Math 132 (Complex Analysis for Applications)
University of California, Los Angeles

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These are my lecture notes for Math 132 (Complex Analysis for Applications), which is taught by Tyler James Arant. The textbook for this class is *Complex Analysis*, by Theodore W. Gamelin.

Contents

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1.1 What are the Complex Numbers?

We first recall the basic algebraic properties of the real numbers, $\mathbb{R}$. For all $a, b, c \in \mathbb{R}$,

1. (Commutative law of addition): $a + b = b + a$
2. (Commutative law of multiplication): $a \cdot b = b \cdot a$
3. (Associative law of addition): $(a + b) + c = a + (b + c)$
4. (Associative law of multiplication): $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
5. (Distributive law): $a(b + c) = a \cdot b + a \cdot c$

The system of real numbers $\mathbb{R}$ has many more (non-algebraic) properties which make it suitable for calculus. However, it lacks a particular desirable property: $\mathbb{R}$ does not contain roots for all of its polynomial equations, e.g., there is not a solution to the equation

$$x^2 + 1 = 0 \quad \text{in } \mathbb{R}.$$

It turns out (by the non-trivial fundamental theorem of algebra) that we can get a number system for which every polynomial equation has a root by "appending" $i = \sqrt{-1}$ to $\mathbb{R}$.

Definition 1.1 (Complex number)
A complex number is an expression of the form

$$x + iy \quad \text{where } x, y \in \mathbb{R},$$

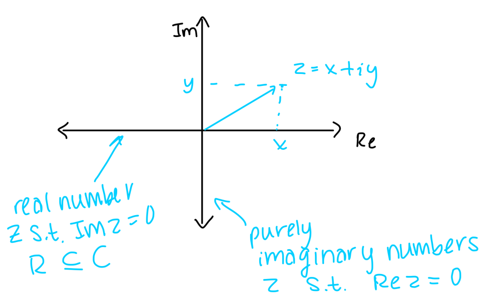
Two complex numbers $a + ib$ and $c + id$ are equal if and only if $a = c$ and $b = d$
We denote by $\mathbb{C}$ the set of all complex numbers.
For a complex number $z = x + iy$, we define its real and imaginary parts as follows:

$$\operatorname{Re} z = x$$
$$\operatorname{Im} z = y$$

There is a one-to-one correspondence between $\mathbb{C}$ and $\mathbb{R}^2$:

$$z \mapsto (\operatorname{Re} z, \operatorname{Im} z)$$

This can be visualized as the *complex plane*, where we can identify the real numbers and the *purely imaginary numbers*.



Example 1.2 (Addition and multiplication on $\mathbb{C}$)
We can define operations of addition and multiplication on $\mathbb{C}$ as follows:

$$z = x + iy, \quad w = a + ib$$

$z + w = (x + iy) + (a + ib) = (x + a) + i(y + b)$
$zw = (x + iy)(a + ib) = xa + ixb + iya + i^2yb$
$= (xa - yb) + i(xb + ya)$

Example 1.3 (Multiplicative inverse in $\mathbb{C}$)
Every nonzero complex number $z = x + iy \neq 0$ has a multiplicative inverse,

$$\frac{1}{z} = \frac{x - iy}{x^2 + y^2}.$$

Need to check $z \cdot \frac{1}{z} = 1$

$$z \cdot \frac{1}{z} = (x + iy)\left(\frac{x - iy}{x^2 + y^2}\right) = \left(\frac{x^2 - ixy + ixy - i^2y^2}{x^2 + y^2}\right) = \left(\frac{x^2 + y^2}{x^2 + y^2}\right) = 1$$

In addition to having additive and multiplicative inverses, the complex numbers also have the following algebraic properties:
For all $z_1, z_2, z_3 \in \mathbb{C}$,

1. (Commutative law of addition): $z_1 + z_2 = z_2 + z_1$
2. (Commutative law of multiplication): $z_1 \cdot z_2 = z_2 \cdot z_1$
3. (Associative law of addition): $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
4. (Associative law of multiplication): $z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$
5. (Distributive law): $z_1(z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$

1.2 Complex Conjugates and the Modulus

Definition 1.4 (Complex conjugate)
The complex conjugate of the number $z = x + iy$ is the number

$$\overline{z} = x - iy.$$

Some basic facts about complex conjugation. All are simple to prove, so we only discuss the proof of a few.

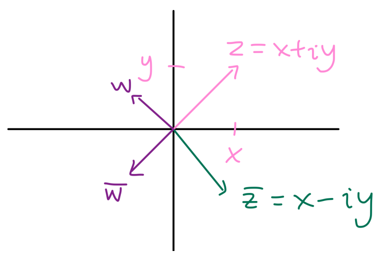
- $\overline{\overline{z}} = z$
- $z = \overline{z}$ if and only if z is a real number
- $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$
- $\overline{z_1 z_2} = \overline{z}_1 \cdot \overline{z}_2$
- $\overline{\left(\frac{1}{z}\right)} = 1/\overline{z}$

Proof. We want to show that $\overline{\left(\frac{1}{z}\right)} = 1/\overline{z}$.

$$\frac{1}{\overline{z}} = \frac{1}{x-iy} = \frac{x-(-iy)}{x^2+y^2} = \frac{x+iy}{x^2+y^2} = \overline{\left(\frac{x-iy}{x^2+y^2}\right)} = \overline{\left(\frac{1}{z}\right)}$$

□

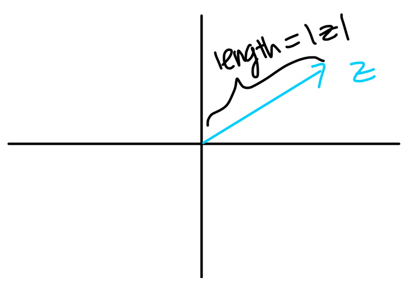
Geometrically, conjugation reflects z across the real axis:



Definition 1.5 (Absolute value/modulus)
The absolute value or modulus of $z = x + iy$ is

$$|z| = \sqrt{x^2+y^2}$$

Geometrically, $|z|$ is the length of z as a vector in the complex plane:



Some properties relating complex conjugation and absolute value:

- $|z|^2 = z\overline{z}$

$$\begin{aligned} z\overline{z} = (x+iy)(x-iy) &= x^2 - ixy + ixy - i^2y^2 \\ &= x^2+y^2 \\ &= |z|^2 \end{aligned}$$

- $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$

$$\frac{1}{z} = \frac{x-iy}{x^2+y^2} = \frac{\overline{z}}{|z|^2}$$

- We have

$$\operatorname{Re} z = \frac{z+\overline{z}}{2}, \quad \operatorname{Im} z = \frac{z-\overline{z}}{2i}$$

$$\frac{z+\overline{z}}{2} = \frac{x+iy+x-iy}{2} = \frac{2x}{2} = x$$

Note: $\frac{1}{i} = -i$

- For $z, w \in \mathbb{C}, |zw| = |z| \cdot |w|$.

$$\begin{aligned} |z|^2 \cdot |w|^2 = z\overline{z} \cdot w\overline{w} &= (zw)(\overline{z} \cdot \overline{w}) \\ &= (zw)\overline{(zw)} = |zw|^2 \end{aligned}$$

Then take a square root.

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2.1 Distance in the Complex Plane

We can use absolute value to measure the distance between complex numbers (thought of as vectors in the complex plane).

The distance between complex numbers z_1, z_2 is $|z_1 - z_2|$:

$$z_1 = x_1 + iy_1, \quad z_2 = x_2 + iy_2$$
$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

A crucial fact about working with absolute values is

Proposition 2.1 (Triangle Inequality)
For any two complex numbers z_1, z_2,

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Some corollaries of the Triangle Inequality:

Corollary 2.2
For any complex numbers z_1, z_2, w,

- $|z_1 - z_2| \leq |z_1 - w| + |w - z_2|$.
- $|z_2| - |z_1| \leq |z_2 - z_1|$ (Reverse triangle inequality)

Proof.

- $$\begin{aligned} |z_1 - z_2| &= |(z_1 - w) + (w - z_2)| \\ &\leq |z_1 - w| + |w - z_2| \qquad \text{(Triangle Inequality)} \end{aligned}$$
- $$\begin{aligned} |z_2| = |(z_2 - z_1) + z_1| &\leq |z_2 - z_1| + |z_1| \\ \implies |z_2| - |z_1| &\leq |z_2 - z_1| \end{aligned}$$

  By symmetry, also $|z_1| - |z_2| \leq |z_2 - z_1|$

$\square$

2.2 Complex Polynomials

Definition 2.3 (Complex polynomial)
A complex polynomial of degree $n \geq 0$ is a function of the form

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 + a_0, \quad z \in \mathbb{C},$$

where the coefficients $a_0, a_1, \ldots, a_n$ are complex numbers with $a_n \neq 0$.

Theorem 2.4 (Fundamental Theorem of Algebra)
Every complex polynomial $p(z)$ of degree $n \geq 1$ has a factorization

$$p(z) = c(z - z_1)^{m_1}(z - z_2)^{m_2} \cdots (z - z_k)^{m_k},$$

where $c \in \mathbb{C}$, the roots $z_1, \ldots, z_n$ are distinct complex numbers, and $m_1, \ldots, m_k \geq 1$. This factorization is unique, up to permutation of the factors. Also note that

$$\sum_{j=1}^{k} m_k = n = \text{the degree of the polynomial}$$

Proof. We will prove this later. $\square$

Example 2.5
Consider $p(z) = iz^2 + i$.
$p(z) = iz^2 + i = i(z^2 + 1) = i(z - i)(z + i)$

2.3 Polar Representation

A nonzero complex number $z \in \mathbb{C}$ can be described by two quantities:

- its length $|z| = r$
- The angle z makes with the positive real axis.

Definition 2.6 (Polar representation)
The polar representation of $z \neq 0$ is:

$$z = r(\cos(\theta) + i\sin(\theta))$$

Given $z = r(\cos(\theta) + i\sin(\theta))$ we can recover the Cartesian coordinates for z:

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

Definition 2.7 (Argument)
For a nonzero $z = r(\cos(\theta) + i\sin(\theta))$, the angle θ is called the argument of z, and is denoted $\theta = \arg z$. But the argument of $z, \arg z$, is actually a multivalued function:

$$\arg z + 2\pi k, \quad k \text{ an integer}$$
$$\text{also represents the same angle}$$

Definition 2.8 (Principal argument)
The principal argument of $z \neq 0$, denoted $\operatorname{Arg} z$, is the unique argument θ which is in $(-\pi, \pi]$.

$$\arg z = \{\operatorname{Arg} z + 2k\pi \colon k \text{ integer}\} \qquad \text{“multivalued function”}$$
$$\operatorname{Arg} z \qquad \text{“single-valued”}$$

Example 2.9
Consider the complex number $1+i$.

$$|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$$
$$1+i = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$$
$$\operatorname{Arg}(1+i) = \frac{\pi}{4}$$

We introduce the notation

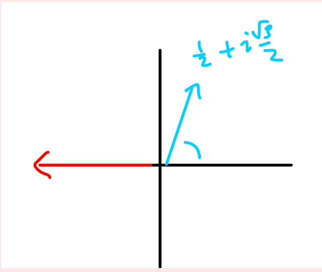
$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \quad \text{for } \theta \in \mathbb{R}.$$

We will see that this equality does follow from how we define complex exponentiation, but in the meantime it is just some convenient notation since the polar form for $z \in \mathbb{C}$ becomes:

$$z \neq 0, z = re^{i\theta},$$
$$\text{where } \theta = \operatorname{Arg} z, r = |z|$$

Example 2.10
We write the polar form using complex exponentials for the numbers -1 and $\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

$$\operatorname{Arg}(-1) = \pi, |-1| = 1$$
$$-1 = e^{i\pi}$$
$$\frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{i\frac{\pi}{3}}$$



Some helpful identities for $e^{i\theta}, \theta \in \mathbb{R}$:

$$|e^{i\theta}| = 1, \quad \overline{e^{i\theta}} = e^{-i\theta}, \quad \frac{1}{e^{i\theta}} = e^{-i\theta}.$$

Proof.

$$\begin{aligned}\overline{e^{i\theta}} &= \cos(\theta) - i\sin(\theta) = \cos(-\theta) + i\sin(-\theta)\\ &= e^{-i\theta}\end{aligned}$$

Check $e^{i\theta}e^{-i\theta} = 1$
use $\cos^2(\theta) + \sin^2(\theta) = 1$ □

Another very important identity for $\theta, \varphi \in \mathbb{R}$:

$$e^{i(\theta+\varphi)} = e^{i\theta}e^{i\varphi}.$$

Just use angle sum formulas

$$\cos(\theta+\varphi) = \cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi)$$
$$\sin(\theta+\varphi) = \cos(\theta)\sin(\varphi) + \sin(\theta)\cos(\varphi)$$

Example 2.11
If $z = re^{i\theta}$ is a nonzero complex number, then

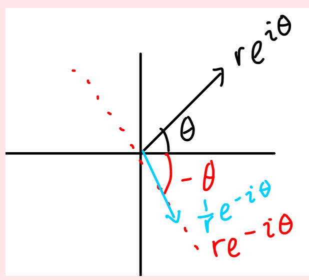
$$\frac{1}{z} = \frac{1}{r}e^{-\theta}, \quad \overline{z} = re^{-i\theta}.$$

$$\left(\frac{1}{r}e^{-i\theta}\right) = \left(r \cdot \frac{1}{r}\right) e^{i\theta + (-i\theta)} = 1$$
$$\implies \frac{1}{r}e^{-i\theta} = \frac{1}{z}$$

Polar form can also help us understand multiplication of complex numbers geometrically in the complex plane:

Example 2.12 (Inversion)

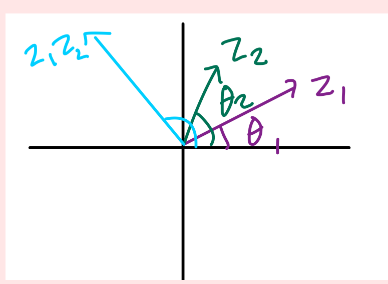
$$z = re^{i\theta} \neq 0 \quad \frac{1}{z} = \frac{1}{r}e^{-i\theta}$$



Example 2.13 (Multiplication)

$$z_1 = r_1e^{i\theta_1}, \quad z_2 = r_2e^{i\theta_2}$$
$$z_1 \cdot z_2 = (r_1e^{i\theta_1})(r_2e^{i\theta_2}) = r_1r_2e^{i(\theta_1+\theta_2)}$$

To multiply two complex numbers, multiply their lengths and add their arg's.



2.4 De Moivre's Formulae

Theorem 2.14 (De Moivre's Formulae)
For any natural number n and any $\theta \in \mathbb{R}$, we have

$$\cos(n\theta) + i\sin(n\theta) = e^{in\theta} = (e^{i\theta})^n = (\cos(\theta) + i\sin(\theta))^n$$

Once the right-hand-side is expanded, we can obtain expressions for $\cos(n\theta)$ and $\sin(n\theta)$ as polynomials is $\cos(\theta)$ and $\sin(\theta)$. These qualities are known as de Moivre's formulae.

Example 2.15
We obtain de Moivre's formulae for the case $n = 2$.

$$\begin{aligned}\cos(2\theta) + i\sin(2\theta) &= (\cos(\theta) + i\sin(\theta))^2 \\ &= \cos^2(\theta) - \sin^2(\theta) + i\,(2\cos(\theta)\sin(\theta))\end{aligned}$$

$$\xRightarrow{\text{equating Re=Im}} \begin{cases} \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) \\ \sin(2\theta) = 2\cos(\theta)\sin(\theta)\end{cases}$$

Exercise. See the textbook for a derivation of the formulae

$$\cos(3\theta) = \cos^3(\theta) + 3\cos(\theta)\sin^2(\theta), \quad \sin(3\theta) = 3\cos^2(\theta)\sin(\theta) - \sin^3(\theta).$$

2.5 nth Roots

Definition 2.16 (nth roots)
A complex number z is an nth root of w if $z^n = w$. In other words, the nth roots of w are precisely the roots of the polynomial $p(z) = z^n - w$. As an immediate consequence: any $z \in \mathbb{C}$ has at most n many distinct nth roots.

For a nonzero complex number w, we can find its nth roots as follows:
Let $w = \rho e^{i\varphi}$ $\quad z = re^{i\theta}$.
For z to be an nth root of w, we need $z^n = w$

$$\implies r^n e^{in\theta} = \rho e^{i\varphi}$$

$$\implies r^n = \rho, \quad n\theta = \varphi$$

or,

$$n\theta = \varphi + 2k\pi \quad k \in \mathbb{Z}.$$

$$\implies r = \rho^{1/n}, \quad \theta = \frac{\varphi + 2k\pi}{n},$$

distinct angles are

$$\theta = \frac{\varphi + 2k\pi}{n}, \quad k = 0, 1, 2, \ldots, n-1$$

Therefore,

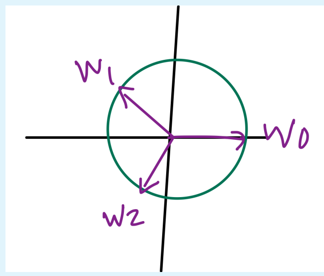
$$n\text{th roots of w:} \quad z_k = \rho^{1/n} e^{i\left(\frac{\varphi+2k\pi}{n}\right)}, \quad k = 0, 1, \ldots, n-1$$

Exercise. Find and plot the 3rd roots of $w = 9i$.

