Math 120A (Differential Geometry)
University of California, Los Angeles

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These are my lecture notes for Math 120A (Differential Geometry), which is taught by Fumiaki Suzuki. The textbook for this class is *Differential Geometry of Curves and Surfaces*, by Kristopher Tapp. Many of the figures I include in these notes are taken from Tapp's book.

Contents

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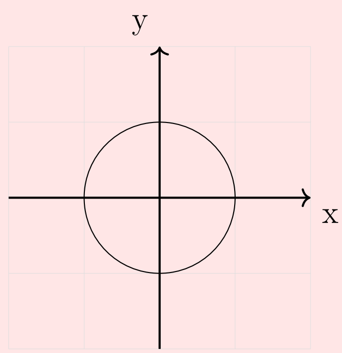
1.1 What is Differential Geometry?

Differential geometry studies geometry via analysis and linear algebra.

Geometry	Analysis	Linear Algebra
Intuitive	Rigorous	Computable
Curved	$\xrightarrow{\text{tangent space}}$	Linear
Global	Local	

1.2 Parametrized Curves

Example 1.1
A unit circle $S' = \{\vec{x} \text{ in } \mathbb{R}^2 \mid |\vec{x}| = 1\}$



$$\vec{\gamma} : [0, 2\pi) \to \mathbb{R}^2$$

$$t \mapsto (\cos t, \sin t)$$

$\vec{\gamma}[0, 2\pi) = S'$

Definition 1.2 (Parametrized curve and Trace)
A <u>(parametrized) curve</u> is a smooth function $\vec{\gamma}\colon I \to \mathbb{R}^n$, where I is an interval in $\mathbb{R}$. The image

$$\vec{\gamma}(I) = \{\vec{\gamma}(t) \mid t \in I\}$$

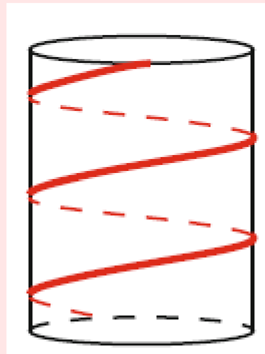
is called the <u>trace</u> of $\vec{\gamma}$.

Recall 1.3 An interval is a subset of $\mathbb{R}$ that has one of the following forms:

$$(a, b), [a, b], (a, b], [a, b), (-\infty, b), (-\infty, b], (a, \infty), [a, \infty), (-\infty, \infty) = \mathbb{R}.$$

A function $\vec{\gamma}\colon I \to \mathbb{R}^n$ is called <u>smooth</u> if $\vec{\gamma}$ is infinitely differentiable, or equivalently, each of the component functions $x_i\colon I \to \mathbb{R}$ is infinitely differentiable.

Example 1.4
$\vec{\gamma}(t) = (\cos t, \sin t, t)$, $t \in (-\infty, \infty)$ is a curve, called a helix.



Definition 1.5 (Derivative)
Let $\vec{\gamma}\colon I \to \mathbb{R}^n$ be a curve. The derivative of $\vec{\gamma}$ at t is defined as

$$\vec{\gamma}'(t) = \lim_{h \to 0} \frac{\vec{\gamma}(t+h) - \vec{\gamma}(t)}{h}$$

If t is on the boundaries of I, then use the left- or right-hand limit.

Remarks 1.6

i. If $\vec{\gamma}(t) = (x_1(t), x_2(t), \cdots, x_n(t))$, then $\vec{\gamma}'(t) = (x_1'(t), x_2'(t), \cdots, x_n'(t))$.

ii. The tangent line to the curve at $\vec{\gamma}'(t_0)$ is defined as

$$\vec{L}(t) = \vec{\gamma}(t_0) + t\vec{\gamma}'(t_0), \quad t \in (-\infty, \infty),$$

as soon as $\vec{\gamma}'(t) \neq \vec{0}$.

Definition 1.7 (Regular)
A curve $\vec{\gamma} : I \to \mathbb{R}^n$ is called regular if $\forall t \in I, \vec{\gamma}'(t) \neq \vec{0}$.

Remark 1.8 regular = tangent line is defined everywhere = the trace is smooth

Example 1.9

$$\vec{\gamma}(t) = (t^2, t^3), \quad t \in (-\infty, \infty)$$

Then $\vec{\gamma}$ is a curve that is not regular.
Indeed, $\vec{\gamma}'(t) = (2t, 3t^2)$, so $\vec{\gamma}'(0) = \vec{0}$.
Notice, $x(t) = t^2, y(t) = t^3$, so $x(t) = y(t)^{2/3}$. Hence, the trace is given by $x = y^{2/3}$ in $\mathbb{R}^2$.

Remark 1.10 The analogy with the physics is useful. If $\vec{\gamma} : I \to \mathbb{R}^n$ is a curve, then $\vec{\gamma}(t)$ is the position of a moving particle at time t in $\mathbb{R}^2$.

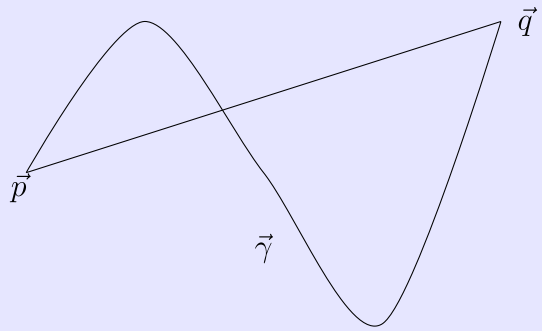
- $\vec{\gamma}'(t)$ velocity
- $\vec{\gamma}''(t)$ acceleration
- $|\vec{\gamma}'(t)|$ speed

In this analogy, regular = the speed is always nonzero = the particle never stops (hence no "corners" on the trace)

Definition 1.11 (Arc length)
Let $\vec{\gamma}(t)\colon I \to \mathbb{R}^n$ be a regular curve. Then the arc length between times t_1, t_2 is defined as

$$\int_{t_1}^{t_2} |\vec{\gamma}'(t)|\, dt$$

Proposition 1.12
Let $\vec{\gamma}\colon [a,b] \to \mathbb{R}^n$ be a regular curve with the arc length L, $\vec{p} = \vec{\gamma}(a), \vec{q} = \vec{\gamma}(b)$. Then $L \geq |\vec{q} - \vec{p}|$.
Moreover, the equality holds if and only if $\vec{\gamma}$ parametrizes the line segment between $\vec{p}, \vec{q}$.



For the proof, we use the inner-product:
for $\vec{x} = (x_1, x_2, \cdots, x_n), \vec{y} = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$,

$$\langle x, y \rangle \coloneqq x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

Basic properties:

i. The inner product $\langle -, - \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is symmetric and bilinear.
ii. $\langle \vec{x}, \vec{y} \rangle = |\vec{x}||\vec{y}| \cos\theta$, where θ is the angle between $\vec{x}, \vec{y}$. ($\theta \in [0, 2\pi]$)
iii. $\langle \vec{x}, \vec{y} \rangle = 0 \iff \vec{x}, \vec{y}$ are orthogonal to each other.
iv. $\langle \vec{x}, \vec{x} \rangle = |\vec{x}|^2$
v. $\langle \vec{x}, \vec{y} \rangle \leq |\vec{x}||\vec{y}|$ (Schwartz Inequality) and the equality holds if and only if $\theta = 0$.

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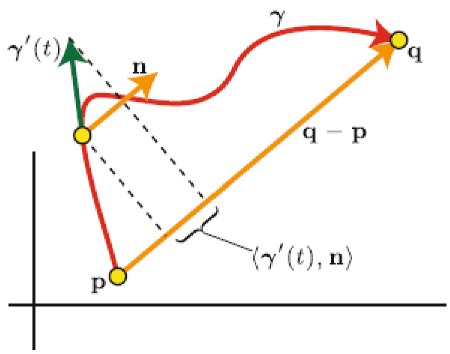
We start with the proof of Proposition 1.12.

Proof. <u>Idea:</u> Compare $\vec{\gamma}'(t)$ and its projection onto $\vec{q}-\vec{p}$. Set $\vec{n} = \frac{\vec{q}-\vec{p}}{|\vec{q}-\vec{p}|}$; $\vec{n}$ is unit.



Tapp Pg.15

Then $|\vec{\gamma}'(t)| \geq \langle \vec{\gamma}'(t), \vec{n} \rangle$ by Schwartz inequality.
Now,

$$\begin{aligned} L &= \int_a^b |\vec{\gamma}'(t)|\, dt \geq \int_a^b \langle \vec{\gamma}'(t), \vec{n} \rangle\, dt \\ &= [\langle \boldsymbol{\gamma}(t), \vec{n} \rangle]_a^b = \langle \boldsymbol{\gamma}(b), \vec{n} \rangle - \langle \boldsymbol{\gamma}(a), \vec{h} \rangle \\ &= \left\langle \vec{q}-\vec{p}, \frac{\vec{q}-\vec{p}}{|\vec{q}-\vec{p}|} \right\rangle = |\vec{q}-\vec{p}| \end{aligned}$$

If the equality holds, then $\forall t \in [a, b], \boldsymbol{\gamma}'(t), \vec{n}$ are in the same direction. So,

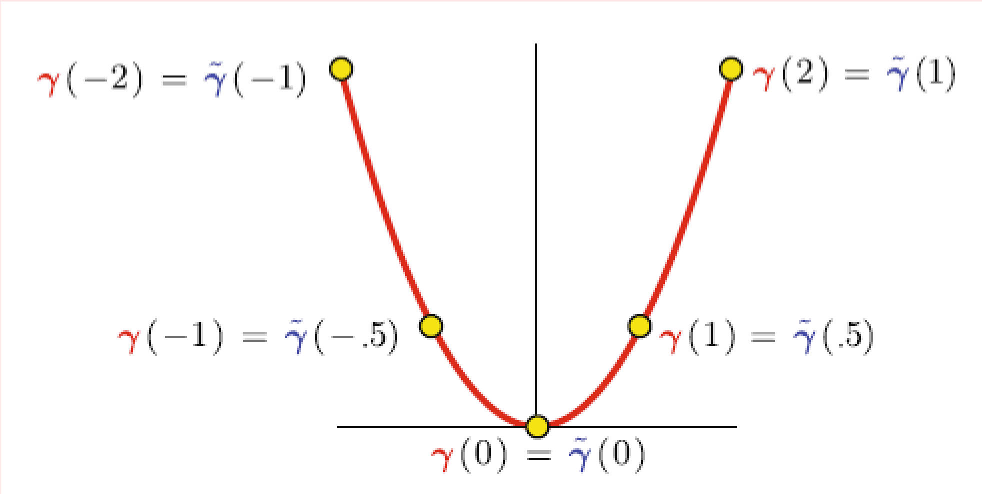
$$\begin{aligned} \boldsymbol{\gamma}'(t) &= \langle \boldsymbol{\gamma}'(t), \vec{n} \rangle \vec{n}. \\ \boldsymbol{\gamma}(t) &= \boldsymbol{\gamma}(a) + \int_a^t \boldsymbol{\gamma}'(u) du \\ &= \vec{p} + \left(\int_a^t \langle \boldsymbol{\gamma}'(u), \vec{n} \rangle dt \right) \vec{n} \end{aligned}$$

parametrizes the line segment between $\vec{p}, \vec{q}$. $\square$

2.1 Reparametrization

There are regular curves that share common properties. Which regular curves should we identify?

Example 2.1
$\boldsymbol{\gamma}(t) = (t, t^2), \quad t \in [-2, 2]$
$\tilde{\boldsymbol{\gamma}}(t) = (-2t, (-2t)^2), t \in [-1, 1].$
Then $\boldsymbol{\gamma}[-2, 2] = \tilde{\boldsymbol{\gamma}}[-1, 1] =$



$\boldsymbol{\gamma}, \tilde{\boldsymbol{\gamma}}$ are the same, up to change in time:
Let $\phi \colon [-1, 1] \to [-2, 2], \quad t \mapsto -2t.$
Then $\tilde{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \circ \phi$

Definition 2.2 (Reparametrization)
Let $\boldsymbol{\gamma} \colon I \to \mathbb{R}^n$ be a regular curve. A reparametrization of $\boldsymbol{\gamma}$ is a function of the form

$$\tilde{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \circ \phi : \tilde{I} \to \mathbb{R}^n,$$

where $\tilde{I}$ is an interval, $\phi \colon \tilde{I} \to I$ is a smooth bijection such that $\forall t \in \tilde{I}$, $\phi'(t) \neq 0$

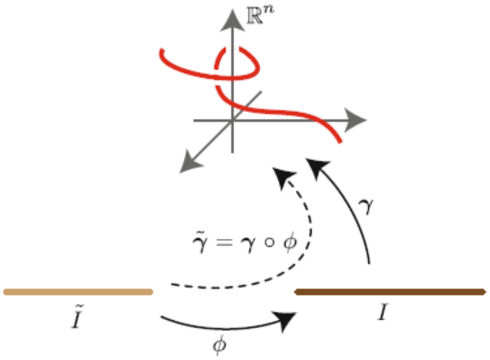


Figure 1: Kapp pg.19

Proposition 2.3
A reparametrization of a regular curve is a regular curve.

Proof. We use the same notations as the definition.
$\tilde{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \circ \phi \colon \tilde{I} \to \mathbb{R}^n$ is the composition of smooth functions, so smooth.
Moreover, $\forall t \in \tilde{I}, \tilde{\boldsymbol{\gamma}}'(t) = \boldsymbol{\gamma}'(\phi(t)) \cdot \phi'(t) \neq 0$ □

We will be interested in regular curves up to reparametrizations.

Remarks 2.4

1. $\boldsymbol{\gamma}, \tilde{\boldsymbol{\gamma}}$ have the same trace.
2. There are regular curves with the same trace that cannot be reparametrized to each other. For instance,
$$\boldsymbol{\gamma}_1(t) = (\cos(t), \sin(t)), t \in [0, 2\pi),$$
$$\boldsymbol{\gamma}_2(t) = (\cos(t), \sin(t)), t \in [0, 4\pi),$$

Question 2.5: Is there a canonical reparametrization of a given regular curve?

