Math 132 (Complex Analysis for Applications)
University of California, Los Angeles

Aaron Chao

Winter 2022

These are my lecture notes for Math 132 (Complex Analysis for Applications), which is taught by Tyler James Arant. The textbook for this class is *Complex Analysis*, by Theodore W. Gamelin.

Contents

1 Jan 3, 2022

1.1 What are the Complex Numbers?

We first recall the basic algebraic properties of the real numbers, $\mathbb{R}$. For all $a, b, c \in \mathbb{R}$,

1. (Commutative law of addition): $a + b = b + a$
2. (Commutative law of multiplication): $a \cdot b = b \cdot a$
3. (Associative law of addition): $(a + b) + c = a + (b + c)$
4. (Associative law of multiplication): $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
5. (Distributive law): $a(b + c) = a \cdot b + a \cdot c$

The system of real numbers $\mathbb{R}$ has many more (non-algebraic) properties which make it suitable for calculus. However, it lacks a particular desirable property: $\mathbb{R}$ does not contain roots for all of its polynomial equations, e.g., there is not a solution to the equation

$$x^2 + 1 = 0 \quad \text{in } \mathbb{R}.$$

It turns out (by the non-trivial fundamental theorem of algebra) that we can get a number system for which every polynomial equation has a root by "appending" $i = \sqrt{-1}$ to $\mathbb{R}$.

Definition 1.1 (Complex number)
A complex number is an expression of the form

$$x + iy \quad \text{where } x, y \in \mathbb{R},$$

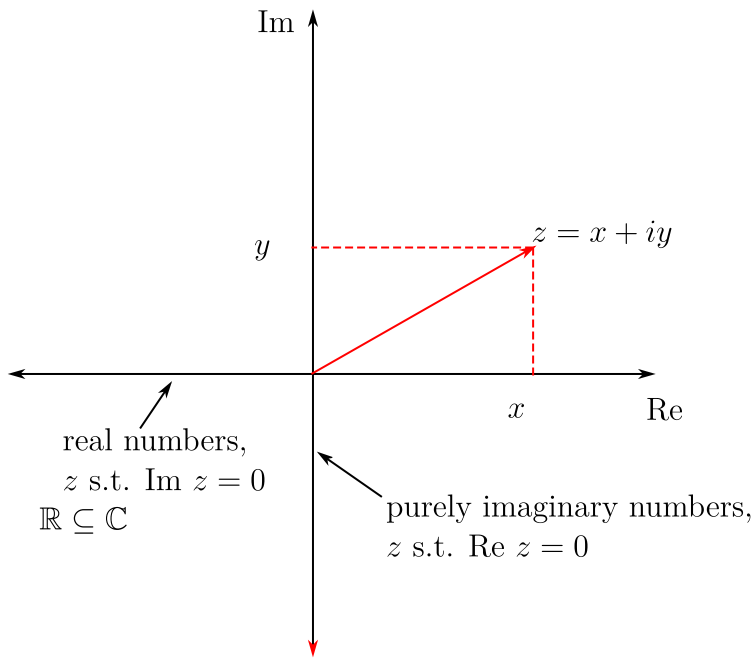
Two complex numbers $a + ib$ and $c + id$ are equal if and only if $a = c$ and $b = d$
We denote by $\mathbb{C}$ the set of all complex numbers.
For a complex number $z = x + iy$, we define its real and imaginary parts as follows:

$$\operatorname{Re} z = x$$
$$\operatorname{Im} z = y$$

There is a one-to-one correspondence between $\mathbb{C}$ and $\mathbb{R}^2$:

$$z \mapsto (\operatorname{Re} z, \operatorname{Im} z)$$

This can be visualized as the *complex plane*, where we can identify the real numbers and the *purely imaginary numbers*.



Example 1.2 (Addition and multiplication on $\mathbb{C}$)
We can define operations of addition and multiplication on $\mathbb{C}$ as follows:

$$\begin{aligned} z &= x + iy, \quad w = a + ib \\ z + w &= (x + iy) + (a + ib) = (x + a) + i(y + b) \\ zw &= (x + iy)(a + ib) \\ &= xa + ixb + iya + i^2 yb \\ &= (xa - yb) + i(xb + ya) \end{aligned}$$

Example 1.3 (Multiplicative inverse in $\mathbb{C}$)
Every nonzero complex number $z = x + iy \neq 0$ has a multiplicative inverse,

$$\frac{1}{z} = \frac{x - iy}{x^2 + y^2}.$$

Need to check $z \cdot \frac{1}{z} = 1$

$$z \cdot \frac{1}{z} = (x + iy)\left(\frac{x - iy}{x^2 + y^2}\right) = \left(\frac{x^2 - ixy + ixy - i^2 y^2}{x^2 + y^2}\right) = \left(\frac{x^2 + y^2}{x^2 + y^2}\right) = 1$$

In addition to having additive and multiplicative inverses, the complex numbers also have the following algebraic properties:
For all $z_1, z_2, z_3 \in \mathbb{C}$,

1. (Commutative law of addition): $z_1 + z_2 = z_2 + z_1$
2. (Commutative law of multiplication): $z_1 \cdot z_2 = z_2 \cdot z_1$

3. (Associative law of addition): $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
4. (Associative law of multiplication): $z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$
5. (Distributive law): $z_1(z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$

1.2 Complex Conjugates and the Modulus

Definition 1.4 (Complex conjugate)
The complex conjugate of the number $z = x + iy$ is the number

$$\overline{z} = x - iy.$$

Some basic facts about complex conjugation. All are simple to prove, so we only discuss the proof of a few.

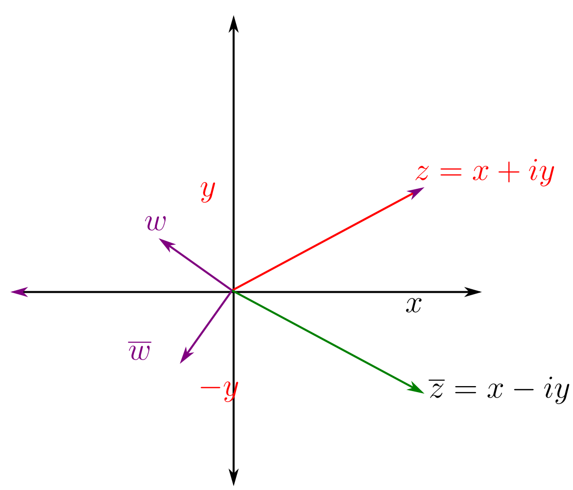
- $\overline{\overline{z}} = z$
- $z = \overline{z}$ if and only if z is a real number
- $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$
- $\overline{z_1 z_2} = \overline{z}_1 \cdot \overline{z}_2$
- $\overline{\left(\frac{1}{z}\right)} = 1/\overline{z}$

Proof. We want to show that $\overline{\left(\frac{1}{z}\right)} = 1/\overline{z}$.

$$\frac{1}{\overline{z}} = \frac{1}{x - iy} = \frac{x - (-iy)}{x^2 + y^2} = \frac{x + iy}{x^2 + y^2} = \overline{\left(\frac{x - iy}{x^2 + y^2}\right)} = \overline{\left(\frac{1}{\overline{z}}\right)}$$

□

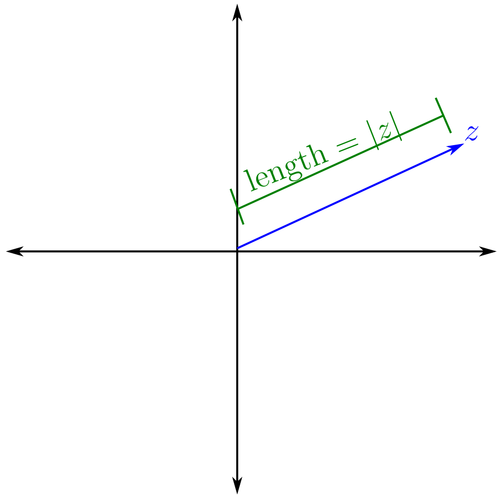
Geometrically, conjugation reflects z across the real axis:



Definition 1.5 (Absolute value/modulus)
The absolute value or modulus of $z = x + iy$ is

$$|z| = \sqrt{x^2 + y^2}$$

Geometrically, $|z|$ is the length of z as a vector in the complex plane:



Some properties relating complex conjugation and absolute value:

- $|z|^2 = z\overline{z}$

$$\begin{aligned} z\overline{z} = (x+iy)(x-iy) &= x^2 - ixy + ixy - i^2y^2 \\ &= x^2 + y^2 \\ &= |z|^2 \end{aligned}$$

- $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$

$$\frac{1}{z} = \frac{x - iy}{x^2 + y^2} = \frac{\overline{z}}{|z|^2}$$

- We have

$$\operatorname{Re} z = \frac{z + \overline{z}}{2}, \quad \operatorname{Im} z = \frac{z - \overline{z}}{2i}$$

$$\frac{z + \overline{z}}{2} = \frac{x + iy + x - iy}{2} = \frac{2x}{2} = x$$

  Note: $\frac{1}{i} = -i$

- For $z, w \in \mathbb{C}$, $|zw| = |z| \cdot |w|$.

$$\begin{aligned} |z|^2 \cdot |w|^2 = z\overline{z} \cdot w\overline{w} &= (zw)(\overline{z} \cdot \overline{w}) \\ &= (zw)\overline{(zw)} = |zw|^2 \end{aligned}$$

  Then take a square root.

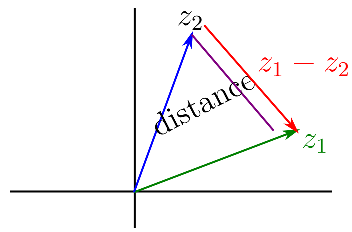
2 Jan 5, 2022

2.1 Distance in the Complex Plane

We can use absolute value to measure the distance between complex numbers (thought of as vectors in the complex plane).

The distance between complex numbers z_1, z_2 is $|z_1 - z_2|$:

$$z_1 = x_1 + iy_1, \quad z_2 = x_2 + iy_2$$
$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$



A crucial fact about working with absolute values is

Proposition 2.1 (Triangle Inequality)
For any two complex numbers z_1, z_2,

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Some corollaries of the Triangle Inequality:

Corollary 2.2
For any complex numbers z_1, z_2, w,

- $|z_1 - z_2| \leq |z_1 - w| + |w - z_2|$.
- $|z_2| - |z_1| \leq |z_2 - z_1|$ (Reverse triangle inequality)

Proof.

- For the first inequality,

$$\begin{aligned} |z_1 - z_2| &= |(z_1 - w) + (w - z_2)| \\ &\leq |z_1 - w| + |w - z_2| && \text{(Triangle Inequality)} \end{aligned}$$

- For the second inequality,

$$\begin{aligned} |z_2| = |(z_2 - z_1) + z_1| &\leq |z_2 - z_1| + |z_1| \\ \implies |z_2| - |z_1| &\leq |z_2 - z_1| \end{aligned}$$

By symmetry, also $|z_1| - |z_2| \leq |z_2 - z_1|$

$\square$

2.2 Complex Polynomials

Definition 2.3 (Complex polynomial)
A complex polynomial of degree $n \geq 0$ is a function of the form

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 + a_0, \quad z \in \mathbb{C},$$

where the coefficients $a_0, a_1, \ldots, a_n$ are complex numbers with $a_n \neq 0$.

Theorem 2.4 (Fundamental Theorem of Algebra)
Every complex polynomial $p(z)$ of degree $n \geq 1$ has a factorization

$$p(z) = c(z - z_1)^{m_1}(z - z_2)^{m_2} \cdots (z - z_k)^{m_k},$$

where $c \in \mathbb{C}$, the roots $z_1, \ldots, z_n$ are distinct complex numbers, and $m_1, \ldots, m_k \geq 1$. This factorization is unique, up to permutation of the factors. Also note that

$$\sum_{j=1}^{k} m_k = n = \text{the degree of the polynomial}$$

Proof. We will prove this here. □

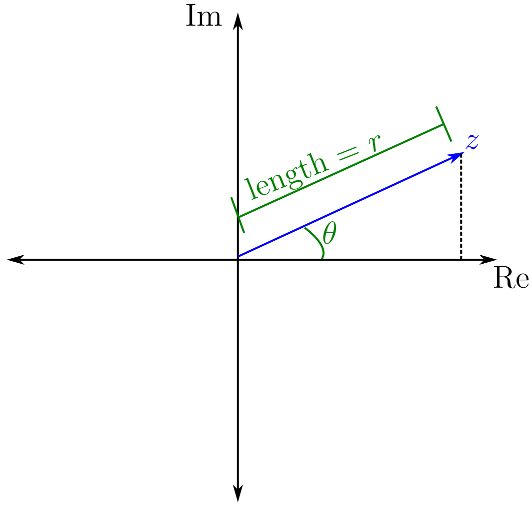
Example 2.5
Consider $p(z) = iz^2 + i$.

$$p(z) = iz^2 + i = i(z^2 + 1) = i(z - i)(z + i)$$

2.3 Polar Representation

A nonzero complex number $z \in \mathbb{C}$ can be described by two quantities:

- its length $|z| = r$
- The angle z makes with the positive real axis.



Definition 2.6 (Polar representation)
The polar representation of $z \neq 0$ is:

$$z = r(\cos(\theta) + i\sin(\theta))$$

Given $z = r(\cos(\theta) + i\sin(\theta))$ we can recover the Cartesian coordinates for z:

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$$

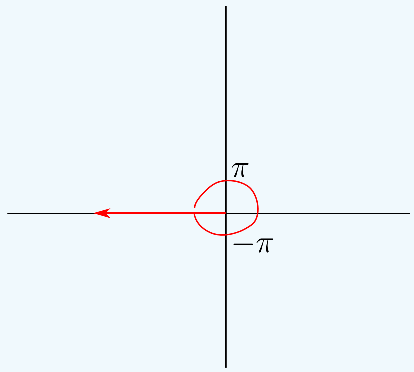
Definition 2.7 (Argument)
For a nonzero $z = r(\cos(\theta) + i\sin(\theta))$, the angle θ is called the argument of z, and is denoted $\theta = \arg z$. But the argument of $z, \arg z$, is actually a multivalued function:

$$\arg z + 2\pi k, \quad k \text{ an integer}$$

also represents the same angle

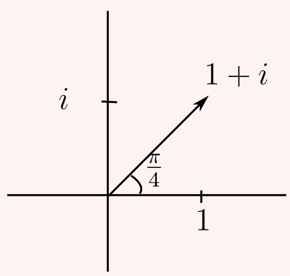
Definition 2.8 (Principal argument)
The principal argument of $z \neq 0$, denoted $\operatorname{Arg} z$, is the unique argument θ which is in $(-\pi, \pi]$.

$$\arg z = \{\operatorname{Arg} z + 2k\pi : k \text{ integer}\} \qquad \text{“multivalued function”}$$
$$\operatorname{Arg} z \qquad \text{“single-valued”}$$



Example 2.9
Consider the complex number $1 + i$.

$$|1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2}$$
$$1 + i = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right)\right)$$
$$\operatorname{Arg}(1 + i) = \frac{\pi}{4}$$



We introduce the notation

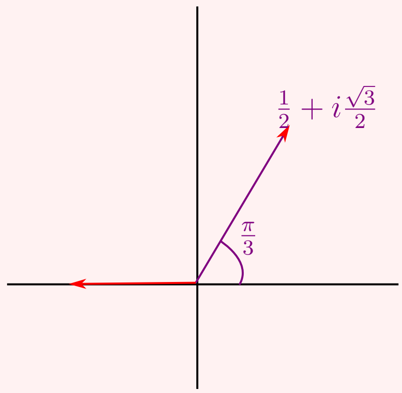
$$e^{i\theta} = \cos(\theta) + i \sin(\theta) \quad \text{for } \theta \in \mathbb{R}.$$

We will see that this equality does follow from how we define complex exponentiation, but in the meantime it is just some convenient notation since the polar form for $z \in \mathbb{C}$ becomes:

$$z \neq 0, z = re^{i\theta},$$
$$\text{where } \theta = \operatorname{Arg} z, r = |z|$$

Example 2.10
We write the polar form using complex exponentials for the numbers -1 and $\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

$$\operatorname{Arg}(-1) = \pi, |-1| = 1$$
$$-1 = e^{i\pi}$$
$$\frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{i\frac{\pi}{3}}$$



Some helpful identities for $e^{i\theta}, \theta \in \mathbb{R}$:

$$|e^{i\theta}| = 1, \quad \overline{e^{i\theta}} = e^{-i\theta}, \quad \frac{1}{e^{i\theta}} = e^{-i\theta}.$$

Proof.

$$\begin{aligned}\overline{e^{i\theta}} &= \cos(\theta) - i\sin(\theta) = \cos(-\theta) + i\sin(-\theta) \\ &= e^{-i\theta}\end{aligned}$$

Check $e^{i\theta}e^{-i\theta} = 1$ using $\cos^2(\theta) + \sin^2(\theta) = 1$ $\square$

Another very important identity for $\theta, \varphi \in \mathbb{R}$:

$$e^{i(\theta+\varphi)} = e^{i\theta}e^{i\varphi}.$$

Just use angle sum formulas

$$\cos(\theta + \varphi) = \cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi)$$

$$\sin(\theta + \varphi) = \cos(\theta)\sin(\varphi) + \sin(\theta)\cos(\varphi)$$

Example 2.11

If $z = re^{i\theta}$ is a nonzero complex number, then

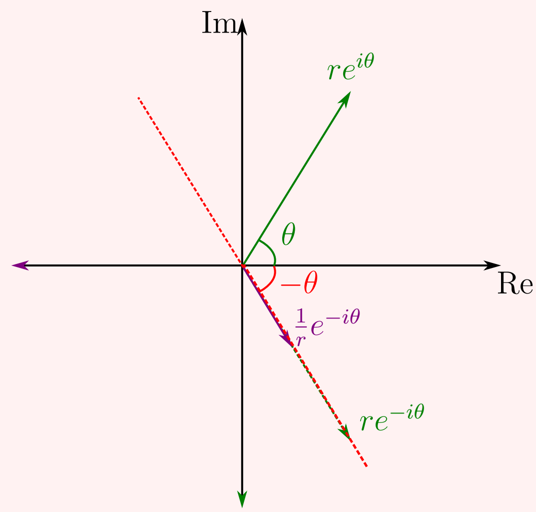
$$\frac{1}{z} = \frac{1}{r}e^{-\theta}, \quad \overline{z} = re^{-i\theta}.$$

$$\left(\frac{1}{r}e^{-i\theta}\right) = \left(r \cdot \frac{1}{r}\right) e^{i\theta + (-i\theta)} = 1$$
$$\implies \frac{1}{r}e^{-i\theta} = \frac{1}{z}$$

Polar form can also help us understand multiplication of complex numbers geometrically in the complex plane:

Example 2.12 (Inversion)

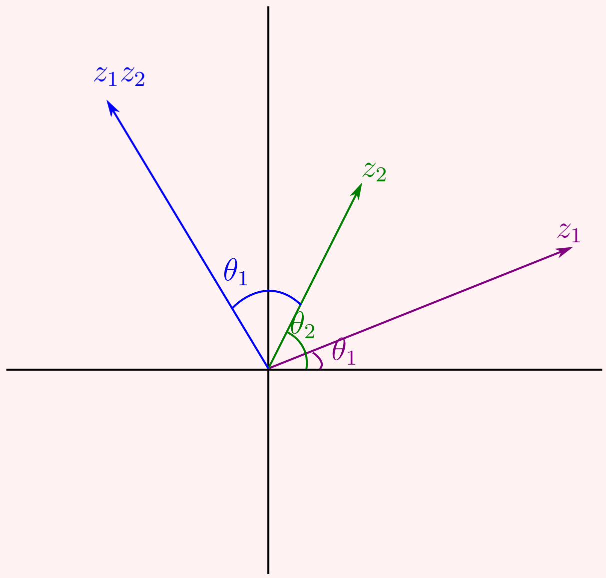
$$z = re^{i\theta} \neq 0 \quad \frac{1}{z} = \frac{1}{r}e^{-i\theta}$$



Example 2.13 (Multiplication)

$$z_1 = r_1 e^{i\theta_1}, \quad z_2 = r_2 e^{i\theta_2}$$
$$z_1 \cdot z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

To multiply two complex numbers, multiply their lengths and add their arg's.



2.4 De Moivre's Formulae

Theorem 2.14 (De Moivre's Formulae)
For any natural number n and any $\theta \in \mathbb{R}$, we have

$$\cos(n\theta) + i\sin(n\theta) = e^{in\theta} = (e^{i\theta})^n = (\cos(\theta) + i\sin(\theta))^n$$

Once the right-hand-side is expanded, we can obtain expressions for $\cos(n\theta)$ and $\sin(n\theta)$ as polynomials is $\cos(\theta)$ and $\sin(\theta)$. These qualities are known as <u>de Moivre's formulae</u>.

Example 2.15
We obtain de Moivre's formulae for the case $n = 2$.

$$\begin{aligned}\cos(2\theta) + i\sin(2\theta) &= (\cos(\theta) + i\sin(\theta))^2 \\ &= \cos^2(\theta) - \sin^2(\theta) + i\,(2\cos(\theta)\sin(\theta))\end{aligned}$$

$$\overset{\text{equating Re=Im}}{\Longrightarrow} = \begin{cases} \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) \\ \sin(2\theta) = 2\cos(\theta)\sin(\theta) \end{cases}$$

Exercise. See the textbook for a derivation of the formulae

$$\cos(3\theta) = \cos^3(\theta) + 3\cos(\theta)\sin^2(\theta), \quad \sin(3\theta) = 3\cos^2(\theta)\sin(\theta) - \sin^3(\theta).$$

2.5 nth Roots

Definition 2.16 (nth roots)
A complex number z is an <u>nth root</u> of w if $z^n = w$. In other words, the nth roots of w are precisely the roots of the polynomial $p(z) = z^n - w$. As an immediate consequence: any $z \in \mathbb{C}$ has at most n many distinct nth roots.

For a nonzero complex number w, we can find its nth roots as follows: Let

$$w = \rho e^{i\varphi} \quad z = re^{i\theta}$$

For z to be an nth root of w, we need

$$z^n = w$$

$$\implies r^n e^{in\theta} = \rho e^{i\varphi}$$

$$\implies r^n = \rho, \quad n\theta = \varphi$$

or,

$$n\theta = \varphi + 2k\pi \quad k \in \mathbb{Z}.$$

$$\implies r = \rho^{1/n}, \quad \theta = \frac{\varphi + 2k\pi}{n},$$

distinct angles are

$$\theta = \frac{\varphi + 2k\pi}{n}, \quad k = 0, 1, 2, \ldots, n-1$$

Therefore,

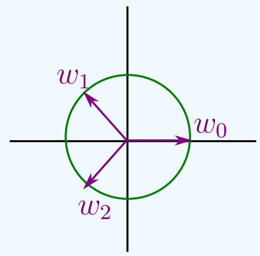
$$n\text{th roots of w:} \quad z_k = \rho^{1/n} e^{i\left(\frac{\varphi + 2k\pi}{n}\right)}, \quad k = 0, 1, \ldots, n-1$$

Exercise. Find and plot the 3rd roots of $w = 9i$.

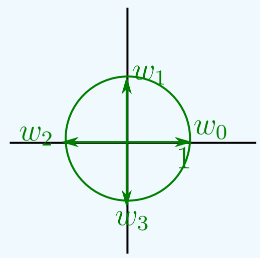
Definition 2.17 (nth roots of unity)
The nth roots of 1 have a special name: they are called the <u>nth roots of unity</u>. Using the same procedure as above, we can find that the nth roots of unity are

$$w_k = e^{2\pi ik/n}, \quad \text{for } k = 0, 1, \ldots, n-1.$$

3rd roots of unity:



4th roots of unity:



nth roots of unity can also be used to find nth roots of complex numbers other than 1. For a nonzero $z = re^{i\theta}$, we can find the first nth root of z to be $z_0 = r^{1/n}e^{i\theta/n}$. Then, if $w_0, \ldots, w_{n-1}$ are the nth root of unity, then the nth roots of z are exactly

$$z_k = z_0 w_k, \quad k = 0, \ldots, n-1.$$

$$z_k^n = (z_0 w_k) = z_0^n \cdot w_k^n = z \cdot 1 = z$$

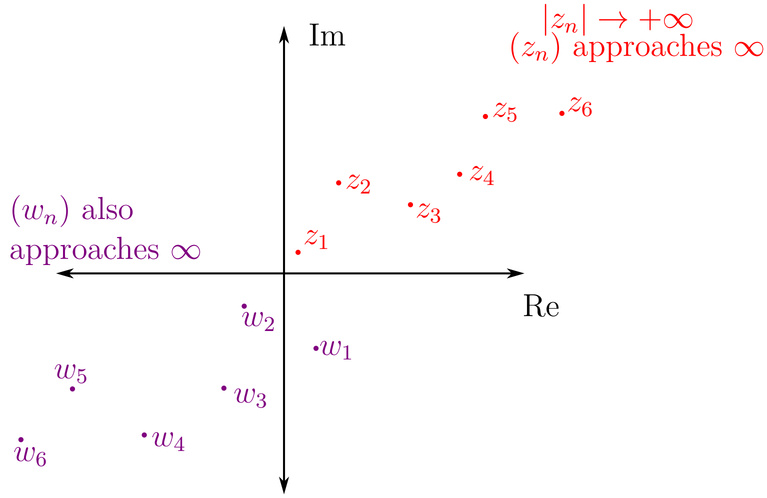
3 Jan 7, 2022

3.1 Stereographic Projection

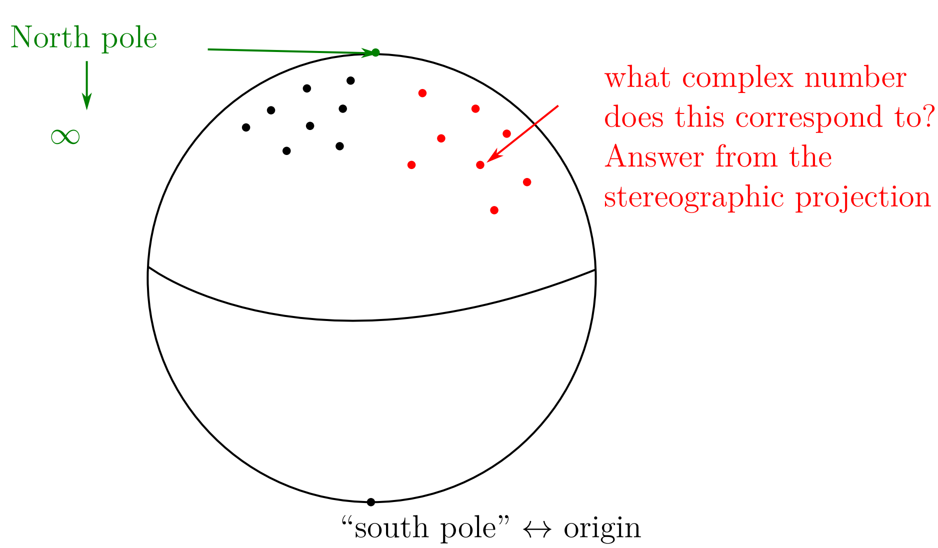
In complex analysis, it is often useful to imagine that the complex plane has an "ideal point" at infinity, denoted ∞.

Definition 3.1 (Extended complex plane)
The extended complex plane is $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$.

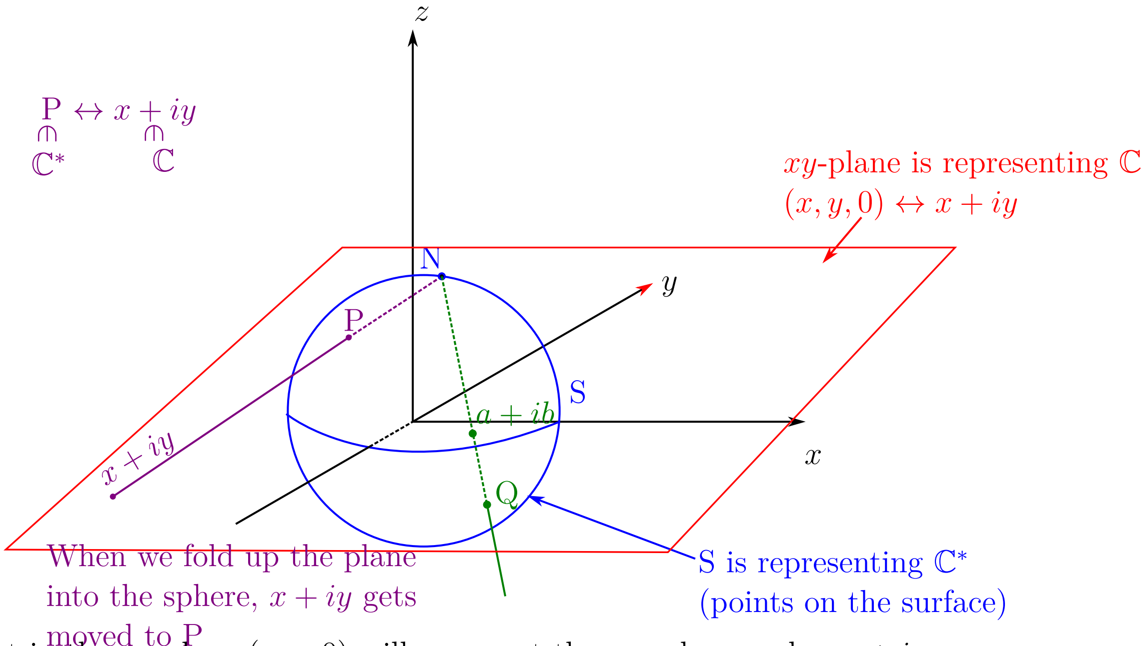
∞ is not a complex number. Instead, it represents what becomes of a complex number if its modulus grows without bound. in other words, a sequence of complex numbers (z_n) is considered to approach ∞ if and only if the moduli $|z_n|$ are diverging to $+\infty$ (as a sequence of reals).



This leaves us with a problem: how can we visualize $\mathbb{C}^*$? $\mathbb{C}^*$ will be visualized as a sphere:



We will use stereographic projection to visualize $\mathbb{C}^*$ as a sphere. Start with 3-space, $\{(X, Y, Z)\colon X, Y, Z \in \mathbb{R}\}$, and consider the unit sphere.



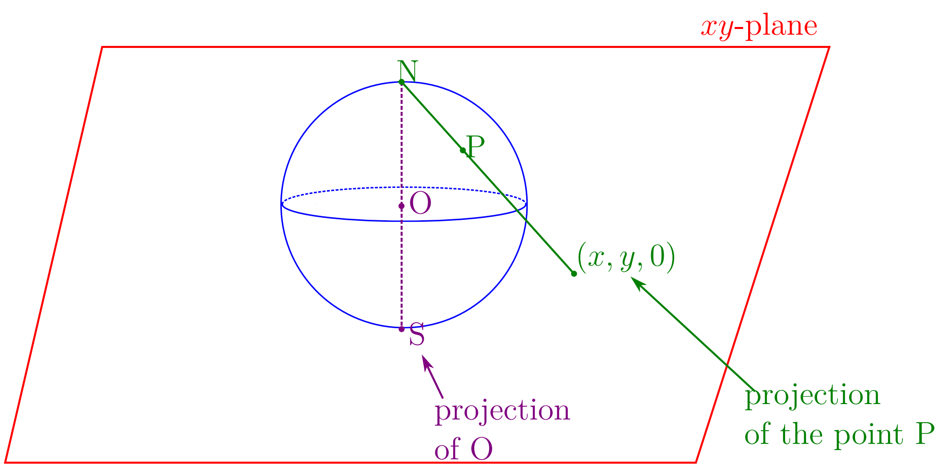
Any point in the xy-plane $(x, y, 0)$ will represent the complex number $x + iy$,

$$x + iy \sim (x, y, 0).$$

The unit sphere S represents $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$. with the north pole $N = (0, 0, 1)$ representing ∞.

But in what manner do points in S represent complex numbers in $\mathbb{C}$? To answer this, we will give a correspondence between points in $S \setminus \{N\}$ and the xy-plane.

The correspondence is called the stereographic projection, and is defined as follows:

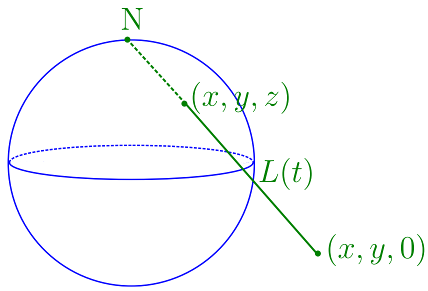


This projection gives a one-to-one correspondence between $S \setminus \{N\}$ and xy-plane.

Note that points on the lower half of the sphere correspond to complex numbers whose moduli are ≤ 1, while points on the upper half of the sphere correspond to complex numbers whose moduli are ≥ 1.

We have given a geometric description of the stereographic projection, but now we describe how to find an explicit formula for it.

We start with a given complex number $z = x + iy(\sim (x, y, 0))$ and seek a formula for the coordinates (X, Y, Z) for the point on the sphere which corresponds to $(x, y, 0)$ via the stereographic projection.



$$\begin{aligned} L(t) &= (0,0,1) + t\Big((X,Y,Z) - (0,0,1)\Big) \\ &= \Big(tX, tY, t(Z-1)+1\Big) \end{aligned}$$

There is t such that $L(t) = (X, Y, 0)$ i.e.

$$\Big(tX, tY, t(Z-1)+1\Big) = (x, y, 0)$$

$$\implies t(Z-1)+1 = 0 \iff t = \frac{1}{1-Z}$$

Note that:

$$Z = \frac{t-1}{t}, \quad tX = x, \quad tY = y$$

Since (X, Y, Z) is on S,

$$X^2 + Y^2 + Z^2 = 1$$

Multiply both sides by t^2

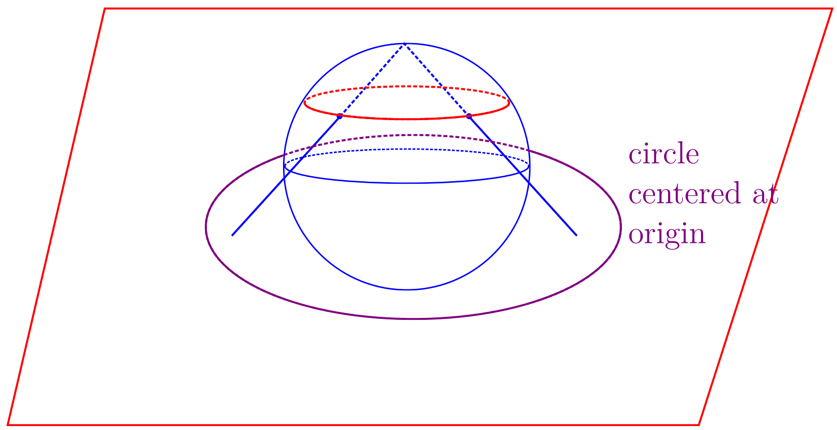
$$\begin{aligned} &\implies (tX)^2 + (tY)^2 + (t-1)^2 = t^2 \\ &\implies |W|^2 + t^2 - 2t + 1 = t^2 \\ &\implies t = \frac{|W|^2+1}{2} \end{aligned}$$

Thus,

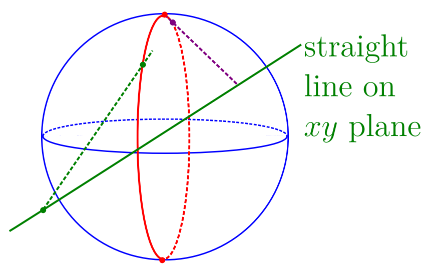
$$\begin{cases} X = \dfrac{1}{t}x = \dfrac{2x}{|w|^2+1} \\ Y = \dfrac{1}{t}y = \dfrac{2y}{|w|^2-1} \\ Z = \dfrac{t-1}{t} = \dfrac{|w|^2-1}{|w|^2+1} \end{cases}$$

$\left(\dfrac{2x}{|w|^2+1}, \dfrac{2y}{|w|^2-1}, \dfrac{|w|^2-1}{|w|^2+1}\right)$ is the unique point on the sphere that gets projected to $w = x + iy$.

We now explore what the stereographic projection does to geometric objects on the sphere. For example, it is not hard to see that lines of latitude on the sphere correspond to circles centered at 0 on the xy-plane.



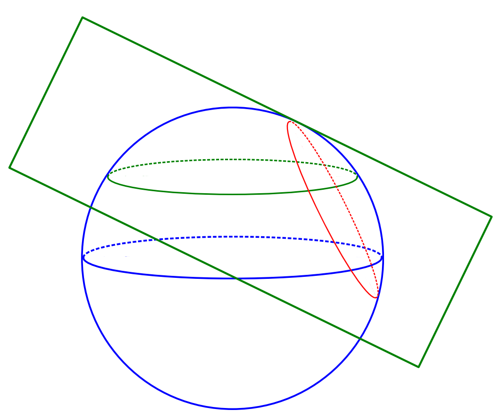
Moreover, lines of longitude on the sphere correspond to straight lines on the xy-plane.



Next, we will prove more general theorem:

> **Theorem 3.2**
> Under the stereographic projection, circles on the sphere correspond to circles and lines on the xy-plane.

Here, "circles on the sphere" means any intersection of a plane in 3-space with the unit sphere.



In our proof, we will also use the fact that the set of points in the plane which satisfy an equation of the form

$$x^2 + y^2 + ax + by + c = 0 \quad (a, b, c, \in \mathbb{R})$$

is either a circle, a point, or empty.

[To see this, start by completing the square to get

$$(x + a/2)^2 + (y + b/2)^2 = (a^2 + b^2)/4 - c$$

Consider the three cases where the right-hand-side is $> 0, = 0, < 0$.]

Proof. Consider a circle on the sphere which is the intersection of the sphere with the plane $AX + BY + CZ = D$.
The projection of this curve in all $z = x + iy \mapsto (x, y, 0)$ which satisfy

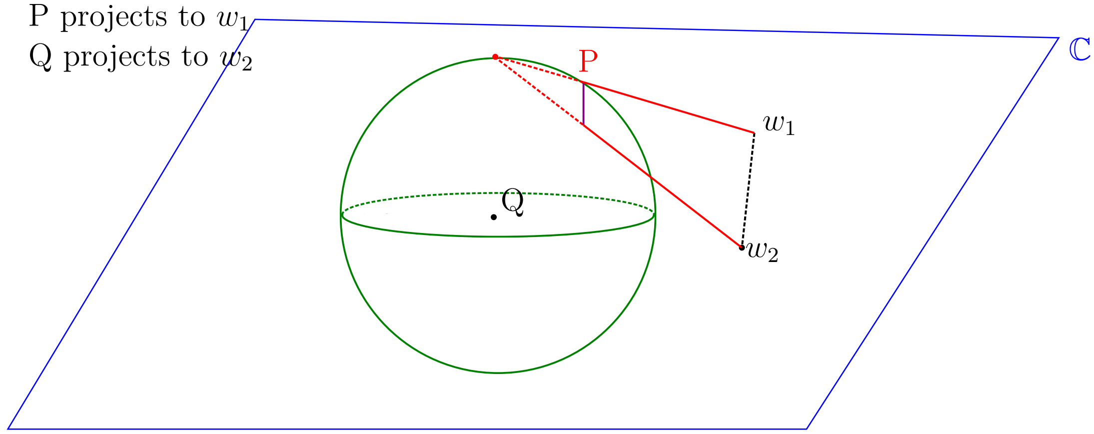
$$\begin{aligned} A\frac{2x}{|z|^2+1} + B\frac{2y}{|z|^2+1} + C\frac{|z|^2-1}{|z|^2+1} &= D \\ \implies 2Ax + 2By + C(|z|^2 - 1) &= D(|z|^2 + 1) \\ \implies 2Ax + 2By + Cx^2 + Cy^2 - C &= Dx^2 + Dy^2 + D \\ \implies (C - D)x^2 + (C - D)y^2 + 2Ax + 2By + (C - D) &= 0 \end{aligned}$$

If $C = D$, then this defines a line.
If $C \neq D$, then by our previous remark (after dividing by $(-D)$), we can conclude it defines a circle. $\square$

3.2 Chordal Distance

This is a notion of distance for C^*.

P projects to w_1
Q projects to w_2



$d(w_1, w_2) =$ length of chord that connects P, Q (chordal)

$\mathbb{C}^*$ with d is a metric space:

1. $d(w_1, w_2) = d(w_2, w_1)$

2. $d(w_1, w_2) \geq 0$,

   $d(w_1, w_2) = 0 \iff w_1 = w_2$

3. $d(w_1, w_2) \leq d(w_1, z) + d(z, w_2)$

Formula:

$$d(w_1, w_2) = \frac{2|w_1 - w_2|}{\sqrt{(|w_1|^2 + 1)(|w_2|^2 + 1)}} \quad w_1, w_2 \in \mathbb{C}$$
$$d(w_1, \infty) = \frac{2}{\sqrt{(|w_1|^2 + 1)}} \quad w_1 \in \mathbb{C}$$

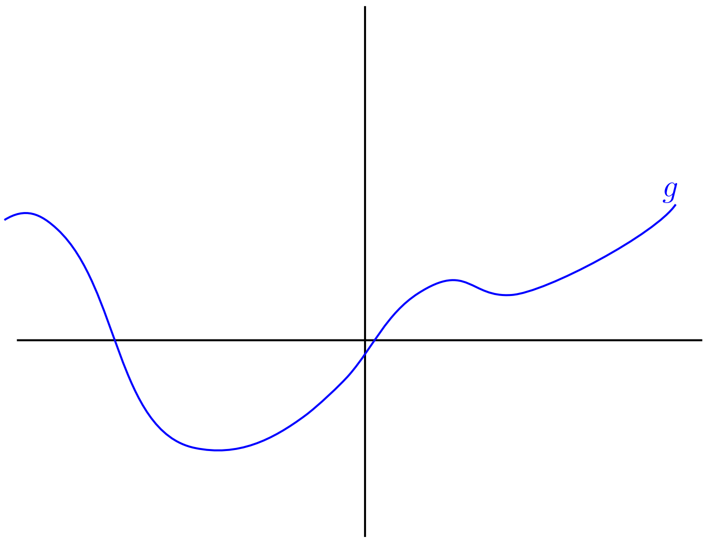
4 Jan 10, 2022

4.1 Visualizing Complex Functions

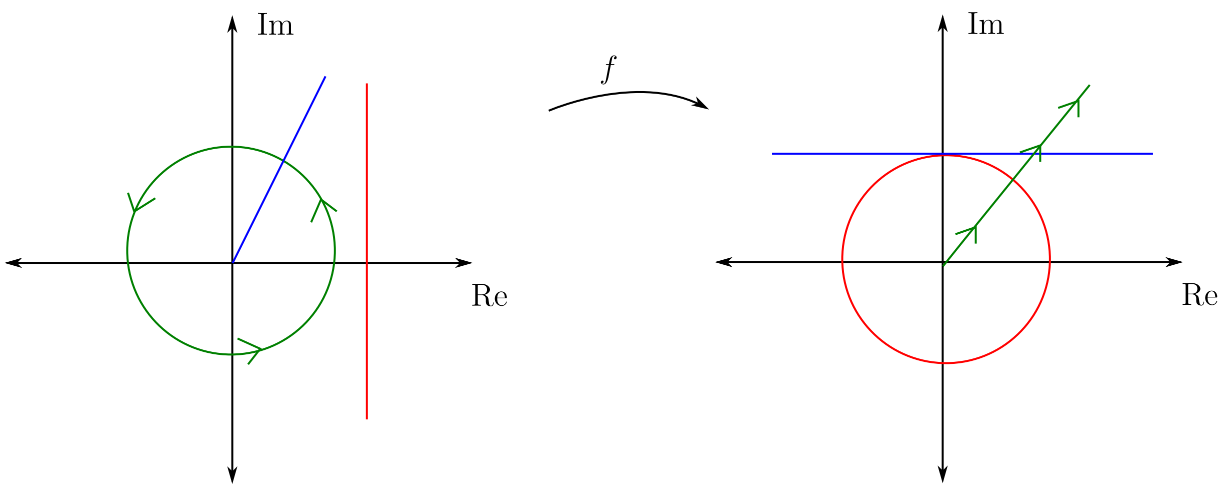
A function $\mathbb{C}$ to $\mathbb{C}$, $f\colon \mathbb{C} \to \mathbb{C}$, maps a two dimensional space to a two dimensional space. Thus, the graph of $f\colon \mathbb{C} \to \mathbb{C}$,

$$G = \{(z, f(z)) \in \mathbb{C}^2 \colon z \in \mathbb{C}\}$$

is not as easy to visualize as a graph of a function $g\colon \mathbb{R} \to \mathbb{R}$



To visualize $f\colon \mathbb{C} \to \mathbb{C}$, our best tool is to analyze how f transforms various geometric objects.

