

Math 167 (Mathematical Game Theory)

University of California, Los Angeles

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Course description: Quantitative modeling of strategic interaction. Topics include extensive and normal form games, background probability, lotteries, mixed strategies, pure and mixed Nash equilibria and refinements, bargaining; emphasis on economic examples. Optional topics include repeated games and evolutionary game theory. More information can be found on Math UCLA website.

These are my lecture notes for Math 167 (Mathematical Game Theory) taught by Oleg Gleizer. The main textbook for this class is *Game Theory, Alive* by Anna Karlin and Yuval Peres and the supplementary textbook is *A Course in Game Theory* by Thomas Ferguson.

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1 Mar 28, 2022

1.1 Impartial Combinatorial Games

Definition 1.1 (Impartial combinatorial game)

In an impartial combinatorial game,

- Two-person
- Perfect information
- No chance moves
- Win-or-lose outcome

Example 1.2

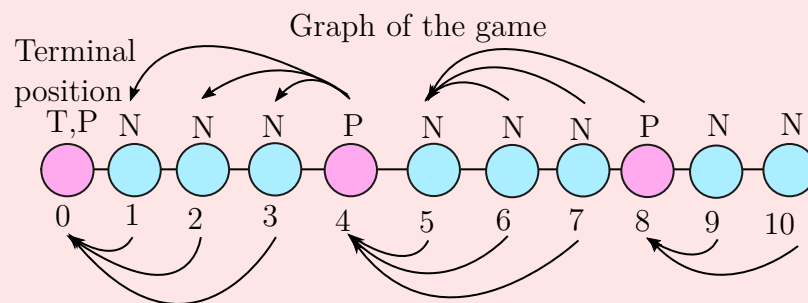
Suppose

- A pile of n chips on the table
- Two players: P1 and P2
- A move consists of removing one, two, or three chips from the pile
- P1 makes the first move, players alternate then
- The player to remove the last chip wins (the last player to move wins. If a player can't move, they lose.)

Method to analyze: backward induction.

Positions:

- **N**, next player to take a move wins.
- **P**, previous (second) player to take a move wins.



Any move from a **P** position leads to an **N** position. There always exists a move from an **N** position to a **P** position.

Ending condition: the game ends in a finite number of moves, no matter how played.

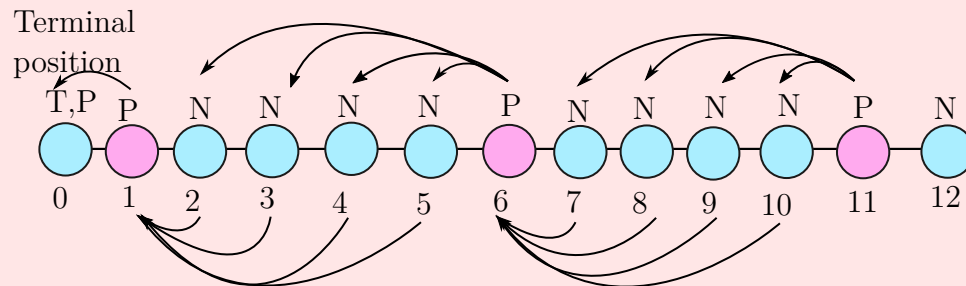
A **T** position is a **P** position.

Definition 1.3 (Normal play vs. misère play)

In a normal play, the last player to move wins. In a misère play, the last player to move loses.

Example 1.4

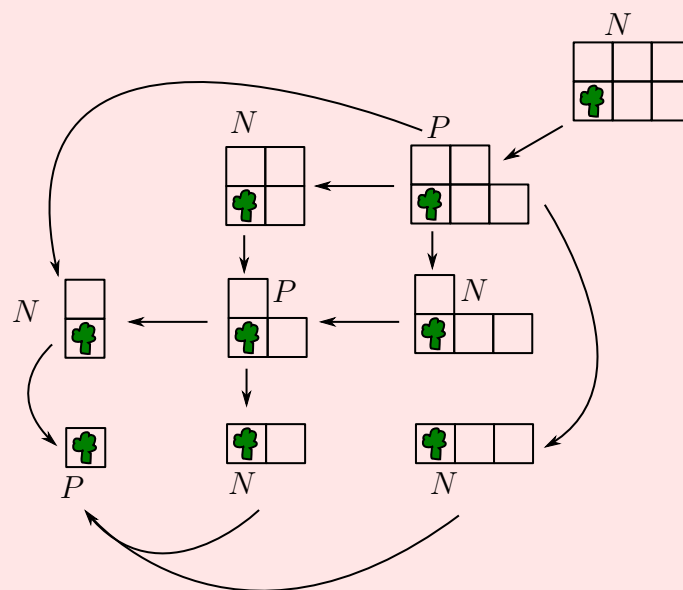
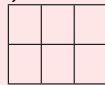
A misère game, a player can take 1-4 chips.



Note 1.5: Every position is either **N** or **P**, but not nothing or both.

Example 1.6 (The game of Chomp)

Let's play the game starting from



Graph of the game:

- Positions correspond to vertices
- Moves correspond to oriented edges

Definition 1.7 (Strategy)

A function that assigns a move to each position, except for the terminal.

Definition 1.8 (Winning strategy from a position x)

A winning strategy from a position x is a sequence of moves, starting from x , that guarantees a win.

Consider a normal game. Let $\mathbf{N}_i/\mathbf{P}_i$ be the set of positions from which P1/P2 can win (reach the nearest terminal vertex of the same graph) in at most i moves.

$$\mathbf{P}_0 = \mathbf{P}_1 = \{\text{terminal positions}\}$$

$$\mathbf{N}_{i+1} = \{x: \text{there is a move from } x \text{ to } \mathbf{P}_i\}$$

$$\mathbf{P}_{i+1} = \{y: \text{each move leads to } \mathbf{N}_i\}$$

Note 1.9: $\mathbf{P}_0 = \mathbf{P}_1 \subseteq \mathbf{P}_2 \subseteq \mathbf{P}_3 \dots$

$$\mathbf{N}_1 \subseteq \mathbf{N}_2 \subseteq \mathbf{N}_3 \dots$$

$$\mathbf{N} = \bigcup_{i=1} \mathbf{N}_i, \quad \mathbf{P} = \bigcup_{i=0} \mathbf{P}_i$$

Definition 1.10 (Progressively bounded)

A game is called progressively bounded if for every position x there exists an upper bound $B(x)$ on the number of moves until the game terminates.

2 Mar 30, 2022

2.1 Combinatorial Games (Cont'd)

Recall 2.1 • $P_0 = P_1 = \{\text{terminal positions}\}$

- $N_{n+1} = \{x: \text{there is a move from } x \text{ to } P_n\}$
- $P_{n+1} = \{y: \text{each move from } y \text{ leads to } N_n\}$
- $P_0 = P_1 \subseteq P_2 \subseteq \dots$
- $N_1 \subseteq N_2 \subseteq N_3 \subseteq \dots$
- $P = \bigcup_{n=0} P_n$
- $N = \bigcup_{n=1} N_n$
- A game is called progressively bounded if for every position x there exists an upper bound $B(x)$ on the number of moves until the game stops.

Theorem 2.2

In a progressively bounded impartial full information combinatorial game, all positions are in $N \cup P$. Thus, for every position there exists a winning strategy.

Proof. Let $B(x) \leq n$. Let us prove by induction that $x \in N_n \cup P_n$.

Base: $n = 0$

x is a terminal vertex $\implies x \in P_0 = P_1$.

Inductive hypothesis by P_0 : $B(x) \leq n \implies x \in N_n \cup P_n$.

Inductive step: Show that $B(x) \leq n + 1 \implies x \in N_{n+1} \cup P_{n+1}$

Consider a move $x \rightarrow y$ and $B(y) \leq n$. Hence, $y \in N_n \cup P_n$. So either

Case 1: Each move from x leads to $y \in N_n \implies x \in P_{n+1}$.

Case 2: There exists a move from x to $y \notin N_n$. Thanks to the inductive typo, $y \in N_n \cup P_n$ so $y \in P_n \implies x \in N_{n+1}$. \square

2.2 The Game of Nim

- Several piles, each containing finitely many chips.
- A move: a player can remove any number of chips, from one to all from any pile
- P1 and P2 alternate taking moves
- The player to take the last chip wins

Consider $x \oplus y$. We rewrite x and y as binary numbers and perform long addition of x_2 and y_2 without carry-over, i.e. mod 2.

$$5 \oplus 7 = \begin{array}{r} 1 \ 0 \ 1 \\ \oplus \ 1 \ 1 \ 1 \\ \hline 0 \ 1 \ 0 \end{array} = 2$$

A position $x = (x_1, x_2, \dots, x_k)$ is a **P** position $\iff x_1 \oplus x_2 \oplus \dots \oplus x_k = 0$.

3 April 1, 2022

3.1 The Game of Nim (Cont'd)

Recall 3.1 $x = (x_1, x_2, \dots, x_k)$

Theorem (Bouton) says $x \in \mathbf{P} \iff x_1 \oplus x_2 \oplus \dots \oplus x_k = 0$.

Proof of Theorem 2.6. We have

Terminal position: $x = (0, 0, \dots, 0) \in \mathbf{P}$ Let $x \in \mathbf{N}$. Then there exists a move $x \rightarrow y \in \mathbf{P}$.

$$x_1 \oplus x_2 \oplus \dots \oplus x_k = \oplus \begin{array}{cccccc} 1 & * & * & \dots & \dots & * & * \\ & & 1 & * & \dots & * & * \\ & & \vdots & \vdots & & \vdots & \vdots \\ 1 & * & * & \dots & \dots & \dots & * & * \end{array}$$

Find the left-most (most significant) column with an odd number of 1's. Change any number that has a 1 in the column so that there is an even number of 1's in every column. The 1 in the most significant position becomes a 0 which implies the number becomes smaller. So this is a legal move.

We have $x \in \mathbf{P} \implies$ any move $x \rightarrow y \in \mathbf{N}$ where

$$x = (x_1, x_2, \dots, x_k) \mapsto y = (x'_1, x_2, \dots, x_k)$$

such that

$$x'_1 < x_1 \text{ and } x_1 \oplus x_2 \oplus \dots \oplus x_k = 0.$$

If

$$x'_1 \oplus x_2 \oplus \dots \oplus x_k = 0$$

then

$$x'_1 \oplus x_2 \oplus \dots \oplus x_k = 0$$

then $x'_1 = x_0$, a contradiction. Hence

$$x'_1 \oplus x_2 \oplus \dots \oplus x_k \neq 0 \implies y \in \mathbf{N}.$$

□

Example 3.2

$$x_1 = 7$$

$$x_2 = 10$$

$$x_3 = 15$$

$$\begin{array}{ccc|ccc} & 0 & 1 & 1 & 1 & 1 \\ \oplus & 1 & 0 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 1 & 1 \\ \hline & 0 & 0 & 1 & 0 & 0 \end{array} \implies \begin{array}{ccc|ccc} & 0 & 1 & 0 & 1 & 1 \\ \oplus & 1 & 0 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 1 & 1 \\ \hline & 0 & 0 & 0 & 0 & 0 \end{array}$$

So we have that $(7, 10, 15) \mapsto (5, 10, 15)$

3.2 Subtraction Nim

Extra condition: A player can remove at most n chips.

We find pile sizes mod $n + 1$, i.e.

$$(x_1, x_2, \dots, x_k) \mapsto (x_1 \bmod n + 1, x_2 \bmod n + 1, \dots, x_k \bmod n + 1)$$

Now we find the Nim-sum and make a move.

$$x \bmod n + 1 = \underbrace{(x_1 \bmod n + 1, x_2 \bmod n + 1, \dots, x_k \bmod n + 1)}_{(x_1 \bmod n+1)_2 \oplus (x_2 \bmod n+1)_2 \oplus \dots \oplus (x_k \bmod n+1)_2} \implies \begin{cases} = 0 \iff \mathbf{P} \\ \neq 0 \iff \mathbf{N} \end{cases}$$

Example 3.3

We have $x = (12, 13, 14)$ and $n = 3$. So,

$$(12 \bmod 4, 13 \bmod 4, 14 \bmod 4) \equiv (0, 1, 2) = (0_2, 1_2, 10_2)$$

So

$$\begin{array}{cc} & 0 & 0 \\ \oplus & 0 & 1 \\ & 1 & 0 \\ \hline & 1 & 1 \end{array} \neq 0$$

so we take away one chip from the third pile

$$\begin{array}{cc} & 0 & 0 \\ \oplus & 0 & 1 \\ & 0 & 1 \\ \hline & 0 & 0 \end{array}$$

So we have that $(12, 13, 14) \mapsto (12, 13, 13)$.

Note 3.4: You can always make a legal move $\mathbf{N} \rightarrow \mathbf{P}$ by removing $i \leq n$ chips from a pile.

Note 3.5: To move from \mathbf{P} to \mathbf{P} , you need to remove $n + 1$ chips from a pile. Not allowed! Hence, any move from \mathbf{P} is to \mathbf{N} .

Example 3.6

We have $x = (12, 13, 13)$, with $n = 3$. So

$$x \bmod 4 = (0, 1, 1)$$

therefore

$$\begin{array}{r} 0 \\ \oplus \quad 1 \\ 1 \\ \hline 0 \end{array}$$

3.3 Two-Person Zero Sum Games (Strategic Form)

We have

- P1: a non-empty set of strategies S1
- P2: a non-empty set of strategies S2
- $A: S1 \times S2 \rightarrow \mathbb{R}$, the min function for P1 (payoff matrix)

Note 3.7: Since the game is zero-sum, a win for P1 is a loss for P2. $A(i, j)$ can be ≤ 0 , so works both ways.

Pure strategies:

		P2			
		S21	S22	...	S2n
P1	S11	a_{11}	a_{12}	...	a_{1n}
	S12	a_{21}	a_{22}	...	a_{2n}
	\vdots	\vdots	\vdots	\ddots	\vdots
	S1m	a_{m1}	a_{m2}	...	a_{mn}

A game. P1 chooses the strategy S1*i*. Simultaneously, P2 chooses the strategy S2*j*. P1 wins a_{ij} .

Lemma 3.8

$$\min_j \max_i a_{ij} \geq \max_i \min_j a_{ij}$$

We will continue in the next lecture.

