Math 170E (Introduction to Probability)
University of California, Los Angeles

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Summer 2020

Course description: Introduction to probability theory with emphasis on topics relevant to applications. Topics include discrete (binomial, Poisson, etc.) and continuous (exponential, gamma, chi-square, normal) distributions, bivariate distributions, distributions of functions of random variables (including moment generating functions and central limit theorem).

These are my lecture notes for Math 170E (Introduction to Probability and Statistics: Part 1 Probability) taught by Enes Ozel. The main textbook for this class is *Probability and Statistical Inference (10th Edition)* by Robert V. Hogg, Elliot Tanis, and Dale Zimmerman.

Contents

1 June 21, 2020

1.1 Properties of Probability

Definition 1.1 (Sample space)
The sample space, denoted by S, is the whole set of possible outcomes.

Definition 1.2 (Event)
Any subset of S is called an event.

Example 1.3
Let A be the event we will get ≥ 1 head. Then

$$A = \{HH, HT, TH\} \subset S$$

Recall 1.4 (Sets) Let A, B be two subsets of S.

- $A \cap B$: intersection $\{x \in S \colon x \in A \wedge x \in B\}$
- $A \cup B$: union $\{x \in S \colon x \in A \vee x \in B\}$
- $A \subset B$: subset $\forall x \in S, x \in A \implies x \in B$
- $\emptyset$: empty set $\forall x \in S, x \notin \emptyset$
- $A' = \overline{A} = A^C$: complement $\{x \in S \colon x \notin A\}$
- $A \setminus B = A \cap B^C = \{x \in S, x \in A \wedge x \notin B\}$
- $A \cap B = \emptyset$ "mutually exclusive"

Example 1.5
Let A be the event we will get ≥ 1 head, B be the event we get both tails. Then

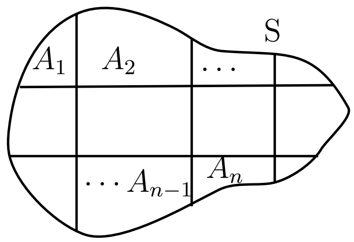
$$A = \{HH, TH, HT\} \quad B = \{TT\}$$

Then

$$A \cup B = S$$

Definition 1.6 (Exhaustive)
$A_1, \dots, A_n \subseteq S$ are exhaustive if $A_1 \cup A_2 \cup \cdots \cup A_n = S$, i.e. $\bigcup_{i=1}^n A_i = S$.

Here, we have mutually exclusive and exhaustive events.



Definition 1.7 (Probability)
Probability is a real-valued set function P that assigns, to each event A in the sample space S, a number $P(A)$, called the probability of the event A, such that the following properties are satisfied:

(a) $P(A) \geq 0$

(b) $P(S) = 1$

(c) For $\{A_i\}_{i=1}^{\infty} \subseteq S$ such that the sets are mutually exclusive,

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Note, $P(A) \in \mathbb{R}$ and $P(S) \to \mathbb{R}$.

Theorem 1.8
$\forall A \subseteq S, P(A') = 1 - P(A)$.

Proof. Let $A \subseteq S$. Then $S = A \cup A' \implies$ mutually exclusive and exhaustive

$$\begin{aligned} \implies P(A \cup A') &= P(A) + P(A') \\ &= P(S) \\ &= 1 \end{aligned}$$

So

$$P(A) + P(A') = 1 \implies P(A') = 1 - P(A)$$

□

Corollary 1.9
$P(\emptyset) = 0$.

