Math 110B (Algebra)
University of California, Los Angeles

Aaron Chao

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These are my lecture notes for Math 110B (Algebra), which is the second course in Algebra taught by Nicolle Gonzales. The textbook for this class is *Abstract Algebra: An Introduction, 3rd edition* by Hungerford.

Contents

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1.1 Groups

- Algebra $\to$ study of mathematical structure.
- Rings $\leftrightarrow$ "numbers" e.g. $\mathbb{R}, \mathbb{Z}, \mathbb{C}, \mathbb{Z}_p$
  2 operations $(+, \cdot)$

Question 1.1: What happens if we have only 1 operation (either $\cdot$ or $+$ but not both)? What kind of structure is this more basic setup?

Answer: Groups! It turns out groups encode the mathematical structures of the symmetries in nature.

Definition 1.2 (Group)
A group $(G, *)$ is a nonempty set with a binary operation $* : G \times G \to G$ that satisfies

1. (Closure): $a * b \in G \quad \forall a, b \in G$
2. (Associativity): $(a * b) * c = a * (b * c) \quad \forall a, b, c \in G$
3. (Identity): $\exists e \in G$ such that $e * a = a = a * e \quad \forall a \in G$
4. (Inverse): $\forall a \in G, \exists d \in G$ such that $d * a = e = a * d$

Note:

- If * is addition, we just divide * by the usual + sign. In this case
$$e = 0 \quad \text{and} \quad d = -a$$
- If the operation * is multiplication, we just divide * by the usual $\cdot$ sign. In this case
$$e = 1 \quad \text{and} \quad d = a^{-1}$$
- Be aware that sometimes * is neither.

Definition 1.3 (Abelian)
If the * operation is commutative, i.e. $a * b = b * a$, then we say that G is abelian (named after the mathematician N.H. Abel)

Definition 1.4 (Order, Finite Group vs. Infinite Group)
The order of a group G, denoted $|G|$, is the number of elements it contains (as a set). Thus, G is a finite group if $|G| < \infty$
and G is an infinite group if $|G| = \infty$

Examples 1.5 (Examples of a group)

1. Rings where you "forget" multiplication.
   $\to (\mathbb{Z}, +)$ integers with $* = +, (\mathbb{R}[X], +)$, etc.
   Note: $(\mathbb{Z}, *)$ with $* = \cdot$ is not a group. Why?

Theorem 1.6
Every ring is an abelian group under addition.

Proof. $e = 0$, inverse $= -a$ for each $a \in R$. $\square$

Fact: If $R \neq 0$ then $(R, \cdot)$ is never a group since 0 has no multiplicative inverse.

Examples 1.7 (More examples of a group)

2. Fields without zero.

Theorem 1.8
Let $\mathbb{F}^*$ denote the nonzero elements of a field $\mathbb{F}$. Then $(\mathbb{F}^*, \cdot)$ is an abelian group.

Recall: A unit in a ring R is an element $a \in R$ with a multiplicative inverse $a^{-1} \in R$ such that $aa^{-1} = 1 = a^{-1}a$.

Theorem 1.9
The set of units $\mathcal{U}$ inside a ring R is a group under multiplication.

Examples 1.10 (More examples of a group cont.)

3. $\mathcal{U}_n = \{m | (m, n) = 1\} \subseteq \mathbb{Z}_n$ is also a group, but under multiplication,

   $\underline{n = 4}$ $\mathbb{Z}_4 = \{0, 1, 2, 3\}$, $\mathcal{U}_4 = \{1, 3\}$
   $(\mathbb{Z}_4, +)$ and $(\mathcal{U}_4, \cdot)$ are groups with different binary operation!

   $\underline{n = 6}$ $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$, $\mathcal{U}_6 = \{1, 5\}$
   $(\mathcal{U}_6, \cdot)$ is a group
   - $1 \cdot 5 = 5 \pmod 6 \in \mathcal{U}_6$ (closure)
   - $1 = e$ (identity)
   - $1 \cdot 1 = 1$, $5 \cdot 5 = 25 \equiv 1 \pmod 6$ (inverse)
   - Associativity is clear

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2.1 Groups (Cont'd)

Examples 2.1

4. $(M_{n\times m}(\mathbb{F}), +) = m \times n$ matrices over $\mathbb{F}$ under addition
   $e =$ zero matrix, inverse of a matrix $-M$

Definition 2.2 (General linear group)
Denote by $GL_n(\mathbb{F})$ the set of nxn invertible matrices under multiplication. $(\det(A) \neq 0 \quad \forall A \in GL_n)$

- Closed: $\det(A \cdot B) = \det(A) \cdot \det(B) \neq 0 \implies AB \in GL_n \quad \forall A, B \in GL_N$
- Associativity: Obvious.
- Identity: $\det(I) = 1 \neq 0 \implies I \in GL_n(\mathbb{F})$
- Inverse: $A \in GL_n; \det(A^{-1}) = \frac{1}{\det(A)} \neq 0 \implies A^{-1} \in GL_n(\mathbb{F})$

Fact: $GL_n(\mathbb{F})$ is a group for any field $\mathbb{F}$.
Comment:

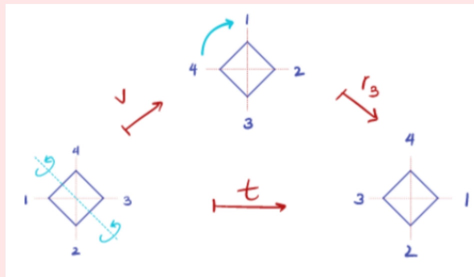
- $\det(A + B) \neq \det(A) + \det(B)$
- $\det(AB) = \det(A) \cdot \det(B)$

Definition 2.3 (Special linear group)
Let $SL_n(\mathbb{F})$ denote the set of invertible matrices over $\mathbb{F}$ with $\det = 1$

Exercise. Show that $SL_n(\mathbb{F})$ is a group.

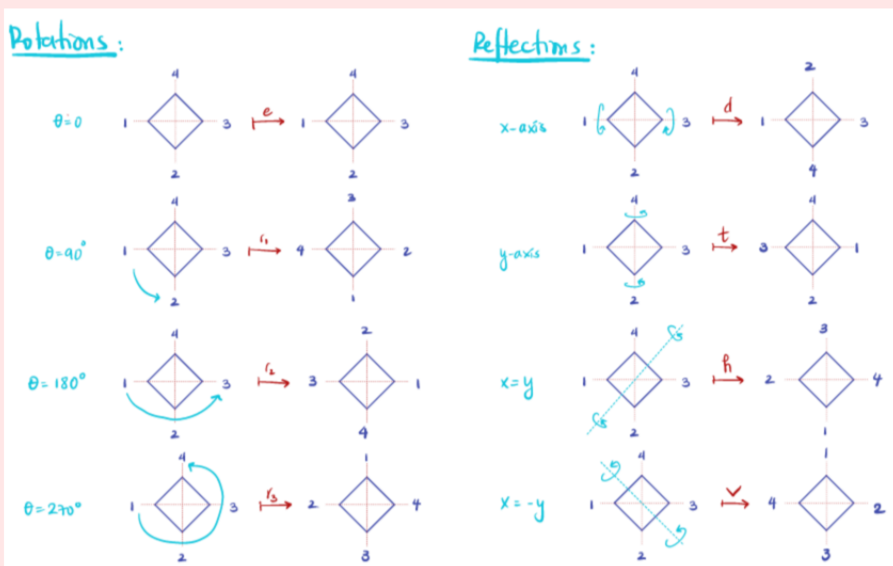
2.2 Symmetries

Example 2.4 (Symmetries over a square)
Rotations and reflection These operations (maps) form a group under composition. So $* = 0$. For instance, suppose
$r_3 \circ t = h$



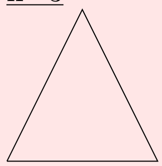
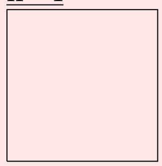
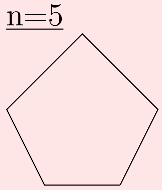
The group of rotations/reflections of a square is called <u>Dihedral Group of degree 4</u>, denoted D_4.

$$D_4 = \{r_1, r_2, r_3, r_4, d, t, h, v \mid \text{under } \circ\}$$



These are Professor Gonzales's lovely drawings.

