

Math 110B (Algebra)

University of California, Los Angeles

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These are my lecture notes for Math 110B (Algebra), which is the second course in Algebra taught by Nicolle Gonzales. The textbook for this class is *Abstract Algebra: An Introduction, 3rd edition* by Hungerford.

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1 Jan 3, 2022

1.1 Groups

- Algebra \rightarrow study of mathematical structure.
- Rings \leftrightarrow “numbers” e.g. $\mathbb{R}, \mathbb{Z}, \mathbb{C}, \mathbb{Z}_p$
2 operations $(+, \cdot)$

Question 1.1: What happens if we have only 1 operation (either \cdot or $+$ but not both)?
What kind of structure is this more basic setup?

Answer: Groups! It turns out groups encode the mathematical structures of the symmetries in nature.

Definition 1.2 (Group)

A group $(G, *)$ is a nonempty set with a binary operation $*$: $G \times G \rightarrow G$ that satisfies

1. (Closure): $a * b \in G \quad \forall a, b \in G$
2. (Associativity): $(a * b) * c = a * (b * c) \quad \forall a, b, c \in G$
3. (Identity): $\exists e \in G$ such that $e * a = a = a * e \quad \forall a \in G$
4. (Inverse): $\forall a \in G, \exists d \in G$ such that $d * a = e = a * d$

Note:

- If $*$ is addition, we just divide $*$ by the usual $+$ sign. In this case

$$e = 0 \quad \text{and} \quad d = -a$$

- If the operation $*$ is multiplication, we just divide $*$ by the usual \cdot sign. In this case

$$e = 1 \quad \text{and} \quad d = a^{-1}$$

- Be aware that sometimes $*$ is neither.

Definition 1.3 (Abelian)

If the $*$ operation is commutative, i.e. $a * b = b * a$, then we say that G is abelian (named after the mathematician N.H. Abel)

Definition 1.4 (Order, Finite Group vs. Infinite Group)

The order of a group G , denoted $|G|$, is the number of elements it contains (as a set).
Thus, G is a finite group if $|G| < \infty$
and G is an infinite group if $|G| = \infty$

Examples 1.5 (Examples of a group)

1. Rings where you “forget” multiplication.
 $\rightarrow (\mathbb{Z}, +)$ integers with $*$ = $+$, $(\mathbb{R}[X], +)$, etc.
Note: $(\mathbb{Z}, *)$ with $*$ = \cdot is not a group. Why?

Theorem 1.6

Every ring is an abelian group under addition.

Proof. $e = 0$, inverse = $-a$ for each $a \in R$. □

Fact: If $R \neq 0$ then (R, \cdot) is never a group since 0 has no multiplicative inverse.

Examples 1.7 (More examples of a group)

2. Fields without zero.

Theorem 1.8

Let \mathbb{F}^* denote the nonzero elements of a field \mathbb{F} . Then (\mathbb{F}^*, \cdot) is an abelian group.

Recall: A unit in a ring R is an element $a \in R$ with a multiplicative inverse $a^{-1} \in R$ such that $aa^{-1} = 1 = a^{-1}a$.

Theorem 1.9

The set of units \mathcal{U} inside a ring R is a group under multiplication.

Examples 1.10 (More examples of a group cont.)

3. $\mathcal{U}_n = \{m \mid (m, n) = 1\} \subseteq \mathbb{Z}_n$ is also a group, but under multiplication,
 $\underline{n=4}$ $\mathbb{Z}_4 = \{0, 1, 2, 3\}$, $\mathcal{U}_4 = \{1, 3\}$
 $(\mathbb{Z}_4, +)$ and (\mathcal{U}_4, \cdot) are groups with different binary operation!

$$\underline{n=6} \quad \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}, \quad \mathcal{U}_6 = \{1, 5\}$$

(\mathcal{U}_6, \cdot) is a group

- $1 \cdot 5 = 5 \pmod{6} \in \mathcal{U}_6$ (closure)
- $1 = e$ (identity)
- $1 \cdot 1 = 1, \quad 5 \cdot 5 = 25 \equiv 1 \pmod{6}$ (inverse)
- Associativity is clear

2 Jan 5, 2022

2.1 Groups (Cont'd)

Examples 2.1

4. $(M_{n \times m}(\mathbb{F}), +) = m \times n$ matrices over \mathbb{F} under addition
 e = zero matrix, inverse of a matrix $-M$

Definition 2.2 (General linear group)

Denote by $GL_n(\mathbb{F})$ the set of $n \times n$ invertible matrices under multiplication. ($\det(A) \neq 0 \quad \forall A \in GL_n$)

- Closed: $\det(A \cdot B) = \det(A) \cdot \det(B) \neq 0 \implies AB \in GL_n \quad \forall A, B \in GL_n$
- Associativity: Obvious.
- Identity: $\det(I) = 1 \neq 0 \implies I \in GL_n(\mathbb{F})$
- Inverse: $A \in GL_n; \det(A^{-1}) = \frac{1}{\det(A)} \neq 0 \implies A^{-1} \in GL_n(\mathbb{F})$

Fact: $GL_n(\mathbb{F})$ is a group for any field \mathbb{F} .

Comment:

- $\det(A + B) \neq \det(A) + \det(B)$
- $\det(AB) = \det(A) \cdot \det(B)$

Definition 2.3 (Special linear group)

Let $SL_n(\mathbb{F})$ denote the set of invertible matrices over \mathbb{F} with $\det = 1$

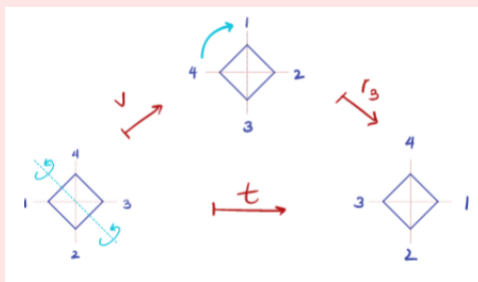
Exercise. Show that $SL_n(\mathbb{F})$ is a group.

2.2 Symmetries

Example 2.4 (Symmetries over a square)

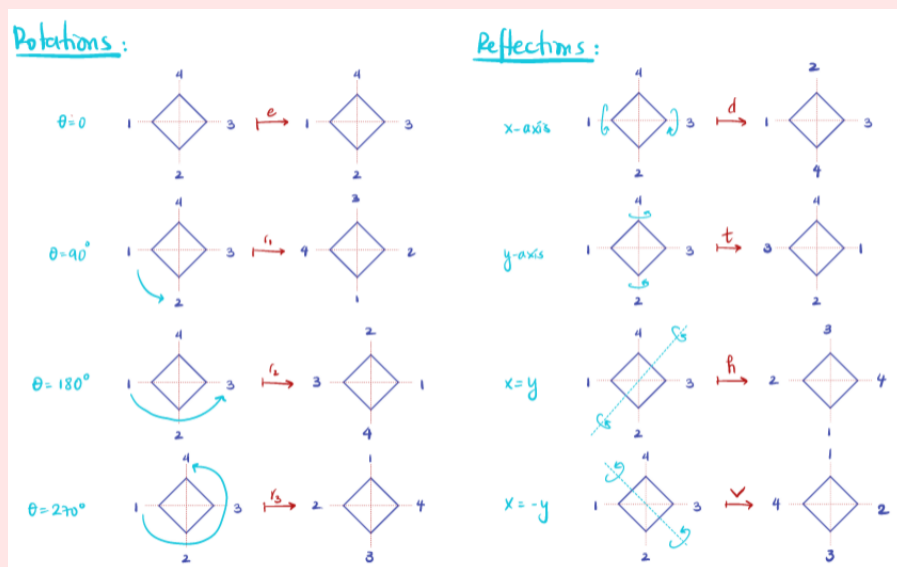
Rotations and reflection These operations (maps) form a group under composition. So $*$ = 0. For instance, suppose

$$r_3 \circ t = h$$



The group of rotations/reflections of a square is called Dihedral Group of degree 4, denoted D_4 .

$$D_4 = \{r_1, r_2, r_3, r_4, d, t, h, v \mid \text{under } \circ\}$$

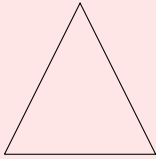


These are Professor Gonzales's lovely drawings.

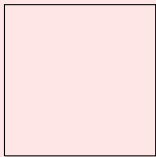
Example 2.5 (Symmetries of a regular polygon with n sides)

Called the dihedral groups of degree n , D_n .

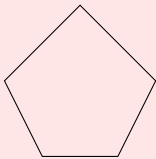
- $n=3$



- $n=4$



- $n=5$



- $n=6$

etc...

Observe: $|D_n| = 2n$ because you have n -axes of reflection and n -angles of notation.

Example 2.6 (The symmetric group)

Let $n \in \mathbb{N}$, and S_n be the set of all permutations of the numbers $\{1, \dots, n\}$.

Note: any permutation of $\{1, \dots, n\}$ can be thought of as a bijection $\{1, \dots, n\} \rightarrow \{1, \dots, n\}$.

\implies This allows us to compose permutations just like functions.

$\implies S_n$ is a group!

Definition 2.7 (Symmetric group)

The symmetric group S_n is the group of permutations of the integers of the integers $\{1, \dots, n\}$.

Given any permutation $\sigma \in S_n$,

$$\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\},$$

$$i \mapsto \sigma_i$$

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_{n-1} & \sigma_n \end{pmatrix} \rightarrow e = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}$$

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma_1^{-1} & \sigma_2^{-1} & \cdots & \sigma_n^{-1} \end{pmatrix}$$

Group operation: function composition.

Example 2.8

n=2:

$$e = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \tau = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$\tau \circ \tau = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = e$$

$$\tau \circ e = \tau$$

$$e \circ \tau = \tau$$

$$e \circ e = e$$

$$\implies S_2 = \{e, \tau\} \text{ is a group}$$

$$e^{-1} = e$$

$$\tau^{-1} = \tau$$

Associativity: obvious because of function composition

3 Jan 7, 2022

3.1 Symmetries (Cont'd)

Example 3.1

$n=3$ S_3 : permutations of $\{1, 2, 3\}$

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \tau_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \tau_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$\tau_{21} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \tau_{12} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \tau_{121} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

So,

$$\begin{aligned} \tau_1 \circ \tau_2 \circ \tau_1 &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \tau_{121} \end{aligned}$$

Note: $\tau_{21} = \tau_2 \circ \tau_1$, $\tau_{12} = \tau_1 \circ \tau_2$

$\tau_{21} \neq \tau_{12} \implies S_3$ is not abelian!

Exercise. τ_{212} ?

3.2 Direct Product of Groups

Definition 3.2 (Direct product)

Given $(G, *)$, (H, \star) both groups define the binary operation:

$$\begin{aligned} \square: (G \times H) \times (G \times H) &\rightarrow G \times H \\ (g, h) \square (g', h') &\mapsto (g * g', h \star h') \end{aligned}$$

Side note: (S, \odot)

$\odot: S \times S \rightarrow S \implies S$ group

Example 3.3

$S_2 \times D_4$:

$$(\tau_1, r_{270^\circ}) \square (\tau_1, v) = (\tau_1 \circ \tau_1, r_{270^\circ} v) = (e, t)$$

Example 3.4

$(\mathbb{R}, +) \times (\mathbb{R}^*, \cdot)$

$$(5, 2) \square (-5, \pi) = (0, 2\pi)$$

Example 3.5
 $\mathbb{Z}_n \times \mathbb{Z}_m \quad n, m \in \mathbb{N}.$

$$(a, b) \square (a', b') = (\underbrace{a + a'}_{\text{mod } n}, \underbrace{b + b'}_{\text{mod } m})$$

$$\begin{aligned} (5, 5) \square (2, 2) &= (5 + 2, 5 + 2) \\ &= (7, 1) \end{aligned}$$

3.3 Properties of Groups

Notation: Going forward, we omit $*$ in the notation: $(G, *) \rightarrow G$. Use multiplicative notation for abstract groups. Instead $a * b \rightarrow ab$.

$$\underbrace{a * a * a * a \cdots * a}_{n \text{ times}} \rightarrow a^n$$

However, for very explicit groups like

$(\mathbb{Z}, +), (\mathbb{R}, +), (\mathbb{Z}_n, +)$, etc, we use additive notation. ($*$ = +)

$$a * b \rightarrow a + b$$

$$\underbrace{a * \cdots * a}_{n \text{ times}} \rightarrow n \cdot a$$

(Review notation on page 198 of book)

Theorem 3.6

G group, $a, b, c \in G$. Then

1. $e \in G$ is unique
2. if $ab = ac$ or $ba = ca \implies b = c$
3. $\forall a \in G : a^{-1}$ is unique.

