Math 132 (Complex Analysis for Applications)
University of California, Los Angeles

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Winter 2022

These are my lecture notes for Math 132 (Complex Analysis for Applications), which is taught by Tyler James Arant. The textbook for this class is *Complex Analysis*, by Theodore W. Gamelin.

Contents

1 Jan 3, 2022

1.1 What are the Complex Numbers?

We first recall the basic algebraic properties of the real numbers, $\mathbb{R}$. For all $a, b, c \in \mathbb{R}$,

1. (Commutative law of addition): $a + b = b + a$
2. (Commutative law of multiplication): $a \cdot b = b \cdot a$
3. (Associative law of addition): $(a + b) + c = a + (b + c)$
4. (Associative law of multiplication): $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
5. (Distributive law): $a(b + c) = a \cdot b + a \cdot c$

The system of real numbers $\mathbb{R}$ has many more (non-algebraic) properties which make it suitable for calculus. However, it lacks a particular desirable property: $\mathbb{R}$ does not contain roots for all of its polynomial equations, e.g., there is not a solution to the equation

$$x^2 + 1 = 0 \quad \text{in } \mathbb{R}.$$

It turns out (by the non-trivial fundamental theorem of algebra) that we can get a number system for which every polynomial equation has a root by "appending" $i = \sqrt{-1}$ to $\mathbb{R}$.

Definition 1.1 (Complex number)

A <u>complex number</u> is an expression of the form

$$x + iy \quad \text{where } x, y \in \mathbb{R},$$

Two complex numbers $a + ib$ and $c + id$ are equal if and only if $a = c$ and $b = d$

We denote by $\mathbb{C}$ the set of all complex numbers.

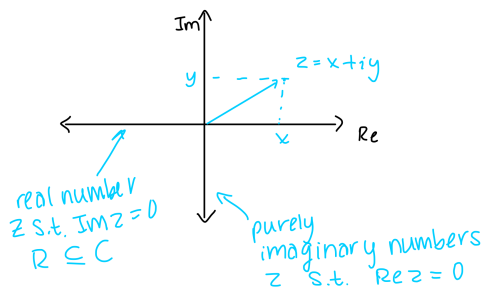For a complex number $z = x + iy$, we define its real and imaginary parts as follows:

$$\operatorname{Re} z = x$$
$$\operatorname{Im} z = y$$

There is a one-to-one correspondence between $\mathbb{C}$ and $\mathbb{R}^2$:

$$z \mapsto (\operatorname{Re} z, \operatorname{Im} z)$$

This can be visualized as the *complex plane*, where we can identify the real numbers and the *purely imaginary numbers*.

> **Example 1.2** (Addition and multiplication on $\mathbb{C}$)
> We can define operations of addition and multiplication on $\mathbb{C}$ as follows:
> $$z = x + iy, \quad w = a + ib$$
>
> $z + w = (x + iy) + (a + ib) = (x + a) + i(y + b)$
> $zw = (x + iy)(a + ib) = xa + ixb + iya + i^2yb$
> $= (xa - yb) + i(xb + ya)$

> **Example 1.3** (Multiplicative inverse in $\mathbb{C}$)
> Every nonzero complex number $z = x + iy \neq 0$ has a multiplicative inverse,
> $$\frac{1}{z} = \frac{x - iy}{x^2 + y^2}.$$
> Need to check $z \cdot \frac{1}{z} = 1$
> $$z \cdot \frac{1}{z} = (x + iy)\left(\frac{x - iy}{x^2 + y^2}\right) = \left(\frac{x^2 - ixy + ixy - i^2y^2}{x^2 + y^2}\right) = \left(\frac{x^2 + y^2}{x^2 + y^2}\right) = 1$$

In addition to having additive and multiplicative inverses, the complex numbers also have the following algebraic properties:

For all $z_1, z_2, z_3 \in \mathbb{C}$,

1. (Commutative law of addition): $z_1 + z_2 = z_2 + z_1$
2. (Commutative law of multiplication): $z_1 \cdot z_2 = z_2 \cdot z_1$
3. (Associative law of addition): $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
4. (Associative law of multiplication): $z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$
5. (Distributive law): $z_1(z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$

1.2 Complex Conjugates and the Modulus

> **Definition 1.4** (Complex conjugate)
> The <u>complex conjugate</u> of the number $z = x + iy$ is the number
> $$\overline{z} = x - iy.$$

Some basic facts about complex conjugation. All are simple to prove, so we only discuss the proof of a few.
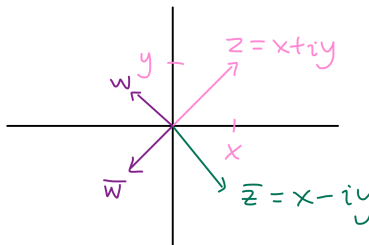
- $\overline{\overline{z}} = z$
- $z = \overline{z}$ if and only if z is a real number
- $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$
- $\overline{z_1 z_2} = \overline{z}_1 \cdot \overline{z}_2$
- $\overline{\left(\frac{1}{z}\right)} = 1/\overline{z}$

Proof. We want to show that $\overline{\left(\frac{1}{z}\right)} = 1/\overline{z}$.

$$\frac{1}{\overline{z}} = \frac{1}{x - iy} = \frac{x - (-iy)}{x^2 + y^2} = \frac{x + iy}{x^2 + y^2} = \overline{\left(\frac{x - iy}{x^2 + y^2}\right)} = \overline{\left(\frac{1}{z}\right)}$$

$\square$
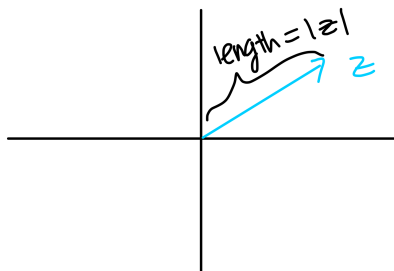
Geometrically, conjugation reflects z across the real axis:

Definition 1.5 (Absolute value/modulus)
The absolute value or modulus of $z = x + iy$ is
$$|z| = \sqrt{x^2 + y^2}$$

Geometrically, $|z|$ is the length of z as a vector in the complex plane:

Some properties relating complex conjugation and absolute value:

- $|z|^2 = z\overline{z}$

$$
\begin{aligned}
z\overline{z} &= (x + iy)(x - iy) = x^2 - ixy + ixy - i^2 y^2 \\
&= x^2 + y^2 \\
&= |z|^2
\end{aligned}
$$

- $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$

$$\frac{1}{z} = \frac{x - iy}{x^2 + y^2} = \frac{\overline{z}}{|z|^2}$$

- We have

$$\operatorname{Re} z = \frac{z + \overline{z}}{2}, \quad \operatorname{Im} z = \frac{z - \overline{z}}{2i}$$

$$\frac{z + \overline{z}}{2} = \frac{x + iy + x - iy}{2} = \frac{2x}{2} = x$$

Note: $\frac{1}{i} = -i$

- For $z, w \in \mathbb{C}, |zw| = |z| \cdot |w|$.

$$|z|^2 \cdot |w|^2 = z\overline{z} \cdot w\overline{w} = (zw)(\overline{z} \cdot \overline{w})$$
$$= (zw)\overline{(zw)} = |zw|^2$$

Then take a square root.

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2.1　Distance in the Complex Plane

We can use absolute value to measure the distance between complex numbers (thought of as vectors in the complex plane).

