Math 120A (Differential Geometry)
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Winter 2022

These are my lecture notes for Math 120A (Differential Geometry), which is taught by Fumiaki Suzuki. The textbook for this class is *Differential Geometry of Curves and Surfaces*, by Kristopher Tapp. Many of the figures I include in these notes are taken from Tapp's book.

Contents

1   Jan 3, 2022

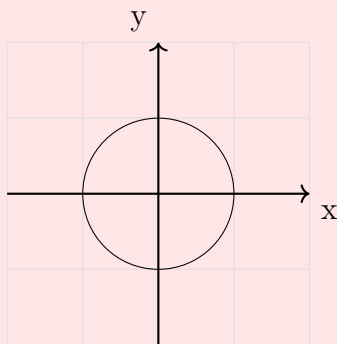## 1.1   What is Differential Geometry?

Differential geometry studies geometry via analysis and linear algebra.

Geometry	Analysis	Linear Algebra
Intuitive	Rigorous	Computable
Curved	$\xrightarrow{\text{tangent space}}$	Linear
Global	Local	

1.2   Parametrized Curves

Example 1.1
A unit circle $S' = \{\vec{x} \text{ in } \mathbb{R}^2 \mid |\vec{x}| = 1\}$

$$\vec{\gamma} : [0, 2\pi) \to \mathbb{R}^2$$
$$t \mapsto (\cos t, \sin t)$$

$\vec{\gamma}[0, 2\pi) = S'$

Definition 1.2 (Parametrized curve and Trace)
A (parametrized) curve is a smooth function $\vec{\gamma} \colon I \to \mathbb{R}^n$, where I is an interval in $\mathbb{R}$. The image

$$\vec{\gamma}(I) = \{\vec{\gamma}(t) \mid t \in I\}$$

is called the trace of $\vec{\gamma}$.

Recall 1.3 An interval is a subset of $\mathbb{R}$ that has one of the following forms:

$$(a, b), [a, b], (a, b], [a, b), (-\infty, b), (-\infty, b], (a, \infty), [a, \infty), (-\infty, \infty) = \mathbb{R}.$$

A function $\vec{\gamma} \colon I \to \mathbb{R}^n$ is called smooth if $\vec{\gamma}$ is infinitely differentiable, or equivalently, each of the component functions $x_i \colon I \to \mathbb{R}$ is infinitely differentiable.

Example 1.4
$\vec{\gamma}(t) = (\cos t, \sin t, t)$, $t \in (-\infty, \infty)$ is a curve, called a helix.
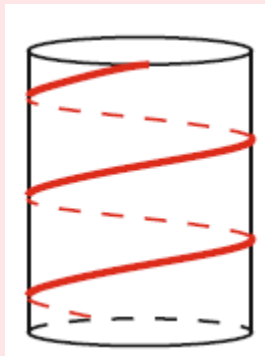
Definition 1.5 (Derivative)
Let $\vec{\gamma} \colon I \to \mathbb{R}^n$ be a curve. The <u>derivative</u> of $\vec{\gamma}$ at t is defined as
$$\vec{\gamma}'(t) = \lim_{h \to 0} \frac{\vec{\gamma}(t+h) - \vec{\gamma}(t)}{h}$$
If t is on the boundaries of I, then use the left- or right-hand limit.

Remarks 1.6
  i. If $\vec{\gamma}(t) = (x_1(t), x_2(t), \cdots, x_n(t))$, then $\vec{\gamma}'(t) = (x_1'(t), x_2'(t), \cdots, x_n'(t))$.
  ii. The tangent line to the curve at $\vec{\gamma}'(t_0)$ is defined as
$$\vec{L}(t) = \vec{\gamma}(t_0) + t\vec{\gamma}'(t_0), \quad t \in (-\infty, \infty),$$
  as soon as $\vec{\gamma}'(t) \neq \vec{0}$.

Definition 1.7 (Regular)
A curve $\vec{\gamma} : I \to \mathbb{R}^n$ is called <u>regular</u> if $\forall t \in I, \vec{\gamma}'(t) \neq \vec{0}$.

Remark 1.8 regular = the tangent line is defined everywhere = the trace is "smooth".