Proof. $\emptyset^C = S$, so

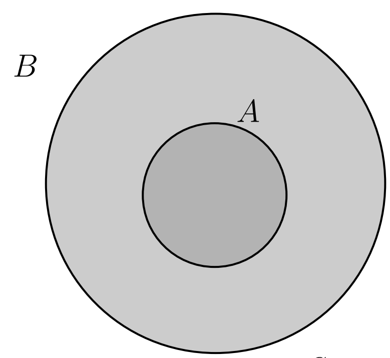
$$\begin{aligned} P(\emptyset) &= 1 - P(S) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

□

Example 1.10
Flip two coins,
$$S = \{HH, HT, TT, TH\}$$
assuming a fair coin toss, all have 1/4 probability.
$A =$ "both heads" so
$$P(A) = \frac{1}{4}$$
$$P(A') = \frac{3}{4}$$
$$A' = \{HT, TT, TH\}$$
where A' is "NOT both heads" = "at least one tail"

Theorem 1.11
Let $A, B \subseteq S$ such that $A \subseteq B$. Then $P(A) < P(B)$.

Proof. $A \subseteq B \implies B = A \cup (B \cap A')$



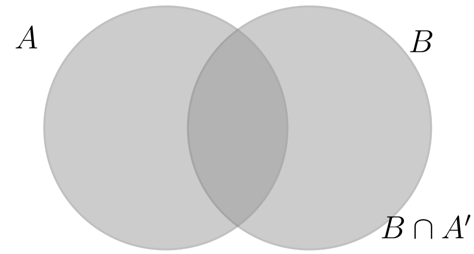
$B \setminus A = B \cap A^C$

$$P(B) = P(A) + \underbrace{P(B \cap A')}_{\geq 0} \geq P(A)$$
$$\implies P(B) \geq P(A)$$
□

Corollary 1.12
$\forall A \subseteq S, 0 \leq P(A) \leq 1.$

Theorem 1.13
If A and B are any two events, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof. Decompose the units into two disjoint parts



$$P(A \cup B) = P(A) + P(B \cap A')$$

But

$$P(B) = P(A \cap B) + P(A' \cap B)$$

So

$$P(B \cap A') = P(B) - P(A \cap B)$$

Hence

$$\begin{aligned} P(A \cup B) &= P(A) + P(B \cap A') \\ &= P(A) + P(B) - P(A \cap B) \end{aligned}$$

□

Example 1.14
We have a fair die, which we roll once. So

$$S = \{1, 2, \ldots, 6\}$$

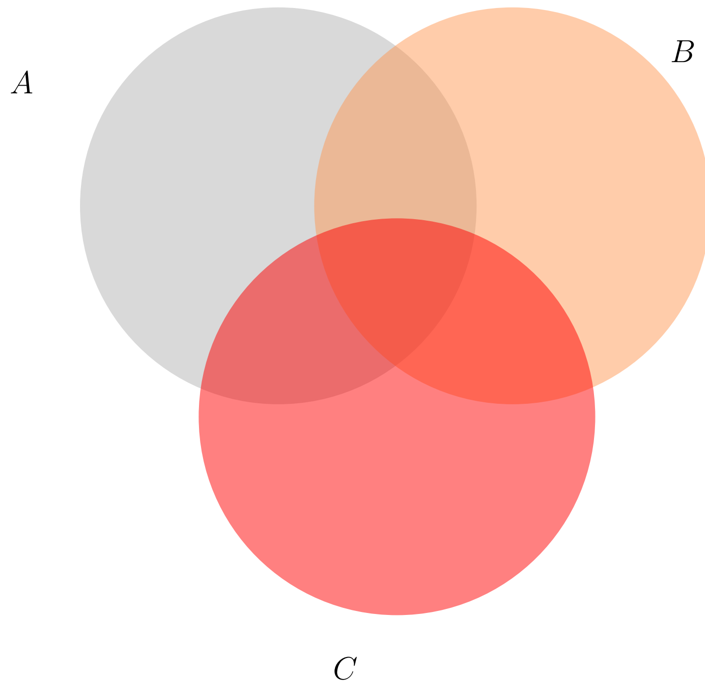
with probability $1/6$ each. We call the outcome X. So

$$\begin{aligned} P(2|X \text{ or } 3|X) &= P(2|X) + P(3|X) - P(6|X) \\ &= P(\{2,4,6\}) + P(\{3,6\}) - P(\{6\}) \\ &= \frac{3}{6} + \frac{2}{6} - \frac{1}{6} \\ &= \frac{2}{3} \end{aligned}$$

