

Math 167 (Mathematical Game Theory)

University of California, Los Angeles

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These are my lecture notes for Math 167 (Mathematical Game Theory) taught by Oleg Gleizer. The main textbook for this class is *Game Theory, Alive* by Anna Karlin and Yuval Peres and the supplementary textbook is *A Course in Game Theory* by Thomas Ferguson.

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1 Mar 28, 2022

1.1 Impartial Combinatorial Games

Definition 1.1 (Impartial combinatorial game)

In an impartial combinatorial game,

- Two-person
- Perfect information
- No chance moves
- Win-or-lose outcome

Example 1.2

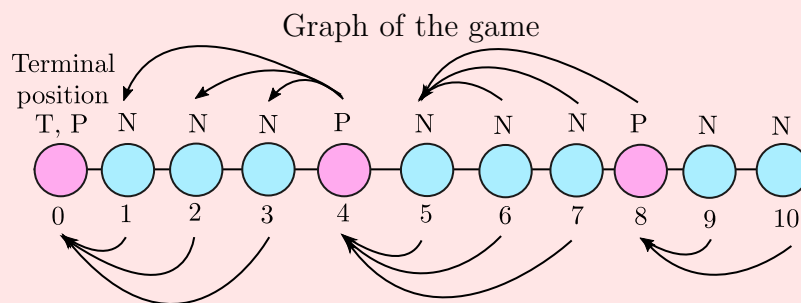
Suppose

- A pile of n chips on the table
- Two players: P1 and P2
- A move consists of removing one, two, or three chips from the pile
- P1 makes the first move, players alternate then
- The player to remove the last chip wins (the last player to move wins. If a player can't move, they lose.)

Method to analyze: backward induction.

Positions:

- **N**, next player to take a move wins.
- **P**, previous (second) player to take a move wins.



Any move from a **P** position leads to an **N** position. There always exists a move from an **N** position to a **P** position.

Ending condition: the game ends in a finite number of moves, no matter how played.

A **T** position is a **P** position.

Definition 1.3 (Normal play vs. misère play)

In a normal play, the last player to move wins. In a misère play, the last player to move loses.

Example 1.4

A misère game, a player can take 1-4 chips.

Every position is either **N** or **P**, but not nothing or both.

Example 1.5 (The game of Chomp)

Graph of the game:

- Positions correspond to vertices
- Moves correspond to oriented edges

**Definition 1.6** (Strategy)

A function that assigns a move to each position, except for the terminal.

Definition 1.7 (Winning strategy from a position x)

A winning strategy from a position x is a sequence of moves, starting from x , that guarantees a win.

Consider a normal game. Let $\mathbf{N}_i/\mathbf{P}_i$ be the set of positions from which P1/P2 can win (reach the nearest terminal vertex of the same graph) in at most i moves.

$$\mathbf{P}_0 = \mathbf{P}_1 = \{\text{terminal positions}\}$$

$$\mathbf{N}_{i+1} = \{x: \text{there is a move from } x \text{ to } \mathbf{P}_i\}$$

$$\mathbf{P}_{i+1} = \{y: \text{each move leads to } \mathbf{N}_i\}$$

Note 1.8: $\mathbf{P}_0 = \mathbf{P}_1 \subseteq \mathbf{P}_2 \subseteq \mathbf{P}_3 \dots$

$$\mathbf{N}_1 \subseteq \mathbf{N}_2 \subseteq \mathbf{N}_3 \dots$$

$$\mathbf{N} = \bigcup_{i=1} \mathbf{N}_i, \quad \mathbf{P} = \bigcup_{i=0} \mathbf{P}_i$$

Definition 1.9 (Progressively bounded)

A game is called progressively bounded if for every position x there exists an upper bound $B(x)$ on the number of moves until the game terminates.

2 Mar 30, 2022

2.1 Combinatorial Games (Cont'd)

Recall 2.1 • $P_0 = P_1 = \{\text{terminal positions}\}$

- $N_{n+1} = \{x: \text{there is a move from } x \text{ to } P_n\}$
- $P_{n+1} = \{y: \text{each move from } y \text{ leads to } N_n\}$
- $P_0 = P_1 \subseteq P_2 \subseteq \dots$
- $N_1 \subseteq N_2 \subseteq N_3 \subseteq \dots$
- $P = \bigcup_{n=0} P_n$
- $N = \bigcup_{n=1} N_n$
- A game is called progressively bounded if for every position x there exists an upper bound $B(x)$ on the number of moves until the game stops.

Theorem 2.2

In a progressively bounded impartial full information combinatorial game, all positions are in $N \cup P$. Thus, for every position there exists a winning strategy.

Proof. Let $B(x) \leq n$. Let us prove by induction that $x \in N_n \cup P_n$.

Base: $n = 0$

x is a terminal vertex $\implies x \in P_0 = P_1$.

Inductive hypothesis by P_0 : $B(x) \leq n \implies x \in N_n \cup P_n$.

Inductive step: Show that $B(x) \leq n + 1 \implies x \in N_{n+1} \cup P_{n+1}$

Consider a move $x \rightarrow y$ and $B(y) \leq n$. Hence, $y \in N_n \cup P_n$. So either

Case 1: Each move from x leads to $y \in N_n \implies x \in P_{n+1}$.

