Math 120A (Differential Geometry)
University of California, Los Angeles

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These are my lecture notes for Math 120A (Differential Geometry), which is taught by Fumiaki Suzuki. The textbook for this class is *Differential Geometry of Curves and Surfaces*, by Kristopher Tapp. Many of the figures I include in these notes are taken from Tapp's book.

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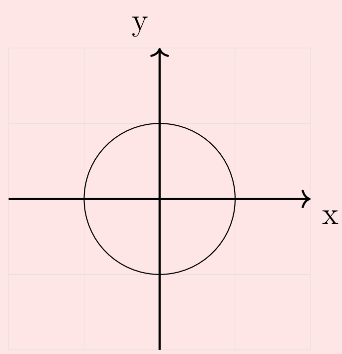
1.1 What is Differential Geometry?

Differential geometry studies geometry via analysis and linear algebra.

Geometry	Analysis	Linear Algebra
Intuitive	Rigorous	Computable
Curved	$\xrightarrow{\text{tangent space}}$	Linear
Global	Local	

1.2 Parametrized Curves

Example 1.1
A unit circle $S' = \{\vec{x} \text{ in } \mathbb{R}^2 \mid |\vec{x}| = 1\}$



$$\vec{\gamma} : [0, 2\pi) \to \mathbb{R}^2$$
$$t \mapsto (\cos t, \sin t)$$

$\vec{\gamma}[0, 2\pi) = S'$

Definition 1.2 (Parametrized curve and Trace)
A <u>(parametrized) curve</u> is a smooth function $\vec{\gamma} \colon I \to \mathbb{R}^n$, where I is an interval in $\mathbb{R}$. The image

$$\vec{\gamma}(I) = \{\vec{\gamma}(t) \mid t \in I\}$$

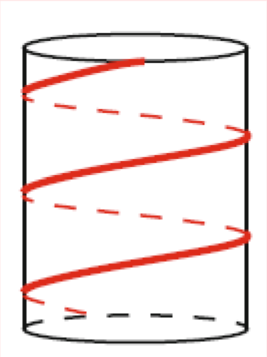
is called the <u>trace</u> of $\vec{\gamma}$.

Recall 1.3 An interval is a subset of $\mathbb{R}$ that has one of the following forms:

$$(a, b), [a, b], (a, b], [a, b), (-\infty, b), (-\infty, b], (a, \infty), [a, \infty), (-\infty, \infty) = \mathbb{R}.$$

A function $\vec{\gamma} \colon I \to \mathbb{R}^n$ is called <u>smooth</u> if $\vec{\gamma}$ is infinitely differentiable, or equivalently, each of the component functions $x_i \colon I \to \mathbb{R}$ is infinitely differentiable.

Example 1.4
$\vec{\gamma}(t) = (\cos t, \sin t, t)$, $t \in (-\infty, \infty)$ is a curve, called a helix.



Definition 1.5 (Derivative)
Let $\vec{\gamma}\colon I \to \mathbb{R}^n$ be a curve. The derivative of $\vec{\gamma}$ at t is defined as

$$\vec{\gamma}'(t) = \lim_{h\to 0} \frac{\vec{\gamma}(t+h) - \vec{\gamma}(t)}{h}$$

If t is on the boundaries of I, then use the left- or right-hand limit.

Remarks 1.6

i. If $\vec{\gamma}(t) = (x_1(t), x_2(t), \cdots, x_n(t))$, then $\vec{\gamma}'(t) = (x_1'(t), x_2'(t), \cdots, x_n'(t))$.

ii. The tangent line to the curve at $\vec{\gamma}'(t_0)$ is defined as

$$\vec{L}(t) = \vec{\gamma}(t_0) + t\vec{\gamma}'(t_0), \quad t \in (-\infty, \infty),$$

as soon as $\vec{\gamma}'(t) \neq \vec{0}$.

Definition 1.7 (Regular)
A curve $\vec{\gamma} : I \to \mathbb{R}^n$ is called regular if $\forall t \in I, \vec{\gamma}'(t) \neq \vec{0}$.

Remark 1.8 regular = tangent line is defined everywhere = the trace is smooth

Example 1.9

$$\vec{\gamma}(t) = (t^2, t^3), \quad t \in (-\infty, \infty)$$

Then $\vec{\gamma}$ is a curve that is not regular.
Indeed, $\vec{\gamma}'(t) = (2t, 3t^2)$, so $\vec{\gamma}'(0) = \vec{0}$.
Notice, $x(t) = t^2, y(t) = t^3$, so $x(t) = y(t)^{2/3}$. Hence, the trace is given by $x = y^{2/3}$ in $\mathbb{R}^2$.

Remark 1.10 The analogy with the physics is useful. If $\vec{\gamma} : I \to \mathbb{R}^n$ is a curve, then $\vec{\gamma}(t)$ is the position of a moving particle at time t in $\mathbb{R}^2$.

- $\vec{\gamma}'(t)$ velocity

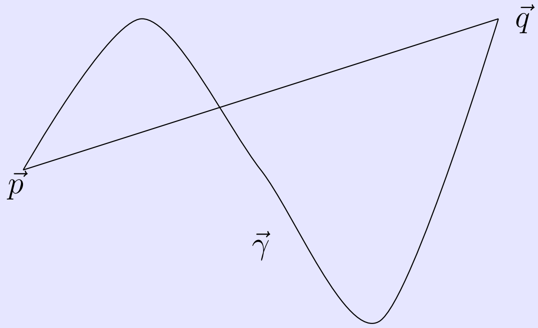
- $\vec{\gamma}''(t)$ acceleration
- $|\vec{\gamma}'(t)|$ speed

In this analogy, regular = the speed is always nonzero = the particle never stops (hence no "corners" on the trace)

Definition 1.11 (Arc length)
Let $\vec{\gamma}(t)\colon I \to \mathbb{R}^n$ be a regular curve. Then the arc length between times t_1, t_2 is defined as

$$\int_{t_1}^{t_2} |\vec{\gamma}'(t)|\, dt$$

Proposition 1.12
Let $\vec{\gamma}\colon [a,b] \to \mathbb{R}^n$ be a regular curve with the arc length L, $\vec{p} = \vec{\gamma}(a), \vec{q} = \vec{\gamma}(b)$. Then $L \geq |\vec{q} - \vec{p}|$.
Moreover, the equality holds if and only if $\vec{\gamma}$ parametrizes the line segment between $\vec{p}, \vec{q}$.



