Math 170E (Introduction to Probability)
University of California, Los Angeles

Aaron Chao

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Course description: Introduction to probability theory with emphasis on topics relevant to applications. Topics include discrete (binomial, Poisson, etc.) and continuous (exponential, gamma, chi-square, normal) distributions, bivariate distributions, distributions of functions of random variables (including moment generating functions and central limit theorem).

These are my lecture notes for Math 170E (Introduction to Probability and Statistics: Part 1 Probability) taught by Enes Ozel. The main textbook for this class is *Probability and Statistical Inference (10th Edition)* by Robert V. Hogg, Elliot Tanis, and Dale Zimmerman.

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1.1 Properties of Probability

Definition 1.1 (Sample space)
The sample space, denoted by S, is the whole set of possible outcomes.

Definition 1.2 (Event)
Any subset of S is called an event.

Example 1.3
Let A be the event we will get ≥ 1 head. Then

$$A = \{HH, HT, TH\} \subset S$$

Recall 1.4 (Sets) Let A, B be two subsets of S.

- $A \cap B$: intersection $\{x \in S \colon x \in A \wedge x \in B\}$
- $A \cup B$: union $\{x \in S \colon x \in A \vee x \in B\}$
- $A \subset B$: subset $\forall x \in S, x \in A \implies x \in B$
- $\emptyset$: empty set $\forall x \in S, x \notin \emptyset$
- $A' = \overline{A} = A^C$: complement $\{x \in S \colon x \notin A\}$
- $A \setminus B = A \cap B^C = \{x \in S, x \in A \wedge x \notin B\}$
- $A \cap B = \emptyset$ "mutually exclusive"

Example 1.5
Let A be the event we will get ≥ 1 head, B be the event we get both tails. Then

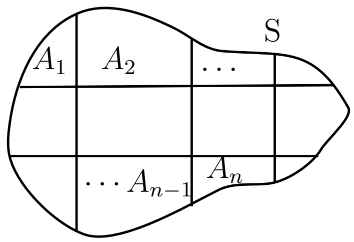
$$A = \{HH, TH, HT\} \quad B = \{TT\}$$

Then

$$A \cup B = S$$

Definition 1.6 (Exhaustive)
$A_1, \ldots, A_n \subseteq S$ are exhaustive if $A_1 \cup A_2 \cup \cdots \cup A_n = S$, i.e. $\bigcup_{i=1}^{n} A_i = S$.

Here, we have mutually exclusive and exhaustive events.



Definition 1.7 (Probability)
Probability is a real-valued set function P that assigns, to each event A in the sample space S, a number $P(A)$, called the probability of the event A, such that the following properties are satisfied:

(a) $P(A) \geq 0$

(b) $P(S) = 1$

(c) For $\{A_i\}_{i=1}^{\infty} \subseteq S$ such that the sets are mutually exclusive,

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Note, $P(A) \in \mathbb{R}$ and $P(S) \to \mathbb{R}$.

Theorem 1.8
$\forall A \subseteq S, P(A') = 1 - P(A)$.

Proof. Let $A \subseteq S$. Then $S = A \cup A' \implies$ mutually exclusive and exhaustive

$$\begin{aligned} \implies P(A \cup A') &= P(A) + P(A') \\ &= P(S) \\ &= 1 \end{aligned}$$

So

$$P(A) + P(A') = 1 \implies P(A') = 1 - P(A)$$

□

Corollary 1.9
$P(\emptyset) = 0$.

Proof. $\emptyset^C = S$, so

$$\begin{aligned} P(\emptyset) &= 1 - P(S) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

□

Example 1.10
Flip two coins,

$$S = \{HH, HT, TT, TH\}$$

assuming a fair coin toss, all have 1/4 probability.
$A =$ "both heads" so

$$P(A) = \frac{1}{4}$$

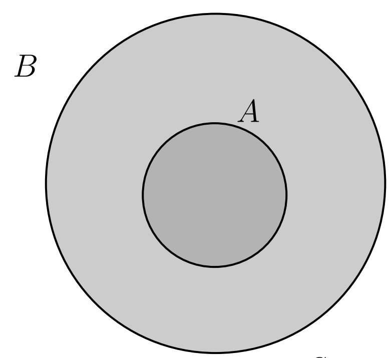
$$P(A') = \frac{3}{4}$$

$$A' = \{HT, TT, TH\}$$

where A' is "NOT both heads" = "at least one tail"

Theorem 1.11
Let $A, B \subseteq S$ such that $A \subseteq B$. Then $P(A) < P(B)$.