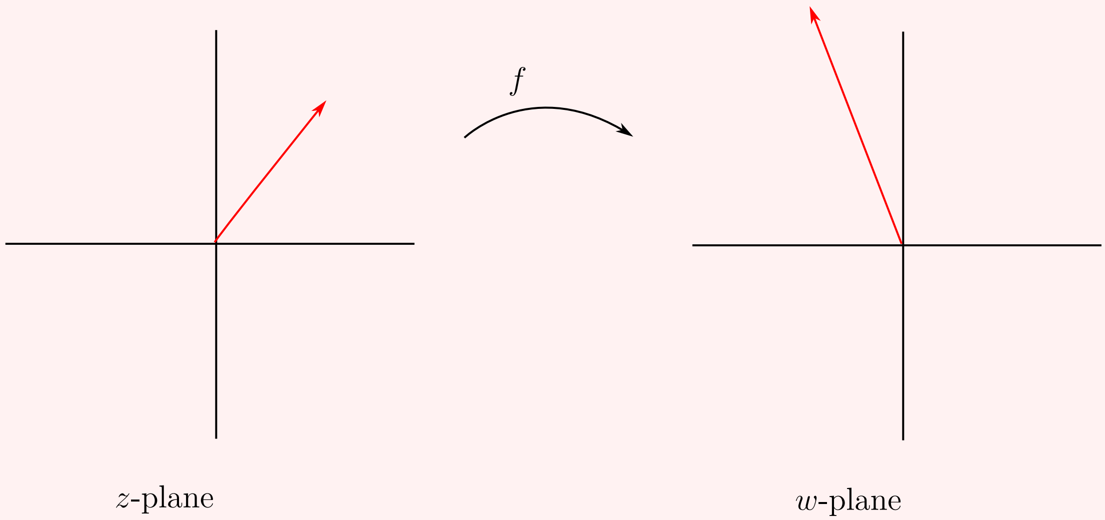


4.2 The Square Function

Consider the square function $f\colon \mathbb{C} \to \mathbb{C}, f(z) = z^2$.

Example 4.1

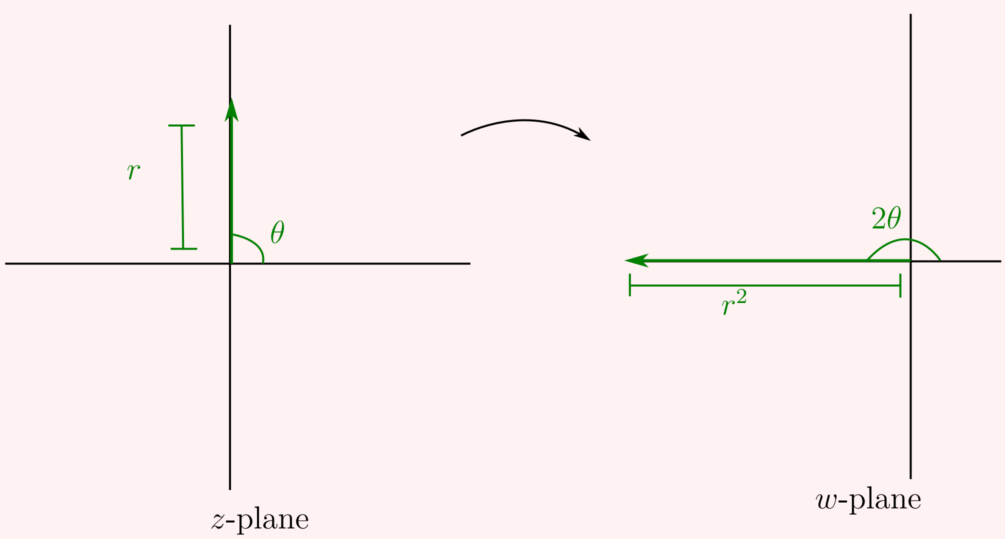
We will also think of this function as being given by $w = z^2$, where the domain space is the z-plane and the codomain space is the w-plane.



Example 4.2

Using polar representation, we can get an idea of what the square function does to a nonzero input $z = re^{i\theta}$:

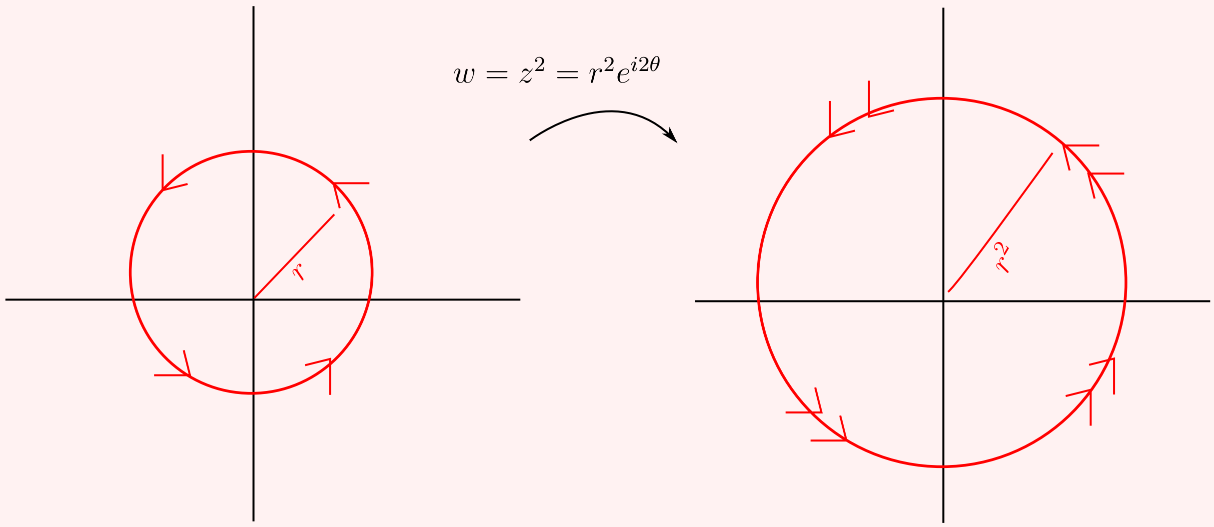
$$z^2 = (re^{i\theta} \cdot re^{i\theta}) = r^2 e^{i2\theta}$$



We can extend our understanding by examining what the square function does the geometric objects in the $z-$plane.

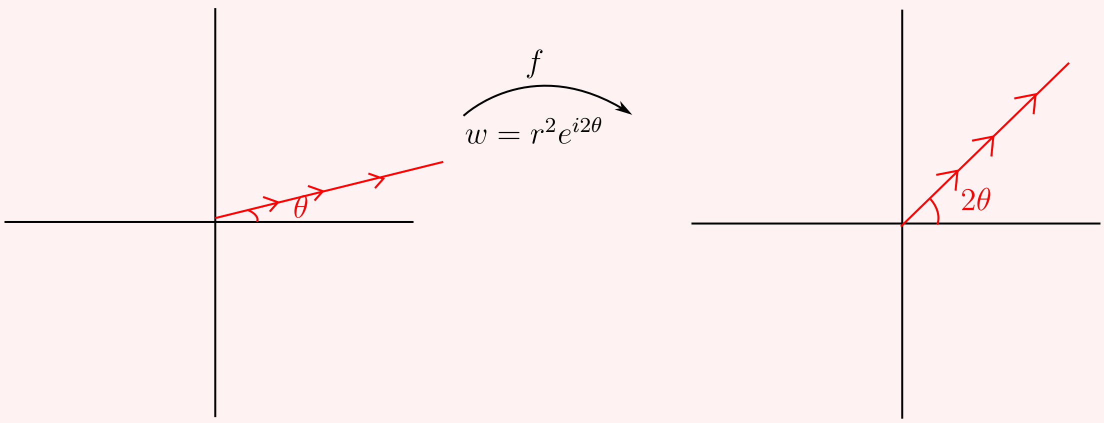
Example 4.3

The square function transforms a circle centered at 0 to another circle centered at 0:



Example 4.4

A ray anchored at 0 is transformed into another ray anchored at 0:

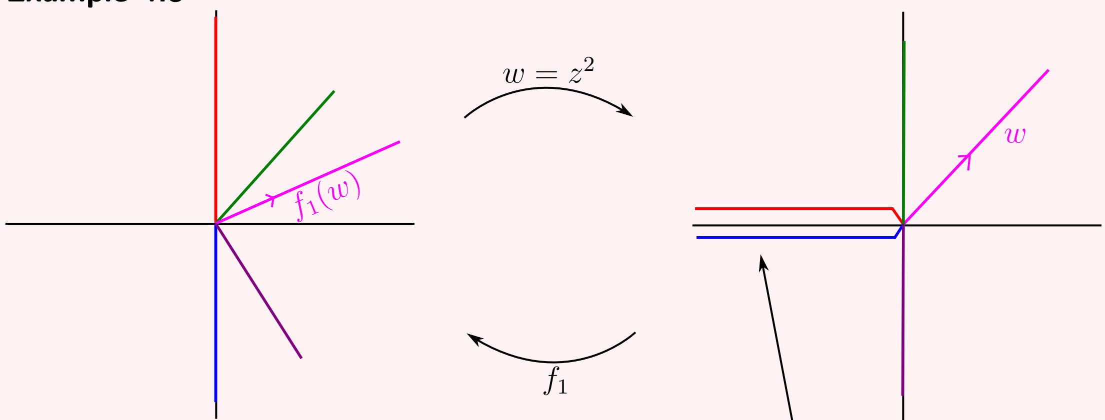


We will now consider the problem of finding an inverse function for $w = z^2$, i.e., a square root function $z = \sqrt{w}$.

Just as in the case of the real numbers, ever nonzero complex number has two distinct square roots; as a result, there are many different ways of defining a square root operation.

We begin by considering how $w = z^2$ transforms the open half-plane $\{z \in \mathbb{C} \colon \operatorname{Re} z > 0\}$.

Example 4.5



negative real axis: $(-\infty, 0]$
this gets "cut" from the domain
of f_1

f_1 does not get defined here
since each number on the negative
real axis has two numbers that
map to it

Definition 4.6 (Principal branch)
Our description above gives us what we will call the <u>principal branch</u> of $\sqrt{w}$,

$$f_1 \colon \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$$

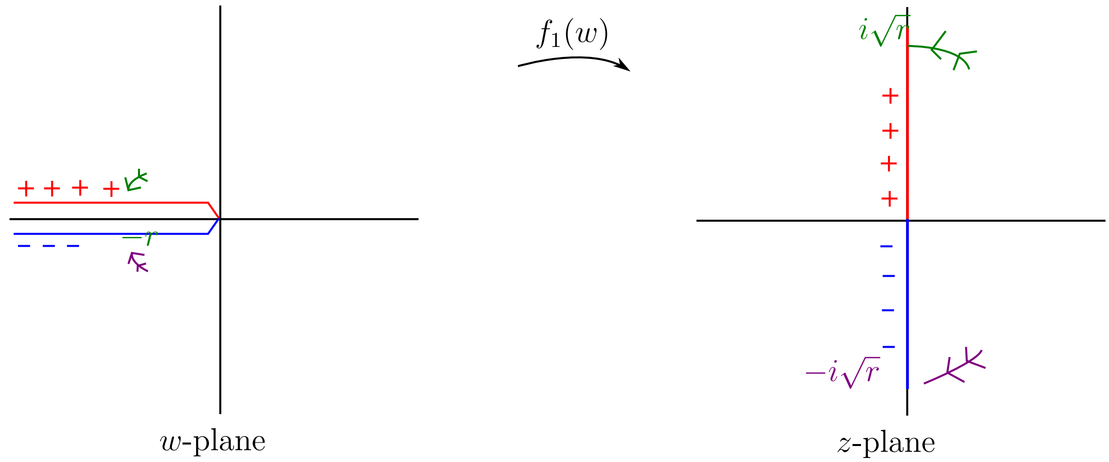
which is algebraically given by

$$f_1(w) = \sqrt{\rho} e^{i\varphi/2}, \quad w = \rho e^{i\varphi}, \quad -\pi < \varphi < \pi.$$

We can also write this equation using Arg:

$$f_1(w) = \sqrt{|w|} e^{i \operatorname{Arg} w/2}$$

The principal branch f_1 is not defined on $(-\infty, 0]$, but it does have the following limit behavior there:



Notation:

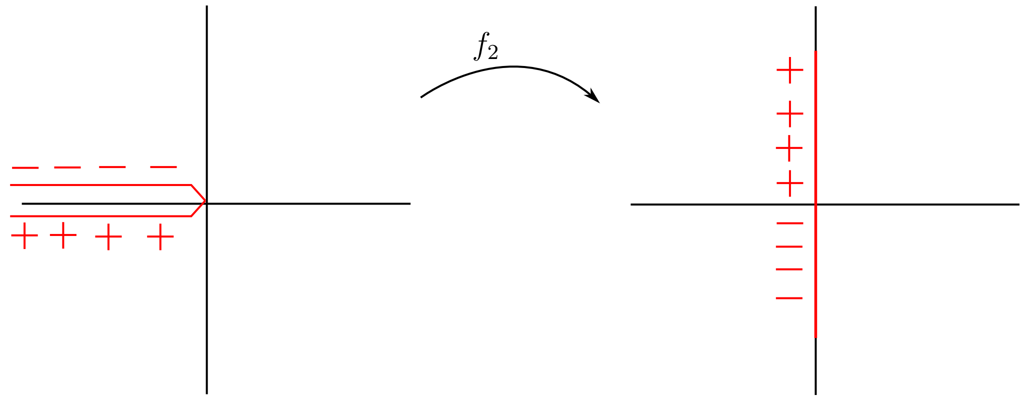
$$f_1(-r + i0) = i\sqrt{r}$$
$$f_1(-r - i0) = -i\sqrt{r}$$

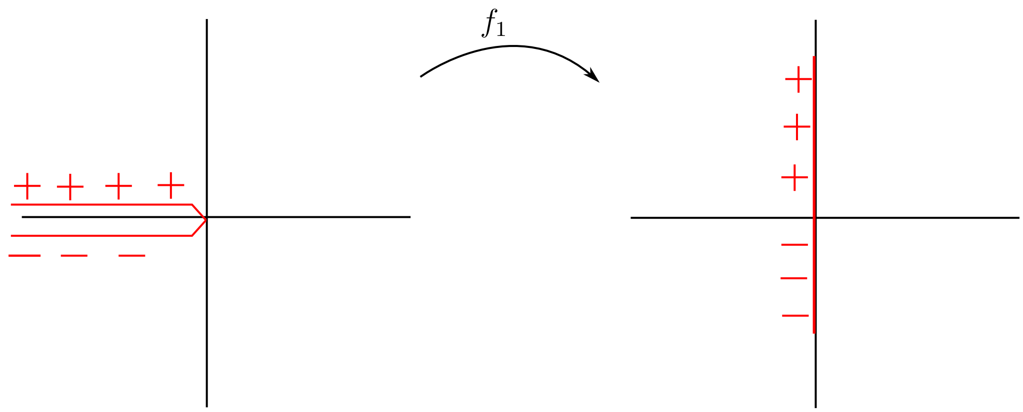
$f_1(w)$ chooses one of the possible square roots of w, but we can define another branch of $\sqrt{w}$ which selects the other square root.

We define the branch $f_2 \colon \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ simply by

$$f_2(w) = -f_1(w)$$

This function is still not defined on $(-\infty, 0]$, but it has the following (slightly different) limit behavior there:



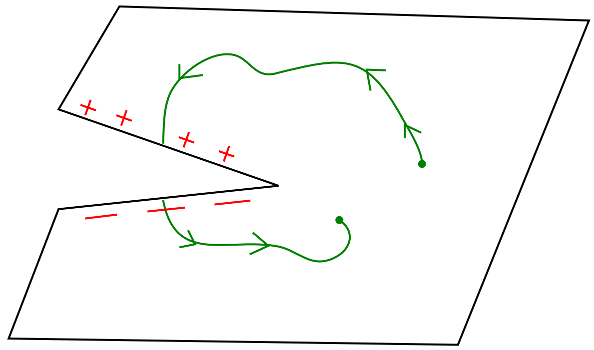


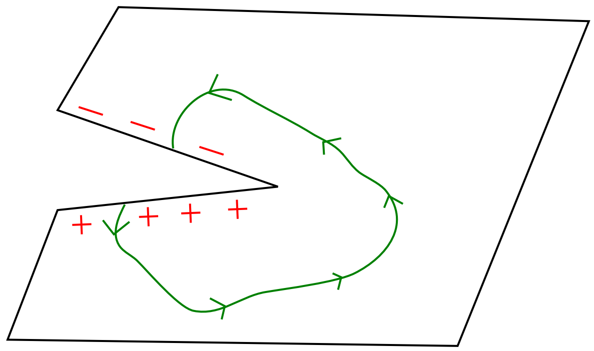
4.3 The Riemann Surface of $\sqrt{w}$

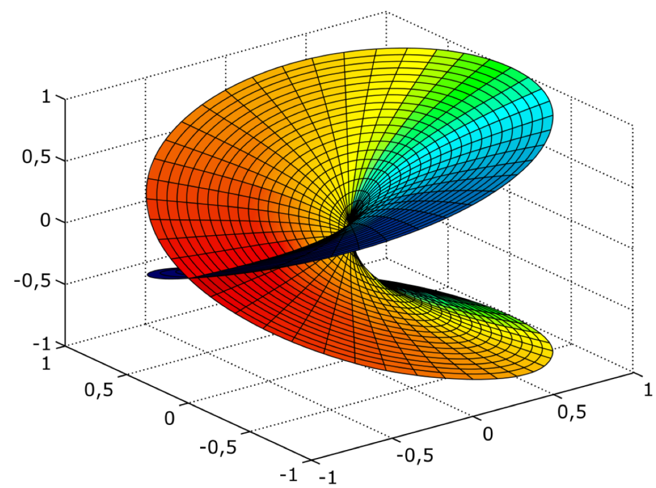
As we have said, the square root operation is multi-valued.

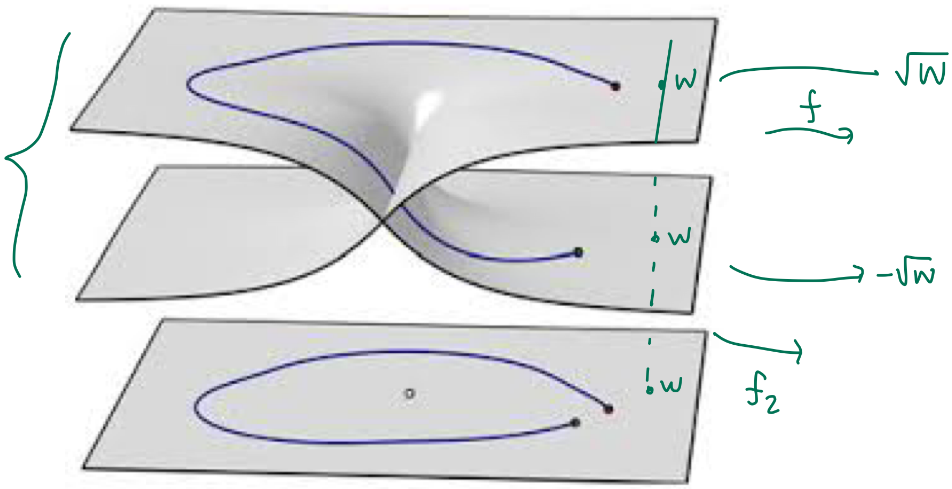
Book: "multi-valued function"

However, we can use our two branches above to construct a surface which can serve as a domain which makes this multi-valued operation a true function.






1
0,5
0
-0,5
-1
1
0,5
0
-0,5
-1
-1
-0,5
0
0,5
1



4.4 The Exponential Function

Definition 4.7 (Exponential function)
We define the exponential function for all $z = x + iy \in \mathbb{C}$ by

$$e^z = e^x \cos(y) + ie^x \sin(y)$$

This, of course, is an extension of our previously used

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \quad (\theta \in \mathbb{R}),$$

and we can write

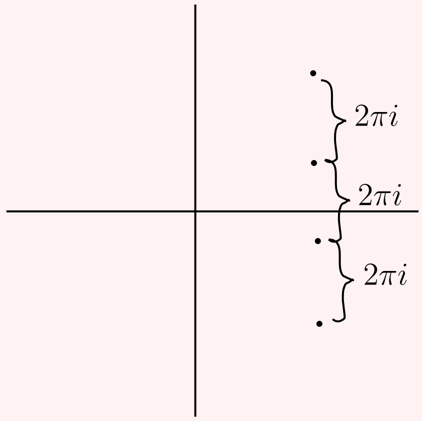
$$e^z = e^x e^{iy}.$$

Definition 4.8 (λ-periodic)
Let $\lambda \in \mathbb{C}$. A complex function $f(z)$ is said to be λ-periodic if

$$f(z + \lambda) = f(z) \quad \forall z \in \mathbb{C}.$$

Example 4.9
The exponential function is $2\pi i$ periodic:

$$e^{z+2\pi i} = e^z \cdot e^{2\pi i} = e^z \cdot 1 = e^z$$



$2\pi i$ periodicity means all these complex numbers have some value under e^z.

Other fundamental properties of the exponential function (which are easy to verify): for all $z, w \in \mathbb{C}$,

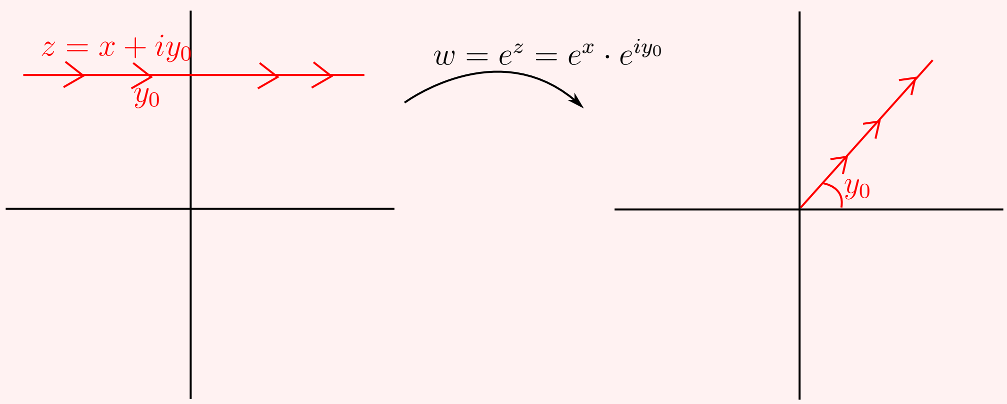
i. $e^{z+w} = e^z e^w$

ii. $1/e^z = e^{-z}$

iii. $\overline{e^z} = e^{\overline{z}}$

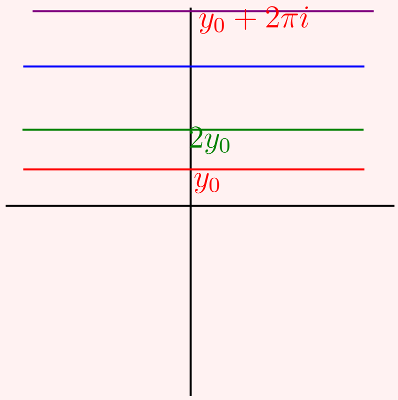
We now examine how $w = e^z$ transforms certain geometric objects.

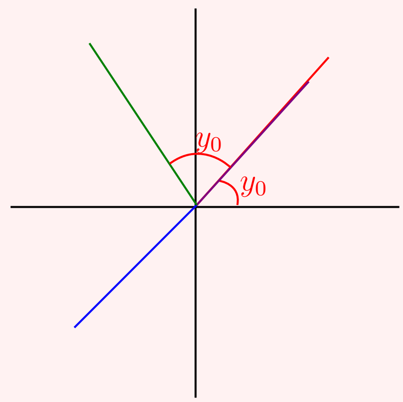
Example 4.10
Horizontal lines are mapped to rays anchored at 0:



Example 4.11

As we move the horizontal line, we can see the geometric meaning of the $2\pi i$ periodicity of $w = e^z$.

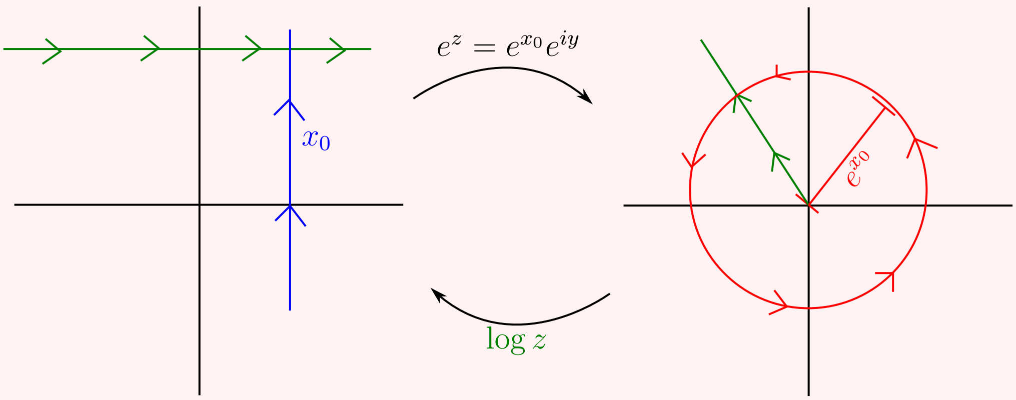




5 Jan 12, 2022

5.1 The Exponential Function (Cont'd)

Example 5.1
Vertical lines are transformed into circles centered at 0:



5.2 The Logarithm Function

Definition 5.2 (Logarithm Function)
We define $\log z, z \neq 0$, to be the multi-valued function

$$\log z = \log|z| + i \arg z = \log|z| + i \operatorname{Arg} z + 2\pi i m, \quad m \in \{0, \pm 1, \pm 2, \dots\}.$$

It is easy to check that any value w of log satisfies $e^w = z$:

$$\begin{aligned} e^{\log z} = e^{\log|z| + i \operatorname{Arg} z + 2\pi i m} &= e^{\log|z|}e^{i \operatorname{Arg} z}e^{2\pi i m} \\ &= |z|e^{i \operatorname{Arg} z} = z \end{aligned}$$

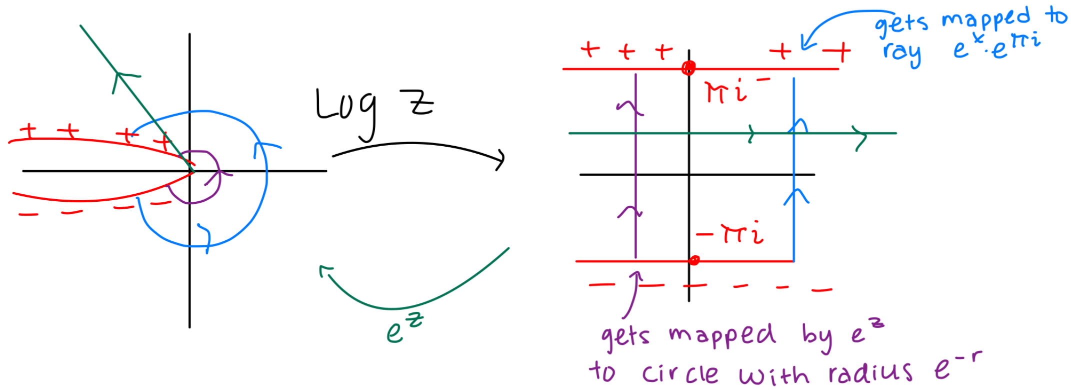
Definition 5.3 (Principal value)
The <u>principal value</u> of $\log z, z \neq 0$, is

$$\operatorname{Log} z = \log|z| + i \operatorname{Arg} z.$$

Note that for $z \neq 0$,

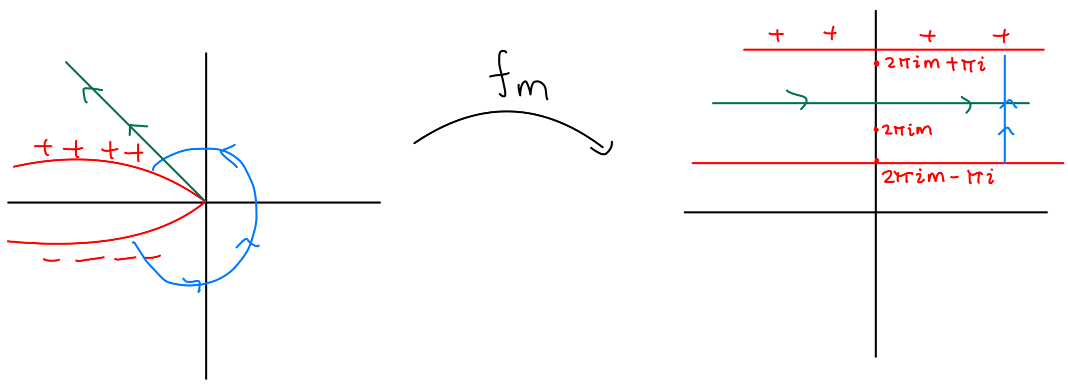
$$\log z = \operatorname{Log} z + 2\pi i m, \quad m \in \{0, \pm 1, \pm 2, \dots\}$$

We can visualize Log as a function with domain $\mathbb{C} \setminus (-\infty, 0]$ as follows:

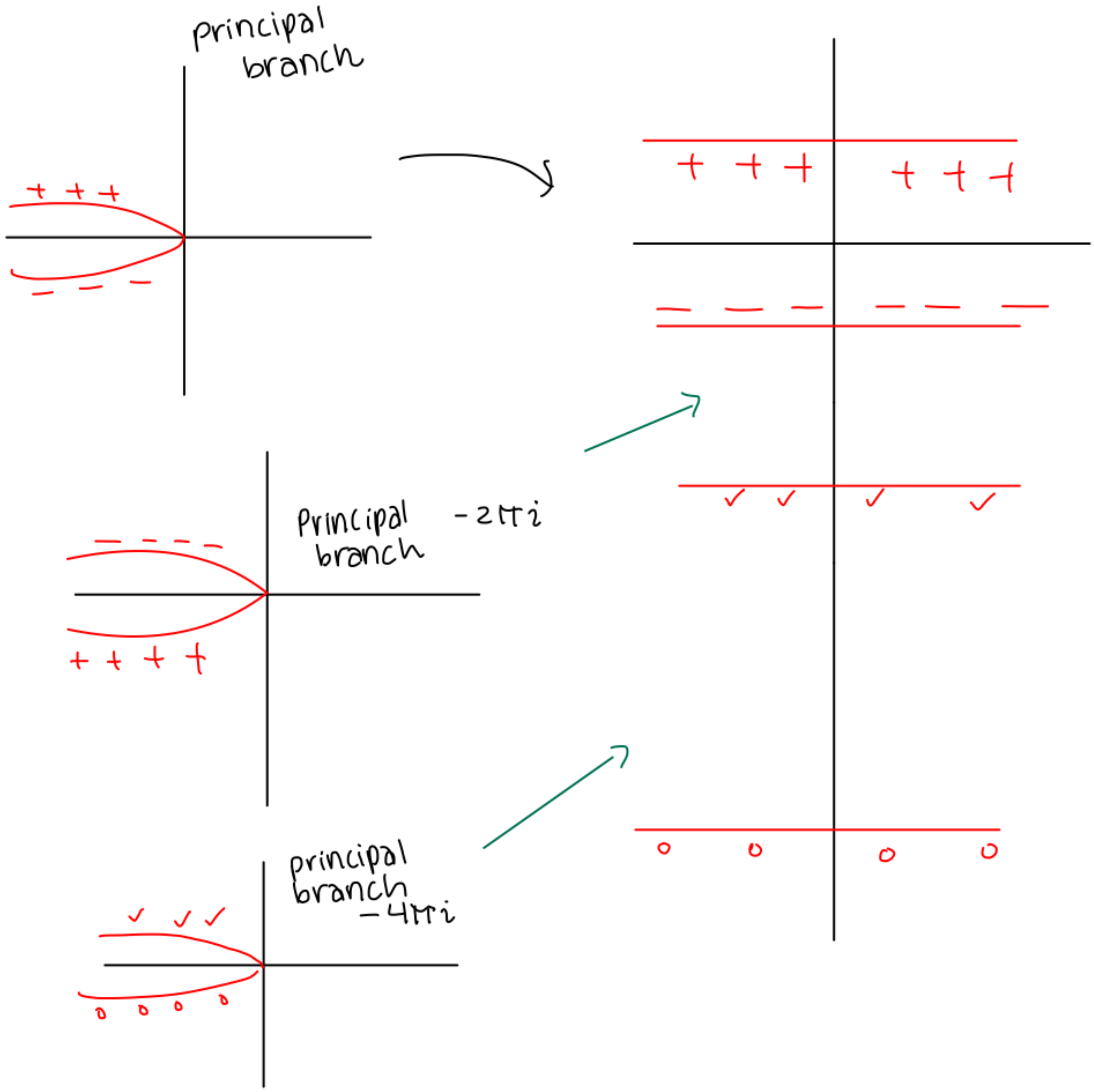


And we have other branches as well,

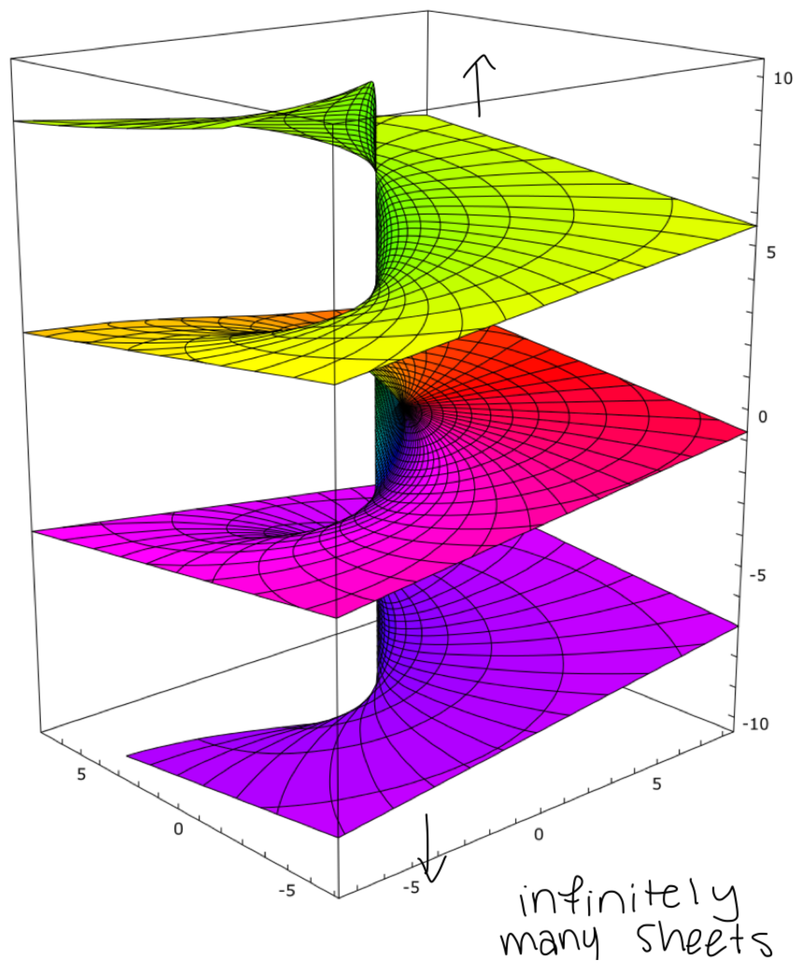
$$f_m(z) = \operatorname{Log} z + 2\pi i m, \qquad \text{where } m \text{ is an integer.}$$



Just as we did with $\sqrt{w}$, we can represent the multi-valued function $\log z$ as a single-valued (i.e., actual) function on a Riemann surface.



Glue these together to get Riemann surface of $\log z$.


10
5
0
-5
-10
5
0
-5
-5
0
5
infinitely
many sheets

5.3 Power Functions and Phase Factors

Definition 5.4 (Power function)
Let $\alpha \in \mathbb{C}$. We define the power function z^α to be the multi-valued function

$$z^\alpha = e^{\alpha \log z}, \quad z \neq 0.$$

Thus, the values of z^α are

$$z^\alpha = e^{\alpha[\log|z| + i \operatorname{Arg} z + 2\pi i m]} = e^{\alpha \operatorname{Log} z} e^{2\pi i \alpha m}, \quad m \text{ an integer.}$$

Note that if α is an integer, then z^α is single-valued:

$$\begin{aligned} z^\alpha &= e^{\alpha \operatorname{Log} z} e^{2\pi i \alpha m} \leftarrow \text{ integer multiple of } 2\pi i \text{ when } \alpha \text{ is an integer} \\ &= e^{\alpha \operatorname{Log} z} \end{aligned}$$

n a positive integer

$$z^n = \underbrace{z \cdot z \cdot z \cdot z \cdots z}_{n \text{ times}}$$

$$z^{-n} = \underbrace{\frac{1}{z} \cdot \frac{1}{z} \cdots \frac{1}{z}}_{n \text{ times}}$$

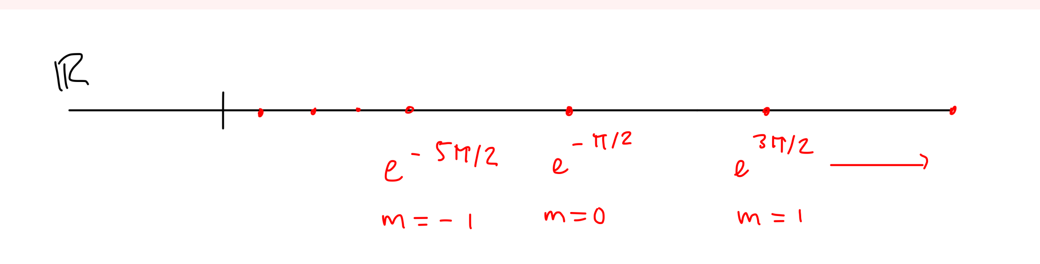
If $\alpha = 1/n$ for an integer n, then $z^\alpha = z^{1/n}$ are the nth roots of z:

$$z^{1/n} = e^{\frac{1}{n} \operatorname{Log} z} e^{\frac{2\pi i m}{n}}$$

Example 5.5
We find the values of i^i.

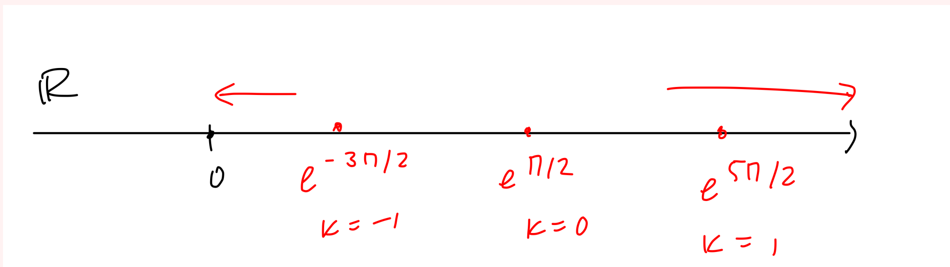
$$\begin{aligned} i^i &= e^{i[\log|i| + i \operatorname{Arg} i + 2\pi i m]} \\ &= e^{0 + i^2 \frac{\pi}{2} + 2\pi i^2 m} \\ &= e^{\frac{-\pi}{2}} e^{-2\pi m} \\ &= e^{-\pi(1-4m)/2} \end{aligned}$$

m integer



Example 5.6
We will find all the values of i^{-i}.

$$\begin{aligned} i^{-i} &= e^{-i[\log|i| + i \operatorname{Arg} i + 2\pi i k]} \\ &= e^{\frac{\pi}{2}} e^{2\pi k} = e^{\pi(1+4k)/2} \end{aligned}$$



Warning: When we multiply the values of i^i with those of i^{-i} we get infinitely many values:

$$\begin{aligned} (i^i)(i^{-i}) &= (e^{-\pi/2} \cdot e^{-2\pi m})(e^{\pi/2} e^{2\pi k}) \\ &= e^{2\pi(k-m)} \\ &= e^{2\pi j} \quad \text{ j integer} \end{aligned}$$

In other words,

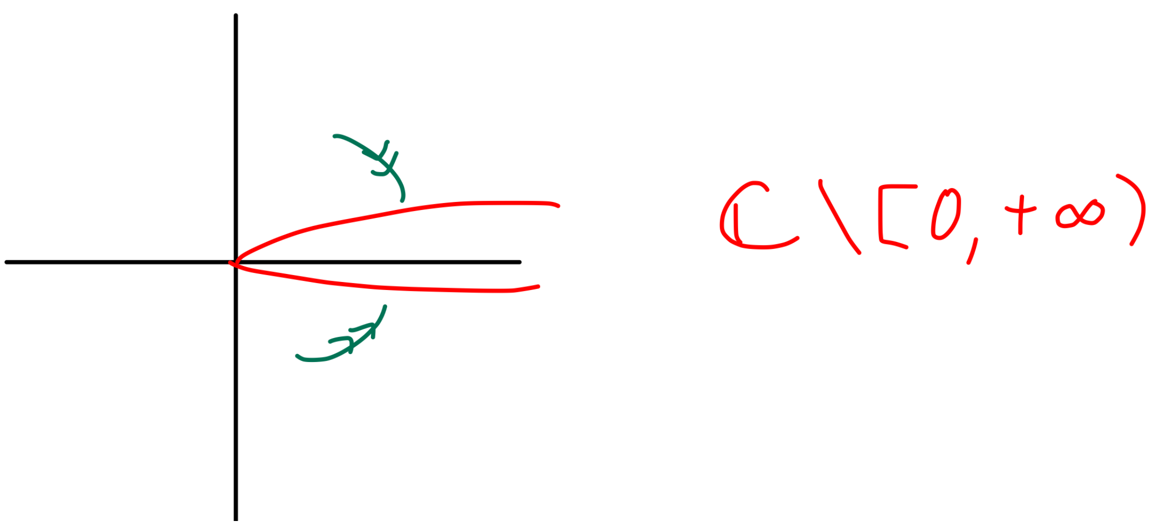
$$(i^i)(i^{-i})$$

is itself multi-valued, and we do not have the familiar looking identity

$$(i^i)(i^{-i}) = 1$$

Instead: 1 is one of the values of $(i^i) \cdot (i^{-i})$ but $e^{2\pi}, e^{-4\pi} \ldots$ are also values.

