

# Math 167 (Mathematical Game Theory)

## *University of California, Los Angeles*

Aaron Chao

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**Course description:** Quantitative modeling of strategic interaction. Topics include extensive and normal form games, background probability, lotteries, mixed strategies, pure and mixed Nash equilibria and refinements, bargaining; emphasis on economic examples. Optional topics include repeated games and evolutionary game theory. More information can be found on Math UCLA website.

These are my lecture notes for Math 167 (Mathematical Game Theory) taught by Oleg Gleizer. The main textbook for this class is *Game Theory, Alive* by Anna Karlin and Yuval Peres and the supplementary textbook is *A Course in Game Theory* by Thomas Ferguson.

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# 1 Mar 28, 2022

## 1.1 Impartial Combinatorial Games

**Definition 1.1** (Impartial combinatorial game)

In an impartial combinatorial game,

- Two-person
- Perfect information
- No chance moves
- Win-or-lose outcome

**Example 1.2**

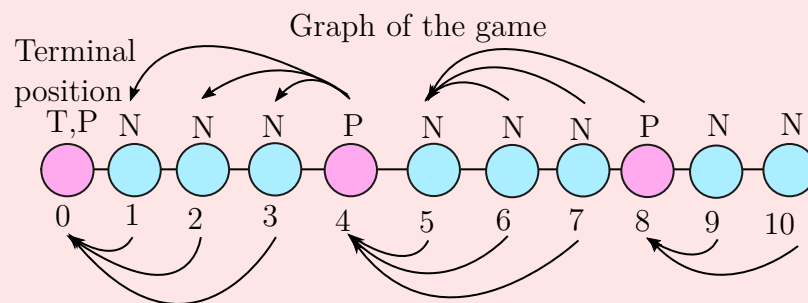
Suppose

- A pile of  $n$  chips on the table
- Two players: P1 and P2
- A move consists of removing one, two, or three chips from the pile
- P1 makes the first move, players alternate then
- The player to remove the last chip wins (the last player to move wins. If a player can't move, they lose.)

Method to analyze: backward induction.

Positions:

- **N**, next player to take a move wins.
- **P**, previous (second) player to take a move wins.



Any move from a **P** position leads to an **N** position. There always exists a move from an **N** position to a **P** position.

Ending condition: the game ends in a finite number of moves, no matter how played.

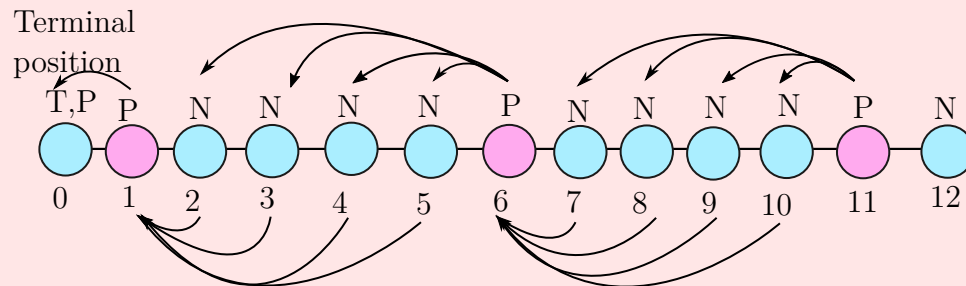
A **T** position is a **P** position.

**Definition 1.3** (Normal play vs. misère play)

In a normal play, the last player to move wins. In a misère play, the last player to move loses.

**Example 1.4**

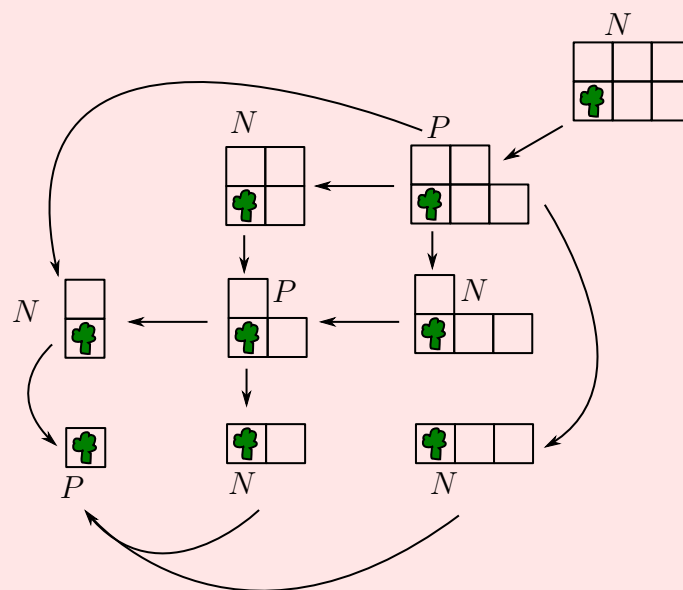
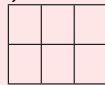
A misère game, a player can take 1-4 chips.



**Note 1.5:** Every position is either **N** or **P**, but not nothing or both.

**Example 1.6** (The game of Chomp)

Let's play the game starting from



Graph of the game:

- Positions correspond to vertices
- Moves correspond to oriented edges

**Definition 1.7** (Strategy)

A function that assigns a move to each position, except for the terminal.

**Definition 1.8** (Winning strategy from a position  $x$ )

A winning strategy from a position  $x$  is a sequence of moves, starting from  $x$ , that guarantees a win.

Consider a normal game. Let  $\mathbf{N}_i/\mathbf{P}_i$  be the set of positions from which P1/P2 can win (reach the nearest terminal vertex of the same graph) in at most  $i$  moves.

$$\mathbf{P}_0 = \mathbf{P}_1 = \{\text{terminal positions}\}$$

$$\mathbf{N}_{i+1} = \{x: \text{there is a move from } x \text{ to } \mathbf{P}_i\}$$

$$\mathbf{P}_{i+1} = \{y: \text{each move leads to } \mathbf{N}_i\}$$

**Note 1.9:**  $\mathbf{P}_0 = \mathbf{P}_1 \subseteq \mathbf{P}_2 \subseteq \mathbf{P}_3 \dots$

$$\mathbf{N}_1 \subseteq \mathbf{N}_2 \subseteq \mathbf{N}_3 \dots$$

$$\mathbf{N} = \bigcup_{i=1} \mathbf{N}_i, \quad \mathbf{P} = \bigcup_{i=0} \mathbf{P}_i$$

**Definition 1.10** (Progressively bounded)

A game is called progressively bounded if for every position  $x$  there exists an upper bound  $B(x)$  on the number of moves until the game terminates.

## 2 Mar 30, 2022

### 2.1 Combinatorial Games (Cont'd)

**Recall 2.1** •  $P_0 = P_1 = \{\text{terminal positions}\}$

- $N_{n+1} = \{x: \text{there is a move from } x \text{ to } P_n\}$
- $P_{n+1} = \{y: \text{each move from } y \text{ leads to } N_n\}$
- $P_0 = P_1 \subseteq P_2 \subseteq \dots$
- $N_1 \subseteq N_2 \subseteq N_3 \subseteq \dots$
- $P = \bigcup_{n=0} P_n$
- $N = \bigcup_{n=1} N_n$
- A game is called progressively bounded if for every position  $x$  there exists an upper bound  $B(x)$  on the number of moves until the game stops.

#### Theorem 2.2

In a progressively bounded impartial full information combinatorial game, all positions are in  $N \cup P$ . Thus, for every position there exists a winning strategy.

**Proof.** Let  $B(x) \leq n$ . Let us prove by induction that  $x \in N_n \cup P_n$ .

Base:  $n = 0$

$x$  is a terminal vertex  $\implies x \in P_0 = P_1$ .

Inductive hypothesis by  $P_0$ :  $B(x) \leq n \implies x \in N_n \cup P_n$ .

Inductive step: Show that  $B(x) \leq n + 1 \implies x \in N_{n+1} \cup P_{n+1}$

Consider a move  $x \rightarrow y$  and  $B(y) \leq n$ . Hence,  $y \in N_n \cup P_n$ . So either

Case 1: Each move from  $x$  leads to  $y \in N_n \implies x \in P_{n+1}$ .

Case 2: There exists a move from  $x$  to  $y \notin N_n$ . Thanks to the inductive typo,  $y \in N_n \cup P_n$  so  $y \in P_n \implies x \in N_{n+1}$ .  $\square$

### 2.2 The Game of Nim

- Several piles, each containing finitely many chips.
- A move: a player can remove any number of chips, from one to all from any pile
- P1 and P2 alternate taking moves
- The player to take the last chip wins



Consider  $x \oplus y$ . We rewrite  $x$  and  $y$  as binary numbers and perform long addition of  $x_2$  and  $y_2$  without carry-over, i.e. mod 2.

$$5 \oplus 7 = \begin{array}{r} \phantom{0} 1 \phantom{0} 1 \\ \oplus \phantom{0} 1 \phantom{0} 1 \\ \hline \phantom{0} 0 \phantom{0} 1 \phantom{0} 0 \end{array} = 2$$

A position  $x = (x_1, x_2, \dots, x_k)$  is a **P** position  $\iff x_1 \oplus x_2 \oplus \dots \oplus x_k = 0$ .

## 3 April 1, 2022

### 3.1 The Game of Nim (Cont'd)

**Recall 3.1**  $x = (x_1, x_2, \dots, x_k)$

Theorem (Bouton) says  $x \in \mathbf{P} \iff x_1 \oplus x_2 \oplus \dots \oplus x_k = 0$ .

**Proof of Theorem 2.6.** We have

Terminal position:  $x = (0, 0, \dots, 0) \in \mathbf{P}$  Let  $x \in \mathbf{N}$ . Then there exists a move  $x \rightarrow y \in \mathbf{P}$ .

$$x_1 \oplus x_2 \oplus \dots \oplus x_k = \oplus \begin{array}{cccccc} 1 & * & * & \dots & \dots & * & * \\ & & 1 & * & \dots & * & * \\ & & \vdots & \vdots & & \vdots & \vdots \\ 1 & * & * & \dots & \dots & \dots & * & * \end{array}$$

Find the left-most (most significant) column with an odd number of 1's. Change any number that has a 1 in the column so that there is an even number of 1's in every column. The 1 in the most significant position becomes a 0 which implies the number becomes smaller. So this is a legal move.

We have  $x \in \mathbf{P} \implies$  any move  $x \rightarrow y \in \mathbf{N}$  where

$$x = (x_1, x_2, \dots, x_k) \mapsto y = (x'_1, x_2, \dots, x_k)$$

such that

$$x'_1 < x_1 \text{ and } x_1 \oplus x_2 \oplus \dots \oplus x_k = 0.$$

If

$$x'_1 \oplus x_2 \oplus \dots \oplus x_k = 0$$

then

$$x'_1 \oplus x_2 \oplus \dots \oplus x_k = 0$$

then  $x'_1 = x_1$ , a contradiction. Hence

$$x'_1 \oplus x_2 \oplus \dots \oplus x_k \neq 0 \implies y \in \mathbf{N}.$$

□

**Example 3.2**

$$x_1 = 7$$

$$x_2 = 10$$

$$x_3 = 15$$

$$\begin{array}{ccc|ccc} & 0 & 1 & 1 & 1 & 1 \\ \oplus & 1 & 0 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 1 & 1 \\ \hline & 0 & 0 & 1 & 0 & 0 \end{array} \implies \begin{array}{ccc|ccc} & 0 & 1 & 0 & 1 & 1 \\ \oplus & 1 & 0 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 1 & 1 \\ \hline & 0 & 0 & 0 & 0 & 0 \end{array}$$

So we have that  $(7, 10, 15) \mapsto (5, 10, 15)$

**3.2 Subtraction Nim**

Extra condition: A player can remove at most  $n$  chips.

We find pile sizes mod  $n + 1$ , i.e.

$$(x_1, x_2, \dots, x_k) \mapsto (x_1 \bmod n + 1, x_2 \bmod n + 1, \dots, x_k \bmod n + 1)$$

Now we find the Nim-sum and make a move.

$$x \bmod n + 1 = \underbrace{(x_1 \bmod n + 1, x_2 \bmod n + 1, \dots, x_k \bmod n + 1)}_{(x_1 \bmod n+1)_2 \oplus (x_2 \bmod n+1)_2 \oplus \dots \oplus (x_k \bmod n+1)_2} \implies \begin{cases} = 0 \iff \mathbf{P} \\ \neq 0 \iff \mathbf{N} \end{cases}$$

**Example 3.3**

We have  $x = (12, 13, 14)$  and  $n = 3$ . So,

$$(12 \bmod 4, 13 \bmod 4, 14 \bmod 4) \equiv (0, 1, 2) = (0_2, 1_2, 10_2)$$

So

$$\begin{array}{cc} & 0 & 0 \\ \oplus & 0 & 1 \\ & 1 & 0 \\ \hline & 1 & 1 \end{array} \neq 0$$

so we take away one chip from the third pile

$$\begin{array}{cc} & 0 & 0 \\ \oplus & 0 & 1 \\ & 0 & 1 \\ \hline & 0 & 0 \end{array}$$

So we have that  $(12, 13, 14) \mapsto (12, 13, 13)$ .

**Note 3.4:** You can always make a legal move  $\mathbf{N} \rightarrow \mathbf{P}$  by removing  $i \leq n$  chips from a pile.

**Note 3.5:** To move from  $\mathbf{P}$  to  $\mathbf{P}$ , you need to remove  $n + 1$  chips from a pile. Not allowed! Hence, any move from  $\mathbf{P}$  is to  $\mathbf{N}$ .

**Example 3.6**

We have  $x = (12, 13, 13)$ , with  $n = 3$ . So

$$x \bmod 4 = (0, 1, 1)$$

therefore

$$\begin{array}{r} 0 \\ \oplus \quad 1 \\ 1 \\ \hline 0 \end{array}$$

### 3.3 Two-Person Zero Sum Games (Strategic Form)

We have

- P1: a non-empty set of strategies S1
- P2: a non-empty set of strategies S2
- $A: S1 \times S2 \rightarrow \mathbb{R}$ , the min function for P1 (payoff matrix)

**Note 3.7:** Since the game is zero-sum, a win for P1 is a loss for P2.  $A(i, j)$  can be  $\leq 0$ , so works both ways.

Pure strategies:

		P2			
		S21	S22	...	S2n
P1	S11	$a_{11}$	$a_{12}$	$\cdots$	$a_{1n}$
	S12	$a_{21}$	$a_{22}$	$\cdots$	$a_{2n}$
	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	S1m	$a_{m1}$	$a_{m2}$	$\cdots$	$a_{mn}$

A game. P1 chooses the strategy S1*i*. Simultaneously, P2 chooses the strategy S2*j*. P1 wins  $a_{ij}$ .

**Lemma 3.8**

$$\min_j \max_i a_{ij} \geq \max_i \min_j a_{ij}$$

We will continue in the next lecture.

# 4 Apr 4, 2022

## 4.1 Two-Person Zero Sum Games in Strategic Form (Cont'd)

**Recall 4.1** Recall that

		P2			
		S21	S22	...	S2n
P1	S11	$a_{11}$	$a_{12}$	$\cdots$	$a_{1n}$
	S12	$a_{21}$	$a_{22}$	$\cdots$	$a_{2n}$
	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	S1m	$a_{m1}$	$a_{m2}$	$\cdots$	$a_{mn}$

P1 has a non-empty set of pure strategies

$$S1 = \{S11, S12, \dots, S1m\}$$

P2 has a non-empty set of pure strategies

$$S2 = \{S21, S22, \dots, S2n\}$$

$A: S1 \times S2 \rightarrow \mathbb{R}$ , payoff matrix

P1,  $S1i: a_{i1}, a_{i2}, \dots, a_{in}$

Betting on the worst possible outcome, P1 bets on  $\min_{1 \leq j \leq n} a_{ij}$ . Being intelligent, P1 chooses

$$\max_{1 \leq i \leq m} \min_{1 \leq j \leq n} a_{ij}.$$

Betting on the worst possible loss, P2 bets on  $\max_{1 \leq i \leq m} a_{ij}$ . Being intelligent, P2 chooses

$$\min_{1 \leq j \leq n} \max_{1 \leq i \leq m} a_{ij}$$

### Lemma 4.2

$$\max_{1 \leq i \leq m} \min_{1 \leq j \leq n} a_{ij} \leq \min_{1 \leq i \leq m} \max_{1 \leq j \leq n} a_{ij}$$

**Proof.** Let

$$\max_i \min_j a_{ij} = a_{pq}$$

$$\min_j \max_i a_{ij} = a_{rs}$$

		$q$		$s$	
$p$		$a_{pq}$	$\leq$	$a_{ps}$	
				$\setminus \nearrow$	
$r$		$a_{rq}$		$a_{rs}$	

□

**Example 4.3**

Chooser (P1), Hider (P2). Hider hides behind their back

- Either left hand with one coin
- or right hand with two coins

Chooser chooses L or R,

		P2	
		L1	R2
P1	L	1	0
	R	0	2

$$P1: \max_j \min_i a_{ij} = 0$$

$$P2: \min_j \max_i a_{ij} = 1$$

Mixed strategies

			P2	
			L1, $q$	R2, $1 - q$
P1	L, $p$		1	0
	R, $1 - p$		0	2

P1: if P2 chooses the strategy L1, the expected gain is

$$1 \cdot p + 0 \cdot (1 - p) = p$$

If P2 chooses R2, the expected gain is

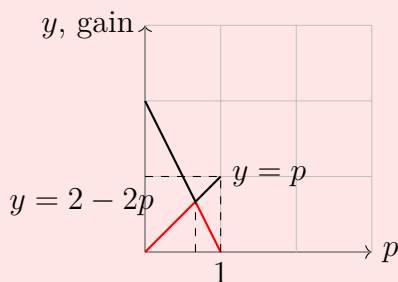
$$0 \cdot p + 2(1 - p) = 2 - 2p.$$

If P1 is out of luck, then the expected gain is

$$\min\{p, 2 - 2p\}$$

Since P1 is intelligent, they choose  $p$  s.t. the gain is

$$\max_{0 \leq p \leq 1} \min\{p, 2 - 2p\}$$



$$2 - 2p = p$$

$$2 = 3p$$

$$p = \frac{2}{3}$$

The optimal strategy is

$$\frac{2}{3}L + \frac{1}{3}R$$

With expected gain  $\geq \frac{2}{3}$ .

P2 is thinking. If P1 chooses  $L$ , my expected loss is

$$1 \cdot q + 0 \cdot (1 - q) = q$$

If P1 chooses  $R$ , my expected loss is

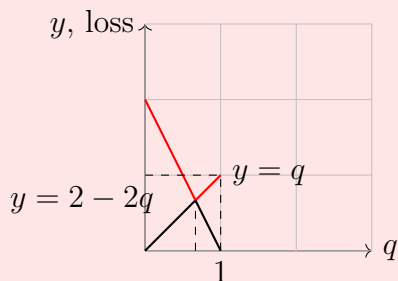
$$0 \cdot q + 2(1 - q) = 2 - 2q$$

Suppose I'm out of luck. Then my expected loss is

$$\max\{q, 2 - 2q\}$$

Being my very smart self,

$$\min_{0 \leq q \leq 1} \max\{q, 2 - 2q\}$$



The optimal strategy is

$$\frac{2}{3}L1 + \frac{1}{3}R2$$

With expected loss  $\leq \frac{2}{3} = V$ , the value of the game.