Case 2: There exists a move from x to $y \notin N_n$. Thanks to the inductive typo, $y \in N_n \cup P_n$ so $y \in P_n \implies x \in N_{n+1}$. \square

2.2 The Game of Nim

- Several piles, each containing finitely many chips.
- A move: a player can remove any number of chips, from one to all from any pile
- P1 and P2 alternate taking moves
- The player to take the last chip wins

Consider $x \oplus y$. We rewrite x and y as binary numbers and perform long addition of x_2 and y_2 without carry-over, i.e. mod 2.

$$5 \oplus 7 = \begin{array}{r} 1 1 \\ \oplus 1 1 \\ \hline 0 1 0 \end{array} = 2$$

A position $x = (x_1, x_2, \dots, x_k)$ is a **P** position $\iff x_1 \oplus x_2 \oplus \dots \oplus x_k = 0$.

3 April 1, 2022

3.1 The Game of Nim (Cont'd)

Recall 3.1 $x = (x_1, x_2, \dots, x_k)$

Theorem (Bouton) says $x \in \mathbf{P} \iff x_1 \oplus x_2 \oplus \dots \oplus x_k = 0$.

Proof of Theorem 2.6. We have

Terminal position: $x = (0, 0, \dots, 0) \in \mathbf{P}$ Let $x \in \mathbf{N}$. Then there exists a move $x \rightarrow y \in \mathbf{P}$.

$$x_1 \oplus x_2 \oplus \dots \oplus x_k = \oplus \begin{array}{cccccc} 1 & * & * & \dots & \dots & * & * \\ & & 1 & * & \dots & * & * \\ & & \vdots & \vdots & & \vdots & \vdots \\ 1 & * & * & \dots & \dots & \dots & * & * \end{array}$$

Find the left-most (most significant) column with an odd number of 1's. Change any number that has a 1 in the column so that there is an even number of 1's in every column. The 1 in the most significant position becomes a 0 which implies the number becomes smaller. So this is a legal move.

We have $x \in \mathbf{P} \implies$ any move $x \rightarrow y \in \mathbf{N}$ where

$$x = (x_1, x_2, \dots, x_k) \mapsto y = (x'_1, x_2, \dots, x_k)$$

such that

$$x'_1 < x_1 \text{ and } x_1 \oplus x_2 \oplus \dots \oplus x_k = 0.$$

If

$$x'_1 \oplus x_2 \oplus \dots \oplus x_k = 0$$

then

$$x'_1 \oplus x_2 \oplus \dots \oplus x_k = 0$$

then $x'_1 = x_1$, a contradiction. Hence

$$x'_1 \oplus x_2 \oplus \dots \oplus x_k \neq 0 \implies y \in \mathbf{N}.$$

□

Example 3.2

$$x_1 = 7$$

$$x_2 = 10$$

$$x_3 = 15$$

$$\begin{array}{ccc|ccc} & 0 & 1 & 1 & 1 & 1 \\ \oplus & 1 & 0 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 1 & 1 \\ \hline & 0 & 0 & 1 & 0 & 0 \end{array} \implies \begin{array}{ccc|ccc} & 0 & 1 & 0 & 1 & 1 \\ \oplus & 1 & 0 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 1 & 1 \\ \hline & 0 & 0 & 0 & 0 & 0 \end{array}$$

So we have that $(7, 10, 15) \mapsto (5, 10, 15)$

3.2 Subtraction Nim

Extra condition: A player can remove at most n chips.

We find pile sizes mod $n + 1$, i.e.

$$(x_1, x_2, \dots, x_k) \mapsto (x_1 \bmod n + 1, x_2 \bmod n + 1, \dots, x_k \bmod n + 1)$$

Now we find the Nim-sum and make a move.

$$x \bmod n + 1 = \underbrace{(x_1 \bmod n + 1, x_2 \bmod n + 1, \dots, x_k \bmod n + 1)}_{(x_1 \bmod n+1)_2 \oplus (x_2 \bmod n+1)_2 \oplus \dots \oplus (x_k \bmod n+1)_2} \implies \begin{cases} = 0 \iff \mathbf{P} \\ \neq 0 \iff \mathbf{N} \end{cases}$$

Example 3.3

We have $x = (12, 13, 14)$ and $n = 3$. So,

$$(12 \bmod 4, 13 \bmod 4, 14 \bmod 4) \equiv (0, 1, 2) = (0_2, 1_2, 10_2)$$

So

$$\begin{array}{cc} & 0 & 0 \\ \oplus & 0 & 1 \\ & 1 & 0 \\ \hline & 1 & 1 \end{array} \neq 0$$

so we take away one chip from the third pile

$$\begin{array}{cc} & 0 & 0 \\ \oplus & 0 & 1 \\ & 0 & 1 \\ \hline & 0 & 0 \end{array}$$

So we have that $(12, 13, 14) \mapsto (12, 13, 13)$.

Note 3.4: You can always make a legal move $\mathbf{N} \rightarrow \mathbf{P}$ by removing $i \leq n$ chips from a pile.

Note 3.5: To move from \mathbf{P} to \mathbf{P} , you need to remove $n + 1$ chips from a pile. Not allowed! Hence, any move from \mathbf{P} is to \mathbf{N} .