Definition 2.6 (Unit-speed)
A regular curve $\boldsymbol{\gamma} \colon I \to \mathbb{R}^n$ is called unit-speed (or parametrized by arc length) if $\forall t \in I$, $|\boldsymbol{\gamma}'(t)| = 1$.

Remark 2.7 If $\boldsymbol{\gamma} \colon I \to \mathbb{R}^n$ is unit-speed, then,
$$\text{Arc length between } t_1, t_2 = \int_{t_1}^{t_2} |\boldsymbol{\gamma}'(t)| dt = \int_{t_1}^{t_2} dt = t_2 - t_1$$

Proposition 2.8
A regular curve always has a unit-speed reparametrization.

Proof. Let $\boldsymbol{\gamma} \colon I \to \mathbb{R}^n$ be a regular curve. Fix $t_0 \in I$. Define $s \colon I \to \mathbb{R}$ by
$$s(t) = \int_{t_0}^{t} |\boldsymbol{\gamma}'(u)|\, du.$$
Let $\tilde{I} = s(I) \subset \mathbb{R}$. Then $\tilde{I}$ is an interval by IVT.
Since $s'(t) = |\boldsymbol{\gamma}'(t)| > 0$ by FTC, regularity, $s \colon I \to \tilde{I}$ is a smooth bijection. Then, $\phi = s^{-1} \colon \tilde{I} \to I$ is a smooth bijection,
$$\phi'(t) = \frac{1}{s'(\phi(t))} = \frac{1}{|\boldsymbol{\gamma}'(\phi(t))|} \neq 0.$$
Now $\tilde{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \circ \phi \colon \tilde{I} \to \mathbb{R}^n$ is a reparametrization of $\boldsymbol{\gamma}$, that is unit-speed:
$$\begin{aligned} |\tilde{\boldsymbol{\gamma}}'(t)| &= |\boldsymbol{\gamma}'(\phi(t)) \cdot \phi'(t)| \\ &= |\boldsymbol{\gamma}'(\phi(t))| \cdot 1/|\boldsymbol{\gamma}'(\phi(t))| \\ &= 1 \end{aligned}$$
□

Note:

$$
\begin{aligned}
s^{-1} \cdot s(t) &= t \\
(s^{-1})'(s(t)) \cdot s'(t) &= 1 \\
(s^{-1})'(s(t)) &= 1/s'(t)
\end{aligned}
$$

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3.1 Reparametrization (Cont'd)

Example 3.1

$\boldsymbol{\gamma}(t) = (\cos(t), \sin(t), t), \quad t \in (-\infty, \infty)$ How can we find a unit-speed reparametrization of $\boldsymbol{\gamma}$? Compute the arc length function $S\colon (-\infty, \infty) \to \mathbb{R}$:

$$\begin{aligned} s(t) = \int_0^t |\boldsymbol{\gamma}'(u)|\, du &= \int_0^t |(-\sin(u), \cos(u), 1)|\, du \\ &= \int_0^t \sqrt{2}\, du = \sqrt{2}t \end{aligned}$$

Set $\phi = s^{-1}$, then $\phi(t) = t/\sqrt{2}$

$$\tilde{\boldsymbol{\gamma}}(t) = \boldsymbol{\gamma}(t) \circ \phi(t) = \left(\cos\left(t/\sqrt{2}\right), \sin\left(t/\sqrt{2}\right), t/\sqrt{2}\right)$$

$t \in (-\infty, \infty)$, is a unit speed reparametrization of $\boldsymbol{\gamma}$.

We will be interested in invariants for a regular curve that are unchanged under any reparametrizations.
Examples include:

- trace
- arc-length
- curvature
- torsion

Non-examples include:

- position
- velocity
- speed
- acceleration

Sometimes we consider more specific reparametrization.

Proposition 3.2

If $\tilde{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \cdot \phi\colon \tilde{I} \to \mathbb{R}^n$ is a reparametrization of a regular curve $\boldsymbol{\gamma}\colon I \to \mathbb{R}^n$, then one of the following holds:

i. $\forall t \in \tilde{I}, \phi'(t) > 0$ i.e. ϕ is strictly increasing

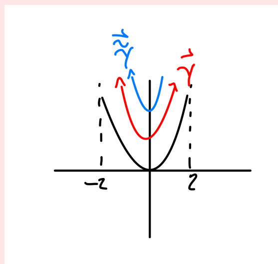
ii. $\forall t \in \tilde{I}, \phi'(t) < 0$ i.e. ϕ is strictly decreasing

Proof. Otherwise $\exists t \in \tilde{I}, \phi'(t) = 0$ by IVT. This contradicts the assumption on ϕ. $\square$

Definition 3.3 (Orientation-preserving vs. orientation-reversing)
Under the setting of the proposition, we say $\tilde{\gamma}$ is <u>orientation-preserving</u> if (i) occurs, or <u>orientation-reversing</u> if (ii) occurs.

Example 3.4 (Orientation-preserving)
The arc length reparametrization of a regular curve $\phi\colon I \to \tilde{I}$ is orientation-preserving, because $\phi'(t) = 1/|\boldsymbol{\gamma}'(\phi(t))| > 0 \quad \forall t \in I$
This shows an orientation-preserving unit-speed. Reparametrization always exists.

Example 3.5 (Orientation-reversing)
$\boldsymbol{\gamma}(t) = (t, t^2), \quad t \in [-2, 2]$
$\vec{\tilde{\gamma}}(t) = (-t, (-t)^2), \quad t \in [-2, 2]$



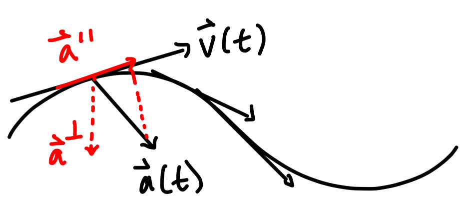
$\vec{\tilde{\gamma}}$ is an orientation-reversing reparametrization of $\boldsymbol{\gamma}$ by $\phi\colon [-2, 2] \to [-2, 2], \quad t \mapsto -t$
(Indeed, $\phi' = -1 < 0$).

We will be interested in invariants that are unchanged under any orientation-preserving reparametrization.

- Signed curvature
- Rotation index

3.2 Curvature

The curvature measures how sharply the trace bends. What is a plausible definition of the curvature?
Let $\boldsymbol{\gamma}\colon I \to \mathbb{R}^n$ be a regular curve. Set $\vec{v} = \boldsymbol{\gamma}', \vec{a} = \boldsymbol{\gamma}''$



$\vec{v}$ knows speed, direction of the motion
$\implies$ $\vec{a}$ should know the change in speed, direction $\to$ curvature.

We write

$$\vec{a} = \vec{a}^{\parallel} + \vec{a}^{\perp}$$

where

$$\vec{a}^{\parallel} = \left\langle \vec{a}, \frac{\vec{v}}{|\vec{v}|} \right\rangle \frac{\vec{v}}{|\vec{v}|}: \quad \text{parallel to } \vec{v}$$
$$\vec{a}^{\perp} = \vec{a} - \vec{a}'': \quad \text{orthogonal to } \vec{v}$$

Proposition 3.6

$$\begin{aligned} \frac{d}{dt}|\vec{v}(t)| &= \left\langle \vec{a}, \frac{\vec{v}}{|\vec{v}|} \right\rangle \\ &= \text{ the parallel component of } \vec{a} \text{ with respect to } \vec{v} \end{aligned}$$

Proof.

$$\begin{aligned} \frac{d}{dt}|\vec{v}(t)| &= \frac{d}{dt}\langle \vec{v}(t), \vec{v}(t)\rangle^{1/2} \\ &= \frac{1}{2}\frac{1}{\langle \vec{v}(t), \vec{v}(t)\rangle^{1/2}} \cdot 2\langle \vec{v}(t), \vec{v}'(t)\rangle \\ &= \left\langle \frac{\vec{v}(t)}{|\vec{v}(t)|}, \vec{a}(t) \right\rangle \end{aligned}$$

Note: $\langle v, v\rangle' = \langle v', v\rangle + \langle v, v'\rangle = 2\langle v', v\rangle$ □

So $|\vec{a}^{\perp}(t)|$ would be a plausible definition of the curvature. However this depends on $|\vec{t}|$. (Imagine a centripetal force for a car turning a corner.)

Definition 3.7 (Curvature)
Let $\boldsymbol{\gamma}\colon I \to \mathbb{R}^n$ be a regular curve. The curvature function $\kappa\colon I \to [0, \infty)$ is defined as

$$\kappa(t) = \frac{|\vec{a}^{\perp}(t)|}{|\vec{v}(t)|^2}$$

Proposition 3.8
Curvature is independent of parametrizations.

Proof. Let γ be a regular curve. $\tilde{\gamma} = \gamma \circ \phi$ is a reparametrization of γ.
Denote:

$$\kappa\colon \text{curvature function for } \gamma$$
$$\tilde{\kappa}\colon \text{curvature function for } \tilde{\gamma}$$

We need to show $\tilde{\kappa} = \kappa \circ \phi$
Denote:

$$v, a\colon \ \text{velocity, acceleration of } \gamma$$
$$\tilde{v}, \tilde{a}\colon \ \text{velocity, acceleration of } \tilde{\gamma}.$$

Then,

$$\begin{aligned}
\tilde{\gamma} &= \gamma \circ \phi \\
\tilde{v} &= \gamma' \circ \phi \cdot \phi' = v \circ \phi \cdot \phi' \\
\tilde{a} &= \gamma'' \circ \phi \cdot (\phi')^2 + \gamma' \circ \phi \cdot \phi' \\
&= a \circ \phi \cdot (\phi')^2 + v \circ \phi \cdot \phi'
\end{aligned}$$

So, $\tilde{v}$ is parallel to v,

$$\tilde{a}^\perp = a^\perp \circ \phi \cdot (\phi')^2$$

Therefore,

$$\begin{aligned}
\tilde{\kappa} = \frac{\tilde{a}^\perp}{|\tilde{v}|^2} = \frac{|a^\perp \circ \phi \cdot (\phi')^2|}{|v \circ \phi \cdot \phi'|^2} = \frac{|a^\perp \circ \phi|}{|v \circ \phi|^2} \\
= \kappa \circ \phi
\end{aligned}$$

$\square$

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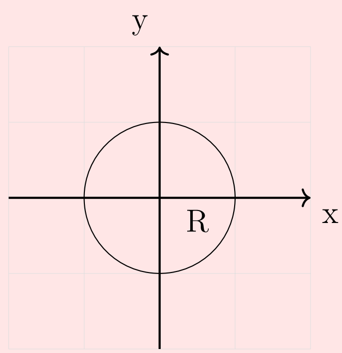
Note: From now on, I will bold my vectors like this $\mathbf{n}$ instead of $\vec{n}$.

4.1 Curvature (Cont'd)

Recall 4.1

$$\kappa(t) = \frac{|\mathbf{a}^\perp(t)|}{|\mathbf{v}(t)|^2}$$

Example 4.2
$\gamma(t) = (R\cos(t), R\sin(t)), \quad t \in (-\infty, \infty)$



$\mathbf{v}(t) = (-R\sin(t), R\cos(t))$
$\mathbf{a}(t) = (-R\cos(t), -R\sin(t))$
Here,

$$\langle \mathbf{v}(t), \mathbf{a}(t)\rangle = -R^2\sin(t)\cos(t) + R^2\cos(t)\sin(t) = 0;$$

So,

$$\mathbf{v}(t) \perp \mathbf{a}(t) \implies \mathbf{a}(t) = \mathbf{a}^\perp(t).$$

Therefore,

$$\kappa(t) = \frac{|\mathbf{a}(t)|}{|\mathbf{v}(t)|^2} = \frac{R}{R^2} = \frac{1}{R} \overset{R\to+\infty}{\to} 0 \text{ (flat)}$$

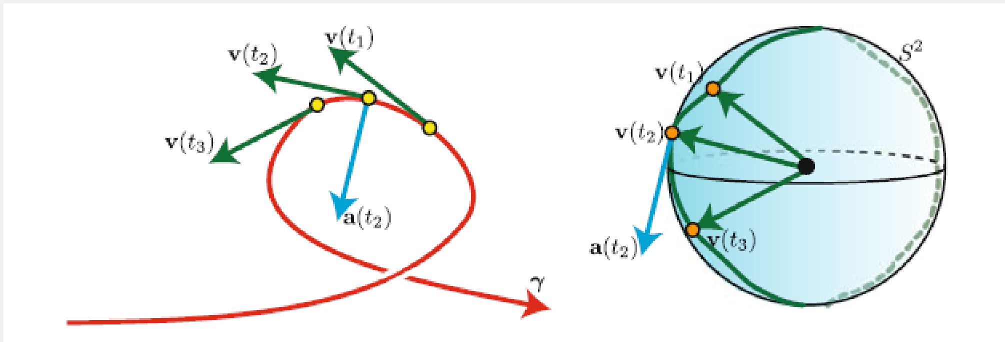
Historically, the curvature of a regular curve was first defined by $\kappa(t) = \frac{1}{R(t)}$, where $R(t)$ is the radius of the circle that best approximates the trace at t (The osculating circle; Read Tapp). Here we give another interpretation of the curvature using the osculating parabola.