For the proof, we use the inner-product:
for $\vec{x} = (x_1, x_2, \cdots, x_n), \vec{y} = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$,

$$\langle x, y \rangle \coloneqq x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

Basic properties:

i. The inner product $\langle -, - \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is symmetric and bilinear.
ii. $\langle \vec{x}, \vec{y} \rangle = |\vec{x}||\vec{y}| \cos\theta$, where θ is the angle between $\vec{x}, \vec{y}$. ($\theta \in [0, 2\pi]$)
iii. $\langle \vec{x}, \vec{y} \rangle = 0 \Leftrightarrow \vec{x}, \vec{y}$ are orthogonal to each other.
iv. $\langle \vec{x}, \vec{x} \rangle = |\vec{x}|^2$
v. $\langle \vec{x}, \vec{y} \rangle \leq |\vec{x}||\vec{y}|$ (Schwartz Inequality) and the equality holds if and only if $\theta = 0$.

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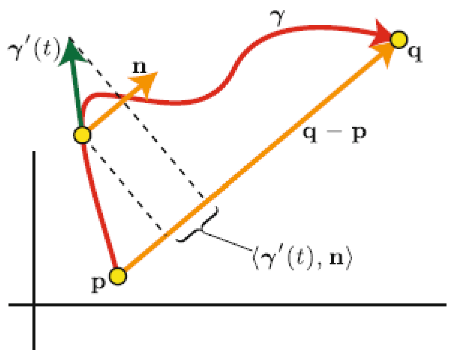
2.1 Proof of Proposition 1.12

Proof. Idea: Compare $\vec{\gamma}'(t)$ and its projection onto $\vec{q}-\vec{p}$. Set $\vec{n} = \frac{\vec{q}-\vec{p}}{|\vec{q}-\vec{p}|}$; $\vec{n}$ is unit.



Tapp Pg.15

Then $|\vec{\gamma}'(t)| \geq \langle \vec{\gamma}'(t), \vec{n} \rangle$ by Schwartz inequality.
Now,

$$\begin{aligned} L &= \int_a^b |\vec{\gamma}'(t)|\,dt \geq \int_a^b \langle \vec{\gamma}'(t), \vec{n} \rangle\,dt \\ &= [\langle \vec{\gamma}(t), \vec{n} \rangle]_a^b = \langle \vec{\gamma}(b), \vec{n} \rangle - \langle \vec{\gamma}(a), \vec{h} \rangle \\ &= \left\langle \vec{q}-\vec{p}, \frac{\vec{q}-\vec{p}}{|\vec{q}-\vec{p}|} \right\rangle = |\vec{q}-\vec{p}| \end{aligned}$$

If the equality holds, then $\forall t \in [a,b], \vec{\gamma}'(t), \vec{n}$ are in the same direction. So,

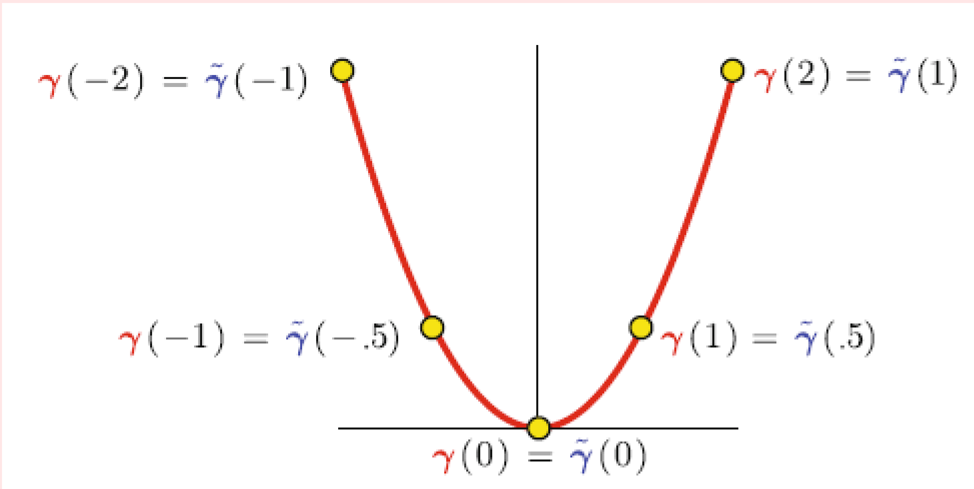
$$\begin{aligned} \vec{\gamma}'(t) &= \langle \vec{\gamma}'(t), \vec{n} \rangle \vec{n}. \\ \vec{\gamma}(t) &= \vec{\gamma}(a) + \int_a^t \vec{\gamma}'(u)du \\ &= \vec{p} + \left(\int_a^t \langle \vec{\gamma}'(u), \vec{n} \rangle dt \right) \vec{n} \end{aligned}$$

parametrizes the line segment between $\vec{p}, \vec{q}$. □

2.2 Reparametrization

There are regular curves that share common properties. Which regular curves should we identify?

