

Math 120A (Differential Geometry)

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These are my lecture notes for Math 120A (Differential Geometry), which is taught by Fumiaki Suzuki. The textbook for this class is *Differential Geometry of Curves and Surfaces*, by Kristopher Tapp. Many of the figures I include in these notes are taken from Tapp's book.

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1 Jan 3, 2022

1.1 What is Differential Geometry?

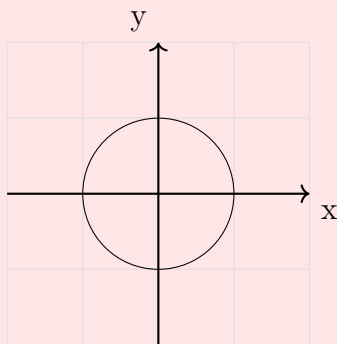
Differential geometry studies geometry via analysis and linear algebra.

Geometry	Analysis	Linear Algebra
Intuitive	Rigorous	Computable
Curved	$\xrightarrow{\text{tangent space}}$	Linear
Global	Local	

1.2 Parametrized Curves

Example 1.1

A unit circle $S' = \{\vec{x} \text{ in } \mathbb{R}^2 \mid |\vec{x}| = 1\}$



$$\begin{aligned}\vec{\gamma} &: [0, 2\pi) \rightarrow \mathbb{R}^2 \\ t &\mapsto (\cos t, \sin t)\end{aligned}$$

$$\vec{\gamma}[0, 2\pi) = S'$$

Definition 1.2 (Parametrized curve and Trace)

A (parametrized) curve is a smooth function $\vec{\gamma}: I \rightarrow \mathbb{R}^n$, where I is an interval in \mathbb{R} . The image

$$\vec{\gamma}(I) = \{\vec{\gamma}(t) \mid t \in I\}$$

is called the trace of $\vec{\gamma}$.

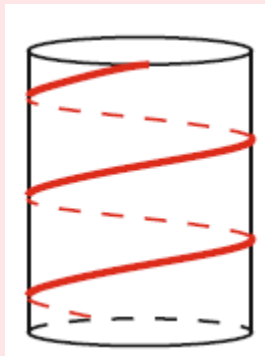
Recall 1.3 An interval is a subset of \mathbb{R} that has one of the following forms:

$$(a, b), [a, b], (a, b], [a, b), (-\infty, b), (-\infty, b], (a, \infty), [a, \infty), (-\infty, \infty) = \mathbb{R}.$$

A function $\vec{\gamma}: I \rightarrow \mathbb{R}^n$ is called smooth if $\vec{\gamma}$ is infinitely differentiable, or equivalently, each of the component functions $x_i: I \rightarrow \mathbb{R}$ is infinitely differentiable.

Example 1.4

$\vec{\gamma}(t) = (\cos t, \sin t, t)$, $t \in (-\infty, \infty)$ is a curve, called a helix.

**Definition 1.5** (Derivative)

Let $\vec{\gamma}: I \rightarrow \mathbb{R}^n$ be a curve. The derivative of $\vec{\gamma}$ at t is defined as

$$\vec{\gamma}'(t) = \lim_{h \rightarrow 0} \frac{\vec{\gamma}(t+h) - \vec{\gamma}(t)}{h}$$

If t is on the boundaries of I , then use the left- or right-hand limit.

Remarks 1.6

- i. If $\vec{\gamma}(t) = (x_1(t), x_2(t), \dots, x_n(t))$, then $\vec{\gamma}'(t) = (x_1'(t), x_2'(t), \dots, x_n'(t))$.
- ii. The tangent line to the curve at $\vec{\gamma}(t_0)$ is defined as

$$\vec{L}(t) = \vec{\gamma}(t_0) + t\vec{\gamma}'(t_0), \quad t \in (-\infty, \infty),$$

as soon as $\vec{\gamma}'(t) \neq \vec{0}$.

