

# Math 120A (Differential Geometry)

## *University of California, Los Angeles*

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These are my lecture notes for Math 120A (Differential Geometry), which is taught by Fumiaki Suzuki. The textbook for this class is *Differential Geometry of Curves and Surfaces*, by Kristopher Tapp. Many of the figures I include in these notes are taken from Tapp's book.

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# 1 Jan 3, 2022

## 1.1 What is Differential Geometry?

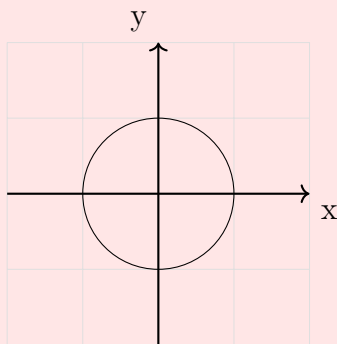
Differential geometry studies geometry via analysis and linear algebra.

Geometry	Analysis	Linear Algebra
Intuitive	Rigorous	Computable
Curved	$\xrightarrow{\text{tangent space}}$	Linear
Global	Local	

## 1.2 Parametrized Curves

### Example 1.1

A unit circle  $S' = \{\vec{x} \text{ in } \mathbb{R}^2 \mid |\vec{x}| = 1\}$



$$\vec{\gamma} : [0, 2\pi) \rightarrow \mathbb{R}^2$$

$$t \mapsto (\cos t, \sin t)$$

$$\vec{\gamma}[0, 2\pi) = S'$$

### Definition 1.2 (Parametrized curve and Trace)

A (parametrized) curve is a smooth function  $\vec{\gamma} : I \rightarrow \mathbb{R}^n$ , where  $I$  is an interval in  $\mathbb{R}$ . The image

$$\vec{\gamma}(I) = \{\vec{\gamma}(t) \mid t \in I\}$$

is called the trace of  $\vec{\gamma}$ .

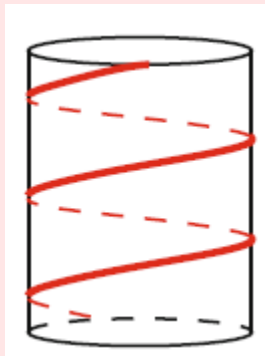
**Recall 1.3** An interval is a subset of  $\mathbb{R}$  that has one of the following forms:

$$(a, b), [a, b], (a, b], [a, b), (-\infty, b), (-\infty, b], (a, \infty), [a, \infty), (-\infty, \infty) = \mathbb{R}.$$

A function  $\vec{\gamma} : I \rightarrow \mathbb{R}^n$  is called smooth if  $\vec{\gamma}$  is infinitely differentiable, or equivalently, each of the component functions  $x_i : I \rightarrow \mathbb{R}$  is infinitely differentiable.

**Example 1.4**

$\vec{\gamma}(t) = (\cos t, \sin t, t)$ ,  $t \in (-\infty, \infty)$  is a curve, called a helix.

**Definition 1.5** (Derivative)

Let  $\vec{\gamma}: I \rightarrow \mathbb{R}^n$  be a curve. The derivative of  $\vec{\gamma}$  at  $t$  is defined as

$$\vec{\gamma}'(t) = \lim_{h \rightarrow 0} \frac{\vec{\gamma}(t+h) - \vec{\gamma}(t)}{h}$$

If  $t$  is on the boundaries of  $I$ , then use the left- or right-hand limit.

**Remarks 1.6**

- i. If  $\vec{\gamma}(t) = (x_1(t), x_2(t), \dots, x_n(t))$ , then  $\vec{\gamma}'(t) = (x_1'(t), x_2'(t), \dots, x_n'(t))$ .
- ii. The tangent line to the curve at  $\vec{\gamma}(t_0)$  is defined as

$$\vec{L}(t) = \vec{\gamma}(t_0) + t\vec{\gamma}'(t_0), \quad t \in (-\infty, \infty),$$

as soon as  $\vec{\gamma}'(t) \neq \vec{0}$ .

**Definition 1.7** (Regular)

A curve  $\vec{\gamma}: I \rightarrow \mathbb{R}^n$  is called regular if  $\forall t \in I, \vec{\gamma}'(t) \neq \vec{0}$ .

**Remark 1.8** regular = tangent line is defined everywhere = the trace is smooth

**Example 1.9**

$$\vec{\gamma}(t) = (t^2, t^3), \quad t \in (-\infty, \infty)$$

Then  $\vec{\gamma}$  is a curve that is not regular.

Indeed,  $\vec{\gamma}'(t) = (2t, 3t^2)$ , so  $\vec{\gamma}'(0) = \vec{0}$ .

Notice,  $x(t) = t^2, y(t) = t^3$ , so  $x(t) = y(t)^{2/3}$ . Hence, the trace is given by  $x = y^{2/3}$  in  $\mathbb{R}^2$ .

**Remark 1.10** The analogy with the physics is useful. If  $\vec{\gamma}: I \rightarrow \mathbb{R}^n$  is a curve, then  $\vec{\gamma}(t)$  is the position of a moving particle at time  $t$  in  $\mathbb{R}^2$ .

- $\vec{\gamma}'(t)$  velocity

- $\vec{\gamma}''(t)$  acceleration
- $|\vec{\gamma}'(t)|$  speed

In this analogy, regular = the speed is always nonzero = the particle never stops (hence no "corners" on the trace)

**Definition 1.11** (Arc length)

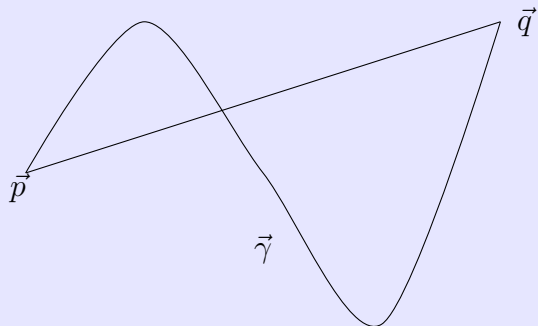
Let  $\vec{\gamma}(t): I \rightarrow \mathbb{R}^n$  be a regular curve. Then the arc length between times  $t_1, t_2$  is defined as

$$\int_{t_1}^{t_2} |\vec{\gamma}'(t)| dt$$

**Proposition 1.12**

Let  $\vec{\gamma}: [a, b] \rightarrow \mathbb{R}^n$  be a regular curve with the arc length  $L$ ,  $\vec{p} = \vec{\gamma}(a), \vec{q} = \vec{\gamma}(b)$ . Then  $L \geq |\vec{q} - \vec{p}|$ .

Moreover, the equality holds if and only if  $\vec{\gamma}$  parametrizes the line segment between  $\vec{p}, \vec{q}$ .



For the proof, we use the inner-product:

for  $\vec{x} = (x_1, x_2, \dots, x_n), \vec{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ ,

$$\langle x, y \rangle := x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

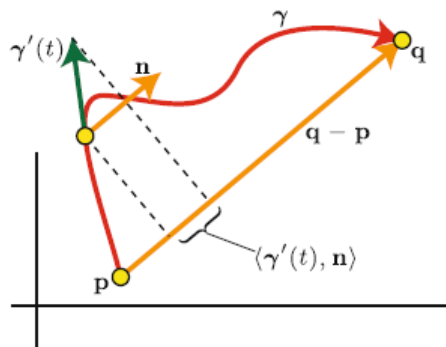
Basic properties:

- The inner product  $\langle -, - \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  is symmetric and bilinear.
- $\langle \vec{x}, \vec{y} \rangle = |\vec{x}| |\vec{y}| \cos \theta$ , where  $\theta$  is the angle between  $\vec{x}, \vec{y}$ . ( $\theta \in [0, 2\pi]$ )
- $\langle \vec{x}, \vec{y} \rangle = 0 \Leftrightarrow \vec{x}, \vec{y}$  are orthogonal to each other.
- $\langle \vec{x}, \vec{x} \rangle = |\vec{x}|^2$
- $\langle \vec{x}, \vec{y} \rangle \leq |\vec{x}| |\vec{y}|$  (Schwartz Inequality) and the equality holds if and only if  $\theta = 0$ .

## 2 Jan 5, 2022

### 2.1 Proof of Proposition 1.12

**Proof.** Idea: Compare  $\vec{\gamma}'(t)$  and its projection onto  $\vec{q} - \vec{p}$ . Set  $\vec{n} = \frac{\vec{q} - \vec{p}}{|\vec{q} - \vec{p}|}$ ;  $\vec{n}$  is unit.



*Tapp Pg.15*

Then  $|\vec{\gamma}'(t)| \geq \langle \vec{\gamma}'(t), \vec{n} \rangle$  by Schwartz inequality.

Now,

$$\begin{aligned} L &= \int_a^b |\vec{\gamma}'(t)| dt \geq \int_a^b \langle \vec{\gamma}'(t), \vec{n} \rangle dt \\ &= [\langle \gamma(t), \vec{n} \rangle]_a^b = \langle \gamma(b), \vec{n} \rangle - \langle \gamma(a), \vec{n} \rangle \\ &= \left\langle \vec{q} - \vec{p}, \frac{\vec{q} - \vec{p}}{|\vec{q} - \vec{p}|} \right\rangle = |\vec{q} - \vec{p}| \end{aligned}$$

If the equality holds, then  $\forall t \in [a, b], \vec{\gamma}'(t), \vec{n}$  are in the same direction. So,

$$\vec{\gamma}'(t) = \langle \vec{\gamma}'(t), \vec{n} \rangle \vec{n}.$$

$$\begin{aligned} \gamma(t) &= \gamma(a) + \int_a^t \gamma'(u) du \\ &= \vec{p} + \left( \int_a^t \langle \gamma'(u), \vec{n} \rangle dt \right) \vec{n} \end{aligned}$$

parametrizes the line segment between  $\vec{p}, \vec{q}$ . □

### 2.2 Reparametrization

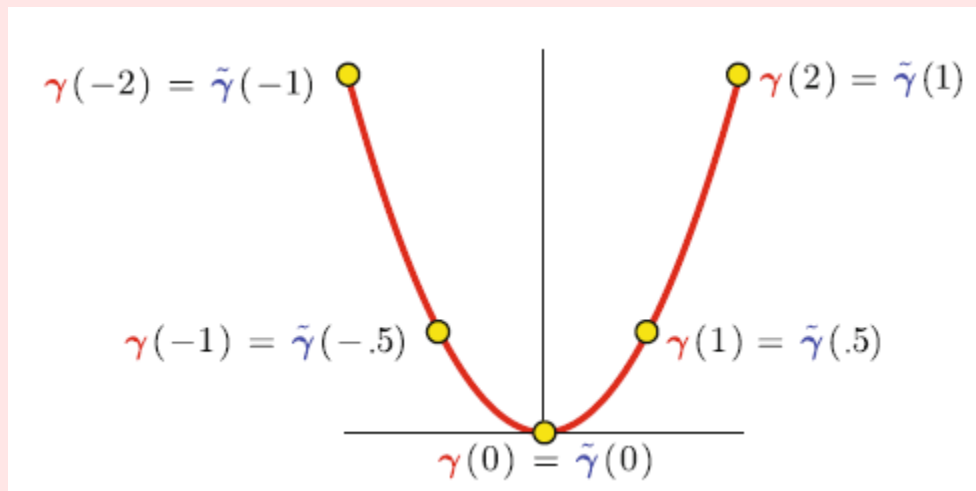
There are regular curves that share common properties. Which regular curves should we identify?