Example 2.1
$\vec{\gamma}(t) = (t, t^2), \quad t \in [-2, 2]$
$\tilde{\vec{\gamma}}(t) = (-2t, (-2t)^2), t \in [-1, 1]$.
Then $\vec{\gamma}[-2, 2] = \tilde{\vec{\gamma}}[-1, 1] =$



$\vec{\gamma}, \tilde{\vec{\gamma}}$ are the same, up to change in time:
Let $\phi\colon [-1, 1] \to [-2, 2], \quad t \mapsto -2t$.
Then $\tilde{\vec{\gamma}} = \vec{\gamma} \circ \phi$

Definition 2.2 (Reparametrization)
Let $\vec{\gamma}\colon I \to \mathbb{R}^n$ be a regular curve. A reparametrization of $\vec{\gamma}$ is a function of the form

$$\tilde{\vec{\gamma}} = \vec{\gamma} \circ \phi : \tilde{I} \to \mathbb{R}^n,$$

where $\tilde{I}$ is an interval, $\phi\colon \tilde{I} \to I$ is a smooth bijection such that $\forall t \in \tilde{I}$, $\phi'(t) \neq 0$

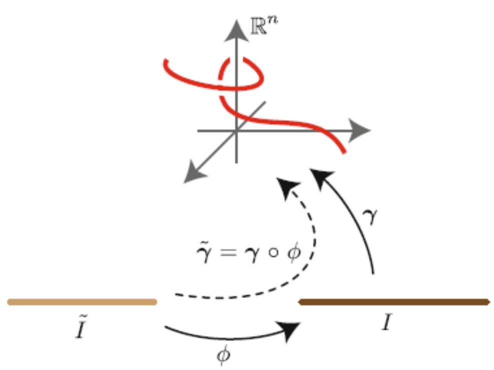


Figure 1: Kapp pg.19

Proposition 2.3
A reparametrization of a regular curve is a regular curve.

Proof. We use the same notations as the definition.
$\tilde{\vec{\gamma}} = \vec{\gamma} \circ \phi \colon \tilde{I} \to \mathbb{R}^n$ is the composition of smooth functions, so smooth.
Moreover, $\forall t \in \tilde{I}, \tilde{\vec{\gamma}}'(t) = \vec{\gamma}'(\phi(t)) \cdot \phi'(t) \neq 0$ □

We will be interested in regular curves up to reparametrizations.

Remarks 2.4

1. $\vec{\gamma}, \tilde{\vec{\gamma}}$ have the same trace.
2. There are regular curves with the same trace that cannot be reparametrized to each other. For instance,
$$\vec{\gamma}_1(t) = (\cos(t), \sin(t)), t \in [0, 2\pi),$$
$$\vec{\gamma}_2(t) = (\cos(t), \sin(t)), t \in [0, 4\pi),$$

Question 2.5: Is there a canonical reparametrization of a given regular curve?

Definition 2.6 (Unit-speed)
A regular curve $\vec{\gamma} \colon I \to \mathbb{R}^n$ is called unit-speed (or parametrized by arc length) if $\forall t \in I$, $|\vec{\gamma}'(t)| = 1$.

Remark 2.7 If $\vec{\gamma} \colon I \to \mathbb{R}^n$ is unit-speed, then,
$$\text{Arc length between } t_1, t_2 = \int_{t_1}^{t_2} |\vec{\gamma}'(t)| dt = \int_{t_1}^{t_2} dt = t_2 - t_1$$

Proposition 2.8
A regular curve always has a unit-speed reparametrization.

Proof. Let $\vec{\gamma} \colon I \to \mathbb{R}^n$ be a regular curve. Fix $t_0 \in I$. Define $s \colon I \to \mathbb{R}$ by
$$s(t) = \int_{t_0}^{t} |\vec{\gamma}'(u)|\, du.$$
Let $\tilde{I} = s(I) \subset \mathbb{R}$. Then $\tilde{I}$ is an interval by IVT.
Since $s'(t) = |\vec{\gamma}'(t)| > 0$ by FTC, regularity, $s \colon I \to \tilde{I}$ is a smooth bijection. Then, $\phi = s^{-1} \colon \tilde{I} \to I$ is a smooth bijection,
$$\phi'(t) = \frac{1}{s'(\phi(t))} = \frac{1}{|\vec{\gamma}'(\phi(t))|} \neq 0.$$
Now $\tilde{\vec{\gamma}} = \vec{\gamma} \circ \phi \colon \tilde{I} \to \mathbb{R}^n$ is a reparametrization of $\vec{\gamma}$, that is unit-speed:
$$\begin{aligned} |\tilde{\vec{\gamma}}'(t)| &= |\vec{\gamma}'(\phi(t)) \cdot \phi'(t)| \\ &= |\vec{\gamma}'(\phi(t))| \cdot 1/|\vec{\gamma}'(\phi(t))| \\ &= 1 \end{aligned}$$
□

<u>Note:</u>

$$
\begin{gathered}
s^{-1} \cdot s(t) = t \\
(s^{-1})'(s(t)) \cdot s'(t) = 1 \\
(s^{-1})'(s(t)) = 1/s'(t)
\end{gathered}
$$

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3.1 Reparametrization (Cont'd)

Example 3.1
$\vec{\gamma}(t) = (\cos(t), \sin(t), t), \quad t \in (-\infty, \infty)$ How can we find a unit-speed reparametrization of $\vec{\gamma}$? Compute the arc length function $S\colon (-\infty, \infty) \to \mathbb{R}$:

$$\begin{aligned} s(t) = \int_0^t |\vec{\gamma}'(u)|\, du &= \int_0^t |(-\sin(u), \cos(u), 1)|\, du \\ &= \int_0^t \sqrt{2}\, du = \sqrt{2}t \end{aligned}$$

Set $\phi = s^{-1}$, then $\phi(t) = t/\sqrt{2}$

$$\tilde{\vec{\gamma}}(t) = \vec{\gamma}(t) \circ \phi(t) = \left(\cos\left(t/\sqrt{2}\right), \sin\left(t/\sqrt{2}\right), t/\sqrt{2}\right)$$

$t \in (-\infty, \infty)$, is a unit speed reparametrization of $\vec{\gamma}$.

We will be interested in invariants for a regular curve that are unchanged under any reparametrizations.

Examples include:

- trace
- arc-length
- curvature
- torsion

Non-examples include:

- position
- velocity
- speed
- acceleration

Sometimes we consider more specific reparametrization.

