Math 167 (Mathematical Game Theory)
University of California, Los Angeles

Aaron Chao

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These are my lecture notes for Math 167 (Mathematical Game Theory) taught by Oleg Gleizer. The main textbook for this class is *Game Theory, Alive* by Anna Karlin and Yuval Peres and the supplementary textbook is *A Course in Game Theory* by Thomas Ferguson.

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1 Mar 28, 2022

1.1 Impartial Combinatorial Games

Definition 1.1 (Impartial combinatorial game)
In an impartial combinatorial game,

- Two-person
- Perfect information
- No chance moves
- Win-or-lose outcome

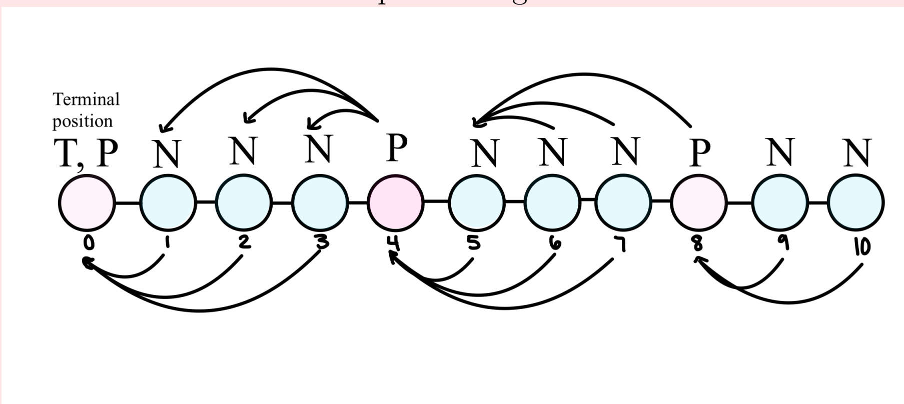
Example 1.2
Suppose

- A pile of n chips on the table
- Two players: P1 and P2
- A move consists of removing one, two, or three chips from the pile
- P1 makes the first move, players alternate then
- The player to remove the last chip wins (the last player to move wins. If a player can't move, they lose.)

Method to analyze: backward induction.
Positions:

- **N**, next player to take a move wins.
- **P**, previous (second) player to take a move wins.

Graph of the game



Any move from a **P** position leads to an **N** position. There always exists a move from an **N** position to a **P** position.
Ending condition: the game ends in a finite number of moves, no matter how played.
A **T** position is a **P** position.

Definition 1.3 (Normal play vs. misère play)
In a normal play, the last player to move wins. In a misère play, the last player to move loses.

Example 1.4
A misère game, a player can take 1-4 chips.
Every position is either **N** or **P**, but not nothing or both.

Example 1.5 (The game of Chomp)
Graph of the game:

- Positions correspond to vertices
- Moves correspond to oriented edges

Definition 1.6 (Strategy)
A function that assigns a move to each position, except for the terminal.

Definition 1.7 (Winning strategy from a position x)
A winning strategy from a position x is a sequence of moves, starting from x, that guarantees a win.

Consider a normal game. Let $\mathbf{N}_i/\mathbf{P}_i$ be the set of positions from which P1/P2 can win (reach the nearest terminal vertex of the same graph) in at most i moves.

$$\mathbf{P}_0 = \mathbf{P}_1 = \{\text{terminal positions}\}$$

$$\mathbf{N}_{i+1} = \{x\colon \text{ there is a move from } x \text{ to } \mathbf{P}_i\}$$

$$\mathbf{P}_{i+1} = \{y\colon \text{ each move leads to } \mathbf{N}_i\}$$

Note 1.8: $\mathbf{P}_0 = \mathbf{P}_1 \subseteq \mathbf{P}_2 \subseteq \mathbf{P}_3 \dots$
$\mathbf{N}_1 \subseteq \mathbf{N}_2 \subseteq \mathbf{N}_3 \dots$

$$\mathbf{N} = \bigcup_{i=1} \mathbf{N}_i, \quad \mathbf{P} = \bigcup_{i=0} \mathbf{P}_i$$

Definition 1.9 (Progressively bounded)
A game is called progressively bounded if for every position x there exists an upper bound $B(x)$ on the number of moves until the game terminates.

