

Math 132 (Complex Analysis for Applications)

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These are my lecture notes for Math 132 (Complex Analysis for Applications), which is taught by Tyler James Arant. The textbook for this class is *Complex Analysis*, by Theodore W. Gamelin.

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1 Jan 3, 2022

1.1 What are the Complex Numbers?

We first recall the basic algebraic properties of the real numbers, \mathbb{R} . For all $a, b, c \in \mathbb{R}$,

1. (Commutative law of addition): $a + b = b + a$
2. (Commutative law of multiplication): $a \cdot b = b \cdot a$
3. (Associative law of addition): $(a + b) + c = a + (b + c)$
4. (Associative law of multiplication): $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
5. (Distributive law): $a(b + c) = a \cdot b + a \cdot c$

The system of real numbers \mathbb{R} has many more (non-algebraic) properties which make it suitable for calculus. However, it lacks a particular desirable property: \mathbb{R} does not contain roots for all of its polynomial equations, e.g., there is not a solution to the equation

$$x^2 + 1 = 0 \quad \text{in } \mathbb{R}.$$

It turns out (by the non-trivial fundamental theorem of algebra) that we can get a number system for which every polynomial equation has a root by "appending" $i = \sqrt{-1}$ to \mathbb{R} .

Definition 1.1 (Complex number)

A complex number is an expression of the form

$$x + iy \quad \text{where } x, y \in \mathbb{R},$$

Two complex numbers $a + ib$ and $c + id$ are equal if and only if $a = c$ and $b = d$

We denote by \mathbb{C} the set of all complex numbers.

For a complex number $z = x + iy$, we define its real and imaginary parts as follows:

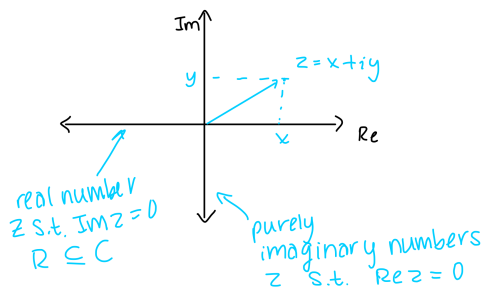
$$\operatorname{Re} z = x$$

$$\operatorname{Im} z = y$$

There is a one-to-one correspondence between \mathbb{C} and \mathbb{R}^2 :

$$z \mapsto (\operatorname{Re} z, \operatorname{Im} z)$$

This can be visualized as the *complex plane*, where we can identify the real numbers and the *purely imaginary numbers*.



Example 1.2 (Addition and multiplication on \mathbb{C})

We can define operations of addition and multiplication on \mathbb{C} as follows:

$$z = x + iy, \quad w = a + ib$$

$$z + w = (x + iy) + (a + ib) = (x + a) + i(y + b)$$

$$zw = (x + iy)(a + ib) = xa + ixb + iya + i^2yb$$

$$= (xa - yb) + i(xb + ya)$$

Example 1.3 (Multiplicative inverse in \mathbb{C})

Every nonzero complex number $z = x + iy \neq 0$ has a multiplicative inverse,

$$\frac{1}{z} = \frac{x - iy}{x^2 + y^2}.$$

Need to check $z \cdot \frac{1}{z} = 1$

$$z \cdot \frac{1}{z} = (x + iy) \left(\frac{x - iy}{x^2 + y^2} \right) = \left(\frac{x^2 - ixy + ixy - i^2y^2}{x^2 + y^2} \right) = \left(\frac{x^2 + y^2}{x^2 + y^2} \right) = 1$$

In addition to having additive and multiplicative inverses, the complex numbers also have the following algebraic properties:

For all $z_1, z_2, z_3 \in \mathbb{C}$,

1. (Commutative law of addition): $z_1 + z_2 = z_2 + z_1$
2. (Commutative law of multiplication): $z_1 \cdot z_2 = z_2 \cdot z_1$
3. (Associative law of addition): $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
4. (Associative law of multiplication): $z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$
5. (Distributive law): $z_1(z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$

1.2 Complex Conjugates and the Modulus

Definition 1.4 (Complex conjugate)

The complex conjugate of the number $z = x + iy$ is the number

$$\bar{z} = x - iy.$$

Some basic facts about complex conjugation. All are simple to prove, so we only discuss the proof of a few.

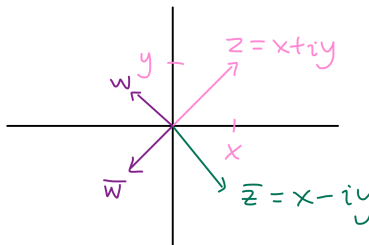
- $\bar{\bar{z}} = z$
- $z = \bar{z}$ if and only if z is a real number
- $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
- $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$
- $\overline{\left(\frac{1}{z}\right)} = 1/\bar{z}$

Proof. We want to show that $\overline{\left(\frac{1}{z}\right)} = 1/\bar{z}$.

$$\frac{1}{\bar{z}} = \frac{1}{x - iy} = \frac{x - (-iy)}{x^2 + y^2} = \frac{x + iy}{x^2 + y^2} = \overline{\left(\frac{x - iy}{x^2 + y^2}\right)} = \overline{\left(\frac{1}{z}\right)}$$

□

Geometrically, conjugation reflects z across the real axis:

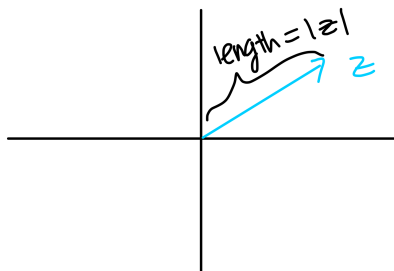


Definition 1.5 (Absolute value/modulus)