Definition 1.7 (Regular)

A curve $\vec{\gamma}: I \rightarrow \mathbb{R}^n$ is called regular if $\forall t \in I, \vec{\gamma}'(t) \neq \vec{0}$.

Remark 1.8 regular = tangent line is defined everywhere = the trace is smooth

Example 1.9

$$\vec{\gamma}(t) = (t^2, t^3), \quad t \in (-\infty, \infty)$$

Then $\vec{\gamma}$ is a curve that is not regular.

Indeed, $\vec{\gamma}'(t) = (2t, 3t^2)$, so $\vec{\gamma}'(0) = \vec{0}$.

Notice, $x(t) = t^2, y(t) = t^3$, so $x(t) = y(t)^{2/3}$. Hence, the trace is given by $x = y^{2/3}$ in \mathbb{R}^2 .

Remark 1.10 The analogy with the physics is useful. If $\vec{\gamma}: I \rightarrow \mathbb{R}^n$ is a curve, then $\vec{\gamma}(t)$ is the position of a moving particle at time t in \mathbb{R}^2 .

- $\vec{\gamma}'(t)$ velocity

- $\vec{\gamma}''(t)$ acceleration
- $|\vec{\gamma}'(t)|$ speed

In this analogy, regular = the speed is always nonzero = the particle never stops (hence no "corners" on the trace)

Definition 1.11 (Arc length)

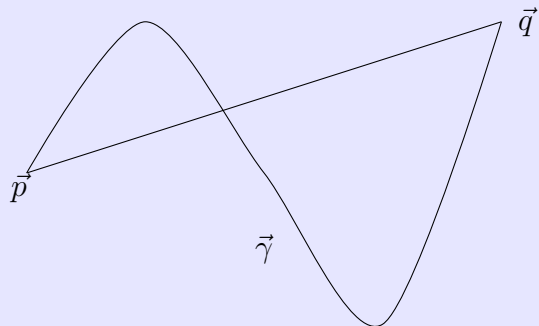
Let $\vec{\gamma}(t): I \rightarrow \mathbb{R}^n$ be a regular curve. Then the arc length between times t_1, t_2 is defined as

$$\int_{t_1}^{t_2} |\vec{\gamma}'(t)| dt$$

Proposition 1.12

Let $\vec{\gamma}: [a, b] \rightarrow \mathbb{R}^n$ be a regular curve with the arc length L , $\vec{p} = \vec{\gamma}(a), \vec{q} = \vec{\gamma}(b)$. Then $L \geq |\vec{q} - \vec{p}|$.

Moreover, the equality holds if and only if $\vec{\gamma}$ parametrizes the line segment between \vec{p}, \vec{q} .



For the proof, we use the inner-product:

for $\vec{x} = (x_1, x_2, \dots, x_n), \vec{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$,

$$\langle x, y \rangle := x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

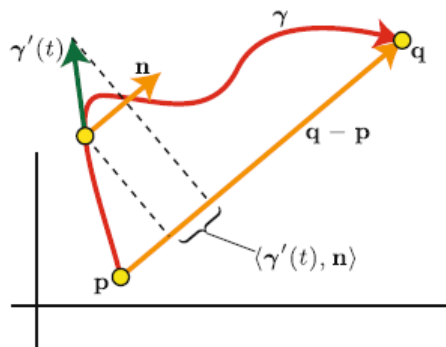
Basic properties:

- The inner product $\langle -, - \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is symmetric and bilinear.
- $\langle \vec{x}, \vec{y} \rangle = |\vec{x}| |\vec{y}| \cos \theta$, where θ is the angle between \vec{x}, \vec{y} . ($\theta \in [0, 2\pi]$)
- $\langle \vec{x}, \vec{y} \rangle = 0 \Leftrightarrow \vec{x}, \vec{y}$ are orthogonal to each other.
- $\langle \vec{x}, \vec{x} \rangle = |\vec{x}|^2$
- $\langle \vec{x}, \vec{y} \rangle \leq |\vec{x}| |\vec{y}|$ (Schwartz Inequality) and the equality holds if and only if $\theta = 0$.