Example 2.5 (Symmetries of a regular polygon with n sides)
Called the dihedral groups of degree n, D_n.

- n=3

- n=4

- n=5

- n=6
etc...

Observe: $|D_n| = 2n$ because you have n-axes of reflection and n-angles of notation.

Example 2.6 (The symmetric group)
Let $n \in N$, and S_n be the set of all permutations of the numbers $\{1, ..., n\}$.
Note: any permutation of $\{1, ..., n\}$ can be thought of as a bijection $\{1, ..., n\} \to \{1, ..., n\}$.
$\implies$ This allows us to compose permutations just like functions.
$\implies$ S_n is a group!

Definition 2.7 (Symmetric group)
The symmetric group S_n is the group of permutations of the integers of the integers $\{1, ..., n\}$.
Given any permutation $\sigma \in S_n$,

$$\begin{aligned} \sigma :&\{1, ..., n\} \to \{1, ..., n\}, \\ &i \mapsto \sigma_i \end{aligned}$$

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_{n-1} & \sigma_n \end{pmatrix} \to e = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}$$

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma_1^{-1} & \sigma_2^{-1} & \cdots & \sigma_n^{-1} \end{pmatrix}$$

Group operation: function composition.

Example 2.8

n=2:

$e = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ $\tau = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$\tau \circ \tau = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = e$

$\tau \circ e = \tau$

$e \circ \tau = \tau$

$e \circ \tau = e$

$\implies S_2 = \{e, \tau\}$ is a group

$e^{-1} = e$

$\tau^{-1} = \tau$

Associativity: obvious because of function composition

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3.1 Symmetries (Cont'd)

Example 3.1
n=3 S_3: permutations of $\{1,2,3\}$

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \tau_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \tau_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$\tau_{21} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \tau_{12} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \tau_{121} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\begin{aligned} \tau_1 \circ \tau_2 \circ \tau_1 &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \tau_{121} \end{aligned}$$

Note: $\tau_{21} = \tau_2 \circ \tau_1$, $\tau_{12} = \tau_1 \circ \tau_2$
$\tau_{21} \neq \tau_{12} \implies S_3$ is not abelian!

Exercise. τ_{212}?

3.2 Direct Product of Groups

Definition 3.2 (Direct product)
Given $(G, *), (H, \star)$ both groups define the binary operation:

$$\begin{aligned} \square \colon (G \times H) \times (G \times H) &\to G \times H \\ (g, h) \,\square\, (g', h') &\mapsto (g * g', h \star h') \end{aligned}$$

Side note: $(S, ☺)$
$☺\colon S \times S \to S \implies S$ group

Example 3.3
$S_2 \times D_4$:
$(\tau_1, r_{270^\circ}) \,\square\, (\tau_1, v) = (\tau_1 \circ \tau_1, r_{270^\circ} v) = (e, t)$

Example 3.4
$(\mathbb{R}, +) \times (\mathbb{R}^*, \cdot)$
$(5, 2) \,\square\, (-5, \pi) = (0, 2\pi)$

Example 3.5
$\mathbb{Z}_n \times \mathbb{Z}_m \quad n, m \in \mathbb{N}.$
$(a, b) \,\square\, (a', b') = (\underbrace{a + a'}_{\text{mod } n}, \underbrace{b + b'}_{\text{mod } m})$

$$\begin{aligned} (5,5) \overset{\mathbb{Z}_8 \times \mathbb{Z}_6}{\square} (2,2) &= (5+2, 5+2) \\ &= (7,1) \end{aligned}$$

3.3 Properties of Groups

Notation: Going forward, we omit $*$ in the notation: $(G, *) \to G$. Use multiplicative notation for abstract groups. Instead $a * b \to ab$.

$$\underbrace{a * a * a * a \cdots * a}_{n \text{ times}} \to a^n$$

However, for very explicit groups like
$(\mathbb{Z}, +), (\mathbb{R}, +), (\mathbb{Z}_n, +)$, etc, we use additive notation. $(* = +)$

$$a * b \to a + b$$

$$\underbrace{a * \cdots * a}_{n \text{ times}} \to n \cdot a$$

(Review notation on page 198 of book)

Theorem 3.6
G group, $a, b, c \in G$. Then

1. $e \in G$ is unique
2. if $ab = ac$ or $ba = ca \implies b = c$
3. $\forall a \in G : a^{-1}$ is unique.

Proof.

1. Suppose $\exists e' \in G$ s.t $e \neq e'$ but $e'a = a = ae'$ $\forall a \in G$. $\implies$ let $a = e \implies e'e = e = ee'$
   On the other hand $e \cdot e' = e' = e'e$
   $\implies e = e'$
2. $ab = ac, \quad a, b, c \in G.$
   Since $a^{-1} \in G$
   $$\begin{aligned} \implies \underbrace{a^{-1}a}_{e} b &= \underbrace{a^{-1}a}_{e} c \\ \implies e \cdot b &= e \cdot c \\ \implies b &= c \end{aligned}$$
3. Suppose $a \in G$ $\exists$ two distinct inverses.
   $d_1, d_2 \in G$.
   $d_1 a = e = a d_1$
   $d_2 a = e = a d_2$
   $\implies d_1 = d_1 e = d_1 a d_2 = e \cdot d_2 = d_2$

$\square$

Corollary 3.7
G group, $a, b \in G$. Then
1. $(ab)^{-1} = b^{-1}a^{-1}$
2. $(a^{-1})^{-1} = a$

Proof. Exercise. □

Note: $ab = ba$ (G is abelian)
$$\implies (ab)^{-1} = a^{-1}b^{-1} = b^{-1}a^{-1}$$
Generally: $ab \neq ba \implies a^{-1}b^{-1} \neq b^{-1}a^{-1}$

3.4 Order of an Element

Definition 3.8 (Order (of an element) and Finite vs. Infinite order)
The order of an element $a \in G$ is the smallest $k \in \mathbb{N}$ such that $a^k = e$. We denote this by $|a|$.
If k is finite $\implies$ a has finite order.
If k is infinite $\implies$ a has infinite order.

Example 3.9
S_2; $e, \tau_1 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$
$|e| = 1; e^1 = e$
$|\tau_1| = 2 \quad \tau_1^2 = \tau_1 \circ \tau_1 = e$
$\tau_1^4 = \tau_1^2 \circ \tau_1^2 = e \circ e = e$

Example 3.10
$\mathbb{Z} \leftarrow e = 0.$
$|1| = ?$
$1 \cdot n = 0$ for which n?
Answer none!
$\implies |1| = \infty$

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4.1 Order of an Element (Cont'd)

Theorem 4.1
G-group, $a \in G$

1. If $|a| = \infty$, then $a^i \neq a^j$ for any $i, j \in \mathbb{Z}$ with $i \neq j$.
2. If $\exists i \neq j$ such that $a^i = a^j \implies |a| < \infty$.

