned} \tau_2 e = \tau_2 = e\tau_2 \\ \tau_2 \cdot \tau_2 = e \end{aligned}\right\} \implies \underbrace{\{e, \tau_2\}}_{H} \text{ is closed.}$$

By Theorem 4.7, $H$ is a subgroup because $H$ is finite, nonempty, and closed.

## 5.2 Center of a Group

**Definition 5.3** (Center of a group)
The <u>center</u> of a group $G$ is the subset

$$Z(G) \coloneqq \{a \in G \mid ag = ga \quad \forall g \in G\}$$

**Note 5.4:** When $G$ is abelian $\implies Z(G) = G$

**Question 5.5:** Is $Z(G) = \emptyset$? No, because $e \in Z(G)$

**Examples 5.6**

- $Z(S_n) = e$
- $Z(D_4) = \{e, r_{180}\}$
- $Z(GL_n) = \{aI \mid a \in \mathbb{F}\}$ $\qquad \begin{pmatrix} a & & 0 \\ & \ddots & \\ 0 & & a \end{pmatrix}$
- $Z(SL_n) = \{I\} = e$

**Theorem 5.7**
$Z(G)$ is a subgroup of $G$.

**Proof.** By Theorem 4.6, since $Z(G) \neq \emptyset$, we need only show closure and inverses.

1. $a, b \in Z(G) \overset{?}{\implies} ab \in Z(G), \forall g \in G.$
   $$(ab)g \overset{\text{b/c } g\in Z(G)}{=} a(gb) \underset{\text{by assoc.}}{=} (ag)b \overset{a\in Z(G)}{=} (ga)b = g(ab)$$
   $$\implies ab \in Z(G)$$
2. $a \in Z(G), ag = ga \quad \forall g \in G.$
   $$\implies a^{-1}(ag)a^{-1} = a^{-1}(ga)a^{-1}$$
   $$\implies ga^{-1} = a^{-1}g \implies a^{-1} \in Z(G)$$

□

## 5.3 Cyclic Group

**Definition 5.8** (Cyclic group)
For any $a \in G$, the set
$$\langle a \rangle = \{a^n \mid n \in \mathbb{Z}\}$$
is a subgroup of $G$. We say $\langle a \rangle$ is the cyclic subgroup generated by $a$.

**Note 5.9:** Cyclic groups are always abelian.

If $G = \langle a \rangle$ for some $a \in G$, then $G$ is a cyclic group.

**Example 5.10**
$\langle r_{90} \rangle \subseteq D_4$
$\langle r_{90} \rangle = \{e, r_{90}, r_{180}, r_{270}\} \leftarrow$ is a cyclic subgroup of $G$.

**Note 5.11:** In additive notation: $a * a = a + a$ (not $a \cdot a = a^2$)
$$\langle a \rangle = \{n \cdot a \mid n \in \mathbb{Z}\} \quad n \cdot a = \underbrace{a + a + \cdots + a}_{n \text{ times}}$$
$$a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ times}}$$

**Example 5.12**
$(\mathbb{Z}, +) = \langle 1 \rangle = \langle -1 \rangle$

**Note 5.13:** The generating element $a$ is not unique.

**Example 5.14**
$(\mathbb{Z}_3, +) = \langle 1 \rangle = \underbrace{\langle 2 \rangle}_{=-1}$

**Exercise.** Which elements generate $\mathbb{Z}_n$ for $n \in \mathbb{N}$?
Hint: Look at units (i.e. relatively prime) of $\mathbb{Z}_n$

**Example 5.15**
$\mathbb{Z}_n = \langle 1 \rangle$
$\implies$ All $\mathbb{Z}_n$ are cyclic groups of order $n$

**Theorem 5.16**
Let $a \in G$

1. If $|a| = \infty$, then $\langle a \rangle = \{a^k \mid k \in \mathbb{Z}\}$ is an infinite group.
2. If $|a| = n < \infty$, then $\langle a \rangle$ is a finite group. In fact, $\langle a \rangle = \langle e, a, a^2, a^3, \ldots, a^{n-1}\} \implies |\langle a \rangle| = |a| = n$.

**Proof (Sketch).**

$$
\begin{aligned}
|a| = \infty &\implies a^i \neq a^j \text{ for } i \neq j \\
&\implies \{a^k \mid k \in \mathbb{Z}\} \implies \text{ infinite set.} \\
|a| = n &\implies \langle a, a^2, \ldots, a^{n-1}, a^n = e\}
\end{aligned}
$$

Since: $a \cdot a^{n-1} = a^n = e = a^{n-1} \cdot a$

$$\implies a^{n-1} = a^{-1}$$

$a^2 a^{n-2} = a^n = e = a^{n-2} a^2$

$$\implies a^{-2} = a^{n-2}$$

$\square$

**Theorem 5.17**
Let $\mathbb{F}$ be any field. Then any finite subgroup $G \subseteq \mathbb{F}^*$ is cyclic.

**Recall 5.18** $\mathbb{F}^* = \mathbb{F} - \{0\}$ is a group under multiplication.

**Proof.** Since $|G| < \infty$, $\exists c \in G$ such that order of $c$ is maximal ($|a| \leq |c| \quad \forall a \in G$). By Corollary 4.3, $\forall a \in G, |a| \mid |c|$ so if $|c| = m \implies a^m = 1$
Consider $p(x) = x^m - 1$. Since $p(a) = 0 \quad \forall a \in G$.
Since $p(x)$ has degree $m$ it can have at most $m$ solutions $\implies |G| \leq m$.
Since $|c| = m$ so $|\langle c \rangle| = m$.
$\implies \langle c \rangle \subseteq G \implies \langle c \rangle = G$.
$\implies G$ is cyclic. $\square$

# 6 Jan 14, 2022

## 6.1 Cyclic Group (Cont'd)

**Recall 6.1** $a \in G$

$$\underbrace{\langle a \rangle}_{\text{cyclic group gen. by a}} := \{a^n \mid n \in \mathbb{Z}\} = \{\dots a^{-2}, a^{-1}, e, a, a^2, \dots\}$$

$G = \langle a \rangle \leftarrow G$ is cyclic group

**Recall 6.2** Thm:

$$|a| = \infty \to |\langle a \rangle| = \infty$$
$$|a| = n < \infty \to |\langle a \rangle| = n$$

**Recall 6.3** $\mathbb{F}$-field, $G \subseteq \mathbb{F}^*$ if $G$ finite $\implies$ $G$ is cyclic. (G is any subgroup)

**Theorem 6.4**
Subgroups of cyclic groups are cyclic.

**Proof.** Suppose $G = \langle a \rangle$ and $H \subseteq G$. We want to show that $H = \underbrace{\langle b \rangle}_{b=a^j \text{ for some } j}$ for some $b \in G$.

If $H = e \implies H = \langle e \rangle$ we're done.
If $H \neq e$, then we can find $k$-smallest positive integer such that $a^k \in H$
Suppose $b \in H$. Then,

$$b = a^i \text{ for some } i \text{ then } i = kd + r \quad 0 \leq r < k.$$

$\implies a^r = a^{i-kd} = b(a^k)^{-d} \in H$ by closure.
If

$$r \neq 0 \implies \begin{cases} a^r \in H \\ a^k \in H \end{cases}$$

with $0 < r < k$ which is a contradiction because $k$ was supposed to be smallest positive integer with $a^k \in H$.

$$\begin{aligned} \implies r = 0 \implies b &= a^i = a^{kd+r} = a^{kd} = (a^k)^d \\ \implies b &\in \langle a^k \rangle \\ \implies H &\subseteq \langle a^k \rangle \end{aligned}$$

$$\begin{aligned} \text{Since } a^k \in H &\implies \langle a^k \rangle \subseteq H \\ &\implies \langle a^k \rangle = H \end{aligned}$$

□

## 6.2 Generating Sets for Groups

**Definition 6.5**
Given a subset $S$ of $G$, let $\langle S\rangle$ denote the set of all possible products of all elements of $S$ and their inverses.

**Note 6.6:** $S \subseteq \langle S\rangle$

**Example 6.7**
$a, b \in G, \quad S = \{a, b\}$

$$\begin{aligned}\langle S\rangle &= \langle a, b\rangle \\ &= \{a^n, b^m, a^n b^m, a^{n_1} b^{m_1} a^{n_2} b^{m_2}, b^m a^n, b^{m_1} a^{n_2} b^{m_2} a^{n_1}, \dots\} \\ &= \left\{\prod_{i=0}^{k} a^{n_i} b^{m_i}, \prod_{i=0}^{k} b^{n_i} a^{m_i} \mid k \in \mathbb{N}, n_i, m_i \in \mathbb{Z}\right\}\end{aligned}$$

**Theorem 6.8**
$S$- any subset of G.

1. $\langle S\rangle$ is always a subgroup of G.
2. If $H$ is any other subgroup of $G$ such that $S \subseteq H \implies \langle S\rangle \subseteq H$.

**Proof (Sketch).**

1. Use the fact that very definition of $\langle S\rangle$ ensures closure and inverses $\implies$ $\langle S\rangle$ is a subgroup.
2. Again follows from closure and inverses contained in $H$ because $H$ is a subgroup.

□

**Definition 6.9** (Generators)
For any $S \subseteq G$, the group $\langle S\rangle$ is called the subgroup generated by $S$. If $G = \langle S\rangle$, then we call elements in $S$, the generators of $G$ and $S$ the generating set of G

**Example 6.10** (Symmetric group)
$S_3 = \{e, \tau_1, \tau_2, \tau_{121}, \tau_{21}, \tau_{12}\}$
$\tau_{121} = \tau_1 \circ \tau_2 \circ \tau_1$
$\tau_{21} = \tau_2 \circ \tau_1$
$\tau_{12} = \tau_1 \circ \tau_2$
$e = \tau_1 \circ \tau_1 = \tau_2 \circ \tau_2$

$$S_3 = \langle \underbrace{\tau_1}_{\begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix}}, \underbrace{\tau_2}_{\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix}} \rangle$$

$S_n \leftarrow$ order $n!$

$$S_n = \langle \underbrace{\tau_1}_{\text{flips } 1-2}, \underbrace{\tau_2}_{2-3}, \tau_3, \ldots, \underbrace{\tau_{n-1}}_{\text{flips } n, n-1} \rangle$$

$S_4 = \langle \tau_1, \tau_2, \tau_3 \rangle$
$S_5 = \langle \tau_1, \tau_2, \tau_3, \tau_4 \rangle$

## 6.3 Isomorphisms and Homomorphisms

**Definition 6.11** (Homomorphism (of groups))
$G, H$ are groups. A homomorphism of groups is a map $\varphi\colon G \to H$ such that $\forall a, b \in G$

$$\varphi(\underbrace{ab}_{ab \text{ prod in } G}) = \varphi(a)\underbrace{\cdot}_{\text{prod in } H}\varphi(b)$$

**Note 6.12:** This means that the "multiplication" table for $G$ is mapped onto "multiplication" table for $H$ i.e. $\varphi$ preserves group structures.

**Note 6.13:** $\varphi(a) = \varphi(e_G \cdot a) = \varphi(e_G)\varphi(a)$
$\implies \varphi(e_G) = e_H$
$\implies \varphi$ takes identities to identities.

**Definition 6.14** (Isomorphism (of groups))
An isomorphism of groups $G$ and $H$ is a homomorphism of $\varphi\colon G \to H$ that is also a bijection, i.e. an isomorphism is an invertible homomorphism.
If $G$ is isomorphic to H, then $G \cong H$, which is the same as writing $\exists \varphi\colon G \to H$ with $\varphi$ one-to-one and onto. Alternatively, $\tilde{\varphi}\colon H \to G$ is also one-to-one and onto.

**Example 6.15**

$\mathbb{Z}_8 = \{0, \dots, 7\}$

$\mathcal{U}_8$ of units $\implies \mathcal{U}_8 = \{\underset{e=}{1}, 3, 5, 7\}$

Consider $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{\underset{e=}{(0,0)}, (1,0), (0,1), (1,1)\}$

Claim: $\mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathcal{U}_8$

Let

$$\begin{aligned} \varphi \colon \mathcal{U}_8 &\to \mathbb{Z}_2 \times \mathbb{Z}_2 \\ \varphi(1) &= (0,0) \\ \varphi(3) &= (1,0) \\ \varphi(5) &= (0,1) \\ \varphi(7) &= (1,1) \end{aligned}$$

$\varphi(ab) = \varphi(a) + \varphi(b)$

Check,

- $\varphi$ is a homomorphism
- multiplication table is preserved
- $\varphi$ is one to one and onto

# 7 Jan 19, 2022

## 7.1 Isomorphisms and Homomorphisms (Cont'd)

**Example 7.1** (Example 6.15 Cont'd)
Let

$$\begin{aligned}\varphi\colon \mathcal{U}_8 &\to \mathbb{Z}_2 \times \mathbb{Z}_2 \\ \varphi(1) &= (0,0) \leftarrow \text{ fixed} \\ \varphi(3) &= (1,0) \\ \varphi(5) &= (0,1) \\ \varphi(7) &= (1,1)\end{aligned}$$

Check,

$$\begin{aligned}(0,0)+(1,0) = \varphi(1)+\varphi(3) &\stackrel{\checkmark}{=} \varphi(1\cdot 3) = \varphi(3) = (1,0) \\ (0,0) = 2(0,1) = \varphi(5)+\varphi(5) &\stackrel{\checkmark}{=} \varphi(5\cdot 5) = \varphi(1) = (0,0) \\ &\vdots\end{aligned}$$

Verify every time $\varphi(ab) = \varphi(a)+\varphi(b) \implies \varphi$ is a homomorphism.
$\varphi$ is one-to-one and onto $\implies$ DONE.
Iso's are not unique. In fact,

$$\begin{aligned}\varphi(1) &= (0,0) \\ \varphi(3) &= (0,1) \\ \varphi(5) &= (1,0) \\ \varphi(7) &= (1,1)\end{aligned}$$

is also an iso. However,

$$\begin{aligned}\varphi(1) &= (0,0) \\ \varphi(3) &= (1,1)\end{aligned}$$

Does it work? Why? (Exercise)

**Example 7.2**
$\mathbb{Z} \to Z_5$

$n \overset{\varphi}{\mapsto} [n] \mod 5$
Let's construct a homomorphism.

1. Check $\varphi$ is well defined.

   $n \equiv m \mod 5 \overset{?}{\implies} \varphi(n) = \varphi(m).$✓

2. $\varphi$ is a homomorphism.

$$\varphi(a+b) = \varphi(a) + \varphi(b)$$
$$[a+b] \underset{\text{true}}{=} [a] + [b]$$
$$\implies \varphi \text{ is a homomorphism}$$

   Note: $\varphi$ is not injective because $|\mathbb{Z}| > |\mathbb{Z}_5|$

   $\varphi$ is not an iso.