Example 3.6

We have $x = (12, 13, 13)$, with $n = 3$. So

$$x \bmod 4 = (0, 1, 1)$$

therefore

$$\begin{array}{r} 0 \\ \oplus \quad 1 \\ 1 \\ \hline 0 \end{array}$$

3.3 Two-Person Zero Sum Games (Strategic Form)

We have

- P1: a non-empty set of strategies S1
- P2: a non-empty set of strategies S2
- $A: S1 \times S2 \rightarrow \mathbb{R}$, the min function for P1 (payoff matrix)

Note 3.7: Since the game is zero-sum, a win for P1 is a loss for P2. $A(i, j)$ can be ≤ 0 , so works both ways.

Pure strategies:

		P2			
		S21	S22	...	S2n
P1	S11	a_{11}	a_{12}	...	a_{1n}
	S12	a_{21}	a_{22}	...	a_{2n}
	\vdots	\vdots	\vdots	\ddots	\vdots
	S1m	a_{m1}	a_{m2}	...	a_{mn}

A game. P1 chooses the strategy S1*i*. Simultaneously, P2 chooses the strategy S2*j*. P1 wins a_{ij} .

Lemma 3.8

$$\min_j \max_i a_{ij} \geq \max_i \min_j a_{ij}$$

We will continue in the next lecture.

4 Apr 4, 2022

4.1 Two-Person Zero Sum Games in Strategic Form (Cont'd)

Recall 4.1 Recall that

		P2			
		S21	S22	...	S2n
P1	S11	a_{11}	a_{12}	\cdots	a_{1n}
	S12	a_{21}	a_{22}	\cdots	a_{2n}
	\vdots	\vdots	\vdots	\ddots	\vdots
	S1m	a_{m1}	a_{m2}	\cdots	a_{mn}

P1 has a non-empty set of pure strategies

$$S1 = \{S11, S12, \dots, S1m\}$$

P2 has a non-empty set of pure strategies

$$S2 = \{S21, S22, \dots, S2n\}$$

$A: S1 \times S2 \rightarrow \mathbb{R}$, payoff matrix

P1, $S1i: a_{i1}, a_{i2}, \dots, a_{in}$

Betting on the worst possible outcome, P1 bets on $\min_{1 \leq j \leq n} a_{ij}$. Being intelligent, P1 chooses

$$\max_{1 \leq i \leq m} \min_{1 \leq j \leq n} a_{ij}.$$

Betting on the worst possible loss, P2 bets on $\max_{1 \leq i \leq m} a_{ij}$. Being intelligent, P2 chooses

$$\min_{1 \leq j \leq n} \max_{1 \leq i \leq m} a_{ij}$$

Lemma 4.2

$$\max_{1 \leq i \leq m} \min_{1 \leq j \leq n} a_{ij} \leq \min_{1 \leq i \leq m} \max_{1 \leq j \leq n} a_{ij}$$

Proof. Let

$$\max_i \min_j a_{ij} = a_{pq}$$

$$\min_j \max_i a_{ij} = a_{rs}$$

		q		s	
p		a_{pq}	\leq	a_{ps}	
				$\setminus \nearrow$	
r		a_{rq}		a_{rs}	

□

Example 4.3

Chooser (P1), Hider (P2). Hider hides behind their back

- Either left hand with one coin
- or right hand with two coins

Chooser chooses L or R,

		P2	
		L1	R2
P1	L	1	0
	R	0	2

$$P1: \max_j \min_i a_{ij} = 0$$

$$P2: \min_j \max_i a_{ij} = 1$$

Mixed strategies

		P2	
		L1, q	R2, $1 - q$
P1	L, p	1	0
	R, $1 - p$	0	2

P1: if P2 chooses the strategy L1, the expected gain is

$$1 \cdot p + 0 \cdot (1 - p) = 0$$

If P2 chooses R2, the expected gain is

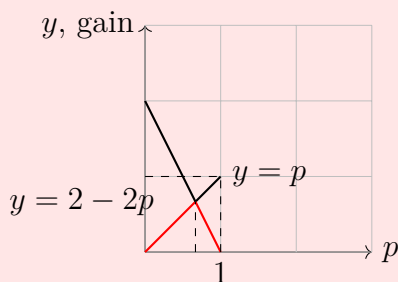
$$0 \cdot p + 2(1 - p) = 2 - 2p.$$

If P1 is out of luck, then the expected gain is

$$\min\{p, 2 - 2p\}$$

Since P1 is intelligent, they choose p s.t. the gain is

$$\max_{0 \leq p \leq 1} \min\{p, 2 - 2p\}$$



$$2 - 2p = p$$

$$2 = 3p$$

$$p = \frac{2}{3}$$

The optimal strategy is

$$\frac{2}{3}L + \frac{1}{3}R$$

With expected gain $\geq \frac{2}{3}$.

