Math 132 (Complex Analysis for Applications)
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These are my lecture notes for Math 132 (Complex Analysis for Applications), which is taught by Tyler James Arant. The textbook for this class is *Complex Analysis*, by Theodore W. Gamelin.

Contents

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1.1 What are the Complex Numbers?

We first recall the basic algebraic properties of the real numbers, $\mathbb{R}$. For all $a, b, c \in \mathbb{R}$,

1. (Commutative law of addition): $a + b = b + a$
2. (Commutative law of multiplication): $a \cdot b = b \cdot a$
3. (Associative law of addition): $(a + b) + c = a + (b + c)$
4. (Associative law of multiplication): $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
5. (Distributive law): $a(b + c) = a \cdot b + a \cdot c$

The system of real numbers $\mathbb{R}$ has many more (non-algebraic) properties which make it suitable for calculus. However, it lacks a particular desirable property: $\mathbb{R}$ does not contain roots for all of its polynomial equations, e.g., there is not a solution to the equation

$$x^2 + 1 = 0 \quad \text{in } \mathbb{R}.$$

It turns out (by the non-trivial fundamental theorem of algebra) that we can get a number system for which every polynomial equation has a root by "appending" $i = \sqrt{-1}$ to $\mathbb{R}$.

Definition 1.1 (Complex number)
A complex number is an expression of the form

$$x + iy \quad \text{where } x, y \in \mathbb{R},$$

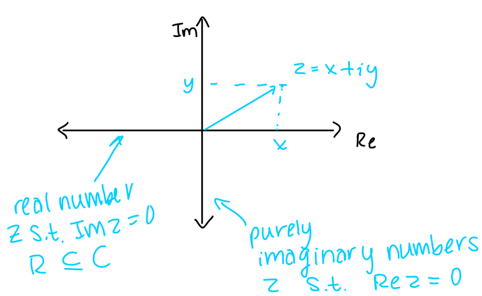
Two complex numbers $a + ib$ and $c + id$ are equal if and only if $a = c$ and $b = d$
We denote by $\mathbb{C}$ the set of all complex numbers.
For a complex number $z = x + iy$, we define its real and imaginary parts as follows:

$$\operatorname{Re} z = x$$
$$\operatorname{Im} z = y$$

There is a one-to-one correspondence between $\mathbb{C}$ and $\mathbb{R}^2$:

$$z \mapsto (\operatorname{Re} z, \operatorname{Im} z)$$

This can be visualized as the *complex plane*, where we can identify the real numbers and the *purely imaginary numbers*.



Example 1.2 (Addition and multiplication on $\mathbb{C}$)
We can define operations of addition and multiplication on $\mathbb{C}$ as follows:

$$z = x + iy, \quad w = a + ib$$

$z + w = (x + iy) + (a + ib) = (x + a) + i(y + b)$
$zw = (x + iy)(a + ib) = xa + ixb + iya + i^2 yb$
$= (xa - yb) + i(xb + ya)$

Example 1.3 (Multiplicative inverse in $\mathbb{C}$)
Every nonzero complex number $z = x + iy \neq 0$ has a multiplicative inverse,

$$\frac{1}{z} = \frac{x - iy}{x^2 + y^2}.$$

Need to check $z \cdot \frac{1}{z} = 1$

$$z \cdot \frac{1}{z} = (x + iy)\left(\frac{x - iy}{x^2 + y^2}\right) = \left(\frac{x^2 - ixy + ixy - i^2y^2}{x^2 + y^2}\right) = \left(\frac{x^2 + y^2}{x^2 + y^2}\right) = 1$$

In addition to having additive and multiplicative inverses, the complex numbers also have the following algebraic properties:
For all $z_1, z_2, z_3 \in \mathbb{C}$,

1. (Commutative law of addition): $z_1 + z_2 = z_2 + z_1$
2. (Commutative law of multiplication): $z_1 \cdot z_2 = z_2 \cdot z_1$
3. (Associative law of addition): $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
4. (Associative law of multiplication): $z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$
5. (Distributive law): $z_1(z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$

1.2 Complex Conjugates and the Modulus

Definition 1.4 (Complex conjugate)
The complex conjugate of the number $z = x + iy$ is the number

$$\overline{z} = x - iy.$$

Some basic facts about complex conjugation. All are simple to prove, so we only discuss the proof of a few.