4 Apr 4, 2022

4.1 Two-Person Zero Sum Games in Strategic Form (Cont'd)

Recall 4.1 Recall that

		P2			
		S21	S22	...	S2n
P1	S11	a_{11}	a_{12}	\cdots	a_{1n}
	S12	a_{21}	a_{22}	\cdots	a_{2n}
	\vdots	\vdots	\vdots	\ddots	\vdots
	S1m	a_{m1}	a_{m2}	\cdots	a_{mn}

P1 has a non-empty set of pure strategies

$$S1 = \{S11, S12, \dots, S1m\}$$

P2 has a non-empty set of pure strategies

$$S2 = \{S21, S22, \dots, S2n\}$$

$A: S1 \times S2 \rightarrow \mathbb{R}$, payoff matrix

P1, $S1i: a_{i1}, a_{i2}, \dots, a_{in}$

Betting on the worst possible outcome, P1 bets on $\min_{1 \leq j \leq n} a_{ij}$. Being intelligent, P1 chooses

$$\max_{1 \leq i \leq m} \min_{1 \leq j \leq n} a_{ij}.$$

Betting on the worst possible loss, P2 bets on $\max_{1 \leq i \leq m} a_{ij}$. Being intelligent, P2 chooses

$$\min_{1 \leq j \leq n} \max_{1 \leq i \leq m} a_{ij}$$

Lemma 4.2

$$\max_{1 \leq i \leq m} \min_{1 \leq j \leq n} a_{ij} \leq \min_{1 \leq i \leq m} \max_{1 \leq j \leq n} a_{ij}$$

Proof. Let

$$\max_i \min_j a_{ij} = a_{pq}$$

$$\min_j \max_i a_{ij} = a_{rs}$$

		q		s	
p		a_{pq}	\leq	a_{ps}	
				$\setminus \nearrow$	
r		a_{rq}		a_{rs}	

□

Example 4.3

Chooser (P1), Hider (P2). Hider hides behind their back

- Either left hand with one coin
- or right hand with two coins

Chooser chooses L or R,

		P2	
		L1	R2
P1	L	1	0
	R	0	2

$$P1: \max_j \min_i a_{ij} = 0$$

$$P2: \min_j \max_i a_{ij} = 1$$

Mixed strategies

		P2	
		L1, q	R2, $1 - q$
P1	L, p	1	0
	R, $1 - p$	0	2

P1: if P2 chooses the strategy L1, the expected gain is

$$1 \cdot p + 0 \cdot (1 - p) = p$$

If P2 chooses R2, the expected gain is

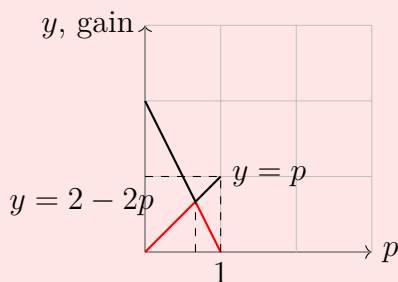
$$0 \cdot p + 2(1 - p) = 2 - 2p.$$

If P1 is out of luck, then the expected gain is

$$\min\{p, 2 - 2p\}$$

Since P1 is intelligent, they choose p s.t. the gain is

$$\max_{0 \leq p \leq 1} \min\{p, 2 - 2p\}$$



$$2 - 2p = p$$

$$2 = 3p$$

$$p = \frac{2}{3}$$

The optimal strategy is

$$\frac{2}{3}L + \frac{1}{3}R$$

With expected gain $\geq \frac{2}{3}$.

P2 is thinking. If P1 chooses L , my expected loss is

$$1 \cdot q + 0 \cdot (1 - q) = q$$

If P1 chooses R , my expected loss is

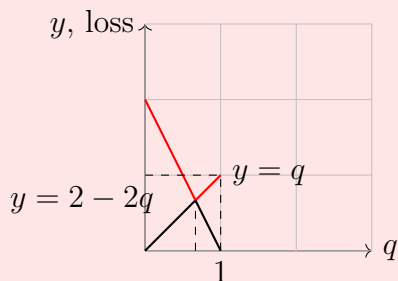
$$0 \cdot q + 2(1 - q) = 2 - 2q$$

Suppose I'm out of luck. Then my expected loss is

$$\max\{q, 2 - 2q\}$$

Being my very smart self,

$$\min_{0 \leq q \leq 1} \max\{q, 2 - 2q\}$$



The optimal strategy is

$$\frac{2}{3}L1 + \frac{1}{3}R2$$

With expected loss $\leq \frac{2}{3} = V$, the value of the game.

Let us generalize $A \in \mathbb{R}^{n \times m}$, an $n \times m$ payoff matrix.

$$\Delta_m = \left\{ \mathbf{p} \in \mathbb{R}^m : p_1 \geq 0, p_2 \geq 0, \dots, p_m \geq 0, \sum_{i=1}^m p_i = 1 \right\}$$

$$\Delta_n = \left\{ \mathbf{q} \in \mathbb{R}^n : q_1 \geq 0, q_2 \geq 0, \dots, q_n \geq 0, \sum_{j=1}^n q_j = 1 \right\}$$

A mixed strategy for P1 is determined by

$$\mathbf{p} \in \Delta_m$$

A mixed strategy for P2 is determined by

$$\mathbf{q} \in \Delta_n$$

Expected gain for P1 (expected loss for P2) = $(\mathbf{p})^T A \mathbf{q}$

		P2			
		q_1	q_2	\cdots	q_n
P1	p_1	a_{11}	a_{12}	\cdots	a_{1n}
	p_2	a_{21}	a_{22}	\cdots	a_{2n}
	\vdots	\vdots	\vdots	\ddots	\vdots
	p_m	a_{m1}	a_{m2}	\cdots	a_{mn}

So

$$(\mathbf{p})^t A \mathbf{q} = p_i(a_{i1}q_1 + a_{i2}q_2 + \cdots + a_{in}q_n)$$

If P1 employs the strategy \mathbf{P} , then in the worst case their payoff is

$$\min_{\mathbf{q} \in \Delta_n} (\mathbf{p})^t A \mathbf{q} = \min_{1 \leq j \leq n} \sum_{i=1}^m a_{ij} p_i$$

Hence, P1's winning strategy is

$$\max_{\mathbf{p} \in \Delta_m} \min_{\mathbf{q} \in \Delta_n} \mathbf{p}^T A \mathbf{q}$$

5 Apr 6, 2022

5.1 General Two-Person Zero-Sum Games in Strategic Form

Recall 5.1 Recall

		P2			
		q_1	q_2	\cdots	q_n
P1	p_1	a_{11}	a_{12}	\cdots	a_{1n}
	p_2	a_{21}	a_{22}	\cdots	a_{2n}
	\vdots	\vdots	\vdots	\ddots	\vdots
	p_m	a_{m1}	a_{m2}	\cdots	a_{mn}

With set of mixed strategies given by,

$$\Delta_m = \left\{ \mathbf{p} \in \mathbb{R}^m : \mathbf{p} \geq 0, \sum_{i=1}^m p_i = 1 \right\}$$

$$\Delta_n = \left\{ \mathbf{q} \in \mathbb{R}^n : \mathbf{q} \geq 0, \sum_{j=1}^n q_j = 1 \right\}$$

where $p_1 \geq 0, p_2 \geq 0, \dots, p_m \geq 0$.

We have

$$\text{Expected gain of P1} = (\mathbf{p})^t A \mathbf{q}$$

with $\mathbf{p} \in \Delta_m$ and $\mathbf{q} \in \Delta_n$.

The winning strategy for P1:

- Worst case: $\min_{\mathbf{q} \in \Delta_n} (\mathbf{p})^t A \mathbf{q}$
- Smart choice: $\max_{\mathbf{p} \in \Delta_m} \min_{\mathbf{q} \in \Delta_n} (\mathbf{p})^t A \mathbf{q}$

$$\begin{aligned} \min_{\mathbf{q} \in \Delta_n} (\mathbf{p})^t A \mathbf{q} &= \min_{\mathbf{q} \in \Delta_n} \sum_{j=1}^n q_j \sum_{i=1}^m a_{ij} p_i \\ &= \min_{1 \leq j \leq n} \sum_{i=1}^m a_{ij} p_i \end{aligned}$$

The winning strategy for P2:

- Worst case: $\max_{\mathbf{p} \in \Delta_m} (\mathbf{p})^t A \mathbf{q}$
- Smart choice: $\min_{\mathbf{q} \in \Delta_n} \max_{\mathbf{p} \in \Delta_m} (\mathbf{p})^t A \mathbf{q}$

$$\begin{aligned}\max_{\mathbf{p} \in \Delta_m} (\mathbf{p})^t A \mathbf{q} &= \max_{\mathbf{p} \in \Delta_m} \sum_{i=1}^m p_i \sum_{j=1}^n a_{ij} q_j \\ &= \max_{1 \leq i \leq m} \sum_{j=1}^n a_{ij} q_j\end{aligned}$$

Definition 5.2 (Safety value for P1 vs. P2)

The value $\hat{\mathbf{p}}$ at which

$$\max_{\mathbf{p} \in \Delta_m} \min_{\mathbf{q} \in \Delta_n} (\mathbf{p})^t A \mathbf{q}$$

is attained is called the safety value for P1. The value $\hat{\mathbf{q}}$ at which

$$\min_{\mathbf{q} \in \Delta_n} \max_{\mathbf{p} \in \Delta_m} (\mathbf{p})^t A \mathbf{q}$$

is attained is called the safety value for P2.

Theorem 5.3 (Von Neumann Minimax Theorem)

For any two-person zero-sum game with $m \times n$ payoff matrix A , there is a number V , called the value of the game, satisfying

$$\min_{\mathbf{q} \in \Delta_n} \max_{\mathbf{p} \in \Delta_m} (\mathbf{p})^t A \mathbf{q} = \max_{\mathbf{p} \in \Delta_m} \min_{\mathbf{q} \in \Delta_n} (\mathbf{p})^t A \mathbf{q} = V$$

Let $\hat{\mathbf{p}}$ be an optimal solution for P1. Let $\hat{\mathbf{q}}$ be an optimal solution for P2. Then

$$\min_{\mathbf{q} \in \Delta_n} (\hat{\mathbf{p}})^t A \mathbf{q} = \max_{\mathbf{p} \in \Delta_m} (\hat{\mathbf{p}})^t A \hat{\mathbf{q}}$$

Proof. Proof seen here. □

Definition 5.4 (Value of the game)

Given conditions from Von Neumann Minimax Theorem, V is the value of the game.

Example 5.5 (Odd or Even) • P1 and P2 simultaneously call out one of the numbers, 1 or 2.

- If the sum is odd, P1 wins and gets the sum of the numbers in \$
- If the sum is even, P2 wins and gets the sum of the numbers in \$

		P2	
		1, q	2, $1 - q$
P1	1, p	-2	3
	2, $1 - p$	3	-4

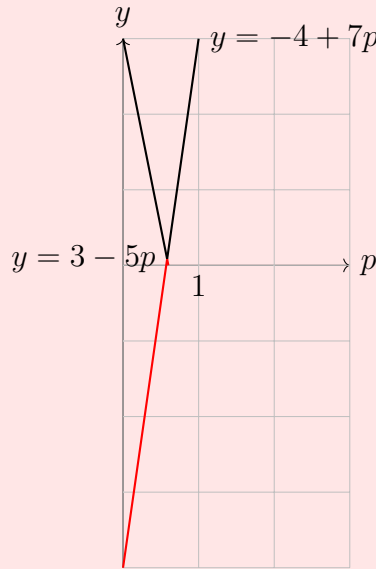
So P1's expected win (P2's expected loss) is

$$\begin{aligned} (\mathbf{p})^t A \mathbf{q} &= \begin{bmatrix} p & 1-p \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} q \\ 1-q \end{bmatrix} \\ &= -12pq + 7p + 7q - 4 \end{aligned}$$

P1's worst possible case:

$$f(p) = \min_{0 \leq q \leq 1} \{-12pq + 7p + 7q - 4\}$$

$$\begin{aligned} q, S21: \quad & -2p + 3(1-p) = 3 - 5p \\ 1-q, S22: \quad & 3p - 4(1-p) = -4 + 7p \end{aligned}$$



- If $3 - 5p \geq -4 + 7p$, then $q = 0$.
- If $3 - 5p < -4 + 7p$, then $q = 1$.

Hence,

$$f(p) = \min\{3 - 5p, -4 + 7p\}$$

Note that

$$\begin{aligned} (-12pq + 7p + 7q - 4) \Big|_{q=0} &= -4 + 7p \\ (-12pq + 7p + 7q - 4) \Big|_{q=1} &= 3 - 5p \end{aligned}$$

$$\text{P1: } \max_{0 \leq p \leq 1} \min_{0 \leq q \leq 1} q(-2p + 3(1-p)) + (1-q)(3p - 4(1-p)) = \max_{0 \leq p \leq 1} \min\{3 - 5p, -4 + 7p\}$$

$$3 - 5p = -4 + 7p$$

$$7 = 12p$$

$$p = \frac{7}{12}, \quad q = \frac{5}{12}$$

Now from P2:

$$\text{P2: } \min_{0 \leq q \leq 1} \max_{0 \leq p \leq 1} p(-2q + 3(1 - q)) + (1 - p)(3q - 4(1 - q)) = \min_{0 \leq q \leq 1} \max\{3 - 5q, -4 + 7q\}$$

6 Apr 8, 2022

6.1 Solving Small-Dimensional Two-Person Zero-Sum Games Pen-and-Paper

Definition 6.1 (Saddle point)

An element of A , a_{ij} is called a saddle point if

- a_{ij} is the min of the i -th row
- a_{ij} is the max of the j -th column

Then $p_i = 1$, $q_j = 1$, $V = a_{ij}$

Example 6.2

Given

$$\begin{array}{ccccc} & & & & \min \\ & & & & -3 \\ & & & & \textcircled{2} \\ & & & & 0 \\ \max & \begin{bmatrix} 4 & 1 & -3 \\ 3 & \textcircled{2} & 5 \\ 0 & 1 & 6 \\ 4 & \textcircled{2} & 6 \end{bmatrix} & & & \end{array}$$

So $p_2 = q_2 = 1$ and $V = 2$.

Lemma 6.3

Let a_{pq} and a_{rs} be saddle points of a payoff matrix A . Then $a_{pq} = a_{rs}$.

		q		s	
p		a_{pq}	\leq	a_{ps}	
		\vee		\wedge	
r		a_{rq}	\geq	a_{rs}	

6.2 Domination

Rows:

$$\begin{array}{ccccccc} i\text{-th row} & a_{i1} & a_{i2} & \cdots & a_{in} \\ & \vee & \vee & & \vee \\ k\text{-th row} & a_{k1} & a_{k2} & \cdots & a_{kn} \end{array}$$

So $p_k = 0$ so k -th row can be removed from A .

Strict domination: for $j = 1, 2, \dots, n$,

$$a_{ij} > a_{kj}$$

Columns: the k -th column dominates the j -th column

$$\begin{array}{ccc} a_{1j} & \geq & a_{1k} \\ a_{2j} & \geq & a_{2k} \\ \vdots & & \vdots \\ a_{mj} & \geq & a_{mk} \end{array}$$

Strict domination: for $i = 1, 2, \dots, m$,

$$a_{ij} > a_{ik}$$

where a_{ik} is dominant.

- Removing a dominant row or column does not change the value of the game, but may remove an optimal strategy.
- Removing a strictly dominant row or column does not change the set of optimal strategies.