Fix a non-integer $\alpha \in \mathbb{C}$. Since z^α is multi-valued, we find a way to define a single-valued branch of z^α. To do this, we first remove the positive real axis $[0, +\infty)$ from $\mathbb{C}$:



Then, define the function $f\colon \mathbb{C} \setminus [0, +\infty) \to \mathbb{C}$ by

$$z = re^{i\theta} \quad f(z) = r^\alpha e^{i\alpha\theta}, \qquad \text{where } 0 < \theta < 2\pi \qquad \text{and} \qquad r^\alpha = e^{\alpha \log r}$$

At the top edge of the slit, we can compute $f(r + i0)$:

$$f(r + i0) = \lim_{\substack{\theta \to 0 \\ \text{from above}}} f(r + i\theta) = r^\alpha e^{i\alpha \cdot 0} = r^\alpha$$

At the bottom edge of the slit, we can compute $f(r - i0)$:

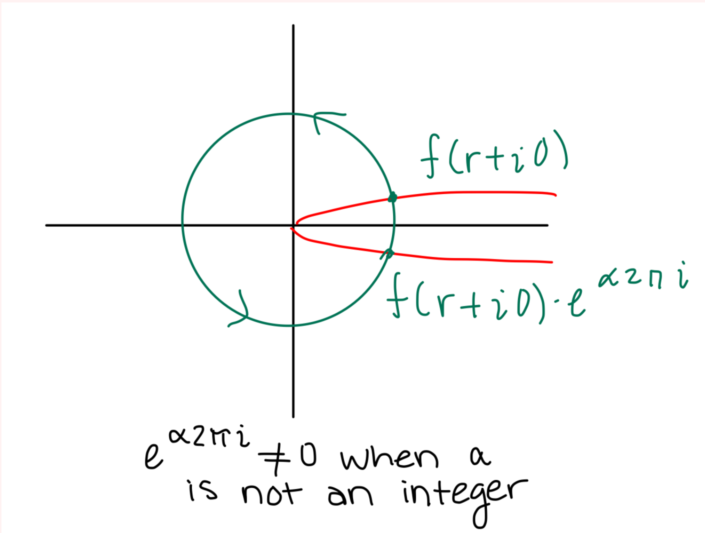
$$f(r - i0) = \lim_{\substack{\theta \to 2\pi \\ \text{from below}}} f(r + i\theta) = r^\alpha e^{i\alpha 2\pi} = e^{i\alpha 2\pi} \cdot f(r + i0)$$

In particular, $f(r + i0) \neq f(r - i0)$

6 Jan 14, 2022

6.1 Power Functions and Phase Factors (Cont'd)

Example 6.1
If we move continuously around the origin starting at a point on the slit, the values of $f(z)$ will move continuously from $f(r+i0)$ to $f(r-i0)$:



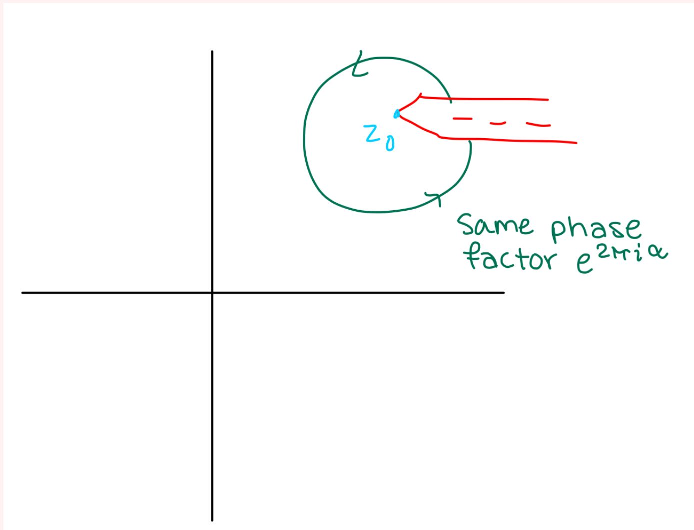
But $f(r+i0) \neq f(r-i0)$, so we cannot continuously extend this function to be defined on the slit. In fact, there is no way to define a continuous branch of z^α on the whole complex plane $\mathbb{C}$ (for α not an integer).

Definition 6.2 (Phase factor)
We showed that for $r > 0$,

$$f(r-i0) = e^{2\pi i\alpha} f(r+i0)$$

The value $e^{2\pi i\alpha}$ is called the phase factor of z^α at $z = 0$.

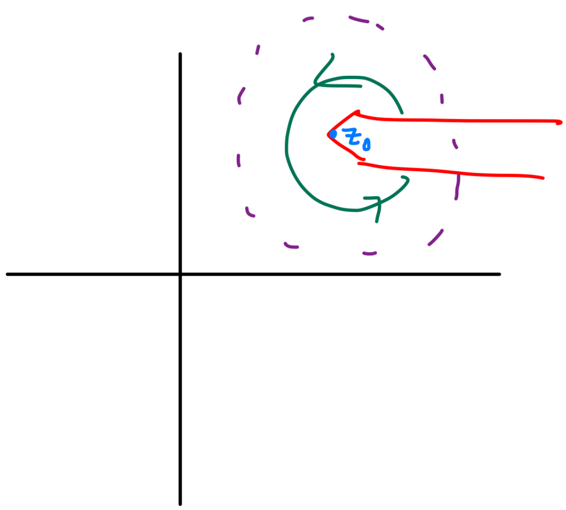
Example 6.3
For the multi-valued function $(z-z_0)^\alpha$, we could perform the same analysis:



Theorem 6.4 (Phase change lemma)
Let $g(z)$ be a (single-valued) function which is defined and continuous near z_0 in some disk centered at z_0. For any continuous varying branch of $(z-z_0)^\alpha$, the function

$$f(z) = (z-z_0)^\alpha g(z)$$

is multiplied by a phase factor of $e^{2\pi i\alpha}$ when z traverses a complete circle around z_0 in the counterclockwise direction.



Example 6.5
If α is an integer, then the phase factor of z^α at $z = 0$ is

$$e^{2\pi i\alpha} = 1$$

This means that traverses the circular path multiplies the value of $f(z)$ by 1, i.e., does not change the value. This is another way of saying z^α, α integer, is single-valued.

Example 6.6
Consider the function $\sqrt{z(1-z)}$.
Since

$$\sqrt{z(1-z)} = \sqrt{z} \cdot \sqrt{1-z},$$

this function actually has two branch points:
$\sqrt{z}$ has a branch point at $z = 0$.
$\sqrt{1-z}$ has a branch point at $z = 1$.
$\implies \sqrt{z(1-z)}$ has branch points at $z = 0, 1$.

$$\sqrt{z} \cdot \underbrace{\sqrt{(1-z)}}_{\substack{\text{have branch of this} \\ \text{is defined and continuous} \\ \text{at } z=0}}$$

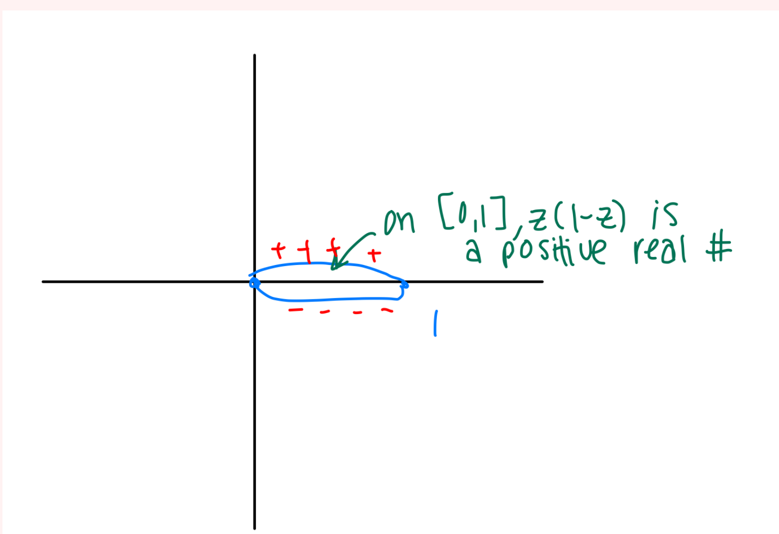
Phase change lemma $\implies$ phase factor at $z = 0$ is

$$e^{2\pi i\cdot\frac{1}{2}} = e^{\pi i} = -1$$

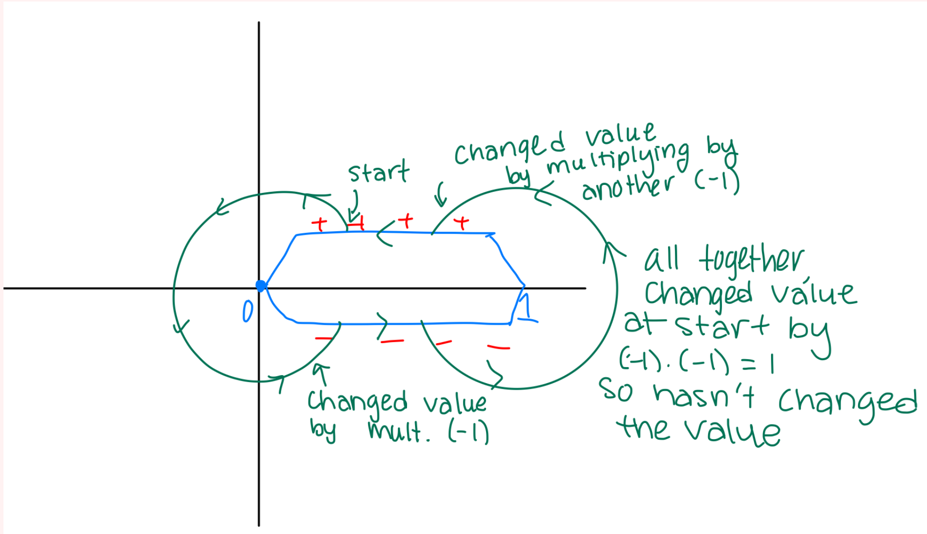
Similarly, phase factor at $z = 1$ is

$$e^{2\pi i\cdot\frac{1}{2}} = -1$$

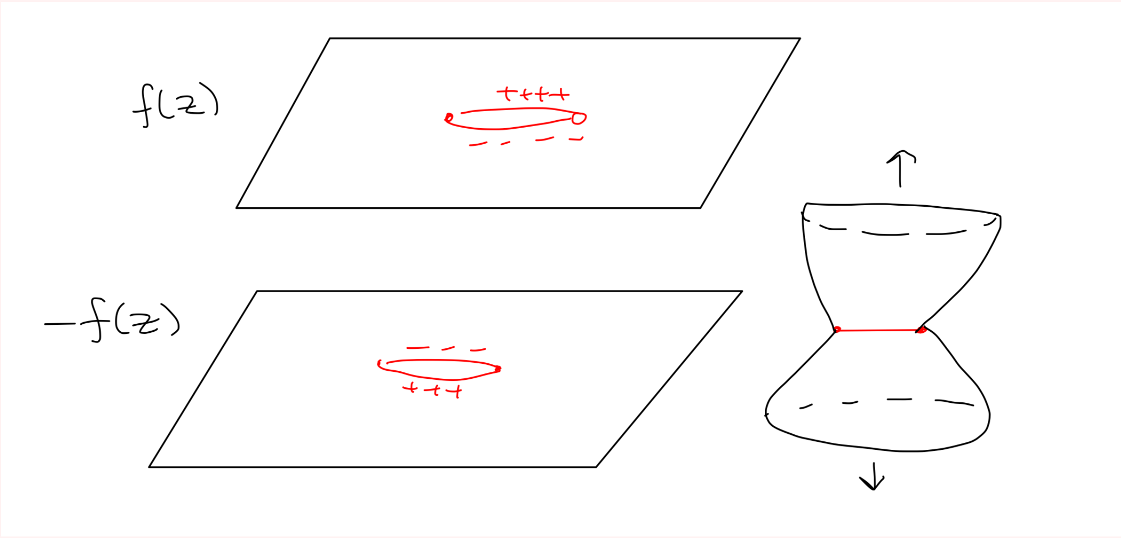
We draw a branch cut 0 to 1 and consider the branch $f(z)$ of $\sqrt{z(1-z)}$ which is positive along the top edge of the cut and negative along the bottom edge.



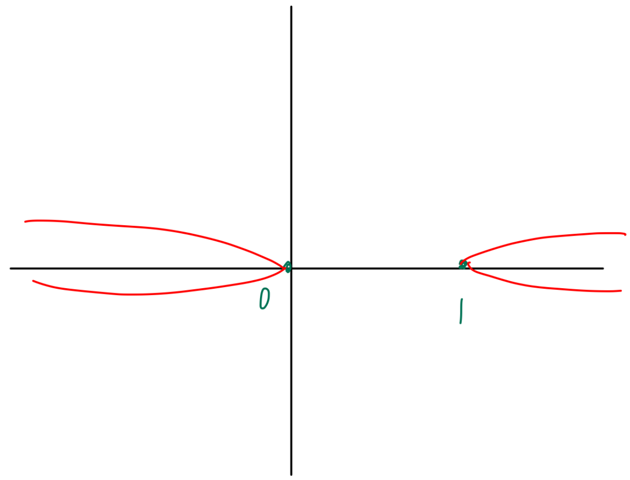
Example 6.7
We can check that this branch is actually single-valued on $\mathbb{C} \setminus [0, 1]$:



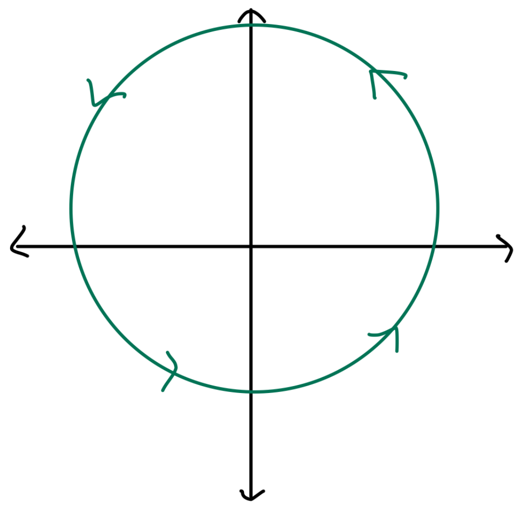
And we can visualize the Riemann surface:
$f(z)$ branch of $\sqrt{z(1-z)}$ on $\mathbb{C} \setminus [0, 1]$
Other branch is $-f(z)$



Note: for the single-valued branch of $\sqrt{z(1-z)}$, we just need enough cuts to ensure that we cannot traverse a closed path around a single branch point.



We will see in the next example that sometimes we need more than just one cut. Also, we will see that ∞ can be a branch point. This happens when traversing very large circles centered at 0 has a phase factor.



Example 6.8

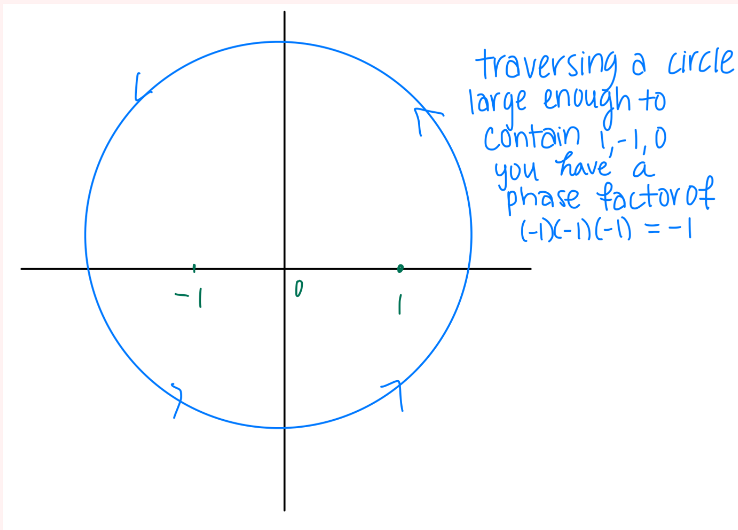
Consider the function $\sqrt{z - 1/z}$. Rewriting this as

$$\sqrt{z - 1/z} = \frac{\sqrt{z-1}\sqrt{z+1}}{\sqrt{z}} = (z-1)^{\frac{1}{2}}(z+1)^{\frac{1}{2}}(z)^{-\frac{1}{2}}$$

we can see that the function has three finite branch points.
$1, -1, 0$, all have phase factors

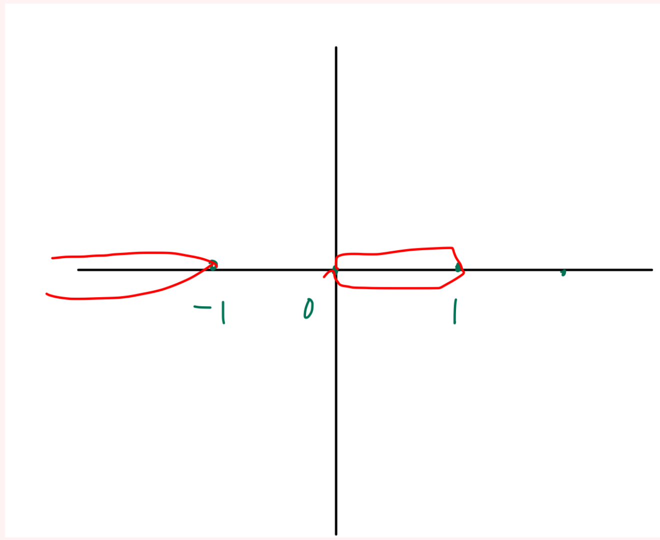
$$-1 = e^{2\pi i \cdot \frac{1}{2}}$$

$$-1 = e^{2\pi i(-\frac{1}{2})} = e^{-\pi i}$$

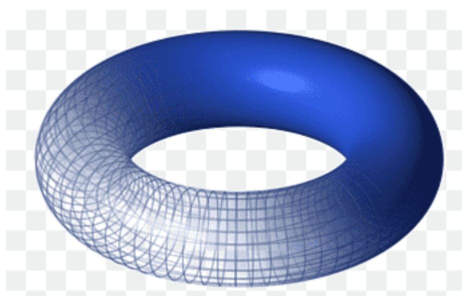


Moreover, ∞ is also a branch point.

Example 6.9
This time, we need two branch cuts: (Again, we want to avoid the possibility of traversing a path around a single branch point, including the one at ∞.)



There is a branch of $\dfrac{\sqrt{z-1}\sqrt{z+1}}{\sqrt{z}}$ defined on $\mathbb{C} \setminus ((-\infty, -1] \cup [0, 1])$
Riemann surface is a torus.



6.2 Trigonometric and Hyperbolic Functions

Definition 6.10 (Trigonometric functions)
We define $\cos z$ and $\sin z$ by setting

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad z \in \mathbb{C},$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad z \in \mathbb{C}$$

It is easy to check that these are indeed an extension of cos, sin on the real line, but this definition can be further motivated by solving the system

$$\begin{cases} e^{i\theta} = \cos\theta + i\sin\theta \\ e^{-i\theta} = \cos\theta - i\sin\theta \end{cases} \quad \text{for } \cos\theta \text{ and } \sin\theta$$

It is easy to check that $\cos z, \sin z$ have many of the familiar properties:

i. $\cos z$ is an even function: $\cos(-z) = \cos(z)$.

ii. $\sin z$ is an odd function: $\sin(-z) = -z\sin(z)$.

iii. $\cos z, \sin z$ are both 2π periodic: $\cos(z + 2\pi) = \cos z, \sin(z + 2\pi) = \sin(z)$.

iv. The sum formulas are valid:

$$\cos(z + w) = \cos z\cos w - \sin z\sin w,$$
$$\sin(z + w) = \sin z\cos w + \cos z\sin w.$$

v. For any $z \in \mathbb{C}, \cos^2 z + \sin^2 z = 1$.

Definition 6.11 (Hyperbolic functions)
The hyperbolic functions are extended to $\mathbb{C}$ in the obvious way:

$$\cosh z = \frac{e^z + e^{-z}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2} \quad z \in \mathbb{C}$$

These functions also have their familiar properties:

i. $\cosh, \sinh$ are both $2\pi i$ periodic.

ii. cosh, is even, sinh is odd.

The hyperbolic functions are closely related to the trig functions. Both sets of functions are obtained from each other by rotating the complex plane by $\pi/2$ clockwise (i.e., by multiplications by i):

$$\begin{cases} \cosh(iz) = \cos z, \\ \sinh(iz) = i\sin z, \end{cases} \quad \begin{cases} \cos(iz) = \cosh z, \\ \sin(iz) = i\sinh z, \end{cases}$$

We can use these equations, and the addition formula to obtain the Cartesian representation for $\sin z$:

$$\sin z = \sin x\cosh y + i\cos x\sinh y \quad (z = x + iy).$$

Using some more trig identities we can then get the formula

$$|\sin z|^2 = \sin^2 x + \sinh^2 y$$

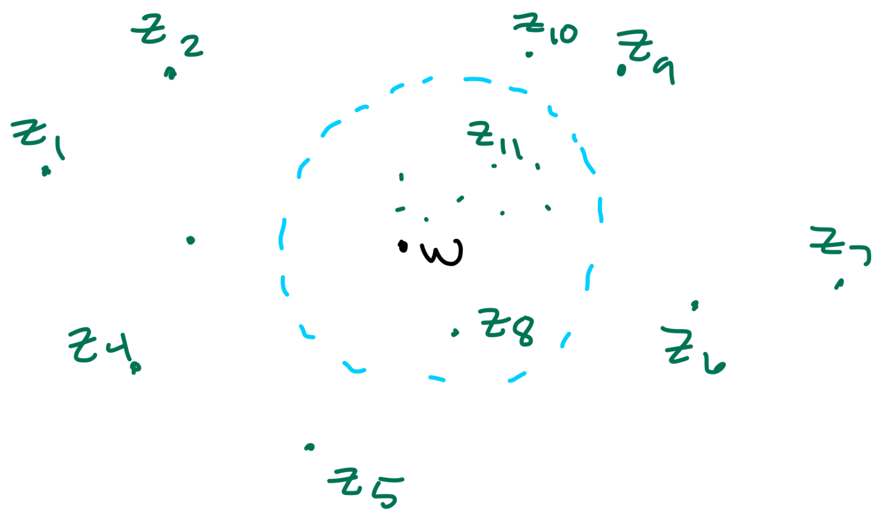
$\implies$ zeroes of $\sin z$ are exactly the zeros of $\sin x$: $k\pi,\ k = 0, \pm 1, \pm 2, \ldots$

7 Jan 19, 2022

7.1 Limits of Sequences of Complex Numbers

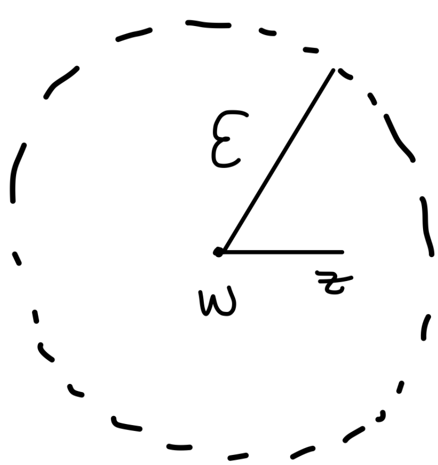
Our first goal is to define the notion of convergent sequence of complex numbers. To build up some intuition, we first give an informal definition.

A sequence of complex numbers (z_n) converges to a complex number w if for any disk centered at w, the sequence eventually enters and remains in that disk.



An algebraic way of expressing that z is in the disk centered at w with radius ε:

$$|z - w| < \varepsilon$$



A way of expressing that the sequence enters and remain in the disk:

$$\forall \varepsilon > 0 \, \exists N \forall n > N [|z_n - w| < \varepsilon]$$

Definition 7.1 (Convergence of a sequence)
A sequence (z_n) in $\mathbb{C}$ converges to $w \in \mathbb{C}$ if

$$\text{for all } \varepsilon > 0, \text{ there exists an integer } N \text{ s.t.}$$

$$\text{for all } n \in \mathbb{N}, n > N \implies |z_n - w| < \varepsilon$$

Notation and terminology. The w in the definition (if it exists) is called the limit of (z_n) and we write

$$z_n \to w \text{ or } \lim z_n = w$$

to mean that (z_n) converges to w.
The theory of sequences of complex numbers has many similarities to the theory of sequences of real numbers. Sometimes, the proofs of theorems that apply to both theories are identical, except for the interpretation of $+, \cdot$ as the complex operations instead of the real ones.
Also, some limits from real analysis will also be helpful to remember:

1. $\lim_n \frac{1}{n^p} = 0$, when $0 < p < +\infty$;
2. $\lim_n |z|^n = 0$, when $|z| < 1$;
3. $\lim_n n^{1/n} = 1$.

Complex sequences obey the same limit laws as real sequences:

Theorem 7.2
Let (z_n) and (w_n) be complex sequences and assume $z_n \to z$ and $w_n \to w$. Then,

i. for $\lambda \in \mathbb{C}, \lambda z_n \to \lambda z$;
ii. $z_n + w_n \to z + w$;
iii. $z_n w_n \to zw$;
iv. $z_n / w_n \to z/w$, provided $w \neq 0$ and $w_n \neq 0$ for any $n \in \mathbb{N}$.

Proof. The proofs are nearly identical to their real counterparts. For example, consider the proof of the sum law:
Let $\varepsilon > 0$ be given. Since $z_n \to z$ and $w_n \to w$, there exists N_1 and N_2 such that

$$n > N_1 \implies |z_n - z| < \varepsilon/2, \quad n > N_2 \implies |w_n - w| < \varepsilon/2.$$

Let $N = \max\{N_1, N_2\}$. Then, for any $n > N$, we have

$$|(z_n + w_n) - (z + w)| = |(z_n - z) + (w_n - w)| \leq |z_n - z| + |w_n - w| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

$\square$

We have the following limit law for complex sequences:

Theorem 7.3
Let (z_n) be a sequence of complex numbers and let $z \in \mathbb{C}$. If $z_n \to z$, then $\overline{z_n} \to \overline{z}$.

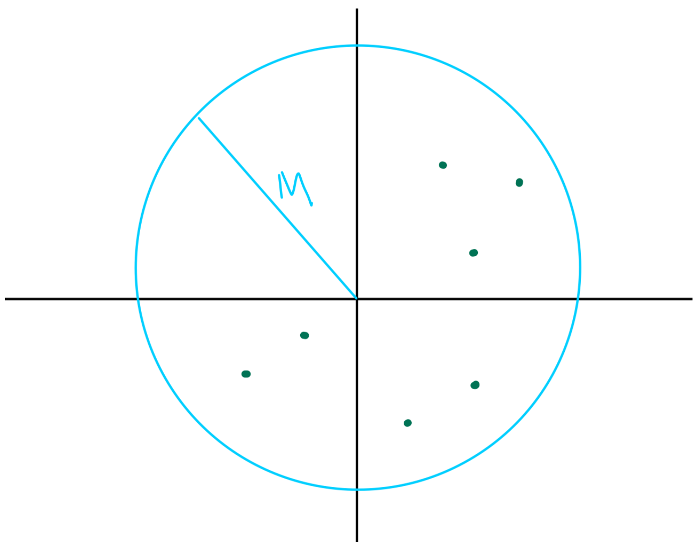
Proof. The proof hinges on the observation that

$$|\overline{z_n} - \overline{z}| = |\overline{z_n - z}| = |z_n - z|$$

$\square$

Definition 7.4 (Bounded sequence)
A sequence (z_n) of complex numbers is <u>bounded</u> if there exists a real number $M > 0$ such that

$$|z_n| \leq M \quad \forall n$$



We have the following theorem (whose real version is familiar).

Theorem 7.5
Any convergent sequence of complex numbers is bounded.

7.2 Alternative Characterizations of Convergence

Theorem 7.6
Let (z_n) be a sequence in $\mathbb{C}$. (z_n) is convergent if and only if the real sequences $(\operatorname{Re} z_n)$ and $(\operatorname{Im} z_n)$ are both convergent.

Proof. For the ($\Longrightarrow$) direction, use the equalities

$$\operatorname{Re} z_n = \frac{z_n + \overline{z_n}}{2}, \quad \operatorname{Im} z_n = \frac{z_n - \overline{z_n}}{2i},$$

and apply limit laws.

For the ($\Longleftarrow$) direction, use

$$z_n = \operatorname{Re} z_n + i \operatorname{Im} z_n$$

and apply limit laws. $\square$

Definition 7.7 (Cauchy sequence)
A sequence of complex numbers (z_n) is Cauchy if

$$\text{for all } \varepsilon > 0, \text{ there exists an integer } N \text{ s.t.}$$

$$\text{for all } n, m > N, \quad |z_n - z_m| < \varepsilon.$$

Theorem 7.8
Let (z_n) be a sequence in $\mathbb{C}$. (z_n) is convergent if and only if (z_n) is Cauchy. i.e. "$\mathbb{C}$ is complete". "$\mathbb{C}$ has no holes."

7.3 Limits of Functions

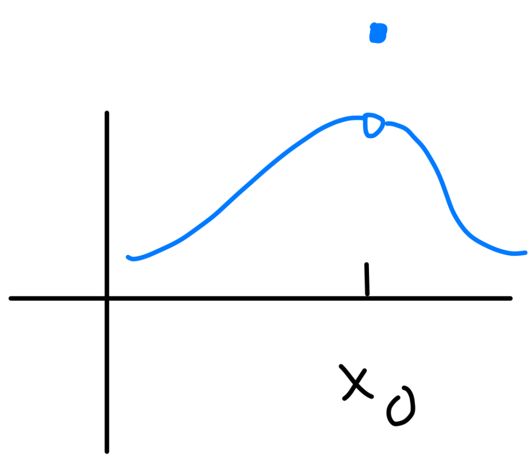
Definition 7.9 (Limit)
Let $f(z)$ be a complex-valued function on its domain $D \subseteq \mathbb{C}$, and let $z_0, L \in \mathbb{C}$. We say that $f(x)$ has limit L when z tends to z_0, and we write

$$\lim_{z \to z_0} f(z) = L,$$

if

$$\text{for all } \varepsilon > 0, \text{ there exists } \delta > 0 \text{ s.t.}$$

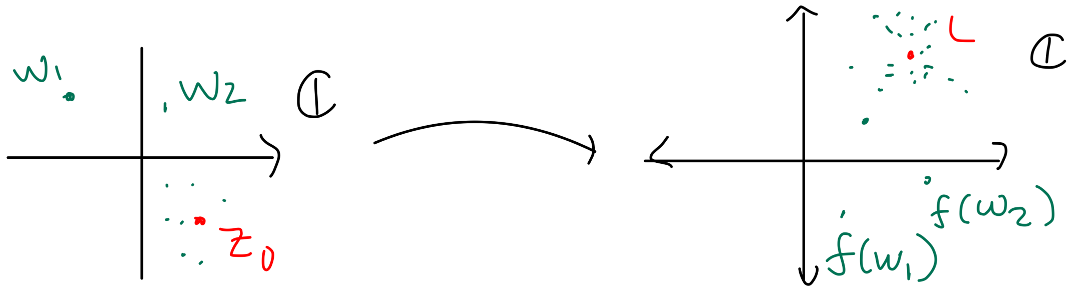
$$\text{for all } z \in D, \quad 0 < |z - z_0| < \delta \implies |f(z) - L| < \varepsilon.$$



Note that the definition of $\lim_{z \to z_0} f(z) = L$ does not require $f(z_0)$ to be defined. Also, even if $\lim_{z \to z_0} f(z)$ and $f(z_0)$ both exist, it may be the case that $\lim_{z \to z_0} f(z) \neq f(z_0)$. The above definition is equivalent to a condition on sequences.

Theorem 7.10
Let $f(z)$ be a complex-valued function on a domain $D \subseteq \mathbb{C}$, and let $z_0, L \in \mathbb{C}$. $\lim_{z\to z_0} f(z) = L$ if and only if for any sequence (w_n) in D with $w_n \neq z_0$ for all n, we have

$$w_n \to z_0 \implies f(w_n) \to L.$$



The sequence characterization of function limits along with the sequence limit laws immediately give us

Theorem 7.11
If a function has a limit at z_0, then the function is bounded near z_0. Moreover, if $\lim\limits_{z\to z_0} f(z) = L$ and $\lim\limits_{z\to z_0} g(z) = M$, then

i. $\lim\limits_{z\to z_0} (\lambda f)(z) = \lambda L$

ii. $\lim\limits_{z\to z_0} [f(z) + g(z)] = L + M$;

iii. $\lim\limits_{z\to z_0} f(z)g(z) = LM$;

iv. $\lim\limits_{z\to z_0} f(z)/g(z) = L/M$, provided $M \neq 0$ and $g(z) \neq 0$ near z_0.

Definition 7.12 (Continuous)
Let $f(z)$ be a complex-valued function of $D \subseteq \mathbb{C}$, and let $z_0 \in D$. We say that f is <u>continuous at z_0</u> if

$$\lim_{z\to z_0} f(z) = f(z_0)$$

f is called <u>continuous</u> if f is continuous at every z_0 in its domain D.

Example 7.13
Any polynomial function is continuous.

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

- $g(z) = z$ is continuous.
- scalar multiples of continuous functions are continuous.
- constant functions are continuous
- sums of continuous functions are continuous
- products of continuous functions are continuous.

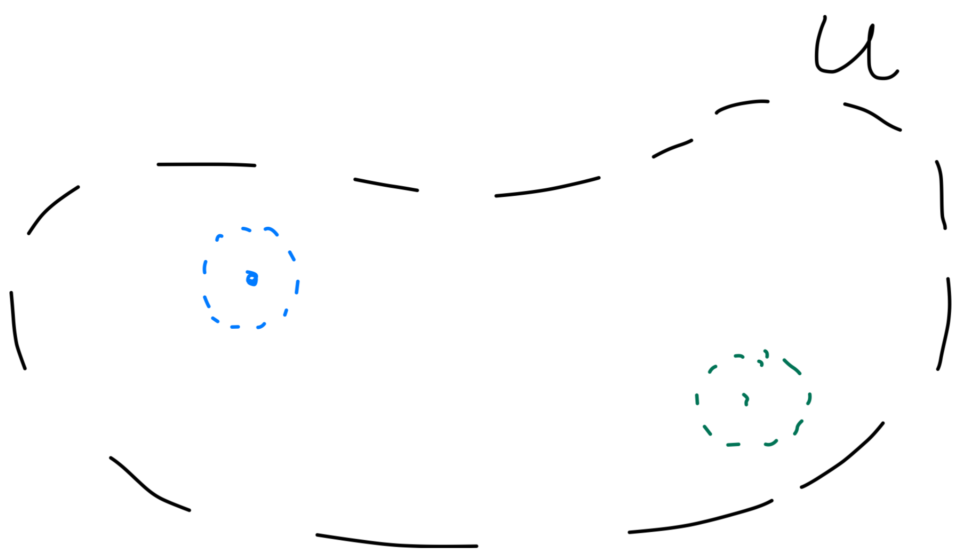
Example 7.14
The functions $\operatorname{Re} z, \operatorname{Im} z, |z|, \overline{z}$ are all continuous. $\overline{z}$ is continuous since we showed

$$z_n \to z_0 \implies \overline{z_n} \to \overline{z_0}$$

$\operatorname{Re} z$: use $|\operatorname{Re} z_n - \operatorname{Re} z_0| \leq |z_n - z_0|$

7.4 Open Sets in $\mathbb{C}$

Definition 7.15 (Open set)
A subset $U \subseteq \mathbb{C}$ is open if for every $z \in U$, there exists a disk D with nonzero radius centered at z with $\overline{D} \subseteq U$.

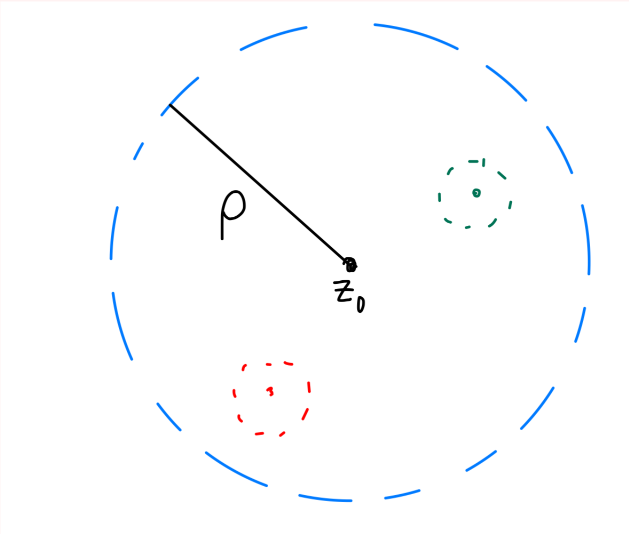


Example 7.16

Any open disk,

$$\{z \in \mathbb{C} \colon |z - z_0| < \rho\}, \qquad \text{(centered at } z_0\text{, with radius } \rho\text{)}$$

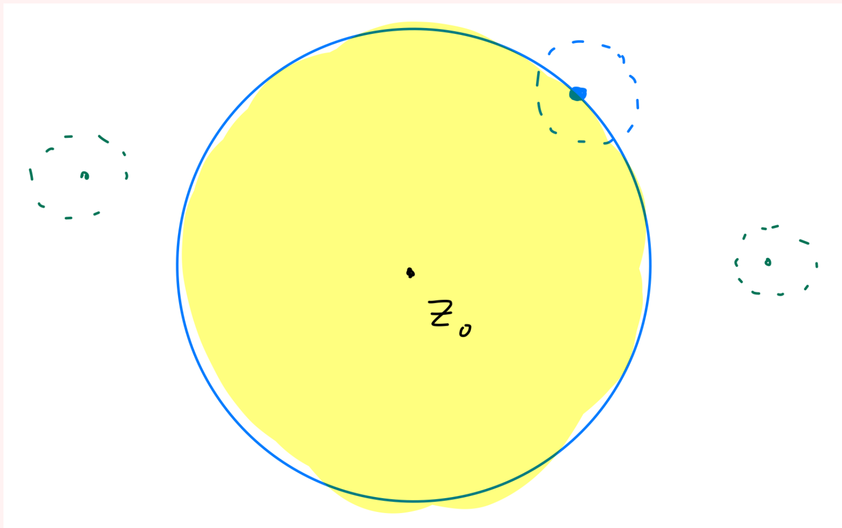
is open.



Example 7.17
Any closed disk,

$$\{z \in \mathbb{C} \colon |z - z_0| \le \rho\},$$

is not open.



The complement of this set is

$$\{z \in \mathbb{C} \colon |z - z_0| > \rho\},$$

which is open.

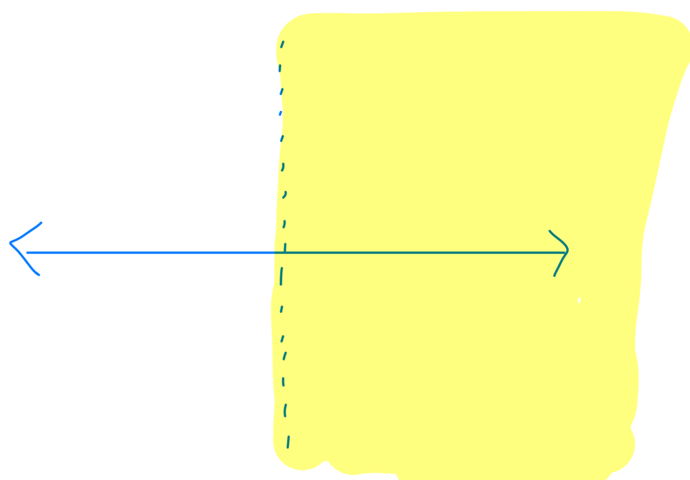
Definition 7.18 (Closed set)
A set $F \subseteq \mathbb{C}$ is closed if its complement

$$F^C = \mathbb{C} \setminus F$$

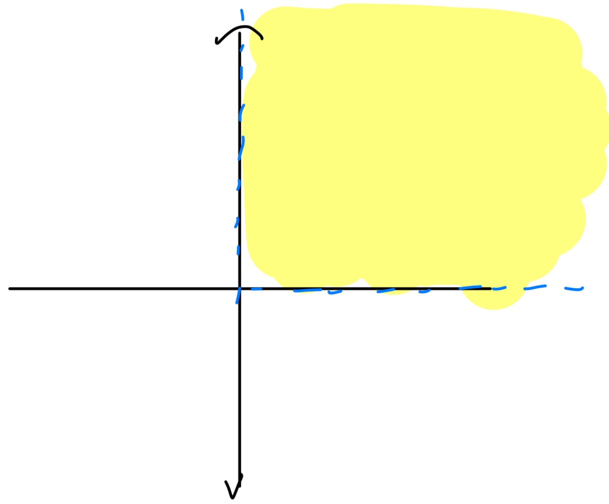
is open.

Example 7.19
Any set defined by strict inequalities and continuous functions is an open set.

E.g. the set $\{z \in \mathbb{C} \colon \operatorname{Re} z > 0\}$ is open.



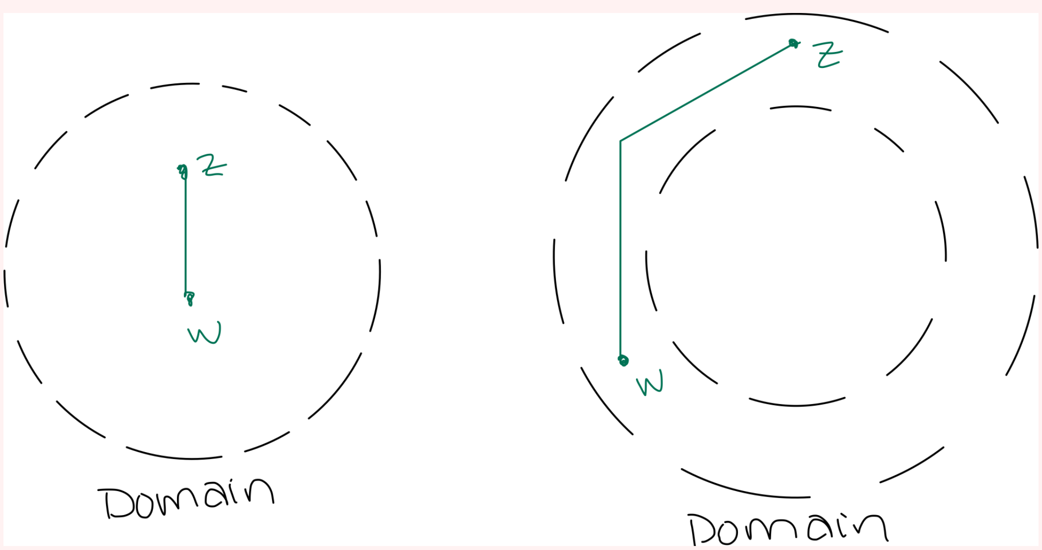
E.g. the set $\{z \in \mathbb{C} \colon 0 < \arg z < \pi/2\}$ is open.