The distance between complex numbers z_1, z_2 is $|z_1 - z_2|$:

$$z_1 = x_1 + iy_1, \quad z_2 = x_2 + iy_2$$

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

A crucial fact about working with absolute values is

Proposition 2.1 (Triangle Inequality)
For any two complex numbers z_1, z_2,

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Some corollaries of the Triangle Inequality:

Corollary 2.2
For any complex numbers z_1, z_2, w,
- $|z_1 - z_2| \leq |z_1 - w| + |w - z_2|$.
- $|z_2| - |z_1| \leq |z_2 - z_1|$　　　(Reverse triangle inequality)

Proof.

-
$$\begin{aligned}
|z_1 - z_2| &= |(z_1 - w) + (w - z_2)| \\
&\leq |z_1 - w| + |w - z_2| \quad\quad\quad\quad\quad\quad\quad\quad \text{(Triangle Inequality)}
\end{aligned}$$

-
$$|z_2| = |(z_2 - z_1) + z_1| \leq |z_2 - z_1| + |z_1|$$
$$\implies |z_2| - |z_1| \leq |z_2 - z_1|$$

By symmetry, also $|z_1| - |z_2| \leq |z_2 - z_1|$

$\square$

2.2　Complex Polynomials

Definition 2.3 (Complex polynomial)
A complex polynomial of degree $n \geq 0$ is a function of the form

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 + a_0, \quad z \in \mathbb{C},$$

where the coefficients $a_0, a_1, \ldots, a_n$ are complex numbers with $a_n \neq 0$.

> **Theorem 2.4** (Fundamental Theorem of Algebra)
> Every complex polynomial $p(z)$ of degree $n \geq 1$ has a factorization
> $$p(z) = c(z - z_1)^{m_1}(z - z_2)^{m_2} \cdots (z - z_k)^{m_k},$$
> where $c \in \mathbb{C}$, the roots $z_1, \ldots, z_n$ are distinct complex numbers, and $m_1, \ldots, m_k \geq 1$. This factorization is unique, up to permutation of the factors. Also note that
> $$\sum_{j=1}^{k} m_k = n = \text{the degree of the polynomial}$$

| **Proof.** We will prove this later.   □

> **Example 2.5**
> Consider $p(z) = iz^2 + i$.
> $p(z) = iz^2 + i = i(z^2 + 1) = i(z - i)(z + i)$

2.3 Polar Representation

A nonzero complex number $z \in \mathbb{C}$ can be described by two quantities:
- its length $|z| = r$
- The angle z makes with the positive real axis.

> **Definition 2.6** (Polar representation)
> The <u>polar representation</u> of $z \neq 0$ is:
> $$z = r(\cos(\theta) + i\sin(\theta))$$
> Given $z = r(\cos(\theta) + i\sin(\theta))$ we can recover the Cartesian coordinates for z:
> $$x = r\cos(\theta)$$
> $$y = r\sin(\theta)$$

> **Definition 2.7** (Argument)
> For a nonzero $z = r(\cos(\theta) + i\sin(\theta))$, the angle θ is called the <u>argument</u> of z, and is denoted $\theta = \arg z$. But the argument of $z, \arg z$, is actually a multivalued function:
> $$\arg z + 2\pi k, \quad k \text{ an integer}$$
> also represents the same angle

> **Definition 2.8** (Principal argument)
> The <u>principal argument</u> of $z \neq 0$, denoted $\text{Arg } z$, is the unique argument θ which is in $(-\pi, \pi]$.
> $$\arg z = \{\text{Arg } z + 2k\pi \colon k \text{ integer}\} \qquad \text{``multivalued function''}$$
> $$\text{Arg } z \qquad \text{``single-valued''}$$

Example 2.9
Consider the complex number $1 + i$.
$$|1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2}$$
$$1 + i = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$$
$$\operatorname{Arg}(1 + i) = \frac{\pi}{4}$$

We introduce the notation
$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \quad \text{for } \theta \in \mathbb{R}.$$

We will see that this equality does follow from how we define complex exponentiation, but in the meantime it is just some convenient notation since the polar form for $z \in \mathbb{C}$ becomes:
$$z \neq 0, z = re^{i\theta},$$
$$\text{where } \theta = \operatorname{Arg} z, r = |z|$$

Example 2.10
We write the polar form using complex exponentials for the numbers -1 and $\frac{1}{2} + i\frac{\sqrt{3}}{2}$.
$$\operatorname{Arg}(-1) = \pi, |-1| = 1$$
$$-1 = e^{i\pi}$$
$$\frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{i\frac{\pi}{3}}$$
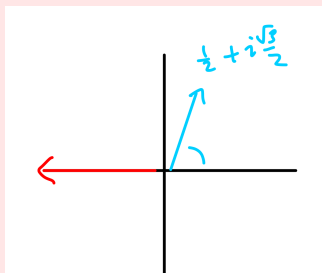
Some helpful identities for $e^{i\theta}, \theta \in \mathbb{R}$:
$$|e^{i\theta}| = 1, \quad \overline{e^{i\theta}} = e^{-i\theta}, \quad \frac{1}{e^{i\theta}} = e^{-i\theta}.$$

Proof.
$$\overline{e^{i\theta}} = \cos(\theta) - i\sin(\theta) = \cos(-\theta) + i\sin(-\theta)$$
$$= e^{-i\theta}$$

Check $e^{i\theta}e^{-i\theta} = 1$
use $\cos^2(\theta) + \sin^2(\theta) = 1$ □

Another very important identity for $\theta, \varphi \in \mathbb{R}$:
$$e^{i(\theta + \varphi)} = e^{i\theta}e^{i\varphi}.$$

Just use angle sum formulas

$$\cos(\theta + \varphi) = \cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi)$$
$$\sin(\theta + \varphi) = \cos(\theta)\sin(\varphi) + \sin(\theta)\cos(\varphi)$$

Example 2.11

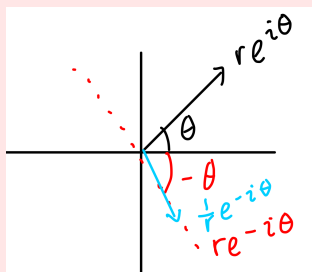If $z = re^{i\theta}$ is a nonzero complex number, then

$$\frac{1}{z} = \frac{1}{r}e^{-\theta}, \quad \overline{z} = re^{-i\theta}.$$

$$\left(\frac{1}{r}e^{-i\theta}\right) = \left(r \cdot \frac{1}{r}\right)e^{i\theta + (-i\theta)} = 1$$

$$\implies \frac{1}{r}e^{-i\theta} = \frac{1}{z}$$

Polar form can also help us understand multiplication of complex numbers geometrically in the complex plane:

Example 2.12 (Inversion)

$$z = re^{i\theta} \neq 0 \quad \frac{1}{z} = \frac{1}{r}e^{-i\theta}$$
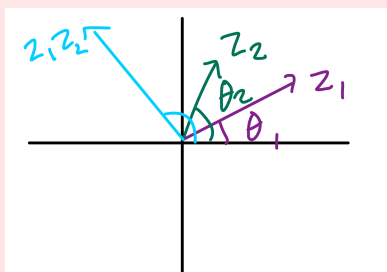

Example 2.13 (Multiplication)

$$z_1 = r_1 e^{i\theta_1}, \quad z_2 = r_2 e^{i\theta_2}$$
$$z_1 \cdot z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

To multiply two complex numbers, multiply their lengths and add their arg's.

2.4 De Moivre's Formulae

Theorem 2.14 (De Moivre's Formulae)
For any natural number n and any $\theta \in \mathbb{R}$, we have
$$\cos(n\theta) + i\sin(n\theta) = e^{in\theta} = (e^{i\theta})^n = (\cos(\theta) + i\sin(\theta))^n$$
Once the right-hand-side is expanded, we can obtain expressions for $\cos(n\theta)$ and $\sin(n\theta)$ as polynomials is $\cos(\theta)$ and $\sin(\theta)$. These qualities are known as <u>de Moivre's formulae</u>.