Proof.

1. Suppose $\exists e' \in G$ s.t $e \neq e'$ but $e'a = a = ae' \forall a \in G$. \implies let $a = e \implies e'e = e = ee'$

On the other hand $e \cdot e' = e' = e'e$

$$\implies e = e'$$

2. $ab = ac, \quad a, b, c \in G$.

Since $a^{-1} \in G$

$$\implies \underbrace{a^{-1}a}_e b = \underbrace{a^{-1}a}_e c$$

$$\implies e \cdot b = e \cdot c$$

$$\implies b = c$$

3. Suppose $a \in G \exists$ two distinct inverses.

$$d_1, d_2 \in G.$$

$$d_1 a = e = a d_1$$

$$d_2 a = e = a d_2$$

$$\implies d_1 = d_1 e = d_1 a d_2 = e \cdot d_2 = d_2$$

□

Corollary 3.7

G group, $a, b \in G$. Then

$$1. (ab)^{-1} = b^{-1}a^{-1}$$

$$2. (a^{-1})^{-1} = a$$

Proof. Exercise. □

Note: $ab = ba$ (G is abelian)

$$\implies (ab)^{-1} = a^{-1}b^{-1} = b^{-1}a^{-1}$$

Generally: $ab \neq ba \implies a^{-1}b^{-1} \neq b^{-1}a^{-1}$

3.4 Order of an Element

Definition 3.8 (Order (of an element) and Finite vs. Infinite order)

The order of an element $a \in G$ is the smallest $k \in \mathbb{N}$ such that $a^k = e$. We denote this by $|a|$.

If k is finite $\implies a$ has finite order.

If k is infinite $\implies a$ has infinite order.

Example 3.9

$$S_2; e, \tau_1 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

Notice,

$$|e| = 1; e^1 = e$$

$$|\tau_1| = 2 \quad \tau_1^2 = \tau_1 \circ \tau_1 = e$$

$$\tau_1^4 = \tau_1^2 \circ \tau_1^2 = e \circ e = e$$

Example 3.10 $\mathbb{Z} \leftarrow e = 0.$ $|1| = ?$ $1 \cdot n = 0$ for which n ?None, so $\implies |1| = \infty$

4 Jan 10, 2022

4.1 Order of an Element (Cont'd)

Theorem 4.1

G -group, $a \in G$

1. If $|a| = \infty$, then $a^i \neq a^j$ for any $i, j \in \mathbb{Z}$ with $i \neq j$.
2. If $\exists i \neq j$ such that $a^i = a^j \implies |a| < \infty$.

Proof. We prove (2) (because $1 \iff 2$).

WLOG suppose $i > j$, then if $a^i = a^j \implies a^{i-j} = a^i a^{-j} \implies a^j a^{-j} = a^0 = e \implies |a| \leq i - j < \infty$ □

Theorem 4.2

G group, $a \in G \quad |a| = n$

1. $a^k = e \iff n \mid k \quad (n \leq k)$
2. $a^i = a^j \iff i \equiv j \pmod{n}$
3. if $n = td \quad d \geq 1 \implies |a^t| = d$.

Proof.

1. If $a^k = e$ and since $a^n = e$ with n -smallest such integer, then $k > n$, and so $k = nd + r$ with $0 \leq r < n$

$$a^k = a^{nd+r} = (a^n)^d a^r = e^d a^r = a^r$$

$$\begin{aligned} \text{If } 0 < r < n &\implies a^r \neq e \implies a^k \neq e \\ &\implies r = 0 \implies k = nd \implies n \mid k. \end{aligned}$$

2. If $a^i = a^j \implies a^{i-j} = e$

$$\begin{aligned} &\implies n \mid i - j \text{ by (1).} \\ &\implies i - j \equiv 0 \pmod{n} \\ &\implies i \equiv j \pmod{n} \end{aligned}$$

3. If $n = td \quad (d \geq 1) \stackrel{?}{\implies} |a^t| = d$

$$\text{Since } a^n = e \implies (a^t)^d = e \implies |a^t| \leq d.$$

$$\text{If } |a^t| = k < d \implies (a^t)^k = a^{tk} = e$$

But $tk < td = n \implies a^{tk} = e$ for $tk < n \implies \neq$ because n is the smallest positive integer such that $a^n = e$.

$$\implies k = d \implies |a^t| = d.$$

□

Corollary 4.3

G - abelian group with $|a| < \infty \quad \forall a \in G$. Suppose $c \in G$ such that $|a| \leq |c| \quad \forall a \in G$. Then $|a| \mid |c|$.

Proof. Suppose not. \exists some $a \in G$ such that $|a| \nmid |c|$. Consider prime factorizations of $|a|$ and $|c|$.

\implies Then \exists some prime p such that $|a| = p^r m \quad |c| = p^s n$ where $r > s$ (s might be zero) and $(p_1 m) = 1 = (p_1 n)$.

Then by (3) of Theorem 4.2,

$$|a^m| = p^r \quad \text{and} \quad |c^{p^s}| = n$$

$$\xRightarrow{\text{because } (p^r, n)=1} \underbrace{|a^m \cdot c^{p^s}|}_{\in G} = p^r \cdot n$$

Note: $|a| = n, |b| = m, |a \cdot b| \neq n \cdot m$ unless $(n, m) = 1$

Recall: $|c| = p^s \cdot n$ where $s < r$

$$\implies p^r > p^s$$

$$\implies p^r n > p^s n$$

$$\implies |a^m \cdot c^{p^s}| > |c|$$

$\implies \neq$ because c is the element in G with maximal order! So $a^m c^{p^s} \in G$ cannot have order larger than c . \square

4.2 Subgroups

Definition 4.4 (Subgroup)

A subset $H \subseteq G$ is a subgroup of $(G, *)$ if it is also a group under $*$.

Note:

$G \subseteq G \implies G$ is always a subgroup of itself (Improper subgroup)

$\{e\} \subseteq G \implies \{e\}$ is always a subgroup of G (Trivial subgroup of G)

\implies Any subgroup $e \neq H \neq G$ is called a nontrivial proper subgroup.

Examples 4.5

- $(\mathbb{Z}, +) \subseteq (\mathbb{Q}, +)$
- $\{e, r_{90}, r_{180}, r_{270}\} \subseteq D_4$
- $SL_n(\mathbb{F}) \subseteq GL_n(\mathbb{F})$

Note: any subgroup always contains e .

Theorem 4.6

A nonempty subset H of G is a subgroup if:

1. $ab \in H \quad \forall a, b \in H$
2. $a^{-1} \in H \quad \forall a \in H$

Proof. Since $H \neq \emptyset \quad \exists a \in H$. By (2), $\exists a^{-1} \in H \implies$ By (1) $aa^{-1} = e \in H \implies e \in H$. \square

Theorem 4.7

Any closed nonempty finite subset H of G is a subgroup.

Proof. By Theorem 4.6, we need only show that H contains inverses.

If $a \in H \quad a^k \in H \quad \forall k \in \mathbb{Z}$.

Since H is finite, not all a^k can be distinct.

$\implies |a| = n < \infty$ for some $n \in \mathbb{N}$.

$\implies a^n = e$

$\implies a^{n-1} \cdot a = e = a \cdot a^{n-1}$

$\implies a^{n-1} = a^{-1}$

If $n > 1 \implies a^{-1} \in H$

If $n = 1 \implies a^{-1} = e \implies a = e \implies a^{-1} = e \in H$. \square

5 Jan 12, 2022

5.1 Subgroups (Cont'd)

Example 5.1

$\mathbb{Z}_5 \leftarrow$ group under addition = $\{0, 1, 2, 3, 4\}$

Units of \mathbb{Z}_5 : $\mathcal{U}_5 = \{1, 2, 3, 4\}$

Clearly, $\mathcal{U}_5 \subseteq \mathbb{Z}_5$

Question: Is \mathcal{U}_5 a subgroup of \mathbb{Z}_5

No, because \mathcal{U}_5 is a group under multiplication.

Example 5.2

S_3 : set of permutations that fix 1.

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad \tau_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$\left. \begin{array}{l} \tau_2 e = \tau_2 = e \tau_2 \\ \tau_2 \cdot \tau_2 = e \end{array} \right\} \implies \underbrace{\{e, \tau_2\}}_H \text{ is closed.}$$

By Theorem 4.7, H is a subgroup because H is finite, nonempty, and closed.

5.2 Center of a Group

Definition 5.3 (Center of a group)

The center of a group G is the subset

$$Z(G) := \{a \in G \mid ag = ga \quad \forall g \in G\}$$

Note 5.4: When G is abelian $\implies Z(G) = G$

Question 5.5: Is $Z(G) = \emptyset$? No, because $e \in Z(G)$

Examples 5.6

- $Z(S_n) = e$

- $Z(D_4) = \{e, r_{180}\}$

- $Z(GL_n) = \{aI \mid a \in \mathbb{F}\}$ $\begin{pmatrix} a & & 0 \\ & \ddots & \\ 0 & & a \end{pmatrix}$

- $Z(SL_n) = \{I\} = e$

Theorem 5.7

$Z(G)$ is a subgroup of G .

Proof. By Theorem 4.6, since $Z(G) \neq \emptyset$, we need only show closure and inverses.

1. $a, b \in Z(G) \xRightarrow{?} ab \in Z(G), \forall g \in G$.

$$(ab)g \stackrel{b/c \ g \in Z(G)}{=} a(gb) \stackrel{\text{by assoc.}}{=} (ag)b \stackrel{a \in Z(G)}{=} (ga)b = g(ab)$$

$$\implies ab \in Z(G)$$
2. $a \in Z(G), ag = ga \quad \forall g \in G$.

$$\implies a^{-1}(ag)a^{-1} = a^{-1}(ga)a^{-1}$$

$$\implies ga^{-1} = a^{-1}g \implies a^{-1} \in Z(G)$$

□

5.3 Cyclic Group

Definition 5.8 (Cyclic group)

For any $a \in G$, the set

$$\langle a \rangle = \{a^n \mid n \in \mathbb{Z}\}$$

is a subgroup of G . We say $\langle a \rangle$ is the cyclic subgroup generated by a .

Note 5.9: Cyclic groups are always abelian.

If $G = \langle a \rangle$ for some $a \in G$, then G is a cyclic group.

Example 5.10

$$\langle r_{90} \rangle \subseteq D_4$$

$\langle r_{90} \rangle = \{e, r_{90}, r_{180}, r_{270}\} \leftarrow$ is a cyclic subgroup of G .

Note 5.11: In additive notation: $a * a = a + a$ (not $a \cdot a = a^2$)

$$\langle a \rangle = \{n \cdot a \mid n \in \mathbb{Z}\} \quad n \cdot a = \underbrace{a + a + \cdots + a}_{n \text{ times}}$$

$$a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ times}}$$

Example 5.12

$$(\mathbb{Z}, +) = \langle 1 \rangle = \langle -1 \rangle$$

Note 5.13: The generating element a is not unique.

Example 5.14

$$(\mathbb{Z}_3, +) = \langle 1 \rangle = \langle 2 \rangle$$

$\quad \quad \quad = -1$

Exercise. Which elements generate \mathbb{Z}_n for $n \in \mathbb{N}$?

Hint: Look at units (i.e. relatively prime) of \mathbb{Z}_n

Example 5.15

$$\mathbb{Z}_n = \langle 1 \rangle$$

\implies All \mathbb{Z}_n are cyclic groups of order n

Theorem 5.16

Let $a \in G$

1. If $|a| = \infty$, then $\langle a \rangle = \{a^k \mid k \in \mathbb{Z}\}$ is an infinite group.
2. If $|a| = n < \infty$, then $\langle a \rangle$ is a finite group. In fact, $\langle a \rangle = \langle e, a, a^2, a^3, \dots, a^{n-1} \rangle \implies |\langle a \rangle| = |a| = n$.

Proof (Sketch).

$$\begin{aligned} |a| = \infty &\implies a^i \neq a^j \text{ for } i \neq j \\ &\implies \{a^k \mid k \in \mathbb{Z}\} \implies \text{infinite set.} \\ |a| = n &\implies \langle a, a^2, \dots, a^{n-1}, a^n = e \rangle \end{aligned}$$

$$\text{Since: } a \cdot a^{n-1} = a^n = e = a^{n-1} \cdot a$$

$$\implies a^{n-1} = a^{-1}$$

$$a^2 a^{n-2} = a^n = e = a^{n-2} a^2$$

$$\implies a^{-2} = a^{n-2}$$

□

Theorem 5.17

Let \mathbb{F} be any field. Then any finite subgroup $G \subseteq \mathbb{F}^*$ is cyclic.