Let us generalize  $A \in \mathbb{R}^{n \times m}$ , an  $n \times m$  payoff matrix.

$$\Delta_m = \left\{ \mathbf{p} \in \mathbb{R}^m : p_1 \geq 0, p_2 \geq 0, \dots, p_m \geq 0, \sum_{i=1}^m p_i = 1 \right\}$$

$$\Delta_n = \left\{ \mathbf{q} \in \mathbb{R}^n : q_1 \geq 0, q_2 \geq 0, \dots, q_n \geq 0, \sum_{j=1}^n q_j = 1 \right\}$$

A mixed strategy for P1 is determined by

$$\mathbf{p} \in \Delta_m$$

A mixed strategy for P2 is determined by

$$\mathbf{q} \in \Delta_n$$

Expected gain for P1 (expected loss for P2) =  $(\mathbf{p})^T A \mathbf{q}$

		P2			
		$q_1$	$q_2$	$\cdots$	$q_n$
P1	$p_1$	$a_{11}$	$a_{12}$	$\cdots$	$a_{1n}$
	$p_2$	$a_{21}$	$a_{22}$	$\cdots$	$a_{2n}$
	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	$p_m$	$a_{m1}$	$a_{m2}$	$\cdots$	$a_{mn}$

So

$$(\mathbf{p})^t A \mathbf{q} = p_i(a_{i1}q_1 + a_{i2}q_2 + \cdots + a_{in}q_n)$$

If P1 employs the strategy  $\mathbf{P}$ , then in the worst case their payoff is

$$\min_{\mathbf{q} \in \Delta_n} (\mathbf{p})^t A \mathbf{q} = \min_{1 \leq j \leq n} \sum_{i=1}^m a_{ij} p_i$$

Hence, P1's winning strategy is

$$\max_{\mathbf{p} \in \Delta_m} \min_{\mathbf{q} \in \Delta_n} \mathbf{p}^T A \mathbf{q}$$



## 5 Apr 6, 2022

### 5.1 General Two-Person Zero-Sum Games in Strategic Form

**Recall 5.1** Recall

		P2			
		$q_1$	$q_2$	$\cdots$	$q_n$
P1	$p_1$	$a_{11}$	$a_{12}$	$\cdots$	$a_{1n}$
	$p_2$	$a_{21}$	$a_{22}$	$\cdots$	$a_{2n}$
	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	$p_m$	$a_{m1}$	$a_{m2}$	$\cdots$	$a_{mn}$

With set of mixed strategies given by,

$$\Delta_m = \left\{ \mathbf{p} \in \mathbb{R}^m : \mathbf{p} \geq 0, \sum_{i=1}^m p_i = 1 \right\}$$

$$\Delta_n = \left\{ \mathbf{q} \in \mathbb{R}^n : \mathbf{q} \geq 0, \sum_{j=1}^n q_j = 1 \right\}$$

where  $p_1 \geq 0, p_2 \geq 0, \dots, p_m \geq 0$ .

We have

$$\text{Expected gain of P1} = (\mathbf{p})^t A \mathbf{q}$$

with  $\mathbf{p} \in \Delta_m$  and  $\mathbf{q} \in \Delta_n$ .

The winning strategy for P1:

- Worst case:  $\min_{\mathbf{q} \in \Delta_n} (\mathbf{p})^t A \mathbf{q}$
- Smart choice:  $\max_{\mathbf{p} \in \Delta_m} \min_{\mathbf{q} \in \Delta_n} (\mathbf{p})^t A \mathbf{q}$

$$\begin{aligned} \min_{\mathbf{q} \in \Delta_n} (\mathbf{p})^t A \mathbf{q} &= \min_{\mathbf{q} \in \Delta_n} \sum_{j=1}^n q_j \sum_{i=1}^m a_{ij} p_i \\ &= \min_{1 \leq j \leq n} \sum_{i=1}^m a_{ij} p_i \end{aligned}$$

The winning strategy for P2:

- Worst case:  $\max_{\mathbf{p} \in \Delta_m} (\mathbf{p})^t A \mathbf{q}$
- Smart choice:  $\min_{\mathbf{q} \in \Delta_n} \max_{\mathbf{p} \in \Delta_m} (\mathbf{p})^t A \mathbf{q}$

$$\begin{aligned}\max_{\mathbf{p} \in \Delta_m} (\mathbf{p})^t A \mathbf{q} &= \max_{\mathbf{p} \in \Delta_m} \sum_{i=1}^m p_i \sum_{j=1}^n a_{ij} q_j \\ &= \max_{1 \leq i \leq m} \sum_{j=1}^n a_{ij} q_j\end{aligned}$$

**Definition 5.2** (Safety value for P1 vs. P2)

The value  $\hat{\mathbf{p}}$  at which

$$\max_{\mathbf{p} \in \Delta_m} \min_{\mathbf{q} \in \Delta_n} (\mathbf{p})^t A \mathbf{q}$$

is attained is called the safety value for P1. The value  $\hat{\mathbf{q}}$  at which

$$\min_{\mathbf{q} \in \Delta_n} \max_{\mathbf{p} \in \Delta_m} (\mathbf{p})^t A \mathbf{q}$$

is attained is called the safety value for P2.

**Theorem 5.3** (Von Neumann Minimax Theorem)

For any two-person zero-sum game with  $m \times n$  payoff matrix  $A$ , there is a number  $V$ , called the value of the game, satisfying

$$\min_{\mathbf{q} \in \Delta_n} \max_{\mathbf{p} \in \Delta_m} (\mathbf{p})^t A \mathbf{q} = \max_{\mathbf{p} \in \Delta_m} \min_{\mathbf{q} \in \Delta_n} (\mathbf{p})^t A \mathbf{q} = V$$

Let  $\hat{\mathbf{p}}$  be an optimal solution for P1. Let  $\hat{\mathbf{q}}$  be an optimal solution for P2. Then

$$\min_{\mathbf{q} \in \Delta_n} (\hat{\mathbf{p}})^t A \mathbf{q} = \max_{\mathbf{p} \in \Delta_m} (\hat{\mathbf{p}})^t A \hat{\mathbf{q}}$$

**Proof.** Proof seen here. □

**Definition 5.4** (Value of the game)

Given conditions from Von Neumann Minimax Theorem,  $V$  is the value of the game.

**Example 5.5** (Odd or Even) • P1 and P2 simultaneously call out one of the numbers, 1 or 2.

- If the sum is odd, P1 wins and gets the sum of the numbers in \$
- If the sum is even, P2 wins and gets the sum of the numbers in \$

		P2	
		1, $q$	2, $1 - q$
P1	1, $p$	-2	3
	2, $1 - p$	3	-4

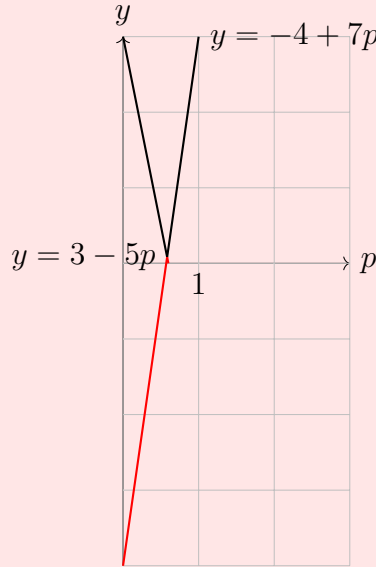
So P1's expected win (P2's expected loss) is

$$\begin{aligned} (\mathbf{p})^t A \mathbf{q} &= \begin{bmatrix} p & 1-p \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} q \\ 1-q \end{bmatrix} \\ &= -12pq + 7p + 7q - 4 \end{aligned}$$

P1's worst possible case:

$$f(p) = \min_{0 \leq q \leq 1} \{-12pq + 7p + 7q - 4\}$$

$$\begin{aligned} q, S21: \quad & -2p + 3(1-p) = 3 - 5p \\ 1-q, S22: \quad & 3p - 4(1-p) = -4 + 7p \end{aligned}$$



- If  $3 - 5p \geq -4 + 7p$ , then  $q = 0$ .
- If  $3 - 5p < -4 + 7p$ , then  $q = 1$ .

Hence,

$$f(p) = \min\{3 - 5p, -4 + 7p\}$$

Note that

$$\begin{aligned} (-12pq + 7p + 7q - 4) \Big|_{q=0} &= -4 + 7p \\ (-12pq + 7p + 7q - 4) \Big|_{q=1} &= 3 - 5p \end{aligned}$$

$$\text{P1: } \max_{0 \leq p \leq 1} \min_{0 \leq q \leq 1} q(-2p + 3(1-p)) + (1-q)(3p - 4(1-p)) = \max_{0 \leq p \leq 1} \min\{3 - 5p, -4 + 7p\}$$

$$3 - 5p = -4 + 7p$$

$$7 = 12p$$

$$p = \frac{7}{12}, \quad q = \frac{5}{12}$$

Now from P2:

$$\text{P2: } \min_{0 \leq q \leq 1} \max_{0 \leq p \leq 1} p(-2q + 3(1 - q)) + (1 - p)(3q - 4(1 - q)) = \min_{0 \leq q \leq 1} \max\{3 - 5q, -4 + 7q\}$$

## 6 Apr 8, 2022

### 6.1 Solving Small-Dimensional Two-Person Zero-Sum Games Pen-and-Paper

**Definition 6.1** (Saddle point)

An element of  $A$ ,  $a_{ij}$  is called a saddle point if

- $a_{ij}$  is the min of the  $i$ -th row
- $a_{ij}$  is the max of the  $j$ -th column

Then  $p_i = 1$ ,  $q_j = 1$ ,  $V = a_{ij}$

**Example 6.2**

Given

$$\begin{array}{ccccc} & & & & \min \\ & & & & -3 \\ & & & & \textcircled{2} \\ & & & & 0 \\ \max & \begin{bmatrix} 4 & 1 & -3 \\ 3 & \textcircled{2} & 5 \\ 0 & 1 & 6 \\ 4 & \textcircled{2} & 6 \end{bmatrix} & & & \end{array}$$

So  $p_2 = q_2 = 1$  and  $V = 2$ .

**Lemma 6.3**

Let  $a_{pq}$  and  $a_{rs}$  be saddle points of a payoff matrix  $A$ . Then  $a_{pq} = a_{rs}$ .

		$q$		$s$	
$p$		$a_{pq}$	$\leq$	$a_{ps}$	
		$\vee$		$\wedge$	
$r$		$a_{rq}$	$\geq$	$a_{rs}$	

### 6.2 Domination

Rows:

$$\begin{array}{ccccccc} i\text{-th row} & a_{i1} & a_{i2} & \cdots & a_{in} \\ & \vee & \vee & & \vee \\ k\text{-th row} & a_{k1} & a_{k2} & \cdots & a_{kn} \end{array}$$

So  $p_k = 0$  so  $k$ -th row can be removed from  $A$ .

Strict domination: for  $j = 1, 2, \dots, n$ ,

$$a_{ij} > a_{kj}$$

Columns: the  $k$ -th column dominates the  $j$ -th column

$$\begin{array}{ccc} a_{1j} & \geq & a_{1k} \\ a_{2j} & \geq & a_{2k} \\ \vdots & & \vdots \\ a_{mj} & \geq & a_{mk} \end{array}$$

Strict domination: for  $i = 1, 2, \dots, m$ ,

$$a_{ij} > a_{ik}$$

where  $a_{ik}$  is dominant.

- Removing a dominant row or column does not change the value of the game, but may remove an optimal strategy.
- Removing a strictly dominant row or column does not change the set of optimal strategies.

### Example 6.4

$$A_1 = \begin{bmatrix} 2 & 0 & 4 \\ 1 & 2 & 3 \\ 4 & 1 & 2 \end{bmatrix}$$

Note:  $\left. \begin{array}{l} 0 < 4 \\ 2 < 3 \\ 1 < 2 \end{array} \right\}$ , strict domination.

$$A_2 = \begin{bmatrix} 2 & 0 \\ 1 & 2 \\ 4 & 1 \end{bmatrix}$$

Note:  $\begin{array}{cc} 2 & 0 \\ \wedge & \wedge \\ 4 & 1 \end{array}$ , strict domination

$$\max \begin{array}{cc} & \min \\ \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix} & \begin{array}{l} 1 \\ 1 \end{array} \\ 4 & 2 \end{array}$$

Note: No saddle point

**Remark 6.5** A row/column can be dominated by a weighted sum of rows columns. For

example,

$$\begin{array}{cccc} a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{array}$$

For  $\ell = 1, 2, \dots, n$ , we have

$$\alpha a_{i\ell} + (1 - \alpha)a_{j\ell} \geq a_{k\ell}$$

### Example 6.6

$$\begin{bmatrix} 0 & 4 & 6 \\ 5 & 7 & 4 \\ 9 & 6 & 3 \end{bmatrix}$$

$$4 > 3$$

$$7 > 4.5$$

$$5 \geq 6$$

# 7 Apr 11, 2022

## 7.1 Principle of Indifference

		P2			
		$q_1$	$q_2$	$\cdots$	$q_n$
P1	$p_1$	$a_{11}$	$a_{12}$	$\cdots$	$a_{1n}$
	$p_2$	$a_{21}$	$a_{22}$	$\cdots$	$a_{2n}$
	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	$p_m$	$a_{m1}$	$a_{m2}$	$\cdots$	$a_{mn}$

Let  $\hat{\mathbf{p}} = (p_1, p_2, \dots, p_m)^t$  be an optimal strategy for P1 and let  $q_j = 1$  be a pure strategy for P2.

$$\sum_{i=1}^m a_{ij} p_i \geq V \quad (1)$$

Let  $\hat{\mathbf{q}} = (q_1, q_2, \dots, q_n)^t$  be an optimal strategy for P2 and let  $p_i = 1$  be a pure strategy for P1. Then

$$\sum_{j=1}^n a_{ij} q_j \leq V \quad (2)$$

**Note 7.1:** If both players use optimal strategies, then

$$\sum_{i=1}^m \sum_{j=1}^n a_{ij} p_i q_j = V$$

**Proof.** We have

$$\begin{aligned}
 V &\leq \sum_{i=1}^m a_{ij} p_i = 1 \cdot \sum_{i=1}^m a_{ij} p_i = \left( \sum_{j=1}^n q_j \right) \sum_{i=1}^m a_{ij} p_i \\
 &= \sum_{i=1}^m \sum_{j=1}^n a_{ij} p_i q_j \\
 &= \sum_{i=1}^m p_i \underbrace{\sum_{j=1}^n a_{ij} q_j}_{=1} \leq V \\
 &= V
 \end{aligned}$$

□



**Theorem 7.2** (The Equilibrium Theorem)

Let  $\hat{\mathbf{p}} = (p_1, p_2, \dots, p_m)$  and  $\hat{\mathbf{q}} = (q_1, q_2, \dots, q_n)$  be optimal strategies for P1 and P2 respectively. Then

$$\sum_{j=1}^n a_{ij}q_j = V \quad \forall i \text{ s.t. } p_i > 0$$

$$\sum_{i=1}^m a_{ij}p_i = V \quad \forall j \text{ s.t. } q_j > 0$$

**Proof.** Let  $p_k > 0$  and let  $\sum_{j=1}^n a_{kj}q_j \neq V \implies \sum_{j=1}^n a_{kj}q_j < V$ . We have

$$V \leq \sum_{i=1}^m p_i \sum_{j=1}^n a_{ij}q_j < V$$

a contradiction. □

**Example 7.3** (The game of Odd-and-Even)

Played with three numbers: 0, 1, and 2.

		P2, even		
		0	1	2
P1, Odd	0, $p_1$	0	1	-2
	1, $p_2$	1	-2	3
	2, $p_3$	-2	3	-4

$p_1 \geq 0, p_2 \geq 0, p_3 \geq 0$ , and  $p_1 + p_2 + p_3 = 1$ . Then

$$\begin{cases} p_2 - 2p_3 - V = 0 \\ p_1 - 2p_2 + 3p_3 - V = 0 \\ -2p_1 + 3p_2 - 4p_3 - V = 0 \\ p_1 + p_2 + p_3 = 1 \end{cases}$$

## 7.2 Symmetric Games

**Definition 7.4** (Symmetric Game)

The rules are the same for P1 and P2. So  $A^t = -A$ .

**Theorem 7.5**

The value of a finite size symmetric game is zero.

**Proof.** Note  $V^t = V$ . And

$$V = (\hat{\mathbf{p}})^t A \hat{\mathbf{p}} = [(\hat{\mathbf{p}})^t A \hat{\mathbf{p}}]^t = -\hat{\mathbf{p}} A \hat{\mathbf{p}} = -V$$

So

$$V = -V \implies V = 0$$

□

### Example 7.6 (Rock, Paper, Scissors)

We have

		P2		
		Rock	Paper	Scissors
P1	Rock	0	-1	1
	Paper	1	0	-1
	Scissors	-1	1	0

So

$$\begin{cases} p_2 - p_3 = 0 \\ -p_1 + p_3 = 0 \\ p_1 - p_2 = 0 \\ p_1 + p_2 + p_3 = 1 \end{cases}$$

Therefore,

$$p_1 = p_2 = p_3 = \frac{1}{3}$$

# 8 Apr 13, 2022

## 8.1 The Equilibrium Theorem and Symmetric Games (Cont'd)

**Recall 8.1** The Equilibrium Theorem:

Let  $\hat{\mathbf{p}}, \hat{\mathbf{q}}$  be optimal solutions for P1 and P2 respectively. Then

$$\sum_{j=1}^n a_{ij}q_j = V \quad \forall i \text{ s.t. } p_i > 0$$

$$\sum_{i=1}^m a_{ij}p_i = V \quad \forall j \text{ s.t. } q_j > 0$$

**Recall 8.2** A game is called a symmetric game if  $A^t = -A$ .

**Example 8.3** (A Mendelson Game)

	1	2	3	4	5	6		99	100
1	0	-1	2	2	2	2	...	2	2
2	1	0	-1	2	2	2	...	2	2
3	-2	1	0	-1	2	2	...	2	2
4	-2	-2	1	0	-1	2	...	2	2
5	-2	-2	-2	1	0	-1	...	2	2
6	-2	-2	-2	-2	1	0	...	2	2
$\vdots$	-2	-2	-2	-2	-2	1	...		
99	-2	-2	-2	-2	-2	-2	...	0	-1
100	-2	-2	-2	-2	-2	-2	...		0

So

$$\begin{cases} p_2 - 2p_3 = 0 \\ -p_1 + p_3 = 0 \\ 2p_1 - p_2 = 0 \\ p_1 + p_2 + p_3 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} p_3 = p_1 \\ p_2 = 2p_1 \end{cases} \Rightarrow p_1 + 2p_1 + p_1 = 1$$

$$\Rightarrow p_1 = \frac{1}{4} = p_3, \quad p_2 = \frac{1}{2}$$

So

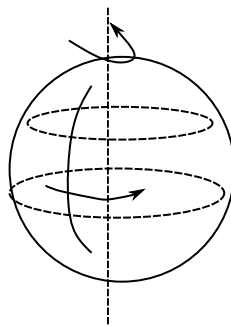
$$\hat{\mathbf{p}} = \left( \frac{1}{4}, \frac{1}{2}, \frac{1}{4}, 0, 0, \dots, 0 \right)$$

## 8.2 Invariance or Games with Symmetries

		P2		
		Rock	Paper	Scissors
P1	Rock	0	-1	1
	Paper	1	0	-1
	Scissors	-1	1	0

We shift to

		P2		
		Paper	Rock	Scissors
P1	Paper	0	1	1
	Rock	-1	0	-1
	Scissors	1	1	0



A group =  $G$

$$G = \{\sigma: \sigma_1 \circ \sigma_2 \in G, e \in G, \sigma^{-1} \circ \sigma = \sigma \circ \sigma^{-1} = e\}$$

Let  $S1 = (1, 2, \dots, m)$  and  $S2 = (1, 2, \dots, n)$  be the sets of pure strategies for P1 and P2. Let  $\sigma$  be a permutation of S1.

$$\begin{array}{cccc} 1 & 2 & \dots & m \\ \sigma(1) & \sigma(2) & \dots & \sigma(m) \end{array}$$

### Definition 8.4 (Invariant)

A finite two-person zero-sum game is invariant under a permutation  $\sigma \in S_m$  if  $\forall i \in S1$ , and  $\forall j \in S2$ , there exists a unique  $j' \in S2$  such that

$$A(i, j) = A(\sigma(i), j')$$

## 9 Apr 15, 2022

### 9.1 Invariance (Cont'd)

**Recall 9.1**  $S1 = \{1, 2, \dots, m\}, S2 = \{1, 2, \dots, n\}$ .

**Recall 9.2**  $\sigma$ , a permutation of  $S1$ , a bijection  $S1 \rightarrow S1$ :

$$\begin{pmatrix} 1 & 2 & \cdots & m \\ \sigma(1) & \sigma(2) & \cdots & \sigma(m) \end{pmatrix}$$

**Definition 9.3** (Invariant under a permutation  $\sigma$ )

A finite two-person zero-sum game is invariant under a permutation  $\sigma$  of  $S1$  if  $\forall i \in S1, j \in S2$ , there exists a unique  $j' \in S2$  such that

$$A(i, j) = A(\sigma(i), j')$$

The uniqueness requirement: if  $\forall i \in S1, \forall j \in S2, \exists j', j'' \in S2$  such that  $j' \neq j''$  and

$$A(i, j) = A(\sigma(i), j') = A(\sigma(i), j'')$$

then the strategies  $j'$  and  $j''$  have identical payoffs. Remembering the rows, we have

	$j'$	$\cdots$	$j''$
$\sigma(1)$	$a_{\sigma(1)j'}$	$\cdots$	$a_{\sigma(1)j''}$
$\sigma(2)$	$a_{\sigma(2)j'}$	$\cdots$	$a_{\sigma(2)j''}$
$\vdots$	$\vdots$		$\vdots$
$\sigma(m)$	$a_{\sigma(m)j'}$	$\cdots$	$a_{\sigma(m)j''}$

A duplicate strategy can be removed without loss of generality.