Example 2.15
We obtain de Moivre's formulae for the case $n = 2$.
$$\begin{aligned}
\cos(2\theta) + i\sin(2\theta) &= (\cos(\theta) + i\sin(\theta))^2 \\
&= \cos^2(\theta) - \sin^2(\theta) + i\left(2\cos(\theta)\sin(\theta)\right)
\end{aligned}$$

$$\overset{\text{equating Re=Im}}{\Longrightarrow} = \begin{cases} \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) \\ \sin(2\theta) = 2\cos(\theta)\sin(\theta) \end{cases}$$

Exercise. See the textbook for a derivation of the formulae
$$\cos(3\theta) = \cos^3(\theta) + 3\cos(\theta)\sin^2(\theta), \quad \sin(3\theta) = 3\cos^2(\theta)\sin(\theta) - \sin^3(\theta).$$

2.5 nth Roots

Definition 2.16 (nth roots)
A complex number z is an <u>nth root</u> of w if $z^n = w$. In other words, the nth roots of w are precisely the roots of the polynomial $p(z) = z^n - w$. As an immediate consequence: any $z \in \mathbb{C}$ has at most n many distinct nth roots.

For a nonzero complex number w, we can find its nth roots as follows:
Let $w = \rho e^{i\varphi} \quad z = re^{i\theta}$.
For z to be an nth root of w, we need $z^n = w$

$$\Longrightarrow r^n e^{in\theta} = \rho e^{i\varphi}$$

$$\Longrightarrow r^n = \rho, \quad n\theta = \varphi$$

or,

$$n\theta = \varphi + 2k\pi \quad k \in \mathbb{Z}.$$

$$\Longrightarrow r = \rho^{1/n}, \quad \theta = \frac{\varphi + 2k\pi}{n},$$

distinct angles are
$$\theta = \frac{\varphi + 2k\pi}{n}, \quad k = 0, 1, 2, \ldots, n-1$$

Therefore,
$$n\text{th roots of w:} \quad z_k = \rho^{1/n} e^{i\left(\frac{\varphi + 2k\pi}{n}\right)}, \quad k = 0, 1, \ldots, n-1$$
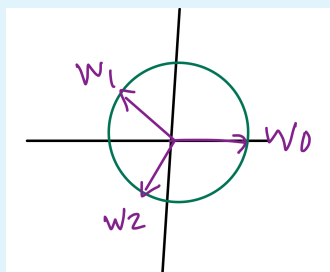
Exercise. Find and plot the 3rd roots of $w = 9i$.
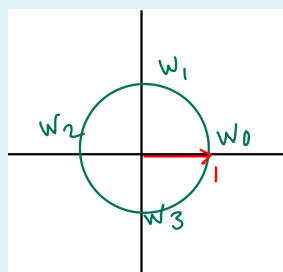

Definition 2.17 (nth roots of unity)

The nth roots of 1 have a special name: they are called the nth roots of unity. Using the same procedure as above, we can find that the nth roots of unity are

$$w_k = e^{2\pi i k/n}, \quad \text{for } k = 0, 1, \ldots, n-1.$$

3rd roots of unity:

4th roots of unity:

---

nth roots of unity can also be used to find nth roots of complex numbers other than 1. For a nonzero $z = re^{i\theta}$, we can find the first nth root of z to be $z_0 = r^{1/n}e^{i\theta/n}$. Then, if $w_0, \ldots, w_{n-1}$ are the nth root of unity, then the nth roots of z are exactly

$$z_k = z_0 w_k, \quad k = 0, \ldots, n-1.$$
$$z_k^n = (z_0 w_k) = z_0^n \cdot w_k^n = z \cdot 1 = z$$

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3.1   Stereographic Projection

In complex analysis, it is often useful to imagine that the complex plane has an "ideal point" at infinity, denoted ∞.
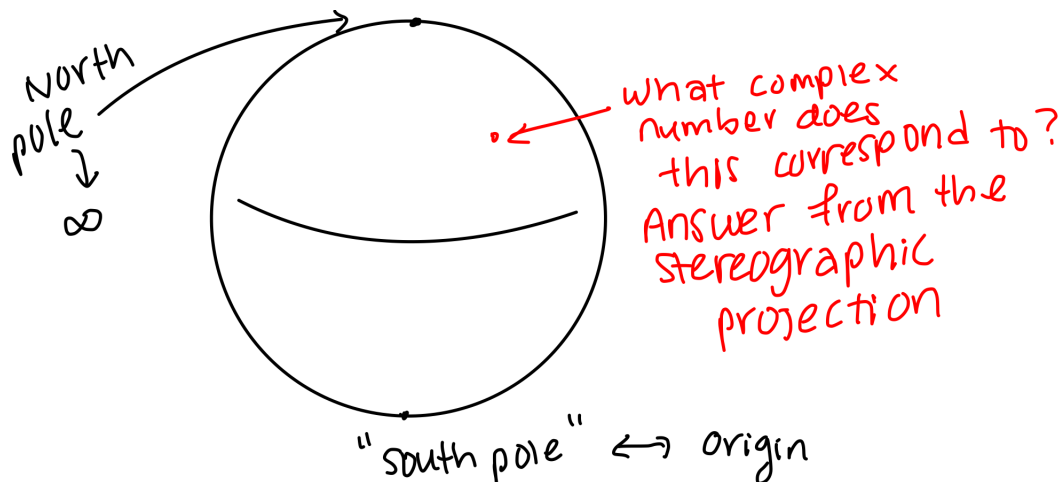
> **Definition 3.1** (Extended complex plane)
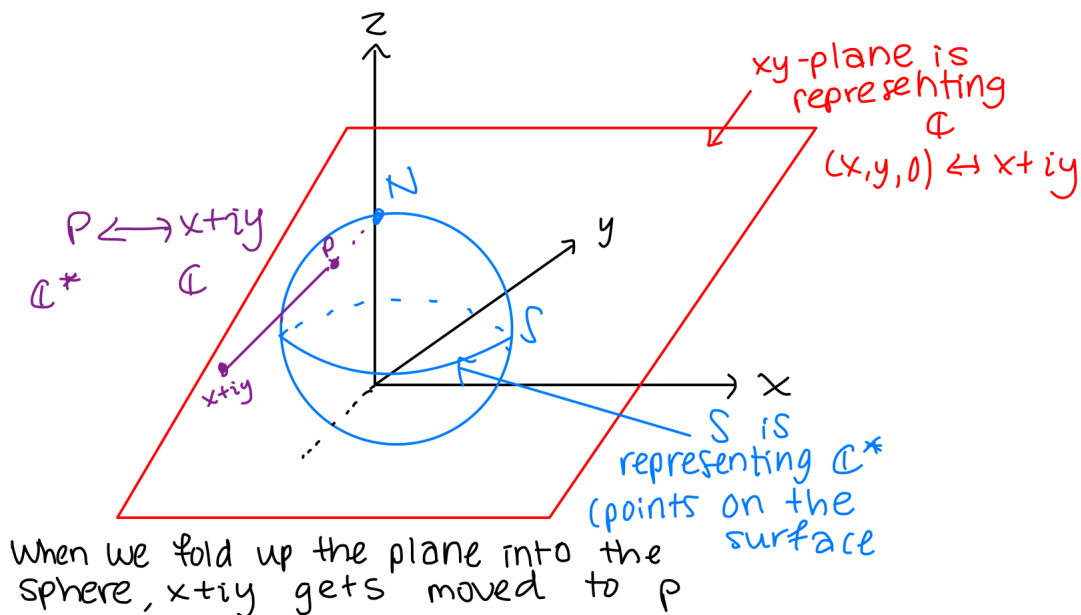> The <u>extended complex plane</u> is $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$.