Proof. $A \subseteq B \implies B = A \cup (B \cap A')$



$B \setminus A = B \cap A^C$

$$P(B) = P(A) + \underbrace{P(B \cap A')}_{\geq 0} \geq P(A)$$

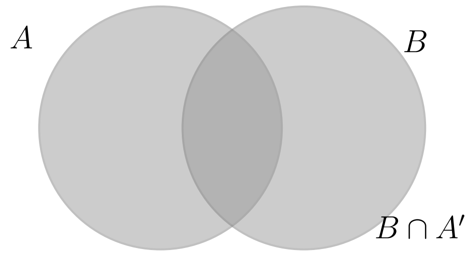
$$\implies P(B) \geq P(A)$$

□

Corollary 1.12
$\forall A \subseteq S, 0 \leq P(A) \leq 1$.

Theorem 1.13
If A and B are any two events, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof. Decompose the units into two disjoint parts



$$P(A \cup B) = P(A) + P(B \cap A')$$

But

$$P(B) = P(A \cap B) + P(A' \cap B)$$

So

$$P(B \cap A') = P(B) - P(A \cap B)$$

Hence

$$\begin{aligned} P(A \cup B) &= P(A) + P(B \cap A') \\ &= P(A) + P(B) - P(A \cap B) \end{aligned}$$

□

Example 1.14
We have a fair die, which we roll once. So

$$S = \{1, 2, \ldots, 6\}$$

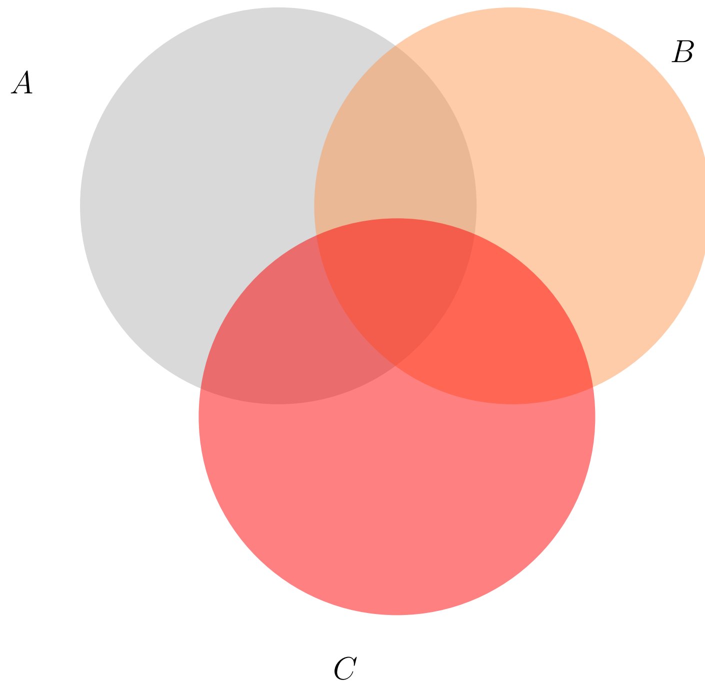
with probability 1/6 each. We call the outcome X. So

$$\begin{aligned} P(2 \mid X \text{ or } 3 \mid X) &= P(2 \mid X) + P(3 \mid X) - P(6 \mid X) \\ &= P(\{2,4,6\}) + P(\{3,6\}) - P(\{6\}) \\ &= \frac{3}{6} + \frac{2}{6} - \frac{1}{6} \\ &= \frac{2}{3} \end{aligned}$$

Theorem 1.15
If A, B, and C are any three events, then

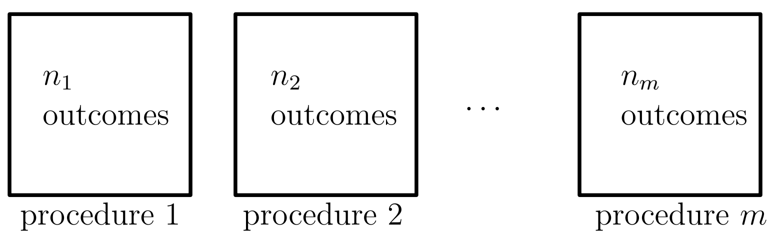
$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Proof.