Proposition 3.2
If $\tilde{\vec{\gamma}} = \vec{\gamma} \cdot \phi\colon \tilde{I} \to \mathbb{R}^n$ is a reparametrization of a regular curve $\vec{\gamma}\colon I \to \mathbb{R}^n$, then one of the following holds:

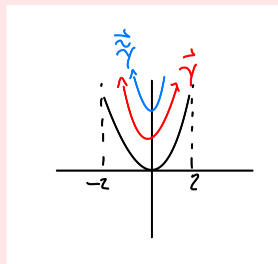
i. $\forall t \in \tilde{I}, \phi'(t) > 0$ i.e. ϕ is strictly increasing
ii. $\forall t \in \tilde{I}, \phi'(t) < 0$ i.e. ϕ is strictly decreasing

Proof. Otherwise $\exists t \in \tilde{I}, \phi'(t) = 0$ by IVT. This contradicts the assumption on ϕ. $\square$

Definition 3.3 (Orientation-preserving vs. orientation-reserving)
Under the setting of the proposition, we say $\tilde{\vec{\gamma}}$ is <u>orientation-preserving</u> if (i) occurs, or <u>orientation-reversing</u> if (ii) occurs.

Example 3.4 (Orientation-preserving)
The arc length reparametrization of a regular curve $\phi\colon I \to \tilde{I}$ is orientation-preserving, because $\phi'(t) = 1/|\vec{\gamma}'(\phi(t))| > 0 \quad \forall t \in I$
This shows an orientation=preserving unit-speed. Reparametrization always exists.

Example 3.5 (Orientation-reversing)
$\vec{\gamma}(t) = (t, t^2), \quad t \in [-2, 2]$
$\vec{\tilde{\gamma}}(t) = (-t, (-t)^2), \quad t \in [-2, 2]$



$\vec{\tilde{\gamma}}$ is an orientation-reserving reparametrization of $\vec{\gamma}$ by $\phi\colon [-2,2] \to [-2,2], \quad t \mapsto -t$
(Indeed, $\phi' = -1 < 0$).

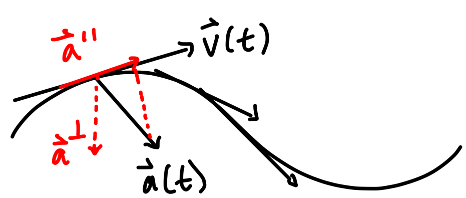
We will be interested in invariants that are unchanged under any orientation-preserving reparametrization.

- Signed curvature
- Rotation index

3.2 Curvature

The curvature measures how sharply the trace bends. What is a plausible definition of the curvature?

Let $\vec{\gamma}\colon I \to R^n$ be a regular curve. Set $\vec{v} = \vec{\gamma}', \vec{a} = \vec{\gamma}''$



$\vec{v}$ knows speed, direction of the motion

$\implies$ $\vec{a}$ should know the change in speed, direction $\to$ curvature.

We write

$$\vec{a} = \vec{a}'' + \vec{a}^{\perp}$$

where

$$\vec{a}'' = \left\langle \vec{a}, \frac{\vec{v}}{|\vec{v}|} \right\rangle : \quad \text{parallel to } \vec{v}$$

$$\vec{a}^{\perp} = \vec{a} - \vec{a}'' : \quad \text{orthogonal to } \vec{v}$$

Proposition 3.6
$\dfrac{d}{dt}|\vec{v}(t)| = \left\langle \vec{a}, \dfrac{\vec{v}}{|\vec{v}|} \right\rangle =$ the parallel component of $\vec{a}$ with respect to $\vec{v}$

Proof.

$$\begin{aligned}
\frac{d}{dt}|\vec{v}(t)| &= \frac{d}{dt}\langle \vec{v}(t), \vec{v}(t)\rangle^{1/2} \\
&= \frac{1}{2}\frac{1}{\langle \vec{v}(t), \vec{v}(t)\rangle^{1/2}} \cdot 2\langle \vec{v}(t), \vec{v}'(t)\rangle \\
&= \left\langle \frac{\vec{v}(t)}{|\vec{v}(t)|}, \vec{a}(t) \right\rangle
\end{aligned}$$

Note: $\langle v, v\rangle' = \langle v', v\rangle + \langle v, v'\rangle = 2\langle v', v\rangle$ □

So $|\vec{a}^{\perp}(t)|$ would be a plausible definition of the curvature. however this depends on $|\vec{t}|$. (Imagine a centripetal force for a car turning a corner.)

Definition 3.7 (Curvature)
Let $\vec{\gamma}\colon I \to \mathbb{R}^n$ be a regular curve. The <u>curvature function</u> $\kappa\colon I \to [0, \infty)$ is defined as

$$\kappa(t) = \frac{|\vec{a}^{\perp}(t)|}{|\vec{v}(t)|^2}$$

Proposition 3.8
Curvature is independent of parametrizations.

Proof. Let γ be a regular curve. $\tilde{\gamma} = \gamma \cdot \phi$ is a reparametrization of γ.
Denote:

$$\begin{aligned}
\kappa&\colon \text{curvature function for } \gamma \\
\tilde{\kappa}&\colon \text{curvature function for } \tilde{\gamma}
\end{aligned}$$

We need to show $\tilde{\kappa} = \kappa \circ \phi$
Denote:

$$\begin{aligned}
v, a&: \text{ velocity, acceleration of } \gamma \\
\tilde{v}, \tilde{a}&: \text{ velocity, acceleration of } \tilde{\gamma}.
\end{aligned}$$

Then,

$$\begin{aligned}
\tilde{\gamma} &= \gamma \cdot \phi \\
\tilde{v} &= \gamma' \cdot \phi \cdot \phi' = v \circ \phi \cdot \phi' \\
\tilde{a} &= \gamma'' \circ \phi \cdot (\phi')^2 + \gamma' \circ \phi \cdot \phi' \\
&= a \circ \phi \cdot (\phi')^2 + v \circ \phi \cdot \phi'
\end{aligned}$$

So, $\tilde{v}$ is parallel to v,

$$\tilde{a}^{\perp} = a^{\perp} \circ \phi \cdot (\phi')^2$$

Therefore,

$$\tilde{\kappa} = \frac{\tilde{a}^{\perp}}{|\tilde{v}|^2} = \frac{|a^{\perp} \circ \phi \cdot (\phi')^2|}{|v \circ \phi \cdot \phi|^2} = \frac{|a^{\perp} \cdot \phi|}{|v \cdot \phi|^2}$$

$$= \kappa \circ \phi$$

$\square$

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Note: From now on, I will bold my vectors like this $\mathbf{n}$ instead of $\vec{n}$.