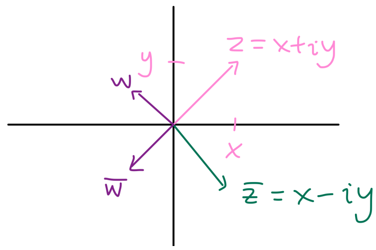
- $\overline{\overline{z}} = z$
- $z = \overline{z}$ if and only if z is a real number
- $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$
- $\overline{z_1 z_2} = \overline{z}_1 \cdot \overline{z}_2$
- $\overline{\left(\frac{1}{z}\right)} = 1/\overline{z}$

Proof. We want to show that $\overline{\left(\frac{1}{z}\right)} = 1/\overline{z}$.

$$\frac{1}{\overline{z}} = \frac{1}{x - iy} = \frac{x - (-iy)}{x^2 + y^2} = \frac{x + iy}{x^2 + y^2} = \overline{\left(\frac{x - iy}{x^2 + y^2}\right)} = \overline{\left(\frac{1}{z}\right)}$$

$\square$

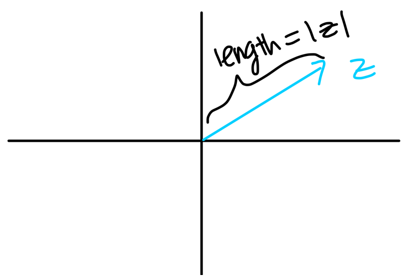
Geometrically, conjugation reflects z across the real axis:



Definition 1.5 (Absolute value/modulus)
The <u>absolute value</u> or <u>modulus</u> of $z = x + iy$ is

$$|z| = \sqrt{x^2 + y^2}$$

Geometrically, $|z|$ is the length of z as a vector in the complex plane:



Some properties relating complex conjugation and absolute value:

- $|z|^2 = z\overline{z}$

$$\begin{aligned} z\overline{z} = (x+iy)(x-iy) &= x^2 - ixy + ixy - i^2y^2 \\ &= x^2 + y^2 \\ &= |z|^2 \end{aligned}$$

- $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$

$$\frac{1}{z} = \frac{x - iy}{x^2 + y^2} = \frac{\overline{z}}{|z|^2}$$

- We have

$$\operatorname{Re} z = \frac{z + \overline{z}}{2}, \quad \operatorname{Im} z = \frac{z - \overline{z}}{2i}$$

$$\frac{z + \overline{z}}{2} = \frac{x + iy + x - iy}{2} = \frac{2x}{2} = x$$

Note: $\frac{1}{i} = -i$

- For $z, w \in \mathbb{C}$, $|zw| = |z| \cdot |w|$.

$$\begin{aligned} |z|^2 \cdot |w|^2 = z\overline{z} \cdot w\overline{w} &= (zw)(\overline{z} \cdot \overline{w}) \\ &= (zw)\overline{(zw)} = |zw|^2 \end{aligned}$$

Then take a square root.

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2.1 Distance in the Complex Plane

We can use absolute value to measure the distance between complex numbers (thought of as vectors in the complex plane).

The distance between complex numbers z_1, z_2 is $|z_1 - z_2|$:

$$z_1 = x_1 + iy_1, \quad z_2 = x_2 + iy_2$$

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

A crucial fact about working with absolute values is

Proposition 2.1 (Triangle Inequality)
For any two complex numbers z_1, z_2,

$$|z_1 + z_2| \le |z_1| + |z_2|$$

Some corollaries of the Triangle Inequality:

Corollary 2.2
For any complex numbers z_1, z_2, w,

- $|z_1 - z_2| \le |z_1 - w| + |w - z_2|$.
- $|z_2| - |z_1| \le |z_2 - z_1|$ (Reverse triangle inequality)

Proof.

- $$\begin{aligned} |z_1 - z_2| &= |(z_1 - w) + (w - z_2)| \\ &= |z_1 - w| + |w - z_2| \qquad \text{(Triangle Inequality)} \end{aligned}$$
- $$\begin{aligned} |z_2| = |(z_2 - z_1) + z_1| &\le |z_2 - z_1| + |z_1| \\ \implies |z_2| - |z_1| &\le |z_2 - z_1| \end{aligned}$$

  By symmetry, also $|z_1| - |z_2| \le |z_2 - z_1|$

$\square$

2.2 Complex Polynomials

Definition 2.3 (Complex polynomial)
A complex polynomial of degree $n \ge 0$ is a function of the form

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 + a_0, \quad z \in \mathbb{C},$$

where the coefficients $a_0, a_1, \ldots, a_n$ are complex numbers with $a_n \ne 0$.

Theorem 2.4 (Fundamental Theorem of Algebra)
Every complex polynomial $p(z)$ of degree $n \geq 1$ has a factorization

$$p(z) = c(z - z_1)^{m_1}(z - z_2)^{m_2} \cdots (z - z_k)^{m_k},$$

where $c \in \mathbb{C}$, the roots $z_1, \ldots, z_n$ are distinct complex numbers, and $m_1, \ldots, m_k \geq 1$. This factorization is unique, up to permutation of the factors. Also note that

$$\sum_{j=1}^{k} m_k = n = \text{the degree of the polynomial}$$

Example 2.5
Consider $p(z) = iz^2 + i$.
$p(z) = iz^2 + i = i(z^2 + 1) = i(z - i)(z + i)$

2.3 Polar Representation

A nonzero complex number $z \in \mathbb{C}$ can be described by two quantities:

- its length $|z| = r$
- The angle z makes with the positive real axis.