**Fact 7.3:** Isomorphic groups always have the same order.
Converse? $|G| = |H| \implies G \cong H$?
FALSE!

**Example 7.4**
Consider $S_3$ and $\mathbb{Z}_6$.

$$|S_3| = 3! = 6 \qquad\qquad |\mathbb{Z}_6| = 6$$

Not isomorphic. Let's suppose $\varphi\colon S_3 \to \mathbb{Z}_6$ an isomorphism.

$$\varphi(ab) = \varphi(a) + \varphi(b) \tag{1}$$

So,

$$\begin{aligned} \varphi(a) + \varphi(b) &= \varphi(b) + \varphi(a) && \text{(because } \mathbb{Z}_6 \text{ is abelian)} \\ &= \varphi(ab) \end{aligned}$$

$\implies$ if (1) holds since $\mathbb{Z}_6$ is abelian
$\implies \varphi(ab) = \varphi(ba) \quad \forall b, a \in S_3$
$\implies S_3$ is abelian
False, $S_3$ is not abelian, so you can't define such an iso $\varphi$.

**Theorem 7.5**
If $G$ is abelian, $H$ is not abelian $\implies G \ncong H$.

**Fact 7.6:** Isomorphisms preserve order of elements, i.e.

$$|a| = |\varphi(a)|$$

**Definition 7.7** (Automorphism)
An automorphism is an isomorphism from $G \to G$. They capture internal symmetries of a group.

**Example 7.8**
identity:

$$\begin{aligned} i_G \colon G &\to G \\ g &\mapsto g \end{aligned}$$

Clearly: $i(ab) = i(a)i(b) = ab \overset{\checkmark}{=} ab$

**Definition 7.9** (Inner automorphism of $G$ induced by $c$)
For any $c \in G$, the inner automorphism of $G$ induced by $c$ is:

$$\varphi_c \colon G \to G; \quad \varphi_c(g) = c^{-1}gc \leftarrow \text{ conjugation by } c.$$

1. Then $\varphi_c$ is a homomorphism:
   $\varphi_c(ab) = c^{-1}abc = (c^{-1}ac)(c^{-1}bc) = \varphi_c(a)\varphi_c(b)$
2. $\varphi$ is surjective: Given any $g \in G$.
   $\varphi_c(cgc^{-1}) = c^{-1}(cgc^{-1})c = g$
3. $\varphi$ is injective: $\varphi_c(a) = \varphi_c(b)$ for some $a, b \in G$
   $\implies c^{-1}ac = c^{-1}bc$
   $\implies a = b$
   $\implies \varphi$ is an isomorphism.

## 7.2 Classification of Cyclic Groups

**Theorem 7.10**
Suppose $G$ is a cyclic group.

1. $|G| = \infty \implies G \cong (\mathbb{Z}, +)$
2. $|G| = n < \infty \implies G \cong (\mathbb{Z}_n, +)$

**Proof.**

1. If $G = \langle a \rangle$ infinite. Then $G = \{a^n \mid n \in \mathbb{Z}\}$. So let

$$\begin{aligned} \varphi \colon G &\to \mathbb{Z} \\ a^n &\mapsto n \end{aligned}$$

   So $\varphi$ is one-to-one and onto by definition.

Then,

$$n + m = \varphi(a^{n+m}) = \varphi(a^n a^m) \stackrel{?}{=} \varphi(a^n) + \varphi(a^m) = n + m$$

$\implies \varphi$ is a homomorphism and $\varphi$ is bijection.

$\implies \varphi$ is an isomorphism.

2. $|G| = n \implies G = \{e, a, a^2, \ldots, a^{n-1}\}$

$$\begin{aligned} \varphi \colon G &\to \mathbb{Z}_n = \{0, 1, \ldots, n-1\} \\ a^i &\mapsto i \end{aligned}$$

Exactly for the same reasons: check $\varphi$ is an isomorphism.

$$k = \underbrace{\varphi(a^k)}_{i+j \equiv k \mod n} = \varphi(a^{i+j}) = \underbrace{\varphi(a^i) + \varphi(a^j)}_{i+j \equiv k \mod n}$$

$\varphi$ is an isomorphism.

$\square$

# 8 Jan 21, 2022

## 8.1 Homomorphisms

**Recall 8.1** Let $\varphi\colon G \to H$ any map. Then

$$\operatorname{Im}\varphi = \{h \in H \mid h = \varphi(g) \text{ some } g \in G\}$$

**Theorem 8.2**
If $\varphi\colon G \to H$ is a homomorphism, then:

1. $\varphi(e_G) = e_H$
2. $\varphi(a^{-1}) = (\varphi(a))^{-1}$
3. $\operatorname{Im}\varphi$ is a subgroup of $H$
4. If $\varphi$ is injective, then $G \cong \operatorname{Im}\varphi$

**Note 8.3:** If $\varphi$ is surjective, then $\operatorname{Im}\varphi = H$

**Proof.**

1. Did before.
2. By (1), $e_H = \varphi(e_G) = \varphi(aa^{-1}) = \varphi(a)\cdot\varphi(a^{-1}) \stackrel{?}{=} e_H \stackrel{?}{=} \varphi(a^{-1})\varphi(a) = \varphi(a^{-1}a) = \varphi(e_G) = e_H$ by (1).
3. Claim $\operatorname{Im}\varphi$ subgroup of $H$. Since $\varphi(e_G) = e_H$ by (1) $\implies e_H \in \operatorname{Im}\varphi$. If $a, b \in \operatorname{Im}\varphi \implies \exists\, a', b' \in G$ s.t. $\varphi(a') = a, \varphi(b') = b \implies ab = \varphi(a')\varphi(b') = \varphi(a'b')$ since $G$ is closed, $a'b' \in G \implies ab \in \operatorname{Im}\varphi \implies \operatorname{Im}\varphi$ is closed.
4. By (2), if $\varphi(g) = a$ then
$$a^{-1} = \varphi(g)^{-1} = \varphi(g^{-1})$$
$\implies a^{-1} = \varphi(g^{-1})$ but $g^{-1} \in G \implies a^{-1} \in \operatorname{Im}\varphi$

   $\operatorname{Im}\varphi$ has inverses $\implies \operatorname{Im}\varphi$ is subgroup.
5. $\varphi$ injective $\implies G \cong \operatorname{Im}\varphi$. Since $\varphi\colon G \to \operatorname{Im}\varphi$ is surjective by construction, if $\varphi$ is also injective, then $\varphi\colon G \to \operatorname{Im}\varphi$ is a bijection and a homomorphism $\implies \varphi\colon G \to \operatorname{Im}\varphi$ is an isomorphism $\implies G \cong \operatorname{Im}\varphi$.

$\square$

**Example 8.4**
$\varphi \colon G \to H$ where $\varphi$ is an injective homomorphism and $H$ is abelian.
Question: Is $G$ abelian?
Yes, because $G \cong \operatorname{Im}\varphi$ by bijectivity, and $\operatorname{Im}\varphi$ subgroup of $H$ and subgroups of abelian groups are abelian $\implies G$ has to be abelian.

## 8.2 Congruence

**Definition 8.5** (Congruence of a group)
Suppose $H$ is a subgroup of $G$. Let $a, b \in G$. We say $a \equiv b \pmod{H}$ if $ab^{-1} \in H$.

**Recall 8.6** An equivalence relation on a set $S$ is a relation $a \sim b$ for $a, b \in S$ that is:
reflexive: $a \sim a \quad \forall a \in S$
transitive: $a \sim b$ and $b \sim c \implies a \sim c$
symmetric: $a \sim b \implies b \sim a$.

**Theorem 8.7**
The congruence relation $a \equiv b \pmod{H}$ is an equivalence relation for any subgroup $H \subseteq G$.

**Definition 8.8** (Right coset (and left coset))
Given any $a \in G$, the right coset of $H$ in $G$ is:

$$Ha = \{ha \in G \mid h \in H\} \text{ where } a \text{ is any } a \in G \text{ fixed}$$

This is a right coset because $a$ is multiplied on the right.
The left coset of $H$ in $G$ is:

$$aH = \{ah \in G \mid h \in H\} \text{ where } a \text{ is any } a \in G \text{ fixed}$$

**Note 8.9:** $Ha$ is just the congruence class of $a$ in $G \mod H$.
For any $a \in G$,

$$\begin{aligned}
[a] &= \{b \in G \mid b \equiv a \mod H\} \\
&= \{b \in G \mid ba^{-1} \in H\} \\
&= \{b \in G \mid \underbrace{ba^{-1} = h}_{b = ha} \text{ for some } h \in H\} \\
&= \{ha \in G \mid h \in H\} = Ha.
\end{aligned}$$

**Theorem 8.10**
1. $Ha = Hb$ iff $ab^{-1} \in H$ (i.e. $a \equiv b \mod H$)
2. Given $a \neq b$ either $Ha = Hb$ or $Ha \cap Hb = \emptyset$.

**Proof.** Analogous as for rings (seen this in 110A). $\square$

## 8.3 Lagrange's Theorem

**Theorem 8.11**
$H$-subgroup of $G$ then:

1. $G = \bigcup_{a \in G} Ha$
2. $\forall a \in G, \exists$ bijection between $H \to Ha$. So if $|H| < \infty$, then $|Ha| = |Hb| \, \forall a, b \in G$.

**Proof.**

1. $\bigcup_{a \in G} Ha \subseteq G$ obvious. Given $g \in G, g = eg$ where since $e \in H \implies eg \in Hg \implies g \in Hg \implies G \subseteq \bigcup_{g \in G} Hg$
2. Consider

$$\begin{aligned} \psi \colon H &\to Ha = \{ha \mid h \in H\} \\ h &\mapsto ha \end{aligned}$$

$\psi$ is surjective by definition. If $\psi(h) = \psi(h') \implies ha = h'a \implies h = h' \implies \psi$ is injective $\implies \psi$ is a bijection.

$\square$

**Definition 8.12** (Index)
Given any subgroup $H$ of $G$, the <u>index of $H$ in $G$</u> denoted $[G{:}H]$ is the number of distinct right cosets of $H$ in $G$.

**Theorem 8.13** (Lagrange's Theorem)
If $H \subseteq G$ is a finite subgroup, then:

$$[G{:}H] = \frac{|G|}{|H|}$$

# 9 Jan 24, 2022

## 9.1 Lagrange's Theorem (Cont'd)

**Proof of Lagrange's Theorem.** Suppose $[G{:}H] = n$ and denote the cosets by $Hg_i$ for $i = 1, \dots, n$.
Recall: $Hg_i \cap Hg_j = \emptyset \quad i \neq j$, also

$$G = \bigcup_{i=1}^{n} Hg_i = Hg_1 \cup Hg_2 \cup \ldots \cup Hg_n$$

$$\implies |G| = |Hg_1| + |Hg_2| + \cdots + |Hg_n|$$

Also know by previous theorem $|Hg_i| = |H| < \infty$

$$\implies |G| = n \cdot |H|$$
$$\implies \frac{|G|}{|H|} = n = [G{:}H]$$

□

**Question 9.1:** What fails when $|H| = \infty$?

**Example 9.2**
$n\mathbb{Z} = \langle n \rangle$ inside $\mathbb{Z}$.
Then for $a \in \mathbb{Z}$,

$$[a] = \underbrace{a + n\mathbb{Z}}_{Ha} = \{a + ni \mid i \in \mathbb{Z}\} = \{a, a+n, a+2n, \dots\}$$

where $Ha = \{ha \mid h \in H\}$ with $H = n\mathbb{Z} \to Ha = Hb \iff ab^{-1} \in H$ and $a \equiv b \mod H$

$$a + n\mathbb{Z} = \underbrace{(a+n)}_{b} + n\mathbb{Z}$$

$-n = a-(a+n) \in n\mathbb{Z} \iff a \equiv a+n \pmod n \implies$ exist exactly $n$ cosets $[0], [1], \dots, [n-1]$

$$[\mathbb{Z}{:}n\mathbb{Z}] = n$$

Lagrange's Theorem $\implies |H|$ divides $|G|$ for any $H$ subgroup of $G$.

**Example 9.3**
If $G$ has order 15.
$G$ can only have subgroups of orders 1, 3, 5, 15.

**Note 9.4:** Lagrange does not imply that subgroups exist for every number dividing $|G|$. In Example 9.3, there may not exist a subgroup of order 5 or 3.

**Corollary 9.5**
$|G| < \infty$

1. $\forall a \in G \implies |a| \Big| |G|$
2. If $|G| = n \implies a^n = e \quad \forall a \in G$.

**Proof.**

1. Consider $H = \langle a \rangle \subseteq G$. $|\langle a \rangle| = |a| \implies$ Since $|G| < \infty$

$$\begin{aligned} &\implies H < \infty \text{ we can use Lagrange} \\ &\implies |H| = |\langle a \rangle| = |a| \Big| |G|. \end{aligned}$$

2. Suppose $|a| = m$. Then by (1), $m \mid n \implies n = md$ for some $d \in \mathbb{Z}$. So then

$$a^n = a^{md} = (a^m)^d = e^d = e$$

$\square$

## 9.2 Classification of Groups of Prime Order

**Theorem 9.6**
Suppose $p > 0$ prime. If $|G| = p \implies G \cong \mathbb{Z}_p$.

**Proof.** By Theorem 7.10, all cyclic groups of order $n$ are isomorphic to $\mathbb{Z}_n$. $\implies$ We only need to show $G$ is cyclic. Consider $a \in G$ with $a \neq e$. Then $|\langle a \rangle| \neq 1 \implies$ by Lagrange, since $|\langle a \rangle| \mid p$. Since only 1 or $p$ divides $p \implies |\langle a \rangle| = p$. Since $|G| = p$ and $\langle a \rangle \subseteq G$

$$\begin{aligned} \implies G = \langle a \rangle \implies &G \text{ is cylic of order } p \\ \implies &G \cong \mathbb{Z}_p \text{ by previous theorem} \end{aligned}$$

$\square$

## 9.3 Classification of Groups of Order $\leq 8$

We know $\cancel{1}, \underbrace{\cancel{2}, \cancel{3}}_{\text{prime}}, 4, \underbrace{\cancel{5}}, 6, \underbrace{\cancel{7}}, 8$

**Theorem 9.7**
If $|G| = 4 \implies$ either $G \cong \underbrace{\mathbb{Z}_4}_{\substack{\text{cyclic} \\ \text{abelian}}}$ or $G \cong \underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2}_{\text{abelian}}$.

**Proof.** If $|G| = 4$, then either $\exists a \in G$ with $|a| = 4$ or not.

- If yes, then $G = \langle a \rangle \implies G$ is cyclic $\implies G \cong \mathbb{Z}_4$.
- If not, then $G = \{e, a, b, c\}$, since only $e$ can have order 1, then $|a| = |b| = |c| = 2$

$$\begin{aligned} &\implies a^2 = b^2 = c^2 = e \\ &\implies a = a^{-1}, b = b^{-1}, c = c^{-1} \end{aligned}$$

If $|ab| = 1 \implies a = b^{-1} \implies$ contradiction $|ab| = 2$.

So either

$$\begin{aligned} &ab = a \implies b = e \text{ contradiction} \\ &ab = b \implies a = e \text{ contradiction} \\ &ab = c\checkmark \end{aligned}$$

Repeat this for $ac, ca, ba, bc, cb$ to find entire multiplication table. Then construct an explicit isomorphism to

$$\mathbb{Z}_2 \times \mathbb{Z}_2 : \begin{aligned} e &\mapsto (0,0) \\ a &\mapsto (1,0) \\ b &\mapsto (0,1) \\ c &\mapsto (1,1) \end{aligned}$$

$\square$

**Theorem 9.8**
$|G| = 6 \implies G \cong \mathbb{Z}_6$ or $S_3$.

# 10 Jan 26, 2022

## 10.1 Normal Subgroups

**Recall 10.1** For $a \in G, H \subseteq G$ subgroup. Right coset $Ha = \{ha \in G \mid h \in H\}$. Left coset $aH = \{ah \in G \mid h \in H\}$.