**Example 2.1**

$$\gamma(t) = (t, t^2), \quad t \in [-2, 2]$$

$$\tilde{\gamma}(t) = (-2t, (-2t)^2), \quad t \in [-1, 1].$$

Then  $\gamma[-2, 2] = \tilde{\gamma}[-1, 1] =$



$\gamma, \tilde{\gamma}$  are the same, up to change in time:

Let  $\phi: [-1, 1] \rightarrow [-2, 2], \quad t \mapsto -2t.$

Then  $\tilde{\gamma} = \gamma \circ \phi$

**Definition 2.2** (Reparametrization)

Let  $\gamma: I \rightarrow \mathbb{R}^n$  be a regular curve. A reparametrization of  $\gamma$  is a function of the form

$$\tilde{\gamma} = \gamma \circ \phi: \tilde{I} \rightarrow \mathbb{R}^n,$$

where  $\tilde{I}$  is an interval,  $\phi: \tilde{I} \rightarrow I$  is a smooth bijection such that  $\forall t \in \tilde{I}, \phi'(t) \neq 0$

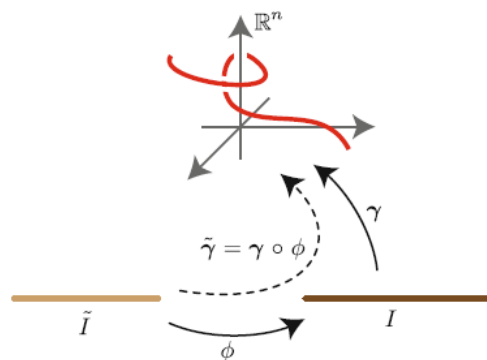


Figure 1: Kapp pg.19

**Proposition 2.3**

A reparametrization of a regular curve is a regular curve.

**Proof.** We use the same notations as the definition.

$\tilde{\gamma} = \gamma \circ \phi: \tilde{I} \rightarrow \mathbb{R}^n$  is the composition of smooth functions, so smooth.

Moreover,  $\forall t \in \tilde{I}, \tilde{\gamma}'(t) = \gamma'(\phi(t)) \cdot \phi'(t) \neq 0$  □

We will be interested in regular curves up to reparametrizations.

**Remarks 2.4**

1.  $\gamma, \tilde{\gamma}$  have the same trace.
2. There are regular curves with the same trace that cannot be reparametrized to each other. For instance,

$$\begin{aligned}\gamma_1(t) &= (\cos(t), \sin(t)), t \in [0, 2\pi), \\ \gamma_2(t) &= (\cos(t), \sin(t)), t \in [0, 4\pi),\end{aligned}$$

**Question 2.5:** Is there a canonical reparametrization of a given regular curve?

**Definition 2.6 (Unit-speed)**

A regular curve  $\gamma: I \rightarrow \mathbb{R}^n$  is called unit-speed (or parametrized by arc length) if  $\forall t \in I, |\gamma'(t)| = 1$ .

**Remark 2.7** If  $\gamma: I \rightarrow \mathbb{R}^n$  is unit-speed, then,

$$\text{Arc length between } t_1, t_2 = \int_{t_1}^{t_2} |\gamma'(t)| dt = \int_{t_1}^{t_2} dt = t_2 - t_1$$

**Proposition 2.8**

A regular curve always has a unit-speed reparametrization.

**Proof.** Let  $\gamma: I \rightarrow \mathbb{R}^n$  be a regular curve. Fix  $t_0 \in I$ . Define  $s: I \rightarrow \mathbb{R}$  by

$$s(t) = \int_{t_0}^t |\gamma'(u)| du.$$

Let  $\tilde{I} = s(I) \subset \mathbb{R}$ . Then  $\tilde{I}$  is an interval by IVT.

Since  $s'(t) = |\gamma'(t)| > 0$  by FTC, regularity,  $s: I \rightarrow \tilde{I}$  is a smooth bijection. Then,  $\phi = s^{-1}: \tilde{I} \rightarrow I$  is a smooth bijection,

$$\phi'(t) = \frac{1}{s'(\phi(t))} = \frac{1}{|\gamma'(\phi(t))|} \neq 0.$$

Now  $\tilde{\gamma} = \gamma \circ \phi: \tilde{I} \rightarrow \mathbb{R}^n$  is a reparametrization of  $\gamma$ , that is unit-speed:

$$\begin{aligned}|\tilde{\gamma}'(t)| &= |\gamma'(\phi(t)) \cdot \phi'(t)| \\ &= |\gamma'(\phi(t))| \cdot 1/|\gamma'(\phi(t))| \\ &= 1\end{aligned}$$

□



Note:

$$\begin{aligned}s^{-1} \cdot s(t) &= t \\ (s^{-1})'(s(t)) \cdot s'(t) &= 1 \\ (s^{-1})'(s(t)) &= 1/s'(t)\end{aligned}$$

## 3 Jan 7, 2022

### 3.1 Reparametrization (Cont'd)

#### Example 3.1

$\gamma(t) = (\cos(t), \sin(t), t)$ ,  $t \in (-\infty, \infty)$  How can we find a unit-speed reparametrization of  $\gamma$ ? Compute the arc length function  $S: (-\infty, \infty) \rightarrow \mathbb{R}$ :

$$\begin{aligned} s(t) &= \int_0^t |\gamma'(u)| du = \int_0^t |(-\sin(u), \cos(u), 1)| du \\ &= \int_0^t \sqrt{2} du = \sqrt{2}t \end{aligned}$$

Set  $\phi = s^{-1}$ , then  $\phi(t) = t/\sqrt{2}$

$$\tilde{\gamma}(t) = \gamma(\phi(t)) = (\cos(t/\sqrt{2}), \sin(t/\sqrt{2}), t/\sqrt{2})$$

$t \in (-\infty, \infty)$ , is a unit speed reparametrization of  $\gamma$ .

We will be interested in invariants for a regular curve that are unchanged under any reparametrizations.

Examples include:

- trace
- arc-length
- curvature
- torsion

Non-examples include:

- position
- velocity
- speed
- acceleration

Sometimes we consider more specific reparametrization.

#### Proposition 3.2

If  $\tilde{\gamma} = \gamma \circ \phi: \tilde{I} \rightarrow \mathbb{R}^n$  is a reparametrization of a regular curve  $\gamma: I \rightarrow \mathbb{R}^n$ , then one of the following holds:

- i.  $\forall t \in \tilde{I}, \phi'(t) > 0$  i.e.  $\phi$  is strictly increasing
- ii.  $\forall t \in \tilde{I}, \phi'(t) < 0$  i.e.  $\phi$  is strictly decreasing

**Proof.** Otherwise  $\exists t \in \tilde{I}, \phi'(t) = 0$  by IVT. This contradicts the assumption on  $\phi$ .  $\square$

#### Definition 3.3 (Orientation-preserving vs. orientation-reversing)

Under the setting of the proposition, we say  $\tilde{\gamma}$  is orientation-preserving if (i) occurs, or orientation-reversing if (ii) occurs.

**Example 3.4** (Orientation-preserving)

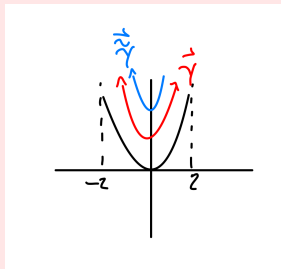
The arc length reparametrization of a regular curve  $\phi: I \rightarrow \tilde{I}$  is orientation-preserving, because  $\phi'(t) = 1/|\gamma'(\phi(t))| > 0 \quad \forall t \in I$

This shows an orientation-preserving unit-speed. Reparametrization always exists.

**Example 3.5** (Orientation-reversing)

$$\gamma(t) = (t, t^2), \quad t \in [-2, 2]$$

$$\tilde{\gamma}(t) = (-t, (-t)^2), \quad t \in [-2, 2]$$



$\tilde{\gamma}$  is an orientation-reversing reparametrization of  $\gamma$  by  $\phi: [-2, 2] \rightarrow [-2, 2], \quad t \mapsto -t$  (Indeed,  $\phi' = -1 < 0$ ).

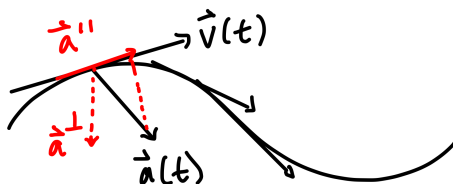
We will be interested in invariants that are unchanged under any orientation-preserving reparametrization.

- Signed curvature
- Rotation index

## 3.2 Curvature

The curvature measures how sharply the trace bends. What is a plausible definition of the curvature?

Let  $\gamma: I \rightarrow \mathbb{R}^n$  be a regular curve. Set  $\vec{v} = \gamma', \vec{a} = \gamma''$



$\vec{v}$  knows speed, direction of the motion

$\implies \vec{a}$  should know the change in speed, direction  $\rightarrow$  curvature.