Definition 2.6 (Polar representation)
The polar representation of $z \neq 0$ is:

$$z = r(\cos(\theta) + i\sin(\theta))$$

Given $z = r(\cos(\theta) + i\sin(\theta))$ we can recover the Cartesian coordinates for z:

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

Definition 2.7 (Argument)
For a nonzero $z = r(\cos(\theta) + i\sin(\theta))$, the angle θ is called the argument of z, and is denoted $\theta = \arg z$. But the argument of $z, \arg z$, is actually a multivalued function:

$$\arg z + 2\pi k, \quad k \text{ an integer}$$

also represents the same angle

Definition 2.8 (Principal argument)
The principal argument of $z \neq 0$, denoted Arg z, is the unique argument θ which is in $(-\pi, \pi]$.

$$\arg z = \{\operatorname{Arg} z + 2k\pi : k \text{ integer}\} \qquad \text{“multivalued function”}$$
$$\operatorname{Arg} z \qquad \text{“single-valued”}$$

Example 2.9
Consider the complex number $1+i$.

$$|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$$
$$1+i = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$$
$$\operatorname{Arg}(1+i) = \frac{\pi}{4}$$

We introduce the notation

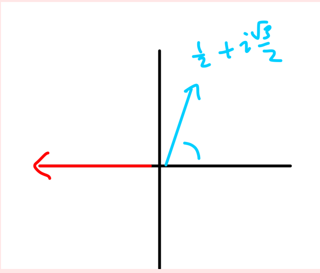
$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \quad \text{for } \theta \in \mathbb{R}.$$

We will see that this equality does follow from how we define complex exponentiation, but in the meantime it is just some convenient notation since the polar form for $z \in \mathbb{C}$ becomes:

$$z \neq 0, z = re^{i\theta},$$
$$\text{where } \theta = \operatorname{Arg} z, r = |z|$$

Example 2.10
We write the polar form using complex exponentials for the numbers -1 and $\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

$$\operatorname{Arg}(-1) = \pi, |-1| = 1$$
$$-1 = e^{i\pi}$$
$$\frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{i\frac{\pi}{3}}$$



Some helpful identities for $e^{i\theta}, \theta \in \mathbb{R}$:

$$|e^{i\theta}| = 1, \quad \overline{e^{i\theta}} = e^{-i\theta}, \quad \frac{1}{e^{i\theta}} = e^{-i\theta}.$$

Proof.

$$\overline{e^{i\theta}} = \cos(\theta) - i\sin(\theta) = \cos(-\theta) + i\sin(-\theta)$$
$$= e^{-i\theta}$$

Check $e^{i\theta}e^{-i\theta} = 1$
use $\cos^2(\theta) + \sin^2(\theta) = 1$ $\square$

Another very important identity for $\theta, \varphi \in \mathbb{R}$:

$$e^{i(\theta+\varphi)} = e^{i\theta}e^{i\varphi}.$$

Just use angle sum formulas

$$\cos(\theta+\varphi) = \cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi)$$
$$\sin(\theta+\varphi) = \cos(\theta)\sin(\varphi) + \sin(\theta)\cos(\varphi)$$

Example 2.11
If $z = re^{i\theta}$ is a nonzero complex number, then

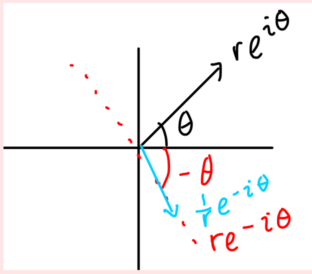
$$\frac{1}{z} = \frac{1}{r}e^{-\theta}, \quad \overline{z} = re^{-i\theta}.$$

$$\left(\frac{1}{r}e^{-i\theta}\right) = \left(r \cdot \frac{1}{r}\right) e^{i\theta + (-i\theta)} = 1$$
$$\implies \frac{1}{r}e^{-i\theta} = \frac{1}{z}$$

Polar form can also help us understand multiplication of complex numbers geometrically in the complex plane:

Example 2.12 (Inversion)

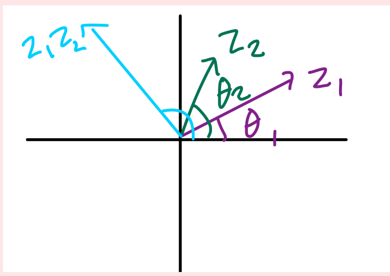
$$z = re^{i\theta} \neq 0 \quad \frac{1}{z} = \frac{1}{r}e^{-i\theta}$$



Example 2.13 (Multiplication)

$$z_1 = r_1 e^{i\theta_1}, \quad z_2 = r_2 e^{i\theta_2}$$
$$z_1 \cdot z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

To multiply two complex numbers, multiply their lengths and add their arg's.