Definition 7.20 (Domain)
A set $D \subseteq \mathbb{C}$ is called a <u>domain</u> if it is

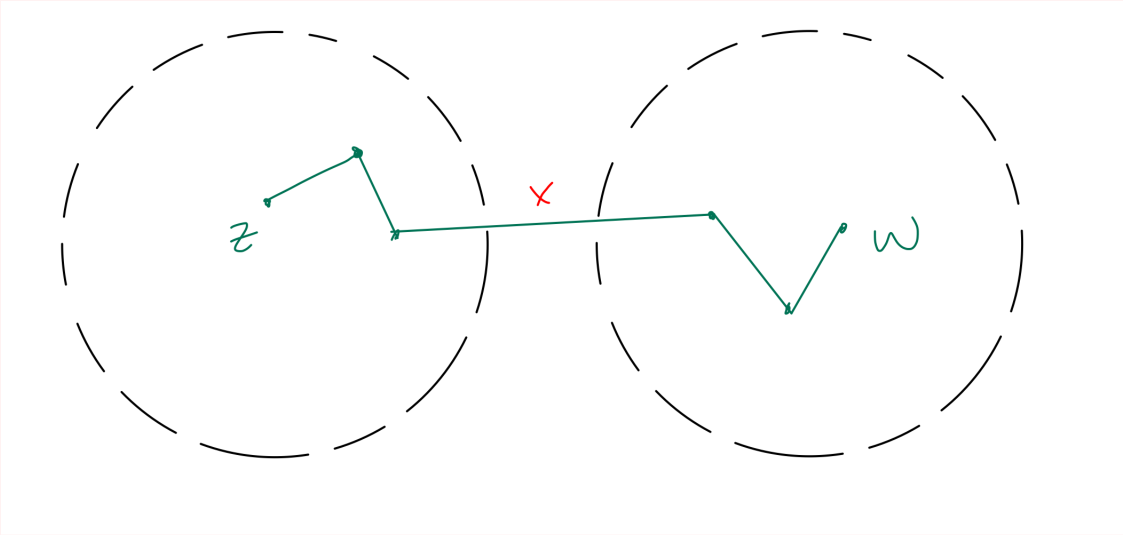
1. open; and
2. any two points $z, w \in D$ can be connected to each other by a finite series of line segments in D, where the end of one line segment is the start of the next.

Example 7.21
Open disks and open annuli $\{z \in \mathbb{C} \colon \rho_1 < |z| < \rho_2\}$ are domains.



Example 7.22

Disconnected sets are not domains



8 Jan 21, 2022

8.1 Analytic Functions

Definition 8.1 (Differentiable and complex derivative)
Let $f(z)$ be a complex-valued function defined in a disk centered at $z_0 \in \mathbb{C}$. $f(z)$ is differentiable at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. When this limit does exist, we denote it by

$$f'(z_0) \quad \text{or} \quad \frac{df}{dz}(z_0),$$

and call it the complex derivative of $f(z)$ at z_0.

The limit that defines the derivative can be equivalently expressed by

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Notation: $\Delta z = \Delta x + i\Delta y$
The simplest example: a constant function $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = \lambda$ for all $z \in \mathbb{C}$, is differentiable at every $z_0 \in \mathbb{C}$.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim \frac{\lambda - \lambda}{z - z_0} = 0$$

Example 8.2

Let m be a positive integer. The power function $f(z) = z^m$ is differentiable at every $z \in \mathbb{C}$. We show $f'(z) = (z^m)' = mz^{m-1}$.

We use the binomial expansion

$$(z+\Delta z)^m = \binom{m}{0} z^m + \binom{m}{1} z^{m-1}\Delta z + \cdots + \binom{m}{m-1} z\Delta z^{m-1} + \binom{m}{m}\Delta z^m$$

Note: $\binom{m}{0} = \binom{m}{m} = 1, \binom{m}{1} = m$ It follows that,

$$\begin{aligned}\frac{(z+\Delta z)^m - z^m}{\Delta z} &= \frac{1}{\Delta z}\left(mz^{m-1}\Delta z + \binom{m}{z} z^{m-2}\Delta z^2 + \cdots + \Delta z^m\right)\\ &= mz^{m-1} + \underbrace{\binom{m}{z} z^{m-2}\Delta z + \cdots + \Delta z^{m-1}}_{\to 0 \text{ as } \Delta z \to 0}\end{aligned}$$

Then,

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^m - z^m}{\Delta z} = mz^{m-1}$$

Next, we look at a non-example:

Example 8.3

The complex conjugation function $f(z) = \overline{z}$ is not differentiable anywhere.

$$\begin{aligned}\lim_{\Delta z \to 0} \frac{\overline{(z+\Delta z)} - \overline{z}}{\Delta z} &= \lim_{\Delta \to 0} \frac{\overline{z} + \overline{\Delta z} - \overline{z}}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}\end{aligned}$$

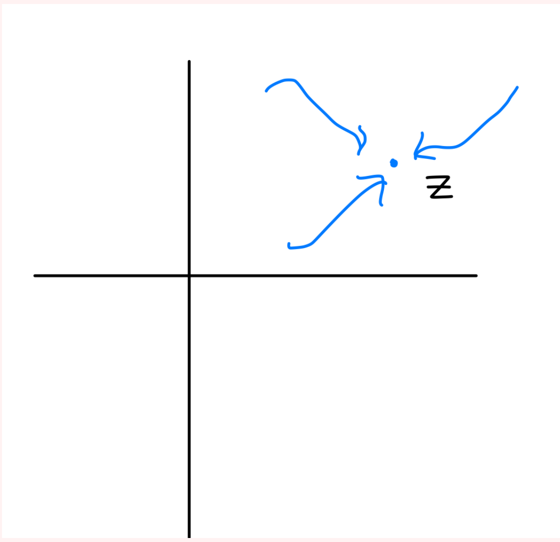
To show this limit does not exist, it suffices to show that $\dfrac{\overline{\Delta z}}{\Delta z}$ has two different limits as we approach from different directions.

1. If $\Delta z = \Delta x + i \cdot 0$ (approaching via horizontal line)

$$\lim_{\Delta x \to 0} \frac{\overline{\Delta x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

2. If $\Delta z = 0 + i\Delta y$ (approaching via vertical line)

$$\lim_{\Delta y \to 0} \frac{\overline{i\Delta y}}{i\Delta y} = \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1$$



Recall that many properties about limits of complex sequences and functions are proved with arguments that are nearly identical to the proofs from real analysis. The same is true for complex derivatives and real derivatives. Using this observation, one can quickly establish the following properties of the complex derivative:

1. If $f(z)$ is differentiable at z_0, then $f(z)$ is continuous at z_0. Just use,

$$f(z) = f(z_0) + \underbrace{\left(\frac{f(z) - f(z_0)}{z - z_0}\right)(z - z_0)}_{f'(z)\cdot 0 \text{ as } z \to z_0}$$

Shows $f(z) \to f(z_0)$ as $z \to z_0$.

2. Differentiation rules:

$$\begin{aligned}
(\lambda f)'(z) &= \lambda f'(z) \quad (\lambda \in \mathbb{C}), \\
(f+g)'(z) &= f'(z) + g'(z), \\
(fg)'(z) &= f(z)g'(z) + f'(z)g(z), \\
(f/g)'(z) &= \frac{g(z)f'(z) - f(z)g'(z)}{[g(z)]^2}, \quad \text{provided}
\end{aligned}$$

g is nonzero on some disk centered at z

3. The chain rule is valid for complex derivatives:

Theorem 8.4
Suppose $g(z)$ is differentiable at z_0 and $f(w)$ is differentiable at $w_0 = f(z_0)$. Then the composition function $(f \circ g)(z) = f(g(z))$ is differentiable at z_0 and

$$(f \circ g)'(z_0) = f'(g(z_0))g'(z_0).$$

Example 8.5
Any polynomial function

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + z_0$$

is everywhere differentiable, and its derivative is

$$n a_n z^{n-1} + (n-1) a_{n-1} z^{n-2} + \cdots + a_1$$

Proved just using the power rule, constant multiple rule, sum rule for derivatives.

Example 8.6
For any integer m (even $m < 0$),

$$\frac{d}{dz} z^m = mz^{m-1}$$

If $m < 0$, then $m = -n$ for some $n = 1, 2, 3, \ldots$

$$z^m = z^{-n} = \frac{1}{z^n}.$$

Then by quotient rule,

$$\left(\frac{1}{z^n}\right)' = \frac{z^n \cdot 0 - 1 \cdot nz^{n-1}}{z^{2n}} = \frac{-nz^{n-1}}{z^{2n}}$$
$$= -nz^{-n-1} = mz^{m-1}$$

Example 8.7
We can differentiate $(z^2 - i)^{-1}$ using the chain rule:

$$\frac{d}{dz}(z^2 - i) = -(z^2 - i)^{-2} \cdot 2z$$

Definition 8.8 (Analytic on the open set)
A complex-valued function $f(z)$ is analytic on the open set $U \subseteq \mathbb{C}$ if

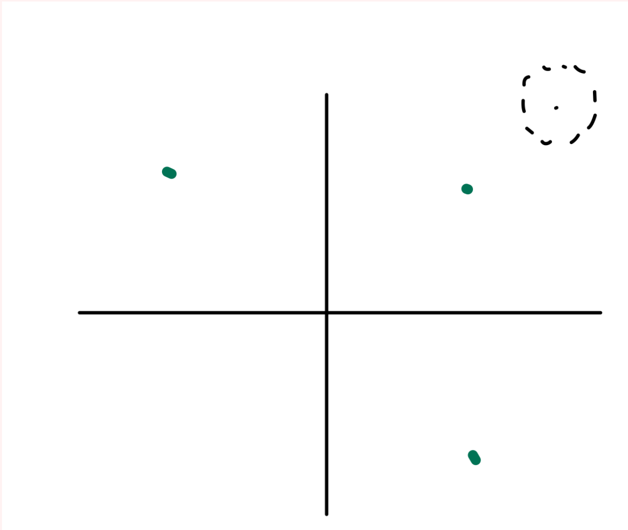
1. $f(z)$ is differentiable at every $z_0 \in U$; and, moreover,
2. $f'(z)$ is continuous on U.

Example 8.9
Polynomial functions are analytic on $\mathbb{C}$.
Polynomials are differentiable on $\mathbb{C}$, and their derivatives are polynomials, which are continuous everywhere.

Example 8.10
Rational functions are analytic wherever their denominators do not vanish.

$$f(z) = \frac{p(z)}{q(z)} \quad p(z), q(z) \text{ are polynomials}$$

f is not defined where $q(z) = 0$.



$U =$ all $\mathbb{C}$ except the roots of $q(z)$
U is open
$f(z)$ is differentiable on U by quotient rule.
$f'(z)$ is again a rational function, so continuous.

Definition 8.11 (Analytic at z_0)
$f(z)$ is analytic at $z_0 \in \mathbb{C}$ if there is an open disk centered at z_0 on which $f(z)$ is analytic.

Later we will prove one of the most important facts of complex analysis:
If condition (1) is satisfied in the definition of analytic on U, then condition (2) is automatically satisfied, i.e.,
If $f(z)$ is differentiable at every z_0 in an open set U, then $f'(z)$ is continuous on U.
This is very different than in real analysis:
There exists $g\colon \mathbb{R} \to \mathbb{R}$ which is differentiable on $\mathbb{R}$, but g' is not continuous on $\mathbb{R}$.

8.2 Cauchy-Riemann Equations

Much like a complex number, a complex valued function f(z) can be split into two parts: its real part and its imaginary part. Moreover, these two parts can be seen as functions of two real variables x and y, where $z = x + iy$. In other words, we can express

$$f(z) = u(x, y) + iv(x, y),$$

where $u, v \colon \mathbb{R}^2 \to \mathbb{R}$ are real-valued functions on the real plane $\mathbb{R}^2$. i.e.

$$u(x,y) = \operatorname{Re} f(x+iy) \qquad v(x,y) = \operatorname{Im} f(x+iy)$$

Definition 8.12 (Cauchy-Riemann equations)
The functions $u(x,y), v(x,y)$ can help us establish when a complex function is analytic. The Cauchy-Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{*}$$

Theorem 8.13
A complex-valued function $f(z) = u(x,y) + iv(x,y)$ is analytic on a domain D if and only if the partial derivatives of $u(x,y)$ and $v(x,y)$ exists and are continuous on D and satisfy the Cauchy-Riemann equations (*) on D.

We will also see that, when the equivalent conditions in the theorem are satisfied, we have the equalities

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) \\ &= \frac{\partial v}{\partial y}(x,y) - i\frac{\partial u}{\partial y}(x,y) \end{aligned}$$

Proof. To prove the theorem, we must establish both "directions" of the "if and only if".
($\Longrightarrow$) Assume f is analytic on the domain D.
We need to show the partials exist, continuous on D and satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Let $z \in D$. We know

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = f'(z)$$

exists so taking $\Delta z = \Delta x$ and $\Delta z = i\Delta y$ will yield the same limit $f'(z)$.

$$\begin{aligned} \frac{f(z+\Delta x) - f(z)}{\Delta x} &= \frac{1}{\Delta x}\left[u(x+\Delta x, y) + iv(x+\Delta x, y) - \big(u(x,y) + iv(x,y)\big)\right] \\ &= \left(\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x}\right) + i\left(\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}\right) \end{aligned}$$

Taking a $\lim_{\Delta x \to 0}$ on both sides, we have

$$f'(z) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y)$$

Since $f'(z)$ is continuous on D, it follows that $\dfrac{\partial u}{\partial x}, \dfrac{\partial v}{\partial x}$ are also continuous on D. Next, we take $\Delta z = i\Delta y$

$$\begin{aligned}\frac{f(z+i\Delta y)-f(z)}{i\Delta y} &= \frac{1}{i\Delta y}\Big[u(x,y+\Delta y)+iv(x,y+\Delta y)-\big(u(x,y)+iv(x,y)\big)\Big]\\ &= \frac{u(x,y+\Delta y)-u(x,y)}{i\Delta y}+\frac{i\big(v(x,y+\Delta y)-v(x,y)\big)}{i\Delta y}\\ &= \left(\frac{v(x,y+\Delta y)-v(x,y)}{\Delta y}\right)-i\left(\frac{u(x,y+\Delta y)-u(x,y)}{\Delta y}\right)\end{aligned}$$

Take limits as $\Delta y \to 0$:

$$\left.\begin{aligned} f'(z) &= \frac{\partial v}{\partial y}(x,y)-i\frac{\partial u}{\partial y}(x,y)\\ &= \frac{\partial u}{\partial x}(x,y)+i\frac{\partial v}{\partial x}(x,y)\end{aligned}\right\}\text{equating real and imaginary points gives C-R equations}$$

We will prove ($\Longleftarrow$) in the next lecture. $\square$

Example 8.14
We can use the Cauchy-Riemann equations to show that e^z is analytic on all of $\mathbb{C}$.

$$e^z = \underbrace{e^x\cos y}_{u} + i\,\underbrace{e^x\sin y}_{v}$$

$$\frac{\partial u}{\partial x} = e^x\cos y = \frac{\partial v}{\partial y} = e^x\cos y$$

$$\frac{\partial u}{\partial y} = -e^x\sin y \qquad \frac{\partial v}{\partial x} = e^x\sin y$$

Since C-R equations satisfied, all partials are continuous functions, it follows by theorem that e^z is analytic on $\mathbb{C}$.

9 Jan 24, 2022

9.1 Cauchy-Riemann Equations (Cont'd)

Proof of Theorem 8.13. Next, we prove the ($\Longleftarrow$) direction.
So, assume the partial derivatives of $u(x, y)$ and $v(x, y)$ exists and are continuous on D and satisfy the Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

We need to show $f(z)$ is analytic on D. We will use Taylor's Theorem on $u, v\colon \mathbb{R}^2 \to \mathbb{R}$ to get

$$u(x + \Delta x, y + \Delta y) = u(x, y) + \frac{\partial u}{\partial x}(x, y)\Delta x + \frac{\partial u}{\partial y}(x, y)\Delta y + R(\Delta x, \Delta y)$$

$$v(x + \Delta x, y + \Delta y) = v(x, y) + \frac{\partial v}{\partial x}(x, y)\Delta x + \frac{\partial v}{\partial y}(x, y)\Delta y + S(\Delta x, \Delta y)$$

where $\dfrac{R(\Delta x, \Delta y)}{|\Delta z|} \to 0, \quad \dfrac{S(\Delta x, \Delta y)}{|\Delta z|} \to 0 \quad \text{as } \Delta z \to 0$

$$\begin{aligned} f(z + \Delta z) &= u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) \\ &= f(z) + \frac{\partial u}{\partial x}(x, y)\Delta x + \frac{\partial u}{\partial y}(x, y)\Delta y + R(\Delta x, \Delta y) + i\frac{\partial v}{\partial x}(x, y)\Delta x + i\frac{\partial v}{\partial y}(x, y)\Delta y \\ &\quad + iS(\Delta x, \Delta y) \end{aligned}$$

Use C-R to replace partial y derivatives

$$= f(z) + \frac{\partial u}{\partial x}(x, y)\Delta x - \frac{\partial v}{\partial x}(x, y)\Delta y + R(\Delta x, \Delta y) + i\frac{\partial v}{\partial x}(x, y)\Delta x + i\frac{\Delta u}{\Delta x}(x, y)\Delta y + iS(\Delta x, \Delta y)$$
$$= f(z) + \frac{\partial u}{\partial x}(x, y)\Delta z + i\frac{\partial v}{\partial x}(x, y)\Delta z + R(\Delta x, \Delta y) + iS(\Delta x, \Delta y)$$

Next, we subtract $f(z)$ from both sides and divide by Δz:

$$\frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{\partial u}{\partial x}(x, y) + i\frac{\partial v}{\partial x}(x, y) + \frac{R(\Delta x, \Delta y) + iS(\Delta x, \Delta y)}{\Delta z}$$

where $\dfrac{R(\Delta x, \Delta y) + iS(\Delta x, \Delta y)}{\Delta z} \to 0$ as $\Delta z \to 0$ So taking limit as $\Delta z \to 0$ on both sides, we get LHS limit exists and

$$f'(z) = \frac{\partial u}{\partial x}(x, y) + i\frac{\partial v}{\partial x}(x, y)$$

Note: If you had used C-R to replace the x-derivatives, would have shown

$$f'(z) = \frac{\partial v}{\partial y}(x, y) - i\frac{\partial u}{\partial y}(x, y)$$

$\square$

Example 9.1
We can use C-R to show (again) that $f(z) = \overline{z}$ is not analytic on any open set.

$$f(z) = \overline{z} = x - iy$$

$$u(x, y) = x, \quad v(x, y) = -y$$

$$\begin{aligned} \frac{\partial u}{\partial x} &= 1 \quad \frac{\partial v}{\partial y} = -1 \\ \frac{\partial u}{\partial y} &= 0 \quad \frac{\partial v}{\partial x} = 0 \end{aligned}$$

Since $\dfrac{\partial u}{\partial x} = 1 \neq -1 = \dfrac{\partial v}{\partial y}$ at any $z \in \mathbb{C}$, we conclude that $\overline{z}$ is not analytic anywhere.

Using the same method we used to show $(e^z)' = e^z$, we can also establish

$$(\sin z)' = \cos z, \quad (\cos z)' = -\sin z,$$

$$(\sinh z)' = \cosh z, \quad (\cosh z)' = \sinh z.$$

Since the Cauchy-Riemann equations are about real functions, we can sometimes use results from real analysis to get conclusions about complex analytic functions.

Theorem 9.2
If $f(z)$ is analytic on a domain D and $f'(z) = 0$ for all $z \in D$, the $f(z)$ is constant on D.

Proof. We know $f'(z) = 0$ for all $z \in D$ and we know

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0 \\ &= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = 0 \end{aligned}$$

on D. Then,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$$

on all of D. From real analysis (calculus) u, v constant functions on D. Then $f = u + iv$ is constant on D. $\square$

We can similarly prove:

Theorem 9.3
If $f(z)$ is analytic on a domain $D \subseteq \mathbb{C}$ and $f(z) \in \mathbb{R}$ for all $z \in D$, then $f(z)$ is constant on D.

Proof. Write $f(z) = u(x, y) + iv(x, y)$. Since $f(z) \in \mathbb{R}$ for all $z \in D$, it follows $v(x, y) = 0$

for all $z = x + iy \in D$. Then, $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$ on D. Since f is analytic, it satisfies C-R on D. Then,

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} = 0 \quad \text{on } D \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} = 0 \quad \text{on } D\end{aligned}$$

Then, u is constant on D so $f = u + i \cdot 0$ is constant on D. □

What if we are expressing z in polar coordinates?

Proposition 9.4
The polar form of the Cauchy-Riemann equations is:

$$\frac{\partial u}{\partial r} = \frac{1}{r} \cdot \frac{\partial v}{\partial \theta}, \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

Proof. To show this, assume $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

$$\begin{aligned} u &= u\Big(x(r,\theta), y(r,\theta)\Big) \quad x(r,\theta) = r\cos\theta \\ v &= v\Big(x(r,\theta), y(r,\theta)\Big) \quad y(r,\theta) = r\sin\theta \end{aligned}$$

Then we can use the multivariable chain rule:

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial \theta} = \frac{\partial v}{\partial x} \cdot (-r\sin\theta) + \frac{\partial v}{\partial y}(r\cos\theta)$$

$$\begin{aligned}\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x} \cdot \cos\theta + \frac{\partial u}{\partial y}\sin\theta \\ &\overset{\text{(C-R)}}{=} \frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\sin\theta \\ &= \frac{1}{r}\left(\frac{\partial v}{\partial y} r\cos\theta - \frac{\partial v}{\partial x} r\sin\theta\right) \\ &= \frac{1}{r}\frac{\partial v}{\partial \theta}\end{aligned}$$

And,

$$\begin{aligned}
\frac{\partial v}{\partial r} &= \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta \\
\frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta) \\
&= \frac{\partial v}{\partial y}(-r\sin\theta) - \frac{\partial v}{\partial x}(r\cos\theta) \\
&= -r\left(\frac{\partial v}{\partial y}\sin\theta + \frac{\partial v}{\partial x}\cos\theta\right) \\
&= -r\frac{\partial v}{\partial r}
\end{aligned}$$

So,

$$\begin{cases} \dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} \\ \dfrac{\partial u}{\partial \theta} = -r\dfrac{\partial v}{\partial r} \end{cases}$$

$\square$

Example 9.5

Consider $\operatorname{Log} z = \log|z| + i\operatorname{Arg} z = \log r + i\theta$.

$$\frac{\partial u}{\partial r} = \frac{1}{r} \quad \frac{\partial v}{\partial \theta} = 1 \quad \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial u}{\partial \theta} = 0 = \frac{\partial v}{\partial r}$$

Homework problem II. 3.3

Show that if $f, \overline{f}$ are both analytic on D, then f is constant.

$$f = u(x,y) + iv(x,y)$$

$$\overline{f} = u(x,y) - iv(x,y)$$

Since f is analytic, the Cauchy-Riemann equations for f are satisfied on D

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Same for the C-R equations for $\overline{f}$

$$\frac{\partial u}{\partial x} = \frac{\partial(-v)}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial(-v)}{\partial x}$$

So, $\dfrac{\partial u}{\partial x} = -\left(\dfrac{\partial v}{\partial y}\right) = -\dfrac{\partial u}{\partial x} \implies \dfrac{\partial u}{\partial x} = 0$ on D.

Can similarly show that

$$\frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$$

are all 0 on D. $\square$

10 Jan 26, 2022

10.1 Midterm 1

11 Jan 28, 2022

11.1 Inverse Mappings

Definition 11.1 (Jacobian matrix)
Let $f = u + iv$ be analytic on a domain D. The Jacobian matrix of f at z is

$$J_f(z) = \begin{bmatrix} \dfrac{\partial u}{\partial x}(x,y) & \dfrac{\partial u}{\partial y}(x,y) \\ \dfrac{\partial v}{\partial x}(x,y) & \dfrac{\partial v}{\partial y}(x,y) \end{bmatrix}$$

and its determinant is

$$\det J_f(z) = \frac{\partial u}{\partial x}(x,y) \cdot \frac{\partial v}{\partial y}(x,y) - \frac{\partial u}{\partial y}(x,y) \cdot \frac{\partial v}{\partial x}(x,y)$$

Of course, since f is analytic on D, it satisfies the Cauchy-Riemann equations on D, hence the formula for the determinant becomes

$$\begin{aligned} \det J_f = \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial x} - \left(-\frac{\partial v}{\partial x}\right)\left(\frac{\partial v}{\partial x}\right) &= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 \\ &= \left|\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right|^2 = |f'(z)|^2 \end{aligned}$$

This proves the following theorem:

Theorem 11.2
If f is analytic on D, then for any $z \in D$ we have

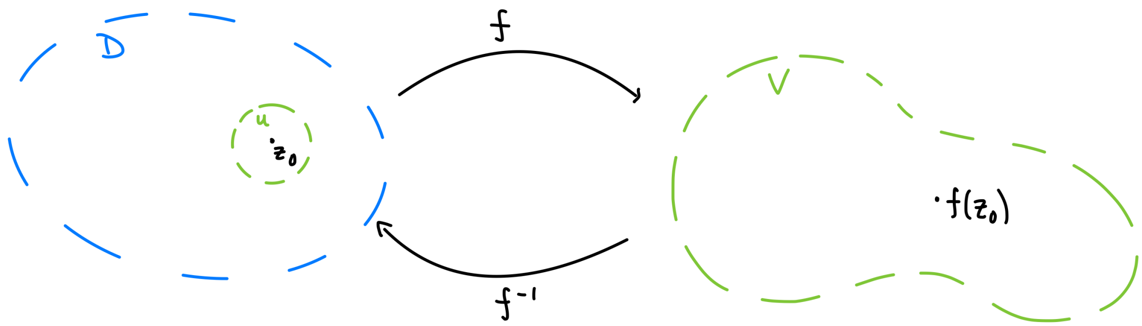
$$\det J_f(z) = |f'(z)|^2$$

We can apply the inverse function theorem from calculus to get

Theorem 11.3
If f is analytic on D, $z_0 \in D$, and $f'(z_0) \neq 0$, then there is an open disk $U \subseteq D$ containing z_0 such that

- f is one-to-one on U, i.e., $f(z) \neq f(w)$ for distinct $z, w \in U$;
- $V = f(U) = \{f(z) \colon z \in U\}$ is open; and
- $f^{-1} \colon V \to U$ exists and is analytic on V and satisfies

$$(f^{-1})'(f(z)) = \frac{1}{f'(z)}, \quad z \in U \tag{*}$$



Note 11.4: Once we know that f^{-1} is analytic, the chain rule gives us the formula in (*):

$$f^{-1}(f(z)) = z$$

Take derivative on both sides,

$$(f^{-1})'(f(z)) \cdot f'(z) = 1$$

$$\implies (f^{-1})'(f(z)) = \frac{1}{f'(z)} \qquad f'(z) \neq 0$$

Example 11.5

$\operatorname{Log} w$ is a continuous inverse for e^z for $-\pi < \operatorname{Arg} z < \pi$. The previous theorem will allow us to compute the derivative on $\operatorname{Log} w$.
e^z is analytic on $\mathbb{C}$ and $(e^z)' = e^z \neq 0 \quad \forall z \in \mathbb{C}$.
Method 1. Set $w = e^z$

$$\operatorname{Log}(e^z) = z$$

$$\overset{\text{take deriv.}}{\implies} \operatorname{Log}'(e^z) \cdot e^z = 1 \overset{w=e^z}{\implies} \operatorname{Log}'(w) = \frac{1}{w}$$

Method 2. $\operatorname{Log} w = z$ take exponent on both sides:
$w = e^z$ then take $\frac{d}{dw}$ on both sides:

$$1 = e^z \frac{dz}{dw} \implies 1 = w \cdot \frac{dz}{dw} \implies \frac{d \operatorname{Log} w}{dw} = \frac{1}{w}$$

Note than any other branch of log differs from Log by a constant; thus,

$$\log z = \log |z| + i \operatorname{Arg} z + i2\pi k$$

Any two branches of log have the same derivative.

Example 11.6

We can use the theorem to differentiate power functions like $w^{1/2}, w^{1/3}$, etc.

$$w^{1/2} = \sqrt{w} \text{ inverse of } z^2$$

$(z^2)' = 2z \neq 0$ as long as $z \neq 0$. But branches of $\sqrt{w}$ are not defined at 0, $\sqrt{0} = 0$. So theorem guarantees any branch of $\sqrt{w}$ is differentiable. $z = \sqrt{w}$

$$\begin{aligned} z^2 = w \text{ take } \frac{d}{dw} &\implies 2z \cdot \frac{dz}{dw} = 1 \\ &\implies \frac{d\sqrt{w}}{dw} = \frac{1}{2z} = \frac{1}{2\sqrt{w}} \text{ the same branch of } \sqrt{w} \end{aligned}$$

$(\sqrt{w})' = \frac{1}{2\sqrt{w}}$

Example 11.7

$\cos^{-1} z = -i \log[z \pm \sqrt{z^2 - 1}]$. Use theorem to take the derivative.

$$w = \cos^{-1} z$$

$$\cos w = z$$

Next, take $\frac{d}{dz}$ on both sides to get

$$-\sin w \cdot \frac{dw}{dz} = 1 \implies \frac{dw}{dz} = \frac{-1}{\sin w}$$

Now use trig identity $\sin w = \sqrt{1 - \cos^2 w}$. So,

$$\frac{d \cos^{-1} z}{dz} = \frac{-1}{\sqrt{1 - \cos^2 w}} = \frac{-1}{\sqrt{1 - z^2}} = \frac{\pm 1}{\sqrt{1 - z^2}}$$

Derivative depends on choice of branch for $\cos^{-1}$.

Note 11.8: $\cos^{-1}(\cos w)$ exists when $\cos' w = -\sin w \neq 0$.

11.2 Harmonic Functions

Definition 11.9 (Laplace operator and Laplace's equation)
Let $u(x_1, \ldots, w_n)$ be a real-valued function of n real variables. The Laplace operator is

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

and Laplace's equation is

$$\Delta u = 0$$

In a less abbreviated notation, Laplace's equation is

$$\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = 0$$

Infinitely differentiable functions u which satisfy Laplace's equation are called harmonic functions.
The first harmonic functions studied were those that describe the motion of a vibrating string (undergoing harmonic motion).
It was found that such functions satisfy Laplace's equation, and eventually harmonic was used to describe any function that satisfies Laplace's equation.
We will focus on harmonic functions of two variables x, y.

Definition 11.10 (Harmonic)
A function $u(x, y)$ is harmonic if its first- and second-order partial derivatives exist, are continuous, and satisfy

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Harmonic functions are closely related to complex analytic functions:

Theorem 11.11
If $f = u + iv$ is analytic (and u and v have continuous second-order partial derivatives), then both u and v are harmonic.

Note 11.12: We will prove later that the condition in parentheses is redundant since it is implied by f being analytic.

Proof. Check that u satisfies Laplace's equation:

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) &\underset{\text{C-R}}{=} \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) \\ &\underset{\text{C-R}}{=} \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2}\end{aligned}$$

Can similarly show v satisfies Laplace's equation. □

Definition 11.13 (Harmonic conjugate)
If $u(x, y)$ is harmonic on D, and $v(x, y)$ is a harmonic function on D such that $f = u + iv$ is analytic on D, then we say v is a harmonic conjugate of u.

Harmonic conjugates are not unique. It is easy to show that if v is a harmonic conjugate of u, then for any real constant C the function $v + C$ is also a harmonic conjugate of u. However, all other harmonic conjugates are of the form $v + C$.
Suppose v, v_0 are both harmonic conjugates of u.

$$\begin{aligned}
\substack{u+iv \\ u+iv_0} \text{ both analytic } &\implies (u + iv) - (u + iv_0) \text{ analytic} \\
&\implies i(v - v_0) \text{ analytic} \\
&\implies \frac{1}{i}\Big(i(v - v_0)\Big) \text{ analytic} \\
&\implies v - v_0 \text{ is analytic and real-valued} \\
&\implies v - v_0 = C \text{ for some } C \in \mathbb{C}
\end{aligned}$$

The harmonic conjugate of u (if it exists) is unique up to a constant.

Example 11.14
Consider the function $u(x, y) = x^2 - y^2$. Show that u is harmonic everywhere and find all of the harmonic conjugates of u.
Easy to show u is harmonic.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 1 - 1 = 0$$

We find a harmonic conjugate v. v must satisfy $C - R$ equations.

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x \implies v = 2xy + h(x)$$

$$2y + h'(x) = \frac{\partial v}{\partial x} \underset{\text{C-R}}{=} -\frac{\partial u}{\partial y} = 2y$$

$$\implies h'(x) = 0 \implies h(x) = C$$

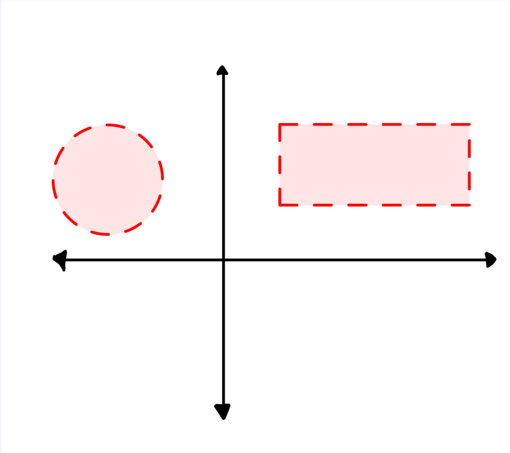
$$v = 2xy + C, \quad C \in \mathbb{R}$$

$$f = u + iv = (x^2 - y^2) + i2xy = z^2$$

Exercise. See the textbook for a similar example: $u(x, y) = xy$.

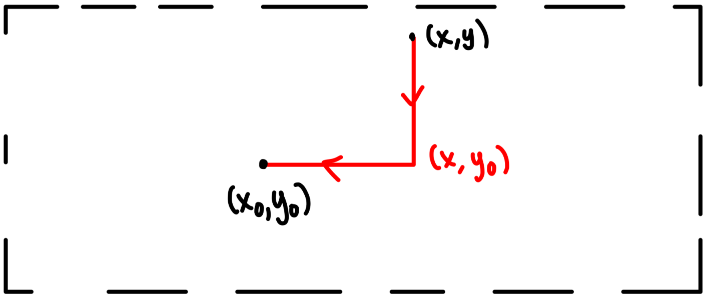
The following theorem guarantees that we can always find harmonic conjugates in certain domains.

Theorem 11.15
Let D be either an open disk or an open rectangle whose sides are parallel to the coordinate axes. If $u(x, y)$ is harmonic on D, then it has a harmonic conjugate on D.



Proof (Sketch). Fix a point $(x_0, y_0) \in D$

$$v(x, y) = \int_{y_0}^{y} \frac{\partial u}{\partial x}(x, t)\, dt - \int_{x_0}^{x} \frac{\partial u}{\partial y}(s, y_0)\, ds + C$$



□

Remark 11.16 In homework problems II.5.6-7, you will show that $\log|z|$ is harmonic on $\mathbb{C} \setminus \{0\}$, but it has no harmonic conjugate on $\mathbb{C} \setminus \{0\}$. It does however, have a harmonic conjugate on $\mathbb{C} \setminus (-\infty, 0]$

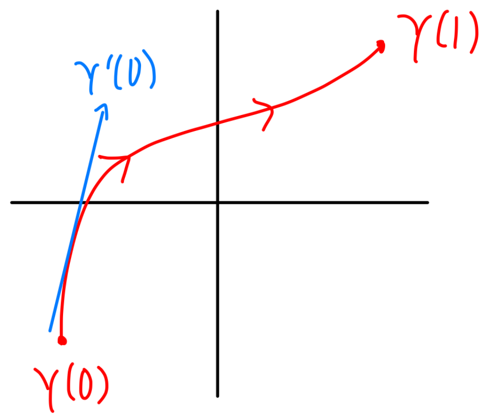
$$\operatorname{Log} z = \log|z| + i \operatorname{Arg} z$$

12 Jan 31, 2022

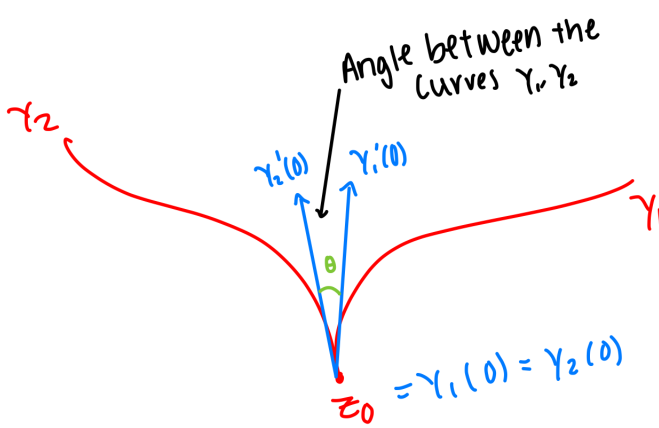
12.1 Conformal Mappings

Definition 12.1 (Tangent Vector)
Let $\gamma(t) = x(t) + iy(t), 0 \le t \le 1$, be a smooth parametrized curve in the complex plane. Let $z_0 = \gamma(0)$. We say that the curve γ starts at z_0. The tangent vector to the curve γ at z_0 is

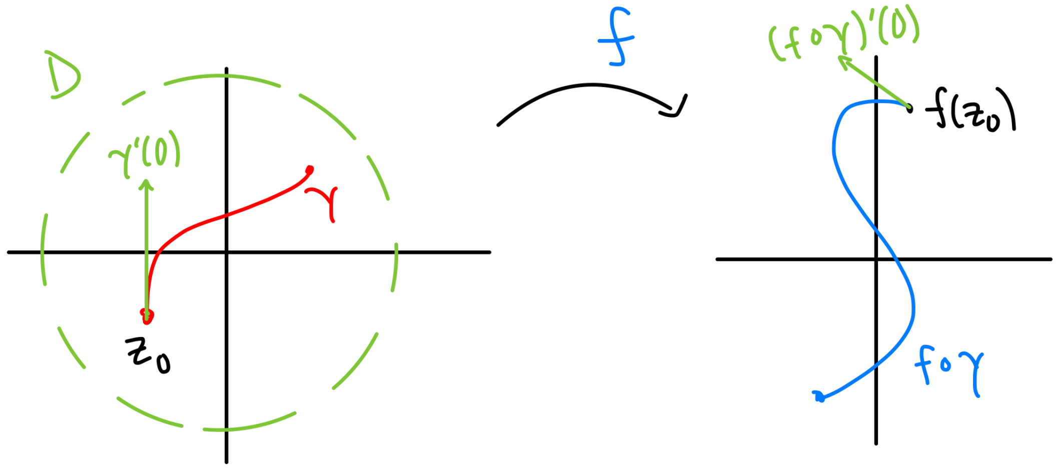
$$\gamma'(0) = x'(0) + iy'(0).$$



We define the angle between two curves starting at z_0 to be the angle between the tangent vectors at z_0.



If $f(z)$ is an analytic function, then $f \circ \gamma$ is a curve which starts at $f(\gamma(0)) = f(z_0)$.



Theorem 12.2
If $\gamma(t), 0 \le t \le 1$, is a smooth curve starting at $z_0 = \gamma(0)$, and let f be analytic at z_0. Then, the tangent to the curve $f(\gamma(t))$ at $f(z_0) = f(\gamma(0))$ is

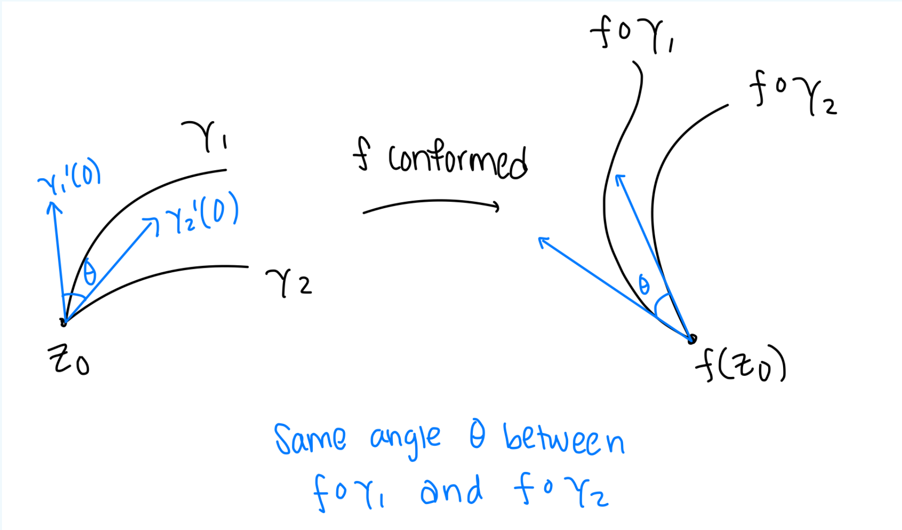
$$(f \circ \gamma)'(0) = f'(z_0)\gamma'(0).$$

Proof (Sketch).

$$\frac{f(\gamma(t)) - f(\gamma(0))}{t} = \underbrace{\left(\frac{f(\gamma(t)) - f(\gamma(0))}{\gamma(t) - \gamma(0)}\right)}_{f'(\gamma(0)) = f'(z_0)} \cdot \underbrace{\left(\frac{\gamma(t) - \gamma(0)}{t}\right)}_{\gamma'(0)}$$

as $t \to 0$ from the right. This works when $\gamma'(0) \neq 0$, since then $\gamma(t) \neq \gamma(0)$ for $0 < t$ near 0. Special case when $\gamma'(0) = 0$ needs a slightly different argument $\square$

Definition 12.3 (Conformal)
A function is conformal if it preserves angles. More precisely, a function $f(z)$ is conformal at z_0 if



Note: The angle between the curves $f \circ \gamma_1, f \circ \gamma_2$ should be measured with the same orientation that you used to measure the angle between γ_1, γ_2.

Example 12.4
Translations $f(z) = z + b$ are conformal everywhere.

Example 12.5
Dilations $g(z) = az, a \neq 0$, are conformal everywhere (tangent vectors get rotated by $\arg a$)

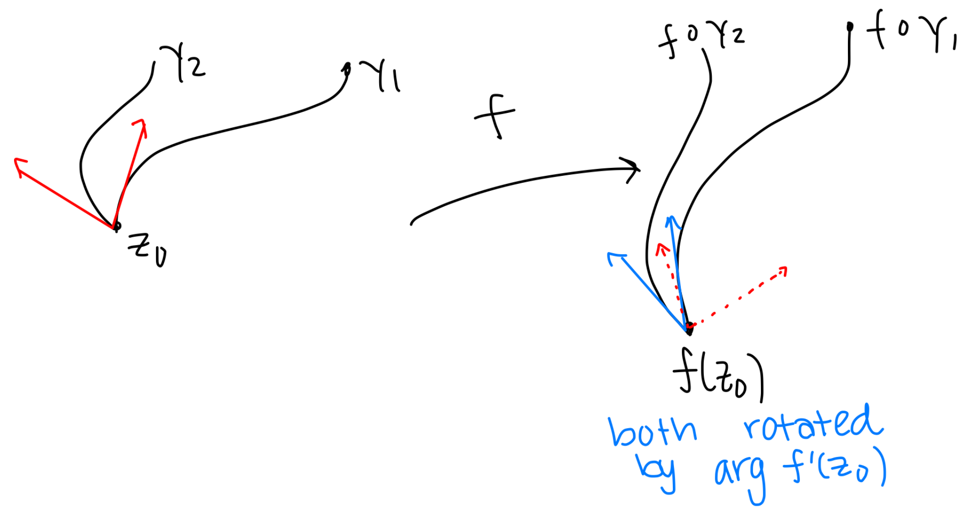
A very important property of complex functions

Theorem 12.6
If $f(z)$ is analytic at z_0 and $f'(z_0) \neq 0$, then $f(z)$ is conformal at z_0.