Proof. We prove (2) (because $1 \iff 2$).
WLOG suppose $i > j$, then if $a^i = a^j \implies a^{i-j} = a^i a^{-j} \implies = a^j a^{-j} = a^0 = e$
$\implies |a| \leq i - j < \infty$ □

Theorem 4.2
G group, $a \in G \quad |a| = n$

1. $a^k = e \iff n \mid k \quad (n \leq k)$
2. $a^i = a^j \iff i \equiv j \pmod{n}$
3. if $n = td \quad d \geq 1 \implies |a^t| = d$.

Proof.

1. If $a^k = e$ and since $a^n = e$ with n-smallest such integer, then $k > n$, and so $k = nd + r$ with $0 \leq r < n$
$$a^k = a^{nd+r} = (a^n)^d a^r = e^d a^r = a^r$$
If $0 < r < n \implies a^r \neq e \implies a^k \neq e$
$\implies r = 0 \implies k = nd \implies n \mid k$.
2. If $a^i = a^j \implies a^{i-j} = e$
$\implies n \mid i - j$ by (1).
$\implies i - j \equiv 0 \pmod{n}$
$\implies i \equiv j \pmod{n}$
3. If $n = td \ (d \geq 1) \overset{?}{\implies} |a^t| = d$
Since $a^n = e \implies (a^t)^d = e \implies |a^t| \leq d$.
If $|a^t| = k < d \implies (a^t)^k = a^{tk} = e$
But $tk < td = n \implies a^{tk} = e$ for $tk < n \implies \neq$ because n is the smallest positive integer such that $a^n = e$.
$\implies k = d \implies |a^t| = d$.

□

Corollary 4.3
G- abelian group with $|a| < \infty \quad \forall a \in G$. Suppose $c \in G$ such that $|a| \leq |c| \quad \forall a \in G$. Then $|a| \mid |c|$.

Proof. Suppose not. $\exists$ some $a \in G$ such that $|a| \nmid |c|$. Consider prime factorizations of $|a|$ and $|c|$.
$\implies$ Then $\exists$ some prime p such that $|a| = p^r m \quad |c| = p^s n$ where $r > s$ (s might be zero) and $(p_1 m) = 1 = (p_1 n)$.
Then by (3) of Theorem 4.2,

$$|a^m| = p^r \quad \text{and} \quad |c^{p^s}| = n$$
$$\underset{\text{because } (p^r,n)=1}{\implies} |\underbrace{a^m \cdot c^{p^s}}_{\in G}| = p^r \cdot n$$

Note: $|a| = n, |b| = m, |a \cdot b| \neq n \cdot m$ unless $(n, m) = 1$
Recall: $|c| = p^s \cdot n$ where $s < r$
$\implies p^r > p^s$
$\implies p^r n > p^s n$
$\implies |a^m \cdot c^{p^s}| > |c|$
$\implies \neq$ because c is the element in G with maximal order! So $a^m c^{p^s} \in G$ cannot have order larger than c. $\square$

4.2 Subgroups

Definition 4.4 (Subgroup)
A subset $H \subseteq G$ is a subgroup of $(G, *)$ if it is also a group under $*$.
Note:
$G \subseteq G \implies G$ is always a subgroup of itself (Improper subgroup)
$\{e\} \subseteq G \implies \{e\}$ is always a subgroup of G (Trivial subgroup of G)
$\implies$ Any subgroup $e \neq H \neq G$ is called a nontrivial proper subgroup.

Examples 4.5

- $(\mathbb{Z}, +) \subseteq (\mathbb{Q}, +)$
- $\{e, r_{90}, r_{180}, r_{270}\} \subseteq D_4$
- $SL_n(\mathbb{F}) \subseteq GL_n(\mathbb{F})$

Note: any subgroup always contains e.

Theorem 4.6
A nonempty subset H of G is a subgroup if:

1. $ab \in H \quad \forall a, b \in H$
2. $a^{-1} \in H \quad \forall a \in H$

Proof. Since $H \neq \emptyset \quad \exists a \in H$. By (2), $\exists a^{-1} \in H$. $\implies$ By (1) $aa^{-1} = e \in H \implies e \in H$. $\square$

Theorem 4.7
Any closed nonempty finite subset H of G is a subgroup.

Proof. By Theorem 4.6, we need only show that H contains inverses.
If $a \in H \quad a^k \in H \quad \forall k \in \mathbb{Z}$.
Since H is finite, not all a^k can be distinct.
$\implies |a| = n < \infty$ for some $n \in \mathbb{N}$.
$\implies a^n = e$
$\implies a^{n-1} \cdot a = e = a \cdot a^{n-1}$
$\implies a^{n-1} = a^{-1}$
If $n > 1 \implies a^{-1} \in H$
If $n = 1 \implies a^{-1} = e \implies a = e \implies a^{-1} = e \in H$. $\square$

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5.1 Subgroups (Cont'd)

Example 5.1
$\mathbb{Z}_5 \leftarrow$ group under addition $= \{0, 1, 2, 3, 4\}$
Units of $\mathbb{Z}_5$: $\mathcal{U}_5 = \{1, 2, 3, 4\}$
Clearly, $\mathcal{U}_5 \subseteq \mathbb{Z}_5$
Question: Is $\mathcal{U}_5$ a subgroup of $\mathbb{Z}_5$
No, because $\mathcal{U}_5$ is a group under multiplication.

Example 5.2
S_3: set of permutations that fix 1.
$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad \tau_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$

$$\left.\begin{aligned} \tau_2 e = \tau_2 = e\tau_2 \\ \tau_2 \cdot \tau_2 = e \end{aligned}\right\} \implies \underbrace{\{e, \tau_2\}}_{H} \text{ is closed.}$$

By Theorem 4.7, H is a subgroup because H is finite, nonempty, and closed.

5.2 Center of a Group

Definition 5.3 (Center of a group)
The center of a group G is the subset

$$Z(G) \coloneqq \{a \in G \mid ag = ga \quad \forall g \in G\}$$

Note 5.4: When G is abelian $\implies Z(G) = G$

Question 5.5: Is $Z(G) = \emptyset$? No, because $e \in Z(G)$

Examples 5.6

- $Z(S_n) = e$
- $Z(D_4) = \{e, r_{180}\}$
- $Z(GL_n) = \{aI \mid a \in \mathbb{F}\} \qquad \begin{pmatrix} a & & 0 \\ & \ddots & \\ 0 & & a \end{pmatrix}$
- $Z(SL_n) = \{I\} = e$

Theorem 5.7
$Z(G)$ is a subgroup of G.

Proof. By Theorem 4.6, since $Z(G) \neq \emptyset$, we need only show closure and inverses.

1. $a, b \in Z(G) \overset{?}{\implies} ab \in Z(G), \forall g \in G.$
   $(ab)g \overset{\text{b/c } g \in Z(G)}{=} a(gb) \underset{\text{by assoc.}}{=} (ag)b \overset{a \in Z(G)}{=} (ga)b = g(ab)$
   $\implies ab \in Z(G)$
2. $a \in Z(G), ag = ga \quad \forall g \in G.$
   $\implies a^{-1}(ag)a^{-1} = a^{-1}(ga)a^{-1}$
   $\implies ga^{-1} = a^{-1}g \implies a^{-1} \in Z(G)$

$\square$

5.3 Cyclic Group

Definition 5.8 (Cyclic group)
For any $a \in G$, the set
$$\langle a \rangle = \{a^n \mid n \in \mathbb{Z}\}$$
is a subgroup of G. We say $\langle a \rangle$ is the cyclic subgroup generated by a.

Note 5.9: Cyclic groups are always abelian.

If $G = \langle a \rangle$ for some $a \in G$, then G is a cyclic group.

Example 5.10
$\langle r_{90} \rangle \subseteq D_4$
$\langle r_{90} \rangle = \{e, r_{90}, r_{180}, r_{270}\} \leftarrow$ is a cyclic subgroup of G.