**Notation 9.4:**  $\bar{\sigma}(j) = j'$

**Lemma 9.5**

$\bar{\sigma}$  is a permutation of  $S2$ .

**Proof.**

- $\bar{\sigma}$  is defined for any  $j \in S2$  by definition.
- To show that  $\bar{\sigma} \in S_n$ , let us show that  $\bar{\sigma}$  is one-to-one.

Assume the opposite:  $j, k \in S2, j \neq k, \bar{\sigma}(j) = \bar{\sigma}(k)$ . Then

$$A(i, j) = A(\sigma(i), \bar{\sigma}(j)) = A(\sigma(i), \bar{\sigma}(k)) = A(i, k) \quad \forall i \in S1$$

Hence, the strategies  $S2j$  and  $S2k$  are duplicates which implies assumption is incorrect.  $\square$

**Lemma 9.6**

A game invariant under  $\sigma$  is invariant under  $\sigma^{-1}$

**Proof.**

$$A(i, j) = A(\sigma(i), \bar{\sigma}(j)) \quad \forall i \in S1, j \in S2$$

Then

$$A(\sigma(i), \bar{\sigma}(j)) = A(i, j)$$

Let  $i' = \sigma(i), j' = \sigma(j)$ . Then

$$A(i', j') = A(\sigma^{-1}(i'), \bar{\sigma}^{-1}(j')) \quad \forall i' \in S1, j' \in S2$$

□

## 9.2 Multiplication of Permutation

**Note 9.7:**

$$\sigma_2 \circ \sigma_1(*) = \sigma_2(\sigma_1(*))$$

**Example 9.8**

Suppose

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$$

Then,

$$\sigma_2 \circ \sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 2 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$$

**Lemma 9.9**

A game invariant under  $\sigma_1$  and  $\sigma_2$  is invariant under  $\sigma_2 \circ \sigma_1$ .

**Proof.**  $\forall i \in S1, j \in S2$ ,

$$\begin{aligned} A(i, j) &= A(\sigma(i), \bar{\sigma}(j)) = A(i', j') = A(\sigma_2(i'), \bar{\sigma}_2(j')) \\ &= A(\sigma_2 \circ \sigma_1(i), \bar{\sigma}_2 \circ \bar{\sigma}_1(j)) \end{aligned}$$

□

**Conclusion:** Invariant permutations of pure strategies form a group.

**Definition 9.10** (Invariant under a group  $\sigma$ )

A game  $(S1, S2, A)$  is invariant under a group  $\sigma$  if it is invariant under any  $\sigma \in G$

# **10 Apr 18, 2022**

## **10.1 Midterm 1**

# 11 Apr 20, 2022

## 11.1 Invariance (Cont'd)

**Recall 11.1** A group  $G$  is a set such that

1.  $\forall \sigma_1, \sigma_2 \in G, \sigma_2 \circ \sigma_1 \in G$ , i.e. there exists an operation of group multiplication.
2.  $\exists e \in G$  such that  $e \circ \sigma = \sigma \circ e = \sigma, \forall \sigma \in G$ .
3.  $\forall \sigma \in G, \exists \sigma^{-1} \in G$  such that  $\sigma^{-1} \circ \sigma = \sigma \circ \sigma^{-1} = e$

Let  $S1 = \{1, \dots, m\}, S2 = \{1, \dots, n\}$ . Let  $G$  be a subgroup of  $S_m$ .

**Definition 11.2** (Homomorphism)

A homomorphism of groups  $G$  and  $\bar{G}$  is a map  $h: G \rightarrow \bar{G}$  preserving the group structure, i.e.

1.  $h(\sigma_2 \circ \sigma_1) = \bar{\sigma}_2 \circ \bar{\sigma}_1, \forall \sigma_1, \sigma_2 \in G$
2.  $h(e) = \bar{e}$
3.  $h(\sigma^{-1}) = \bar{\sigma}^{-1}, \forall \sigma \in G$

**Definition 11.3** (Invariant under a game)

Let  $G$  be a subgroup of  $S_m$ . A game  $(S1, S2, A)$  is invariant under  $G$  if it is invariant  $\forall \sigma \in G$ .

**Recall 11.4**  $\forall i \in S1, j \in S2$ , there exists a unique  $\bar{\sigma} \in S_n$  such that  $A(i, j) = A(\sigma(i), \bar{\sigma}(j))$ . In other words, there exists a homomorphism

$$\begin{aligned} h: G &\rightarrow S_n \\ \sigma &\mapsto \bar{\sigma} \end{aligned}$$

such that

$$a_{ij} = a_{\sigma(i), \bar{\sigma}(j)} \quad \forall i \in S1, j \in S2$$

**Definition 11.5** (Orbit of a group action)

The orbit of a group action is the set

$$O_i = \{\sigma(i) : i \in S1, \sigma \in G\}$$

**Note 11.6:** The relation “being in the same orbit” is an equivalence relation.

Action of  $G$  on mixed strategies

$$p_1 S11 + p_2 S12 + \dots + p_m S1m$$



$$\begin{array}{c} \sigma \\ \downarrow \\ p_1 S 1 \sigma(1) + p_2 S 1 \sigma(2) + \cdots + p_m S 1 \sigma(m) \end{array}$$

so some  $\sigma(k)$  in the sum above equals 1

$$\begin{aligned} \sigma(k) = 1 & \sim k = \sigma^{-1}(1) \\ p_k S 1 \sigma(k) &= p_{\sigma^{-1}(1)} S 1 1 \end{aligned}$$

Similarly for  $2, 3, \dots, m$ ,

$$p_{\sigma^{-1}(1)} S 1 1 + p_{\sigma^{-1}(2)} S 1 2 + \cdots + p_{\sigma^{-1}(m)} S 1 m$$

**Definition 11.7** ( $G$ -invariant)

A mixed strategy  $\mathbf{p}$  is  $G$ -invariant if  $\sigma(\mathbf{p}) = \mathbf{p}$  for all  $\sigma \in G$ .

**Theorem 11.8**

If a game (S1, S2, A) is invariant under  $G$ , then each player has a  $G$ -invariant optimal strategy.

**Proof.** Let  $\hat{\mathbf{p}} = (p_1, p_2, \dots, p_m)^t$  be an optimal strategy for P1. Then

$$\sum_{i=1}^m p_i A(i, j) \geq V \quad \forall j \in S2$$

Let

$$\hat{\mathbf{p}}_G = \frac{1}{|G|} \sum_{\sigma \in G} \sigma(\hat{\mathbf{p}}),$$

i.e.  $\forall i = 1, 2, \dots, m$ ,

$$(\hat{\mathbf{p}}_G)_i = \frac{1}{|G|} \sum_{\sigma \in G} p_{\sigma^{-1}(i)},$$

the average of the  $i$ th coordinate over the orbit  $O$ .

Invariance of  $\hat{\mathbf{p}}_G$ :

$$\forall \alpha \in G, \quad \alpha(\hat{\mathbf{p}}_G) = \frac{1}{|G|} \sum_{\sigma \in G} \alpha \circ \sigma(\hat{\mathbf{p}})$$

- $\alpha \circ \sigma$  is defined  $\forall \sigma \in G$
- Assume  $\sigma_1 \neq \sigma_2$ ,  $\alpha \circ \sigma_1 = \alpha \circ \sigma_2$ . Then

$$\alpha^{-1} \circ \alpha \circ \sigma_1 = \alpha^{-1} \circ \alpha \circ \sigma_2 \sim \sigma_1 = \sigma_2$$

Contradiction implies multiplication by  $\alpha$  is a one-to-one map  $G \rightarrow G$ .

Hence, multiplication by  $\alpha$  is a bijection  $G \rightarrow G$ .

$$\frac{1}{|G|} \sum_{\sigma \in G} \sigma(\hat{\mathbf{p}}) = \hat{\mathbf{p}}_G$$

We shall continue this proof in the next lecture. □

# 12 Apr 22, 2022

## 12.1 Invariance (Cont'd)

### Recall 12.1

$$\sigma(\mathbf{p} = (p_1, p_2, \dots, p_m)^t) = (p_{\sigma^{-1}(1)}, p_{\sigma^{-1}(2)}, \dots, p_{\sigma^{-1}(m)})$$

A mixed strategy  $\mathbf{p}$  is  $G$ -invariant if  $\sigma(\mathbf{p}) = \mathbf{p}$  for all  $\sigma \in G$ .

**Proof of Theorem 11.8.**  $\alpha: G \rightarrow G$ ,  $\alpha(\sigma) = \alpha \circ \sigma$  is a bijection of  $G$ . Thus,

$$\hat{\mathbf{p}}_G = \frac{1}{|G|} \sum_{\sigma \in G} \sigma(\hat{\mathbf{p}}) = \frac{1}{|G|} \sum_{\sigma \in G} \alpha \circ \sigma(\hat{\mathbf{p}}) = \alpha(\hat{\mathbf{p}}_G)$$

Optimality of  $\hat{\mathbf{p}}_G$ :

$$\begin{aligned} \sum_{i=1}^m (\hat{\mathbf{p}}_G)_i A(i, j) &= \sum_{i=1}^m \frac{1}{|G|} \sum_{\sigma \in G} (\hat{\mathbf{p}})_{\sigma^{-1}(i)} A(i, j) \\ &= \frac{1}{|G|} \sum_{\sigma \in G} \sum_{i=1}^m (\hat{\mathbf{p}})_{\sigma^{-1}(i)} A(i, j) \\ &= \frac{1}{|G|} \sum_{\sigma \in G} \sum_{i=1}^m p_i A(\sigma(i), \bar{\sigma}(j)) \\ &= \frac{1}{|G|} \sum_{\sigma \in G} \sum_{i=1}^m p_i A(i, j) \geq V \end{aligned}$$

because

$$\sum_{i=1}^m p_i A(i, j) \geq V \quad \forall j \in S_2$$

□

### Example 12.2 (Battleships)

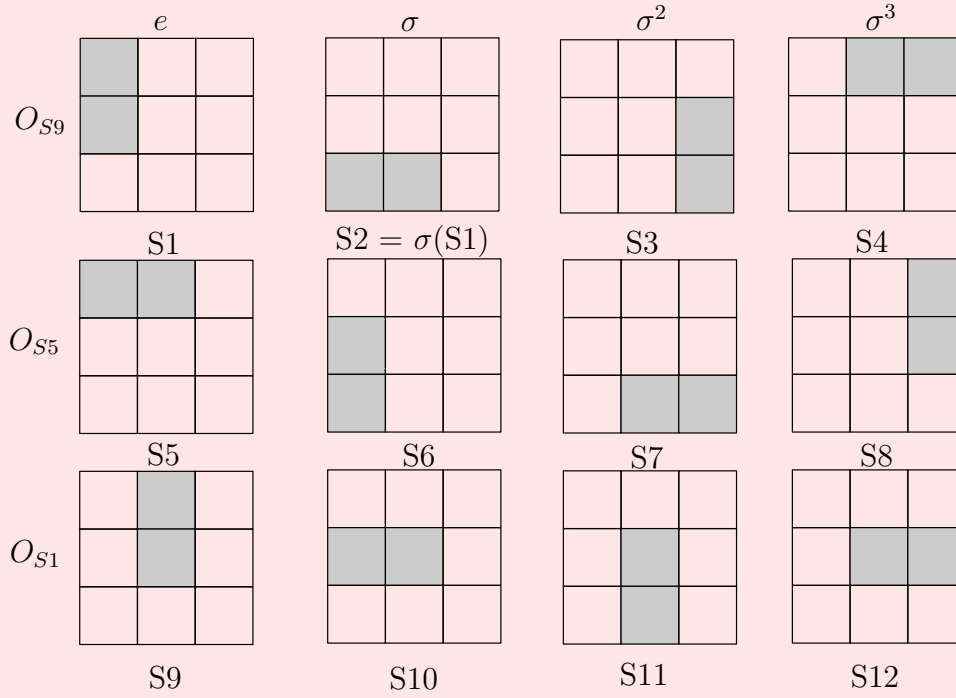
We have a  $3 \times 3$  grid, a submarine, P1 is the bomber,  $G$ -counter-clockwise rotations of the grid.

$G$  is cyclic:  $\{e, \sigma, \sigma^2, \sigma^3\}$  and note  $\sigma^4 = e$ .

P1:

S1	S2	S3
S4	S5	S6
S7	S8	S9

P2: sub



$A$ , a  $9 \times 12$  matrix

Let  $\hat{\mathbf{p}}_G = (p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9)$  be a  $G$ -invariant optimal strategy for P1. Then

$$p_1 = p_3 = p_7 = p_9 = \tilde{p}_1$$

$$p_2 = p_4 = p_6 = p_8 = \tilde{p}_2$$

$$p_5 = \tilde{p}_5$$

Similarly,

$$q_1 = q_2 = q_3 = q_4 = \tilde{q}_1$$

$$q_5 = q_6 = q_7 = q_8 = \tilde{q}_2$$

$$q_9 = q_{10} = q_{11} = q_{12} = \tilde{q}_3$$

		$O_1, \tilde{q}_1$	$O_5, \tilde{q}_2$	$O_9, \tilde{q}_3$	min
P1	$O_1, \tilde{p}_1$	$\frac{1}{4}$	$\frac{1}{4}$	0	0
	$O_2, \tilde{p}_2$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
	$O_5, \tilde{p}_3$	0	0	1	0
max		$\frac{1}{4}$	$\frac{1}{4}$	1	

Note:  $A_{21} = A_{22} = \frac{1}{4}$  are saddle points.

$$C1 = C2 \implies \tilde{q}_1 = \tilde{q}_2$$

Solution of the game:  $V = \frac{1}{4}$ ,

$$\tilde{p}_2 = 1, \quad \tilde{q}_1 = \tilde{q}_2 = \frac{1}{2}$$

i.e.,

$$p_2 = p_4 = p_6 = p_8 = \frac{1}{4}$$

$$q_1 = q_2 = q_3 = q_4 = q_5 = q_6 = q_7 = q_8 = \frac{1}{8}$$

# 13 Apr 25, 2022

## 13.1 Colonel Blotto Games

### Example 13.1

Colonel Blotto has 4 regiments with which to occupy two posts. Lieutenant Kije has 3 regiments for the task.

Payoffs:

- The side sending more units to a post captures it as well as the enemy units sent to the post. The payoff is 1 point for the post and 1 point for each captured enemy unit.
- If both sides send the same number of units, the forces withdraw with no payoff.

	$(3, 0), q_1$	$(2, 1), q_2$	$(1, 2), q_2$	$(0, 3), q_1$
$p_1, (4, 0)$	4	2	1	0
$p_2, (3, 1)$	1	3	0	-1
$p_3, (2, 2)$	-2	2	2	-2
$p_2, (1, 3)$	-1	0	3	1
$p_1, (0, 4)$	0	1	2	4

So  $G = \mathbb{Z}_2 = \{e, \sigma : \sigma^2 = e\}$ , acts swapping posts.