We write

$$\vec{a} = \vec{a}^\parallel + \vec{a}^\perp$$

where

$$\vec{a}^\parallel = \left\langle \vec{a}, \frac{\vec{v}}{|\vec{v}|} \right\rangle \frac{\vec{v}}{|\vec{v}|}: \quad \text{parallel to } \vec{v}$$

$$\vec{a}^\perp = \vec{a} - \vec{a}'': \quad \text{orthogonal to } \vec{v}$$

**Proposition 3.6**

$$\begin{aligned} \frac{d}{dt}|\vec{v}(t)| &= \left\langle \vec{a}, \frac{\vec{v}}{|\vec{v}|} \right\rangle \\ &= \text{the parallel component of } \vec{a} \text{ with respect to } \vec{v} \end{aligned}$$

**Proof.**

$$\begin{aligned} \frac{d}{dt}|\vec{v}(t)| &= \frac{d}{dt} \langle \vec{v}(t), \vec{v}(t) \rangle^{1/2} \\ &= \frac{1}{2} \frac{1}{\langle \vec{v}(t), \vec{v}(t) \rangle^{1/2}} \cdot 2 \langle \vec{v}(t), \vec{v}'(t) \rangle \\ &= \left\langle \frac{\vec{v}(t)}{|\vec{v}(t)|}, \vec{a}(t) \right\rangle \end{aligned}$$

Note:  $\langle v, v \rangle' = \langle v', v \rangle + \langle v, v' \rangle = 2\langle v', v \rangle$  □

So  $|\vec{a}^\perp(t)|$  would be a plausible definition of the curvature. However this depends on  $|\vec{t}|$ . (Imagine a centripetal force for a car turning a corner.)

**Definition 3.7 (Curvature)**

Let  $\gamma: I \rightarrow \mathbb{R}^n$  be a regular curve. The curvature function  $\kappa: I \rightarrow [0, \infty)$  is defined as

$$\kappa(t) = \frac{|\vec{a}^\perp(t)|}{|\vec{v}(t)|^2}$$

**Proposition 3.8**

Curvature is independent of parametrizations.

**Proof.** Let  $\gamma$  be a regular curve.  $\tilde{\gamma} = \gamma \circ \phi$  is a reparametrization of  $\gamma$ .

Denote:

$\kappa$ : curvature function for  $\gamma$

$\tilde{\kappa}$ : curvature function for  $\tilde{\gamma}$

We need to show  $\tilde{\kappa} = \kappa \circ \phi$

Denote:

$v, a$ : velocity, acceleration of  $\gamma$

$\tilde{v}, \tilde{a}$ : velocity, acceleration of  $\tilde{\gamma}$ .

Then,

$$\begin{aligned} \tilde{\gamma} &= \gamma \circ \phi \\ \tilde{v} &= \gamma' \circ \phi \cdot \phi' = v \circ \phi \cdot \phi' \\ \tilde{a} &= \gamma'' \circ \phi \cdot (\phi')^2 + \gamma' \circ \phi \cdot \phi'' \\ &= a \circ \phi \cdot (\phi')^2 + v \circ \phi \cdot \phi'' \end{aligned}$$

So,  $\tilde{v}$  is parallel to  $v$ ,

$$\tilde{a}^\perp = a^\perp \circ \phi \cdot (\phi')^2$$

Therefore,

$$\begin{aligned}\tilde{\kappa} &= \frac{\tilde{a}^\perp}{|\tilde{v}|^2} = \frac{|a^\perp \circ \phi \cdot (\phi')^2|}{|v \circ \phi \cdot \phi'|^2} = \frac{|a^\perp \circ \phi|}{|v \circ \phi|^2} \\ &= \kappa \circ \phi\end{aligned}$$

□

## 4 Jan 10, 2022

Note: From now on, I will bold my vectors like this  $\mathbf{n}$  instead of  $\vec{n}$ .

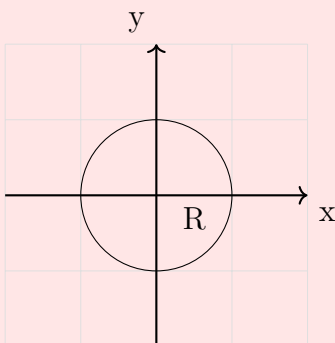
### 4.1 Curvature (Cont'd)

#### Recall 4.1

$$\kappa(t) = \frac{|\mathbf{a}^\perp(t)|}{|\mathbf{v}(t)|^2}$$

#### Example 4.2

$$\gamma(t) = (R \cos(t), R \sin(t)), \quad t \in (-\infty, \infty)$$



$$\mathbf{v}(t) = (-R \sin(t), R \cos(t))$$

$$\mathbf{a}(t) = (-R \cos(t), -R \sin(t))$$

Here,

$$\langle \mathbf{v}(t), \mathbf{a}(t) \rangle = -R^2 \sin(t) \cos(t) + R^2 \cos(t) \sin(t) = 0;$$

So,

$$\mathbf{v}(t) \perp \mathbf{a}(t) \implies \mathbf{a}(t) = \mathbf{a}^\perp(t).$$

Therefore,

$$\kappa(t) = \frac{|\mathbf{a}(t)|}{|\mathbf{v}(t)|^2} = \frac{R}{R^2} = \frac{1}{R} \xrightarrow{R \rightarrow +\infty} 0 \text{ (flat)}$$

Historically, the curvature of a regular curve was first defined by  $\kappa(t) = \frac{1}{R(t)}$ , where  $R(t)$  is the radius of the circle that best approximates the trace at  $t$  (The osculating circle; Read Tapp). Here we give another interpretation of the curvature using the osculating parabola.

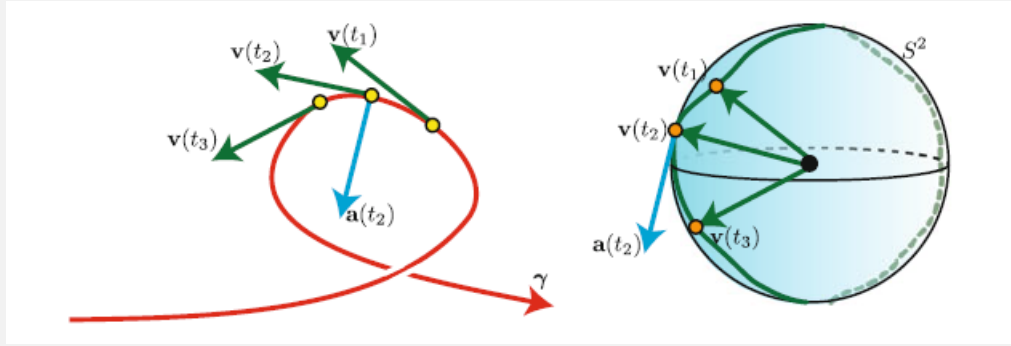
#### Definition 4.3 (Unit tangent and normal vectors)

Let  $\gamma: I \rightarrow \mathbb{R}^n$  be a regular curve. Define the unit tangent and normal vectors as

$$\mathbf{t}(t_0) = \frac{\mathbf{v}(t_0)}{|\mathbf{v}(t_0)|}, \quad \underbrace{\mathbf{n}(t_0) = \frac{\mathbf{a}^\perp(t_0)}{|\mathbf{a}^\perp(t_0)|}}_{\text{defined only if } \kappa(t_0) \neq 0}$$

**Remarks 4.4**

- i.  $\mathbf{t}(t_0), \mathbf{n}(t_0)$  are orthonormal, i.e. unit, orthogonal to each other



Tapp Page 27

- ii. The osculating plane at  $t_0$  is the plane through  $\mathbf{t}_0$  spanned by  $\mathbf{t}(t_0), \mathbf{n}(t_0)$ . The osculating plane is the plane that  $\gamma$  is the closest to begin in, and contains the directions where the curve is heading and bending.

**Proposition 4.5**

Let  $\gamma: I \rightarrow \mathbb{R}^n$  be a regular curve. Then  $|\mathbf{t}'| = \kappa|\mathbf{v}|^2$ , and  $\mathbf{t}' = \kappa|\mathbf{v}|\mathbf{n}$  if  $\mathbf{n}$  is defined. In particular, if  $\gamma$  is unit-speed, then

$$|\mathbf{t}'| = \kappa, \quad \text{and } \mathbf{t}' = \kappa\mathbf{n} \text{ if } \mathbf{n} \text{ is defined.}$$

**Proof.**

$$\mathbf{t}' = \left( \frac{\mathbf{v}}{|\mathbf{v}|} \right)' = \frac{\mathbf{a}}{|\mathbf{v}|} - \mathbf{v} \frac{\langle \mathbf{a}, \mathbf{v} \rangle}{|\mathbf{v}|^3} = \frac{\mathbf{a} - \mathbf{a}^\parallel}{|\mathbf{v}|} = \frac{\mathbf{a}^\perp}{|\mathbf{v}|}$$

Hence  $|\mathbf{t}'| = \frac{|\mathbf{a}^\perp|}{|\mathbf{v}|^2} \cdot |\mathbf{v}| = \kappa|\mathbf{v}|$ , and

$$\mathbf{t}' = \frac{|\mathbf{a}^\perp|}{|\mathbf{v}|^2} |\mathbf{v}| \frac{\mathbf{a}^\perp}{|\mathbf{a}^\perp|} = \kappa|\mathbf{v}|\mathbf{n} \text{ if } \mathbf{n} \text{ is defined.}$$

□

**Remark 4.6** Let  $\gamma: I \rightarrow \mathbb{R}^n$  be a unit-speed curve,  $t_0 \in I$  with  $\kappa(t_0) \neq 0$ .