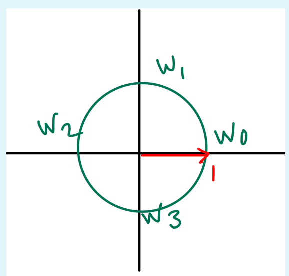
Definition 2.17 (nth roots of unity)
The nth roots of 1 have a special name: they are called the nth roots of unity. Using the same procedure as above, we can find that the nth roots of unity are

$$w_k = e^{2\pi i k/n}, \quad \text{for } k = 0, 1, \ldots, n-1.$$

3rd roots of unity:



4th roots of unity:



nth roots of unity can also be used to find nth roots of complex numbers other than 1. For a nonzero $z = re^{i\theta}$, we can find the first nth root of z to be $z_0 = r^{1/n}e^{i\theta/n}$. Then, if $w_0, \ldots, w_{n-1}$ are the nth root of unity, then the nth roots of z are exactly

$$z_k = z_0 w_k, \quad k = 0, \ldots, n-1.$$
$$z_k^n = (z_0 w_k) = z_0^n \cdot w_k^n = z \cdot 1 = z$$

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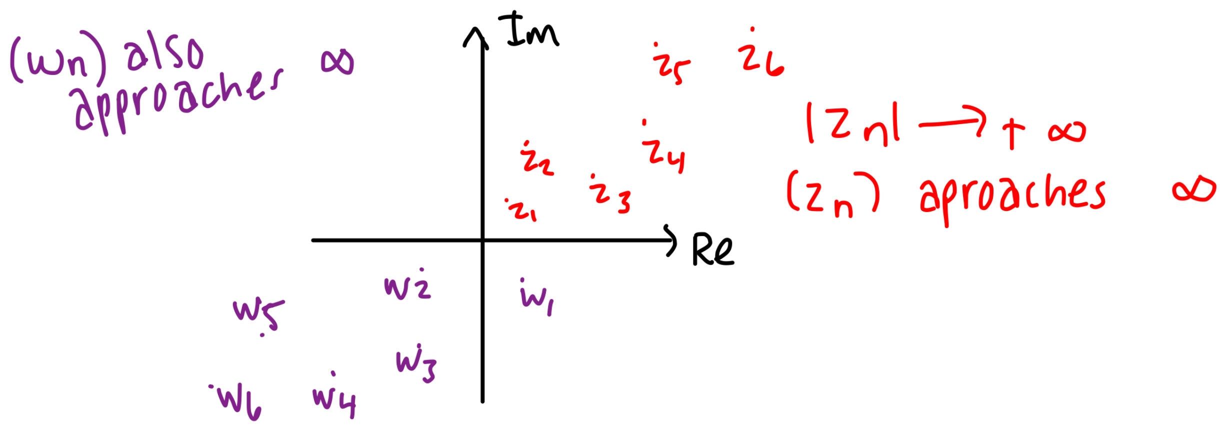
3.1 Stereographic Projection

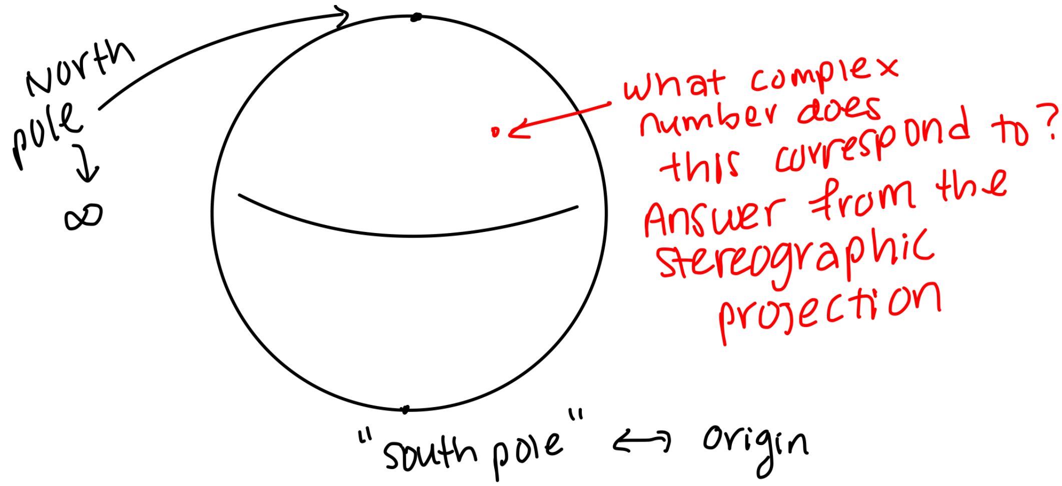
In complex analysis, it is often useful to imagine that the complex plane has an "ideal point" at infinity, denoted ∞.

Definition 3.1 (Extended complex plane)
The extended complex plane is $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$.

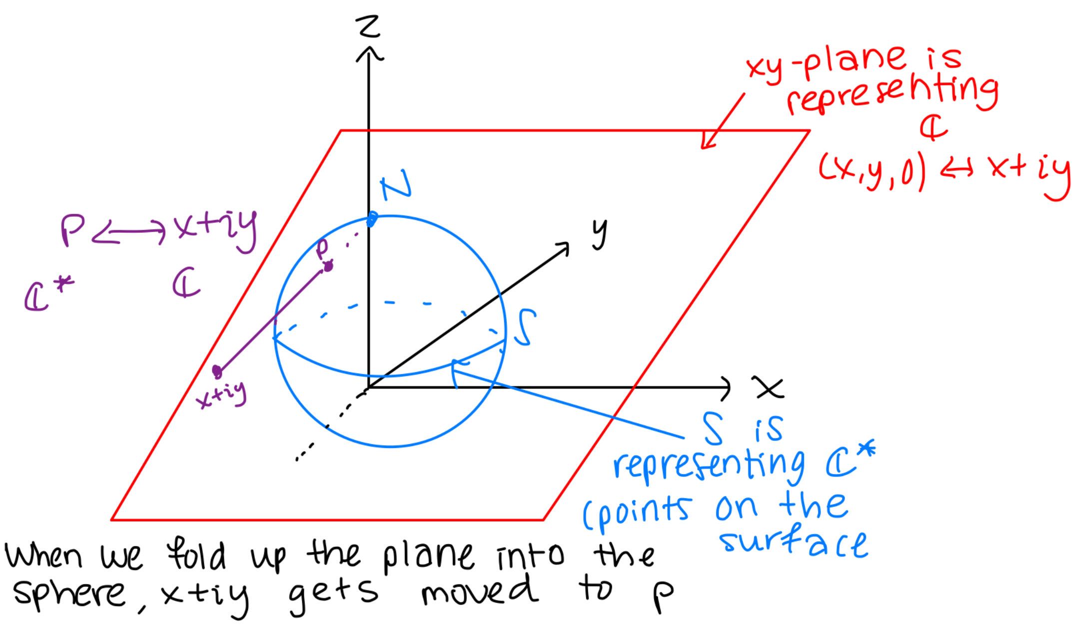
∞ is not a complex number. Instead, it represents what becomes of a complex number if its modulus grows without bound. in other words, a sequence of complex numbers (z_n) is considered to approach ∞ if and only if the moduli $|z_n|$ are diverging to $+\infty$ (as a sequence of reals).



This leaves us with a problem: how can we visualize $\mathbb{C}^*$?
$\mathbb{C}^*$ will be visualized as a sphere:



We will use stereographic projection to visualize $\mathbb{C}^*$ as a sphere.
Start with 3-space, $\{(X, Y, Z)\colon X, Y, Z \in \mathbb{R}\}$, and consider the unit sphere.



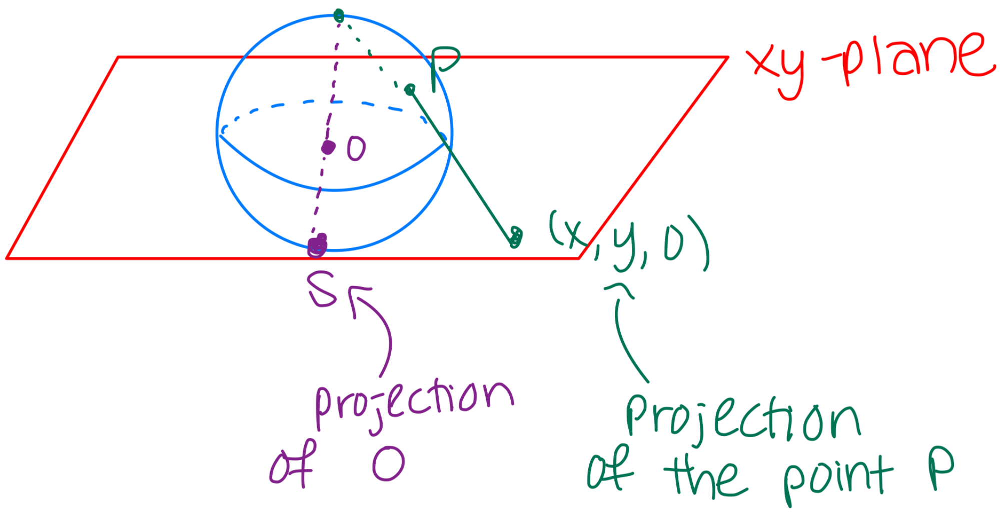
Any point in the xy-plane $(x, y, 0)$ will represent the complex number $x + iy$,

$$x + iy \sim (x, y, 0).$$

The unit sphere S represents $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$. with the north pole $N = (0, 0, 1)$ representing ∞.

But in what manner do points in S represent complex numbers in $\mathbb{C}$? To answer this, we will give a correspondence between points in $S \setminus \{N\}$ and the xy-plane.

The correspondence is called the stereographic projection, and is defined as follows:

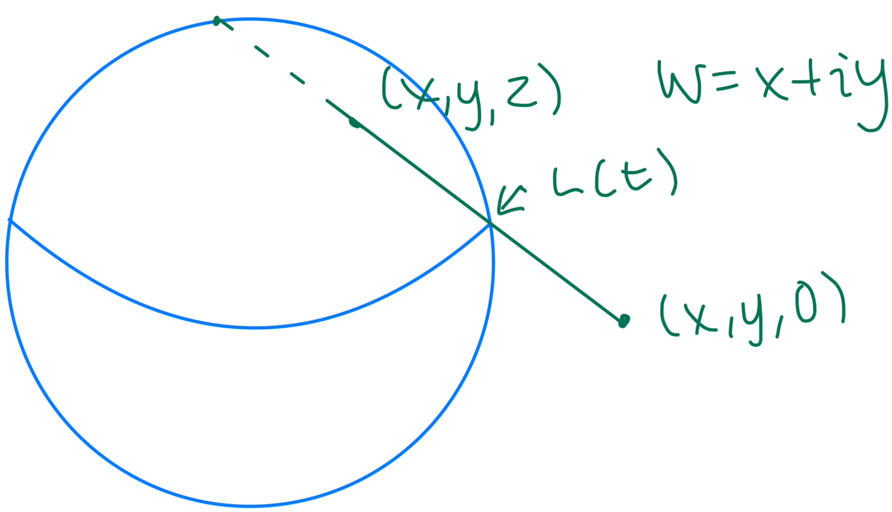


This projection gives a one-to-one correspondence between $S \setminus \{N\}$ and xy-plane.

Note that points on the lower half of the sphere correspond to complex numbers whose moduli are ≤ 1, while points on the upper half of the sphere correspond to complex numbers whose moduli are ≥ 1.

We have given a geometric description of the stereographic projection, but now we describe how to find an explicit formula for it.

We start with a given complex number $z = x + iy(\sim (x, y, 0))$ and seek a formula for the coordinates (X, Y, Z) for the point on the sphere which corresponds to $(x, y, 0)$ via the stereographic projection.



$$\begin{aligned} L(t) &= (0,0,1) + t\left((X,Y,Z) - (0,0,1)\right) \\ &= (tX, tY, t(Z-1)+1) \end{aligned}$$

There is t such that $L(t) = (X, Y, 0)$ i.e. $(tX, tY, t(Z-1)+1) = (x, y, 0)$

$$\implies t(Z-1) + 1 = 0 \Leftrightarrow t = \frac{1}{1-Z}$$

Note that:

$$Z = \frac{t-1}{t}, \quad tX = x, \quad tY = y$$

Since (X, Y, Z) is on S,

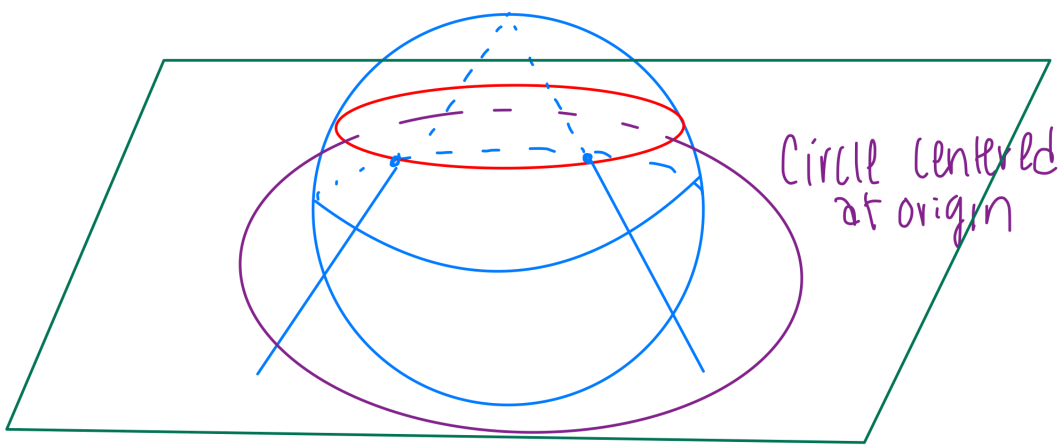
$$\begin{aligned} & X^2 + Y^2 + Z^2 = 1 && \text{Multiply both sides by } t^2 \\ \implies & (tX)^2 + (tY)^2 + (t-1)^2 = t^2 \\ \implies & |W|^2 + t^2 - 2t + 1 = t^2 \\ \implies & t = \frac{|W|^2 + 1}{2} && \text{solved for } t \text{ in terms of given } w \end{aligned}$$

Thus,

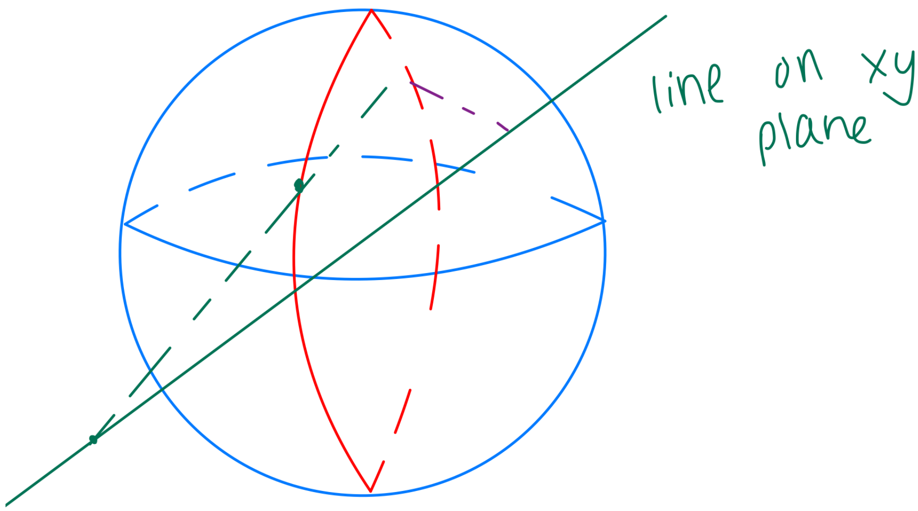
$$\begin{cases} X = \dfrac{1}{t}x = \dfrac{2x}{|w|^2+1} \\ Y = \dfrac{1}{t}y = \dfrac{2y}{|w|^2-1} \\ Z = \dfrac{t-1}{t} = \dfrac{|w|^2-1}{|w|^2+1} \end{cases}$$

$\left(\dfrac{2x}{|w|^2+1}, \dfrac{2y}{|w|^2-1}, \dfrac{|w|^2-1}{|w|^2+1}\right)$ is the unique point on the sphere that gets projected to $w = x + iy$.

We now explore what the stereographic projection does to geometric objects on the sphere. For example, it is not hard to see that lines of latitude on the sphere correspond to circles centered at 0 on the xy-plane.



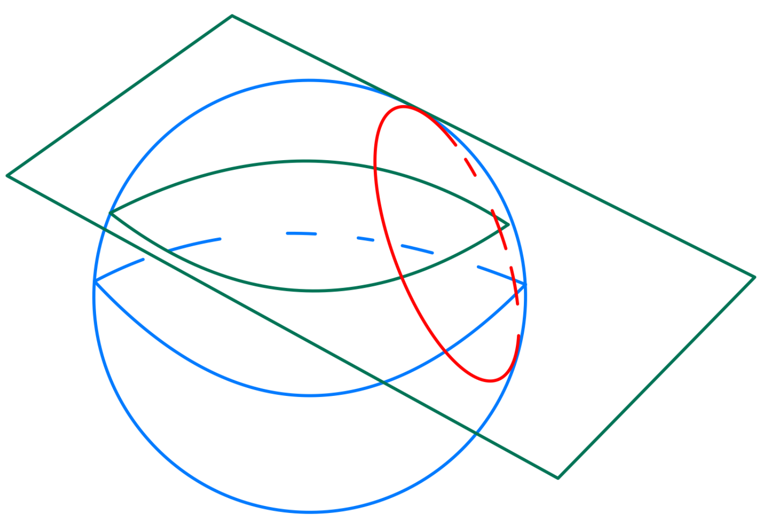
Moreover, lines of longitude on the sphere correspond to straight lines on the xy-plane.



Next, we will prove more general theorem:

Theorem 3.2
Under the stereographic projection, circles on the sphere correspond to circles and lines on the xy-plane.

Here, "circles on the sphere" means any intersection of a plane in 3-space with the unit sphere.



In our proof, we will also use the fact that the set of points in the plane which satisfy an equation of the form

$$x^2 + y^2 + ax + by + c = 0 \quad (a, b, c, \in \mathbb{R})$$

is either a circle, a point, or empty.

[To see this, start by completing the square to get

$$(x + a/2)^2 + (y + b/2)^2 = (a^2 + b^2)/4 - c$$

Consider the three cases where the right-hand-side is $> 0, = 0, < 0$.]