P2 is thinking. If P1 chooses L , my expected loss is

$$1 \cdot q + 0 \cdot (1 - q) = q$$

If P1 chooses R , my expected loss is

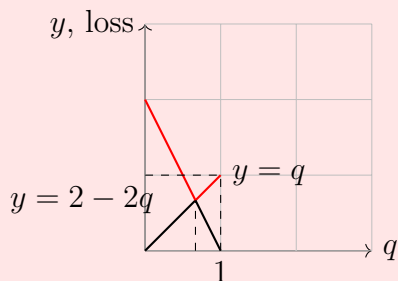
$$0 \cdot q + 2(1 - q) = 2 - 2q$$

Suppose I'm out of luck. Then my expected loss is

$$\max\{q, 2 - 2q\}$$

Being my very smart self,

$$\min_{0 \leq q \leq 1} \max\{q, 2 - 2q\}$$



The optimal strategy is

$$\frac{2}{3}L1 + \frac{1}{3}R2$$

With expected loss $\leq \frac{2}{3} = V$, the value of the game.

Let us generalize $A \in \mathbb{R}^{n \times m}$, an $n \times m$ payoff matrix.

$$\Delta_m = \left\{ \mathbf{p} \in \mathbb{R}^m : p_1 \geq 0, p_2 \geq 0, \dots, p_m \geq 0, \sum_{i=1}^m p_i = 1 \right\}$$

$$\Delta_n = \left\{ \mathbf{q} \in \mathbb{R}^n : q_1 \geq 0, q_2 \geq 0, \dots, q_n \geq 0, \sum_{j=1}^n q_j = 1 \right\}$$

A mixed strategy for P1 is determined by

$$\mathbf{p} \in \Delta_m$$

A mixed strategy for P2 is determined by

$$\mathbf{q} \in \Delta_n$$

Expected gain for P1 (expected loss for P2) = $(\mathbf{p})^T A \mathbf{q}$

		P2			
		q_1	q_2	\cdots	q_n
P1	p_1	a_{11}	a_{12}	\cdots	a_{1n}
	p_2	a_{21}	a_{22}	\cdots	a_{2n}
	\vdots	\vdots	\vdots	\ddots	\vdots
	p_m	a_{m1}	a_{m2}	\cdots	a_{mn}

So

$$(\mathbf{p})^t A \mathbf{q} = p_i(a_{i1}q_1 + a_{i2}q_2 + \cdots + a_{in}q_n)$$

If P1 employs the strategy \mathbf{P} , then in the worst case their payoff is

$$\min_{\mathbf{q} \in \Delta_n} (\mathbf{p})^t A \mathbf{q} = \min_{1 \leq j \leq n} \sum_{i=1}^m a_{ij} p_i$$

Hence, P1's winning strategy is

$$\max_{\mathbf{p} \in \Delta_m} \min_{\mathbf{q} \in \Delta_n} \mathbf{p}^T A \mathbf{q}$$

5 Apr 6, 2022

5.1 General Two-Person Zero-Sum Games in Strategic Form

Recall 5.1 Recall

		P2			
		q_1	q_2	\cdots	q_n
P1	p_1	a_{11}	a_{12}	\cdots	a_{1n}
	p_2	a_{21}	a_{22}	\cdots	a_{2n}
	\vdots	\vdots	\vdots	\ddots	\vdots
	p_m	a_{m1}	a_{m2}	\cdots	a_{mn}

With set of mixed strategies given by,

$$\Delta_m = \left\{ \mathbf{p} \in \mathbb{R}^m : \mathbf{p} \geq 0, \sum_{i=1}^m p_i = 1 \right\}$$

$$\Delta_n = \left\{ \mathbf{q} \in \mathbb{R}^n : \mathbf{q} \geq 0, \sum_{j=1}^n q_j = 1 \right\}$$

where $p_1 \geq 0, p_2 \geq 0, \dots, p_m \geq 0$.

We have

$$\text{Expected gain of P1} = (\mathbf{p})^t A \mathbf{q}$$

with $\mathbf{p} \in \Delta_m$ and $\mathbf{q} \in \Delta_n$.

The winning strategy for P1:

- Worst case: $\min_{\mathbf{q} \in \Delta_n} (\mathbf{p})^t A \mathbf{q}$
- Smart choice: $\max_{\mathbf{p} \in \Delta_m} \min_{\mathbf{q} \in \Delta_n} (\mathbf{p})^t A \mathbf{q}$

$$\begin{aligned} \min_{\mathbf{q} \in \Delta_n} (\mathbf{p})^t A \mathbf{q} &= \min_{\mathbf{q} \in \Delta_n} \sum_{j=1}^n q_j \sum_{i=1}^m a_{ij} p_i \\ &= \min_{1 \leq j \leq n} \sum_{i=1}^m a_{ij} p_i \end{aligned}$$

The winning strategy for P2:

- Worst case: $\max_{\mathbf{p} \in \Delta_m} (\mathbf{p})^t A \mathbf{q}$
- Smart choice: $\min_{\mathbf{q} \in \Delta_n} \max_{\mathbf{p} \in \Delta_m} (\mathbf{p})^t A \mathbf{q}$

$$\begin{aligned}\max_{\mathbf{p} \in \Delta_m} (\mathbf{p})^t A \mathbf{q} &= \max_{\mathbf{p} \in \Delta_m} \sum_{i=1}^m p_i \sum_{j=1}^n a_{ij} q_j \\ &= \max_{1 \leq i \leq m} \sum_{j=1}^n a_{ij} q_j\end{aligned}$$

Definition 5.2 (Safety value for P1 vs. P2)

The value $\hat{\mathbf{p}}$ at which

$$\max_{\mathbf{p} \in \Delta_m} \min_{\mathbf{q} \in \Delta_n} (\mathbf{p})^t A \mathbf{q}$$

is attained is called the safety value for P1. The value $\hat{\mathbf{q}}$ at which

$$\min_{\mathbf{q} \in \Delta_n} \max_{\mathbf{p} \in \Delta_m} (\mathbf{p})^t A \mathbf{q}$$

is attained is called the safety value for P2.