The absolute value or modulus of $z = x + iy$ is

$$|z| = \sqrt{x^2 + y^2}$$

Geometrically, $|z|$ is the length of z as a vector in the complex plane:



Some properties relating complex conjugation and absolute value:

- $|z|^2 = z\bar{z}$

$$\begin{aligned} z\bar{z} &= (x + iy)(x - iy) = x^2 - ixy + ixy - i^2y^2 \\ &= x^2 + y^2 \\ &= |z|^2 \end{aligned}$$

- $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$

$$\frac{1}{z} = \frac{x - iy}{x^2 + y^2} = \frac{\bar{z}}{|z|^2}$$

- We have

$$\begin{aligned} \operatorname{Re} z &= \frac{z + \bar{z}}{2}, \quad \operatorname{Im} z = \frac{z - \bar{z}}{2i} \\ \frac{z + \bar{z}}{2} &= \frac{x + iy + x - iy}{2} = \frac{2x}{2} = x \end{aligned}$$

Note: $\frac{1}{i} = -i$

- For $z, w \in \mathbb{C}$, $|zw| = |z| \cdot |w|$.

$$\begin{aligned}|z|^2 \cdot |w|^2 &= z\bar{z} \cdot w\bar{w} = (zw)(\bar{z} \cdot \bar{w}) \\ &= (zw)\overline{(zw)} = |zw|^2\end{aligned}$$

Then take a square root.

2 Jan 5, 2022

2.1 Distance in the Complex Plane

We can use absolute value to measure the distance between complex numbers (thought of as vectors in the complex plane).

The distance between complex numbers z_1, z_2 is $|z_1 - z_2|$:

$$z_1 = x_1 + iy_1, \quad z_2 = x_2 + iy_2$$

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

A crucial fact about working with absolute values is

Proposition 2.1 (Triangle Inequality)

For any two complex numbers z_1, z_2 ,

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Some corollaries of the Triangle Inequality:

Corollary 2.2

For any complex numbers z_1, z_2, w ,

- $|z_1 - z_2| \leq |z_1 - w| + |w - z_2|$.
- $|z_2| - |z_1| \leq |z_2 - z_1|$ (Reverse triangle inequality)

Proof.

•

$$\begin{aligned} |z_1 - z_2| &= |(z_1 - w) + (w - z_2)| \\ &\leq |z_1 - w| + |w - z_2| \end{aligned} \quad (\text{Triangle Inequality})$$

•

$$\begin{aligned} |z_2| &= |(z_2 - z_1) + z_1| \leq |z_2 - z_1| + |z_1| \\ \implies |z_2| - |z_1| &\leq |z_2 - z_1| \end{aligned}$$

By symmetry, also $|z_1| - |z_2| \leq |z_2 - z_1|$

□

2.2 Complex Polynomials

Definition 2.3 (Complex polynomial)

A complex polynomial of degree $n \geq 0$ is a function of the form

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 + a_0, \quad z \in \mathbb{C},$$

where the coefficients a_0, a_1, \dots, a_n are complex numbers with $a_n \neq 0$.

Theorem 2.4 (Fundamental Theorem of Algebra)

Every complex polynomial $p(z)$ of degree $n \geq 1$ has a factorization

$$p(z) = c(z - z_1)^{m_1}(z - z_2)^{m_2} \cdots (z - z_k)^{m_k},$$

where $c \in \mathbb{C}$, the roots z_1, \dots, z_n are distinct complex numbers, and $m_1, \dots, m_k \geq 1$. This factorization is unique, up to permutation of the factors. Also note that

$$\sum_{j=1}^k m_k = n = \text{the degree of the polynomial}$$

Example 2.5

Consider $p(z) = iz^2 + i$.

$$p(z) = iz^2 + i = i(z^2 + 1) = i(z - i)(z + i)$$

2.3 Polar representation

A nonzero complex number $z \in \mathbb{C}$ can be described by two quantities:

- its length $|z| = r$
- The angle z makes with the positive real axis.

Definition 2.6 (Polar representation)

The polar representation of $z \neq 0$ is:

$$z = r(\cos(\theta) + i \sin(\theta))$$

Given $z = r(\cos(\theta) + i \sin(\theta))$ we can recover the Cartesian coordinates for z :

$$x = r \cos(\theta)$$

$$y = r \sin(\theta)$$

Definition 2.7 (Argument)

For a nonzero $z = r(\cos(\theta) + i \sin(\theta))$, the angle θ is called the argument of z , and is denoted $\theta = \arg z$. But the argument of z , $\arg z$, is actually a multivalued function:

$$\arg z + 2\pi k, \quad k \text{ an integer}$$

also represents the same angle

Definition 2.8 (Principal argument)

The principal argument of $z \neq 0$, denoted $\text{Arg } z$, is the unique argument θ which is in $(-\pi, \pi]$.

$$\arg z = \{\text{Arg } z + 2k\pi : k \text{ integer}\}$$

"multivalued function"

$\text{Arg } z$ is single-valued

Example 2.9

Consider the complex number $1 + i$.

$$\begin{aligned}|1 + i| &= \sqrt{1^2 + 1^2} = \sqrt{2} \\ 1 + i &= \sqrt{2} \left(\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) \right) \\ \text{Arg}(1 + i) &= \frac{\pi}{4}\end{aligned}$$

We introduce the notation

$$e^{i\theta} = \cos(\theta) + i \sin(\theta) \quad \text{for } \theta \in \mathbb{R}.$$

We will see that this equality does follow from how we define complex exponentiation, but in the meantime it is just some convenient notation since the polar form for $z \in \mathbb{C}$ becomes:

$$\begin{aligned}z \neq 0, z &= re^{i\theta}, \\ \text{where } \theta &= \text{Arg}z, r = |z|\end{aligned}$$