**Definition 10.2** (Normal subgroup)
A subgroup $N$ of $G$ is <u>normal</u> if $Na = aN \; \forall a \in G$.

**Note 10.3:** $Na = aN \;\not\Longrightarrow\; an = na$. Rather, it means that $an = n'a$ for some $n, n' \in N$.

**Notation 10.4:** Whenever $N$ is normal in $G$, we write $N \lhd G$.

**Example 10.5**
Consider $G = D_4$ (not abelian).
Let $M = \{e, r_{180}\}$ then you can show

$$r_{180} \cdot a = a \cdot r_{180} \quad \forall a \in D_4$$

$$\implies Ma = aM \implies M \lhd D_4$$

**Theorem 10.6**
If $G$ is abelian, then all subgroups are normal.

**Recall 10.7** The center $Z(G) = \{a \in G \mid ag = ga\}$.

**Proposition 10.8**
For any $G$, the center $Z(G)$ is always normal.

**Proof.** Using the definition of $Z(G)$, we notice that for any $g \in G$,

$$Z(G)g = gZ(G)$$

For any $a \in Z(G), ag \in Z(G)g$. Since $ag = ga$ because $a \in Z(G)$ (by definition), then $ga \in gZ(G)$. $\square$

**Example 10.9**
$S_3 = \{e, \tau_1, \tau_2, \tau_{12}, \tau_{21}, \tau_{121}\}$.
Let $A_3 := \{e, \tau_{12}, \tau_{21}\}$.
Then

$$A_3 a = \left\{ \begin{array}{c} \tau_{12} \circ \tau_1 = \tau_{121} = \tau_1 \circ \tau_{21} \\ \tau_{12} \circ \tau_2 = \tau_1 = \tau_2 \circ \tau_{21} \\ \underbrace{\tau_{12} \circ \tau_{121}}_{\in A\tau_{121}} = \tau_2 = \underbrace{\tau_{121} \circ \tau_{21}}_{\in \tau_{121} A} \end{array} \right\} = aA_3$$

Recall $(a \in N, aN = N = Na)$

$$\implies A_3 a = aA_3 \quad \forall a \in S_3 \implies A_3 \text{ is normal}$$

**Theorem 10.10**
For $N \lhd G$, if $Na = Nb$ and $Nd = Nc \implies Nad = Nbc$ (Analogously, $Nda = Ncb$).

**Proof.** Direct from set definitions of cosets. □

**Definition 10.11**
Given $a, b \in G, N \subseteq G$,

$$aNb := \{anb \in G \mid n \in N\}$$

**Theorem 10.12**
TFAE:

1. $N \lhd G$.
2. $a^{-1}Na \subseteq N \quad \forall a \in G$.
3. $aNa^{-1} \subseteq N \quad \forall a \in G$.
4. $a^{-1}Na = N \quad \forall a \in G$.
5. $aNa^{-1} = N \quad \forall a \in G$.

**Proof.** 1) $\implies$ 3) $N$ normal $\implies aN = Na \implies \forall a \in G$ and $n \in N$

$$\begin{aligned} \exists n' \in N \text{ such that } an = n'a &\implies ana^{-1} = n' \\ &\implies aNa^{-1} \subseteq N \end{aligned}$$

3) $\implies$ 2) Since if $aNa^{-1} \subseteq N$ $\forall a \in G$ and $a^{-1} \in G$

$$(a^{-1})N(a^{-1})^{-1} = a^{-1}Na \subseteq N$$

2) $\implies$ 3) analogous.
4) $\iff$ 5) proved the same way.

3) $\implies$ 4) If $aNa^{-1} \subseteq N$ then since $ana^{-1} \in N \quad \forall a \in G, \forall n \in N$

$$\begin{aligned}
&\overset{\text{by 2)}}{\implies} a^{-1}\underbrace{(ana^{-1})}_{n'}a \in a^{-1}Na \\
&\implies n \in a^{-1}Na \implies N \subseteq \underbrace{a^{-1}Na}_{\Longleftrightarrow \text{by 3}} \\
&\implies N \subseteq aNa^{-1} \implies N = aNa^{-1}
\end{aligned}$$

2) $\implies$ 5) same proof as 3) $\implies$ 4).
5) $\implies$ 1)

$$\begin{aligned}
aNa^{-1} = N &\implies ana^{-1} = n' \text{ for some } n' \in N \\
&\implies an = n'a \\
&\implies aN \subseteq Na
\end{aligned}$$

Use the fact 4) $\Longleftrightarrow$ 5) to show $Na \subseteq aN$.
$\implies Na = aN \implies N \lhd G$. $\square$

# 11 Jan 28, 2022

## 11.1 Quotient Groups

Given $N \lhd G$, let $G/N := \{Na \mid a \in G\}$.

**Recall 11.1** If $N \lhd G, Na = Nb$ and $Nc = Nd$, then $\implies Nac = Nbd$.

**Theorem 11.2**
$N \lhd G$, then

1. $G/N$ is a group with operation $\underbrace{Na \cdot Nb}_{*\text{ operation in } G/N} \overset{\text{product inside } G}{:=} Nab$
2. If $|G| < \infty \implies |G/N| = |G|/|N|$
3. If $G$ is abelian $\implies G/N$ is abelian.

We call $G/N$ the quotient group of $G$ by $N$.

**Proof.** 1) Check each axiom of groups:

- $id := N$
- Inverse $:= Na^{-1} \implies (Na)(Na^{-1}) = Naa^{-1} = Ne = N$
- etc.

2) $|G/N| = [G{:}N] = |G|/|N|$
3) $\underbrace{(Na)(Nb)}_{Nab} = \underbrace{(Nb)(Na)}_{Nba}$
because $G$ is abelian, $N\underline{ab} = N\underline{ba}$. ☐

**Example 11.3**
Consider

$$2\mathbb{Z} = \langle 2 \rangle \subseteq \mathbb{Z}.$$

$\mathbb{Z}$ abelian $\implies 2\mathbb{Z}$ normal.

$$|\mathbb{Z}/2\mathbb{Z}| = [\mathbb{Z}{:}2\mathbb{Z}] = 2$$

$$2\mathbb{Z} = \{-4, -2, 0, 2, 4, \dots\} = \text{ evens}$$

$2\mathbb{Z} + 1 = \text{odds} \implies \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}_2$
Generally,

$$\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$$

**Example 11.4**
$A_3 \lhd S_3$
$A_3 = \{e, \tau_{12}, \tau_{21}\}$
$|S_3| = 6, |A_3| = 3$, so

$$|S_3/A_3| = \frac{6}{3} = 2$$
$$\implies S_3/A_3 \cong \mathbb{Z}_2$$

**Example 11.5**
$N = \langle 4 \rangle = \{0, 4, 8\} \subseteq \mathbb{Z}_{12}$

$$\begin{aligned}
[0] &= N + 0 = N \\
[1] &= N + 1 = \{1, 5, 9\} \\
[2] &= N + 2 = \{2, 6, 10\} \\
[3] &= N + 3 = \{3, 7, 11\}
\end{aligned}$$

$$\begin{aligned}
\implies N + a = N + b &\iff a \equiv b \mod 4 \\
\text{i.e: } N + 6 = \{6, 10, 2\} &\quad 6 \equiv 2 \mod 4
\end{aligned}$$

$\mathbb{Z}_{12}/N \cong$ ? where $|\mathbb{Z}_{12}/N| = 4$
So either

$$\mathbb{Z}_{12}/N \cong \mathbb{Z}_4 \text{ or } \mathbb{Z}_2 \times \mathbb{Z}_2$$

$[4] = [1] + [1] + [1] + [1] = [0]$
$(N + 1) + (N + 1) + (N + 1) + (N + 1) = N + 4 = N$, because $4 \equiv 0 \mod 4$.
So,

$$|N + 1| = 4 \implies \mathbb{Z}_{12}/N \cong \mathbb{Z}_4$$

**Theorem 11.6**
$N \lhd G$. Then $G/N$ is abelian if and only if $aba^{-1}b^{-1} \in N \; \forall a, b \in G$.

**Proof.** $G/N$ is abelian iff $Nab = Nba \; \forall a, b \in G$

$$\begin{aligned}
&\iff ab \equiv ba \mod N \; \forall a, b \in G \\
&\iff aba^{-1}b^{-1} \equiv e \mod N \iff aba^{-1}b^{-1} \in N
\end{aligned}$$

□

**Theorem 11.7**
$G$ any group. $G/Z(G)$ is cyclic $\implies G$ abelian.

**Proof.** If $G/Z(G)$ is cyclic, then $G/Z = \langle Zg \rangle$ for some $g \in G \implies$ every other coset $Zg' = (Zg)^k = Zg^k$. So then if $a, b \in G$, then

$a \in Za = Zg^k$ for some $k$,
$b \in Zb = Zg^j$ for some $j$.

$$\begin{aligned}
&\implies a = c \cdot g^k \text{ and } b = c'g^j \text{ for some } c, c' \in Z \\
&\implies ab = cg^k \cdot c'g^j = c'g^jcg^k = ba \\
&\implies G \text{ is abelian.}
\end{aligned}$$

$\square$

## 11.2 Quotient Groups and Homomorphisms

**Definition 11.8** (Kernel)
Let $\varphi\colon G \to H$ be a homomorphism. The kernel of $\varphi$ is the set

$$\ker \varphi := \{g \in G \mid \varphi(g) = e_H\}$$

**Example 11.9**
Consider

$$\begin{aligned}
\varphi\colon \mathbb{Z} &\to \mathbb{Z}_5 \\
n &\mapsto [n]
\end{aligned}$$

Then,

$$\begin{aligned}
\ker \varphi &= \{n \in \mathbb{Z} \mid [n] = [0]\} = \{n \mid n \equiv 0 \mod 5\} \\
&= 5\mathbb{Z}
\end{aligned}$$

**Theorem 11.10**
Suppose $\varphi\colon G \to H$ is a homomorphism. Then $\ker \varphi \lhd G$ is a normal subgroup of G.

**Proof.** Subgroup:

- (Identity): Since $\varphi(e) = e \implies e \in \ker \varphi$
- (Closure): If $a, b \in \ker \varphi$,

$$\begin{aligned}
\varphi(ab) = \varphi(a) \cdot \varphi(b) = e \cdot e = e \\
\implies ab \in \ker \varphi.
\end{aligned}$$

- (Inverse): If $a \in \ker \varphi$, then $\varphi(a^{-1}) = (\varphi(a))^{-1} = e^{-1} = e$

  $\implies \ker \varphi$ is a subgroup.

Normal: We will show $g \ker \varphi g^{-1} \subseteq \ker \varphi \; \forall g \in G$.
Let $a \in \ker \varphi$, so $\varphi(a) = e$. Then any $g \in G$:

$$g\varphi(a)g^{-1} = g \cdot e \cdot g^{-1} = e \in \ker \varphi$$

$$\implies g \cdot \ker \varphi g^{-1} \subseteq \ker \varphi$$

$\square$

# 12 Jan 31, 2022

## 12.1 Quotient Groups and Homomorphisms (Cont'd)

**Example 12.1**
Let

$$\begin{aligned} \varphi\colon S_3 &\to \mathbb{Z}_2 \text{ given by} \\ e, \tau_{21}, \tau_{12} &\mapsto 0 \\ \tau_1, \tau_2, \tau_{121} &\mapsto 1 \end{aligned}$$

- Is a homomorphism? Yes. (Check this).
- Kernel of $\varphi$? $\ker\varphi = \{e, \tau_{12}, \tau_{21}\} = A_3$
  By theorem $A_3$ is normal in $S_3$. $S_3/A_3 \cong \mathbb{Z}_2$

**Theorem 12.2**
A homomorphism $\varphi$ is injective if and only if $\ker\varphi = e$.

**Proof.** Standard. □

**Theorem 12.3**
If $N \lhd G$, then

$$\begin{aligned} \pi\colon G &\to G/N \\ a &\mapsto Na \end{aligned}$$

is surjective group homomorphism with $\ker\pi = N$.

**Proof.** $\pi$ is surjective: To every coset $Na$ $\exists a \in G$ such that $a \mapsto Na$.
$\pi$ is homomorphic: $\pi(ab) = Nab = (Na)\cdot(Nb) = \pi(a)\cdot\pi(b)$
$e = N$ if $\pi(a) = N \implies Na = N \iff a \in N$ So,

$$\ker\varphi = \{a \in G \mid a \in N\} = N$$

□

**Lemma 12.4**
Suppose $\varphi\colon G \to H$ is a homomorphism with $\ker\varphi = K$. Then $\forall a, b \in G, \varphi(a) = \varphi(b)$ if and only if $Ka = Kb$.

**Proof.** $\varphi(a) = \varphi(b) \iff \varphi(a)\varphi(b)^{-1} = e \iff \varphi(a)\varphi(b^{-1}) = \varphi(ab^{-1}) = e \iff ab^{-1} \in \ker\varphi = K \iff a \equiv b \mod K \iff Ka = Kb$ □

## 12.2 The Isomorphism Theorems

**Theorem 12.5** (First Isomorphism Theorem)
Let $\varphi\colon G \to H$ be a surjective homomorphism. Then

$$G/\ker\varphi \cong H$$

**Proof.** Let

$$\begin{aligned} \pi\colon G/\ker\varphi &\to H \\ Ka &\mapsto \varphi(a) \end{aligned}$$

where $K = \ker\varphi$. We need to show $\pi$ is a well-defined isomorphism

1. Well-defined: Let $Ka = Kb$ for $a \neq b$. Then $ab^{-1} \in K = \ker\varphi \implies \varphi(ab^{-1}) = e \implies \varphi(a) = \varphi(b)$
2. Homomorphism:

$$\begin{aligned} \pi(Ka \cdot Kb) &= \pi(Kab) \\ &= \varphi(ab) = \varphi(a) \cdot \varphi(b) \\ &= \pi(Ka) \cdot \pi(Kb) \end{aligned}$$

3. Surjective: $\pi\colon G/K \to H$. Let $h \in H$, then $\exists g \in G$ such that $\varphi(g) = h$ because $\varphi$ is surjective. Consider $Kg \in G/\ker\varphi$. Then $\pi(Kg) = \varphi(g) = h$.
4. Injective: Suppose $\pi(Ka) = \pi(Kb)$

$$\begin{aligned} &\implies \varphi(a) = \varphi(b) \\ &\implies ab^{-1} \in \ker\varphi \\ &\implies Ka = Kb \implies \pi \text{ is 1-1} \end{aligned}$$

□

**Theorem 12.6** (Second Isomorphism Theorem)
Suppose $N$ and $K$ are subgroups of $G$, with $N \lhd G$. Then

$$NK := \{nk \mid n \in N, k \in K\}$$

is a subgroup of $G$ containing both $N$ and $K$.