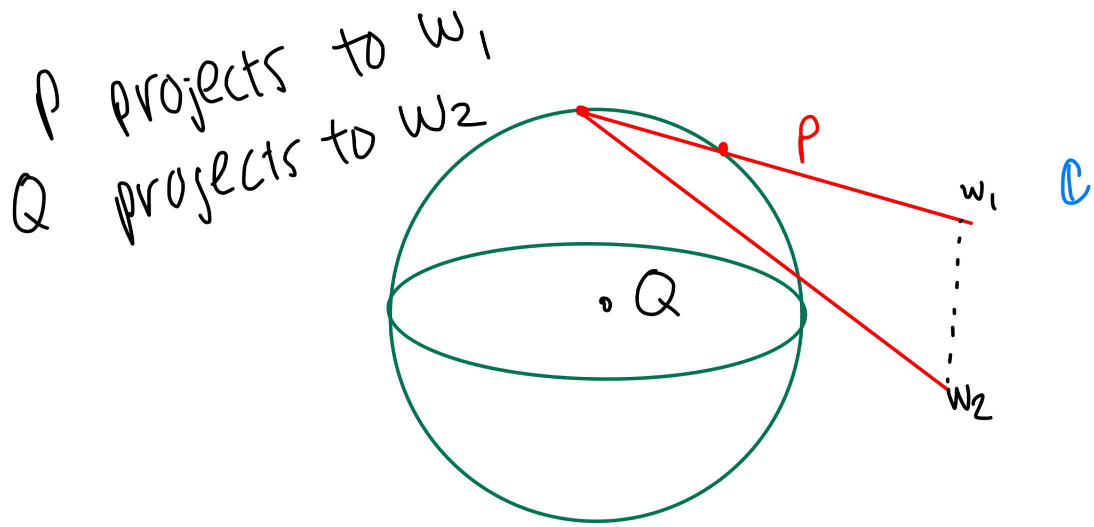
Proof. Consider a circle on the sphere which is the intersection of the sphere with the plane $AX + BY + CZ = D$.
The projection of this curve in all $z = x + iy \mapsto (x, y, 0)$ which satisfy

$$A\frac{2x}{|z|^2+1} + B\frac{2y}{|z|^2+1} + C\frac{|z|^2-1}{|z|^2+1} = D$$
$$\implies 2Ax + 2By + C(|z|^2 - 1) = D(|z|^2 + 1)$$
$$\implies 2Ax + 2By + Cx^2 + Cy^2 - C = Dx^2 + Dy^2 + D$$
$$\implies (C - D)x^2 + (C - D)y^2 + 2Ax + 2By + (C - D) = 0$$

If $C = D$, then this defines a line.
If $C \neq D$, then by our previous remark (after dividing by $(-D)$), we can conclude it defines a circle. □

3.2 Chordal Distance

This is a notion of distance for C^*.



$d(w_1, w_2)$ = length of chord that connects P, Q (chordal)

$\mathbb{C}^*$ with d is a metric space:

1. $d(w_1, w_2) = d(w_2, w_1)$
2. $d(w_1, w_2) \geq 0$,
   $d(w_1, w_2) = 0 \Leftrightarrow w_1 = w_2$
3. $d(w_1, w_2) \leq d(w_1, z) + d(z, w_2)$

Formula:

$$d(w_1, w_2) = \frac{2|w_1 - w_2|}{\sqrt{(|w_1|^2+1)(|w_2|^2+1)}} \quad w_1, w_2 \in \mathbb{C}$$
$$d(w_1, \infty) = \frac{2}{\sqrt{(|w_1|^2+1)}} \quad w_1 \in \mathbb{C}$$

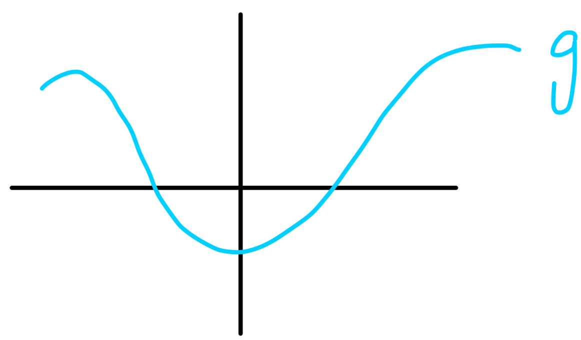
4 Jan 10, 2022

4.1 Visualizing Complex Functions

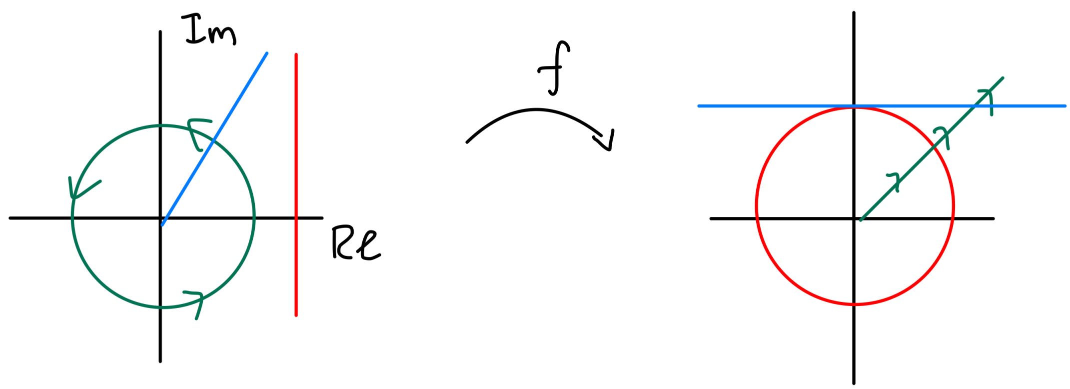
A function $\mathbb{C}$ to $\mathbb{C}$, $f\colon \mathbb{C} \to \mathbb{C}$, maps a two dimensional space to a two dimensional space. Thus, the graph of $f\colon \mathbb{C} \to \mathbb{C}$,

$$G = \{(z, f(z)) \in \mathbb{C}^2 \colon z \in \mathbb{C}\}$$

is not as easy to visualize as a graph of a function $g\colon \mathbb{R} \to \mathbb{R}$



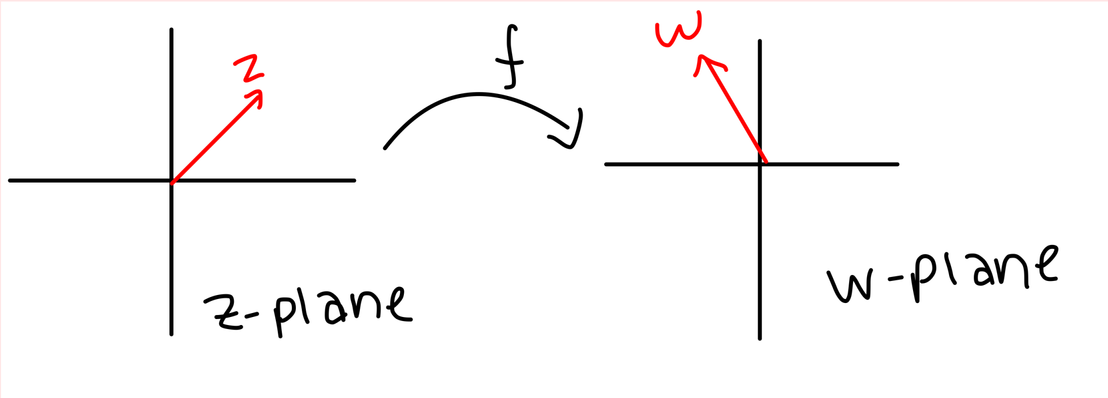
To visualize $f\colon \mathbb{C} \to \mathbb{C}$, our best tool is to analyze how f transforms various geometric objects.



4.2 The Square Function

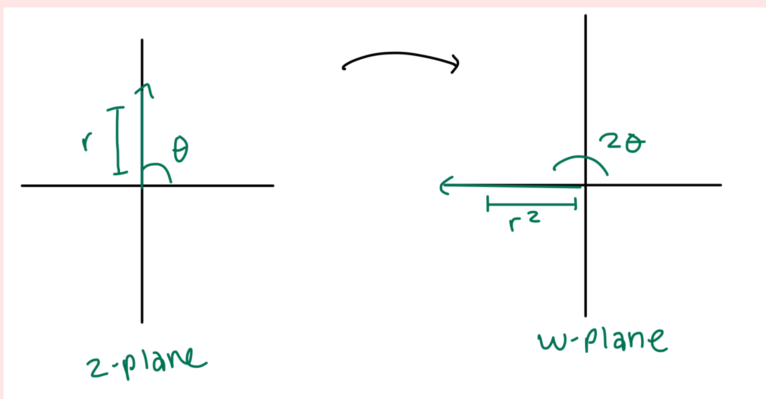
Consider the square function $f\colon \mathbb{C} \to \mathbb{C}, f(z) = z^2$.

Example 4.1
We will also think of this function as being given by $w = z^2$, where the domain space is the z-plane and the codomain space is the w-plane.



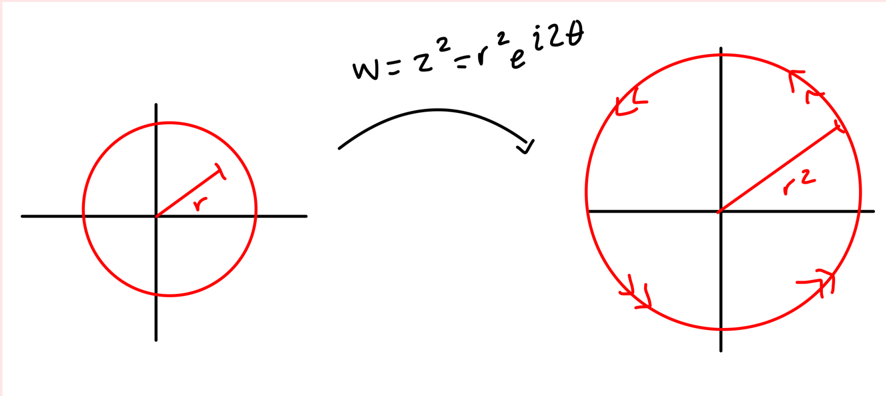
Example 4.2
Using polar representation, we can get an idea of what the square function does to a nonzero input $z = re^{i\theta}$:

$$z^2 = (re^{i\theta} \cdot re^{i\theta}) = r^2 e^{i2\theta}$$

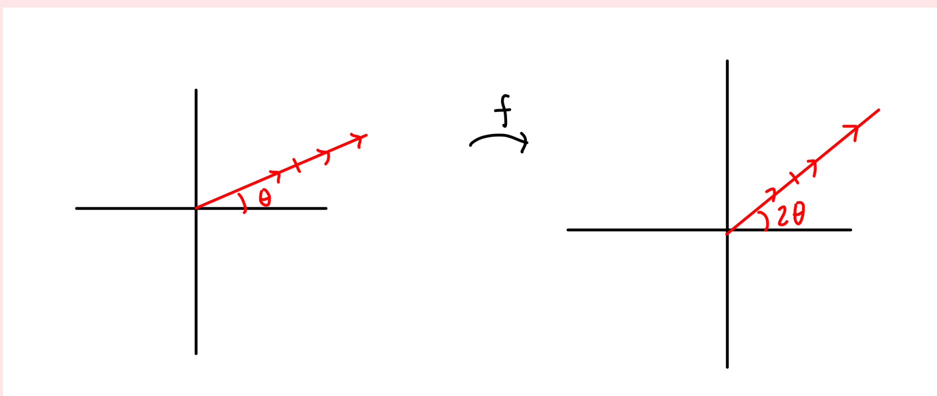


We can extend our understanding by examining what the square function does the geometric objects in the $z-$plane.

Example 4.3
The square function transforms a circle centered at 0 to another circle centered at 0:



Example 4.4
A ray anchored at 0 is transformed into another ray anchored at 0:

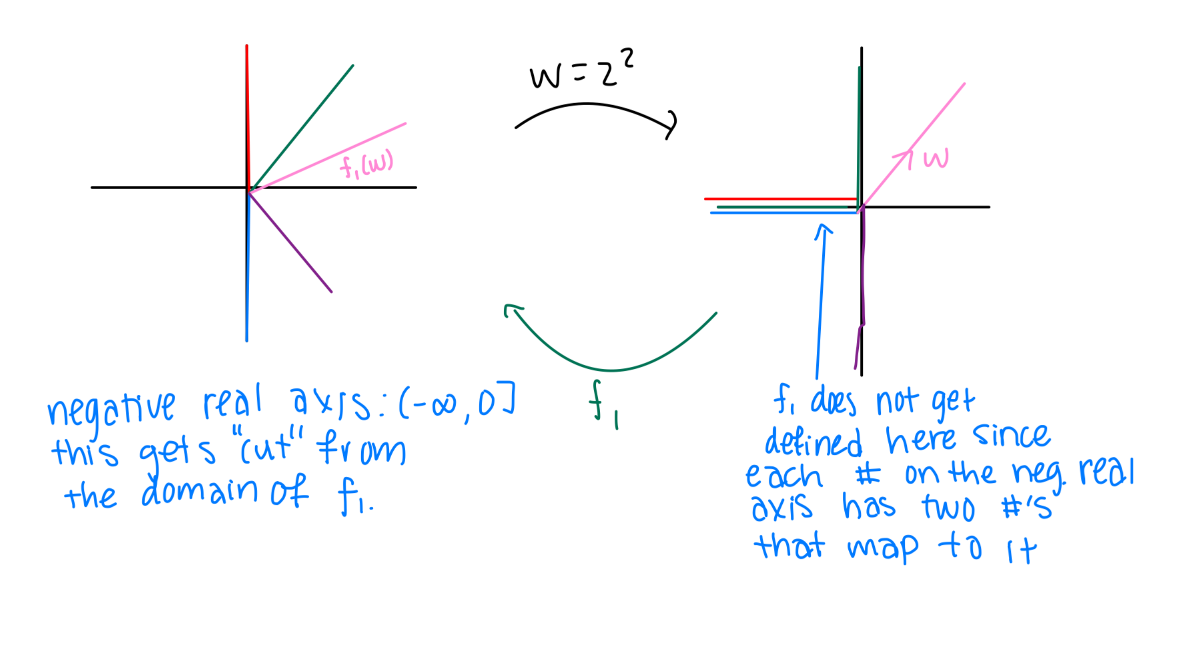


We will now consider the problem of finding an inverse function for $w = z^2$, i.e., a square root function $z = \sqrt{w}$.

Just as in the case of the real numbers, ever nonzero complex number has two distinct square roots; as a result, there are many different ways of defining a square root operation.

We begin by considering how $w = z^2$ transforms the open half-plane $\{z \in \mathbb{C} \colon \operatorname{Re} z > 0\}$.

Example 4.5



Definition 4.6 (Principal branch)
Our description above gives us what we will call the principal branch of $\sqrt{w}$,

$$f_1 \colon \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$$

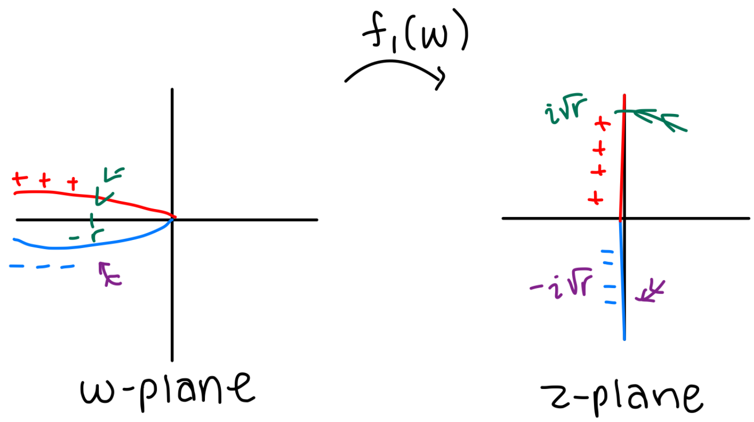
which is algebraically given by

$$f_1(w) = \sqrt{\rho} e^{i\varphi/2}, \quad w = \rho e^{i\varphi}, \quad -\pi < \varphi < \pi.$$

We can also write this equation using Arg:

$$f_1(w) = \sqrt{|w|} e^{i \operatorname{Arg} w/2}$$

The principal branch f_1 is not defined on $(-\infty, 0]$, but it does have the following limit behavior there:



<u>Notation:</u>

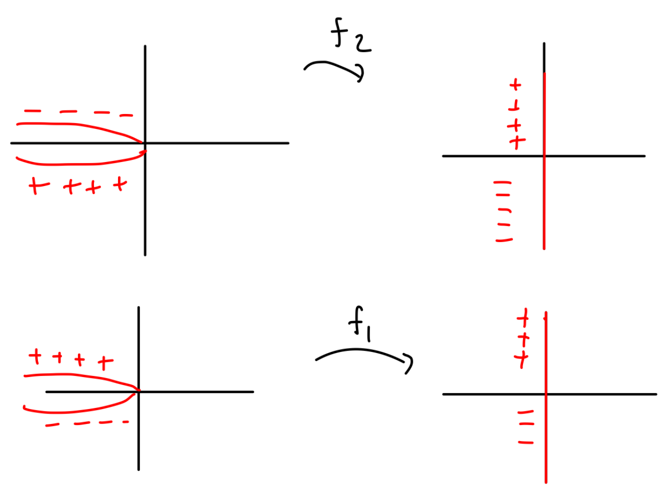
$$f_1(-r + i0) = i\sqrt{r}$$
$$f_1(-r - i0) = -i\sqrt{r}$$

$f_1(w)$ chooses one of the possible square roots of w, but we can define another branch of $\sqrt{w}$ which selects the other square root.

We define the branch $f_2\colon \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ simply by

$$f_2(w) = -f_1(w)$$

This function is still not defined on $(-\infty, 0]$, but it has the following (slightly different) limit behavior there:

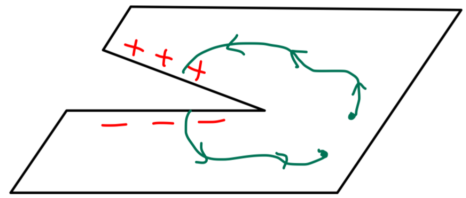


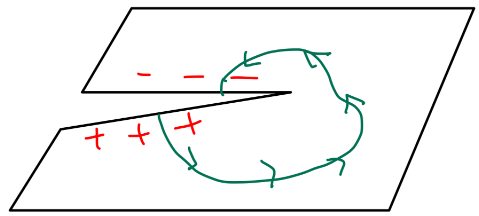
4.3 The Riemann Surface of $\sqrt{w}$

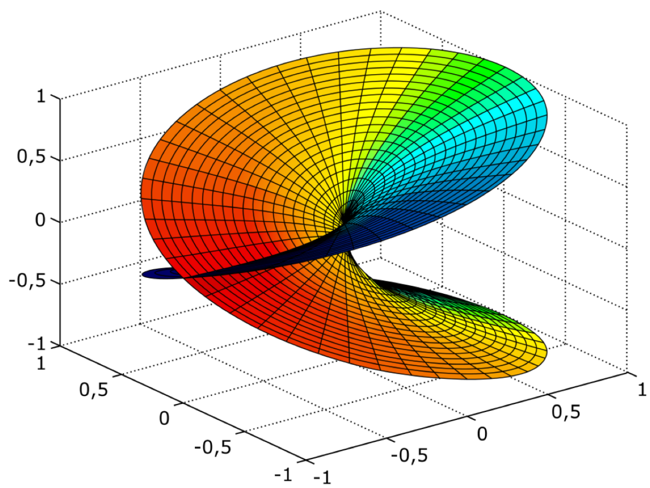
As we have said, the square root operation is multi-valued.