Example 6.4

$$A_1 = \begin{bmatrix} 2 & 0 & 4 \\ 1 & 2 & 3 \\ 4 & 1 & 2 \end{bmatrix}$$

Note: $\left. \begin{array}{l} 0 < 4 \\ 2 < 3 \\ 1 < 2 \end{array} \right\}$, strict domination.

$$A_2 = \begin{bmatrix} 2 & 0 \\ 1 & 2 \\ 4 & 1 \end{bmatrix}$$

Note: $\begin{array}{cc} 2 & 0 \\ \wedge & \wedge \\ 4 & 1 \end{array}$, strict domination

$$\max \begin{array}{cc} & \min \\ \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix} & \begin{array}{l} 1 \\ 1 \end{array} \\ 4 & 2 \end{array}$$

Note: No saddle point

Remark 6.5 A row/column can be dominated by a weighted sum of rows columns. For

example,

$$\begin{array}{cccc} a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{array}$$

For $\ell = 1, 2, \dots, n$, we have

$$\alpha a_{i\ell} + (1 - \alpha)a_{j\ell} \geq a_{k\ell}$$

Example 6.6

$$\begin{bmatrix} 0 & 4 & 6 \\ 5 & 7 & 4 \\ 9 & 6 & 3 \end{bmatrix}$$

$$4 > 3$$

$$7 > 4.5$$

$$5 \geq 6$$

7 Apr 11, 2022

7.1 Principle of Indifference

		P2			
		q_1	q_2	\cdots	q_n
P1	p_1	a_{11}	a_{12}	\cdots	a_{1n}
	p_2	a_{21}	a_{22}	\cdots	a_{2n}
	\vdots	\vdots	\vdots	\ddots	\vdots
	p_m	a_{m1}	a_{m2}	\cdots	a_{mn}

Let $\hat{\mathbf{p}} = (p_1, p_2, \dots, p_m)^t$ be an optimal strategy for P1 and let $q_j = 1$ be a pure strategy for P2.

$$\sum_{i=1}^m a_{ij} p_i \geq V \quad (1)$$

Let $\hat{\mathbf{q}} = (q_1, q_2, \dots, q_n)^t$ be an optimal strategy for P2 and let $p_i = 1$ be a pure strategy for P1. Then

$$\sum_{j=1}^n a_{ij} q_j \leq V \quad (2)$$

Note 7.1: If both players use optimal strategies, then

$$\sum_{i=1}^m \sum_{j=1}^n a_{ij} p_i q_j = V$$

Proof. We have

$$\begin{aligned}
 V &\leq \sum_{i=1}^m a_{ij} p_i = 1 \cdot \sum_{i=1}^m a_{ij} p_i = \left(\sum_{j=1}^n q_j \right) \sum_{i=1}^m a_{ij} p_i \\
 &= \sum_{i=1}^m \sum_{j=1}^n a_{ij} p_i q_j \\
 &= \sum_{i=1}^m p_i \underbrace{\sum_{j=1}^n a_{ij} q_j}_{=1} \leq V \\
 &= V
 \end{aligned}$$

□

Theorem 7.2 (The Equilibrium Theorem)

Let $\hat{\mathbf{p}} = (p_1, p_2, \dots, p_m)$ and $\hat{\mathbf{q}} = (q_1, q_2, \dots, q_n)$ be optimal strategies for P1 and P2 respectively. Then

$$\sum_{j=1}^n a_{ij}q_j = V \quad \forall i \text{ s.t. } p_i > 0$$

$$\sum_{i=1}^m a_{ij}p_i = V \quad \forall j \text{ s.t. } q_j > 0$$

Proof. Let $p_k > 0$ and let $\sum_{j=1}^n a_{kj}q_j \neq V \implies \sum_{j=1}^n a_{kj}q_j < V$. We have

$$V \leq \sum_{i=1}^m p_i \sum_{j=1}^n a_{ij}q_j < V$$

a contradiction. □

Example 7.3 (The game of Odd-and-Even)

Played with three numbers: 0, 1, and 2.

		P2, even		
		0	1	2
P1, Odd	0, p_1	0	1	-2
	1, p_2	1	-2	3
	2, p_3	-2	3	-4

$p_1 \geq 0, p_2 \geq 0, p_3 \geq 0$, and $p_1 + p_2 + p_3 = 1$. Then

$$\begin{cases} p_2 - 2p_3 - V = 0 \\ p_1 - 2p_2 + 3p_3 - V = 0 \\ -2p_1 + 3p_2 - 4p_3 - V = 0 \\ p_1 + p_2 + p_3 = 1 \end{cases}$$

7.2 Symmetric Games

Definition 7.4 (Symmetric Game)

The rules are the same for P1 and P2. So $A^t = -A$.

Theorem 7.5

The value of a finite size symmetric game is zero.

Proof. Note $V^t = V$. And

$$V = (\hat{\mathbf{p}})^t A \hat{\mathbf{p}} = [(\hat{\mathbf{p}})^t A \hat{\mathbf{p}}]^t = -\hat{\mathbf{p}} A \hat{\mathbf{p}} = -V$$

So

$$V = -V \implies V = 0$$

□

Example 7.6 (Rock, Paper, Scissors)

We have

		P2		
		Rock	Paper	Scissors
P1	Rock	0	-1	1
	Paper	1	0	-1
	Scissors	-1	1	0

So

$$\begin{cases} p_2 - p_3 = 0 \\ -p_1 + p_3 = 0 \\ p_1 - p_2 = 0 \\ p_1 + p_2 + p_3 = 1 \end{cases}$$

Therefore,

$$p_1 = p_2 = p_3 = \frac{1}{3}$$

8 Apr 13, 2022

8.1 The Equilibrium Theorem and Symmetric Games (Cont'd)

Recall 8.1 The Equilibrium Theorem:

Let $\hat{\mathbf{p}}, \hat{\mathbf{q}}$ be optimal solutions for P1 and P2 respectively. Then

$$\sum_{j=1}^n a_{ij}q_j = V \quad \forall i \text{ s.t. } p_i > 0$$

$$\sum_{i=1}^m a_{ij}p_i = V \quad \forall j \text{ s.t. } q_j > 0$$

Recall 8.2 A game is called a symmetric game if $A^t = -A$.

Example 8.3 (A Mendelson Game)

	1	2	3	4	5	6		99	100
1	0	-1	2	2	2	2	...	2	2
2	1	0	-1	2	2	2	...	2	2
3	-2	1	0	-1	2	2	...	2	2
4	-2	-2	1	0	-1	2	...	2	2
5	-2	-2	-2	1	0	-1	...	2	2
6	-2	-2	-2	-2	1	0	...	2	2
\vdots	-2	-2	-2	-2	-2	1	...		
99	-2	-2	-2	-2	-2	-2	...	0	-1
100	-2	-2	-2	-2	-2	-2	...		0

So

$$\begin{cases} p_2 - 2p_3 = 0 \\ -p_1 + p_3 = 0 \\ 2p_1 - p_2 = 0 \\ p_1 + p_2 + p_3 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} p_3 = p_1 \\ p_2 = 2p_1 \end{cases} \Rightarrow p_1 + 2p_1 + p_1 = 1$$

$$\Rightarrow p_1 = \frac{1}{4} = p_3, \quad p_2 = \frac{1}{2}$$

So

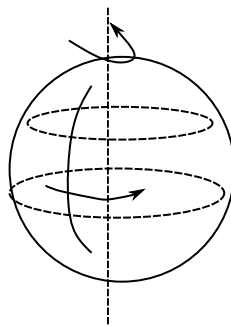
$$\hat{\mathbf{p}} = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, 0, 0, \dots, 0 \right)$$

8.2 Invariance or Games with Symmetries

		P2		
		Rock	Paper	Scissors
P1	Rock	0	-1	1
	Paper	1	0	-1
	Scissors	-1	1	0

We shift to

		P2		
		Paper	Rock	Scissors
P1	Paper	0	1	1
	Rock	-1	0	-1
	Scissors	1	1	0



A group = G

$$G = \{\sigma: \sigma_1 \circ \sigma_2 \in G, e \in G, \sigma^{-1} \circ \sigma = \sigma \circ \sigma^{-1} = e\}$$

Let $S1 = (1, 2, \dots, m)$ and $S2 = (1, 2, \dots, n)$ be the sets of pure strategies for P1 and P2. Let σ be a permutation of S1.

$$\begin{array}{cccc} 1 & 2 & \dots & m \\ \sigma(1) & \sigma(2) & \dots & \sigma(m) \end{array}$$

Definition 8.4 (Invariant)

A finite two-person zero-sum game is invariant under a permutation $\sigma \in S_m$ if $\forall i \in S1$, and $\forall j \in S2$, there exists a unique $j' \in S2$ such that

$$A(i, j) = A(\sigma(i), j')$$

9 Apr 15, 2022

9.1 Invariance (Cont'd)

Recall 9.1 $S1 = \{1, 2, \dots, m\}, S2 = \{1, 2, \dots, n\}$.

Recall 9.2 σ , a permutation of $S1$, a bijection $S1 \rightarrow S1$:

$$\begin{pmatrix} 1 & 2 & \dots & m \\ \sigma(1) & \sigma(2) & \dots & \sigma(m) \end{pmatrix}$$

Definition 9.3 (Invariant under a permutation σ)

A finite two-person zero-sum game is invariant under a permutation σ of $S1$ if $\forall i \in S1, j \in S2$, there exists a unique $j' \in S2$ such that

$$A(i, j) = A(\sigma(i), j')$$

The uniqueness requirement: if $\forall i \in S1, \forall j \in S2, \exists j', j'' \in S2$ such that $j' \neq j''$ and

$$A(i, j) = A(\sigma(i), j') = A(\sigma(i), j'')$$

then the strategies j' and j'' have identical payoffs. Remembering the rows, we have

	j'	\dots	j''
$\sigma(1)$	$a_{\sigma(1)j'}$	\dots	$a_{\sigma(1)j''}$
$\sigma(2)$	$a_{\sigma(2)j'}$	\dots	$a_{\sigma(2)j''}$
\vdots	\vdots		\vdots
$\sigma(m)$	$a_{\sigma(m)j'}$	\dots	$a_{\sigma(m)j''}$

A duplicate strategy can be removed without loss of generality.

Notation 9.4: $\bar{\sigma}(j) = j'$

Lemma 9.5

$\bar{\sigma}$ is a permutation of $S2$.

Proof.

- $\bar{\sigma}$ is defined for any $j \in S2$ by definition.
- To show that $\bar{\sigma} \in S_n$, let us show that $\bar{\sigma}$ is one-to-one.

Assume the opposite: $j, k \in S2, j \neq k, \bar{\sigma}(j) = \bar{\sigma}(k)$. Then

$$A(i, j) = A(\sigma(i), \bar{\sigma}(j)) = A(\sigma(i), \bar{\sigma}(k)) = A(i, k) \quad \forall i \in S1$$

Hence, the strategies $S2j$ and $S2k$ are duplicates which implies assumption is incorrect. \square

Lemma 9.6

A game invariant under σ is invariant under σ^{-1}

Proof.

$$A(i, j) = A(\sigma(i), \bar{\sigma}(j)) \quad \forall i \in S1, j \in S2$$

Then

$$A(\sigma(i), \bar{\sigma}(j)) = A(i, j)$$

Let $i' = \sigma(i), j' = \sigma(j)$. Then

$$A(i', j') = A(\sigma^{-1}(i'), \bar{\sigma}^{-1}(j')) \quad \forall i' \in S1, j' \in S2$$

□

9.2 Multiplication of Permutation

Note 9.7:

$$\sigma_2 \circ \sigma_1(*) = \sigma_2(\sigma_1(*))$$

Example 9.8

Suppose

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$$

Then,

$$\sigma_2 \circ \sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 2 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$$

Lemma 9.9

A game invariant under σ_1 and σ_2 is invariant under $\sigma_2 \circ \sigma_1$.

Proof. $\forall i \in S1, j \in S2$,

$$\begin{aligned} A(i, j) &= A(\sigma(i), \bar{\sigma}(j)) = A(i', j') = A(\sigma_2(i'), \bar{\sigma}_2(j')) \\ &= A(\sigma_2 \circ \sigma_1(i), \bar{\sigma}_2 \circ \bar{\sigma}_1(j)) \end{aligned}$$

□

Conclusion: Invariant permutations of pure strategies form a group.

Definition 9.10 (Invariant under a group σ)

A game $(S1, S2, A)$ is invariant under a group σ if it is invariant under any $\sigma \in G$

10 Apr 18, 2022

10.1 Midterm 1

11 Apr 20, 2022

11.1 Invariance (Cont'd)

Recall 11.1 A group G is a set such that

1. $\forall \sigma_1, \sigma_2 \in G, \sigma_2 \circ \sigma_1 \in G$, i.e. there exists an operation of group multiplication.
2. $\exists e \in G$ such that $e \circ \sigma = \sigma \circ e = \sigma, \forall \sigma \in G$.
3. $\forall \sigma \in G, \exists \sigma^{-1} \in G$ such that $\sigma^{-1} \circ \sigma = \sigma \circ \sigma^{-1} = e$

Let $S1 = \{1, \dots, m\}, S2 = \{1, \dots, n\}$. Let G be a subgroup of S_m .

Definition 11.2 (Homomorphism)

A homomorphism of groups G and \bar{G} is a map $h: G \rightarrow \bar{G}$ preserving the group structure, i.e.

1. $h(\sigma_2 \circ \sigma_1) = \bar{\sigma}_2 \circ \bar{\sigma}_1, \forall \sigma_1, \sigma_2 \in G$
2. $h(e) = \bar{e}$
3. $h(\sigma^{-1}) = \bar{\sigma}^{-1}, \forall \sigma \in G$

Definition 11.3 (Invariant under a game)

Let G be a subgroup of S_m . A game $(S1, S2, A)$ is invariant under G if it is invariant $\forall \sigma \in G$.

Recall 11.4 $\forall i \in S1, j \in S2$, there exists a unique $\bar{\sigma} \in S_n$ such that $A(i, j) = A(\sigma(i), \bar{\sigma}(j))$. In other words, there exists a homomorphism

$$\begin{aligned} h: G &\rightarrow S_n \\ \sigma &\mapsto \bar{\sigma} \end{aligned}$$

such that

$$a_{ij} = a_{\sigma(i), \bar{\sigma}(j)} \quad \forall i \in S1, j \in S2$$

Definition 11.5 (Orbit of a group action)

The orbit of a group action is the set

$$O_i = \{\sigma(i) : i \in S1, \sigma \in G\}$$

Note 11.6: The relation “being in the same orbit” is an equivalence relation.

Action of G on mixed strategies

$$p_1 S11 + p_2 S12 + \dots + p_m S1m$$

$$\begin{array}{c} \sigma \\ \downarrow \\ p_1 S 1 \sigma(1) + p_2 S 1 \sigma(2) + \cdots + p_m S 1 \sigma(m) \end{array}$$

so some $\sigma(k)$ in the sum above equals 1

$$\begin{aligned} \sigma(k) = 1 & \sim k = \sigma^{-1}(1) \\ p_k S 1 \sigma(k) &= p_{\sigma^{-1}(1)} S 1 1 \end{aligned}$$

Similarly for $2, 3, \dots, m$,

$$p_{\sigma^{-1}(1)} S 1 1 + p_{\sigma^{-1}(2)} S 1 2 + \cdots + p_{\sigma^{-1}(m)} S 1 m$$

Definition 11.7 (G -invariant)

A mixed strategy \mathbf{p} is G -invariant if $\sigma(\mathbf{p}) = \mathbf{p}$ for all $\sigma \in G$.

Theorem 11.8

If a game $(S1, S2, A)$ is invariant under G , then each player has a G -invariant optimal strategy.