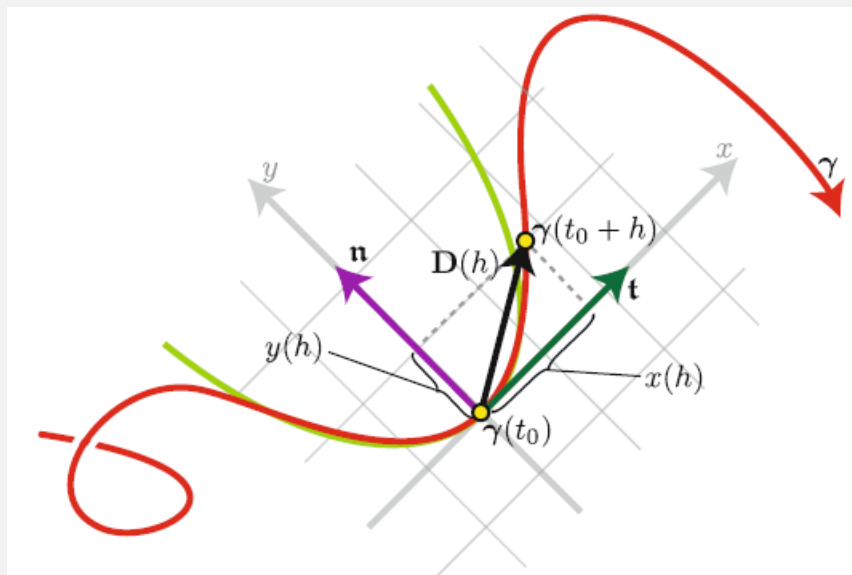
Then  $\gamma'(t_0) = \mathbf{t}$ ,  $\gamma''(t_0) = \mathbf{t}' = \kappa\mathbf{n}$ , and the 2nd order Taylor approximation at  $\gamma$  at  $t_0$  is

$$\begin{aligned} \gamma(t_0 + h) &\approx \gamma(t_0) + h\gamma'(t_0) + \frac{h^2}{2}\gamma''(t_0) \\ &= \gamma(t_0) + h\mathbf{t} + \frac{\kappa h^2}{2}\mathbf{n} \end{aligned}$$

Set  $\mathbf{D}(h) = \gamma(t_0 + h) - \gamma(t_0) \approx h\mathbf{t} + \frac{\kappa h^2}{2}\mathbf{n}$ : displacement.

Then,

$$\left. \begin{aligned} x(t) &:= \langle \mathbf{D}(h), \mathbf{t} \rangle \approx h \\ y(t) &:= \langle \mathbf{D}(h), \mathbf{n} \rangle \approx \frac{\kappa h^2}{2} \end{aligned} \right\} \text{ the parabola } y = \frac{\kappa}{2}x^2 \text{ in the osculating plane}$$



Tapp Page 30

$\kappa(t_0)$  = the concavity of the parabola that best approximates the trace at  $t_0$

### Proposition 4.7

Let  $\gamma: I \rightarrow \mathbb{R}^n$  be a regular curve. If  $\forall t \in I, \kappa(t) = 0$ , then  $\gamma$  parametrizes a straight line.

**Proof.**

$$\begin{aligned}
 |\mathbf{t}'| = \kappa|\mathbf{v}| = 0 &\implies \mathbf{t}' = \mathbf{0} \\
 &\implies \mathbf{t} = \mathbf{0} \text{ constant} \\
 &\implies \mathbf{v} = |\mathbf{v}|\mathbf{c} \\
 &\implies \text{fixing } t_0 \in I, \\
 \gamma(t) &= \gamma(t_0) + \int_{t_0}^t \mathbf{v}(u) du \\
 &= \gamma(t_0) + \left( \int_{t_0}^t |\mathbf{v}(u)| du \right) \mathbf{c}
 \end{aligned}$$

□

## 4.2 Plane Curves

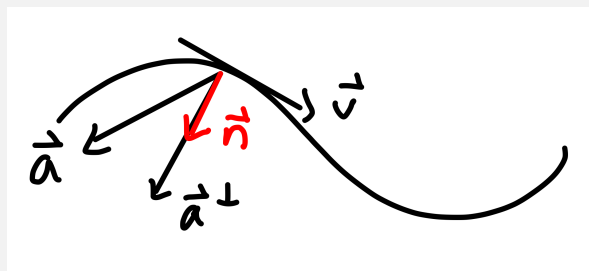
$\mathbb{R}^2$  is the only  $\mathbb{R}^n$  where the terms “clockwise” and “counter-clockwise” makes sense. This allows us to define

“signed curvature” = curvature + turning direction with respect to  $\mathbf{v}$

### Recall 4.8

$$\kappa = \frac{|\mathbf{a}^\perp|}{|\mathbf{v}|^2} = \frac{\langle \mathbf{a}, \mathbf{n} \rangle}{|\mathbf{v}|^2}$$



**Definition 4.9** (Signed curvature)

Let  $\gamma: I \rightarrow \mathbb{R}^2$  be a regular plane curve. Then the signed curvature  $\kappa_s: I \rightarrow \mathbb{R}$  is defined as

$$\kappa_s = \frac{\langle \mathbf{a}, \mathbf{n}_s \rangle}{|\mathbf{v}|^2},$$

where,

$$\begin{aligned} \mathbf{n}_s &= R_{90} \mathbf{t} \\ &= \text{the counterclockwise } 90^\circ \text{ rotation of } \mathbf{t} \end{aligned}$$

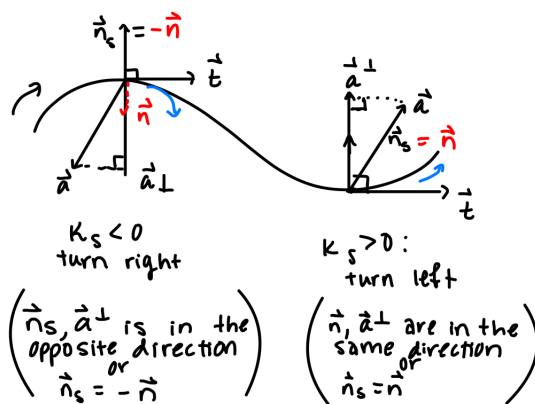
# 5 Jan 12, 2022

## 5.1 Plane Curves (Cont'd)

### Recall 5.1

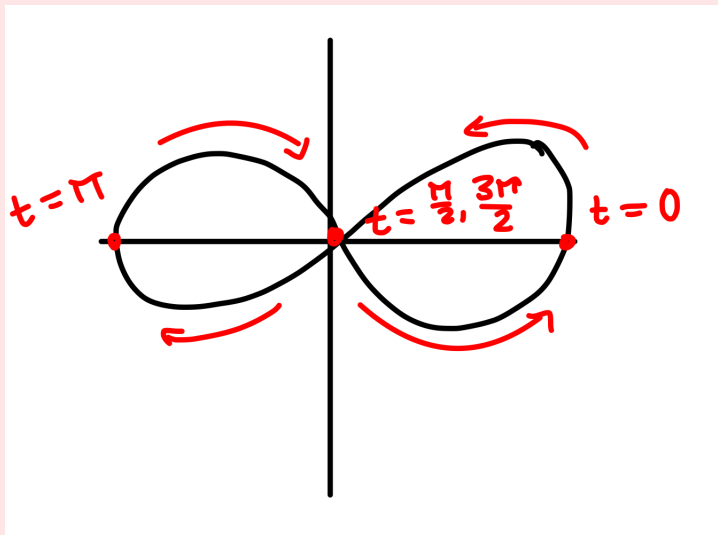
$$\kappa_s = \frac{\langle \mathbf{a}, \mathbf{n}_s \rangle}{|\mathbf{v}|^2}$$

where,  $\mathbf{n}_s = R_{90^\circ} \mathbf{t}$



**Example 5.2**

$$\gamma(t) = (\cos(t), \sin(2t)), \quad t \in [0, 2\pi]$$



Lissajous curve

$$\mathbf{v}(t) = (-\sin t, 2 \cos 2t)$$

$$\mathbf{a}(t) = (-\cos t, -4 \sin 2t)$$

$$|\mathbf{v}(t)| = \sqrt{\sin^2 t + 4 \cos^2 2t}$$

$$\mathbf{t}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = (-\sin t, 2 \cos 2t) \frac{1}{\sqrt{\sin^2 t + 4 \cos^2 2t}}$$

$$\mathbf{n}_s = R_{90} \mathbf{t} = (-2 \cos 2t, -\sin t) \frac{1}{\sqrt{\sin^2 t + 4 \cos^2 2t}}$$

$$\kappa_s = \frac{\langle \mathbf{a}, \mathbf{n}_s \rangle}{|\mathbf{v}|^2} = \frac{2 \cos t \cos 2t + 4 \sin t \sin 2t}{(\sin^3 t + 4 \cos^2 2t)^{3/2}}$$

$$\kappa_s(0) = \frac{2}{4^{3/2}} = \frac{2}{8} = \frac{1}{4} > 0$$

$$\kappa_s\left(\frac{\pi}{2}\right) = 0$$

$$\kappa_s(\pi) = \frac{-1}{4} < 0$$

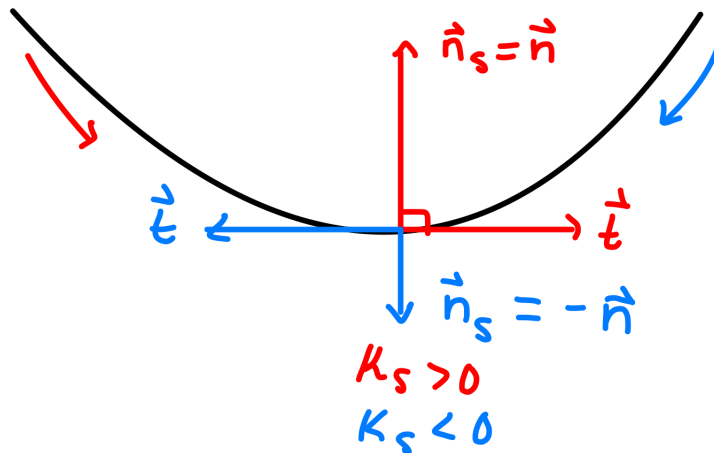
$$\kappa_s\left(\frac{3\pi}{2}\right) = 0$$

**Proposition 5.3**

Let  $\gamma: I \rightarrow \mathbb{R}^2$  be a plane curve. Then  $|\kappa_s| = \kappa$ .

**Proof.** Compare  $\kappa = \frac{\langle \mathbf{a}, \mathbf{n} \rangle}{|\mathbf{v}|^2}$ ,  $\kappa_s = \frac{\langle \mathbf{a}, \mathbf{n}_s \rangle}{|\mathbf{v}|^2}$

$\mathbf{n}_s = \pm \mathbf{n}$ , because they are both unit, orthogonal to  $\mathbf{t}$ . Hence  $\kappa_s$  coincides with  $\kappa_{\geq 0}$  up to signs.