Proof. Let γ_1, γ_2 both start at z_0. Then by Theorem 12.2,

$$\text{new tangent vectors } \begin{cases} (f \circ \gamma_1)'(0) = f'(z_0) \cdot \gamma_1'(0) \\ (f \circ \gamma_2)'(0) = f'(z_0) \cdot \gamma_2'(0) \end{cases}$$

The original tangent vectors both get multiplied by the same nonzero constant $f'(z_0)$. Thus, the tangent vectors are changed by f by both being rotated by $\arg(f'(z_0))$ (and dilated by $|f'(z_0)|$.)

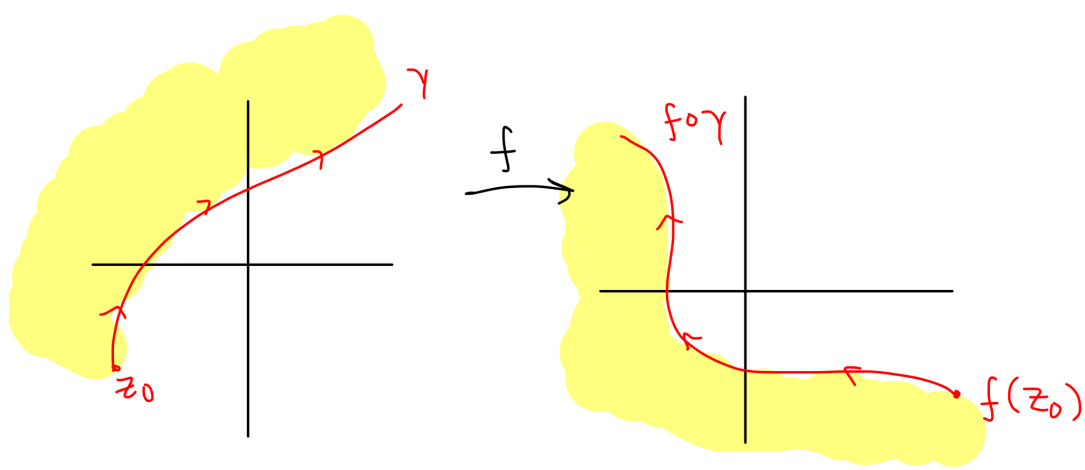


$\square$

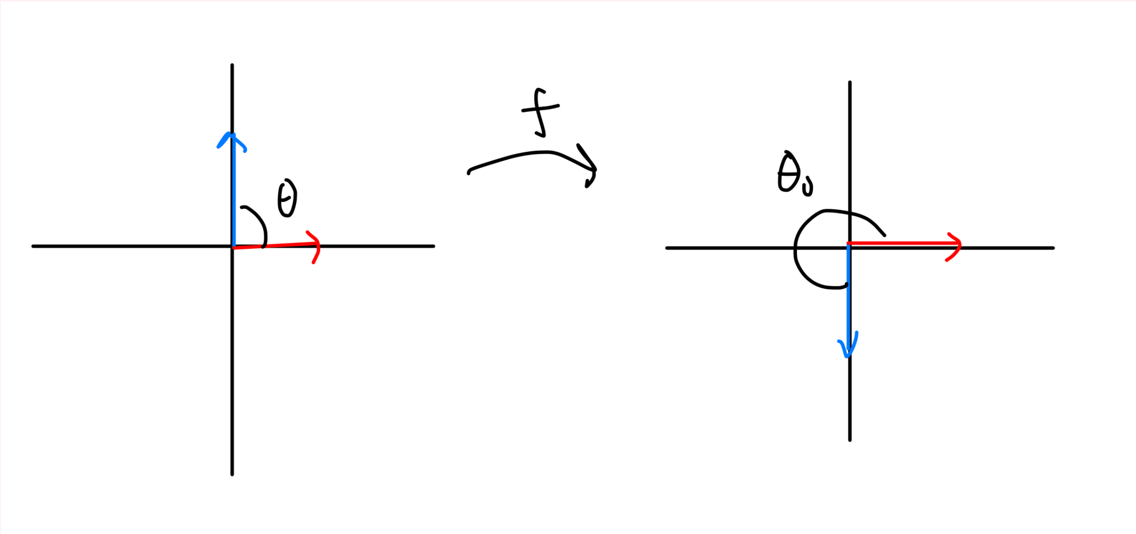
Definition 12.7 (Conformal mapping)
A conformal mapping from one domain D to another domain V is a function $f\colon D \to V$ such that

- f is continuously differentiable on D;
- f is conformal at every point in D;
- f is one-to-one (i.e., $f(z) \neq f(w)$ for distinct $z, w \in D$ and
- f is onto V (i.e. for every $w \in V$ there exists $z \in D$ s.t. $f(z) = w$).

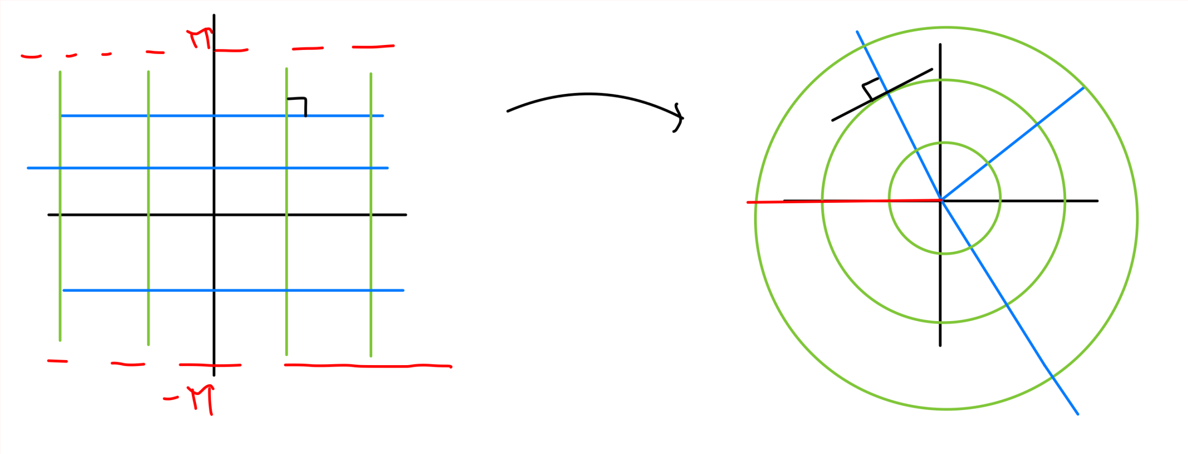
Conformal maps also preserve orientation.



Example 12.8
$f(z) = \overline{z}$ does not preserve orientation.



Example 12.9
Consider the exponential function $f(z) = e^z$.
Everywhere analytic and $f'(z) = e^z \neq 0$. Thus e^z is conformal at every $z_0 \in \mathbb{C}$. The image of e^z is $f(\mathbb{C}) = \mathbb{C} \setminus \{0\}$. But $f\colon \mathbb{C} \to \mathbb{C} \setminus \{0\}$ is not conformal since f is not one-to-one ($e^{z+i2\pi} = e^z$).
However, we can get a conformal map by restricting the domain to a set on which e^z is one-to-one.
e.g. e^z is a conformal map from $\{z \in \mathbb{C} \colon |\operatorname{Im} z| < \pi\}$ to $\mathbb{C} \setminus (-\infty, 0]$.



12.2 Fractional Linear Transformations

Definition 12.10 (Fractional linear transformation)
A fractional linear transformation (or Mörbius transformation) is a function of the form

$$f(z) = \frac{az+b}{cz+d},$$

where a, b, c, d are complex constants such that

$$ad - bc \neq 0.$$

The condition $ad - bc \neq 0$ guarantees that $f(z)$ is not a constant function:

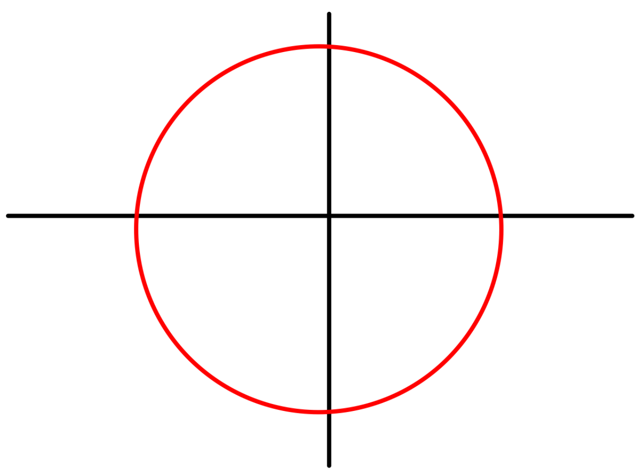
$$f'(z) = \frac{(cz+d)a - (az+b)c}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2} \neq 0$$

when $ad - bc \neq 0$.
Note that different choices of constants may define the same fractional linear transformation: For $\lambda \neq 0$, then

$$\frac{az+b}{cz+d} = \frac{(\lambda a)z + \lambda b}{(\lambda c)z + \lambda d}.$$

Special classes of fractional linear transformations get their own names:

- a fractional linear transformation of the form $f(z) = az + b$ is called an affine transformation.
- a translation is of the form $f(z) = z + b$ and a dilation is of the form $f(z) = az$.
- $f(z) = 1/z$ is a fractional linear transformation, called an inversion.



We can regard a fractional linear transformation $f(z)$ as a map

$$f\colon \mathbb{C}^* \to \mathbb{C}^* \quad (\mathbb{C}^* = \mathbb{C} \cup \{\infty\}).$$

If $f(z) = az + b, a \neq 0$ is affine, then we define

$$f(\infty) = \lim_{z \to \infty} (az + b) = \infty$$

If $f(z) = \frac{az+b}{cz+d}$ is not affine (so that $c \neq 0$), then we can make the following definitions:

$$f\left(\frac{-d}{c}\right) = \infty \quad \frac{a(\frac{-d}{c}) + b}{c(\frac{-d}{c}) + d} \text{``} = \text{''} \frac{-ad + bc}{0}$$

$$f(\infty) = \frac{a}{c} \quad \lim_{z \to \infty} \frac{az + b}{cz + d} = \frac{a}{c}$$

13 Feb 2, 2022

13.1 Fractional Linear Transformations (Cont'd)

Proposition 13.1
The inverse function of a fractional linear transformation

$$w = f(z) = \frac{az+b}{cz+d} \quad (ad - bc \neq 0)$$

exists and is also a fractional linear transformation.

Proof. To invert $w = \frac{az+b}{cz+d}$, just solve for z:

$$\begin{aligned} &\implies w(cz+d) = az+b \\ &\implies cwz + dw = az + b \\ &\implies dw - b = az - cwz = z(a - cw) \\ &\implies z = \frac{dw-b}{-cw+a} = g(w) \end{aligned}$$

Then can check that $g = f^{-1}$, i.e. $g(f(z)) = z = f(g(z))$ for all $z \in \mathbb{C}^*$.
Note: The existence of an inverse $f^{-1} \colon \mathbb{C}^* \to \mathbb{C}^*$ implies that any fractional linear transformation is one-to-one and onto $\mathbb{C}^*$. □

$$\begin{tikzcd} \mathbb{C}^* \arrow[r, "g"] \arrow[rr, bend right, "f \circ g"'] & \mathbb{C}^* \arrow[r, "f"] & \mathbb{C}^* \end{tikzcd}$$

Proposition 13.2
The composition $f \circ g$ of two fractional linear transformations

$$f(z) = \frac{az+b}{cz+d}, \quad g(z) = \frac{\alpha z + \beta}{\gamma z + \delta},$$

is also a fractional linear transformation.

Proof. For any $z \in \mathbb{C}$,

$$\begin{aligned}(f \circ g)(z) = f(g(z)) &= f\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right) \\ &= \frac{a\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right) + b}{c\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right) + d} \cdot \left(\frac{\gamma z + \delta}{\gamma z + \delta}\right) \\ &= \frac{a(\alpha z + \beta) + b(\gamma z + \delta)}{c(\alpha z + \beta) + d(\gamma z + \delta)} \\ &= \frac{(a\alpha + b\gamma)z + (a\beta + b\delta)}{(c\alpha + d\gamma)z + (c\beta + d\delta)}\end{aligned}$$

But have to check that the "determinant" is non-zero:

$$(a\alpha + b\gamma)(c\beta + d\delta) - (a\beta + b\gamma)(c\alpha + d\delta) \neq 0$$

Note that the coefficients of the composition $f \circ g$ can be obtained from matrix multiplication:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{bmatrix}$$

Thus, this has nonzero determinant since it is the product of nonzero determinant matrices. $\square$

The next theorem is a powerful existence and uniqueness result for fractional linear transformations.

Theorem 13.3

Given any three distinct points z_0, z_1, z_2 in $\mathbb{C}^*$, and any three distinct points w_0, w_1, w_2 in $\mathbb{C}^*$, there exists a unique linear fractional transformation $f(z)$ satisfying

$$f(z_0) = w_0, \quad f(z_1) = w_1, \quad f(z_2) = w_2.$$

Proof. We will prove existence: Claim: For any three distinct $z_0, z_1, z_2 \in \mathbb{C}^*$, there is a fractional linear transformation $g(z)$ such that

$$g(z_0) = 0 \quad g(z_1) = 1 \quad g(z_2) = \infty$$

Proof of claim:

- Case 1: None of z_0, z_1, z_2 is ∞. Then check

$$g(z) = \left(\frac{z - z_0}{z - z_2}\right)\left(\frac{z_1 - z_2}{z_1 - z_0}\right)$$

satisfies the claim.

- Case 2: one of the z_i, say z_0, is ∞. We use the idea of the definition in case 1 with

z_0 finite, but take a limit $z_0 \to \infty$:

$$\left(\frac{z-z_0}{z-z_2}\right)\left(\frac{z_1-z_2}{z_1-z_0}\right) = \left(\frac{\frac{z}{z_0}-1}{z-z_2}\right)\left(\frac{z_1-z_2}{\frac{z_1}{z_0}-1}\right) \xrightarrow{z_0\to\infty} \left(\frac{-1}{z-z_2}\right)\left(\frac{z_1-z_2}{-1}\right)$$

Check $g(z) = \frac{z_1-z_2}{z-z_2}$ satisfies the claim for this case. $\square$

Now, we use the claim to prove the theorem. Let $z_0, z_1, z_2 \in \mathbb{C}^*$ be distinct and let $w_0, w_1, w_2 \in \mathbb{C}^*$ be distinct. By claim (applied twice) there are fractional linear transformations g, h such that

$$\begin{aligned} z_0 &\overset{g}{\longmapsto} 0 \\ z_1 &\longmapsto 1 \\ z_2 &\longmapsto \infty, \end{aligned} \qquad\qquad \begin{aligned} w_o &\overset{h}{\longmapsto} 0 \\ w_1 &\longmapsto 1 \\ w_2 &\longmapsto \infty \end{aligned}$$

By previous proposition, h^{-1} is a fractional linear transformation, satisfies

$$\begin{aligned} 0 &\overset{h^{-1}}{\longmapsto} w_0 \\ 1 &\longmapsto w_1 \\ \infty &\longmapsto w_2 \end{aligned}$$

Then by our other proposition, the composition $f = h^{-1} \circ g$ is a fractional linear transformation and satisfies

$$\begin{aligned} z_0 &\overset{g}{\longmapsto} 0 \overset{h^{-1}}{\longmapsto} w_0 \\ z_1 &\longmapsto 1 \longmapsto w_1 \\ z_2 &\longmapsto \infty \longmapsto w_2 \end{aligned}$$

$\square$

Example 13.4
Find a formula for the linear transformation $f(z)$ which maps

$$\begin{aligned} 0 &\mapsto 1 \\ -1 &\mapsto 0 \\ \infty &\mapsto i \end{aligned}$$

We have

$$f(z) = \frac{a(z+1)}{cz+d}$$

We want

$$\begin{aligned} 1 &= f(0) = \frac{a(0+1)}{c \cdot 0 + d} = \frac{a}{d} \\ i &= f(\infty) = \frac{a}{c} \end{aligned}$$

$$\implies \begin{cases} \frac{a}{d} = 1 \implies a = d \\ \frac{a}{c} = i \implies a = ic \end{cases}$$

Try $c = 1$ then $a = i, d = i$. So,

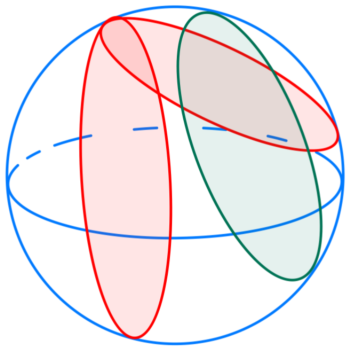
$$f(z) = \frac{i(z+1)}{z+i}$$

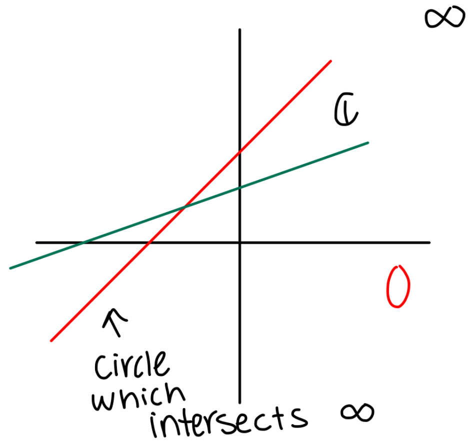
Check:

$$\begin{aligned} f(0) &= \frac{i(0+1)}{0+i} = 1 \\ f(-1) &= \frac{i(-1+i)}{(-1)+i} = 0 \\ f(\infty) &= \frac{i}{1} = i \end{aligned}$$

Theorem 13.5
A fractional linear transformation maps circles in the extended complex plane to circles in the extended complex plane.

Proof. Read text for proof.
Outline:
Every fractional linear transformation is a composition of translations, dilations, inversions. Then just check the translations, dilations, inversions have to satisfy property of the theorem.
Circles in $\mathbb{C}^*$





□

Fractional linear transformations have non-zero derivatives, so they are conformal (preserves angles) at every point in their domain. hence, they also preserve orientation. In particular, orthogonal circles are mapped to orthogonal circles.

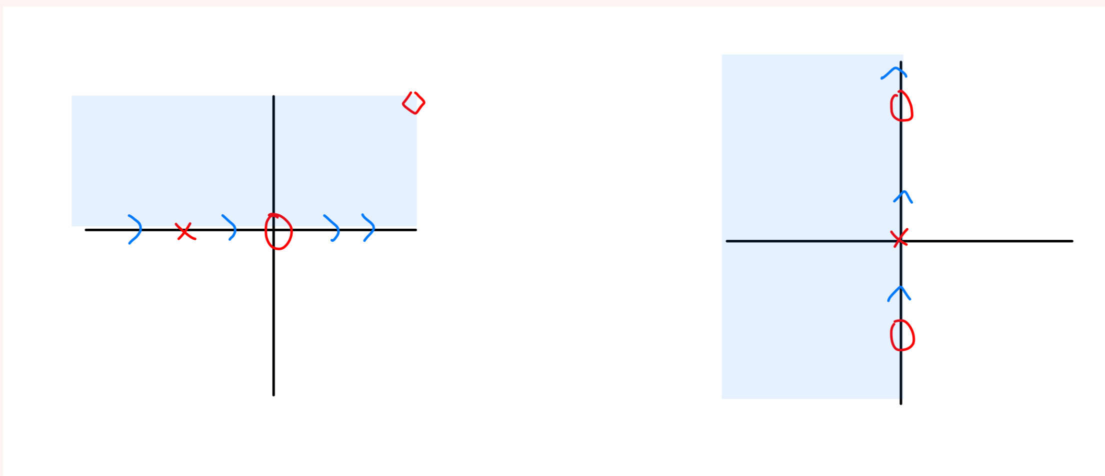
Example 13.6

Consider the linear fractional linear transformation $f(z)$ which maps

$$\begin{aligned} 0 &\mapsto i \\ -1 &\mapsto 0 \\ \infty &\mapsto -i \end{aligned}$$

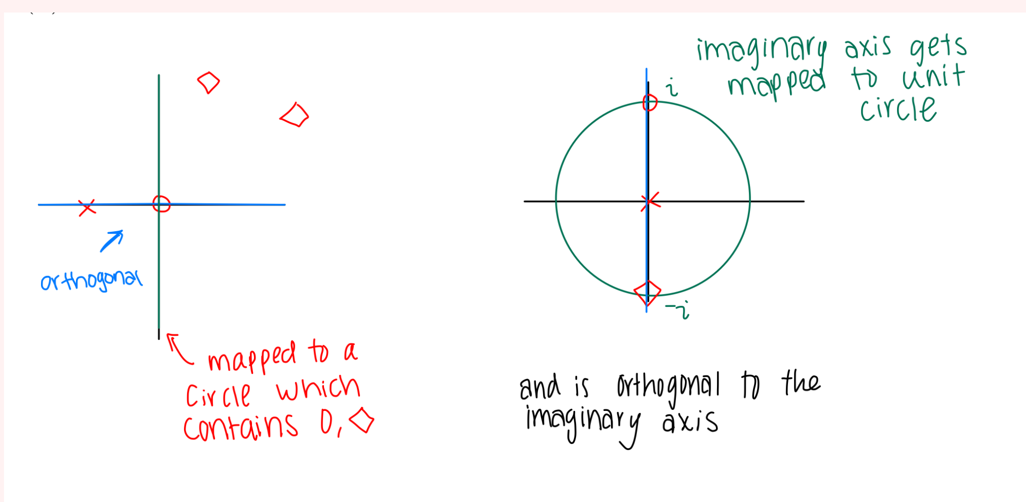
Without finding a formula for $f(z)$, determine where each of the following sets is mapped to under $f(z)$.

(a) The real axis and the upper half-plane $\operatorname{Im} z > 0$.



The circle in $\mathbb{C}^*$ which contains $-1, 0, \infty$ gets mapped to the circle in $\mathbb{C}^*$ which contains $-i, 0, i$, i.e. the imaginary axis. Using that fractional linear transformations preserve orientation, we conclude $\{\operatorname{Im} z > 0\}$ gets mapped to $\{\operatorname{Re} z < 0\}$

(b) The imaginary axis



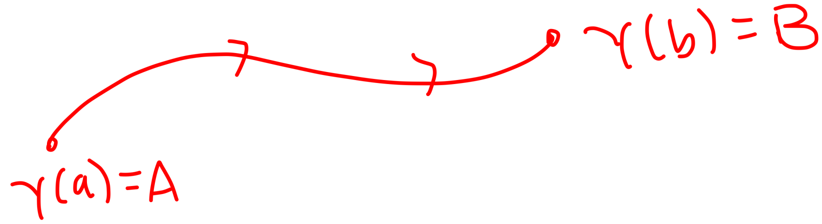
14 Feb 4, 2022

14.1 Review of Basic Definitions

We will study line integrals in the complex plane, but first we (quickly) review the basics of line integrals in the real plane from multivariable calculus.

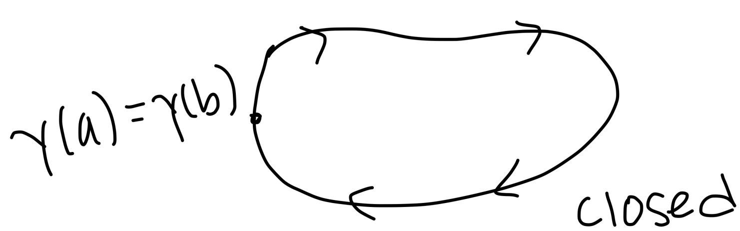
Definition 14.1 (Path)
A <u>path</u> from points A, B in the plane is a continuous function $\gamma(t), a \leq t \leq b$, such that

$$\gamma(a) = A, \quad \gamma(b) = B.$$

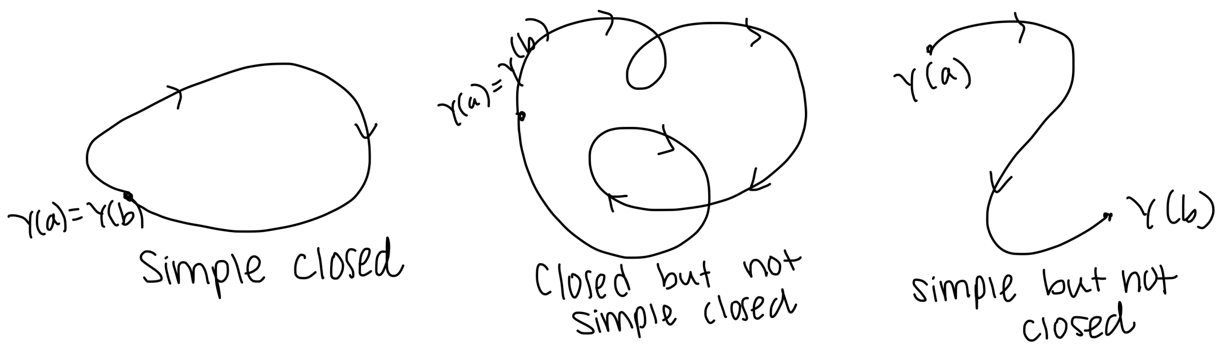


Definition 14.2 (Simple path)
The path $\gamma(t)$ is <u>simple</u> if it is $\gamma(t) \neq \gamma(s)$ when $t \neq s$.

Definition 14.3 (Closed path)
The path $\gamma(t)$ is <u>closed</u> if it starts and ends at the same point, i.e., if $\gamma(a) = \gamma(b)$.

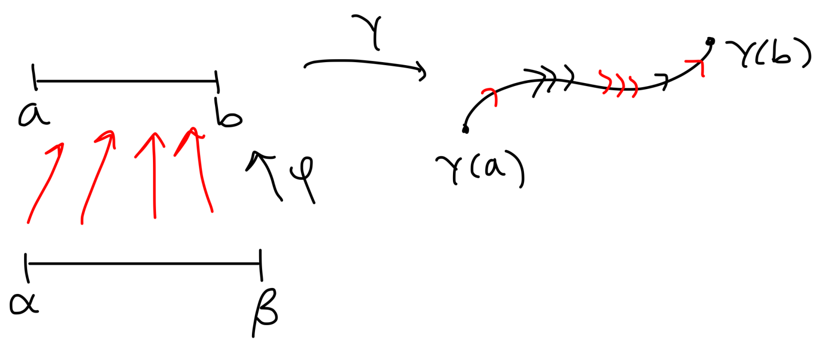


Definition 14.4 (Simple closed path)
The path $\gamma(t)$ is a <u>simple closed path</u> if it is closed and $\gamma(s) \neq \gamma(t)$ for $a \leq s < t < b$.



14.2 Reparametrization

Let $\gamma\colon [a,b] \to \mathbb{R}^2$ be a path, and let $\phi\colon [\alpha,\beta] \to [a,b]$ be strictly increasing function with $\phi(\alpha) = a, \phi(\beta) = b$.



Then the composition $\gamma(\phi(s)), \alpha \le s \le \beta$ is a path from $A = \gamma(a)$ to $B = \gamma(b)$.
We call $\gamma(\phi(s))$ a reparametrization of $\gamma(t)$.

Example 14.5
We have

$$\gamma(t) = (\cos t, \sin t), \quad 0 \le t \le 2\pi$$

If

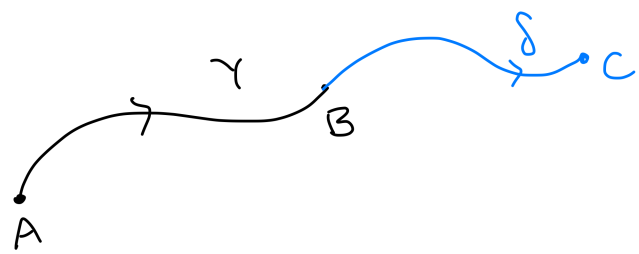
$$\phi(s) = 2s, \quad 0 \le s \le \pi$$

then $\gamma(\phi(s))$ traverses the unit circle once at double the speed of γ.

Note that $\gamma(\phi(s))$ traverses the points on the path in the same order as $\gamma(t)$ does, i.e., the reparametrization is orientation preserving (this is because $\phi(s)$ is increasing.

Definition 14.6 (Trace)
The image of a path, $\gamma([a,b]) = \{\gamma(t)\colon a \le t \le b\}$, is called its trace.

Often, we will be a bit sloppy and use γ to denote both the path and its trace.
If one path $\gamma(t)$ ends where another path $\delta(t)$ begins, then we can concatenate them into a single path.



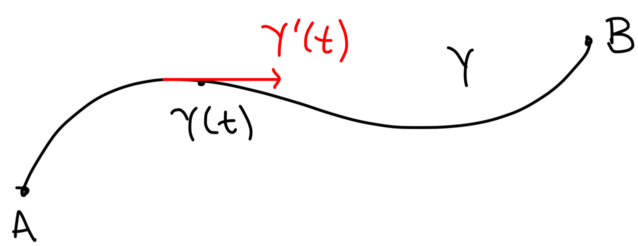
The operation of concatenation is well-defined up to the choice of parametrization of the new path.

Definition 14.7 (Smooth path)
A path $\gamma(t)$ can be represented with coordinate functions

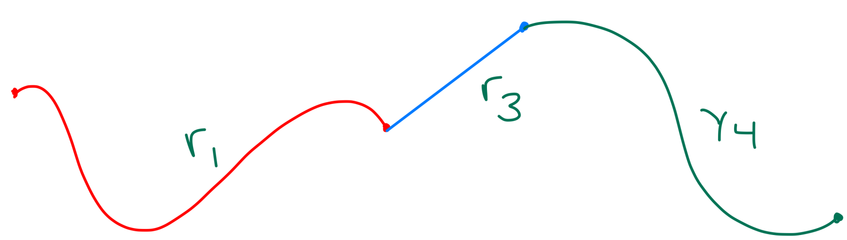
$$\gamma(t) = (x(t), y(t))$$

$\gamma(t)$ is smooth if both $x(t), y(t)$ are smooth, i.e., infinitely differentiable.

Of course, this implies that we can take derivatives of the path itself, e.g., $\gamma'(t) = (x'(t), y'(t))$ (tangent vector).



Definition 14.8 (Piecewise smooth path)
A piecewise smooth path is a concatenation of finitely many smooth paths. By a curve we (usually) mean a smooth or piecewise smooth path.

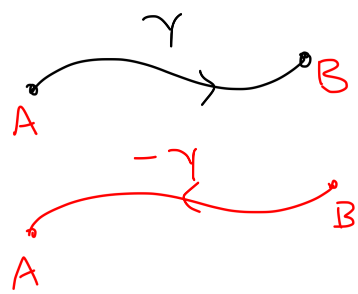


14.3 Line Integrals of Differential Forms

Definition 14.9 (Line integral)
Let $\gamma(t) = (x(t), y(t))$ be a smooth path from A to B, and let $P(x, y)$ and $Q(x, y)$ be continuous functions which are defined on all the points of the trace of γ. The line integral of $Pdx + Qdy$ along γ can be computed by the equation

$$\int_\gamma Pdx + Qdy = \int_a^b P(x(t), y(t))\frac{dx}{dt}dt + \int_a^b Q(x(t), y(t))\frac{dy}{dt}dt \tag{*}$$

Fact: The value of the line integral does not change if we use a reparametrization $\gamma(\phi(s))$ of γ. It does, however, change by a minus sign if we reverse the orientation of γ.



$$\int_{-\gamma} Pdx + Qdy = -\int_\gamma Pdx + Qdy$$

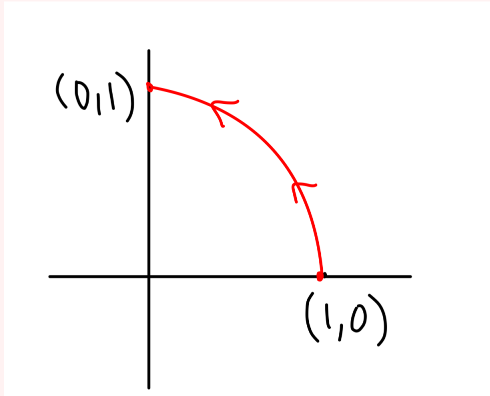
If δ is the concatenation of smooth paths $\gamma_1, \ldots, \gamma_n$, then

$$\int_\delta Pdx + Qdy = \sum_{i=1}^n \int_{\gamma_i} (Pdx + Qdy),$$

where each of the summand integrals is evaluated using the formula in (*).

Example 14.10

Evaluate $\int_\gamma xydx + dy$, where γ is the quarter unit circle from (1,0) to (0,1).



First, we need to parametrize γ.

$$\gamma(t) = (\cos t, \sin t), \quad 0 \le t \le \frac{\pi}{2}$$

So

$$\frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = \cos t$$

Therefore,

$$\begin{aligned}
\int_\gamma xydx + dy &= \int_0^{\frac{\pi}{2}} \cos t \cdot \sin t \cdot \frac{dx}{dt}dt + \int_0^{\frac{\pi}{2}} \frac{dy}{dt}dt \\
&= \int_0^{\frac{\pi}{2}} -\cos t \sin^2 tdt + \int_0^{\frac{\pi}{2}} \cos tdt \\
&= \left[-\frac{\sin^3 t}{3}\right]_0^{\frac{\pi}{2}} + [\sin t]_0^{\frac{\pi}{2}} \\
&= \frac{-1}{3} + 1 = \frac{2}{3}
\end{aligned}$$

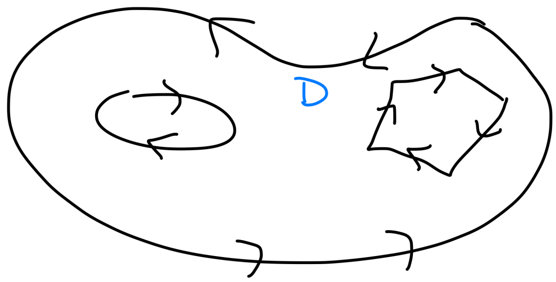
14.4 Green's Theorem

Green's theorem gives us a very useful way to evaluate line integrals. The theorem shows that, in certain cases, the line integral is equal to the area integral.

Theorem 14.11 (Green's Theorem)
Let D be a bounded domain in the plane whose boundary ∂D consists of a finite number of disjoint piecewise smooth pieces. Let P and Q be continuously differentiable functions on $D \cup \partial D$. Then,

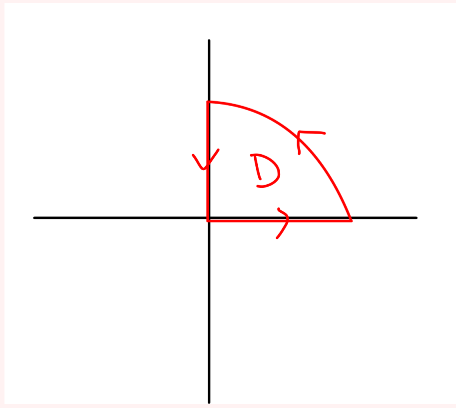
$$\int_{\partial D} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

where in the line integral the boundary ∂D is traversed positively (as you traverse the path, D should always be on your left).



Example 14.12

Compute $\int_{\partial D} xydx + dy$, where D is the quarter disk in the first quadrant.



$$P(x, y) = xy$$
$$Q(x, y) = 1$$

Therefore,

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 - x$$

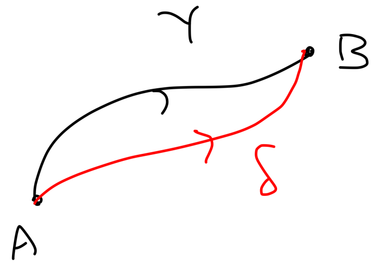
By Green's theorem,

$$\int_{\partial D} xydx + dy = \iint_D -xdxdy$$

to evaluate, change to polar coordinates.

$$\begin{aligned} &= \int_0^{\frac{\pi}{2}} \int_0^1 -r\cos\theta r dr d\theta \\ &= -\int_0^{\frac{\pi}{2}} \cos\theta d\theta \int_0^1 r^2 dr \\ &= -[\sin\theta]_0^{\frac{\pi}{2}} \left[\frac{r^3}{3}\right]_0^1 \\ &= -\frac{1}{3} \end{aligned}$$

14.5 Independence of Path



Definition 14.13 (Differential)
Let $h(x,y)$ be a continuously differentiable function. The differential of h is the differentiable form

$$dh = \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy.$$

We have the following version of the fundamental theorem of calculus for line integrals of differentials:

Theorem 14.14
If γ is a piecewise smooth path from A to B and $h(x,y)$ is continuously differentiable on the trace of γ, then

$$\int_\gamma dh = h(B) - h(A).$$

Thus, the value of $\int_\gamma dh$ only depends on the endpoints A, B. So, the line integral of dh is independent of the path in the sense that if δ is another path from A to B, then

$$\int_\delta dh = \int_\gamma dh = h(B) - h(A).$$

What about other differentiable forms $Pdx + Qdy$ with this property?

14.6 Exact Forms

Definition 14.15 (Exact form)
A differential form $Pdx+Qdy$ is exact if there exists a continuously differentiable function $h(x,y)$ such that

$$Pdx + Qdy = dh = \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy.$$

It turns out that the exact differential forms are precisely the ones show line integrals have path independence:

Theorem 14.16
Let P, Q be continuous functions on a domain D. Then, $\int Pdx + Qdy$ is independent of path in D if and only if $Pdx + Qdy$ is exact.

14.7 Closed Forms

Definition 14.17 (Closed form)
Let P and Q be continuously differentiable on a domain D. The differential form $Pdx + Qdy$ is closed if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{on } D.$$

Proposition 14.18
Exact differential forms are closed.

Proof. Consider an exact form

$$Pdx + Qdy = \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy$$

Then, we check that $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$:

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\frac{\partial h}{\partial x} = \frac{\partial}{\partial x}\frac{\partial h}{\partial y} = \frac{\partial}{\partial x}Q$$

Note: if you apply Green's theorem to a closed form, then

$$\int_{\partial D} Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \iint_D 0dxdy = 0$$

□

In summary, for a general domain D,

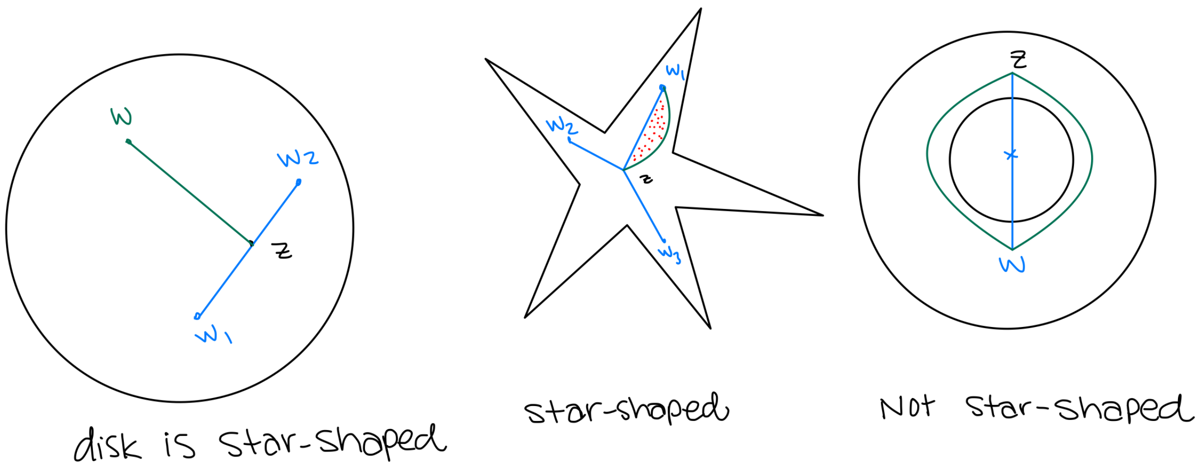
$$\text{(independent of path)} \iff \text{exact} \implies \text{closed.}$$

It turns out that the " $\implies$ " becomes a " $\iff$ " for certain types of domains.

15 Feb 7, 2022

15.1 Star-Shaped Domains

Definition 15.1 (Star-shaped domain)
A domain D is star-shaped if there is a point $z \in D$ such that for any $w \in D$, the straight line connecting z and w is completely contained in D.



Theorem 15.2
For a star-shaped domain D, a differential form in D is

$$\text{(independent of path)} \iff \text{exact} \iff \text{closed}$$

So to check if $Pdx + Qdy$ is exact in a star shaped D, only need to verify

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \text{ on } D$$

which is much easier than trying to solve

$$\begin{cases} \dfrac{\partial h}{\partial x} = P \\ \dfrac{\partial h}{\partial y} = Q. \end{cases} \text{ in } D$$

15.2 Harmonic Conjugates

We can apply the theorems from the previous section to get a new theorem on the existence of harmonic conjugates on star-shaped domains.

Lemma 15.3
If $u(x, y)$ is harmonic on a domain D, then the differential form

$$-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

is closed.

Proof. u harmonic means

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{*}$$

on D. Now, we show that

$$Pdx + Qdy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

is closed:

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2} \overset{(*)}{=} \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial Q}{\partial x}$$

Thus, it is closed. □

Theorem 15.4
Any harmonic function $u(x, y)$ on a star-shaped domain D has a harmonic conjugate $v(x, y)$ on D.

Proof. Since $u(x, y)$ is harmonic, it follows from Lemma 15.3 that

$$-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

is closed. Since D is star-shaped, closed implies exact, so there is a smooth $v(x, y)$ such that

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

But also

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$\implies \begin{cases} -\dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \end{cases}$$

So $u + iv$ is analytic on D since they satisfy the Cauchy-Riemann equations on D. □

15.3 Complex Line Integrals

First, we discuss how to integrate a continuous function of the form

$$\phi\colon \mathbb{R} \to \mathbb{C}, \quad \phi(t) = u(t) + iv(t).$$

The integral of ϕ from a to b is defined to be

$$\int_a^b \phi(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt. \tag{*}$$

Definition 15.5 (Line integral)
Now, consider a function $f(z)$ on some domain $D \subseteq \mathbb{C}$ and a smooth path $\gamma(t) = x(t) + iy(t), a \leq t \leq b$ in D. Recall that

$$\gamma'(t) = x'(t) + iy'(t).$$

The line integral of $f(z)$ along γ is

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

Note that the RHS integral is an integral of type (*) above, where $\phi(t) = f(\gamma(t))\gamma'(t)$.

Example 15.6
Let γ be the unit circle traversed counter-clockwise. Compute the line integral

$$\int_\gamma \frac{1}{z}dz.$$

Notation:

$$\int_\gamma \frac{1}{z}dz = \oint_{|z|=1} \frac{1}{z}dz.$$

We parametrize γ by

$$\gamma(t) = \cos(t) + i\sin(t) = e^{it}, \quad 0 \leq t \leq 2\pi$$

Then,

$$\begin{aligned}\oint_{|z|=1} \frac{1}{z}dz &= \int_0^{2\pi} \frac{1}{\gamma(t)}\gamma'(t)dt \\ &= \int_0^{2\pi} \frac{1}{e^{it}} ie^{it}dt \\ &= \int_0^{2\pi} idt = 2\pi i\end{aligned}$$

Note that if we write $dz = dx + idy$, then we can think of $\int_\gamma f(z)dz$ as a line integral of the differential form $f(z)dx + if(z)$ since

$$\begin{aligned}\int_\gamma f(z)dx + if(z)dy &= \int_a^b f(\gamma(t))x'(t)dt + if(\gamma(t))y'(t)dt \\ &= \int_a^b f(\gamma(t))(x'(t) + iy'(t))dt \\ &= \int_a^b f(\gamma(t))\gamma'(t)dt\end{aligned}$$

agrees with out previous definition of $\int_\gamma f(z)dz$.