Recall 5.18 $\mathbb{F}^* = \mathbb{F} - \{0\}$ is a group under multiplication.

Proof. Since $|G| < \infty$, $\exists c \in G$ such that order of c is maximal ($|a| \leq |c| \quad \forall a \in G$). By Corollary 4.3, $\forall a \in G$, $|a| \mid |c|$ so if $|c| = m \implies a^m = 1$

Consider $p(x) = x^m - 1$. Since $p(a) = 0 \quad \forall a \in G$.

Since $p(x)$ has degree m it can have at most m solutions $\implies |G| \leq m$.

Since $|c| = m$ so $|\langle c \rangle| = m$.

$$\implies \langle c \rangle \subseteq G \implies \langle c \rangle = G.$$

$$\implies G \text{ is cyclic.}$$

□

6 Jan 14, 2022

6.1 Cyclic Group (Cont'd)

Recall 6.1 $a \in G$

$$\underbrace{\langle a \rangle}_{\text{cyclic group gen. by } a} := \{a^n \mid n \in \mathbb{Z}\} = \{\dots a^{-2}, a^{-1}, e, a, a^2, \dots\}$$

$G = \langle a \rangle \leftarrow G$ is cyclic group

Recall 6.2 Thm:

$$\begin{aligned} |a| = \infty &\rightarrow |\langle a \rangle| = \infty \\ |a| = n < \infty &\rightarrow |\langle a \rangle| = n \end{aligned}$$

Recall 6.3 \mathbb{F} -field, $G \subseteq \mathbb{F}^*$ if G finite $\implies G$ is cyclic. (G is any subgroup)

Theorem 6.4

Subgroups of cyclic groups are cyclic.

Proof. Suppose $G = \langle a \rangle$ and $H \subseteq G$. We want to show that $H = \underbrace{\langle b \rangle}_{b=a^j \text{ for some } j}$ for some $b \in G$.

If $H = e \implies H = \langle e \rangle$ we're done.

If $H \neq e$, then we can find k -smallest positive integer such that $a^k \in H$

Suppose $b \in H$. Then,

$$b = a^i \text{ for some } i \text{ then } i = kd + r \quad 0 \leq r < k.$$

$$\implies a^r = a^{i-kd} = b(a^k)^{-d} \in H \text{ by closure.}$$

If

$$r \neq 0 \implies \begin{cases} a^r \in H \\ a^k \in H \end{cases}$$

with $0 < r < k$ which is a contradiction because k was supposed to be smallest positive integer with $a^k \in H$.

$$\begin{aligned} \implies r = 0 &\implies b = a^i = a^{kd+r} = a^{kd} = (a^k)^d \\ &\implies b \in \langle a^k \rangle \\ &\implies H \subseteq \langle a^k \rangle \end{aligned}$$

Since $a^k \in H \implies \langle a^k \rangle \subseteq H$
 $\implies \langle a^k \rangle = H$

□

6.2 Generating Sets for Groups

Definition 6.5

Given a subset S of G , let $\langle S \rangle$ denote the set of all possible products of all elements of S and their inverses.

Note 6.6: $S \subseteq \langle S \rangle$

Example 6.7

$$\begin{aligned} a, b \in G, \quad S &= \{a, b\} \\ \langle S \rangle &= \langle a, b \rangle \\ &= \{a^n, b^m, a^n b^m, a^{n_1} b^{m_1} a^{n_2} b^{m_2}, b^m a^n, b^{m_1} a^{n_2} b^{m_2} a^{n_1}, \dots\} \\ &= \left\{ \prod_{i=0}^k a^{n_i} b^{m_i}, \prod_{i=0}^k b^{n_i} a^{m_i} \mid k \in \mathbb{N}, n_i, m_i \in \mathbb{Z} \right\} \end{aligned}$$

Theorem 6.8

S - any subset of G .

1. $\langle S \rangle$ is always a subgroup of G .
2. If H is any other subgroup of G such that $S \subseteq H \implies \langle S \rangle \subseteq H$.

Proof (Sketch).

1. Use the fact that very definition of $\langle S \rangle$ ensures closure and inverses $\implies \langle S \rangle$ is a subgroup.
2. Again follows from closure and inverses contained in H because H is a subgroup.

□

Definition 6.9 (Generators)

For any $S \subseteq G$, the group $\langle S \rangle$ is called the subgroup generated by S . If $G = \langle S \rangle$, then we call elements in S , the generators of G and S the generating set of G .

Example 6.10 (Symmetric group)

$$S_3 = \{e, \tau_1, \tau_2, \tau_{121}, \tau_{21}, \tau_{12}\}$$

$$\tau_{121} = \tau_1 \circ \tau_2 \circ \tau_1$$

$$\tau_{21} = \tau_2 \circ \tau_1$$

$$\tau_{12} = \tau_1 \circ \tau_2$$

$$e = \tau_1 \circ \tau_1 = \tau_2 \circ \tau_2$$

$$S_3 = \left\langle \underbrace{\tau_1}_{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}}, \underbrace{\tau_2}_{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}} \right\rangle$$

$$S_n \leftarrow \text{order } n!$$

$$S_n = \left\langle \underbrace{\tau_1}_{\text{flips } 1-2}, \underbrace{\tau_2}_{2-3}, \tau_3, \dots, \underbrace{\tau_{n-1}}_{\text{flips } n, n-1} \right\rangle$$

$$S_4 = \langle \tau_1, \tau_2, \tau_3 \rangle$$

$$S_5 = \langle \tau_1, \tau_2, \tau_3, \tau_4 \rangle$$

6.3 Isomorphisms and Homomorphisms

Definition 6.11 (Homomorphism (of groups))

G, H are groups. A homomorphism of groups is a map $\varphi: G \rightarrow H$ such that $\forall a, b \in G$

$$\varphi(\underbrace{ab}_{\text{ab prod in } G}) = \varphi(\underbrace{a}_{\text{prod in } G}) \cdot \varphi(\underbrace{b}_{\text{prod in } H})$$

Note 6.12: This means that the “multiplication” table for G is mapped onto “multiplication” table for H i.e. φ preserves group structures.

Note 6.13: $\varphi(a) = \varphi(e_G \cdot a) = \varphi(e_G)\varphi(a)$

$$\implies \varphi(e_G) = e_H$$

$$\implies \varphi \text{ takes identities to identities.}$$

Definition 6.14 (Isomorphism (of groups))

An isomorphism of groups G and H is a homomorphism of $\varphi: G \rightarrow H$ that is also a bijection, i.e. an isomorphism is an invertible homomorphism.

If G is isomorphic to H , then $G \cong H$, which is the same as writing $\exists \varphi: G \rightarrow H$ with φ one-to-one and onto. Alternatively, $\tilde{\varphi}: H \rightarrow G$ is also one-to-one and onto.

Example 6.15

$$\mathbb{Z}_8 = \{0, \dots, 7\}$$

$$\mathcal{U}_8 \text{ of units} \implies \mathcal{U}_8 = \{1, 3, 5, 7\}$$

$$\text{Consider } \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$$

$$\text{Claim: } \mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathcal{U}_8$$

Let

$$\varphi: \mathcal{U}_8 \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2$$

$$\varphi(1) = (0, 0)$$

$$\varphi(3) = (1, 0)$$

$$\varphi(5) = (0, 1)$$

$$\varphi(7) = (1, 1)$$

$$\varphi(ab) = \varphi(a) + \varphi(b)$$

Check,

- φ is a homomorphism
- multiplication table is preserved
- φ is one to one and onto

7 Jan 19, 2022

7.1 Isomorphisms and Homomorphisms (Cont'd)

Example 7.1 (Example 6.15 Cont'd)

Let

$$\begin{aligned}\varphi: \mathcal{U}_8 &\rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \\ \varphi(1) &= (0, 0) \leftarrow \text{fixed} \\ \varphi(3) &= (1, 0) \\ \varphi(5) &= (0, 1) \\ \varphi(7) &= (1, 1)\end{aligned}$$

Check,

$$\begin{aligned}(0, 0) + (1, 0) &= \varphi(1) + \varphi(3) \stackrel{\checkmark}{=} \varphi(1 \cdot 3) = \varphi(3) = (1, 0) \\ (0, 0) + 2(0, 1) &= \varphi(5) + \varphi(5) \stackrel{\checkmark}{=} \varphi(5 \cdot 5) = \varphi(1) = (0, 0) \\ &\vdots\end{aligned}$$

Verify every time $\varphi(ab) = \varphi(a) + \varphi(b) \implies \varphi$ is a homomorphism.

φ is one-to-one^a and onto^b \implies DONE.

Iso's are not unique. In fact,

$$\begin{aligned}\varphi(1) &= (0, 0) \\ \varphi(3) &= (0, 1) \\ \varphi(5) &= (1, 0) \\ \varphi(7) &= (1, 1)\end{aligned}$$

is also an iso. However,

$$\begin{aligned}\varphi(1) &= (0, 0) \\ \varphi(3) &= (1, 1)\end{aligned}$$

Does it work? Why? (Exercise)

$$^a \varphi(x) = \varphi(y) \implies x = y$$

$$^b \forall y \in \mathbb{Z}_2 \times \mathbb{Z}_2 \exists x \in \mathcal{U}_8 \text{ s.t. } \varphi(x) = y$$

Example 7.2

$$\mathbb{Z} \rightarrow \mathbb{Z}_5$$

$$n \mapsto [n] \pmod{5}$$

Let's construct a homomorphism.

1. Check φ is well defined.

$$n \equiv m \pmod{5} \stackrel{?}{\implies} \varphi(n) = \varphi(m). \checkmark$$

2. φ is a homomorphism.

$$\varphi(a + b) = \varphi(a) + \varphi(b)$$

$$[a + b] \stackrel{\text{true}}{=} [a] + [b]$$

$$\implies \varphi \text{ is a homomorphism}$$

Note: φ is not injective because $|\mathbb{Z}| > |\mathbb{Z}_5|$

φ is not an iso.

Fact 7.3: Isomorphic groups always have the same order.

Converse? $|G| = |H| \implies G \cong H$?

FALSE!

Example 7.4

Consider S_3 and \mathbb{Z}_6 .

$$|S_3| = 3! = 6$$

$$|\mathbb{Z}_6| = 6$$

Not isomorphic. Let's suppose $\varphi: S_3 \rightarrow \mathbb{Z}_6$ an isomorphism.

$$\varphi(ab) = \varphi(a) + \varphi(b) \tag{1}$$

So,

$$\begin{aligned} \varphi(a) + \varphi(b) &= \varphi(b) + \varphi(a) && \text{(because } \mathbb{Z}_6 \text{ is abelian)} \\ &= \varphi(ab) \end{aligned}$$

$$\implies \text{if (1) holds since } \mathbb{Z}_6 \text{ is abelian}$$

$$\implies \varphi(ab) = \varphi(ba) \quad \forall b, a \in S_3$$

$$\implies S_3 \text{ is abelian}$$

False, S_3 is not abelian, so you can't define such an iso φ .

Theorem 7.5

If G is abelian, H is not abelian $\implies G \not\cong H$.

Fact 7.6: Isomorphisms preserve order of elements, i.e.

$$|a| = |\varphi(a)|$$

Definition 7.7 (Automorphism)

An automorphism is an isomorphism from $G \rightarrow G$. They capture internal symmetries of a group.

Example 7.8

identity:

$$\begin{aligned} i_G: G &\rightarrow G \\ g &\mapsto g \end{aligned}$$

Clearly: $i(ab) = i(a)i(b) = ab \stackrel{\checkmark}{=} ab$

Definition 7.9 (Inner automorphism of G induced by c)

For any $c \in G$, the inner automorphism of G induced by c is:

$$\varphi_c: G \rightarrow G; \quad \varphi_c(g) = c^{-1}gc \leftarrow \text{conjugation by } c.$$

1. Then φ_c is a homomorphism:

$$\varphi_c(ab) = c^{-1}abc = (c^{-1}ac)(c^{-1}bc) = \varphi_c(a)\varphi_c(b)$$

2. φ is surjective: Given any $g \in G$.

$$\varphi_c(cgc^{-1}) = c^{-1}(cgc^{-1})c = g$$

3. φ is injective: $\varphi_c(a) = \varphi_c(b)$ for some $a, b \in G$

$$\implies c^{-1}ac = c^{-1}bc$$

$$\implies a = b$$

$$\implies \varphi \text{ is an isomorphism.}$$

7.2 Classification of Cyclic Groups

Theorem 7.10

Suppose G is a cyclic group.