**Proof.** Homework. ☺ □

**Lemma 12.7**
Let $N \lhd G$, and $K$ is any subgroup of $G$ such that $N \subseteq K$. Then $N \lhd K$ and $K/N$ is a subgroup of $G/N$.

**Proof.** Since $aN = Na \; \forall a \in G$ so then if $a \in K$, then $aN = Na \; \forall a \in K$
$\implies N \lhd K \implies K/N$ is a subgroup.
Since

$$K/N = \{Na \mid a \in K\}$$

and since $K \subseteq G \implies K/N \subseteq G/N$. $\square$

**Theorem 12.8** (Third Isomorphism Theorem)
Let $K \lhd G, N \lhd G, N \subseteq K \subseteq G$. Then,

1. $K/N \lhd G/N$ and
2. $(G/N)/(K/N) \cong G/K$

# 13 Feb 2, 2022

## 13.1 The Isomorphism Theorems (Cont'd)

**Proof of Third Isomorphism Theorem.** Since $K \lhd G$ and $N \lhd G \implies G/N$ and $G/K$ are groups. Consider

$$\begin{aligned}\varphi \colon G/N &\to G/K \\ Ng &\mapsto Kg\end{aligned}$$

Well-defined:
If $Ng = Ng'$ with $g \neq g'$

$$\begin{aligned}\implies g'g^{-1} \in N \subseteq K &\implies Kg = Kg' \\ &\implies \varphi(Ng) = \varphi(Ng')\end{aligned}$$

Homomorphism:

$$\begin{aligned}\varphi(Ng \cdot Ng') &= \varphi(Ngg') = Kgg' \\ &= Kg \cdot Kg' = \varphi(Ng) \cdot \varphi(Ng')\end{aligned}$$

Surjective: Obvious by definition of the map

$$\varphi \colon G/N \to G/K \quad \forall Kg \to \exists Ng \text{ s.t. } \varphi(Ng) = Kg$$

$\implies$ We can apply the First Isomorphism Theorem so that

$$(G/N)/\ker\varphi \cong G/K$$

We show $\ker\varphi = K/N$: Now, $\varphi(Ng) = K = Ke \iff g \in K$. Then,

$$\ker\varphi = \{Ng \mid g \in K\}$$

By Lemma 12.7, $N \lhd K$ so $K/N$ makes sense. Also, $\ker\varphi = K/N$.
Since by previous theorem, since $\ker\varphi \lhd G/N$ then this means that $K/N \lhd G/N$ and

$$(G/N)/(K/N) \cong G/K.$$

$\square$

**Corollary 13.1**
Suppose $N \lhd G$ and $K$ is any subgroup of $G$ such that $N \subseteq K \subseteq G$. Then $K \lhd G$ if and only if $K/N \lhd G/N$.

**Proof.** $(\implies) K \lhd G \implies K/N \lhd G/N$ (by Third Isomorphism Theorem).

( $\Longleftarrow$ ) Suppose $K/N \lhd G/N$. For any $Na \in G/N$, we know

$$(Na)^{-1}(Nk)(Na) \in \underbrace{K/N}_{\ni Nk} \lhd \underbrace{G/N}_{\ni Na}$$

Then $\forall a \in G$ and $k \in K$,

$$Na^{-1}ka = (Na^{-1})(Nk)(Na) \in K/N$$

$$\implies Na^{-1}ka \in K/N$$

So this means $\exists t \in K$ such that

$$Na^{-1}ka = Nt$$

$$\implies \forall n \in N \; \exists n' \in N$$

$$na^{-1}ka = n't$$

$$\implies a^{-1}ka = \underbrace{n^{-1}n't}_{\substack{n,n' \in N \subseteq K \\ t \in K}} \in K.$$

<u>Recall:</u> $K \lhd G$ if and only if $aKa^{-1} \subseteq K \; \forall a \in G$.
Equivalently: $aka^{-1} \in K \quad \forall a \in G \quad \forall k \in K$
$\implies K$ is normal. $\square$

**Theorem 13.2** (The Correspondence Theorem)
Suppose $T \subseteq G/N$ is a subgroup. Then there exists some subgroup $H \subseteq G$ with $N \subseteq H$ such that

$$T = H/N$$

i.e. There exists a correspondence between

$$N \subseteq H \subseteq G \longleftrightarrow T \subseteq G/N$$

This theorem classifies all subgroups of $G/N$.

**Proof.** Given $T \subseteq G/N$ subgroup. Let $H \coloneqq \{a \in G \mid Na \in T\}$.

- $N \in T$ since $T$ is a subgroup of $G/N \implies e \in H$.
- If $Na$ and $Nb \in T$ then

  $$Nab = Na \cdot Nb \in T$$

  Since $T$ is closed $\implies ab \in H$.
- If $Na \in T$ then $(Na)^{-1} = Na^{-1} \in T \implies a^{-1} \in H \implies H$ is a subgroup of $G$.

Now, $\forall a \in N, Na = N$ and since $N \in T$

$$\implies a \in H \; \forall a \in N \implies N \subseteq H.$$

Thus, $N \subseteq H \subseteq G$.

Finally, we must show $T = H/N$. (By Lemma 12.7, $N \lhd G \implies N \lhd H$ so $H/N$ makes sense).
Using the fact that $H = \{a \in G \mid Na \in T\}$,

$$H/N = \{Na \mid a \in H\} = \{Na \in T \mid a \in G\} = T$$

$\square$

# 14 Feb 4, 2022

## 14.1 Simple Groups

**Definition 14.1** (Simple group)
A group is simple if it has no nontrivial proper subgroups, i.e. the only subgroups it has are $e$ and $G$.

**Example 14.2**
$\mathbb{Z}_2, \mathbb{Z}_3, \ldots, \mathbb{Z}_p$

By Lagrange, $1 \mid p$ and $p \mid p \implies$ only subgroups of $\mathbb{Z}_p$ are $e$ and $\mathbb{Z}_p$
$\implies \mathbb{Z}_p$ is simple if and only if $p$ is prime.

**Theorem 14.3**
$G$ is a simple abelian group if and only if $G \cong \mathbb{Z}_p$ for $p$ prime.

**Proof.** ( $\impliedby$ ) done.
( $\implies$ ) Suppose $G$ is a simple abelian group. Then $\forall a \in G$ with $a \neq e$; $G = \langle a \rangle$. Then $G$ is cyclic $\implies G \cong \mathbb{Z}$ or $G \cong \mathbb{Z}_n$ for some $n \in \mathbb{N}$.
If $G \cong \mathbb{Z}$, $G$ cannot be simple since $\mathbb{Z}$ has infinitely many subgroups (i.e. $n\mathbb{Z}$) $\implies G \cong \mathbb{Z}_n$.
If $n$ is not prime, then $n = kd$ for $k, d \in \mathbb{N}$.
$\implies \langle a^d \rangle \subseteq G$ is a proper subgroup of order $k$ which is a contradiction because $G$ is simple $\implies n$ is prime. So $G \cong \mathbb{Z}_p$. $\square$

$$\longrightarrow \text{Midterm is up to here!} \longleftarrow$$

---

## 14.2 The Symmetric Group

**Definition 14.4** (Symmetric group)
The symmetric group $S_n$ is the group of permutations of $\{1, \ldots, n\}$ where group operation corresponds to composition of permutations. It has order $n!$

Permutation $\implies$ assignment of entry to position $a_i$

$$\begin{pmatrix} 1 & \ldots & i & \ldots & n \\ \downarrow & & \downarrow & & \downarrow \\ a_1 & & a_i & & a_n \end{pmatrix}$$

So each permutation is just a bijection $\{1 \ldots n\} \to \{1 \ldots n\}$.

## 14.3 Cycle Notation

**Example 14.5**

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \implies \begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[ll, bend right] \end{tikzcd} \implies \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

**Example 14.6**

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 4 & 2 \end{pmatrix} \implies \begin{pmatrix} 1 & 3 & 5 & 2 \end{pmatrix}(4) = \begin{pmatrix} 1 & 3 & 5 & 2 \end{pmatrix}$$

**Example 14.7**

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = 1 \quad 2 \quad 3 = (1)(2)(3) = e$$

**Example 14.8**

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 1 & 7 & 2 & 4 & 6 & 3 \end{pmatrix}$$

$$\begin{aligned} (1542)(37)(6) &= (1542)(6)(37) \\ &= (6)(37)(1542) = (37)(1542) \\ &= (1542)(37) \end{aligned}$$

Note: $(2154) = (1542) \neq (5142)$

## 14.4 Multiplying in Cycle Notation

To compose in cycle notation you "trace" each entry from right to left. Always start with the first entry of the right most cycle.

**Example 14.9**

$(243)(1243) = (1423)$

**Example 14.10**

$(12)(34) = (34)(12)$

Can't merge this because the cycles are disjoint

**Example 14.11**

$(12)(23)(34) = (3412) = (4123) = (1234)$

Check this:

$$\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 2 & 4 & 3\end{pmatrix}$$

$$\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 3 & 2 & 4\end{pmatrix}$$

$$\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 3 & 4\end{pmatrix}$$

$$= \begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 3 & 4 & 1\end{pmatrix}$$

$$= (1234)$$

# 15 Feb 7, 2022

## 15.1 The Symmetric Group (Cont'd)

**Definition 15.1** (Disjoint cycle)
We say two cycles are disjoint if they have no entries in common.

**Example 15.2**
(12)(358) are disjoint
(132)(358) not disjoint

**Theorem 15.3**
Disjoint cycles commute.

**Proof.** Easy and straightforward. □

**Theorem 15.4**
Every permutation in $S_n$ is a product of disjoint cycles.

**Example 15.5**
From above (1542)(37)(6) product of disjoint cycles.

**Recall 15.6** Order of $g \in G$ is the smallest positive integer $k$ s.t. $g^k = e$.

**Theorem 15.7**
The order of any $w \in S_n$ is the least common multiple of the lengths of the disjoint cycles of $w$.

**Example 15.8**
$w = (1542)(37)$ So,

$$|w| = \operatorname{lcm}(4, 2) = 4$$

Check this by computing $w \neq e$, $\underbrace{w^2, w^3, w^4}_{\text{which of these } = e?}$.

$$\begin{aligned} w^2 &= (1542)(37)(1542)(37) \\ &= (1542)(1542)(37)(37) \end{aligned}$$

And

$$w^4 = \underbrace{(1542)\dots(1542)}_{4\times}\underbrace{(37)(37)(37)(37)}$$

**Example 15.9**
We have $w = (1243)(243)$. What is $|w|$?
Because $(1243)(243)$ are not disjoint, we need to make them disjoint. By multiplying,

$$(2341) \implies w = (2341) \implies |w| = 4$$

**Definition 15.10** (Transposition)
A cycle of length 2 is called a transposition, i.e. $(ab)$ for any $a, b \in \{1 \dots n\}$.

**Definition 15.11** (Simple transposition)
A transposition is simple when $b = a \pm 1$, i.e. $(a\ a+1)$ or $(a-1\ a)$. Or

$$(12), (23), (34), \dots \text{ etc}$$

**Fact 15.12:** Simple transpositions generate all of $S_n$.

**Example 15.13**
$(1 \quad 5) = (12)(23)(34)(45)$

**Proposition 15.14**
For any $a, b = \{1 \dots n\}$

1. (Self-inverses) $(ab)(ab) = e$.
2. Suppose $\sigma_1 \dots \sigma_k$ are all transpositions.

$$[\sigma_1 \dots \sigma_k]^{-1} = \sigma_k \sigma_{k-1} \dots \sigma_1$$

3. Every cycle is a product of (not necessarily disjoint) transpositions, i.e.

$$(a_1 \dots a_k) = (a_1 a_2)(a_2 a_3) \dots (a_{k-1} a_k)$$

**Example 15.15**
$S_3 = \{e, \tau_1, \tau_2, \tau_{21}, \tau_{12}, \tau_{121}\}$.

$$\tau_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (12)$$

$$\tau_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (23)$$

$$\begin{aligned} \implies \tau_{21} &= (23)(12) = (132) \\ \tau_{12} &= (12)(23) = (231) = (123) \\ \tau_{21} &= (12)(23)(12) = (12)(132) = (13)(2) = (13) \end{aligned}$$

<u>Multiplication is easy using this notation:</u>

$$\begin{aligned} \tau_{12} \cdot \tau_{21} &= (23)(12) \cdot (12)(23) \\ &= (23)(23) = e \end{aligned}$$

$$\tau_{212} = (23)(12)(23) = \tau_{121}$$

**Theorem 15.16**
Every permutation $w \in S_n$ is a product of (not necessarily disjoint) transpositions.

**Proof.** Combine (3) above in proposition with the fact that every permutation $w \in S_n$ is a product of cycles. $\square$

**Corollary 15.17**
$S_n$ is generated by simple transpositions, i.e.

$$S_n = \langle (12), (23), (34), \ldots, (n-2, n-1), (n-1, n) \rangle$$

Note there are $n-1$ generators.

**Definition 15.18** (Odd vs. even)
A permutation $w \in S_n$ is:

- <u>Odd</u> if $w =$ product of an odd number of transpositions.
- <u>Even</u> if $w =$ product of an even number of transpositions.

This is known as the <u>parity</u> of a permutation.

**Example 15.19**
$w = (1542)(37) = (15)(54)(42)(37)$.
Since $w$ is a product of 4 transpositions and 4 is even $\implies$ $w$ is even.

**Example 15.20**
$w = (2341) = (23)(34)(41)$
$\implies w$ is odd
Notice $|w| = 4$ and $w$ is a product of 3 transpositions. So the order $\neq$ parity. Notice we could have written

$$w = (2341) = (12)(24)(43)(24)(43)$$

$\implies w$ is now a product of 5 transpositions.

# 16 Feb 9, 2022

## 16.1 The Symmetric Group (Cont'd)

The parity of a permutation is independent of the choice of decomposition into transpositions.

**Lemma 16.1**
The identity $e \in S_n$ is even not odd.

**Proof.** Tedious. Please read in book. $\square$

**Theorem 16.2**
Every permutation is either even or odd, not both.

**Proof.** Suppose not. Then $\exists w \in S_n$ such that

$$w = \sigma_1 \dots \sigma_n \quad w = \tau_1 \dots \tau_m$$

where $n$ is even and $m$ is odd

$$\begin{aligned} e = w \dots w^{-1} &= \sigma_1 \dots \sigma_n (\tau_1 \dots \tau_m)^{-1} \\ &= \sigma_1 \dots \sigma_n \tau_m \dots \tau_1 \end{aligned}$$

$\implies$ $e$ is a product of $n + m$ transpositions.
$\implies$ $e$ is odd because $n + m$ is odd.
Which is a contradiction. Thus $w$ is either even or odd, not both. $\square$

## 16.2 The Alternating Group

**Definition 16.3** (Alternating group)
For any given $S_n$, define the alternating group $A_n$ as the set of all even permutations in $S_n$.

**Theorem 16.4**
$A_n$ is a subgroup of $S_n$ of order $\dfrac{n!}{2}$.

**Proof.** Products and inverses of even permutations remain even. Because $e \in A_n$ by Lemma 16.1. $\square$

**Note 16.5:** $A_n$ is almost always the only normal simple subgroup of $S_n$! This fact is crucial to trying to solve quintic and higher order polynomial equations.

