Math 120A (Differential Geometry)
University of California, Los Angeles

Aaron Chao

Winter 2022

These are my lecture notes for Math 120A (Differential Geometry), which is taught by Fumiaki Suzuki. The textbook for this class is *Differential Geometry of Curves and Surfaces*, by Kristopher Tapp. Many of the figures I include in these notes are taken from Tapp's book.

Contents

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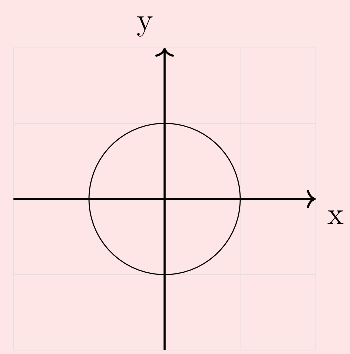
1.1 What is Differential Geometry?

Differential geometry studies geometry via analysis and linear algebra.

Geometry	Analysis	Linear Algebra
Intuitive	Rigorous	Computable
Curved	$\xrightarrow{\text{tangent space}}$	Linear
Global	Local	

1.2 Parametrized Curves

Example 1.1
A unit circle $S' = \{\vec{x} \text{ in } \mathbb{R}^2 \mid |\vec{x}| = 1\}$



$$\vec{\gamma} : [0, 2\pi) \to \mathbb{R}^2$$
$$t \mapsto (\cos t, \sin t)$$

$\vec{\gamma}[0, 2\pi) = S'$

Definition 1.2 (Parametrized curve and Trace)
A (parametrized) curve is a smooth function $\vec{\gamma} \colon I \to \mathbb{R}^n$, where I is an interval in $\mathbb{R}$. The image

$$\vec{\gamma}(I) = \{\vec{\gamma}(t) \mid t \in I\}$$

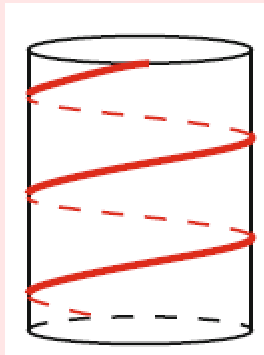
is called the trace of $\vec{\gamma}$.

Recall 1.3 An interval is a subset of $\mathbb{R}$ that has one of the following forms:

$$(a, b), [a, b], (a, b], [a, b), (-\infty, b), (-\infty, b], (a, \infty), [a, \infty), (-\infty, \infty) = \mathbb{R}.$$

A function $\vec{\gamma} \colon I \to \mathbb{R}^n$ is called smooth if $\vec{\gamma}$ is infinitely differentiable, or equivalently, each of the component functions $x_i \colon I \to \mathbb{R}$ is infinitely differentiable.

Example 1.4
$\vec{\gamma}(t) = (\cos t, \sin t, t)$, $t \in (-\infty, \infty)$ is a curve, called a helix.



Definition 1.5 (Derivative)
Let $\vec{\gamma}\colon I \to \mathbb{R}^n$ be a curve. The derivative of $\vec{\gamma}$ at t is defined as

$$\vec{\gamma}'(t) = \lim_{h \to 0} \frac{\vec{\gamma}(t+h) - \vec{\gamma}(t)}{h}$$

If t is on the boundaries of I, then use the left- or right-hand limit.

Remarks 1.6

i. If $\vec{\gamma}(t) = (x_1(t), x_2(t), \cdots, x_n(t))$, then $\vec{\gamma}'(t) = (x_1'(t), x_2'(t), \cdots, x_n'(t))$.
ii. The tangent line to the curve at $\vec{\gamma}'(t_0)$ is defined as
$$\vec{L}(t) = \vec{\gamma}(t_0) + t\vec{\gamma}'(t_0), \quad t \in (-\infty, \infty),$$
as soon as $\vec{\gamma}'(t) \neq \vec{0}$.

Definition 1.7 (Regular)
A curve $\vec{\gamma} : I \to \mathbb{R}^n$ is called regular if $\forall t \in I, \vec{\gamma}'(t) \neq \vec{0}$.

Remark 1.8 regular = tangent line is defined everywhere = the trace is smooth

Example 1.9

$$\vec{\gamma}(t) = (t^2, t^3), \quad t \in (-\infty, \infty)$$

Then $\vec{\gamma}$ is a curve that is not regular.
Indeed, $\vec{\gamma}'(t) = (2t, 3t^2)$, so $\vec{\gamma}'(0) = \vec{0}$.
Notice, $x(t) = t^2, y(t) = t^3$, so $x(t) = y(t)^{2/3}$. Hence, the trace is given by $x = y^{2/3}$ in $\mathbb{R}^2$.

Remark 1.10 The analogy with the physics is useful. If $\vec{\gamma} : I \to \mathbb{R}^n$ is a curve, then $\vec{\gamma}(t)$ is the position of a moving particle at time t in $\mathbb{R}^2$.

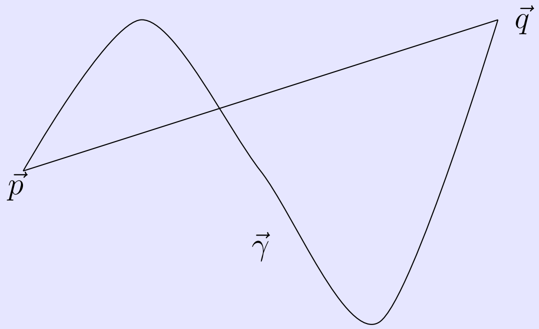
- $\vec{\gamma}'(t)$ velocity

- $\vec{\gamma}''(t)$ acceleration
- $|\vec{\gamma}'(t)|$ speed

In this analogy, regular = the speed is always nonzero = the particle never stops (hence no ”corners” on the trace)

Definition 1.11 (Arc length)
Let $\vec{\gamma}(t)\colon I \to \mathbb{R}^n$ be a regular curve. Then the arc length between times t_1, t_2 is defined as
$$\int_{t_1}^{t_2} |\vec{\gamma}'(t)|\, dt$$

Proposition 1.12
Let $\vec{\gamma}\colon [a,b] \to \mathbb{R}^n$ be a regular curve with the arc length L, $\vec{p} = \vec{\gamma}(a), \vec{q} = \vec{\gamma}(b)$. Then $L \geq |\vec{q} - \vec{p}|$.
Moreover, the equality holds if and only if $\vec{\gamma}$ parametrizes the line segment between $\vec{p}, \vec{q}$.



For the proof, we use the inner-product:
for $\vec{x} = (x_1, x_2, \cdots, x_n), \vec{y} = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$,
$$\langle x, y\rangle \coloneqq x_1y_1 + x_2y_2 + \cdots + x_ny_n$$
Basic properties:

i. The inner product $\langle -, -\rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is symmetric and bilinear.
ii. $\langle \vec{x}, \vec{y}\rangle = |\vec{x}||\vec{y}|\cos\theta$, where θ is the angle between $\vec{x}, \vec{y}$. ($\theta \in [0, 2\pi]$)
iii. $\langle \vec{x}, \vec{y}\rangle = 0 \Leftrightarrow \vec{x}, \vec{y}$ are orthogonal to each other.
iv. $\langle \vec{x}, \vec{x}\rangle = |\vec{x}|^2$
v. $\langle \vec{x}, \vec{y}\rangle \leq |\vec{x}||\vec{y}|$ (Schwartz Inequality) and the equality holds if and only if $\theta = 0$.