Proof. Let $\hat{\mathbf{p}} = (p_1, p_2, \dots, p_m)^t$ be an optimal strategy for P1. Then

$$\sum_{i=1}^m p_i A(i, j) \geq V \quad \forall j \in S2$$

Let

$$\hat{\mathbf{p}}_G = \frac{1}{|G|} \sum_{\sigma \in G} \sigma(\hat{\mathbf{p}}),$$

i.e. $\forall i = 1, 2, \dots, m$,

$$(\hat{\mathbf{p}}_G)_i = \frac{1}{|G|} \sum_{\sigma \in G} p_{\sigma^{-1}(i)},$$

the average of the i th coordinate over the orbit O .

Invariance of $\hat{\mathbf{p}}_G$:

$$\forall \alpha \in G, \quad \alpha(\hat{\mathbf{p}}_G) = \frac{1}{|G|} \sum_{\sigma \in G} \alpha \circ \sigma(\hat{\mathbf{p}})$$

- $\alpha \circ \sigma$ is defined $\forall \sigma \in G$
- Assume $\sigma_1 \neq \sigma_2$, $\alpha \circ \sigma_1 = \alpha \circ \sigma_2$. Then

$$\alpha^{-1} \circ \alpha \circ \sigma_1 = \alpha^{-1} \circ \alpha \circ \sigma_2 \sim \sigma_1 = \sigma_2$$

Contradiction implies multiplication by α is a one-to-one map $G \rightarrow G$.

Hence, multiplication by α is a bijection $G \rightarrow G$.

$$\frac{1}{|G|} \sum_{\sigma \in G} \sigma(\hat{\mathbf{p}}) = \hat{\mathbf{p}}_G$$

We shall continue this proof in the next lecture. □

12 Apr 22, 2022

12.1 Invariance (Cont'd)

Recall 12.1

$$\sigma(\mathbf{p} = (p_1, p_2, \dots, p_m)^t) = (p_{\sigma^{-1}(1)}, p_{\sigma^{-1}(2)}, \dots, p_{\sigma^{-1}(m)})$$

A mixed strategy \mathbf{p} is G -invariant if $\sigma(\mathbf{p}) = \mathbf{p}$ for all $\sigma \in G$.

Proof of Theorem 11.8. $\alpha: G \rightarrow G$, $\alpha(\sigma) = \alpha \circ \sigma$ is a bijection of G . Thus,

$$\hat{\mathbf{p}}_G = \frac{1}{|G|} \sum_{\sigma \in G} \sigma(\hat{\mathbf{p}}) = \frac{1}{|G|} \sum_{\sigma \in G} \alpha \circ \sigma(\hat{\mathbf{p}}) = \alpha(\hat{\mathbf{p}}_G)$$

Optimality of $\hat{\mathbf{p}}_G$:

$$\begin{aligned} \sum_{i=1}^m (\hat{\mathbf{p}}_G)_i A(i, j) &= \sum_{i=1}^m \frac{1}{|G|} \sum_{\sigma \in G} (\hat{\mathbf{p}})_{\sigma^{-1}(i)} A(i, j) \\ &= \frac{1}{|G|} \sum_{\sigma \in G} \sum_{i=1}^m (\hat{\mathbf{p}})_{\sigma^{-1}(i)} A(i, j) \\ &= \frac{1}{|G|} \sum_{\sigma \in G} \sum_{i=1}^m p_i A(\sigma(i), \bar{\sigma}(j)) \\ &= \frac{1}{|G|} \sum_{\sigma \in G} \sum_{i=1}^m p_i A(i, j) \geq V \end{aligned}$$

because

$$\sum_{i=1}^m p_i A(i, j) \geq V \quad \forall j \in S_2$$

□

Example 12.2 (Battleships)

We have a 3×3 grid, a submarine, P1 is the bomber, G -counter-clockwise rotations of the grid.

G is cyclic: $\{e, \sigma, \sigma^2, \sigma^3\}$ and note $\sigma^4 = e$.

P1:

S1	S2	S3
S4	S5	S6
S7	S8	S9

P2: sub

	e	σ	σ^2	σ^3
O_{S9}				
	S1	S2 = $\sigma(S1)$	S3	S4
O_{S5}				
	S5	S6	S7	S8
O_{S1}				
	S9	S10	S11	S12

A , a 9×12 matrix

Let $\hat{\mathbf{p}}_G = (p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9)$ be a G -invariant optimal strategy for P1. Then

$$p_1 = p_3 = p_7 = p_9 = \tilde{p}_1$$

$$p_2 = p_4 = p_6 = p_8 = \tilde{p}_2$$

$$p_5 = \tilde{p}_5$$

Similarly,

$$q_1 = q_2 = q_3 = q_4 = \tilde{q}_1$$

$$q_5 = q_6 = q_7 = q_8 = \tilde{q}_2$$

$$q_9 = q_{10} = q_{11} = q_{12} = \tilde{q}_3$$

		O_1, \tilde{q}_1	O_5, \tilde{q}_2	O_9, \tilde{q}_3	min
P1	O_1, \tilde{p}_1	$\frac{1}{4}$	$\frac{1}{4}$	0	0
	O_2, \tilde{p}_2	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
	O_5, \tilde{p}_3	0	0	1	0
	max	$\frac{1}{4}$	$\frac{1}{4}$	1	

Note: $A_{21} = A_{22} = \frac{1}{4}$ are saddle points.

$$C1 = C2 \implies \tilde{q}_1 = \tilde{q}_2$$

Solution of the game: $V = \frac{1}{4}$,

$$\tilde{p}_2 = 1, \quad \tilde{q}_1 = \tilde{q}_2 = \frac{1}{2}$$

i.e.,

$$p_2 = p_4 = p_6 = p_8 = \frac{1}{4}$$

$$q_1 = q_2 = q_3 = q_4 = q_5 = q_6 = q_7 = q_8 = \frac{1}{8}$$

13 Apr 25, 2022

13.1 Colonel Blotto Games

Example 13.1

Colonel Blotto has 4 regiments with which to occupy two posts. Lieutenant Kije has 3 regiments for the task.

Payoffs:

- The side sending more units to a post captures it as well as the enemy units sent to the post. The payoff is 1 point for the post and 1 point for each captured enemy unit.
- If both sides send the same number of units, the forces withdraw with no payoff.

	$(3, 0), q_1$	$(2, 1), q_2$	$(1, 2), q_2$	$(0, 3), q_1$
$p_1, (4, 0)$	4	2	1	0
$p_2, (3, 1)$	1	3	0	-1
$p_3, (2, 2)$	-2	2	2	-2
$p_2, (1, 3)$	-1	0	3	1
$p_1, (0, 4)$	0	1	2	4

So $G = \mathbb{Z}_2 = \{e, \sigma : \sigma^2 = e\}$, acts swapping posts.

	$O_1: (3, 0) \sim (0, 3)$ $\tilde{q}_1 = 2q_1$	$O_2: (2, 1) \sim (1, 2)$ $\tilde{q}_2 = 2q_1$
$O_1: (4, 0) \sim (0, 4)$ $\tilde{p}_1 = 2p_1$	$\frac{4 + 0 + 4 + 0}{4} = 2$	$\frac{2 + 1 + 1 + 2}{4} = \frac{3}{2}$
$O_2: (3, 1) \sim (1, 3)$ $\tilde{p}_2 = 2p_2$	0	$\frac{3}{2}$
$O_3: (2, 2)$ $\tilde{p}_3 = p_3$	$\frac{-2 - 2}{2} = -2$	$\frac{2 + 2}{2} = 2$

$$\begin{array}{c} 2 \\ \vee \\ 0 \end{array} \begin{array}{c} \frac{3}{2} \\ \vee \\ \frac{3}{2} \end{array} \implies \tilde{p}_2 = 0 \implies p_2 = 0$$

	\tilde{q}_1	$1 - \tilde{q}_1 = \tilde{q}_2$
\tilde{p}_1	2	$\frac{3}{2}$
$\tilde{p}_3 = 1 - \tilde{p}_1$	-2	2

$$\text{P1: } \max_{0 \leq p \leq 1} \min_{0 \leq q \leq 1} q(2p - 2(1-p)) + (1-q) \left(\frac{3}{2}p + 2(1-p) \right) = \max_{0 \leq p \leq 1} \min \left\{ 4p - 2, -\frac{p}{2} + 2 \right\}$$

$$\implies 4p - 2 = -\frac{p}{2} + 2 \implies \frac{9p}{2} = 4 \implies 9p = 8 \implies \boxed{\tilde{p}_1 = \frac{8}{9}}$$

$$V = 2 - \frac{\tilde{p}_1}{2} = 2 - \frac{4}{9} = \boxed{\frac{14}{9}}$$

Since $0 < \tilde{p}_1 < 1$, we can use the indifference principle to find \tilde{q}_1 .

$$2q + \frac{3}{2}(1 - q) = \frac{14}{9}$$

$$\frac{q}{2} + \frac{3}{2} = \frac{14}{9}$$

$$q + 3 = \frac{28}{9}$$

$$q = \frac{28}{9} - 3 = \frac{1}{9}$$

$$\implies \boxed{\tilde{q}_1 = \frac{1}{9}}$$

$$(\hat{\mathbf{p}})^t = \left(\frac{4}{9}, 0, \frac{1}{9}, 0, \frac{4}{9} \right)$$

$$(\hat{\mathbf{q}})^t = \left(\frac{1}{18}, \frac{4}{9}, \frac{4}{9}, \frac{1}{18} \right)$$

14 Apr 27, 2022

14.1 Linear Programming

Example 14.1

We want to maximize

$$x + 5y$$

such that

$$5x + 6y \leq 30$$

$$3x + 2y \leq 12$$

$$x \geq 0, y \geq 0$$

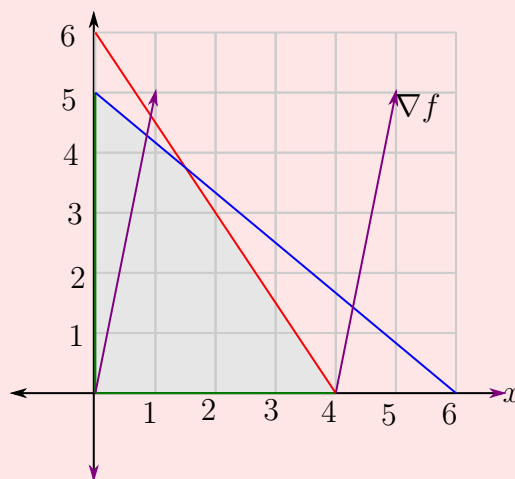
So

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

And

$$6y = 30 - 5x$$

$$\Rightarrow \boxed{y = 6 - \frac{5}{2}x} \quad \boxed{y = 5 - \frac{5}{6}x}$$



Standard form of a LP

We want to minimize

$$v = c^t x = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

such that

$$Ax \geq b$$

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \geq b_1$$

$$a_{12}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \geq b_2$$

$$\begin{aligned} & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \geq b_m \\ & x \geq 0 \end{aligned}$$

i.e.

$$x_1 \geq 0, \dots, x_n \geq 0$$

Example 14.2 (Example 14.1 in the standard form)

We want to minimize

$$-x - 5y$$

such that

$$-5x - 6y \geq -30$$

$$-3x - 2y \geq 0$$

$$x \geq 0$$

$$y \geq 0$$

14.2 Two-Person Zero-Sum Games as LP Programs in the Standard Form

We have

$$\begin{aligned} \text{P1: } \max_{\mathbf{p} \in \Delta_m} V(p_1, p_2, \dots, p_m) &= \max_{\mathbf{p} \in \Delta_m} \min_{\mathbf{q} \in \Delta_n} (\mathbf{p})^t A \mathbf{q} \\ &= \max_{\mathbf{p} \in \Delta_m} \left(\min_{\mathbf{q} \in \Delta_n} q_1 \sum_{i=1}^m p_i a_{i1} + q_2 \sum_{i=1}^m p_i a_{i2} + \cdots + q_n \sum_{i=1}^m p_i a_{in} \right) \\ &= \max_{\mathbf{p} \in \Delta_m} \min \left\{ \sum_{i=1}^m p_i a_{ij}, j = 1, \dots, n \right\} \end{aligned}$$

Example 14.3

P1 is solving the following problem:

Choose p_1, p_2, \dots, p_m, V , to maximize V such that

$$V \leq \sum_{i=1}^m p_i a_{i1}, V \leq \sum_{i=1}^m p_i a_{i2}, \dots, V \leq \sum_{i=1}^m p_i a_{in}$$

$$\implies 0 \leq \sum_{i=1}^m p_i a_{i1} - V, 0 \leq \sum_{i=1}^m p_i a_{i2} - V, \dots, 0 \leq \sum_{i=1}^m p_i a_{in} - V$$

$$\mathbf{p} \in \Delta_m \quad \boxed{p_1 \geq 0, p_2 \geq 0, \dots, p_m \geq 0, p_1 + p_2 + \dots + p_m = 1}$$

So

$$V = V_1 - V_2, V_1 \geq 0, V_2 \geq 0$$

$$p_1 + p_2 + \dots + p_m \geq 1$$

$$-p_1 - p_2 - \dots - p_m \geq -1$$

Example 14.4 (Colonel Blotto)

We have

	$(3, 0), q$	$(2, 1), q_2$	$(1, 2), q_2$	$(0, 3), q_4$
$(4, 0), p_1$	4	2	1	0
$(3, 1), p_2$	1	3	0	-1
$(2, 2), p_3$	-2	2	2	-2
$(1, 3), p_4$	-1	0	3	1
$(0, 4), p_5$	0	1	2	4

So

$$p_1 + p_2 + p_3 + p_4 + p_5 = 1$$

$$p_1 \geq 0, p_2 \geq 0, p_3 \geq 0, p_4 \geq 0, p_5 \geq 0$$

LP in the standard form

$$V = V_1 - V_2, V_1 \geq 0, V_2 \geq 0$$

$$-V \rightarrow \min \sim V_2 - V_1 \rightarrow \min \text{ s.t.}$$

$$\begin{cases} 4p_1 + p_2 - 2p_3 - p_4 - V_1 + V_2 \geq 0 \\ 2p_1 + 3p_2 + 2p_3 + p_5 - V_1 + V_2 \geq 0 \\ p_1 + 2p_3 + 3p_4 + 2p_5 - V_1 + V_2 \geq 0 \\ -p_2 - 2p_3 + p_4 + 4p_5 - V_1 + V_2 \geq 0 \\ p_1 + p_2 + p_3 + p_4 + p_5 = 1 \end{cases}$$

So

$$\left\{ p_1 = \frac{4}{9}, p_2 = 0, p_3 = \frac{1}{9}, p_4 = 0, p_5 = \frac{4}{9}, V_1 = \frac{14}{9}, V_2 = 0 \right\}$$

15 Apr 29, 2022

15.1 LP Duality

We have the problem

$$\begin{aligned} V = c^t x &\rightarrow \min \text{ s.t.} \\ Ax &\geq b \\ x &\geq 0 \end{aligned}$$

And the **Dual problem** is

$$\begin{aligned} w = b^t y &\rightarrow \max \text{ s.t.} \\ A^t y &\leq c \\ y &\geq 0 \end{aligned}$$

Lemma 15.1

Dual to dual = primary

Proof.