∞ is not a complex number. Instead, it represents what becomes of a complex number if its modulus grows without bound. in other words, a sequence of complex numbers (z_n) is considered to approach ∞ if and only if the moduli $|z_n|$ are diverging to $+\infty$ (as a sequence of reals).
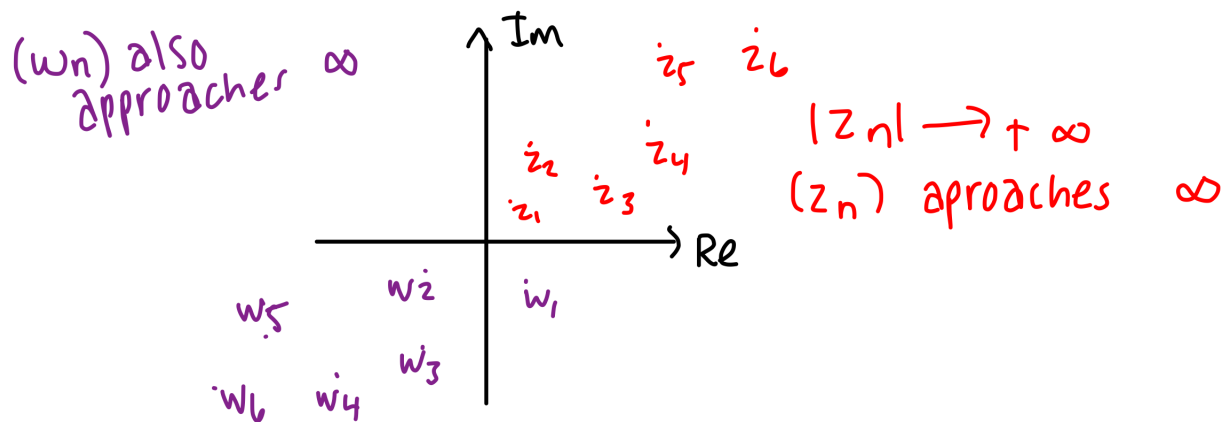
This leaves us with a problem: how can we visualize $\mathbb{C}^*$?
$\mathbb{C}^*$ will be visualized as a sphere:

North
pole
↓
∞

what complex
number does
this correspond to?
Answer from the
stereographic
projection

"south pole" ⟷ origin

We will use stereographic projection to visualize $\mathbb{C}^*$ as a sphere.
Start with 3-space, $\{(X, Y, Z) \colon X, Y, Z \in \mathbb{R}\}$, and consider the unit sphere.

xy-plane is
representing
$\mathbb{C}$
$(x, y, 0) \leftrightarrow x + iy$

$P \longleftrightarrow x + iy$
$\mathbb{C}^*$ $\mathbb{C}$

$x + iy$

N

y

S

x

S is
representing $\mathbb{C}^*$
(points on the
surface

When we fold up the plane into the
sphere, $x + iy$ gets moved to p

Any point in the xy-plane $(x, y, 0)$ will represent the complex number $x + iy$,
$$x + iy \sim (x, y, 0).$$
The unit sphere S represents $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$. with the north pole $N = (0, 0, 1)$ representing ∞.

But in what manner do points in S represent complex numbers in $\mathbb{C}$? To answer this, we will give a correspondence between points in $S \setminus \{N\}$ and the xy-plane.

The correspondence is called the stereographic projection, and is defined as follows:
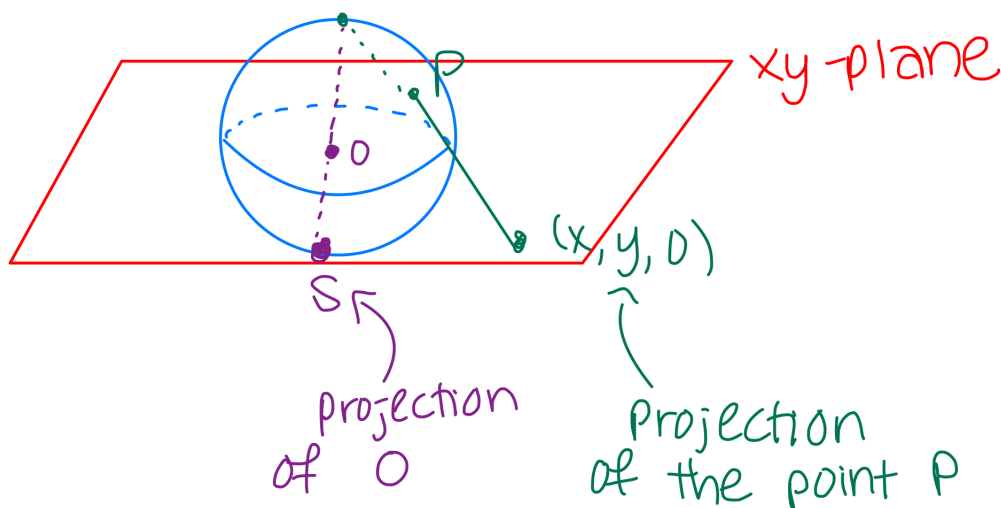
This projection gives a one-to-one correspondence between $S \setminus \{N\}$ and xy-plane.

Note that points on the lower half of the sphere correspond to complex numbers whose moduli are ≤ 1, while points on the upper half of the sphere correspond to complex numbers whose moduli are ≥ 1.

We have given a geometric description of the stereographic projection, but now we describe how to find an explicit formula for it.

We start with a given complex number $z = x + iy (\sim (x, y, 0))$ and seek a formula for the coordinates (X, Y, Z) for the point on the sphere which corresponds to $(x, y, 0)$ via the stereographic projection.
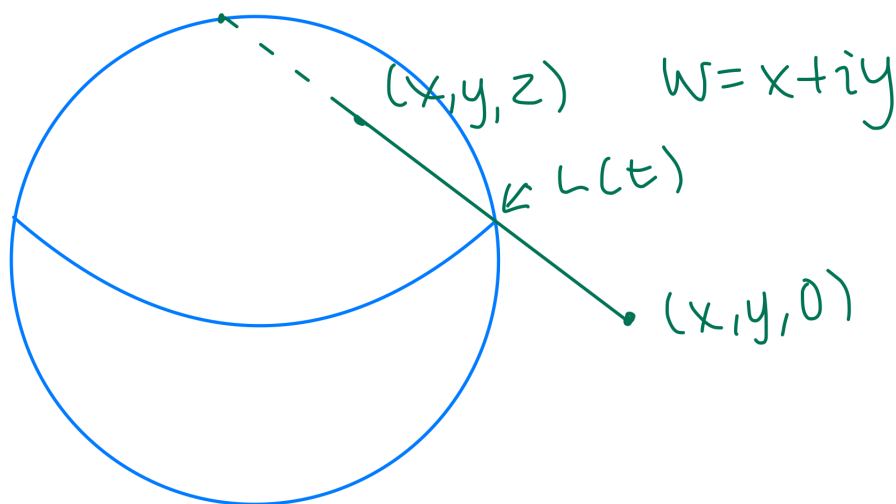
$$L(t) = (0, 0, 1) + t\left((X, Y, Z) - (0, 0, 1)\right)$$
$$= (tX, tY, t(Z-1) + 1)$$