2 Mar 30, 2022

2.1 Combinatorial Games (Cont'd)

Recall 2.1
- $\mathbf{P}_0 = \mathbf{P}_1 = \{\text{terminal positions}\}$
- $\mathbf{N}_{n+1} = \{x\colon \text{there is a move from } x \text{ to } \mathbf{P}_n\}$
- $\mathbf{P}_{n+1} = \{y\colon \text{each move from } y \text{ leads to } \mathbf{N}_n\}$
- $\mathbf{P}_0 = \mathbf{P}_1 \subseteq \mathbf{P}_2 \subseteq \dots$
- $\mathbf{N}_1 \subseteq \mathbf{N}_2 \subseteq \mathbf{N}_3 \subseteq$
- $\mathbf{P} = \bigcup\limits_{n=0} \mathbf{P}_n$
- $\mathbf{N} = \bigcup\limits_{n=1} \mathbf{N}_n$
- A game is called progressively bounded if for every position x there exists an upper bound $B(x)$ on the number of moves until the game stops.

Theorem 2.2
In a progressively bounded impartial full information combinatorial game, all positions are in $\mathbf{N} \cup \mathbf{P}$. Thus, for every position there exists a winning strategy.

Proof. Let $B(x) \leq n$. Let us prove by induction that $x \in \mathbf{N}_n \cup \mathbf{P}_n$.
Base: $n = 0$
x is a terminal vertex $\implies x \in \mathbf{P}_0 = \mathbf{P}_1$.
Inductive hypothesis by $\mathbf{P}_0$: $B(x) \leq n \implies x \in \mathbf{N}_n \cup \mathbf{P}_n$.
Inductive step: Show that $B(x) \leq n+1 \implies x \in \mathbf{N}_{n+1} \cup \mathbf{P}_{n+1}$
Consider a move $x \to y$ and $B(y) \leq n$. Hence, $y \in \mathbf{N}_n \cup \mathbf{P}_n$. So either
Case 1: Each move from x leads to $y \in \mathbf{N}_n \implies x \in \mathbf{P}_{n+1}$.
Case 2: There exists a move from x to $y \notin \mathbf{N}_n$. Thanks to the inductive typo, $y \in \mathbf{N}_n \cup \mathbf{P}_n$ so $y \in \mathbf{P}_n \implies x \in \mathbf{N}_{n+1}$. □

2.2 The Game of Nim

- Several piles, each containing finitely many chips.
- A move: a player can remove any number of chips, from one to all from any pile
- P1 and P2 alternate taking moves
- The player to take the last chip wins

Example 2.3

We have two piles. The general case for k piles, we state: $(x_1, x_2, \ldots, x_k)$.
Nim-Sum:
Consider $x \oplus y$. We rewrite x and y as binary numbers and perform long addition of x_2 and y_2 without carry-over, i.e. mod 2.

Example 2.4

$$5 \oplus 7 = \begin{array}{cccc} & 1 & 0 & 1 \\ \oplus & 1 & 1 & 1 \\ \hline & 0 & 1 & 0 \end{array} = 2$$

Note 2.5: Nim is a progressively bounded game.

Theorem 2.6 (Bouton)

A position $x = (x_1, x_2, \ldots, x_k)$ is a $\mathbf{P}$ position $\iff x_1 \oplus x_2 \oplus \cdots \oplus x_k = 0$.

3 April 1, 2022

3.1 The Game of Nim (Cont'd)

Recall 3.1 $x = (x_1, x_2, \dots, x_k)$
Theorem (Bouton) says $x \in \mathbf{P} \iff x_1 \oplus x_2 \oplus \cdots \oplus x_k = 0$.

Proof of Theorem 2.6. We have
Terminal position: $x = (0, 0, \dots 0) \in \mathbf{P}$ Let $x \in \mathbf{N}$. Then there exists a move $x \to y \in \mathbf{P}$.

$$x_1 \oplus x_2 \oplus \cdots \oplus x_k = \begin{array}{ccccccccc} & & 1 & * & * & \dots & \dots & * & * \\ \bigoplus & & & & 1 & * & \dots & * & * \\ & & & & \vdots & \vdots & & \vdots & \vdots \\ & 1 & * & * & \dots & \dots & \dots & * & * \\ \hline \end{array}$$

