Math 136 (Partial Differential Equations)
University of California, Los Angeles

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These are my lecture notes for Math 136 (Partial Differential Equations) taught by Marcus Roper. The main textbook for this class is *Partial Differential Equations: An Introduction* by Walter Strauss.

Contents

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1.1 Motivation

Motivating example: Suppose we want to describe where the gas molecules are in a room.

Approach 1: Label every gas molecule and give x, y, z coordinates for each.

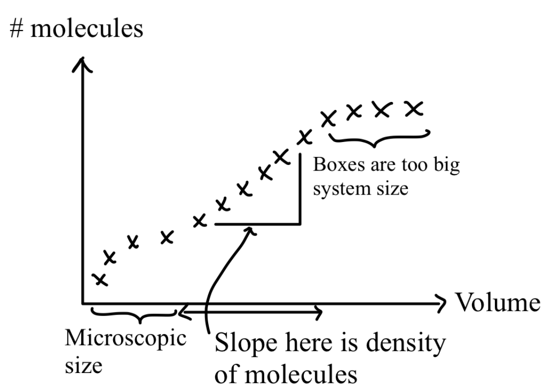
- Too much data.

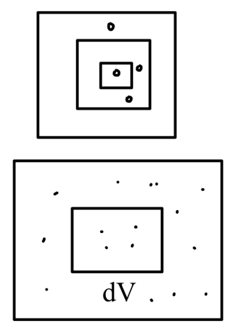
Approach 2: Divide the room up into small volumes (boxes). Count the number of molecules in each volume.

- Number of molecules depends on the size of the box
- Take density/concentration:

$$\frac{\text{\# molecules in the box}}{\text{volume of the box}}$$

We assume the distribution of molecules obeys the Continuum Hypothesis.





We assume our box sizes are in a region in which the number of molecules or volume, so density is well-defined. We defined. We define a field $u(\mathbf{x}, t)$ that describes the density of molecules.

At each point u counts molecules at $(\mathbf{x}, t)$; in the sense that if I make a box, volume dV, at $(\mathbf{x}, t)$; the number of molecules in box is: $u(\mathbf{x}, t)\mathrm{dV}$.

Note 1.1: u is dependent variable, and there are multiple independent variables; x, y, z, t

u (density) is one example of a field - a dependent variable that is defined at different points:

$$u\colon \mathbb{R}^3 \subset D \times \underbrace{[0, T]}_{\text{time interval}} \to \mathbb{R}$$

$$u\colon (\mathbf{x}, t) \mapsto u(\mathbf{x}, t)$$

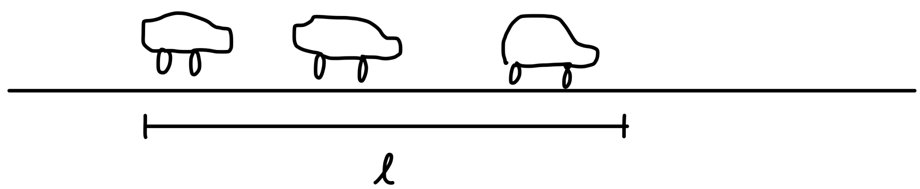
Other examples:

- Velocity (vector field); $\mathbf{v}\colon D \times [0, T] \to \mathbb{R}^3$
- temperature (scalar field); $\theta\colon D \times [0, T] \to \mathbb{R}$
- Electric field/magnetic field
- Distribution/density of cars
- Displacement of the ocean surface

We will derive (and solve) Partial Differential Equations (PDEs) as mathematical models for scalar and vector fields that depend on position (and in many cases, time).

1.2 Example of a PDE

We are modeling the density/distribution of cars on a freeway (looking at only one direction).



Count number of cars in some length ℓ of freeway.

$$\text{density of cars } = \frac{\#\text{ in length } \ell}{\ell}$$

e.g. if $\ell = 1$ km; and 1 count 30 cars $\implies u = 30/\text{km}$ given this density.