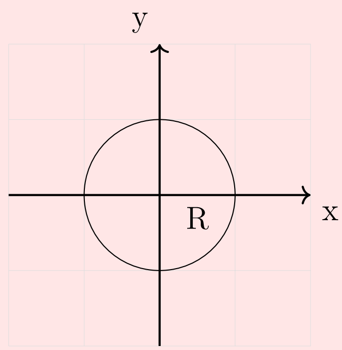
4.1 Curvature (Cont'd)

Recall 4.1

$$\kappa(t) = \frac{|\mathbf{a}^{\perp}(t)|}{|\mathbf{v}(t)|^2}$$

Example 4.2

$\gamma(t) = (R\cos(t), R\sin(t)), \quad t \in (-\infty, \infty)$



$\mathbf{v}(t) = (-R\sin(t), R\cos(t))$
$\mathbf{a}(t) = (-R\cos(t), -R\sin(t))$
Here $\langle \mathbf{v}(t), \mathbf{a}(t)\rangle = -R^2\sin(t)\cos(t) + R^2\cos(t)\sin(t) = 0$;
So $\mathbf{v}(t) \perp \mathbf{a}(t) \implies \mathbf{a}(t) = \mathbf{a}^{\perp}(t)$.
Therefore,

$$\kappa(t) = \frac{|\mathbf{a}(t)|}{|\mathbf{v}(t)|^2} = \frac{R}{R^2} = \frac{1}{R} \overset{R\to+\infty}{\to} 0 \text{ (flat)}$$

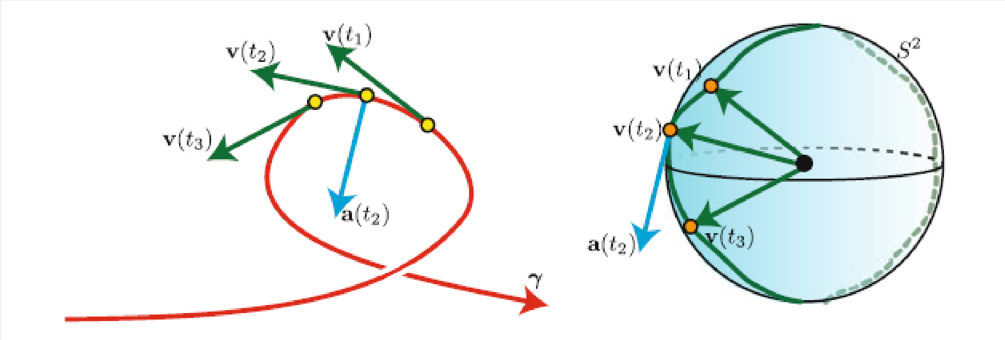
Historically, the curvature of a regular curve was first defined by $\kappa(t) = \frac{1}{R(t)}$, where $R(t)$ is the radius of the circle that best approximates the trace at t (The osculating circle; Read Tapp). Here we give another interpretation of the curvature using the osculating parabola.

Definition 4.3 (Unit tangent and normal vectors)
Let $\gamma\colon I \to \mathbb{R}^n$ be a regular curve. Define the unit tangent and normal vectors as

$$\mathbf{t}(t_0) = \frac{\mathbf{v}(t_0)}{|\mathbf{v}(t_0)|}, \qquad \underbrace{\mathbf{n}(t_0) = \frac{\mathbf{a}^{\perp}(t_0)}{|\mathbf{a}^{\perp}(t_0)|}}_{\text{defined only if } \kappa(t_0)\neq 0}$$

Remarks 4.4

i. $\mathbf{t}(t_0), \mathbf{n}(t_0)$ are orthonormal, i.e. unit, orthogonal to each other



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ii. The osculating plane at t_0 is the plane through $\mathbf{t_0}$ spanned by $\mathbf{t}(t_0), \mathbf{n}(t_0)$. The osculating plane is the plane that γ is the closest to begin in, and contains the directions where the curve is $\underset{(\mathbf{t})}{\text{heading}}$ and $\underset{(\mathbf{n})}{\text{bending}}$.

Proposition 4.5
Let $\gamma\colon I \to \mathbb{R}^n$ be a regular curve. Then $|\mathbf{t}'| = \kappa|\mathbf{v}|^2$, and $\mathbf{t}' = \kappa|\mathbf{v}|\mathbf{n}$ if $\mathbf{n}$ is defined. In particular, if γ is unit-speed, then

$$|\mathbf{t}'| = \kappa, \quad \text{and } \mathbf{t}' = \kappa\mathbf{n} \text{ if } \mathbf{n} \text{ is defined.}$$

Proof.

$$\mathbf{t}' = \left(\frac{\mathbf{v}}{|\mathbf{v}|}\right)' = \frac{\mathbf{a}}{|\mathbf{v}|} - \mathbf{v}\frac{\langle \mathbf{a}, \mathbf{v}\rangle}{|\mathbf{v}|^3} = \frac{\mathbf{a} - \mathbf{a}''}{|\mathbf{v}|} = \frac{\mathbf{a}^\perp}{|\mathbf{v}|}$$

Hence $|\mathbf{t}'| = \frac{|\mathbf{a}|^\perp}{|\mathbf{v}|^2} \cdot |\mathbf{v}| = \kappa|\mathbf{v}|$, and

$$\mathbf{t}' = \frac{|\mathbf{a}^\perp|}{|\mathbf{v}|^2}|\mathbf{v}|\frac{\mathbf{a}^\perp}{|\mathbf{a}^\perp|} = \kappa|\mathbf{v}|\mathbf{n} \text{ if } \mathbf{n} \text{ is defined.}$$