1. $|G| = \infty \implies G \cong (\mathbb{Z}, +)$
2. $|G| = n < \infty \implies G \cong (\mathbb{Z}_n, +)$

Proof.

1. If $G = \langle a \rangle$ infinite. Then $G = \{a^n \mid n \in \mathbb{Z}\}$. So let

$$\begin{aligned} \varphi: G &\rightarrow \mathbb{Z} \\ a^n &\mapsto n \end{aligned}$$

So φ is one-to-one and onto by definition.

Then,

$$n + m = \varphi(a^{n+m}) = \varphi(a^n a^m) \stackrel{?}{=} \varphi(a^n) + \varphi(a^m) = n + m$$

$\implies \varphi$ is a homomorphism and φ is bijection.

$\implies \varphi$ is an isomorphism.

$$2. |G| = n \implies G = \{e, a, a^2, \dots, a^{n-1}\}$$

$$\begin{aligned} \varphi: G &\rightarrow \mathbb{Z}_n = \{0, 1, \dots, n-1\} \\ a^i &\mapsto i \end{aligned}$$

Exactly for the same reasons: check φ is an isomorphism.

$$k = \underbrace{\varphi(a^k)}_{i+j \equiv k \pmod n} = \varphi(a^{i+j}) = \underbrace{\varphi(a^i) + \varphi(a^j)}_{i+j \equiv k \pmod n}$$

φ is an isomorphism.

□

8 Jan 21, 2022

8.1 Homomorphisms

Recall 8.1 Let $\varphi: G \rightarrow H$ any map. Then

$$\text{Im } \varphi = \{h \in H \mid h = \varphi(g) \text{ some } g \in G\}$$

Theorem 8.2

If $\varphi: G \rightarrow H$ is a homomorphism, then:

1. $\varphi(e_G) = e_H$
2. $\varphi(a^{-1}) = (\varphi(a))^{-1}$
3. $\text{Im } \varphi$ is a subgroup of H
4. If φ is injective, then $G \cong \text{Im } \varphi$

Note 8.3: If φ is surjective, then $\text{Im } \varphi = H$

Proof.

1. Did before.
2. By (1), $e_H = \varphi(e_G) = \varphi(aa^{-1}) = \varphi(a) \cdot \varphi(a^{-1}) \stackrel{?}{=} e_H \stackrel{?}{=} \varphi(a^{-1})\varphi(a) = \varphi(a^{-1}a) = \varphi(e_G) = e_H$ by (1).
3. Claim $\text{Im } \varphi$ subgroup of H . Since $\varphi(e_G) = e_H$ by (1) $\implies e_H \in \text{Im } \varphi$. If $a, b \in \text{Im } \varphi \implies \exists a', b' \in G$ s.t. $\varphi(a') = a, \varphi(b') = b \implies ab = \varphi(a')\varphi(b') = \varphi(a'b')$ since G is closed, $a'b' \in G \implies ab \in \text{Im } \varphi \implies \text{Im } \varphi$ is closed.
4. By (2), if $\varphi(g) = a$ then

$$a^{-1} = \varphi(g)^{-1} = \varphi(g^{-1})$$

$$\implies a^{-1} = \varphi(g^{-1}) \text{ but } g^{-1} \in G \implies a^{-1} \in \text{Im } \varphi$$

$$\text{Im } \varphi \text{ has inverses} \implies \text{Im } \varphi \text{ is subgroup.}$$
5. φ injective $\implies G \cong \text{Im } \varphi$. Since $\varphi: G \rightarrow \text{Im } \varphi$ is surjective by construction, if φ is also injective, then $\varphi: G \rightarrow \text{Im } \varphi$ is a bijection and a homomorphism $\implies \varphi: G \rightarrow \text{Im } \varphi$ is an isomorphism $\implies G \cong \text{Im } \varphi$.

□

Example 8.4

$\varphi: G \rightarrow H$ where φ is an injective homomorphism and H is abelian.

Question: Is G abelian?

Yes, because $G \cong \text{Im } \varphi$ by bijectivity, and $\text{Im } \varphi$ subgroup of H and subgroups of abelian groups are abelian $\implies G$ has to be abelian.

8.2 Congruence

Definition 8.5 (Congruence of a group)

Suppose H is a subgroup of G . Let $a, b \in G$. We say $a \equiv b \pmod{H}$ if $ab^{-1} \in H$.

Recall 8.6 An equivalence relation on a set S is a relation $a \sim b$ for $a, b \in S$ that is:

reflexive: $a \sim a \quad \forall a \in S$

transitive: $a \sim b$ and $b \sim c \implies a \sim c$

symmetric: $a \sim b \implies b \sim a$.

Theorem 8.7

The congruence relation $a \equiv b \pmod{H}$ is an equivalence relation for any subgroup $H \subseteq G$.

Definition 8.8 (Right coset (and left coset))

Given any $a \in G$, the right coset of H in G is:

$$Ha = \{ha \in G \mid h \in H\} \text{ where } a \text{ is any } a \in G \text{ fixed}$$

This is a right coset because a is multiplied on the right.

The left coset of H in G is:

$$aH = \{ah \in G \mid h \in H\} \text{ where } a \text{ is any } a \in G \text{ fixed}$$

Note 8.9: Ha is just the congruence class of a in $G \pmod{H}$.

For any $a \in G$,

$$\begin{aligned} [a] &= \{b \in G \mid b \equiv a \pmod{H}\} \\ &= \{b \in G \mid ba^{-1} \in H\} \\ &= \{b \in G \mid \underbrace{ba^{-1}}_{b=ha} = h \text{ for some } h \in H\} \\ &= \{ha \in G \mid h \in H\} = Ha. \end{aligned}$$

Theorem 8.10 1. $Ha = Hb$ iff $ab^{-1} \in H$ (i.e. $a \equiv b \pmod{H}$)

2. Given $a \neq b$ either $Ha = Hb$ or $Ha \cap Hb = \emptyset$.

Proof. Analogous as for rings (seen this in 110A). □

8.3 Lagrange's Theorem

Theorem 8.11

H -subgroup of G then:

1. $G = \bigcup_{a \in G} Ha$
2. $\forall a \in G, \exists$ bijection between $H \rightarrow Ha$. So if $|H| < \infty$, then $|Ha| = |H| \forall a, b \in G$.

Proof.

1. $\bigcup_{a \in G} Ha \subseteq G$ obvious. Given $g \in G, g = eg$ where since $e \in H \implies eg \in Hg \implies g \in Hg \implies G \subseteq \bigcup_{g \in G} Hg$

2. Consider

$$\begin{aligned} \psi: H &\rightarrow Ha = \{ha \mid h \in H\} \\ h &\mapsto ha \end{aligned}$$

ψ is surjective by definition. If $\psi(h) = \psi(h') \implies ha = h'a \implies h = h' \implies \psi$ is injective $\implies \psi$ is a bijection.

□

Definition 8.12 (Index)

Given any subgroup H of G , the index of H in G denoted $[G:H]$ is the number of distinct right cosets of H in G .

Theorem 8.13 (Lagrange's Theorem)

If $H \subseteq G$ is a finite subgroup, then:

$$[G:H] = \frac{|G|}{|H|}$$

9 Jan 24, 2022

9.1 Lagrange's Theorem (Cont'd)

Proof of Lagrange's Theorem. Suppose $[G:H] = n$ and denote the cosets by Hg_i for $i = 1, \dots, n$.

Recall: $Hg_i \cap Hg_j = \emptyset$ $i \neq j$, also

$$G = \bigcup_{i=1}^n Hg_i = Hg_1 \cup Hg_2 \cup \dots \cup Hg_n$$

$$\implies |G| = |Hg_1| + |Hg_2| + \dots + |Hg_n|$$

Also know by previous theorem $|Hg_i| = |H| < \infty$

$$\implies |G| = n \cdot |H|$$

$$\implies \frac{|G|}{|H|} = n = [G:H]$$

□

Question 9.1: What fails when $|H| = \infty$?

Example 9.2

$n\mathbb{Z} = \langle n \rangle$ inside \mathbb{Z} .

Then for $a \in \mathbb{Z}$,

$$[a] = \underbrace{a + n\mathbb{Z}}_{Ha} = \{a + ni \mid i \in \mathbb{Z}\} = \{a, a + n, a + 2n, \dots\}$$

where $Ha = \{ha \mid h \in H\}$ with $H = n\mathbb{Z} \rightarrow Ha = Hb \iff ab^{-1} \in H$ and $a \equiv b \pmod{H}$

$$a + n\mathbb{Z} = \underbrace{(a + n)}_b + n\mathbb{Z}$$

$-n = a - (a + n) \in n\mathbb{Z} \iff a \equiv a + n \pmod{n} \implies$ exist exactly n cosets $[0], [1], \dots, [n-1]$

$$[\mathbb{Z}:n\mathbb{Z}] = n$$

Lagrange's Theorem $\implies |H|$ divides $|G|$ for any H subgroup of G .

Example 9.3

If G has order 15.

G can only have subgroups of orders 1, 3, 5, 15.

Note 9.4: Lagrange does not imply that subgroups exist for every number dividing $|G|$. In Example 9.3, there may not exist a subgroup of order 5 or 3.

Corollary 9.5

$$|G| < \infty$$

1. $\forall a \in G \implies |a| \mid |G|$
2. If $|G| = n \implies a^n = e \quad \forall a \in G$.

Proof.

1. Consider $H = \langle a \rangle \subseteq G$. $|\langle a \rangle| = |a| \implies$ Since $|G| < \infty$

$$\implies H < \infty \text{ we can use Lagrange}$$

$$\implies |H| = |\langle a \rangle| = |a| \mid |G|.$$

2. Suppose $|a| = m$. Then by (1), $m \mid n \implies n = md$ for some $d \in \mathbb{Z}$. So then

$$a^n = a^{md} = (a^m)^d = e^d = e$$

□

9.2 Classification of Groups of Prime Order**Theorem 9.6**

Suppose $p > 0$ prime. If $|G| = p \implies G \cong \mathbb{Z}_p$.

Proof. By Theorem 7.10, all cyclic groups of order n are isomorphic to \mathbb{Z}_n . \implies We only need to show G is cyclic. Consider $a \in G$ with $a \neq e$. Then $|\langle a \rangle| \neq 1 \implies$ by Lagrange, since $|\langle a \rangle| \mid p$. Since only 1 or p divides $p \implies |\langle a \rangle| = p$. Since $|G| = p$ and $\langle a \rangle \subseteq G$

$$\implies G = \langle a \rangle \implies G \text{ is cyclic of order } p$$

$$\implies G \cong \mathbb{Z}_p \text{ by previous theorem}$$

□

9.3 Classification of Groups of Order ≤ 8

We know $1, \underbrace{2, 3}_{\text{prime}}, 4, \underbrace{5}, 6, \underbrace{7}, 8$

Theorem 9.7

If $|G| = 4 \implies$ either $G \cong \underbrace{\mathbb{Z}_4}_{\text{cyclic abelian}}$ or $G \cong \underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2}_{\text{abelian}}$.

Proof. If $|G| = 4$, then either $\exists a \in G$ with $|a| = 4$ or not.

- If yes, then $G = \langle a \rangle \implies G$ is cyclic $\implies G \cong \mathbb{Z}_4$.
- If not, then $G = \{e, a, b, c\}$, since only e can have order 1, then $|a| = |b| = |c| = 2$

$$\begin{aligned} \implies a^2 &= b^2 = c^2 = e \\ \implies a &= a^{-1}, b = b^{-1}, c = c^{-1} \end{aligned}$$

If $|ab| = 1 \implies a = b^{-1} \implies$ contradiction $|ab| = 2$.

So either

$$\begin{aligned} ab &= a \implies b = e \text{ contradiction} \\ ab &= b \implies a = e \text{ contradiction} \\ ab &= c \checkmark \end{aligned}$$

Repeat this for ac, ca, ba, bc, cb to find entire multiplication table. Then construct an explicit isomorphism to

$$\mathbb{Z}_2 \times \mathbb{Z}_2: \begin{array}{l} e \mapsto (0,0) \\ a \mapsto (1,0) \\ b \mapsto (0,1) \\ c \mapsto (1,1) \end{array}$$

□

Theorem 9.8

$|G| = 6 \implies G \cong \mathbb{Z}_6$ or S_3 .

10 Jan 26, 2022

10.1 Normal Subgroups

Recall 10.1 For $a \in G, H \subseteq G$ subgroup. Right coset $Ha = \{ha \in G \mid h \in H\}$. Left coset $aH = \{ah \in G \mid h \in H\}$.