□

**Proposition 5.4**

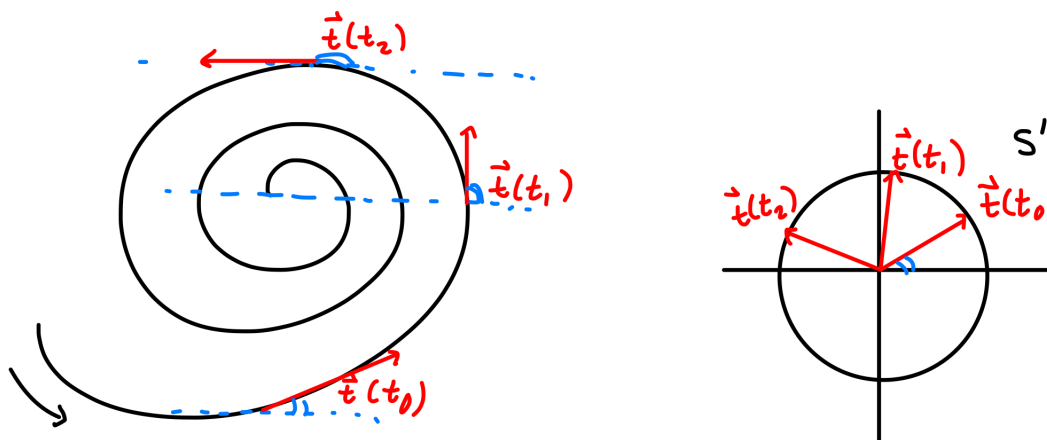
Signed curvature is unchanged by any orientation-preserving reparametrizations.

**Proof.** Exercise.

□

**Proposition 5.5**

Let  $\gamma: I \rightarrow \mathbb{R}^2$  be a plane curve. Then there exists a smooth function  $\theta: I \rightarrow \mathbb{R}$  such that  $\forall t \in I, \mathbf{t}(t) = (\cos \theta(t), \sin \theta(t))$ .



What should  $\theta$  be?

$$\mathbf{t}' = \theta'(-\sin \theta, \cos \theta) = \theta' R_{90} \mathbf{t} = \theta' \mathbf{n}_s.$$

On the other hand,

$$\mathbf{t}' = \left( \frac{\mathbf{v}}{|\mathbf{v}|} \right)' = \frac{\mathbf{a}^\perp}{|\mathbf{v}|} = \frac{\langle \mathbf{a}, \mathbf{n}_s \rangle}{|\mathbf{v}|} \mathbf{n}_s = \kappa_s |\mathbf{v}| \mathbf{n}_s$$

By comparing the two formulas,  $\theta' = \kappa_s |\mathbf{v}|$ . In the proof, we solve this differential equation.

**Remark 5.6** If  $\gamma$  is unit-speed,  $\theta' = \kappa_s$ . This shows:

$$\begin{aligned} \text{signed curvature} &= \text{the rate of change of the angle} \\ \text{curvature} &= |\text{the rate of change of the angle}| \end{aligned}$$

**Proof.** Fix  $t_0 \in I, \theta_0 \in \mathbb{R}$  such that  $\mathbf{t}(t_0) = (\cos \theta_0, \sin \theta_0)$ .

Define

$$\theta(t) = \theta_0 + \int_{t_0}^t \kappa_s(u) |\mathbf{v}(u)| du$$

We will show this  $\theta(t)$  works.

$\theta: I \rightarrow \mathbb{R}$  is a smooth function

$$\theta' = \kappa_s |\mathbf{v}|, \theta(t_0) = \theta_0.$$

Set  $\mathbf{t}_\theta = (\cos \theta, \sin \theta)$

We need to show  $\mathbf{t} = \mathbf{t}_\theta$ .

Observe  $\mathbf{t}, \mathbf{t}_\theta$  are unit.

Enough to show  $\langle \mathbf{t}, \mathbf{t}_\theta \rangle = 1$

On the other hand,

$$\begin{aligned} \mathbf{t}_\theta(t_0) &= (\cos \theta(t_0), \sin \theta(t_0)) \\ &= (\cos \theta_0, \sin \theta_0) \\ &= \mathbf{t}(t_0) \end{aligned}$$

So,

$$\langle \mathbf{t}(t_0), \mathbf{t}_\theta(t_0) \rangle = 1$$

Enough to show  $\langle \mathbf{t}, \mathbf{t}_\theta \rangle' = 0$

$$\begin{aligned} \mathbf{t}' &= \kappa_s |\mathbf{v}| \mathbf{n}_s = \kappa_s |\mathbf{v}| R_{90} \mathbf{t} \\ \mathbf{t}'_\theta &= \theta' (-\sin \theta, \cos \theta) = \kappa_s |\mathbf{v}| R_{90} \mathbf{t}_\theta \end{aligned}$$

Therefore,

$$\begin{aligned} \langle \mathbf{t}, \mathbf{t}_\theta \rangle' &= \langle \mathbf{t}', \mathbf{t}_\theta \rangle + \langle \mathbf{t}, \mathbf{t}'_\theta \rangle \\ &= \kappa_s |\mathbf{v}| (\langle R_{90} \mathbf{t}, \mathbf{t}_\theta \rangle + \langle \mathbf{t}, R_{90} \mathbf{t}_\theta \rangle) \\ &= \kappa_s |\mathbf{v}| (\langle R_{90} \mathbf{t}, \mathbf{t}_\theta \rangle + \langle R_{90} \mathbf{t}, R_{90} (R_{90} \mathbf{t}_\theta) \rangle) && R_{90} \text{ is orthogonal} \\ &= \kappa_s |\mathbf{v}| (\langle R_{90} \mathbf{t}, \mathbf{t}_\theta \rangle - \langle R_{90} \mathbf{t}, \mathbf{t}_\theta \rangle) && R_{90} \circ R_{90} = R_{180} = -1 \\ &= 0 \end{aligned}$$

□

**Remark 5.7** The angle function  $\theta$  is unique up to an integer multiple of  $2\pi$ .

Indeed if  $\Theta: I \rightarrow \mathbb{R}$  is a smooth function such that  $\forall t \in I, \gamma = (\cos \Theta, \sin \Theta)$ , then,

$$\begin{aligned} \Theta' &= \theta' = \kappa_s |\mathbf{v}| \\ \implies |\Theta - \theta|' &= 0 \\ \implies \Theta - \theta &= \text{constant} \end{aligned}$$

On the other hand,

$$(\cos \theta, \sin \theta) = (\cos \Theta, \sin \Theta) = \mathbf{t}$$

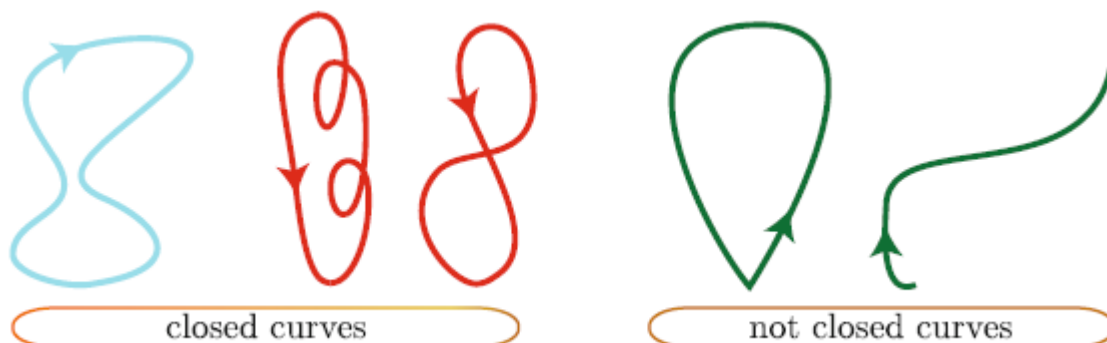
So  $\Theta - \theta \in 2\pi \cdot \mathbb{Z}$

## 6 Jan 14, 2022

### 6.1 Plane Curves(Cont'd)

**Definition 6.1** (Closed curve)

A regular curve  $\gamma: [a, b] \rightarrow \mathbb{R}^n$  is called closed if  $\gamma(a) = \gamma(b)$ , and  $\forall n \in \mathbb{N}, \gamma^{(n)}(a) = \gamma^{(n)}(b)$


**Definition 6.2** (Rotation index)

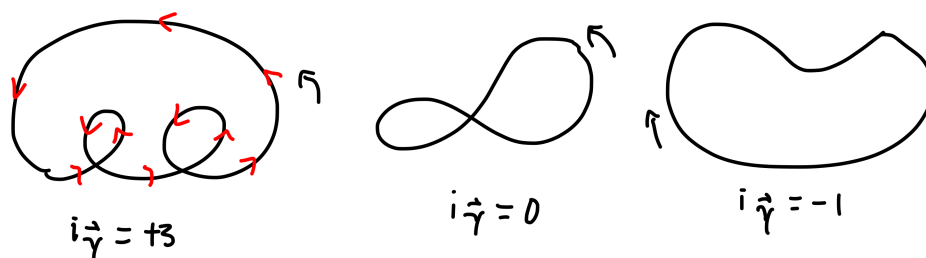
Let  $\gamma: [a, b] \rightarrow \mathbb{R}^2$  be a closed plane curve. The rotation index of  $\gamma$  is defined as

$$i_\gamma = \frac{1}{2\pi}(\theta(b) - \theta(a)),$$

where  $\theta$  is the angle function from proposition 5.5.

**Remarks 6.3**

- i.  $i_\gamma \in \mathbb{Z}$ , because  $\mathbf{t}(a) = \mathbf{t}(b)$ , so  $\theta(b) - \theta(a) \in 2\pi\mathbb{Z}$
- ii. Later on, we will show  $i_\gamma = \pm 1$  if  $\gamma$  has no self-intersection.