Note 5.11: In additive notation: $a * a = a + a$ (not $a \cdot a = a^2$)
$$\langle a \rangle = \{n \cdot a \mid n \in \mathbb{Z}\} \quad n \cdot a = \underbrace{a + a + \cdots + a}_{n \text{ times}}$$
$$a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ times}}$$

Example 5.12
$(\mathbb{Z}, +) = \langle 1 \rangle = \langle -1 \rangle$

Note 5.13: The generating element a is not unique.

Example 5.14
$(\mathbb{Z}_3, +) = \langle 1 \rangle = \underbrace{\langle 2 \rangle}_{=-1}$

Exercise. Which elements generate $\mathbb{Z}_n$ for $n \in \mathbb{N}$?
Hint: Look at units (i.e. relatively prime) of $\mathbb{Z}_n$

Example 5.15
$\mathbb{Z}_n = \langle 1 \rangle$
$\implies$ All $\mathbb{Z}_n$ are cyclic groups of order n

Theorem 5.16
Let $a \in G$

1. If $|a| = \infty$, then $\langle a \rangle = \{a^k \mid k \in \mathbb{Z}\}$ is an infinite group.
2. If $|a| = n < \infty$, then $\langle a \rangle$ is a finite group. In fact, $\langle a \rangle = \langle e, a, a^2, a^3, \dots, a^{n-1}\} \implies |\langle a \rangle| = |a| = n$.

Proof (Sketch).

$$\begin{aligned} |a| = \infty &\implies a^i \neq a^j \text{ for } i \neq j \\ &\implies \{a^k \mid k \in \mathbb{Z}\} \implies \text{ infinite set.} \\ |a| = n &\implies \langle a, a^2, \dots, a^{n-1}, a^n = e\} \end{aligned}$$

Since: $a \cdot a^{n-1} = a^n = e = a^{n-1} \cdot a$

$$\implies a^{n-1} = a^{-1}$$

$a^2 a^{n-2} = a^n = e = a^{n-2} a^2$

$$\implies a^{-2} = a^{n-2}$$

$\square$

Theorem 5.17
Let $\mathbb{F}$ be any field. Then any finite subgroup $G \subseteq \mathbb{F}^*$ is cyclic.

Recall 5.18 $\mathbb{F}^* = \mathbb{F} - \{0\}$ is a group under multiplication.

Proof. Since $|G| < \infty$, $\exists c \in G$ such that order of c is maximal ($|a| \leq |c| \quad \forall a \in G$). By Corollary 4.3, $\forall a \in G, |a| \mid |c|$ so if $|c| = m \implies a^m = 1$
Consider $p(x) = x^m - 1$. Since $p(a) = 0 \quad \forall a \in G$.
Since $p(x)$ has degree m it can have at most m solutions $\implies |G| \leq m$.
Since $|c| = m$ so $|\langle c \rangle| = m$.
$\implies \langle c \rangle \subseteq G \implies \langle c \rangle = G$.
$\implies G$ is cyclic. $\square$

6 Jan 14, 2022

6.1 Cyclic Group (Cont'd)

Recall 6.1 $a \in G$

$$\underbrace{\langle a \rangle}_{\text{cyclic group gen. by a}} := \{a^n \mid n \in \mathbb{Z}\} = \{\dots a^{-2}, a^{-1}, e, a, a^2, \dots\}$$

$G = \langle a \rangle \leftarrow G$ is cyclic group

Recall 6.2 Thm:

$$|a| = \infty \to |\langle a \rangle| = \infty$$
$$|a| = n < \infty \to |\langle a \rangle| = n$$

Recall 6.3 $\mathbb{F}$-field, $G \subseteq \mathbb{F}^*$ if G finite $\implies G$ is cyclic. (G is any subgroup)

Theorem 6.4
Subgroups of cyclic groups are cyclic.

Proof. Suppose $G = \langle a \rangle$ and $H \subseteq G$. We want to show that $H = \underbrace{\langle b \rangle}_{b=a^j \text{ for some } j}$ for some $b \in G$.

If $H = e \implies H = \langle e \rangle$ we're done.
If $H \neq e$, then we can find k-smallest positive integer such that $a^k \in H$
Suppose $b \in H$. Then,

$$b = a^i \text{ for some } i \text{ then } i = kd + r \quad 0 \leq r < k.$$

$\implies a^r = a^{i-kd} = b(a^k)^{-d} \in H$ by closure.
If

$$r \neq 0 \implies \begin{cases} a^r \in H \\ a^k \in H \end{cases}$$

with $0 < r < k$ which is a contradiction because k was supposed to be smallest positive integer with $a^k \in H$.

$$\begin{aligned} \implies r = 0 &\implies b = a^i = a^{kd+r} = a^{kd} = (a^k)^d \\ &\implies b \in \langle a^k \rangle \\ &\implies H \subseteq \langle a^k \rangle \end{aligned}$$

Since $a^k \in H \implies \langle a^k \rangle \subseteq H$
$\implies \langle a^k \rangle = H$ $\square$

6.2 Generating Sets for Groups

Definition 6.5
Given a subset S of G, let $\langle S \rangle$ denote the set of all possible products of all elements of S and their inverses.

Note 6.6: $S \subseteq \langle S \rangle$

Example 6.7
$a, b \in G, \quad S = \{a, b\}$

$$\begin{aligned}\langle S \rangle &= \langle a, b \rangle \\ &= \{a^n, b^m, a^n b^m, a^{n_1} b^{m_1} a^{n_2} b^{m_2}, b^m a^n, b^{m_1} a^{n_2} b^{m_2} a^{n_1}, \dots\} \\ &= \left\{ \prod_{i=0}^{k} a^{n_i} b^{m_i}, \prod_{i=0}^{k} b^{n_i} a^{m_i} \mid k \in \mathbb{N}, n_i, m_i \in \mathbb{Z} \right\}\end{aligned}$$

Theorem 6.8
S- any subset of G.

1. $\langle S \rangle$ is always a subgroup of G.
2. If H is any other subgroup of G such that $S \subseteq H \implies \langle S \rangle \subseteq H$.

Proof (Sketch).

1. Use the fact that very definition of $\langle S \rangle$ ensures closure and inverses $\implies$ $\langle S \rangle$ is a subgroup.
2. Again follows from closure and inverses contained in H because H is a subgroup.

$\square$

Definition 6.9 (Generators)
For any $S \subseteq G$, the group $\langle S \rangle$ is called the <u>subgroup generated by S</u>. If $G = \langle S \rangle$, then we call elements in S, the <u>generators of G</u> and S the <u>generating set of G</u>

Example 6.10 (Symmetric group)
$S_3 = \{e, \tau_1, \tau_2, \tau_{121}, \tau_{21}, \tau_{12}\}$
$\tau_{121} = \tau_1 \circ \tau_2 \circ \tau_1$
$\tau_{21} = \tau_2 \circ \tau_1$
$\tau_{12} = \tau_1 \circ \tau_2$
$e = \tau_1 \circ \tau_1 = \tau_2 \circ \tau_2$

$$S_3 = \langle \underbrace{\tau_1}_{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}}, \underbrace{\tau_2}_{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}} \rangle$$

$S_n \leftarrow$ order $n!$

$$S_n = \langle \underbrace{\tau_1}_{\text{flips } 1-2}, \underbrace{\tau_2}_{2-3}, \tau_3, \ldots, \underbrace{\tau_{n-1}}_{\text{flips } n, n-1} \rangle$$

$S_4 = \langle \tau_1, \tau_2, \tau_3 \rangle$
$S_5 = \langle \tau_1, \tau_2, \tau_3, \tau_4 \rangle$

6.3 Isomorphisms and Homomorphisms

Definition 6.11 (Homomorphism (of groups))
G, H are groups. A homomorphism of groups is a map $\varphi\colon G \to H$ such that $\forall a, b \in G$

$$\varphi(\underbrace{ab}_{ab \text{ prod in } G}) = \varphi(a) \underbrace{\cdot}_{\text{prod in } H} \varphi(b)$$

Note 6.12: This means that the "multiplication" table for G is mapped onto "multiplication" table for H i.e. φ preserves group structures.