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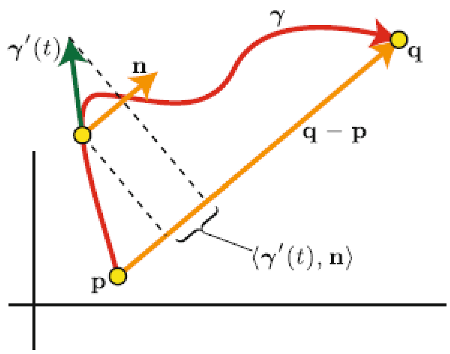
2.1 Proof of Proposition 1.12

Proof. <u>Idea:</u> Compare $\vec{\gamma}'(t)$ and its projection onto $\vec{q}-\vec{p}$. Set $\vec{n} = \frac{\vec{q}-\vec{p}}{|\vec{q}-\vec{p}|}$; $\vec{n}$ is unit.



Tapp Pg.15

Then $|\vec{\gamma}'(t)| \geq \langle\vec{\gamma}'(t), \vec{n}\rangle$ by Schwartz inequality.
Now,

$$\begin{aligned}
L &= \int_a^b |\vec{\gamma}'(t)|\,dt \geq \int_a^b \langle\vec{\gamma}'(t), \vec{n}\rangle\,dt \\
&= [\langle\vec{\gamma}(t), \vec{n}\rangle]_a^b = \langle\vec{\gamma}(b), \vec{n}\rangle - \langle\vec{\gamma}(a), \vec{h}\rangle \\
&= \left\langle \vec{q}-\vec{p}, \frac{\vec{q}-\vec{p}}{|\vec{q}-\vec{p}|}\right\rangle = |\vec{q}-\vec{p}|
\end{aligned}$$

If the equality holds, then $\forall t \in [a,b], \vec{\gamma}'(t), \vec{n}$ are in the same direction. So,

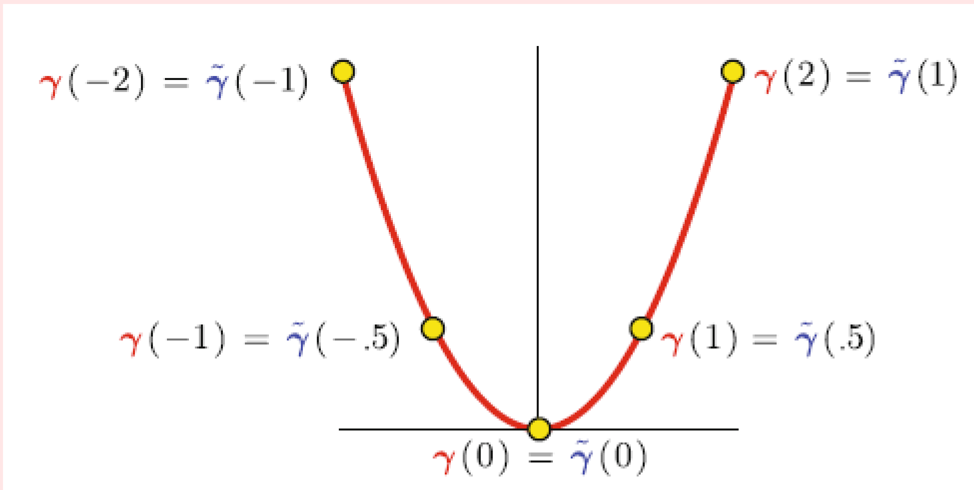
$$\begin{aligned}
\vec{\gamma}'(t) &= \langle\vec{\gamma}'(t), \vec{n}\rangle\vec{n}. \\
\vec{\gamma}(t) &= \vec{\gamma}(a) + \int_a^t \vec{\gamma}'(u)du \\
&= \vec{p} + \left(\int_a^t \langle\vec{\gamma}'(u), \vec{n}\rangle dt\right)\vec{n}
\end{aligned}$$

parametrizes the line segment between $\vec{p}, \vec{q}$. □

2.2 Reparametrization

There are regular curves that share common properties. Which regular curves should we identify?

Example 2.1
$\vec{\gamma}(t) = (t, t^2), \quad t \in [-2, 2]$
$\tilde{\vec{\gamma}}(t) = (-2t, (-2t)^2), t \in [-1, 1]$.
Then $\vec{\gamma}[-2, 2] = \tilde{\vec{\gamma}}[-1, 1] =$



$\vec{\gamma}, \tilde{\vec{\gamma}}$ are the same, up to change in time:
Let $\phi\colon [-1, 1] \to [-2, 2], \quad t \mapsto -2t$.
Then $\tilde{\vec{\gamma}} = \vec{\gamma} \circ \phi$

Definition 2.2 (Reparametrization)
Let $\vec{\gamma}\colon I \to \mathbb{R}^n$ be a regular curve. A reparametrization of $\vec{\gamma}$ is a function of the form

$$\tilde{\vec{\gamma}} = \vec{\gamma} \circ \phi : \tilde{I} \to \mathbb{R}^n,$$

where $\tilde{I}$ is an interval, $\phi\colon \tilde{I} \to I$ is a smooth bijection such that $\forall t \in \tilde{I}$, $\phi'(t) \neq 0$

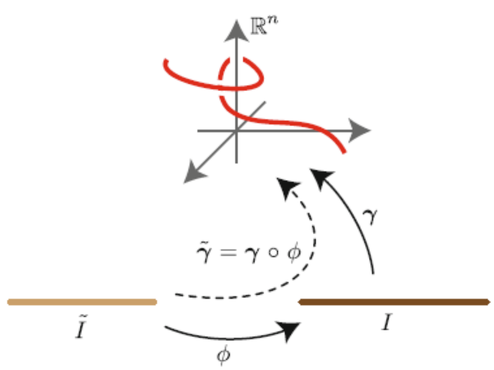


Figure 1: Kapp pg.19

Proposition 2.3
A reparametrization of a regular curve is a regular curve.

Proof. We use the same notations as the definition.
$\tilde{\vec{\gamma}} = \vec{\gamma} \circ \phi \colon \tilde{I} \to \mathbb{R}^n$ is the composition of smooth functions, so smooth.
Moreover, $\forall t \in \tilde{I}, \tilde{\vec{\gamma}}'(t) = \vec{\gamma}'(\phi(t)) \cdot \phi'(t) \neq 0$ □

We will be interested in regular curves up to reparametrizations.

Remarks 2.4

1. $\vec{\gamma}, \tilde{\vec{\gamma}}$ have the same trace.
2. There are regular curves with the same trace that cannot be reparametrized to each other. For instance,
$$\vec{\gamma}_1(t) = (\cos(t), \sin(t)), t \in [0, 2\pi),$$
$$\vec{\gamma}_2(t) = (\cos(t), \sin(t)), t \in [0, 4\pi),$$

Question 2.5: Is there a canonical reparametrization of a given regular curve?

Definition 2.6 (Unit-speed)
A regular curve $\vec{\gamma} \colon I \to \mathbb{R}^n$ is called unit-speed (or parametrized by arc length) if $\forall t \in I$, $|\vec{\gamma}'(t)| = 1$.

Remark 2.7 If $\vec{\gamma} \colon I \to \mathbb{R}^n$ is unit-speed, then,
$$\text{Arc length between } t_1, t_2 = \int_{t_1}^{t_2} |\vec{\gamma}'(t)| dt = \int_{t_1}^{t_2} dt = t_2 - t_1$$

Proposition 2.8
A regular curve always has a unit-speed reparametrization.