Theorem 5.3 (Von Neumann Minimax Theorem)

For any two-person zero-sum game with $m \times n$ payoff matrix A , there is a number V , called the value of the game, satisfying

$$\min_{\mathbf{q} \in \Delta_n} \max_{\mathbf{p} \in \Delta_m} (\mathbf{p})^t A \mathbf{q} = \max_{\mathbf{p} \in \Delta_m} \min_{\mathbf{q} \in \Delta_n} (\mathbf{p})^t A \mathbf{q} = V$$

Let $\hat{\mathbf{p}}$ be an optimal solution for P1. Let $\hat{\mathbf{q}}$ be an optimal solution for P2. Then

$$\min_{\mathbf{q} \in \Delta_n} (\hat{\mathbf{p}})^t A \mathbf{q} = \max_{\mathbf{p} \in \Delta_m} (\hat{\mathbf{p}})^t A \hat{\mathbf{q}}$$

Definition 5.4 (Value of the game)

Given conditions from Von Neumann Minimax Theorem, V is the value of the game.

Example 5.5 (Odd or Even) • P1 and P2 simultaneously call out one of the numbers, 1 or 2.

- If the sum is odd, P1 wins and gets the sum of the numbers in \$
- If the sum is even, P2 wins and gets the sum of the numbers in \$

		P2	
		1, q	2, $1 - q$
P1	1, p	-2	3
	2, $1 - p$	3	-4

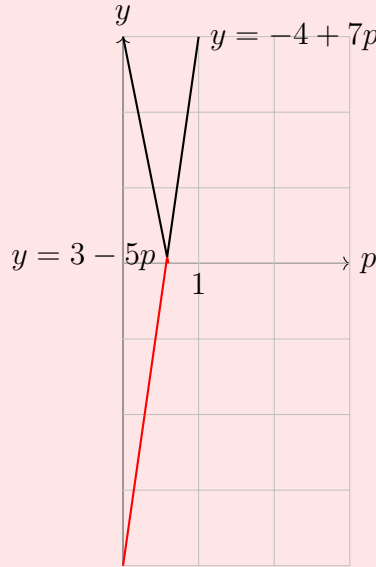
So P1's expected win (P2's expected loss) is

$$\begin{aligned} (\mathbf{p})^t A \mathbf{q} &= \begin{bmatrix} p & 1-p \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} q \\ 1-q \end{bmatrix} \\ &= -12pq + 7p + 7q - 4 \end{aligned}$$

P1's worst possible case:

$$f(p) = \min_{0 \leq q \leq 1} \{-12pq + 7p + 7q - 4\}$$

$$\begin{aligned} q, S21: \quad & -2p + 3(1-p) = 3 - 5p \\ 1-q, S22: \quad & 3p - 4(1-p) = -4 + 7p \end{aligned}$$



- If $3 - 5p \geq -4 + 7p$, then $q = 0$.
- If $3 - 5p < -4 + 7p$, then $q = 1$.

Hence,

$$f(p) = \min\{3 - 5p, -4 + 7p\}$$

Note that

$$\begin{aligned} (-12pq + 7p + 7q - 4) \Big|_{q=0} &= -4 + 7p \\ (-12pq + 7p + 7q - 4) \Big|_{q=1} &= 3 - 5p \end{aligned}$$

$$\text{P1: } \max_{0 \leq p \leq 1} \min_{0 \leq q \leq 1} q(-2p + 3(1-p)) + (1-q)(3p - 4(1-p)) = \max_{0 \leq p \leq 1} \min\{3 - 5p, -4 + 7p\}$$

$$3 - 5p = -4 + 7p$$

$$7 = 12p$$

$$p = \frac{7}{12}, \quad q = \frac{5}{12}$$

Now from P2:

$$\text{P2: } \min_{0 \leq q \leq 1} \max_{0 \leq p \leq 1} p(-2q + 3(1 - q)) + (1 - p)(3q - 4(1 - q)) = \min_{0 \leq q \leq 1} \max\{3 - 5q, -4 + 7q\}$$

6 Apr 8, 2022

6.1 Solving Small-Dimensional Two-Person Zero-Sum Games Pen-and-Paper

Definition 6.1 (Saddle point)

An element of A , a_{ij} is called a saddle point if

- a_{ij} is the min of the i -th row
- a_{ij} is the max of the j -th column

Then $p_i = 1$, $q_j = 1$, $V = a_{ij}$

Example 6.2

Given

$$\begin{array}{ccccc} & & & & \min \\ & & & & -3 \\ & & & & \textcircled{2} \\ & & & & 0 \\ \max & \begin{bmatrix} 4 & 1 & -3 \\ 3 & \textcircled{2} & 5 \\ 0 & 1 & 6 \\ 4 & \textcircled{2} & 6 \end{bmatrix} & & & \end{array}$$

So $p_2 = q_2 = 1$ and $V = 2$.