	$O_1: (3, 0) \sim (0, 3)$ $\tilde{q}_1 = 2q_1$	$O_2: (2, 1) \sim (1, 2)$ $\tilde{q}_2 = 2q_1$
$O_1: (4, 0) \sim (0, 4)$ $\tilde{p}_1 = 2p_1$	$\frac{4 + 0 + 4 + 0}{4} = 2$	$\frac{2 + 1 + 1 + 2}{4} = \frac{3}{2}$
$O_2: (3, 1) \sim (1, 3)$ $\tilde{p}_2 = 2p_2$	0	$\frac{3}{2}$
$O_3: (2, 2)$ $\tilde{p}_3 = p_3$	$\frac{-2 - 2}{2} = -2$	$\frac{2 + 2}{2} = 2$

$$\begin{array}{c} 2 \\ \vee \\ 0 \end{array} \begin{array}{c} \frac{3}{2} \\ \vee \\ \frac{3}{2} \end{array} \implies \tilde{p}_2 = 0 \implies p_2 = 0$$

	$\tilde{q}_1$	$1 - \tilde{q}_1 = \tilde{q}_2$
$\tilde{p}_1$	2	$\frac{3}{2}$
$\tilde{p}_3 = 1 - \tilde{p}_1$	-2	2

$$\text{P1: } \max_{0 \leq p \leq 1} \min_{0 \leq q \leq 1} q(2p - 2(1-p)) + (1-q) \left( \frac{3}{2}p + 2(1-p) \right) = \max_{0 \leq p \leq 1} \min \left\{ 4p - 2, -\frac{p}{2} + 2 \right\}$$

$$\implies 4p - 2 = -\frac{p}{2} + 2 \implies \frac{9p}{2} = 4 \implies 9p = 8 \implies \boxed{\tilde{p}_1 = \frac{8}{9}}$$

$$V = 2 - \frac{\tilde{p}_1}{2} = 2 - \frac{4}{9} = \boxed{\frac{14}{9}}$$

Since  $0 < \tilde{p}_1 < 1$ , we can use the indifference principle to find  $\tilde{q}_1$ .

$$2q + \frac{3}{2}(1 - q) = \frac{14}{9}$$

$$\frac{q}{2} + \frac{3}{2} = \frac{14}{9}$$

$$q + 3 = \frac{28}{9}$$

$$q = \frac{28}{9} - 3 = \frac{1}{9}$$

$$\implies \boxed{\tilde{q}_1 = \frac{1}{9}}$$

$$(\hat{\mathbf{p}})^t = \left( \frac{4}{9}, 0, \frac{1}{9}, 0, \frac{4}{9} \right)$$

$$(\hat{\mathbf{q}})^t = \left( \frac{1}{18}, \frac{4}{9}, \frac{4}{9}, \frac{1}{18} \right)$$

# 14 Apr 27, 2022

## 14.1 Linear Programming

### Example 14.1

We want to maximize

$$x + 5y$$

such that

$$5x + 6y \leq 30$$

$$3x + 2y \leq 12$$

$$x \geq 0, y \geq 0$$

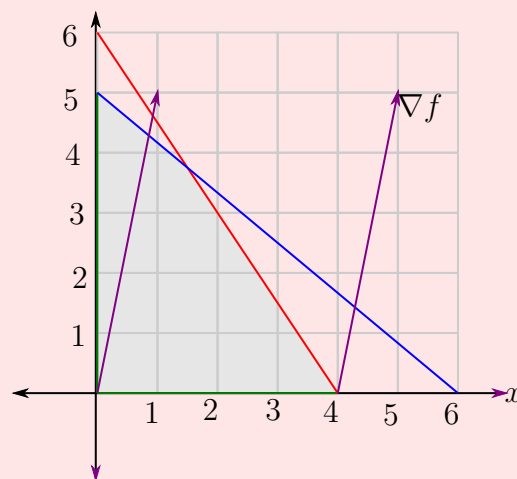
So

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

And

$$6y = 30 - 5x$$

$$\Rightarrow \boxed{y = 6 - \frac{5}{2}x} \quad \boxed{y = 5 - \frac{5}{6}x}$$



### Standard form of a LP

We want to minimize

$$v = c^t x = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

such that

$$Ax \geq b$$

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \geq b_1$$

$$a_{12}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \geq b_2$$

$$\begin{aligned} & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \geq b_m \\ & x \geq 0 \end{aligned}$$

i.e.

$$x_1 \geq 0, \dots, x_n \geq 0$$

**Example 14.2** (Example 14.1 in the standard form)

We want to minimize

$$-x - 5y$$

such that

$$-5x - 6y \geq -30$$

$$-3x - 2y \geq 0$$

$$x \geq 0$$

$$y \geq 0$$

## 14.2 Two-Person Zero-Sum Games as LP Programs in the Standard Form

We have

$$\begin{aligned} \text{P1: } \max_{\mathbf{p} \in \Delta_m} V(p_1, p_2, \dots, p_m) &= \max_{\mathbf{p} \in \Delta_m} \min_{\mathbf{q} \in \Delta_n} (\mathbf{p})^t A \mathbf{q} \\ &= \max_{\mathbf{p} \in \Delta_m} \left( \min_{\mathbf{q} \in \Delta_n} q_1 \sum_{i=1}^m p_i a_{i1} + q_2 \sum_{i=1}^m p_i a_{i2} + \cdots + q_n \sum_{i=1}^m p_i a_{in} \right) \\ &= \max_{\mathbf{p} \in \Delta_m} \min \left\{ \sum_{i=1}^m p_i a_{ij}, j = 1, \dots, n \right\} \end{aligned}$$



**Example 14.3**

P1 is solving the following problem:

Choose  $p_1, p_2, \dots, p_m, V$ , to maximize  $V$  such that

$$V \leq \sum_{i=1}^m p_i a_{i1}, V \leq \sum_{i=1}^m p_i a_{i2}, \dots, V \leq \sum_{i=1}^m p_i a_{in}$$

$$\implies 0 \leq \sum_{i=1}^m p_i a_{i1} - V, 0 \leq \sum_{i=1}^m p_i a_{i2} - V, \dots, 0 \leq \sum_{i=1}^m p_i a_{in} - V$$

$$\mathbf{p} \in \Delta_m \quad \boxed{p_1 \geq 0, p_2 \geq 0, \dots, p_m \geq 0, p_1 + p_2 + \dots + p_m = 1}$$

So

$$V = V_1 - V_2, V_1 \geq 0, V_2 \geq 0$$

$$p_1 + p_2 + \dots + p_m \geq 1$$

$$-p_1 - p_2 - \dots - p_m \geq -1$$

**Example 14.4 (Colonel Blotto)**

We have

	$(3, 0), q$	$(2, 1), q_2$	$(1, 2), q_2$	$(0, 3), q_4$
$(4, 0), p_1$	4	2	1	0
$(3, 1), p_2$	1	3	0	-1
$(2, 2), p_3$	-2	2	2	-2
$(1, 3), p_4$	-1	0	3	1
$(0, 4), p_5$	0	1	2	4

So

$$p_1 + p_2 + p_3 + p_4 + p_5 = 1$$

$$p_1 \geq 0, p_2 \geq 0, p_3 \geq 0, p_4 \geq 0, p_5 \geq 0$$

LP in the standard form

$$V = V_1 - V_2, V_1 \geq 0, V_2 \geq 0$$

$$-V \rightarrow \min \sim V_2 - V_1 \rightarrow \min \text{ s.t.}$$

$$\begin{cases} 4p_1 + p_2 - 2p_3 - p_4 - V_1 + V_2 \geq 0 \\ 2p_1 + 3p_2 + 2p_3 + p_5 - V_1 + V_2 \geq 0 \\ p_1 + 2p_3 + 3p_4 + 2p_5 - V_1 + V_2 \geq 0 \\ -p_2 - 2p_3 + p_4 + 4p_5 - V_1 + V_2 \geq 0 \\ p_1 + p_2 + p_3 + p_4 + p_5 = 1 \end{cases}$$

So

$$\left\{ p_1 = \frac{4}{9}, p_2 = 0, p_3 = \frac{1}{9}, p_4 = 0, p_5 = \frac{4}{9}, V_1 = \frac{14}{9}, V_2 = 0 \right\}$$

# 15 Apr 29, 2022

## 15.1 LP Duality

We have the problem

$$\begin{aligned} V = c^t x &\rightarrow \min \text{ s.t.} \\ Ax &\geq b \\ x &\geq 0 \end{aligned}$$

And the **Dual problem** is

$$\begin{aligned} w = b^t y &\rightarrow \max \text{ s.t.} \\ A^t y &\leq c \\ y &\geq 0 \end{aligned}$$

### Lemma 15.1

Dual to dual = primary

**Proof.**

$$\begin{aligned} (-b)^t y &\rightarrow \min \text{ s.t.} \\ (-A)^t y &\geq -c \\ y &\geq 0 \end{aligned}$$

Next,

$$\begin{aligned} (-c)^t x &\rightarrow \max \text{ s.t.} \\ -Ax &\leq -b \\ x &\geq 0 \end{aligned}$$

Finally,

$$\begin{aligned} c^t x &\rightarrow \min \text{ s.t.} \\ Ax &\geq b \\ x &\geq 0 \end{aligned}$$

□

### Lemma 15.2

Dual to LP for P1 = LP for P2

**Proof.** We have

$$\begin{aligned}
 a_{11}p_1 + a_{21}p_2 + \cdots + a_{m1}p_m - V_1 + V_2 &\geq 0 \\
 a_{12}p_1 + a_{22}p_2 + \cdots + a_{m2}p_m - V_1 + V_2 &\geq 0 \\
 &\vdots \\
 a_{1n}p_1 + a_{2n}p_2 + \cdots + a_{mn}p_m - V_1 + V_2 &\geq 0 \\
 -p_1 - p_2 + \cdots - p_m &\geq -1 \\
 p_1 + p_2 + \cdots + p_m &\geq 1
 \end{aligned}$$

With

$$p_1 \geq 0$$

$$p_2 \geq 0$$

$$\vdots$$

$$p_m \geq 0$$

$$v_1 \geq 0$$

$$v_2 \geq 0$$

$$p_1 + p_2 + \cdots + p_m = 1$$

So

$$V = V_1 - V_2$$

$$\mathbf{x}^t = (p_1, p_2, \dots, p_m, V_1, V_2)$$

$$V = V(p_1, p_2, \dots, p_m) \rightarrow \max$$

$$V_2 - V_1 \rightarrow \min$$

So

$$Ax = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} & -1 & 1 \\ a_{12} & a_{22} & \cdots & a_{m2} & -1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} & -1 & 1 \\ -1 & -1 & \cdots & -1 & 0 & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ v_1 \\ v_2 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

and

$$c^t = (\underbrace{0, \dots, 0}_m, -1, 1) \quad x \geq 0$$

So

$$(\mathbf{x})^t = (q_1, q_2, \dots, q_n, W_1, W_2)$$

So the objective function is

$$W_2 - W_1 \rightarrow \max \text{ s.t.}$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & -1 & 1 \\ a_{21} & a_{22} & \dots & a_{2n} & -1 & 1 \\ \vdots & \vdots & & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} & -1 & 1 \\ -1 & -1 & \dots & -1 & 0 & 0 \\ 1 & 1 & \dots & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \\ W_1 \\ W_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

with

$$q_1 \geq 0, W_1 \geq 0, W_2 \geq 0$$

Then

$$-q_1 - q_2 - \dots + q_n \leq -1$$

$$q_1 + q_2 + \dots + q_n \leq 1$$

$$\underline{q_1 + q_2 + \dots + q_n \geq 1}$$

$$q_1 + q_2 + \dots + q_n = 1$$

So

$$a_{11}q_1 + a_{12}q_2 + \dots + a_{m1}q_n \leq W = W_1 - W_2$$

$$\vdots$$

$$a_{m1}q_1 + a_{12}q_2 + \dots + a_{1n}q_n \leq W \rightarrow \min$$

Furthermore, we are maximizing

$$V(p_1, p_2, \dots, p_m)$$

and minimizing

$$W(q_1, q_2, \dots, q_n)$$

□

### Theorem 15.3 (Weak Duality)

Let  $x$  be a feasible point for the primary problem:

$$V = c^t x \rightarrow \min \text{ subject to}$$

$$Ax \geq b, x \geq 0$$

Let  $y$  be a feasible point for the dual primary problem

$$W = b^t y \rightarrow \max \text{ subject to}$$

$$A^t y \leq c, y \geq 0.$$

Then

$$c^t x \geq b^t y$$

**Proof.** Write

$$c^t \geq y^t A$$

So

$$-V = c^t x \geq y^t A x \geq v^t b = -W$$

$$-V \geq -W \implies W \geq V$$

So

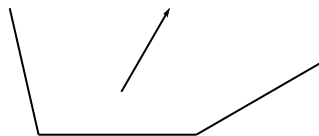
$$\boxed{\overbrace{\max_{p \in \Delta_m} \min_{q \in \Delta_n} p^t A q}^V \leq \overbrace{\min_{q \in \Delta_n} \max_{p \in \Delta_m} p^t A q}^W}$$

□

**Corollary 15.4** • If the primary problem is unbounded, then the dual problem is infeasible.

- If the dual problem is unbounded, then the problem is infeasible.

**Proof.**  $c^t x \geq b^t y$



□

# 16 May 2, 2022

## 16.1 Using LP Duality to Prove Von Neumann's Theorem

**Recall 16.1** (Weak Duality Theorem) Let  $x$  be a feasible point for a linear program.

$$-V = c^t x \rightarrow \min \text{ subject to}$$

$$Ax \geq b, x \geq 0$$

where

$$x^t = (p_1, p_2, \dots, p_m, V_1, V_2)$$

$$V = V_1 - V_2$$

Let  $y$  be a feasible point for the dual LP

$$-W = b^t y \rightarrow \max \text{ subject to}$$

$$A^t y \leq c, y \geq 0$$

where

$$y^t = (q_1, q_2, \dots, q_n, W_1, W_2)$$

$$W = W_1 - W_2$$

Then

$$c^t x \geq b^t y$$

$$-V \geq -W$$

### Corollary 16.2

If  $x$  is a feasible solution to the primary problem and  $y$  is a feasible solution to the dual problem  $c^t x = b^t y$ , then  $x$  and  $y$  are optimal.

### Lemma 16.3

$\max_{p \in \Delta_m} \min_{q \in \Delta_n} p^t A q$  exists.

**Proof.**

$$\begin{aligned} \max_{p \in \Delta_m} \min_{q \in \Delta_n} p^t A q &= \max_{p \in \Delta_m} \min \left\{ \sum_{i=1}^m a_{ij} p_i, j = 1, \dots, n \right\} \\ &= \max_{\underbrace{p \in \Delta_m}_{\text{compact set}}} \underbrace{V(p_1, p_2, \dots, p_m)}_{\text{piecewise linear, hence continuous function}} \end{aligned}$$

□

**Recall 16.4** (Weierstrass Extreme Value Theorem) A real-valued function continuous on a compact set attains its min and max on the set.

**Theorem 16.5** (LP: Strong Duality Theorem)

If one of the two dual linear programs has an optimal solution, then so does the other and the values of the objective functions at the optimal points are equal.

## 16.2 General Sum Games

### Prisoner's Dilemma

Two members of a criminal organization are arrested and put into solitary confinement. They have no means to communicate with each other. Prosecution lacks evidence to convict each on the principal charge. Prosecution has enough evidence to convict each on a lesser charge. Each prisoner is given a possibility to cooperate (Rat) or stay silent (Mum). The payoff bimatrix is given by

		S21	S2
	P1	R	M
	P2		
S11	R	(-8, -8)	(0, -10)
S12	M	(-10, 0)	(-1, 1)

$$A = \begin{bmatrix} -8 & 0 \\ -10 & -1 \end{bmatrix}, \quad B^t = \begin{bmatrix} -8 & -10 \\ 0 & -1 \end{bmatrix}$$

Dominant strategy: Whatever P2 chooses, P1 is better off

**Definition 16.6** (Maxmin strategy)

Let  $A$  and  $B$  be  $m \times n$  payoff matrices for P1 and P2.

$$V_1 = \max_{p \in \Delta_m} \min_{q \in \Delta_n} p^t A q = \max_{p \in \Delta_m} \min_{j=1, \dots, n} \sum_{i=1}^m p_i a_{ij} = \text{Val}(A)$$

where  $\text{Val}(A) =$  safety level for P1. And

$$V_2 = \max_{q \in \Delta_n} \min_{p \in \Delta_m} p^t B q = \max_{q \in \Delta_n} \min_{i=1, \dots, m} \sum_{j=1}^n b_{ij} q_j = \text{Val}(B^t)$$

Noting that

$$p^t B q = q^t B^t p \quad V_2^t = V_2$$

Also note that safety level strategies disregard payoff for the other player. A strategy  $p$  that achieves  $\text{Val}(A)$  for P1 is called a maxmin strategy.

**Note 16.7:** This game is symmetric if  $A = B^t$ .



**Example 16.8**

Recall our payoff matrix

P1/P2	R	M
R	-8, -8	0, -10
M	-10, 0	-1, -1

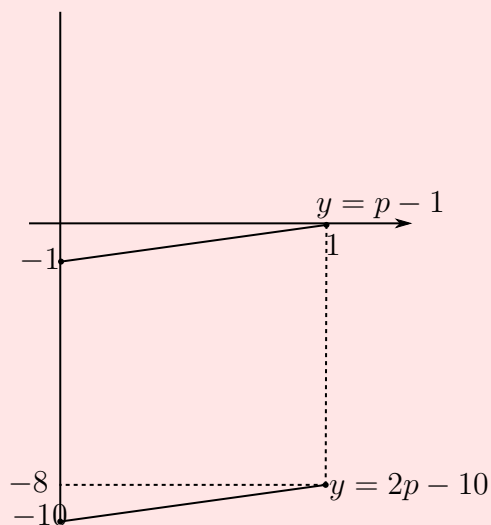
 $\rightarrow$ 

P1/P2	R	M
$p$	-8	0
$1-p$	-10	-1

The safety strategy is given by

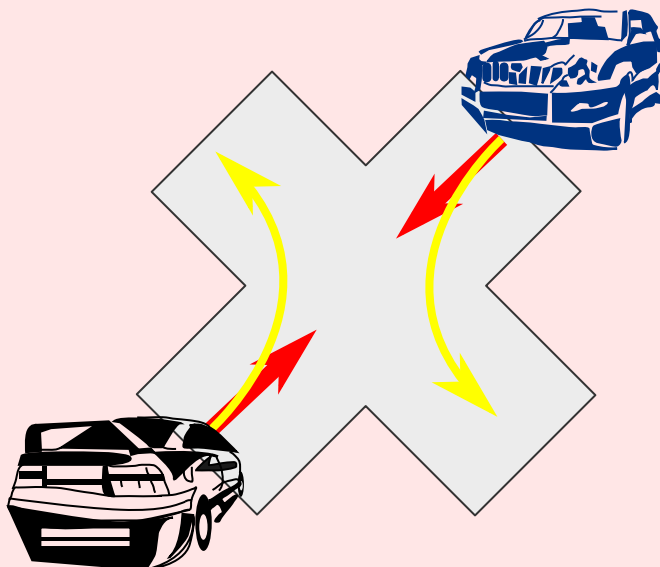
$$\begin{aligned} \text{Val}(A) &= \max_{0 \leq p \leq 1} \min\{-8p - 10(1-p), p-1\} = \max_{0 \leq p \leq 1} \min\{2p-10, p-1\} \\ &= \max_{0 \leq p \leq 1} (2p-10) = -8 \end{aligned}$$

$$p = 1, 1-p = 0$$



**Example 16.9** (The Game of Chicken)

Two drivers speed head-on towards each other at crossroads.



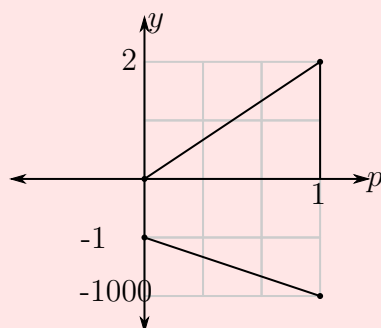
		P2	
		Drive	Chicken
P1	Drive, $p$	$-1000, 1000$	$2, -1$
	Chicken, $1 - p$	$-1, 2$	$0, 0$

$$\begin{aligned}\text{Val}(A) &= \max_{0 \leq p \leq 1} \min\{-1000p - (1 - p), 2p\} \\ &= \max_{0 \leq p \leq 1} \min\{-999p - 1, 2p\}\end{aligned}$$

So

$$p = 0, \quad 1 - p = 1$$

$$\text{Val}(A) = -1$$



# 17 May 4, 2022

## 17.1 Nash Equilibrium in General Sum Games

**Recall 17.1** For an  $m \times n$  matrix  $A$ ,

$$\text{Val}(A) = \max_{p \in \Delta_m} \min_{q \in \Delta_n} p^t A q$$

which is the safety level of P1.

### Definition 17.2 (Nash Equilibrium (NE))

A pair  $(\hat{p}, \hat{q})$  is called a Nash Equilibrium, if

$$* (\hat{p})^t A \hat{q} \geq p A \hat{q} \quad \forall p \in \Delta_m. \text{ i.e. } \hat{p} \text{ is the best response to } \hat{q}$$

$$** (\hat{p})^t B \hat{q} \geq \hat{p} B q \quad \forall q \in \Delta_n. \text{ i.e. } \hat{q} \text{ is the best response to } \hat{p}.$$

### Definition 17.3 (Pure Nash Equilibrium (PNE))

A Pure Nash Equilibrium is a Nash Equilibrium such that  $\hat{p} = e_i \in \mathbb{R}^m$  and  $\hat{q} = e_j \in \mathbb{R}^n$ , where

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{th position}$$

**Note 17.4:** Pure Nash Equilibriums are easy to find.

### Example 17.5

We have

	S21	S22	S23	S24	S25	S26
S11	2,1	4,3	7*,2	7*,4	0,5*	3,2
S12	4*,0	5*,4	1,6*	0,4	0,3	5*,1
S13	1,3*	5*,3*	3,2	4,1	1*,0	4,3*
S14	4*,3	2,5*	4,0	1,0	1*,5*	2,1

So

$$\text{PNE} = \{(S13, S22), (S14, S25)\}$$

### Lemma 17.6

Nash equilibria are “individually rational”, i.e. delivery at least the safety value.