Definition 10.2 (Normal subgroup)

A subgroup N of G is normal if $Na = aN \forall a \in G$.

Note 10.3: $Na = aN \not\Rightarrow an = na$. Rather, it means that $an = n'a$ for some $n, n' \in N$.

Notation 10.4: Whenever N is normal in G , we write $N \triangleleft G$.

Example 10.5

Consider $G = D_4$ (not abelian).

Let $M = \{e, r_{180}\}$ then you can show

$$\begin{aligned} r_{180} \cdot a &= a \cdot r_{180} \quad \forall a \in D_4 \\ \implies Ma &= aM \implies M \triangleleft D_4 \end{aligned}$$

Theorem 10.6

If G is abelian, then all subgroups are normal.

Recall 10.7 The center $Z(G) = \{a \in G \mid ag = ga\}$.

Proposition 10.8

For any G , the center $Z(G)$ is always normal.

Proof. Using the definition of $Z(G)$, we notice that for any $g \in G$,

$$Z(G)g = gZ(G)$$

For any $a \in Z(G)$, $ag \in Z(G)g$. Since $ag = ga$ because $a \in Z(G)$ (by definition), then $ga \in gZ(G)$. \square

Example 10.9

$S_3 = \{e, \tau_1, \tau_2, \tau_{12}, \tau_{21}, \tau_{121}\}.$

Let $A_3 := \{e, \tau_{12}, \tau_{21}\}.$

Then

$$A_3 a = \left\{ \begin{array}{l} \tau_{12} \circ \tau_1 = \tau_{121} = \tau_1 \circ \tau_{21} \\ \tau_{12} \circ \tau_2 = \tau_1 = \tau_2 \circ \tau_{21} \\ \underbrace{\tau_{12} \circ \tau_{121}}_{\in A_{\tau_{121}}} = \tau_2 = \underbrace{\tau_{121} \circ \tau_{21}}_{\in \tau_{121} A} \end{array} \right\} = a A_3$$

Recall $(a \in N, aN = N = Na)$

$$\implies A_3 a = a A_3 \quad \forall a \in S_3 \implies A_3 \text{ is normal}$$

Theorem 10.10

For $N \triangleleft G$, if $Na = Nb$ and $Nd = Nc \implies Nad = Nbc$ (Analogously, $Nda = Ncb$).

■ **Proof.** Direct from set definitions of cosets. □

Definition 10.11

Given $a, b \in G, N \subseteq G$,

$$aNb := \{anb \in G \mid n \in N\}$$

Theorem 10.12

TFAE:

1. $N \triangleleft G$.
2. $a^{-1}Na \subseteq N \quad \forall a \in G$.
3. $aNa^{-1} \subseteq N \quad \forall a \in G$.
4. $a^{-1}Na = N \quad \forall a \in G$.
5. $aNa^{-1} = N \quad \forall a \in G$.

Proof. 1) \implies 3) N normal $\implies aN = Na \implies \forall a \in G$ and $n \in N$

$$\begin{aligned} \exists n' \in N \text{ such that } an = n'a &\implies ana^{-1} = n' \\ &\implies aNa^{-1} \subseteq N \end{aligned}$$

3) \implies 2) Since if $aNa^{-1} \subseteq N \quad \forall a \in G$ and $a^{-1} \in G$

$$(a^{-1})N(a^{-1})^{-1} = a^{-1}Na \subseteq N$$

2) \implies 3) analogous.

4) \iff 5) proved the same way.

3) \implies 4) If $aNa^{-1} \subseteq N$ then since $ana^{-1} \in N \quad \forall a \in G, \forall n \in N$

$$\begin{aligned}
 &\stackrel{\text{by 2)}}{\implies} a^{-1} \underbrace{(ana^{-1})}_{n'} a \in a^{-1}Na \\
 &\implies n \in a^{-1}Na \implies N \subseteq \underbrace{a^{-1}Na}_{\iff \text{by 3}} \\
 &\implies N \subseteq aNa^{-1} \implies N = aNa^{-1}
 \end{aligned}$$

2) \implies 5) same proof as 3) \implies 4).

5) \implies 1)

$$\begin{aligned}
 aNa^{-1} = N &\implies ana^{-1} = n' \text{ for some } n' \in N \\
 &\implies an = n'a \\
 &\implies aN \subseteq Na
 \end{aligned}$$

Use the fact 4) \iff 5) to show $Na \subseteq aN$.

$$\implies Na = aN \implies N \triangleleft G.$$

□

11 Jan 28, 2022

11.1 Quotient Groups

Given $N \triangleleft G$, let $G/N := \{Na \mid a \in G\}$.

Recall 11.1 If $N \triangleleft G$, $Na = Nb$ and $Nc = Nd$, then $\implies Nac = Nbd$.

Theorem 11.2

$N \triangleleft G$, then

1. G/N is a group with operation $Na \cdot Nb := \overset{\text{product inside } G}{Na \cdot Nb} = \overset{* \text{ operation in } G/N}{Nab}$
2. If $|G| < \infty \implies |G/N| = |G|/|N|$
3. If G is abelian $\implies G/N$ is abelian.

We call G/N the quotient group of G by N .

Proof. 1) Check each axiom of groups:

- $id := N$
- Inverse $:= Na^{-1} \implies (Na)(Na^{-1}) = Naa^{-1} = Ne = N$
- etc.

$$2) |G/N| = [G:N] = |G|/|N|$$

$$3) \underbrace{(Na)(Nb)}_{Nab} = \underbrace{(Nb)(Na)}_{Nba}$$

because G is abelian, $Nab = Nba$. □

Example 11.3

Consider

$$2\mathbb{Z} = \langle 2 \rangle \subseteq \mathbb{Z}.$$

\mathbb{Z} abelian $\implies 2\mathbb{Z}$ normal.

$$|\mathbb{Z}/2\mathbb{Z}| = [\mathbb{Z}:2\mathbb{Z}] = 2$$

$$2\mathbb{Z} = \{-4, -2, 0, 2, 4, \dots\} = \text{evens}$$

$$2\mathbb{Z} + 1 = \text{odds} \implies \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}_2$$

Generally,

$$\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$$

Example 11.4

$$A_3 \triangleleft S_3$$

$$A_3 = \{e, \tau_{12}, \tau_{21}\}$$

$$|S_3| = 6, |A_3| = 3, \text{ so}$$

$$|S_3/A_3| = \frac{6}{3} = 2$$

$$\implies S_3/A_3 \cong \mathbb{Z}_2$$

Example 11.5

$$N = \langle 4 \rangle = \{0, 4, 8\} \subseteq \mathbb{Z}_{12}$$

$$[0] = N + 0 = N$$

$$[1] = N + 1 = \{1, 5, 9\}$$

$$[2] = N + 2 = \{2, 6, 10\}$$

$$[3] = N + 3 = \{3, 7, 11\}$$

$$\implies N + a = N + b \iff a \equiv b \pmod{4}$$

$$\text{i.e: } N + 6 = \{6, 10, 2\} \quad 6 \equiv 2 \pmod{4}$$

$$\mathbb{Z}_{12}/N \cong ? \text{ where } |\mathbb{Z}_{12}/N| = 4$$

So either

$$\mathbb{Z}_{12}/N \cong \mathbb{Z}_4 \text{ or } \mathbb{Z}_2 \times \mathbb{Z}_2$$

$$[4] = [1] + [1] + [1] + [1] = [0]$$

$$(N + 1) + (N + 1) + (N + 1) + (N + 1) = N + 4 = N, \text{ because } 4 \equiv 0 \pmod{4}.$$

So,

$$|N + 1| = 4 \implies \mathbb{Z}_{12}/N \cong \mathbb{Z}_4$$

Theorem 11.6

$N \triangleleft G$. Then G/N is abelian if and only if $aba^{-1}b^{-1} \in N \forall a, b \in G$.

Proof. G/N is abelian iff $Nab = Nba \forall a, b \in G$

$$\iff ab \equiv ba \pmod{N} \forall a, b \in G$$

$$\iff aba^{-1}b^{-1} \equiv e \pmod{N} \iff aba^{-1}b^{-1} \in N$$

□

Theorem 11.7

G any group. $G/Z(G)$ is cyclic $\implies G$ abelian.

Proof. If $G/Z(G)$ is cyclic, then $G/Z = \langle Zg \rangle$ for some $g \in G \implies$ every other coset $Zg' = (Zg)^k = Zg^k$. So then if $a, b \in G$, then

$a \in Za = Zg^k$ for some k ,
 $b \in Zb = Zg^j$ for some j .

$$\begin{aligned} \implies a &= c \cdot g^k \text{ and } b = c'g^j \text{ for some } c, c' \in Z \\ \implies ab &= cg^k \cdot c'g^j = c'g^jcg^k = ba \\ \implies G &\text{ is abelian.} \end{aligned}$$

□

11.2 Quotient Groups and Homomorphisms

Definition 11.8 (Kernel)

Let $\varphi: G \rightarrow H$ be a homomorphism. The kernel of φ is the set

$$\ker \varphi := \{g \in G \mid \varphi(g) = e_H\}$$

Example 11.9

Consider

$$\begin{aligned} \varphi: \mathbb{Z} &\rightarrow \mathbb{Z}_5 \\ n &\mapsto [n] \end{aligned}$$

Then,

$$\begin{aligned} \ker \varphi &= \{n \in \mathbb{Z} \mid [n] = [0]\} = \{n \mid n \equiv 0 \pmod{5}\} \\ &= 5\mathbb{Z} \end{aligned}$$

Theorem 11.10

Suppose $\varphi: G \rightarrow H$ is a homomorphism. Then $\ker \varphi \triangleleft G$ is a normal subgroup of G .

Proof. Subgroup:

- (Identity): Since $\varphi(e) = e \implies e \in \ker \varphi$
- (Closure): If $a, b \in \ker \varphi$,

$$\begin{aligned} \varphi(ab) &= \varphi(a) \cdot \varphi(b) = e \cdot e = e \\ \implies ab &\in \ker \varphi. \end{aligned}$$

- (Inverse): If $a \in \ker \varphi$, then $\varphi(a^{-1}) = (\varphi(a))^{-1} = e^{-1} = e$
 $\implies \ker \varphi$ is a subgroup.

Normal: We will show $g \ker \varphi g^{-1} \subseteq \ker \varphi \forall g \in G$.

Let $a \in \ker \varphi$, so $\varphi(a) = e$. Then any $g \in G$:

$$g\varphi(a)g^{-1} = g \cdot e \cdot g^{-1} = e \in \ker \varphi$$

$$\implies g \cdot \ker \varphi g^{-1} \subseteq \ker \varphi$$

□

12 Jan 31, 2022

12.1 Quotient Groups and Homomorphisms (Cont'd)

Example 12.1

Let

$$\begin{aligned}\varphi: S_3 &\rightarrow \mathbb{Z}_2 \text{ given by} \\ e, \tau_{21}, \tau_{12} &\mapsto 0 \\ \tau_1, \tau_2, \tau_{121} &\mapsto 1\end{aligned}$$

- Is a homomorphism? Yes. (Check this).
- Kernel of φ ? $\ker \varphi = \{e, \tau_{12}, \tau_{21}\} = A_3$
By theorem A_3 is normal in S_3 . $S_3/A_3 \cong \mathbb{Z}_2$

Theorem 12.2

A homomorphism φ is injective if and only if $\ker \varphi = e$.

Proof. Standard. □

Theorem 12.3

If $N \triangleleft G$, then

$$\begin{aligned}\pi: G &\rightarrow G/N \\ a &\mapsto Na\end{aligned}$$

is surjective group homomorphism with $\ker \pi = N$.

Proof. π is surjective: To every coset $Na \exists a \in G$ such that $a \mapsto Na$.

π is homomorphic: $\pi(ab) = Nab = (Na) \cdot (Nb) = \pi(a) \cdot \pi(b)$

$e = N$ if $\pi(a) = N \implies Na = N \iff a \in N$ So,

$$\ker \varphi = \{a \in G \mid a \in N\} = N$$

□

Lemma 12.4

Suppose $\varphi: G \rightarrow H$ is a homomorphism with $\ker \varphi = K$. Then $\forall a, b \in G, \varphi(a) = \varphi(b)$ if and only if $Ka = Kb$.