Exercise. □

1.2 Methods of Enumeration



We are interested in the number of ways the overall outcome may be formed, which we calculate using the formula $n_1 \times n_2 \times \cdots \times n_m$.

Example 1.16 (Cafe, deli sandwich)
Suppose a cafe has

	Bread	Meat	Cheese	Garnishes
	6	4	4	12
# that can be chosen	1	0,1	0,1	$0, 1, 2, \cdots, 12$

So

$$\# \text{ different sandwiches } = 6 \times 5 \times 5 \times 2^{12}$$

Suppose we have n people to be placed. Then the number of ways to arrange them is given by

$$\begin{aligned}\#\text{ ways } &= \underset{1^{st}}{n} \quad \underset{2^{nd}}{n-1} \quad \underset{3^{rd}}{n-2} \quad \cdots \quad \underset{n-1^{st}}{2} \quad \underset{n^{th}}{1} \\ &= n!\end{aligned}$$

Example 1.17

Suppose $S = \{a, b, c, d\}$. Then the number of permutations is $4! = 24$.

$$\left.\begin{array}{l} abcd \\ acbd \\ \vdots \end{array}\right\} 24$$

If we allow repetitions,

$$4 \times 4 \times 4 \times 4 = 4^4 = 256$$

Definition 1.18 (Permutation)

Suppose we have n objects/people and $r \leq n$ positions, then the number of ways to arrange them is given by

$$\begin{aligned}\#\text{ of ways } &= \underset{1^{st}}{n} \quad \underset{2^{nd}}{n-1} \quad \underset{3^{rd}}{n-2} \quad \cdots \quad \underset{r-1^{st}}{n-r+2} \quad \underset{r^{th}}{n-r+1} \\ &= n \times (n-1) \times (n-2) \times \cdots \times (n-r+2) \times (n-r+1) \\ &= {}_nP_r\end{aligned}$$

which is the <u>number of permutations of n objects taken r at a time</u>.
We define the <u>r^{th} falling factorial of n</u> as

$$(n)_r = n \cdot (n-1) \cdots (n-r+1)$$

and the <u>factorial of n</u> as

$$(n)_n = n \cdot (n-1) \cdots (n-n+1) = n!$$

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2.1 Binomial Coefficients

Definition 2.1 (Binomial coefficient)
We are interested in the number of ways to choose r objects out of n objects. The number is given by

$$\begin{aligned}
\# = \frac{{}_nP_r}{r!} = \frac{(n)_r}{r!} \\
&= \frac{n(n-1)\cdots(n-r+1)}{r!} \\
&= \frac{n(n-1)\cdots(n-r+1)}{r!}\frac{(n-r)!}{(n-r)!} \\
&= \frac{n!}{r!(n-r)!} \\
&= \binom{n}{r}
\end{aligned}$$

which we call the binomial coefficient.

Example 2.2
We have a group of 5 people. We want to choose 2 to govern, and it does not matter who is president/treasurer.

$$\#\text{ ways } = \binom{5}{2} = \frac{5!}{2!3!} = 10$$

Example 2.3 (Five-card Poker Hands)
We have a deck of 52 cards, where there are 4 suits and 13 denominations. The number of possible five card Poker hands is

$$\binom{52}{5} = \frac{52 \times 51 \times 50 \times 49 \times 48}{1 \times 2 \times 3 \times 4 \times 5} = 2598960$$

4	4	K	K	K	Full House
8	8	8	8	A	Four of a Kind

Which one should beat the other? The one less likely.

$$P(\text{full house}) = \frac{\#\text{ full house hands}}{\binom{52}{5}}$$

$$P(4\text{ of a kind}) = \frac{\#4\text{ kind}}{\binom{52}{5}}$$

So

$$P(\text{full house}) = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}} \approx 0.00144$$

$$P(4\text{ of a kind}) = \frac{\binom{13}{1}\binom{4}{4}\binom{12}{1}\binom{4}{1}}{\binom{52}{5}} \approx 0.00024$$