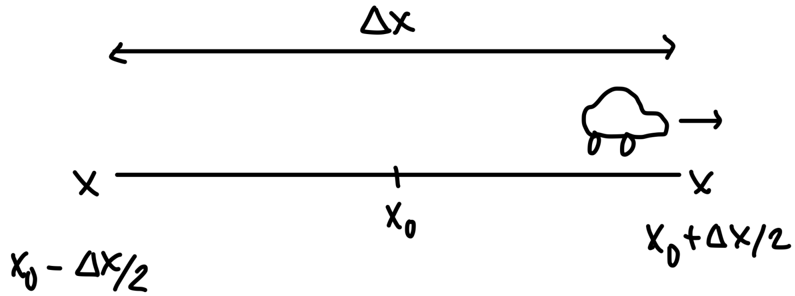
If $\ell = 500m$,

$$\begin{aligned}\# &= 30 \text{ km } \times 500 \text{ m} \\ &= 30/\text{km } \times 0.5 \text{ km} \\ &= 15 \text{ cars}\end{aligned}$$

$u \equiv u(x, t)$; u depends on x (distance along freeway) and t (time).

I will assume (Continuum-Hypothesis) that densities calculated at each poinn in the freeway give rise to a C^1 (continuously differentiable) field u. I want to derive an equation for u.

<u>Calculus idea:</u> If I know $u(x, t)$, I want to calculate the density shortly after; $u(x, t + \Delta t)$. If I can do this then I can calculate $u(x, t + 2\Delta t), u(x, t + 3\Delta t), \ldots$. Imagine that all cars drive at the same speed, c. Consider # cars in some interval.



At time t, there are $u(x_0, t)\Delta x$ cars in the interval.

$$\begin{aligned}\#\text{ cars at time } t + \Delta t = \#\text{ cars at time } t + \#\text{ entering at } x_0 - \Delta x/2 \\ - \#\text{ leaving at } x_0 + \Delta x/2\end{aligned}$$

This is the word statement of conservation of mass/cars.

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2.1 Example of a PDE (Cont'd)

Recall 2.1

$$\begin{aligned}\#\text{ cars at time } t+\Delta t = \#\text{ cars at time } t + \#\text{ entering at } x_0 - \Delta x/2\\ - \#\text{ leaving at } x_0 + \Delta x/2\end{aligned}$$

Therefore,

$$u(x_0, t+\Delta t)\Delta x = u(x_0, t)\Delta x + \underline{\qquad\qquad\qquad\qquad}$$

Let's fill in the # cars entering or leaving.

Consider station at $x = x_0 + \Delta x/2$, how many cars pass this station in time Δt?

All of the cars to my left, that are within distance $c\Delta t$ of me, will pass in time Δt. In time Δt a car travels distance $c\Delta t$, so # cars in the interval is

$$\underbrace{u(x_0 + \Delta x/2, t)}_{\text{density}} \times \underbrace{c\Delta t}_{\text{length}}$$

We will show it doesn't change anything if we use $u(x_0 + \frac{\Delta x}{2} - \frac{1}{2}c\Delta t, t)$ instead.

Returning to conservation of cars:

$$u(x_0, t+\Delta t)\Delta x = u(x_0, t)\Delta x + \underbrace{u(x_0 - \Delta x/2, t)c\Delta t}_{\#\text{ entering}} - \underbrace{u(x_0 + \Delta x/2, t)c\Delta t}_{\#\text{ leaving}}$$

$$\Big(u(x_0, t+\Delta t) - u(x_0, t)\Big)\Delta x = -\Big(u(x_0 + \Delta x/2, t) - u(x_0 - \Delta x/2, t)\Big)c\Delta t \tag{1}$$

Recall

$$\frac{\partial u}{\partial t}(x,t) = \lim_{h\to 0}\left(\frac{u(x, t+h) - u(x,t)}{h}\right)$$

So,

$$\begin{aligned}\frac{\partial u}{\partial x}(x,t) &= \lim_{h\to 0}\left(\frac{u(x+h, t) - u(x,t)}{h}\right)\\ &= \lim_{h\to 0}\left(\frac{u(x+h/2, t) - u(x-h/2,t)}{h}\right)\end{aligned}$$

Now dividing (1) by $\Delta x \Delta t$

$$\frac{u(x_0, t+\Delta t) - u(x_0, t)}{\Delta t} = -c\left(\frac{u(x_0 + \Delta x/2, t) - u(x_0 - \Delta t/2, t)}{\Delta x}\right)$$

let $\Delta x \to 0, \Delta t \to 0$, then

$$\begin{aligned}\frac{\partial u}{\partial t} &= -c\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} &= 0\end{aligned} \tag{2}$$

u is a dependent variable, it depends on x and t as independent variables.

There is also a constant, parameter c.