Definition 4.3 (Unit tangent and normal vectors)
Let $\gamma\colon I \to \mathbb{R}^n$ be a regular curve. Define the unit tangent and normal vectors as

$$\mathbf{t}(t_0) = \frac{\mathbf{v}(t_0)}{|\mathbf{v}(t_0)|}, \quad \underbrace{\mathbf{n}(t_0) = \frac{\mathbf{a}^\perp(t_0)}{|\mathbf{a}^\perp(t_0)|}}_{\text{defined only if } \kappa(t_0)\neq 0}$$

Remarks 4.4

i. $\mathbf{t}(t_0), \mathbf{n}(t_0)$ are orthonormal, i.e. unit, orthogonal to each other



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ii. The osculating plane at t_0 is the plane through $\mathbf{t_0}$ spanned by $\mathbf{t}(t_0), \mathbf{n}(t_0)$. The osculating plane is the plane that γ is the closest to begin in, and contains the directions where the curve is heading and bending.
($\mathbf{t}$) ($\mathbf{n}$)

Proposition 4.5
Let $\gamma\colon I \to \mathbb{R}^n$ be a regular curve. Then $|\mathbf{t}'| = \kappa|\mathbf{v}|^2$, and $\mathbf{t}' = \kappa|\mathbf{v}|\mathbf{n}$ if $\mathbf{n}$ is defined. In particular, if γ is unit-speed, then

$$|\mathbf{t}'| = \kappa, \quad \text{and } \mathbf{t}' = \kappa\mathbf{n} \text{ if } \mathbf{n} \text{ is defined.}$$

Proof.

$$\mathbf{t}' = \left(\frac{\mathbf{v}}{|\mathbf{v}|}\right)' = \frac{\mathbf{a}}{|\mathbf{v}|} - \mathbf{v}\frac{\langle \mathbf{a}, \mathbf{v}\rangle}{|\mathbf{v}|^3} = \frac{\mathbf{a} - \mathbf{a}^{\parallel}}{|\mathbf{v}|} = \frac{\mathbf{a}^{\perp}}{|\mathbf{v}|}$$

Hence $|\mathbf{t}'| = \frac{|\mathbf{a}|^{\perp}}{|\mathbf{v}|^2} \cdot |\mathbf{v}| = \kappa|\mathbf{v}|$, and

$$\mathbf{t}' = \frac{|\mathbf{a}^{\perp}|}{|\mathbf{v}|^2}|\mathbf{v}|\frac{\mathbf{a}^{\perp}}{|\mathbf{a}^{\perp}|} = \kappa|\mathbf{v}|\mathbf{n} \text{ if } \mathbf{n} \text{ is defined.}$$

□

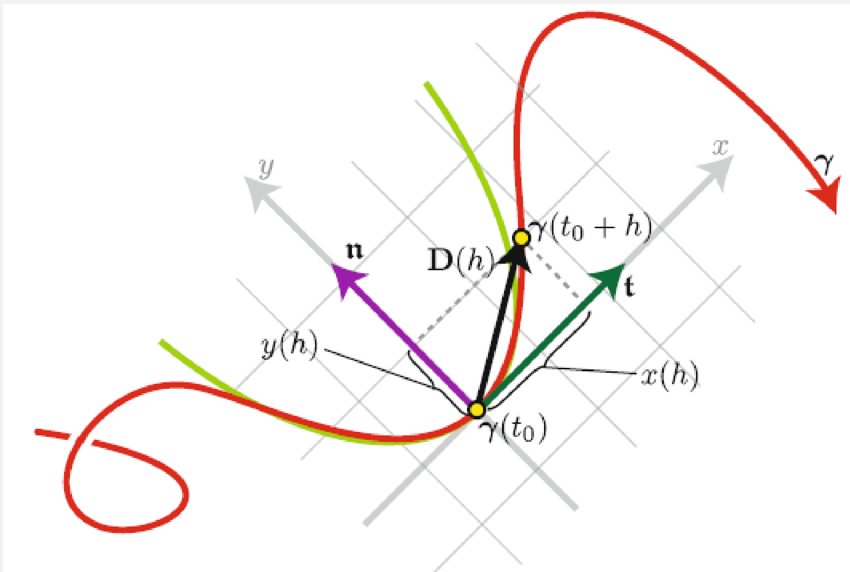
Remark 4.6 Let $\gamma\colon I \to \mathbb{R}^n$ be a unit-speed curve, $t_0 \in I$ with $\kappa(t_0) \neq 0$.
Then $\gamma'(t_0) = \mathbf{t}, \gamma''(t_0) = \mathbf{t}' = \kappa\mathbf{n}$, and the 2nd order Taylor approximation at γ at t_0 is

$$\begin{aligned}\gamma(t_0 + h) &\approx \gamma(t_0) + h\gamma'(t_0) + \frac{h^2}{2}\gamma''(t_0)\\ &= \gamma(t_0) + h\mathbf{t} + \frac{\kappa h^2}{2}\mathbf{n}\end{aligned}$$

Set $\mathbf{D}(h) = \gamma(t_0 + h) - \gamma(t_0) \approx h\mathbf{t} + \frac{\kappa h^2}{2}\mathbf{n}$: displacement.

Then,

$$\left.\begin{aligned} x(t) &\coloneqq \langle \mathbf{D}(h), \mathbf{t}\rangle \approx h \\ y(t) &\coloneqq \langle \mathbf{D}(h), \mathbf{n}\rangle \approx \tfrac{\kappa h^2}{2} \end{aligned}\right\} \text{ the parabola } y = \frac{\kappa}{2}x^2 \text{ in the osculating plane}$$



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$\kappa(t_0) =$ the concavity of the parabola that best approximates the trace at t_0

Proposition 4.7
Let $\gamma\colon I \to \mathbb{R}^n$ be a regular curve. If $\forall t \in I, \kappa(t) = 0$, then γ parametrizes a straight line.

Proof.

$$\begin{aligned} |\mathbf{t}'| = \kappa|\mathbf{v}| = 0 &\implies \mathbf{t}' = \mathbf{0} \\ &\implies \mathbf{t} = \mathbf{c} \text{ constant} \\ &\implies \mathbf{v} = |\mathbf{v}|\mathbf{c} \\ &\implies \text{ fixing } t_0 \in I, \\ \gamma(t) = \gamma(t_0) &+ \int_{t_0}^{t} \mathbf{v}(u)\, du \\ = \gamma(t_0) &+ \left(\int_{t_0}^{t} |\mathbf{v}(u)|\, du\right)\mathbf{c} \end{aligned}$$

□

4.2 Plane Curves

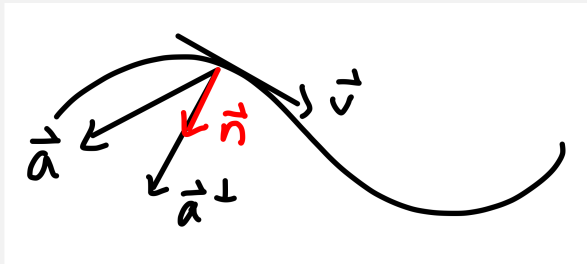
$\mathbb{R}^2$ is the only $\mathbb{R}^n$ where the terms “clockwise” and “counter-clockwise” makes sense.

This allows us to define

$$\text{“signed curvature”} = \text{curvature} + \text{turning direction with respect to } \mathbf{v}$$

Recall 4.8

$$\kappa = \frac{|\mathbf{a}^{\perp}|}{|\mathbf{v}|^2} = \frac{\langle \mathbf{a}, \mathbf{n} \rangle}{|\mathbf{v}|^2}$$



Definition 4.9 (Signed curvature)
Let $\gamma\colon I \to \mathbb{R}^2$ be a regular plane curve. Then the signed curvature $\kappa_s\colon I \to \mathbb{R}$ is defined as

$$\kappa_s = \frac{\langle \mathbf{a}, \mathbf{n}_s \rangle}{|\mathbf{v}|^2},$$

where,

$$\begin{aligned} \mathfrak{n}_s &= R_{90}\mathbf{t} \\ &= \text{the counterclockwise } 90^\circ \text{ rotation of } \mathbf{t} \end{aligned}$$

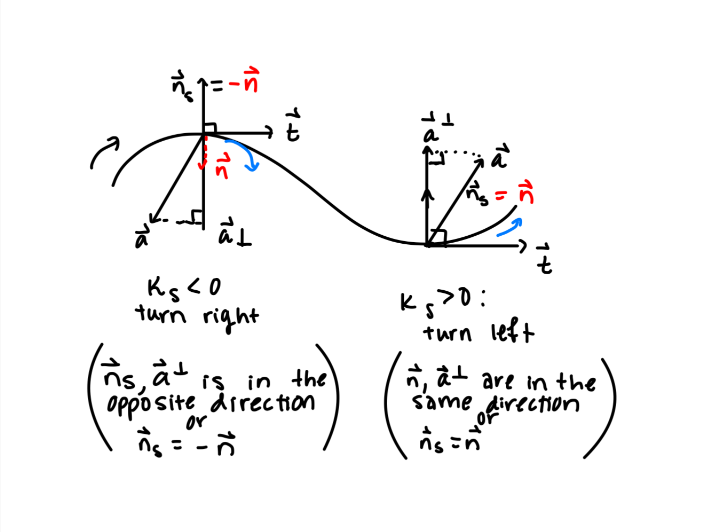
5 Jan 12, 2022

5.1 Plane Curves (Cont'd)

Recall 5.1

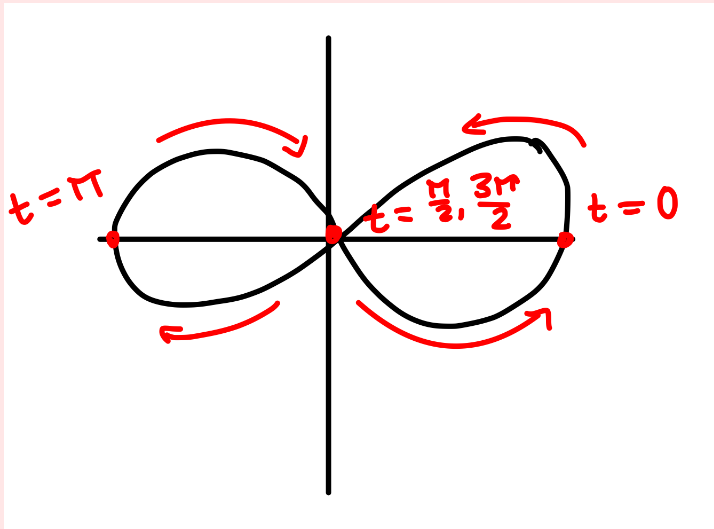
$$\kappa_s = \frac{\langle \mathbf{a}, \mathbf{n}_s \rangle}{|\mathbf{v}|^2}$$

where, $\mathbf{n}_s = R_{90^\circ}\mathbf{t}$



Example 5.2

$$\gamma(g) = (\cos(t), \sin(2t)), \quad t \in [0, 2\pi]$$



Lissajous curve

$$\mathbf{v}(t) = (-\sin t, 2\cos 2t)$$

$$\mathbf{a}(t) = (-\cos t, -4\sin 2t)$$

$$|\mathbf{v}(t)| = \sqrt{\sin^2 t + 4\cos^2 2t}$$

$$\mathbf{t}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = (-\sin t, 2\cos 2t)\frac{1}{\sqrt{\sin^2 t + 4\cos^2 2t}}$$

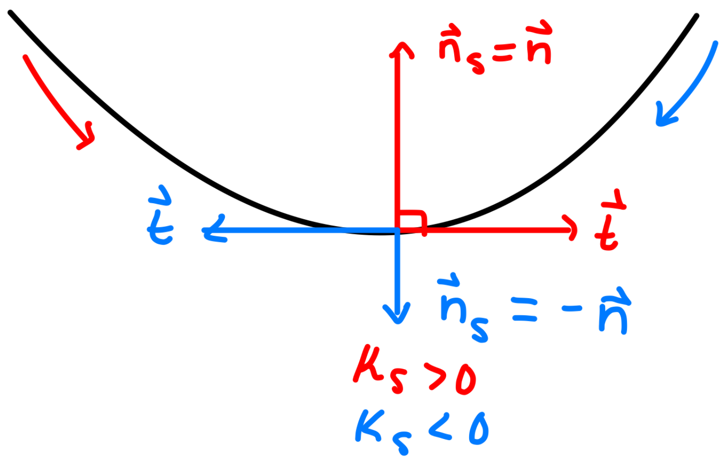
$$\mathbf{n}_s = R_{90}\mathbf{t} = (-2\cos 2t, -\sin t)\frac{1}{\sqrt{\sin^2 t + 4\cos^2 2t}}$$

$$\kappa_s = \frac{\langle \mathbf{a}, \mathbf{n}_s \rangle}{|\mathbf{v}|^2} = \frac{2\cos t\cos 2t + 4\sin t\sin 2t}{(\sin^3 t + 4\cos^2 2t)^{3/2}}$$

$$\kappa_s(0) = \frac{2}{4^{3/2}} = \frac{2}{8} = \frac{1}{4} > 0$$
$$\kappa_s\left(\frac{\pi}{2}\right) = 0$$
$$\kappa_s(\pi) = \frac{-1}{4} < 0$$
$$\kappa_s\left(\frac{3\pi}{2}\right) = 0$$

Proposition 5.3
Let $\gamma \colon I \to \mathbb{R}^2$ be a plane curve. Then $|\kappa_s| = \kappa$.

Proof. Compare $\kappa = \dfrac{\langle \mathbf{a}, \mathbf{n} \rangle}{|\mathbf{v}|^2}, \kappa_s = \dfrac{\langle \mathbf{a}, \mathbf{n}_s \rangle}{|\mathbf{v}|^2}$
$\mathbf{n}_s = \pm \mathbf{n}$, because they are both unit, orthogonal to $\mathbf{t}$. Hence κ_s coincides with $\underset{\geq 0}{\kappa}$ up to signs.