**Proof.** Let  $p^*$  be a solution to the problem

$$\max_{p \in \Delta_n} \min_{q \in \Delta_n} p^t A q = \text{Val}(A)$$

i.e.

$$(p^*)^t A q \geq \text{Val}(A)_q \quad \forall q \in \Delta_n$$

So,

$$(\hat{p})^t A \hat{q} \geq (p^*)^t A q \geq \text{Val}(A)$$

$$(\hat{p})^t A \hat{q} \geq p A \hat{q}$$

□

### Lemma 17.7

Nash Equilibria survive elimination of strongly dominated strategies.

**Proof.**

$$\left[ \begin{array}{cc|cc} a_{11} & a_{12} & a_{1j} & a_{1n} \\ a_{21} & a_{22} & a_{2j} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \hline a_{i1} & a_{i2} & a_{ij} & a_{in} \\ \hline \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{mj} & a_{mn} \end{array} \right]$$

Suppose the  $i$ th row is strictly domination by other rows, then

$$\exists \alpha_1 \geq 0, \alpha_2 \geq 0, \dots, \alpha_{i-1} \geq 0$$

$$\alpha_{i+1} \geq 0, \dots, \alpha_m \geq 0$$

$$\sum_{\substack{k=1 \\ k \neq i}}^m \alpha_k = 1$$

and

$$a_{ij} < \sum_{\substack{k=1 \\ k \neq i}} \alpha_k a_{kj} \quad \text{for } j = 1, 2, \dots, n$$

Suppose that  $(\hat{p}, \hat{q})$  is a Nash Equilibrium such that  $\hat{p}_i \neq 0$ . So

$$\begin{aligned}
 (\hat{p})^t A \hat{q} &= \sum_{i,j} a_{ij} \hat{p}_i \hat{q}_j = \hat{p}_1 \sum_{j=1}^n a_{1j} \hat{q}_j + \cdots + \hat{p}_i \sum_{j=1}^n a_{ij} \hat{q}_j + \cdots + \hat{p}_m \sum_{j=1}^n a_{mj} \hat{q}_j \\
 &< \hat{p}_1 \sum_{j=1}^n a_{1j} \hat{q}_j + \cdots + \hat{p}_i \sum_{j=1}^n \sum_{\substack{k=1 \\ k \neq i}}^m \alpha_k a_{kj} \hat{q}_j + \cdots + \hat{p}_m \sum_{j=1}^n a_{mj} \hat{q}_j \\
 &= \hat{p}_1 \sum_{j=1}^n a_{1j} \hat{q}_j + \cdots + \sum_{\substack{k=1 \\ k \neq i}}^n \hat{p}_i \alpha_k \sum_{j=1}^n a_{kj} \hat{q}_j + \cdots + \hat{p}_n \sum_{j=1}^n a_{nj} \hat{q}_j \\
 &= (\hat{p}_1 + \hat{p}_i \alpha_1) \sum_{j=1}^n a_{1j} \hat{q}_j + \cdots + 0 + \cdots + (\hat{p}_m + \hat{p}_i \alpha_n) \sum_{j=1}^n a_{mj} \hat{q}_j
 \end{aligned}$$

This way, the vector,

$$(p^*)^t = (\hat{p}_1 + \hat{p}_i \alpha_1, \hat{p}_2 + \hat{p}_i \alpha_2, \dots, 0, \dots, \hat{p}_m + \hat{p}_i \alpha_m)$$

is a better response to  $\hat{q}$  than

$$(\hat{p})^t = (\hat{p}_1, \hat{p}_2, \dots, \hat{p}_i, \dots, \hat{p}_m)$$

Thus, contrary to the assumption,  $\hat{p}_i = 0$ . □

# 18 May 6, 2022

## 18.1 Equalizing Strategies

P1 plays a strategy equalizing payoff of P2. P2 has no incentive to leave whatever strategy they are playing. A pair of equalizing strategies forms a Nash Equilibrium. We return back to the game of Chicken.

### Example 18.1 (Game of Chicken (Cont'd))

We have two Pure Nash Equilibriums

		P2	
		D	C
P1	D, $p$	$-1000, -1000$	$2^*, -1^*$
	C, $1-p$	$-1^*, 2^*$	$0, 0$

So

$$\text{PNE} = \{(D, C), (C, D)\}$$

$$-1000p + 2(1-p) = -p$$

$$\implies 2 = 1001p$$

$$\implies \begin{cases} p = \frac{2}{1001} \\ 1-p = \frac{999}{1001} \end{cases}$$

So,

		P2		
		D, $\frac{2}{1001}$	C, $\frac{999}{1001}$	
P1	D, $\frac{2}{1001}$	$-1000, -1000$	$2^*, -1^*$	The expected payoff is $\frac{2}{1001}$ .
	C, $\frac{999}{1001}$	$-1^*, 2^*$	$0, 0$	

### Example 18.2 (Cheetahs and Antelopes)

$S$

- Two cheetahs chasing two antelopes,  $L$  and  $S$
- Each cheetah can catch either antelope
- If catch same, have to share

		P2	
		L	S
P1	L	$\frac{\ell}{2}, \frac{\ell}{2}$	$\ell, s$
	S	$s, \ell$	$\frac{s}{2}, \frac{s}{2}$

and  $\ell > s$ .

Case 1:  $\frac{\ell}{2} > s \implies \ell > 2s$  (Strict domination)

		P2	
		L	S
P1	L	$\frac{\ell^*}{2}, \frac{\ell^*}{2}$	$\ell^*, s$
	S	$s, \ell^*$	$\frac{s}{2}, \frac{s}{2}$

PNE:  $(L, L)$

Case 2:  $\frac{\ell}{2} = s \sim \ell = 2s$

		P2	
		L	S
P1	L	$\frac{\ell^*}{2}, \frac{\ell^*}{2}$	$\ell^*, s^*$
	S	$s^*, \ell^*$	$\frac{s}{2}, \frac{s}{2}$

PNE:  $\{(L, L), (L, S), (S, L)\}$

Case 3:  $s < \ell < 2s$

		P2	
		L	S
P1	L	$\frac{\ell}{2}, \frac{\ell}{2}$	$\ell^*, s^*$
	S	$s^*, \ell^*$	$\frac{s}{2}, \frac{s}{2}$

PNE:  $\{(L, S), (S, L)\}$

Equalizing strategy for Cases 2 and 3:

$$s < \ell \leq 2s$$

		P2	
		L	S
P1	L, $p$	$\frac{\ell}{2}, \frac{\ell}{2}$	$\ell, s$
	S, $1 - p$	$s, \ell$	$\frac{s}{2}, \frac{s}{2}$

$$p\frac{\ell}{2} + (1-p)\ell = ps + (1-p)\frac{s}{2}$$

$$\frac{\ell}{2}p + \ell - \ell p = sp + \frac{s}{2} - \frac{s}{2}p$$

$$-\frac{\ell}{2}p + \ell = \frac{s}{2}p + \frac{s}{2}$$

$$\ell - \frac{s}{2} = \frac{\ell + s}{2}p$$

$$2\ell - s = (\ell + s)p$$

$\hat{p} = \frac{2\ell - s}{\ell + s},$	$1 - \hat{p} = \frac{2s - \ell}{\ell + s}$
---	--

$$\begin{aligned}
 \text{payoff} &= \frac{s}{2}(\hat{p} + 1) = \frac{s}{2} \left( \frac{2\ell - s}{\ell + s} + 1 \right) \\
 &= \frac{s}{2} \frac{2\ell - s + \ell + s}{\ell + s} \\
 &= \frac{3\ell s}{2(\ell + s)}
 \end{aligned}$$

So we have the payoff for the  $L$  strategy:

$$L(p) = \ell - \frac{\ell}{1}p; \quad S(p) = \frac{s}{2} + \frac{s}{2}p$$

$$\begin{aligned}
 \hat{p} &= \frac{2\ell - s}{\ell + s} \\
 p &< \hat{p}
 \end{aligned}$$

Evolutionary stability:

Let  $0 \leq p \leq 1$  be the part of greedy cheetahs (always choose  $L$ ) in the population.

- For  $0 \leq p < \hat{p}$ ,  $L(p) > S(p)$ , so greedy cheetahs have an advantage
- For  $\hat{p} < p \leq 1$ ,  $L(p) < S(p)$ , so non-greedy cheetahs have an advantage.

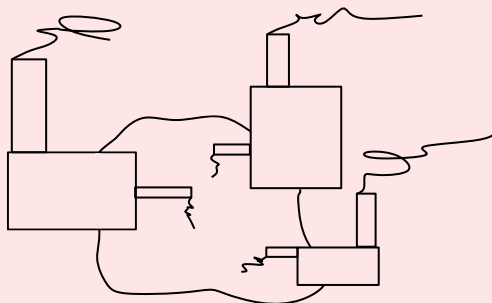
So, evolution pushes  $p$  to  $\hat{p}$ .



## 18.2 General Sum Games with More Than Two Players

### Example 18.3

We have



P3: Purify

P2

S21: purify S22: pollute

P1	S11	1,1,1	1,0,1
	S12	0,1,1	3,3,4

P3: pollute

P2

S21 S22

P1	S11	1,1,0	4,3,3
	S12	3,4,3	3,3,3

# 19 May 9, 2022

## 19.1 Review for Midterm 2

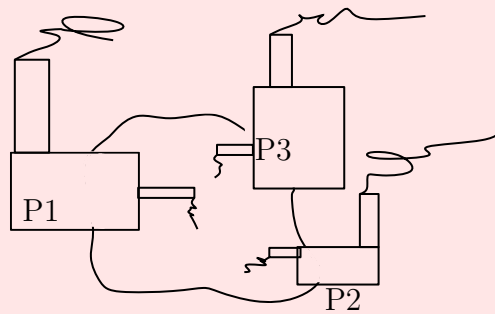
Questions include:

1. Dual problem
2. Pure Nash Equilibrium for general sum matrix
3. Word problem: Find all Nash Equilibrium (2 part problem)
4. Extra Credit

## 19.2 General Sum Games with More Than Two Players (Cont'd)

### Example 19.1

We continue from Example 18.3.

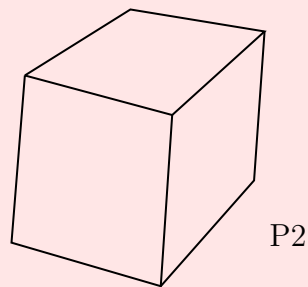


cost  $\rightarrow$  min

payoff  $\rightarrow$  max

payoff = -cost

P3



P1

P2

	P3 purifies	
	S21, P2 purifies	S22, P2 pollutes
S11, P1 purifies, $p$	1,1,1	1*,0*,1*
S12, P1 pollutes, $1-p$	0*,1*,1*	3,3,4

	P3 pollutes	
	S21, $p_2$	S22, $1-p_2$
S11	1*,1*,0	4,3,3
S12	3,4,3	3*,3*,3*

$$\begin{aligned}
\text{PNE} &= \{(1, 0, 1), (0, 1, 1), (1, 1, 0), (3, 3, 3)\} \\
&= \left\{ \left( (1, 0), (0, 1), (1, 0) \right), \dots \right\} \\
&= \{(S11, S22, S31), \dots\}
\end{aligned}$$

Now

$P3$  purifies :  $(1, 0)$

$$P1 : (p, 1-p) = \left( \frac{2}{3}, \frac{1}{3} \right)$$

$$1 = 3(1-p) \implies 3p = 2 \implies p = \frac{2}{3}$$

So

$$\text{NE} \stackrel{?}{=} \left( \left( \frac{2}{3}, \frac{1}{3} \right), \left( \frac{2}{3}, \frac{1}{3} \right), (1, 0) \right) \quad (*)$$

$$\stackrel{?}{=} \left( \left( \frac{2}{3}, \frac{1}{3} \right), \left( \frac{2}{3}, \frac{1}{3} \right), (0, 1) \right) \quad (**)$$

So for (\*)

$$\text{Cost} = 1 \cdot \frac{2}{3} \cdot \frac{2}{3} + 1 \cdot \frac{2}{3} \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} \cdot \frac{2}{3} + 4 \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{12}{9} = \frac{4}{3}$$

and for (\*\*)

$$\text{Cost} = \frac{6}{9} + \frac{6}{9} + \frac{3}{9} = \frac{15}{9} = \frac{5}{3}$$

So

$$\text{NE} = \left\{ \left( (1, 0), \left( \frac{2}{3}, \frac{1}{3} \right), \left( \frac{2}{3}, \frac{1}{3} \right) \right), \left( \left( \frac{2}{3}, \frac{1}{3} \right), (1, 0), \left( \frac{2}{3}, \frac{1}{3} \right) \right), \left( \left( \frac{2}{3}, \frac{1}{3} \right), \left( \frac{2}{3}, \frac{1}{3} \right), (1, 0) \right) \right\}$$

Now

$$C(P1 : \text{purify}) = C(P2 : \text{pollute})$$

$$p_2 p_3 + (1-p_2)p_3 + p_2(1-p_3) + 4(1-p_2)(1-p_3) = 3(1-p_2)p_3 + 3p_2(1-p_3) + 3(1-p_2)(1-p_3)$$

# **20 May 11, 2022**

## **20.1 Midterm 2**

# 21 May 13, 2022

## 21.1 General Sum Games with More Than Two Players (Cont'd)

**Recall 21.1** Recall from Example 19.1,

	P3 purifies	
	S21, P2 purifies	S22, P2 pollutes
S11, P1 purifies, $p$	1,1,1	1,0,1
S12, P1 pollutes, $1 - p$	0,1,1	3,3,4

	P3 pollutes	
	S21, $p_2$	S22, $1 - p_2$
S11	1,1,0	4,3,3
S12	3,4,3	3,3,3

**Example 21.2** (Fully Mixed NE)

Continuing from the example,

$$C((1, 0), (p_2, 1 - p_2), (p_3, 1 - p_3)) = C((0, 1), (p_2, 1 - p_2), (p_3, 1 - p_3))$$

$$1 \cdot p_2 \cdot p_3 + 1 \cdot (1 - p_2)p_3 + 1 \cdot p_2 \cdot (1 - p_3) + 4(1 - p_2)(1 - p_3) = 0 \cdot p_2 \cdot p_3 + 3 \cdot (1 - p_2)p_3 + 3p_2(1 - p_3) + 3(1 - p_2)(1 - p_3)$$

$$p_2p_3 + (1 - p_2)p_3 + p_2(1 - p_3) + 4(1 - p_2)(1 - p_3) = 3(1 - p_2)p_3 + 3p_2(1 - p_3) + 3(1 - p_2)(1 - p_3)$$

$$1 = 3(p_2 + p_3 - 2p_2p_3) \quad (*)$$

$$1 = 3(p_1 + p_3 - 2p_1p_3) \quad (**)$$

$$1 = 3(p_1 + p_2 - 2p_1p_2) \quad (***)$$

$$(***) - (**) = 3(p_2 - p_3 + 2p_1p_3 - 2p_1p_2) = 0$$

$$p_2 - p_3 - 2p_1(p_2 - p_3) = 0$$

$$(p_2 - p_3)(1 - 2p_1) = 0$$

So

$$p_2 = p_3 \quad p_1 = \frac{1}{2}$$

By (\*),

$$\begin{aligned}
 1 &= 3(2x - 2x^2) \\
 6x^2 - 6x + 1 &= 0 \\
 x &= \frac{6 \pm \sqrt{36 - 24}}{12} \\
 x &= \frac{6 \pm \sqrt{12}}{12} \\
 x &= \frac{3 \pm \sqrt{3}}{6}
 \end{aligned}$$

And by (\*\*),

$$\begin{aligned}
 1 &= 3 \left( \frac{1}{2} + p_2 - 2 \cdot \frac{1}{2} \cdot p_2 \right) \\
 1 &= 3 \cdot \frac{1}{2} = \frac{3}{2} \\
 1 &= \frac{3}{2}
 \end{aligned}$$

Note,

$$\begin{aligned}
 0 &< \frac{3 - \sqrt{3}}{6}, \frac{3 + \sqrt{3}}{6} < 1 \\
 \frac{3 - \sqrt{3}}{6} + \frac{3 + \sqrt{3}}{6} &= 1
 \end{aligned}$$

Two more NE:

$$\left( \left( \frac{3 - \sqrt{3}}{6}, \frac{3 + \sqrt{3}}{6} \right), \dots \right), \left( \left( \frac{3 + \sqrt{3}}{6}, \frac{3 - \sqrt{3}}{6} \right), \dots \right)$$

So

$$\begin{aligned}
 C \left( \frac{\sqrt{3} + 3}{6}, \frac{\sqrt{3} + 3}{6} \right) &= 2 - \frac{\sqrt{3}}{2} \\
 C \left( \frac{3 - \sqrt{3}}{6}, \frac{3 - \sqrt{3}}{6} \right) &= 2 + \frac{\sqrt{3}}{2}
 \end{aligned}$$

**Notation 21.3** (Multi-index Notations):

$$\begin{aligned}
 S1 &= (S11, S12, \dots, S1m_1) \\
 S2 &= (S21, S22, \dots, S2m_2) \\
 &\vdots \\
 Sn &= (Sn1, Sn2, \dots, Snm_n)
 \end{aligned}$$

A mixed strategy for  $\mathbf{P}_j$ :

$$(\mathbf{p}^j)^t = (p_1^j, p_2^j, \dots, p_{m_j}^j) \text{ s.t.}$$

$$p_1^j \geq 0, p_2^j \geq 0, \dots, p_{m_j}^j \geq 0, \quad \sum_{k=1}^{m_j} p_k^j = 1, \text{ i.e. } \mathbf{p}_j \in \Delta_{m_j}$$

**Definition 21.4**

The list  $(\hat{\mathbf{p}}^1, \hat{\mathbf{p}}^2, \dots, \hat{\mathbf{p}}^n)$  is a Nash Equilibrium if

$$u(\hat{\mathbf{p}}^1, \hat{\mathbf{p}}^2, \dots, \hat{\mathbf{p}}^j, \dots, \hat{\mathbf{p}}^n) \geq u(\hat{p}^1, \hat{p}^2, \dots, \hat{p}^j, \dots, \hat{p}^n) \quad \forall \mathbf{p}^j \in \Delta_{m_j} \quad \forall j = 1, 2, \dots, n$$

where

$$u(\mathbf{p}^1, \mathbf{p}^2, \dots, \mathbf{p}^n) = \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} \cdots \sum_{i_n=1}^{m_n} p_{i_1}^1 p_{i_2}^2 \cdots p_{i_n}^n u(S1i_1, S2i_2, \dots, Sn i_n)$$

# 22 May 16, 2022

## 22.1 Nash Equilibrium Theorem

### Theorem 22.1

Every finite general sum game has a NE (Nash Equilibrium).