Notation 2.2: Other notations are used for partial derivatives.

$$\underbrace{u_t + cu_x = 0}_{\text{Strauss}} \qquad \text{or} \qquad \underbrace{u_{,t} + cu_{,x}}_{\text{Roper}}$$

There are some solutions of (2).

$$\begin{aligned} u &= 1 + \frac{1}{2}\sin(x - ct) \\ u &= \frac{1}{2}(x - ct)^2 \\ u &= e^{-x+ct} \end{aligned}$$

We can check these are solutions

$$\begin{aligned} u(x,t) &= e^{-x+ct} \\ \frac{\partial u}{\partial t} &= ce^{-x+ct} \\ \frac{\partial u}{\partial x} &= -e^{-x+ct} \end{aligned}$$

So

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = ce^{-x+ct} + c(-e^{-x+ct}) = 0$$

Compare with ODEs:

$$\frac{dy}{dx} = y$$

has solution $y(x) = Ce^x$ which has a constant of integration.

2.2 Linearity

In ODEs we use initial conditions to find our constants. To solve a PDE completely, we need both the PDE and an auxiliary or side condition. That is, we need either initial conditions or boundary conditions (or both) on u. (2) is an example of a PDE.

Definition 2.3 (Operator)
Most generally, a PDE takes the form:

$$\mathcal{L}[u] = 0$$

We call $\mathcal{L}$ an <u>operator</u>.

In this case:

$$\mathcal{L}[u] = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}$$

$\mathcal{L}$ includes derivatives of u, more derivatives are possible: e.g.:

$$\mathcal{L}[u] = \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}$$

Definition 2.4 (Linear operator)
We say an operator is linear if it has the following properties:

1. If $\mathcal{L}[u] = 0$ and a is constant, then $\mathcal{L}[au] = 0$.
2. If u_1, u_2 solve the PDE, $\mathcal{L}[u_1] = 0, \mathcal{L}[u_2] = 0$, then $v = u_1 + u_2$ also solves the PDE $\mathcal{L}[u_1 + u_2] = 0$.

For (2),

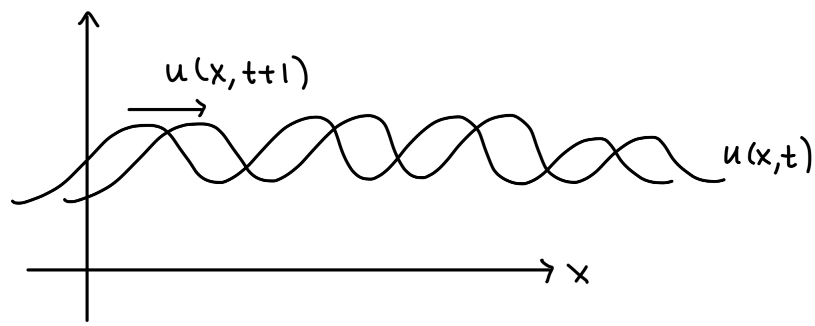
$$\begin{aligned}
\mathcal{L}[u_1 + u_2] &= \frac{\partial}{\partial t}(u_1 + u_2) + c\frac{\partial}{\partial x}(u_1 + u_2) \\
&= u_{1,t} + u_{2,t} + c(u_{1,x} + u_{2,x}) \\
&= (u_{1,t} + cu_{1,x}) + (u_{2,t} + cu_{2,x}) \\
&= \mathcal{L}[u_1] + \mathcal{L}[u_2]
\end{aligned}$$

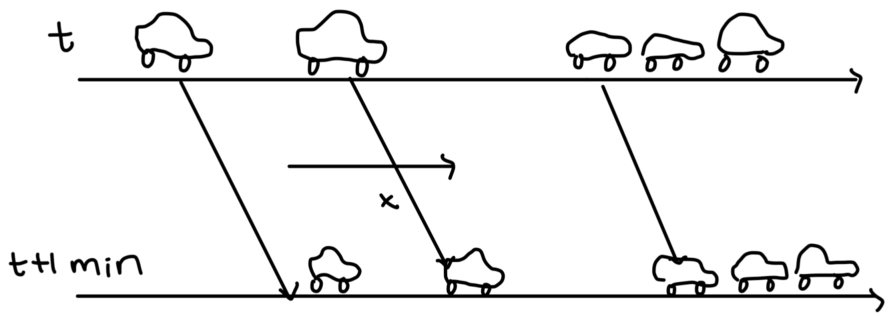
Example 2.5
Examples of linear versus non-linear PDEs