2.4 De Moivre's Formulae

Theorem 2.14 (De Moivre's Formulae)
For any natural number n and any $\theta \in \mathbb{R}$, we have
$$\cos(n\theta) + i\sin(n\theta) = e^{in\theta} = (e^{i\theta})^n = (\cos(\theta) + i\sin(\theta))^n$$
Once the right-hand-side is expanded, we can obtain expressions for $\cos(n\theta)$ and $\sin(n\theta)$ as polynomials is $\cos(\theta)$ and $\sin(\theta)$. These qualities are known as de Moivre's formulae.

Example 2.15
We obtain de Moivre's formulae for the case $n = 2$.
$$\begin{aligned}\cos(2\theta) + i\sin(2\theta) &= (\cos(\theta) + i\sin(\theta))^2 \\ &= \cos^2(\theta) - \sin^2(\theta) + i\,(2\cos(\theta)\sin(\theta))\end{aligned}$$
$$\overset{\text{equating Re=Im}}{\Longrightarrow} = \begin{cases}\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) \\ \sin(2\theta) = 2\cos(\theta)\sin(\theta)\end{cases}$$

Exercise. See the textbook for a derivation of the formulae
$$\cos(3\theta) = \cos^3(\theta) + 3\cos(\theta)\sin^2(\theta), \quad \sin(3\theta) = 3\cos^2(\theta)\sin(\theta) - \sin^3(\theta).$$

2.5 nth Roots

Definition 2.16 (nth roots)
A complex number z is an nth root of w if $z^n = w$. In other words, the nth roots of w are precisely the roots of the polynomial $p(z) = z^n - w$. As an immediate consequence: any $z \in \mathbb{C}$ has at most n many distinct nth roots.

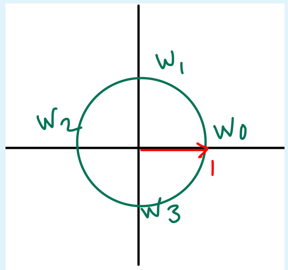
For a nonzero complex number w, we can find its nth roots as follows:
Let $w = \rho e^{i\varphi} \quad z = re^{i\theta}$.
For z to be an nth root of w, we need $z^n = w$
$$\Longrightarrow r^n e^{in\theta} = \rho e^{i\varphi}$$
$$\Longrightarrow r^n = \rho, \quad n\theta = \varphi$$
or,
$$n\theta = \varphi + 2k\pi \quad k \in \mathbb{Z}.$$
$$\Longrightarrow r = \rho^{1/n}, \quad \theta = \frac{\varphi + 2k\pi}{n},$$
distinct angles are
$$\theta = \frac{\varphi + 2k\pi}{n}, \quad k = 0, 1, 2, \ldots, n-1$$
Therefore,
$$n\text{th roots of w}: \quad z_k = \rho^{1/n} e^{i\left(\frac{\varphi+2k\pi}{n}\right)}, \quad k = 0, 1, \ldots, n-1$$

Exercise. Find and plot the 3rd roots of $w = 9i$.

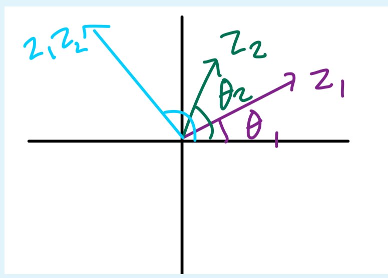
Definition 2.17 (nth roots of unity)
The nth roots of 1 have a special name: they are called the nth roots of unity. Using the same procedure as above, we can find that the nth roots of unity are

$$w_k = e^{2\pi ik/n}, \quad \text{for } k = 0, 1, \dots, n-1.$$

3rd roots of unity:



4th roots of unity:



nth roots of unity can also be used to find nth roots of complex numbers other than 1. For a nonzero $z = re^{i\theta}$, we can find the first nth root of z to be $z_0 = r^{1/n}e^{i\theta/n}$. Then, if $w_0, \dots, w_{n-1}$ are the nth root of unity, then the nth roots of z are exactly

$$z_k = z_0 w_k, \quad k = 0, \dots, n-1.$$

$$z_k^n = (z_0 w_k) = z_0^n \cdot w_k^n = z \cdot 1 = z$$