Proof. Let $\vec{\gamma} \colon I \to \mathbb{R}^n$ be a regular curve. Fix $t_0 \in I$. Define $s \colon I \to \mathbb{R}$ by
$$s(t) = \int_{t_0}^{t} |\vec{\gamma}'(u)| \, du.$$
Let $\tilde{I} = s(I) \subset \mathbb{R}$. Then $\tilde{I}$ is an interval by IVT.
Since $s'(t) = |\vec{\gamma}'(t)| > 0$ by FTC, regularity, $s \colon I \to \tilde{I}$ is a smooth bijection. Then, $\phi = s^{-1} \colon \tilde{I} \to I$ is a smooth bijection,
$$\phi'(t) = \frac{1}{s'(\phi(t))} = \frac{1}{|\vec{\gamma}'(\phi(t))|} \neq 0.$$
Now $\tilde{\vec{\gamma}} = \vec{\gamma} \circ \phi \colon \tilde{I} \to \mathbb{R}^n$ is a reparametrization of $\vec{\gamma}$, that is unit-speed:
$$\begin{aligned} |\tilde{\vec{\gamma}}'(t)| &= |\vec{\gamma}'(\phi(t)) \cdot \phi'(t)| \\ &= |\vec{\gamma}'(\phi(t))| \cdot 1/|\vec{\gamma}'(\phi(t))| \\ &= 1 \end{aligned}$$
□

Note:

$$\begin{gathered} s^{-1} \cdot s(t) = t \\ (s^{-1})'(s(t)) \cdot s'(t) = 1 \\ (s^{-1})'(s(t)) = 1/s'(t) \end{gathered}$$

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3.1 Reparametrization (Cont'd)

Example 3.1
$\vec{\gamma}(t) = (\cos(t), \sin(t), t), \quad t \in (-\infty, \infty)$ How can we find a unit-speed reparametrization of $\vec{\gamma}$? Compute the arc length function $S\colon (-\infty, \infty) \to \mathbb{R}$:

$$\begin{aligned} s(t) = \int_0^t |\vec{\gamma}'(u)|\, du &= \int_0^t |(-\sin(u), \cos(u), 1)|\, du \\ &= \int_0^t \sqrt{2}\, du = \sqrt{2}t \end{aligned}$$

Set $\phi = s^{-1}$, then $\phi(t) = t/\sqrt{2}$

$$\tilde{\vec{\gamma}}(t) = \vec{\gamma}(t) \circ \phi(t) = \left(\cos\left(t/\sqrt{2}\right), \sin\left(t/\sqrt{2}\right), t/\sqrt{2}\right)$$

$t \in (-\infty, \infty)$, is a unit speed reparametrization of $\vec{\gamma}$.

We will be interested in invariants for a regular curve that are unchanged under any reparametrizations.

Examples include:

- trace
- arc-length
- curvature
- torsion

Non-examples include:

- position
- velocity
- speed
- acceleration

Sometimes we consider more specific reparametrization.

Proposition 3.2
If $\tilde{\vec{\gamma}} = \vec{\gamma} \cdot \phi\colon \tilde{I} \to \mathbb{R}^n$ is a reparametrization of a regular curve $\vec{\gamma}\colon I \to \mathbb{R}^n$, then one of the following holds:

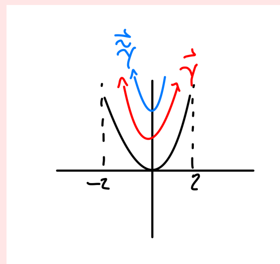
i. $\forall t \in \tilde{I}, \phi'(t) > 0$ i.e. ϕ is strictly increasing
ii. $\forall t \in \tilde{I}, \phi'(t) < 0$ i.e. ϕ is strictly decreasing

Proof. Otherwise $\exists t \in \tilde{I}, \phi'(t) = 0$ by IVT. This contradicts the assumption on ϕ. □

Definition 3.3 (Orientation-preserving vs. orientation-reserving)
Under the setting of the proposition, we say $\tilde{\vec{\gamma}}$ is <u>orientation-preserving</u> if (i) occurs, or <u>orientation-reversing</u> if (ii) occurs.

Example 3.4 (Orientation-preserving)
The arc length reparametrization of a regular curve $\phi\colon I \to \tilde{I}$ is orientation-preserving, because $\phi'(t) = 1/|\vec{\gamma}'(\phi(t))| > 0 \quad \forall t \in I$
This shows an orientation=preserving unit-speed. Reparametrization always exists.

Example 3.5 (Orientation-reversing)
$\vec{\gamma}(t) = (t, t^2), \quad t \in [-2, 2]$
$\tilde{\vec{\gamma}}(t) = (-t, (-t)^2), \quad t \in [-2, 2]$



$\tilde{\vec{\gamma}}$ is an orientation-reserving reparametrization of $\vec{\gamma}$ by $\phi\colon [-2, 2] \to [-2, 2], \quad t \mapsto -t$
(Indeed, $\phi' = -1 < 0$).

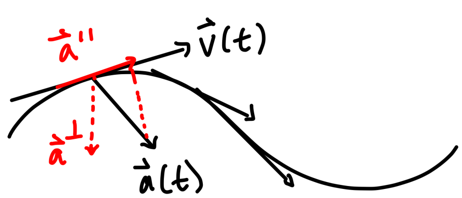
We will be interested in invariants that are unchanged under any orientation-preserving reparametrization.

- Signed curvature
- Rotation index

3.2 Curvature

The curvature measures how sharply the trace bends. What is a plausible definition of the curvature?

Let $\vec{\gamma}\colon I \to R^n$ be a regular curve. Set $\vec{v} = \vec{\gamma}', \vec{a} = \vec{\gamma}''$



$\vec{v}$ knows speed, direction of the motion
$\implies$ $\vec{a}$ should know the change in speed, direction $\to$ curvature.
We write

$$\vec{a} = \vec{a}'' + \vec{a}^{\perp}$$

where

$$\vec{a}'' = \left\langle \vec{a}, \frac{\vec{v}}{|\vec{v}|} \right\rangle : \quad \text{parallel to } \vec{v}$$

$$\vec{a}^{\perp} = \vec{a} - \vec{a}'' : \quad \text{orthogonal to } \vec{v}$$

Proposition 3.6
$\dfrac{d}{dt}|\vec{v}(t)| = \left\langle \vec{a}, \dfrac{\vec{v}}{|\vec{v}|} \right\rangle =$ the parallel component of $\vec{a}$ with respect to $\vec{v}$

Proof.

$$\begin{aligned}\frac{d}{dt}|\vec{v}(t)| &= \frac{d}{dt}\langle \vec{v}(t), \vec{v}(t)\rangle^{1/2} \\ &= \frac{1}{2}\frac{1}{\langle \vec{v}(t), \vec{v}(t)\rangle^{1/2}} \cdot 2\langle \vec{v}(t), \vec{v}'(t)\rangle \\ &= \left\langle \frac{\vec{v}(t)}{|\vec{v}(t)|}, \vec{a}(t) \right\rangle\end{aligned}$$

Note: $\langle v, v\rangle' = \langle v', v\rangle + \langle v, v'\rangle = 2\langle v', v\rangle$ □

So $|\vec{a}^{\perp}(t)|$ would be a plausible definition of the curvature. however this depends on $|\vec{t}|$. (Imagine a centripetal force for a car turning a corner.)

Definition 3.7 (Curvature)
Let $\vec{\gamma} \colon I \to \mathbb{R}^n$ be a regular curve. The <u>curvature function</u> $\kappa \colon I \to [0, \infty)$ is defined as

$$\kappa(t) = \frac{|\vec{a}^{\perp}(t)|}{|\vec{v}(t)|^2}$$

Proposition 3.8
Curvature is independent of parametrizations.