Example 1.9

$$\vec{\gamma}(t) = (t^2, t^3), \quad t \in (-\infty, \infty)$$
Then $\vec{\gamma}$ is a curve that is not regular.
Indeed, $\vec{\gamma}'(t) = (2t, 3t^2)$, so $\vec{\gamma}'(0) = \vec{0}$.
Notice, $x(t) = t^2, y(t) = t^3$, so $x(t) = y(t)^{2/3}$. Hence, the trace is given by $x = y^{2/3}$ in $\mathbb{R}^2$.

Remark 1.10 The analogy with the physics is useful. If $\vec{\gamma} : I \to \mathbb{R}^n$ is a curve, then $\vec{\gamma}(t)$ is the position of a moving particle at time t in $\mathbb{R}^2$.
  • $\vec{\gamma}'(t)$ velocity

- $\vec{\gamma}''(t)$ acceleration
- $|\vec{\gamma}'(t)|$ speed

In this analogy, regular = the speed is always nonzero = the particle never stops (hence no "corners" on the trace)
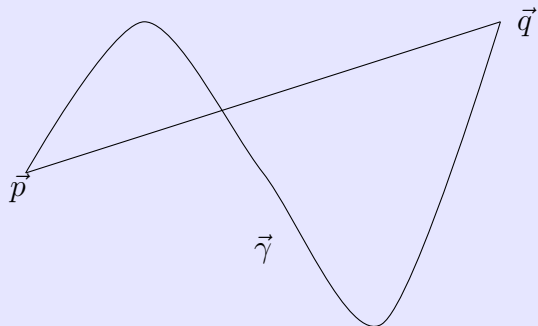

Definition 1.11 (Arc length)

Let $\vec{\gamma}(t)\colon I \to \mathbb{R}^n$ be a regular curve. Then the arc length between times t_1, t_2 is defined as

$$\int_{t_1}^{t_2} |\vec{\gamma}'(t)|\, dt$$

Proposition 1.12

Let $\vec{\gamma}\colon [a, b] \to \mathbb{R}^n$ be a regular curve with the arc length L, $\vec{p} = \vec{\gamma}(a), \vec{q} = \vec{\gamma}(b)$. Then $L \geq |\vec{q} - \vec{p}|$.

Moreover, the equality holds if and only if $\vec{\gamma}$ parametrizes the line segment between $\vec{p}, \vec{q}$.

For the proof, we use the inner-product:

for $\vec{x} = (x_1, x_2, \cdots, x_n), \vec{y} = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$,

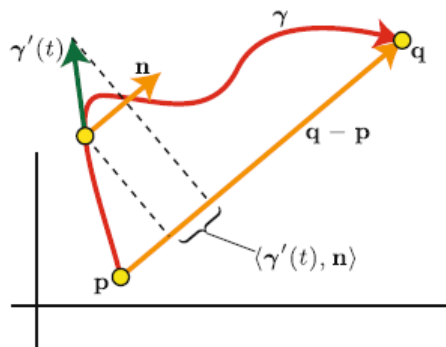$$\langle x, y \rangle := x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

Basic properties:

i. The inner product $\langle -, - \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is symmetric and bilinear.

ii. $\langle \vec{x}, \vec{y} \rangle = |\vec{x}||\vec{y}| \cos\theta$, where θ is the angle between $\vec{x}, \vec{y}$. ($\theta \in [0, 2\pi]$)

iii. $\langle \vec{x}, \vec{y} \rangle = 0 \Leftrightarrow \vec{x}, \vec{y}$ are orthogonal to each other.

iv. $\langle \vec{x}, \vec{x} \rangle = |\vec{x}|^2$

v. $\langle \vec{x}, \vec{y} \rangle \leq |\vec{x}||\vec{y}|$ (Schwartz Inequality) and the equality holds if and only if $\theta = 0$.

2 Jan 5, 2022

2.1 Proof of Proposition 1.12

Proof. Idea: Compare $\vec{\gamma}'(t)$ and its projection onto $\vec{q} - \vec{p}$. Set $\vec{n} = \frac{\vec{q}-\vec{p}}{|\vec{q}-\vec{p}|}$; $\vec{n}$ is unit.

Tapp Pg.15

Then $|\vec{\gamma}'(t)| \geq \langle \vec{\gamma}'(t), \vec{n} \rangle$ by Schwartz inequality.