$$\begin{aligned} (-b)^t y &\rightarrow \min \text{ s.t.} \\ (-A)^t y &\geq -c \\ y &\geq 0 \end{aligned}$$

Next,

$$\begin{aligned} (-c)^t x &\rightarrow \max \text{ s.t.} \\ -Ax &\leq -b \\ x &\geq 0 \end{aligned}$$

Finally,

$$\begin{aligned} c^t x &\rightarrow \min \text{ s.t.} \\ Ax &\geq b \\ x &\geq 0 \end{aligned}$$

□

Lemma 15.2

Dual to LP for P1 = LP for P2

Proof. We have

$$\begin{aligned}
 a_{11}p_1 + a_{21}p_2 + \cdots + a_{m1}p_m - V_1 + V_2 &\geq 0 \\
 a_{12}p_1 + a_{22}p_2 + \cdots + a_{m2}p_m - V_1 + V_2 &\geq 0 \\
 &\vdots \\
 a_{1n}p_1 + a_{2n}p_2 + \cdots + a_{mn}p_m - V_1 + V_2 &\geq 0 \\
 -p_1 - p_2 + \cdots - p_m &\geq -1 \\
 p_1 + p_2 + \cdots + p_m &\geq 1
 \end{aligned}$$

With

$$p_1 \geq 0$$

$$p_2 \geq 0$$

$$\vdots$$

$$p_m \geq 0$$

$$v_1 \geq 0$$

$$v_2 \geq 0$$

$$p_1 + p_2 + \cdots + p_m = 1$$

So

$$V = V_1 - V_2$$

$$\mathbf{x}^t = (p_1, p_2, \dots, p_m, V_1, V_2)$$

$$V = V(p_1, p_2, \dots, p_m) \rightarrow \max$$

$$V_2 - V_1 \rightarrow \min$$

So

$$Ax = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} & -1 & 1 \\ a_{12} & a_{22} & \cdots & a_{m2} & -1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} & -1 & 1 \\ -1 & -1 & \cdots & -1 & 0 & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ v_1 \\ v_2 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

and

$$c^t = (\underbrace{0, \dots, 0}_m, -1, 1) \quad x \geq 0$$

So

$$(\mathbf{x})^t = (q_1, q_2, \dots, q_n, W_1, W_2)$$

So the objective function is

$$W_2 - W_1 \rightarrow \max \text{ s.t.}$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & -1 & 1 \\ a_{21} & a_{22} & \dots & a_{2n} & -1 & 1 \\ \vdots & \vdots & & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} & -1 & 1 \\ -1 & -1 & \dots & -1 & 0 & 0 \\ 1 & 1 & \dots & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \\ W_1 \\ W_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

with

$$q_1 \geq 0, W_1 \geq 0, W_2 \geq 0$$

Then

$$-q_1 - q_2 - \dots + q_n \leq -1$$

$$q_1 + q_2 + \dots + q_n \leq 1$$

$$\underline{q_1 + q_2 + \dots + q_n \geq 1}$$

$$q_1 + q_2 + \dots + q_n = 1$$

So

$$a_{11}q_1 + a_{12}q_2 + \dots + a_{m1}q_n \leq W = W_1 - W_2$$

$$\vdots$$

$$a_{m1}q_1 + a_{12}q_2 + \dots + a_{1n}q_n \leq W \rightarrow \min$$

Furthermore, we are maximizing

$$V(p_1, p_2, \dots, p_m)$$

and minimizing

$$W(q_1, q_2, \dots, q_n)$$

□

Theorem 15.3 (Weak Duality)

Let x be a feasible point for the primary problem:

$$V = c^t x \rightarrow \min \text{ subject to}$$

$$Ax \geq b, x \geq 0$$

Let y be a feasible point for the dual primary problem

$$W = b^t y \rightarrow \max \text{ subject to}$$

$$A^t y \leq c, y \geq 0.$$

Then

$$c^t x \geq b^t y$$

Proof. Write

$$c^t \geq y^t A$$

So

$$-V = c^t x \geq y^t A x \geq v^t b = -W$$

$$-V \geq -W \implies W \geq V$$

So

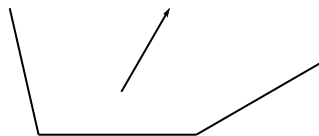
$$\boxed{\overbrace{\max_{p \in \Delta_m} \min_{q \in \Delta_n} p^t A q}^V \leq \overbrace{\min_{q \in \Delta_n} \max_{p \in \Delta_m} p^t A q}^W}$$

□

Corollary 15.4 • If the primary problem is unbounded, then the dual problem is infeasible.

- If the dual problem is unbounded, then the problem is infeasible.

Proof. $c^t x \geq b^t y$



□

16 May 2, 2022

16.1 Using LP Duality to Prove Von Neumann's Theorem

Recall 16.1 (Weak Duality Theorem) Let x be a feasible point for a linear program.

$$-V = c^t x \rightarrow \min \text{ subject to}$$

$$Ax \geq b, x \geq 0$$

where

$$x^t = (p_1, p_2, \dots, p_m, V_1, V_2)$$

$$V = V_1 - V_2$$

Let y be a feasible point for the dual LP

$$-W = b^t y \rightarrow \max \text{ subject to}$$

$$A^t y \leq c, y \geq 0$$

where

$$y^t = (q_1, q_2, \dots, q_n, W_1, W_2)$$

$$W = W_1 - W_2$$

Then

$$c^t x \geq b^t y$$

$$-V \geq -W$$

Corollary 16.2

If x is a feasible solution to the primary problem and y is a feasible solution to the dual problem $c^t x = b^t y$, then x and y are optimal.

Lemma 16.3

$\max_{p \in \Delta_m} \min_{q \in \Delta_n} p^t A q$ exists.

Proof.

$$\begin{aligned} \max_{p \in \Delta_m} \min_{q \in \Delta_n} p^t A q &= \max_{p \in \Delta_m} \min \left\{ \sum_{i=1}^m a_{ij} p_i, j = 1, \dots, n \right\} \\ &= \max_{\underbrace{p \in \Delta_m}_{\text{compact set}}} \underbrace{V(p_1, p_2, \dots, p_m)}_{\text{piecewise linear, hence continuous function}} \end{aligned}$$

□

Recall 16.4 (Weierstrass Extreme Value Theorem) A real-valued function continuous on a compact set attains its min and max on the set.

Theorem 16.5 (LP: Strong Duality Theorem)

If one of the two dual linear programs has an optimal solution, then so does the other and the values of the objective functions at the optimal points are equal.

16.2 General Sum Games

Prisoner's Dilemma

Two members of a criminal organization are arrested and put into solitary confinement. They have no means to communicate with each other. Prosecution lacks evidence to convict each on the principal charge. Prosecution has enough evidence to convict each on a lesser charge. Each prisoner is given a possibility to cooperate (Rat) or stay silent (Mum). The payoff bimatrix is given by

		S21	S2
	P1	R	M
	P2		
S11	R	(-8, -8)	(0, -10)
S12	M	(-10, 0)	(-1, 1)

$$A = \begin{bmatrix} -8 & 0 \\ -10 & -1 \end{bmatrix}, \quad B^t = \begin{bmatrix} -8 & -10 \\ 0 & -1 \end{bmatrix}$$

Dominant strategy: Whatever P2 chooses, P1 is better off

Definition 16.6 (Maxmin strategy)

Let A and B be $m \times n$ payoff matrices for P1 and P2.

$$V_1 = \max_{p \in \Delta_m} \min_{q \in \Delta_n} p^t A q = \max_{p \in \Delta_m} \min_{j=1, \dots, n} \sum_{i=1}^m p_i a_{ij} = \text{Val}(A)$$

where $\text{Val}(A) =$ safety level for P1. And

$$V_2 = \max_{q \in \Delta_n} \min_{p \in \Delta_m} p^t B q = \max_{q \in \Delta_n} \min_{i=1, \dots, m} \sum_{j=1}^n b_{ij} q_j = \text{Val}(B^t)$$

Noting that

$$p^t B q = q^t B^t p \quad V_2^t = V_2$$

Also note that safety level strategies disregard payoff for the other player. A strategy p that achieves $\text{Val}(A)$ for P1 is called a maxmin strategy.

Note 16.7: This game is symmetric if $A = B^t$.

Example 16.8

Recall our payoff matrix

P1/P2	R	M
R	-8, -8	0, -10
M	-10, 0	-1, -1

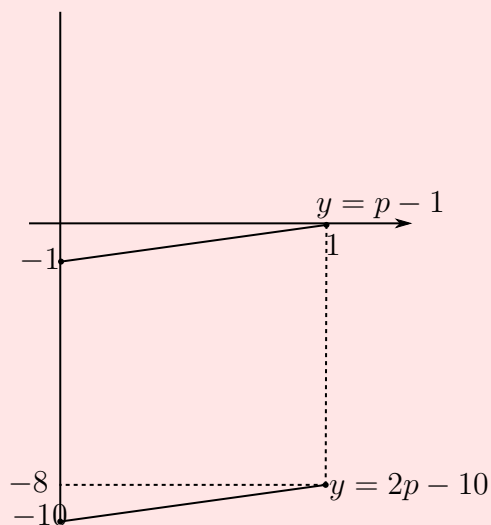
 \rightarrow

P1/P2	R	M
p	-8	0
$1-p$	-10	-1

The safety strategy is given by

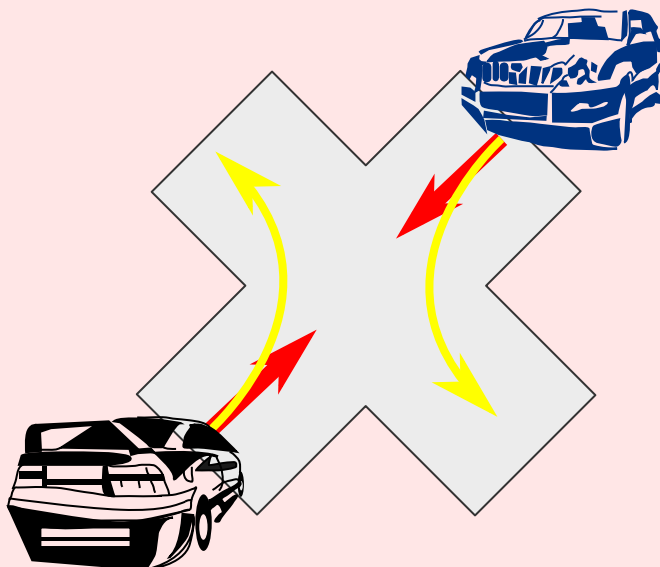
$$\begin{aligned} \text{Val}(A) &= \max_{0 \leq p \leq 1} \min\{-8p - 10(1-p), p-1\} = \max_{0 \leq p \leq 1} \min\{2p-10, p-1\} \\ &= \max_{0 \leq p \leq 1} (2p-10) = -8 \end{aligned}$$

$$p = 1, 1-p = 0$$



Example 16.9 (The Game of Chicken)

Two drivers speed head-on towards each other at crossroads.



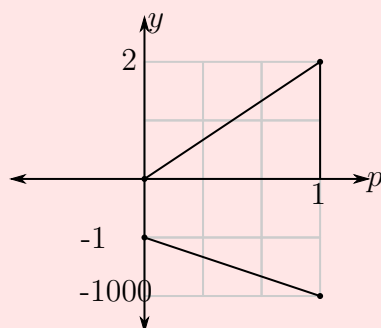
		P2	
		Drive	Chicken
P1	Drive, p	$-1000, 1000$	$2, -1$
	Chicken, $1 - p$	$-1, 2$	$0, 0$

$$\begin{aligned}\text{Val}(A) &= \max_{0 \leq p \leq 1} \min\{-1000p - (1 - p), 2p\} \\ &= \max_{0 \leq p \leq 1} \min\{-999p - 1, 2p\}\end{aligned}$$

So

$$p = 0, \quad 1 - p = 1$$

$$\text{Val}(A) = -1$$



17 May 4, 2022

17.1 Nash Equilibrium in General Sum Games

Recall 17.1 For an $m \times n$ matrix A ,

$$\text{Val}(A) = \max_{p \in \Delta_m} \min_{q \in \Delta_n} p^t A q$$

which is the safety level of P1.

Definition 17.2 (Nash Equilibrium (NE))

A pair (\hat{p}, \hat{q}) is called a Nash Equilibrium, if

$$* (\hat{p})^t A \hat{q} \geq p A \hat{q} \quad \forall p \in \Delta_m. \text{ i.e. } \hat{p} \text{ is the best response to } \hat{q}$$

$$** (\hat{p})^t B \hat{q} \geq \hat{p} B q \quad \forall q \in \Delta_n. \text{ i.e. } \hat{q} \text{ is the best response to } \hat{p}.$$

Definition 17.3 (Pure Nash Equilibrium (PNE))

A Pure Nash Equilibrium is a Nash Equilibrium such that $\hat{p} = e_i \in \mathbb{R}^m$ and $\hat{q} = e_j \in \mathbb{R}^n$, where

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{th position}$$

Note 17.4: Pure Nash Equilibriums are easy to find.

Example 17.5

We have

	S21	S22	S23	S24	S25	S26
S11	2,1	4,3	7*,2	7*,4	0,5*	3,2
S12	4*,0	5*,4	1,6*	0,4	0,3	5*,1
S13	1,3*	5*,3*	3,2	4,1	1*,0	4,3*
S14	4*,3	2,5*	4,0	1,0	1*,5*	2,1

So

$$\text{PNE} = \{(S13, S22), (S14, S25)\}$$

Lemma 17.6

Nash equilibria are “individually rational”, i.e. delivery at least the safety value.

Proof. Let p^* be a solution to the problem

$$\max_{p \in \Delta_n} \min_{q \in \Delta_n} p^t A q = \text{Val}(A)$$

i.e.

$$(p^*)^t A q \geq \text{Val}(A) \quad \forall q \in \Delta_n$$

So,

$$(\hat{p})^t A \hat{q} \geq (p^*)^t A q \geq \text{Val}(A)$$

$$(\hat{p})^t A \hat{q} \geq p A \hat{q}$$

□

Lemma 17.7

Nash Equilibria survive elimination of strongly dominated strategies.