There is t such that $L(t) = (X, Y, 0)$ i.e. $(tX, tY, t(Z-1)+1) = (x, y, 0)$

$$\implies t(Z-1) + 1 = 0 \Leftrightarrow t = \frac{1}{1-Z}$$

Note that:
$$Z = \frac{t-1}{t}, \quad tX = x, \quad tY = y$$

Since (X, Y, Z) is on S,

$$X^2 + Y^2 + Z^2 = 1 \qquad\qquad\qquad \text{Multiply both sides by } t^2$$
$$\implies (tX)^2 + (tY)^2 + (t-1)^2 = t^2$$
$$\implies |W|^2 + t^2 - 2t + 1 = t^2$$
$$\implies t = \frac{|W|^2 + 1}{2} \qquad\qquad\qquad \text{solved for } t \text{ in terms of given } w$$

Thus,

$$\begin{cases} X = \dfrac{1}{t}x = \dfrac{2x}{|w|^2 + 1} \\[2mm] Y = \dfrac{1}{t}y = \dfrac{2y}{|w|^2 - 1} \\[2mm] Z = \dfrac{t-1}{t} = \dfrac{|w|^2 - 1}{|w|^2 + 1} \end{cases}$$

$\left(\dfrac{2x}{|w|^2 + 1}, \dfrac{2y}{|w|^2 - 1}, \dfrac{|w|^2 - 1}{|w|^2 + 1} \right)$ is the unique point on the sphere that gets projected to $w = x + iy$.

We now explore what the stereographic projection does to geometric objects on the sphere. For example, it is not hard to see that lines of latitude on the sphere correspond to circles centered at 0 on the xy-plane.
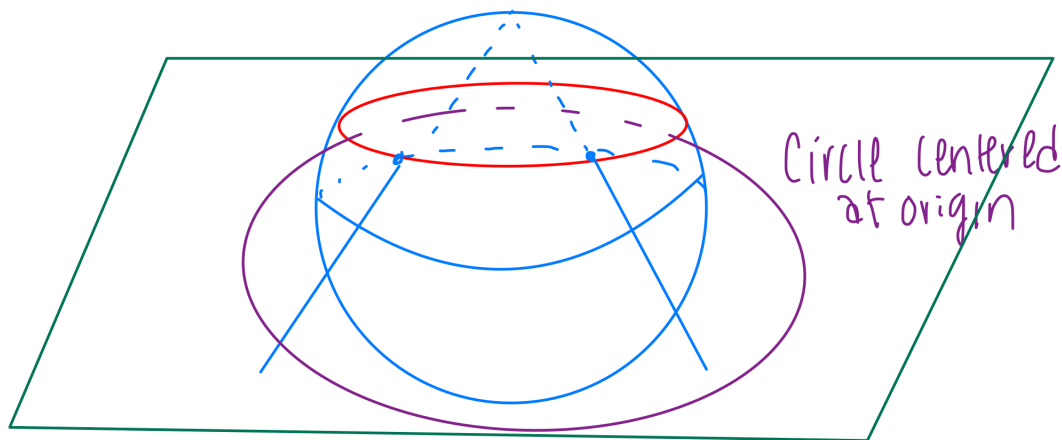
Moreover, lines of longitude on the sphere correspond to straight lines on the xy-plane.
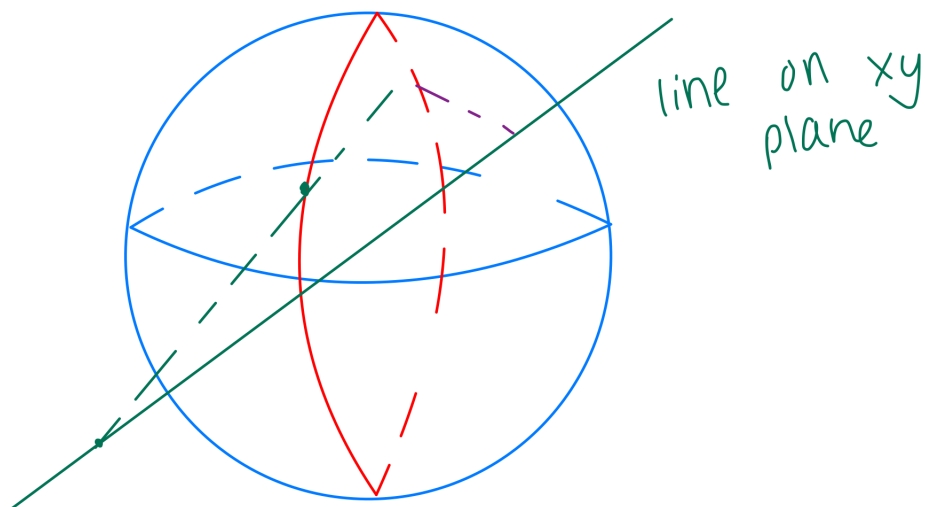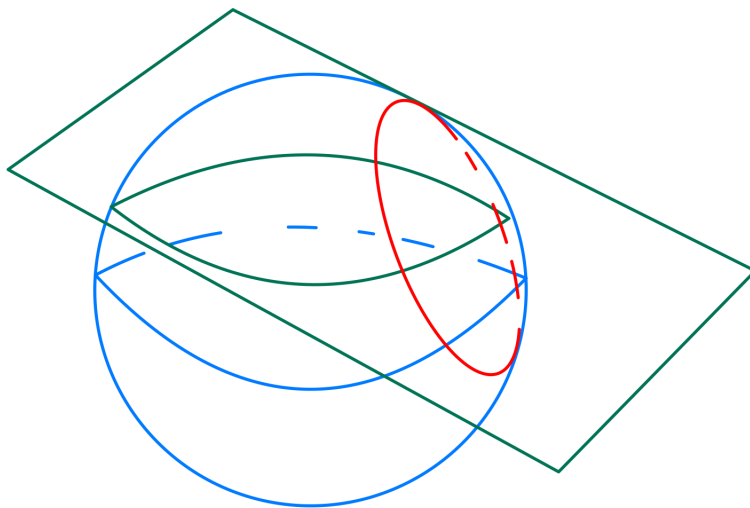
Next, we will prove more general theorem:

Theorem 3.2
Under the stereographic projection, circles on the sphere correspond to circles and lines on the xy-plane.

Here, "circles on the sphere" means any intersection of a plane in 3-space with the unit sphere.

In our proof, we will also use the fact that the set of points in the plane which satisfy an equation of the form
$$x^2 + y^2 + ax + by + c = 0 \quad (a, b, c, \in \mathbb{R})$$
is either a circle, a point, or empty.

[To see this, start by completing the square to get
$$(x + a/2)^2 + (y + b/2)^2 = (a^2 + b^2)/4 - c$$
Consider the three cases where the right-hand-side is $> 0, = 0, < 0$.]

Proof. Consider a circle on the sphere which is the intersection of the sphere with the plane $AX + BY + CZ = D$.

The projection of this curve in all $z = x + iy \mapsto (x, y, 0)$ which satisfy

$$A\frac{2x}{|z|^2 + 1} + B\frac{2y}{|z|^2 + 1} + C\frac{|z|^2 - 1}{|z|^2 + 1} = D$$

$$\implies 2Ax + 2By + C(|z|^2 - 1) = D(|z|^2 + 1)$$

$$\implies 2Ax + 2By + Cx^2 + Cy^2 - C = Dx^2 + Dy^2 + D$$

$$\implies (C - D)x^2 + (C - D)y^2 + 2Ax + 2By + (C - D) = 0$$

If $C = D$, then this defines a line.