### Lemma 22.2 (The Indifference Principle)

Let  $(\hat{p}^1, \hat{p}^2, \dots, \hat{p}^j, \dots, \hat{p}^n)$  be a NE and let  $\hat{p}_{k_1}^j > 0, \hat{p}_{k_2}^j > 0, \hat{p}_{k_3}^j = 0$ . Then

$$\begin{aligned} u(\hat{\mathbf{p}}^1, \hat{\mathbf{p}}^2, \dots, \underbrace{Sj k_1}_{=\mathbf{e}_{k_1} \in \mathbb{R}^{m_j}}, \dots, \hat{\mathbf{p}}^n) &= u(\hat{\mathbf{p}}^1, \hat{\mathbf{p}}^2, \dots, \underbrace{Sj k_2}_{=\mathbf{e}_{k_2} \in \mathbb{R}^{m_j}}, \dots, \hat{\mathbf{p}}^n) \\ &\geq u(\hat{\mathbf{p}}^1, \hat{\mathbf{p}}^2, \dots, \underbrace{Sj k_3}_{=\mathbf{e}_{k_3} \in \mathbb{R}^{m_j}}, \dots, \hat{\mathbf{p}}^n) \end{aligned}$$

$\forall j = 1, 2, \dots, n$

$\forall k_1, k_2, k_3$  s.t.  $1 \leq k_1, k_2, k_3 \leq m_j$ ,

$k_1 \neq k_2 \neq k_3$

### Proof.

$$\begin{aligned} u(\hat{\mathbf{p}}^1, \hat{\mathbf{p}}^2, \dots, \hat{\mathbf{p}}^j, \dots, \hat{\mathbf{p}}^n) &= \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} \cdots \sum_{i_{j-1}=1}^{m_{j-1}} \sum_{i_{j+1}=1}^{m_{j+1}} \cdots \sum_{i_n=1}^{m_n} \hat{p}_{i_1}^1 \hat{p}_{i_2}^2 \cdots \hat{p}_{i_{j-1}}^{j-1} \cdot \hat{p}_{i_{j+1}}^{j+1} \cdots \hat{p}_{i_n}^n \\ &\quad \cdot (\hat{p}_1^j u(*, Sj1, *) + \hat{p}_2^j u(*, Sj2, *) + \cdots + \hat{p}_{m_j}^j u(*, Sj m_j, *)) \end{aligned}$$

Moving a weight to a larger  $u$  increases the sum. □

### Proof of Theorem 22.1. Step 1: The case of two players.

Let the payoff be given by two  $m \times n$  matrices,  $A$  and  $B$ . Let

$$K = \Delta_m \times \Delta_n = \{(\mathbf{p}, \mathbf{q}) : \mathbf{p} \in \Delta_m, \mathbf{q} \in \Delta_n\}$$

Let us construct a continuous map

$$\begin{aligned} T: K &\rightarrow K \\ (p, q) &\mapsto (\hat{p}, \hat{q}) \end{aligned}$$

such that

\*  $\hat{p}$  is a better response to  $q$  if such exists,  $\hat{p} = p$  otherwise.

\*\*  $\hat{q}$  is a better response to  $p$  if such exists,  $\hat{q} = q$  otherwise.

$$c_i(\mathbf{p}, \mathbf{q}) := \max\{\mathbf{e}_i^t A \mathbf{q} - \mathbf{p}^t A \mathbf{q}, 0\}$$

where  $\mathbf{e}_i$  is the  $i$ -th standard basis vector in  $\mathbb{R}^m$ .



Note:  $\mathbf{e}_i^t A \mathbf{q} - \mathbf{p}^t A \mathbf{q}$  shows the change in P1's gain if they switch from  $\mathbf{p}$  to  $S1i$ . Let

$$\hat{\mathbf{p}}^t = (\hat{p}_1, \hat{p}_2, \dots, \hat{p}_m)$$

where

$$\hat{p}_i = \frac{p_i + c_i}{1 + \sum_{K=1}^m c_K}$$

Note:  $\sum_{i=1}^m \hat{p}_i = 1, \hat{p}_i \geq 0$ .

Similarly, let

$$d_j(\mathbf{p}, \mathbf{q}) = \max\{\mathbf{p} B \mathbf{e}_j - \mathbf{p} B \mathbf{q}, 0\}$$

and

$$\hat{q}_j = \frac{q_j + d_j}{1 + \sum_{K=1}^n d_K}$$

Finally, let us define

$$T(\mathbf{p}, \mathbf{q}) = (\hat{\mathbf{p}}, \hat{\mathbf{q}})$$

Case 1:  $c_i = 0$  for  $i = 1, \dots, m$ .

Then

$$\mathbf{p}^t A \mathbf{q} \geq \mathbf{e}_i^t A \mathbf{q} \text{ for } i = 1, \dots, m$$

and  $\hat{\mathbf{p}} = \mathbf{p}$  is the best response to  $\mathbf{q}$ .

Case 2: Not all  $c_i = 0$ . Then

$$S = \sum_{i=1}^m c_i > 0$$

Need to show that

$$\begin{aligned} \sum_{i=1}^m \hat{p}_i \mathbf{e}_i^t A \mathbf{q} &> \mathbf{p}^t A \mathbf{q} \sim (1 + S) \sum_{i=1}^m \frac{p_i + c_i}{1 + S} \mathbf{e}_i^t A \mathbf{q} > \mathbf{p}^t A \mathbf{q} (1 + S) \\ &\sim \sum_{i=1}^m (p_i + c_i) \mathbf{e}_i^t A \mathbf{q} > (1 + S) \mathbf{p}^t A \mathbf{q} \\ &\sim \sum_{i=1}^m p_i \mathbf{e}_i^t A \mathbf{q} + \sum_{i=1}^m c_i \underbrace{\mathbf{e}_i^t A \mathbf{q}}_{\mathbf{p}^t A \mathbf{q}} > \mathbf{p}^t A \mathbf{q} + S \mathbf{p}^t A \mathbf{q} \\ &\sim \mathbf{p}^t A \mathbf{q} + \mathbf{p}^t A \mathbf{q} \underbrace{\sum_{i=1}^m c_i}_{=S} > \mathbf{p}^t A \mathbf{q} + S \mathbf{p}^t A \mathbf{q} \\ &\sim \mathbf{p}^t A \mathbf{q} S = S \mathbf{p}^t A \mathbf{q} \end{aligned}$$

Step 2:  $m > 2$  players, let

$$c_i^j = \max\{u(\mathbf{p}^1, \dots, \mathbf{p}^{j-1}, \mathbf{e}_i, \mathbf{p}^{j+1}, \dots, \mathbf{p}^n) - u(\mathbf{p}^1, \dots, \mathbf{p}^{j-1}, \mathbf{p}^j, \mathbf{p}^{j+1}, \dots, \mathbf{p}^n), 0\}$$

$j = 1, 2, \dots, n$  and  $i = 1, 2, \dots, m_j$ . The rest of the proof does not change.

Step 3: The map

$$\begin{aligned} T: \Delta_{m_1} \times \Delta_{m_2} \times \dots \times \Delta_{m_n} &\rightarrow \Delta_{m_1} \times \Delta_{m_2} \times \dots \times \Delta_{m_n} \\ (\mathbf{p}^1, \mathbf{p}^2, \dots, \mathbf{p}^n) &\mapsto (\hat{p}_1, \hat{p}_2, \dots, \hat{p}_n) \end{aligned}$$

is continuous.

Step 4:

**Theorem 22.3** (Brouwer's Fixed Point Theorem)

Let  $K$  be a closed, bounded, convex subset of  $\mathbb{R}^N$ . Let  $T$  be a continuous map  $T: K \rightarrow K$ . Then  $\exists x \in K$  such that  $T(x) = x$ .

□

# 23 May 18, 2022

## 23.1 Games in Extensive Form

**Definition 23.1** (Kuhn Tree) • Vertices  $\sim$  states of the game

- Directed edges  $\sim$  legal moves
- Weights of edges  $\sim$  probabilities
- Leaves  $\sim$  terminal states, marked with payoffs

**Definition 23.2** (Noncooperative Games)

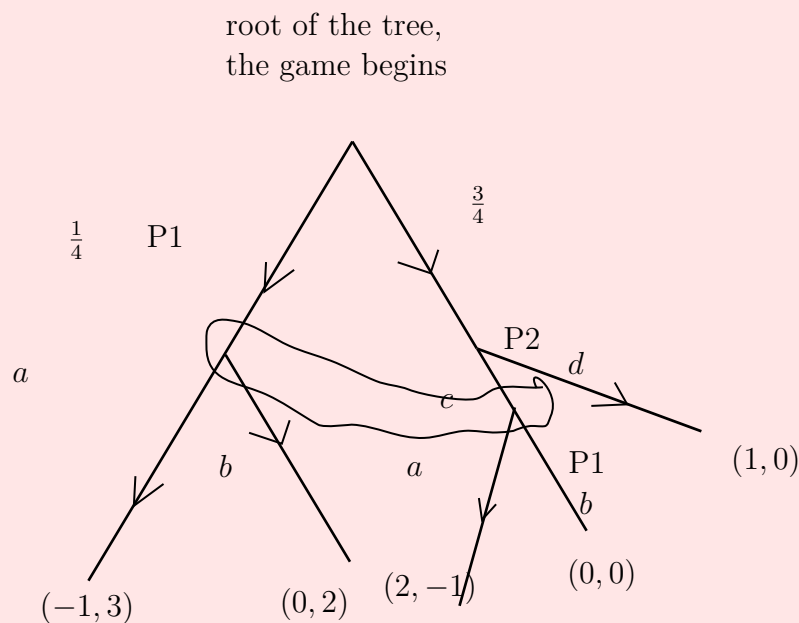
Players do not communicate before making moves.

**Definition 23.3** (Cooperative Games)

Players can communicate before making moves.

### Example 23.4

We have



Information set, describe incomplete information

**Example 23.5** (Extensive  $\rightarrow$  Strategic)Case 1:  $\frac{3}{4}$ 

		P2	
		$c$	$d$
P1	$a$	2, -1	1, 0
	$b$	0, 0	1, 0

Case 2:  $\frac{1}{4}$ 

		P2	
		$c$	$d$
P1	$a$	-1, 3	-1, 3
	$b$	0, 2	0, 2

Expected payoffs:<sup>a</sup>

		P2	
		$c$	$d$
P1	$a$	$\frac{5}{4}^*, 0$	$\frac{1}{2}, \frac{3}{4}^*$
	$b$	$0, \frac{1}{2}^*$	$\frac{3}{4}^*, \frac{1}{2}$

So

PNE :  $\{(b, d)\}$ 

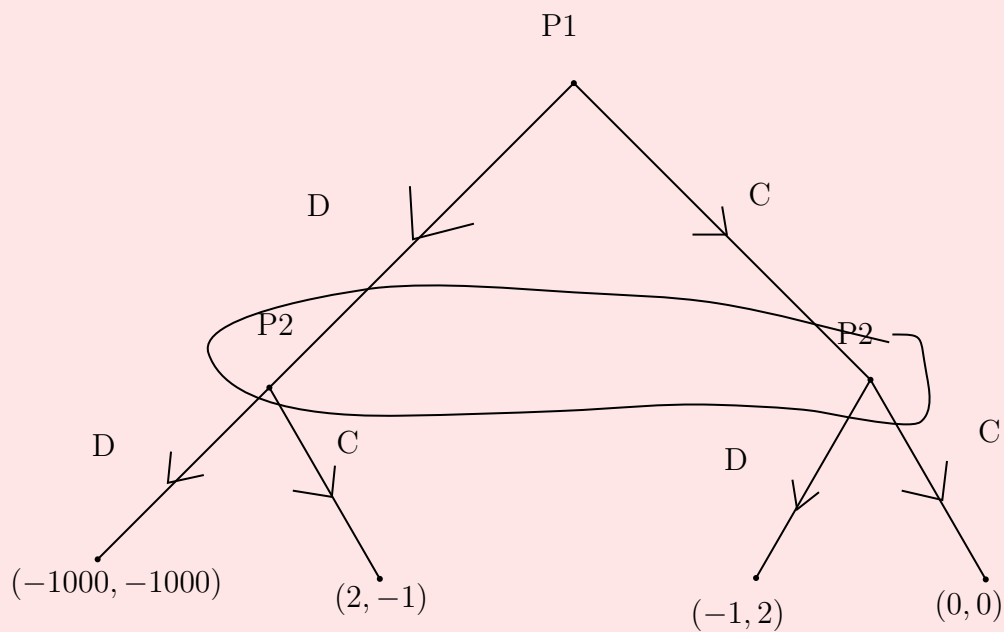

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<sup>a</sup>Expect to see on final

**Example 23.6** (Game of Chicken (Strategic  $\rightarrow$  Extensive))

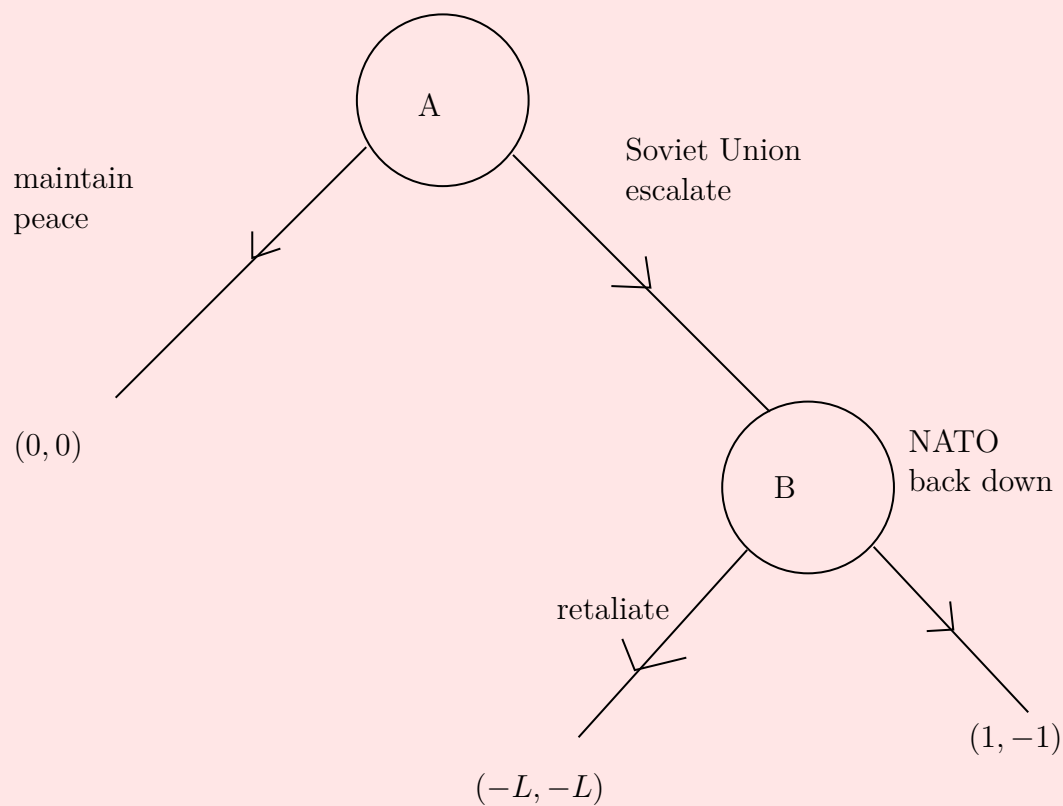
We have

		P2	
		Drive	Chicken
P1	Drive	$-1000, -1000$	$2, -1$
	Chicken	$-1, 2$	$0, 0$

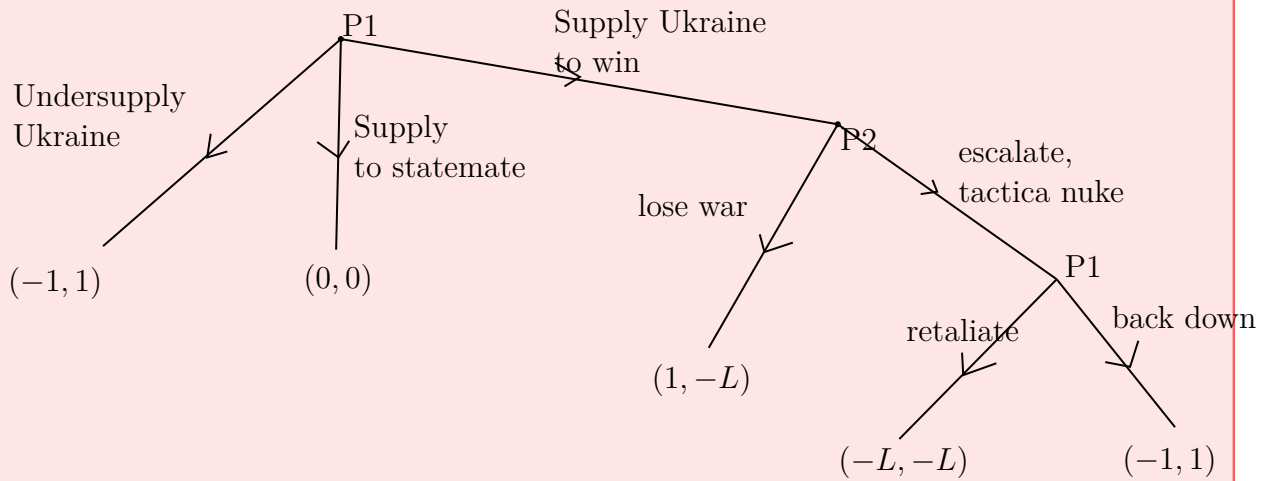


**Example 23.7** (MAD (Mutually Assured Destruction))

We have



		B	
		retaliate	back down
A	peace	$0^*, 0^*$	$0, 0^*$
	escalate	$-L, -L$	$1^*, -1^*$

**Example 23.8** $P1 \sim \text{West}, P2 \sim \text{Russia}$ 

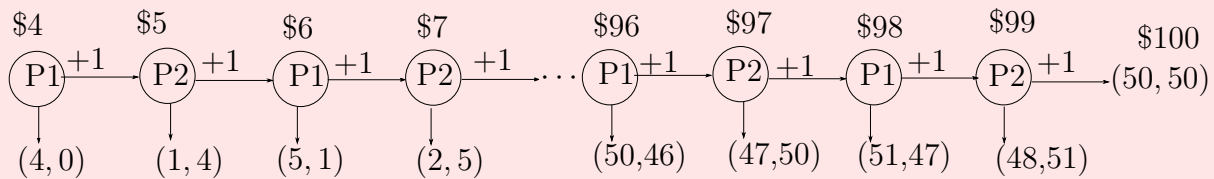
## 24 May 20, 2022

### 24.1 Games in Extensive Form (Cont'd)

#### Example 24.1 (The Centipede Game)

The Centipede Game is as follows:

- Initially, there are \$4 on the table
- There are two players, alternating moves
- When there are \$ $n$  on the table, the player to move can take  $\lfloor \frac{n+4}{2} \rfloor$  or choose to continue. In the first case, the other player takes the rest of the pot.
- If a player chooses to continue, \$1 is added to the pot
- If the pot reaches \$100, then each player gets \$50 and the game ends

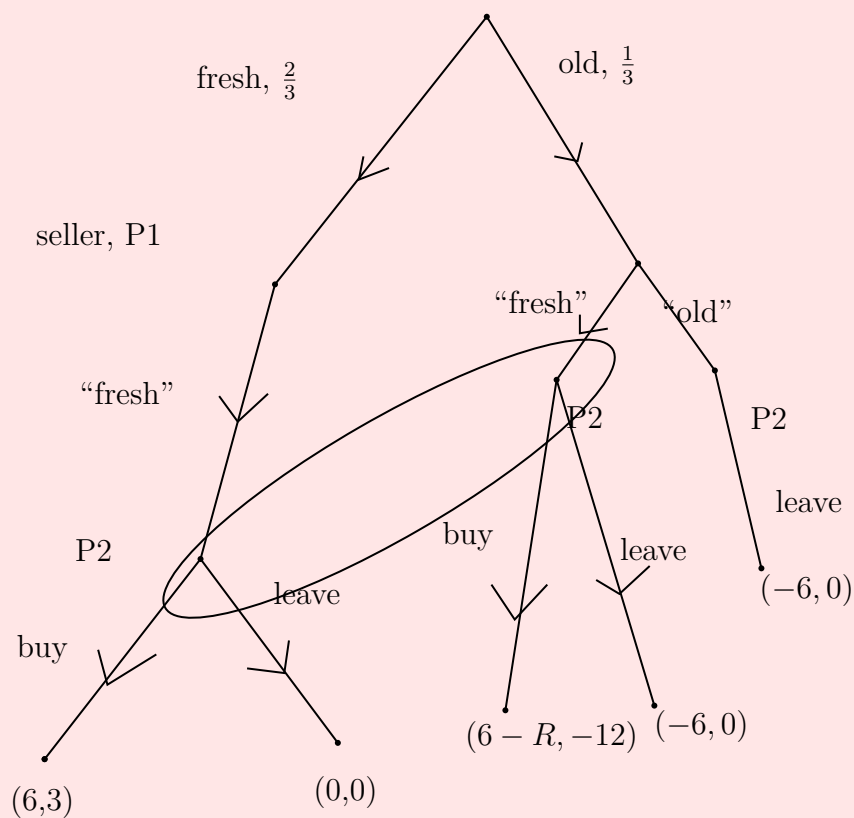


#### Example 24.2 (Fish-Selling Game)

The game is as follows:

- Fish sold at the market is either fresh with  $p = \frac{2}{3}$  or old with  $1 - p = \frac{1}{3}$ .
- The seller knows whether the fish he is selling is fresh or old.
- The customer asks whether the fish is fresh or old. The seller answers.
- Based on the answer, the customer decides whether to buy or leave without buying
- The selling price of the fish is \$12. The value of the fish for the customer is \$15 if the fish is fresh and \$0 if the fish is old.
- The seller bought the fish for \$6. If the fish is fresh, he can sell it to another seller for \$6. If the fish is old, he can throw it in trash.
- If the seller sells the old fish as fresh, he gains \$6, but he loses \$ $R$  in reputation.