**Theorem 16.6**
$\forall n \neq 4$, $A_n$ is simple.

**Proof (Sketch).** Idea: decompose permutations in $A_n$ into case by case analysis of the cycle lengths of each one. Basically follows from the next two lemmas. $\square$

**Lemma 16.7**
For $n \geq 3$, every nonidentity element of $A_n$ is a product of cycles of length 3.

**Proof.** Consider any pair of transpositions $(ab)(cd)$.

- If $a = c, b = d$: $(ab)(ab) = e$.
- If $a = c$: $(ab)(ad) = (adb)$
- Else: $(ab)(cd) = (ab)(bc)(bc)(cd) = (abc)(bcd)$

Since any $w \in A_n$ is a product of an even number of transpositions. $\implies$ This allows you to then write $w =$ product of cycles of length 3. □

**Lemma 16.8**
If $N \triangleleft A_n$ and $N$ contains a 3-cycle $\implies N = A_n$. So $A_n$ is simple.

**Corollary 16.9**
For $n \geq 5$, $A_n$ is the only proper nontrivial normal subgroup of $S_n$.

**Proof.** If $N \triangleleft S_n$ then one can show $N \cap A_n \triangleleft A_n$. Since $A_n$ is simple either:

- $N \cap A_n = A_n \implies N = A_n$ or $N = S_n$
- $N \cap A_n = e \implies N = e \cup \{w \in S_n \mid w \text{ odd}\}$.

But $N$ is a subgroup and if $w, w'$ are odd, then $w \cdot w'$ is even.
$\implies N$ is not closed if $N$ contains odd permutations and $N = e \cup \{w \in S_n \mid w \text{ odd}\}$.
$\implies N = e$. □

**Theorem 16.10** (Cayley's Theorem)
Every group $G$ (finite) is isomorphic to a group of permutations.

**Proof.** Consider $G$ as a set, let $S(G)$ denote the group of all permutations of the set $G$. Then,

$$S(G) = \{\text{bijections from } G \to G \text{ under composition}\}$$

Define:

$$\begin{aligned} \varphi \colon G &\to S(G) \\ a &\mapsto \varphi(a) \end{aligned}$$

where

$$\begin{aligned} \varphi(a) \colon G &\to G \\ g &\mapsto ag \end{aligned}$$

The map $\varphi(a)$ is a bijection:

$$\begin{aligned} g_1, g_2 \in G \implies \varphi(a)(g_1) &= \varphi(a)(g_2) \\ ag_1 &= ag_2 \\ g_1 &= g_2 \implies \varphi(a) \text{ is 1-1.} \end{aligned}$$

Given $g \in G$,

$$\varphi(a)(a^{-1}g) = a \cdot (a^{-1}g) = g$$

$\implies \varphi(a)$ is onto.
$\implies \varphi(a) \colon G \to G$ is a bijection $\forall a \in G$.
$\implies \varphi \colon G \to S(G)$ is well-defined.
<u>$\varphi$ is a homomorphism:</u> Let $a, b, g \in G$.
<u>Want:</u> $\varphi(ab) = \varphi(a) \circ \varphi(b)$

$$\begin{aligned} \varphi(ab)(g) &= \big(\varphi(a) \circ \varphi(b)\big)(g) \quad \forall g \in G \\ abg &= \underbrace{\varphi(a)(bg)}_{abg} \end{aligned}$$

<u>$\varphi$ is injective:</u> $\forall g \in G$,

$$\begin{aligned} \varphi(a) = \varphi(b) &\implies \varphi(a)(g) = \varphi(b)(g) \\ &\implies ag = bg \\ &\implies a = b \end{aligned}$$

$\implies \varphi$ is injective.
$\implies \varphi \colon G \to S(G)$ is an injective group homomorphism such that $G \cong \operatorname{Im} \varphi \subseteq S(G)$.
$\implies G$ is isomorphic to a group of permutations. $\square$

**Corollary 16.11**
If $|G| < \infty$ then $G$ is isomorphic to a subgroup of $S_n$ with $n = |G|$.

# 17 Feb 11, 2022

## 17.1 Direct Products

**Definition 17.1** (Direct product)
Given $G_1, \dots, G_k$ groups, the direct product $G_1 \times \cdots \times G_k$ is the group with elements $(g_1, \dots, g_k)$ with $g_i \in G_i \ \forall i$ and with binary operation:

$$(g_1, \dots, g_k)(g_1', \dots, g_k') = (g_1 g_1', \dots, g_k g_k')$$

**Notation 17.2:** In additive notation, we denote it instead as direct sum and write it as $G_1 \oplus \cdots \oplus G_k$.

**Fact 17.3:** $|G_1 \times \dots G_k| = \prod_{i=1}^{k} |G_i|$

**Example 17.4**
$\mathcal{U}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(1,0,0),(1,1,0),(1,0,1),(1,1,1),(2,0,0),(2,1,0),(2,0,1),(2,1,1)\}$
So,

$$|\mathcal{U}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2| = 2 \times 2 \times 2 = 8$$

**Example 17.5**
$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 = \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3$.
In this case either notation is fine.

**Theorem 17.6**
Given $N_i \lhd G$ for $i = 1, \dots, k$. Suppose each $g \in G$ can be uniquely written as a product $g = n_1 \dots n_k$ with $n_i \in N_i \ \forall i$. Then $G \cong N_1 \times N_2 \cdots \times N_k$.

To prove this theorem, we need a lemma:

**Lemma 17.7**
Suppose $M, N \lhd G$ such that $M \cap N = \{e\}$. If $a \in M$ and $b \in N$ then $ab = ba$.

**Proof.** Let $a \in M, b \in N$. We need to show that $aba^{-1}b^{-1} \in M \cap N$.

1. Since $M \lhd G$ then $ba^{-1}b^{-1} \in M \implies \underbrace{a}_{\in M} \cdot \underbrace{ba^{-1}b^{-1}}_{\in M} \in M$ because $M$ is closed.
2. Since, $N \lhd G$ then $aba^{-1} \in N \implies \underbrace{aba^{-1}}_{\in N} \cdot \underbrace{b^{-1}}_{\in N} \in N$
3. $aba^{-1}b^{-1} \in M \cap N = \{e\} \implies aba^{-1}b^{-1} = e \implies ab = ba$

□

**Proof of Theorem 17.6.** Define a map

$$\begin{aligned}\varphi\colon N_1 \times \dots N_k &\to G\\ (n_1, \dots, n_k) &\mapsto n_1 \dots n_k = g\end{aligned}$$

- $\varphi$ is surjective: follows from the fact that $\forall g$ can be written as $n_1 \dots n_k$.
- $\varphi$ is injective: follows from the product $n_1 \dots n_k = g$ being unique.
- $\varphi$ is a homomorphism:
  $(n_1, \dots, n_k), (n_1', \dots, n_k') \in N_1 \times \dots N_k$
  So

$$\begin{aligned}\varphi\Big((n_1, \dots, n_k) \cdot (n_1', \dots, n_k')\Big) &= \varphi\Big((n_1n_1', \dots, n_kn_k')\Big)\\ &= n_1n_1' \cdot n_2n_2' \cdot \dots n_{k-1}n_{k-1}' \cdot n_kn_k'\\ &= n_1n_1' \cdot n_2n_2' \cdot \dots n_{k-2}'n_{k-1}n_kn_{k-1}'n_k'\\ &= n_1n_1' \cdot n_2n_2' \cdot \dots n_{k-1}n_{k-2}'n_kn_{k-1}'n_k'\\ &= n_1n_1' \cdot n_2n_2' \cdot \dots n_{k-1}n_kn_{k-2}'n_{k-1}'n_k'\\ &\vdots\\ &= n_1n_2 \dots n_k \cdot n_1' \dots n_{k-1}'n_k'\\ &= \varphi(n_1, \dots, n_k) \cdot \varphi(n_1', \dots, n_k')\end{aligned}$$

$\implies \varphi$ is an isomorphism $\implies G \cong N_1 \times \dots \times N_k$. □

**Definition 17.8** (Direct product and direct factor)
Whenever $G = N_1 \times \dots \times N_k$ we say $G$ is the <u>direct product</u> of the $N_1's$ and each $N_1$ is a <u>direct factor</u>.

**Definition 17.9**
Given $N, M \subseteq G$, subgroups, let $MN = \{mn \in G \mid m \in M, n \in N\}$. Note $MN$ is not necessarily a group.

**Theorem 17.10**
If $M, N \lhd G$ such that $G = MN$ and $M \cap N = \{e\}$. Then $G = M \times N$.

**Proof.** By Theorem 17.6 we only need to show uniqueness.
Suppose $g \in G$ such that $g = mn$ and $g = m'n'$ and $m, m' \in M, n, n' \in N$.
$\implies mn = m'n' \implies \underbrace{(m')^{-1} \cdot m}_{\in M} = \underbrace{n'n^{-1}}_{\in N}$
$\implies (m')^{-1}m$ and $n'n^{-1} \in M \cap N = \{e\}$
$\implies n'n^{-1} = e \implies n' = n$ and $(m')^{-1}m = e \implies m' = m$
$\implies g = mn$ is a unique decomposition. □

# 18 Feb 14, 2022

## 18.1 Midterm

# 19 Feb 16, 2022

## 19.1 Direct Products (Cont'd)

**Recall 19.1** If $M, N \lhd G, G = MN, M \cap N = \{e\} \implies G = M \times N$.

**Example 19.2**
Consider $\underbrace{\mathcal{U}_{15}}_{G} = \{1, 2, 4, 7, 8, 11, 13, 14\}$

$$N = \{1, 2, 4, 8\} \quad M = \{1, 11\}$$

- $N, M$ are normal subgroups of $\mathcal{U}_{15}$.
- $M \cap N = \{e\}$
- $\mathcal{U}_{15} = MN$
  We want to show that 7, 13, 14 are products $mn$ for some $m \in M, n \in N$.

$$\begin{aligned} 7 &\equiv 11 \cdot 2 \mod 15 \\ 13 &\equiv 11 \cdot 8 \mod 15 \\ 14 &\equiv 11 \cdot 4 \mod 15 \end{aligned}$$

$\implies \mathcal{U}_{15} = MN$. By theorem, $U_{15} \cong M \times N$.

$$\underbrace{N = \{1, 2, 4, 8\}}_{|N|=4} \quad \underbrace{M = \{1, 11\}}_{|M|=2 \implies M = \mathbb{Z}_2}$$

Check whether $N$ has an element of order 4.
$|1| = 1, |2| = 4$ We know

$$2 \cdot 2 \cdot 2 \cdot 2 = 16 \equiv 1 \mod 15$$

$$2^4 \equiv 1 \mod 15$$

$$\implies \mathcal{U}_{15} \cong \mathbb{Z}_2 \times \mathbb{Z}_4$$

## 19.2 Finite Abelian Groups

<u>Step 1:</u> Change to additive notation.

$$\begin{aligned} ab &\longmapsto a + b \\ a^k &\longmapsto k \cdot a \\ e &\longmapsto 0 \\ MN &\longmapsto M + N = \{m + n \mid m \in M, n \in N\} \\ M \times N &\longmapsto M \oplus N \\ \text{direct factors } &\longmapsto \text{ direct summands} \end{aligned}$$

**Definition 19.3**
$G$-abelian group, $p$-prime.

$$G(p) := \{a \in G \mid \underbrace{|a| = p^n}_{p^n \cdot a = 0} \text{ some } n \geq 0\}$$

**Proposition 19.4**
$G(p)$ is a subgroup of $G$.

We will show: $G = \bigoplus_{p_i} G(p_i)$

**Lemma 19.5**
$G$-abelian group, $a \in G$ with $|a| = p_1^{n_1} \dots p_k^{n_k} < \infty$ with $p_i$ prime, $p_i \neq p_j, i \neq j$. Then, $a = a_1 + \cdots + a_k$ with $a_i \in G(p_i)$ each $i$.

**Proof.** Proof by induction on $k$.
Base case: $(k = 1)$ $|a| = p_1^{n_1}$.
By definition of $G(p_1) \implies a \in G(p_1)$.
Inductive step: Suppose statement is true for elements that are divisible by at most $k-1$ distinct primes. Then if $|a| = p_1^{n_1} \dots p_k^{n_k}$, let $m = p_1^{n_1} \cdots p_{k-1}^{n_{k-1}}$

$$\implies (m, p_k^{n_k}) = 1$$

$$\implies \exists u_1 v \text{ such that } 1 = um + vp_k^{n_k}$$

Rewrite

$$a = 1 \cdot a = \underbrace{(um)a}_{\in G(p_k)} + (vp_k^{n_k})a$$

$$\implies p_k^{n_k}(uma) = u \cdot (p_k^{n_k} ma) = 0$$

because

$$|a| = p_1^{n_1} \cdots p_k^{n_k} = mp_k^{n_k} \implies (p_k^{n_k} \cdot ma) = 0 \implies uma \in G(p_k).$$

Likewise:

$$m(vp_k^{n_k} a) = v(mp_k^{n_k} a) = 0 \implies vp_k^{n_k} a \text{ has order } m.$$

Since $m$ is a product of $k-1$ primes $\implies$ induction hypothesis applied to $vp_k^{n_k} a$

$$\implies vp_k^{n_k} a = a_1 + \cdots + a_{k-1} \text{ with } a_i \in G(p_i) \quad 1 \leq i \leq k-1$$

$$a = 1 \cdot a = \underbrace{(um)a}_{\in G(p_k)} + \underbrace{(vp_k^{n_k})a}_{\implies a_1 + \cdots + a_{k-1}}$$

$$\implies a = \underbrace{a_1}_{G(p_1)} + \cdots + \underbrace{a_{k-1}}_{G(p_{k-1})} + \underbrace{a_k}_{G(p_k)}$$

$\square$

**Theorem 19.6**
$G$-abelian group with $|G| = p_1^{n_1} \cdots p_k^{n_k} < \infty$, $p_i$ are distinct primes. Then

$$G = G(p_1) \oplus \cdots \oplus G(p_k)$$

**Proof.** If $a \in G, |a|\Big||G|$, so we can write

$$a = a_1 + \cdots + a_k \text{ with } a_i \in G(p_i) \text{ each } i$$

where some $a_i$ may be zero. This tells us that there exists such a decomposition. We need to prove uniqueness.
Suppose it's not unique.

$$a = a_1 + \cdots + a_k = b_1 + \cdots + b_k$$

where $a_i, b_i \in G(p_i)$ each $i$.

$$\implies (a_1 - b_1) + (a_2 - b_2) + \cdots + (a_k - b_k) = 0$$

$$\implies \underbrace{(a_1 - b_1)}_{\in G(p_1)} = \underbrace{(b_2 - a_2)}_{\in G(p_2)} + \cdots + \underbrace{(b_k - a_k)}_{\in G(p_k)}$$

$$\implies p_2^{n_1} \cdots p_k^{n_k}(a_1 - b_1) = 0 \text{ for some } n_i \in N$$

$$\implies \underbrace{|a_1 - b_1|}_{\in G(p_1)}\Big|p_2^{n_2} \ldots p_k^{n_k}$$

$$\implies p_1^{n_1}\Big|p_2^{n_2} \ldots p_k^{n_k}$$

which is impossible for $p_1^{n_1} \geq p_1 \implies n_1 = 0$

$$\implies |a_1 - b_1| = p_1^{n_1} = p_1^0 = 1$$

$a_1 - b_1 = 0 \implies a_1 = b_1 \implies$ by doing the same thing for each

$$(a_1 - b_1) \implies a_1 = b_1 \quad \forall i$$

This proves uniqueness.

$$\implies G = G(p_1) \oplus \cdots \oplus G(p_k)$$

$\square$

# 20 Feb 18, 2022

## 20.1 Finite Abelian Groups (Cont'd)

**Recall 20.1** $G$-abelian with $|G| = p_1^{n_1} \dots p_k^{n_k} < \infty$

$$G = G(p_1) \oplus \cdots \oplus G(p_k)$$

**Definition 20.2** ($p$-group)
For $p$-prime, a $p$-group is a group $G$ such that $G = G(p)$.

**Fact 20.3:** If $G$-$p$ group, $a \in G$ has maximal order with $|a| = p^n$ then $\forall b \in G, b \neq a$

- $|b| = p^j$ with $j \leq n$
- $p^n \cdot b = 0$ (additive notation)

If $|G| < \infty$ then maximal order elements always exist.