2 Jan 5, 2022

2.1 Proof of Proposition 1.12

Proof. Idea: Compare $\vec{\gamma}'(t)$ and its projection onto $\vec{q} - \vec{p}$. Set $\vec{n} = \frac{\vec{q} - \vec{p}}{|\vec{q} - \vec{p}|}$; \vec{n} is unit.



Tapp Pg.15

Then $|\vec{\gamma}'(t)| \geq \langle \vec{\gamma}'(t), \vec{n} \rangle$ by Schwartz inequality.

Now,

$$\begin{aligned} L &= \int_a^b |\vec{\gamma}'(t)| dt \geq \int_a^b \langle \vec{\gamma}'(t), \vec{n} \rangle dt \\ &= [\langle \vec{\gamma}(t), \vec{n} \rangle]_a^b = \langle \vec{\gamma}(b), \vec{n} \rangle - \langle \vec{\gamma}(a), \vec{n} \rangle \\ &= \left\langle \vec{q} - \vec{p}, \frac{\vec{q} - \vec{p}}{|\vec{q} - \vec{p}|} \right\rangle = |\vec{q} - \vec{p}| \end{aligned}$$

If the equality holds, then $\forall t \in [a, b]$, $\vec{\gamma}'(t), \vec{n}$ are in the same direction. So,

$$\begin{aligned} \vec{\gamma}'(t) &= \langle \vec{\gamma}'(t), \vec{n} \rangle \vec{n}. \\ \vec{\gamma}(t) &= \vec{\gamma}(a) + \int_a^t \vec{\gamma}'(u) du \\ &= \vec{p} + \left(\int_a^t \langle \vec{\gamma}'(u), \vec{n} \rangle dt \right) \vec{n} \end{aligned}$$

parametrizes the line segment between \vec{p}, \vec{q} . □

2.2 Reparametrization

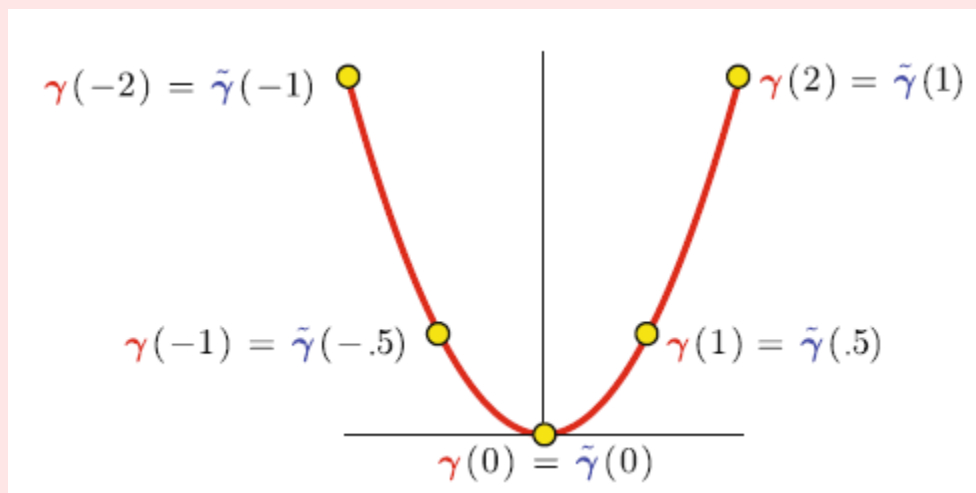
There are regular curves that share common properties. Which regular curves should we identify?