Example 2.4
Choose from orchestra from 100 students who can play cello, violin, trumpet, clarinet, and ney. We want 10 cellists, 55 violinists, 15 trumpeteers, 18 clarinetists, the rest will be neyzens. How many different orchestras may be formed?

$$\binom{100}{10}\binom{90}{55}\binom{35}{15}\binom{20}{18}\binom{2}{2}$$

The answer may be reformed in terms of a multinomial coefficient:

$$\binom{100}{10, 55, 15, 18, 2} = \frac{100!}{10! \cdot 55! \cdot 15! \cdot 18! \cdot 2!}$$

Definition 2.5 (Multinomial coefficient)
Let $n_1 + \cdots + n_k = n$, then the mutinomial coefficient is

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! n_2! \cdots n_k!}$$

Order matters	Repetition allowed	
Yes	Yes	$P(n,r) = n^r$
Yes	No	$P(n,r) = \dfrac{n!}{(n-r)!}$
No	No	$C(n,r) = \dfrac{n!}{r!(n-r)!}$
No	Yes	$C(n+r-1,r) = \dfrac{(n+r-1)!}{r!(n-1)!}$

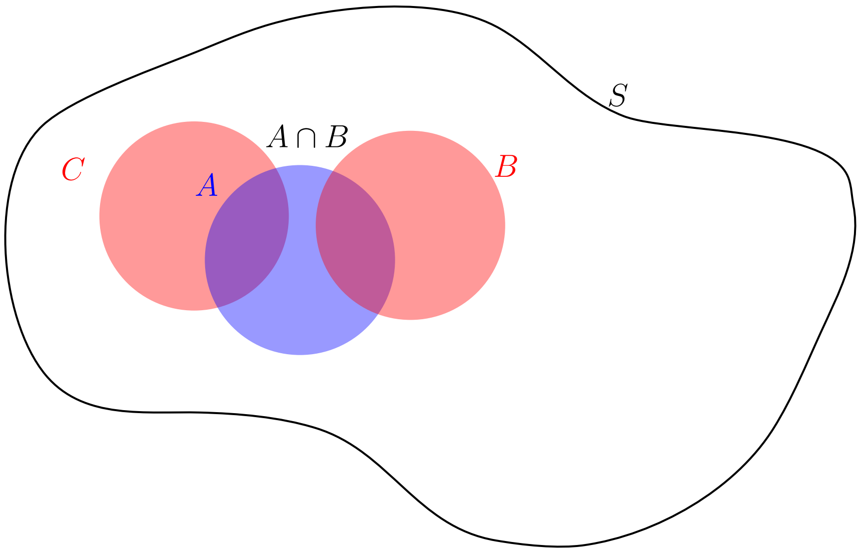
Theorem 2.6 (Binomial and Multinomial Theorem)
If $n \geq 0$ and $s \geq 0$, then

$$(a+b)^n = \sum_{k=0}^{n} a^k b^{n-k} \binom{n}{k}$$

$$(a_1 + a_2 + \cdots + a_s)^n = \sum_{n_1+n_2+\cdots+n_s=n} a_1^{n_1} a_2^{n_2} \cdots a_s^{n_s} \cdot \binom{n}{n_1, n_2, \ldots, n_s}$$

2.2 Conditional Probability

Recall the sample space S and let $A \subseteq S$ be any event with probability of occurrence $P(A)$. Assume $B \subseteq S$ is another event and we know B has happened. Should that affect chances of A?



Notice $B \cap C = \varnothing$. So B happens implies C cannot happen.

Definition 2.7 (Conditional probability)
The conditional probability of an event A given that B has occurred is defined by

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

given that $P(B) > 0$.

Note 2.8: If we condition on B, B becomes the new sample space.

Exercise. Prove $P(A \cap B^C \mid B) = 0$.