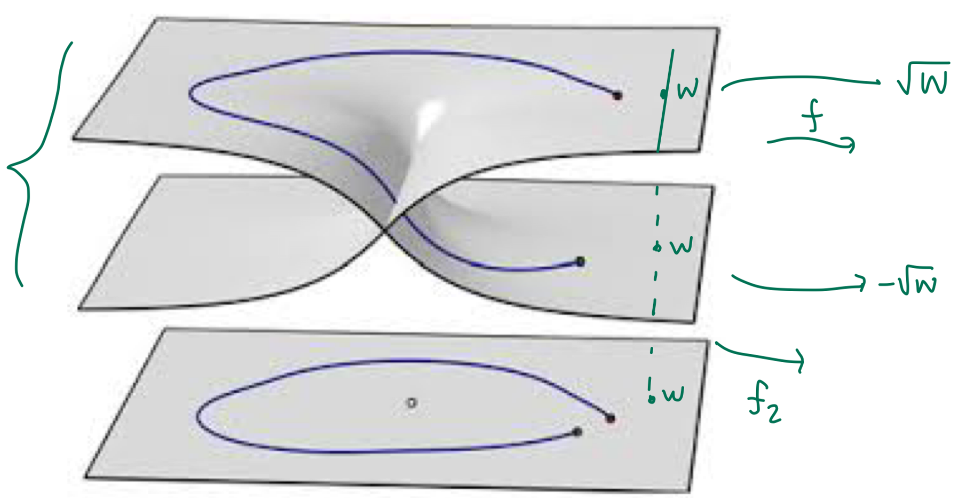
Book: "multi-valued function"

However, we can use our two branches above to construct a surface which can serve as a domain which makes this multi-valued operation a true function.






1
0,5
0
-0,5
-1
1
0,5
0
-0,5
-1
-1
-0,5
0
0,5
1

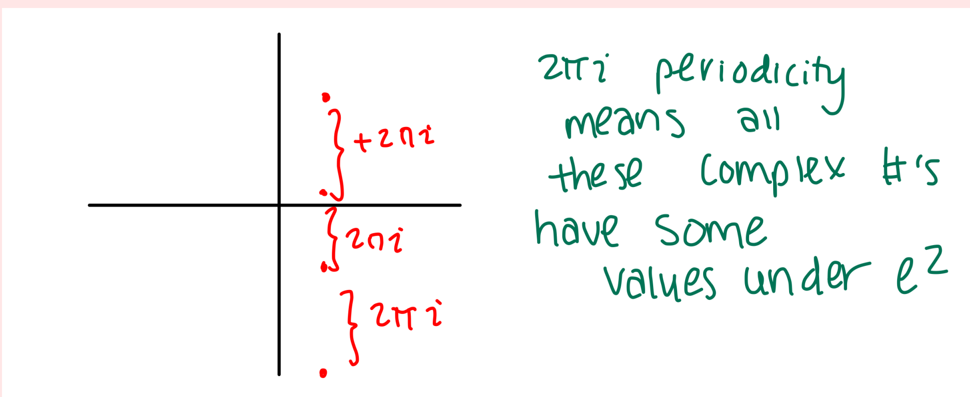

w
$\sqrt{w}$
f
w
$-\sqrt{w}$
w
f_2

4.4 The Exponential Function

Definition 4.7 (Exponential function)
We define the exponential function for all $z = x + iy \in \mathbb{C}$ by
$$e^z = e^x \cos(y) + ie^x \sin(y)$$
This, of course, is an extension of our previously used
$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \quad (\theta \in \mathbb{R}),$$
and we can write
$$e^z = e^x e^{iy}.$$

Definition 4.8 (λ-periodic)
Let $\lambda \in \mathbb{C}$. A complex function $f(z)$ is said to be λ-periodic if
$$f(z + \lambda) = f(z) \quad \forall z \in \mathbb{C}.$$

Example 4.9
The exponential function is $2\pi i$ periodic:
$$e^{z+2\pi i} = e^z \cdot e^{2\pi i} = e^z \cdot 1 = e^z$$



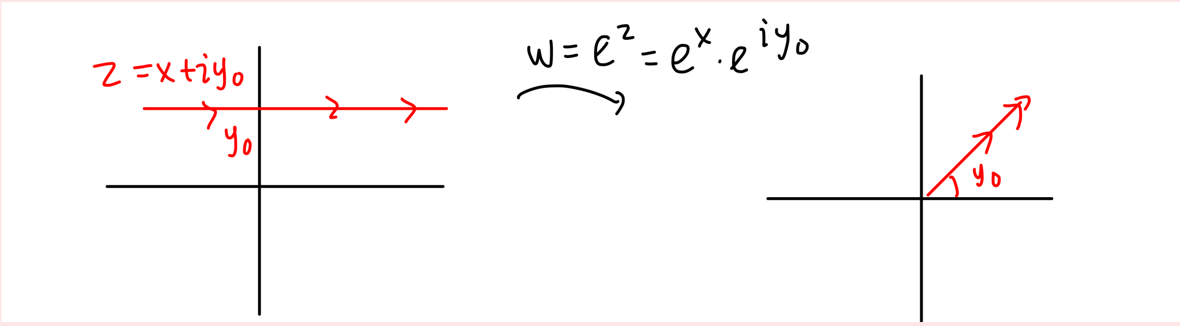
Other fundamental properties of the exponential function (which are easy to verify): for all $z, w \in \mathbb{C}$,

i. $e^{z+w} = e^z e^w$
ii. $1/e^z = e^{-z}$
iii. $\overline{e^z} = e^{\overline{z}}$

We now examine how $w = e^z$ transforms certain geometric objects.

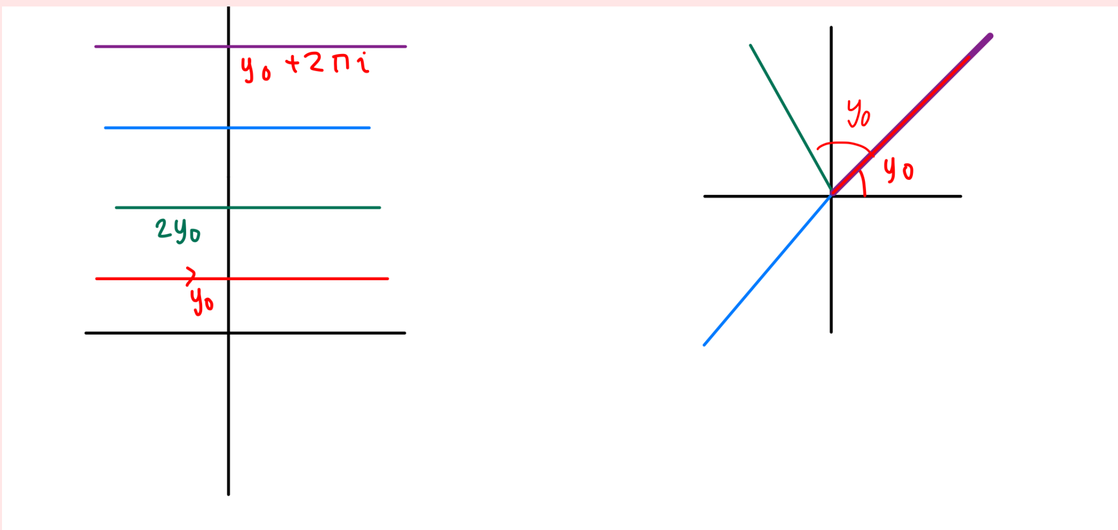
Example 4.10

Horizontal lines are mapped to rays anchored at 0:



Example 4.11

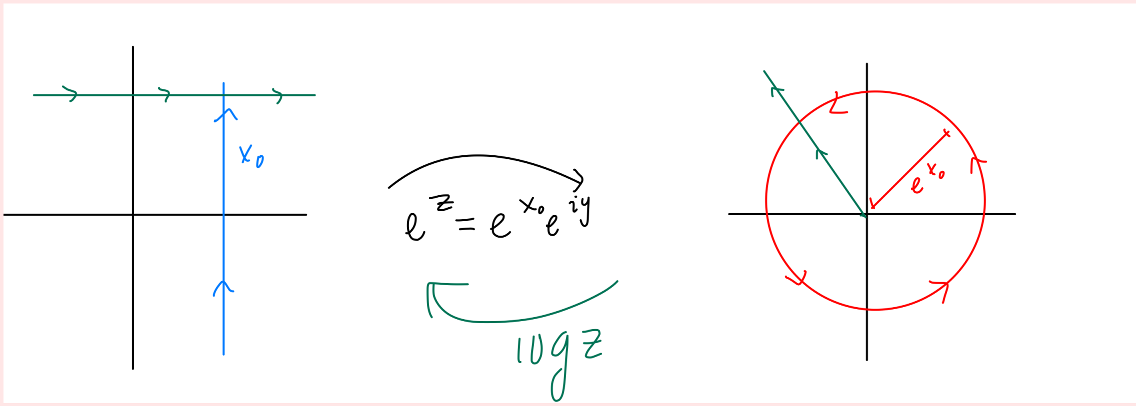
As we move the horizontal line, we can see the geometric meaning of the $2\pi i$ periodicity of $w = e^z$.



5 Jan 12, 2022

5.1 The Exponential Function (Cont'd)

Example 5.1
Vertical lines are transformed into circles centered at 0:



5.2 The Logarithm Function

Definition 5.2 (Logarithm Function)
We define $\log z$, $z \neq 0$, to be the multi-valued function

$$\log z = \log|z| + i \arg z = \log|z| + i \operatorname{Arg} z + 2\pi i m, \quad m \in \{0, \pm 1, \pm 2, \dots\}.$$

It is easy to check that any value w of log satisfies $e^w = z$:

$$\begin{aligned} e^{\log z} &= e^{\log|z| + i \operatorname{Arg} z + 2\pi i m} = e^{\log|z|} e^{i \operatorname{Arg} z} e^{2\pi i m} \\ &= |z| e^{i \operatorname{Arg} z} = z \end{aligned}$$

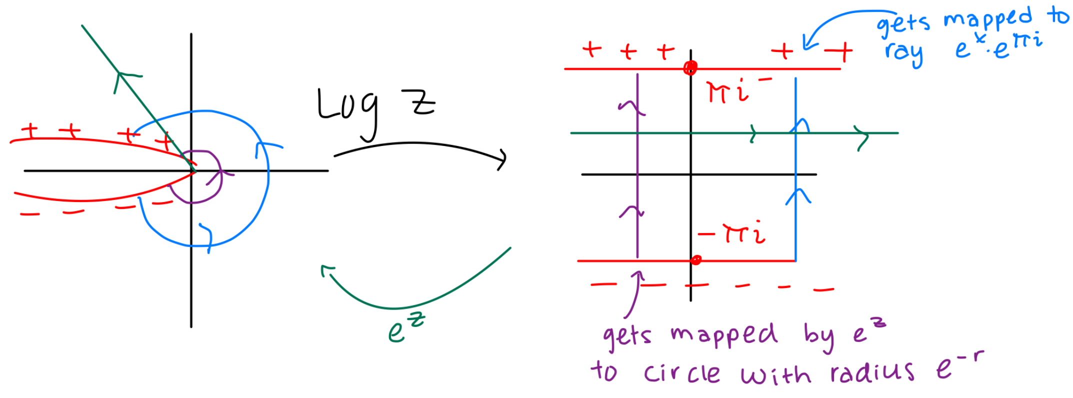
Definition 5.3 (Principal value)
The principal value of $\log z$, $z \neq 0$, is

$$\operatorname{Log} z = \log|z| + i \operatorname{Arg} z.$$

Note that for $z \neq 0$,

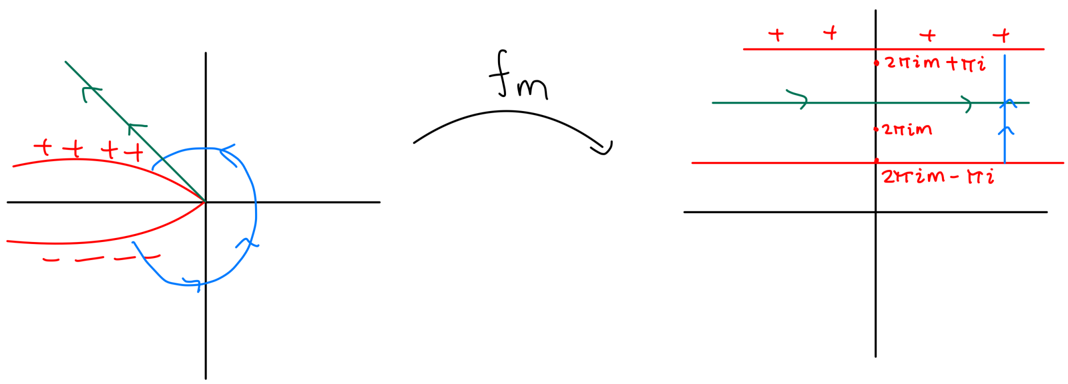
$$\log z = \operatorname{Log} z + 2\pi i m, \quad m \in \{0, \pm 1, \pm 2, \dots\}$$

We can visualize Log as a function with domain $\mathbb{C} \setminus (-\infty, 0]$ as follows:

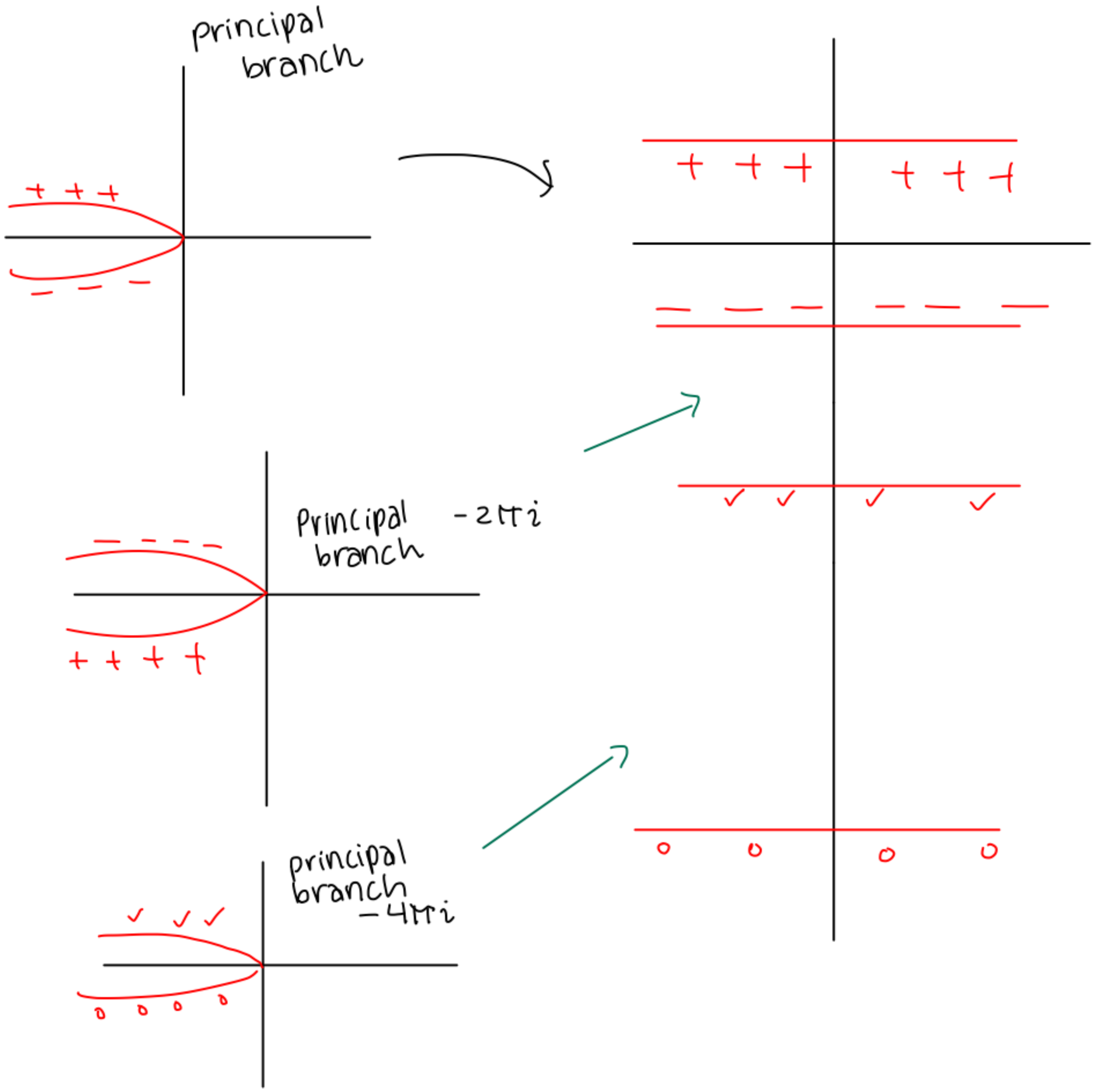


And we have other branches as well,

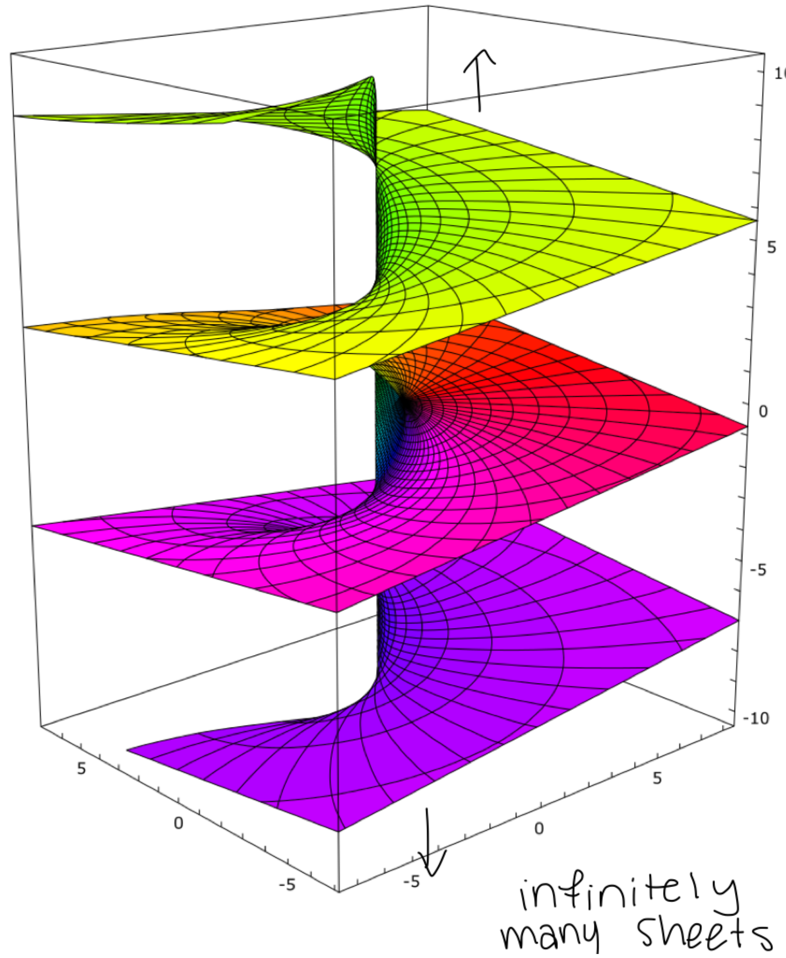
$$f_m(z) = \operatorname{Log} z + 2\pi i m, \qquad \text{where } m \text{ is an integer.}$$



Just as we did with $\sqrt{w}$, we can represent the multi-valued function $\log z$ as a single-valued (i.e., actual) function on a Riemann surface.