**Example 20.4**

- $G = \mathbb{Z}_{p^k} \implies |G| = p^k$
- $G = \mathbb{Z}_2 \times Z_2$
- $G = D_4, \mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

**Lemma 20.5**
$G$-finite abelian $p$-group and $a \in G$ with maximal order. Then there exists a subgroup $K \subseteq G$ such that $G = \langle a \rangle \oplus K$.

**Proof.** Let $K$ be the largest subgroup of $G$ such that $K \cap \langle a \rangle = 0$. ($G$ finite $\implies$ $K$ exists). $G$ abelian $\implies K \lhd G$ and $\langle a \rangle \lhd G$.
By Theorem 17.1, since $K \lhd G, \langle a \rangle \lhd G$ and $K \cap \langle a \rangle = 0$, to show that $G = K \oplus \langle a \rangle$ we need only show that $G = K + \langle a \rangle$.
Suppose that $G \neq K + \langle a \rangle \implies \exists b \in G$ s.t $b \notin K + \langle a \rangle$.
In particular $b \neq a$ where $a$ is the max order element. Then since $G$ is a $p$-group and $b$ is not a max element then

$$p^j b = 0 \text{ for some } j \quad (|b| = p^j).$$

Let $r =$ smallest possible integer such that

$$\left.\begin{matrix} p^r b \in \langle a \rangle + K \\ 0 \in \langle a \rangle + K, r \leq j \end{matrix}\right\} \implies p^r b = ta + k \quad (t \in \mathbb{Z}, k \in K)$$

Suppose $|a| = p^n$ so that $p^n a = 0$ then

$$p^n b = 0 \quad \forall b \in G \text{ since } a \text{ is maximal}$$

So,

$$\begin{aligned} p^n(p^{r-1}b) &= p^{n-1}(p^r b) \\ &= p^{n-1}(ta+k) = 0 \end{aligned}$$

Therefore,

$$\begin{aligned} &\implies p^{n-1}ta + p^{n-1}k = 0 \\ &\implies \underbrace{p^{n-1}ta}_{\langle a \rangle} = \underbrace{-p^{n-1}k}_{\in K} \in \langle a \rangle \cap K = 0 \\ &\implies p^{n-1}ta = 0 \text{ and } -p^{n-1}k = 0 \\ &\implies |a| = p^n \implies p^n \mid p^{n-1}t \\ &\implies p \mid t \implies t = pm \text{ for some } m \end{aligned}$$

So

$$\left.\begin{aligned} p^r b &= ta + k \\ p^r b &= pma + k \end{aligned}\right\} \implies \begin{cases} k = p^r b - pma \\ \quad = p(p^{r-1}b - ma) \in K \end{cases}$$

If $p^{r-1}b - ma \in K$

$$\begin{aligned} &\implies p^{r-1}b - ma = k' \text{ with } k' \in K \\ &\implies p^{r-1}b = \underbrace{k'}_{\in K} + \underbrace{ma}_{\in \langle a \rangle} \in K + \langle a \rangle \end{aligned}$$

But $r-1 < r$ and we said that $r$ was the smallest integer such that $p^r b \in K + \langle a \rangle$, a contradiction

$$\implies p^{r-1} - ma \notin K.$$

Let $H \coloneqq \{x + z(p^{r-1}b - ma) \mid x \in K, z \in \mathbb{Z}\}$. Then

1. $H$ is a subgroup of $G$
2. $K \subseteq H$ (take $z = 0$)
3. $K \neq H$ because $z = 1, x = 0$ then $p^{r-1}b - ma \in H$ not in $K$

Since $K$ is the largest subgroup of $G$ such that $K \cap \langle a \rangle = 0$

$$\implies H \cap \langle a \rangle \neq 0$$

$$\implies \exists w \in H \cap \langle a \rangle \text{ s.t. } w \neq 0$$

Note that $K \cap \langle a \rangle = 0 \implies w \notin K$

$$\implies w = sa = x + z(p^{r-1}b - ma) \text{ for some } s, z \in \mathbb{Z}, x \in K.$$

If $p \mid z$ then $z = qp \implies z(p^{r-1}b - ma) \in K$ since $p(p^{r-1}b - ma) \in K$

$$\implies w = \underbrace{x}_{\in K} + \underbrace{z(p^{r-1}b - ma)}_{\in K} \in K$$

A contradiction, so $p \nmid z \implies (p, z) = 1$ this means $1 = up + vz$ for some $u, v \in \mathbb{Z}$ So

$$\begin{aligned}
p^{r-1}b &= (up + vz)p^{r-1}b \\
&= up^r b + vzp^{r-1}b \\
&= u(pma + k) + v(sa - x + zma) \\
&= \underbrace{\underbrace{(upm + vs + zm)}_{\text{numbers}} a}_{\text{multiple of } a \in \langle a \rangle} + \underbrace{(uk - vx)}_{\in K} \\
&\implies p^{r-1}b \in K + \langle a \rangle
\end{aligned}$$

a contradiction because $r$ was the smallest integer such that $p^r b \in K + \langle a \rangle \implies b$ cannot exist. So we're done and

$$G = K + \langle a \rangle$$

By previous theorem, since $\langle a \rangle, K \lhd G$ and $\langle a \rangle \cap K = 0$ and $G = K + \langle a \rangle \implies \langle a \rangle \oplus K = G$. □

**Theorem 20.6** (The Fundamental Theorem of Finite Abelian Groups (I) (Existence))
Every finite abelian group $G$ is the direct sum of cyclic $p$-groups i.e. there exists such a decomposition for G s.t.

$$G \cong \bigoplus_i \mathbb{Z}_{p_i^{r_i}} \text{ over some primes (possibly repeated)}$$

# 21 Feb 23, 2022

## 21.1 Finite Abelian Groups (Cont'd)

**Proof of The Fundamental Theorem of Finite Abelian Groups (I).** By Theorem 19.6:

$$G \cong \underbrace{G(p_1)}_{p\text{-groups}} \oplus \cdots \oplus \underbrace{G(p_k)}_{p\text{-groups}}$$

with $|G| = p_1^{n_1} \cdots p_k^{n_k}$. So we only need to show that each $p$-group $G(p_i)$ = direct sum of cyclic groups.
Proof is by induction on $|G(p_i)| = n$.
Base case: $n = 2$. Then by previous theorem $|G(p_i)| = 2 \implies G(p_i) = \mathbb{Z}_2$.
Inductive Step: Suppose the result holds for any finite abelian group of order $< n$. Let $a \in G(p_i)$ with maximal order. Let $|a| = p_i^m$ for some $m > 0$.
Since $G(p_i)$ is a finite abelian $p$-group, so by Lemma 20.5 $\implies G(p_i) = \langle a \rangle + K$ for some $K \subseteq G(p_i)$.
Note: $|K| < |G(p_i)| = n$. By induction hypothesis: $K$ = direct sum of cyclic groups.

$$\langle a \rangle = \mathbb{Z}_{p_i^m} \text{ and } K = \bigoplus_i \mathbb{Z}_{p_i^{s_i}}$$

$$\implies G(p_1) = \langle a \rangle \oplus K = \bigoplus_i \mathbb{Z}_{p_i^{s_i}} \oplus \mathbb{Z}_{p_i^m}$$

for possibly repeated primes. □

**Example 21.1**
Suppose $G$ is a finite abelian group.

1. $|G| = 42 = 7 \cdot 3 \cdot 2$
   $\implies G \cong \mathbb{Z}_7 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$
2. $|G| = 72 = 3^2 \cdot 2^3$
   - $\mathbb{Z}_{3^2} \oplus \mathbb{Z}_{2^3}$
   - $\mathbb{Z}_{3^2} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{2^2}$
   - $\mathbb{Z}_{3^2} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$
   - $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{2^3}$
   - $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{2^2}$
   - $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

   Note that these are all different and $G$ could be any of these!

**Question 21.2:** Why does $\mathbb{Z}_6 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_2$ not show up?

$6 \neq p^n$ but $6 = 3 \cdot 2$ different primes. So, $\mathbb{Z}_6 \cong \mathbb{Z}_3 \oplus \mathbb{Z}_2 = \mathbb{Z}_2 \oplus \mathbb{Z}_3$

$$\begin{aligned} \mathbb{Z}_6 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_2 &= \mathbb{Z}_3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\ &= \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\ &= (\mathbb{Z}_3)^2 \oplus (\mathbb{Z}_2)^3 \end{aligned}$$

**Lemma 21.3**
If $(m, k) = 1$. Then $\mathbb{Z}_m \oplus \mathbb{Z}_k \cong \mathbb{Z}_{mk}$.

**Proof.** $\mathbb{Z}_{mk} = \langle 1 \rangle$ where 1 has order $mk$. By previous theorem, if $G$ is cyclic of order $n \implies G \cong \mathbb{Z}_n \implies$ suffices to show $\mathbb{Z}_m \oplus \mathbb{Z}_k$ is cyclic of order $mk$.

1. $\mathbb{Z}_m \oplus \mathbb{Z}_k = \langle (1,1) \rangle$ since by Chinese Remainder Theorem,

$$\begin{aligned} (m, k) = 1 \implies \exists r \text{ s.t. } r &\equiv a \mod m \\ r &\equiv b \mod k \\ \implies (a, b) &= r(1,1) \end{aligned}$$

   $\implies$ any $(a, b) \in \mathbb{Z}_m \oplus \mathbb{Z}_k$ can be expressed $r(1,1) \implies \langle (1,1) \rangle = \mathbb{Z}_m \oplus \mathbb{Z}_k$.

2. If $t(1,1) = 0, t \equiv 0 \mod m, t \equiv 0 \mod k$

   But $(m, k) = 1 \implies t = \text{lcm}(m, k) = mk \implies |(1,1)| = mk \implies \mathbb{Z}_m \oplus \mathbb{Z}_k$ has order $mk$, is cyclic $\implies \mathbb{Z}_m \oplus \mathbb{Z}_k \cong \mathbb{Z}_{mk}$.

□

**Example 21.4**
- $\mathbb{Z}_6 \oplus \mathbb{Z}_4 \ncong \mathbb{Z}_{24}$ because $(6, 4) \neq 1$.

  Note $\mathbb{Z}_3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{2^2} \ncong \mathbb{Z}_3 \oplus \mathbb{Z}_{2^3}$
- $\mathbb{Z}_9 \ncong \mathbb{Z}_3 \oplus \mathbb{Z}_3$ because $(3, 3) \neq 1$.

**Theorem 21.5**
Suppose $n = p_1^{n_1} \cdots p_k^{n_k}$ with $p_i$ are distinct primes. Then $\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{n_k}}$.

**Proof.** Induction on number of factors $k$. □

# 22 Feb 25, 2022

## 22.1 Finite Abelian Groups (Cont'd)

**Corollary 22.1**
If $G$ is any finite abelian group then $G$ is the direct sum of cyclic groups of orders $m_1, \ldots, m_t$ such that $m_i \mid m_{i+1}$ for all $i$.

**Idea of proof.** $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_{49} \oplus \mathbb{Z}_5$.
$\implies G = \mathbb{Z}_2 \oplus \mathbb{Z}_{30} \oplus \mathbb{Z}_{8820}$
So, $2 \mid 30$ and $30 \mid 8820$. $\square$

**Corollary 22.2**
Suppose $\mathbb{F}$ is a field. If $G$ is a finite subgroup of $\mathbb{F}^*$, then $G$ is cyclic.

**Proof.** Exercise. $\square$

**Definition 22.3** (Invariant factors and elementary divisors)
The numbers $m_i$ in Corollary 22.1 are the <u>invariant factors</u> of $G$. The prime powers $p_i^{n_i}$ arise in the fundamental theorem of finite abelian groups are the <u>elementary divisors</u>.
Suppose

$$G = \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \cdots \oplus \mathbb{Z}_{m_t}$$

then $m_i \mid m_{m+1}$ are the invariant factors and

$$G = \bigoplus_i \mathbb{Z}_{p_i^{n_i}}$$

are the elementary divisors which are potentially repeated primes.

**Fact 22.4:** The elementary divisors (and the invariant factors) uniquely determine the group (finite abelian) up to isomorphism.

**Theorem 22.5** (Fundamental Theorem of Finite Abelian Groups II (Uniqueness))
Suppose $G$ and $H$ are finite abelian groups. Then $G \cong H$ if and only if $G$ and $H$ have the same elementary divisors.

**Proof.** " $\impliedby$ " $G \cong \bigoplus \mathbb{Z}_{p_i^{n_i}}$ and $H \cong \bigoplus \mathbb{Z}_{p_i^{n_i}} \implies$ obviously $G \cong H$.
" $\implies$ " Suppose $\varphi \colon G \to H$ is an isomorphism $\implies \forall a \in G$ then $|a| = |\varphi(a)| \implies \forall$ primes $p \Big| |G|$ we must have $\varphi(G(p)) = H(p)$.
Thus, we can assume that $G$ and $H$ have the same $p$-groups. So we only need to prove that $p$-groups have the same elementary divisors.
Suppose $G = G(p)$ and $|G| = n$. Induct on $n$.
<u>Base case:</u> $n = 2, G \cong \mathbb{Z}_2 \implies H \cong \mathbb{Z}_2$.
<u>Inductive step:</u> Suppose it's true for all groups of orders less than n.