Example 2.9
Let (d_1, d_2) be two fair dice with 36 outcomes, all with probability $1/36$.
So $P(d_1 = 5) = 1/6$.

$$\begin{aligned} P(d_1 = 5 \mid d_1 + d_2 = 7) &= \frac{P(d_1 = 5 \cap d_1 + d_2 = 7)}{P(d_1 + d_2 = 7)} \\ &= \frac{P(d_1 = 5 \cap d_2 = 2)}{P(d_1 + d_2 = 7)} \\ &= \frac{1/36}{6/36} = 1/6 \end{aligned}$$

The event of $d_1 + d_2 = 7$ did not change the probability of $d_1 = 5$. Now,

$$\begin{aligned} P(d_1 = 5 \mid d_1 + d_2 = 6) &= \frac{P(d_1 = 5 \cap d_1 + d_2 = 6)}{P(d_1 + d_2 = 6)} \\ &= \frac{P(d_1 = 5 \cap d_2 = 1)}{P(d_1 + d_2 = 6)} \\ &= \frac{1/36}{5/36} = 1/5 \end{aligned}$$

The event of $d_1 + d_2 = 6$ changed the probability of $d_1 = 5$.

Exercise. $P(d_1 = 5 \mid d_1 + d_2 = 5) = 0$

Definition 2.10 (Multiplication Rule)
The probability that two events, A and B, both occur is given by the Multiplication Rule:

$$P(A \cap B) = P(A \mid B)P(B) = P(B \mid A)P(A)$$

assuming $P(A) > 0$ and $P(B) > 0$.

Example 2.11 (Card example)
Assuming a deck of 52 cards:

$$P(\text{1st card is a queen}) = P(\text{1st} = \text{Q}) = \frac{4}{52}$$

$$P(\text{2nd} = \text{queen} \mid \text{1st} = \text{queen}) = \frac{3}{51}$$
$$P(\text{2nd} = \text{queen} \mid \text{1st} \neq \text{queen}) = \frac{4}{51}$$
$$P(\text{2nd} = \text{queen}) = \frac{4}{52}$$
$$P(\text{26th} = \text{H} \mid \text{1st} = \text{Queen of Hearts, 5th} = \text{Ace of spades}) = \frac{12}{50}$$
$$P(\text{1st} = \text{A} \cap \text{50th} = \text{Queen of H}) = P(\text{50th} = \text{Queen of H} \mid \text{1st} = \text{A}) \cdot P(\text{1st} = \text{A}) = \frac{1}{51} \cdot \frac{4}{52}$$

Theorem 2.12

$$\begin{aligned} P(A \cap B \cap C) &= P(C \mid A \cap B) \cdot P(A \cap B) \\ &= P(C \mid A \cap B) \cdot P(B \mid A) \cdot P(A) \\ &= \underbrace{P(A) \cdot P(B \mid A)}_{P(A \cap B)} \cdot P(C \mid A \cap B) \end{aligned}$$

given $P(A), P(A \cap B) > 0$.

Example 2.13 (Card example (Cont'd))

$$\begin{aligned} &P(\text{1st} = \text{J} \cap \text{2nd} = \text{J} \cap \text{4th} = \text{J of Spades}) = P(\text{4th} = \text{J of Spades}) \\ &\quad \cdot P(\text{2nd} = \text{J} \mid \text{4th} = \text{J of Spades}) \cdot P(\text{1st} = \text{J} \mid \text{2nd} = \text{J} \cap \text{4th} = \text{J of Spades}) \\ &\qquad = \frac{1}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} \end{aligned}$$

2.3 Independent Events

Suppose

$$P(d_1 = 5) = 1/6$$
$$P(d_1 = 5 \mid d_1 + d_2 = 7) = 1/6$$

but

$$P(d_1 = 5 \mid d_1 + d_2 = 6) = 1/5$$

Then,

- $\{d_1 = 5\}$ and $\{d_1 + d_2 = 7\}$ are independent

- $\{d_1 = 5\}$ and $\{d_1 + d_2 = 6\}$ are dependent

Definition 2.14 (Independent vs. dependent events)
Events A and B are independent if and only if $P(A \cap B) = P(A) \cdot P(B)$, otherwise A and B are called dependent events.