Find the left-most (most significant) column with an odd number of 1's. Change any number that has a 1 in the column so that there is an even number of 1's in every column. The 1 in the most significant position becomes a 0 which implies the number becomes smaller. So this is a legal move.
We have $x \in \mathbf{P} \implies$ any move $x \to y \in \mathbf{N}$ where

$$x = (x_1, x_2, \dots, x_k) \mapsto y = (x_1', x_2, \dots, x_k)$$

such that

$$x_1' < x_1 \text{ and } x_1 \oplus x_2 \oplus \cdots \oplus x_k = 0.$$

If

$$x_1' \oplus x_2 \oplus \cdots \oplus x_k = 0$$

then

$$x_1' \oplus x_2 \oplus \cdots \oplus x_k = 0$$

then $x_1' = x_0$, a contradiction. Hence

$$x_1' \oplus x_2 \oplus \cdots \oplus x_k \neq 0 \implies y \in \mathbf{N}.$$

□

Example 3.2

$$x_1 = 7$$
$$x_2 = 10$$
$$x_3 = 15$$

$$\begin{array}{cc|c|c} & 0 & 1 & 1 & 1 \\ \oplus & 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 & 1 \\ \hline & 0 & 0 & 1 & 0 \end{array} \implies \begin{array}{ccccc} & 0 & 1 & 0 & 1 \\ \oplus & 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 & 1 \\ \hline & 0 & 0 & 0 & 0 \end{array}$$

So we have that $(7, 10, 15) \mapsto (5, 10, 15)$

3.2 Subtraction Nim

Extra condition: A player can remove at most n chips.

We find pile sizes mod $n + 1$, i.e.

$$(x_1, x_2, \ldots, x_k) \mapsto (x_1 \bmod n + 1, x_2 \bmod n + 1, \ldots, x_k \bmod n + 1)$$

Now we find the Nim-sum and make a move.

$$x \bmod n + 1 = \underbrace{(x_1 \bmod n + 1, x_2 \bmod n + 1, \ldots, x_k \bmod n + 1)}_{(x_1 \bmod n+1)_2 \oplus (x_2 \bmod n+1)_2 \oplus \cdots \oplus (x_k \bmod n+1)_2} \implies \begin{cases} = 0 \iff \mathbf{P} \\ \neq 0 \iff \mathbf{N} \end{cases}$$

Example 3.3

We have $x = (12, 13, 14)$ and $n = 3$. So,

$$(12 \bmod 4, 13 \bmod 4, 14 \bmod 4) \equiv (0, 1, 2) = (0_2, 1_2, 10_2)$$

So

$$\begin{array}{ccc} & 0 & 0 \\ \oplus & 0 & 1 \\ & 1 & 0 \\ \hline & 1 & 1 \end{array} \neq 0$$

so we take away one chip from the third pile

$$\begin{array}{ccc} & 0 & 0 \\ \oplus & 0 & 1 \\ & 0 & 1 \\ \hline & 0 & 0 \end{array}$$

So we have that $(12, 13, 14) \mapsto (12, 13, 13)$.

Note 3.4: You can always make a legal move $\mathbf{N} \to \mathbf{P}$ by removing $i \leq n$ chips from a pile.

Note 3.5: To move from $\mathbf{P}$ to $\mathbf{P}$, you need to remove $n + 1$ chips from a pile. Not allowed! Hence, any move from $\mathbf{P}$ is to $\mathbf{N}$.

Example 3.6
We have $x = (12, 13, 13)$, with $n = 3$. So

$$x \mod 4 = (0, 1, 1)$$

therefore

$$\begin{array}{rr} & 0 \\ \bigoplus & 1 \\ & 1 \\ \hline & 0 \end{array}$$

3.3 Two-Person Zero Sum Games (Strategic Form)

We have

- P1: a non-empty set of strategies S1
- P2: a non-empty set of strategies S2
- A: S1 $\times$ S2 $\to \mathbb{R}$, the min function for P1 (payoff matrix)

Note 3.7: Since the game is zero-sum, a win for P1 is a loss for P2. $A(i, j)$ can be ≤ 0, so works both ways.

Pure strategies:

$$\begin{array}{cc|c|c|c|c|} & & \multicolumn{4}{c}{\text{P2}} \\ & & \text{S21} & \text{S22} & \cdots & \text{S2}n \\ \cline{3-6} & \text{S11} & a_{11} & a_{12} & \cdots & a_{1n} \\ \cline{3-6} \text{P1} & \text{S12} & a_{21} & a_{22} & \cdots & a_{2n} \\ \cline{3-6} & \vdots & \vdots & \vdots & \ddots & \vdots \\ \cline{3-6} & \text{S1}m & a_{m1} & a_{m2} & \cdots & a_{mn} \\ \cline{3-6} \end{array}$$

A game. P1 chooses the strategy S1i. Simultaneously, P2 chooses the strategy S2j. P1 wins a_{ij}.