Now,

$$L = \int_a^b |\vec{\gamma}'(t)|\, dt \geq \int_a^b \langle \vec{\gamma}'(t), \vec{n} \rangle\, dt$$
$$= [\langle \vec{\gamma}(t), \vec{n} \rangle]_a^b = \langle \vec{\gamma}(b), \vec{n} \rangle - \langle \vec{\gamma}(a), \vec{h} \rangle$$
$$= \langle \vec{q} - \vec{p}, \frac{\vec{q} - \vec{p}}{|\vec{q} - \vec{p}|} \rangle = |\vec{q} - \vec{p}|$$

If the equality holds, then $\forall t \in [a, b], \vec{\gamma}'(t), \vec{n}$ are in the same direction. So,

$$\vec{\gamma}'(t) = \langle \vec{\gamma}'(t), \vec{n} \rangle \vec{n}.$$
$$\vec{\gamma}(t) = \vec{\gamma}(a) + \int_a^t \vec{\gamma}'(u) du$$
$$= \vec{p} + \left(\int_a^t \langle \vec{\gamma}'(u), \vec{n} \rangle dt \right) \vec{n}$$

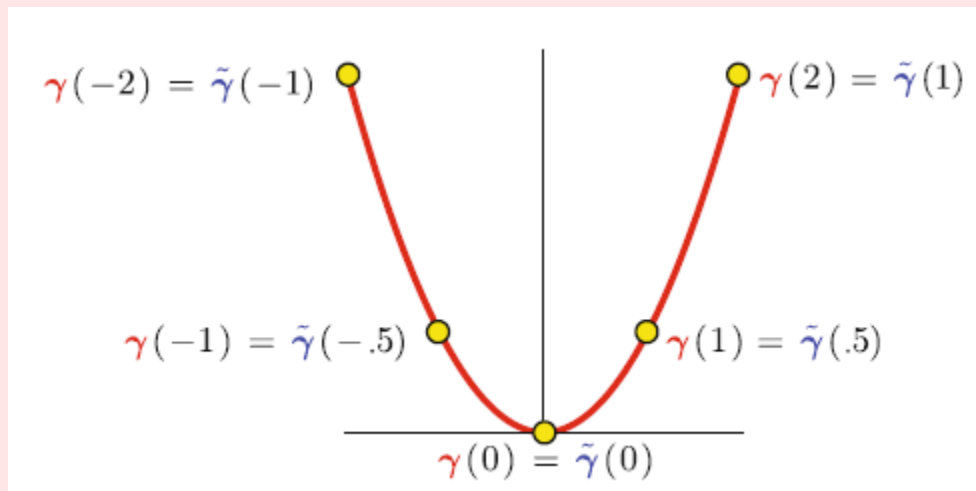parametrizes the line segment between $\vec{p}, \vec{q}$. □

2.2 Reparametrization

There are regular curves that share common properties. Which regular curves should we identify?

Example 2.1
$\vec{\gamma}(t) = (t, t^2), \quad t \in [-2, 2]$
$\tilde{\vec{\gamma}}(t) = (-2t, (-2t)^2), t \in [-1, 1].$
Then $\vec{\gamma}[-2, 2] = \tilde{\vec{\gamma}}[-1, 1] =$

$\boldsymbol{\gamma}(-2) = \tilde{\boldsymbol{\gamma}}(-1)$ $\boldsymbol{\gamma}(2) = \tilde{\boldsymbol{\gamma}}(1)$

$\boldsymbol{\gamma}(-1) = \tilde{\boldsymbol{\gamma}}(-.5)$ $\boldsymbol{\gamma}(1) = \tilde{\boldsymbol{\gamma}}(.5)$

$\boldsymbol{\gamma}(0) = \tilde{\boldsymbol{\gamma}}(0)$

$\vec{\gamma}, \tilde{\vec{\gamma}}$ are the same, up to change in time:
Let $\phi \colon [-1, 1] \to [-2, 2], \quad t \mapsto -2t.$
Then $\tilde{\vec{\gamma}} = \vec{\gamma} \circ \phi$