Theorem 2.15
Assume A and B are independent and both have nonzero probabilities. Then,

$$\begin{aligned} P(A \cap B) &= P(A \mid B)P(B) = P(B \mid A)P(A) \\ &= P(A)P(B) \end{aligned}$$

As a result, we get

$$P(A \mid B) = P(A)$$

and

$$P(B \mid A) = P(B)$$

In the case $P(A) = 0$ (or $P(B) = 0$) as $A \cap B \subseteq A$,

$$P(A \cap B) \leq P(A) = 0$$

$$P(A \cap B) = 0 = P(A)P(B)$$

As a result,

Theorem 2.16
A trivial event (meaning probability 0) is always independent from any other event.

$$0 < P(A) < 1 \implies 0 < P(A') < 1$$

Theorem 2.17
If A and B are independent events, then the following events are also independent:

a) A and B'

b) A' and B

c) A' and B'

Definition 2.18 (Mutually independent)
Events A, B, and C are mutually independent if and only if the following two conditions hold:

i. Pairwise independent

$$P(A \cap B) = P(A)P(B), P(A \cap C) = P(A)P(C), P(B \cap C) = P(B)P(C)$$

ii. Triple wise independent

$$P(A \cap B \cap C) = P(A)P(B)P(C)$$

2.4 Bayes' Theorem

Example 2.19
Consider a very rare disease and a diagnosis test proposed by a very famous pharmaceutical company.

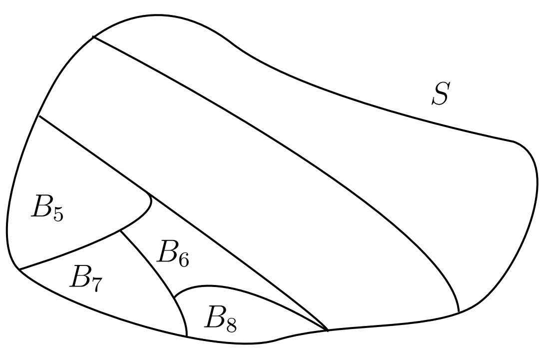
D: have the disease
D^C: not have the disease
T: test positive
T^C: test negative

Suppose $P(D) = \dfrac{1}{1000000}$. The company assures us the test is accurate.

$$P(T \mid D) = 0.999$$

$$P(T^C \mid D^C) = 0.9999$$

Let A be any event. Let $\{B_i\}_{i=1}^n$ be mutually exclusive and exhaustive events.



Theorem 2.20 (Bayes' Theorem)
Let A and $\{B_i\}_{i=1}^n$ be defined as above. Then for each $k \in \{1, 2, \ldots, n\}$:

$$P(B_k \mid A) = \frac{P(A \mid B_k) \cdot P(B_k)}{\sum_{i=1}^n P(A \mid B_i) \cdot P(B_i)}$$

Proof.

$$P(B_k \mid A) = \frac{P(A \cap B_k)}{P(A)}$$

$$A = \underbrace{(A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n)}_{\text{mutually exclusive}}$$

Substitute in the denominator,

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i)$$

So,

$$P(B_k \mid A) = \frac{P(A \cap B_k)}{\sum_{i=1}^{n} P(A \cap B_i)} = \frac{P(A \mid B_k)P(B_k)}{\sum_{i=1}^{n} P(A \mid B_i)P(B_i)}$$

□

Example 2.21 (Company testing (Cont'd))
The company's test:

$$P(T \mid D) = 0.999$$

$$P(T^C \mid D^C) = 0.9999$$

We conduct the test and it says positive. What is the probability that they are really sick?

$$\begin{aligned} P(D \mid T) &= \frac{P(T \mid D)P(D)}{P(T \mid D)P(D) + P(T \mid D^C)P(D^C)} \\ &= \frac{0.999(0.000001)}{0.999(0.000001) + 0.0001(0.999999)} \\ &\approx 0.00989 < 1\% \end{aligned}$$

Example 2.22
A life insurance company has the following policies: standard, preferred, ultrapreferred. We have a fixed age x: Policyholders who are of x years old:

$$P(S) = 0.6$$

$$P(P) = 0.3$$

$$P(U) = 0.1$$

1-year mortality probabilities for x year old policyholders:

S	0.01
P	0.008
U	0.007

$D = \{x \text{ dies within a year}\}$.

$$P(D \mid S) = 0.01$$

$$P(D \mid P) = 0.008$$

$$P(D \mid U) = 0.007$$

Given that a policyholder dies within a year, what are the probabilities that their policy was of each type?