Lemma 3.8
$\min_j \max_i a_{ij} \geq \max_i \min_j a_{ij}$

We will continue in the next lecture.

4 Apr 4, 2022

4.1 Two-Person Zero Sum Games in Strategic Form (Cont'd)

Recall 4.1 Recall that

$$\begin{array}{ccc|c|c|c|c|}
 & & \multicolumn{4}{c}{\text{P2}} \\
 & & \text{S21} & \text{S22} & \cdots & \text{S2}n \\
\hline
 & \text{S11} & a_{11} & a_{12} & \cdots & a_{1n} \\
\hline
\text{P1} & \text{S12} & a_{21} & a_{22} & \cdots & a_{2n} \\
\hline
 & \vdots & \vdots & \vdots & \ddots & \vdots \\
\hline
 & \text{S1}m & a_{m1} & a_{m2} & \cdots & a_{mn} \\
\hline
\end{array}$$

P1 has a non-empty set of pure strategies

$$S1 = \{S11, S12, \dots, S1m\}$$

P2 has a non-empty set of pure strategies

$$S2 = \{S21, S22, \dots, S2n\}$$

$A\colon S1 \times S2 \to \mathbb{R}$, payoff matrix
P1, $S1i\colon a_{i1}, a_{i2}, \dots, a_{in}$

Betting on the worst possible outcome, P1 bets on $\min_{1 \le j \le n} a_{ij}$. Being intelligent, P1 chooses

$$\max_{1 \le i \le m} \min_{1 \le j \le m} a_{ij}.$$

Betting on the worst possible loss, P2 bets on $\max_{1 \le i \le m} a_{ij}$. Being intelligent, P2 chooses

$$\min_{1 \le j \le n} \max_{1 \le i \le m} a_{ij}$$

.

Lemma 4.2
$\max\limits_{1 \le i \le m} \min\limits_{1 \le j \le n} a_{ij} \le \min\limits_{1 \le i \le m} \max\limits_{1 \le j \le n} a_{ij}$

Proof. Let

$$\max_i \min_j a_{ij} = a_{pq}$$

$$\min_j \max_i a_{ij} = a_{rs}$$

		q		s	
p		a_{pq}	$\leq$	a_{ps}	
				$\leq$	
r		a_{rq}		a_{rs}	

□

Example 4.3
Chooser (P1), Hider (P2). Hider hides behind their back

- Either left hand with one coin
- or right hand with two coins

Chooser chooses L or R,

		P2	
		L1	R2
P1	L	1	0
	R	0	2

$$P1\colon \max_j \min_i a_{ij} = 0$$

$$P2\colon \min_j \max_i a_{ij} = 1$$

<u>Mixed strategies</u>

		P2	
		L1, q	R2, $1-q$
P1	L, p	1	0
	R, $1-p$	0	2

P1: if P2 chooses the strategy L1, the expected gain is

$$1 \cdot p + 0 \cdot (1-p) = 0$$

If P2 chooses R2, the expected gain is

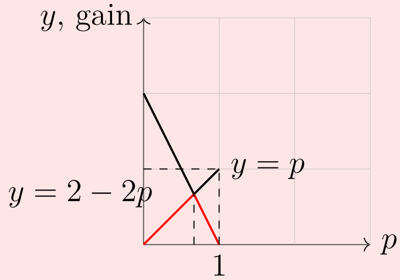
$$0 \cdot p + 2(1-p) = 2 - 2p.$$

If P1 is out of luck, then the expected gain is

$$\min\{p, 2-2p\}$$

Since P1 is intelligent, they choose p s.t. the gain is

$$\max_{0 \leq p \leq 1} \min\{p, 2-2p\}$$



$$2 - 2p = p$$

$$2 = 3p$$

$$p = \frac{2}{3}$$

The optimal strategy is

$$\frac{2}{3}L + \frac{1}{3}R$$

With expected gain $\geqslant \frac{2}{3}$.
P2 is thinking. If P1 chooses L, my expected loss is

$$1 \cdot q + 0 \cdot (1 - q) = q$$

If P1 chooses R, my expected loss is

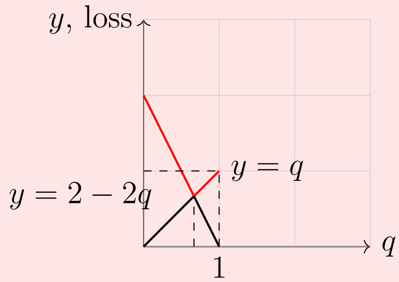
$$0 \cdot q + 2(1 - q) = 2 - 2q$$

Suppose I'm out of luck. Then my expected loss is

$$\max\{q, 2 - 2q\}$$

Being my very smart self,

$$\min_{0 \le q \le 1} \max\{q, 2 - 2q\}$$



The optimal strategy is

$$\frac{2}{3}L1 + \frac{1}{3}R2$$

With expected loss $\le \frac{2}{3} = V$, the value of the game.