If $C \neq D$, then by our previous remark (after dividing by $(-D)$), we can conclude it defines a circle. $\qquad\square$

3.2 Chordal Distance

This is a notion of distance for C^*.
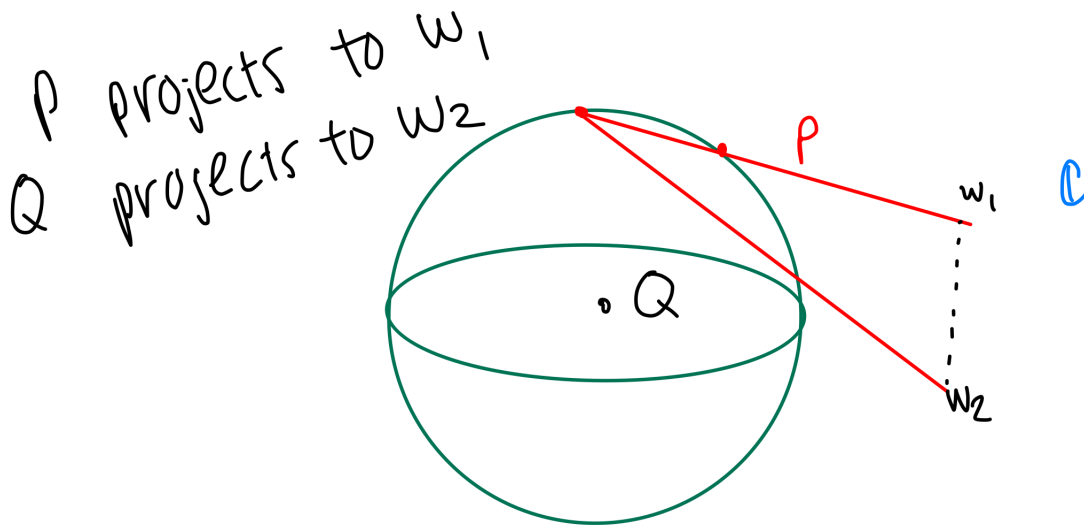
$d(w_1, w_2) = $ length of chord that connects P, Q (chordal)

$\mathbb{C}^*$ with d is a metric space:

1. $d(w_1, w_2) = d(w_2, w_1)$
2. $d(w_1, w_2) \geq 0$,
    $d(w_1, w_2) = 0 \Leftrightarrow w_1 = w_2$
3. $d(w_1, w_2) \leq d(w_1, z) + d(z, w_2)$

Formula:

$$d(w_1, w_2) = \frac{2|w_1 - w_2|}{\sqrt{(|w_1|^2 + 1)(|w_2|^2 + 1)}} \quad w_1, w_2 \in \mathbb{C}$$

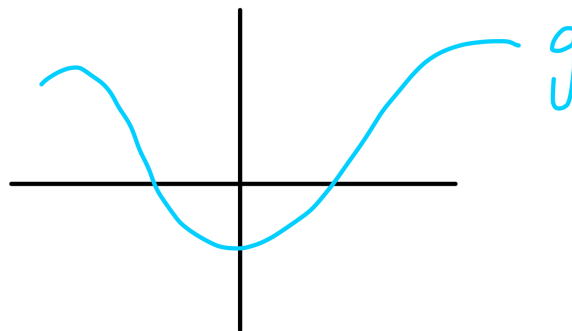$$d(w_1, \infty) = \frac{2}{\sqrt{(|w_1|^2 + 1)}} \quad w_1 \in \mathbb{C}$$

4　Jan 10, 2022

4.1　Visualizing Complex Functions

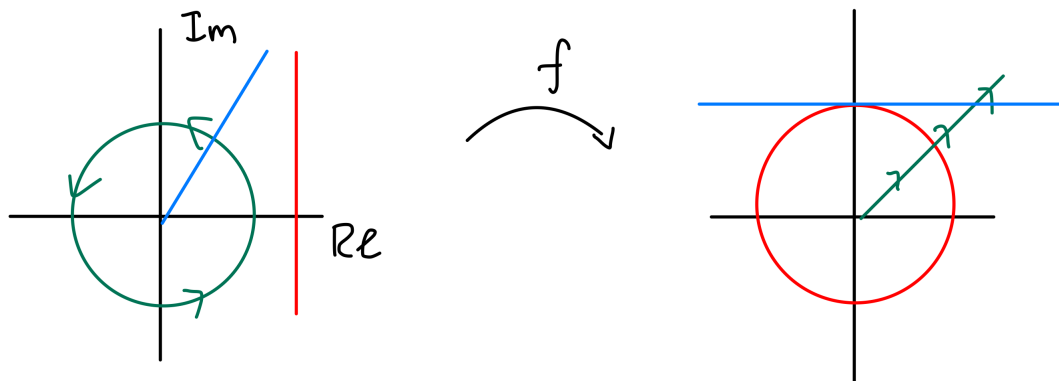A function $\mathbb{C}$ to $\mathbb{C}$, $f\colon \mathbb{C} \to \mathbb{C}$, maps a two dimensional space to a two dimensional space. Thus, the graph of $f\colon \mathbb{C} \to \mathbb{C}$,

$$G = \{(z, f(z)) \in \mathbb{C}^2 \colon z \in \mathbb{C}\}$$

is not as easy to visualize as a graph of a function $g\colon \mathbb{R} \to \mathbb{R}$

To visualize $f\colon \mathbb{C} \to \mathbb{C}$, our best tool is to analyze how f transforms various geometric objects.

4.2　The Square Function

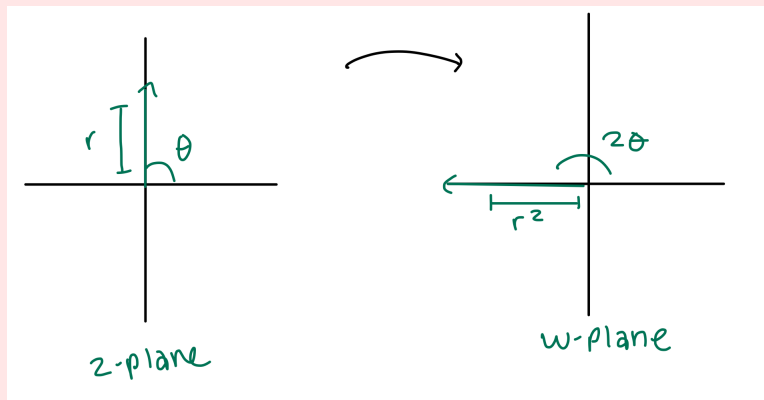Consider the square function $f\colon \mathbb{C} \to \mathbb{C}, f(z) = z^2$.