Example 15.7 (Important Example)
Fix an integral m and $R > 0$, and let γ be the circle centered at z_o with radius R traversed counter-clockwise. Then, we will show

$$\int_\gamma (z - z_0)^m dz = \begin{cases} 0 & \text{if } m \neq -1 \\ 2\pi i & \text{if } m = -1. \end{cases}$$

So,

$$\gamma(t) = Re^{it} + z_0 \quad 0 \leq t \leq 2\pi$$

Then,

$$\begin{aligned}\int_\gamma (z - z_0)^m dz &= \int_0^{2\pi} (Re^{it} + z_0 - z_0)^m iRe^{it} dt \\ &= \int_0^{2\pi} R^m e^{imt} iRe^{it} dt \\ &= iR^{m+1} \int_0^{2\pi} e^{it(m+1)} dt \end{aligned} \tag{*}$$

<u>Case 1:</u> $m \neq -1$. Then, $m + 1 \neq 0$, so

$$(*) = iR^{m+1} \frac{e^{it(m+1)}}{i(m+1)}\Bigg|_{t=0}^{2\pi} = \frac{R^{m+1}}{m+1}\left(e^{i(m+1)2\pi} - e^0\right) = 0$$

<u>Case 2:</u> $m = -1$. Then $m + 1 = 0$, so

$$(*) = iR^0 \int_0^{2\pi} e^{it\cdot 0} dt = i\int_0^{2\pi} dt = 2\pi i$$

15.4 Arclength

We denote infinitesimal arclength ds by $|dz|$,

$$|dz| = ds = \sqrt{(dx)^2 + (dy)^2}.$$

Then, if $\gamma(t) = x(t) + iy(t)$, then

$$\int_\gamma h(z)|dz| = \int_a^b h(\gamma(t))\sqrt{(x'(t))^2 + (y'(t))^2} = \int_a^b h(\gamma(t))|\gamma'(t)|dt$$

In particular, the arclength of (the trace of) γ is

$$L = \int_\gamma ds = \int_\gamma |dz|$$

Example 15.8
We can use the arclength formula to compute the circumference of a circle: We have

$$\gamma(t) = Re^{it}, \quad 0 \le t \le 2\pi$$

Then,

$$\begin{aligned} L &= \int_0^{2\pi} |\gamma'(t)|dt = \int_0^{2\pi} \left|Rie^{it}\right|dt \\ &= \int_0^{2\pi} |R| \cdot |i| \cdot |e^{it}|dt = \int_0^{2\pi} Rdt = 2\pi R. \end{aligned}$$

For $R > 0$.

Theorem 15.9 (ML estimate)
Suppose γ is a piecewise smooth curve, and let $h(z)$ be a continuous function on the trace of γ. Then,

$$\left|\int_\gamma h(z)dz\right| \le \int_\gamma |h(z)||dz|.$$

Moreover, if γ has arclength L and $|h(z)| \le M$ on γ, then

$$\left|\int_\gamma h(z)dz\right| \le ML.$$

Example 15.10

Let γ be the straight line from 0 to $2+i$. Use the ML estimate to find an upper bound for

$$\left|\int_\gamma z^3 dz\right|.$$

First, find L, the arc length of γ

$$L = |2+i| = \sqrt{5}$$

Next, we find a bound M for $|z^3|$ on γ.

$$|z^3| = |z|^3$$

and the cube function is increasing, so larger $|z|$ gives larger $|z|^3$. Thus the largest value $|z^3|$ takes on γ is

$$M = |(2+i)^3| = |2+i|^3 = (\sqrt{5})^3 = 5^{3/2}$$

Thus,

$$\left|\int_\gamma z^3 dz\right| \leq ML = 5^{3/2} \cdot 5^{1/2} = 5^2 = 25.$$

Definition 15.11 (Sharp/not sharp)

If

$$\left|\int_\gamma h(z)dz\right| < ML,$$

we say that the ML estimate is <u>not sharp</u> for $\int_\gamma h(z)dz$.

If

$$\left|\int_\gamma h(z)dz\right| = ML,$$

then the ML estimate is <u>sharp</u>. (Still does not tell you the exact value of $\int_\gamma h(z)dz$.)

Example 15.12
We know

$$\oint_{|z|=1} \frac{1}{z} dz = 2\pi i$$

Using ML estimate,

$$L = 2\pi, \quad \left|\frac{1}{z}\right| = \frac{1}{|z|} = 1$$

for all z on unit circle. So ML estimate gives

$$\left|\oint_{|z|=1} \frac{1}{z} dz\right| \leq M \cdot L = 2\pi$$

So the estimate is sharp since

$$\left|\oint_{|z|=1} \frac{1}{z} dz\right| = |2\pi i| = 2\pi$$

16 Feb 9, 2022

16.1 Fundamental Theorem of Calculus for Analytic Functions

Definition 16.1 (Complex primitive)
Let $f(z)$ be a continuous function on D. A function $F(z)$ on D is a (complex) primitive for $f(z)$ if

- $F(z)$ is analytic on D, and
- $F'(z) = f(z)$ for all $z \in D$.

Example 16.2
Consider the function $1/z$.
$\frac{1}{z}$ has $F(z) = \operatorname{Log} z$ as a primitive on $D = \mathbb{C} \setminus (-\infty, 0]$. $\operatorname{Log} z$ is analytic on D and

$$\frac{d}{dz} \operatorname{Log} z = \frac{1}{z}.$$

Does $\frac{1}{z}$ have a primitive on $\mathbb{C} \setminus \{0\}$? Seems like the answer is no, and we will be able to prove this later.

Theorem 16.3 (FTC Part I)
If $f(z)$ is continuous on D, $F(z)$ is a primitive for $f(z)$ on D, and A and B are points in D, then

$$\int_A^B f(z)dz = F(B) - F(A),$$

where the line integral can be taken over any path in D from A to B. In particular, this line integral has path independence in D.

$$\text{(Primitive on } D \implies \text{ path independence on } D.)$$

Proof. Recall that for line integrals for $dh = \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy$,

$$\int_\gamma dh = h(A) - h(B).$$

Write $F = u + iv$. Then

$$\frac{\partial F}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = F'(z)$$

and

$$\frac{\partial F}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = i\left(-\frac{\partial v}{\partial y} + i\frac{\partial u}{\partial y}\right) = iF'(z)$$

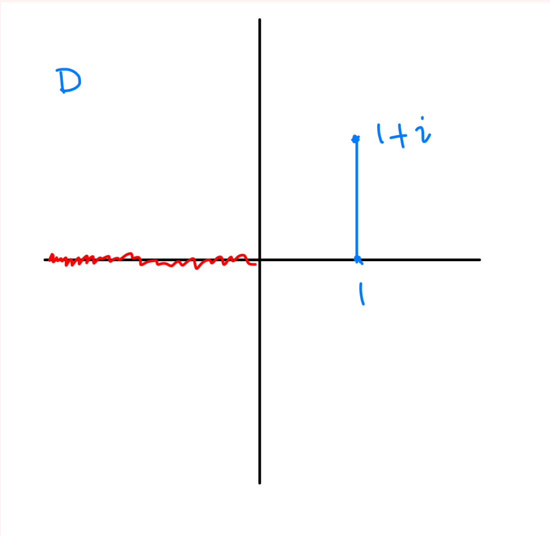
Thus,

$$\begin{aligned} F(B) - F(A) = \int_A^B df &= \int_A^B \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy \\ &= \int_A^B F'(z) dx + i F'(z) dy \\ &= \int_A^B F'(z)(dx + i dy) = \int_A^B F'(z) dz. \end{aligned}$$

□

Example 16.4
We evaluate

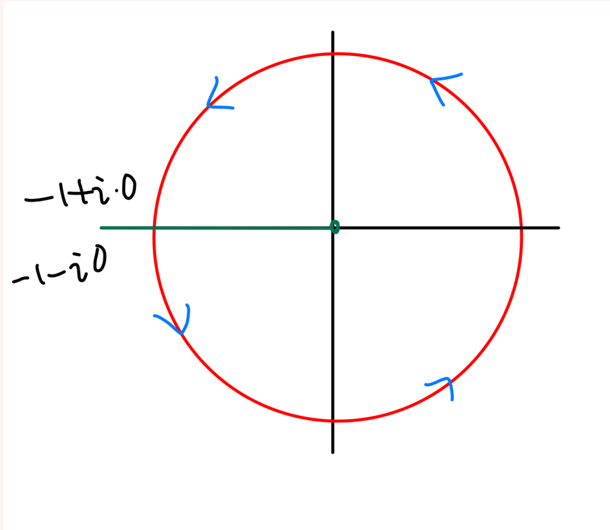
$$\int_1^{1+i} \frac{dz}{z}.$$



$\operatorname{Log} z$ is a primitive on D.
$\gamma =$ straight line from 1 to $1+i$ is contained in D. Thus, by FTC

$$\begin{aligned} \int_1^{1+i} \frac{dz}{z} &= \operatorname{Log} z \Big|_{z=1}^{1+i} \\ &= \operatorname{Log}(1+i) - \operatorname{Log} 1 \\ &= \operatorname{Log}(1+i) = \log \sqrt{2} + i \operatorname{Arg}(1+i) \\ &= \log \sqrt{2} + \frac{\pi i}{4}. \end{aligned}$$

Example 16.5
We can also evaluate
$$\oint_{|z|=1} \frac{dz}{z}.$$



There is a primitive for $\frac{1}{z}$ on a D which contains all of the unit circle. But we can still use FTC to evaluate this integral. $\operatorname{Log} z$ is primitive on $\mathbb{C} \setminus (-\infty, 0]$. Pick a point $-1+i0$ which is on the unit circle, slightly above -1 and $-1-i0$, a point on circle slightly below -1. The path from $-1-i0$ to $-1+i0$ along the unit circle is in D. So by FTC,
$$\int_{-1-i0}^{-1+i0} \frac{dz}{z} = \operatorname{Log}(-1+i0) - \operatorname{Log}(-1-i0)$$
Now, take a limit as
$$-1+i0 \to -1 \text{ from above}$$
$$-1-i0 \to -1 \text{ from below}$$
So,
$$\oint_{|z|=1} \frac{dz}{z} = \lim \Big(\operatorname{Log}(-1+i0) - \operatorname{Log}(-1-i0) \Big) = i\pi - (-i\pi) = 2\pi i$$

Theorem 16.6 (FTC Part II)
Let D be a star-shaped domain, and let $f'(z)$ be analytic on D. Then, $f(z)$ has a primitive on D, which is unique up to adding a constant. Moreover, a primitive for $f(z)$ is given explicitly by
$$F(z) = \int_{z_0}^{z} f(w)dw,$$
where z_0 is any fixed point in D and the line integral can be taken along any path in D from z_0 to z.

Proof. Write $f = u+iv$, which is analytic on D. Consider the differential form $udx-vdy$. Since f is analytic,

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(-v),$$

which shows $udx - vdy$ is closed. Since D is star-shaped, $udx - vdy$ closed implies that it is exact, so, there is a continuously differentiable U on D such that

$$dU = udx - vdy \implies \begin{cases} \dfrac{\partial U}{\partial x} = u \\ \dfrac{\partial U}{\partial y} = -v \end{cases}$$

Then, we can check that U is harmonic on D:

$$\begin{aligned} \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} &= \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial U}{\partial y}\right) \\ &= \frac{\partial u}{\partial x} + \frac{\partial}{\partial y}(-v) \\ &= \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \\ &= 0 \text{ by Cauchy-Riemann} \end{aligned}$$

Since U is harmonic on D and D is star-shaped, it follows that U has a harmonic conjugate V on D. So, $G = U + iV$ is analytic on D and

$$G' = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x}$$

so applying Cauchy-Riemann to G,

$$= \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y} = u - i(-v) = u + iv = f$$

So, G is indeed a primitive for f on D.
<u>Unique up to adding a constant:</u>
If $G_0(z)$ is another primitive for f on D, then

$$(G - G_0)' = G' - G_0' = f - f = 0$$

hence

$$\begin{gathered} G - G_0 = C, \quad C \in \mathbb{C} \\ G = G_0 + C \end{gathered}$$

For our explicit primitive $F(z)$, fix $z_0 \in D$ and define $F(z) = G(z) - G(z_0)$, and

$$\int_{z_0}^{z} f(w)dw = G(z) - G(z_0) = F(z)$$

□

Corollary 16.7
The line integrals of analytic functions in star-shaped domains are independent of path.

$$\begin{aligned}\text{analytic star-shaped } D &\overset{\text{FTC II}}{\Longrightarrow} \text{have primitive on } D\\ &\overset{\text{FTC I}}{\Longrightarrow} \text{independent of path on } D\end{aligned}$$

Note 16.8: If f has a primitive F in D, γ is a closed path ($A = B$) in D, then

$$\int_\gamma f(z)dz = 0.$$

We can use the above results to show that $1/z$ does not have a primitive in the punctured plane $\mathbb{C} \setminus \{0\}$.

Suppose by contradiction that $F(z)$ is a primitive for $1/z$ on $\mathbb{C} \setminus \{0\}$. Then,

$$\oint_{|z|=1} \frac{1}{z} dz = F(-1) - F(-1) = 0.$$

However, this is not the case since we already showed

$$\oint_{|z|=1} \frac{dz}{z} = 2\pi i \neq 0.$$

□

16.2 Cauchy's Theorem

Let $f = u + iv$ be a smooth-valued function. We express $f(z)dz$ as a differential form in the plane:

$$f(z)dz = (u+iv)(dx+idy) = \underbrace{(u+iv)}_{P=u+iv} dx + \underbrace{(-v+iu)}_{Q=-v+iu} dy$$

Since f is analytic, we can show that the differential form $f(z)dz$ is closed:

$$\frac{\partial P}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} \underset{\text{C-R}}{=} -\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(-v+iu) = \frac{\partial}{\partial x}Q.$$

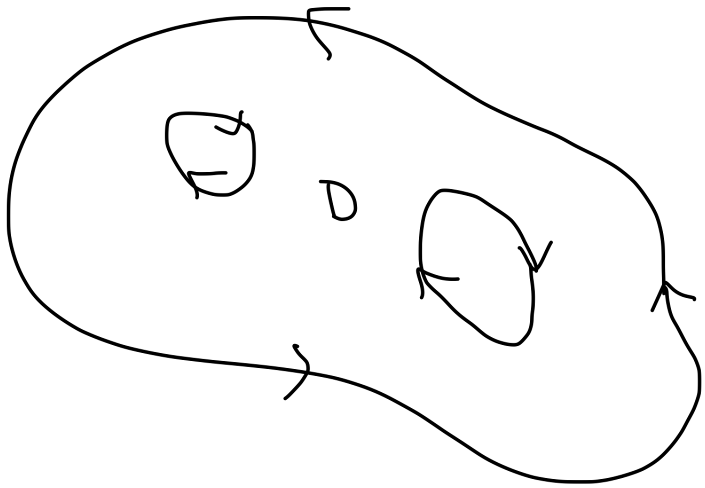
Conversely, if $f(z)dz$ is closed, then f is analytic since it satisfies the Cauchy-Riemann equations:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \implies f \text{ satisfies C-R equations.}$$

Thus, we have established

Theorem 16.9
A continuously differentiable function $f(z)$ is analytic if and only if the differential form $f(z)dz$ is closed.

Recall the consequence of Green's theorem: any line integral of a closed differential form along the boundary domain D with piecewise smooth boundary is equal to 0.



If $Pdx + Qdy$ is closed, then

$$\begin{aligned}\int_{\partial D} Pdx + Qdy &\underset{\text{Green's}}{=} \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy \\ &= \iint_D 0 dxdy = 0.\end{aligned}$$

17 Feb 11, 2022

17.1 Cauchy's Theorem (Cont'd)

Thus, from Green's theorem we get:

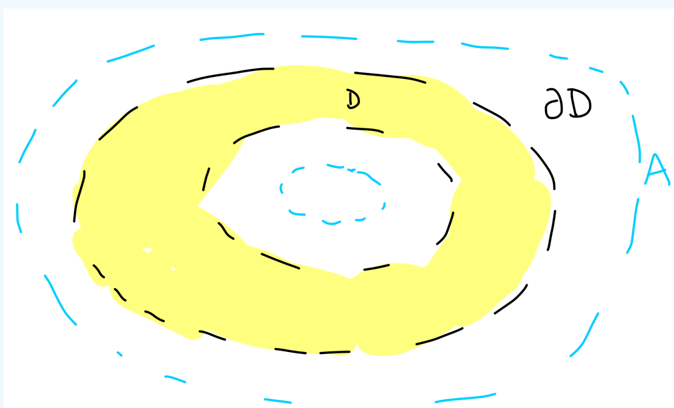
Theorem 17.1 (Cauchy's Theorem)
Let D be a bounded domain with a piecewise smooth boundary. If $f(z)$ is an analytic function on D which extends smoothly to ∂D, then

$$\int_{\partial D} f(z)dz = 0.$$

A remark on "extends smoothly to ∂D":

Definition 17.2 (Extends smoothly to ∂D)
$D \cup \partial D$ is not an open set. When we say "f extends smoothly to ∂D" we mean there is another domain A such that

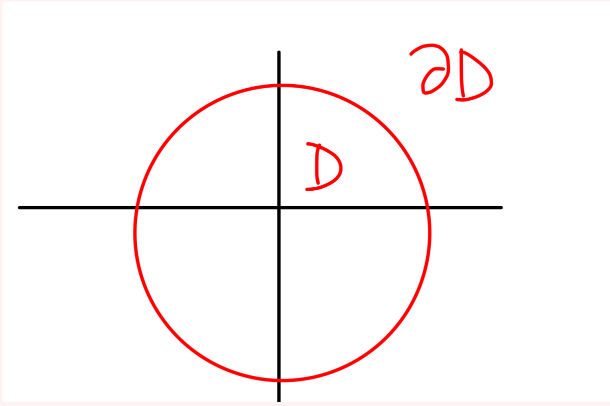
1. $D \cup \partial D \subseteq A$ and
2. there is an extension of f which is analytic on A.



Example 17.3

Compute

$$\oint_{|z|=1} z^3 dz.$$

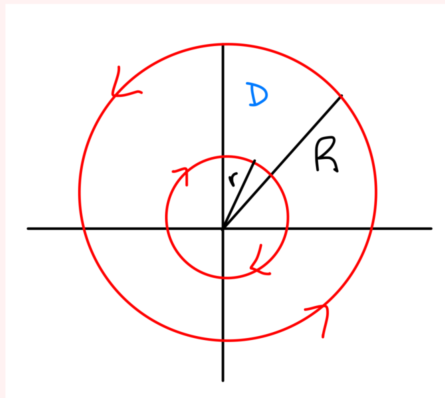


z^3 extends smoothly to ∂D since it is analytic on all of $\mathbb{C}$ and it is analytic on D. Thus, by Cauchy's Theorem,

$$\int_{|z|=1} z^3 dz = 0.$$

Example 17.4

Let D be the boundary of the annulus $D = \{r < |r| < R\}$, and suppose $f(z)$ is analytic on D and extends smoothly to the boundary ∂D.



By Cauchy's Theorem,

$$\int_{\partial D} f(z)dz = 0.$$

On the other hand since ∂D consists of

- $|z| = R$ traversed counterclockwise
- $|z| = r$ traversed clockwise,

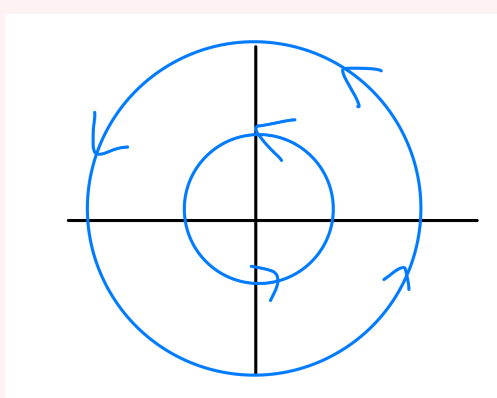
so we have

$$\int_{\partial D} f(z)dz = \int_{|z|=R} f(z)dz - \int_{|z|=r} f(z)dz$$

(The integrals are subtracted because of reversed orientation)

$$\implies 0 = \int_{|z|=R} f(z)dz - \int_{|z|=r} f(z)dz$$

$$\implies \int_{|z|=R} f(z)dz = \int_{|z|=r} f(z)dz$$



17.2 The Cauchy Integral Formula

The Cauchy integral formula represents the value of an analytic function as an integral. This is often helpful since we can use integration theory (ML-estimates, fundamental theorem of calculus, etc.) to better understand the original function.

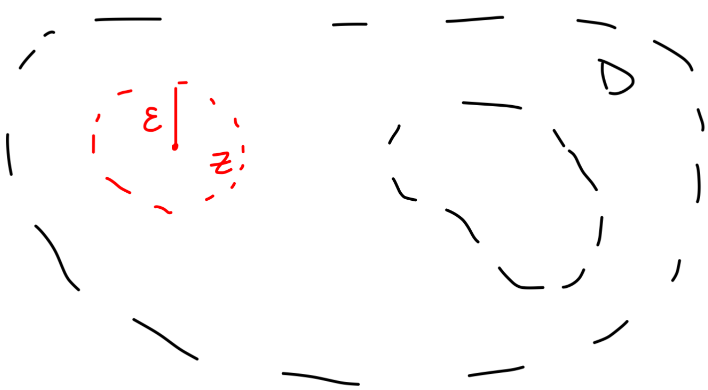
Theorem 17.5 (Cauchy Integral Formula)
Let D be a bounded domain with piecewise smooth boundary. If $f(z)$ is analytic on D and it extends smoothly to ∂D, then

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-z}dw. \quad (z \in D)$$

Proof. Fix $z \in D$. Fix $\varepsilon > 0$ small enough so that

$$\{|w-z| \leq \varepsilon\} \subseteq D$$

Let $D_\varepsilon = D \setminus \{|w-z| \leq \varepsilon\}$.



Note that

$$\partial D_\varepsilon = \partial D \cup \{|w-z| = \varepsilon\},$$

and note that $\dfrac{f(w)}{w-z}$ is analytic on $D_\varepsilon \subseteq D$ and extends smoothly to ∂D_ε. So, by Cauchy's theorem,

$$\int_{\partial D_\varepsilon} \frac{f(w)}{w-z}dw = 0.$$

But also,

$$0 = \int_{\partial D_\varepsilon} \frac{f(w)}{w-z}dw = \int_{\partial D} \frac{f(w)}{w-z}dw - \oint_{|w-z|=\varepsilon} \frac{f(w)}{w-z}dw$$

$$\implies \int_{\partial D} \frac{f(w)}{w-z}dw = \oint_{|w-z|=\varepsilon} \frac{f(w)}{w-z}dw \overset{\text{WTS}}{=} 2\pi i f(z)$$

In other words, we want to show

$$f(z) = \frac{1}{2\pi i} \oint_{|w-z|=\varepsilon} \frac{f(w)}{w-z} dw.$$

We want to parametrize $|w-z| = \varepsilon$ with $\gamma(t) = \varepsilon e^{it} + z, 0 \le t \le 2\pi$

$$\begin{aligned}
\frac{1}{2\pi i} \oint_{|w-z|=\varepsilon} \frac{f(w)}{w-z} dw &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z+\varepsilon e^{it})}{\varepsilon e^{it}} \varepsilon i e^{it} dt \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(z+\varepsilon e^{it}) dt \\
&= \text{ average value of } f \text{ on } \{|w-z| = \varepsilon\}
\end{aligned}$$

But since f is analytic at z, it is continuous at z. Thus, as $\varepsilon \to 0$, the average value of f on $\{|w-z| = \varepsilon\}$ tends to $f(z)$. Thus, after taking a limit $\varepsilon \to 0$,

$$\frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w-z} dw = \frac{1}{2\pi i} \oint_{|w-z|=\varepsilon} \frac{f(w)}{w-z} dw \underset{\text{on } \varepsilon \to 0}{\longrightarrow} f(z)$$

Then,

$$\frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w-z} dw = f(z).$$

$\square$

Theorem 17.6
Let D be a bounded domain with piecewise smooth boundary. If $f(z)$ is analytic on D and it extends smoothly to ∂D, then $f(z)$ has complex derivatives of all orders on D and they are given by

$$f^{(m)}(z) = \frac{m!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^{m+1}} dw,$$

where m is a non-negative integer.

Proof (Sketch). We sketch the proof for the first derivative ($m = 1$) case. The other derivative formulas are established similarly.
We prove that

$$f'(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^2} dw.$$

We compute the difference quotient using Cauchy's formulas:

$$\begin{aligned}
\frac{f(z+\Delta z)-f(z)}{\Delta z} &= \frac{1}{\Delta z}\left[\frac{1}{2\pi i}\int_{\partial D}\frac{f(w)}{w-(z+\Delta z)}dw - \frac{1}{2\pi i}\int_{\partial D}\frac{f(w)}{w-z}dw\right] \\
&= \frac{1}{2\pi i\Delta z}\left[\int_{\partial D}\left(\frac{f(w)}{w-(z+\Delta z)} - \frac{f(w)}{w-z}\right)dw\right] \\
&= \frac{1}{2\pi i\Delta z}\int_{\partial D}\frac{\Delta z f(w)}{\big(w-(z+\Delta z)\big)\big(w-z\big)}dw \\
&= \frac{1}{2\pi i}\int_{\partial D}\frac{f(w)}{\big(w-(z+\Delta z)\big)\big(w-z\big)}dw
\end{aligned}$$

Taking $\Delta z \to 0$ on both sides, we have

$$f'(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(w)}{(w-z)^2}dw$$

Note the integrals converge as $\Delta z \to 0$ since

$$\frac{f(w)}{\big(w-(z+\Delta z)\big)\big(w-z\big)} \to \frac{f(w)}{(w-z)^2} \text{ uniformly as } \Delta z \to 0.$$

□

Whether a function is analytic on z is determined by the values of f close to z, i.e., analyticity at a point is a "local condition".

Thus, the condition that D have piecewise smooth boundary is not really relevant since we could restrict to a small disk around z which is contained in D.

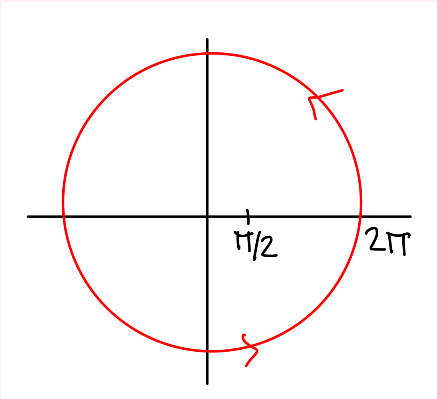
Corollary 17.7

If $f(z)$ is analytic on a domain D, then $f(z)$ is infinitely differentiable on D, and the complex derivatives $f', f'', f^{(3)}, \ldots$ are all analytic on D.

Example 17.8

Evaluate the integral

$$\oint_{|z|=2\pi} \frac{z^2 \cos z}{(z-\pi/2)^2} dz.$$



We can use Cauchy's formula to express this integral as a derivative:

$$f(w) = w^2 \cos w.$$

Note f is analytic on the disk $|w| < 2\pi$. Then by Cauchy's formula

$$f'(w) = \frac{1}{2\pi i} \oint_{|z|=2\pi} \frac{f(z)}{(z-w)^2} dz = \frac{1}{2\pi i} \oint_{|z|=2\pi} \frac{z^2 \cos z}{(z-w)^2} dz$$

Thus, if we set $w = \dfrac{\pi}{2}$, then

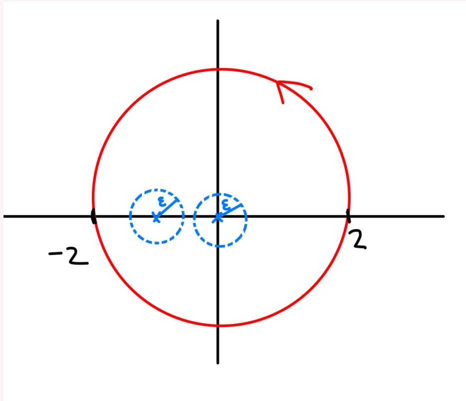
$$\begin{aligned}
\oint_{|z|=2\pi} \frac{z^2 \cos z}{(z-\frac{\pi}{2})^2} dz &= 2\pi i f'(\frac{\pi}{2}) \\
&= 2\pi i \Big(2w \cos w - w^2 \sin w\Big)\Big|_{w=\frac{\pi}{2}} \\
&= 2\pi i \left(-\left(\frac{\pi}{2}\right)^2 \sin \frac{\pi}{2}\right) = \frac{-\pi^3 i}{2}
\end{aligned}$$

18 Feb 14, 2022

18.1 The Cauchy Integral Formula (Cont'd)

Example 18.1
Consider the integral

$$\oint_{|z|=2} \frac{e^z}{z(z+1)} dz.$$



Note, this is not of the form

$$\frac{f(z)}{(z-w)}$$

for some $w \in \{|z| < 2\}$ and $f(z)$ analytic in $\{|z| < 2\}$.
But we can still use Cauchy's formula to evaluate the integral, by splitting into two integrals.
Find $\varepsilon > 0$ small enough so that the disks centered at 0 and -1 don't overlap and are contained in $\{|z| < 2\}$. Then by Cauchy's Theorem,

$$0 = \oint_{|z|=2} \frac{e^z}{z(z+1)} dz - \underbrace{\oint_{|z|=\varepsilon} \frac{e^z}{z(z+1)} dz - \oint_{|z+1|=\varepsilon} \frac{e^z}{z(z+1)} dz}_{\text{We can use Cauchy's integral formula to evaluate these}}$$

Note that $\dfrac{e^z}{z+1}$ is analytic on $\{|z| \leq \varepsilon\}$ since this does not contain -1. So

$$\begin{aligned} \oint_{|z|=\varepsilon} \frac{e^z}{z(z+1)} dz &= \oint_{|z|=\varepsilon} \frac{e^z/(z+1)}{z} dz \\ &= 2\pi i \left(\frac{e^z}{z+1} \right) \Bigg|_{z=0} = 2\pi i \end{aligned}$$

And

$$\begin{aligned}\oint_{|z+1|=\varepsilon} \frac{e^z}{z(z+1)}dz &= \oint_{|z+1|=\varepsilon} \frac{e^z/z}{z+1}dz \\ &\overset{\text{Cauchy}}{\underset{\text{formula}}{=}} 2\pi i \left(\frac{e^z}{z}\right)\Bigg|_{z=-1} \\ &= -2\pi i e^{-1}\end{aligned}$$

Thus,

$$\begin{aligned}\oint_{|z|=2} \frac{e^z}{z(z+1)}dz &= \oint_{|z|=\varepsilon} \frac{e^z}{z(z+1)}dz + \oint_{|z+1|=\varepsilon} \frac{e^z}{z(z+1)}dz \\ &= 2\pi i - 2\pi i e^{-1}.\end{aligned}$$

18.2 Liouville's Theorem

We can use Cauchy's formula and integral bounds to get the following bounds for the derivatives of analytic functions:

Theorem 18.2 (Cauchy estimates)
Suppose $f(z)$ is analytic for $|z-z_0| \le \rho$. If $|f(z)| \le M$ for all z with $|z-z_0| = \rho$, then

$$\left|f^{(m)}(z_0)\right| \le \frac{m!}{\rho^m}M$$

for any non-negative integer m.

Proof. We can use Cauchy's formula and the ML estimate for integrals. By Cauchy's formula,

$$\left|f^{(m)}(z_0)\right| = \left|\frac{m!}{2\pi i}\int_{|z-z_0|=\rho} \frac{f(z)}{(z-z_0)^{m+1}}dz\right|$$

The arclength of the curve we integrate over is

$$L = 2\pi\rho.$$

We can also get a bound for

$$\left|\frac{f(z)}{(z-z_0)^{m+1}}\right|$$

on the boundary,

$$\left|\frac{f(z)}{(z-z_0)^{m+1}}\right| = \frac{|f(z)|}{|z-z_0|^{m+1}} = \frac{|f(z)|}{\rho^{m+1}} \le \frac{M}{\rho^{m+1}}$$

So, by ML estimate,

$$|f^{(m+1)}(z_0)| = \left|\frac{m!}{2\pi i}\right| \cdot \left|\int_{|z-z_0)|=\rho} \frac{f(z)}{(z-z_0)^{m+1}}dz\right| = \frac{m!}{2\pi} \cdot 2\pi\rho \cdot \frac{M}{\rho^{m+1}} = \frac{m!}{\rho^m}M$$

□

We can use the Cauchy estimates to prove an important theorem about entire functions.

Definition 18.3 (Entire function)
An entire function is a complex-valued function $f(z)$ which is analytic on all of $\mathbb{C}$.

We point out some examples and non-examples:

Example 18.4
- Polynomial functions, e^z are entire.
- $\frac{1}{z}$ is not entire.
- $\operatorname{Log} z$ is not entire.

Theorem 18.5 (Liouville's Theorem)
If $f(z)$ is entire and bounded (i.e., there is $M > 0$ with $|f(z)| \leq M$ for all $z \in \mathbb{C}$), then $f(z)$ is constant on $\mathbb{C}$.

Proof. Let $f(z)$ be entire and bounded,

$$|f(z)| \leq M \quad \forall z \in \mathbb{C}.$$

Fix $z_0 \in \mathbb{C}$. Then for any $\rho > 0$, we can apply the Cauchy estimate on the circle $|z - z_0| = \rho$ to get

$$|f'(z_0)| \leq \frac{M}{\rho}$$

But if we let $\rho \to \infty$, then we have

$$|f'(z_0)| \leq 0 \implies f'(z_0) = 0$$

since $z_0 \in \mathbb{C}$ was arbitrary, $f' = 0$ on $\mathbb{C}$, hence f is constant. $\square$

We can use Liouville's Theorem to prove the fundamental theorem of algebra:

Proof of Fundamental Theorem of Algebra. We show that every polynomial with coefficients in $\mathbb{C}$ of degree ≥ 1 has a zero in $\mathbb{C}$. Suppose towards a contradiction that

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

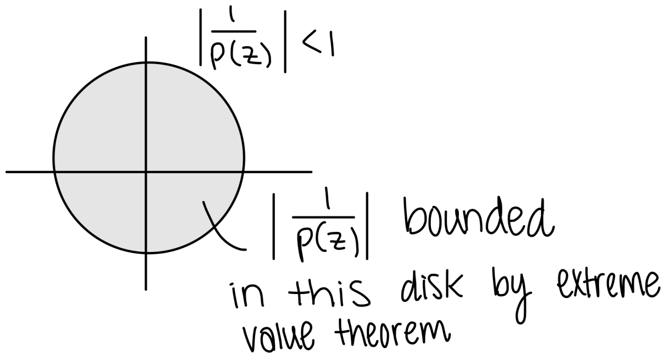
does not have a zero in $\mathbb{C}$. Then $\dfrac{1}{p(z)}$ is defined on all of $\mathbb{C}$ and entire by the quotient rule for derivatives.
Next, we will show it is bounded.
Since

$$\frac{p(z)}{z^n} = 1 + \frac{a_{n-1}}{z} + \cdots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n} \xrightarrow{z \to \infty} 1$$

it follows that $p(z) \to \infty$ as $z \to \infty$. Thus,

$$\frac{1}{p(z)} \to 0 \text{ as } z \to \infty.$$

It follows that $\dfrac{1}{p(z)}$ is bounded, hence by Liouville, $\dfrac{1}{p(z)}$ is constant $\implies p(z)$ is constant, a contradiction.



□

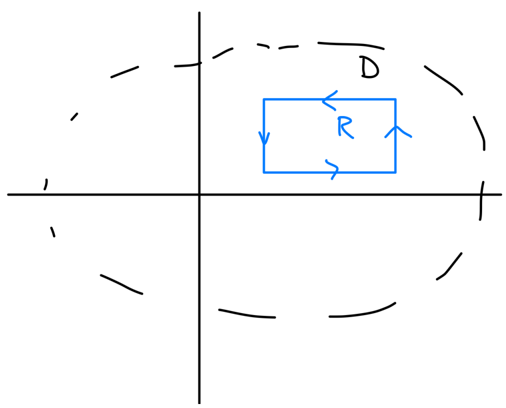
18.3 Morera's Theorem

We state (but do not give a proof sketch for) the following theorem:

Theorem 18.6 (Morera's Theorem)
let $f(z)$ be a continuous function on a domain D. If for any closed rectangle R with sides parallel to the coordinate axis and contained in D we have

$$\int_{\partial R} f(z)dz = 0,$$

then $f(z)$ is analytic on D.



We study one consequence of this theorem:

Theorem 18.7
Suppose $h(t,z)$ is a continuous complex-valued function, defined for $a \le t \le b$ and $z \in D$. If for each fixed t, the function
$$z \mapsto h(t,z)$$
is an analytic function on D, then
$$H(z) = \int_a^b h(t,z)dt$$
is also analytic on D.

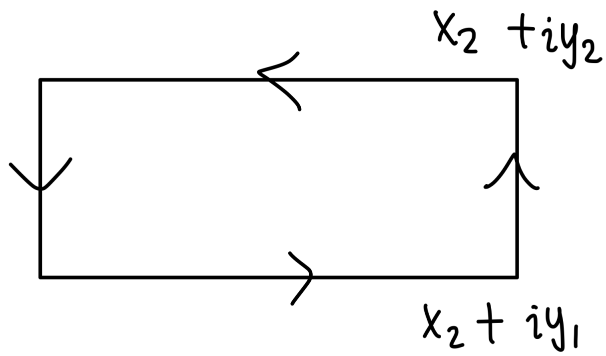
One application of this theorem is to the theory of Fourier transforms:

Theorem 18.8
Let $h(t)$ be a continuous function on $[a,b]$. Then, the Fourier transform
$$H(z) = \int_a^b h(t)e^{-itz}dt$$
is an entire function.

Proof. This follows from Theorem 18.7:
Define
$$\overline{h}(t,z) = h(t)e^{-itz}.$$
For each fixed t,
$$z \longmapsto \overline{h}(t,z) = h(t)e^{-itz}$$
is entire (its derivative $(-it)h(t)e^{-itz}$). Thus by Theorem 18.7,
$$H(z) = \int_a^b \overline{h}(t,z)dt = \int_a^b h(t)e^{-itz}dt$$
is entire. $\square$

Proof of Theorem 18.7. Suppose $h(t,z)$ is continuous and for each fixed t,
$$z \longmapsto h(t,z) \text{ is analytic on } D.$$
We need to show
$$H(z) = \int_a^b h(t,z)dt \text{ is analytic on } D.$$
By Morera's Theorem, it is enough to show that for any closed rectangle R in D with sides parallel to the coordinate axes, we have
$$\int_{\partial R} H(z)dz = 0 = \int_{\partial R}\int_a^b h(t,z)dtdz$$
Let R be such a rectangle. By Cauchy's theorem,
$$\int_{\partial R} h(t,z)dz = 0.$$

Thus,
$$\int_a^b \int_{\partial R} h(t,z)dzdt = \int_a^b 0dt = 0.$$
So, just need to show we can swap the order of the integrals. We can do this since $\int_{\partial R}$ can be expressed as a sum of integrals over real intervals.



$$\int_{\partial R} h(t,z)dz = \int_{x_1}^{x_2} h(t, s+iy_1)ds + \int_{y_1}^{y_2} h(t, x_2+is)ds + \cdots$$
And each of these integrals can swap the order of integration, e.g.:
$$\int_a^b \int_{x_1}^{x_2} h(t, s+iy_1)dsdt = \int_{x_1}^{x_2} \int_a^b h(t, s+iy_2)dtds$$

Thus,
$$\begin{aligned} 0 = \int_a^b \int_{\partial R} h(t,z)dzdt &= \int_{\partial R} \int_a^b h(t,z)dtdz \\ &= \int_{\partial R} H(z)dz. \end{aligned}$$

□

19 Feb 16, 2022

19.1 Basics of Complex Series

Definition 19.1 (Convergence of a series)
Let $a_0, a_1, a_2, \ldots$ be a sequence of complex numbers. The series $\sum_{n=0}^{\infty} a_n$ is said to converge to s if the sequence of partial sums (s_k),

$$s_k = a_0 + a_1 + \cdots + a_k,$$

converges to s.

Example 19.2
Let $z \in \mathbb{C}$ have $|z| < 1$. Consider the geometric series

$$\sum_{n=0}^{\infty} z^n.$$

$$\begin{aligned}(1-z)(1+z+z^2+\cdots+z^n) &= 1+z+z^2+\cdots+z^n \\ &\quad -z-z^2-\cdots-z^n-z^{n+1} \\ &= 1-z^{n+1}\end{aligned}$$

$$\implies 1+z+\cdots+z^n = \frac{1-z^{n+1}}{1-z}$$

Since $|z| < 1$, $z^{n+1} \to 0$ as $n \to \infty$. Thus,

$$1+z+\cdots+z^n = \frac{1-z^{n+1}}{1-z} \to \frac{1}{1-z} \text{ as } n \to \infty.$$

Notation: often, mathematicians are lazy and use the notation

$$\sum a_n = \sum_{n=0}^{\infty} a_n.$$

We quickly review some basic facts.

Theorem 19.3 (Sum laws for series)
Let $\sum a_n$ and $\sum b_n$ be complex series.

1. If $\sum a_n = A$ and $\sum b_n = B$, then $\sum (a_n + b_n) = A + B$.
2. If $\sum a_n = A$ and $c \in C$, then $\sum c a_n = cA$.

Proof. These are proved by applying limit laws to partial sums. □

Also remember an important test for convergence for real series.

Theorem 19.4 (Comparison Test)
If $\sum a_n$ and $\sum r_n$ are real series such that $\sum r_n$ converges and

$$0 \leq a_n \leq r_n \quad \forall n,$$

Then $\sum a_n$ also converges and $0 \leq \sum a_n \leq \sum r_n$.

The following test for divergence applies to complex sequences:

Theorem 19.5 (Test for divergence)
If $\sum a_n$ is a convergence complex series, then $a_n \to 0$. In particular, the contrapositive of this statement is also true:
If $a_n \not\to 0$, then $\sum a_n$ is divergent.

Note 19.6: The converse is false. $a_n \to 0$ does not imply $\sum a_n$ converges, e.g.

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

harmonic series divergent, but $\frac{1}{n} \to 0$ as $n \to \infty$.

Proof. $a_k = s_k - s_{k-1}$. So if $\sum a_k$ convergent, then

$$\begin{aligned} s_k &\longrightarrow s = \sum a_n \\ s_{k-1} &\longrightarrow s \end{aligned}$$

Then by a limit law

$$a_k = s_k - s_{k-1} \to s - s = 0$$

□

In the future, we will refer to complex series simply as series, and only use the phrase real series when we are emphasizing that the series we are working with only has real terms.

Definition 19.7 (Converges absolutely)
A series $\sum a_n$ converges absolutely if $\sum |a_n|$ (real series) converges.