Glue these together to get Riemann surface of $\log z$.


10
5
0
-5
-10
5
0
-5
-5
0
5
infinitely
many sheets

5.3 Power Functions and Phase Factors

Definition 5.4 (Power function)
Let $\alpha \in \mathbb{C}$. We define the <u>power function</u> z^α to be the multi-valued function

$$z^\alpha = e^{\alpha \log z}, \quad z \neq 0.$$

Thus, the values of z^α are

$$z^\alpha = e^{\alpha[\log|z| + i \operatorname{Arg} z + 2\pi i m]} = e^{\alpha \operatorname{Log} z} e^{2\pi i \alpha m}, \quad m \text{ an integer.}$$

Note that if α is an integer, then z^α is single-valued:

$$\begin{aligned} z^\alpha &= e^{\alpha \operatorname{Log} z} e^{2\pi i \alpha m} \leftarrow \text{ integer multiple of } 2\pi i \text{ when } \alpha \text{ is an integer} \\ &= e^{\alpha \operatorname{Log} z} \end{aligned}$$

n a positive integer

$$z^n = \underbrace{z \cdot z \cdot z \cdot z \cdots z}_{n \text{ times}}$$

$$z^{-n} = \underbrace{\frac{1}{z} \cdot \frac{1}{z} \cdots \frac{1}{z}}_{n \text{ times}}$$

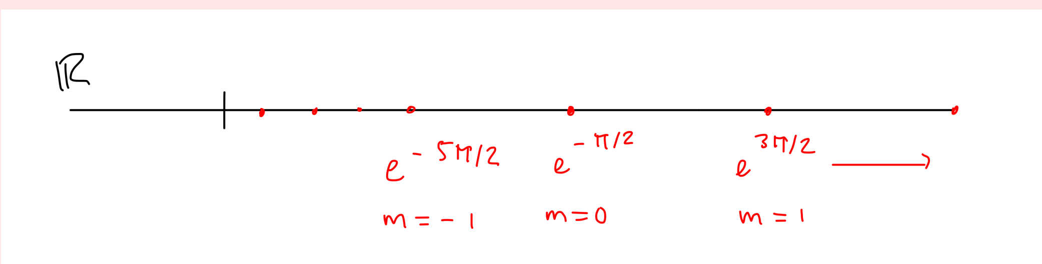
If $\alpha = 1/n$ for an integer n, then $z^\alpha = z^{1/n}$ are the nth roots of z:

$$z^{1/n} = e^{\frac{1}{n} \operatorname{Log} z} e^{\frac{2\pi i m}{n}}$$

Example 5.5
We find the values of i^i.

$$\begin{aligned} i^i &= e^{i[\log|i| + i \operatorname{Arg} i + 2\pi i m]} \\ &= e^{0 + i^2 \frac{\pi}{2} + 2\pi i^2 m} \\ &= e^{\frac{-\pi}{2}} e^{-2\pi m} \\ &= e^{-\pi(1-4m)/2} \end{aligned}$$

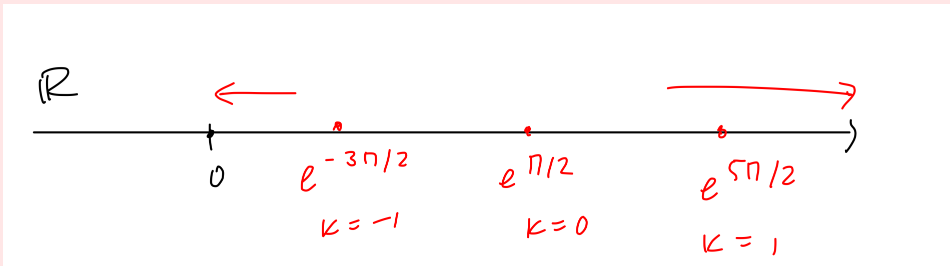
m integer



Example 5.6

We will find all the values of i^{-i}.

$$\begin{aligned} i^{-i} &= e^{-i[\log|i| + i\operatorname{Arg} i + 2\pi i k]} \\ &= e^{\frac{\pi}{2}} e^{2\pi k} = e^{\pi(1+4k)/2} \end{aligned}$$



Warning: When we multiply the values of i^i with those of i^{-i} we get infinitely many values:

$$\begin{aligned} (i^i)(i^{-i}) &= (e^{-\pi/2} \cdot e^{-2\pi m})(e^{\pi/2} e^{2\pi k}) \\ &= e^{2\pi(k-m)} \\ &= e^{2\pi j} \quad \text{j integer} \end{aligned}$$

In other words,

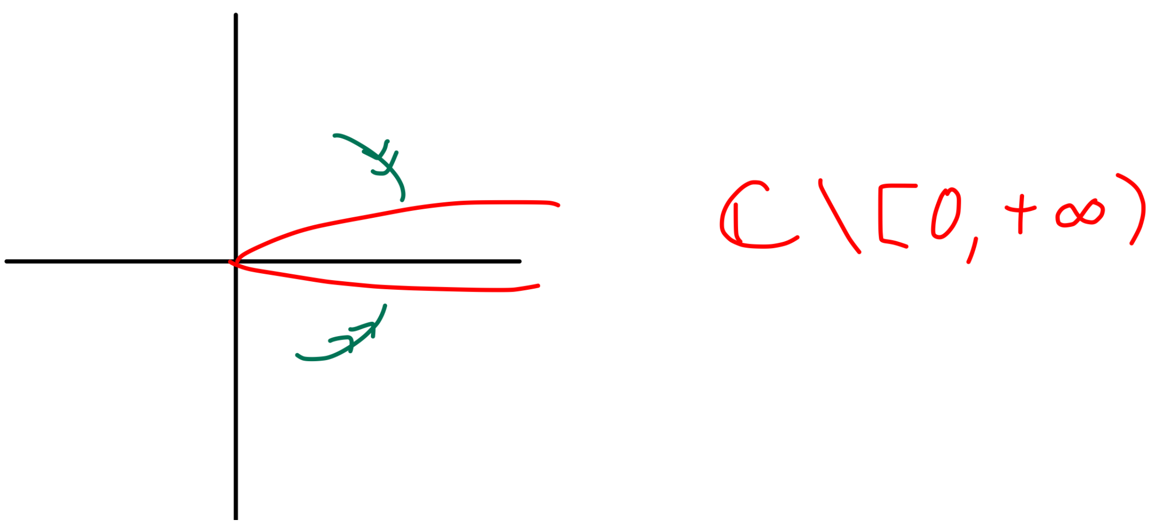
$$(i^i)(i^{-i})$$

is itself multi-valued, and we do not have the familiar looking identity

$$(i^i)(i^{-i}) = 1$$

Instead: 1 is one of the values of $(i^i) \cdot (i^{-i})$ but $e^{2\pi}, e^{-4\pi} \dots$ are also values.

Fix a non-integer $\alpha \in \mathbb{C}$. Since z^α is multi-valued, we find a way to define a single-valued branch of z^α. To do this, we first remove the positive real axis $[0, +\infty)$ from $\mathbb{C}$:



Then, define the function $f\colon \mathbb{C} \setminus [0, +\infty) \to \mathbb{C}$ by

$$z = re^{i\theta} \quad f(z) = r^\alpha e^{i\alpha\theta}, \quad \text{where } 0 < \theta < 2\pi \quad \text{and} \quad r^\alpha = e^{\alpha \log r}$$

At the top edge of the slit, we can compute $f(r + i0)$:

$$f(r+i0) = \lim_{\substack{\theta \to 0 \\ \text{from above}}} f(r+i\theta) = r^\alpha e^{i\alpha \cdot 0} = r^\alpha$$

At the bottom edge of the slit, we can compute $f(r-i0)$:

$$f(r-i0) = \lim_{\substack{\theta \to 2\pi \\ \text{from below}}} f(r+i\theta) = r^\alpha e^{i\alpha 2\pi} = e^{i\alpha 2\pi} \cdot f(r+i0)$$

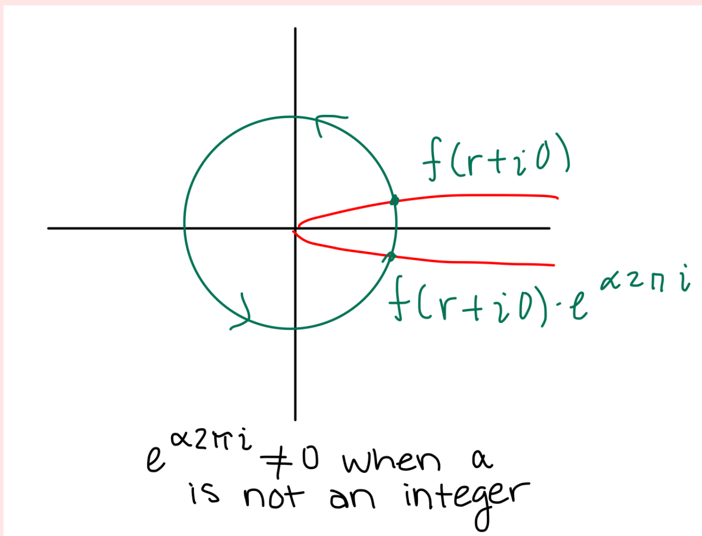
In particular, $f(r+i0) \neq f(r-i0)$

6 Jan 14, 2022

6.1 Power Functions and Phase Factors (Cont'd)

Example 6.1

If we move continuously around the origin starting at a point on the slit, the values of $f(z)$ will move continuously from $f(r+i0)$ to $f(r-i0)$:



But $f(r+i0) \neq f(r-i0)$, so we cannot continuously extend this function to be defined on the slit. In fact, there is no way to define a continuous branch of z^α on the whole complex plane $\mathbb{C}$ (for α not an integer).

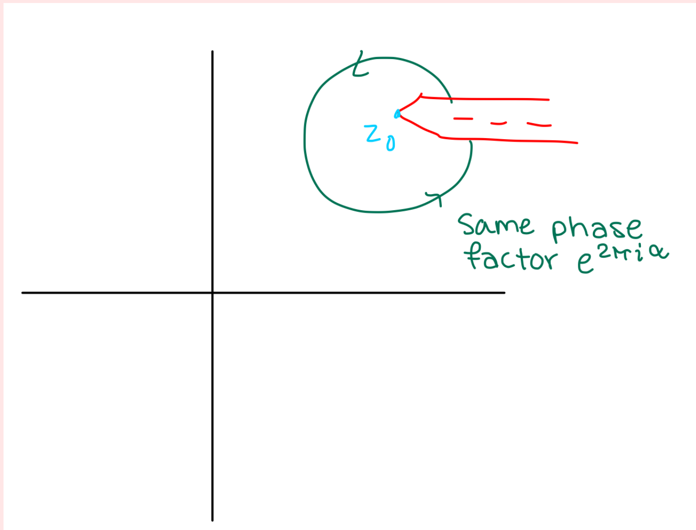
Definition 6.2 (Phase factor)

We showed that for $r > 0$,

$$f(r-i0) = e^{2\pi i\alpha} f(r+i0)$$

The value $e^{2\pi i\alpha}$ is called the phase factor of z^α at $z = 0$.

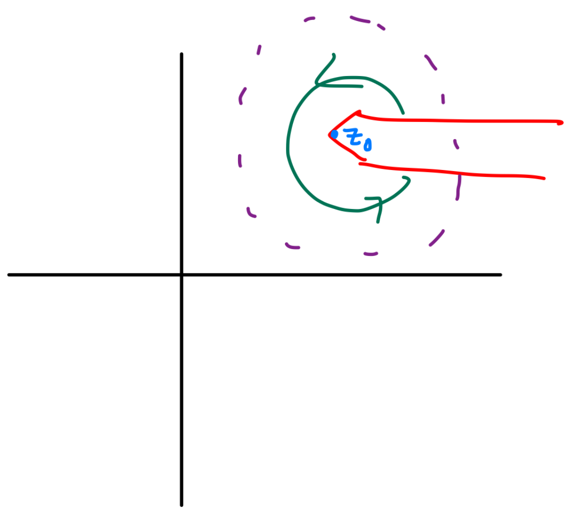
Example 6.3
For the multi-valued function $(z-z_0)^\alpha$, we could perform the same analysis:



Theorem 6.4 (Phase change lemma)
Let $g(z)$ be a (single-valued) function which is defined and continuous near z_0 in some disk centered at z_0. For any continuous varying branch of $(z-z_0)^\alpha$, the function

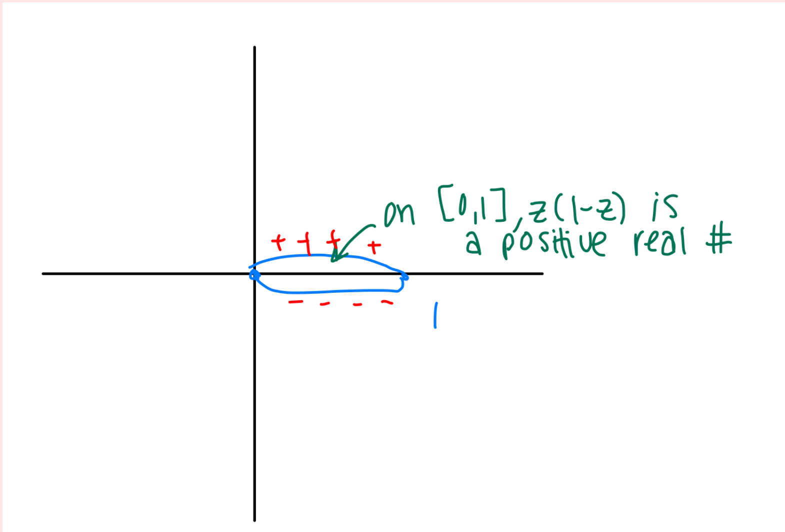
$$f(z) = (z-z_0)^\alpha g(z)$$

is multiplied by a phase factor of $e^{2\pi i\alpha}$ when z traverses a complete circle around z_0 in the counterclockwise direction.

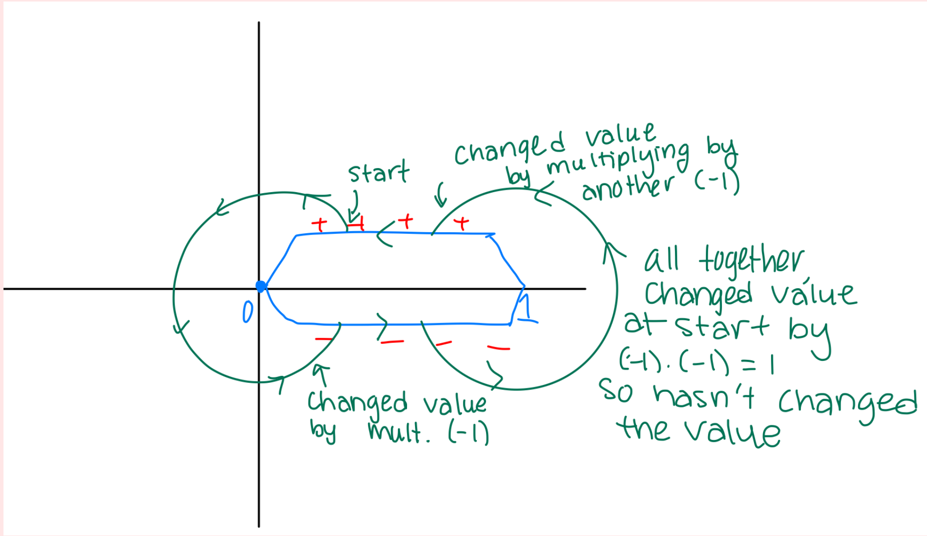


Example 6.5
If α is an integer, then the phase factor of z^α at $z = 0$ is
$$e^{2\pi i\alpha} = 1$$
This means that traverses the circular path multiplies the value of $f(z)$ by 1, i.e., does not change the value. This is another way of saying z^α, α integer, is single-valued.