Proof. Let γ be a regular curve. $\tilde{\gamma} = \gamma \cdot \phi$ is a reparametrization of γ.
Denote:

$$\begin{aligned}\kappa &\colon \text{curvature function for } \gamma \\ \tilde{\kappa} &\colon \text{curvature function for } \tilde{\gamma}\end{aligned}$$

We need to show $\tilde{\kappa} = \kappa \circ \phi$
Denote:

$$\begin{aligned}v, a &\colon \text{ velocity, acceleration of } \gamma \\ \tilde{v}, \tilde{a} &\colon \text{ velocity, acceleration of } \tilde{\gamma}.\end{aligned}$$

Then,

$$\begin{aligned}\tilde{\gamma} &= \gamma \cdot \phi \\ \tilde{v} &= \gamma' \cdot \phi \cdot \phi' = v \circ \phi \cdot \phi' \\ \tilde{a} &= \gamma'' \circ \phi \cdot (\phi')^2 + \gamma' \circ \phi \cdot \phi' \\ &= a \circ \phi \cdot (\phi')^2 + v \circ \phi \cdot \phi'\end{aligned}$$

So, $\tilde{v}$ is parallel to v,

$$\tilde{a}^{\perp} = a^{\perp} \circ \phi \cdot (\phi')^2$$

Therefore,

$$\tilde{\kappa} = \frac{\tilde{a}^{\perp}}{|\tilde{v}|^2} = \frac{|a^{\perp} \circ \phi \cdot (\phi')^2|}{|v \circ \phi \cdot \phi|^2} = \frac{|a^{\perp} \cdot \phi|}{|v \cdot \phi|^2}$$

$$= \kappa \circ \phi$$

$\square$

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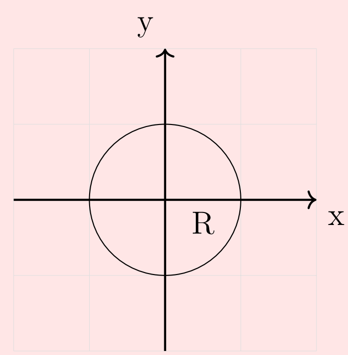
Note: From now on, I will bold my vectors like this $\mathbf{n}$ instead of $\vec{n}$.

4.1 Curvature (Cont'd)

Recall 4.1

$$\kappa(t) = \frac{|\mathbf{a}^{\perp}(t)|}{|\mathbf{v}(t)|^2}$$

Example 4.2
$\gamma(t) = (R\cos(t), R\sin(t)), \quad t \in (-\infty, \infty)$



$\mathbf{v}(t) = (-R\sin(t), R\cos(t))$
$\mathbf{a}(t) = (-R\cos(t), -R\sin(t))$
Here $\langle \mathbf{v}(t), \mathbf{a}(t)\rangle = -R^2\sin(t)\cos(t) + R^2\cos(t)\sin(t) = 0$;
So $\mathbf{v}(t) \perp \mathbf{a}(t) \implies \mathbf{a}(t) = \mathbf{a}^{\perp}(t)$.
Therefore,

$$\kappa(t) = \frac{|\mathbf{a}(t)|}{|\mathbf{v}(t)|^2} = \frac{R}{R^2} = \frac{1}{R} \overset{R\to+\infty}{\to} 0 \text{ (flat)}$$

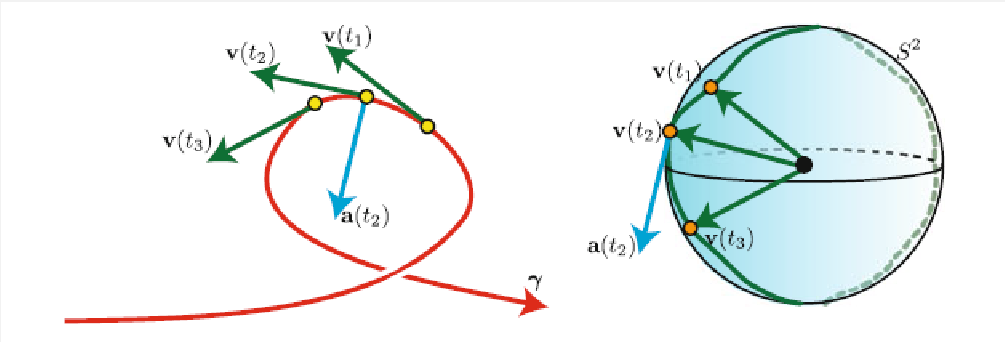
Historically, the curvature of a regular curve was first defined by $\kappa(t) = \frac{1}{R(t)}$, where $R(t)$ is the radius of the circle that best approximates the trace at t (The osculating circle; Read Tapp). Here we give another interpretation of the curvature using the osculating parabola.

Definition 4.3 (Unit tangent and normal vectors)
Let $\gamma\colon I \to \mathbb{R}^n$ be a regular curve. Define the unit tangent and normal vectors as

$$\mathbf{t}(t_0) = \frac{\mathbf{v}(t_0)}{|\mathbf{v}(t_0)|}, \qquad \underbrace{\mathbf{n}(t_0) = \frac{\mathbf{a}^{\perp}(t_0)}{|\mathbf{a}^{\perp}(t_0)|}}_{\text{defined only if } \kappa(t_0)\neq 0}$$

Remarks 4.4

i. $\mathbf{t}(t_0), \mathbf{n}(t_0)$ are orthonormal, i.e. unit, orthogonal to each other



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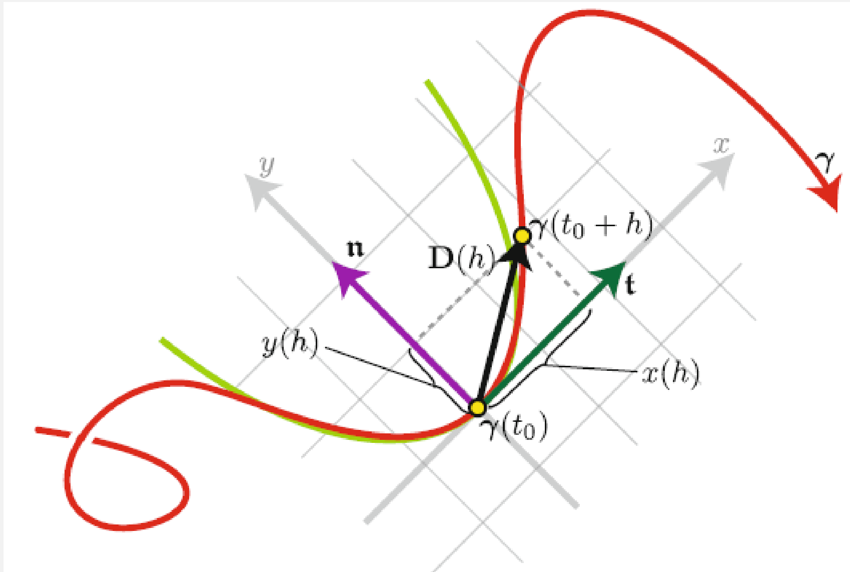
ii. The osculating plane at t_0 is the plane through $\mathbf{t_0}$ spanned by $\mathbf{t}(t_0), \mathbf{n}(t_0)$. The osculating plane is the plane that γ is the closest to begin in, and contains the directions where the curve is heading and bending.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad(\mathbf{t})\qquad\qquad\qquad(\mathbf{n})$