Strategies for P1

S11: always say "fresh"

S12: say the truth

Strategies for P2

S21: buy if "fresh"

S22: leave

Case 1:  $p = \frac{2}{3}$ , the fish is fresh

		P2	
		S21 = buy if "fresh"	S22 = leave
P1	S11 = always "fresh"	6, 3	0, 0
	S12 = truth	6, 3	0, 0

Case 2:  $1 - p = \frac{1}{3}$ , old fish

		P2	
		buy	leave
P1	"fresh"	$6 - R, -12$	-6, 0
	truth	-6, 0	-6, 0

So the expected payoff bimatrix is:

		P2	
		buy	leave
P1	“fresh”	$6 - \frac{R}{3}, -2$	$-2, 0$
	truth	$2, 2$	$-2, 0$

PNEs:

$$6 - \frac{R}{3} \leq 2$$

$$12 \leq R$$

Case 1:  $R < 12$

		P2	
		buy	leave
P1	“fresh”	$6 - \frac{R^*}{3}, -2$	$-2^*, 0^*$
	truth	$2, 2^*$	$-2^*, 0$

Case 2:  $R = 12$

		P2	
		buy	leave
P1	“fresh”	$6 - \frac{R^*}{3}, -2$	$-2^*, 0^*$
	truth	$2^*, 2^*$	$-2^*, 0$

Case 3:  $R > 12$

		P2	
		buy	leave
P1	“fresh”	$6 - \frac{R}{3}, -2$	$-2^*, 0^*$
	truth	$2^*, 2^*$	$-2^*, 0$

Case 1:  $0 \leq R < 12$

PNE = (always “fresh”, leave)

Payoff =  $(-2, 0)$

Conclusion: low reputation, price = (lie when needed, walk away)

Case 2:  $R \geq 12$

PNEs = {(always “fresh”, leave), (truth, buy if fresh)}

Payoffs:  $(-2, 0)$  and  $(2, 2)$

Conclusion: High reputation, price =  $\exists$  a better PNE

		Mixed NE	
		$q$	$1 - q$
$p$	$6 - \frac{R}{3}, -2$	$-2, 0$	
	$2, 2$	$-2, 0$	

$$-2p + 2(1 - p) = 0$$

$$4p = 2$$

$$\boxed{p = \frac{1}{2}}$$

$$\left(6 - \frac{R}{3}\right)q - 2(1 - q) = 2q - 2(1 - q)$$

$$4q - \frac{R}{3}q = 0 \sim (12 - R)q = 0$$

( $R = 12$  not likely)

$$\text{NE} = \left( \left( \frac{1}{2}, \frac{1}{2} \right), (0, 1) \right)$$

$$\text{Payoffs} = (-2, 0)$$

# 25 May 23, 2022

## 25.1 Evolutionary Stable Strategies

### Definition 25.1 (Starting point)

Population with  $n$  types of behavior of frequencies  $p_1, p_2, \dots, p_n$ . A small mutation introduces a new mix

$$\mathbf{q} = (q_1, q_2, \dots, q_n)$$

For  $\varepsilon > 0$ ,

$$(1 - \varepsilon)\mathbf{p} + \varepsilon\mathbf{q}$$

**Question 25.2:** How does the old population fare against the new one?

$$B = A^t$$

$$\begin{array}{c} p_1 \\ p_2 \\ \vdots \\ p_n \end{array} \begin{array}{c} (1 - \varepsilon)\mathbf{p} + \varepsilon\mathbf{q} \\ A \end{array} = (\mathbf{p})^t A [(1 - \varepsilon)\mathbf{p} + \varepsilon\mathbf{q}] = (1 - \varepsilon)(\mathbf{p})^t A \mathbf{p} + \varepsilon(\mathbf{p})^t A \mathbf{q}$$

$$\begin{array}{c} q_1 \\ q_2 \\ \vdots \\ q_n \end{array} \begin{array}{c} (1 - \varepsilon)\mathbf{p} + \varepsilon\mathbf{q} \\ A \end{array} = (\mathbf{q})^t A [(1 - \varepsilon)\mathbf{p} + \varepsilon\mathbf{q}] = (1 - \varepsilon)(\mathbf{q})^t A \mathbf{p} + \varepsilon(\mathbf{q})^t A \mathbf{q}$$

Note:

$$(\mathbf{p})^t A \mathbf{p} \geq (\mathbf{q})^t A \mathbf{p}$$

Condition 1:  $\mathbf{p}$  is a symmetric NE.

$\mathbf{q} = \mathbf{e}_i$  such that  $p_i > 0$

$$(\mathbf{p})^t A \mathbf{p} = (\mathbf{e}_i)^t A \mathbf{p}$$

Condition 2:  $(\mathbf{p})^t A \mathbf{q} > (\mathbf{q})^t A \mathbf{q}$

### Lemma 25.3

A symmetric  $n$ -players general sum game has a symmetric NE.

**Proof, Professor teased us about this being on the exam, maybe it will?** Recall

$$\begin{aligned} T: \Delta_m \times \Delta_n &\rightarrow \Delta_m \times \Delta_n \\ (\mathbf{p}, \mathbf{q}) &\mapsto (\hat{\mathbf{p}}, \hat{\mathbf{q}}) \end{aligned}$$

where

$$\hat{p}_i = \frac{p_i + c_i}{1 + \sum_{n=1}^n c_i} \quad \hat{q}_i = \frac{q_i + d_i}{1 + \sum_{n=1}^n d_i}$$

and

$$c_i = \max\{(\mathbf{e}_i)^t A \mathbf{q} - (\mathbf{p})^t A \mathbf{q}, 0\}$$

$$d_i = \max\{\mathbf{p}^t A^t \mathbf{q} - (\mathbf{p})^t A^t \mathbf{q}, 0\}$$

Game is symmetric, i.e.  $B = A^t$ ,  $\mathbf{p} = \mathbf{q}$ . We need to show

$$(\mathbf{e}_i)^t A \mathbf{p} - (\mathbf{p})^t A \mathbf{p}$$

equals

$$\mathbf{p}^t A^t \mathbf{e}_i - (\mathbf{p})^t A^t \mathbf{p}$$

Which is true.

$$\text{Diagonal } D = \{(\mathbf{p}, \mathbf{p}), \mathbf{p} \in \Delta_n\}$$

$$\underbrace{\Delta_n \times \Delta_n \times \cdots \times \Delta_n}_{k \text{ times}}$$

Now

$$T: D \rightarrow D$$

$$(\mathbf{p}, \mathbf{p}) \mapsto (\hat{\mathbf{p}}, \hat{\mathbf{q}})$$

And

$$(\mathbf{p}, \mathbf{q}) \mapsto (\hat{\mathbf{p}}, \hat{\mathbf{q}})$$

□

#### Example 25.4 (Hawks and Doves)

We have a game of Hawks and Doves

- Species with two behavior types, aggressive (hawks), and aggression averse (doves).
- $V$  = Victory's value
- $C$  = cost of fight

		P2	
		H	D
P1	$p, H$	$\frac{v}{2} - c, \frac{v}{2} - c$	$v, 0$
	$1 - p, D$	$0, v$	$\frac{v}{2}, \frac{v}{2}$

# 26 May 25, 2022

## 26.1 Evolutionary Stable Strategies (Cont'd)

**Recall 26.1** Recall

$$\begin{array}{c} p_1 \\ p_2 \\ \vdots \\ p_n \end{array} \begin{array}{|c|} \hline (1-\varepsilon)\mathbf{p} + \varepsilon\mathbf{q} \\ \hline \mathbf{A} \\ \hline \end{array} = (\mathbf{p})^t \mathbf{A} [(1-\varepsilon)\mathbf{p} + \varepsilon\mathbf{q}] = (1-\varepsilon)(\mathbf{p})^t \mathbf{A} \mathbf{p} + \varepsilon(\mathbf{p})^t \mathbf{A} \mathbf{q}$$

$$\begin{array}{c} q_1 \\ q_2 \\ \vdots \\ q_n \end{array} \begin{array}{|c|} \hline (1-\varepsilon)\mathbf{p} + \varepsilon\mathbf{q} \\ \hline \mathbf{A} \\ \hline \end{array} = (\mathbf{q})^t \mathbf{A} [(1-\varepsilon)\mathbf{p} + \varepsilon\mathbf{q}] = (1-\varepsilon)(\mathbf{q})^t \mathbf{A} \mathbf{p} + \varepsilon(\mathbf{q})^t \mathbf{A} \mathbf{q}$$

For the population to be stable, we need

$$(\mathbf{p})^t \mathbf{A} \mathbf{q} > (\mathbf{q})^t \mathbf{A} \mathbf{q}$$

**Example 26.2** (Hawks and Doves)

The payoff matrix is

		P2	
		H	D
P1	H	$\frac{v}{2} - c, \frac{v}{2} - c$	$v, 0$
	D	$0, v$	$\frac{v}{2}, \frac{v}{2}$

$v > 0, c > 0$ .

Case 1:  $0 < c < \frac{v}{2}$

		P2	
		H	D
P1	H	$\frac{v}{2} - c^*, \frac{v}{2} - c^*$	$v^*, 0$
	D	$0, v$	$\frac{v}{2}, \frac{v}{2}$

NE:  $(H, H)$

Case 2:  $c = \frac{v}{2}$

		P2	
		H	D
P1	H	$0^*, 0^*$	$v^*, 0$
	D	$0^*, v^*$	$\frac{v}{2}, \frac{v}{2}$

Case 3:  $0 < \frac{v}{2} < c$

		P2	
		H	D
P1	$p, H$	$\frac{v}{2} - c, \frac{v}{2} - c$	$v^*, 0^*$
	$1 - p, D$	$0^*, v^*$	$\frac{v}{2}, \frac{v}{2}$

Continue from Case 3,

$$H(p) = p \left( \frac{v}{2} - c \right) + (1 - p)v$$

$$H(p) = \frac{v}{2}p - cp + v - \frac{v}{2}p$$

$$\begin{aligned} H(p) &= v - \frac{v}{2}p - cp \\ &= v - \left( \frac{v}{2} + c \right) p \end{aligned}$$

$$D(p) = \frac{v}{2}(1 - p)$$

The mixed nash equilibrium occurs when  $H(p) = D(p)$ .

$$\begin{aligned} H(p) &= D(p) \\ v - \frac{v}{2}p - cp &= \frac{v}{2} - \frac{v}{2}p \\ \frac{v}{2} &= cp \\ \implies \hat{p} &= \frac{v}{2c} \\ 1 - \hat{p} &= 1 - \frac{v}{2c} \end{aligned}$$

So

$$D(\hat{p}) = \frac{v}{2} \left( 1 - \frac{v}{2c} \right)$$

Suppose  $0 < p < \hat{p}$ , then

$$H \nearrow D \searrow$$

Suppose  $\hat{p} < p < 1$ , then

$$H \searrow D \nearrow$$

### Example 26.3

Now consider,<sup>a</sup>

$$\begin{array}{c}
 \text{P2} \\
 (1 - \varepsilon)\hat{\mathbf{p}} + \varepsilon\mathbf{q} \\
 \text{P1} \quad \begin{array}{c} q \\ 1 - q \end{array} \begin{array}{|c|c|} \hline \frac{v}{2} - c, \frac{v}{2} - c & v, 0 \\ \hline 0, v & \frac{v}{2}, \frac{v}{2} \\ \hline \end{array}
 \end{array}$$

and

$$\begin{array}{c}
 \text{P2} \\
 (1 - \varepsilon)\hat{\mathbf{p}} + \varepsilon\mathbf{q} \\
 A = \quad \begin{array}{c} \mathbf{p} \\ \frac{v}{2c} \\ 1 - \frac{v}{2c} \end{array} \begin{array}{|c|c|} \hline \frac{v}{2} - c & v \\ \hline 0 & \frac{v}{2} \\ \hline \end{array}
 \end{array}$$

So

$$\hat{\mathbf{p}} = \begin{pmatrix} \frac{v}{2c} \\ 1 - \frac{v}{2c} \end{pmatrix} \quad \mathbf{q} = \begin{pmatrix} q \\ 1 - q \end{pmatrix}$$

So

$$(\hat{\mathbf{p}})^t A \mathbf{q} > (\mathbf{q})^t A \mathbf{q}$$

Then

$$\begin{aligned}
 \left[ \frac{v}{2c}, 1 - \frac{v}{2c} \right] \begin{bmatrix} \frac{v}{2} - c & v \\ 0 & \frac{v}{2} \end{bmatrix} &= [q, 1 - q] \begin{bmatrix} \frac{v}{2} - c & v \\ 0 & \frac{v}{2} \end{bmatrix} \begin{bmatrix} q \\ 1 - q \end{bmatrix} \stackrel{?}{>} 0 \\
 &= \left[ \frac{v}{2c} - q, 1 - \frac{v}{2c} - (1 - q) \right] \begin{bmatrix} \frac{v}{2} - c & v \\ 0 & \frac{v}{2} \end{bmatrix} \begin{bmatrix} q \\ 1 - q \end{bmatrix} \stackrel{?}{>} 0 \\
 &= \left[ \frac{v}{2c} - q \right] \begin{bmatrix} \frac{v}{2} - c & v \\ 0 & \frac{v}{2} \end{bmatrix} \begin{bmatrix} q \\ 1 - q \end{bmatrix} \stackrel{?}{>} 0 \\
 &= \left[ \left( \frac{v}{2} - c \right) \left( \frac{v}{2c} - q \right), v \left( \frac{v}{2c} - q \right) + \frac{v}{2} \left( q - \frac{v}{2c} \right) \right] \begin{bmatrix} q \\ 1 - q \end{bmatrix} \stackrel{?}{>} 0 \\
 &= \left[ \left( \frac{v}{2} - c \right) \left( \frac{v}{2c} - q \right), -\frac{v^2}{4c} - \frac{v}{2}q \right] \begin{bmatrix} q \\ 1 - q \end{bmatrix} \stackrel{?}{>} 0 \\
 &= \left[ \left( \frac{v}{2} - c \right) \left( \frac{v}{2c} - q \right), \frac{v}{2} \left( \frac{v}{2c} - q \right) \right] \begin{bmatrix} q \\ 1 - q \end{bmatrix} \stackrel{?}{>} 0 \\
 &= \left( \frac{v}{2} - c \right) \left( \frac{v}{2c} - q \right) q + (1 - q) \frac{v}{2} \left( \frac{v}{2c} - q \right) \stackrel{?}{>} 0 \\
 &= \left( \frac{v}{2c} - q \right) \left[ \frac{v}{2}q - cq + \frac{v}{2} - \frac{v}{2}q \right] \stackrel{?}{>} 0 \\
 &= c \left( \frac{v}{2c} - q \right) \left( \frac{v}{2} - q \right) \stackrel{?}{>} 0 \\
 &= c \left( \frac{v}{2c} - q \right)^2 > 0
 \end{aligned}$$

And

$$\begin{aligned}
 q &\neq \hat{p} \\
 q &\neq \frac{v}{2c}
 \end{aligned}$$



---

<sup>a</sup>Evolutionary stability problem might be on final

**Example 26.4** (Rock, Paper, Scissors)

Recall

$$(\hat{\mathbf{p}})^t = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right)$$

And

		P2		
		R	P	S
P1	R	0	-1	1
	P	1	0	-1
	S	-1	1	0

So check

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = 0$$

And

$$\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} q_2 - q_3 & q_3 - q_1 & q_1 - q_2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = 0$$

# 27 May 25, 2022

## 27.1 Evolutionary Stable Strategies (Cont'd)

**Recall 27.1** Recall

$$\begin{array}{c} p_1 \\ p_2 \\ \vdots \\ p_n \end{array} \begin{array}{|c|} \hline (1-\varepsilon)\mathbf{p} + \varepsilon\mathbf{q} \\ \hline \mathbf{A} \\ \hline \end{array} = (\mathbf{p})^t \mathbf{A} [(1-\varepsilon)\mathbf{p} + \varepsilon\mathbf{q}] = (1-\varepsilon)(\mathbf{p})^t \mathbf{A} \mathbf{p} + \varepsilon(\mathbf{p})^t \mathbf{A} \mathbf{q}$$

$$\begin{array}{c} q_1 \\ q_2 \\ \vdots \\ q_n \end{array} \begin{array}{|c|} \hline (1-\varepsilon)\mathbf{p} + \varepsilon\mathbf{q} \\ \hline \mathbf{A} \\ \hline \end{array} = (\mathbf{q})^t \mathbf{A} [(1-\varepsilon)\mathbf{p} + \varepsilon\mathbf{q}] = (1-\varepsilon)(\mathbf{q})^t \mathbf{A} \mathbf{p} + \varepsilon(\mathbf{q})^t \mathbf{A} \mathbf{q}$$

For the population to be stable, we need

$$(\mathbf{p})^t \mathbf{A} \mathbf{q} > (\mathbf{q})^t \mathbf{A} \mathbf{q}$$

**Example 27.2** (Hawks and Doves)

The payoff matrix is

		P2	
		H	D
P1	H	$\frac{v}{2} - c, \frac{v}{2} - c$	$v, 0$
	D	$0, v$	$\frac{v}{2}, \frac{v}{2}$

$v > 0, c > 0$ .