Since $G$ and $H$ are $p$-groups, then:

$$G = \mathbb{Z}_p^{n_1} \oplus \mathbb{Z}_{p^2}^{n_2} \oplus \cdots \oplus \mathbb{Z}_{p^t}^{n_t} \quad n_i \in \mathbb{Z}_{\geq 0}$$

$$H = \mathbb{Z}_p^{m_1} \oplus \mathbb{Z}_{p^2}^{m_2} \oplus \cdots \oplus \mathbb{Z}_{p^k}^{m_k} \quad m_i \in \mathbb{Z}_{\geq 0}$$

Consider $pG = \{pg \mid g \in G\}$.
$pG \subseteq G$ subgroup of $G$ with direct summands $p\mathbb{Z}_{p^i} = \langle pa \rangle$ where $\mathbb{Z}_{p^i} = \langle a \rangle$. Exercise
Then $|pa| = p^{i-1}$ since $|a| = p^i \implies p\mathbb{Z}_{p^i}$ is cyclic of order $p^{i-1}$

$$\implies p\mathbb{Z}_{p^i} = \mathbb{Z}_{p^{i-1}} \implies pG \cong \mathbb{Z}_p^{n_2} \oplus \cdots \oplus \mathbb{Z}_{p^{i-1}}^{n_t}$$

$$\implies pH \cong \mathbb{Z}_p^{m_2} \oplus \cdots \oplus \mathbb{Z}_{p^{k-1}}^{m_k}$$

<u>Exercise:</u> $\varphi(pG) = pH \implies pH \cong pG$
By the induction hypothesis since $|pH| < |H|$ and $|pG| < |G| \implies$ the elementary divisors of $pH$ and $pG$ are the same.

$$\implies t = k \text{ and } n_i = m_i \quad \forall i \geq 2$$

All we need now is to show that $n_1 = m_1$.
Recall $G \cong H \implies |G| = |H|$

$$\implies p_1^{n_1} p_2^{n_2} \ldots p_t^{n_t} = p_1^{m_1} p_2^{n_2} \ldots p_t^{n_t}$$

$$\implies p^{n_1} = p^{m_1}$$

$$\implies n_1 = m_1$$

□

## 22.2 Sylow Theorems

**Question 22.6:** What if the groups aren't abelian? How do you classify those.

This is hard. We are going to start this theory by looking at the Sylow theorems.

**Definition 22.7** (Conjugate)
Two elements $a, b \in G$ are <u>conjugate</u> if there exists $g \in G$ such that $b = g^{-1}ag$.

**Example 22.8**
$(13) = \underbrace{(12)}_{g^{-1}}(23)\underbrace{(12)}_{g}$
$\implies$ (13) and (23) are conjugate.
$(13) = (23)(12)(23) \implies$ (13) and (12) are conjugate.

**Proposition 22.9**
Conjugacy is an equivalence relation in $G$.

**Definition 22.10** (Conjugacy classes)
Equivalence classes $\implies$ conjugacy classes $\implies$

- Conjugacy classes are either disjoint or the same.
- $G = \bigcup$ conjugacy classes.

**Definition 22.11** (Centralizer)
The centralizer of $a \in G$ is

$$\begin{aligned} C(a) &:= \{g \in G | g^{-1}ag = a\} \\ &= \{g \in G | ga = ag\} \end{aligned}$$

**Note 22.12:** This is similar but different to center

$$Z(G) = \{a \in G | ag = ga, \forall g \in G\}$$

**Theorem 22.13**
$C(a)$ is a subgroup of $G, \forall a \in G$.

**Proof.** Standard. □

# 23 Feb 28, 2022

## 23.1 Sylow Theorems (Cont'd)

By partioning $G$ into conjugacy classes,

**Definition 23.1** (Orbit)
$[a] = [b]$ if and only if $b = g^{-1}ag$ for some $g \in G$. We call the elements $b \in [a]$ the orbit of $a$ under conjugation.

**Theorem 23.2**
Suppose $|G| < \infty, a \in G$. Then

$$|[a]| = [G \colon C(a)] = \frac{|G|}{|C(a)|}$$

**Proof.** Recall $[G \colon C(a)] =$ the number of right cosets of $C(a)$ in $G$. Let $C = C(a)$ and

$$S = \{Cg \mid g \in G\} \text{ be right cosets of } C(a).$$

Define:

$$\begin{aligned} \varphi \colon S &\to [a] \\ Cg &\mapsto g^{-1}ag \end{aligned}$$

$\varphi$ well defined: $Cg_1 = Cg_2 \implies g_1 g_2^{-1} \in C$

$$\begin{aligned} &\implies (g_1g_2^{-1})a(g_1g_2)^{-1} = a \\ &\implies (g_1g_2^{-1})a(g_2g_1^{-1}) = a \\ &\implies g_2^{-1}ag_2 = g_1^{-1}ag_1 \\ &\implies \varphi(Cg_2) = \varphi(Cg_1) \end{aligned}$$

$\varphi$ is injective: Suppose $\varphi(Cg_1) = \varphi(Cg_2)$. If $g_1^{-1}ag_1 = g_2^{-1}ag_2$

$$\begin{aligned} &\implies (g_2g^{-1})a(g_2g_1^{-1})^{-1} = a \\ &\implies Cg_2g_1^{-1} = C \\ &\implies Cg_1 = Cg_2 \end{aligned}$$

$\implies \varphi$ is injective.
$\varphi$ surjective: If $b \in [a]$, there exists $x$ such that $b = x^{-1}ax$ and so then

$$\varphi(Cx) = x^{-1}ax = b.$$

$\implies \varphi$ is a bijection $S \to [a]$

$$\implies [G \colon C(a)] = |S| = |[a]|$$

$\square$

**Corollary 23.3**
$|[a]|\Big||G|$

**Proof.** Lagrange. If $a \in Z(G)$,

$$\implies ag = ga \quad \forall g \in G.$$

$[a] = \{a\}$ since

$$g^{-1}ag = a \quad \forall g \in G$$

$$\implies |[a]| = 1 \quad \forall a \in Z(G).$$

□

**Definition 23.4** (The class equation)
Suppose $|G| < \infty$. Let $a_1, \ldots, a_n$ denote representations for the distinct conjugacy classes of $G$:

1. $|G| = \sum_{i=1}^{n} \Big|[a_i]\Big|$

2. $|G| = \sum_{i=1}^{n} [G\colon C(a_i)]$

3. $|G| = \underbrace{|Z(G)|}_{|[a_i]|=1} + \underbrace{\sum_{a_i \notin Z(G)} \Big|[a_i]\Big|}_{|[a_i]|>1}$

**Theorem 23.5** (Cauchy's Theorem for Finite Abelian Groups)
If $G$ is a finite abelian group and $p$ is prime such that $p\Big||G| \implies \exists g \in G$ such that $|g| = p$.

**Proof.** By Fundamental Theorem of Finite Abelian Groups,

$$G \cong \bigoplus_i \mathbb{Z}_{p_i^{n_i}}$$

where $p_i$ are potentially repeated primes, $n_i \in \mathbb{N}$ and $p_i\Big||G|$.
Then consider $e_i = (0, \ldots, 1, 0 \ldots 0)$ with 1 in position $i$

$$\implies \langle e_i \rangle \cong \mathbb{Z}_{p_i^{n_i}} \text{ cyclic subgroup of } G$$

$$1 = (e_i)^{p_i^{n_i}} = \left(e_i^{p_i^{n_i - 1}}\right)^{p_i} \implies \left|e_i^{p_i^{n_i-1}}\right| = p_i$$

□

## 23.2 First Sylow Theorem

**Theorem 23.6** (First Sylow Theorem)
Suppose $|G| < \infty$, and $p$ is prime with $p^k \Big| |G|$ for some $k$. Then $G$ has a subgroup of order $p^k$.

**Proof.** Induction on $|G|$.
Base case: $|G| = 1 \implies G = e$.
Since

$$p^0 \mid 1 \quad \forall \text{ prime } p$$

and $G \supseteq \langle e \rangle$ where

$$|\langle e \rangle| = p^0 = 1.$$

Inductive Step: Suppose this is true for all groups of orders less than $|G|$.
Let $[a_i]$ denote the conjugacy classes of $G$.
Lagrange $\implies |G| = [G \colon C(a_i)] \cdot |C(a_i)|$ for all i

1. If $\exists a_j \notin Z(G)$ such that $p \nmid [G \colon C(a_j)]$

$$\implies p \mid [C(a_j)]$$

$$\implies \exists k \text{ s.t. } p^k \mid [C(a_j)]$$

Since $a_j \notin Z(G)$

$$\implies [G \colon C(a_j)] > 1$$
$$\implies |C(a_j)| < |G|$$

Then by the induction hypothesis, this implies there exists a subgroup of order $p^k$ inside $C(a_j)$

$$\implies H \subseteq C(a_j) \subseteq G.$$

We will continue this proof in the next lecture. □

# 24 Mar 2, 2022

## 24.1 First Sylow Theorem (Cont'd)

**Proof of First Sylow Theorem (Cont'd).** 2) If $p\Big||G|$ and $p \mid [G\colon C(a_i)]\ \forall a_i \notin Z(G)$. Then by class equation:

$$\implies |Z(G)| = \underbrace{|G|}_{\text{divisible by } p} - \underbrace{\sum_{a_i \notin Z(G)} [G\colon C(a_i)]}_{\text{divisible by } p}$$

$$\implies p\Big||Z(G)|$$

<u>Note:</u> $Z(G)$ is a finite abelian group. Cauchy's Thoerem says that since $p\Big||Z(G)|$ then there exists $x \in z(G)$ such that $|x| = p$.
Consider $\langle x \rangle \lhd G$ with $|\langle x \rangle| = p$, then

$$|G/\langle x \rangle| = |G|/p < |G|$$

also $p^{k-1}\Big||G/\langle x \rangle| \implies$ by the induction hypothesis that $G/\langle x \rangle$ contains a subgroup $T$ of order $p^{k-1}$.
By Correspondence Theorem:

$$\underbrace{T \subseteq G/\langle x \rangle}_{\text{subgroups}} \longleftrightarrow \underbrace{\langle x \rangle \subseteq H \subseteq G}_{\text{subgroups containing } \langle x \rangle}$$

where $T = H/\langle x \rangle$.
Thus,

$$p^{k-1} = |H/\langle x \rangle| = \frac{|H|}{|\langle x \rangle|} = \frac{|H|}{p}$$

$\implies |H| = p^k$. Then $H \subseteq G$ is a subgroup of order $p^k$. □

**Corollary 24.1** (Cauchy's Theorem for finite groups)
Suppose $|G| < \infty$ with $p\Big||G|$, then $\exists g \in G$ such that $|g| = p$.

**Proof.** Immediate from First Sylow Theorem by taking $k = 1$. □

**Definition 24.2** (Sylow-$p$-subgroup)
If $|G| < \infty$ and $p$ is prime, a subgroup $H \subseteq G$ with $|H| = p^n$ is <u>Sylow-$p$-subgroup</u> if $n$ is the largest positive integer such that $p^n\Big||G|$. This subgroup always exists by First Sylow Theorem.

**Example 24.3**
Consider $|S_8| = 8! = 2^7 \cdot 3^2 \cdot 5 \cdot 7$.
First Sylow Theorem guarantees there exists subgroups of order:

- $2, 2^2, 2^3, 2^4, \dots, 2^7$
- $3, 3^2$
- $5$
- $7$

Where

- $2^7$ are Sylow 2 subgroups
- $3^2$ are Sylow 3 subgroups
- $5$ are Sylow 5-subgroups
- $7$ are Sylow 7-subgroups

**Note 24.4:** First Sylow Theorem does not guarantee uniqueness.

**Example 24.5**
Consider $|S_3| = 3! = 3 \cdot 2$
$S_3$ contains subgroups of orders

- $2$
- $3$

Since $S_3$ is very small, then every nontrivial subgroup is a Sylow-$p$-subgroup.

Goal: Play a similar conjugation game with sets instead of elements.

**Proposition 24.6**
Suppose $H$ is a Sylow-$p$-subgroup of $G$. Then

$$g^{-1}Hg \coloneqq \{g^{-1}hg \in G \mid h \in H\}$$

is also a Sylow $p$-subgroup of $G, \forall g \in G$.
So $|g^{-1}Hg| = p^n$.

**Proof.** Recall

$$\begin{aligned} \varphi_g \colon G &\to G \\ x &\mapsto g^{-1}xg \end{aligned}$$

$\varphi_g$ is an isomorphism, that takes

$$\varphi_g(H) = g^{-1}Hg \implies H \cong g^{-1}Hg$$

□

# 25 Mar 4, 2022

## 25.1 $K$-conjugacy and Normalizers

**Definition 25.1** ($K$-conjugate, conjugate to)
For a fixed subgroup $K \subseteq G$, we say two subgroups $H_1$ and $H_2$ are $K$-conjugate if there exists $k \in K$ such that

$$H_1 = k^{-1} H_2 k$$

If $K = G$, then we say $H_1$ is conjugate to $H_2$.

**Theorem 25.2**
Given any subgroup $K \subseteq G$, $K$-conjugacy is an equivalence relation on subgroups of $G$.

**Proof.** Exercise. □

**Definition 25.3** (Normalizer)
The normalizer of a subgroup $A \subseteq G$ is the set

$$N(A) := \{g \in G \mid g^{-1} A g = A\}$$

**Remark 25.4** $N(A)$ is basically the centralizer

$$C(a) = \{g \in G \mid g^{-1} a g = a\}$$

for subgroups instead of elements.

**Note 25.5:** $N(A)$ is the set of elements $g \in G$ with respect to which $A$ is normal.

**Theorem 25.6**
For all subgroups $A \subseteq G$:

1. $N(A)$ is a subgroup of G.
2. $A \lhd N(A)$.

**Proof.** Follows directly from the definition of $N(A)$. □

**Definition 25.7**
Let $[A]_H$ denote the class of all subgroups of $G$ that are $H$-conjugate to $A$, i.e. $[A]_H =$ equivalence class of $A$ under $H$-conjugation.

**Theorem 25.8**
Suppose $|G| < \infty$, and $H$ is a fixed subgroup of $G$.

$$\Big|[A]_H\Big| = [H\colon H \cap N(A)]$$

Compare it with $\Big|[a]\Big| = [G\colon C(a)]$

**Proof.** Proof is analogous to the proof of $\Big|[a]\Big| = [G\colon C(a)]$ in the case of elements instead of subgroups. □

**Lemma 25.9**
Suppose $Q$ is a sylow $p$-subgroup of $G$ with $|G| < \infty$. If $x \in G$ with $|x| = p^r$ for some $r \in \mathbb{N}$, and $x^{-1}Qx = Q \implies x \in Q$.

**Proof.** If $x^{-1}Qx = Q \implies x \in N(Q)$.

$$Q \lhd N(Q) \implies N(Q)/Q \text{ is well defined}$$

Consider $Qx \in N(Q)/Q$

$$|x| = p^r \implies |Qx| = p^r$$

Consider

$$\langle Qx\rangle \subseteq N(Q)/Q \text{ with } |\langle Qx\rangle| = p^r$$

By Correspondence Theorem $\implies$ there exists subgroup $Q \subseteq H \subseteq N(Q)$ such that $H/Q = \langle Qx\rangle$. By Lagrange

$$\implies |H| = |\langle Qx\rangle| \cdot |Q| = p^r p^n = p^{n+r}$$

which contradicts $Q$ being a Sylow $p$-subgroup.

$$r = 0 \implies |\langle Qx\rangle| = p^0 = 1 \implies |H|/|Q| = 1 \implies |H| = |Q| \implies Q \subseteq H \implies H = Q$$

And

$$\langle Qx\rangle = H/Q = \langle e\rangle = Q \implies Qx = Q \implies x \in Q$$

□

## 25.2 Second-Sylow Theorem

**Theorem 25.10** (Second-Sylow Theorem)
Suppose $|G| < \infty$. If $K$ and $P$ are two Sylow-$p$-subgroups of $G$, then there exists $x \in G$ such that $P = x^{-1}Kx$. Hence any two Sylow $p$-subgroups are isomorphic.