Example 4.1
We will also think of this function as being given by $w = z^2$, where the domain space is the z-plane and the codomain space is the w-plane.
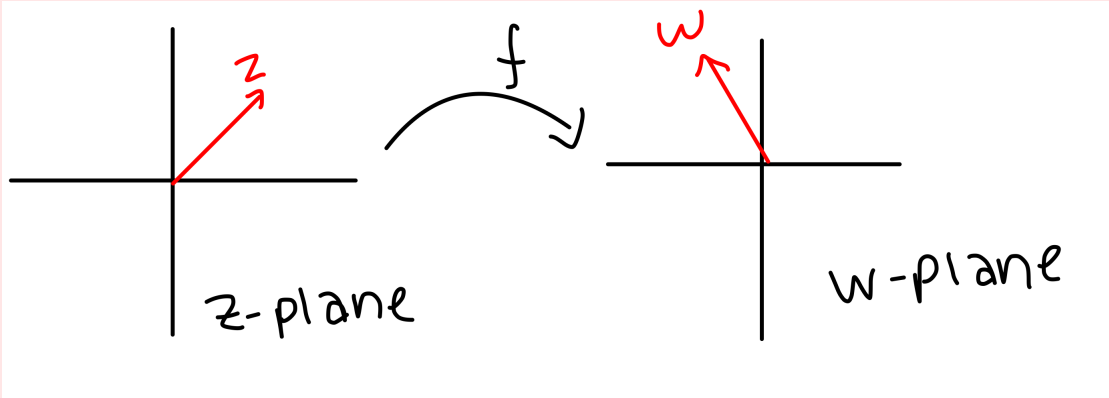
Example 4.2
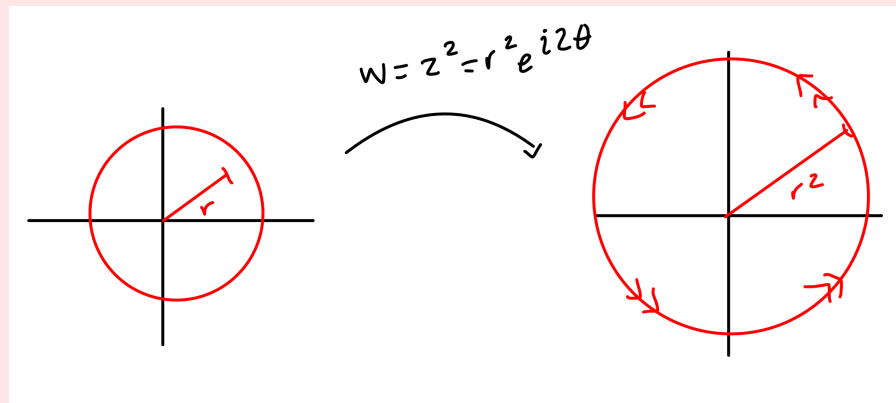Using polar representation, we can get an idea of what the square function does to a nonzero input $z = re^{i\theta}$:
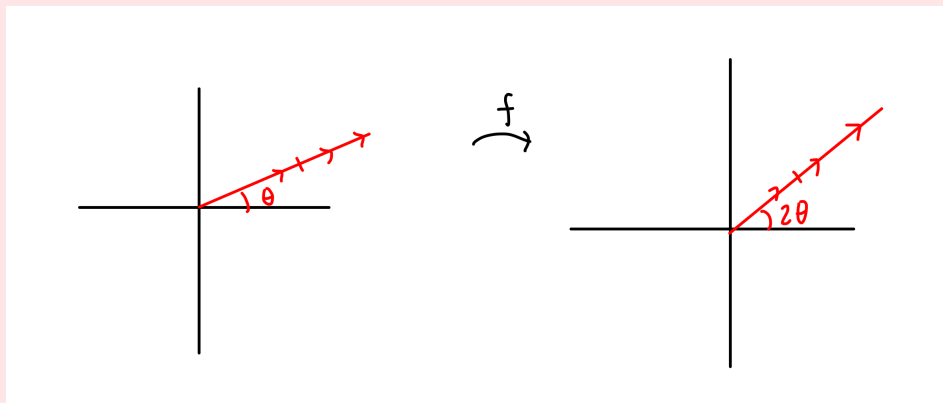$$z^2 = \left(re^{i\theta} \cdot re^{i\theta}\right) = r^2 e^{i2\theta}$$

We can extend our understanding by examining what the square function does the geometric objects in the $z-$plane.

Example 4.3
The square function transforms a circle centered at 0 to another circle centered at 0:
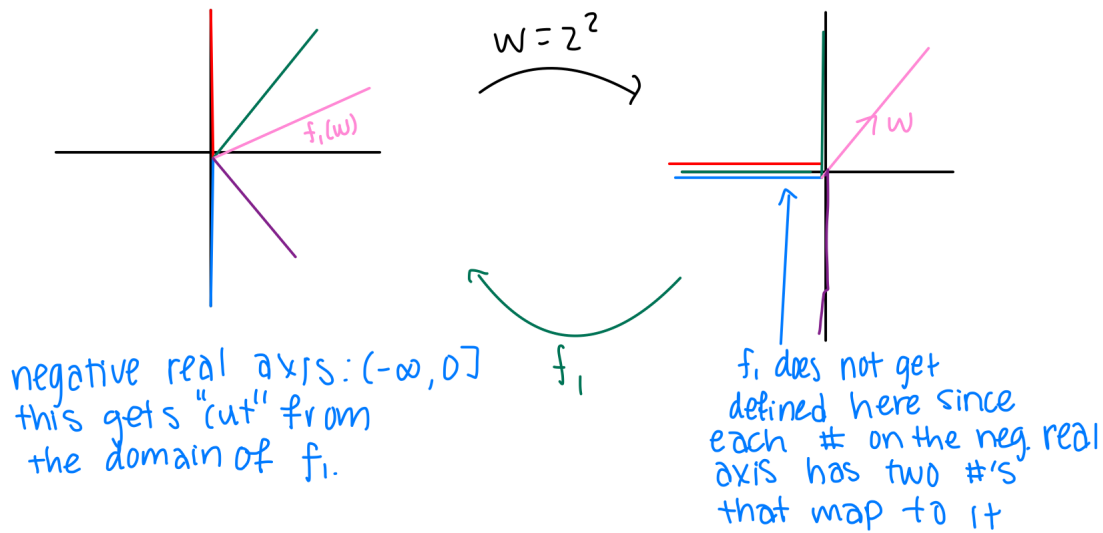

Example 4.4
A ray anchored at 0 is transformed into another ray anchored at 0:

We will now consider the problem of finding an inverse function for $w = z^2$, i.e., a square root function $z = \sqrt{w}$.

Just as in the case of the real numbers, ever nonzero complex number has two distinct square roots; as a result, there are many different ways of defining a square root operation.

We begin by considering how $w = z^2$ transforms the open half-plane $\{z \in \mathbb{C} \colon \operatorname{Re} z > 0\}$.

Example 4.5

$$W = z^2$$

$f_1(w)$

f_1

w

negative real axis: $(-\infty, 0]$
this gets "cut" from
the domain of f_1.

f_1 does not get
defined here since
each # on the neg. real
axis has two #'s
that map to it

Definition 4.6 (Principal branch)

Our description above gives us what we will call the <u>principal branch</u> of $\sqrt{w}$,

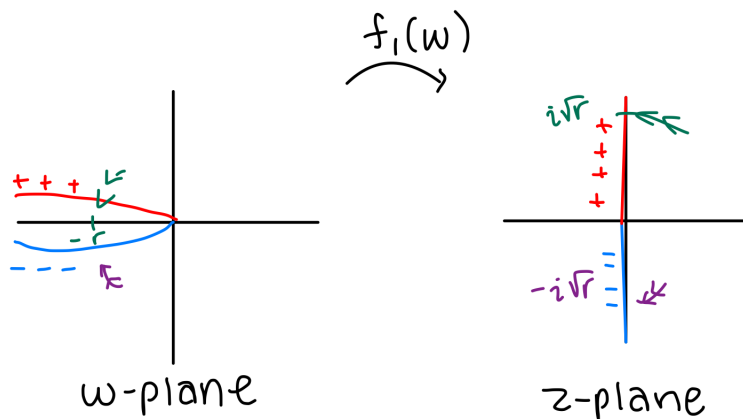$$f_1 \colon \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$$

which is algebraically given by

$$f_1(w) = \sqrt{\rho}e^{i\varphi}, \quad w = \rho e^{i\varphi}, \quad -\pi < \varphi < \pi.$$

We can also write this equation using Arg:

$$f_1(w) = \sqrt{|w|}e^{i\operatorname{Arg}w/2}$$

The principal branch f_1 is not defined on $(-\infty, 0]$, but it does have the following limit behavior there:
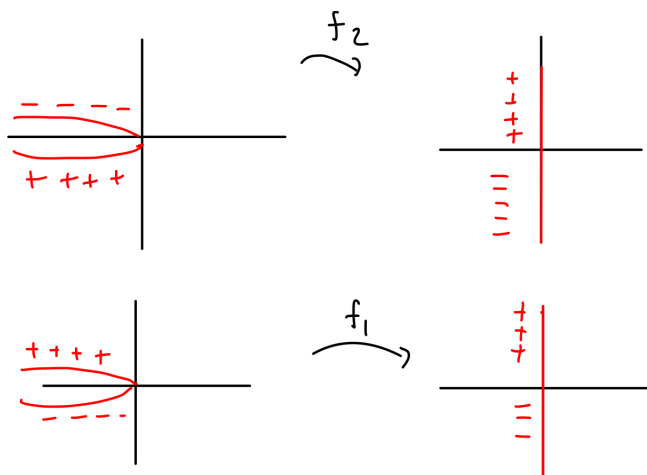
Notation:
$$f_1(-r + i0) = i\sqrt{r}$$
$$f_1(-r - i0) = -i\sqrt{r}$$

$f_1(w)$ chooses one of the possible square roots of w, but we can define another branch of $\sqrt{w}$ which selects the other square root.