Proposition 4.5
Let $\gamma\colon I \to \mathbb{R}^n$ be a regular curve. Then $|\mathbf{t}'| = \kappa|\mathbf{v}|^2$, and $\mathbf{t}' = \kappa|\mathbf{v}|\mathbf{n}$ if $\mathbf{n}$ is defined. In particular, if γ is unit-speed, then
$$|\mathbf{t}'| = \kappa, \quad \text{and } \mathbf{t}' = \kappa\mathbf{n} \text{ if } \mathbf{n} \text{ is defined.}$$

Proof.
$$\mathbf{t}' = \left(\frac{\mathbf{v}}{|\mathbf{v}|}\right)' = \frac{\mathbf{a}}{|\mathbf{v}|} - \mathbf{v}\frac{\langle \mathbf{a}, \mathbf{v}\rangle}{|\mathbf{v}|^3} = \frac{\mathbf{a} - \mathbf{a}''}{|\mathbf{v}|} = \frac{\mathbf{a}^\perp}{|\mathbf{v}|}$$
Hence $|\mathbf{t}'| = \frac{|\mathbf{a}|^\perp}{|\mathbf{v}|^2} \cdot |\mathbf{v}| = \kappa|\mathbf{v}|$, and
$$\mathbf{t}' = \frac{|\mathbf{a}^\perp|}{|\mathbf{v}|^2}|\mathbf{v}|\frac{\mathbf{a}^\perp}{|\mathbf{a}^\perp|} = \kappa|\mathbf{v}|\mathbf{n} \text{ if } \mathbf{n} \text{ is defined.}$$
□

Remark 4.6 Let $\gamma\colon I \to \mathbb{R}^n$ be a unit-speed curve, $t_0 \in I$ with $\kappa(t_0) \neq 0$.
Then $\gamma'(t_0) = \mathbf{t}, \gamma''(t_0) = \mathbf{t}' = \kappa\mathbf{n}$, and the 2nd order Taylor approximation at γ at t_0 is
$$\begin{aligned}\gamma(t_0 + h) &\approx \gamma(t_0) + h\gamma'(t_0) + \frac{h^2}{2}\gamma''(t_0)\\ &= \gamma(t_0) + h\mathbf{t} + \frac{\kappa h^2}{2}\mathbf{n}\end{aligned}$$
Set $\mathbf{D}(h) = \gamma(t_0 + h) - \gamma(t_0) \approx h\mathbf{t} + \frac{\kappa h^2}{2}\mathbf{n}$: displacement.
Then,
$$\left.\begin{aligned} x(t) &:= \langle \mathbf{D}(h), \mathbf{t}\rangle \approx h \\ y(t) &:= \langle \mathbf{D}(h), \mathbf{n}\rangle \approx \tfrac{\kappa h^2}{2}\end{aligned}\right\} \text{ the parabola } y = \frac{\kappa}{2}x^2 \text{ in the osculating plane}$$



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$\kappa(t_0)$ = the concavity of the parabola that best approximates the trace at t_0

Proposition 4.7
Let $\gamma\colon I \to \mathbb{R}^n$ be a regular curve. If $\forall t \in I, \kappa(t) = 0$, then γ parametrizes a straight line.

Proof.

$$\begin{aligned}
|\mathbf{t}'| = \kappa|\mathbf{v}| = 0 &\implies \mathbf{t}' = \mathbf{0} \\
&\implies \mathbf{t} = \mathbf{0} \text{ constant} \\
&\implies \mathbf{v} = |\mathbf{v}|\mathbf{c} \\
&\implies \text{ fixing } t_0 \in I, \\
\gamma(t) &= \gamma(t_0) + \int_{t_0}^{t} \mathbf{v}(u)\, du \\
&= \gamma(t_0) + \left(\int_{t_0}^{t} |\mathbf{v}(u)|\, du\right) \mathbf{c}
\end{aligned}$$

□

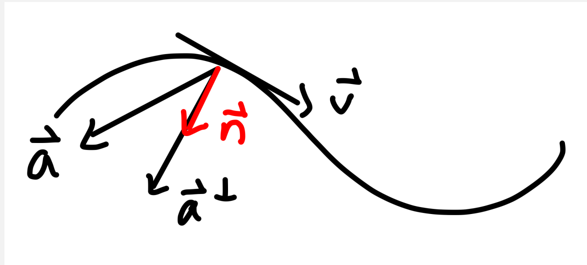
4.2 Plane Curves

$\mathbb{R}^2$ is the only $\mathbb{R}^n$ where the terms "clockwise" and "counter-clockwise" makes sense. This allows us to define

"signed curvature" = curvature + turning direction with respect to $\mathbf{v}$

Recall 4.8

$$\kappa = \frac{|\mathbf{a}^{\perp}|}{|\mathbf{v}|^2} = \frac{\langle \mathbf{a}, \mathbf{n}\rangle}{|\mathbf{v}|^2}$$



Definition 4.9 (Signed curvature)
Let $\gamma\colon I \to \mathbb{R}^2$ be a regular plane curve. Then the signed curvature $\kappa_s\colon I \to \mathbb{R}$ is defined as

$$\kappa_s = \frac{\langle \mathbf{a}, \mathbf{n}_s \rangle}{|\mathbf{v}|^2},$$

where,

$$\begin{aligned} \mathfrak{n}_s &= R_{90}\mathbf{t} \\ &= \text{ the counterclockwise } 90^\circ \text{ rotation of } \mathbf{t} \end{aligned}$$

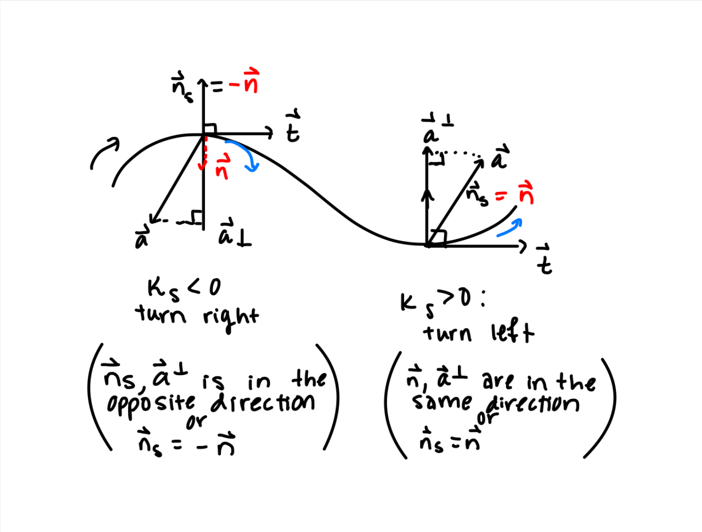
5 Jan 12, 2022

5.1 Plane Curves (Cont'd)

Recall 5.1

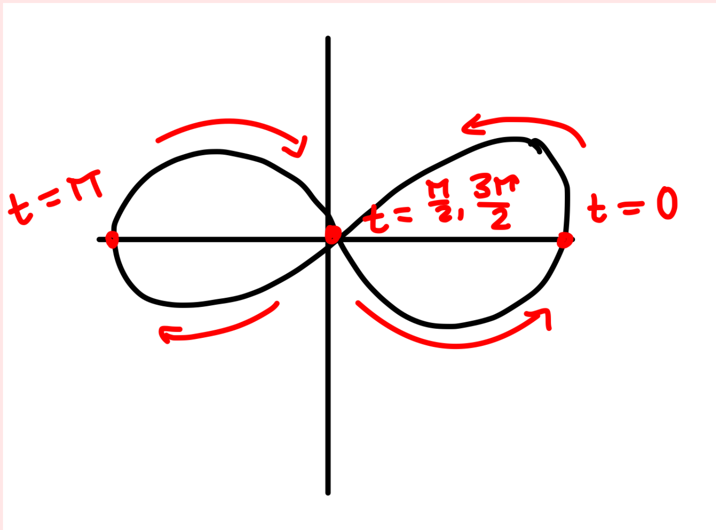
$$\kappa_s = \frac{\langle \mathbf{a}, \mathbf{n}_s \rangle}{|\mathbf{v}|^2}$$