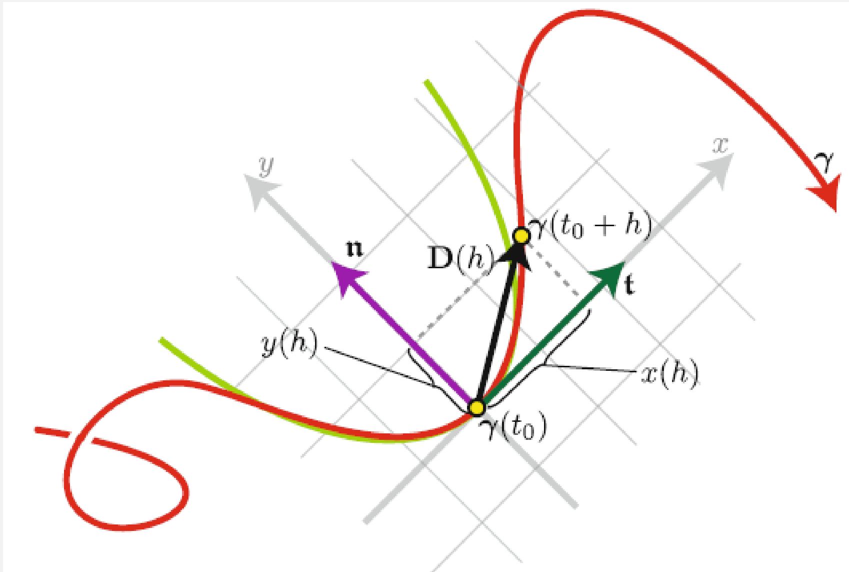
□

Remark 4.6 Let $\gamma\colon I \to \mathbb{R}^n$ be a unit-speed curve, $t_0 \in I$ with $\kappa(t_0) \neq 0$.
Then $\gamma'(t_0) = \mathbf{t}, \gamma''(t_0) = \mathbf{t}' = \kappa\mathbf{n}$, and the 2nd order Taylor approximation at γ at t_0 is

$$\begin{aligned}\gamma(t_0 + h) &\approx \gamma(t_0) + h\gamma'(t_0) + \frac{h^2}{2}\gamma''(t_0)\\ &= \gamma(t_0) + h\mathbf{t} + \frac{\kappa h^2}{2}\mathbf{n}\end{aligned}$$

Set $\mathbf{D}(h) = \gamma(t_0 + h) - \gamma(t_0) \approx h\mathbf{t} + \frac{\kappa h^2}{2}\mathbf{n}$: displacement.
Then,

$$\left.\begin{aligned} x(t) &:= \langle \mathbf{D}(h), \mathbf{t}\rangle \approx h \\ y(t) &:= \langle \mathbf{D}(h), \mathbf{n}\rangle \approx \tfrac{\kappa h^2}{2}\end{aligned}\right\} \text{ the parabola } y = \frac{\kappa}{2}x^2 \text{ in the osculating plane}$$



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$\kappa(t_0) =$ the concavity of the parabola that best approximates the trace at t_0

Proposition 4.7
Let $\gamma \colon I \to \mathbb{R}^n$ be a regular curve. If $\forall t \in I, \kappa(t) = 0$, then γ parametrizes a straight line.

Proof.

$$\begin{aligned} |\mathbf{t}'| = \kappa|\mathbf{v}| = 0 &\implies \mathbf{t}' = \mathbf{0} \\ &\implies \mathbf{t} = \mathbf{0} \text{ constant} \\ &\implies \mathbf{v} = |\mathbf{v}|\mathbf{c} \\ &\implies \text{ fixing } t_0 \in I, \end{aligned}$$

$$\begin{aligned} \gamma(t) &= \gamma(t_0) + \int_{t_0}^{t} \mathbf{v}(u)\, du \\ &= \gamma(t_0) + \left(\int_{t_0}^{t} |\mathbf{v}(u)|\, du\right)\mathbf{c} \end{aligned}$$

□

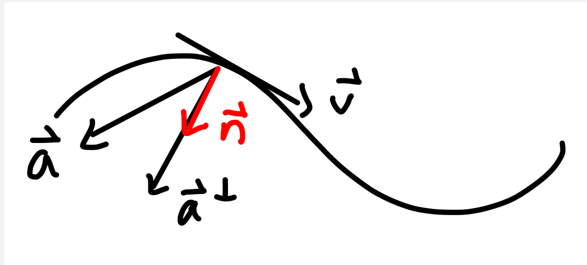
4.2 Plane Curves

$\mathbb{R}^2$ is the only $\mathbb{R}^n$ where the terms "clockwise" and "counter-clockwise" makes sense. This allows us to define

"signed curvature" = curvature + turning direction with respect to $\mathbf{v}$

Recall 4.8

$$\kappa = \frac{|\mathbf{a}^{\perp}|}{|\mathbf{v}|^2} = \frac{\langle \mathbf{a}, \mathbf{n}\rangle}{|\mathbf{v}|^2}$$



Definition 4.9 (Signed curvature)
Let $\gamma \colon I \to \mathbb{R}^2$ be a regular plane curve. Then the signed curvature $\kappa_s \colon I \to \mathbb{R}$ is defined as

$$\kappa_s = \frac{\langle \mathbf{a}, \mathbf{n}_s \rangle}{|\mathbf{v}|^2},$$

where,

$$\begin{aligned} \mathfrak{n}_s &= R_{90}\mathbf{t} \\ &= \text{ the counterclockwise } 90^\circ \text{ rotation of } \mathbf{t} \end{aligned}$$