Note 6.13: $\varphi(a) = \varphi(e_G \cdot a) = \varphi(e_G)\varphi(a)$
$\implies \varphi(e_G) = e_H$
$\implies \varphi$ takes identities to identities.

Definition 6.14 (Isomorphism (of groups))
An isomorphism of groups G and H is a homomorphism of $\varphi\colon G \to H$ that is also a bijection, i.e. an isomorphism is an invertible homomorphism.
If G is isomorphic to H, then $G \cong H$, which is the same as writing $\exists \varphi\colon G \to H$ with φ one-to-one and onto. Alternatively, $\tilde{\varphi}\colon H \to G$ is also one-to-one and onto.

Example 6.15
$\mathbb{Z}_8 = \{0, \ldots, 7\}$
$\mathcal{U}_8$ of units $\implies \mathcal{U}_8 = \{\underset{e=}{1}, 3, 5, 7\}$
Consider $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{\underset{e=}{(0,0)}, (1,0), (0,1), (1,1)\}$
Claim: $\mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathcal{U}_8$
Let

$$\begin{aligned} \varphi\colon \mathcal{U}_8 &\to \mathbb{Z}_2 \times \mathbb{Z}_2 \\ \varphi(1) &= (0,0) \\ \varphi(3) &= (1,0) \\ \varphi(5) &= (0,1) \\ \varphi(7) &= (1,1) \end{aligned}$$

$\varphi(ab) = \varphi(a) + \varphi(b)$
Check,

- φ is a homomorphism
- multiplication table is preserved
- φ is one to one and onto

7 Jan 19, 2022

7.1 Isomorphisms and Homomorphisms (Cont'd)

Example 7.1 (Example 6.15 Cont'd)
Let

$$\begin{aligned}\varphi\colon \mathcal{U}_8 &\to \mathbb{Z}_2\times\mathbb{Z}_2\\ \varphi(1) &= (0,0) \leftarrow \text{ fixed}\\ \varphi(3) &= (1,0)\\ \varphi(5) &= (0,1)\\ \varphi(7) &= (1,1)\end{aligned}$$

Check,

$$(0,0)+(1,0) = \varphi(1)+\varphi(3) \overset{\checkmark}{=} \varphi(1\cdot 3) = \varphi(3) = (1,0)$$

$$(0,0) = 2(0,1) = \varphi(5)+\varphi(5) \overset{\checkmark}{=} \varphi(5\cdot 5) = \varphi(1) = (0,0)$$

$$\vdots$$

Verify every time $\varphi(ab) = \varphi(a)+\varphi(b) \implies \varphi$ is a homomorphism.
φ is one-to-one[a] and onto[b] $\implies$ DONE.
Iso's are not unique. In fact,

$$\begin{aligned}\varphi(1) &= (0,0)\\ \varphi(3) &= (0,1)\\ \varphi(5) &= (1,0)\\ \varphi(7) &= (1,1)\end{aligned}$$

is also an iso. However,

$$\begin{aligned}\varphi(1) &= (0,0)\\ \varphi(3) &= (1,1)\end{aligned}$$

Does it work? Why? (Exercise)

[a] $\varphi(x) = \varphi(y) \implies x = y$
[b] $\forall y \in Z_2\times Z_2\ \exists x \in \mathcal{U}_8$ s.t. $\varphi(x) = y$

Example 7.2
$\mathbb{Z} \to Z_5$

$n \overset{\varphi}{\mapsto} [n] \mod 5$

Let's construct a homomorphism.

1. Check φ is well defined.
   $n \equiv m \mod 5 \overset{?}{\implies} \varphi(n) = \varphi(m).\checkmark$
2. φ is a homomorphism.

$$\varphi(a+b) = \varphi(a) + \varphi(b)$$
$$[a+b] \underset{\text{true}}{=} [a] + [b]$$

$$\implies \varphi \text{ is a homomorphism}$$

Note: φ is not injective because $|\mathbb{Z}| > |\mathbb{Z}_5|$
φ is not an iso.

Fact 7.3: Isomorphic groups always have the same order.
Converse? $|G| = |H| \implies G \cong H$?
FALSE!

Example 7.4
Consider S_3 and $\mathbb{Z}_6$.

$$|S_3| = 3! = 6 \qquad\qquad |\mathbb{Z}_6| = 6$$

Not isomorphic. Let's suppose $\varphi\colon S_3 \to \mathbb{Z}_6$ an isomorphism.

$$\varphi(ab) = \varphi(a) + \varphi(b) \tag{1}$$

So,

$$\begin{aligned} \varphi(a) + \varphi(b) &= \varphi(b) + \varphi(a) && \text{(because } \mathbb{Z}_6 \text{ is abelian)} \\ &= \varphi(ab) \end{aligned}$$

$\implies$ if (1) holds since $\mathbb{Z}_6$ is abelian
$\implies \varphi(ab) = \varphi(ba) \quad \forall b, a \in S_3$
$\implies S_3$ is abelian
False, S_3 is not abelian, so you can't define such an iso φ.

Theorem 7.5
If G is abelian, H is not abelian $\implies G \ncong H$.

Fact 7.6: Isomorphisms preserve order of elements, i.e.

$$|a| = |\varphi(a)|$$

Definition 7.7 (Automorphism)
An automorphism is an isomorphism from $G \to G$. They capture internal symmetries of a group.

Example 7.8
identity:

$$\begin{aligned} i_G \colon G &\to G \\ g &\mapsto g \end{aligned}$$

Clearly: $i(ab) = i(a)i(b) = ab \overset{\checkmark}{=} ab$

Definition 7.9 (Inner automorphism of G induced by c)
For any $c \in G$, the inner automorphism of G induced by c is:

$$\varphi_c \colon G \to G; \quad \varphi_c(g) = c^{-1}gc \leftarrow \text{ conjugation by } c.$$

1. Then φ_c is a homomorphism:

   $\varphi_c(ab) = c^{-1}abc = (c^{-1}ac)(c^{-1}bc) = \varphi_c(a)\varphi_c(b)$

2. φ is surjective: Given any $g \in G$.

   $\varphi_c(cgc^{-1}) = c^{-1}(cgc^{-1})c = g$

3. φ is injective: $\varphi_c(a) = \varphi_c(b)$ for some $a, b \in G$
   $\implies c^{-1}ac = c^{-1}bc$
   $\implies a = b$
   $\implies \varphi$ is an isomorphism.

7.2 Classification of Cyclic Groups

Theorem 7.10
Suppose G is a cyclic group.

1. $|G| = \infty \implies G \cong (\mathbb{Z}, +)$
2. $|G| = n < \infty \implies G \cong (\mathbb{Z}_n, +)$

Proof.

1. If $G = \langle a \rangle$ infinite. Then $G = \{a^n \mid n \in \mathbb{Z}\}$. So let

   $$\begin{aligned} \varphi \colon G &\to \mathbb{Z} \\ a^n &\mapsto n \end{aligned}$$

   So φ is one-to-one and onto by definition.
   Then,

$$n + m = \varphi(a^{n+m}) = \varphi(a^n a^m) \stackrel{?}{=} \varphi(a^n) + \varphi(a^m) = n + m$$

$\implies$ φ is a homomorphism and φ is bijection.
$\implies$ φ is an isomorphism.