Lemma 6.3

Let a_{pq} and a_{rs} be saddle points of a payoff matrix A . Then $a_{pq} = a_{rs}$.

		q		s	
p		a_{pq}	\leq	a_{ps}	
		\vee		\wedge	
r		a_{rq}	\geq	a_{rs}	

6.2 Domination

Rows:

$$\begin{array}{ccccccc} i\text{-th row} & a_{i1} & a_{i2} & \cdots & a_{in} \\ & \vee & \vee & & \vee \\ k\text{-th row} & a_{k1} & a_{k2} & \cdots & a_{kn} \end{array}$$

So $p_k = 0$ so k -th row can be removed from A .

Strict domination: for $j = 1, 2, \dots, n$,

$$a_{ij} > a_{kj}$$

Columns: the k -th column dominates the j -th column

$$\begin{array}{ccc} a_{1j} & \geq & a_{1k} \\ a_{2j} & \geq & a_{2k} \\ \vdots & & \vdots \\ a_{mj} & \geq & a_{mk} \end{array}$$

Strict domination: for $i = 1, 2, \dots, m$,

$$a_{ij} > a_{ik}$$

where a_{ik} is dominant.

- Removing a dominant row or column does not change the value of the game, but may remove an optimal strategy.
- Removing a strictly dominant row or column does not change the set of optimal strategies.

Example 6.4

$$A_1 = \begin{bmatrix} 2 & 0 & 4 \\ 1 & 2 & 3 \\ 4 & 1 & 2 \end{bmatrix}$$

Note: $\left. \begin{array}{l} 0 < 4 \\ 2 < 3 \\ 1 < 2 \end{array} \right\}$, strict domination.

$$A_2 = \begin{bmatrix} 2 & 0 \\ 1 & 2 \\ 4 & 1 \end{bmatrix}$$

Note: $\begin{array}{cc} 2 & 0 \\ \wedge & \wedge \\ 4 & 1 \end{array}$, strict domination

$$\max \begin{array}{cc} & \min \\ \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix} & \begin{matrix} 1 \\ 1 \end{matrix} \\ 4 & 2 \end{array}$$

Note: No saddle point

Remark 6.5 A row/column can be dominated by a weighted sum of rows columns. For

example,

$$\begin{array}{cccc} a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{array}$$

For $\ell = 1, 2, \dots, n$, we have

$$\alpha a_{i\ell} + (1 - \alpha)a_{j\ell} \geq a_{k\ell}$$

Example 6.6

$$\begin{bmatrix} 0 & 4 & 6 \\ 5 & 7 & 4 \\ 9 & 6 & 3 \end{bmatrix}$$

$$4 > 3$$

$$7 > 4.5$$

$$5 \geq 6$$

7 Apr 11, 2022

7.1 Principle of Indifference

		P2			
		q_1	q_2	\cdots	q_n
P1	p_1	a_{11}	a_{12}	\cdots	a_{1n}
	p_2	a_{21}	a_{22}	\cdots	a_{2n}
	\vdots	\vdots	\vdots	\ddots	\vdots
	p_m	a_{m1}	a_{m2}	\cdots	a_{mn}

Let $\hat{\mathbf{p}} = (p_1, p_2, \dots, p_m)^t$ be an optimal strategy for P1 and let $q_j = 1$ be a pure strategy for P2.

$$\sum_{i=1}^m a_{ij} p_i \geq V \quad (1)$$

Let $\hat{\mathbf{q}} = (q_1, q_2, \dots, q_n)^t$ be an optimal strategy for P2 and let $p_i = 1$ be a pure strategy for P1. Then

$$\sum_{j=1}^n a_{ij} q_j \leq V \quad (2)$$

Note 7.1: If both players use optimal strategies, then

$$\sum_{i=1}^m \sum_{j=1}^n a_{ij} p_i q_j = V$$

Proof. We have

$$\begin{aligned}
 V &\leq \sum_{i=1}^m a_{ij} p_i = 1 \cdot \sum_{i=1}^m a_{ij} p_i = \left(\sum_{j=1}^n q_j \right) \sum_{i=1}^m a_{ij} p_i \\
 &= \sum_{i=1}^m \sum_{j=1}^n a_{ij} p_i q_j \\
 &= \sum_{i=1}^m p_i \underbrace{\sum_{j=1}^n a_{ij} q_j}_{=1} \leq V \\
 &= V
 \end{aligned}$$

□

Theorem 7.2 (The Equilibrium Theorem)

Let $\hat{\mathbf{p}} = (p_1, p_2, \dots, p_m)$ and $\hat{\mathbf{q}} = (q_1, q_2, \dots, q_n)$ be optimal strategies for P1 and P2 respectively. Then

$$\sum_{j=1}^n a_{ij}q_j = V \quad \forall i \text{ s.t. } p_i > 0$$

$$\sum_{i=1}^m a_{ij}p_i = V \quad \forall j \text{ s.t. } q_j > 0$$

Proof. Let $p_k > 0$ and let $\sum_{j=1}^n a_{kj}q_j \neq V \implies \sum_{j=1}^n a_{kj}q_j < V$. We have

$$V \leq \sum_{i=1}^m p_i \sum_{j=1}^n a_{ij}q_j < V$$

a contradiction. □

Example 7.3 (The game of Odd-and-Even)

Played with three numbers: 0, 1, and 2.