Theorem 1.15
If A, B, and C are any three events, then

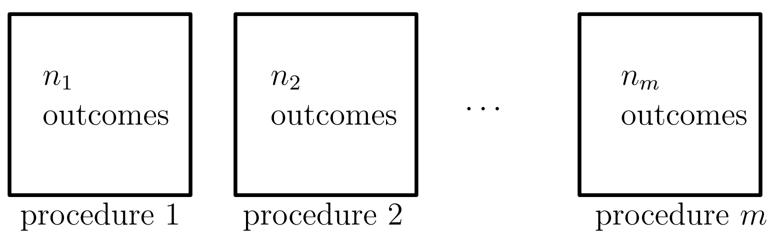
$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Proof.



Exercise. □

1.2 Methods of Enumeration



We are interested in the number of ways the overall outcome may be formed, which we calculate using the formula $n_1 \times n_2 \times \cdots \times n_m$.

Example 1.16 (Cafe, deli sandwich)
Suppose a cafe has

	Bread	Meat	Cheese	Garnishes
	6	4	4	12
# that can be chosen	1	0,1	0,1	$0, 1, 2, \cdots, 12$

So

$$\# \text{ different sandwiches } = 6 \times 5 \times 5 \times 2^{12}$$

Suppose we have n people to be placed. Then the number of ways to arrange them is given by

$$\begin{aligned} \#\text{ ways } &= \underset{1^{st}}{n} \qquad \underset{2^{nd}}{n-1} \qquad \underset{3^{rd}}{n-2} \qquad \cdots \qquad \underset{n-1^{st}}{2} \qquad \underset{n^{th}}{1} \\ &= n! \end{aligned}$$

Example 1.17
Suppose $S = \{a, b, c, d\}$. Then the number of permutations is $4! = 24$.

$$\left.\begin{matrix} abcd \\ acbd \\ \vdots \end{matrix}\right\} 24$$

If we allow repetitions,

$$4 \times 4 \times 4 \times 4 = 4^4 = 256$$

Definition 1.18 (Permutation)
Suppose we have n objects/people and $r \leq n$ positions, then the number of ways to arrange them is given by

$$\begin{aligned} \#\text{ of ways } &= \underset{1^{st}}{n} \qquad \underset{2^{nd}}{n-1} \qquad \underset{3^{rd}}{n-2} \qquad \cdots \qquad \underset{r-1^{st}}{n-r+2} \qquad \underset{r^{th}}{n-r+1} \\ &= n \times (n-1) \times (n-2) \times \cdots \times (n-r+2) \times (n-r+1) \\ &= {}_nP_r \end{aligned}$$

which is the <u>number of permutations of n objects taken r at a time</u>.
We define the <u>r^{th} falling factorial of n</u> as

$$(n)_r = n \cdot (n-1) \cdots (n-r+1)$$

and the <u>factorial of n</u> as

$$(n)_n = n \cdot (n-1) \cdots (n-n+1) = n!$$

2 June 24, 2020

2.1 Binomial Coefficients

Definition 2.1 (Binomial coefficient)
We are interested in the number of ways to choose r objects out of n objects. The number is given by

$$\begin{aligned}\# &= \frac{{}_nP_r}{r!} = \frac{(n)_r}{r!} \\ &= n(n-1)\cdots(n-r+1) \\ &= \frac{n(n-1)\cdots(n-r+1)}{r!}\frac{(n-r)!}{(n-r)!} \\ &= \frac{n!}{r!(n-r)!} \\ &= \binom{n}{r}\end{aligned}$$

which we call the binomial coefficient.

Example 2.2
We have a group of 5 people. We want to choose 2 to govern, and it does not matter who is president/treasurer.

$$\#\text{ ways } = \binom{5}{2} = \frac{5!}{2!3!} = 10$$