Example 2.1

$$\vec{\gamma}(t) = (t, t^2), \quad t \in [-2, 2]$$

$$\tilde{\gamma}(t) = (-2t, (-2t)^2), \quad t \in [-1, 1].$$

Then $\vec{\gamma}[-2, 2] = \tilde{\gamma}[-1, 1] =$



$\vec{\gamma}, \tilde{\gamma}$ are the same, up to change in time:

Let $\phi: [-1, 1] \rightarrow [-2, 2], \quad t \mapsto -2t.$

Then $\tilde{\gamma} = \vec{\gamma} \circ \phi$

Definition 2.2 (Reparametrization)

Let $\vec{\gamma}: I \rightarrow \mathbb{R}^n$ be a regular curve. A reparametrization of $\vec{\gamma}$ is a function of the form

$$\tilde{\gamma} = \vec{\gamma} \circ \phi: \tilde{I} \rightarrow \mathbb{R}^n,$$

where \tilde{I} is an interval, $\phi: \tilde{I} \rightarrow I$ is a smooth bijection such that $\forall t \in \tilde{I}, \phi'(t) \neq 0$

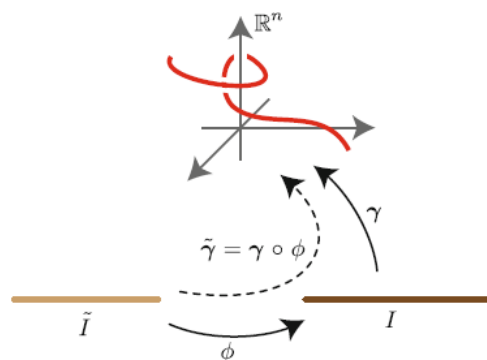


Figure 1: Kapp pg.19

Proposition 2.3

A reparametrization of a regular curve is a regular curve.

Proof. We use the same notations as the definition.

$\tilde{\gamma} = \gamma \circ \phi: \tilde{I} \rightarrow \mathbb{R}^n$ is the composition of smooth functions, so smooth.

Moreover, $\forall t \in \tilde{I}, \tilde{\gamma}'(t) = \gamma'(\phi(t)) \cdot \phi'(t) \neq 0$ □

We will be interested in regular curves up to reparametrizations.

Remarks 2.4

1. $\gamma, \tilde{\gamma}$ have the same trace.
2. There are regular curves with the same trace that cannot be reparametrized to each other. For instance,

$$\begin{aligned}\gamma_1(t) &= (\cos(t), \sin(t)), t \in [0, 2\pi), \\ \gamma_2(t) &= (\cos(t), \sin(t)), t \in [0, 4\pi),\end{aligned}$$

Question 2.5: Is there a canonical reparametrization of a given regular curve?

Definition 2.6 (Unit-speed)

A regular curve $\gamma: I \rightarrow \mathbb{R}^n$ is called unit-speed (or parametrized by arc length) if $\forall t \in I, |\gamma'(t)| = 1$.

Remark 2.7 If $\gamma: I \rightarrow \mathbb{R}^n$ is unit-speed, then,

$$\text{Arc length between } t_1, t_2 = \int_{t_1}^{t_2} |\gamma'(t)| dt = \int_{t_1}^{t_2} dt = t_2 - t_1$$

Proposition 2.8

A regular curve always has a unit-speed reparametrization.

Proof. Let $\gamma: I \rightarrow \mathbb{R}^n$ be a regular curve. Fix $t_0 \in I$. Define $s: I \rightarrow \mathbb{R}$ by $s(t) = \int_{t_0}^t \gamma'(u) du$.

Let $\tilde{I} = s(I) \subset \mathbb{R}$. Then \tilde{I} is an interval by IVT.

Since $s'(t) = |\gamma'(t)| > 0$ by FTC, regularity, $s: I \rightarrow \tilde{I}$ is a smooth bijection. Then, $\phi = s^{-1}: \tilde{I} \rightarrow I$ is a smooth bijection,

$$\phi'(t) = \frac{1}{s'(\phi(t))} = \frac{1}{|\gamma'(\phi(t))|} \neq 0.$$

Now $\tilde{\gamma} = \gamma \circ \phi: \tilde{I} \rightarrow \mathbb{R}^n$ is a reparametrization of γ , that is unit-speed:

$$\begin{aligned}|\tilde{\gamma}'(t)| &= |\gamma'(\phi(t)) \cdot \phi'(t)| \\ &= |\gamma'(\phi(t))| \cdot 1/|\gamma'(\phi(t))| \\ &= 1\end{aligned}$$