Definition 2.2 (Reparametrization)
Let $\vec{\gamma} \colon I \to \mathbb{R}^n$ be a regular curve. A <u>reparametrization</u> of $\vec{\gamma}$ is a function of the form
$$\tilde{\vec{\gamma}} = \vec{\gamma} \circ \phi : \tilde{I} \to \mathbb{R}^n,$$
where $\tilde{I}$ is an interval, $\phi \colon \tilde{I} \to I$ is a smooth bijection such that $\forall t \in \tilde{I}, \phi'(t) \neq 0$
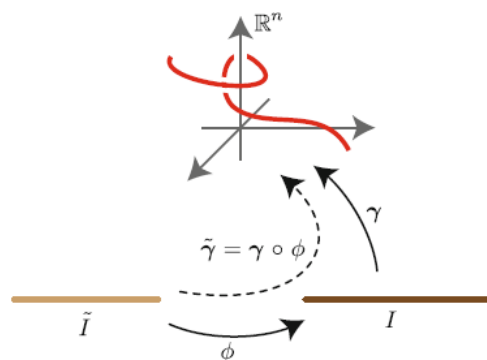

Figure 1: Kapp pg.19

> **Proposition 2.3**
> A reparametrization of a regular curve is a regular curve.

Proof. We use the same notations as the definition.
$\tilde{\vec{\gamma}} = \vec{\gamma} \circ \phi \colon \tilde{I} \to \mathbb{R}^n$ is the composition of smooth functions, so smooth.
Moreover, $\forall t \in \tilde{I}, \tilde{\vec{\gamma}}'(t) = \vec{\gamma}'(\phi(t)) \cdot \phi'(t) \neq 0$ $\qquad\square$

We will be interested in regular curves up to reparametrizations.

> **Remarks 2.4**
> 1. $\vec{\gamma}, \tilde{\vec{\gamma}}$ have the same trace.
> 2. There are regular curves with the same trace that cannot be reparametrized to each other. For instance,
> $$\vec{\gamma}_1(t) = (\cos(t), \sin(t)), t \in [0, 2\pi),$$
> $$\vec{\gamma}_2(t) = (\cos(t), \sin(t)), t \in [0, 4\pi),$$

Question 2.5: Is there a canonical reparametrization of a given regular curve?

> **Definition 2.6** (Unit-speed)
> A regular curve $\vec{\gamma} \colon I \to \mathbb{R}^n$ is called <u>unit-speed</u> (or parametrized by arc length) if $\forall t \in I$,
> $|\vec{\gamma}'(t)| = 1$.

Remark 2.7 If $\vec{\gamma} \colon I \to \mathbb{R}^n$ is unit-speed, then,
$$\text{Arc length between } t_1, t_2 = \int_{t_1}^{t_2} |\vec{\gamma}'(t)| dt = \int_{t_1}^{t_2} dt = t_2 - t_1$$

> **Proposition 2.8**
> A regular curve always has a unit-speed reparametrization.

Proof. Let $\vec{\gamma} \colon I \to \mathbb{R}^n$ be a regular curve. Fix $t_0 \in I$. Define $s \colon I \to \mathbb{R}$ by $s(t) = \int_{t_0}^{t} \vec{\gamma}'(u) du$.
Let $\tilde{I} = s(I) \subset \mathbb{R}$. Then $\tilde{I}$ is an interval by IVT.
Since $s'(t) = |\vec{\gamma}'(t)| > 0$ by FTC, regularity, $s \colon I \to \tilde{I}$ is a smooth bijection. Then, $\phi = s^{-1} \colon \tilde{I} \to I$ is a smooth bijection,
$$\phi'(t) = \frac{1}{s'(\phi(t))} = \frac{1}{|\vec{\gamma}'(\phi(t))|} \neq 0.$$
Now $\tilde{\vec{\gamma}} = \vec{\gamma} \circ \phi \colon \tilde{I} \to \mathbb{R}^n$ is a reparametrization of $\vec{\gamma}$, that is unit-speed:
$$|\tilde{\vec{\gamma}}'(t)| = |\vec{\gamma}'(\phi(t)) \cdot \phi'(t)|$$
$$= |\vec{\gamma}'(\phi(t))| \cdot 1/|\vec{\gamma}'(\phi(t))|$$
$$= 1$$
$\qquad\square$

Note:
$$s^{-1} \cdot s(t) = t$$

$$(s^{-1})'(s(t)) \cdot s'(t) = 1$$
$$(s^{-1})'(s(t)) = 1/s'(t)$$

$$(s^{-1})'(s(t)) \cdot s'(t) = 1$$
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