Theorem 19.8
If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.

Proof.

$$\begin{aligned} |\operatorname{Re} a_k| \leq |a_k| &\implies -|a_k| \leq \operatorname{Re} a_k \leq |a_k| \\ &\implies 0 \leq \operatorname{Re} a_k + |a_k| \leq 2|a_k|, \end{aligned}$$

Since $\sum a_k$ is absolutely convergent, $\sum |a_k|$ convergent, hence $\sum 2|a_k|$ is convergent. Thus, the partial sums of $\sum (\operatorname{Re} a_k + |a_k|)$ are bounded above by $\sum 2|a_k|$. Thus, by comparison test,

$$\sum \Big(\operatorname{Re} a_k + |a_k| \Big) \text{ converges}$$

Now,

$$\sum \operatorname{Re} a_k = \sum \Big(\operatorname{Re} a_k + |a_k| \Big) - \sum |a_k|$$

which is the sum of two convergent series, hence convergent. By a similar argument, $\sum \operatorname{Im} a_k$ is convergent. Since

$$\sum a_k = \sum \operatorname{Re} a_k + \sum i \operatorname{Im} a_k,$$

so $\sum a_k$ is convergent. $\square$

Remark 19.9 $|\sum a_k| \le \sum |a_k|$.
Just use triangle inequality argument.

Remark 19.10 $\sum \frac{(-1)^n}{n}$ converges, but does not absolutely converge.

Example 19.11
For $|z| < 1$, the geometric series $\sum z^n$ converges absolutely.

$$\sum |z^n| = \sum |z|^n$$

where $\sum |z|^n$ is a convergent real geometric series, $0 \le |z| < 1$.

19.2 Sequences and Series of Functions

In this section, we study two different types of convergence of functions.

Definition 19.12 (Converges pointwise)
Let $\{f_i\}$ be sequence of complex-valued functions defined on a set E. The sequence $\{f_i\}$ converges pointwise to a function f on E if

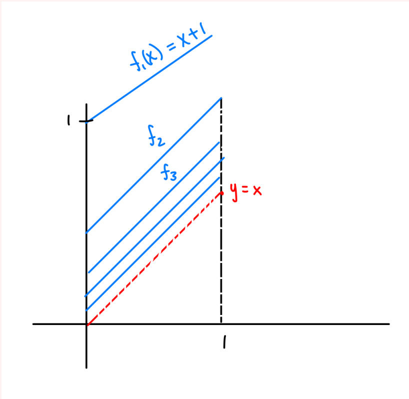
$$\forall x \in E, \lim_i f_i(x) = f(x).$$

Our first several examples will be functions defined on $E = [0,1]$ which only take real values. This is because such functions have graphs which are easy to visualize.

Example 19.13

Let $E = [0, 1]$ and for each $i = 1, 2, 3, \ldots$ define f_i on E by

$$f_i(x) = x + \frac{1}{i}.$$

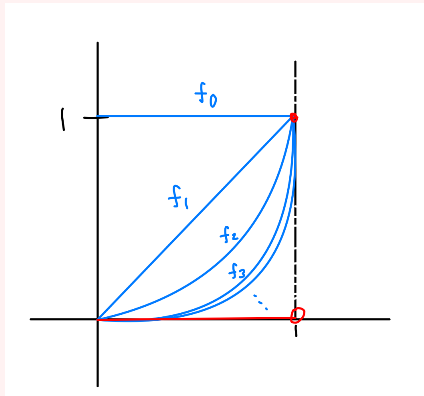


$\{f_i\}$ converges pointwise to $f(x) = x$ on $[0, 1]$. Check that for any $x \in [0, 1]$

$$\lim_i f_i(x) = \lim_i \left(x + \frac{1}{i}\right) = x = f(x)$$

Example 19.14
Let $E = [0, 1]$ and for each $i = 0, 1, 2, 3, \ldots$ define f_i on E by

$$f_i(x) = x^i.$$



For $x \in [0, 1)$,

$$\lim_i f_i(x) = \lim_i x^i = 0$$

For $x = 1$,

$$\lim_i f_i(1) = \lim_i 1^i = 1$$

Thus,

$$\{f_i\} \text{ converges pointwise to } f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

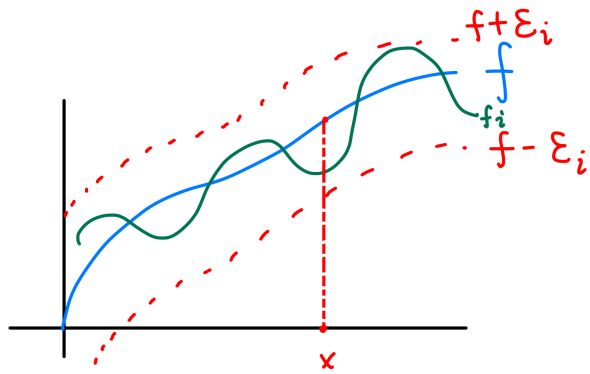
The previous example showed that the pointwise limit of continuous functions need not be a continuous function.

We would like a notion of convergence of functions which has the nicer property that the limit (according to this notion) of continuous functions is always a continuous function.

Definition 19.15 (Converges uniformly)
Let $\{f_i\}$ be sequence of complex-valued functions defined on a set E. The sequence $\{f_i\}$ converges uniformly to a function f on E if there is a sequence of positive reals $\{\epsilon_i\}$ such that

1. $\lim_i \epsilon_i = 0$; and
2. for all $x \in E, |f)_i(x) - f(x)| < \epsilon_i$.

$$\implies 0 \le \lim_i |f_i(x) - f(x)| \le \lim_i \epsilon_i = 0$$



It is often convenient to look at the "worst-case" estimators,

$$\epsilon_i = \sup_{x \in E} |f_i(x) - f(x)|.$$

If these $\epsilon_i \not\to 0$, then $\{f_i\}$ does not converge uniformly to $f(x)$.

Note that

$$\{f_i\} \text{ converges uniformly to } f \implies \{f_i\} \text{ converges pointwise to } f,$$

but the converse is not true.

Example 19.16
The functions $f_i(x) = x + \frac{1}{i}$ from before converges uniformly to $f(x) = x$ on $E = [0, 1]$.

$$|f_i(x) - f(x)| = \left| x + \frac{1}{i} - x \right| = \frac{1}{i}$$

Define $\epsilon_i = \frac{1}{i}$. Then

1. $\lim \epsilon_i = 0$
2. for all $x \in E, |f_i(x) - f(x)| \leq \frac{1}{i} = \epsilon_i$ Thus, $\{f_i\}$ converges uniformly to f.

Example 19.17
The function $f_i(x) = x^i$ do not converge uniformly on $E = [0, 1]$.
The only possible function $\{f_i\}$ can converge uniformly to is their pointwise limit

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}.$$

We can look at the "worst-case estimators" for the difference between f_i and f:

$$\epsilon_i = \sup_{x \in E} |f_i(x) - f(x)| = 1$$

Since the worst case estimators

$$\epsilon_i \not\to 0,$$

$\{f_i\}$ does not converge uniformly to f.

Here are the two most important properties of uniform convergence.

Theorem 19.18
Let $\{f_i\}$ be a sequence of continuous complex-valued functions on $E \subseteq \mathbb{C}$. If $\{f_i\}$ converges uniformly to f on E, then f is continuous on E.

Theorem 19.19
Let γ be a piecewise smooth curve in $\mathbb{C}$, and let $\{f_i\}$ be a sequence of complex-valued continuous functions on γ. If $\{f_i\}$ converges uniformly to f on γ, then

$$\lim_i \int_\gamma f_i(z) dz = \int_\gamma f(z) dz.$$

20 Feb 18, 2022

20.1 Series of Functions

Definition 20.1 (Partial sums, Converges pointwise vs. converges uniformly (for series))
Let $\sum g_i(z)$ be a series of complex-valued functions. The partial sums of $\sum g_i(z)$ are the functions

$$S_k(z) = g_0(z) + g_1(z) + \cdots + g_k(z).$$

We say $\sum g_i(z)$ converges pointwise to $S(z)$ if the sequence $\{S_k(z)\}$ converges pointwise to $S(z)$.
We say $\sum g_i(z)$ converges uniformly to $S(z)$ if the sequence $\{S_k(z)\}$ converges uniformly to $S(z)$.

Of course, the most notable examples of such series of functions are power series, e.g.:

$$\sum_{k=0}^{\infty} a_k z^k, \quad a_k \in \mathbb{C}$$

$$g_i(z) = a_i z^i$$

$$S_k(z) = a_0 z^0 + a_1 z + a_2 z^2 + \cdots + a_k z^k.$$

The following theorem provides a useful criterion for concluding that a series of functions converges uniformly.

Theorem 20.2 (Weierstrass M-test)
Suppose $\{M_k\}$ is a sequence of positive reals with $\sum M_k$ convergent. If $\{g_k\}$ is a sequence of complex-valued functions on a set E such that for each k we have

$$|g_k(z)| < M_k \quad \forall z \in E,$$

then $\sum g_k$ converges uniformly on E.

Proof. Since $\forall x \in E$, $|g_k(x)| < M_k$ and $\sum M_k < +\infty$,

$$\text{comparison test} \implies \forall x \in E \quad \sum g_k(x) \text{ is absolutely convergent.}$$

Define for each $x \in E, g(x) = \sum g_k(x)$.

For any $x \in E$ and any $n \in N$,

$$\begin{aligned}
|g(x) - S_n(x)| &= \left|\sum_{k=0}^{\infty} g_k(x) - \sum_{k=n}^{\infty} g_k(x)\right| \\
&= \left|\sum_{k=n+1}^{\infty} g_k(x)\right| \\
&\leq \sum_{k=n+1}^{\infty} |g_k(x)| \\
&< \sum_{k=n+1}^{\infty} M_k =: \varepsilon_n
\end{aligned}$$

The ε_n satisfy

$$|g(x) - S_n(x)| < \varepsilon_n \quad \forall x \in E$$

and

$$\lim_n \varepsilon_n = \lim_n \sum_{k=n+1}^{\infty} M_k = 0$$

Since $\sum M_k < +\infty$ □

Example 20.3

For $|z| < 1$, consider the geometric series

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad |z| < 1$$

So

$$\left|\frac{1}{1-z} - S_n(z)\right| = \left|\frac{z^{n+1}}{1-z}\right| \longrightarrow +\infty \text{ as } z \to 1$$

So, on $\{|z| < 1\}$, the worst-case estimate is $+\infty$, which shows that $\sum z^n$ does not converge uniformly to $\frac{1}{1-z}$ on $\{|z| < 1\}$.
However, for any $0 \leq r < 1$, we can use Weierstrass M-test to show that $\sum z^n$ does converge uniformly to $\frac{1}{1-z}$ on $\{|z| \leq r\}$. Fix $0 \leq r < 1$. Then for $|z| \leq r$, we have

$$|z^k| = |z|^k \leq r^k =: M_k$$

Since

$$\sum M_k = \sum r^k < +\infty,$$

the result follows by Weierstrass M-test.

Now, we explore some applications of uniform convergence to analytic functions:

Theorem 20.4
If a sequence $\{f_k(z)\}$ of analytic functions on D converge uniformly to $f(z)$ on D, then $f(z)$ is analytic on D.

Proof. By Morera's theorem, enough to show that

$$\int_{\partial D} f(z)dz = 0$$

for any rectangle $R \subseteq D$ which has sides parallel to the coordinate axes. Let R be such a rectangle. Since $f_k(z)$ is analytic on D, from Cauchy's Theorem, we

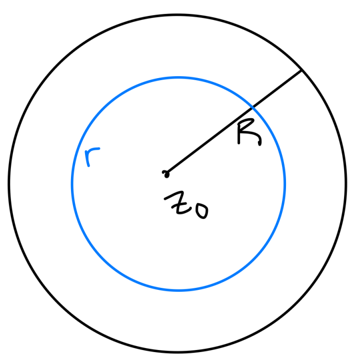
$$\int_{\partial R} f_k(z)dz = 0 \quad R \subseteq D$$

Since $\{f_k(z)\}$ converge uniformly to $f(z)$, we have

$$\int_{\partial R} f(z)dz = \lim \int_{\partial R} f_k(z)dz = \lim_k 0 = 0.$$

□

Theorem 20.5
Suppose that $\{f_k(z)\}$ is a sequence of analytic functions on the closed disk $\{|z-z_0| \leq R\}$, and suppose that the sequence converges uniformly to $f(z)$ on this disk. Then for each $r < R$ and for each $m \geq 1$, the sequence of mth derivative $\{f_k^{(m)}(z)\}$ converges uniformly to $f^{(m)}(z)$ on the disk $\{|z - z_0| \leq r\}$.



Proof. Let $\{\varepsilon\}$ be a sequence with $\lim \varepsilon_k = 0$ and

$$|f_k(z) - f(z)| < \varepsilon_k \quad \forall |z - z_0| \leq R.$$

Fix $0 \leq r < R$ and pick s with $r < s < R$. By Cauchy's integral formula, we have

$$f_k^{(m)}(z) - f^{(m)}(z) = \frac{m!}{2\pi i} \int_{|z-z_0|=s} \frac{f_k(w) - f(w)}{(w-z)^{m+1}} dw \quad \text{when } |z - z_0| \leq s.$$

By the triangle inequality,

$$s = |w - z_0| \leq |w - z| + |z - z_0| \leq |w - z| + r$$

$$\implies s - r \le |w - z| \implies \frac{1}{|w-z|} \le \frac{1}{s-r}$$

Then, on $|w - z_0| = s$, we have by an ML-estimate,

$$\begin{aligned}\left|f_k^{(m)}(z) - f^{(m)}(z)\right| &= \left|\frac{m!}{2\pi i}\int_{|w-z_0|=s} \frac{f_k(w) - f(w)}{(w-z)^{m+1}}dw\right| \\ &= \frac{m!}{2\pi} \cdot \frac{\varepsilon_k}{(s-r)^{m+1}} \cdot 2\pi s \longrightarrow 0 \text{ as } k \to \infty \text{ since } \varepsilon_k \to 0.\end{aligned}$$

Then, $\{f_k^{(m)}\}$ converges uniformly to $f^{(m)}$ on $\{|z| \le r\}$. □

End of midterm 2 material

20.2 Power Series

Definition 20.6 (Power series)
A power series centered at $z_0 \in \mathbb{C}$ is a series of functions of the form

$$\sum_{n=0}^{\infty} a_k(z - z_0)^k$$

Since we can always make the change of variable $w = z - z_0$, when stating and proving theorems about power series, we will always work with power series of the (slightly simpler) form

$$\sum_{n=0}^{\infty} a_k z^k.$$

Theorem 20.7
Let $\sum a_k z^k$ be a power series. Then there is a real R, $0 \le R \le +\infty$, such that

1. $\sum a_k z^k$ converges absolutely for $|z| < R$,
2. $\sum a_k z^k$ diverges for $|z| > R$.

Moreover, for each $r < R$, the series $\sum a_k z^k$ converges uniformly on the disk $\{|z| \le r\}$.

Definition 20.8 (Radius of convergence)
The R in the theorem above is called the radius of convergence

We have already seen in previous examples that the radius of convergence of the power series $\sum z^k$ is $R = 1$ since it converges for $|z| < 1$, diverges for $|z| \ge 1$.

Proof. Consider the set

$$S = \{r \ge 0 \colon \left(|a_k|r^k\right)_k \text{ is bounded}\}$$

Note: If $r_0 \in S$, then for any $0 \le r < r_0$, we have $r \in S$. Define $R = \sup S$. For any

$r < R$, we have that $(|a_k|r^k)$ is bounded.
For any $r > R$, we have $(|a_k|r^k)$ is not bounded. So for any z with $|z| = r < R$, $\left(|a_k| \cdot |z|^k\right)$ is not bounded, hence $|a_k| \cdot |z|^k \not\to 0$. Thus, $a_k z^k \not\to 0$, hence $\sum a_k z^k$ diverges by test for divergence.
Next, let $0 \le r < R$, and we will show uniform convergence on $\{|z| \le r\}$. Pick $r < s < R$, so that $(|a_k|s^k)_k$ is bounded, say $|a_k|s^k \le C, \forall k$. Then for any z with $|z| \le r$, we have

$$|a_k z^k| = |a_k| \cdot |z|^k \le |a_k| \cdot r^k = |a_k| \cdot s^k \left(\frac{r}{s}\right)^k \le C \left(\frac{r}{s}\right)^k$$

Since $0 \le r < s$, we have $0 \le \frac{r}{s} < 1$, hence $\sum C \left(\frac{r}{s}\right)^k$ is convergent. So $M_k = C \left(\frac{r}{s}\right)^k$ satisfies the Weierstrass M-Test on $\{|z| \le r\}$, hence $\sum a_k z^k$ converges uniformly on $\{|z| \le r\}$. $\square$

21 Feb 23, 2022

21.1 Midterm 2

22 Feb 25, 2022

22.1 Power Series (Cont'd)

Example 22.1
What is the radius of convergence for $\sum z^k/k^2$?
Converges for $|z| < 1$: this follows from Weierstrass M-Test,

$$\left|\frac{z^k}{k^2}\right| = \frac{|z|^k}{k^2} < \frac{1}{k^2} = M_k$$

Diverges for $|z| > 1$: this can be shows using the test for divergence since

$$\lim_{k\to\infty} \frac{|z|^k}{k^2} = +\infty$$

Then, radius of convergence is $R = 1$

Example 22.2
What about the radius of convergence for $\sum k^k z^k$?
$R = 0$

For a power series $\sum a_k z^k$, we know that the partial sums

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

are analytic functions which converge uniformly to the full series on disks $|z| \leq r$, for a fixed $r < R$.

Moreover, from the previous sections, we know that the mth derivatives of analytic functions converge to the mth derivative of the uniform limit. Thus, we have

Theorem 22.3
Let $\sum a_k z^k$ be a power series with radius of convergence $R > 0$. Then the function

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad |z| < R$$

is analytic, and the derivatives of $f(z)$ are obtained by differentiating term-by-term:

$$f'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1}$$

$$f''(z) = \sum_{k=2}^{\infty} k(k-1) a_k z^{k-2}$$

$$\vdots$$

Moreover, using the formulas for the derivatives, we can obtain nice formulas for the coefficients of the original power series $\sum a_k z^k$:

$$\begin{aligned} f(z) &= a_0 + a_1 z + a_2 z^2 + \cdots \\ &\implies f(0) = a_0 \\ f'(z) &= a_1 + 2a_2 z + \cdots \\ &\implies f'(0) = a_1 \end{aligned}$$

In general,

$$a_k = \frac{f^{(k)}}{k!}$$

Example 22.4
We can use the above theorem to get a power series representation of the function $\frac{1}{(1-z)^2}$. Notice,

$$\frac{1}{(1-z)^2} = \frac{d}{dz}\left(\frac{1}{1-z}\right)$$

Since

$$\begin{aligned} \frac{1}{1-z} &= \sum_{k=0}^{\infty} z^k, \quad |z| < 1 \\ \implies \frac{1}{(1-z)^2} &= \frac{d}{dz}\left(\frac{1}{1-z}\right) = \frac{d}{dz}\left(\sum_{k=0}^{\infty} z^k\right) \\ &= \sum_{k=1}^{\infty} kz^{k-1} \quad |z| < 1 \end{aligned}$$

Another consequence of uniform convergence is that if $\{f_i\}$ converges uniformly to f on γ, then

$$\lim_i \int_\gamma f_i(z)dz = \int_\gamma f(z)dz.$$

Applied to power series, this means we can integrate the power series term-by-term.

Using this method, we can obtain a power series representation of $\operatorname{Log} z$.

Example 22.5
We obtain a power series representation of $\operatorname{Log} z$.

$$\begin{aligned} -\operatorname{Log}(1-z) = \int_0^z \frac{1}{1-\rho} d\rho &= \int_0^\rho \sum_{k=0}^\infty \rho^k d\rho \\ &= \sum_{k=0}^\infty \int_0^z \rho^k d\rho \\ &= \sum_{k=0}^\infty \frac{z^{k+1}}{k+1} \\ &= \sum_{m=1}^\infty \frac{z^m}{m} \end{aligned}$$

Then, do a change of variable $w = 1-z, z = 1-w$ to get a power series representation of $\operatorname{Log}(z)$.

Luckily, determining the radius of convergence for a power series using old methods from calculus (with real numbers):

Theorem 22.6
Let $\sum a_k z^k$ be a power series with $a_k \neq 0$ for all k. If $\lim_k \frac{|a_k|}{|a_{k+1}|}$ is defined (either finite or $+\infty$), then its limit is the radius of convergence for $\sum a_k z^k$.

Theorem 22.7
If $\lim |a_k|^{1/k}$ is defined (either finite or $+\infty$), then its reciprocal

$$R = \frac{1}{\lim |a_k|^{1/k}}$$

is the radius of convergence for $\sum a_k z^k$

22.2 Power Series Expansions of Analytic Functions

Any analytic function can be represented locally as a power series:

Theorem 22.8
Suppose that $f(z)$ is analytic on $\{|z - z_0| < \rho\}$. Then, $f(z)$ is represented by the power series

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \quad |z - z_0| < \rho,$$

where the coefficients are

$$a_k = \frac{f^{(k)}(z_0)}{k!}, \quad k = 0, 1, 2, \ldots,$$

and the power series has radius of convergence $R \geq \rho$.

We can use Cauchy's theorem to express the coefficients a_k using integrals:
Let r satisfy $0 < r < \rho$. Then, by Cauchy's formula we have

$$f^{(k)}(z_0) = \frac{k!}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z - z_0)^{k+1}} dz$$

$$\implies a_k = \frac{f^{(k)}(z_0)}{k!} = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z - z_0)^{k+1}} dz$$

Remark 22.9 This fact about analytic functions is not true about real smooth (infinitely differentiable) functions. There are real-valued functions $f(x)$ which are smooth at a point x_0 but are **not** represented by a power series near z_0.
For example,

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

is such a function. $f(x)$ is smooth, but not represented by a power series near 0. This is shown by calculating $f^{(k)}(0) = 0$.

Proof of Theorem 22.8. For simpler notation, assume $z_0 = 0$. Let r satisfy $0 < r < \rho$. Then, for any z with $|z| < r$, by Cauchy's formula we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_{|w|=r} \frac{f(w)}{w - z} dw = \frac{1}{2\pi i} \oint_{|w|=r} \frac{f(w)}{w} \left(\frac{1}{1 - \left(\frac{z}{w}\right)} \right) dw \\ &= \frac{1}{2\pi i} \oint_{|w|=r} \frac{f(w)}{w} \sum_{k=0}^{\infty} \left(\frac{z}{w} \right)^k dw \\ &= \frac{1}{2\pi i} \sum_{k=0}^{\infty} \oint_{|w|=r} f(w) \cdot \frac{z^k}{w^{k+1}} dw \\ &= \sum_{k=0}^{\infty} \left(\frac{1}{2\pi i} \oint_{|w|=r} \frac{f(w)}{w^{k_1}} dw \right) z^k = \sum_{k=0}^{\infty} a_k z^k \end{aligned}$$

□

Example 22.10
The exponential function e^z.
For any $k \geq 0$,

$$\frac{d^k}{dz^k}(e^z)\Big|_{k=0} = e^z\Big|_{k=0} = 1$$

$$\implies e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k$$

which is valid on all $\mathbb{C}$ by the theorem, since e^z is entire.

Exercise. The sine function $\sin z$.

This theorem has many important consequences:

Corollary 22.11
Suppose $f(z)$ and $g(z)$ are analytic for $|z - z_0| < r$. If $f^{(k)}(z_0) = g^{(k)}(z_0)$ for all $k \geq 0$, then $f(z) = g(z)$ for $|z - z_0| < r$.

Corollary 22.12
Suppose that $f(z)$ is analytic at z_0, with power series representation $\sum a_k(z-z_0)^k$ centered at z_0. Then, the radius convergence of this power series is the largest R such that f extends to be analytic on the disk $\{|z - z_0| < R\}$.

Example 22.13
Consider the power series expansion of

$$\frac{z+1}{z^2-1}$$

about -2.

$$\frac{z+1}{z^2-1} = \frac{z+1}{(z-1)(z+1)} = \frac{1}{z-1}$$

only non-removable singularity is at $z = 1$. Thus, the radius of convergence for the power series expansion centered at -2 is $R = 3$.

22.3 Power Series Expansion at ∞

In the previous section, we discussed how to expand an analytic function as a power series around a point $z_0 \in \mathbb{C}$. In this section, we will discuss how to expand a function which is "analytic at ∞".

Definition 22.14 (Analytic at ∞)
A function $f(z)$ is analytic at ∞ if the function

$$g(w) = f(1/w)$$

is analytic at 0.

Example 22.15
Consider $f(z) = 1/z^2$.
Then,

$$g(w) = f\left(\frac{1}{w}\right) = \frac{1}{\left(\frac{1}{w}\right)^2} = w^2$$

which is analytic at $w = 0$, hence $f(z)$ is analytic at ∞.

Suppose $f(z)$ is analytic at ∞. Then, by definition, $g(w) = f(1/w)$ is analytic at 0. Thus, we can express $g(w)$ with a power series centered at 0:

$$g(w) = \sum_{k=0}^{\infty} b_k w^k = b_0 + b_1 w + b_2 w^2 + \cdots, \quad |w| < \rho.$$

Then, since $f(z) = g(1/z)$, we have an expansion of f:

$$f(z) = g\left(\frac{1}{z}\right) = \sum_{k=0}^{\infty} b_k \left(\frac{1}{z}\right)^k = b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots$$

which is valid when $\left|\frac{1}{z}\right| < \rho \iff \frac{1}{\rho} < |z|$.

Since the representation for $g(w)$ converges absolutely for $|w| < \rho$ and converges uniformly on $|w| < r$ for $r < \rho$, it follows that

$$f(z) = \sum_{k=0}^{\infty} b_k \left(\frac{1}{z}\right)^k$$

converges absolutely for $|z| > \frac{1}{\rho}$ and converges uniformly on $|w| > r$ for any $r > \frac{1}{\rho}$.

We can use integration to obtain nice formulas for the coefficients b_k of the expansion

$$f(z) = \sum_{k=0}^{\infty} b_k \frac{1}{z^k}.$$

Pick $r > \frac{1}{\rho}$. Then

$$\begin{aligned}
\oint_{|z|=r} f(z)z^m dz &= \oint_{|z|=r} \sum_{k=0}^{\infty} b_k \frac{1}{z^k} z^m dz \\
&= \sum_{k=0}^{\infty} \underbrace{\oint_{|z|=r} b_k z^{m-k} dz}_{\substack{=0 \text{ except when} \\ m-k=-1 \iff k=m+1}} \\
&= \oint_{|z|=r} b_{m+1} z^{-1} dz = b_{m+1} 2\pi i \\
\implies b_{m+1} &= \frac{1}{2\pi i} \oint_{|z|=r} f(z) z^m dz
\end{aligned}$$

Although these are nice formulas for the coefficients, we will later obtain even more useful expressions for the coefficients.

Example 22.16

Consider the function $f(z) = 1/(z^2+1)$.
Easiest to get a separate first for

$$\begin{aligned}
g(w) = f\left(\frac{1}{w}\right) = \frac{1}{\left(\frac{1}{w}\right)^2 + 1} &= \frac{w^2}{1+w^2} \\
&= w^2 \cdot \frac{1}{1-(-w^2)} \\
&= w^2 \cdot \sum_{k=0}^{\infty} (-w^2)^k \\
&= w^2 \sum_{k=0}^{\infty} (-1)^k w^{2k} \\
f(z) = g\left(\frac{1}{z}\right) &= \frac{1}{z^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{z^{2k}}
\end{aligned}$$

Which is valid when $|z| > 1$.

23 Feb 28, 2022

23.1 Manipulation of Power Series

We have already seen how to dfifferentiate and integrate power series:

$$\frac{d}{dz}\sum_{k=0}^{\infty} a_k z^k = \sum_{k=1}^{\infty} a_k k z^{k-1}$$

$$\int \sum_{k=0}^{\infty} z^k = \sum_{k=0}^{\infty} \frac{a_k}{k+1} z^{k+1}$$

We can also add power series. For example, if $f(z)$ and $g(z)$ are analytic at 0 with power series representations

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad g(z) = \sum_{k=0}^{\infty} b_k z^k,$$

then $(f+g)(z)$ has power representation

$$(f+g)(z) = \sum_{k=0}^{\infty} (a_k + b_k) z^k$$

Similarly, for a constant $c \in \mathbb{C}$,

$$(cf)(z) = \sum_{k=0}^{\infty} c a_k z^k.$$

Multiplying power series is a bit more complicated. For $f(z)$ and $g(z)$ are analytic at 0 with power series representations

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad g(z) = \sum_{k=0}^{\infty} b_k z^k,$$

the product $f(z)g(z)$ has power series

$$f(z)g(z) = \sum_{k=1}^{\infty} c_k z^k,$$

where

$$\begin{aligned} c_k &= a_k b_0 + a_{k-1} b_1 + \cdots + a_1 b_{k-1} + a_0 b_k \\ &= \sum_{n=0}^{\infty} a_{k-n} b_n \end{aligned}$$

This also is proved by analyzing partial sums, e.g.

$$(a_0 + a_1 z + a_2 z^2)(b_0 + b_1 z + b_2 z^2) = \underbrace{a_0 b_0}_{c_0} + \underbrace{(a_1 b_0 + a_0 b_1)}_{c_1} z + \underbrace{(a_2 b_0 + a_1 b_1 + a_0 b_2)}_{c_2} z^2$$

Notation: For any m, $\mathcal{O}(z^m)$ ("big $\mathcal{O}$ of z^m") stands for a sum of terms of the form dz^k, where $k \geq m$.

Also, given $g(z) = \sum_{k=0}^{\infty} b_k z^k$, we can also find the power series of

$$\frac{1}{g(z)},$$

assuming $g(0) \neq 0$.

For simplicity, assume $g(0) = 1$. This means that

$$g(z) = 1 + \sum_{k=1}^{\infty} b_k z^k$$

For z close enough to 0, $\sum_{k=1}^{\infty} b_k z^k$ is close to 0, so that in particular its modulus is < 1. Then, we can use a geometric series expansion as follows:

$$\begin{aligned}
\frac{1}{g(z)} = \frac{1}{1 + \sum_{k=1}^{\infty} b_k z^k} &= \frac{1}{1 - \left(-\sum_{k=1}^{\infty} b_k z^k\right)} \\
&= \sum_{n=0}^{\infty} \left(-\sum_{k=1}^{\infty} b_k z^k\right)^n \\
&= \sum_{n=0}^{\infty} (-1)^n \left(\sum_{k=1}^{\infty} b_k z^k\right)^n \\
&= 1 - \underbrace{\left(\sum_{k=1}^{\infty} b_k z^k\right)}_{\mathcal{O}(z)} + \underbrace{\left(\sum_{k=1}^{\infty} b_k z^k\right)^2}_{\mathcal{O}(z^2)} - \underbrace{\left(\sum_{k=1}^{\infty} b_k z^k\right)^3}_{\mathcal{O}(z^3)} + \mathcal{O}(z^4)
\end{aligned}$$

So, if you want to know the coefficients on $1, z, z^2$, we only have to examine the 3 summands.

Example 23.1

Find the first few terms of the power series expansion of $1/\cos(z)$ around 0.

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \mathcal{O}(z^6)$$

So,

$$\begin{aligned}
\frac{1}{\cos(z)} = \frac{1}{1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \mathcal{O}(z^6)} &= \frac{1}{1 - \left(\frac{z^2}{2!} - \frac{z^4}{4!} + \mathcal{O}(z^6)\right)} \\
&= 1 + \left(\frac{z^2}{2!} - \frac{z^4}{4!} + \mathcal{O}(z^6)\right) + \left(\frac{z^2}{2!} - \frac{z^4}{4!} + \mathcal{O}(z^6)\right)^2 + \mathcal{O}(z^6) \\
&= 1 + \frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^4}{(2!)^2} + \mathcal{O}(z^6) \\
&= 1 + \frac{z^2}{2} + \left(\frac{1}{4} - \frac{1}{4!}\right) z^4 + \mathcal{O}(z^6)
\end{aligned}$$

23.2 The Zeros of an Analytic Function

Definition 23.2 (Zero of order N of $f(z)$)
Suppose $f(z)$ is analytic at z_0 and that $f(z)$ is not the constant function with value 0. Assume that z_0 is a zero of $f(z)$, i.e., $f(z_0) = 0$. We say that z_0 is a zero of order N of $f(z)$ if

$$f(z_0) = f'(z_0) = f''(z_0) = \cdots = f^{N-1}(z_0) = 0 \quad \text{and } f^{(N)}(z_0) \neq 0.$$

If we examine the power series expansion of $f(z)$ around z_0,

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k,$$

then we can see that z_0 is a zero of order N if and only if

$$\begin{aligned} f(z) &= \sum_{k=N}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k \\ &= \frac{f^{(N)}(z_0)}{N!}(z - z_0)^N + \frac{f^{(N+1)}(z_0)}{(N+1)!}(z - z_0)^{N+1} + \cdots \\ &= (z - z_0)^N \left(\sum_{k=N}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^{k-N} \right) \end{aligned}$$

This is an analytic function for which z_0 is not a zero. And

$$f(z) = (z - z_0)^N h(z), \quad h(z_0) \neq 0$$

is valid where the power series converges.

Definition 23.3 (Simple zero vs. double zero)
A zero of order 1 is called a simple zero.
A zero of order two is called a double zero.

Example 23.4
Consider the function $f(z) = z^m$, where m is a positive integer.
0 is a zero of f of order m.

Example 23.5
Consider $f(z) = \cos z$.
Consider $\frac{\pi}{2}$, which is a zero of order 1 of $\cos z$,

$$f'\left(\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = -1 \neq 0$$

Proposition 23.6
If $f(z)$ has a zero of order N at z_0 and $g(z)$ has a zero of order M at z_0, then $f(z)g(z)$ has a zero of order $N + M$ at z_0.

Proof. By our hypotheses,

$$f(z) = \sum_{k=N}^{\infty} a_k (z - z_0)^k \quad a_N \neq 0$$

$$g(z) = \sum_{k=M}^{\infty} b_k (z - z_0)^k \quad b_M \neq 0$$

$$\implies f(z)g(z) = a_N b_M (z - z_0)^{N+M} + \mathcal{O}(z^{N+M+1}) \quad a_N b_M \neq 0.$$

$\square$

Definition 23.7 (Zero of order N at ∞)
We say that $f(z)$ has a zero of order N at ∞ if $g(w) = f(1/w)$ has a zero of order N at 0.

We can also characterize when $f(z)$ has a zero of order N at ∞ in terms of its series representation around ∞.

f has a zero of order N at ∞
$\implies$ g has a zero of order N at 0

$$\implies g(w) = \sum_{k=N}^{\infty} b_k w^k \quad |w| < R$$

$$\implies f(z) = g\left(\frac{1}{z}\right) = \sum_{k=N}^{\infty} b_k z^{-k} \quad |z| > R.$$

Example 23.8
The function $f(z) = \frac{1}{z^n}$ has a zero of order n at ∞.

23.3 Isolated Points

Definition 23.9 (Isolated point)
Let $E \subseteq \mathbb{C}$ and let $z_0 \in E$. z_0 is an isolated point of E if there is a disk D centered at z_0 which contains no points from E other than z_0.

Example 23.10
Consider the set

$$E = \{-3\} \cup [-2, -1] \cup \{0\} \cup \left\{ \frac{1}{n} : n = 1, 2, 3, \ldots \right\}$$

- -3 is isolated
- No point in $[-2, -1]$ is isolated
- $\frac{1}{n}, n = 1, 2, 3, \ldots$ isolated
- 0 is not isolated

The zeros of an analytic function which is not constantly 0 are isolated from each other:

Theorem 23.11
Let D be a domain, and let $f(z)$ be analytic on D and not identically 0 on D. Then, the zeros of $f(z)$ are isolated points, i.e., if Z is the set of zeros of $f(z)$ in D, then every point in Z is an isolated point of Z.

Proof (Sketch). Step 1: Show every zero has a finite order.
Step 2: Let $z_0 \in Z$, of order N. Then close to z_0, we have

$$f(z) = (z - z_0)^N h(z),$$

where h is analytic at z_0 but $h(z_0) \neq 0$. By continuity, you can find a small disk on which $h(z) \neq 0$. That disk isolates z_0 from other zeros. $\square$

Corollary 23.12 (Uniqueness Principal)
Let $f(z)$ and $g(z)$ be analytic on a domain D. Let $S \subseteq D$ be a set which contains at least one nonisolated point. If $f(z) = g(z)$ for all $z \in S$, then $f(z) = g(z)$ for all $z \in D$.

Consider $f(z) - g(z)$. Often, $S = \mathbb{R}$.

Example 23.13
We can use the uniqueness principal to establish the identity

$$\cos^2 z + \sin^2 z = 1 \quad \forall z \in \mathbb{C}.$$

24 Mar 2, 2022

24.1 Laurent Decompositions

Suppose $f(z)$ is analytic in an annulus.

This annulus could just be a punctured disk.

Definition 24.1 (Laurent decomposition)
The laurent decomposition of $f(z)$ will express $f(z)$ as a sum of two functions: one that is analytic inside the annulus, and one that is analytic outside the annulus.

Theorem 24.2 (Laurent decomposition)
Suppose $0 \leq \rho < \sigma \leq +\infty$, and suppose $f(z)$ is analytic in the annulus $\{\rho < |z - z_0| < \sigma\}$. Then, $f(z)$ can be decomposed into a sum

$$f(z) = f_0(z) + f_1(z),$$

where

1. $f_0(z)$ is analytic for $|z - z_0| < \sigma$; and
2. $f_1(z)$ is analytic for $|z - z_0| > \rho$.

Moreover, there is only one such decomposition with $f_1(\infty) = 0$, as $z \to \infty$, $f(z) \to 0$.

Proof of the existence of a Laurent decomposition. Pick r, s such that $\rho < r < s < \sigma$. Let z_0 be the center of the annulus.
For any z with $r < |z| < s$, we can use Cauchy's Integral Formula to express

$$\begin{aligned} f(z) &= \frac{1}{2\pi} \int_{\partial \text{ smaller annulus}} \frac{f(w)}{w - z} dw \\ &= \frac{1}{2\pi i} \oint_{|w - z_0| = s} \frac{f(w)}{w - z} dw - \frac{1}{2\pi i} \oint_{|w - z_0| = r} \frac{f(w)}{w - z} dw \end{aligned}$$

Define

$$f_0(z) = \frac{1}{2\pi i} \oint_{|w - z_0| = s} \frac{f(w)}{w - z} dw$$

is analytic on $|z - z_0| < s$ and define

$$f_1(z) = -\frac{1}{2\pi i} \oint_{|w - z_0| = r} \frac{f(w)}{w - z} dw$$

is analytic on $|z - z_0| > r$.
By a simple ML estimate argument,

$$f_(z) \to 0 \text{ as } z \to \infty$$

Then, $f_1(\infty) = 0$. Using the uniqueness part of the theorem, you can show that the definitions of $f_0(z), f_1(z)$ does not depend on $r < s$. $\square$

The following example shows that the Laurent decomposition of a function depends on which annulus domain we want to represent the function on.

Example 24.3
Consider the function

$$f(z) = \frac{1}{(z+1)(z-3)}.$$

Consider the function on the annulus $\{0 < |z| < 1\}$.

$$f(z) = \frac{1}{(z+1)(z-3)}$$

is already analytic on $\{|z| < 1\}$. So, we can just define $f_0(z) = f(z)$ and $f_1(z) = 0$. So

$$f(z) = f(z) + 0$$

is its Laurent decomposition on $\{0 < |z| < 1\}$.

Example 24.4
Next, consider the function $\{3 < |z| < +\infty\}$.
Note that the "outside" of the annulus is just $\{3 < |z| < +\infty\}$.
f is analytic here so just take $f_1(z) = f(z)$ and $f_0(z) = 0$

$$f(z) = 0 + f(z)$$

Example 24.5
Finally, we will see that we get a nontrivial, more interesting decomposition on the annulus $\{1 < |z| < 3\}$.

$$f(z) = \frac{1}{(z+1)(z-3)}$$

So,

$$\frac{1}{(z+1)(z-3)} = \frac{A}{z+1} + \frac{B}{z-3}$$
$$1 = A(z-3) + B(z+1)$$
$$\implies 1 = (A+B)z - 3A + B \implies \begin{cases} A + B = 0 \\ -3A + B = 1 \end{cases}$$

Solution is $B = \frac{1}{4}, A = \frac{-1}{4}$. So

$$f(z) = \frac{-1}{4} \cdot \frac{1}{z+1} + \frac{1}{4} \cdot \frac{1}{z-3}$$

We can take

$$f_0(z) = \frac{1}{4}\left(\frac{1}{z-3}\right)$$

which is analytic on $\{|z| < 3\}$ and

$$f_1(z) = \frac{-1}{4}\left(\frac{1}{z+1}\right)$$

which is analytic on $\{|z| > 1\}$. So,

$$f(z) = f_0(z) + f_1(z)$$

is the Laurent decomposition.

24.2 Laurent Series Expansions

For a function analytic in an annulus centered at z_0, we will show how to obtain a series representation which (unlike power series) may include some negative powers of $(z - z_0)$.

Suppose $f(z)$ is analytic in the annulus $\rho < |z - z_0| < \sigma$, and let

$$f(z) = f_0(z) + f_1(z)$$

be its Laurent decomposition with $f_1(\infty) = 0$.

Since $f_0(z)$ is analytic in $\{|z - z_0| < \sigma\}$, we can represent it there as a power series centered at z_0:

$$f_0(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k, \quad |z - z_0| < \sigma$$

Note that this series converges absolutely on $\{|z-z_0| < \sigma\}$ converges uniformly on $\{|z-z_0| \le s\}$ for any $s < \sigma$.

Also, we can get a series expansion of $f_1(z)$ at ∞, this time with negative powers of $(z-z_0)$:

$$f_1(z) = \sum_{k=-\infty}^{-1} a_k(z-z_0)^k, \quad |z-z_0| > \rho$$

Note $k=0$ not included since $f(\infty)=0$.

This expansion converges on $\{|z-z_0| > p\}$ and converges uniformly for $\{|z-z_0| \geq r\}$ for any $r > \rho$.

If we add these two series together, then we get the Laurent series expansion for $f(z)$:

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z-z_0)^k, \quad \rho < |z-z_0| < \sigma$$

Convergence behavior:

Converges absolutely on

$$\{\rho < |z-z_0| < \sigma\}$$

converges uniformly on

$$\{r \leq |z-z_0| \leq s\}$$

where r, s are any numbers with $\rho < r < s < \sigma$.