$$P(S \mid D) = ? \quad P(P \mid D) = ? \quad P(U \mid D) = ?$$

$$\begin{aligned} P(S \mid D) &= \frac{P(D \mid S)P(S)}{P(D \mid S)P(S) + P(D \mid P)P(P) + P(D \mid U)P(U)} \\ &= \frac{0.01(0.6)}{0.01(0.6) + 0.008(0.3) + 0.007(0.1)} \\ &= \frac{0.006}{0.0091} \end{aligned}$$

so $P(D) = 0.0091$

$$\approx 0.65934$$

$$P(P \mid D) = \frac{P(D \mid P)P(P)}{0.0091} = \frac{0.008(0.3)}{0.0091} \approx 0.26374$$

$$P(U \mid D) = \frac{P(D \mid U)P(U)}{0.0091} = \frac{0.007(0.1)}{0.0091} = 0.07692$$

Example 2.23 (Birthday problem)
We have 80 students in lecture. What is the probabilities that $\geq$ two students have the same birthday?
Assume 365 days in a year, for each student, each day is equally probable, and birthdays are independent. Then

$$\begin{aligned} P(\geq 1 \text{ coincide}) &= 1 - P(\text{all birthdays differ}) \\ &= 1 - \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \cdots \times \frac{286}{365} \\ &\approx 0.99991433 \end{aligned}$$

If there are 23 students,

$$P(\geq 1 \text{ coincide}) > 0.5$$

$$P_{30} > 0.7$$

$$P_{40} > 0.89$$

$$P_{50} > 0.97$$

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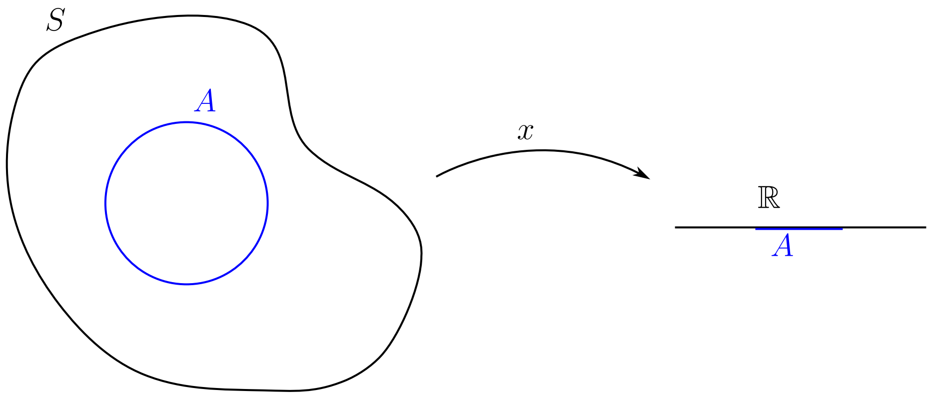
3.1 Discrete Distributions

The sample space S may consist of many strange outcomes wherever numbers are not numerical, analysis becomes difficult. Random variables are functions that carry the events $A \subseteq S$ into subsets of $\mathbb{R}$.



Is any $x\colon S \to \mathbb{R}$ a random variable? No. It must be "measurable".

Definition 3.1 (Random variable and space)
Given a random experiment with a sample space S, a function x that assigns a real number to each element $s \in S$, $X(s) = x \in \mathbb{R}$ is called a random variable. The space of X is the set of real numbers

$$\{x\colon X(s) = x, \text{ for some } s \in S\}$$

Definition 3.2 (Discrete random variable)
If $\exists n \in \mathbb{N}$ such that $|S| = \aleph$, or S can be put into a bijection with $\mathbb{N}$, also called countably infinite, then x is called a discrete random variable.