**Proposition 6.4**

Let  $\gamma: [a, b] \rightarrow \mathbb{R}^2$  be a closed plane curve. Then

$$i_\gamma = \frac{1}{2\pi} \int_a^b \kappa_s(t) |\mathbf{v}(t)| dt$$

**Proof.** This follows from the construction of the angle function.  $\square$

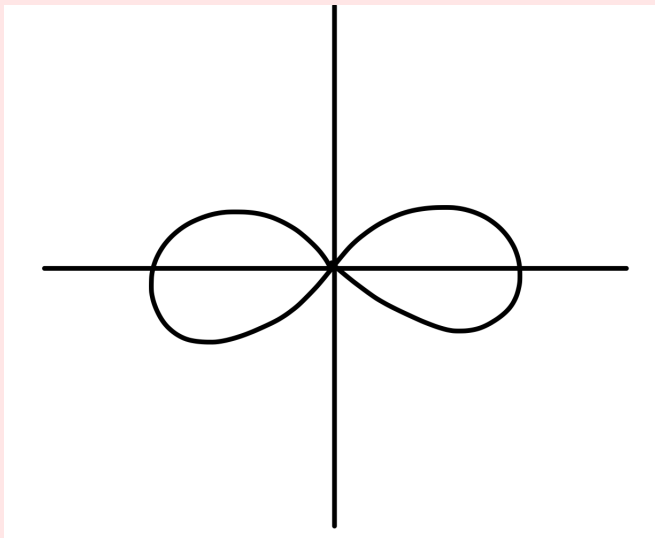
**Proposition 6.5**

Rotation index is unchanged under any orientation-preserving reparametrizations.

**Proof.** Exercise.  $\square$

**Example 6.6**

$$\gamma(t) = (\cos t, \sin 2t), t \in [0, 2\pi]$$



Recall:

$$\kappa_s(t) = \frac{2 \cos t \cos 2t + 4 \sin t \sin 2t}{(\sin^2 t + 4 \cos^2 2t)^{3/2}}$$

$$|\mathbf{v}| = (\sin^2 t + 4 \cos^2 2t)^{1/2}$$

Therefore,

$$\begin{aligned} i_\gamma &= \frac{1}{2\pi} \int_0^{2\pi} \frac{2 \cos t \cos 2t + 4 \sin t \sin 2t}{\sin^2 t + 4 \cos^2 2t} dt \\ &= \frac{1}{2\pi} \left( \int_0^\pi \text{---} dt + \underbrace{\int_\pi^{2\pi} \text{---} dt}_{\substack{t=s+\pi, \\ \text{then the integrand} \\ \text{is multiplied by } -1}} \right) \\ &= 0 \end{aligned}$$

## 6.2 Space Curves

What's special about  $\mathbb{R}^3$ ?

$\mathbb{R}^3$  has the cross product.

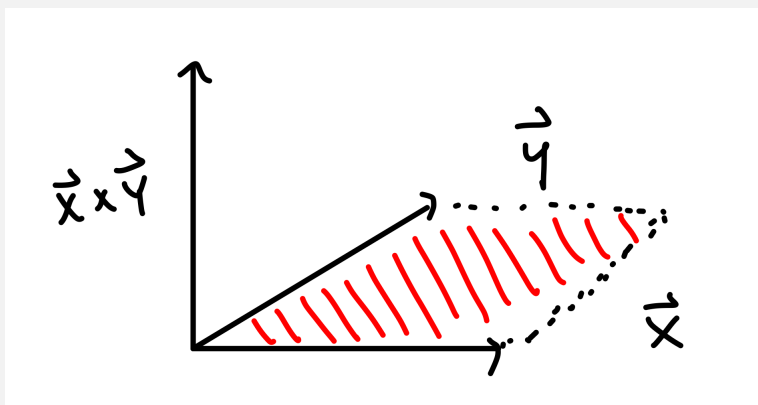
**Recall 6.7**  $\mathbf{x} = (x_1, x_2, x_3), \mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$ ,

$$\mathbf{x} \times \mathbf{y} = (x_2 y_3 - x_3 y_2, -(x_1 y_3 - x_3 y_1), x_1 y_2 - x_2 y_1) \in \mathbb{R}^3$$



Basic properties:

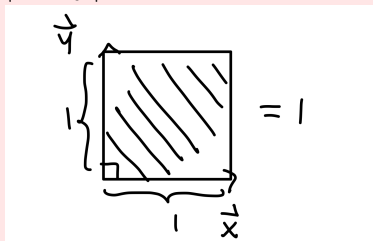
- i.  $\times: \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is bilinear, and antisymmetric.  
(i.e.  $\mathbf{y} \times \mathbf{x} = -\mathbf{x} \times \mathbf{y}$ )
- ii.  $|\mathbf{x} \times \mathbf{y}| = |\mathbf{x}||\mathbf{y}|\sin(\theta)$ , where  $\theta$  is the angle between  $\mathbf{x}, \mathbf{y}$   
= the area of the parallelogram spanned by  $\mathbf{x}, \mathbf{y}$
- iii.  $\mathbf{x} \times \mathbf{y}$  is orthogonal to  $\mathbf{x}, \mathbf{y}$ ;  
 $\{\mathbf{x}, \mathbf{y}, \mathbf{x} \times \mathbf{y}\}$  is a right-handed system.



### Example 6.8

If  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$  are orthonormal, then  $\{\mathbf{x}, \mathbf{y}, \mathbf{x} \times \mathbf{y}\}$  is an orthonormal basis for  $\mathbb{R}^3$ :

- $\mathbf{x} \times \mathbf{y}$  is orthogonal to  $\mathbf{x}, \mathbf{y}$ , and
- $|\mathbf{x} \times \mathbf{y}| =$



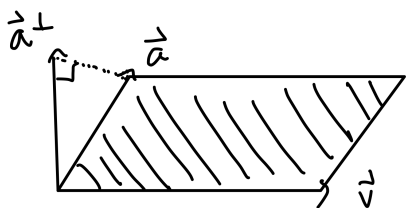
= 1

### Proposition 6.9

Let  $\gamma: I \rightarrow \mathbb{R}^3$  be a space curve, then

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$$

**Proof.**  $|\mathbf{v} \times \mathbf{a}| =$



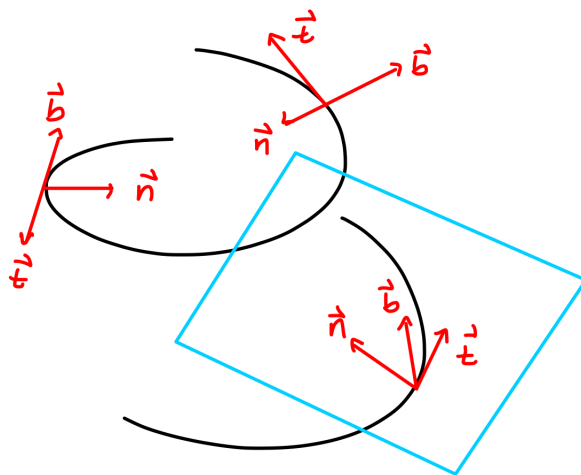
$$= |\mathbf{v}| |\mathbf{a}^\perp|$$

$$\implies \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{|\mathbf{a}^\perp|}{|\mathbf{v}|^2} = \kappa$$

□

**Definition 6.10** (Unit binormal vector and Frenet frame)

Let  $\gamma: I \rightarrow \mathbb{R}^3$  be a space curve. The unit binormal vector for  $\gamma$  at  $t \in I$  is defined as  $\mathbf{b}(t) = \mathbf{t}(t) \times \mathbf{n}(t)$  (only if  $\kappa(t) \neq 0$ ). The orthonormal basis  $\{\mathbf{t}(t), \mathbf{n}(t), \mathbf{b}(t)\}$  for  $\mathbb{R}^3$  is called the Frenet frame for  $\gamma$  at  $t$ .



**Remark 6.11**  $\mathbf{b}(t)$  is a unit normal vector to the osculating plane of  $\gamma$  at  $t$ .

$\implies \mathbf{b}$  encodes the tilt of the osculating plane of  $\gamma$ .

We want to define the “torsion” as the measurement of the change of the tilt of the osculating plane.

**Definition 6.12** (Torsion)

Let

$$\gamma: I \rightarrow \mathbb{R}^3 \text{ be a space curve,}$$

$$t \in I \text{ s.t. } \kappa(t) \neq 0$$

The torsion of  $\gamma$  at  $t$  is defined as

$$\tau(t) = -\frac{\langle \mathbf{b}'(t), \mathbf{n}(t) \rangle}{|\mathbf{v}(t)|}$$

**Remark 6.13** Why is this definition plausible?

- i.  $\mathbf{b}'(t)$  is parallel to  $\mathbf{n}(t)$  (later).  
So  $\langle \mathbf{b}'(t), \mathbf{n}(t) \rangle = \pm |\mathbf{b}'(t)|$
- ii.  $\langle \mathbf{b}'(t), \mathbf{n}(t) \rangle$  depends on parametrizations.

**Proposition 6.14**

Torsion is independent of parametrizations.

**Proof.** Read Tapp for the details.

Sketch:

$\varphi$  is orientation-preserving.

$$\tilde{t} = t \circ \varphi, \tilde{n} = n \circ \varphi$$

$$\implies \tilde{b} = b \circ \varphi$$

$$\implies \tilde{b}' = b' \circ \varphi \cdot \varphi'$$

□

# 7 Jan 19, 2022

## 7.1 Space Curves (Cont'd)

**Recall 7.1**  $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ ,  $\tau = -\frac{\langle \mathbf{b}', \mathbf{n} \rangle}{|\mathbf{v}|}$

Note:  $\mathbf{b}' = -\tau|\mathbf{v}|\mathbf{n}$

### Proposition 7.2

Let  $\gamma: I \rightarrow \mathbb{R}^3$  be a space curve such that  $\forall t \in I, \kappa(t) \neq 0$ . Then the following conditions are equivalent:

- i. The trace of  $\gamma$  is contained in a plane in  $\mathbb{R}^3$ .
- ii.  $\forall t \in I, \tau(t) = 0$ .