□

Note:

$$s^{-1} \cdot s(t) = t$$

$$(s^{-1})'(s(t)) \cdot s'(t) = 1$$

$$(s^{-1})'(s(t)) = 1/s'(t)$$

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3.1 Reparametrization (Cont'd)

Example 3.1

$\vec{\gamma}(t) = (\cos(t), \sin(t), t)$, $t \in (-\infty, \infty)$ How can we find a unit-speed reparametrization of $\vec{\gamma}$? Compute the arc length function $S: (-\infty, \infty) \rightarrow \mathbb{R}$:

$$\begin{aligned} s(t) &= \int_0^t |\vec{\gamma}'(u)| \, du = \int_0^t |(-\sin(u), \cos(u), 1)| \, du \\ &= \int_0^t \sqrt{2} \, du = \sqrt{2}t \end{aligned}$$

Set $\phi = s^{-1}$, then $\phi(t) = t/\sqrt{2}$

$$\tilde{\gamma}(t) = \vec{\gamma}(\phi(t)) = (\cos(t/\sqrt{2}), \sin(t/\sqrt{2}), t/\sqrt{2})$$

$t \in (-\infty, \infty)$, is a unit speed reparametrization of $\vec{\gamma}$.

We will be interested in invariants for a regular curve that are unchanged under any reparametrizations.

Examples include:

- trace
- arc-length
- curvature
- torsion

Non-examples include:

- position
- velocity
- speed
- acceleration

Sometimes we consider more specific reparametrization.

Proposition 3.2

If $\tilde{\gamma} = \vec{\gamma} \cdot \phi: \tilde{I} \rightarrow \mathbb{R}^n$ is a reparametrization of a regular curve $\vec{\gamma}: I \rightarrow \mathbb{R}^n$, then one of the following holds:

- i. $\forall t \in \tilde{I}, \phi'(t) > 0$ i.e. ϕ is strictly increasing
- ii. $\forall t \in \tilde{I}, \phi'(t) < 0$ i.e. ϕ is strictly decreasing

Proof. Otherwise $\exists t \in \tilde{I}, \phi'(t) = 0$ by IVT. This contradicts the assumption on ϕ . \square

Definition 3.3 (Orientation-preserving vs. orientation-reversing)

Under the setting of the proposition, we say $\tilde{\gamma}$ is orientation-preserving if (i) occurs, or orientation-reversing if (ii) occurs.

Example 3.4 (Orientation-preserving)

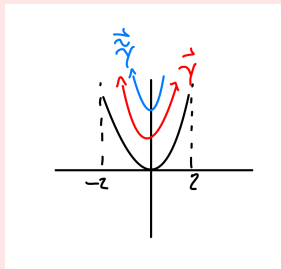
The arc length reparametrization of a regular curve $\phi: I \rightarrow \tilde{I}$ is orientation-preserving, because $\phi'(t) = 1/|\tilde{\gamma}'(\phi(t))| > 0 \quad \forall t \in I$

This shows an orientation=preserving unit-speed. Reparametrization always exists.

Example 3.5 (Orientation-reversing)

$$\tilde{\gamma}(t) = (t, t^2), \quad t \in [-2, 2]$$

$$\tilde{\gamma}(t) = (-t, (-t)^2), \quad t \in [2, 2]$$



$\tilde{\gamma}$ is an orientation-reversing reparametrization of $\tilde{\gamma}$ by $\phi: [-2, 2] \rightarrow [-2, 2], \quad t \mapsto -t$ (Indeed, $\phi' = -1 < 0$).

We will be interested in invariants that are unchanged under any orientation-preserving reparametrization.

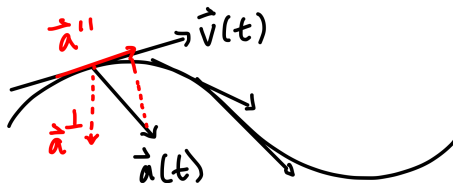
Example 3.6

signed curvature, rotation index

3.2 Curvature

The curvature measures how sharply the trace bends. What is a plausible definition of the curvature?