1. $\mathcal{L}[u] = u_{,x} + xu_{,y}$ (linear)
2. $\mathcal{L}[u] = u_{,t} + \underbrace{u_{,xxx}}_{\partial^3 u/\partial x^3}$ (linear)
3. $\mathcal{L}[u] = u_{,x} + \alpha u_{,y}, \alpha$ is a constant. (linear)
4. $\mathcal{L}[u] = u_{,x} + \sqrt{x^2 + y^2}e^{-x}u_{,y}$ (linear)
5. $\mathcal{L}[u] = u_{,x} + \sqrt{x^2 + u^2}u_{,xx}$ (non-linear)

1)

$$\begin{aligned}
\mathcal{L}[au] &= au_{,x} + xau_{,y} \\
&= a(u_{,x} + xu_{,y}) \\
&= a\mathcal{L}[u] \\
\mathcal{L}[u_1 + u_2] &= (u_1 + u_2)_{,x} + x(u_1 + u_2)_{,y} \\
&= (u_{1,x} + xu_{1,y}) + (u_{2,x} + xu_{2,y}) \\
&= \mathcal{L}[u_1] + \mathcal{L}[u_2]
\end{aligned}$$





The solution of the PDE is a traveling/shifting copy of $u(x,t)$; we call these solutions traveling waves.

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3.1 Characteristics

We are studying the PDE:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \tag{3}$$

Based on our understanding of how cars move, if all travel at same speed c, then the solution will be a traveling wave. So, if at time $t = 0$, $u(x, 0) = g(x)$, we expect $u(x, t)$ to be the same graph shifted by ct units to the right.

> **Recall 3.1** If $y = f(x)$ has a certain graph $y = f(x - a)$ is the same graph shifted by a to the right.

So

$$u(x, t) = g(x - ct) \tag{4}$$

E.g. if $u(x, 0) = e^{-x}$ then at time t, $u(x, t) = e^{-(x-ct)} = e^{-x+ct}$. We can check that any function of the form (4) solves our PDE.

$$u_{,t} = -cg'(x - ct)$$

$$u_{,x} = (1)g'(x - ct)$$

Hence,

$$u_{,t} + cu_{,x} = -cg' + cg' = 0$$

To understand, mathematically, why (3) has traveling wave solutions, we need to study the advective derivative.

Given a field $\theta(x, t)$ derivatives give us information about the rate of change of θ. E.g.

$$\frac{\partial \theta}{\partial t} = \text{ time rate of change at fixed } x \text{ (Eulerian derivative)}$$

Another derivative comes from sampling θ at different points and times (time rate of change according to a moving observer).

Moving observer has a location $x(t)$, their rate of change is given by the advective derivative or Lagrangian derivative

$$\frac{d}{dt}(\theta(x(t), t)) = \underbrace{\frac{\partial \theta}{\partial x}\frac{dx}{dt}}_{\text{new term}} + \underbrace{\frac{\partial \theta}{\partial t}}_{\text{Eulerian derivative}} \tag{5}$$

> **Example 3.2**
> $\theta(x, t)$ is temperature, x is distance.
> $\theta(x, t) = mx + b$ for some m and b constants.

Now imagine an observer enters room, and walks along x, at speed v so $\frac{dx}{dt} = v$. The time rate of change of temperature is

$$\frac{d\theta}{dt} = mv + 0 = mv$$

Compare (5)

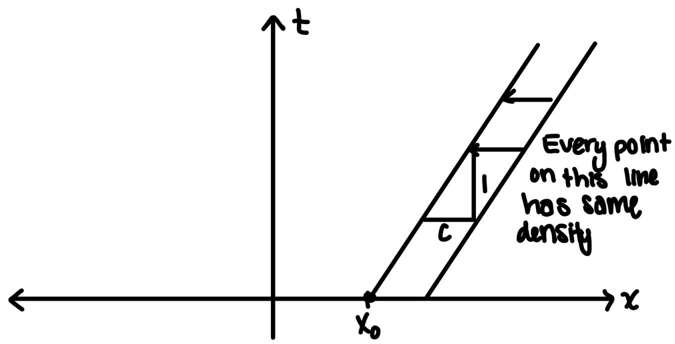
$$\frac{d\theta}{dt} = \frac{\partial\theta}{\partial t} + \left(\frac{dx}{dt}\right)\frac{\partial\theta}{\partial x}$$

with (3)