**Remark 7.3** The torsion measures the failure of a space curve to remain in a plane in  $\mathbb{R}^3$ .

**Proof.** (i.) is equivalent to:

$$(i.)' \exists \mathbf{w} \neq \mathbf{0} \in \mathbb{R}^3, c \in \mathbb{R}, \forall t \in I, \langle \gamma, \mathbf{w} \rangle = c$$

We show (i.)'  $\Leftrightarrow$  (ii.).

$$(\Leftarrow) \mathbf{b}' = -\tau|\mathbf{v}|\mathbf{n} = 0, \text{ so}$$

$$\mathbf{b} = \text{constant} =: \mathbf{w} \neq 0$$

$$\langle \gamma(t), \mathbf{w} \rangle' = \langle \mathbf{v}(t), \mathbf{w} \rangle = \langle |\mathbf{v}(t)|\mathbf{t}(t), \mathbf{b}(t) \rangle = 0, \text{ so}$$

$$\langle \gamma(t), \mathbf{w} \rangle = \text{constant}.$$

$$(\Rightarrow) \langle \gamma(t), \mathbf{w} \rangle = \text{constant, so}$$

$$\langle \gamma(t), \mathbf{w} \rangle = \langle \mathbf{a}(t), \mathbf{w} \rangle = 0$$

$$\mathbf{t}(t), \mathbf{n}(t) \in \text{span}(\mathbf{v}(t), \mathbf{a}(t)), \text{ so}$$

$$\langle \mathbf{t}(t), \mathbf{w} \rangle = \langle \mathbf{n}(t), \mathbf{w} \rangle = 0.$$

This shows that  $\mathbf{w}$  is normal to the osculating plane spanned by  $\mathbf{t}(t), \mathbf{n}(t)$ , so

$$\mathbf{b}(t) = \pm \frac{\mathbf{w}}{|\mathbf{w}|} = \text{constant, so}$$

$$\mathbf{b}'(t) = \mathbf{0}, \text{ so}$$

$$\tau(t) = -\frac{\langle \mathbf{b}'(t), \mathbf{n}(t) \rangle}{|\mathbf{v}(t)|} = 0$$

□

There are differential equations for  $\mathbf{t}, \mathbf{n}, \mathbf{b}$  determined by  $\kappa, \tau$ .

**Proposition 7.4** (Frenet equations)

Let  $\gamma: I \rightarrow \mathbb{R}^3$  be a space curve such that  $\forall t \in I, \kappa(t) \neq 0$ .  
Then,

$$\begin{aligned} \mathbf{t}' &= \kappa|\mathbf{v}|\mathbf{n} \\ \mathbf{n}' &= -\kappa|\mathbf{v}|\mathbf{t} + \tau|\mathbf{v}|\mathbf{b} \\ \mathbf{b}' &= -\kappa|\mathbf{v}|\mathbf{n} \end{aligned}$$

In particular, if  $\gamma$  is unit-speed, then

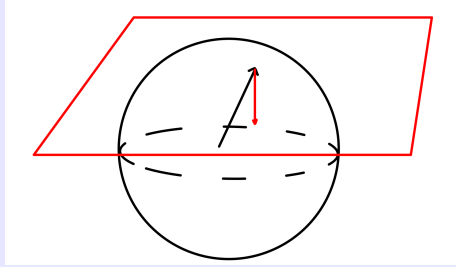
$$\begin{aligned} \mathbf{t}' &= \kappa\mathbf{n} \\ \mathbf{n}' &= -\kappa\mathbf{t} + \tau\mathbf{b} \\ \mathbf{b}' &= -\kappa\mathbf{n} \end{aligned}$$

**Remark 7.5** This suggests that a space curve is completely determined by the functions  $\kappa, \tau$  up to initial conditions. (Fundamental Theorem of Space Curves)

**Lemma 7.6**

Let  $\gamma, \delta: I \rightarrow \mathbb{R}^n$  be curves (not necessarily regular).

- i. If  $\exists c \in \mathbb{R}, \forall t \in I, |\gamma(t)| = c$ , then  $\forall t \in I, \gamma'(t)$  is orthogonal to  $\gamma(t)$ .



- ii. If  $\exists D \in \mathbb{R}, \forall t \in I, \langle \gamma(t), \delta(t) \rangle = D$ , then  $\forall t \in I, \langle \gamma'(t), \delta(t) \rangle = -\langle \gamma(t), \delta'(t) \rangle$ .

**Remark 7.7** Both the assumptions are satisfied if  $\forall t \in I, \gamma(t), \delta(t)$  are orthogonal.

**Proof of Lemma.**

- i.  $c^2 = |\gamma(t)|^2 = \langle \gamma(t), \gamma(t) \rangle$ .  
 $\implies 0 = 2 \langle \gamma(t), \gamma'(t) \rangle$   
 $\implies \langle \gamma(t), \gamma'(t) \rangle = 0$
- ii.  $\langle \gamma(t), \delta(t) \rangle = D$   
 $\implies \langle \gamma'(t), \delta(t) \rangle + \langle \gamma(t), \delta'(t) \rangle = 0$   
 $\implies \langle \gamma'(t), \delta(t) \rangle = -\langle \gamma(t), \delta'(t) \rangle$

□

**Proof of Proposition 7.4.** We have proved  $\mathbf{t}' = \kappa|\mathbf{v}|\mathbf{n}$ . As for  $\mathbf{n}', \mathbf{b}'$ , it is enough to compute their components with respect to the Frenet frame  $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ .

$$\langle \mathbf{n}', \mathbf{t} \rangle = -\langle \mathbf{n}, \mathbf{t}' \rangle = -\langle \mathbf{n}, \kappa|\mathbf{v}|\mathbf{n} \rangle = -\kappa|\mathbf{v}|$$

$$\begin{aligned}\langle \mathbf{n}', \mathbf{n} \rangle &= 0 \\ \langle \mathbf{n}', \mathbf{b} \rangle &= -\langle \mathbf{n}, \mathbf{b}' \rangle = \tau|\mathbf{v}|\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbf{n}' &= -\kappa|\mathbf{v}|\mathbf{t} + \tau|\mathbf{v}|\mathbf{b}. \\ \langle \mathbf{b}', \mathbf{t} \rangle &= -\langle \mathbf{b}, \mathbf{t}' \rangle = -\langle \mathbf{b}, -\kappa|\mathbf{v}|\mathbf{n} \rangle = 0 \\ \langle \mathbf{b}', \mathbf{n} \rangle &= -\tau|\mathbf{v}| \\ \langle \mathbf{b}', \mathbf{b} \rangle &= 0\end{aligned}$$

Therefore,

$$\mathbf{b}' = -\tau|\mathbf{v}|\mathbf{n}$$

□

**Remark 7.8** Another interpretation of the torsion can be given by the Frenet equations. Let  $\gamma: I \rightarrow \mathbb{R}^3$  be a unit-speed space curve. Then,

$$\begin{aligned}\gamma' &= \mathbf{t}, \gamma'' = \mathbf{t}' = \kappa\mathbf{n}, \\ \gamma''' &= (\kappa\mathbf{n})' = \kappa'\mathbf{n} + \kappa\mathbf{n}' = -\kappa^2\mathbf{t} + \kappa'\mathbf{n} + \kappa\tau\mathbf{b}\end{aligned}$$

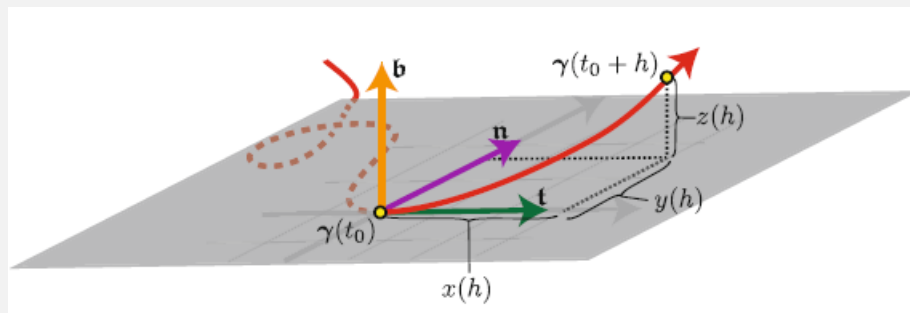
So the 3rd order Taylor approximation at  $t_0 \in I, \kappa(t_0) > 0$  is as follows:

$$\begin{aligned}\mathbf{D}(h) &= \gamma(t_0 + h) - \gamma(t_0) \\ &\approx h\gamma'(t_0) + \frac{h^2}{2}\gamma''(t_0) + \frac{h^3}{6}\gamma'''(t_0) \\ &= \left(h - \frac{\kappa^2 h^3}{6}\right)\mathbf{t} + \left(\frac{\kappa h^2}{2} + \frac{\kappa' h^3}{6}\right)\mathbf{n} + \frac{\kappa\tau h^3}{6}\mathbf{b}\end{aligned}$$

Therefore,

$$\begin{aligned}x(h) &= \langle \mathbf{D}(h), \mathbf{t} \rangle \approx h - \frac{\kappa^2 h^3}{6} \\ y(h) &= \langle \mathbf{D}(h), \mathbf{n} \rangle \approx \frac{\kappa h^2}{2} + \frac{\kappa' h^3}{6} \\ z(h) &= \langle \mathbf{D}(h), \mathbf{b} \rangle \approx \frac{\kappa\tau h^3}{6}\end{aligned}$$

If  $\tau(t_0) > 0$ , then the curve passes through the osculating plane from below.

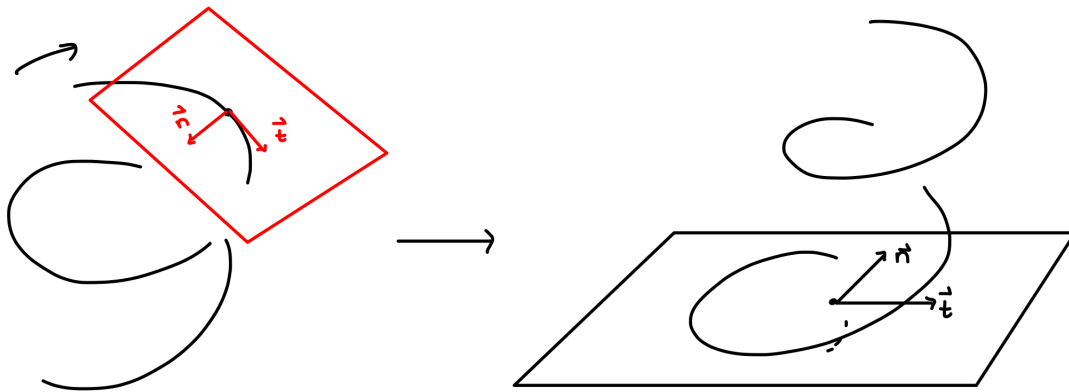


If  $\tau(t_0) < 0$ , then the curve passes through the osculating plane from above.

# 8 Jan 21, 2022

## 8.1 Rigid Motions

In geometry, it is often useful to “tilt your head”, or choose an orthonormal set of vectors at a point, adapted to the problem at hand:



This is achieved by rigid motions.