where, $\mathbf{n}_s = R_{90^\circ}\mathbf{t}$



Example 5.2

$$\gamma(g) = (\cos(t), \sin(2t)), \quad t \in [0, 2\pi]$$



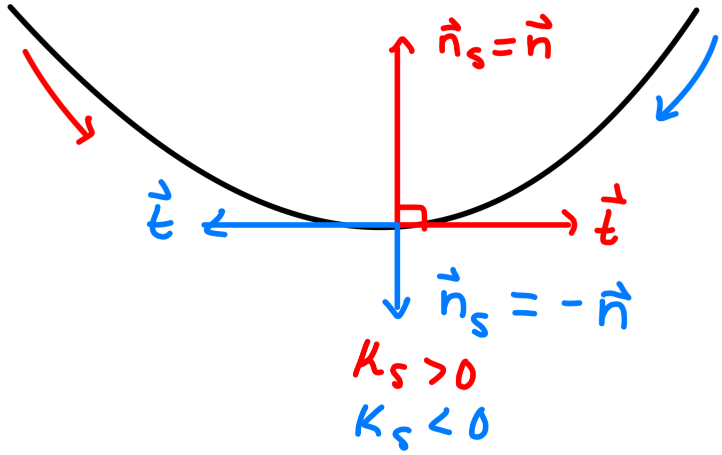
Lissajous curve

$$\mathbf{v}(t) = (-\sin(t), 2\cos(2t))$$
$$\mathbf{a}(t) = (-\cos(t), -4\sin(2t))$$
$$|\mathbf{v}(t)| = \sqrt{\sin^2(t) + 4\cos^2(2t)}$$
$$\mathbf{t}(t) = \frac{\mathbf{t}}{\mathbf{v}(t)} = (-\sin(t), 2\cos(2t))\frac{1}{\sqrt{\sin^2 t + 4\cos^2 2t}}$$
$$\mathbf{n}_s = R_{90}\mathbf{t} = (-2\cos(2t), -\sin(t))\frac{1}{\sqrt{\sin^2 t + 4\cos^2(2t)}}$$
$$\kappa_s = \frac{\langle \mathbf{a}, \mathbf{n}_s \rangle}{|\mathbf{v}|^2} = \frac{2\cos(t)\cos(2t) + 4\sin(t)\sin(2t)}{(\sin(3t) + 4\cos^2(2t))^{3/2}}$$

$$\kappa_s(0) = \frac{2}{4^{3/2}} = \frac{2}{8} = \frac{1}{4} > 0$$
$$\kappa_s\left(\frac{\pi}{2}\right) = 0$$
$$\kappa_s(\pi) = \frac{-1}{4} < 0$$
$$\kappa_s\left(\frac{3\pi}{2}\right) = 0$$

Proposition 5.3
Let $\gamma\colon I \to \mathbb{R}^2$ be a plane curve. Then $|\kappa_s| = \kappa$.

Proof. Compare $\kappa = \frac{\langle \mathbf{a}, \mathbf{n} \rangle}{|\mathbf{v}|^2}, \kappa_s = \frac{\langle \mathbf{a}, \mathbf{n}_s \rangle}{|\mathbf{v}|^2}$ $\mathbf{n}_s = \pm\mathbf{n}$, because they are both unit, orthogonal to $\mathbf{t}$. Hence κ_s coincides with $\underset{\geq 0}{\kappa}$ up to signs.

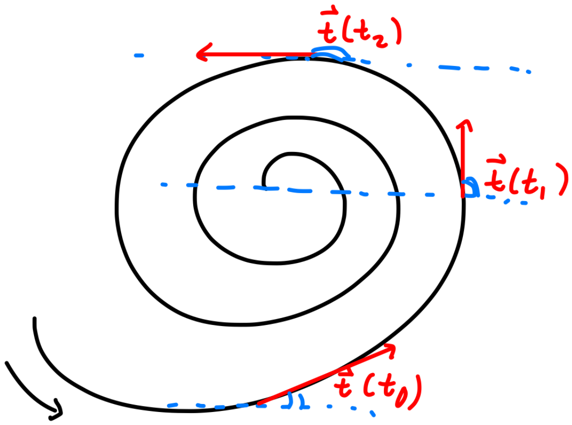


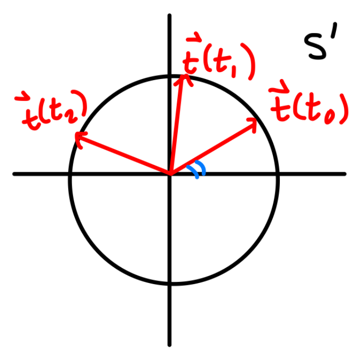
□

Proposition 5.4
Signed curvature is unchanged by any orientation-preserving reparametrizations.

Proof. Exercise. □

Proposition 5.5
Let $\gamma\colon I \to \mathbb{R}^2$ be a plane curve. Then there exists a smooth function $\theta\colon I \to \mathbb{R}$ such that $\forall t \in I, \mathbf{t}(t) = (\cos\theta(t), \sin\theta(t))$.





What should θ be?

$$\mathbf{t}' = \theta'(-\sin\theta, \cos\theta) = \theta' R_{90}\mathbf{t} = \theta'\mathbf{n}_s.$$

On the other hand,

$$\mathbf{t}' = \left(\frac{\mathbf{v}}{|\mathbf{v}|}\right)' = \frac{\mathbf{a}^\perp}{|\mathbf{v}|} = \frac{\langle \mathbf{a}, \mathbf{n}_s\rangle}{|\mathbf{v}|}\mathbf{n}_s = \kappa_s|\mathbf{v}|\mathbf{n}_s$$

By comparing the two formulas, $\theta' = \kappa_s|\mathbf{v}|$. In the proof, we solve this differential equation.

Remark 5.6 If γ is unit-speed, $\theta' = \kappa_s$. This shows:

$$\begin{aligned} \text{signed curvature} &= \text{ the rate of change of the angle} \\ \text{curvature} &= |\text{the rate of change of the angle}| \end{aligned}$$

Proof. Fix $t_0 \in I, \theta_0 \in \mathbb{R}$ such that $\mathbf{t}(t_0) = (\cos\theta_0, \sin\theta_0)$.
Define

$$\theta(t) = \theta_0 + \int_{t_0}^{t} \kappa_s(u)|\mathbf{v}(u)|\, du$$

We will show this $\theta(t)$ works.

$$\begin{gathered} \theta\colon I \to \mathbb{R} \text{ is a smooth function} \\ \theta' = \kappa_s|\mathbf{v}|, \theta(t_0) = \theta_0. \end{gathered}$$

Set $\mathbf{t}_\theta = (\cos\theta, \sin\theta)$ We need to show $\mathbf{t} = \mathbf{t}_\theta$. Observe $\mathbf{t}, \mathbf{t}_\theta$ are unit.
Enough to show $\langle \mathbf{t}, \mathbf{t}_\theta \rangle = 1$
On the other hand,

$$\begin{aligned} \mathbf{t}_\theta(t_0) &= (\cos\theta(t_0), \sin\theta(t_0)) \\ &= (\cos\theta_0, \sin\theta_0) \\ &= \mathbf{t}(t_0) \end{aligned}$$