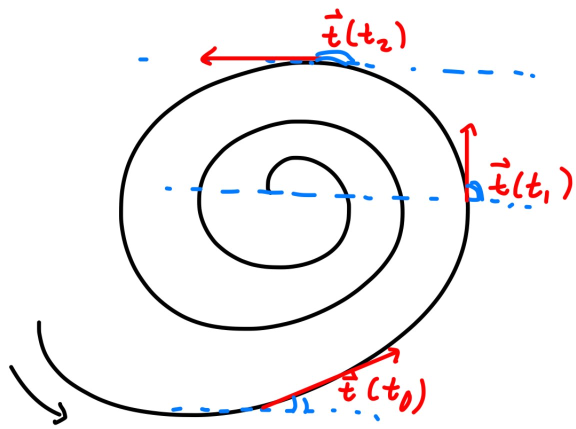


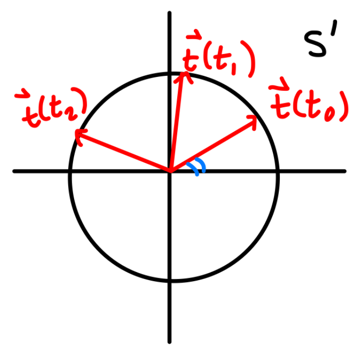
□

Proposition 5.4
Signed curvature is unchanged by any orientation-preserving reparametrizations.

Proof. Exercise. □

Proposition 5.5
Let $\gamma \colon I \to \mathbb{R}^2$ be a plane curve. Then there exists a smooth function $\theta \colon I \to \mathbb{R}$ such that $\forall t \in I, \mathbf{t}(t) = (\cos\theta(t), \sin\theta(t))$.





What should θ be?

$$\mathbf{t}' = \theta'(-\sin\theta, \cos\theta) = \theta' R_{90}\mathbf{t} = \theta'\mathbf{n}_s.$$

On the other hand,

$$\mathbf{t}' = \left(\frac{\mathbf{v}}{|\mathbf{v}|}\right)' = \frac{\mathbf{a}^\perp}{|\mathbf{v}|} = \frac{\langle \mathbf{a}, \mathbf{n}_s\rangle}{|\mathbf{v}|}\mathbf{n}_s = \kappa_s|\mathbf{v}|\mathbf{n}_s$$

By comparing the two formulas, $\theta' = \kappa_s|\mathbf{v}|$. In the proof, we solve this differential equation.

Remark 5.6 If γ is unit-speed, $\theta' = \kappa_s$. This shows:

$$\begin{aligned}\text{signed curvature} &= \text{ the rate of change of the angle}\\ \text{curvature} &= |\text{the rate of change of the angle}|\end{aligned}$$

Proof. Fix $t_0 \in I, \theta_0 \in \mathbb{R}$ such that $\mathbf{t}(t_0) = (\cos\theta_0, \sin\theta_0)$.
Define

$$\theta(t) = \theta_0 + \int_{t_0}^{t} \kappa_s(u)|\mathbf{v}(u)|\, du$$

We will show this $\theta(t)$ works.

$$\theta\colon I \to \mathbb{R} \text{ is a smooth function}$$
$$\theta' = \kappa_s|\mathbf{v}|, \theta(t_0) = \theta_0.$$

Set $\mathbf{t}_\theta = (\cos\theta, \sin\theta)$
We need to show $\mathbf{t} = \mathbf{t}_\theta$.
Observe $\mathbf{t}, \mathbf{t}_\theta$ are unit.
Enough to show $\langle \mathbf{t}, \mathbf{t}_\theta\rangle = 1$
On the other hand,

$$\begin{aligned}\mathbf{t}_\theta(t_0) &= (\cos\theta(t_0), \sin\theta(t_0))\\ &= (\cos\theta_0, \sin\theta_0)\\ &= \mathbf{t}(t_0)\end{aligned}$$

So,

$$\langle \mathbf{t}(t_0), \mathbf{t}_\theta(t_0)\rangle = 1$$

Enough to show $\langle \mathbf{t}, \mathbf{t}_\theta\rangle' = 0$

$$\mathbf{t}' = \kappa_s|\mathbf{v}|\mathbf{n}_s = \kappa_s|\mathbf{v}|R_{90}\mathbf{t}$$
$$\mathbf{t}_\theta' = \theta'(-\sin\theta, \cos\theta) = \kappa_s|\mathbf{v}|R_{90}\mathbf{t}_\theta$$

Therefore,

$$\begin{aligned}\langle \mathbf{t}, \mathbf{t}_\theta\rangle' &= \langle \mathbf{t}', \mathbf{t}_\theta\rangle + \langle \mathbf{t}, \mathbf{t}_\theta'\rangle\\ &= \kappa_s|\mathbf{v}|\left(\langle R_{90}\mathbf{t}, \mathbf{t}_\theta\rangle + \langle \mathbf{t}, R_{90}\mathbf{t}_\theta\rangle\right)\\ &= \kappa_s|\mathbf{v}|\left(\langle R_{90}\mathbf{t}, \mathbf{t}_\theta\rangle + \langle R_{90}\mathbf{t}, R_{90}(R_{90}\mathbf{t}_\theta)\rangle\right) && R_{90} \text{ is orthogonal}\\ &= \kappa_s|\mathbf{v}|\left(\langle R_{90}\mathbf{t}, \mathbf{t}_\theta\rangle - \langle R_{90}\mathbf{t}, \mathbf{t}_\theta\rangle\right) && R_{90}\circ R_{90} = R_{180} = -1\\ &= 0\end{aligned}$$

$\square$

Remark 5.7 The angle function θ is unique up to an integer multiple of 2π.
Indeed if $\Theta\colon I \to \mathbb{R}$ is a smooth function such that $\forall t \in I, \gamma = (\cos\Theta, \sin\Theta)$, then,

$$\begin{aligned}\Theta' = \theta' &= \kappa_s |\mathbf{v}| \\ &\implies |\Theta - \theta|' = 0 \\ &\implies \Theta - \theta = \text{constant}\end{aligned}$$

On the other hand,

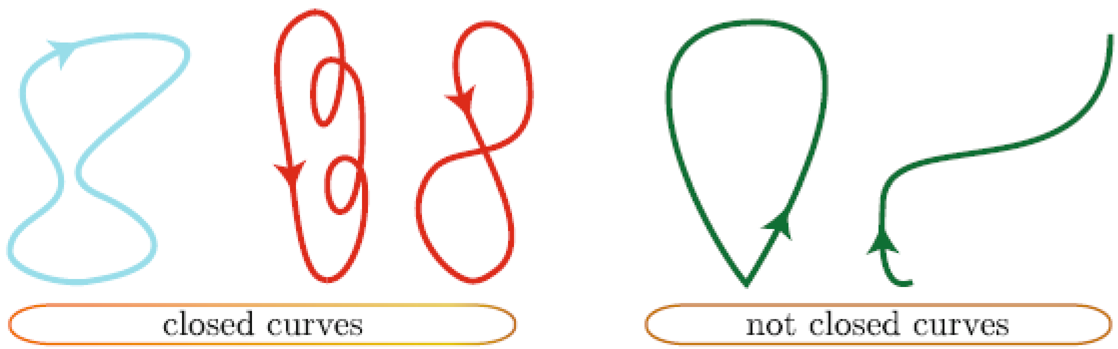
$$(\cos\theta, \sin\theta) = (\cos\Theta, \sin\Theta) = \mathbf{t}$$

So $\Theta - \theta \in 2\pi \cdot \mathbb{Z}$

6 Jan 14, 2022

6.1 Plane Curves(Cont'd)

Definition 6.1 (Closed curve)
A regular curve $\boldsymbol{\gamma}\colon [a,b] \to \mathbb{R}^n$ is called <u>closed</u> if $\boldsymbol{\gamma}(a) = \boldsymbol{\gamma}(b)$, and $\forall n \in \mathbb{N}, \boldsymbol{\gamma}^{(n)}(a) = \boldsymbol{\gamma}^{(n)}(b)$



Definition 6.2 (Rotation index)
Let $\boldsymbol{\gamma}\colon [a,b] \to \mathbb{R}^2$ be a closed plane curve. The <u>rotation index</u> of $\boldsymbol{\gamma}$ is defined as

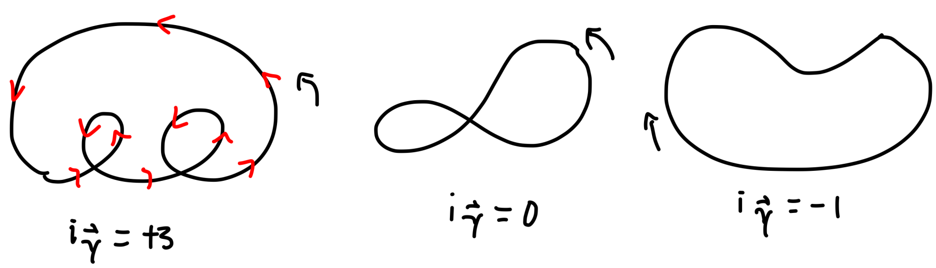
$$i_\gamma = \frac{1}{2\pi}(\theta(b) - \theta(a)),$$

where θ is the angle function from proposition 5.5.

Remarks 6.3

i. $i_\gamma \in \mathbb{Z}$, because $\mathbf{t}(a) = \mathbf{t}(b)$, so $\theta(b) - \theta(a) \in 2\pi\mathbb{Z}$

ii. Later on, we will show $i_\gamma = \pm 1$ if $\boldsymbol{\gamma}$ has no self-intersection.



Proposition 6.4
Let $\boldsymbol{\gamma}\colon [a,b] \to \mathbb{R}^2$ be a closed plane curve. Then

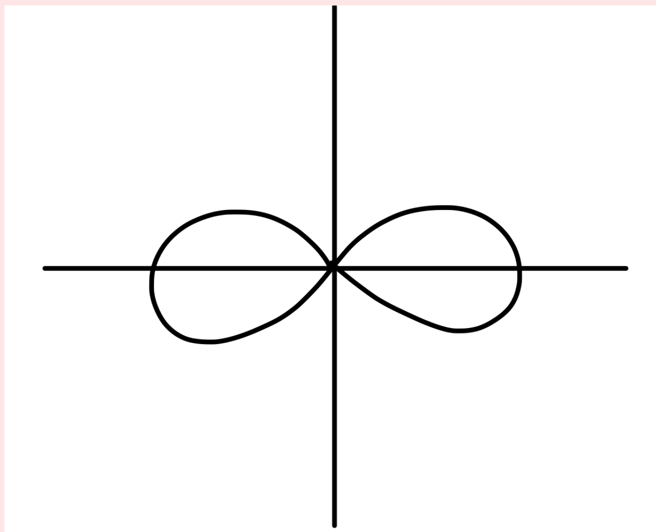
$$i_\gamma = \frac{1}{2\pi}\int_a^b \kappa_s(t)|\mathbf{v}(t)|\, dt$$

Proof. This follows from the construction of the angle function. □

Proposition 6.5
Rotation index is unchanged under any orientation-preserving reparametrizations.

Proof. Exercise. □

Example 6.6
$\boldsymbol{\gamma}(t) = (\cos t, \sin 2t), t \in [0, 2\pi]$



Recall:

$$\kappa_s(t) = \frac{2\cos t \cos 2t + 4 \sin t \sin 2t}{(\sin^2 t + 4\cos^2 2t)^{3/2}}$$

$$|\mathbf{v}| = (\sin^2 t + 4\cos^2 2t)^{1/2}$$

Therefore,

$$\begin{aligned} i_\gamma &= \frac{1}{2\pi} \int_0^{2\pi} \frac{2\cos t \cos 2t + 4 \sin t \sin 2t}{\sin^2 t + 4\cos^2 2t} \, dt \\ &= \frac{1}{2\pi} \left(\int_0^{\pi} - - - \, dt + \underbrace{\int_\pi^{2\pi} - - - \, dt}_{\substack{t = s + \pi, \\ \text{then the integrand} \\ \text{is multiplied by } -1}} \right) \\ &= 0 \end{aligned}$$

6.2 Space Curves

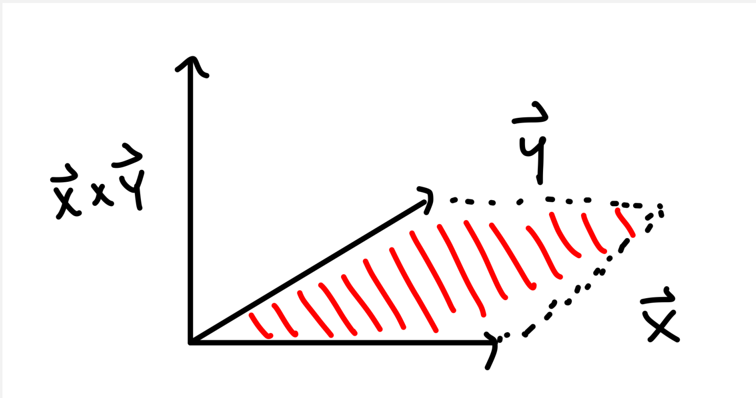
What's special about $\mathbb{R}^3$?
$\mathbb{R}^3$ has the cross product.

Recall 6.7 $\mathbf{x} = (x_1, x_2, x_3), \mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$,
$\mathbf{x} \times \mathbf{y} = (x_2y_3 - x_3y_2, -(x_1y_3 - x_3y_1), x_1y_2 - x_2y_1) \in \mathbb{R}^3$
Basic properties:

i. $\times\colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is bilinear, and antisymmetric.
(i.e. $\mathbf{y} \times \mathbf{x} = -\mathbf{x} \times \mathbf{y}$)

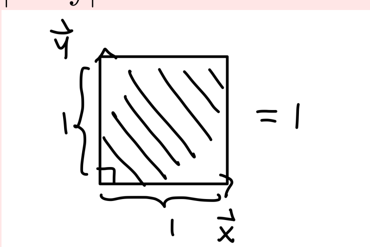
ii. $$\begin{aligned} |\mathbf{x} \times \mathbf{y}| &= |\mathbf{x}||\mathbf{y}|\sin(\theta), \text{ where } \theta \text{ is the angle between } \mathbf{x}, \mathbf{y} \\ &= \text{the area of the parallelogram spanned by } \mathbf{x}, \mathbf{y} \end{aligned}$$

iii. $\mathbf{x} \times \mathbf{y}$ is orthogonal to $\mathbf{x}, \mathbf{y}$;
$\{\mathbf{x}, \mathbf{y}, \mathbf{x} \times \mathbf{y}\}$ is a right-handed system.