Case 1:  $0 < c < \frac{v}{2}$

		P2	
		H	D
P1	H	$\frac{v}{2} - c^*, \frac{v}{2} - c^*$	$v^*, 0$
	D	$0, v$	$\frac{v}{2}, \frac{v}{2}$

NE:  $(H, H)$

Case 2:  $c = \frac{v}{2}$

		P2	
		H	D
P1	H	$0^*, 0^*$	$v^*, 0$
	D	$0^*, v^*$	$\frac{v}{2}, \frac{v}{2}$

Case 3:  $0 < \frac{v}{2} < c$

		P2	
		H	D
P1	$p, H$	$\frac{v}{2} - c, \frac{v}{2} - c$	$v^*, 0^*$
	$1 - p, D$	$0^*, v^*$	$\frac{v}{2}, \frac{v}{2}$

Continue from Case 3,

$$H(p) = p \left( \frac{v}{2} - c \right) + (1 - p)v$$

$$H(p) = \frac{v}{2}p - cp + v - \frac{v}{2}p$$

$$H(p) = v - \frac{v}{2}p - cp$$

$$= v - \left( \frac{v}{2} + c \right) p$$

$$D(p) = \frac{v}{2}(1 - p)$$

The mixed nash equilibrium occurs when  $H(p) = D(p)$ .

$$H(p) = D(p)$$

$$v - \frac{v}{2}p - cp = \frac{v}{2} - \frac{v}{2}p$$

$$\frac{v}{2} = cp$$

$$\implies \hat{p} = \frac{v}{2c}$$

$$1 - \hat{p} = 1 - \frac{v}{2c}$$

So

$$D(\hat{p}) = \frac{v}{2} \left( 1 - \frac{v}{2c} \right)$$

Suppose  $0 < p < \hat{p}$ , then

$$H \nearrow D \searrow$$

Suppose  $\hat{p} < p < 1$ , then

$$H \searrow D \nearrow$$

### Example 27.3

Now consider,<sup>a</sup>

$$\begin{array}{c}
 \text{P2} \\
 (1 - \varepsilon)\hat{\mathbf{p}} + \varepsilon\mathbf{q} \\
 \text{P1} \quad \begin{array}{cc} q & \begin{array}{|c|c|} \hline \frac{v}{2} - c, \frac{v}{2} - c & v, 0 \\ \hline 0, v & \frac{v}{2}, \frac{v}{2} \\ \hline \end{array} \\ 1 - q & \end{array}
 \end{array}$$

and

$$\begin{array}{c}
 \text{P2} \\
 (1 - \varepsilon)\hat{\mathbf{p}} + \varepsilon\mathbf{q} \\
 \mathbf{p} \\
 A = \begin{array}{cc} \frac{v}{2c} & \begin{array}{|c|c|} \hline \frac{v}{2} - c & v \\ \hline 0 & \frac{v}{2} \\ \hline \end{array} \\ 1 - \frac{v}{2c} & \end{array}
 \end{array}$$

So

$$\hat{\mathbf{p}} = \begin{pmatrix} \frac{v}{2c} \\ 1 - \frac{v}{2c} \end{pmatrix} \quad \mathbf{q} = \begin{pmatrix} q \\ 1 - q \end{pmatrix}$$

So

$$(\hat{\mathbf{p}})^t A \mathbf{q} > (\mathbf{q})^t A \mathbf{q}$$

Then

$$\begin{aligned}
 \left[ \frac{v}{2c}, 1 - \frac{v}{2c} \right] \begin{bmatrix} \frac{v}{2} - c & v \\ 0 & \frac{v}{2} \end{bmatrix} &= [q, 1 - q] \begin{bmatrix} \frac{v}{2} - c & v \\ 0 & \frac{v}{2} \end{bmatrix} \begin{bmatrix} q \\ 1 - q \end{bmatrix} \stackrel{?}{>} 0 \\
 &= \left[ \frac{v}{2c} - q, 1 - \frac{v}{2c} - (1 - q) \right] \begin{bmatrix} \frac{v}{2} - c & v \\ 0 & \frac{v}{2} \end{bmatrix} \begin{bmatrix} q \\ 1 - q \end{bmatrix} \stackrel{?}{>} 0 \\
 &= \left[ \frac{v}{2c} - q \right] \begin{bmatrix} \frac{v}{2} - c & v \\ 0 & \frac{v}{2} \end{bmatrix} \begin{bmatrix} q \\ 1 - q \end{bmatrix} \stackrel{?}{>} 0 \\
 &= \left[ \left( \frac{v}{2} - c \right) \left( \frac{v}{2c} - q \right), v \left( \frac{v}{2c} - q \right) + \frac{v}{2} \left( q - \frac{v}{2c} \right) \right] \begin{bmatrix} q \\ 1 - q \end{bmatrix} \stackrel{?}{>} 0 \\
 &= \left[ \left( \frac{v}{2} - c \right) \left( \frac{v}{2c} - q \right), -\frac{v^2}{4c} - \frac{v}{2}q \right] \begin{bmatrix} q \\ 1 - q \end{bmatrix} \stackrel{?}{>} 0 \\
 &= \left[ \left( \frac{v}{2} - c \right) \left( \frac{v}{2c} - q \right), \frac{v}{2} \left( \frac{v}{2c} - q \right) \right] \begin{bmatrix} q \\ 1 - q \end{bmatrix} \stackrel{?}{>} 0 \\
 &= \left( \frac{v}{2} - c \right) \left( \frac{v}{2c} - q \right) q + (1 - q) \frac{v}{2} \left( \frac{v}{2c} - q \right) \stackrel{?}{>} 0 \\
 &= \left( \frac{v}{2c} - q \right) \left[ \frac{v}{2}q - cq + \frac{v}{2} - \frac{v}{2}q \right] \stackrel{?}{>} 0 \\
 &= c \left( \frac{v}{2c} - q \right) \left( \frac{v}{2} - q \right) \stackrel{?}{>} 0 \\
 &= c \left( \frac{v}{2c} - q \right)^2 > 0
 \end{aligned}$$

And

$$\begin{aligned}
 q &\neq \hat{p} \\
 q &\neq \frac{v}{2c}
 \end{aligned}$$

---

<sup>a</sup>Evolutionary stability problem might be on final

**Example 27.4** (Rock, Paper, Scissors)

Recall

$$(\hat{\mathbf{p}})^t = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right)$$

And

		P2		
		R	P	S
P1	R	0	-1	1
	P	1	0	-1
	S	-1	1	0

So check

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = 0$$

And

$$\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} q_2 - q_3 & q_3 - q_1 & q_1 - q_2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = 0$$

# 28 May 27, 2022

## 28.1 Games in Coalition Form

### Definition 28.1 (Coalition)

Let  $n \in \mathbb{N}$ , the number of players. A coalition, a subset  $S \subseteq \{P1, P2, \dots, Pn\}$ . Let  $N$  be the set of all the subsets of  $\{P1, P2, \dots, Pn\}$ , so

$$|N| = 2^n$$

### Definition 28.2 (Characteristic function)

The characteristic function of the game

$$V: N \rightarrow \mathbb{R} \text{ s.t.}$$

$$* V(\emptyset) = 0$$

\*\* Superadditivity:

$$S, T \in N, S \cap T = \emptyset \implies V(S) + V(T) \leq V(S \cup T)$$

Relation to strategic form:

$$P1: S1 = (S11, S12, \dots, S1m_1); u_1$$

$$P2: S2 = (S21, S22, \dots, S2m_2); u_2$$

$$\vdots$$

$$Pn: Sn = (Sn1, Sn2, \dots, Snm_n); u_n$$

Where

$$u_i: S1 \times S2 \times \dots \times S_n \rightarrow \mathbb{R}$$

$$(S1i_1, S2i_2, \dots, Sni_n) \mapsto u(S1i, S2i_2, \dots, Sni_n)$$

with

$$i = 1, \dots, n$$

$$i_1 = 1, \dots, m_1; \quad i_2 = 1, \dots, m_2; \quad \dots \quad ; i_n = 1, \dots, m_n$$

Let  $S \in N$ , a coalition. They play against the “world”

$$= \{P1, P2, \dots, Pn\} - S$$

We have

... p.s. available to the world

all $k$ -tuples of p.s.	•	•	•
	•	•	•
	•	•	•
	•	•	•

$$V(S) = \text{Safety value}$$

$$V(T) = \text{Safety value}$$

$$V(S) + V(T) \leq V(S \cup T)$$

**Example 28.3**

We have

		P3 purifies, S31	
		S21	S22
P1	P1 purifies, S11	1, 1, 1	1, 0, 1
	P2 pollutes, S12	0, 1, 1	3, 3, 4

And

		P3 pollutes, S32	
		S21	S22
P1	S11	1, 1, 0	4, 3, 3
	S12	3, 4, 3	3, 3, 3

$$V(\emptyset) = 0$$

$$S = \{P1\}$$

	(S21, S31)	(S21, S32)	(S22, S31)	(S22, S32)
S11, $p$	-1	-1	-1	-4
S12, $1 - p$	0	-3	-3	-3

$$V(P1) = V(P2) = V(P3) = -3$$

	$q_1$	$q_2$	$q_3$
$p$	-1	-1	-4
$1 - p$	0	-3	-3

So

$$\begin{aligned} \max_{0 \leq p \leq 1} \min_{\mathbf{q} \in \Delta_3} (\mathbf{p})^t A \mathbf{q} &= \max_{0 \leq p \leq 1} \min_{\mathbf{q} \in \Delta_3} \left( q_1(-p) + q_2(-p - 3(1 - p)) + q_3(-4p - 3(1 - p)) \right) \\ &= \max_{0 \leq p \leq 1} \min\{-p, 2p - 3, -p - 3\} \end{aligned}$$

So  $p = 0$ .

# 29 June 1, 2022

## 29.1 Games in Coalition Form (Cont'd)

**Recall 29.1** We have  $n$  players,  $P_1, P_2, \dots, P_n$ . A coalition  $S \subseteq \{P_1, P_2, \dots, P_n\}$ . Characteristic function:  $V(S)$ , s.t.

1.  $V(\emptyset) = 0$
2. Superadditivity:  $S, T \subseteq \{P_1, \dots, P_n\}, S \cap T = \emptyset \implies V(S \cup T) \geq V(S) + V(T)$ .



**Example 29.2** (Example (Cont'd))

We have

	P3 purifies, S31			P3 pollutes, S32	
	S21	S22		S21	S22
P1 purifies, S11	1, 1, 1	1, 0, 1	S11	1, 1, 0	4, 3, 3
P2 pollutes, S12	0, 1, 1	3, 3, 4	S12	3, 4, 3	3, 3, 3

Step 0:  $V(\emptyset) = 0$

Step 1:  $V(\{P1\}) = V(\{P2\}) = V(\{P3\}) = -3$

Step 2: Consider  $\{P1, P2\}$ .

	S31	S32	
(S11, S21)	$-1 - 1 = -2$	$-1 - 1 = -2$	duplicate strategy $\leftarrow \checkmark$
(S11, S22)	$-1$	$-7$	
(S12, S21)	$-1$	$-7$	
(S12, S22)	$-6$	$-6$	

So  $-2 > -6$  and  $-2 > -6$ , so

$$\vec{q}$$

$$\vec{p} \begin{bmatrix} -2 & -2 \\ -1 & -7 \end{bmatrix}$$

with saddle point of  $a_{12} = -2$ .

$$\max_{\vec{p} \in \Delta_m} \min_{\vec{q} \in \Delta_n} (\vec{p})^t A \vec{q} = \min_{\vec{q} \in \Delta_n} \max_{\vec{p} \in \Delta_m} (\vec{p})^t A \vec{q}$$

So

$$V(\{P1, P2\}) = -2 = V(\{P1, P3\}) = V(\{P2, P3\})$$

Step 3:

	P2	
	(S11, S21, S31)	(S11, S21, S32)
World	-3	-2

So

$$V(\{P1, P2, P3\}) = -2$$

$$-3 - 3 \leq -2$$

$$-2 - 3 \leq -2$$

**Note 29.3** (Coalition Form  $\rightarrow$  Strategic Form): Strategy: join this or that coalition

$S_i$  = the set of all coalitions containing  $P_i$

$S_i1$  = the first coalition containing  $P_i$ ,

$S_i2$  = the second coalition etc.

$$u_i(S_1k, S_2k_2, \dots, S_nk_n) = \begin{cases} \frac{V(S_ik_i)_i}{|S_ik_i|} & \text{if } S_ik_i = S_jk_j \quad \forall P_j \in S_ik_i \\ V(\{P_i\}) & \text{otherwise} \end{cases}$$

#### Example 29.4

$$V(\emptyset) = 0$$

$$V(\{P1\}) = V(\{P2\}) = V(\{P3\}) = -3$$

$$V(\{P1, P2\}) = V(\{P1, P3\}) = V(\{P2, P3\}) = -2$$

$$V(\{P1, P2, P3\}) = -2$$

	{P3}			
	{P2}	{P2, P1}	{P2, P3}	{P1, P2, P3}
{P1}	-3	-3	-3	-3
{P1, P2}	-3	-1	-3	-3
{P1, P3}	-3	-3	-3	-3
{P1, P2, P3}	-3	-3	-3	-3

# 30 June 3, 2022

## 30.1 Final Exam Information

1. Problem 1 is a Nim.
2. Problem 2: Zero sum game and symmetry on a two person game
3. Problem 3: Game in Extensive Form. Fish market example. Translate. Draw a graph of the game. Pay attention to the information set. Second problem ask to present game in strategic form.
4. Problem 4: Stability in equilibrium **OR** game in coalition form.
5. Problem 5 (Extra credit) on Cauchy induction.

## 30.2 Review for Final Exam

(Quite likely) the Main result in combinatorial game theory.

### Theorem 30.1 (Sprague-Grundy Theorem)

Any two-person progressively bounded impartial game under normal play is equivalent to the game of Nim, including the subtraction form.

**Note 30.2:** This theorem should help for the first problem of the final.

### Example 30.3

We have  $n = 100$ .

$m = 3 \implies m + 1 = 4$ .

$$\begin{array}{r}
 0 \ 0 \ 0 \\
 0 \ 0 \ 0 \\
 \vdots \\
 0 \ 0 \ 0 \\
 \hline
 0 \ 0 \ 0
 \end{array}$$

**Example 30.4** (Nim Example (This will be on the final))Given  $(12, 19, 27)$ .

- (a) How would you play the game under Normal play?
- (b) How would you play the game as subtraction Nim with the maximal number of chips to remove = 6
- (c) How would you play the misere version of the game?

(a)

$$12_{10} = 8 + 4 = 1100_2$$

$$19_{10} = 16 + 2 + 1 = 10011_2$$

$$27_{10} = 16 + 8 + 2 + 1 = 11011_2$$

Then

$$\begin{array}{r} \phantom{\oplus} \phantom{1} \phantom{0} \phantom{0} \phantom{1} \phantom{1} \\ \phantom{\oplus} \phantom{1} \phantom{0} \phantom{0} \phantom{1} \phantom{1} \\ \oplus \phantom{1} \phantom{0} \phantom{0} \phantom{1} \phantom{1} \phantom{1} \\ \phantom{\oplus} \phantom{1} \phantom{0} \phantom{0} \phantom{1} \phantom{1} \\ \hline \phantom{\oplus} \phantom{1} \phantom{0} \phantom{0} \phantom{1} \phantom{1} \\ \phantom{\oplus} \phantom{1} \phantom{0} \phantom{0} \phantom{1} \phantom{1} \\ \phantom{\oplus} \phantom{1} \phantom{0} \phantom{0} \phantom{1} \phantom{1} \\ \phantom{\oplus} \phantom{1} \phantom{0} \phantom{0} \phantom{1} \phantom{1} \end{array}$$

So

$$12 \oplus 19 \oplus 27 = 4 + 0$$

(b)

$$12 \bmod 7 = 5 = 4 + 1 = 101_2$$

$$19 \bmod 7 = 5 = 101_2$$

$$27 \bmod 7 = 6 = 110_2$$

So

$$\begin{array}{r} \phantom{1} \phantom{0} \phantom{1} \\ \phantom{1} \phantom{0} \phantom{1} \\ \phantom{1} \phantom{0} \phantom{1} \\ \oplus \phantom{1} \phantom{0} \phantom{1} \\ \hline \phantom{1} \phantom{0} \phantom{1} \\ \phantom{1} \phantom{0} \phantom{1} \\ \phantom{1} \phantom{0} \phantom{1} \end{array}$$

(c)

$$\underbrace{\square \quad \square \quad \dots \quad \square \quad \square \quad \square}$$

**30.2.1 Cauchy Induction**Problem: prove a family  $S(n)$  of statements depending on  $n \in \mathbb{N}$ ,  $n \geq 2$ .

Before we start the next example, let's define AM and GM.

**Definition 30.5** (Arithmetic Mean vs. Geometric Mean)

Suppose  $a_1 > 0, a_2 > 0, \dots, a_n > 0$ . Then the arithmetic mean (AM) is defined as

$$\text{AM} = \frac{a_1 + a_2 + \dots + a_n}{n}$$

and the geometric mean (GM) is defined as

$$\text{GM} = \sqrt[n]{a_1 a_2 \dots a_n}$$

**Example 30.6** (AM-GM)

Algorithm:

**Step 1:** Base of induction

Check if  $S(2)$  is true

$$\begin{aligned} \frac{a_1 + a_2}{2} &\stackrel{?}{\geq} \sqrt{a_1 a_2} \\ \frac{a_1^2 - 2a_1 a_2 + a_2^2}{4} &\stackrel{?}{\geq} 0 \\ \frac{(a_1 - a_2)^2}{4} &\geq 0 \end{aligned}$$

**Step 2:**  $S(n)$  true  $\implies S(2n)$ .

$$\begin{aligned} \frac{a_1 + a_2 + \dots + a_n + a_{n+1} + a_{n+2} + \dots + a_{2n}}{2n} &= \frac{\frac{a_1 + a_2 + \dots + a_n}{n} + \frac{a_{n+1} + a_{n+2} + \dots + a_{2n}}{n}}{2} \\ &\geq \frac{\sqrt[n]{a_1 a_2 \dots a_n} + \sqrt[n]{a_{n+1} a_{n+2} \dots a_{2n}}}{2} \\ &\geq \sqrt{\sqrt[n]{a_1 a_2 \dots a_n} \cdot \sqrt[n]{a_{n+1} a_{n+2} \dots a_{2n}}} \\ &= \sqrt[2n]{a_1 a_2 \dots a_{2n}} \end{aligned}$$

**Step 3:**  $S(n) \implies S(n-1)$ .

$$\begin{aligned} \frac{a_1 + a_2 + \dots + a_{n-1} + \frac{a_1 + a_2 + \dots + a_{n-1}}{n-1}}{n} &\geq \sqrt[n]{a_1 a_2 \dots a_{n-1} \cdot \frac{(a_1 + a_2 + \dots + a_{n-1})}{n-1}} \\ \left( \frac{a_1 + a_2 + \dots + a_{n-1}}{n-1} \right)^n &\geq a_1 \cdot a_2 \dots a_{n-1} \cdot \frac{a_1 + a_2 + \dots + a_{n-1}}{n-1} \end{aligned}$$

Hence

$$\left( \frac{a_1 + a_2 + \dots + a_{n-1}}{n-1} \right)^{n-1} \geq a_1 \cdot a_2 \dots a_{n-1}$$

□