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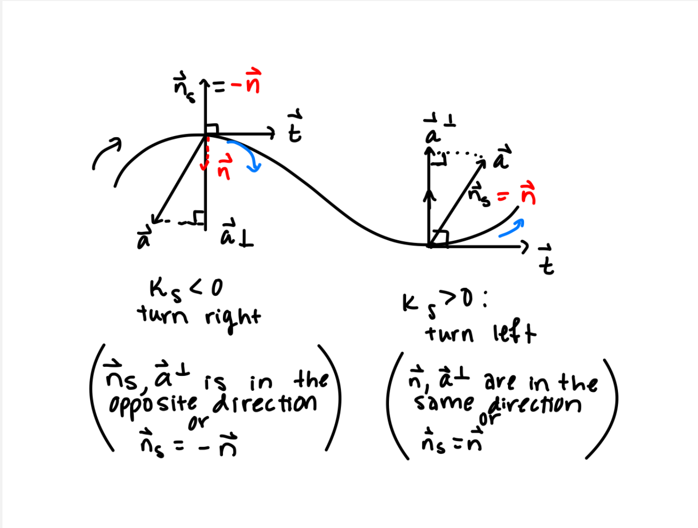
5.1 Plane Curves (Cont'd)

Recall 5.1

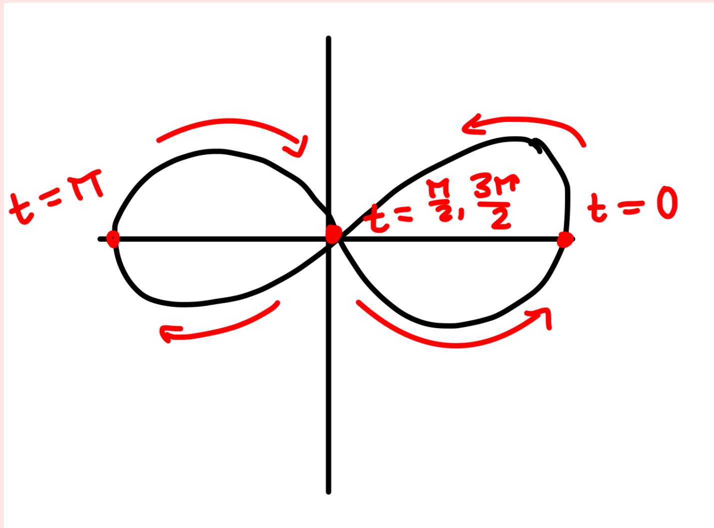
$$\kappa_s = \frac{\langle \mathbf{a}, \mathbf{n}_s \rangle}{|\mathbf{v}|^2}$$

where, $\mathbf{n}_s = R_{90^\circ}\mathbf{t}$



Example 5.2

$$\gamma(g) = (\cos(t), \sin(2t)), \quad t \in [0, 2\pi]$$



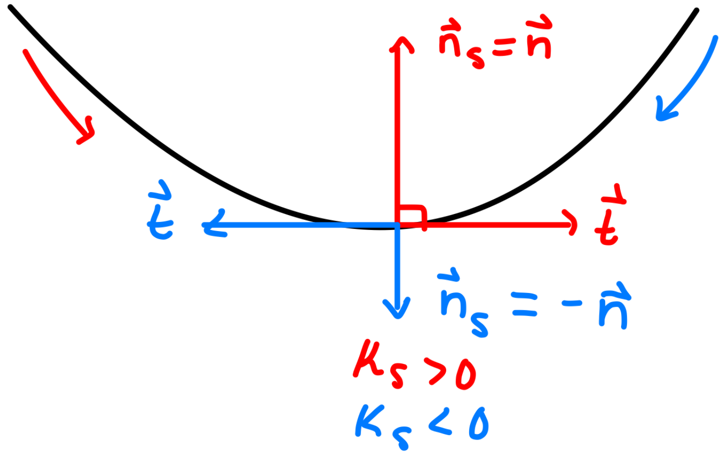
Lissajous curve

$$\mathbf{v}(t) = (-\sin(t), 2\cos(2t))$$
$$\mathbf{a}(t) = (-\cos(t), -4\sin(2t))$$
$$|\mathbf{v}(t)| = \sqrt{\sin^2(t) + 4\cos^2(2t)}$$
$$\mathbf{t}(t) = \frac{\mathbf{t}}{\mathbf{v}(t)} = (-\sin(t), 2\cos(2t))\frac{1}{\sqrt{\sin^2 t + 4\cos^2 2t}}$$
$$\mathbf{n}_s = R_{90}\mathbf{t} = (-2\cos(2t), -\sin(t))\frac{1}{\sqrt{\sin^2 t + 4\cos^2(2t)}}$$
$$\kappa_s = \frac{\langle \mathbf{a}, \mathbf{n}_s \rangle}{|\mathbf{v}|^2} = \frac{2\cos(t)\cos(2t) + 4\sin(t)\sin(2t)}{(\sin(3t) + 4\cos^2(2t))^{3/2}}$$

$$\kappa_s(0) = \frac{2}{4^{3/2}} = \frac{2}{8} = \frac{1}{4} > 0$$
$$\kappa_s\left(\frac{\pi}{2}\right) = 0$$
$$\kappa_s(\pi) = \frac{-1}{4} < 0$$
$$\kappa_s\left(\frac{3\pi}{2}\right) = 0$$

Proposition 5.3
Let $\gamma\colon I \to \mathbb{R}^2$ be a plane curve. Then $|\kappa_s| = \kappa$.

Proof. Compare $\kappa = \frac{\langle \mathbf{a}, \mathbf{n} \rangle}{|\mathbf{v}|^2}$, $\kappa_s = \frac{\langle \mathbf{a}, \mathbf{n}_s \rangle}{|\mathbf{v}|^2}$ $\mathbf{n}_s = \pm\mathbf{n}$, because they are both unit, orthogonal to $\mathbf{t}$. Hence κ_s coincides with $\underset{\geq 0}{\kappa}$ up to signs.