		P2, even		
		0	1	2
P1, Odd	0, p_1	0	1	-2
	1, p_2	1	-2	3
	2, p_3	-2	3	-4

$p_1 \geq 0, p_2 \geq 0, p_3 \geq 0$, and $p_1 + p_2 + p_3 = 1$. Then

$$\begin{cases} p_2 - 2p_3 - V = 0 \\ p_1 - 2p_2 + 3p_3 - V = 0 \\ -2p_1 + 3p_2 - 4p_3 - V = 0 \\ p_1 + p_2 + p_3 = 1 \end{cases}$$

7.2 Symmetric Games

Definition 7.4 (Symmetric Game)

The rules are the same for P1 and P2. So $A^t = -A$.

Theorem 7.5

The value of a finite size symmetric game is zero.

Proof. Note $V^t = V$. And

$$V = (\hat{\mathbf{p}})^t A \hat{\mathbf{p}} = [(\hat{\mathbf{p}})^t A \hat{\mathbf{p}}]^t = -\hat{\mathbf{p}} A \hat{\mathbf{p}} = -V$$

So

$$V = -V \implies V = 0$$

□

Example 7.6 (Rock, Paper, Scissors)

We have

		P2		
		Rock	Paper	Scissors
P1	Rock	0	-1	1
	Paper	1	0	-1
	Scissors	-1	1	0

So

$$\begin{cases} p_2 - p_3 = 0 \\ -p_1 + p_3 = 0 \\ p_1 - p_2 = 0 \\ p_1 + p_2 + p_3 = 1 \end{cases}$$

Therefore,

$$p_1 = p_2 = p_3 = \frac{1}{3}$$

8 Apr 13, 2022

8.1 The Equilibrium Theorem and Symmetric Games (Cont'd)

Recall 8.1 The Equilibrium Theorem:

Let $\hat{\mathbf{p}}, \hat{\mathbf{q}}$ be optimal solutions for P1 and P2 respectively. Then

$$\sum_{j=1}^n a_{ij}q_j = V \quad \forall i \text{ s.t. } p_i > 0$$

$$\sum_{i=1}^m a_{ij}p_i = V \quad \forall j \text{ s.t. } q_j > 0$$

Recall 8.2 A game is called a symmetric game if $A^t = -A$.

Example 8.3 (A Mendelson Game)

	1	2	3	4	5	6		99	100
1	0	-1	2	2	2	2	...	2	2
2	1	0	-1	2	2	2	...	2	2
3	-2	1	0	-1	2	2	...	2	2
4	-2	-2	1	0	-1	2	...	2	2
5	-2	-2	-2	1	0	-1	...	2	2
6	-2	-2	-2	-2	1	0	...	2	2
\vdots	-2	-2	-2	-2	-2	1	...		
99	-2	-2	-2	-2	-2	-2	...	0	-1
100	-2	-2	-2	-2	-2	-2	...		0

So

$$\begin{cases} p_2 - 2p_3 = 0 \\ -p_1 + p_3 = 0 \\ 2p_1 - p_2 = 0 \\ p_1 + p_2 + p_3 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} p_3 = p_1 \\ p_2 = 2p_1 \end{cases} \Rightarrow p_1 + 2p_1 + p_1 = 1$$

$$\Rightarrow p_1 = \frac{1}{4} = p_3, \quad p_2 = \frac{1}{2}$$

So

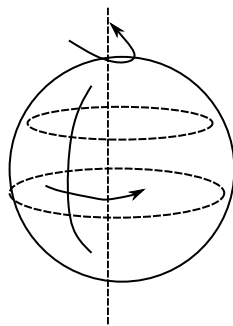
$$\hat{\mathbf{p}} = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, 0, 0, \dots, 0 \right)$$

8.2 Invariance or Games with Symmetries

		P2		
		Rock	Paper	Scissors
P1	Rock	0	-1	1
	Paper	1	0	-1
	Scissors	-1	1	0

We shift to

		P2		
		Paper	Rock	Scissors
P1	Paper	0	1	1
	Rock	-1	0	-1
	Scissors	1	1	0



A group = G

$$G = \{\sigma: \sigma_1 \circ \sigma_2 \in G, e \in G, \sigma^{-1} \circ \sigma = \sigma \circ \sigma^{-1} = e\}$$

Let $S1 = (1, 2, \dots, m)$ and $S2 = (1, 2, \dots, n)$ be the sets of pure strategies for P1 and P2. Let σ be a permutation of S1.

$$\begin{array}{cccc} 1 & 2 & \dots & m \\ \sigma(1) & \sigma(2) & \dots & \sigma(m) \end{array}$$

Definition 8.4 (Invariant)

A finite two-person zero-sum game is invariant under a permutation $\sigma \in S_m$. If $\forall i \in S1$, and $\forall j \in S2$, there exists a unique $j' \in S2$ such that

$$A(i, j) = A(\sigma(i), j')$$

9 Apr 15, 2022

9.1 Invariance (Cont'd)

Recall 9.1 $S1 = \{1, 2, \dots, m\}, S2 = \{1, 2, \dots, n\}$.