2. $|G| = n \implies G = \{e, a, a^2, \ldots, a^{n-1}\}$

$$\begin{aligned} \varphi\colon G &\to \mathbb{Z}_n = \{0, 1, \ldots, n-1\} \\ a^i &\mapsto i \end{aligned}$$

Exactly for the same reasons: check φ is an isomorphism.

$$k = \underbrace{\varphi(a^k)}_{i+j \equiv k \mod n} = \varphi(a^{i+j}) = \underbrace{\varphi(a^i) + \varphi(a^j)}_{i+j \equiv k \mod n}$$

φ is an isomorphism.

$\square$

8 Jan 21, 2022

8.1 Homomorphisms

Recall 8.1 Let $\varphi\colon G \to H$ any map. Then

$$\operatorname{Im}\varphi = \{h \in H \mid h = \varphi(g) \text{ some } g \in G\}$$

Theorem 8.2
If $\varphi\colon G \to H$ is a homomorphism, then:

1. $\varphi(e_G) = e_H$
2. $\varphi(a^{-1}) = (\varphi(a))^{-1}$
3. $\operatorname{Im}\varphi$ is a subgroup of H
4. If φ is injective, then $G \cong \operatorname{Im}\varphi$

Note 8.3: If φ is surjective, then $\operatorname{Im}\varphi = H$

Proof.

1. Did before.
2. By (1), $e_H = \varphi(e_G) = \varphi(aa^{-1}) = \varphi(a)\cdot\varphi(a^{-1}) \stackrel{?}{=} e_H \stackrel{?}{=} \varphi(a^{-1})\varphi(a) = \varphi(a^{-1}a) = \varphi(e_G) = e_H$ by (1).
3. Claim $\operatorname{Im}\varphi$ subgroup of H. Since $\varphi(e_G) = e_H$ by (1) $\implies$ $e_H \in \operatorname{Im}\varphi$. If $a, b \in \operatorname{Im}\varphi \implies \exists\, a', b' \in G$ s.t. $\varphi(a') = a, \varphi(b') = b \implies ab = \varphi(a')\varphi(b') = \varphi(a'b')$ since G is closed, $a'b' \in G \implies ab \in \operatorname{Im}\varphi \implies \operatorname{Im}\varphi$ is closed.
4. By (2), if $\varphi(g) = a$ then

$$a^{-1} = \varphi(g)^{-1} = \varphi(g^{-1})$$

$\implies a^{-1} = \varphi(g^{-1})$ but $g^{-1} \in G \implies a^{-1} \in \operatorname{Im}\varphi$
$\operatorname{Im}\varphi$ has inverses $\implies \operatorname{Im}\varphi$ is subgroup.
5. φ injective $\implies G \cong \operatorname{Im}\varphi$. Since $\varphi\colon G \to \operatorname{Im}\varphi$ is surjective by construction, if φ is also injective, then $\varphi\colon G \to \operatorname{Im}\varphi$ is a bijection and a homomorphism $\implies \varphi\colon G \to \operatorname{Im}\varphi$ is an isomorphism $\implies G \cong \operatorname{Im}\varphi$.

$\square$

Example 8.4
$\varphi\colon G \to H$ where φ is an injective homomorphism and H is abelian.
Question: Is G abelian?
Yes, because $G \cong \operatorname{Im}\varphi$ by bijectivity, and $\operatorname{Im}\varphi$ subgroup of H and subgroups of abelian groups are abelian $\implies G$ has to be abelian.

8.2 Congruence

Definition 8.5 (Congruence of a group)
Suppose H is a subgroup of G. Let $a, b \in G$. We say $a \equiv b \pmod{H}$ if $ab^{-1} \in H$.

Recall 8.6 An equivalence relation on a set S is a relation $a \sim b$ for $a, b \in S$ that is:
reflexive: $a \sim a \quad \forall a \in S$
transitive: $a \sim b$ and $b \sim c \implies a \sim c$
symmetric: $a \sim b \implies b \sim a$.

Theorem 8.7
The congruence relation $a \equiv b \pmod{H}$ is an equivalence relation for any subgroup $H \subseteq G$.

Definition 8.8 (Right coset (and left coset))
Given any $a \in G$, the right coset of H in G is:

$$Ha = \{ha \in G \mid h \in H\} \text{ where } a \text{ is any } a \in G \text{ fixed}$$

This is a right coset because a is multiplied on the right.
The left coset of H in G is:

$$aH = \{ah \in G \mid h \in H\} \text{ where } a \text{ is any } a \in G \text{ fixed}$$

Note 8.9: Ha is just the congruence class of a in $G \mod H$.
For any $a \in G$,

$$\begin{aligned}
[a] &= \{b \in G \mid b \equiv a \mod H\} \\
&= \{b \in G \mid ba^{-1} \in H\} \\
&= \{b \in G \mid \underbrace{ba^{-1} = h}_{b = ha} \text{ for some } h \in H\} \\
&= \{ha \in G \mid h \in H\} = Ha.
\end{aligned}$$

Theorem 8.10
1. $Ha = Hb$ iff $ab^{-1} \in H$ (i.e. $a \equiv b \mod H$)
2. Given $a \neq b$ either $Ha = Hb$ or $Ha \cap Hb = \emptyset$.

Proof. Analogous as for rings (seen this in 110A). $\square$

8.3 Lagrange's Theorem

Theorem 8.11
H-subgroup of G then:

1. $G = \bigcup_{a \in G} Ha$
2. $\forall a \in G, \exists$ bijection between $H \to Ha$. So if $|H| < \infty$, then $|Ha| = |Hb| \, \forall a, b \in G$.

Proof.

1. $\bigcup_{a \in G} Ha \subseteq G$ obvious. Given $g \in G, g = eg$ where since $e \in H \implies eg \in Hg \implies g \in Hg \implies G \subseteq \bigcup_{g \in G} Hg$
2. Consider

$$\begin{aligned} \psi \colon H &\to Ha = \{ha \mid h \in H\} \\ h &\mapsto ha \end{aligned}$$

ψ is surjective by definition. If $\psi(h) = \psi(h') \implies ha = h'a \implies h = h' \implies \psi$ is injective $\implies \psi$ is a bijection.

$\square$

Definition 8.12 (Index)
Given any subgroup H of G, the index of H in G denoted $[G{:}H]$ is the number of distinct right cosets of H in G.

Theorem 8.13 (Lagrange's Theorem)
If $H \subseteq G$ is a finite subgroup, then:

$$[G{:}H] = \frac{|G|}{|H|}$$

9 Jan 24, 2022

9.1 Lagrange's Theorem (Cont'd)

Proof of Lagrange's Theorem. Suppose $[G:H] = n$ and denote the cosets by Hg_i for $i = 1, \dots, n$.
Recall: $Hg_i \cap Hg_j = \emptyset \quad i \neq j$, also

$$G = \bigcup_{i=1}^{n} Hg_i = Hg_1 \cup Hg_2 \cup \ldots \cup Hg_n$$

$$\implies |G| = |Hg_1| + |Hg_2| + \cdots + |Hg_n|$$

Also know by previous theorem $|Hg_i| = |H| < \infty$

$$\implies |G| = n \cdot |H|$$
$$\implies \frac{|G|}{|H|} = n = [G:H]$$

$\square$

Question 9.1: What fails when $|H| = \infty$?