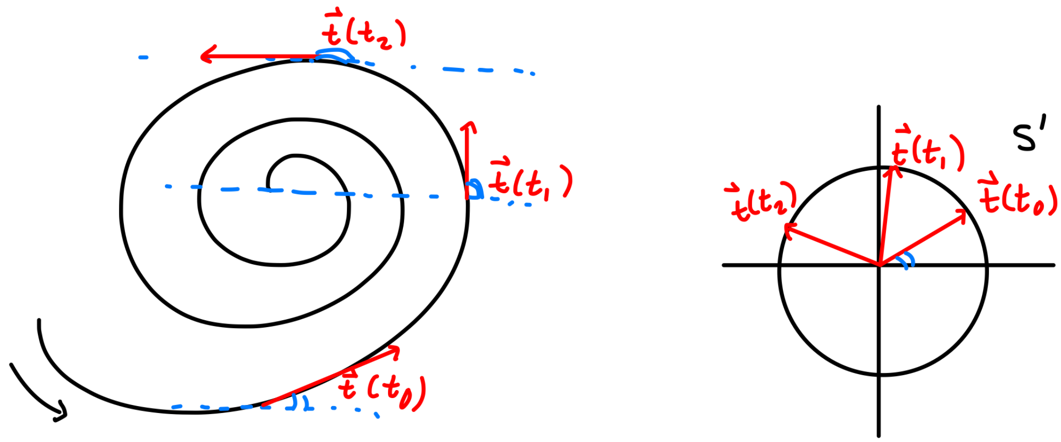


□

Proposition 5.4
Signed curvature is unchanged by any orientation-preserving reparametrizations.

Proof. Exercise. □

Proposition 5.5
Let $\gamma\colon I \to \mathbb{R}^2$ be a plane curve. Then there exists a smooth function $\theta\colon I \to \mathbb{R}$ such that $\forall t \in I, \mathbf{t}(t) = (\cos\theta(t), \sin\theta(t))$.



What should θ be?

$$\mathbf{t}' = \theta'(-\sin\theta, \cos\theta) = \theta' R_{90}\mathbf{t} = \theta' \mathbf{n}_s.$$

On the other hand,

$$\mathbf{t}' = \left(\frac{\mathbf{v}}{|\mathbf{v}|}\right)' = \frac{\mathbf{a}^\perp}{|\mathbf{v}|} = \frac{\langle \mathbf{a}, \mathbf{n}_s\rangle}{|\mathbf{v}|}\mathbf{n}_s = \kappa_s|\mathbf{v}|\mathbf{n}_s$$

By comparing the two formulas, $\theta' = \kappa_s|\mathbf{v}|$. In the proof, we solve this differential equation.

Remark 5.6 If γ is unit-speed, $\theta' = \kappa_s$. This shows:

$$\begin{aligned} \text{signed curvature} &= \text{ the rate of change of the angle} \\ \text{curvature} &= |\text{the rate of change of the angle}| \end{aligned}$$

Proof. Fix $t_0 \in I, \theta_0 \in \mathbb{R}$ such that $\mathbf{t}(t_0) = (\cos\theta_0, \sin\theta_0)$.
Define

$$\theta(t) = \theta_0 + \int_{t_0}^{t} \kappa_s(u)|\mathbf{v}(u)|\,du$$

We will show this $\theta(t)$ works.

$$\theta\colon I \to \mathbb{R} \text{ is a smooth function}$$
$$\theta' = \kappa_s|\mathbf{v}|, \theta(t_0) = \theta_0.$$

Set $\mathbf{t}_\theta = (\cos\theta, \sin\theta)$ We need to show $\mathbf{t} = \mathbf{t}_\theta$. Observe $\mathbf{t}, \mathbf{t}_\theta$ are unit.
Enough to show $\langle \mathbf{t}, \mathbf{t}_\theta \rangle = 1$
On the other hand,

$$\begin{aligned} \mathbf{t}_\theta(t_0) &= (\cos\theta(t_0), \sin\theta(t_0)) \\ &= (\cos\theta_0, \sin\theta_0) \\ &= \mathbf{t}(t_0) \end{aligned}$$

Enough to show $\langle \mathbf{t}, \mathbf{t}_\theta \rangle' = 0$

$$\begin{aligned} \mathbf{t}' &= \kappa_s|\mathbf{v}|\mathbf{n}_s = \kappa_s|\mathbf{v}|R_{90}\mathbf{t} \\ \mathbf{t}_\theta' &= \theta'(-\sin\theta, \cos\theta) = \kappa_s|\mathbf{v}|R_{90}\mathbf{t}_\theta \end{aligned}$$

Therefore,

$$\begin{aligned} \langle \mathbf{t}, \mathbf{t}_\theta \rangle' &= \langle \mathbf{t}', \mathbf{t}_\theta \rangle + \langle \mathbf{t}, \mathbf{t}_\theta' \rangle \\ &= \kappa_s|\mathbf{v}|\left(\langle R_{90}\mathbf{t}, \mathbf{t}_\theta \rangle + \langle \mathbf{t}, R_{90}\mathbf{t}_\theta \rangle\right) \\ &= \kappa_s|\mathbf{v}|\left(\langle R_{90}\mathbf{t}, \mathbf{t}_\theta \rangle + \langle R_{90}\mathbf{t}, R_{90}(R_{90}\mathbf{t}_\theta) \rangle\right) \\ & R_{90} \text{ is orthogonal} \\ &= \kappa_s|\mathbf{v}|\left(\langle R_{90}\mathbf{t}, \mathbf{t}_\theta \rangle - \langle R_{90}\mathbf{t}, \mathbf{t}_\theta \rangle\right) \\ & R_{90} \circ R_{90} = R_{180} = -1 \\ &= 0 \end{aligned}$$

□

Remark 5.7 The angle function θ is unique up to an integer multiple of 2π.
Indeed if $\Theta\colon I \to \mathbb{R}$ is a smooth function such that $\forall t \in I, \gamma = (\cos\Theta, \sin\Theta)$, then,

$$\begin{aligned} \Theta' &= \theta' = \kappa_s|\mathbf{v}| \\ &\implies |\Theta - \theta|' = 0 \\ &\implies \Theta - \theta = \text{constant} \end{aligned}$$

On the other hand,

$$(\cos\theta, \sin\theta) = (\cos\Theta, \sin\Theta) = \mathbf{t}$$

So $\Theta - \theta \in 2\pi \cdot \mathbb{Z}$