Proof.

$$\left[\begin{array}{cc|cc} a_{11} & a_{12} & a_{1j} & a_{1n} \\ a_{21} & a_{22} & a_{2j} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \hline a_{i1} & a_{i2} & a_{ij} & a_{in} \\ \hline \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{mj} & a_{mn} \end{array} \right]$$

Suppose the i th row is strictly domination by other rows, then

$$\exists \alpha_1 \geq 0, \alpha_2 \geq 0, \dots, \alpha_{i-1} \geq 0$$

$$\alpha_{i+1} \geq 0, \dots, \alpha_m \geq 0$$

$$\sum_{\substack{k=1 \\ k \neq i}}^m \alpha_k = 1$$

and

$$a_{ij} < \sum_{\substack{k=1 \\ k \neq i}} \alpha_k a_{kj} \quad \text{for } j = 1, 2, \dots, n$$

Suppose that (\hat{p}, \hat{q}) is a Nash Equilibrium such that $\hat{p}_i \neq 0$. So

$$\begin{aligned}
 (\hat{p})^t A \hat{q} &= \sum_{i,j} a_{ij} \hat{p}_i \hat{q}_j = \hat{p}_1 \sum_{j=1}^n a_{1j} \hat{q}_j + \cdots + \hat{p}_i \sum_{j=1}^n a_{ij} \hat{q}_j + \cdots + \hat{p}_m \sum_{j=1}^n a_{mj} \hat{q}_j \\
 &< \hat{p}_1 \sum_{j=1}^n a_{1j} \hat{q}_j + \cdots + \hat{p}_i \sum_{j=1}^n \sum_{\substack{k=1 \\ k \neq i}}^m \alpha_k a_{kj} \hat{q}_j + \cdots + \hat{p}_m \sum_{j=1}^n a_{mj} \hat{q}_j \\
 &= \hat{p}_1 \sum_{j=1}^n a_{1j} \hat{q}_j + \cdots + \sum_{\substack{k=1 \\ k \neq i}}^n \hat{p}_i \alpha_k \sum_{j=1}^n a_{kj} \hat{q}_j + \cdots + \hat{p}_n \sum_{j=1}^n a_{nj} \hat{q}_j \\
 &= (\hat{p}_1 + \hat{p}_i \alpha_1) \sum_{j=1}^n a_{1j} \hat{q}_j + \cdots + 0 + \cdots + (\hat{p}_m + \hat{p}_i \alpha_n) \sum_{j=1}^n a_{mj} \hat{q}_j
 \end{aligned}$$

This way, the vector,

$$(p^*)^t = (\hat{p}_1 + \hat{p}_i \alpha_1, \hat{p}_2 + \hat{p}_i \alpha_2, \dots, 0, \dots, \hat{p}_m + \hat{p}_i \alpha_m)$$

is a better response to \hat{q} than

$$(\hat{p})^t = (\hat{p}_1, \hat{p}_2, \dots, \hat{p}_i, \dots, \hat{p}_m)$$

Thus, contrary to the assumption, $\hat{p}_i = 0$. □

18 May 6, 2022

18.1 Equalizing Strategies

P1 plays a strategy equalizing payoff of P2. P2 has no incentive to leave whatever strategy they are playing. A pair of equalizing strategies forms a Nash Equilibrium. We return back to the game of Chicken.

Example 18.1 (Game of Chicken (Cont'd))

We have two Pure Nash Equilibriums

		P2	
		D	C
P1	D, p	$-1000, -1000$	$2^*, -1^*$
	C, $1-p$	$-1^*, 2^*$	$0, 0$

So

$$\text{PNE} = \{(D, C), (C, D)\}$$

$$-1000p + 2(1-p) = -p$$

$$\implies 2 = 1001p$$

$$\implies \begin{cases} p = \frac{2}{1001} \\ 1-p = \frac{999}{1001} \end{cases}$$

So,

		P2		
		D, $\frac{2}{1001}$	C, $\frac{999}{1001}$	
P1	D, $\frac{2}{1001}$	$-1000, -1000$	$2^*, -1^*$	The expected payoff is $\frac{2}{1001}$.
	C, $\frac{999}{1001}$	$-1^*, 2^*$	$0, 0$	

Example 18.2 (Cheetahs and Antelopes)

S

- Two cheetahs chasing two antelopes, L and S
- Each cheetah can catch either antelope
- If catch same, have to share

		P2	
		L	S
P1	L	$\frac{\ell}{2}, \frac{\ell}{2}$	ℓ, s
	S	s, ℓ	$\frac{s}{2}, \frac{s}{2}$

and $\ell > s$.

Case 1: $\frac{\ell}{2} > s \implies \ell > 2s$ (Strict domination)

		P2	
		L	S
P1	L	$\frac{\ell^*}{2}, \frac{\ell^*}{2}$	ℓ^*, s
	S	s, ℓ^*	$\frac{s}{2}, \frac{s}{2}$

PNE: (L, L)

Case 2: $\frac{\ell}{2} = s \sim \ell = 2s$

		P2	
		L	S
P1	L	$\frac{\ell^*}{2}, \frac{\ell^*}{2}$	ℓ^*, s^*
	S	s^*, ℓ^*	$\frac{s}{2}, \frac{s}{2}$

PNE: $\{(L, L), (L, S), (S, L)\}$

Case 3: $s < \ell < 2s$

		P2	
		L	S
P1	L	$\frac{\ell}{2}, \frac{\ell}{2}$	ℓ^*, s^*
	S	s^*, ℓ^*	$\frac{s}{2}, \frac{s}{2}$

PNE: $\{(L, S), (S, L)\}$

Equalizing strategy for Cases 2 and 3:

$$s < \ell \leq 2s$$

		P2	
		L	S
P1	L, p	$\frac{\ell}{2}, \frac{\ell}{2}$	ℓ, s
	S, $1 - p$	s, ℓ	$\frac{s}{2}, \frac{s}{2}$

$$p\frac{\ell}{2} + (1-p)\ell = ps + (1-p)\frac{s}{2}$$

$$\frac{\ell}{2}p + \ell - \ell p = sp + \frac{s}{2} - \frac{s}{2}p$$

$$-\frac{\ell}{2}p + \ell = \frac{s}{2}p + \frac{s}{2}$$

$$\ell - \frac{s}{2} = \frac{\ell + s}{2}p$$

$$2\ell - s = (\ell + s)p$$

$$\hat{p} = \frac{2\ell - s}{\ell + s}, \quad 1 - \hat{p} = \frac{2s - \ell}{\ell + s}$$

$$\begin{aligned}
 \text{payoff} &= \frac{s}{2}(\hat{p} + 1) = \frac{s}{2} \left(\frac{2\ell - s}{\ell + s} + 1 \right) \\
 &= \frac{s}{2} \frac{2\ell - s + \ell + s}{\ell + s} \\
 &= \frac{3\ell s}{2(\ell + s)}
 \end{aligned}$$

So we have the payoff for the L strategy:

$$L(p) = \ell - \frac{\ell}{1}p; \quad S(p) = \frac{s}{2} + \frac{s}{2}p$$

$$\begin{aligned}
 \hat{p} &= \frac{2\ell - s}{\ell + s} \\
 p &< \hat{p}
 \end{aligned}$$

Evolutionary stability:

Let $0 \leq p \leq 1$ be the part of greedy cheetahs (always choose L) in the population.

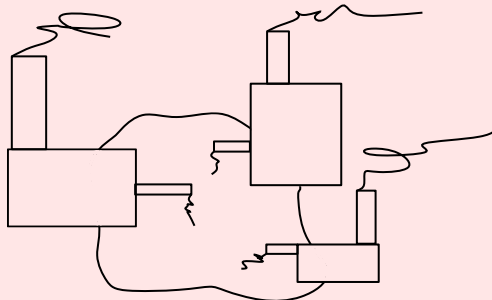
- For $0 \leq p < \hat{p}$, $L(p) > S(p)$, so greedy cheetahs have an advantage
- For $\hat{p} < p \leq 1$, $L(p) < S(p)$, so non-greedy cheetahs have an advantage.

So, evolution pushes p to \hat{p} .

18.2 General Sum Games with More Than Two Players

Example 18.3

We have



P3: Purify

P2

S21: purify S22: pollute

P1	S11	1,1,1	1,0,1
	S12	0,1,1	3,3,4

P3: pollute

P2

S21 S22

P1	S11	1,1,0	4,3,3
	S12	3,4,3	3,3,3

19 May 9, 2022

19.1 Review for Midterm 2

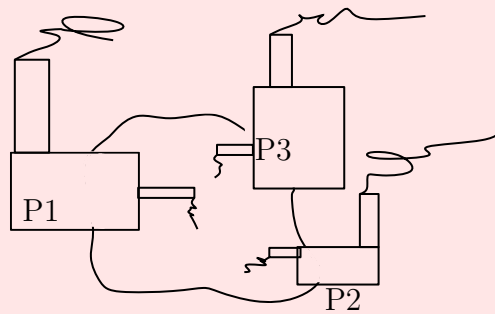
Questions include:

1. Dual problem
2. Pure Nash Equilibrium for general sum matrix
3. Word problem: Find all Nash Equilibrium (2 part problem)
4. Extra Credit

19.2 General Sum Games with More Than Two Players (Cont'd)

Example 19.1

We continue from Example 18.3.

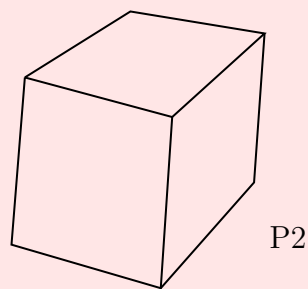


cost \rightarrow min

payoff \rightarrow max

payoff $= -\text{cost}$

P3



P1

P2

	P3 purifies	
	S21, P2 purifies	S22, P2 pollutes
S11, P1 purifies, p	1,1,1	1*,0*,1*
S12, P1 pollutes, $1-p$	0*,1*,1*	3,3,4

	P3 pollutes	
	S21, p_2	S22, $1-p_2$
S11	1*,1*,0	4,3,3
S12	3,4,3	3*,3*,3*

$$\begin{aligned}
\text{PNE} &= \{(1, 0, 1), (0, 1, 1), (1, 1, 0), (3, 3, 3)\} \\
&= \left\{ \left((1, 0), (0, 1), (1, 0) \right), \dots \right\} \\
&= \{(S11, S22, S31), \dots\}
\end{aligned}$$

Now

$P3$ purifies : $(1, 0)$

$$P1 : (p, 1-p) = \left(\frac{2}{3}, \frac{1}{3} \right)$$

$$1 = 3(1-p) \implies 3p = 2 \implies p = \frac{2}{3}$$

So

$$\text{NE} \stackrel{?}{=} \left(\left(\frac{2}{3}, \frac{1}{3} \right), \left(\frac{2}{3}, \frac{1}{3} \right), (1, 0) \right) \quad (*)$$

$$\stackrel{?}{=} \left(\left(\frac{2}{3}, \frac{1}{3} \right), \left(\frac{2}{3}, \frac{1}{3} \right), (0, 1) \right) \quad (**)$$

So for (*)

$$\text{Cost} = 1 \cdot \frac{2}{3} \cdot \frac{2}{3} + 1 \cdot \frac{2}{3} \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} \cdot \frac{2}{3} + 4 \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{12}{9} = \frac{4}{3}$$

and for (**)

$$\text{Cost} = \frac{6}{9} + \frac{6}{9} + \frac{3}{9} = \frac{15}{9} = \frac{5}{3}$$

So

$$\text{NE} = \left\{ \left((1, 0), \left(\frac{2}{3}, \frac{1}{3} \right), \left(\frac{2}{3}, \frac{1}{3} \right) \right), \left(\left(\frac{2}{3}, \frac{1}{3} \right), (1, 0), \left(\frac{2}{3}, \frac{1}{3} \right) \right), \left(\left(\frac{2}{3}, \frac{1}{3} \right), \left(\frac{2}{3}, \frac{1}{3} \right), (1, 0) \right) \right\}$$

Now

$$C(P1 : \text{purify}) = C(P2 : \text{pollute})$$

$$p_2 p_3 + (1-p_2)p_3 + p_2(1-p_3) + 4(1-p_2)(1-p_3) = 3(1-p_2)p_3 + 3p_2(1-p_3) + 3(1-p_2)(1-p_3)$$

20 May 11, 2022

20.1 Midterm 2

21 May 13, 2022

21.1 General Sum Games with More Than Two Players (Cont'd)

Recall 21.1 Recall from Example 19.1,

	P3 purifies	
	S21, P2 purifies	S22, P2 pollutes
S11, P1 purifies, p	1,1,1	1,0,1
S12, P1 pollutes, $1 - p$	0,1,1	3,3,4

	P3 pollutes	
	S21, p_2	S22, $1 - p_2$
S11	1,1,0	4,3,3
S12	3,4,3	3,3,3

Example 21.2 (Fully Mixed NE)

Continuing from the example,

$$C((1, 0), (p_2, 1 - p_2), (p_3, 1 - p_3)) = C((0, 1), (p_2, 1 - p_2), (p_3, 1 - p_3))$$

$$1 \cdot p_2 \cdot p_3 + 1 \cdot (1 - p_2)p_3 + 1 \cdot p_2 \cdot (1 - p_3) + 4(1 - p_2)(1 - p_3) = 0 \cdot p_2 \cdot p_3 + 3 \cdot (1 - p_2)p_3 + 3p_2(1 - p_3) + 3(1 - p_2)(1 - p_3)$$

$$p_2p_3 + (1 - p_2)p_3 + p_2(1 - p_3) + 4(1 - p_2)(1 - p_3) = 3(1 - p_2)p_3 + 3p_2(1 - p_3) + 3(1 - p_2)(1 - p_3)$$

$$1 = 3(p_2 + p_3 - 2p_2p_3) \quad (*)$$

$$1 = 3(p_1 + p_3 - 2p_1p_3) \quad (**)$$

$$1 = 3(p_1 + p_2 - 2p_1p_2) \quad (***)$$

$$(***) - (**) = 3(p_2 - p_3 + 2p_1p_3 - 2p_1p_2) = 0$$

$$p_2 - p_3 - 2p_1(p_2 - p_3) = 0$$

$$(p_2 - p_3)(1 - 2p_1) = 0$$

So

$$p_2 = p_3 \quad p_1 = \frac{1}{2}$$

By (*),

$$\begin{aligned}
 1 &= 3(2x - 2x^2) \\
 6x^2 - 6x + 1 &= 0 \\
 x &= \frac{6 \pm \sqrt{36 - 24}}{12} \\
 x &= \frac{6 \pm \sqrt{12}}{12} \\
 x &= \frac{3 \pm \sqrt{3}}{6}
 \end{aligned}$$

And by (**),

$$\begin{aligned}
 1 &= 3 \left(\frac{1}{2} + p_2 - 2 \cdot \frac{1}{2} \cdot p_2 \right) \\
 1 &= 3 \cdot \frac{1}{2} = \frac{3}{2} \\
 1 &= \frac{3}{2}
 \end{aligned}$$

Note,

$$\begin{aligned}
 0 &< \frac{3 - \sqrt{3}}{6}, \frac{3 + \sqrt{3}}{6} < 1 \\
 \frac{3 - \sqrt{3}}{6} + \frac{3 + \sqrt{3}}{6} &= 1
 \end{aligned}$$

Two more NE:

$$\left(\left(\frac{3 - \sqrt{3}}{6}, \frac{3 + \sqrt{3}}{6} \right), \dots \right), \left(\left(\frac{3 + \sqrt{3}}{6}, \frac{3 - \sqrt{3}}{6} \right), \dots \right)$$

So

$$\begin{aligned}
 C \left(\frac{\sqrt{3} + 3}{6}, \frac{\sqrt{3} + 3}{6} \right) &= 2 - \frac{\sqrt{3}}{2} \\
 C \left(\frac{3 - \sqrt{3}}{6}, \frac{3 - \sqrt{3}}{6} \right) &= 2 + \frac{\sqrt{3}}{2}
 \end{aligned}$$

Notation 21.3 (Multi-index Notations):

$$\begin{aligned}
 S1 &= (S11, S12, \dots, S1m_1) \\
 S2 &= (S21, S22, \dots, S2m_2) \\
 &\vdots \\
 Sn &= (Sn1, Sn2, \dots, Snm_n)
 \end{aligned}$$

A mixed strategy for \mathbf{P}_j :

$$(\mathbf{p}^j)^t = (p_1^j, p_2^j, \dots, p_{m_j}^j) \text{ s.t.}$$

$$p_1^j \geq 0, p_2^j \geq 0, \dots, p_{m_j}^j \geq 0, \quad \sum_{k=1}^{m_j} p_k^j = 1, \text{ i.e. } \mathbf{p}_j \in \Delta_{m_j}$$

Definition 21.4

The list $(\hat{\mathbf{p}}^1, \hat{\mathbf{p}}^2, \dots, \hat{\mathbf{p}}^n)$ is a Nash Equilibrium if

$$u(\hat{\mathbf{p}}^1, \hat{\mathbf{p}}^2, \dots, \hat{\mathbf{p}}^j, \dots, \hat{\mathbf{p}}^n) \geq u(\hat{p}^1, \hat{p}^2, \dots, \hat{p}^j, \dots, \hat{p}^n) \quad \forall \mathbf{p}^j \in \Delta_{m_j} \quad \forall j = 1, 2, \dots, n$$

where

$$u(\mathbf{p}^1, \mathbf{p}^2, \dots, \mathbf{p}^n) = \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} \cdots \sum_{i_n=1}^{m_n} p_{i_1}^1 p_{i_2}^2 \cdots p_{i_n}^n u(S1i_1, S2i_2, \dots, Sn i_n)$$

22 May 16, 2022

22.1 Nash Equilibrium Theorem

Theorem 22.1

Every finite general sum game has a NE (Nash Equilibrium).