To get a formula for the coefficient a_n in the Laurent expansion, we divide $f(z)$ by $(z-z_0)^{n+1}$ and integrate around the circle $|z-z_0| = r$, where $\rho < r < \sigma$.

$$\begin{aligned}
\oint_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}} dz &= \oint_{|z-z_0|=r} \frac{1}{(z-z_0)^{n+1}} \sum_{k=-\infty}^{\infty} a_k(z-z_0)^k dz \\
&= \sum_{k=\infty}^{\infty} a_k \underbrace{\oint_{|z-z_0|=r} (z-z_0)^{k-n-1} dz}_{\neq 0 \text{ only when } k-n-1=-1 \iff k=n} \\
&= a_n 2\pi i \\
\implies a_n &= \frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}} dz
\end{aligned}$$

We summarize the above conclusion with the following:

Theorem 24.6
Suppose $0 \leq \rho < \sigma \leq +\infty$, and let $f(z)$ be analytic on the annulus $\{\rho < |z - z_0| < \sigma\}$. Then, $f(z)$ has a Laurent expansion

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k, \quad \rho < |z - z_0| < \sigma,$$

which converges uniformly on each annulus

$$\{r \leq |z - z_0| \leq s\}, \quad \text{where } \rho < r < s < \sigma.$$

The coefficients a_n are uniquely determined and given by

$$a_n = \frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{f(z)}{(z - z_0)^{n+1}} dz, \quad \text{where } \rho < r < \sigma.$$

Proof. Above. $\square$

Example 24.7
We expand the function

$$f(z) = \frac{1}{(z-1)(z-2)}$$

in a Laurent series for the annulus $\{1 < |z| < 2\}$.
Again, start by using partial fractions,

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

$f_0 = \frac{1}{z-2}$ should have a power series representation
$f_1(z) = \frac{1}{z-1}$ so this will give us our negative powers of z.

$$\frac{1}{z-2} = \frac{-1}{2} \cdot \frac{1}{1-\frac{z}{2}} = -\frac{1}{2}\sum_{k=0}^{\infty} \left(\frac{z}{2}\right)^k$$

which is valid since

$$\left|\frac{z}{2}\right| < 1 \iff |z| < 2$$

So,

$$\begin{aligned} \frac{-1}{z-1} &= \frac{1}{1-z} \\ &= \frac{-1}{z} \cdot \frac{1}{1-\frac{1}{z}} = \frac{-1}{z}\sum_{k=0}^{\infty} \left(\frac{1}{z}\right)^k \end{aligned}$$

which is valid since

$$\left|\frac{1}{z}\right| < 1 \iff 1 < |z|$$

So,

$$f(z) = \frac{-1}{z}\sum_{k=0}^{\infty} \frac{1}{z^k} - \sum_{k=0}^{\infty} \frac{z^k}{2^{k+1}} \quad 1 < |z| < 2$$

25 Mar 4, 2022

25.1 Laurent Series Expansions (Cont'd)

Example 25.1
We can also expand

$$f(z) = \frac{1}{(z-1)(z-2)}$$

in a Laurent series centered at 2.
The annulus is a punctured disk $0 < |z-2| < 1$.

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

where $\frac{1}{z-2}$ is already centered at 2 and we need to turn $\frac{1}{z-1}$ into a series centered at 2.

$$\frac{-1}{z-1} = \frac{-1}{z-2+2-1} = \frac{-1}{(z-2)+1} = \frac{-1}{1-\big(-(z-2)\big)}$$
$$= -\sum_{k=0}^{\infty}(-1)^k(z-2)^k$$

which is valid when $|z-2| < 1$, which covers our annulus.

$$f(z) = \frac{1}{z-2} - \sum_{k=0}^{\infty}(-1)^k(z-2)^k$$

25.2 Isolated Singularities

Definition 25.2 (Isolated singularity)
A point z_0 is an isolated singularity of $f(z)$ if $f(z)$ is analytic in some punctured disk

$$\{0 < |z-z_0| < r\}$$

centered at z_0.

Example 25.3
$1/\sin(z)$ has isolated singularities at each $\pm k\pi$, k an integer

Example 25.4
$\mathrm{Log}(z)$ has a singularity at $z = 0$, but is not isolated.

25.3 Types of Singularities

If $f(z)$ has an isolated singularity at z_0, then it is analytic on

$$\{0 < |z - z_0| < r\}$$

Thus, $f(z)$ has a Laurent decomposition centered at z_0:

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k, \quad 0 < |z - z_0| < r.$$

Definition 25.5 (Removable singularity)
If $a_k = 0$ for $k < 0$, i.e., if the Laurent series is actually a power series, then we say that z_0 is a removable singularity.

Example 25.6
$\sin z/z$ has a removable singularity at $z_0 = 0$.

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots$$

We can remove the singularity by defining

$$\left.\frac{\sin z}{z}\right|_{z=0} = 1$$

The following theorem gives a useful criterion to conclude that z_0 is a removable singularity of $f(z)$:

Theorem 25.7 (Riemann's Theorem on removable singularities)
Let z_0 be an isolated singularity of $f(z)$. If $f(z)$ is bounded in some disk centered at z_0, then z_0 is a removable singularity of $f(z)$.

Proof. Let M satisfy $|f(z)| < M$ for z near z_0. Pick $r > 0$ small enough so that this bound is valid on the circle $|z - z_0| = r$. Then by a formula for the coefficients a_n of the Laurent series proved last time,

$$a_n = \frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Then we use an ML-estimate to get

$$|a_n| \leq \frac{1}{2\pi} \cdot 2\pi r \cdot \frac{M}{r^{n+1}} = \frac{M}{r^n}$$

When $n < 0$, then $\frac{M}{r^n} \to 0$ as $r \to 0$. Thus, $a_n = 0$ for $n < 0$. □

Definition 25.8 (Pole, order of the pole)
An isolated singularity z_0 of $f(z)$ is a pole if there is a negative integer $-N < 0$ such that $a_{-N} \neq 0$ and $a_k = 0$ for all $k < -N$.
$N > 0$ is called the order of the pole.

When z_0 is a pole of order N, the Laurent series has the form:

$$f(z) = \frac{a_{-N}}{(z-z_0)^N} + \frac{a_{-N+1}}{(z-z_0)^{N-1}} + \cdots + a_0 + a_1(z-z_0) + \cdots$$

where $a_{-N} \neq 0$.

Definition 25.9 (Principal part)
The sum of the negative powers of $(z - z_0)$ is called the principal part of $f(z)$ at the pole z_0, and is denoted $P(z)$.

$$P(z) = \frac{a_{-N}}{(z-z_0)^N} + \cdots + \frac{a_{-1}}{(z-z_0)}$$

Near z_0,

$$f(z) - P(z) = a_0 + a_1(z-z_0) + \cdots$$

the principal part $P(z)$ incorporates all the "bad" behavior of $f(z)$ at its pole z_0. In particular, $f(z) - P(z)$ is analytic at z_0.

The next two theorems give a criterion for concluding that a singularity is a pole:

Theorem 25.10
Let z_0 be an isolated singularity of $f(z)$. Then z_0 is a pole for $f(z)$ of order N if and only if

$$f(z) = \frac{g(z)}{(z-z_0)^N}, \quad \text{near } z_0$$

where $g(z)$ is analytic at z_0, and $g(z_0) \neq 0$.

Proof. " $\implies$ " Suppose z_0 is a pole of order N, so then

$$\begin{aligned} f(z) &= \frac{a_{-N}}{(z-z_0)^N} + \cdots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) = \cdots \text{ with } a_{-N} \neq 0 \\ &= \frac{1}{(z-z_0)^n} \underbrace{\left[a_{-N} + \cdots + a_{-1}(z-z_0)^{N-1} + a_0(z-z_0)^N + \cdots\right]}_{g(z)} \end{aligned}$$

so $g(z_0) = a_{-N} \neq 0$, where g is analytic near z_0.
" $\impliedby$ " Suppose

$$f(z) = \frac{g(z)}{(z-z_0)^n}$$

where g is analytic and nonzero at z_0. Then express g as a power series near z_0:

$$g(z) = b_0 + b_1(z-z_0) + \cdots \quad b_0 = g(z_0) \neq 0$$

so,
$$f(z) = \frac{b_0}{(z-z_0)^N} + \frac{b_1}{(z-z_0)^{N-1}} + \cdots, \quad b \neq 0$$
□

Theorem 25.11
Let z_0 be an isolated singularity of $f(z)$. Then z_0 is a pole for $f(z)$ of order N if and only if $\frac{1}{f(z)}$ is analytic at z_0 and has a zero of order N.

Proof. " $\Longrightarrow$ " Suppose z_0 is a pole of order N. Then by previous theorem,
$$f(z) = \frac{g(z)}{(z-z_0)^N}$$
where g is analytic and nonzero at z_0
$$\Longrightarrow \frac{1}{f(z)} = \frac{(z-z_0)^N}{g(z)}$$
which is a function that has a zero of order N at z_0.
" $\Longleftarrow$ " Suppose $\frac{1}{f(z)}$ is analytic at z_0 and has a zero of order N.
$$\begin{aligned}\frac{1}{f(z)} &= b_N(z-z_0)^N + b_{N+1}(z-z_0)^{N+1} + \cdots \\ &= (z-z_0)^N \underbrace{\left[b_N + b_{N+1}(z-z_0)^{N+1} + \cdots\right]}_{g(z)} \\ \Longrightarrow f(z) &= \frac{(1/g(z))}{(z-z_0)^N}\end{aligned}$$
so $\frac{1}{g(z)}$ is analytic and nonzero at z_0. □

Example 25.12
Classify the singularities of $\frac{1}{\sin z}$.
We already showed $\sin z$ has simple zeros at $k\pi, k \in \mathbb{Z} \implies \frac{1}{\sin z}$ has simple poles at $k\pi, k \in \mathbb{Z}$.

One more characterization of poles:

Theorem 25.13
Let z_0 be an isolated singularity of $f(z)$. Then, z_0 is a pole if and only if $|f(z)| \to +\infty$ as $z \to z_0$.

Proof. Compare to z_0 a removable singularity $\iff |f(z)|$ bounded near z_0
" $\implies$ " Suppose z_0 is a pole of order N. Then

$$f(z) = \frac{g(z)}{(z-z_0)^N}$$

where g is analytic and nonzero at z_0.
Since $g(z)$ is continuous at $z_0 \implies |g(z)|$ is bounded near z_0, nonzero. Then,

$$|f(z)| = \frac{|g(z)|}{|z-z_0|^N} \to \infty \text{ as } z \to z_0$$

" $\impliedby$ " Suppose $|f(z)| \to +\infty$ as $z \to z_0$. Then $|f(z)|$ is bounded away from 0 close to z_0. Thus, $\frac{1}{f(z)}$ is defined and analytic near z_0 and also bounded near z_0. Thus, z_0 is a removable singularity of $\frac{1}{f(z)}$. Then

$$h(z) = \begin{cases} \frac{1}{f(z)} & z \neq z_0 \\ 0 & z = z_0 \end{cases} \text{ is analytic at } z_0 \text{ and } z_0 \text{ is a zero}$$

$\implies$ there is N such that $h(z) = (z-z_0)^N g(z)$, where g is analytic and nonzero at z_0.
So, near z_0,

$$\frac{1}{f(z)} = (z-z_0)^N g(z) \implies f(z) = \frac{(1/g(z))}{(z-z_0)^N}$$

$\implies$ z_0 is a pole of order N. □

Definition 25.14 (Essential singularity)
An isolated singularity z_0 is said to be an essential singularity if $a_k \neq 0$ for infinitely many $k < 0$. In other words, the essential singularities are the isolated singularities that are neither removable nor poles.

Example 25.15
Consider $e^{1/z}$.

$$e^w = 1 + w + \frac{1}{2!}w^2 + \frac{1}{3!}w^3 + \cdots$$
$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!}\cdot\frac{1}{z^2} + \frac{1}{3!}\cdot\frac{1}{z^3} + \cdots$$

so 0 is an essential singularity.

Let z_0 be an isolated singularity of $f(z)$. We know that if z_0 is removable, then $|f(z)|$ is bounded near z_0. Also, if z_0 is a pole, then $|f(z)| \to +\infty$ as z tends to z_0. Essential singularities have different behavior near z_0:

Theorem 25.16
Suppose that z_0 is an essential isolated singularity of $f(z)$. Then, for every $w_0 \in \mathbb{C}$, there is a sequence (z_n) limiting to z_0 such that $\lim f(z_n) = w_0$.

Proof. We prove the contrapositive. So, assume there is $w_0 \in \mathbb{C}$ such that no sequences (z_n) converging to z_0 have $\lim f(z_0) = w_0$. We want to show that z_0 is not essential. By our hypothesis, there is $\varepsilon > 0$ such that $|f(z) - w_0| > \varepsilon$ for all z near z_0.

$$\implies h(z) = \frac{1}{f(z) - w_0} \text{ is bounded near } z_0$$

$\implies h(z)$ has a removable singularity at z_0.

$$\implies h(z) = (z - z_0)^N g(z), \quad g \text{ is analytic and nonzero at } z_0 \text{ and } N \geq 0$$

so near z_0,

$$\implies \frac{1}{f(z) - w_0} = (z - z_0)^N g(z)$$

$$\implies f(z) - w_0 = \frac{(1/g(z))}{(z - z_0)^N}$$

If $N = 0$, then z_0 is removable for f.
If $N > 0$, then z_0 is a pole of order N for f. $\square$

26 Mar 7, 2022

26.1 Isolated Singularities at ∞

Definition 26.1 (Isolated singularity at ∞)
$f(z)$ has an isolated singularity at ∞ if there is $R > 0$ such that $f(z)$ is analytic on $\{|z| > R\}$.
It follows immediately from the definition that $f(z)$ has an isolated singularity at ∞ if and only if $g(w) = f(1/w)$ has an isolated singularity at 0.

We can classify different types of isolated singularities at ∞ be examining the Laurent expansions at ∞.

$f(z)$ will have an isolated singularity at ∞ of type – if and only if $g(w) = f(1/w)$ has an isolated singularity at 0 of that same type –.

$$g(w) = \sum_{k=-\infty}^{\infty} b_k w^k \to f(z) = g\left(\frac{1}{z}\right) = \sum_{k=-\infty}^{\infty} b_k z^{-k}$$

Let $f(z)$ have an isolated singularity at ∞, and let

$$f(z) = \sum_{k=-\infty}^{\infty} b_k z^k, \quad |z| > R$$

be the Laurent expansion of $f(z)$ on $\{|z| > R\}$.

Definition 26.2 (Removable singularity)
∞ is a removable singularity if $b_k = 0$ for all $k > 0$. In this case, it follows that $f(z)$ is analytic at ∞, i.e., that $g(w) = f(1/w)$ is analytic at 0.

$$f(z) = b_0 + b_{-1}z^{-1} + \cdots$$

$$\implies g(w) = f(1/w) = b_0 + b_{-1}w + \cdots$$

a power series, hence analytic at 0.

Definition 26.3 (Essential singularity)
∞ is an essential singularity if $b_k \neq 0$ for infinitely many $k > 0$.

Definition 26.4 (Pole of order N, princiap part of $f(z)$ at ∞)
For $N \geq 1, \infty$ is a pole of order N if $b_N \neq 0$ and $b_k = 0$ for $k > N$.

$$f(z) = b_N z^N + \cdots + b_1 z + b_0 + b_{-1}z^{-1} + b_{-z}z^{-2} + \cdots$$

For a pole of order N at ∞, we can also define the principal part of $f(z)$ at ∞:

$$P(z) = b_N z^N + \cdots + b_1 z + b_0$$

Example 26.5
$f(z) = 1/z$ has a removable singularity at ∞.
This is the Laurent series and it has no positive powers of z.

Example 26.6
A polynomial of degree $N \geq 1$ has a pole of order N at ∞.

$$f(z) = b_N z^N + \cdots + b_1 z + b_0, \quad b_N \neq 0$$

Again, this is the Laurent series.

Example 26.7
e^z and $\cos z$ both have essential singularities at ∞.

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$g(w) = e^{1/w} = 1 + \frac{1}{w} + \frac{1}{2!}\frac{1}{w^2} + \cdots$$

26.2 The Residue Theorem

Suppose z_0 is an isolated singularity of $f(z)$. Then, $f(z)$ has a Laurent series expansion

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k, \quad 0 < |z - z_0| < \rho.$$

Definition 26.8 (Residue)
We define the residue of $f(z)$ at z_0 to be the coefficient a_{-1} of the term $1/(z - z_0)$ in the Laurent series:

$$\operatorname{Res}[f(z), z_0] = a_{-1} = \frac{1}{2\pi i} \oint_{|z-z_0|=r} f(z)dz,$$

where $0 < r < \rho$.

Note:

$$\oint_{|z-z_0|=r} f(z)dz = 2\pi i \operatorname{Res}[f(z), z_0]$$

Example 26.9
Find
$$\text{Res}\left[\frac{3}{1-z}, 1\right]$$
So,
$$\frac{3}{1-z} = \frac{-3}{z-1} = (-3)\cdot\frac{1}{z-1}$$
$$\implies \text{Res}\left[\frac{3}{1-z}, 1\right] = -3$$

Example 26.10
Find
$$\text{Res}\left[\frac{1}{z^2+1}, -i\right]$$
Use partial fractions
$$\frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)} = \frac{A}{z-i} + \frac{B}{z+i}$$
$$= \frac{1}{2\pi}\left(\frac{1}{z-i}\right) + \left(\frac{-1}{2i}\right)\left(\frac{1}{z+i}\right)$$
$$= \frac{1}{2i}\left(\frac{1}{z-i}\right) + \left(\frac{-1}{2i}\right)\left(\frac{1}{z-(-i)}\right)$$
where
$$\frac{1}{2i}\left(\frac{1}{z-i}\right)$$
is analytic at $-i$ so it expands as a power series around $-i$. Thus no $(z-(-i))^{-1}$ term
$$\implies \text{Res}\left[\frac{1}{z^2+1}, -i\right] = \frac{-1}{2i}$$
Similar reasoning gets you
$$\text{Res}\left[\frac{1}{z^2+1}, i\right] = \frac{1}{2i}$$

Residues are very helpful when computing line integrals of functions with isolated singularities:

Theorem 26.11
Let D be a bounded domain in $\mathbb{C}$ with piecewise smooth boundary. Suppose $f(z)$ is analytic on $D\cup\partial D$, except possibly at a finite number of isolated singularities $z_1,\dots,z_m$ in D. Then,
$$\int_{\partial D} f(z)dz = 2\pi i\sum_{j=1}^{m}\text{Res}[f(z), z_j].$$

Proof. Let D_ε be D with small disk centered at $z_1, \ldots, z_n$ removed, as in the picture. Let U_j be the disk around z_j. We know

$$\oint_{\partial U_j} f(z)dz = 2\pi i \operatorname{Res}[f(z), z_j]$$

Moreover, by Cauchy's Theorem,

$$0 = \int_{\partial D_\varepsilon} f(z)dz = \int_{\partial D} f(z)dz - \sum \oint_{\partial U_j} f(z)dz$$

$$\implies \int_{\partial D} f(z)dz = \sum_{j=1}^{m} \oint_{\partial U_j} f(z)dz = 2\pi i \sum_{j=1}^{m} \operatorname{Res}[f(z), z_j]$$

□

So, evaluating line integrals of functions with finitely many isolated singularities is reduced to just computing residues. We give some helpful rules for computing these residues,

Theorem 26.12 (Rule 1)
If $f(z)$ has a simple pole at z_0, then

$$\operatorname{Res}[f(z), z_0] = \lim_{z \to z_0} (z - z_0) f(z).$$

Proof.

$$f(z) = a_{-1}(z - z_0)^{-1} + a_0 + a_1(z - z_0) + \cdots$$

$$(z - z_0)f(z) = a_{-1} + a_0(z - z_0) + a_1(z - z_0)^2 + \cdots$$

which is analytic at z_0, hence continuous at z_0

$$\implies \lim_{z \to z_0} f(z) = a_{-1} + a_0(z - z_0) + \cdots \Big|_{z = z_0} = a_{-1}$$

□

Example 26.13
We use rule 1 to evaluate $\operatorname{Res}\left[\frac{1}{z^2+1}, -i\right]$ again.

$$\frac{1}{z^2 + 1} = \frac{1}{(z - i)(z + i)} = \frac{1/(z - i)}{(z - (-i))}$$

so $\dfrac{1}{z^2 + 1}$ has a simple pole at $-i$. By Rule 1,

$$\operatorname{Res}\left[\frac{1}{z^2 + 1}, -i\right] = \lim_{z \to -i} \frac{(z + i)}{z^2 + 1} = \lim_{z \to -i} \frac{1}{z - i} = \frac{1}{z - i}\Big|_{z = -i} = \frac{-1}{2i}$$

Theorem 26.14 (Rule 2)
If $f(z)$ has a double pole at z_0, then

$$\operatorname{Res}[f(z), z_0] = \lim_{z\to z_0} \frac{d}{dz}[(z-z_0)^2 f(z)].$$

Proof.

$$f(z) = a_{-2}(z-z_0)^{-2} + a_{-1}(z-z_0)^{-1} + a_0 + a_1(z-z_0) + \cdots$$
$$(z-z_0)^2 f(z) = a_{-2} + a_{-1}(z-z_0) + a_0(z-z_0)^2 + \cdots$$
$$\frac{d}{dz}(z-z_0)^2 f(z) = a_{-1} + 2a_0(z-z_0) + \cdots$$
$$\implies \lim_{z\to z_0} \frac{d}{dz}(z-z_0)^2 f(z) = (a_{-1} + 2a_0(z-z_0) + \cdots)\Big| z = z_0 = a_{-1}$$

$\square$

Example 26.15
Compute

$$\operatorname{Res}\left[\frac{1}{(z^2+1)^2}, -i\right]$$

So

$$\frac{1}{(z^2+1)^2} = \frac{(1/(z-i)^2)}{(z+i)^2}$$

which shows that we have a double pole at $z_0 = -i$. So,

$$\begin{aligned}
\operatorname{Res}\left[\frac{1}{(z^2+1)^2}, -i\right] &= \lim_{z\to -i} \frac{d}{dz}\frac{(z+i)^2}{(z^2+1)^2} = \lim_{z\to -i}\frac{d}{dz}\cdot\frac{1}{(z-i)^2} \\
&= \lim_{z\to -i}\frac{-2}{(z-i)^3} = \frac{-2}{(z-i)^3}\bigg|_{z=-i} \\
&= \frac{-2}{(-2i)^3} = \frac{-1}{4i}
\end{aligned}$$

Theorem 26.16 (Rule 3)
If $f(z)$ and $g(z)$ are both analytic at z_0, and $g(z)$ has a simple zero at z_0, then

$$\operatorname{Res}\left[\frac{f(z)}{g(z)}, z_0\right] = \frac{f(z_0)}{g'(z_0)}.$$

Proof. In this case, $\frac{f(z)}{g(z)}$ has a simple pole at z_0. By Rule 1,

$$
\begin{aligned}
\operatorname{Res}\left[\frac{f(z)}{g(z)}, z_0\right] &= \lim_{z\to z_0}(z-z_0)\cdot\frac{f(z)}{g(z)} \\
&= \lim_{z\to z_0}(z-z_0)\frac{f(z)}{g(z)-g(z_0)} \\
&= \lim_{z\to z_0}\frac{f(z)}{\left(\frac{g(z)-g(z_0)}{z-z_0}\right)} \\
&= \frac{f(z_0)}{g'(z_0)}
\end{aligned}
$$

□

Example 26.17
Find the residue of

$$\frac{z^3}{z^2+4}$$

at $z_0 = 2i$.

$$z^2+4 = (z+2i)(z-2i)$$

So denominator has a simple zero at $2i$ and z^3 is analytic at $z_0 = 2i$. So we can apply Rule 3

$$\operatorname{Res}\left[\frac{z^3}{z^2+4}, 2i\right] = \left.\frac{z^3}{2z}\right|_{z=2i} = \frac{(2i)^3}{2(2i)} = -2$$

The last rule is just a special case of rule 3, but it is particularly useful.

Theorem 26.18 (Rule 4)
If $g(z)$ has a simple zero at z_0, then,

$$\operatorname{Res}\left[\frac{1}{g(z)}, z_0\right] = \frac{1}{g'(z_0)},$$

Example 26.19

Compute

$$\oint_{|z|=2} \frac{1}{z^2+1} dz$$

So

$$\frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}$$

By rule 4,

$$\operatorname{Res}\left[\frac{1}{z^2+1}, i\right] = \frac{1}{2z}\bigg|_{z=i} = \frac{1}{2i}$$

$$\operatorname{Res}\left[\frac{1}{z^2+1}, -i\right] = \frac{1}{2z}\bigg|_{z=-i} = \frac{-1}{2i}$$

$$\begin{aligned}\implies \oint_{|z|=2} \frac{1}{z^2+1} dz &= 2\pi i \left[\operatorname{Res}\left[\frac{1}{z^2+1}, i\right] + \operatorname{Res}\left[\frac{1}{z^2+1}, -i\right]\right] \\ &= 2\pi i \left[\frac{1}{2i} - \frac{1}{2i}\right] = 0\end{aligned}$$

Example 26.20

Compute

$$\oint_{|z|=2} \frac{e^z}{z^2+1}$$

So

$$\frac{e^z}{z^2-1} = \frac{e^z/(z+1)}{z-1}$$

So 1 is a simple pole. Similarly, -1 is a simple pole. By Rule 3,

$$\operatorname{Res}\left[\frac{e^z}{z^2-1}, 1\right] = \frac{e^z}{2z}\bigg|_{z=1} = \frac{e}{2}$$

$$\operatorname{Res}\left[\frac{e^z}{z^2-1}, -1\right] = \frac{e^z}{2z}\bigg|_{z=-1} = \frac{e^{-1}}{-2} = \frac{-1}{2e}$$

$$\implies \oint_{|z|=2} \frac{e^z}{z^2-1} dz = 2\pi i \left[\frac{e}{2} - \frac{1}{2e}\right]$$

27 Mar 9, 2022

27.1 Integrals Featuring Rational Functions

In this section, we will explore how to use residue theory to evaluate real integrals.

Example 27.1

Integrate

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{R\to+\infty} \int_{-R}^{R} \frac{dx}{1+x^2}$$

using residue theory. So we will start by trying to compute

$$\int_{-R}^{R} \frac{dx}{1+x^2}$$

for large R. Consider

$$f(z) = \frac{1}{1+z^2}$$

Note on $\mathbb{R}$,

$$f(x) = \frac{1}{1+x^2}$$

Consider

$$\int_{\partial D_R} \frac{1}{1+z^2} dz$$

We can compute this using Residue theory:

$\dfrac{1}{1+z^2}$ has two singularities: $\pm i$

i is a simple zero of $1+z^2$, so by Rule 4,

$$\operatorname{Res}\left[\frac{1}{1+z^2}, i\right] = \left.\frac{1}{2z}\right|_{z=i} = \frac{1}{2i}$$

Thus,

$$\int_{\partial D_R} \frac{1}{1+z^2} dz = 2\pi i \cdot \operatorname{Res}\left[\frac{1}{1+z^2}, i\right] = \frac{2\pi i}{2i} = \pi$$

On the other hand,

$$\pi = \int_{\partial D_R} \frac{dz}{1+z^2} = \int_{-R}^{R} \frac{dx}{1+x^2} + \int_{\Gamma_R} \frac{dz}{1+z^2}$$

We show

$$\int_{\Gamma_R} \frac{dz}{1+z^2} \to 0 \text{ as } R \to +\infty$$

using an ML estimate argument. The arclength of Γ_R is $\frac{1}{2}2\pi R = \pi R$. We need an upper bound for $\left|\dfrac{1}{1+z^2}\right|$ on Γ_R. By the reverse triangle inequality,

$$\left|1+z^2\right| \geq |z^2| - 1 = |z|^2 - 1 = R^2 - 1 \text{ for } z \text{ on } \Gamma_R.$$

Thus,

$$\left|\frac{1}{1+z^2}\right| \leq \frac{1}{R^2-1} \text{ on } \Gamma_R$$

By ML,

$$\left|\int_{\Gamma_R} \frac{dz}{1+z^2}\right| \leq \frac{\pi R}{R^2-1} \to 0 \text{ as } R \to +\infty$$

So, taking a limit as $R \to +\infty$ of

$$\pi = \int_{-R}^{R} \frac{dx}{1+x^2} + \int_{\Gamma_R} \frac{dz}{1+z^2}$$

we get

$$\pi = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}.$$

We can use this same method to evaluate integrals of the form

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx,$$

where $P(x)$ and $Q(x)$ are polynomials, $Q(x)$ has no zeros on the real line, and

$$\deg(Q) \geq \deg(P) + 2.$$

$$\int_{\partial D_R} \frac{P(z)}{Q(z)} dz = \int_{-R}^{R} \frac{P(x)}{Q(x)} dx + \int_{\Gamma_R} \frac{P(z)}{Q(z)} dz$$

Where we compute

$$\int_{\partial D_R} \frac{P(z)}{Q(z)} dz$$

using Residue theory. And use ML argument to show that

$$\int_{\Gamma_R} \frac{P(z)}{Q(z)} dz \to 0 \text{ as } R \to +\infty$$

This is where the condition $\deg(Q) \geq \deg(P) + 2$ is used.

We can extend this method to evaluate integrals of the form

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(ax) dx,$$

where $a > 0$ is positive real constant. In this method, we will not use $\cos(az)$, because cosine is not well behaved in the upper half-plane of $\mathbb{C}$. Instead, we will consider e^{iaz}. We have two reasons for this:

1. When $z = x$ is real,

$$e^{iaz} = e^{iax} = \cos(ax) + i\sin(ax),$$

2. e^{iaz} has the nice property that

$$\left|e^{iaz}\right| \leq 1 \text{ on } \{z \in \mathbb{C} \colon \operatorname{Im} z > 0\}$$

Let $z = x + iy$, where $y = \operatorname{Im} z > 0$

$$\left|e^{iaz}\right| = \left|e^{iax-ay}\right| = \left|e^{iax}\right| \cdot \left|e^{-ay}\right| = 1e^{-ay} \leq 1$$

since $-ay < 0$.

Example 27.2
We compute
$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{1+x^2}dx,$$
where $a > 0$. Consider
$$f(z) = \frac{e^{iaz}}{1+z^2}$$
Compute
$$\int_{\partial D_R} \frac{e^{iaz}}{1+z^2}dz$$
using residue theory. The only singularity in D_R is i (when $R > 1$) and it is a simple zero of the denominator, not a zero of numerator. So, by Rule 3,
$$\text{Res}\left[\frac{e^{iaz}}{1+z^2}, i\right] = \left.\frac{e^{iaz}}{2z}\right|_{z=i} = \frac{e^{-a}}{2i}$$
Thus,
$$\int_{\partial D_R} \frac{e^{iaz}}{1+z^2}dz = 2\pi i \cdot \frac{e^{-a}}{2i} = \pi e^{-a}$$
Next, we notice
$$\pi e^{-a} = \int_{\partial D_R} \frac{e^{iaz}}{1+z^2}dz = \int_{-R}^{R} \frac{e^{iax}}{1+x^2}dz + \int_{\Gamma_R} \frac{e^{iaz}}{1+z^2}dz \tag{*}$$
By our previous remark, $|e^{iaz}| \leq 1$ on Γ_R and $|1+z^2| \geq |z^2| - 1 = R^2 - 1$ on Γ_R. Thus,
$$\left|\int_{\Gamma_R} \frac{e^{iaz}}{1+z^2}dz\right| \leq \pi R \cdot \frac{1}{R^2-1} \to 0 \text{ as } R \to +\infty.$$
Taking limit $R \to +\infty$ in $(*)$, we have
$$\begin{aligned}\pi e^{-a} &= \int_{-\infty}^{\infty} \frac{e^{iax}}{1+x^2}dx \\ &= \int_{-\infty}^{\infty} \frac{\cos(ax) + i\sin(ax)}{1+x^2}dx \\ &= \int_{-\infty}^{\infty} \frac{\cos(ax)}{1+x^2}dx + i\int_{-\infty}^{\infty} \frac{\sin(ax)}{1+x^2}dx\end{aligned}$$
equating real and imaginary parts, we get
$$\pi e^{-a} = \int_{-\infty}^{\infty} \frac{\cos(ax)}{1+x^2}dx$$
and also
$$0 = \int_{-\infty}^{\infty} \frac{\sin(ax)}{1+x^2}dx$$

27.2 Integrals of Trigonometric Functions

In this section, we will see how definite integrals of trigonometric functions can be solved by converting them into complex contour integrals and evaluating using residue calculus.

Example 27.3
We compute

$$\int_0^{2\pi} \frac{d\theta}{a+\cos\theta}, \quad a>1.$$

We will show that this integral is equal to a complex line integral around the unit circle in $\mathbb{C}$, then we use residue theory to compute it. We parametrize the unit circle with $z=e^{i\theta}, 0\le\theta\le 2\pi$

$$\implies dz = ie^{i\theta}d\theta = izd\theta \implies d\theta = \frac{dz}{iz}$$

Also:

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2}$$

Note

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-\frac{1}{z}}{2i}$$

We get

$$\begin{aligned}\int_0^{2\pi}\frac{d\theta}{a+\cos\theta} &= \oint_{|z|=1}\frac{1}{a+\frac{1}{2}\left(z+\frac{1}{z}\right)}\cdot\frac{dz}{iz}\\ &= \frac{2}{i}\oint_{|z|=1}\frac{dz}{z\left(2a+z+\frac{1}{z}\right)}\\ &= \frac{2}{i}\oint_{|z|=1}\frac{dz}{z^2+2az+1} \qquad (*)\end{aligned}$$

Now use Residues to compute $(*)$. The function $\dfrac{1}{z^2+2az+1}$ has two singularities:

$$\frac{-2a\pm\sqrt{4a^2-4}}{2} = -a\pm\sqrt{a^2-1} \quad a>1$$

Next, we determine which, if any, are in unit circle. $-a-\sqrt{a^2-1}$ is not in unit circle since

$$-a<-1 \implies -a-\sqrt{a^2-a}<-1$$

$$-a+\sqrt{a^2-1} \text{ is in the unit circle}$$

$$\begin{aligned}&-a+\sqrt{a^2-1}<1\\ \iff& \sqrt{a^2-1}<a+1\\ \iff& a^2-1<(a+1)^2=a^2+2a+1\\ \iff& -2<2a\\ \iff& -1<a\end{aligned}$$

$$\begin{aligned}&-1<-a+\sqrt{a^2-1}\\ \iff& a-1<\sqrt{a^2-1}\\ \iff& (a-1)^2<a^2-1\\ \iff& a^2-2a+1<a^2-1\\ \iff& 2<2a \iff 1<a\end{aligned}$$

So,

$$\begin{aligned}
\int_{|z|=1} \frac{1}{z^2+2az+1} dz &= 2\pi i \operatorname{Res}\left[\frac{1}{z^2+2az+1}, -a+\sqrt{a^2-1}\right] \\
&= 2\pi i \cdot \left.\frac{1}{2z+2a}\right|_{z=-a+\sqrt{a^2-1}} \\
&= 2\pi i \cdot \frac{1}{2\sqrt{a^2-1}} = \frac{\pi i}{\sqrt{a^2-1}} \\
\int_0^{2\pi} \frac{1}{a+\cos(\theta)} d\theta &= \frac{2}{i}\int_{|z|=1} \frac{dz}{z^2+2az+1} \\
&= \frac{2}{i}\frac{\pi i}{\sqrt{a^2-1}} \\
&= \frac{2\pi}{\sqrt{a^2-1}}
\end{aligned}$$

28 Mar 11, 2022

28.1 Integrals of Trigonometric Functions

Se have shown that
$$\int_0^{2\pi} \frac{d\theta}{a+\cos\theta} = \frac{2\pi}{\sqrt{a^2-1}}$$
when $a > 1$. We can use this established fact, along with the uniqueness principal, to prove that, in fact,
$$\int_0^{2\pi} \frac{d\theta}{w+\cos\theta} = \frac{2\pi}{\sqrt{w^2-1}}, \quad w \in \mathbb{C} \setminus [-1,1].$$
which has branch points at $1, -1$. So
$$\frac{2\pi}{\sqrt{w^2-1}} = 2\pi \cdot \frac{1}{\sqrt{w+1}} \cdot \frac{1}{w-1}$$
which is analytic on $\mathbb{C} \setminus [-1,1]$

But now, LHS and RHS of $(*)$ are analytic functions on $\mathbb{C} \setminus [-1,1]$ which agree on $(1,+\infty)$, which has non-isolated points. So by the Uniqueness Principal, Equality $(*)$ holds for all $w \in \mathbb{C} \setminus [-1,1]$.

28.2 Integrands with Branch Points

Integrals involving the real functions x^a and $\log x$ can also be evaluated using residue calculus methods. however, since their complex extensions have large branch cuts, the method gets a bit more complicated. We will illustrate with the following example:

Example 28.1

Compute
$$\int_0^\infty \frac{x^a}{(1+x)^2} dx, \quad -1 < a < 1 \text{ and } a \neq 0$$
Consider the branch of $\dfrac{z^a}{(1+z)^2}$ defined by
$$f(z) = \frac{r^a e^{ia\theta}}{(1+z)^2}, \quad z = re^{i\theta} \text{ and } 0 < \theta < 2\pi$$

$f(z)$ has a double pole singularity at -1. We compute $\int_{\partial D} f(z)dz$ with residue theorem.

$$\begin{aligned}\operatorname{Res}\left[\frac{z^a}{(1+z)^2}, -1\right] &= \lim_{z\to -1} \frac{d}{dz}\frac{z^a}{(1+z)^2}\cdot(1+z)^2 \\ &= \lim_{z\to -1}\frac{az^a}{z} \\ &= \frac{az}{z}\bigg|_{z=-1} \\ &= \frac{a(-1)^a}{-1} = -a[1^a e^{ia\pi} \\ &= -ae^{ia\pi}\end{aligned}$$

Thus,

$$\int_{\partial D} f(z)dz = 2\pi i(-ae^{ia\pi}) = -2\pi a i e^{\pi a i}$$

On the other hand,

$$\int_{\partial D} f(z)dz = \int_\varepsilon^R f(x+i0)dx + \oint_{\Gamma_R} f(z)dz + \int_R^\varepsilon f(x-i0)dx - \oint_{\gamma_\varepsilon} f(z)dz$$

So for $x > 0$,

$$f(x+i0) = \frac{x^a e^{i0a}}{(1+x)^2} = \frac{x^a}{(1+x)^2}$$

$$f(x-i0) = \frac{x^a e^{i2\pi a}}{(1+x)^2} = e^{i2\pi a}\cdot f(x+i0)$$

So note that as $R\to+\infty, \varepsilon\to 0$,

$$\begin{aligned}\int_\varepsilon^R f(x+i0)dx + \int_R^\varepsilon f(x-i0)dx \to \int_0^{+\infty}\frac{x^a}{(1+x)^2}dx - \int_0^\infty \frac{x^a}{(1+x)^2}dx \\ - e^{i2\pi a}\int_0^{+\infty}\frac{x^a}{(1+x)^2}dx \\ = (1-e^{i2\pi a})\int_0^{+\infty}\frac{x^a}{(1+x)^2}dx\end{aligned}$$

Next, we show the "red" integrals limit to 0 by using an ML estimate.

$$\left|\int_{\Gamma_R}\frac{z^a}{(1+z)^2}dz\right| \le 2\pi R\frac{R^a}{(R-1)^2} \sim 2\pi R^{a-1}$$

$$\left|\frac{z^a}{(1+z)^2}\right| = \frac{|z|^a}{|1+z|^2} = \frac{R^a}{(R-1)^2}$$

Since $a-1<0, 2\pi R^{a-1}\to 0$ as $R\to+\infty$.

$$\left|\int_{\gamma_\varepsilon}\frac{z^a}{(1+z)^2}dz\right| \le 2\pi\varepsilon\cdot\frac{\varepsilon^a}{(1-\varepsilon)^2} \sim 2\pi\varepsilon^{a+1}$$

By Reverse Triangle Inequality, $|1+z| \leq 1-|z| = 1-\varepsilon$. Thus $2\pi\varepsilon^{a+1} \to 0$ as $\varepsilon \to 0$. Putting together,

$$\begin{aligned} -2\pi i a e^{\pi i a} &= \int_{\partial D} f(z)dz \\ &= \int_{\varepsilon}^{R} f(x+i0)dx - \int_{\varepsilon}^{R} f(x-i0)dx + \int_{\Gamma_R} f(z)dz + \int_{\gamma_\varepsilon} f(z)dz \end{aligned}$$

Taking limit as $R \to +\infty, \varepsilon \to 0$,

$$-2\pi i a e^{\pi i a} = (1-e^{2\pi i a})\int_0^{+\infty} \frac{x^a}{(1+x)^2}dx + 0 + 0$$

So,

$$\begin{aligned} \int_0^{+\infty} \frac{x^a}{(1+x)^2} &= \frac{-2\pi i e^{\pi i a}}{(1-e^{2\pi i a})} \\ &= \frac{2\pi i}{(e^{\pi i a} - e^{-\pi i a}} \\ &= \frac{\pi a}{\left(\frac{e^{\pi i a}-e^{-\pi i a}}{2i}\right)} \\ &= \frac{\pi a}{\sin(\pi a)} \end{aligned}$$

Example 28.2

Show

$$\int_0^\infty \frac{x^2+1}{x^4+1}dx = \frac{\pi}{\sqrt{2}}$$

This is an even function. Thus,

$$\int_{-\infty}^\infty \frac{x^2+1}{x^4+1}dx = 2\int_0^\infty \frac{x^2+1}{x^4+1}dx$$

Step 1: Use residues to compute

$$\int_{\partial D_R} \frac{z^2+1}{z^4+1}dz$$

The simple poles singularities are the fourth roots of -1, so $e^{\pi i/4}, e^{3\pi i/4}, e^{5\pi/4}, e^{7\pi i/4}$. Only $e^{\pi/4}$ and $e^{3\pi i/4}$ are in D_R. So

$$\operatorname{Res}\left[\frac{z^2+1}{z^4+1}, e^{\pi i/4}\right] = \left.\frac{z^2+1}{4z^3}\right|_{z=e^{\pi i/4}} = \frac{e^{\pi i/2}+1}{4e^{3\pi i/4}} = \frac{i+1}{4e^{3\pi i/4}}$$

$$\operatorname{Res}\left[\frac{z^2+1}{z^4+1}, e^{3\pi i/4}\right] = \left.\frac{z^2+1}{4z^3}\right|_{z=e^{3\pi i/4}} = \frac{e^{6\pi i/4}+1}{4e^{9\pi i/4}} = \frac{-i+1}{4e^{\pi i/4}}$$

And $\dfrac{9\pi i}{4} = 2\pi + \dfrac{\pi i}{4}$, so

$$\implies \int_{\partial D_R} \frac{z^2+1}{z^4+1}dz = 2\pi i\left[\frac{i+1}{4e^{3\pi i/4}} + \frac{1-i}{4e^{\pi i/4}}\right] = \cdots = \frac{2\pi}{\sqrt{2}}$$

So

$$\left|\int_{\Gamma_R} \frac{z^2+1}{z^4+1}dz\right| \to 0 \text{ as } R \to +\infty$$

Use ML argument,

$$\frac{|z^2+1|}{|z^4+1|} \le \frac{|z|^2+1}{|z|^4-1} = \frac{R^2+1}{R^4-1}$$

So,

$$\left|\int_{\Gamma_R} \frac{z^2+1}{z^4+1}dz\right| \le \pi R\left(\frac{R^2+1}{R^4-1}\right) \sim \pi\frac{R^3}{R^4} = \frac{\pi}{R}$$

So $\to 0$ as $R \to \infty$. So taking limit as $R \to \infty$,

$$\frac{2\pi}{\sqrt{2}} = \int_{-R}^R \frac{x^2+1}{x^4+1}dx + \int_{\Gamma_R} \frac{z^2+1}{z^4+1}dz$$

$$\implies \frac{2\pi}{\sqrt{2}} = \int_{-\infty}^\infty \frac{x^2+1}{x^4+1}dz = 2\int_0^\infty \frac{x^2+1}{x^4+1}dx$$