Example 3.3
Let S be the outcomes of 5 coin tosses. So $P(H) = p \in (0,1)$. Then

$$S = \{HHHHH, HTTTH, TTHTH, \dots\}$$

$$|S| = 2^5 = 32$$

$$X: S \to \mathbb{R}$$

$$X(s) = \#H's$$

So

$$X(HHHHH) = 5$$

$$X(THTHT) = 2$$

So $X \sim \text{Binomial}(n, P)$. The support/space is given by $S(x) = \{0, 1, \dots, 5\}$.
Now let

$$Y: S \to \mathbb{R}$$

$$\forall s \in S, Y(S) = \#H - \#T$$

So

$$Y(HHHHH) = 5$$

$$Y(THTHT) = -1$$

Hence,

$$S(Y) \in \{-5, \dots, 5\}$$

Wherever we consider a discrete random variable, the values it may take, its discrete space will be associated with masses. We call $P(X = x)$ the probability mass of X at x.

$$p_x(x) = f(x) = P(X = x)$$

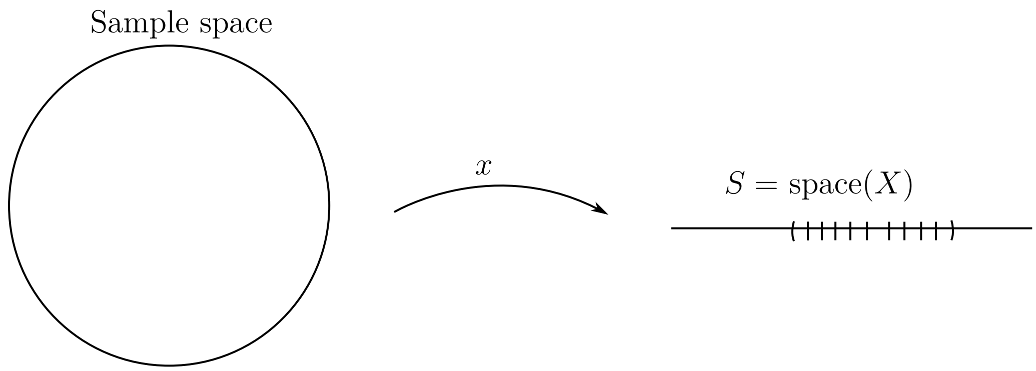
Definition 3.4 (Probability mass function (PMF))
Let X be a given discrete random variable, S be its space, i.e. $x \in S$. Then any function $f: S \to [0,1]$ satisfying the following properties is a well-defined PMF:

a) $f(x) > 0, \forall x \in S$

b) $\sum_{x \in S} f(x) = 1$

c) $P(X \in A) = \sum_{x \in A} f(x)$, for any $A \subseteq S$

Suppose we have our sample space S from Example 3.3, then:



$$S(x) = \{0, 1, \ldots, 5\}$$

Definition 3.5 (Cumulative distribution function (CDF))
The cumulative distribution function, which is the distribution of a random variable, is given by

$$F(x) = P(X \leq x), \quad \forall x \in \mathbb{R}$$

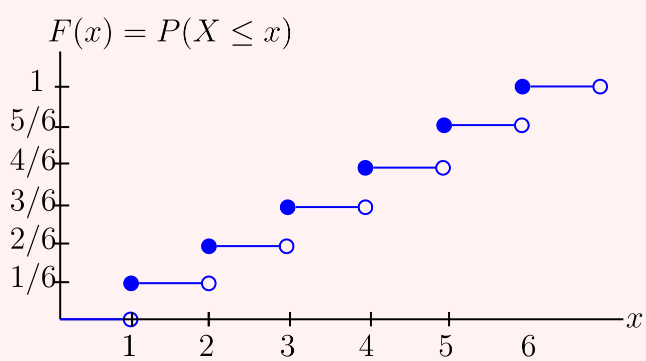
Example 3.6
Recall $\{1, 2, 3, 4, 5, 6\}$, a fair die roll. Then the PMF is

$$f(x) = P(X = x) = 1/6 \text{ for } x \in \{1, 2, 3, \ldots, 6\}$$

$$f(x) = 0 \text{ for all other } x$$

The CDF is

$$F_x(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1/6 & \text{if } 1 \leq x < 2 \\ 2/6 & \text{if } 2 \leq x < 3 \\ 3/6 & \text{if } 3 \leq x < 4 \\ 4/6 & \text{if } 4 \leq x < 5 \\ 5/6 & \text{if } 5 \leq x < 6 \\ 1 & \text{if } x \geq 6 \end{cases}$$



$F(6.5) = P(X \leq 6.5) = P(X = 1, X = 2, \ldots, X = 6)$.