**Proof.** Suppose $|K| = |P| = p^n$ where $p\Big||G|$. Let $K_1, \ldots, K_t$ denote distinct conjugates of $K$, so $|K_i| = p^n$ for all $i$. By previous theorem $t = [G\colon N(K)]$. Since $K_i = x^{-1}Kx$ and

$K_j = y^{-1}Ky$ for some $x, y \in G$

$$\implies K_j = y^{-1}(xK_i x^{-1})y = (x^{-1}y)^{-1}K_i(x^{-1}y)$$

$\implies$ every $K_j$ is conjugate to $K_i$.
We will continue the proof in the next lecture. $\square$

# 26 Mar 7, 2022

## 26.1 Second-Sylow Theorem (Cont'd)

**Second Sylow Theorem Proof (Cont'd).** Restrict to conjugation by $P \subseteq G$. Conjugation by $P$ divides the set $\{K_1, \ldots, K_t\}$ into equivalence classes $[K_i]_p$ with $|[K_i]_p| = [P \colon P \cap N(K_i)]$.
$\implies |[K_i]_p| = p^k$ for some $k \in N$.
Since

$$t = \sum_{i \in I} |[K_i]_p|$$

If $|[K_i]_p| = p^k$ with $k > 0$ for all $i \implies p \mid t$.
But $K \subseteq N(K) \implies p^n \Big| |N(K)|$

$$|G| = |N(K)| \cdot \underbrace{[G \colon N(K)]}_{t} \implies p \nmid t \implies \text{contradiction}$$

Thus, there exists $j$ such that $|[K_j]_p| = p^0 = 1$

$$\implies [K_j]_p = K_j$$

$$\implies \forall x \in P \quad x^{-1} K_j x = K_j$$

Recall by a Lemma 25.9, $Q$ any Sylow $p$-subgroup, $x \in G, |x| = p^r \implies x^{-1}Qx = Q$.

$$\implies \forall x \in P \implies x \in K_j \implies P \subseteq K_j$$

But $|K_j| = |P| \implies P = K_j \implies P$ is conjugate to $K$. □

**Corollary 26.1**
Suppose $|G| < \infty$ and $K \subseteq G$ a Sylow $p$-subgroup. Then $K \lhd G$ if and only if $K$ is the only Sylow $p$-subgroup.

**Proof.** " $\impliedby$ " Suppose $|[K]_G| = 1$, so then $x^{-1}Kx = K$ for all $x \in G \implies Kx = xK$ for all $x \in G \implies K$ is normal.
" $\implies$ " If $K \lhd G$ by the Second Sylow Theorem if $P$ is any other Sylow $p$-subgroup $\implies \exists x \in G$ such that $x^{-1}Kx = P$.
But if $K \lhd G$ then $x^{-1}Kx = K \implies P = K$. □

## 26.2 Third Sylow Theorem

**Recall 26.2** First Sylow Theorem: (Existence) $|G| < \infty$ for all primes $p^k \Big| |G|$, $k$ maximal $\implies H \subseteq G$ such that $|H| = p^k$.
Second Sylow Theorem: (Uniqueness up to isomorphism) $|G| < \infty$, any two Sylow $p$-subgroups are isomorphic.

**Theorem 26.3** (Third Sylow Theorem)
Suppose $|G| < \infty$, $p \Big| |G|$. Let $t =$ number of Sylow $p$-subgroups of $G$. Then:

1. $t \Big| |G|$.
2. $t = 1 + ps$ for some $s \in \mathbb{Z}_{\geq 0}$.

**Proof.**

1. Let $K$ be any Sylow $p$-subgroup. Second Sylow Theorem $\implies$ $P$ is any other Sylow $p$-subgroup then $K$ and $P$ are conjugate

$$P \in [K]_G \implies t = [K]_G$$

$$t = [K]_G = [G \colon N(K)] \Big| |G| \implies t \Big| |G|.$$

2. Consider all Sylow $p$-subgroups under conjugation by $P$

$$\implies [p]_p = p$$

By proof of Second Sylow Theorem,

$$|[K_i]_p| = p^r \quad r > 0$$

All Sylow $p$-subgroups $K_i \neq P$.

$$\begin{aligned} \implies t &= |[P]_P| + \sum_{K_i \neq P} \Big|[K_i]_P\Big| \\ &= 1 + sp \end{aligned}$$

$\square$

## 26.3 Applications of Sylow Theorems

**Note 26.4:** When $t = 1$, the Sylow $p$-subgroup is the only one! And thus is normal $\implies$ $G$ is not simple.

**Example 26.5**
$|G| = 63 = 3^2 \cdot 7$; is $G$ simple?
Let $p = 7$; By First Sylow Theorem $\implies \exists P \subseteq G$ such that $|P| = 7$.
Let $t =$ number of Sylow 7 subgroups.
By Third Sylow Theorem: $t|63$ and $t = 1 + 7s$

$$t = \{1, 3, 7, 9, 21, 63\} \cap \{1, 8, 15, 22, 29, 36, 43, 50, 57, 64\}$$

$$\implies t = 1 \implies P \lhd G \implies G \text{ is not simple.}$$

What if we chose $p = 3$ instead of 7?

$$p = 3 \implies |H| = 3^2$$

$$t_3|63 \quad t_3 = 1 + 3s$$

$$t = \{1, 3, 7, 9, 21, 63\} \cap \{1, 4, 7, \ldots, 61\}$$

this actually needs more work.

# 27 Mar 9, 2022

## 27.1 Application of Sylow Theorems (Cont'd)

**Example 27.1**
Suppose $|G| = 56 = 2^3 \cdot 7$.
Let $p = 7$
Recall $t_7$ = number of Sylow 7 subgroups, so

- $t_7 \mid 56$
- $t_7 = 1 + 7s$

If $t_7 = 1 \implies G$ is not simple. So

$$t_7 = \{1, 2, 4, 7, 8, 14, 28, 56\} \cap \{1, 8, 15, 22, 29, 36, 43, 50, 57\}$$

$\implies t_7 = 1$ or 8.
If $t_7 = 1$: There exists a unique $|P| = 7$ such that $P \lhd G$.
If $t_7 = 8$: This means there exists 8 distinct Sylow 7 subgroups.
Since each Sylow 7 subgroup has 6 non identity elements so there exists $8 \cdot 6 = 48$ distinct non identity elements in $G$, each of order 7. However, since $2^3 \Big| |G| \implies G$ contains at least one Sylow 2-subgroup. Therefore, $G$ has at least $8 - 1 = 7$ elements of order 2, 4, 8.
If this is the only Sylow 2-subgroup, then $G$ is not simple.
If $G$ had more than one Sylow 2-subgroup, then $G$ has at least 14 elements of orders 2, 4, 8. But $48 + 14 + 1 > 56$.
If $t_7 = 8$ and $G$ has 8 Sylow 7-subgroups, then $G$ doesn't have room for more than one Sylow 2-subgroup. So $G$ is not simple.
If instead I choose $p = 2$.

$$t_2 = \{1, 2, \dots, 56\} \cap \{1, 3, 5, 7, \dots\}$$

So $t_2 = \{1, 7\}$.
If $t_2 = 1$, done.
If $t_2 = 7$, then $G$ has $(8 - 1) \cdot 7 = 49$ elements of order $2, 4, 8$, so $56 - 49 = 7$, so we have 6 leftover nonidentity elements, so there is only room for one Sylow 7-subgroup.

**Corollary 27.2**
Suppose $|G| = pq$ with $p, q$ as primes and $p > q$. If $q \nmid (p - 1)$, then $G \cong \mathbb{Z}_{pq}$.

**Proof.** By the Third Sylow Theorem $p$-subgroups.

$$t_p = \{1, q, p, pq\} \cap \{1 + ps \mid s \in \mathbb{Z}_{\geq 0}\}$$

- $q < p$ and $q$ is prime then $q \neq 1 + ps \implies t_p \neq q$.

- $p \neq 1 + ps$ for all $s$, then $t_p \neq p$.
- $pq = 1 + ps$, then $(q - s)p = 1 \implies 1 \equiv 0 \mod p$, a contradiction

$\implies t_p = 1 \implies \exists! P \subseteq G$ such that $|P| = p \implies P \lhd G$.
Now let's do it again with $q$-sylows.

$$t_q = \{1, q, p, pq\} \cap \{1 + qs \mid s \in \mathbb{Z}_{\geq 0}\}$$

- $q \neq 1 + qs \implies t_q \neq q$.
- $p = 1 + qs \implies p - 1 = qs \implies q | p - 1$, a contradiction.
- $pq = 1 + qs \implies q(p - s) = 1 \implies 1 \equiv 0 \mod q$, a contradiction.

$\implies t_q = 1 \implies \exists! Q \subseteq G$ such that $|Q| = q$, so $Q \lhd G$.
Thus, $|G| = |P| \cdot |Q| = |PQ| = pq$. Since $P \cap Q = \{e\}$, because $(p, q) = 1$. So,

$$G = P \times Q \cong \mathbb{Z}_p \times \mathbb{Z}_q \cong \mathbb{Z}_{pq}$$

□

## 27.2 Structure of Finite Groups

For the rest of the notes, suppose $p$ is prime.

**Theorem 27.3**
Suppose $|G| = p^n$ for $n \geq 1$. Then $|Z(G)| = p^k$ for $1 \leq k \leq n$, i.e. $k \neq 0$, so $Z(G)$ is nontrivial.

**Proof.** By Lagrange $Z(G) \subseteq G \implies |Z(G)| = p^k$ where $0 \leq k \leq n$. From Class equation,

$$|G| = Z(G) + \sum_{a_i \notin Z} |C(a_i)|$$

since $|C(a_i)| \Big| |G| \implies |C(a_i)| = p^{\ell_i} \neq 1$ where $\ell_i > 0$.

$$\implies |Z(G)| = |G| - \sum_{a_i \notin Z} |C(a_i)| \text{ divisible by } p$$

$$\implies |Z(G)| = p^k \quad k > 0$$

□

**Corollary 27.4**
Suppose $|G| = p^n$, and $n > 1$. Then $G$ is not simple.

**Proof.** By previous theorem, $Z(G)$ is nontrivial.
If $Z(G) \neq G$, then $Z(G) \lhd G \implies G$ is not simple.

If $Z(G) = G \implies G$ is abelian, then by Cauchy's Theorem for Finite Abelian groups $\implies G$ is not simple. □

**Corollary 27.5**
Suppose $|G| = p^2$, then $G$ is abelian. Thus $G \cong \mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \times \mathbb{Z}_p$.

**Proof.** $Z(G) \subseteq G \implies |Z(G)| = p$ or $|Z(G)| = p^2$.

- If $|Z(G)| = p^2 \implies Z(G) = G \implies G$ is abelian.
- If $|Z(G)| = p \implies Z(G) \lhd G \implies |G/Z(G)| = p \implies G/Z(G) \cong \mathbb{Z}_p$ is cyclic $\implies G$ is abelian by previous theorem.

□

# 28 Mar 11, 2022

## 28.1 Structure of Finite Groups (Cont'd)

**Theorem 28.1**
Suppose $p, q$ are distinct primes, and $q \not\equiv 1 \mod p$ and $p^2 \not\equiv 1 \mod q$. If $|G| = p^2 q$, then $G \cong \mathbb{Z}_{p^2 q}$ or $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q$. (This implies that $G$ is cyclic and/or abelian).

**Proof.** $t_p$ = number of Sylow $p$-subgroups. By Third Sylow Theorem, $t_p \in \{1, q, p, qp, p^2, qp^2\}$ and $t_p = 1 + sp$ for some $s$ ($t_p \equiv 1 \mod p$).
Since $q \not\equiv 1 \mod p$ then $t_p \neq q, qp, p^2, qp^2, p \implies t_p = 1$. So the Sylow $p$-subgroup, $P$ is unique $\implies P \lhd G$ with $|P| = p^2$. By Corollary 27.5, $P \cong \mathbb{Z}_{p^2}$ or $\mathbb{Z}_p \times \mathbb{Z}_p$.
Repeat for $q$: $t_q \in \{1, q, p, pq, p^2, qp^2\}$ and $t_q \equiv 1 \mod q$, $p^2 \not\equiv 1 \mod q \implies t_q \neq p^2 \implies t_q \in \{1, p\}$.
If $t_q = p \implies p \equiv 1 \mod q \implies p^2 = 1 \mod q \implies p \neq 1 \mod q$. So $t_q = 1$.
Then $q$-Sylow subgroup is unique and hence normal. There exists $Q \lhd G$ with $|Q| = q \implies Q \cong \mathbb{Z}_q$.
We want to show $G \cong P \times Q$.
We need to show $G = PQ$ and $P \cap Q = e$.
Since $(p, q) = 1 \implies P \cap Q = e$.
Since $PG \subseteq G$ and $|PQ| = |P| \cdot |Q| = p^2 q = |G| \implies PQ = G$.

$$\implies G \cong P \times Q \cong \begin{cases} \mathbb{Z}_{p^2} \times \mathbb{Z}_q \\ \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q \end{cases}$$

$\square$

**Example 28.2**
Suppose $|G| = 2^2 \cdot 11$.
By Theorem 28.1, $p = 2, q = 11$. Check that $11 \equiv 1 \mod 2$ and $4 \not\equiv 1 \mod 11$, so we can't use the theorem.

**Example 28.3**
Suppose $|G| = 5^2 \cdot 7$, so $p = 5$ and $q = 7$.
So $7 \not\equiv 1 \mod 5$ and $25 \not\equiv 1 \mod 7$, so by Theorem 28.1, $G \cong \mathbb{Z}_{25} \times \mathbb{Z}_7$ or $\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_7$.

**Corollary 28.4**
Suppose $p, q$ are distinct primes. Then $|G| = p^2 q \implies G$ is not simple.

**Proof.** $t_q, t_p \in \{1, q, p, qp, p^2, qp^2\}$.
If $q \not\equiv 1 \mod p \implies P \lhd G \implies G$ is not simple.
If $p^2 \not\equiv 1 \mod q \implies Q \lhd G \implies G$ is not simple.
Suppose $q \equiv 1 \mod p$ and $p^2 \equiv 1 \mod q$.
$\implies q - 1 \equiv 0 \mod p$ and $p^2 - 1 \equiv 0 \mod q$.

$\implies p \mid (q-1)$ and $q \mid (q^2-1)$
$\implies p \leq q-1$, either $q \mid (p-1)$ or $q \mid (p+1)$. But

$$\underbrace{q \leq p-1 < p+1 \leq p}_{\text{contradiction}} \quad q \leq p+1 \leq q \implies p+1 = q$$

But $p$ and $q$ are primes and $p+1=q$, then if $q$ is odd $\implies p$ even $\implies p=2, q=3$. It remains to show that $|G| = 2^2 \cdot 3$ is not simple. $\square$ Exercise

Using these tools, you can classify all groups of order $\leq 15$.