Enough to show $\langle \mathbf{t}, \mathbf{t}_\theta \rangle' = 0$

$$\begin{aligned} \mathbf{t}' &= \kappa_s|\mathbf{v}|\mathbf{n}_s = \kappa_s|\mathbf{v}|R_{90}\mathbf{t} \\ \mathbf{t}_\theta' &= \theta'(-\sin\theta, \cos\theta) = \kappa_s|\mathbf{v}|R_{90}\mathbf{t}_\theta \end{aligned}$$

Therefore,

$$\begin{aligned} \langle \mathbf{t}, \mathbf{t}_\theta \rangle' &= \langle \mathbf{t}', \mathbf{t}_\theta \rangle + \langle \mathbf{t}, \mathbf{t}_\theta' \rangle \\ &= \kappa_s|\mathbf{v}| \left(\langle R_{90}\mathbf{t}, \mathbf{t}_\theta \rangle + \langle \mathbf{t}, R_{90}\mathbf{t}_\theta \rangle \right) \\ &= \kappa_s|\mathbf{v}| \left(\langle R_{90}\mathbf{t}, \mathbf{t}_\theta \rangle + \langle R_{90}\mathbf{t}, R_{90}(R_{90}\mathbf{t}_\theta) \rangle \right) \\ & R_{90} \text{ is orthogonal} \\ &= \kappa_s|\mathbf{v}| \left(\langle R_{90}\mathbf{t}, \mathbf{t}_\theta \rangle - \langle R_{90}\mathbf{t}, \mathbf{t}_\theta \rangle \right) \\ & R_{90} \circ R_{90} = R_{180} = -1 \\ &= 0 \end{aligned}$$

$\square$

Remark 5.7 The angle function θ is unique up to an integer multiple of 2π.
Indeed if $\Theta\colon I \to \mathbb{R}$ is a smooth function such that $\forall t \in I, \gamma = (\cos\Theta, \sin\Theta)$, then,

$$\begin{aligned} \Theta' &= \theta' = \kappa_s|\mathbf{v}| \\ &\implies |\Theta - \theta|' = 0 \\ &\implies \Theta - \theta = \text{constant} \end{aligned}$$

On the other hand,

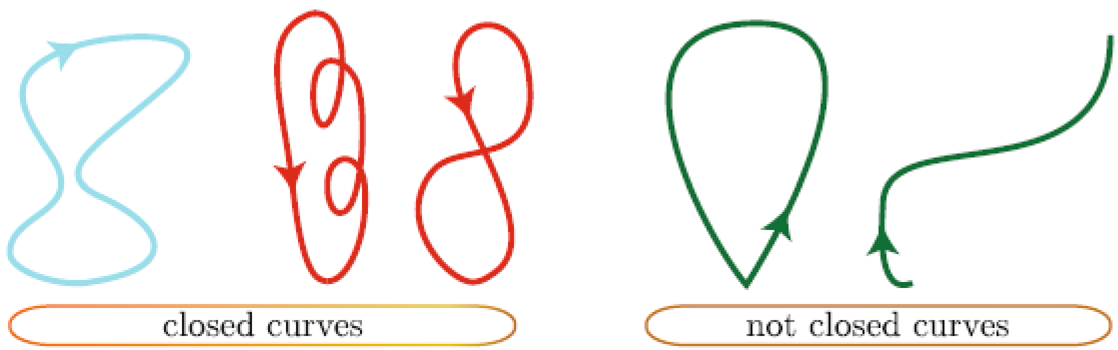
$$(\cos\theta, \sin\theta) = (\cos\Theta, \sin\Theta) = \mathbf{t}$$

So $\Theta - \theta \in 2\pi \cdot \mathbb{Z}$

6 Jan 14, 2022

6.1 Plane Curves(Cont'd)

Definition 6.1 (Closed curve)
A regular curve $\vec{\gamma}\colon [a,b] \to \mathbb{R}^n$ is called <u>closed</u> if $\vec{\gamma}(a) = \vec{\gamma}(b)$, and $\forall n \in \mathbb{N}, \vec{\gamma}^{(n)}(a) = \vec{\gamma}^{(n)}(b)$



Definition 6.2 (Rotation index)
Let $\vec{\gamma}\colon [a,b] \to \mathbb{R}^2$ be a closed plane curve. The <u>rotation index</u> of $\vec{\gamma}$ is defined as

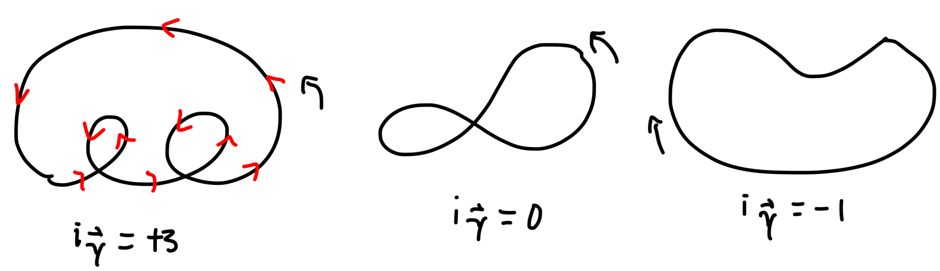
$$i_{\vec{\gamma}} = \frac{1}{2\pi}(\theta(b) - \theta(a)),$$

where θ is the angle function from the proposition.

Remarks 6.3

i. $i_{\vec{\gamma}} \in \mathbb{Z}$, because $\mathbf{t}(a) = \mathbf{t}(b)$, so $\theta(b) - \theta(a) \in 2\pi\mathbb{Z}$

ii. Later on, we will show $i_{\vec{\gamma}} = \pm 1$ if $\vec{\gamma}$ has no self-intersection.



Proposition 6.4
Let $\vec{\gamma}\colon [a,b] \to \mathbb{R}^2$ be a closed plane curve. Then

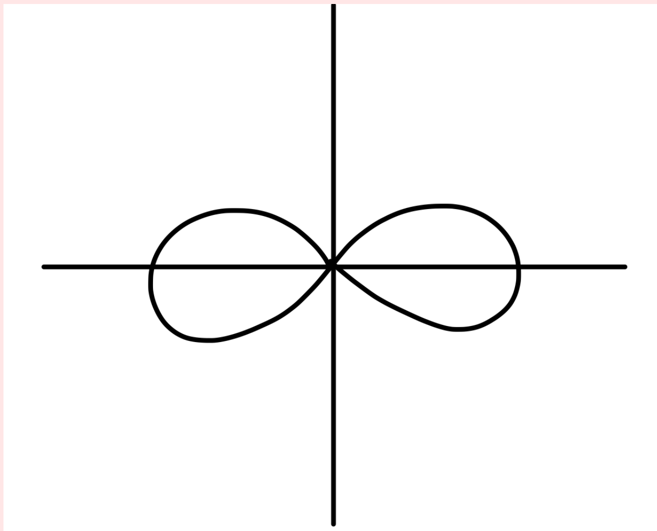
$$i_{\vec{\gamma}} = \frac{1}{2\pi}\int_a^b \kappa_s(t)|\mathbf{v}(t)|\, dt$$

Proof. This follows from the construction of the angle function. □

Proposition 6.5
Rotation index is unchanged under any orientation-preserving reparametrizations.