Proof. $\varphi(a) = \varphi(b) \iff \varphi(a)\varphi(b)^{-1} = e \iff \varphi(a)\varphi(b^{-1}) = \varphi(ab^{-1}) = e \iff ab^{-1} \in \ker \varphi = K \iff a \equiv b \pmod{K} \iff Ka = Kb$ □

12.2 The Isomorphism Theorems

Theorem 12.5 (First Isomorphism Theorem)

Let $\varphi: G \rightarrow H$ be a surjective homomorphism. Then

$$G/\ker \varphi \cong H$$

Proof. Let

$$\begin{aligned}\pi: G/\ker \varphi &\rightarrow H \\ Ka &\mapsto \varphi(a)\end{aligned}$$

where $K = \ker \varphi$. We need to show π is a well-defined isomorphism

1. Well-defined: Let $Ka = Kb$ for $a \neq b$. Then $ab^{-1} \in K = \ker \varphi \implies \varphi(ab^{-1}) = e \implies \varphi(a) = \varphi(b)$

2. Homomorphism:

$$\begin{aligned}\pi(Ka \cdot Kb) &= \pi(Kab) \\ &= \varphi(ab) = \varphi(a) \cdot \varphi(b) \\ &= \pi(Ka) \cdot \pi(Kb)\end{aligned}$$

3. Surjective: $\pi: G/K \rightarrow H$. Let $h \in H$, then $\exists g \in G$ such that $\varphi(g) = h$ because φ is surjective. Consider $Kg \in G/\ker \varphi$. Then $\pi(Kg) = \varphi(g) = h$.

4. Injective: Suppose $\pi(Ka) = \pi(Kb)$

$$\begin{aligned}\implies \varphi(a) &= \varphi(b) \\ \implies ab^{-1} &\in \ker \varphi \\ \implies Ka &= Kb \implies \pi \text{ is 1-1}\end{aligned}$$

□

Theorem 12.6 (Second Isomorphism Theorem)

Suppose N and K are subgroups of G , with $N \triangleleft G$. Then

$$NK := \{nk \mid n \in N, k \in K\}$$

is a subgroup of G containing both N and K .

Proof. Homework. ☺

□

Lemma 12.7

Let $N \triangleleft G$, and K is any subgroup of G such that $N \subseteq K$. Then $N \triangleleft K$ and K/N is a subgroup of G/N .

Proof. Since $aN = Na \forall a \in G$ so then if $a \in K$, then $aN = Na \forall a \in K$
 $\implies N \triangleleft K \implies K/N$ is a subgroup.

Since

$$K/N = \{Na \mid a \in K\}$$

and since $K \subseteq G \implies K/N \subseteq G/N$. □

Theorem 12.8 (Third Isomorphism Theorem)

Let $K \triangleleft G, N \triangleleft G, N \subseteq K \subseteq G$. Then,

1. $K/N \triangleleft G/N$ and
2. $(G/N)/(K/N) \cong G/K$

13 Feb 2, 2022

13.1 The Isomorphism Theorems (Cont'd)

Proof of Third Isomorphism Theorem. Since $K \triangleleft G$ and $N \triangleleft G \implies G/N$ and G/K are groups. Consider

$$\begin{aligned}\varphi: G/N &\rightarrow G/K \\ Ng &\mapsto Kg\end{aligned}$$

Well-defined:

If $Ng = Ng'$ with $g \neq g'$

$$\begin{aligned}\implies g'g^{-1} &\in N \subseteq K \implies Kg = Kg' \\ &\implies \varphi(Ng) = \varphi(Ng')\end{aligned}$$

Homomorphism:

$$\begin{aligned}\varphi(Ng \cdot Ng') &= \varphi(Ngg') = Kgg' \\ &= Kg \cdot Kg' = \varphi(Ng) \cdot \varphi(Ng')\end{aligned}$$

Surjective: Obvious by definition of the map

$$\varphi: G/N \rightarrow G/K \quad \forall Kg \rightarrow \exists Ng \text{ s.t. } \varphi(Ng) = Kg$$

\implies We can apply the First Isomorphism Theorem so that

$$(G/N)/\ker \varphi \cong G/K$$

We show $\ker \varphi = K/N$: Now, $\varphi(Ng) = K = Ke \iff g \in K$. Then,

$$\ker \varphi = \{Ng \mid g \in K\}$$

By Lemma 12.7, $N \triangleleft K$ so K/N makes sense. Also, $\ker \varphi = K/N$.

Since by previous theorem, since $\ker \varphi \triangleleft G/N$ then this means that $K/N \triangleleft G/N$ and

$$(G/N)/(K/N) \cong G/K.$$

□

Corollary 13.1

Suppose $N \triangleleft G$ and K is any subgroup of G such that $N \subseteq K \subseteq G$. Then $K \triangleleft G$ if and only if $K/N \triangleleft G/N$.

Proof. $(\implies) K \triangleleft G \implies K/N \triangleleft G/N$ (by Third Isomorphism Theorem).

(\Leftarrow) Suppose $K/N \triangleleft G/N$. For any $Na \in G/N$, we know

$$(Na)^{-1}(Nk)(Na) \in \underbrace{K/N}_{\ni Nk} \triangleleft \underbrace{G/N}_{\ni Na}$$

Then $\forall a \in G$ and $k \in K$,

$$Na^{-1}ka = (Na^{-1})(Nk)(Na) \in K/N$$

$$\implies Na^{-1}ka \in K/N$$

So this means $\exists t \in K$ such that

$$Na^{-1}ka = Nt$$

$$\implies \forall n \in N \exists n' \in N$$

$$na^{-1}ka = n't$$

$$\implies a^{-1}ka = \underbrace{n^{-1}n't}_{\substack{n, n' \in N \subseteq K \\ t \in K}} \in K.$$

Recall: $K \triangleleft G$ if and only if $aKa^{-1} \subseteq K \forall a \in G$.

Equivalently: $aka^{-1} \in K \quad \forall a \in G \quad \forall k \in K$

$\implies K$ is normal. □

Theorem 13.2 (The Correspondence Theorem)

Suppose $T \subseteq G/N$ is a subgroup. Then there exists some subgroup $H \subseteq G$ with $N \subseteq H$ such that

$$T = H/N$$

i.e. There exists a correspondence between

$$N \subseteq H \subseteq G \longleftrightarrow T \subseteq G/N$$

This theorem classifies all subgroups of G/N .

Proof. Given $T \subseteq G/N$ subgroup. Let $H := \{a \in G \mid Na \in T\}$.

- $N \in T$ since T is a subgroup of $G/N \implies e \in H$.

- If Na and $Nb \in T$ then

$$Nab = Na \cdot Nb \in T$$

Since T is closed $\implies ab \in H$.

- If $Na \in T$ then $(Na)^{-1} = Na^{-1} \in T \implies a^{-1} \in H \implies H$ is a subgroup of G .

Now, $\forall a \in N, Na = N$ and since $N \in T$

$$\implies a \in H \quad \forall a \in N \implies N \subseteq H.$$

Thus, $N \subseteq H \subseteq G$.

Finally, we must show $T = H/N$. (By Lemma 12.7, $N \triangleleft G \implies N \triangleleft H$ so H/N makes sense).

Using the fact that $H = \{a \in G \mid Na \in T\}$,

$$H/N = \{Na \mid a \in H\} = \{Na \in T \mid a \in G\} = T$$

□

14 Feb 4, 2022

14.1 Simple Groups

Definition 14.1 (Simple group)

A group is simple if it has no nontrivial proper subgroups, i.e. the only subgroups it has are e and G .

Example 14.2

$\mathbb{Z}_2, \mathbb{Z}_3, \dots, \mathbb{Z}_p$

By Lagrange, $1 \mid p$ and $p \mid p \implies$ only subgroups of \mathbb{Z}_p are e and \mathbb{Z}_p
 $\implies \mathbb{Z}_p$ is simple if and only if p is prime.

Theorem 14.3

G is a simple abelian group if and only if $G \cong \mathbb{Z}_p$ for p prime.

Proof. (\Leftarrow) done.

(\Rightarrow) Suppose G is a simple abelian group. Then $\forall a \in G$ with $a \neq e$; $G = \langle a \rangle$. Then G is cyclic $\implies G \cong \mathbb{Z}$ or $G \cong \mathbb{Z}_n$ for some $n \in \mathbb{N}$.

If $G \cong \mathbb{Z}$, G cannot be simple since \mathbb{Z} has infinitely many subgroups (i.e. $n\mathbb{Z}$) $\implies G \cong \mathbb{Z}_n$.

If n is not prime, then $n = kd$ for $k, d \in \mathbb{N}$.

$\implies \langle a^d \rangle \subseteq G$ is a proper subgroup of order k which is a contradiction because G is simple $\implies n$ is prime. So $G \cong \mathbb{Z}_p$. \square

————→ Midterm is up to here! ←————

14.2 The Symmetric Group

Definition 14.4 (Symmetric group)

The symmetric group S_n is the group of permutations of $\{1, \dots, n\}$ where group operation corresponds to composition of permutations. It has order $n!$

Permutation \implies assignment of entry to position a_i

$$\begin{pmatrix} 1 & \dots & i & \dots & n \\ \downarrow & & \downarrow & & \downarrow \\ a_1 & & a_i & & a_n \end{pmatrix}$$

So each permutation is just a bijection $\{1 \dots n\} \rightarrow \{1 \dots n\}$.

14.3 Cycle Notation

Example 14.5

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \Rightarrow 1 \xrightarrow{\quad} 2 \xrightarrow{\quad} 3 \xrightarrow{\quad} 1 \Rightarrow (1 \ 2 \ 3)$$

Example 14.6

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 4 & 2 \end{pmatrix} \Rightarrow (1 \ 3 \ 5 \ 2)(4) = (1 \ 3 \ 5 \ 2)$$

Example 14.7

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = 1 \ 2 \ 3 = (1)(2)(3) = e$$

Example 14.8

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 1 & 7 & 2 & 4 & 6 & 3 \end{pmatrix}$$

$$\begin{aligned} (1542)(37)(6) &= (1542)(6)(37) \\ &= (6)(37)(1542) = (37)(1542) \\ &= (1542)(37) \end{aligned}$$

Note: $(2154) = (1542) \neq (5142)$

14.4 Multiplying in Cycle Notation

To compose in cycle notation you “trace” each entry from right to left. Always start with the first entry of the right most cycle.

Example 14.9

$$(243)(1243) = (1423)$$

Example 14.10

$$(12)(34) = (34)(12)$$

Can’t merge this because the cycles are disjoint

Example 14.11

$$(12)(23)(34) = (3412) = (4123) = (1234)$$

Check this:

$$\begin{aligned} & \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix} \\ & \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} \\ & \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \\ &= (1234) \end{aligned}$$

15 Feb 7, 2022

15.1 The Symmetric Group (Cont'd)

Definition 15.1 (Disjoint cycle)

We say two cycles are disjoint if they have no entries in common.

Example 15.2

$(12)(358)$ are disjoint

$(132)(358)$ not disjoint

Theorem 15.3

Disjoint cycles commute.

Proof. Easy and straightforward. □

Theorem 15.4

Every permutation in S_n is a product of disjoint cycles.

Example 15.5

From above $(1542)(37)(6)$ product of disjoint cycles.

Recall 15.6 Order of $g \in G$ is the smallest positive integer k s.t. $g^k = e$.

Theorem 15.7

The order of any $w \in S_n$ is the least common multiple of the lengths of the disjoint cycles of w .

Example 15.8

$w = (1542)(37)$ So,

$$|w| = \text{lcm}(4, 2) = 4$$

Check this by computing $w \neq e$, $\underbrace{w^2, w^3, w^4}_{\text{which of these} = e?}$.

$$\begin{aligned} w^2 &= (1542)(37)(1542)(37) \\ &= (1542)(1542)(37)(37) \end{aligned}$$

And

$$w^4 = \underbrace{(1542) \dots (1542)}_{4 \times} \underbrace{(37)(37)(37)(37)}$$

Example 15.9

We have $w = (1243)(243)$. What is $|w|$?

Because $(1243)(243)$ are not disjoint, we need to make them disjoint. By multiplying,

$$(2341) \implies w = (2341) \implies |w| = 4$$

Definition 15.10 (Transposition)

A cycle of length 2 is called a transposition, i.e. (ab) for any $a, b \in \{1 \dots n\}$.

Definition 15.11 (Simple transposition)

A transposition is simple when $b = a \pm 1$, i.e. $(a \ a + 1)$ or $(a - 1 \ a)$. Or

$$(12), (23), (34), \dots \text{ etc}$$

Fact 15.12: Simple transpositions generate all of S_n .