Lemma 22.2 (The Indifference Principle)

Let $(\hat{p}^1, \hat{p}^2, \dots, \hat{p}^j, \dots, \hat{p}^n)$ be a NE and let $\hat{p}_{k_1}^j > 0, \hat{p}_{k_2}^j > 0, \hat{p}_{k_3}^j = 0$. Then

$$\begin{aligned} u(\hat{\mathbf{p}}^1, \hat{\mathbf{p}}^2, \dots, \underbrace{Sj k_1}_{=\mathbf{e}_{k_1} \in \mathbb{R}^{m_j}}, \dots, \hat{\mathbf{p}}^n) &= u(\hat{\mathbf{p}}^1, \hat{\mathbf{p}}^2, \dots, \underbrace{Sj k_2}_{=\mathbf{e}_{k_2} \in \mathbb{R}^{m_j}}, \dots, \hat{\mathbf{p}}^n) \\ &\geq u(\hat{\mathbf{p}}^1, \hat{\mathbf{p}}^2, \dots, \underbrace{Sj k_3}_{=\mathbf{e}_{k_3} \in \mathbb{R}^{m_j}}, \dots, \hat{\mathbf{p}}^n) \end{aligned}$$

$\forall j = 1, 2, \dots, n$

$\forall k_1, k_2, k_3$ s.t. $1 \leq k_1, k_2, k_3 \leq m_j$,

$k_1 \neq k_2 \neq k_3$

Proof.

$$\begin{aligned} u(\hat{\mathbf{p}}^1, \hat{\mathbf{p}}^2, \dots, \hat{\mathbf{p}}^j, \dots, \hat{\mathbf{p}}^n) &= \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} \cdots \sum_{i_{j-1}=1}^{m_{j-1}} \sum_{i_{j+1}=1}^{m_{j+1}} \cdots \sum_{i_n=1}^{m_n} \hat{p}_{i_1}^1 \hat{p}_{i_2}^2 \cdots \hat{p}_{i_{j-1}}^{j-1} \cdot \hat{p}_{i_{j+1}}^{j+1} \cdots \hat{p}_{i_n}^n \\ &\quad \cdot (\hat{p}_1^j u(*, Sj1, *) + \hat{p}_2^j u(*, Sj2, *) + \cdots + \hat{p}_{m_j}^j u(*, Sj m_j, *)) \end{aligned}$$

Moving a weight to a larger u increases the sum. □

Proof of Theorem 22.1. Step 1: The case of two players.

Let the payoff be given by two $m \times n$ matrices, A and B . Let

$$K = \Delta_m \times \Delta_n = \{(\mathbf{p}, \mathbf{q}) : \mathbf{p} \in \Delta_m, \mathbf{q} \in \Delta_n\}$$

Let us construct a continuous map

$$\begin{aligned} T: K &\rightarrow K \\ (p, q) &\mapsto (\hat{p}, \hat{q}) \end{aligned}$$

such that

* \hat{p} is a better response to q if such exists, $\hat{p} = p$ otherwise.

** \hat{q} is a better response to p if such exists, $\hat{q} = q$ otherwise.

$$c_i(\mathbf{p}, \mathbf{q}) := \max\{\mathbf{e}_i^t A \mathbf{q} - \mathbf{p}^t A \mathbf{q}, 0\}$$

where \mathbf{e}_i is the i -th standard basis vector in \mathbb{R}^m .

Note: $\mathbf{e}_i^t A \mathbf{q} - \mathbf{p}^t A \mathbf{q}$ shows the change in P1's gain if they switch from \mathbf{p} to $S1i$. Let

$$\hat{\mathbf{p}}^t = (\hat{p}_1, \hat{p}_2, \dots, \hat{p}_m)$$

where

$$\hat{p}_i = \frac{p_i + c_i}{1 + \sum_{K=1}^m c_K}$$

Note: $\sum_{i=1}^m \hat{p}_i = 1, \hat{p}_i \geq 0$.

Similarly, let

$$d_j(\mathbf{p}, \mathbf{q}) = \max\{\mathbf{p} B \mathbf{e}_j - \mathbf{p} B \mathbf{q}, 0\}$$

and

$$\hat{q}_j = \frac{q_j + d_j}{1 + \sum_{K=1}^n d_K}$$

Finally, let us define

$$T(\mathbf{p}, \mathbf{q}) = (\hat{\mathbf{p}}, \hat{\mathbf{q}})$$

Case 1: $c_i = 0$ for $i = 1, \dots, m$.

Then

$$\mathbf{p}^t A \mathbf{q} \geq \mathbf{e}_i^t A \mathbf{q} \text{ for } i = 1, \dots, m$$

and $\hat{\mathbf{p}} = \mathbf{p}$ is the best response to \mathbf{q} .

Case 2: Not all $c_i = 0$. Then

$$S = \sum_{i=1}^m c_i > 0$$

Need to show that

$$\begin{aligned} \sum_{i=1}^m \hat{p}_i \mathbf{e}_i^t A \mathbf{q} &> \mathbf{p}^t A \mathbf{q} \sim (1 + S) \sum_{i=1}^m \frac{p_i + c_i}{1 + S} \mathbf{e}_i^t A \mathbf{q} > \mathbf{p}^t A \mathbf{q} (1 + S) \\ &\sim \sum_{i=1}^m (p_i + c_i) \mathbf{e}_i^t A \mathbf{q} > (1 + S) \mathbf{p}^t A \mathbf{q} \\ &\sim \sum_{i=1}^m p_i \mathbf{e}_i^t A \mathbf{q} + \sum_{i=1}^m c_i \underbrace{\mathbf{e}_i^t A \mathbf{q}}_{\mathbf{p}^t A \mathbf{q}} > \mathbf{p}^t A \mathbf{q} + S \mathbf{p}^t A \mathbf{q} \\ &\sim \mathbf{p}^t A \mathbf{q} + \mathbf{p}^t A \mathbf{q} \underbrace{\sum_{i=1}^m c_i}_{=S} > \mathbf{p}^t A \mathbf{q} + S \mathbf{p}^t A \mathbf{q} \\ &\sim \mathbf{p}^t A \mathbf{q} S = S \mathbf{p}^t A \mathbf{q} \end{aligned}$$

Step 2: $m > 2$ players, let

$$c_i^j = \max\{u(\mathbf{p}^1, \dots, \mathbf{p}^{j-1}, \mathbf{e}_i, \mathbf{p}^{j+1}, \dots, \mathbf{p}^n) - u(\mathbf{p}^1, \dots, \mathbf{p}^{j-1}, \mathbf{p}^j, \mathbf{p}^{j+1}, \dots, \mathbf{p}^n), 0\}$$

$j = 1, 2, \dots, n$ and $i = 1, 2, \dots, m_j$. The rest of the proof does not change.

Step 3: The map

$$\begin{aligned} T: \Delta_{m_1} \times \Delta_{m_2} \times \dots \times \Delta_{m_n} &\rightarrow \Delta_{m_1} \times \Delta_{m_2} \times \dots \times \Delta_{m_n} \\ (\mathbf{p}^1, \mathbf{p}^2, \dots, \mathbf{p}^n) &\mapsto (\hat{p}_1, \hat{p}_2, \dots, \hat{p}_n) \end{aligned}$$

is continuous.

Step 4:

Theorem 22.3 (Brouwer's Fixed Point Theorem)

Let K be a closed, bounded, convex subset of \mathbb{R}^N . Let T be a continuous map $T: K \rightarrow K$. Then $\exists x \in K$ such that $T(x) = x$.

□

23 May 18, 2022

23.1 Games in Extensive Form

Definition 23.1 (Kuhn Tree) • Vertices \sim states of the game

- Directed edges \sim legal moves
- Weights of edges \sim probabilities
- Leaves \sim terminal states, marked with payoffs

Definition 23.2 (Noncooperative Games)

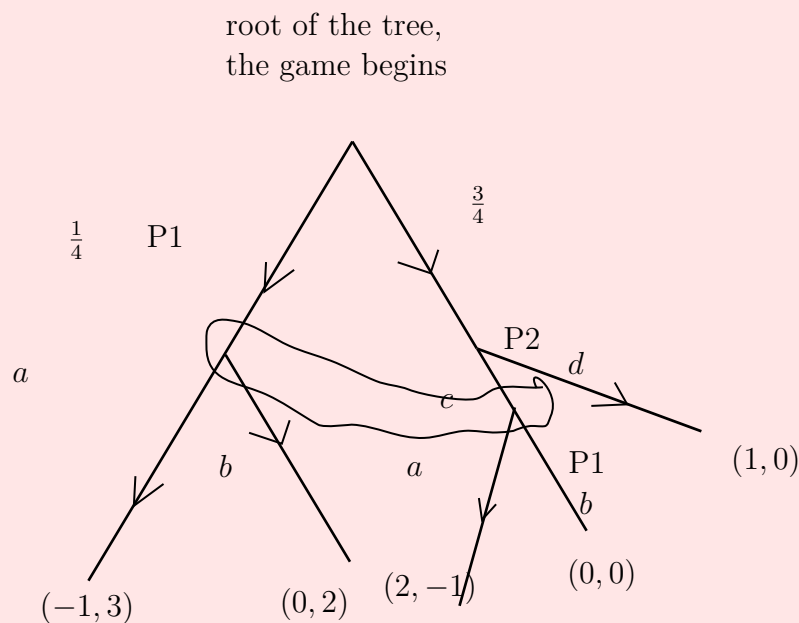
Players do not communicate before making moves.

Definition 23.3 (Cooperative Games)

Players can communicate before making moves.

Example 23.4

We have



Information set, describe incomplete information

Example 23.5 (Extensive \rightarrow Strategic)Case 1: $\frac{3}{4}$

		P2	
		c	d
P1	a	2, -1	1, 0
	b	0, 0	1, 0

Case 2: $\frac{1}{4}$

		P2	
		c	d
P1	a	-1, 3	-1, 3
	b	0, 2	0, 2

Expected payoffs:^a

		P2	
		c	d
P1	a	$\frac{5}{4}^*, 0$	$\frac{1}{2}, \frac{3}{4}^*$
	b	$0, \frac{1}{2}^*$	$\frac{3}{4}^*, \frac{1}{2}$

So

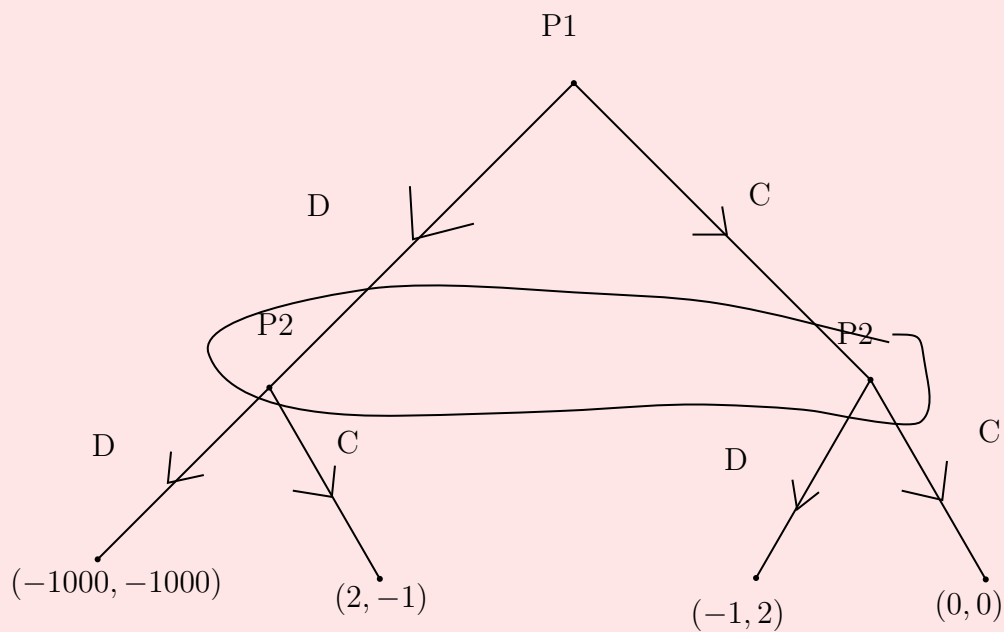
PNE : $\{(b, d)\}$

^aExpect to see on final

Example 23.6 (Game of Chicken (Strategic \rightarrow Extensive))