Example 6.8
If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ are orthonormal, then $\{\mathbf{x}, \mathbf{y}, \mathbf{x} \times \mathbf{y}\}$ is an orthonormal basis for $\mathbb{R}^3$:

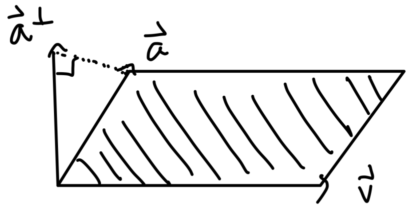
- $\mathbf{x} \times \mathbf{y}$ is orthogonal to $\mathbf{x}, \mathbf{y}$, and
- $|\mathbf{x} \times \mathbf{y}| =$



$= 1$

Proposition 6.9
Let $\boldsymbol{\gamma}\colon I \to \mathbb{R}^3$ be a space curve, then
$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$$

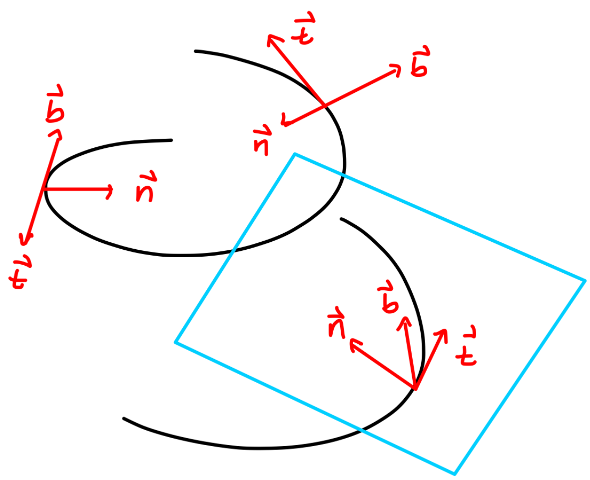
Proof. $|\mathbf{v} \times \mathbf{a}| =$



$= |\mathbf{v}||\mathbf{a}^{\perp}|$

$$\implies \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{|\mathbf{a}^{\perp}|}{|\mathbf{v}|^2} = \kappa \qquad \square$$

Definition 6.10 (Unit binormal vector and Frenet frame)
Let $\boldsymbol{\gamma}\colon I \to \mathbb{R}^3$ be a space curve. The unit binormal vector for $\boldsymbol{\gamma}$ at $t \in I$ is defined as $\mathbf{b}(t) = \mathbf{t}(t) \times \mathbf{n}(t)$ (only if $\kappa(t) \neq 0$). The orthonormal basis $\{\mathbf{t}(t), \mathbf{n}(t), \mathbf{b}(t)\}$ for $\mathbb{R}^3$ is called the Frenet frame for $\boldsymbol{\gamma}$ at t.



Remark 6.11 $\mathbf{b}(t)$ is a unit normal vector to the osculating plane of $\boldsymbol{\gamma}$ at t.
$\implies$ $\mathbf{b}$ encodes the tilt of the osculating plane of $\boldsymbol{\gamma}$.

We want to define the "torsion" as the measurement of the change of the tilt of the osculating plane.

Definition 6.12 (Torsion)
Let

$$\begin{aligned}&\boldsymbol{\gamma}\colon I \to \mathbb{R}^3 \text{ be a space curve,}\\ &t \in I \text{ s.t. } \kappa(t) \neq 0\end{aligned}$$

The torsion of $\boldsymbol{\gamma}$ at t is defined as

$$\tau(t) = -\frac{\langle \mathbf{b}'(t), \mathbf{n}(t)\rangle}{|\mathbf{v}(t)|}$$

Remark 6.13 Why is this definition plausible?

i. $\mathbf{b}'(t)$ is parallel to $\mathbf{n}(t)$ (later).
   So $\langle \mathbf{b}'(t), \mathbf{n}(t)\rangle = \pm|\mathbf{b}'(t)|$
ii. $\langle \mathbf{b}'(t), \mathbf{n}(t)\rangle$ depends on parametrizations.

Proposition 6.14
Torsion is independent of parametrizations.

Proof. Read Tapp for the details.
Sketch:
φ is orientation-preserving.
$\tilde{t} = t \circ \varphi, \tilde{n} = n \circ \varphi$
$\implies \tilde{b} = b \circ \varphi$
$\implies \tilde{b}' = b' \circ \varphi \cdot \varphi'$ ∎

7 Jan 19, 2022

7.1 Space Curves (Cont'd)

Recall 7.1 $\mathbf{b} = \mathbf{t} \times \mathbf{n}$, $\tau = -\dfrac{\langle \mathbf{b}', \mathbf{n} \rangle}{|\mathbf{v}|}$

Note: $\mathbf{b}' = -\tau |\mathbf{v}| \mathbf{n}$

Proposition 7.2
Let $\boldsymbol{\gamma} \colon I \to \mathbb{R}^3$ be a space curve such that $\forall t \in I, \kappa(t) \neq 0$. Then the following conditions are equivalent:

i. The trace of $\boldsymbol{\gamma}$ is contained in a plane in $\mathbb{R}^3$.

ii. $\forall t \in I, \tau(t) = 0$.

Remark 7.3 The torsion measures the failure of a space curve to remain in a plane in $\mathbb{R}^3$.

Proof. (i.) is equivalent to:
(i.)' $\exists \mathbf{w} \neq \mathbf{0} \in \mathbb{R}^3, c \in \mathbb{R}, \forall t \in I, \langle \boldsymbol{\gamma}, \mathbf{w} \rangle = c$
We show (i.)' $\iff$ (ii.).

$$
\begin{aligned}
(\impliedby)\mathbf{b}' &= -\tau |\mathbf{v}| \mathbf{n} = 0, \text{ so} \\
\mathbf{b} &= \text{ constant } =: \mathbf{w} \neq 0 \\
\langle \boldsymbol{\gamma}(t), \mathbf{w} \rangle' &= \langle \mathbf{v}(t), \mathbf{w} \rangle = \langle |\mathbf{v}(t)| \mathbf{t}(t), \mathbf{b}(t) \rangle = 0, \text{ so} \\
\langle \boldsymbol{\gamma}(t), \mathbf{w} \rangle &= \text{ constant.}
\end{aligned}
$$

$$
\begin{aligned}
(\implies)\ \langle \boldsymbol{\gamma}(t), \mathbf{w} \rangle &= \text{ constant, so} \\
\langle \boldsymbol{\gamma}(t), \mathbf{w} \rangle &= \langle \mathbf{a}(t), \mathbf{w} \rangle = 0 \\
\mathbf{t}(t), \mathbf{n}(t) &\in \operatorname{span}(\mathbf{v}(t), \mathbf{a}(t)), \text{ so} \\
\langle \mathbf{t}(t), \mathbf{w} \rangle &= \langle \mathbf{n}(t), \mathbf{w} \rangle = 0.
\end{aligned}
$$

This shows that $\mathbf{w}$ is normal to the osculating plane spanned by $\mathbf{t}(t), \mathbf{n}(t)$, so

$$
\begin{aligned}
\mathbf{b}(t) &= \pm \frac{\mathbf{w}}{|\mathbf{w}|} = \text{ constant, so} \\
\mathbf{b}'(t) &= \mathbf{0}, \text{ so} \\
\tau(t) &= -\frac{\langle \mathbf{b}'(t), \mathbf{n}(t) \rangle}{|\mathbf{v}(t)|} = 0
\end{aligned}
$$

$\square$

There are differential equations for $\mathbf{t}, \mathbf{n}, \mathbf{b}$ determined by κ, τ.

Proposition 7.4 (Frenet equations)
Let $\boldsymbol{\gamma}\colon I \to \mathbb{R}^3$ be a space curve such that $\forall t \in I, \kappa(t) \neq 0$.
Then,

$$\begin{aligned} \mathbf{t}' &= & \kappa|\mathbf{v}|\mathbf{n} & \\ \mathbf{n}' &= -\kappa|\mathbf{v}|\mathbf{t} & & +\tau|\mathbf{v}|\mathbf{b} \\ \mathbf{b}' &= & -\tau|\mathbf{v}|\mathbf{n} & \end{aligned}$$

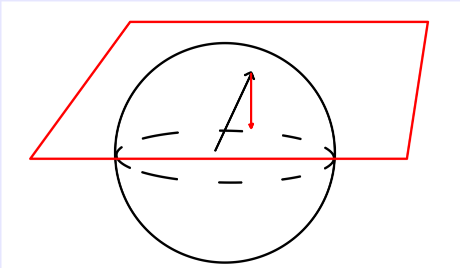
In particular, if $\boldsymbol{\gamma}$ is unit-speed, then

$$\begin{aligned} \mathbf{t}' &= & \kappa\mathbf{n} & \\ \mathbf{n}' &= -\kappa\mathbf{t} & & +\tau\mathbf{b} \\ \mathbf{b}' &= & -\tau\mathbf{n} & \end{aligned}$$

Remark 7.5 This suggests that a space curve is completely determined by the functions κ, τ up to initial conditions. (Fundamental Theorem of Space Curves)

Lemma 7.6
Let $\boldsymbol{\gamma}, \boldsymbol{\delta}\colon I \to \mathbb{R}^n$ be curves (not necessarily regular).

i. If $\exists c \in \mathbb{R}, \forall t \in I, |\boldsymbol{\gamma}(t)| = c$, then $\forall t \in I, \boldsymbol{\gamma}'(t)$ is orthogonal to $\boldsymbol{\gamma}(t)$.



ii. If $\exists D \in \mathbb{R}$, $\forall t \in I, \langle \boldsymbol{\gamma}(t), \boldsymbol{\delta}(t) \rangle = D$, then $\forall t \in I, \langle \boldsymbol{\gamma}'(t), \boldsymbol{\delta}(t) \rangle = -\langle \boldsymbol{\gamma}(t), \boldsymbol{\delta}'(t) \rangle$.

Remark 7.7 Both the assumptions are satisfied if $\forall t \in I$, $\boldsymbol{\gamma}(t), \boldsymbol{\delta}(t)$ are orthogonal.

Proof of Lemma.

i. $c^2 = |\boldsymbol{\gamma}(t)|^2 = \langle \boldsymbol{\gamma}(t), \boldsymbol{\gamma}(t) \rangle$.
$\implies 0 = 2\langle \boldsymbol{\gamma}(t), \boldsymbol{\gamma}'(t) \rangle$
$\implies \langle \boldsymbol{\gamma}(t), \boldsymbol{\gamma}'(t) \rangle = 0$

ii. $\langle \boldsymbol{\gamma}(t), \boldsymbol{\delta}(t) \rangle = D$
$\implies \langle \boldsymbol{\gamma}'(t), \boldsymbol{\delta}(t) \rangle + \langle \boldsymbol{\gamma}(t), \boldsymbol{\delta}'(t) \rangle = 0$
$\implies \langle \boldsymbol{\gamma}'(t), \boldsymbol{\delta}(t) \rangle = -\langle \boldsymbol{\gamma}(t), \boldsymbol{\delta}'(t) \rangle$

□

Proof of Proposition 7.4. We have proved $\mathbf{t}' = \kappa|\mathbf{v}|\mathbf{n}$. As for $\mathbf{n}', \mathbf{b}'$, it is enough to compute their components with respect to the Frenet frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$. $\langle \mathbf{n}', \mathbf{t} \rangle = -\langle \mathbf{n}, \mathbf{t}' \rangle = -\langle \mathbf{n}, \kappa|\mathbf{v}|\mathbf{n} \rangle = -\kappa|\mathbf{v}|$

$\langle \mathbf{n}', \mathbf{n} \rangle = 0$
$\langle \mathbf{n}', \mathbf{b} \rangle = -\langle \mathbf{n}, \mathbf{b}' \rangle = \tau|\mathbf{v}|$
Therefore,
$\mathbf{n}' = -\kappa|\mathbf{v}|\mathbf{t} + \tau|\mathbf{v}|\mathbf{b}$.
$\langle \mathbf{b}', \mathbf{t} \rangle = -\langle \mathbf{b}, \mathbf{t}' \rangle = -\langle \mathbf{b}, -\kappa|\mathbf{v}|\mathbf{n} \rangle = 0$
$\langle \mathbf{b}', \mathbf{n} \rangle = -\tau|\mathbf{v}|$
$\langle \mathbf{b}', \mathbf{b} \rangle = 0$
Therefore,
$\mathbf{b}' = -\tau|\mathbf{v}|\mathbf{n}$ □

Remark 7.8 Another interpretation of the torsion can be given by the Frenet equations. Let $\boldsymbol{\gamma}\colon I \to \mathbb{R}^3$ be a unit-speed space curve.
Then,

$$\boldsymbol{\gamma}' = \mathbf{t}, \boldsymbol{\gamma}'' = \mathbf{t}' = \kappa\mathbf{n},$$
$$\boldsymbol{\gamma}''' = (\kappa\mathbf{n})' = \kappa'\mathbf{n} + \kappa\mathbf{n}' = -\kappa^2\mathbf{t} + \kappa'\mathbf{n} + \kappa\tau\mathbf{b}$$

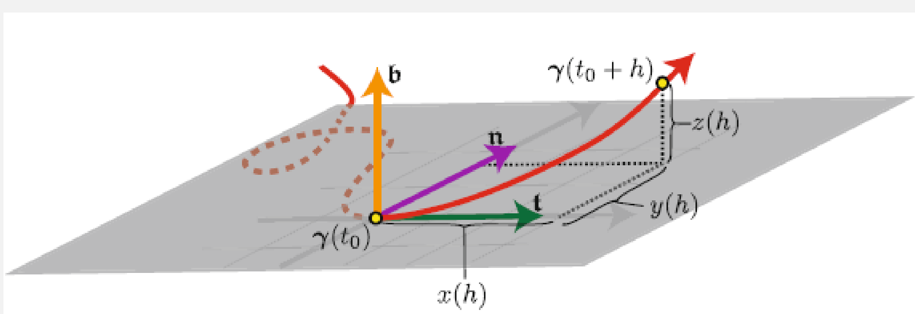
So the 3rd order Taylor approximation at $t_0 \in I, \kappa(t_0) > 0$ is as follows:

$$\begin{aligned}\mathbf{D}(h) &= \boldsymbol{\gamma}(t_0 + h) - \boldsymbol{\gamma}(t_0)\\ &\approx h\boldsymbol{\gamma}'(t_0) + \frac{h^2}{2}\boldsymbol{\gamma}''(t_0) + \frac{h^3}{6}\boldsymbol{\gamma}'''(t_0)\\ &= \left(h - \frac{\kappa^2 h^3}{6}\right)\mathbf{t} + \left(\frac{\kappa h^2}{2} + \frac{\kappa' h^3}{6}\right)\mathbf{n} + \frac{\kappa\tau h^3}{6}\mathbf{b}\end{aligned}$$

Therefore,

$$\begin{aligned}x(h) &= \langle \mathbf{D}(h), \mathbf{t} \rangle \approx h - \frac{\kappa^2 h^3}{6}\\ y(h) &= \langle \mathbf{D}(h), \mathbf{n} \rangle \approx \frac{\kappa h^2}{2} + \frac{\kappa' h^3}{6}\\ z(h) &= \langle \mathbf{D}(h), \mathbf{b} \rangle \approx \frac{\kappa\tau h^3}{6}\end{aligned}$$

If $\tau(t_0) > 0$, then the curve passes through the osculating plane from below.