Proof. Exercise. □

Example 6.6
$\vec{\gamma}(t) = (\cos t, \sin 2t), t \in [0, 2\pi]$



Recall:

$$\kappa_s(t) = \frac{2\cos t \cos 2t + 4\sin t \sin 2t}{(\sin^2 t + 4\cos^2 2t)^{3/2}}$$
$$|\mathbf{v}| = (\sin^2 t + 4\cos^2 2t)^{1/2}$$

Therefore,

$$\begin{aligned} i_{\vec{\gamma}} &= \frac{1}{2\pi}\int_0^{2\pi} \frac{2\cos t \cos 2t + 4\sin t \sin 2t}{\sin^2 t + 4\cos^2 2t}\, dt \\ &= \frac{1}{2\pi}\Big(\int_0^{\pi} - - - \, dt + \underbrace{\int_{\pi}^{2\pi} - - - \, dt}_{\substack{t=s+\pi, \\ \text{then the integrand} \\ \text{is multiplied by } -1}}\Big) \\ &= 0 \end{aligned}$$

6.2 Space Curves

What's special about $\mathbb{R}^3$?
$\mathbb{R}^3$ has the cross product.

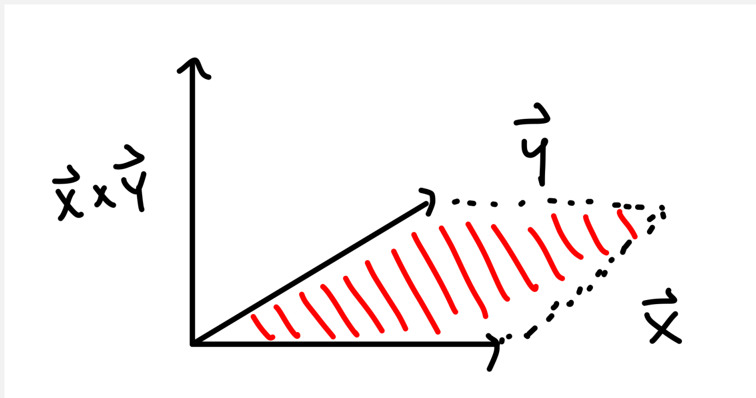
Recall 6.7 $\mathbf{x} = (x_1, x_2, x_3), \mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$,
$\mathbf{x} \times \mathbf{y} = (x_2y_3 - x_3y_2, -(x_1y_3 - x_3y_1), x_1y_2 - x_2y_1) \in \mathbb{R}^3$

Basic properties:

i. $\times \colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is bilinear, and antisymmetric.
(i.e. $\mathbf{y} \times \mathbf{x} = -\mathbf{x} \times \mathbf{y}$)

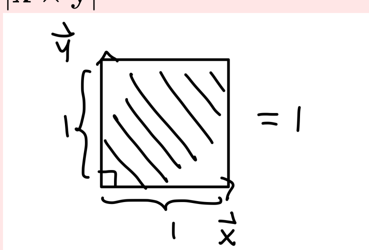
ii. $$\begin{aligned} |\mathbf{x} \times \mathbf{y}| &= |\mathbf{x}||\mathbf{y}|\sin(\theta), \text{ where } \theta \text{ is the angle between } \mathbf{x}, \mathbf{y} \\ &= \text{ the area of the parallelogram spanned by } \mathbf{x}, \mathbf{y} \end{aligned}$$

iii. $\mathbf{x} \times \mathbf{y}$ is orthogonal to $\mathbf{x}, \mathbf{y}$;
$\{\mathbf{x}, \mathbf{y}, \mathbf{x} \times \mathbf{y}\}$ is a right-handed system.



Example 6.8
If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ are orthonormal, then $\{\mathbf{x}, \mathbf{y}, \mathbf{x} \times \mathbf{y}\}$ is an orthonormal basis for $\mathbb{R}^3$:

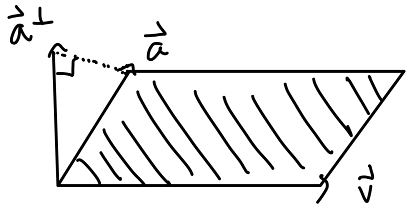
- $\mathbf{x} \times \mathbf{y}$ is orthogonal to $\mathbf{x}, \mathbf{y}$, and
- $|\mathbf{x} \times \mathbf{y}| =$



$= 1$

Proposition 6.9
Let $\vec{\gamma} \colon I \to \mathbb{R}^3$ be a space curve, then
$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$$

Proof. $|\mathbf{v} \times \mathbf{a}| =$

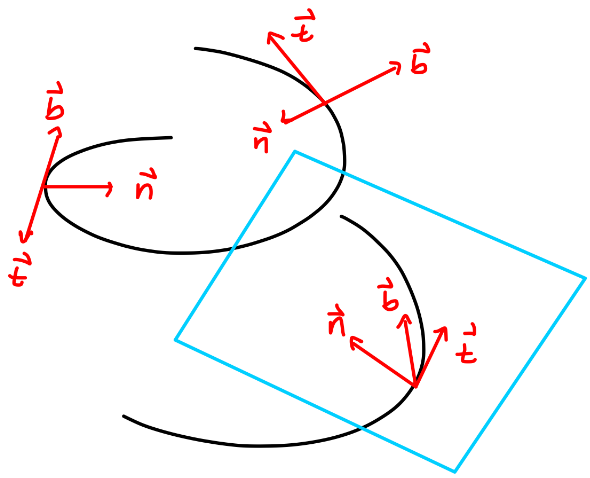


$= |\mathbf{v}||\mathbf{a}^{\perp}|$

$$\implies \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{|\mathbf{a}^{\perp}|}{|\mathbf{v}|^2} = \kappa$$

□

Definition 6.10 (Unit binormal vector and Frenet frame)
Let $\vec{\gamma}\colon I \to \mathbb{R}^3$ be a space curve. The unit binormal vector for $\vec{\gamma}$ at $t \in I$ is defined as $\mathbf{b}(t) = \mathbf{t}(t) \times \mathbf{n}(t)$ (only if $\kappa(t) \neq 0$). The orthonormal basis $\{\mathbf{t}(t), \mathbf{n}(t), \mathbf{b}(t)\}$ for $\mathbb{R}^3$ is called the Frenet frame for $\vec{\gamma}$ at t.



Remark 6.11 $\mathbf{b}(t)$ is a unit normal vector to the osculating plane of $\vec{\gamma}$ at t.
$\implies$ $\mathbf{b}$ encodes the tilt of the osculating plane of $\vec{\gamma}$.

We want to define the "torsion" as the measurement of the change of the tilt of the osculating plane.

Definition 6.12 (Torsion)
Let

$$\vec{\gamma}\colon I \to \mathbb{R}^3 \text{ be a space curve,}$$
$$t \in I \text{ s.t. } \kappa(t) \neq 0$$

The torsion of $\vec{\gamma}$ at t is defined as

$$\tau(t) = -\frac{\langle \mathbf{b}'(t), \mathbf{n}(t) \rangle}{|\mathbf{v}(t)|}$$

Remark 6.13 Why is this definition plausible?

i. $\mathbf{b}'(t)$ is parallel to $\mathbf{n}(t)$ (later).
   So $\langle \mathbf{b}'(t), \mathbf{n}(t) \rangle = \pm |\mathbf{b}'(t)|$

ii. $\langle \mathbf{b}'(t), \mathbf{n}(t) \rangle$ depends on parametrizations.

Proposition 6.14
Torsion is independent of parametrizations.

Proof. Read Tapp for the details. $\square$