Let $\tilde{\gamma}: I \rightarrow R^n$ be a regular curve. Set $\vec{v} = \tilde{\gamma}', \vec{a} = \tilde{\gamma}''$



\vec{v} knows speed, direction of the motion

$\implies \vec{a}$ should know the change in speed, direction \rightarrow curvature.

We write

$$\vec{a} = \vec{a}'' + \vec{a}^\perp$$

where

$$\begin{aligned}\vec{a}'' &= \left\langle \vec{a}, \frac{\vec{v}}{|\vec{v}|} \right\rangle : \text{parallel to } \vec{v} \\ \vec{a}^\perp &= \vec{a} - \vec{a}' : \text{orthogonal to } \vec{v}\end{aligned}$$

Proposition 3.7

$\frac{d}{dt}|\vec{v}(t)| = \left\langle \vec{a}, \frac{\vec{v}}{|\vec{v}|} \right\rangle$ = the parallel component of \vec{a} with respect to \vec{v}

Proof.

$$\begin{aligned}\frac{d}{dt}|\vec{v}(t)| &= \frac{d}{dt} \langle \vec{v}(t), \vec{v}(t) \rangle^{1/2} \\ &= \frac{1}{2} \frac{1}{\langle \vec{v}(t), \vec{v}(t) \rangle^{1/2}} \cdot 2 \langle \vec{v}(t), \vec{v}(t) \rangle' \\ &= \left\langle \frac{\vec{v}(t)}{|\vec{v}(t)|}, \vec{a}(t) \right\rangle\end{aligned}$$

Note: $\langle v, v \rangle' = \langle v', v \rangle + \langle v, v' \rangle = 2\langle v', v \rangle$ □

So $|\vec{a}^\perp(t)|$ would be a plausible definition of the curvature. however this depends on $|\vec{t}|$. (Imagine a centripetal force for a car turning a corner.)

Definition 3.8 (Curvature)

Let $\tilde{\gamma}: I \rightarrow \mathbb{R}^n$ be a regular curve. The curvature function $\kappa: I \rightarrow [0, \infty)$ is defined as

$$\kappa(t) = \frac{|\vec{a}^\perp(t)|}{|\vec{v}(t)|^2}$$

Proposition 3.9

Curvature is independent of parametrizations.

Proof. Let γ be a regular curve. $\tilde{\gamma} = \gamma \circ \phi$ is a reparametrization of γ .

Denote:

κ : curvature function for γ

$\tilde{\kappa}$: curvature function for $\tilde{\gamma}$

We need to show $\tilde{\kappa} = \kappa \circ \phi$

Denote:

v, a : velocity, acceleration of γ

\tilde{v}, \tilde{a} : velocity, acceleration of $\tilde{\gamma}$.

Then,

$$\begin{aligned}\tilde{\gamma} &= \gamma \circ \phi \\ \tilde{v} &= \gamma' \circ \phi \cdot \phi' = v \circ \phi \cdot \phi' \\ \tilde{a} &= \gamma'' \circ \phi \cdot (\phi')^2 + \gamma' \circ \phi \cdot \phi'' \\ &= a \circ \phi \cdot (\phi')^2 + v \circ \phi \cdot \phi''\end{aligned}$$

So, \tilde{v} is parallel to v ,

$$\tilde{a}^\perp = a^\perp \circ \phi \cdot (\phi')^2$$

Therefore,

$$\begin{aligned}\tilde{\kappa} &= \frac{\tilde{a}^\perp}{|\tilde{v}|^2} = \frac{|a^\perp \circ \phi \cdot (\phi')^2|}{|v \circ \phi \cdot \phi|^2} = \frac{|a^\perp \cdot \phi|}{|v \cdot \phi|^2} \\ &= \kappa \circ \phi\end{aligned}$$

□