Example 9.2
$n\mathbb{Z} = \langle n \rangle$ inside $\mathbb{Z}$.
Then for $a \in \mathbb{Z}$,

$$[a] = \underbrace{a + n\mathbb{Z}}_{Ha} = \{a + ni \mid i \in \mathbb{Z}\} = \{a, a+n, a+2n, \dots\}$$

where $Ha = \{ha \mid h \in H\}$ with $H = n\mathbb{Z} \to Ha = Hb \iff ab^{-1} \in H$ and $a \equiv b \mod H$

$$a + n\mathbb{Z} = \underbrace{(a+n)}_{b} + n\mathbb{Z}$$

$-n = a - (a+n) \in n\mathbb{Z} \iff a \equiv a + n \pmod{n} \implies$ exist exactly n cosets $[0], [1], \dots, [n-1]$

$$[\mathbb{Z}:n\mathbb{Z}] = n$$

Lagrange's Theorem $\implies |H|$ divides $|G|$ for any H subgroup of G.

Example 9.3
If G has order 15.
G can only have subgroups of orders 1, 3, 5, 15.

Note 9.4: Lagrange does not imply that subgroups exist for every number dividing $|G|$. In

Example 9.3, there may not exist a subgroup of order 5 or 3.

Corollary 9.5
$|G| < \infty$

1. $\forall a \in G \implies |a| \mid |G|$
2. If $|G| = n \implies a^n = e \quad \forall a \in G$.

Proof.

1. Consider $H = \langle a \rangle \subseteq G$. $|\langle a \rangle| = |a| \implies$ Since $|G| < \infty$

$$\begin{aligned} &\implies H < \infty \text{ we can use Lagrange} \\ &\implies |H| = |\langle a \rangle| = |a| \mid |G|. \end{aligned}$$

2. Suppose $|a| = m$. Then by (1), $m \mid n \implies n = md$ for some $d \in \mathbb{Z}$. So then

$$a^n = a^{md} = (a^m)^d = e^d = e$$

$\square$

9.2 Classification of Groups of Prime Order

Theorem 9.6
Suppose $p > 0$ prime. If $|G| = p \implies G \cong \mathbb{Z}_p$.

Proof. By Theorem 7.10, all cyclic groups of order n are isomorphic to $\mathbb{Z}_n$. $\implies$ We only need to show G is cyclic. Consider $a \in G$ with $a \neq e$. Then $|\langle a \rangle| \neq 1 \implies$ by Lagrange, since $|\langle a \rangle| \mid p$. Since only 1 or p divides $p \implies |\langle a \rangle| = p$. Since $|G| = p$ and $\langle a \rangle \subseteq G$

$$\begin{aligned} \implies G = \langle a \rangle &\implies G \text{ is cylic of order } p \\ &\implies G \cong \mathbb{Z}_p \text{ by previous theorem} \end{aligned}$$

$\square$

9.3 Classification of Groups of Order ≤ 8

We know $\cancel{1}, \underbrace{\cancel{2}, \cancel{3}}_{\text{prime}}, 4, \underbrace{\cancel{5}}, 6, \underbrace{\cancel{7}}, 8$

Theorem 9.7
If $|G| = 4 \implies$ either $G \cong \underbrace{\mathbb{Z}_4}_{\substack{\text{cyclic} \\ \text{abelian}}}$ or $G \cong \underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2}_{\text{abelian}}$.

Proof. If $|G| = 4$, then either $\exists a \in G$ with $|a| = 4$ or not.

- If yes, then $G = \langle a \rangle \implies G$ is cyclic $\implies G \cong \mathbb{Z}_4$.

- If not, then $G = \{e, a, b, c\}$, since only e can have order 1, then $|a| = |b| = |c| = 2$

$$\begin{aligned} &\implies a^2 = b^2 = c^2 = e \\ &\implies a = a^{-1}, b = b^{-1}, c = c^{-1} \end{aligned}$$

If $|ab| = 1 \implies a = b^{-1} \implies$ contradiction $|ab| = 2$.
So either

$$\begin{aligned} &ab = a \implies b = e \text{ contradiction} \\ &ab = b \implies a = e \text{ contradiction} \\ &ab = c\checkmark \end{aligned}$$

Repeat this for ac, ca, ba, bc, cb to find entire multiplication table. Then construct an explicit isomorphism to

$$\mathbb{Z}_2 \times \mathbb{Z}_2 : \begin{smallmatrix} e\mapsto(0,0) \\ a\mapsto(1,0) \\ b\mapsto(0,1) \\ c\mapsto(1,1) \end{smallmatrix}$$

$\square$

Theorem 9.8
$|G| = 6 \implies G \cong \mathbb{Z}_6$ or S_3.

10 Jan 26, 2022

10.1 Normal Subgroups

Recall 10.1 For $a \in G, H \subseteq G$ subgroup.
Right coset $Ha = \{ha \in G \mid h \in H\}$.
Left coset $aH = \{ah \in G \mid h \in H\}$.

Definition 10.2 (Normal subgroup)
A subgroup N of G is normal if $Na = aN \ \forall a \in G$.

Note 10.3: $Na = aN \;\not\Longrightarrow\; an = na$. Rather, it means that $an = n'a$ for some $n, n' \in N$.

Notation 10.4: Whenever N is normal in G, we write $N \lhd G$.

Example 10.5
Consider $G = D_4$ (not abelian).
Let $M = \{e, r_{180}\}$ then you can show

$$r_{180} \cdot a = a \cdot r_{180} \quad \forall a \in D_4$$
$$\Longrightarrow \; Ma = aM \;\Longrightarrow\; M \lhd D_4$$

Theorem 10.6
If G is abelian, then all subgroups are normal.

Recall 10.7 The center $Z(G) = \{a \in G \mid ag = ga\}$.

Proposition 10.8
For any G, the center $Z(G)$ is always normal.

Proof. Using the definition of $Z(G)$, we notice that for any $g \in G$,

$$Z(G)g = gZ(G)$$

For any $a \in Z(G), ag \in Z(G)g$. Since $ag = ga$ because $a \in Z(G)$ (by definition), then $ga \in gZ(G)$. □

Example 10.9
$S_3 = \{e, \tau_1, \tau_2, \tau_{12}, \tau_{21}, \tau_{121}\}$.
Let $A_3 \coloneqq \{e, \tau_{12}, \tau_{21}\}$.
Then

$$A_3 a = \left\{ \begin{aligned} \tau_{12} \cdot \tau_1 &= \tau_{121} = \tau_1 \cdot \tau_{21} \\ \tau_{12} \cdot \tau_2 &= \tau_1 = \tau_2 \cdot \tau_{21} \\ \underbrace{\tau_{12} \cdot \tau_{121}}_{\in A\tau_{121}} &= \tau_2 = \underbrace{\tau_{121} \cdot \tau_{21}}_{\in \tau_{121} A} \end{aligned} \right\} = aA_3$$

Recall $(a \in N, aN = N = Na)$

$$\implies A_3 a = aA_3 \quad \forall a \in S_3 \implies A_3 \text{ is normal}$$

Theorem 10.10
For $N \lhd G$, if $Na = Nb$ and $Nd = Nc \implies Nad = Nbc$ (Analogously, $Nda = Ncb$).

Proof. Direct from set definitions of cosets. □

Definition 10.11
Given $a, b \in G, N \subseteq G$,

$$aNb \coloneqq \{anb \in G \mid n \in N\}$$

Theorem 10.12
TFAE:

1. $N \lhd G$.
2. $a^{-1}Na \subseteq N \quad \forall a \in G$.
3. $aNa^{-1} \subseteq N \quad \forall a \in G$.
4. $a^{-1}Na = N \quad \forall a \in G$.
5. $aNa^{-1} = N \quad \forall a \in G$.