Let us generalize $A \in \mathbb{R}^{n\times m}$, an $n \times m$ payoff matrix.

$$\Delta_m = \left\{ \mathbf{p} \in \mathbb{R}^m \colon p_1 \geq 0, p_2 \geq 0, \dots, p_m \geq 0, \sum_{i=1}^{m} p_i = 1 \right\}$$

$$\Delta_n = \left\{ \mathbf{q} \in \mathbb{R}^n \colon q_1 \geq, q_2 \geq 0, \dots, q_n \geq 0, \sum_{j=1}^{n} q_j = 1 \right\}$$

A mixed strategy for P1 is determined by

$$\mathbf{p} \in \Delta_m$$

A mixed strategy for P2 is determined by

$$\mathbf{q} \in \Delta_n$$

Expected gain for P1 (expected loss for P2) = $(\mathbf{p})^T A\mathbf{q}$

		P2			
		q_1	q_2	$\cdots$	q_n
P1	p_1	a_{11}	a_{12}	$\cdots$	a_{1n}
	p_2	a_{21}	a_{22}	$\cdots$	a_{2n}
	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	p_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}

So

$$(\mathbf{p})^t A\mathbf{q} = p_i(a_i q_1 + a_{i2}q_2 + \cdots + a_{in}q_n)$$

If P1 employs the strategy $\mathbf{P}$, then in the worst case their payoff is

$$\min_{\mathbf{q}\in\Delta_n} (\mathbf{p})^t A\mathbf{q} = \min_{1\leq j\leq n} \sum_{i=1}^{m} a_{ij}p_i$$

Hence, P1's winning strategy is

$$\max_{\mathbf{p}\in\Delta_m} \min_{\mathbf{q}\in\Delta_n} \mathbf{p}^T A\mathbf{q}$$

5 Apr 6, 2022

5.1 General Two-Person Zero-Sum Games in Strategic Form

Recall 5.1 Recall

$$\begin{array}{cc|c|c|c|c|}
 & & \multicolumn{4}{c}{\text{P2}} \\
 & & q_1 & q_2 & \cdots & q_n \\ \hline
 & p_1 & a_{11} & a_{12} & \cdots & a_{1n} \\ \hline
\text{P1} & p_2 & a_{21} & a_{22} & \cdots & a_{2n} \\ \hline
 & \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline
 & p_m & a_{m1} & a_{m2} & \cdots & a_{mn} \\ \hline
\end{array}$$

With set of mixed strategies given by,

$$\Delta_m = \left\{ \mathbf{p} \in \mathbb{R}^m \colon \mathbf{p} \geq 0, \sum_{i=1}^{m} p_i = 1 \right\}$$

$$\Delta_n = \left\{ \mathbf{q} \in \mathbb{R}^n \colon \mathbf{q} \geq 0, \sum_{j=1}^{n} q_j = 1 \right\}$$

where $p_1 \geq 0, p_2 \geq 0, \ldots, p_m \geq 0$.

We have

$$\text{Expected gain of P1 } = (\mathbf{p})^t A\mathbf{q}$$

with $\mathbf{p} \in \Delta_m$ and $\mathbf{q} \in \Delta_n$.

The winning strategy for P1:

- Worst case: $\min\limits_{\mathbf{q} \in \Delta_n} (\mathbf{p})^t A\mathbf{q}$
- Smart choice: $\max\limits_{\mathbf{p} \in \Delta_m} \min\limits_{\mathbf{q} \in \Delta_n} (\mathbf{p})^t A\mathbf{q}$

$$\begin{aligned}
\min_{\mathbf{q} \in \Delta_n} (\mathbf{p})^t A\mathbf{q} &= \min_{\mathbf{q} \in \Delta_m} \sum_{j=1}^{n} q_j \sum_{i=1}^{m} a_{ij} p_i \\
&= \min_{1 \leq j \leq n} \sum_{i=1}^{m} a_{ij} p_i
\end{aligned}$$

The winning strategy for P2:

- Worst case: $\max\limits_{\mathbf{p} \in \Delta_m} (\mathbf{p})^t A\mathbf{q}$
- Smart choice: $\min\limits_{\mathbf{q} \in \Delta_n} \max\limits_{\mathbf{p} \in \Delta_m} (\mathbf{p})^t A\mathbf{q}$

$$\max_{\mathbf{p}\in\Delta_m} (\mathbf{p})^t A\mathbf{q} = \max_{\mathbf{p}\in\Delta_m} \sum_{i=1}^{m} p_i \sum_{j=1}^{n} a_{ij} q_j$$
$$= \max_{1\le i\le m} \sum_{j=1}^{n} a_{ij} q_j$$

Definition 5.2 (Safety value for P1 vs. P2)
The value $\hat{\mathbf{p}}$ at which

$$\max_{\mathbf{p}\in\Delta_m} \min_{\mathbf{q}\in\Delta_n} (\mathbf{p})^t A\mathbf{q}$$

is attained is called the safety value for P1. The value $\hat{\mathbf{q}}$ at which

$$\min_{\mathbf{q}\in\Delta_n} \max_{\mathbf{p}\in\Delta_m} (\mathbf{p})^t A\mathbf{q}$$

is attained is called the safety value for P2.

Theorem 5.3 (Von Neumann Minimax Theorem)
For any two-person zero-sum game with $m \times n$ payoff matrix A, there is a number V, called the value of the game, satisfying

$$\min_{\mathbf{q}\in\Delta_n} \max_{\mathbf{p}\in\Delta_m} (\mathbf{p})^t A\mathbf{q} = \max_{\mathbf{p}\in\Delta_m} \min_{\mathbf{q}\in\Delta_n} (\mathbf{p})^t A\mathbf{q} = V$$

Let $\hat{\mathbf{p}}$ be an optimal solution for P1. Let $\hat{\mathbf{q}}$ be an optimal solution for P2. Then

$$\min_{\mathbf{q}\in\Delta_n} (\hat{\mathbf{p}})^t A\mathbf{q} = \max_{\mathbf{p}\in\Delta_m} (\hat{\mathbf{p}})^t A\hat{\mathbf{q}}$$

Definition 5.4 (Value of the game)
Given conditions from Von Neumann Minimax Theorem, V is the value of the game.

Example 5.5 (Odd or Even)

- P1 and P2 simultaneously call out one of the numbers, 1 or 2.
- If the sum is odd, P1 wins and gets the sum of the numbers in $
- If the sum is even, P2 wins and gets the sum of the numbers in $

		P2: $1, q$	P2: $2, 1-q$
P1	$1, p$	-2	3
	$2, 1-p$	3	-4

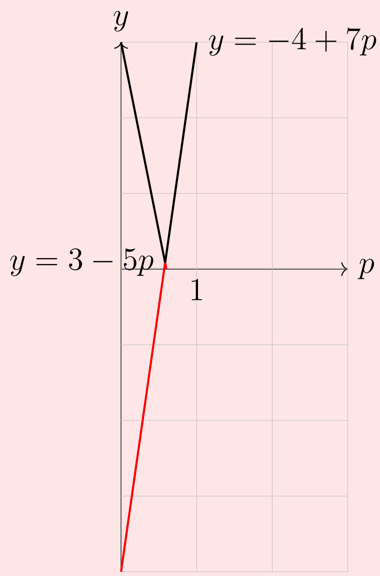
So P1's expected win (P2's expected loss) is

$$\begin{aligned}(\mathbf{p})^t A\mathbf{q} &= \begin{bmatrix} p & 1-p \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} q \\ 1-q \end{bmatrix} \\ &= -12pq + 7p + 7q - 4\end{aligned}$$

P1's worst possible case:

$$f(p) = \min_{0 \le q \le 1} \{-12pq + 7p + 7q - 4\}$$

$$\begin{aligned} q, S21\colon &\quad -2p + 3(1-p) = 3 - 5p \\ 1-q, S22\colon &\quad 3p - 4(1-p) = -4 + 7p \end{aligned}$$



- If $3 - 5p \ge -4 + 7p$, then $q = 0$.
- If $3 - 5p < -4 + 7p$, then $q = 1$.