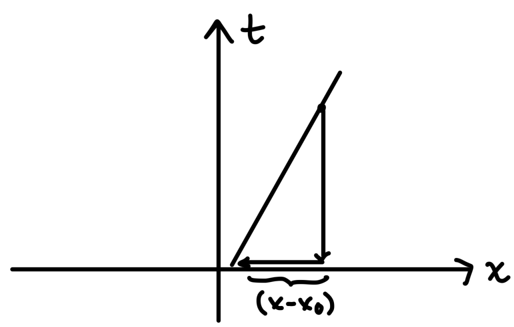
$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

(3) says that $\frac{du}{dt} = 0$ if $\frac{dx}{dt} = c$. In other words, according to an observer moving at speed $\frac{dx}{dt} = c$, the time rate of change is $\frac{du}{dt} = 0$. So u is constant according to this observer.

Suppose we know $u(x, 0)$, the initial density. I can draw a space-time diagram.



According to an observer moving at speed c, the density is constant. Similarly line starting at $x = x_1$. These lines called <u>characteristics</u> must be level curves / contours / isocontours of $u(x, t)$. Using characteristics, we can solve our equation.



Given (x, t) what is $u(x, t)$?

I need the PDE and some initial condition or boundary condition, suppose I know $u(x, 0) = g(x)$.

(x, t), where I want density, lies on a characteristic, every point on characteristic has the same density.

Follow the characteristic back to the x-axis $u(x, t) = u(x_0, 0)$ where x_0 is where the characteristic hits the x-axis $u(x, t) = g(x_0)$.

Given (x, t), I need to find x_0. From the picture,

$$(x - x_0) = ct$$
$$x_0 = x - ct$$

hence

$$u(x, t) = g(x_0) = g(x - ct)$$

3.2 Using Characteristics to Solve More PDEs

Method of characteristics solves PDEs by tracing characteristics; lines or curves on which u is a constant. It can be used to solve PDEs of the form:

$$a(t, x)u_{,t} + b(t, x)u_{,x} = 0$$

where a and b are any functions. Or,

$$a(x, y)u_{,x} + b(x, y)u_{,y} = 0$$

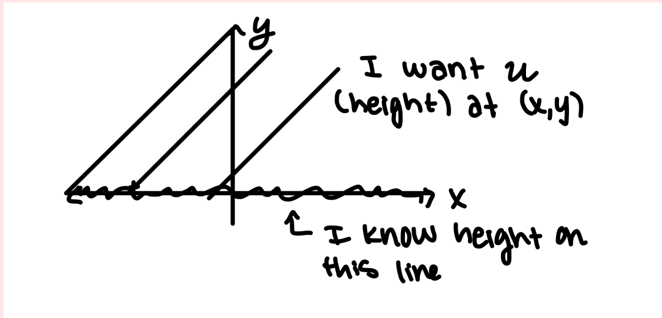
Note 3.3: Now we are using x and y for independent variables.

This is a linear PDE. it is also called <u>first order</u> (only one partial derivative in each term). To use it, start with example:

Example 3.4
We have

$$au_{,x} + bu_{,y} = 0$$

where a, b are both constants (a, b are not both 0). Solution is $u(x, y)$.



Assume I know $u(x, 0)$.

$$au_{,x} + bu_{,y} = 0 \iff (a, b)^T \cdot \nabla u = 0$$

i.e. the directional derivative of u, along $\begin{pmatrix} a \\ b \end{pmatrix}$ is equal to 0, i.e. u is constant in the direction $\begin{pmatrix} a \\ b \end{pmatrix}$.

Characteristics point in the direction of $\begin{pmatrix} a \\ b \end{pmatrix}$.

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4.1 Using Characterizations to Solve More PDEs (Cont'd)

For the PDE:

$$au_{,x} + bu_{,y} = 0$$

Method #1: The lines parallel to $\begin{pmatrix} a \\ b \end{pmatrix}$ are characteristics.

Characteristics are lines $y = \frac{b}{a}x + c$ where c is a constant. Hence the characteristics are lines $ay - bx = ac = C$. Each value of C gives a different straight line, and $u \equiv f(C)$ or $u \equiv f(ay - bx)$. Any function $u \equiv f(ay - bx)$ is a solution of the PDE. E.g.

$$u(x, y) = \sin(ay - bx)$$

$$u(x, y) = (ay - bx)^2$$

Apply the auxiliary condition to find out what f is.