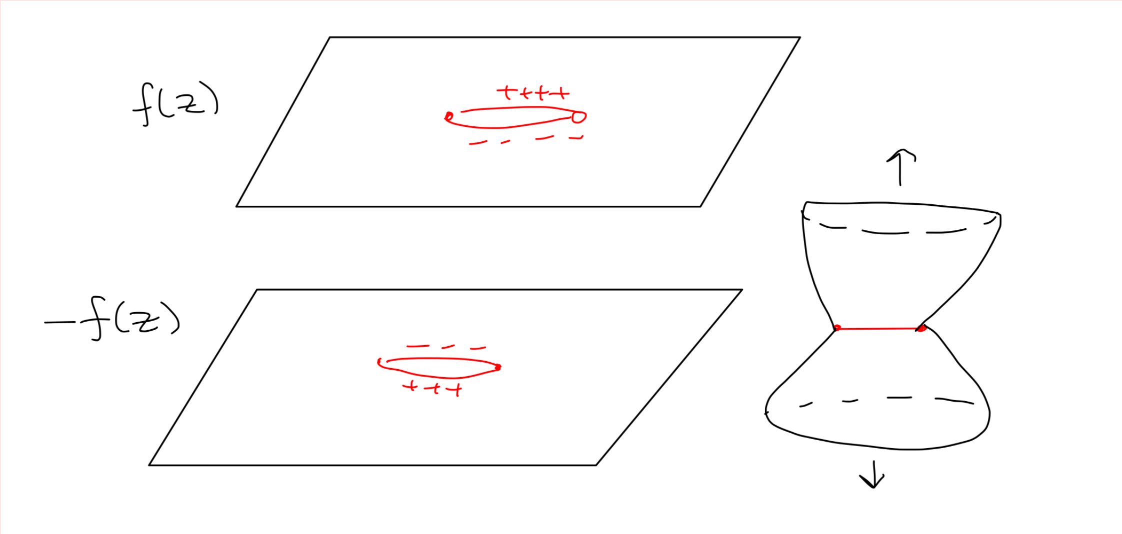
Example 6.6
Consider the function $\sqrt{z(1-z)}$.
Since
$$\sqrt{z(1-z)} = \sqrt{z} \cdot \sqrt{1-z},$$
this function actually has two branch points:
$\sqrt{z}$ has a branch point at $z = 0$.
$\sqrt{1-z}$ has a branch point at $z = 1$.
$\implies \sqrt{z(1-z)}$ has branch points at $z = 0, 1$.
$$\sqrt{z} \cdot \underbrace{\sqrt{(1-z)}}_{\substack{\text{have branch of this} \\ \text{is defined and continuous} \\ \text{at } z=0}}$$
Phase change lemma $\implies$ phase factor at $z = 0$ is
$$e^{2\pi i \cdot \frac{1}{2}} = e^{\pi i} = -1$$
Similarly, phase factor at $z = 1$ is
$$e^{2\pi i \cdot \frac{1}{2}} = -1$$
We draw a branch cut 0 to 1 and consider the branch $f(z)$ of $\sqrt{z(1-z)}$ which is positive along the top edge of the cut and negative along the bottom edge.



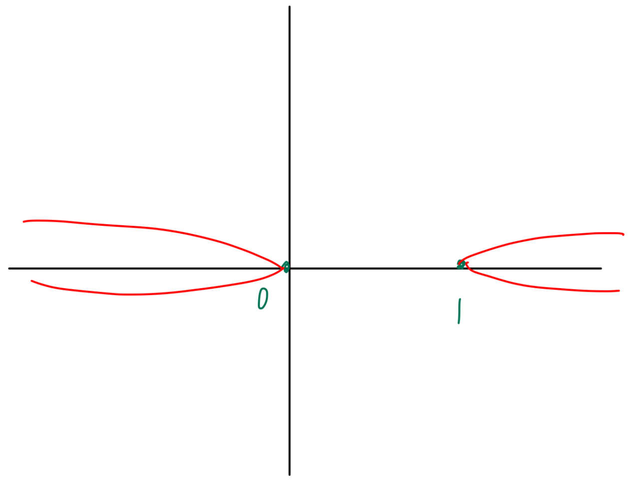
Example 6.7
We can check that this branch is actually single-valued on $\mathbb{C} \setminus [0,1]$:



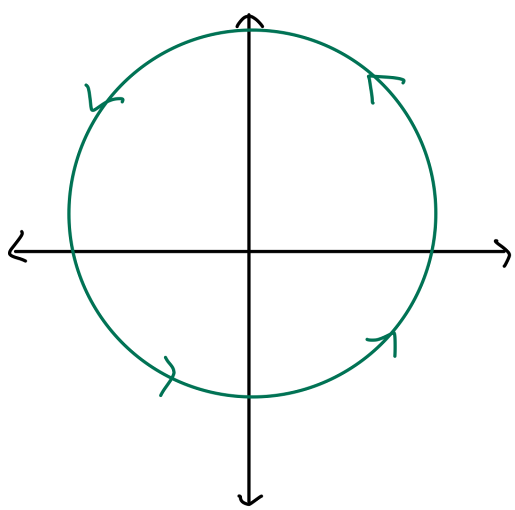
And we can visualize the Riemann surface:
$f(z)$ branch of $\sqrt{z(1-z)}$ on $\mathbb{C} \setminus [0,1]$
Other branch is $-f(z)$



Note: for the single-valued branch of $\sqrt{z(1-z)}$, we just need enough cuts to ensure that we cannot traverse a closed path around a single branch point.



We will see in the next example that sometimes we need more than just one cut. Also, we will see that ∞ can be a branch point. This happens when traversing very large circles centered at 0 has a phase factor.



Example 6.8

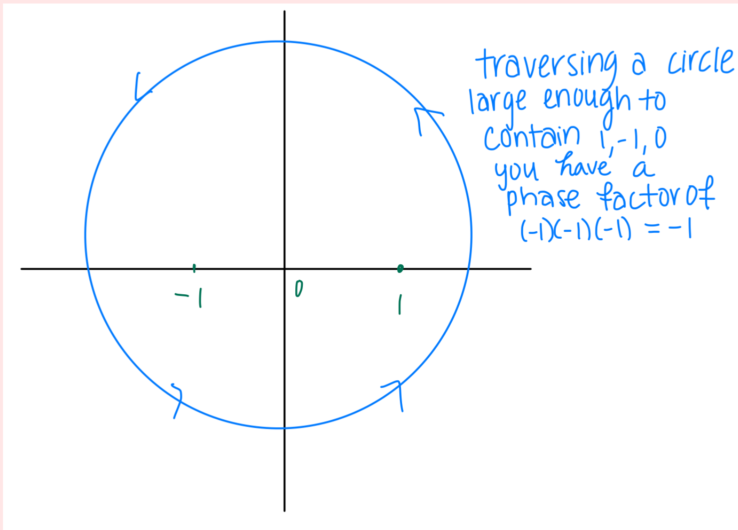
Consider the function $\sqrt{z - 1/z}$. Rewriting this as

$$\sqrt{z-1/z} = \frac{\sqrt{z-1}\sqrt{z+1}}{\sqrt{z}} = (z-1)^{\frac{1}{2}}(z+1)^{\frac{1}{2}}(z)^{-\frac{1}{2}}$$

we can see that the function has three finite branch points.
$1, -1, 0$, all have phase factors

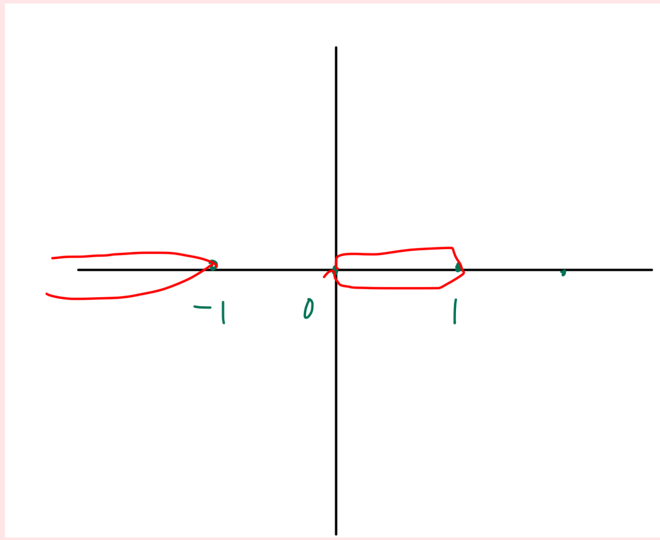
$$-1 = e^{2\pi i \cdot \frac{1}{2}}$$

$$-1 = e^{2\pi i(-\frac{1}{2})} = e^{-\pi i}$$

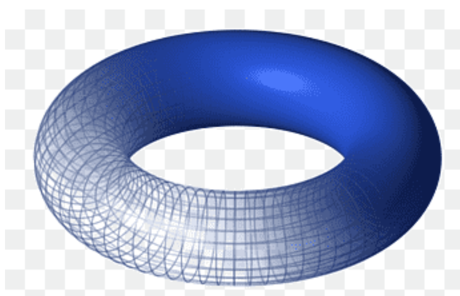


Moreover, ∞ is also a branch point.

Example 6.9
This time, we need two branch cuts: (Again, we want to avoid the possibility of traversing a path around a single branch point, including the one at ∞.)



There is a branch of $\dfrac{\sqrt{z-1}\sqrt{z+1}}{\sqrt{z}}$ defined on $\mathbb{C} \setminus ((-\infty, -1] \cup [0, 1])$
Riemann surface is a torus.



6.2 Trigonometric and Hyperbolic Functions

Definition 6.10 (Trigonometric functions)
We define $\cos z$ and $\sin z$ by setting

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad z \in \mathbb{C},$$
$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad z \in \mathbb{C}$$

It is easy to check that these are indeed an extension of cos, sin on the real line, but this definition can be further motivated by solving the system

$$\begin{cases} e^{i\theta} = \cos\theta + i\sin\theta \\ e^{-i\theta} = \cos\theta - i\sin\theta \end{cases} \quad \text{for } \cos\theta \text{ and } \sin\theta$$

It is easy to check that $\cos z, \sin z$ have many of the familiar properties:

i. $\cos z$ is an even function: $\cos(-z) = \cos(z)$.
ii. $\sin z$ is an odd function: $\sin(-z) = -z \sin(z)$.
iii. $\cos z, \sin z$ are both 2π periodic: $\cos(z + 2\pi) = \cos z, \sin(z + 2\pi) = \sin(z)$.
iv. The sum formulas are valid:
$$\cos(z + w) = \cos z \cos w - \sin z \sin w,$$
$$\sin(z + w) = \sin z \cos w + \cos z \sin w.$$
v. For any $z \in \mathbb{C}, \cos^2 z + \sin^2 z = 1$.

Definition 6.11 (Hyperbolic functions)
The hyperbolic functions are extended to $\mathbb{C}$ in the obvious way:
$$\cosh z = \frac{e^z + e^{-z}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2} \quad z \in \mathbb{C}$$

These functions also have their familiar properties:

i. cosh, sinh are both $2\pi i$ periodic.
ii. cosh, is even, sinh is odd.

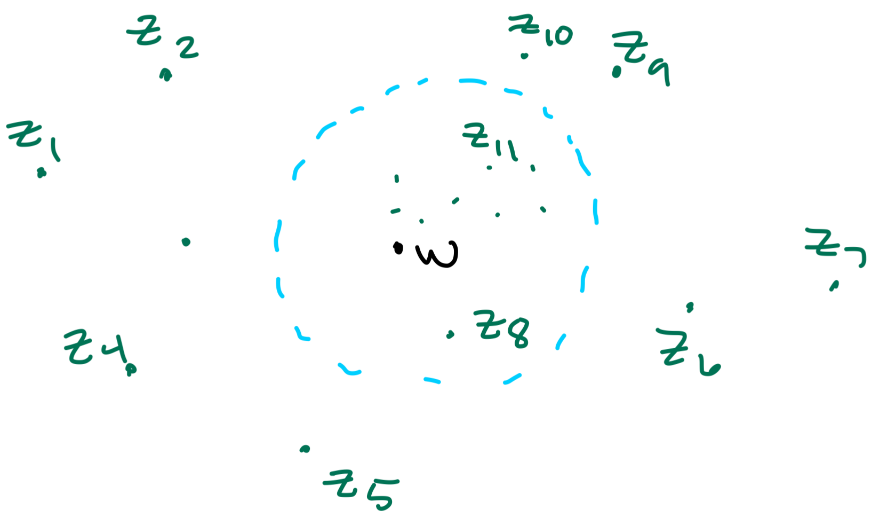
The hyperbolic functions are closely related to the trig functions. Both sets of functions are obtained from each other by rotating the complex plane by $\pi/2$ clockwise (i.e., by multiplications by i):
$$\begin{cases} \cosh(iz) = \cos z, \\ \sinh(iz) = i \sin z, \end{cases} \quad \begin{cases} \cos(iz) = \cosh z, \\ \sin(iz) = i \sinh z, \end{cases}$$
We can use these equations, and the addition formula to obtain the Cartesian representation for $\sin z$:
$$\sin z = \sin x \cosh y + i \cos x \sinh y \quad (z = x + iy).$$
Using some more trig identities we can then get the formula
$$|\sin z|^2 = \sin^2 x + \sinh^2 y$$
$\implies$ zeroes of $\sin z$ are exactly the zeros of $\sin x$: $k\pi,\ k = 0, \pm 1, \pm 2, \ldots$

7 Jan 19, 2022

7.1 Limits of Sequences of Complex Numbers

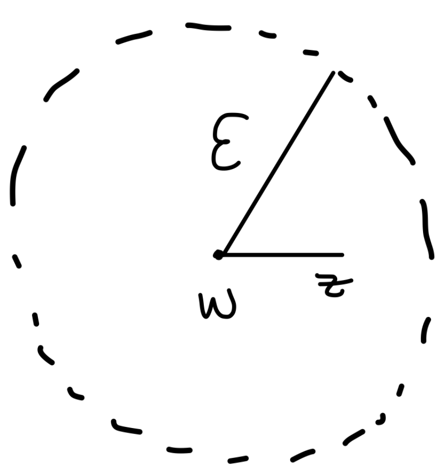
Our first goal is to define the notion of convergent sequence of complex numbers. To build up some intuition, we first give an informal definition.

A sequence of complex numbers (z_n) converges to a complex number w if for any disk centered at w, the sequence eventually enters and remains in that disk.



An algebraic way of expressing that z is in the disk centered at w with radius ε:

$$|z - w| < \varepsilon$$



A way of expressing that the sequence enters and remain in the disk:

$$\forall \varepsilon > 0 \, \exists N \forall n > N [|z_n - w| < \varepsilon]$$

Definition 7.1 (Convergence of a sequence)
A sequence (z_n) in $\mathbb{C}$ converges to $w \in \mathbb{C}$ if

$$\text{for all } \varepsilon > 0, \text{ there exists an integer } N \text{ s.t.}$$
$$\text{for all } n \in \mathbb{N}, n > N \implies |z_n - w| < \varepsilon$$

Notation and terminology. The w in the definition (if it exists) is called the limit of (z_n) and we write

$$z_n \to w \text{ or } \lim z_n = w$$

to mean that (z_n) converges to w.
The theory of sequences of complex numbers has many similarities to the theory of sequences of real numbers. Sometimes, the proofs of theorems that apply to both theories are identical, except for the interpretation of $+, \cdot$ as the complex operations instead of the real ones.
Also, some limits from real analysis will also be helpful to remember:

1. $\lim_n \frac{1}{n^p} = 0$, when $0 < p < +\infty$;
2. $\lim_n |z|^n = 0$, when $|z| < 1$;
3. $\lim_n n^{1/n} = 1$.

Complex sequences obey the same limit laws as real sequences:

Theorem 7.2
Let (z_n) and (w_n) be complex sequences and assume $z_n \to z$ and $w_n \to w$. Then,

i. for $\lambda \in \mathbb{C}, \lambda z_n \to \lambda z$;
ii. $z_n + w_n \to z + w$;
iii. $z_n w_n \to zw$;
iv. $z_n / w_n \to z/w$, provided $w \neq 0$ and $w_n \neq 0$ for any $n \in \mathbb{N}$.

Proof. The proofs are nearly identical to their real counterparts. For example, consider the proof of the sum law:
Let $\varepsilon > 0$ be given. Since $z_n \to z$ and $w_n \to w$, there exists N_1 and N_2 such that

$$n > N_1 \implies |z_n - z| < \varepsilon/2, \quad n > N_2 \implies |w_n - w| < \varepsilon/2.$$

Let $N = \max\{N_1, N_2\}$. Then, for any $n > N$, we have

$$|(z_n + w_n) - (z + w)| = |(z_n - z) + (w_n - w)| \leq |z_n - z| + |w_n - w| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

$\square$

We have the following limit law for complex sequences:

Theorem 7.3
Let (z_n) be a sequence of complex numbers and let $z \in \mathbb{C}$. If $z_n \to z$, then $\overline{z_n} \to \overline{z}$.

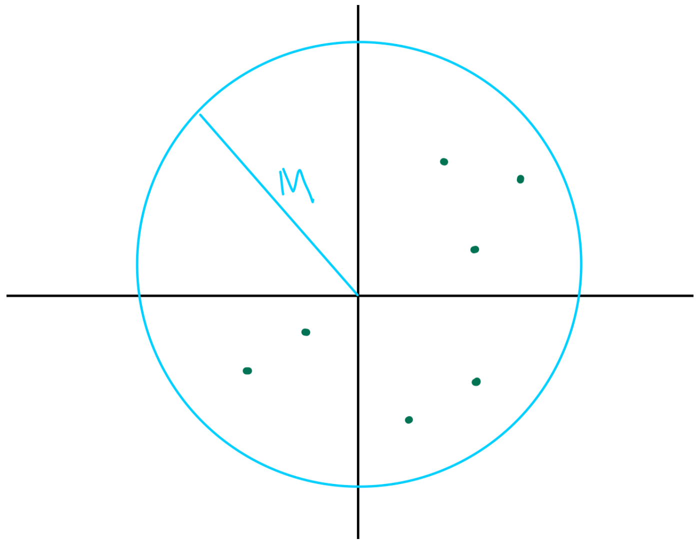
Proof. The proof hinges on the observation that

$$|\overline{z_n} - \overline{z}| = |\overline{z_n - z}| = |z_n - z|$$

□

Definition 7.4 (Bounded sequence)
A sequence (z_n) of complex numbers is bounded if there exists a real number $M > 0$ such that

$$|z_n| \leq M \quad \forall n$$



We have the following theorem (whose real version is familiar).

Theorem 7.5
Any convergent sequence of complex numbers is bounded.

7.2 Alternative Characterizations of Convergence

Theorem 7.6
Let (z_n) be a sequence in $\mathbb{C}$. (z_n) is convergent if and only if the real sequences $(\operatorname{Re} z_n)$ and $(\operatorname{Im} z_n)$ are both convergent.

Proof. For the ($\Longrightarrow$) direction, use the equalities

$$\operatorname{Re} z_n = \frac{z_n + \overline{z_n}}{2}, \quad \operatorname{Im} z_n = \frac{z_n - \overline{z_n}}{2i},$$

and apply limit laws.
For the ($\Longleftarrow$) direction, use

$$z_n = \operatorname{Re} z_n + i \operatorname{Im} z_n$$

and apply limit laws. □

Definition 7.7 (Cauchy sequence)
A sequence of complex numbers (z_n) is Cauchy if

$$\text{for all } \varepsilon > 0, \text{ there exists an integer } N \text{ s.t.}$$
$$\text{for all } n, m > N, \quad |z_n - z_m| < \varepsilon.$$

Theorem 7.8
Let (z_n) be a sequence in $\mathbb{C}$. (z_n) is convergent if and only if (z_n) is Cauchy. i.e. "$\mathbb{C}$ is complete". "$\mathbb{C}$ has no holes."