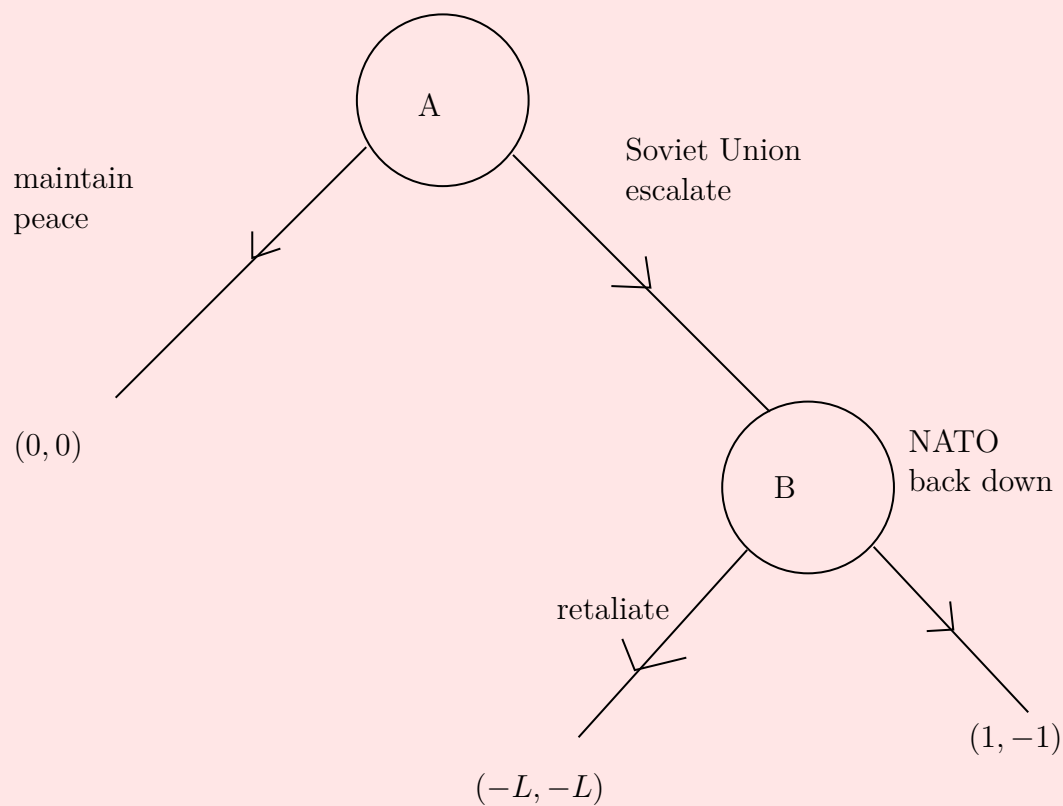
We have

		P2	
		Drive	Chicken
P1	Drive	$-1000, -1000$	$2, -1$
	Chicken	$-1, 2$	$0, 0$

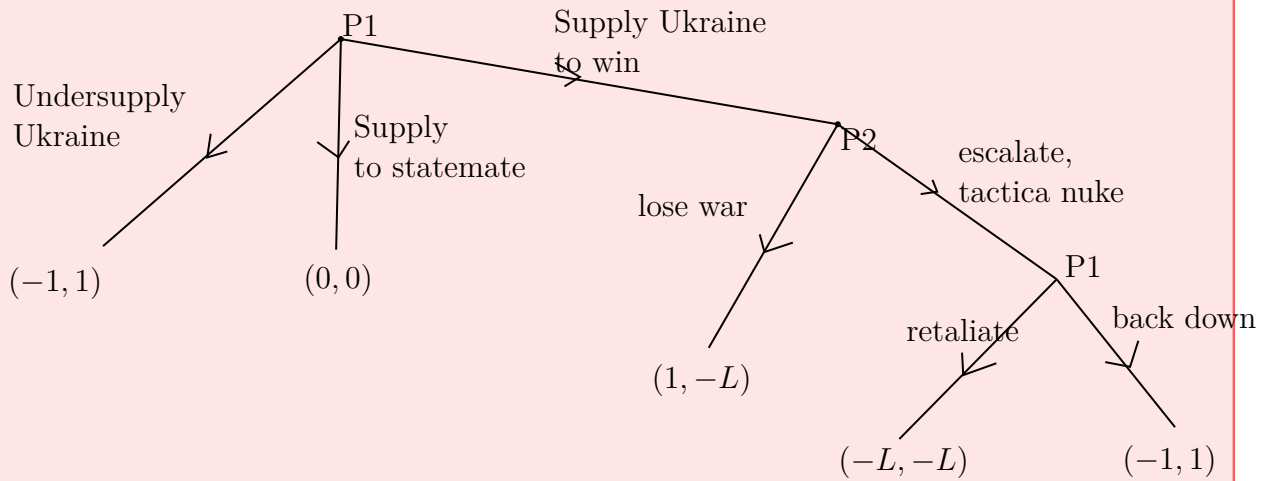


Example 23.7 (MAD (Mutually Assured Destruction))

We have



		B	
		retaliate	back down
A	peace	$0^*, 0^*$	$0, 0^*$
	escalate	$-L, -L$	$1^*, -1^*$

Example 23.8 $P1 \sim \text{West}, P2 \sim \text{Russia}$ 

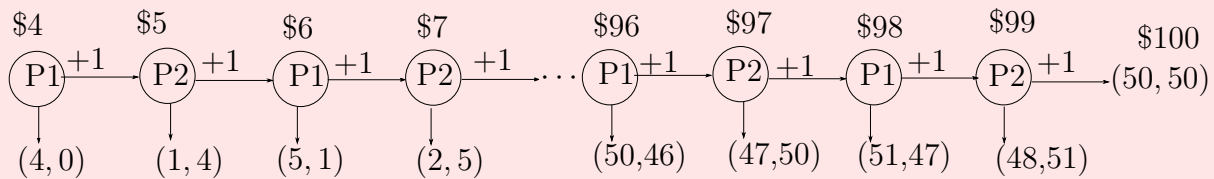
24 May 20, 2022

24.1 Games in Extensive Form (Cont'd)

Example 24.1 (The Centipede Game)

The Centipede Game is as follows:

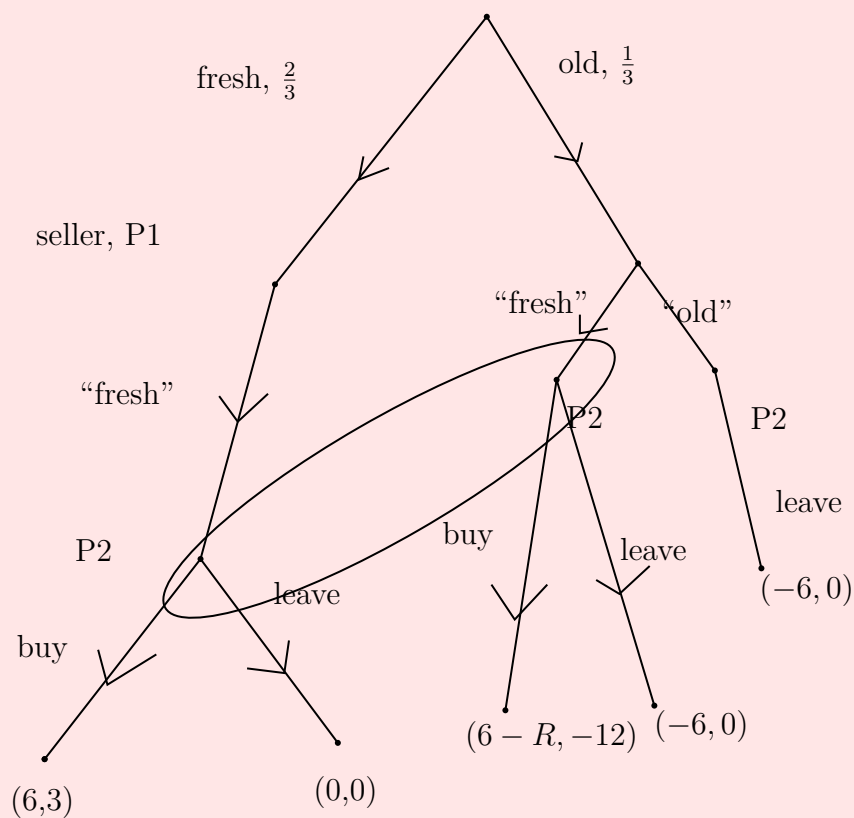
- Initially, there are \$4 on the table
- There are two players, alternating moves
- When there are \$ n on the table, the player to move can take $\lfloor \frac{n+4}{2} \rfloor$ or choose to continue. In the first case, the other player takes the rest of the pot.
- If a player chooses to continue, \$1 is added to the pot
- If the pot reaches \$100, then each player gets \$50 and the game ends



Example 24.2 (Fish-Selling Game)

The game is as follows:

- Fish sold at the market is either fresh with $p = \frac{2}{3}$ or old with $1 - p = \frac{1}{3}$.
- The seller knows whether the fish he is selling is fresh or old.
- The customer asks whether the fish is fresh or old. The seller answers.
- Based on the answer, the customer decides whether to buy or leave without buying
- The selling price of the fish is \$12. The value of the fish for the customer is \$15 if the fish is fresh and \$0 if the fish is old.
- The seller bought the fish for \$6. If the fish is fresh, he can sell it to another seller for \$6. If the fish is old, he can throw it in trash.
- If the seller sells the old fish as fresh, he gains \$6, but he loses \$ R in reputation.



Strategies for P1

S11: always say "fresh"

S12: say the truth

Strategies for P2

S21: buy if "fresh"

S22: leave

Case 1: $p = \frac{2}{3}$, the fish is fresh

		P2	
		S21 = buy if "fresh"	S22 = leave
P1	S11 = always "fresh"	6, 3	0, 0
	S12 = truth	6, 3	0, 0

Case 2: $1 - p = \frac{1}{3}$, old fish

		P2	
		buy	leave
P1	"fresh"	$6 - R, -12$	-6, 0
	truth	-6, 0	-6, 0

So the expected payoff bimatrix is:

		P2	
		buy	leave
P1	“fresh”	$6 - \frac{R}{3}, -2$	$-2, 0$
	truth	$2, 2$	$-2, 0$

PNEs:

$$6 - \frac{R}{3} \leq 2$$

$$12 \leq R$$

Case 1: $R < 12$

		P2	
		buy	leave
P1	“fresh”	$6 - \frac{R^*}{3}, -2$	$-2^*, 0^*$
	truth	$2, 2^*$	$-2^*, 0$

Case 2: $R = 12$

		P2	
		buy	leave
P1	“fresh”	$6 - \frac{R^*}{3}, -2$	$-2^*, 0^*$
	truth	$2^*, 2^*$	$-2^*, 0$

Case 3: $R > 12$

		P2	
		buy	leave
P1	“fresh”	$6 - \frac{R}{3}, -2$	$-2^*, 0^*$
	truth	$2^*, 2^*$	$-2^*, 0$

Case 1: $0 \leq R < 12$

PNE = (always “fresh”, leave)

Payoff = $(-2, 0)$

Conclusion: low reputation, price = (lie when needed, walk away)

Case 2: $R \geq 12$

PNEs = {(always “fresh”, leave), (truth, buy if fresh)}

Payoffs: $(-2, 0)$ and $(2, 2)$

Conclusion: High reputation, price = \exists a better PNE

		Mixed NE	
		q	$1 - q$
p	p	$6 - \frac{R}{3}, -2$	$-2, 0$
	$1 - p$	$2, 2$	$-2, 0$

$$-2p + 2(1 - p) = 0$$

$$4p = 2$$

$$\boxed{p = \frac{1}{2}}$$

$$\left(6 - \frac{R}{3}\right)q - 2(1 - q) = 2q - 2(1 - q)$$

$$4q - \frac{R}{3}q = 0 \sim (12 - R)q = 0$$

($R = 12$ not likely)

$$\text{NE} = \left(\left(\frac{1}{2}, \frac{1}{2} \right), (0, 1) \right)$$

$$\text{Payoffs} = (-2, 0)$$

25 May 23, 2022

25.1 Evolutionary Stable Strategies

Definition 25.1 (Starting point)

Population with n types of behavior of frequencies p_1, p_2, \dots, p_n . A small mutation introduces a new mix

$$\mathbf{q} = (q_1, q_2, \dots, q_n)$$

For $\varepsilon > 0$,

$$(1 - \varepsilon)\mathbf{p} + \varepsilon\mathbf{q}$$

Question 25.2: How does the old population fare against the new one?

$$B = A^t$$

$$\begin{array}{c} p_1 \\ p_2 \\ \vdots \\ p_n \end{array} \begin{array}{c} (1 - \varepsilon)\mathbf{p} + \varepsilon\mathbf{q} \\ A \end{array} = (\mathbf{p})^t A [(1 - \varepsilon)\mathbf{p} + \varepsilon\mathbf{q}] = (1 - \varepsilon)(\mathbf{p})^t A \mathbf{p} + \varepsilon(\mathbf{p})^t A \mathbf{q}$$

$$\begin{array}{c} q_1 \\ q_2 \\ \vdots \\ q_n \end{array} \begin{array}{c} (1 - \varepsilon)\mathbf{p} + \varepsilon\mathbf{q} \\ A \end{array} = (\mathbf{q})^t A [(1 - \varepsilon)\mathbf{p} + \varepsilon\mathbf{q}] = (1 - \varepsilon)(\mathbf{q})^t A \mathbf{p} + \varepsilon(\mathbf{q})^t A \mathbf{q}$$

Note:

$$(\mathbf{p})^t A \mathbf{p} \geq (\mathbf{q})^t A \mathbf{p}$$

Condition 1: \mathbf{p} is a symmetric NE.

$\mathbf{q} = \mathbf{e}_i$ such that $p_i > 0$

$$(\mathbf{p})^t A \mathbf{p} = (\mathbf{e}_i)^t A \mathbf{p}$$

Condition 2: $(\mathbf{p})^t A \mathbf{q} > (\mathbf{q})^t A \mathbf{q}$

Lemma 25.3

A symmetric n -players general sum game has a symmetric NE.

Proof, Professor teased us about this being on the exam, maybe it will? Recall

$$\begin{aligned} T: \Delta_m \times \Delta_n &\rightarrow \Delta_m \times \Delta_n \\ (\mathbf{p}, \mathbf{q}) &\mapsto (\hat{\mathbf{p}}, \hat{\mathbf{q}}) \end{aligned}$$

where

$$\hat{p}_i = \frac{p_i + c_i}{1 + \sum_{n=1}^n c_i} \quad \hat{q}_i = \frac{q_i + d_i}{1 + \sum_{n=1}^n d_i}$$

and

$$c_i = \max\{(\mathbf{e}_i)^t A \mathbf{q} - (\mathbf{p})^t A \mathbf{q}, 0\}$$

$$d_i = \max\{\mathbf{p}^t A^t \mathbf{q} - (\mathbf{p})^t A^t \mathbf{q}, 0\}$$

Game is symmetric, i.e. $B = A^t$, $\mathbf{p} = \mathbf{q}$. We need to show

$$(\mathbf{e}_i)^t A \mathbf{p} - (\mathbf{p})^t A \mathbf{p}$$

equals

$$\mathbf{p}^t A^t \mathbf{e}_i - (\mathbf{p})^t A^t \mathbf{p}$$

Which is true.

$$\text{Diagonal } D = \{(\mathbf{p}, \mathbf{p}), \mathbf{p} \in \Delta_n\}$$

$$\underbrace{\Delta_n \times \Delta_n \times \cdots \times \Delta_n}_{k \text{ times}}$$

Now

$$T: D \rightarrow D$$

$$(\mathbf{p}, \mathbf{p}) \mapsto (\hat{\mathbf{p}}, \hat{\mathbf{q}})$$

And

$$(\mathbf{p}, \mathbf{q}) \mapsto (\hat{\mathbf{p}}, \hat{\mathbf{q}})$$

□

Example 25.4 (Hawks and Doves)

We have a game of Hawks and Doves

- Species with two behavior types, aggressive (hawks), and aggression averse (doves).
- V = Victory's value
- C = cost of fight

		P2	
		H	D
P1	p, H	$\frac{v}{2} - c, \frac{v}{2} - c$	$v, 0$
	$1 - p, D$	$0, v$	$\frac{v}{2}, \frac{v}{2}$