Hence,

$$f(p) = \min\{3 - 5p, 4 - 7p\}$$

Note that

$$(-12pq + 7p + 7q - 4)\Big|_{q=0} = -4 + 7p$$

$$(-12pq + 7p + 7q - 4)\Big|_{q=1} = 3 - 5p$$

P1: $\displaystyle\max_{0 \le p \le 1} \min_{0 \le q \le 1} q\Big(-2p + 3(1-p)\Big) + (1-q)\Big(3p - 4(1-p)\Big) = \max_{0 \le p \le 1} \min\{3 - 5p, -4 + 7p\}$

$$3 - 5p = -4 + 7p$$
$$7 = 12p$$
$$p = \frac{7}{12}, \quad q = \frac{5}{12}$$

Now from P2:

P2: $\min\limits_{0 \le q \le 1} \max\limits_{0 \le p \le 1} p\Big(-2q + 3(1-q)\Big) + (1-p)\Big(3q - 4(1-q)\Big) = \min\limits_{0 \le q \le 1} \max\{3 - 5q, -4 + 7q\}$

6 Apr 8, 2022

6.1 Solving Small-Dimensional Two-Person Zero-Sum Games Pen-and-Paper

Definition 6.1 (Saddle point)
An element of A, a_{ij} is called a saddle point if

- a_{ij} is the min of the i-th row
- a_{ij} is the max of the j-th column

Then $p_i = 1$, $q_j = 1$, $V = a_{ij}$

Example 6.2
Given

$$\begin{array}{cc} & \begin{array}{c} \text{min} \end{array} \\ \begin{bmatrix} 4 & 1 & -3 \\ 3 & 2 & 5 \\ 0 & 1 & 6 \end{bmatrix} & \begin{array}{c} \text{-3} \\ 2 \\ 0 \end{array} \\ \text{max} \quad \begin{array}{ccc} 4 & 2 & 6 \end{array} & \end{array}$$

So $p_2 = q_2 = 1$ and $V = 2$.

Lemma 6.3
Let a_{pq} and a_{rs} be saddle points of a payoff matrix A. Then $a_{pq} = a_{rs}$.

		q		s	
p		a_{pq}	$\leq$	a_{ps}	
		$\geq$		$\leq$	
r		a_{rq}	$\geq$	a_{rs}	

6.2 Domination

<u>Rows:</u>

$$\begin{array}{ccccc} i\text{-th row} & a_{i1} & a_{12} & \cdots & a_{in} \\ & \geq & \geq & & \geq \\ k\text{-th row} & a_{k1} & a_{k2} & \cdots & a_{kn} \end{array}$$

So $p_k = 0$ so k-th row can be removed from A.

Strict domination: for $j = 1, 2, \ldots, n$,

$$a_{ij} > a_{kj}$$

<u>Columns:</u> the k-th column dominates the j-th column

$$\begin{matrix} a_{1j} & \geqslant & a_{1k} \\ a_{2j} & \geqslant & a_{2k} \\ \vdots & \vdots & \vdots \\ a_{mj} & \geqslant & a_{mk} \end{matrix}$$

Strict domination: for $i = 1, 2, \ldots, m$,

$$a_{ij} > a_{ij}$$

where a_{ij} is dominant.

- Removing a dominant row or column does not change the value of the game, but may remove an optimal strategy.
- Removing a strictly dominant row or column does not change the set of optimal strategies.

Example 6.4

$$A_1 = \begin{bmatrix} 2 & 0 & 4 \\ 1 & 2 & 3 \\ 4 & 1 & 2 \end{bmatrix}$$

Note: $\left.\begin{matrix} 0 < 4 \\ 2 < 3 \\ 1 < 2 \end{matrix}\right\}$, strict domination.

$$A_2 = \begin{bmatrix} 2 & 0 \\ 1 & 2 \\ 4 & 1 \end{bmatrix}$$

Note: $\begin{matrix} 2 & 0 \\ \wedge & \wedge \\ 4 & 1 \end{matrix}$, strict domination

$$\begin{array}{c|c|c} & & \text{min} \\ & \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix} & \begin{matrix} 1 \\ 1 \end{matrix} \\ \text{max} & \begin{matrix} 4 & 2 \end{matrix} & \end{array}$$

Note: No saddle point

Remark 6.5 A row/column can be dominated by a weighted sum of rows columns. For example,

$$\begin{matrix} a_{i1} & a_{12} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{matrix}$$