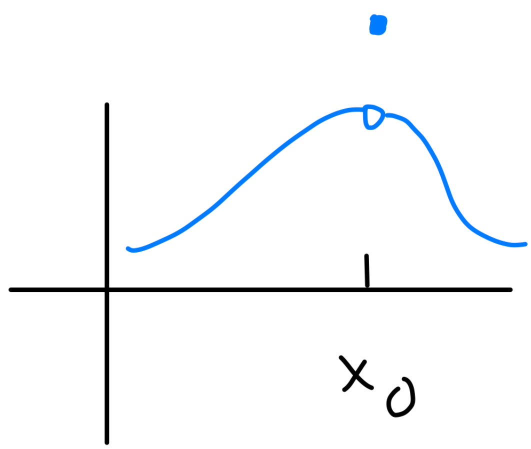
7.3 Limits of Functions

Definition 7.9 (Limit)
Let $f(z)$ be a complex-valued function on its domain $D \subseteq \mathbb{C}$, and let $z_0, L \in \mathbb{C}$. We say that $f(x)$ has limit L when z tends to z_0, and we write

$$\lim_{z \to z_0} f(z) = L,$$

if

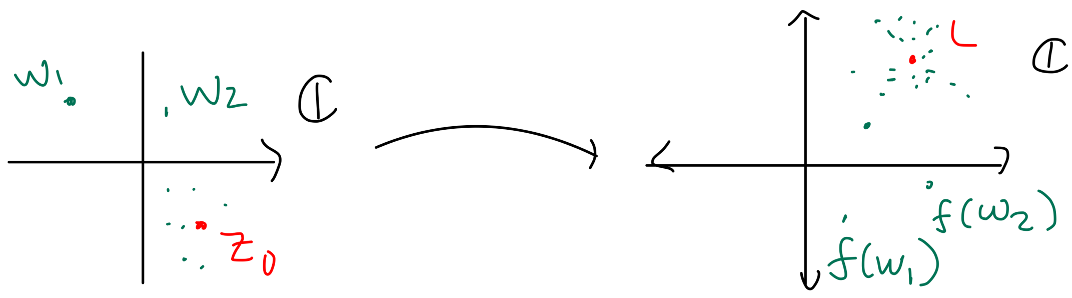
$$\text{for all } \varepsilon > 0, \text{ there exists } \delta > 0 \text{ s.t.}$$
$$\text{for all } z \in D, \quad 0 < |z - z_0| < \delta \implies |f(z) - L| < \varepsilon.$$



Note that the definition of $\lim_{z\to z_0} f(z) = L$ does not require $f(z_0)$ to be defined. Also, even if $\lim_{z\to z_0} f(z)$ and $f(z_0)$ both exist, it may be the case that $\lim_{z\to z_0} f(z) \neq f(z_0)$. The above definition is equivalent to a condition on sequences.

Theorem 7.10
Let $f(z)$ be a complex-valued function on a domain $D \subseteq \mathbb{C}$, and let $z_0, L \in \mathbb{C}$.
$\lim_{z \to z_0} f(z) = L$ if and only if for any sequence (w_n) in D with $w_n \neq z_0$ for all n, we have

$$w_n \to z_0 \implies f(w_n) \to L.$$



The sequence characterization of function limits along with the sequence limit laws immediately give us

Theorem 7.11
If a function has a limit at z_0, then the function is bounded near z_0. Moreover, if $\lim\limits_{z \to z_0} f(z) = L$ and $\lim\limits_{z \to z_0} g(z) = M$, then

i. $\lim\limits_{z \to z_0} (\lambda f)(z) = \lambda L$
ii. $\lim\limits_{z \to z_0} [f(z) + g(z)] = L + M$;
iii. $\lim\limits_{z \to z_0} f(z)g(z) = LM$;
iv. $\lim\limits_{z \to z_0} f(z)/g(z) = L/M$, provided $M \neq 0$ and $g(z) \neq 0$ near z_0.

Definition 7.12 (Continuous)
Let $f(z)$ be a complex-valued function of $D \subseteq \mathbb{C}$, and let $z_0 \in D$. We say that f is continuous at z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

f is called continuous if f is continuous at every z_0 in its domain D.

Example 7.13
Any polynomial function is continuous.

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

- $g(z) = z$ is continuous.
- scalar multiplies of continuous functions are continuous.
- constant functions are continuous
- sums of continuous functions are continuous
- products of continuous functions are continuous.

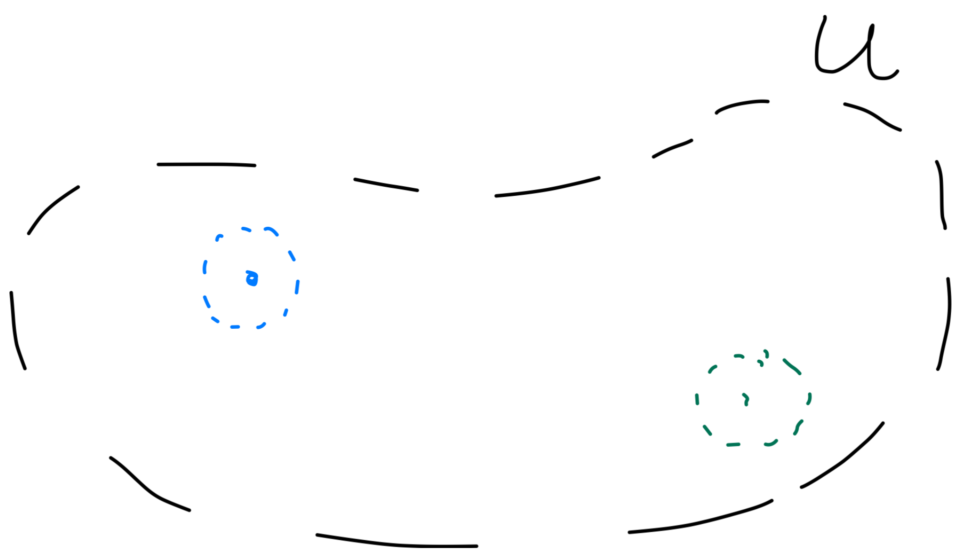
Example 7.14
The functions $\operatorname{Re} z, \operatorname{Im} z, |z|, \overline{z}$ are all continuous.
$\overline{z}$ is continuous since we showed

$$z_n \to z_0 \implies \overline{z_n} \to \overline{z_0}$$

$\operatorname{Re} z$: use $|\operatorname{Re} z_n - \operatorname{Re} z_0| \leq |z_n - z_0|$

7.4 Open Sets in $\mathbb{C}$

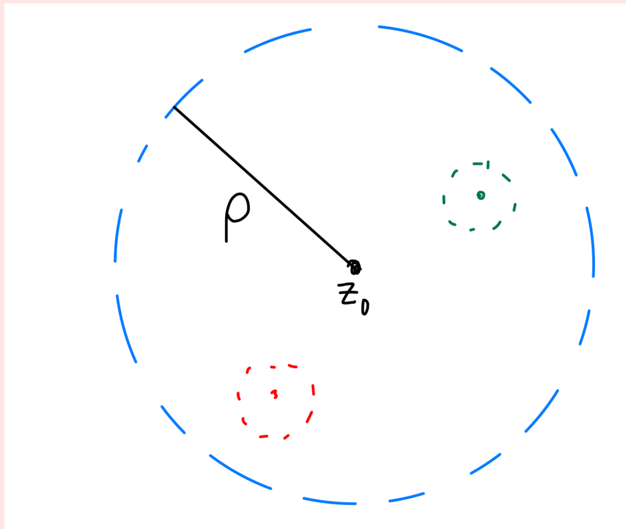
Definition 7.15 (Open set)
A subset $U \subseteq \mathbb{C}$ is <u>open</u> if for every $z \in U$, there exists a disk D with nonzero radius centered at z with $\overline{D} \subseteq U$.



Example 7.16
Any open disk,

$$\{z \in \mathbb{C} \colon |z - z_0| < \rho\}, \qquad \text{(centered at } z_0\text{, with radius } \rho\text{)}$$

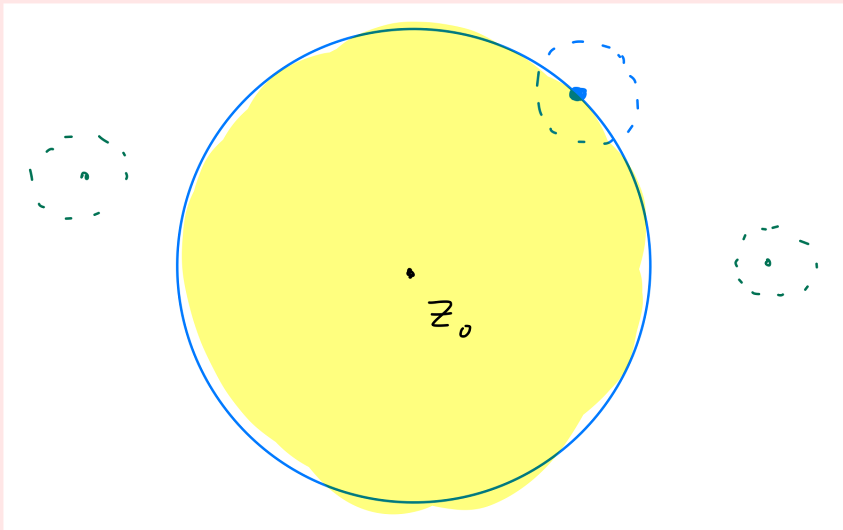
is open.



Example 7.17
Any closed disk,

$$\{z \in \mathbb{C} \colon |z - z_0| \le \rho\},$$

is not open.



The complement of this set is

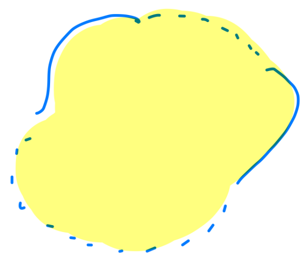
$$\{z \in \mathbb{C} \colon |z - z_0| > \rho\},$$

which is open.

Definition 7.18 (Closed set)
A set $F \subseteq \mathbb{C}$ is closed if its complement

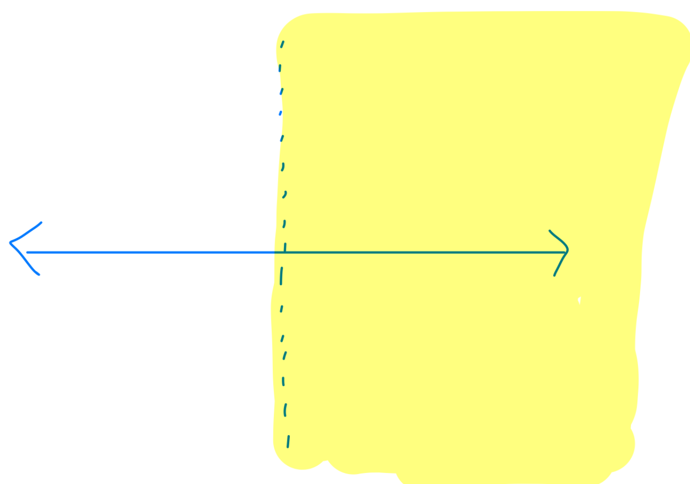
$$F^C = \mathbb{C} \setminus F$$

is open.

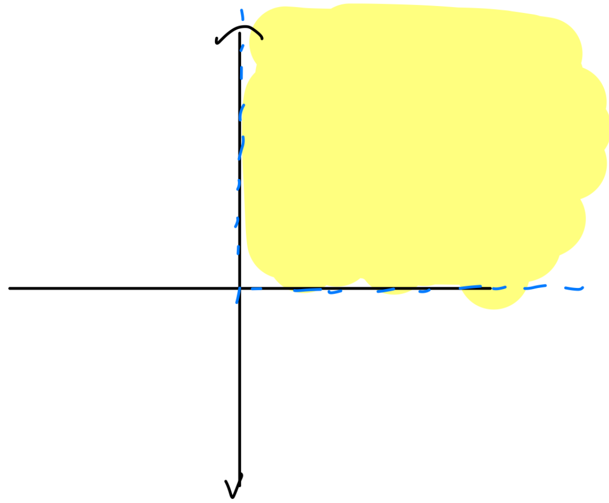


Example 7.19
Any set defined by strict inequalities and continuous functions is an open set.

E.g. the set $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ is open.



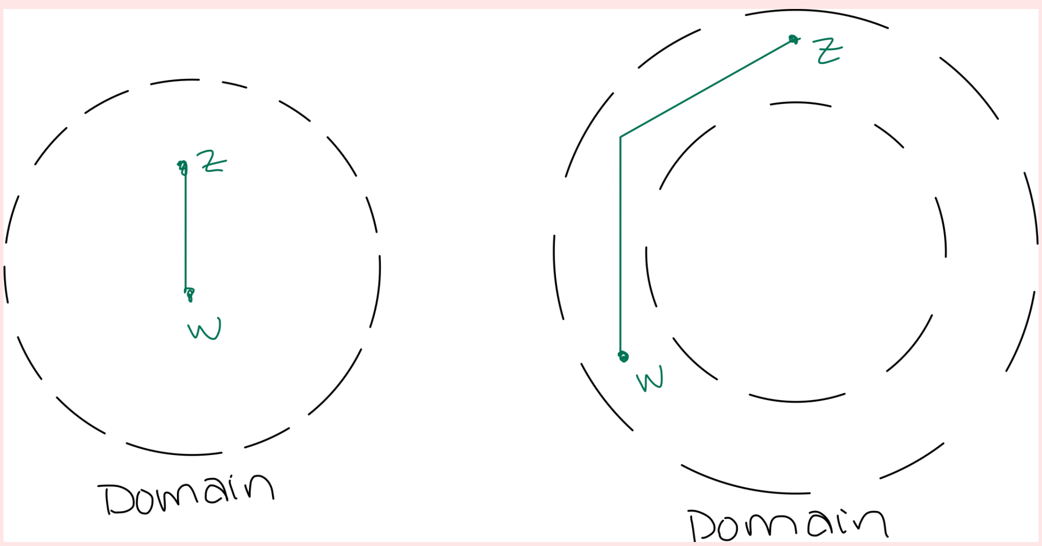
E.g. the set $\{z \in \mathbb{C} : 0 < \arg z < \pi/2\}$ is open.



Definition 7.20 (Domain)
A set $D \subseteq \mathbb{C}$ is called a domain if it is

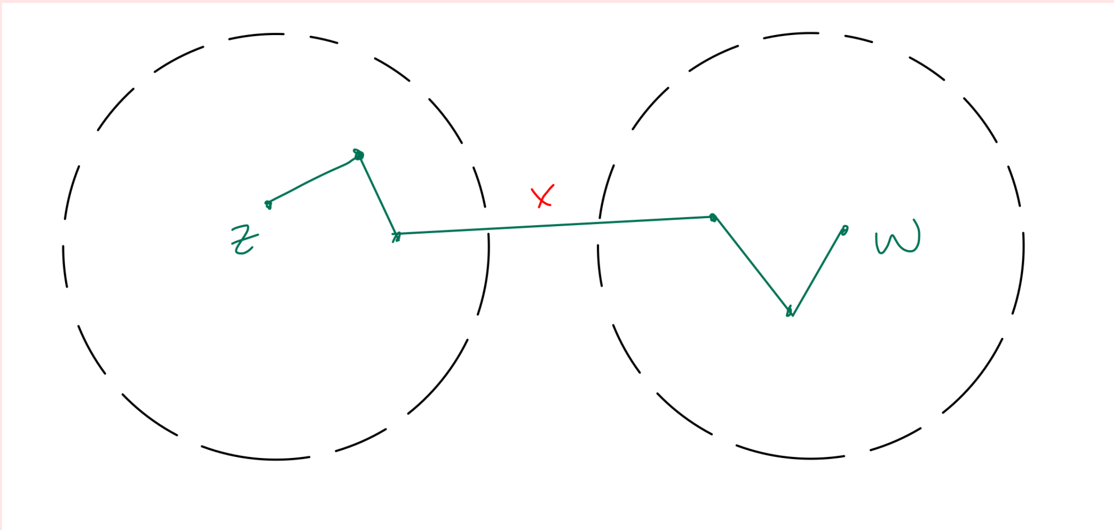
1. open; and
2. any two points $z, w \in D$ can be connected to each other by a finite series of line segments in D, where the end of one line segment is the start of the next.

Example 7.21
Open disks and open annuli $\{z \in \mathbb{C} \colon \rho_1 < |z| < \rho_2\}$ are domains.



Example 7.22

Disconnected sets are not domains



8 Jan 21, 2022

8.1 Analytic Functions

Definition 8.1 (Differentiable and complex derivative)
Let $f(z)$ be a complex-valued function defined in a disk centered at $z_0 \in \mathbb{C}$. $f(z)$ is differentiable at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. When this limit does exist, we denote it by

$$f'(z_0) \quad \text{or} \quad \frac{df}{dz}(z_0),$$

and call it the complex derivative of $f(z)$ at z_0.

The limit that defines the derivative can be equivalently expressed by

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Notation: $\Delta z = \Delta x + i\Delta y$
The simplest example: a constant function $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = \lambda$ for all $z \in \mathbb{C}$, is differentiable at every $z_0 \in \mathbb{C}$.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim \frac{\lambda - \lambda}{z - z_0} = 0$$

Example 8.2

Let m be a positive integer. The power function $f(z) = z^m$ is differentiable at every $z \in \mathbb{C}$. We show $f'(z) = (z^m)' = mz^{m-1}$.