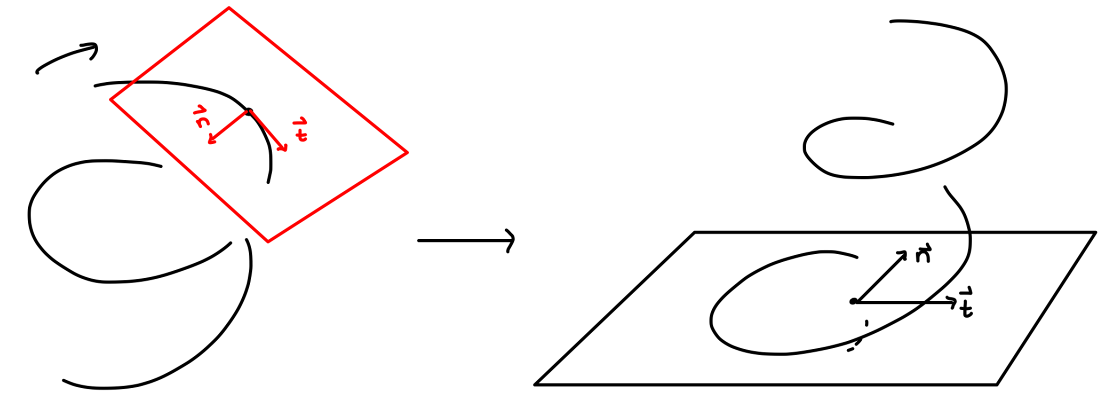


If $\tau(t_0) < 0$, then the curve passes through the osculating plane from above.

8 Jan 21, 2022

8.1 Rigid Motions

In geometry, it is often useful to "tilt your head", or choose an orthonormal set of vectors at a point, adapted to the problem at hand:



This is achieved by rigid motions.

Definition 8.1 (Rigid motion)
A rigid motion in $\mathbb{R}^n$ is a function $f\colon \mathbb{R}^n \to \mathbb{R}^n$ that preserves the distances:

$$\forall \mathbf{x}, \mathbf{y} \in \mathbb{R},\ |f(\mathbf{x}) - f(\mathbf{y})| = |\mathbf{x} - \mathbf{y}|$$

Example 8.2
The translation by $\mathbf{p} \in \mathbb{R}^n$

$$T_{\mathbf{p}}\colon \mathbb{R}^n \to \mathbb{R}^n,\ \mathbf{x} \mapsto \mathbf{x} + \mathbf{p}$$

is a rigid motion. Indeed,

$$\begin{aligned} |T_{\mathbf{p}}(\mathbf{x}) - T_{\mathbf{p}}(\mathbf{y})| &= |\mathbf{x} + \mathbf{p} - (\mathbf{y} + \mathbf{p})| \\ &= |\mathbf{x} - \mathbf{y}| \end{aligned}$$

Note: $T_{\mathbf{p}}$ is never linear if $\mathbf{p} \neq \mathbf{0}$, because $T_{\mathbf{p}}(\mathbf{0}) = \mathbf{p} \neq 0$.

Theorem 8.3
Let $L_A\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation represented by an $n \times n$ matrix A. The following conditions are equivalent:

1. L_A is a rigid motion.
2. $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\langle L_A(\mathbf{x}), L_A(\mathbf{y})\rangle = \langle \mathbf{x}, \mathbf{y}\rangle$.
3. If $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ is an orthonormal basis of $\mathbb{R}^n$, so is $\{L_A\mathbf{x}_1, \cdots, L_A\mathbf{x}_n\}$.
4. The column vectors of A form an orthonormal basis of $\mathbb{R}^n$.
5. $A^T A = I_n$

Definition 8.4
A linear rigid motion and its matrix are called orthogonal.

$$O(n) \coloneqq \text{ the set of all } n \times n \text{ orthogonal matrices}$$

Proposition 8.5
Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a rigid motion. Then,

$$\exists!\, \mathbf{p} \in \mathbb{R}^n,\ \exists!\, A \in O(n),\ f = T_{\mathbf{p}} \circ L_A$$

Sketch of proof. Step 1: $(f(\mathbf{0}) = \mathbf{0})$: $\exists!\, A \in O(n),\ f = L_A$
Step 2: (General Case): Set $\mathbf{p} = f(\mathbf{0})$. Then apply Step 1 to $(T_{\mathbf{p}})^{-1} \circ f = T_{-\mathbf{p}} \circ f$
Indeed,

$$(T_{\mathbf{p}})^{-1} \circ f(\mathbf{0}) = T_{-\mathbf{p}} \circ f(\mathbf{0}) = T_{-\mathbf{p}}(\mathbf{p}) = \mathbf{0},$$

So,

$$\exists!\, A \in O(n),\ (T_{\mathbf{p}})^{-1} \circ f = L_A$$

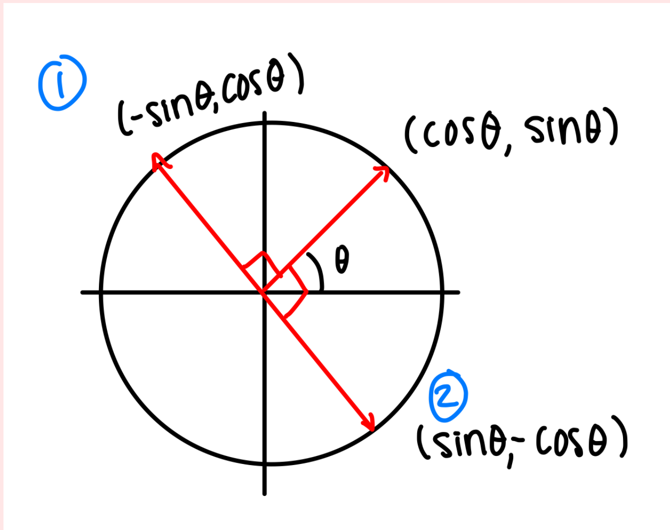
$\implies f = T_{\mathbf{p}} \circ L_A$ Read Tapp for the details. $\square$

We can classify rigid motions:

Lemma 8.6
$A \in O(n) \implies \det(A) = \pm 1$

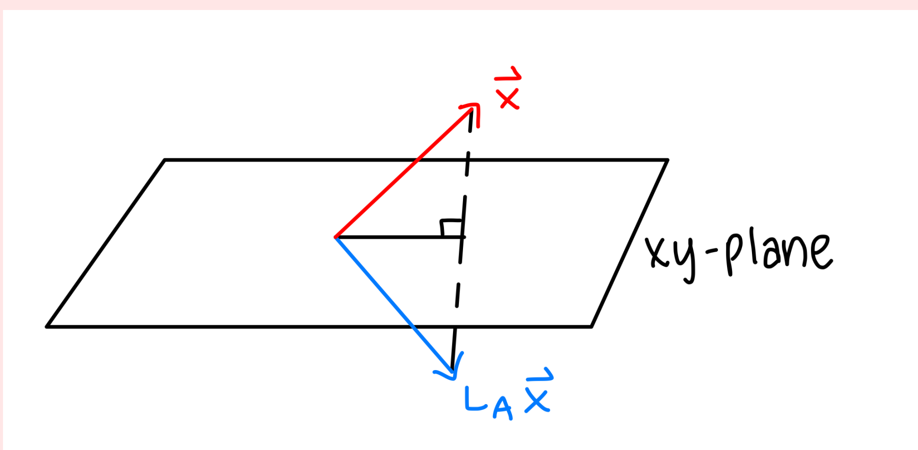
Proof. $A^T A = \mathbb{I}_n$, so $1 = \det\left(A^T A\right) = \det\left(A^T\right)\det(A) = \det(A)^2$ $\square$

Example 8.7
Let $A \in O(2)$. The column vectors of A are orthonormal:



$$A = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{\substack{\text{rotation,} \\ \text{det=1} \\ \text{proper}}} \text{ or } \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}}_{\substack{\text{reflection} \\ \text{det}=-1 \\ \text{improper}}}$$

Example 8.8
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in O(3)$ represents the reflection about the xy plane:



$\det(A) = -1$, so L_A is improper.

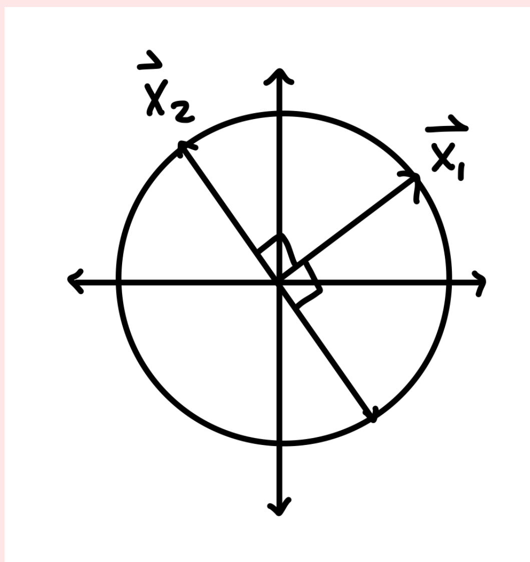
Remark 8.9

$$\begin{aligned} \text{proper} &= \text{physically performable (e.g. rotations)} \\ \text{improper} &= \text{physically unperformable (e.g. reflections)} \end{aligned}$$

Another interpretation of proper (improper rigid motions is given in terms of the orientation of orthonormal basis.

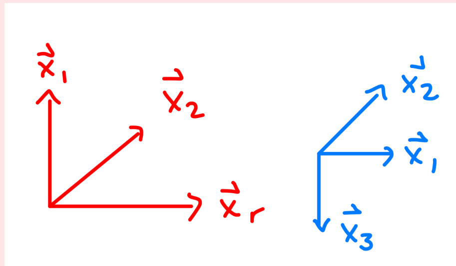
Definition 8.10 (Ordered orthonormal basis and Positively oriented vs. Negatively oriented)
An ordered orthonormal basis (o.o.b.) $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ of $\mathbb{R}^n$ is called positively oriented (p.o.) if the orthogonal matrix whose column vectors are $\mathbf{x}_1, \cdots, \mathbf{x}_n$ has det $= 1$, and negatively oriented (n.o.) if it has det $= -1$.

Example 8.11
$\{\mathbf{x}_1, \mathbf{x}_2\}$ are o.o.b of $\mathbb{R}^2$.



p.o. $\iff \mathbf{x}_2 = R_{90}\mathbf{x}_1$
n.o. $\iff \mathbf{x}_2 = R_{-90}\mathbf{x}_1$

Example 8.12
$\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ o.o.b. of $\mathbb{R}^3$.
p.o. $\iff \mathbf{x}_3 = \mathbf{x}_1 \times \mathbf{x}_2 \iff$ right-hand
n.o. $\iff \mathbf{x}_3 = -\mathbf{x}_1 \times \mathbf{x}_2 \iff$ left-hand



Proposition 8.13
Let $A \in O(n)$. Then A preserves the orientation of any o.o.b. $\iff \det(A) = +1$
A reserves the orientation of any o.o.b. $\iff \det(A) = -1$.

Proof. Let $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ be an o.o.b. of $\mathbb{R}^n$. Set

$$B \coloneqq (\mathbf{x}_1 \cdots \mathbf{x}_n) \in O(n).$$

Then,

$$AB = (L_A\mathbf{x}_1 \cdots L_A\mathbf{x}_n)$$

Note, $\det(AB) = \det(A)\det(B)$.
Therefore,

$$\det(AB) = \begin{cases} \det(B) & \text{if } \det(A) = 1 \\ -\det(B) & \text{if } \det(A) = -1 \end{cases}$$

□

Proposition 8.14
The following functions are unchanged by proper rigid motions:

i. Curvature for a regular curve
ii. Torsion for a space curve
iii. Signed curvature for a plane curve.

By improper rigid motions, (i) is unchanged, (ii) and (iii) are multiplied by -1.