We define the branch $f_2 \colon \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ simply by
$$f_2(w) = -f_1(w)$$

This function is still not defined on $(-\infty, 0]$, but it has the following (slightly different) limit behavior there:
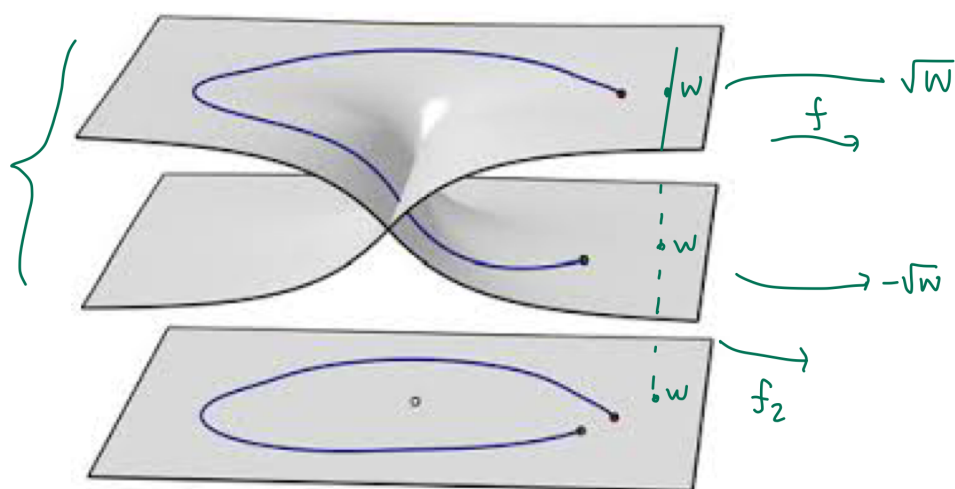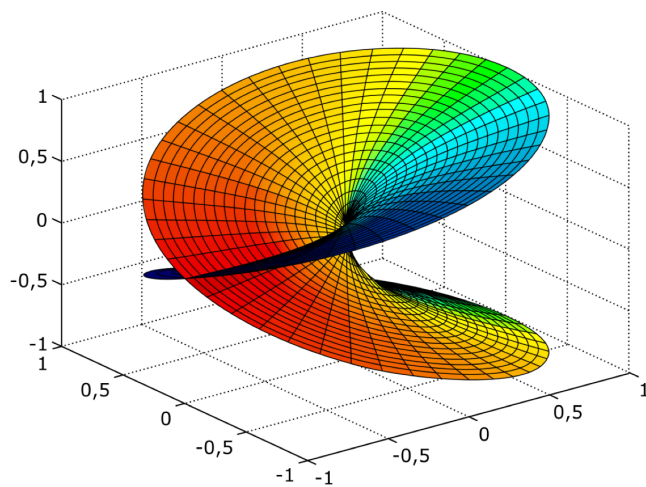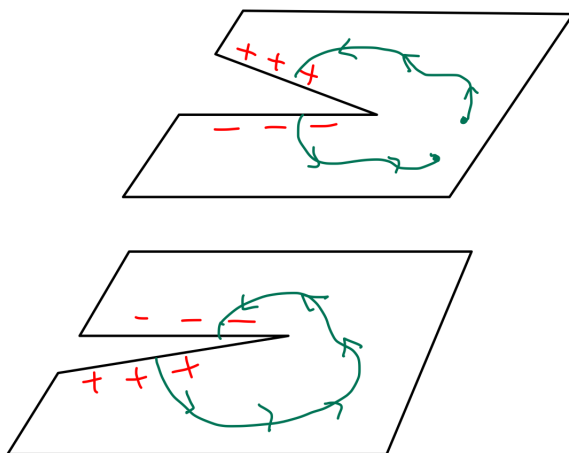

4.3 The Riemann Surface of $\sqrt{w}$

As we have said, the square root operation is multi-valued.

Book: "multi-valued function"

However, we can use our two branches above to construct a surface which can serve as a domain which makes this multi-valued operation a true function.

4.4 The Exponential Function

> **Definition 4.7** (Exponential function)
> We define the exponential function for all $z = x + iy \in \mathbb{C}$ by
> $$e^z = e^x \cos(y) + ie^x \sin(y)$$
> This, of course, is an extension of our previously used
> $$e^{i\theta} = \cos(\theta) + i\sin(\theta) \quad (\theta \in \mathbb{R}),$$
> and we can write
> $$e^z = e^x e^{iy}.$$
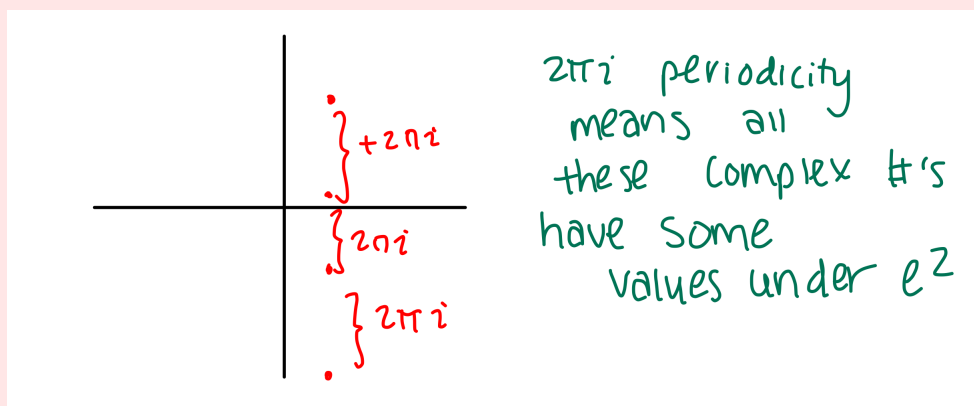
> **Definition 4.8** (λ-periodic)
> Let $\lambda \in \mathbb{C}$. A complex function $f(z)$ is said to be λ-periodic if
> $$f(z + \lambda) = f(z) \quad \forall z \in \mathbb{C}.$$

> **Example 4.9**
> The exponential function is $2\pi i$ periodic:
> $$e^{z+2\pi i} = e^z \cdot e^{2\pi i} = e^z \cdot 1 = e^z$$
>
>

Other fundamental properties of the exponential function (which are easy to verify): for all $z, w \in \mathbb{C}$,

    i. $e^{z+w} = e^z e^w$

   ii. $1/e^z = e^{-z}$

  iii. $\overline{e^z} = e^{\bar{z}}$

We now examine how $w = e^z$ transforms certain geometric objects.

Example 4.10

Horizontal lines are mapped to rays anchored at 0:

$$z = x + iy_0$$

$$w = e^z = e^x \cdot e^{iy_0}$$