Example 15.13

$$(1 \ 5) = (12)(23)(34)(45)$$

Proposition 15.14

For any $a, b \in \{1 \dots n\}$

1. (Self-inverses) $(ab)(ab) = e$.
2. Suppose $\sigma_1 \dots \sigma_k$ are all transpositions.

$$[\sigma_1 \dots \sigma_k]^{-1} = \sigma_k \sigma_{k-1} \dots \sigma_1$$

3. Every cycle is a product of (not necessarily disjoint) transpositions, i.e.

$$(a_1 \dots a_k) = (a_1 a_2)(a_2 a_3) \dots (a_{k-1} a_k)$$

Example 15.15

$$S_3 = \{e, \tau_1, \tau_2, \tau_{21}, \tau_{12}, \tau_{121}\}.$$

$$\tau_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (12)$$

$$\tau_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (23)$$

$$\implies \tau_{21} = (23)(12) = (132)$$

$$\tau_{12} = (12)(23) = (231) = (123)$$

$$\tau_{21} = (12)(23)(12) = (12)(132) = (13)(2) = (13)$$

Multiplication is easy using this notation:

$$\begin{aligned} \tau_{12} \cdot \tau_{21} &= (23)(12) \cdot (12)(23) \\ &= (23)(23) = e \end{aligned}$$

$$\tau_{212} = (23)(12)(23) = \tau_{121}$$

Theorem 15.16

Every permutation $w \in S_n$ is a product of (not necessarily disjoint) transpositions.

Proof. Combine (3) above in proposition with the fact that every permutation $w \in S_n$ is a product of cycles. \square

Corollary 15.17

S_n is generated by simple transpositions, i.e.

$$S_n = \langle (12), (23), (34), \dots, (n-2, n-1), (n-1, n) \rangle$$

Note there are $n - 1$ generators.

Definition 15.18 (Odd vs. even)

A permutation $w \in S_n$ is:

- Odd if w = product of an odd number of transpositions.
- Even if w = product of an even number of transpositions.

This is known as the parity of a permutation.

Example 15.19

$$w = (1542)(37) = (15)(54)(42)(37).$$

Since w is a product of 4 transpositions and 4 is even $\implies w$ is even.

Example 15.20

$$w = (2341) = (23)(34)(41)$$

$\implies w$ is odd

Notice $|w| = 4$ and w is a product of 3 transpositions. So the order \neq parity. Notice we could have written

$$w = (2341) = (12)(24)(43)(24)(43)$$

$\implies w$ is now a product of 5 transpositions.

16 Feb 9, 2022

16.1 The Symmetric Group (Cont'd)

The parity of a permutation is independent of the choice of decomposition into transpositions.

Lemma 16.1

The identity $e \in S_n$ is even not odd.

Proof. Tedious. Please read in book. □

Theorem 16.2

Every permutation is either even or odd, not both.

Proof. Suppose not. Then $\exists w \in S_n$ such that

$$w = \sigma_1 \dots \sigma_n \quad w = \tau_1 \dots \tau_m$$

where n is even and m is odd

$$\begin{aligned} e &= w \dots w^{-1} = \sigma_1 \dots \sigma_n (\tau_1 \dots \tau_m)^{-1} \\ &= \sigma_1 \dots \sigma_n \tau_m \dots \tau_1 \end{aligned}$$

$\implies e$ is a product of $n + m$ transpositions.

$\implies e$ is odd because $n + m$ is odd.

Which is a contradiction. Thus w is either even or odd, not both. □

16.2 The Alternating Group

Definition 16.3 (Alternating group)

For any given S_n , define the alternating group A_n as the set of all even permutations in S_n .

Theorem 16.4

A_n is a subgroup of S_n of order $\frac{n!}{2}$.

Proof. Products and inverses of even permutations remain even. Because $e \in A_n$ by Lemma 16.1. □

Note 16.5: A_n is almost always the only normal simple subgroup of S_n ! This fact is crucial to trying to solve quintic and higher order polynomial equations.

Theorem 16.6

$\forall n \neq 4, A_n$ is simple.

Proof (Sketch). Idea: decompose permutations in A_n into case by case analysis of the cycle lengths of each one. Basically follows from the next two lemmas. □

Lemma 16.7

For $n \geq 3$, every nonidentity element of A_n is a product of cycles of length 3.

Proof. Consider any pair of transpositions $(ab)(cd)$.

- If $a = c, b = d$: $(ab)(ab) = e$.
- If $a = c$: $(ab)(ad) = (adb)$
- Else: $(ab)(cd) = (ab)(bc)(bc)(cd) = (abc)(bcd)$

Since any $w \in A_n$ is a product of an even number of transpositions. \implies This allows you to then write $w =$ product of cycles of length 3. \square

Lemma 16.8

If $N \triangleleft A_n$ and N contains a 3-cycle $\implies N = A_n$. So A_n is simple.

Corollary 16.9

For $n \geq 5$, A_n is the only proper nontrivial normal subgroup of S_n .

Proof. If $N \triangleleft S_n$ then one can show $N \cap A_n \triangleleft A_n$. Since A_n is simple either:

- $N \cap A_n = A_n \implies N = A_n$ or $N = S_n$
- $N \cap A_n = e \implies N = e \cup \{w \in S_n \mid w \text{ odd}\}.$

But N is a subgroup and if w, w' are odd, then $w \cdot w'$ is even.

$\implies N$ is not closed if N contains odd permutations and $N = e \cup \{w \in S_n \mid w \text{ odd}\}.$

$\implies N = e.$ \square

Theorem 16.10 (Cayley's Theorem)

Every group G (finite) is isomorphic to a group of permutations.

Proof. Consider G as a set, let $S(G)$ denote the group of all permutations of the set G . Then,

$$S(G) = \{\text{bijections from } G \rightarrow G \text{ under composition}\}$$

Define:

$$\begin{aligned} \varphi: G &\rightarrow S(G) \\ a &\mapsto \varphi(a) \end{aligned}$$

where

$$\begin{aligned} \varphi(a): G &\rightarrow G \\ g &\mapsto ag \end{aligned}$$

The map $\varphi(a)$ is a bijection:

$$\begin{aligned}
 g_1, g_2 \in G &\implies \varphi(a)(g_1) = \varphi(a)(g_2) \\
 &\implies ag_1 = ag_2 \\
 &\implies g_1 = g_2 \implies \varphi(a) \text{ is 1-1.}
 \end{aligned}$$

Given $g \in G$,

$$\varphi(a)(a^{-1}g) = a \cdot (a^{-1}g) = g$$

$\implies \varphi(a)$ is onto.

$\implies \varphi(a): G \rightarrow G$ is a bijection $\forall a \in G$.

$\implies \varphi: G \rightarrow S(G)$ is well-defined.

φ is a homomorphism: Let $a, b, g \in G$.

Want: $\varphi(ab) = \varphi(a) \circ \varphi(b)$

$$\begin{aligned}
 \varphi(ab)(g) &= (\varphi(a) \circ \varphi(b))(g) \quad \forall g \in G \\
 abg &= \underbrace{\varphi(a)(bg)}_{abg}
 \end{aligned}$$

φ is injective: $\forall g \in G$,

$$\begin{aligned}
 \varphi(a) = \varphi(b) &\implies \varphi(a)(g) = \varphi(b)(g) \\
 &\implies ag = bg \\
 &\implies a = b
 \end{aligned}$$

$\implies \varphi$ is injective.

$\implies \varphi: G \rightarrow S(G)$ is an injective group homomorphism such that $G \cong \text{Im } \varphi \subseteq S(G)$.

$\implies G$ is isomorphic to a group of permutations. \square

Corollary 16.11

If $|G| < \infty$ then G is isomorphic to a subgroup of S_n with $n = |G|$.

17 Feb 11, 2022

17.1 Direct Products

Definition 17.1 (Direct product)

Given G_1, \dots, G_k groups, the direct product $G_1 \times \dots \times G_k$ is the group with elements (g_1, \dots, g_k) with $g_i \in G_i \forall i$ and with binary operation:

$$(g_1, \dots, g_k)(g'_1, \dots, g'_k) = (g_1g'_1, \dots, g_kg'_k)$$

Notation 17.2: In additive notation, we denote it instead as direct sum and write it as $G_1 \oplus \dots \oplus G_k$.

Fact 17.3: $|G_1 \times \dots \times G_k| = \prod_{i=1}^k |G_i|$

Example 17.4

$\mathcal{U}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(1, 0, 0), (1, 1, 0), (1, 0, 1), (1, 1, 1), (2, 0, 0), (2, 1, 0), (2, 0, 1), (2, 1, 1)\}$
So,

$$|\mathcal{U}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2| = 2 \times 2 \times 2 = 8$$

Example 17.5

$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 = \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3$.

In this case either notation is fine.

Theorem 17.6

Given $N_i \triangleleft G$ for $i = 1, \dots, k$. Suppose each $g \in G$ can be uniquely written as a product $g = n_1 \dots n_k$ with $n_i \in N_i \forall i$. Then $G \cong N_1 \times N_2 \times \dots \times N_k$.

To prove this theorem, we need a lemma:

Lemma 17.7

Suppose $M, N \triangleleft G$ such that $M \cap N = \{e\}$. If $a \in M$ and $b \in N$ then $ab = ba$.

Proof. Let $a \in M, b \in N$. We need to show that $aba^{-1}b^{-1} \in M \cap N$.

1. Since $M \triangleleft G$ then $ba^{-1}b^{-1} \in M \implies \underbrace{a}_{\in M} \cdot \underbrace{ba^{-1}b^{-1}}_{\in M} \in M$ because M is closed.
2. Since, $N \triangleleft G$ then $aba^{-1} \in N \implies \underbrace{aba^{-1}}_{\in N} \cdot \underbrace{b^{-1}}_{\in N} \in N$
3. $aba^{-1}b^{-1} \in M \cap N = \{e\} \implies aba^{-1}b^{-1} = e \implies ab = ba$

□

Proof of Theorem 17.6. Define a map

$$\begin{aligned}\varphi: N_1 \times \dots \times N_k &\rightarrow G \\ (n_1, \dots, n_k) &\mapsto n_1 \dots n_k = g\end{aligned}$$

- φ is surjective: follows from the fact that $\forall g$ can be written as $n_1 \dots n_k$.
- φ is injective: follows from the product $n_1 \dots n_k = g$ being unique.
- φ is a homomorphism:

$$(n_1, \dots, n_k), (n'_1, \dots, n'_k) \in N_1 \times \dots \times N_k$$

So

$$\begin{aligned}\varphi((n_1, \dots, n_k) \cdot (n'_1, \dots, n'_k)) &= \varphi((n_1 n'_1, \dots, n_k n'_k)) \\ &= n_1 n'_1 \cdot n_2 n'_2 \cdot \dots \cdot n_{k-1} n'_{k-1} \cdot n_k n'_k \\ &= n_1 n'_1 \cdot n_2 n'_2 \cdot \dots \cdot n'_{k-2} n_{k-1} n_k n'_{k-1} n'_k \\ &= n_1 n'_1 \cdot n_2 n'_2 \cdot \dots \cdot n_{k-1} n'_{k-2} n_k n'_{k-1} n'_k \\ &= n_1 n'_1 \cdot n_2 n'_2 \cdot \dots \cdot n_{k-1} n_k n'_{k-2} n'_{k-1} n'_k \\ &\vdots \\ &= n_1 n_2 \dots n_k \cdot n'_1 \dots n'_{k-1} n'_k \\ &= \varphi(n_1, \dots, n_k) \cdot \varphi(n'_1, \dots, n'_k)\end{aligned}$$

$\implies \varphi$ is an isomorphism $\implies G \cong N_1 \times \dots \times N_k$. □

Definition 17.8 (Direct product and direct factor)

Whenever $G = N_1 \times \dots \times N_k$ we say G is the direct product of the N_i 's and each N_i is a direct factor.

Definition 17.9

Given $N, M \subseteq G$, subgroups, let $MN = \{mn \in G \mid m \in M, n \in N\}$. Note MN is not necessarily a group.

Theorem 17.10

If $M, N \triangleleft G$ such that $G = MN$ and $M \cap N = \{e\}$. Then $G = M \times N$.

Proof. By Theorem 17.6 we only need to show uniqueness.

Suppose $g \in G$ such that $g = mn$ and $g = m'n'$ and $m, m' \in M, n, n' \in N$.

$$\implies mn = m'n' \implies \underbrace{(m')^{-1} \cdot m}_{\in M} = \underbrace{n'n^{-1}}_{\in N}$$

$$\implies (m')^{-1}m \text{ and } n'n^{-1} \in M \cap N = \{e\}$$

$$\implies n'n^{-1} = e \implies n' = n \text{ and } (m')^{-1}m = e \implies m' = m$$

$$\implies g = mn \text{ is a unique decomposition.} \quad \square$$

18 Feb 14, 2022

18.1 Midterm

19 Feb 16, 2022

19.1 Direct Products (Cont'd)

Recall 19.1 If $M, N \triangleleft G, G = MN, M \cap N = \{e\} \implies G = M \times N$.