Proof. 1) $\implies$ 3) N normal $\implies aN = Na \implies \forall a \in G$ and $n \in N$

$$\begin{aligned} \exists n' \in N \text{ such that } an = n'a &\implies ana^{-1} = n' \\ &\implies aNa^{-1} \subseteq N \end{aligned}$$

3) $\implies$ 2) Since if $aNa^{-1} \subseteq N \ \forall a \in G$ and $a^{-1} \in G$

$$(a^{-1})N(a^{-1})^{-1} = a^{-1}Na \subseteq N$$

2) $\implies$ 3) analogous.
4) $\iff$ 5) proved the same way.

3) $\implies$ 4) If $aNa^{-1} \subseteq N$ then since $ana^{-1} \in N \quad \forall A \in G, \forall n \in N$

$$\begin{aligned}
&\overset{\text{by 2)}}{\implies} a^{-1}\underbrace{(ana^{-1})}_{n'}a \in a^{-1}Na \\
&\implies n \in a^{-1}Na \implies N \subseteq \underbrace{a^{-1}Na}_{\iff \text{by 3}} \\
&\implies N \subseteq aNa^{-1} \implies N = aNa^{-1}
\end{aligned}$$

2) $\implies$ 5) same proof as 3) $\implies$ 4).
5) $\implies$ 1)

$$\begin{aligned}
aNa^{-1} = N &\implies ana^{-1} = n' \text{ for some } n' \in N \\
&\implies an = n'a \\
&\implies aN \subseteq Na
\end{aligned}$$

Use the fact 4) $\iff$ 5) to show $Na \subseteq aN$.
$\implies Na = aN \implies N \lhd G$. $\square$

11 Jan 28, 2022

11.1 Quotient Groups

Given $N \lhd G$, let $G/N := \{Na \mid a \in G\}$.

Recall 11.1 If $N \lhd G, Na = Nb$ and $Nc = Nd$, then $\implies Nac = Nbd$.

Theorem 11.2
$N \lhd G$, then

1. G/N is a group with operation $\underbrace{Na \cdot Nb}_{* \text{ operation in } G/N} := \overbrace{Nab}^{\text{product inside } G}$
2. If $|G| < \infty \implies |G/N| = |G|/|N|$
3. If G is abelian $\implies G/N$ is abelian.

We call G/N the quotient group of G by N.

Proof. 1) Check each axiom of groups:

- $id := N$
- Inverse $:= Na^{-1} \implies (Na)(Na^{-1}) = Naa^{-1} = Ne = N$
- etc.

2) $|G/N| = [G{:}N] = |G|/|N|$
3) $\underbrace{(Na)(Nb)}_{Nab} = \underbrace{(Nb)(Na)}_{Nba}$
because G is abelian, $N\underline{ab} = N\underline{ba}$. □

Example 11.3
Consider $2\mathbb{Z} = \langle 2 \rangle \subseteq \mathbb{Z}$.
$\mathbb{Z}$ abelian $\implies 2\mathbb{Z}$ normal.
$|\mathbb{Z}/2\mathbb{Z}| = [Z{:}2\mathbb{Z}] = 2$
$2\mathbb{Z} = \{-4, -2, 0, 2, 4, \dots\} =$ evens
$2\mathbb{Z} + 1 =$ odds $\implies \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}_2$
Generally,

$$\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$$

Example 11.4
$A_3 \lhd S_3$
$A_3 = \{e, \tau_{12}, \tau_{21}\}$
$|S_3| = 6, |A_3| = 3$, so

$$|S_3/A_3| = \frac{6}{3} = 2$$
$$\implies S_3/A_3 \cong \mathbb{Z}_2$$

Example 11.5
$N = \langle 4 \rangle = \{0, 4, 8\} \subseteq \mathbb{Z}_{12}$

$$\begin{aligned}
[0] &= N + 0 = N \\
[1] &= N + 1 = \{1, 5, 9\} \\
[2] &= N + 2 = \{2, 6, 10\} \\
[3] &= N + 3 = \{3, 7, 11\}
\end{aligned}$$

$$\begin{aligned}
&\implies N + a = N + b \iff a \equiv b \mod 4 \\
&\text{i.e: } N + 6 = \{6, 10, 2\} \quad 6 \equiv 2 \mod 4
\end{aligned}$$

$\mathbb{Z}_{12}/N \cong\ ?$ where $|Z_{12}/N| = 4$
So either

$$Z_{12}/N \cong Z_4 \text{ or } \mathbb{Z}_2 \times \mathbb{Z}_2$$

$[4] = [1] + [1] + [1] + [1] = [0]$
$(N + 1) + (N + 1) + (N + 1) + (N + 1) = N + 4 = N$, because $4 \equiv 0 \mod 4$.
So,

$$|N + 1| = 4 \implies Z_{12}/N \cong \mathbb{Z}_4$$

Theorem 11.6
$N \lhd G$. Then G/N is abelian if and only if $aba^{-1}b^{-1} \in N\ \forall a, b \in G$.

Proof. G/N is abelian iff $Nab = Nba\ \forall a, b \in G$

$$\begin{aligned}
&\implies ab \equiv ba \mod N\ \forall a, b \in G \\
&\implies aba^{-1}b^{-1} \equiv e \mod N \iff aba^{-1}b^{-1} \in N
\end{aligned}$$

□

Theorem 11.7
G any group. $G/Z(G)$ is cyclic $\implies$ G abelian.

Proof. If $G/Z(G)$ is cyclic, then $G/Z = \langle Zg \rangle$ for some $g \in G \implies$ every other coset $Zg' = (Zg)^k = Zg^k$. So then if $a, b \in G$, then
$a \in Za = Zg^k$ for some k,

$b \in Zb = Zg^j$ for some j.

$$\begin{aligned} &\implies a = c \cdot g^k \text{ and } b = c'g^j \text{ for some } c, c' \in Z \\ &\implies ab = cg^k \cdot c'g^j = c'g^j cg^k = ba \\ &\implies G \text{ is abelian.} \end{aligned}$$

□

11.2 Quotient Groups and Homomorphisms

Definition 11.8 (Kernel)
Let $\varphi\colon G \to H$ be a homomorphism. The kernel of φ is the set

$$\ker \varphi := \{g \in G \mid \varphi(g) = e_H\}$$

Example 11.9
Consider

$$\begin{aligned} \varphi\colon \mathbb{Z} &\to \mathbb{Z}_5 \\ n &\mapsto [n] \end{aligned}$$

Then,

$$\begin{aligned} \ker \varphi &= \{n \in \mathbb{Z} \mid [n] = [0]\} = \{n \mid n \equiv 0 \mod 5\} \\ &= 5\mathbb{Z} \end{aligned}$$

Theorem 11.10
Suppose $\varphi\colon G \to H$ is a homomorphism. Then $\ker \varphi \lhd G$ is a normal subgroup of G.

Proof. Subgroup:

- (Identity): Since $\varphi(e) = e \implies e \in \ker \varphi$
- (Closure): If $a, b \in \ker \varphi$,
$$\begin{aligned} \varphi(ab) = \varphi(a) \cdot \varphi(b) = e \cdot e = e \\ \implies ab \in \ker \varphi. \end{aligned}$$
- (Inverse): If $a \in \ker \varphi$, then $\varphi(a^{-1}) = (\varphi(a))^{-1} = e^{-1} = e$
  $\implies \ker \varphi$ is a subgroup.

Normal: We will show $g \ker \varphi g^{-1} \subseteq \ker \varphi \; \forall g \in G$.
Let $a \in \ker \varphi$, so $\varphi(a) = e$. Then any $g \in G$:

$$\begin{aligned} g\varphi(a)g^{-1} = g \cdot e \cdot g^{-1} = e \in \ker \varphi \\ \implies g \cdot \ker \varphi g^{-1} \subseteq \ker \varphi \end{aligned}$$

□