We use the binomial expansion

$$(z + \Delta z)^m = \binom{m}{0} z^m + \binom{m}{1} z^{m-1}\Delta z + \cdots + \binom{m}{m-1} z\Delta z^{m-1} + \binom{m}{m}\Delta z^m$$

Note: $\binom{m}{0} = \binom{m}{m} = 1$, $\binom{m}{1} = m$

It follows that,

$$\begin{aligned}\frac{(z+\Delta z)^m - z^m}{\Delta z} &= \frac{1}{\Delta z}\left(mz^{m-1}\Delta z + \binom{m}{z} z^{m-2}\Delta z^2 + \cdots + \Delta z^m\right) \\ &= mz^{m-1} + \underbrace{\binom{m}{z} z^{m-2}\Delta z + \cdots + \Delta z^{m-1}}_{\to 0 \text{ as } \Delta z \to 0}\end{aligned}$$

Then,

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^m - z^m}{\Delta z} = mz^{m-1}$$

Next, we look at a non-example:

Example 8.3

The complex conjugation function $f(z) = \overline{z}$ is not differentiable anywhere.

$$\lim_{\Delta z \to 0} \frac{\overline{(z+\Delta z)} - \overline{z}}{\Delta z} = \lim_{\Delta \to 0} \frac{\overline{z} + \overline{\Delta z} - \overline{z}}{\Delta z}$$
$$= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

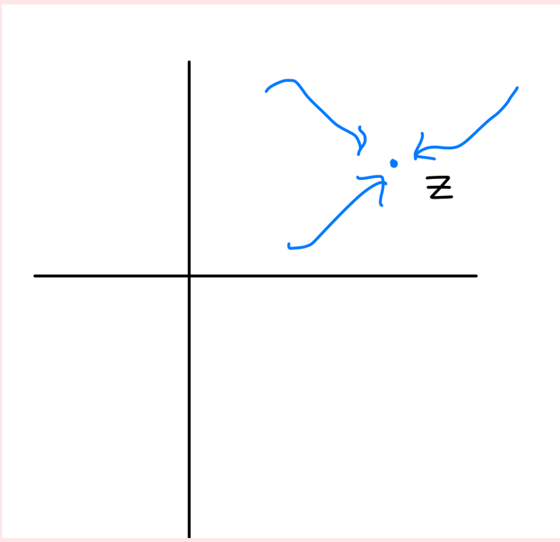
To show this limit does not exist, it suffices to show that $\frac{\overline{\Delta z}}{\Delta z}$ has two different limits as we approach from different directions.

1. If $\Delta z = \Delta x + i \cdot 0$ (approaching via horizontal line)

$$\lim_{\Delta x \to 0} \frac{\overline{\Delta x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

2. If $\Delta z = 0 + i\Delta y$ (approaching via vertical line)

$$\lim_{\Delta y \to 0} \frac{\overline{i\Delta y}}{i\Delta y} = \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1$$



Recall that many properties about limits of complex sequences and functions are proved with arguments that are nearly identical to the proofs from real analysis.

The same is true for complex derivatives and real derivatives. Using this observation, one can quickly establish the following properties of the complex derivative:

1. If $f(z)$ is differentiable at z_0, then $f(z)$ is continuous at z_0.
   Just use,

$$f(z) = f(z_0) + \underbrace{\left(\frac{f(z) - f(z_0)}{z - z_0}\right)(z - z_0)}_{f'(z)\cdot 0 \text{ as } z\to z_0}$$

Shows $f(z) \to f(z_0)$ as $z \to z_0$.

2. Differentiation rules:

$$\begin{aligned}(\lambda f)'(z) &= \lambda f'(z) \quad (\lambda \in \mathbb{C}),\\ (f+g)'(z) &= f'(z) + g'(z),\\ (fg)'(z) &= f(z)g'(z) + f'(z)g(z),\\ (f/g)'(z) &= \frac{g(z)f'(z) - f(z)g'(z)}{[g(z)]^2}, \quad \text{provided}\end{aligned}$$

g is nonzero on some disk centered at z

3. The chain rule is valid for complex derivatives:

Theorem 8.4
Suppose $g(z)$ is differentiable at z_0 and $f(w)$ is differentiable at $w_0 = f(z_0)$. Then the composition function $(f \circ g)(z) = f(g(z))$ is differentiable at z_0 and

$$(f \circ g)'(z_0) = f'(g(z_0))g'(z_0).$$

Example 8.5
Any polynomial function

$$a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + z_0$$

is everywhere differentiable, and its derivative is

$$na_n z^{n-1} + (n-1)a_{n-1}z^{n-2} + \cdots + a_1$$

Proved just using the power rule, constant multiple rule, sum rule for derivatives.

Example 8.6
For any integer m (even $m < 0$),

$$\frac{d}{dz}z^m = mz^{m-1}$$

If $m < 0$, then $m = -n$ for some $n = 1, 2, 3, \ldots$

$$z^m = z^{-n} = \frac{1}{z^n}.$$

Then by quotient rule,

$$\left(\frac{1}{z^n}\right)' = \frac{z^n \cdot 0 - 1 \cdot nz^{n-1}}{z^{2n}} = \frac{-nz^{n-1}}{z^{2n}}$$
$$= -nz^{-n-1} = mz^{m-1}$$

Example 8.7
We can differentiate $(z^2 - i)^{-1}$ using the chain rule:

$$\frac{d}{dz}(z^2 - i) = -(z^2 - i)^{-2} \cdot 2z$$

Definition 8.8 (Analytic on the open set)
A complex-valued function $f(z)$ is analytic on the open set $U \subseteq \mathbb{C}$ if

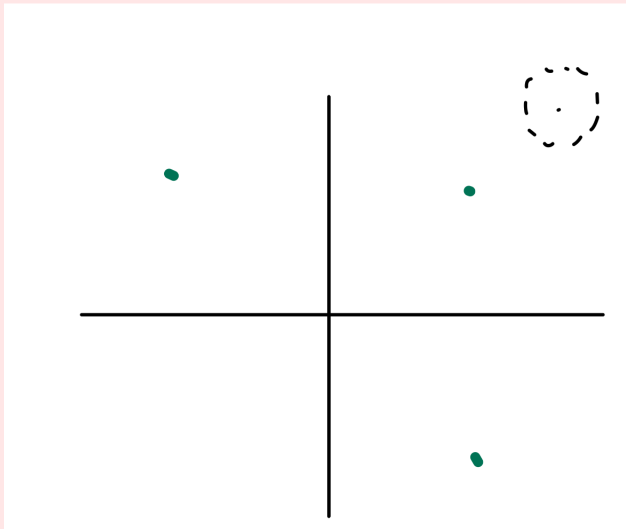
1. $f(z)$ is differentiable at every $z_0 \in U$; and, moreover,
2. $f'(z)$ is continuous on U.

Example 8.9
Polynomial functions are analytic on $\mathbb{C}$.
Polynomials are differentiable on $\mathbb{C}$, and their derivatives are polynomials, which are continuous everywhere.

Example 8.10
Rational functions are analytic wherever their denominators do not vanish.

$$f(z) = \frac{p(z)}{q(z)} \quad p(z), q(z) \text{ are polynomials}$$

f is not defined where $q(z) = 0$.



$U =$ all $\mathbb{C}$ except the roots of $q(z)$
U is open
$f(z)$ is differentiable on U by quotient rule.
$f'(z)$ is again a rational function, so continuous.

Definition 8.11 (Analytic at z_0)
$f(z)$ is analytic at $z_0 \in \mathbb{C}$ if there is an open disk centered at z_0 on which $f(z)$ is analytic.

Later we will prove one of the most important facts of complex analysis:
If condition (1) is satisfied in the definition of analytic on U, then condition (2) is automatically satisfied, i.e.,
If $f(z)$ is differentiable at every z_0 in an open set U, then $f'(z)$ is continuous on U.

This is very different than in real analysis:
There exists $g\colon \mathbb{R} \to \mathbb{R}$ which is differentiable on $\mathbb{R}$, but g' is not continuous on $\mathbb{R}$.

8.2 Cauchy-Riemann Equations

Much like a complex number, a complex valued function f(z) can be split into two parts: its real part and its imaginary part. Moreover, these two parts can be seen as functions of two real variables x and y, where $z = x + iy$.
In other words, we can express

$$f(z) = u(x, y) + iv(x, y),$$

where $u, v \colon \mathbb{R}^2 \to \mathbb{R}$ are real-valued functions on the real plane $\mathbb{R}^2$. i.e.

$$u(x, y) = \operatorname{Re} f(x + iy) \qquad\qquad v(x, y) = \operatorname{Im} f(x + iy)$$

Definition 8.12 (Cauchy-Riemann equations)
The functions $u(x, y), v(x, y)$ can help us establish when a complex function is analytic. The Cauchy-Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{*}$$

Theorem 8.13
A complex-valued function $f(z) = u(x, y) + iv(x, y)$ is analytic on a domain D if and only if the partial derivatives of $u(x, y)$ and $v(x, y)$ exists and are continuous on D and satisfy the Cauchy-Riemann equations (*) on D.

We will also see that, when the equivalent conditions in the theorem are satisfied, we have the equalities

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial x}(x, y) + i\frac{\partial v}{\partial x}(x, y) \\ &= \frac{\partial v}{\partial y}(x, y) - i\frac{\partial u}{\partial y}(x, y) \end{aligned}$$

Proof. To prove the theorem, we must establish both "directions" of the "if and only if".
($\implies$) Assume f is analytic on the domain D.
We need to show the partials exist, continuous on D and satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Let $z \in D$. We know

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = f'(z)$$

exists so taking $\Delta z = \Delta x$ and $\Delta z = i\Delta y$ will yield the same limit $f'(z)$.

$$\begin{aligned} \frac{f(z + \Delta x) - f(z)}{\Delta x} &= \frac{1}{\Delta x}\left[u(x + \Delta x, y) + iv(x + \Delta x, y) - \big(u(x, y) + iv(x, y)\big)\right] \\ &= \left(\frac{u(x + \Delta x, y) - u(x, y)}{\Delta x}\right) + i\left(\frac{v(x + \Delta x, y) - v(x, y)}{\Delta x}\right) \end{aligned}$$

Taking a $\lim_{\Delta x \to 0}$ on both sides, we have

$$f'(z) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y)$$

Since $f'(z)$ is continuous on D, it follows that $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}$ are also continuous on D. Next, we take $\Delta z = i\Delta y$

$$\begin{aligned}
\frac{f(z+i\Delta y) - f(z)}{i\Delta y} &= \frac{1}{i\Delta y}\left[u(x,y+\Delta y) + iv(x,y+\Delta y) - \Big(u(x,y) + iv(x,y)\Big)\right] \\
&= \frac{u(x,y+\Delta y) - u(x,y)}{i\Delta y} + \frac{i\Big(v(x,y+\Delta y) - v(x,y)\Big)}{i\Delta y} \\
&= \left(\frac{v(x,y+\Delta y) - v(x,y)}{\Delta y}\right) - i\left(\frac{u(x,y+\Delta y) - u(x,y)}{\Delta y}\right)
\end{aligned}$$

Take limits as $\Delta y \to 0$:

$$\left.\begin{aligned}
f'(z) &= \frac{\partial v}{\partial y}(x,y) - i\frac{\partial u}{\partial y}(x,y) \\
&= \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y)
\end{aligned}\right\} \text{equating real and imaginary points gives C-R equations}$$

We will prove ($\Longleftarrow$) in the next lecture. $\square$

Example 8.14

We can use the Cauchy-Riemann equations to show that e^z is analytic on all of $\mathbb{C}$.

$$e^z = \underbrace{e^x \cos y}_{u} + i\underbrace{e^x \sin y}_{v}$$

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y} = e^x \cos y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y \qquad \frac{\partial v}{\partial x} = e^x \sin y$$

Since C-R equations satisfied, all partials are continuous functions, it follows by theorem that e^z is analytic on $\mathbb{C}$.

9 Jan 24, 2022

9.1 Cauchy-Riemann Equations (Cont'd)

Proof of Theorem 8.13. Next, we prove the ($\Longleftarrow$) direction.
So, assume the partial derivatives of $u(x, y)$ and $v(x, y)$ exists and are continuous on D and satisfy the Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

We need to show $f(z)$ is analytic on D. We will use Taylor's Theorem on $u, v \colon \mathbb{R}^2 \to \mathbb{R}$ to get

$$\begin{aligned} u(x+\Delta x, y+\Delta y) &= u(x,y) + \frac{\partial u}{\partial x}(x,y)\Delta x + \frac{\partial u}{\partial y}(x,y)\Delta y + R(\Delta x, \Delta y) \\ v(x+\Delta x, y+\Delta y) &= v(x,y) + \frac{\partial v}{\partial x}(x,y)\Delta x + \frac{\partial v}{\partial y}(x,y)\Delta y + S(\Delta x, \Delta y) \end{aligned}$$

where $\dfrac{R(\Delta x, \Delta y)}{|\Delta z|} \to 0, \quad \dfrac{S(\Delta x, \Delta y)}{|\Delta z|} \to 0 \quad \text{as } \Delta z \to 0$

$$\begin{aligned} f(z+\Delta z) &= u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) \\ &= f(z) + \frac{\partial u}{\partial x}(x,y)\Delta x + \frac{\partial u}{\partial y}(x,y)\Delta y + R(\Delta x, \Delta y) + i\frac{\partial v}{\partial x}(x,y)\Delta x + i\frac{\partial v}{\partial y}(x,y)\Delta y \\ &+ iS(\Delta x, \Delta y) \end{aligned}$$

Use C-R to replace partial y derivatives

$$\begin{aligned} &= f(z) + \frac{\partial u}{\partial x}(x,y)\Delta x - \frac{\partial v}{\partial x}(x,y)\Delta y + R(\Delta x, \Delta y) + i\frac{\partial v}{\partial x}(x,y)\Delta x + i\frac{\Delta u}{\Delta x}(x,y)\Delta y + iS(\Delta x, \Delta y) \\ &= f(z) + \frac{\partial u}{\partial x}(x,y)\Delta z + i\frac{\partial v}{\partial x}(x,y)\Delta z + R(\Delta x, \Delta y) + iS(\Delta x, \Delta y) \end{aligned}$$

Next, we subtract $f(z)$ from both sides and divide by Δz:

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) + \frac{R(\Delta x, \Delta y) + iS(\Delta x, \Delta y)}{\Delta z}$$

where $\dfrac{R(\Delta x, \Delta y) + iS(\Delta x, \Delta y)}{\Delta z} \to 0$ as $\Delta z \to 0$ So taking limit as $\Delta z \to 0$ on both sides, we get LHS limit exists and

$$f'(z) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y)$$

Note: If you had used C-R to replace the x-derivatives, would have shown

$$f'(z) = \frac{\partial v}{\partial y}(x,y) - i\frac{\partial u}{\partial y}(x,y)$$

□

Example 9.1
We can use C-R to show (again) that $f(z) = \overline{z}$ is not analytic on any open set.

$$\begin{gathered} f(z) = \overline{z} = x - iy \\ u(x,y) = x, \quad v(x,y) = -y \\ \frac{\partial u}{\partial x} = 1 \quad \frac{\partial v}{\partial y} = -1 \\ \frac{\partial u}{\partial y} = 0 \quad \frac{\partial v}{\partial x} = 0 \end{gathered}$$

Since $\frac{\partial u}{\partial x} = 1 \neq -1 = \frac{\partial v}{\partial y}$ at any $z \in \mathbb{C}$, we conclude that $\overline{z}$ is not analytic anywhere.

Using the same method we used to show $(e^z)' = e^z$, we can also establish

$$\begin{gathered} (\sin z)' = \cos z, \quad (\cos z)' = -\sin z, \\ (\sinh z)' = \cosh z, \quad (\cosh z)' = \sinh z. \end{gathered}$$

Since the Cauchy-Riemann equations are about real functions, we can sometimes use results from real analysis to get conclusions about complex analytic functions.

Theorem 9.2
If $f(z)$ is analytic on a domain D and $f'(z) = 0$ for all $z \in D$, the $f(z)$ is constant on D.

Proof. We know $f'(z) = 0$ for all $z \in D$ and we know

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0 \\ &= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = 0 \end{aligned}$$

on D. Then,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$$

on all of D. From real analysis (calculus) u, v constant functions on D. Then $f = u + iv$ is constant on D. □

We can similarly prove:

Theorem 9.3
If $f(z)$ is analytic on a domain $D \subseteq \mathbb{C}$ and $f(z) \in \mathbb{R}$ for all $z \in D$, then $f(z)$ is constant on D.

Proof. Write $f(z) = u(x,y) + iv(x,y)$. Since $f(z) \in \mathbb{R}$ for all $z \in D$, it follows $v(x,y) = 0$ for all $z = x + iy \in D$. Then, $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$ on D. Since f is analytic, it satisfies C-R on D. Then,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0 \quad \text{on } D$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0 \quad \text{on } D$$

Then, u is constant on D so $f = u + i \cdot 0$ is constant on D. $\square$

What if we are expressing z in polar coordinates?

Proposition 9.4
The polar form of the Cauchy-Riemann equations is:

$$\frac{\partial u}{\partial r} = \frac{1}{r} \cdot \frac{\partial v}{\partial \theta}, \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

Proof. To show this, assume $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

$$u = u\Big(x(r,\theta), y(r,\theta)\Big) \quad x(r,\theta) = r\cos\theta$$
$$v = v\Big(x(r,\theta), y(r,\theta)\Big) \quad y(r,\theta) = r\sin\theta$$

Then we can use the multivariable chain rule:

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial \theta} = \frac{\partial v}{\partial x} \cdot (-r\sin\theta) + \frac{\partial v}{\partial y}(r\cos\theta)$$

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x} \cdot \cos\theta + \frac{\partial u}{\partial y}\sin\theta \\
&\overset{\text{(C-R)}}{=} \frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\sin\theta \\
&= \frac{1}{r}\left(\frac{\partial v}{\partial y} r\cos\theta - \frac{\partial v}{\partial x} r\sin\theta\right) \\
&= \frac{1}{r}\frac{\partial v}{\partial \theta}
\end{aligned}$$

And,

$$\begin{aligned}
\frac{\partial v}{\partial r} &= \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta \\
\frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta) \\
&= \frac{\partial v}{\partial y}(-r\sin\theta) - \frac{\partial v}{\partial x}(r\cos\theta) \\
&= -r\left(\frac{\partial v}{\partial y}\sin\theta + \frac{\partial v}{\partial x}\cos\theta\right) \\
&= -r\frac{\partial v}{\partial r}
\end{aligned}$$

So,

$$\begin{cases} \dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} \\ \dfrac{\partial u}{\partial \theta} = -r\dfrac{\partial v}{\partial r} \end{cases}$$

$\square$

Example 9.5

Consider $\operatorname{Log} z = \log|z| + i\operatorname{Arg} z = \log r + i\theta$.

$$\frac{\partial u}{\partial r} = \frac{1}{r} \quad \frac{\partial v}{\partial \theta} = 1 \quad \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$
$$\frac{\partial u}{\partial \theta} = 0 = \frac{\partial v}{\partial r}$$