Example 19.2

Consider $\underbrace{\mathcal{U}_{15}}_G = \{1, 2, 4, 7, 8, 11, 13, 14\}$

$$N = \{1, 2, 4, 8\} \quad M = \{1, 11\}$$

- N, M are normal subgroups of \mathcal{U}_{15} .
- $M \cap N = \{e\}$
- $\mathcal{U}_{15} = MN$

We want to show that 7, 13, 14 are products mn for some $m \in M, n \in N$.

$$7 \cong 11 \cdot 2 \pmod{15}$$

$$13 \cong 11 \cdot 8 \pmod{15}$$

$$14 \cong 11 \cdot 4 \pmod{15}$$

$\implies \mathcal{U}_{15} = MN$. By theorem, $\mathcal{U}_{15} \cong M \times N$.

$$\underbrace{N = \{1, 2, 4, 8\}}_{|N|=4} \quad \underbrace{M = \{1, 11\}}_{|M|=2 \implies M \cong \mathbb{Z}_2}$$

Check whether N has an element of order 4.

$|1| = 1, |2| = 4$ We know

$$2 \cdot 2 \cdot 2 \cdot 2 = 16 \cong 1 \pmod{15}$$

$$2^4 \cong 1 \pmod{15}$$

$$\implies \mathcal{U}_{15} \cong \mathbb{Z}_2 \times \mathbb{Z}_4$$

19.2 Finite Abelian Groups

Step 1: Change to additive notation.

$$ab \mapsto a + b$$

$$a^k \mapsto k \cdot a$$

$$e \mapsto 0$$

$$MN \mapsto M + N = \{m + n \mid m \in M, n \in N\}$$

$$M \times N \mapsto M \oplus N$$

$$\text{direct factors} \mapsto \text{direct summands}$$

Definition 19.3

G -abelian group, p -prime.

$$G(p) := \{a \in G \mid \underbrace{|a| = p^n}_{p^n \cdot a = 0} \text{ some } n \geq 0\}$$

Proposition 19.4

$G(p)$ is a subgroup of G .

We will show: $G = \bigoplus_{p_i} G(p_i)$

Lemma 19.5

G -abelian group, $a \in G$ with $|a| = p_1^{n_1} \dots p_k^{n_k} < \infty$ with p_i prime, $p_i \neq p_j, i \neq j$. Then, $a = a_1 + \dots + a_k$ with $a_i \in G(p_i)$ each i .

Proof. Proof by induction on k .

Base case: ($k = 1$) $|a| = p_1^{n_1}$.

By definition of $G(p_1) \implies a \in G(p_1)$.

Inductive step: Suppose statement is true for elements that are divisible by at most $k - 1$ distinct primes. Then if $|a| = p_1^{n_1} \dots p_k^{n_k}$, let $m = p_1^{n_1} \dots p_{k-1}^{n_{k-1}}$

$$\implies (m, p_k^{n_k}) = 1$$

$$\implies \exists u, v \text{ such that } 1 = um + vp_k^{n_k}$$

Rewrite

$$a = 1 \cdot a = \underbrace{(um)a}_{\in G(p_k)} + (vp_k^{n_k})a$$

$$\implies p_k^{n_k}(uma) = u \cdot (p_k^{n_k}ma) = 0$$

because

$$|a| = p_1^{n_1} \dots p_k^{n_k} = mp_k^{n_k} \implies (p_k^{n_k} \cdot ma) = 0 \implies uma \in G(p_k).$$

Likewise:

$$m(vp_k^{n_k}a) = v(mp_k^{n_k}a) = 0 \implies vp_k^{n_k}a \text{ has order } m.$$

Since m is a product of $k - 1$ primes \implies induction hypothesis applied to $vp_k^{n_k}a$

$$\implies vp_k^{n_k}a = a_1 + \dots + a_{k-1} \text{ with } a_i \in G(p_i) \quad 1 \leq i \leq k - 1$$

$$a = 1 \cdot a = \underbrace{(um)a}_{\in G(p_k)} + \underbrace{(vp_k^{n_k})a}_{\implies a_1 + \dots + a_{k-1}}$$

$$\implies a = \underbrace{a_1}_{G(p_1)} + \dots + \underbrace{a_{k-1}}_{G(p_{k-1})} + \underbrace{a_k}_{G(p_k)}$$

□

Theorem 19.6

G -abelian group with $|G| = p_1^{n_1} \cdots p_k^{n_k} < \infty$, p_i are distinct primes. Then

$$G = G(p_1) \oplus \cdots \oplus G(p_k)$$

Proof. If $a \in G$, $|a| \mid |G|$, so we can write

$$a = a_1 + \cdots + a_k \text{ with } a_i \in G(p_i) \text{ each } i$$

where some a_i may be zero. This tells us that there exists such a decomposition. We need to prove uniqueness.

Suppose it's not unique.

$$a = a_1 + \cdots + a_k = b_1 + \cdots + b_k$$

where $a_i, b_i \in G(p_i)$ each i .

$$\implies (a_1 - b_1) + (a_2 - b_2) + \cdots + (a_k - b_k) = 0$$

$$\implies \underbrace{(a_1 - b_1)}_{\in G(p_1)} = \underbrace{(b_2 - a_2)}_{\in G(p_2)} + \cdots + \underbrace{(b_k - a_k)}_{\in G(p_k)}$$

$$\implies p_2^{n_2} \cdots p_k^{n_k} (a_1 - b_1) = 0 \text{ for some } n_i \in \mathbb{N}$$

$$\implies \underbrace{|a_1 - b_1|}_{\in G(p_1)} \mid p_2^{n_2} \cdots p_k^{n_k}$$

$$\implies p_1^{n_1} \mid p_2^{n_2} \cdots p_k^{n_k}$$

which is impossible for $p_1^{n_1} \geq p_1 \implies n_1 = 0$

$$\implies |a_1 - b_1| = p_1^{n_1} = p_1^0 = 1$$

$a_1 - b_1 = 0 \implies a_1 = b_1 \implies$ by doing the same thing for each

$$(a_i - b_i) \implies a_i = b_i \quad \forall i$$

This proves uniqueness.

$$\implies G = G(p_1) \oplus \cdots \oplus G(p_k)$$

□

20 Feb 18, 2022

20.1 Finite Abelian Groups (Cont'd)

Recall 20.1 G -abelian with $|G| = p_1^{n_1} \dots p_k^{n_k} < \infty$

$$G = G(p_1) \oplus \dots \oplus G(p_k)$$

Definition 20.2 (p -group)

For p -prime, a p -group is a group G such that $G = G(p)$.

Fact 20.3: If G - p group, $a \in G$ has maximal order with $|a| = p^n$ then $\forall b \in G, b \neq a$

- $|b| = p^j$ with $j \leq n$
- $p^n \cdot b = 0$ (additive notation)

If $|G| < \infty$ then maximal order elements always exist.

Example 20.4 • $G = \mathbb{Z}_{p^k} \implies |G| = p^k$

- $G = \mathbb{Z}_2 \times \mathbb{Z}_2$
- $G = D_4, \mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Lemma 20.5

G -finite abelian p -group and $a \in G$ with maximal order. Then there exists a subgroup $K \subseteq G$ such that $G = \langle a \rangle \oplus K$.

Proof. Let K be the largest subgroup of G such that $K \cap \langle a \rangle = 0$. (G finite $\implies K$ exists). G abelian $\implies K \triangleleft G$ and $\langle a \rangle \triangleleft G$.

By Theorem 17.1, since $K \triangleleft G, \langle a \rangle \triangleleft G$ and $K \cap \langle a \rangle = 0$, to show that $G = K \oplus \langle a \rangle$ we need only show that $G = K + \langle a \rangle$.

Suppose that $G \neq K + \langle a \rangle \implies \exists b \in G$ s.t. $b \notin K + \langle a \rangle$.

In particular $b \neq a$ where a is the max order element. Then since G is a p -group and b is not a max element then

$$p^j b = 0 \text{ for some } j \quad (|b| = p^j).$$

Let r = smallest possible integer such that

$$\left. \begin{array}{l} p^r b \in \langle a \rangle + K \\ 0 \in \langle a \rangle + K, r \leq j \end{array} \right\} \implies p^r b = ta + k \quad (t \in \mathbb{Z}, k \in K)$$

Suppose $|a| = p^n$ so that $p^n a = 0$ then

$$p^n b = 0 \quad \forall b \in G \text{ since } a \text{ is maximal}$$

So,

$$\begin{aligned} p^n(p^{r-1}b) &= p^{n-1}(p^r b) \\ &= p^{n-1}(ta + k) = 0 \end{aligned}$$

Therefore,

$$\begin{aligned} &\implies p^{n-1}ta + p^{n-1}k = 0 \\ &\implies \underbrace{p^{n-1}ta}_{\in \langle a \rangle} = \underbrace{-p^{n-1}k}_{\in K} \in \langle a \rangle \cap K = 0 \\ &\implies p^{n-1}ta = 0 \text{ and } -p^{n-1}k = 0 \\ &\implies |a| = p^n \implies p^n \mid p^{n-1}t \\ &\implies p \mid t \implies t = pm \text{ for some } m \end{aligned}$$

So

$$\left. \begin{aligned} p^r b &= ta + k \\ p^r b &= pma + k \end{aligned} \right\} \implies \begin{cases} k = p^r b - pma \\ = p(p^{r-1}b - ma) \in K \end{cases}$$

If $p^{r-1}b - ma \in K$

$$\begin{aligned} &\implies p^{r-1}b - ma = k' \text{ with } k' \in K \\ &\implies p^{r-1}b = \underbrace{k'}_{\in K} + \underbrace{ma}_{\in \langle a \rangle} \in K + \langle a \rangle \end{aligned}$$

But $r-1 < r$ and we said that r was the smallest integer such that $p^r b \in K + \langle a \rangle$, a contradiction

$$\implies p^{r-1}b - ma \notin K.$$

Let $H := \{x + z(p^{r-1}b - ma) \mid x \in K, z \in \mathbb{Z}\}$. Then

1. H is a subgroup of G
2. $K \subseteq H$ (take $z = 0$)
3. $K \neq H$ because $z = 1, x = 0$ then $p^{r-1}b - ma \in H$ not in K

Since K is the largest subgroup of G such that $K \cap \langle a \rangle = 0$

$$\implies H \cap \langle a \rangle \neq 0$$

$$\implies \exists w \in H \cap \langle a \rangle \text{ s.t. } w \neq 0$$

Note that $K \cap \langle a \rangle = 0 \implies w \notin K$

$$\implies w = sa = x + z(p^{r-1}b - ma) \text{ for some } s, z \in \mathbb{Z}, x \in K.$$

If $p \mid z$ then $z = qp \implies z(p^{r-1}b - ma) \in K$ since $p(p^{r-1}b - ma) \in K$

$$\implies w = \underbrace{x}_{\in K} + \underbrace{z(p^{r-1}b - ma)}_{\in K} \in K$$

A contradiction, so $p \nmid z \implies (p, z) = 1$ this means $1 = up + vz$ for some $u, v \in \mathbb{Z}$ So

$$\begin{aligned}
 p^{r-1}b &= (up + vz)p^{r-1}b \\
 &= up^rb + vzp^{r-1}b \\
 &= u(pma + k) + v(sa - x + zma) \\
 &= \underbrace{(upm + vs + zm)}_{\substack{\text{numbers} \\ \text{multiple of } a \in \langle a \rangle}} a + \underbrace{(uk - vx)}_{\in K} \\
 &\implies p^{r-1}b \in K + \langle a \rangle
 \end{aligned}$$

a contradiction because r was the smallest integer such that $p^rb \in K + \langle a \rangle \implies b$ cannot exist. So we're done and

$$G = K + \langle a \rangle$$

By previous theorem, since $\langle a \rangle, K \triangleleft G$ and $\langle a \rangle \cap K = 0$ and $G = K + \langle a \rangle \implies \langle a \rangle \oplus K = G$. \square

Theorem 20.6 (The Fundamental Theorem of Finite Abelian Groups (I))

Every finite abelian group G is the direct sum of cyclic p -groups i.e. there exists such a decomposition for G s.t.

$$G \cong \bigoplus_i \mathbb{Z}_{p_i^{r_i}} \text{ over some primes (possible repeated)}$$