For $\ell = 1, 2, \ldots, n$, we have

$$\alpha a_{i\ell} + (1-\alpha)a_{j\ell} \geqslant a_{k\ell}$$

Example 6.6

$$\begin{bmatrix} 0 & 4 & 6 \\ 5 & 7 & 4 \\ 9 & 6 & 3 \end{bmatrix}$$

$$4 > 3$$

$$7 > 4.5$$

$$5 \geq 6$$

7 Apr 11, 2022

7.1 Principle of Indifference

$$
\begin{array}{cc|c|c|c|c|}
 & & & \text{P2} & & \\
 & & q_1 & q_2 & \cdots & q_n \\
\hline
 & p_1 & a_{11} & a_{12} & \cdots & a_{1n} \\
\hline
\text{P1} & p_2 & a_{21} & a_{22} & \cdots & a_{2n} \\
\hline
 & \vdots & \vdots & \vdots & \ddots & \vdots \\
\hline
 & p_m & a_{m1} & a_{m2} & \cdots & a_{mn} \\
\hline
\end{array}
$$

Let $\hat{\mathbf{p}} = (p_1, p_2, \ldots, p_m)^t$ be an optimal strategy for P1 and let $q_j = 1$ be a pure strategy for P2.

$$\sum_{i=1}^{m} a_{ij} p_i \geqslant V \tag{1}$$

Let $\hat{\mathbf{q}} = (q_1, q_2, \ldots, q_n)^t$ be an optimal strategy for P2 and let $p_i = 1$ be a pure strategy for P1. Then

$$\sum_{j=1}^{n} a_{ij} q_j \leqslant V \tag{2}$$

Note 7.1: If both players use optimal strategies, then

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j = V$$

Proof. We have

$$
\begin{aligned}
V \leqslant \sum_{i=1}^{m} a_{ij} p_i = 1 \cdot \sum_{i=1}^{m} a_{ij} p_i &= \left(\sum_{j=1}^{m} q_j \right) \sum_{i=1}^{m} a_{ij} p_i \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j \\
&= \underbrace{\sum_{i=1}^{m} p_i}_{=1} \sum_{j=1}^{m} a_{ij} q_j \leqslant V \\
&= V
\end{aligned}
$$

□

Theorem 7.2 (The Equilibrium Theorem)
Let $\hat{\mathbf{p}} = (p_1, p_2, \ldots, p_m)$ and $\hat{\mathbf{q}} = (q_1, q_2, \ldots, q_n)$ be optimal strategies for P1 and P2 respectively. Then

$$\sum_{j=1}^{n} a_{ij} q_j = V \quad \forall i \text{ s.t. } p_i > 0$$

$$\sum_{i=1}^{m} a_{ij} p_i = V \quad \forall j \text{ s.t. } q_j > 0$$

Proof. Let $p_k > 0$ and let $\sum_{j=1}^{n} a_{kj} q_j \neq V \implies \sum_{j=1}^{n} a_{kj} q_j < V$. We have

$$V \leqslant \sum_{i=1}^{m} p_i \sum_{j=1}^{n} a_{ij} q_j < V$$

a contradiction. □

Example 7.3 (The game of Odd-and-Even)
Played with three numbers: 0, 1, and 2.

		P2, even		
		0	1	2
P1, Odd	$0, p_1$	0	1	−2
	$1, p_2$	1	−2	3
	$2, p_3$	−2	3	−4

$p_1 \geqslant 0, p_2 \geqslant 0, p_3 \geqslant 0$, and $p_1 + p_2 + p_3 = 1$. Then

$$\begin{cases} p_2 - 2p_3 - V = 0 \\ p_1 - 2p_2 + 3p_3 - V = 0 \\ -2p_1 + 3p_2 - 4p_3 - V = 0 \\ p_1 + p_2 + p_3 = 1 \end{cases}$$

7.2 Symmetric Games

The rules are the same for P1 and P2. So $A^t = -A$.

Theorem 7.4
The value of a finite size symmetric game is zero.

Proof. Note $V^t = V$. And

$$V = (\hat{\mathbf{p}})^t A \hat{\mathbf{p}} = \left[(\hat{\mathbf{p}})^t A \hat{\mathbf{p}}\right]^t = -\hat{\mathbf{p}} A \hat{\mathbf{p}} = -V$$

So

$$V = -V \implies V = 0$$

□

Example 7.5 (Rock, Paper, Scissors)
We have

		P2		
		Rock	Paper	Scissors
P1	Rock	0	-1	1
	Paper	1	0	-1
	Scissors	-1	1	0

So

$$\begin{cases} \quad\;\; p_2 - p_3 = 0 \\ -p_1 \quad\; + p_3 = 0 \\ p_1 - p_2 \quad\;\; = 0 \\ p_1 + p_2 + p_3 = 1 \end{cases}$$

Therefore,

$$p_1 = p_2 = p_3 = \frac{1}{3}$$

8 Apr 13, 2022