Definition 9.2 (Invariant)

A finite two-person zero-sum game is invariant under a permutation σ of $S1$ if $\forall i \in S1, j \in S2$, there exists a unique $j' \in S2$ such that

$$A(i, j) = A(\sigma(i), j')$$

Uniqueness of j' : $\exists j'' \neq j'$ such that

$$A(i, j) = A(\sigma(i), j') = A(\sigma(i), j'')$$

$\forall i \in S1, j \in S2$.

Lemma 9.3

$\bar{\sigma}$ is a permutation of $S2$.

Proof. $\exists j, k \in S2, j \neq k$ such that $\bar{\sigma}(j) =$ □

Lemma 9.4

A game invariant under σ is invariant under σ^{-1}

Proof.

$$\forall i \in S1, j \in S2, A(i, j) = A(\underbrace{\sigma(i)}_{i'}, \underbrace{\bar{\sigma}(j)}_{j'})$$

$$\forall i' \in S1, j' \in S2$$

□

Lemma 9.5

A game invariant under σ_1 and σ_2 is invariant under $\sigma_2 \circ \sigma_1$.

Proof.

$$\forall i \in S1, j \in S2, A(i, j) = A(\underbrace{\sigma_1(i)}_{i'}, j)$$

□

10 Discussion 3 (Apr 14, 2022)

10.1 Diagonal Games/Pick a Number

Example 10.1 • P1 picks a number from 1 to n .

- P2 does the same.
- If they're equal, P1 receives $a_k > 0$, where k is the number chosen.
- Otherwise, no payoff

$$A = \begin{pmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & a_3 & \\ & & & \ddots \\ 0 & & & & a_n \end{pmatrix}$$

Dominance? No: For any $w_1, \dots, w_n \geq 0$, $w_1 + \dots + w_n = 1$, $w_i = 0$, we have

$$a_{ii} > w_1 a_{1i} + \dots + w_n a_{ni}$$

so the i th row is not dominated. If $w_j > 0$, then

$$a_{ij} < w_1 a_{i1} + \dots + a_{in}$$

Principle of indifference:

Let p, q be optimal strategies for P1, P2. If $q_j > 0$, then

$$\begin{cases} q_j > 0 & \sum_{i=1}^n a_{ij} p_i = V \\ p_i > 0 & \sum_{j=1}^n a_{ij} q_j = V \end{cases} \implies \begin{cases} \sum_{i=1}^n a_{ij} p_i = V \\ \sum_{j=1}^n a_{ij} q_j = V \end{cases}$$

Thus

$$\begin{cases} p_i > 0 \\ q_i > 0 \end{cases} \implies \begin{cases} q_i = \frac{1}{a_i} V > 0 \\ p_i = \frac{1}{a_i} V > 0 \end{cases}$$

So there exists $I \subseteq \{1, \dots, n\}$ such that

$$p_i = q_j = \begin{cases} \frac{1}{a_i} V & \text{if } i \in I \\ 0 & \text{otherwise} \end{cases}$$

Fact 10.2: No rows or columns dominated \implies there exists optimal strategies \mathbf{p}, \mathbf{q} , all positive

Observe directly:

Thus $p_i = a_i = \frac{1}{a_i}V$ for all i . Since

$$\sum_{j=1}^n p_j = 1 \implies \left(\sum_{i=1}^n \frac{1}{a_i} \right) V = 1 \implies V = \frac{1}{\frac{1}{a_1} + \dots + \frac{1}{a_n}}$$

If $a_k \not\geq 0$?

If some $a_i \geq 0$ and some $a_j \leq 0$, game has value zero with $\begin{cases} \text{P1 picks } a_i \geq 0 \\ \text{P2 picks } a_j \leq 0 \end{cases}$.

We proceed to prove the fact,

Proof of fact (10.2). Suppose row i is not dominated by any convex combination of other rows. Suppose \mathbf{p} has $p_i = 0$, then

$$\exists j \text{ s.t. } a_{ij} > p_1 a_{1j} + \dots + p_n a_{nj}$$

so, for any $\varepsilon > 0$, the strategy $\sum \mathbf{e}_i + (1 - \varepsilon)\mathbf{p}$ is better if P2 picks j . (TA did not finish this proof) \square

Proposition 10.3

Suppose the square payoff matrix A has an optimal solution with all $p_i, q_j > 0$ and A is non singular (i.e. invertible). Then ???

Proof. As with diagonal game, get

$$\sum_{j=1}^n a_{ij} p_j = V \quad \text{and} \quad \sum_{j=1}^n a_{ij} q_j = V \quad \forall i, j$$

then, $A\mathbf{q} = V\mathbf{1}$ and $\mathbf{p}^t = V\mathbf{1}^t$, so $\mathbf{q} = VA^{-1}\mathbf{1}$, and $\mathbf{p} = V(A^t)^{-1}\mathbf{1}$, and $V = \frac{1}{\mathbf{1}^t A^{-1} \mathbf{1}}$. \square

Note 10.4: If $\mathbf{1}^t A^{-1} \mathbf{1} = 0$, we can replace A by $A + CJ$, where

$$J = \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix}$$