### Definition 8.1 (Rigid motion)

A rigid motion in  $\mathbb{R}^n$  is a function  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  that preserves the distances:

$$\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n, |f(\mathbf{x}) - f(\mathbf{y})| = |\mathbf{x} - \mathbf{y}|$$

### Example 8.2

The translation by  $\mathbf{p} \in \mathbb{R}^n$

$$T_{\mathbf{p}}: \mathbb{R}^n \rightarrow \mathbb{R}^n, \mathbf{x} \mapsto \mathbf{x} + \mathbf{p}$$

is a rigid motion. Indeed,

$$\begin{aligned} |T_{\mathbf{p}}(\mathbf{x}) - T_{\mathbf{p}}(\mathbf{y})| &= |\mathbf{x} + \mathbf{p} - (\mathbf{y} + \mathbf{p})| \\ &= |\mathbf{x} - \mathbf{y}| \end{aligned}$$

Note:  $T_{\mathbf{p}}$  is never linear if  $\mathbf{p} \neq \mathbf{0}$ , because  $T_{\mathbf{p}}(\mathbf{0}) = \mathbf{p} \neq \mathbf{0}$ .



**Theorem 8.3**

Let  $L_A: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear transformation represented by an  $n \times n$  matrix  $A$ . The following conditions are equivalent:

1.  $L_A$  is a rigid motion.
2.  $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n, \langle L_A(\mathbf{x}), L_A(\mathbf{y}) \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$ .
3. If  $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$  is an orthonormal basis of  $\mathbb{R}^n$ , so is  $\{L_A \mathbf{x}_1, \dots, L_A \mathbf{x}_n\}$ .
4. The column vectors of  $A$  form an orthonormal basis of  $\mathbb{R}^n$ .
5.  $A^T A = I_n$

**Definition 8.4**

A linear rigid motion and its matrix are called orthogonal.

$$O(n) := \text{the set of all } n \times n \text{ orthogonal matrices}$$

**Proposition 8.5**

Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a rigid motion. Then,

$$\exists! \mathbf{p} \in \mathbb{R}^n, \exists! A \in O(n), f = T_{\mathbf{p}} \circ L_A$$

**Sketch of proof.** Step 1: ( $f(\mathbf{0}) = \mathbf{0}$ ):  $\exists! A \in O(n), f = L_A$

Step 2: (General Case): Set  $\mathbf{p} = f(\mathbf{0})$ . Then apply Step 1 to  $(T_{\mathbf{p}})^{-1} \circ f = T_{-\mathbf{p}} \circ f$

Indeed,

$$(T_{\mathbf{p}})^{-1} \circ f(\mathbf{0}) = T_{-\mathbf{p}} \circ f(\mathbf{0}) = T_{-\mathbf{p}}(\mathbf{p}) = \mathbf{0},$$

So,

$$\exists! A \in O(n), (T_{\mathbf{p}})^{-1} \circ f = L_A$$

$$\implies f = T_{\mathbf{p}} \circ L_A$$

Read Tapp for the details. □

We can classify rigid motions:

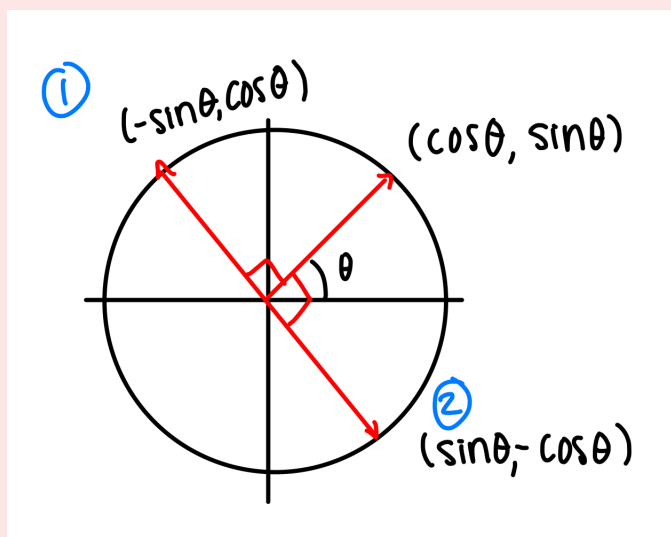
**Lemma 8.6**

$$A \in O(n) \implies \det(A) = \pm 1$$

**Proof.**  $A^T A = I_n$ , so  $1 = \det(A^T A) = \det(A^T) \det(A) = \det(A)^2$  □

**Example 8.7**

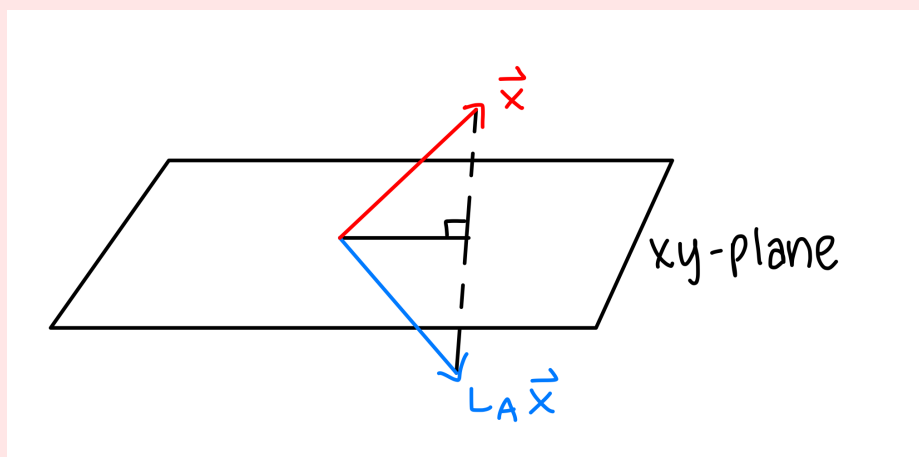
Let  $A \in O(2)$ . The column vectors of  $A$  are orthonormal:



$$A = \underbrace{\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}}_{\substack{\text{rotation,} \\ \det=1 \\ \text{proper}}} \text{ or } \underbrace{\begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}}_{\substack{\text{reflection} \\ \det=-1 \\ \text{improper}}}$$

**Example 8.8**

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in O(3)$  represents the reflection about the  $xy$  plane:



$\det(A) = -1$ , so  $L_A$  is improper.

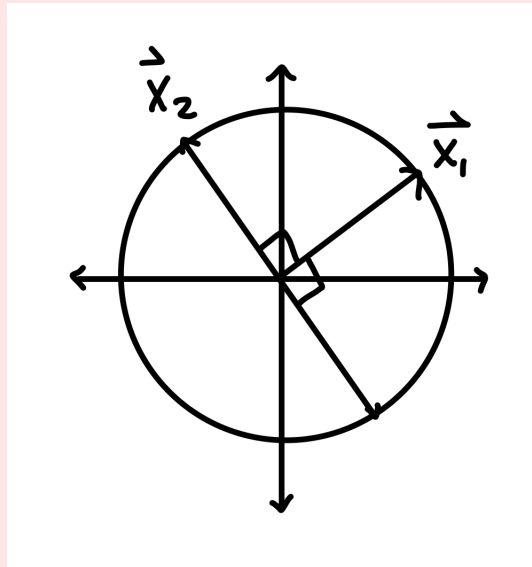
**Remark 8.9**      proper = physically performable (e.g. rotations)  
                          improper = physically unperformable (e.g. reflections)

Another interpretation of proper (improper) rigid motions is given in terms of the orientation of orthonormal basis.

**Definition 8.10** (Ordered orthonormal basis and Positively oriented vs. Negatively oriented)  
 An ordered orthonormal basis (o.o.b.)  $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$  of  $\mathbb{R}^n$  is called positively oriented (p.o.) if the orthogonal matrix whose column vectors are  $\mathbf{x}_1, \dots, \mathbf{x}_n$  has  $\det = 1$ , and negatively oriented (n.o.) if it has  $\det = -1$ .

**Example 8.11**

$\{\mathbf{x}_1, \mathbf{x}_2\}$  are o.o.b of  $\mathbb{R}^2$ .



$$\text{p.o.} \Leftrightarrow \mathbf{x}_2 = R_{90} \mathbf{x}_1$$

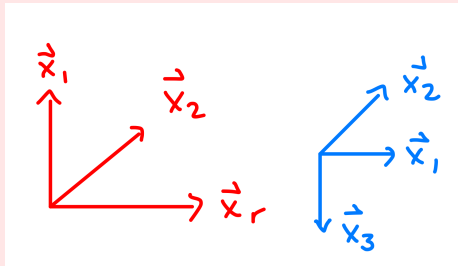
$$\text{n.o.} \Leftrightarrow \mathbf{x}_2 = R_{-90} \mathbf{x}_1$$

**Example 8.12**

$\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$  o.o.b. of  $\mathbb{R}^3$ .

p.o.  $\Leftrightarrow \mathbf{x}_3 = \mathbf{x}_1 \times \mathbf{x}_2 \Leftrightarrow$  right-hand

n.o.  $\Leftrightarrow \mathbf{x}_3 = -\mathbf{x}_1 \times \mathbf{x}_2 \Leftrightarrow$  left-hand

**Proposition 8.13**

Let  $A \in O(n)$ . Then

$A$  preserves the orientation of any o.o.b.  $\Leftrightarrow \det(A) = +1$

$A$  reserves the orientation of any o.o.b.  $\Leftrightarrow \det(A) = -1$ .

**Proof.** Let  $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$  be an o.o.b. of  $\mathbb{R}^n$ . Set

$$B := (\mathbf{x}_1 \cdots \mathbf{x}_n) \in O(n).$$

Then,

$$AB = (L_A \mathbf{x}_1 \cdots L_A \mathbf{x}_n)$$

Note,  $\det(AB) = \det(A) \det(B)$ .

Therefore,

$$\det(AB) = \begin{cases} \det(B) & \text{if } \det(A) = 1 \\ -\det(B) & \text{if } \det(A) = -1 \end{cases}$$

□

**Proposition 8.14**

The following functions are unchanged by proper rigid motions:

- i. Curvature for a regular curve
- ii. Torsion for a space curve
- iii. Signed curvature for a plane curve.

By improper rigid motions, (i) is unchanged, (ii) and (iii) are multiplied by  $-1$ .