9 Jan 24, 2022

9.1 Rigid Motions (Cont'd)

Proof of Proposition 8.14. Let $\boldsymbol{\gamma}\colon I \to \mathbb{R}^n$ be a regular curve, $f = T_p \circ L_A \colon \mathbb{R}^n \to \mathbb{R}^n$ be a rigid motion. Set $\hat{\gamma} = f \circ \gamma \colon I \to \mathbb{R}^n$. Then,

$$\begin{aligned} \hat{\gamma} &= A\gamma(t) + p \\ \hat{v}(t) &= (A\gamma(t) + p)' = Av(t), \\ \hat{a}(t) &= (Av(t))' = Aa(t). \end{aligned}$$

Note: $\hat{\gamma}\colon I \to \mathbb{R}^n$ is a regular curve, because $\hat{\gamma}$ is smooth, and

$$\forall t \in I, \quad |\hat{v}(t)| = |Av(t)| = |v(t)| \neq 0.$$

Moreover,

$$\begin{aligned} \hat{t}(t) &= \frac{\hat{v}(t)}{|\hat{v}(t)|} = \frac{Av(t)}{|Av(t)|} = A\frac{v(t)}{|v(t)|} = At(t), \\ \hat{t}'(t) &= At'(t), \\ \hat{n}(t) &= \frac{\hat{t}'(t)}{\left|\hat{t}'(t)\right|} = \frac{At'(t)}{|At'(t)|} = A\frac{t'(t)}{|t'(t)|} = An(t) \end{aligned}$$

i. $\hat{\kappa} = \dfrac{\left|\hat{t}'\right|}{|\hat{v}|} = \dfrac{|At'|}{|Av|} = \dfrac{|t'|}{|v|} = \kappa$

ii. $\hat{b} \overset{?}{\leftrightarrow} Ab$. Compare $\{\hat{t}, \hat{n}, \hat{b}\}, \{At, An, Ab\}$:

(1) $\forall t \in I, \{\hat{t}(t), \hat{n}(t), \hat{b}(t)\}, \{At(t), An(t), Ab(t)\}$ are o.o.b.

(2) $\hat{t} = At, \hat{n} = An$

(3) $\{\hat{t}(t), \hat{n}(t), \hat{b}(t)\}$ is p.o.,

$$\{At, An, Ab\} \text{ is } \begin{cases} \text{p.o. if } \det(A) = 1, \text{ proper} \\ \text{n.o. if } \det(a) = -1, \text{ improper} \end{cases}$$

Therefore,

$$\hat{b} = \pm Ab, \text{ where } \begin{cases} + & \text{if } \det(A) = 1 \\ - & \text{if } \det(A) = 1 \end{cases}$$

$$\begin{aligned} \implies \hat{\tau} &= -\frac{\left\langle \hat{b}', \hat{n} \right\rangle}{|\hat{v}|^2} = -\frac{\langle \pm Ab', An \rangle}{|Av|^2} \\ &= \pm\left(-\frac{\langle b', n \rangle}{|v|^2}\right) = \pm\tau \end{aligned}$$

iii. Similar.

$\square$

Theorem 9.1 (Fundamental Theorems for Plane and Space Curves)

i. If $\kappa_s \colon I \to \mathbb{R}$ is a smooth function, then there exists a unit-speed plane curve $\gamma \colon I \to \mathbb{R}^2$ whose signed curvature $= \kappa_s$. If $\gamma, \hat{\gamma} \colon I \to \mathbb{R}^2$ are two such curves, then there exists a proper rigid motion $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\hat{\gamma} = f \circ \gamma$.

ii. If $\kappa, \tau \colon I \to \mathbb{R}$ are smooth functions with $\kappa > 0$, then there exists a unit-speed space curve $\gamma \colon I \to \mathbb{R}^3$ whose curvature $= \kappa$, torsion $= \tau$. If $\gamma, \hat{\gamma} \colon I \to \mathbb{R}^3$ are two such curves, then there exists a proper rigid motion $f \colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $\hat{\gamma} = f \circ \gamma$.

Proof.

i. Read the proof in Tapp.

ii. (Sketch, full proof uploaded on Canvas):
Fix $t_0 \in I$. We will show that, given the initial Frenet frame $\{t_0, n_0, b_0\}$, position γ_0, there exists a unique unit-speed space curve $\gamma \colon I \to \mathbb{R}^3$ such that $\gamma(t_0) = \gamma_0$, $\{t(t_0), n(t_0), b(t_0)\} = \{t_0, n_0, b_0\}$.
Step 1: Solve the Frenet equations for t, n, b:

$$\begin{cases} \mathbf{t}' = & & \kappa\mathbf{n} & \\ \mathbf{n}' = & -\kappa\mathbf{t} & & +\tau\mathbf{b} \\ \mathbf{b}' = & & -\tau\mathbf{n} & \end{cases} \qquad \text{the system of } 3 \times 3 = 9 \text{ ODEs}$$

The Picard theorem from the theory of ODEs implies there is a unique solution $\{t, n, b\}$ such that $\{t(t_0), n(t_0), b(t_0)\} = \{t_0, n_0, b_0\}$. (Read textbooks for ODEs)
Step 2: Show $\forall t \in I, \{t(t), n(t), b(t)\}$ is orthonormal. It is important that

$$\begin{pmatrix} 0 & \kappa(t) & 0 \\ -\kappa(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{pmatrix}$$

is skew-symmetric i.e. $c(t)^T = -c(t)$.
Step 3: $\gamma' = t$ has a unique solution such that $\gamma(t_0) = \gamma_0$, namely

$$\gamma(t) = \gamma_0 + \int_{t_0}^{t} t(u)\, du$$

Show $\gamma \colon I \to \mathbb{R}^3$ is a unit-speed space curve whose

$$\begin{aligned} \text{Frenet-frame } &= \{t, n, b\} \\ \text{curvature } &= \kappa \\ \text{torsion } &= \tau \end{aligned}$$

The result follows. Finally, suppose $\gamma, \hat{\gamma} \colon I \to \mathbb{R}^3$ are unit-speed space curves whose curvature $= \kappa$, torsion $= \tau$. Want to find a proper rigid motion $f = T_p \circ L_A \colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $\hat{\gamma} = f \circ \gamma$. Fix $t_0 \in I$. Set

$$A := (\hat{t}(t_0)\ \hat{n}(t_0)\ \hat{b}(t_0))^{-1}(t(t_0)\ n(t_0)\ b(t_0)).$$

Note $A \in O(3), \det(A) = 1$ because $(t(t_0)\ n(t_0)\ b(t_0)), (\hat{t}(t_0)\ \hat{n}(t_0)\ \hat{b}(t_0))$ have the same property. Set

$$\begin{aligned} p &\coloneqq \hat{\gamma}(t_0) - \gamma(t_0) \\ f &\coloneqq T_p \circ L_A \colon \mathbb{R}^3 \to \mathbb{R}^3 \text{ proper rigid motion} \end{aligned}$$

We want to show $\hat{\gamma} = f \circ \gamma$. Enough to show their initial positions and Frenet frames are the same:

$$\begin{aligned} \hat{\gamma}(t_0) &= f \circ \gamma(t_0), & \hat{t}(t_0) &= At(t_0), \\ \hat{n}(t_0) &= An(t_0), & \hat{b}(t_0) &= Ab(t_0). \end{aligned}$$

These are true by the choice of A, p.

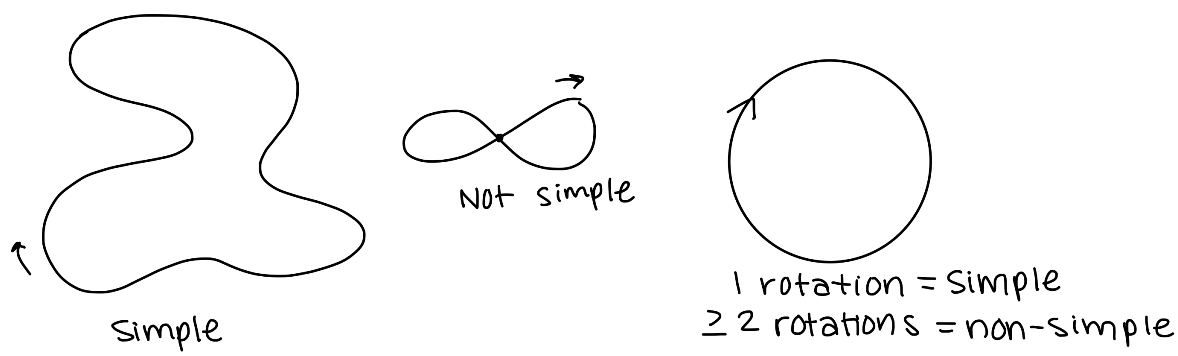
$\square$

10 Jan 26, 2022

10.1 Hopf's Theorem

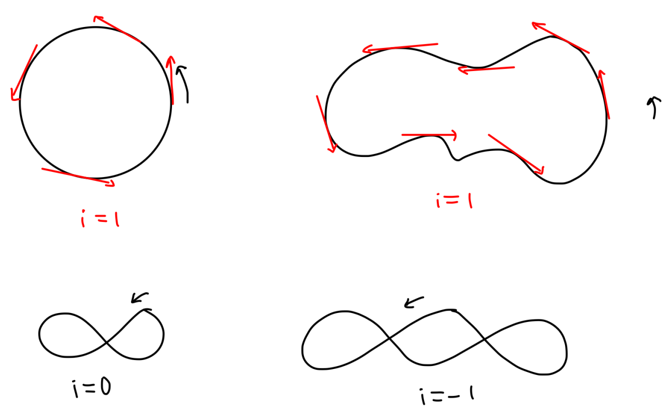
Definition 10.1 (Simple)
A closed regular curve $\gamma\colon [a, b] \to \mathbb{R}^n$ is called simple if γ is one-to-one on $[a, b)$.

Remark 10.2 simple = no self-intersection + 1 full rotation



Theorem 10.3 (Hopf's Umlaufsatz)
Let $\gamma\colon [a, b] \to \mathbb{R}^2$ be a simple closed plane curve. Then $i_\gamma = \pm 1$.

Recall 10.4 $i_\gamma = \frac{1}{2\pi}(\theta(b) - \theta(a))$ = "degree" for t, where θ is a smooth angle function from $[a, b]$ to $\mathbb{R}$ such that $\forall t \in [a, b], t(t) = (\cos\theta(t), \sin\theta(t))$.



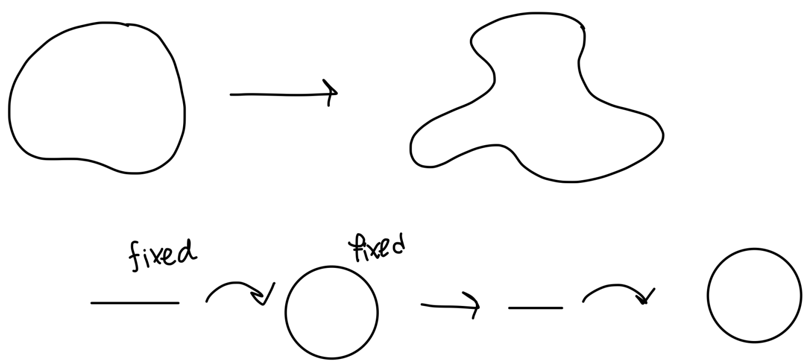
Idea: Deform the unit tangent to another function, while the "degree" is constant in a family of continuous functions.

Proposition 10.5
Let $f\colon [a,b] \to S^1 = \{x \in \mathbb{R}^2 \mid |x| = 1\}$ be a continuous function. Then there exists a continuous angle function $\theta\colon [a,b] \to \mathbb{R}$ such that $\forall t \in [a,b], f(t) = (\cos\theta(t), \sin\theta(t))$. The angle function θ is unique up to adding a multiple of 2π. If $f(a) = f(b)$, then $\frac{1}{2\pi}(\theta(b) - \theta(a)) \in \mathbb{Z}$ and the integer is called the degree of f, denoted by $\deg(f)$.

Remark 10.6 If $\gamma\colon [a,b] \to \mathbb{R}^2$ is a closed plane curve, then the unit tangent gives $t\colon [a,b] \to S^1$, and $\deg(t) = i_\gamma$.

Proof of Proposition (Sketch). Using $\cos^{-1}, \sin^{-1}$, define θ locally, then patch them to define θ globally so that θ is continuous on entire $[a,b]$. $\square$

Proposition 10.7
$\deg(f)$ is locally constant under deformation (continuous change of shapes) of $f\colon [a,b] \to S^1$. Loosely speaking, the proposition says that $\deg(f)$ does not change by small continuous change in f.



This follows from another lemma:

Lemma 10.8
Let $f_1, f_2\colon [a,b] \to S^1$ be continuous functions. If $\deg(f_1) \neq \deg(f_2)$, then $\exists t_0 \in [a,b], f_1(t_0) = -f_2(t_0)$.

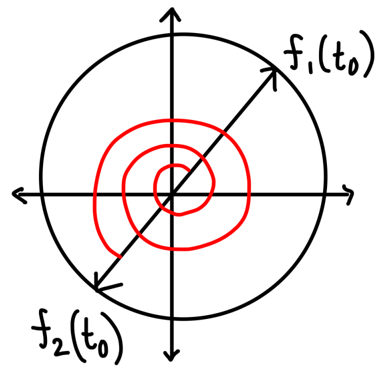
Remark 10.9 If f_1, f_2 never point in the opposite directions, then $\deg(f_1) = \deg(f_2)$.