Example 4.1

Solve $2u_{,x} + 3u_{,y} = 0$ with an auxiliary condition $u(0, y) = y^3$ (u is known on the x-axis). Theory above says

$$u(x, y) = f(2y - 3x)$$

to satisfy the auxiliary condition:

$$\begin{aligned} u(0, y) &= y^3 \\ \implies f(2y) &= y^3 \\ \implies f(y) &= \left(\frac{y}{2}\right)^3 \end{aligned}$$

So

$$u(x, y) = f(2y - 3x) = \left(\frac{2y - 3x}{2}\right)^3$$

Method #2: If we didn't explicitly use the equation of a straight line, we know that the characteristics are parallel to $\begin{pmatrix} a \\ b \end{pmatrix}$. If a characteristic is a curve $(x(s), y(s))$.

As s increases, I go along the curve, where

$$\frac{dx}{ds} = a \qquad \frac{dy}{ds} = b$$

Chain rule says

$$\frac{dy}{dx} = \frac{\frac{dy}{ds}}{\frac{dy}{ds}} = \frac{b}{a}$$

Hence,

$$y = \left(\frac{b}{a}\right)x + c$$

We can follow Method #2, even when a and b are not constants.

Example 4.2

Solve $u_{,x} + yu_{,y} = 0$ with an auxiliary condition $u(0, y) = 5y$.
Our PDE gives:

$$\begin{pmatrix} 1 \\ y \end{pmatrix} \cdot \nabla u = 0$$

which implies ∇u is perpendicular to $\begin{pmatrix} 1 \\ y \end{pmatrix}$. In other words, I can define characteristics (on which u is constant) that are always parallel to $\begin{pmatrix} 1 \\ y \end{pmatrix}$.

The equation of any characteristic is:

$$\frac{dx}{ds} = 1 \qquad\qquad \frac{dy}{ds} = y$$

$$\frac{dy}{dx} = \frac{\frac{dy}{ds}}{\frac{dx}{ds}} = y$$

So the characteristics are lines $y(x) = Ce^x$. Different values of C give different characteristics, and therefore different values of u.

$$\begin{aligned} u &= f(C) \\ u &= f(ye^{-x}) \end{aligned}$$

$$\implies \begin{cases} u(x, y) = (ye^{-x})^2 \\ u(x, y) = \sinh(ye^{-x}) \end{cases}$$

Appeal to the auxiliary condition

$$u(0, y) = 5y \implies \begin{cases} f(y) = 5y \\ f(z) = 5z \end{cases}$$

Hence, $u(x, y) = f(ye^{-x}) = 5ye^{-x}$.

Example 4.3
Solve $yu_{,x} - xu_{,y} = 0$ with auxiliary condition $u(x,0) = e^{-x^2}$.
u is constant on characteristics

$$\left.\begin{aligned} \tfrac{dx}{ds} &= y \\ \tfrac{dy}{ds} &= -x \end{aligned}\right\} \implies \frac{dy}{dx} = \frac{-x}{y}$$

So

$$\begin{aligned} 2\int y dy &= 2\int -x dx \\ y^2 &= -x^2 + C \\ \implies x^2 + y^2 &= C \text{ on characteristics} \end{aligned}$$

So $u \equiv f(x^2 + y^2)$ and

$$\begin{aligned} u(x,0) = f(x^2) &= e^{-x^2} \\ f(z) &= e^{-z} \end{aligned}$$

hence $u(x,y) = e^{-x^2-y^2}$.

Note 4.4: Each point (x,y) lies on
This can be a problem for some kinds of auxiliary conditions. E.g. if $u(x,0) = e^{-x}$.

4.2 PDE Models

We met

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

as an example of conservation of mass. This is an example of a transport model. Other examples start from the following idea:

$$u \text{ changes in an interval } \left(x_0 - \frac{\Delta x}{2}, x_0 + \frac{\Delta x}{2}\right)$$

due to flaws at either end of the interval. We model these flows through a field q. $q(x,t)$ is the rate at which mass (e.g. cars) pass a station x.

$$q = \frac{\#\text{cars / amount of mass passing } x}{\text{time counted over}}$$