Proof (sketch). θ_1, θ_2: angle functions for $f_1, f_2, \theta := \theta_1 - \theta_2$. Then

$$|\theta(a) - \theta(b)| = |\underbrace{(\theta_1(a) - \theta_1(b))}_{2\pi\deg(f_1)} - \underbrace{(\theta_2(a) - \theta_2(b))}_{2\pi\deg(f_2)}|$$
$$\geq 2\pi$$

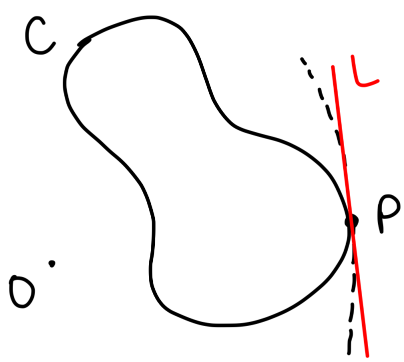
$\implies \exists$ odd multiple of π between $\theta(a), \theta(b), (2n-1)\pi$.

$$\underset{\text{IVT}}{\implies} \exists t_0 \in [a,b], \quad \theta(t_0) = (2n-1)\pi$$
$$\theta_1(t_0) = \theta_2(t_0) + (2n-1)\pi$$

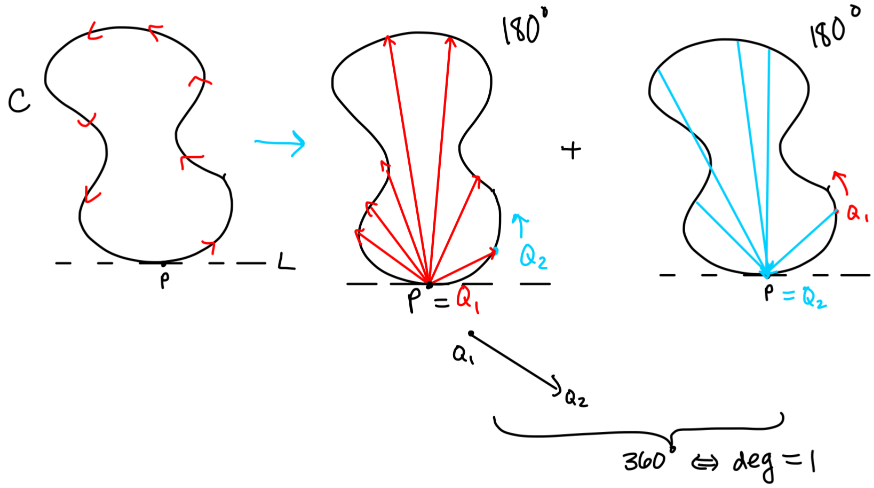


□

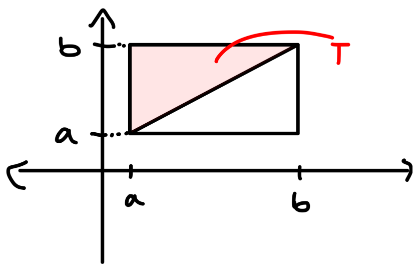
Proof of Hopf's Umlaufsatz. Let $\gamma\colon [a,b] \to \mathbb{R}^2$ be a simple closed curve, $c :=$ the trace of γ. We need to show $i_\gamma = \pm 1$. Let $p \in C$ such that $|\gamma|$ has the maximum at p.



Then C is entirely on one side of the tangent line L to C at p. We may assume γ is unit-speed, $p = \gamma(a)$.



Set $T := \{(t_1, t_2) \in \mathbb{R}^2 \mid a \leq t_1 \leq t_2 \leq b\}$



Define $\psi\colon T \to S^1$ as follows:

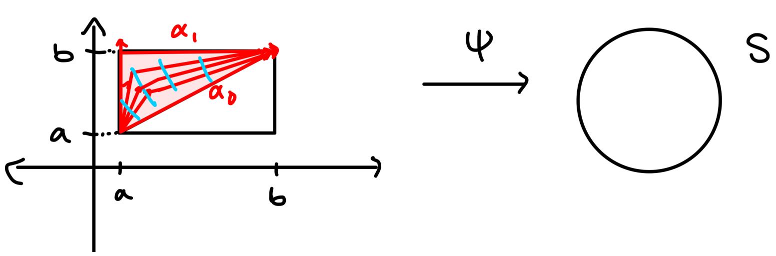
$$\psi(t_1, t_2) := \begin{cases} \gamma'(t_1) = t(t_1) & \text{if } t_1 = t_2 \\ \dfrac{\gamma(t_2) - \gamma(t_1)}{|\gamma(t_2) - \gamma(t_1)|} & \text{if } t_1 \neq t_2 \cap (t_1, t_2) \neq (a, b) \\ -\gamma'(a) & \text{if } (t_1, t_2) = (a, b) \end{cases}$$

ψ is well-defined because γ is simple, and ψ is continuous. For instance,

$$\psi(t_1, t_2) = \gamma'(t_1) = \lim_{t_2 \to t_1} \frac{\gamma(t_2) - \gamma(t_1)}{|\gamma(t_2) - \gamma(t_1)|} = \lim_{t_2 \to t_1} \psi(t_1, t_2)$$

Consider paths:

$$\alpha_0\colon [0, 1] \to T \quad (a, a) \to (b, b)$$
$$\alpha_1\colon [0, 1] \to T \quad (a, a) \to (a, b) \to (b, b)$$

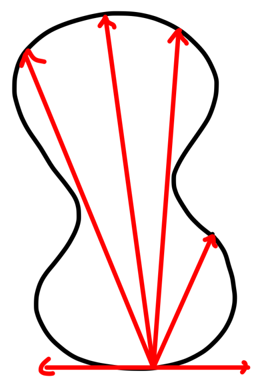


α_0 deforms to α_1 in a family of continuous functions

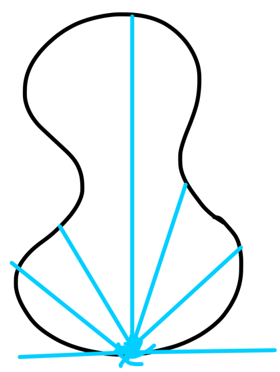
$$\begin{aligned} &\alpha_s = (1-s)\alpha_0 + s\alpha_1, \quad s \in [0,1] \\ &\implies \psi \circ \alpha_0 \text{ deforms to } \psi \circ \alpha_1 \colon [0,1] \to S^1 \\ &\implies \deg(\psi \circ \alpha_0) = \deg(\psi \circ \alpha_1) \\ &\deg(\psi \circ \alpha_0) = \deg(t) = i_\gamma \end{aligned}$$

Enough to show $\deg(\psi \circ \alpha_1) = \pm 1$.

$$(a,a) \to (a,b) \colon \psi(a,t) = \frac{\gamma(t) - \gamma(a)}{|\gamma(t) - \gamma(a)|}$$



$$(a,b) \to (b,b) \colon \psi(t,b) = \frac{\gamma(b) - \gamma(t)}{|\gamma(b) - \gamma(t)|}$$



$\square$

11 Jan 28, 2022

11.1 Midterm 1

12 Jan 31, 2022

12.1 Jordan's Theorem

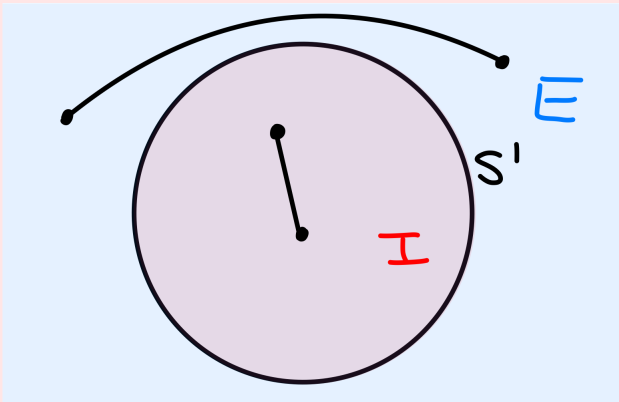
Definition 12.1 (Path-connected)
A subset $S \subseteq \mathbb{R}^n$ is called path-connected if any two points in S are connected by a continuous path in S.

Example 12.2
We have

$$I := \{\vec{x} \in \mathbb{R}^2 \mid |\vec{x}| < 1\}$$

$$E := \{\vec{x} \in \mathbb{R}^2 \mid |\vec{x}| > 1\}$$

I, E are both path-connected.



Definition 12.3 (Path-connected component)
A path-connected component of a subset $S \subseteq \mathbb{R}^n$ is a maximal path-connected subset of S.

Example 12.4
$\mathbb{R}^2 - S^1$ has exactly two connected components, namely I, E.

Theorem 12.5 (Jordan's Theorem)
Let $\gamma \colon [a, b] \to \mathbb{R}^2$ be a simple closed plane curve, and C the trace of γ. Then $\mathbb{R}^2 - C$ has exactly two path-connected components. One is bounded (called the interior), and the other is unbounded (called the exterior).

Remark 12.6 Intuitively clear, but a rigorous proof is not easy.

Recall 12.7 $f\colon [a,b] \to S^1$ continuous, $f(a) = f(b), \forall t \in [a,b]$,

$$f(t) = (\cos\theta(t), \sin\theta(t))$$

where $\theta\colon [a,b] \to \mathbb{R}$ continuous

$$\deg f \coloneqq \frac{1}{2\pi}(\theta(b) - \theta(a)) \in \mathbb{Z}$$

Proposition 12.8
Let $D \subseteq \mathbb{R}^n$ be a subset. Let $\{f_s\}_{s\in D}$ be a continuous family of continuous functions $f_s\colon [a,b] \to S^1$, i.e.

$$\begin{aligned} [a,b] \times D &\to S^1 \quad \text{is continuous} \\ (t,s) &\mapsto f_s(t) \end{aligned}$$

Then,

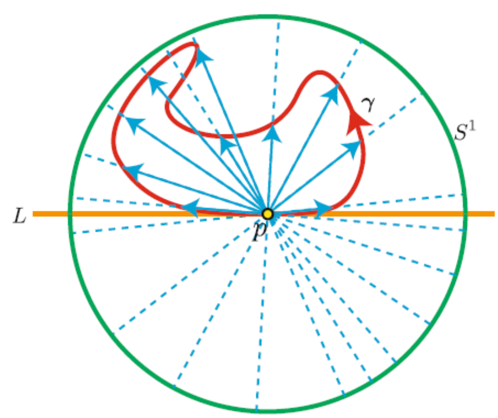
$$\begin{aligned} \deg\colon D &\to \mathbb{Z}, \\ s &\mapsto \deg f_s \end{aligned}$$

is constant on every path-connected component of D.

Proof of Jordan's Theorem (Sketch). Let $\gamma\colon [a,b] \to \mathbb{R}^2$ be simple closed, $C = \operatorname{im}\gamma$. For $p \in \mathbb{R}^2 - C$,

$$f_p(t) \coloneqq \frac{\gamma(t) - p}{|\gamma(t) - p|},$$

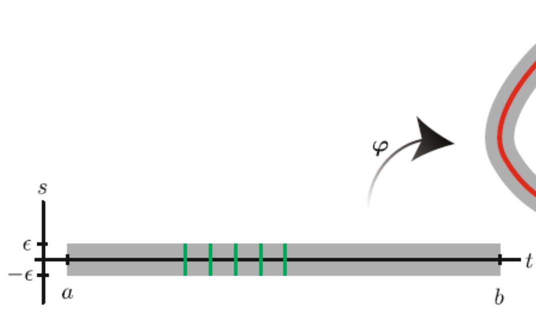
$f_p\colon [a,b] \to S^1$ continuous



$\{f_p\}_{p\in\mathbb{R}^2 - C}$ continuous family.

We show: $\mathbb{R}^2 - C$ has exactly two path-component components, one on which $\underbrace{\deg f_p = 0}_{\text{unbounded}}$, the other on which $\underbrace{\deg f_p = 1 \text{ or } -1}_{\text{bounded}}$.

Idea: Consider a tubular neighborhood of C. A tubular neighborhood being a thickening of C by $\pm\varepsilon$ in the normal direction of C.



The tubular neighborhood has no self-intersection, as C is simple. Take P, Q in the zoom window that are very close to each other, but on the opposites of C.

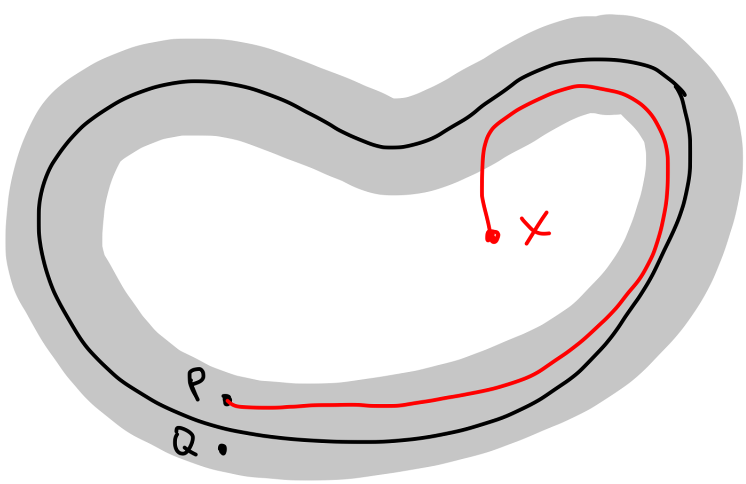
Step 1: Show $|\deg f_P - \deg f_Q| = 1 \implies \mathbb{R}^2 - C$ has at least two components.

$f_P - f_Q$ almost makes ± 1 rotation on $[a, c]$, while $f_P - f_Q$ makes only a very small change in $[c, b]$, too small to contribute to the change of the degree.

Step 2: Show $\mathbb{R}^2 - C$ has exactly two path-connected components.

Let $x \in \mathbb{R}^2 - C$. Then x is connected to either P or Q by a continuous path in $\mathbb{R}^2 - C$ as follows:

i. Choose a shortest path from x to C.
ii. Before reaching C, the path reaches the tubular neighborhood of C.
iii. Then, inside the tubular neighborhood, the path can be connected to either P or Q.

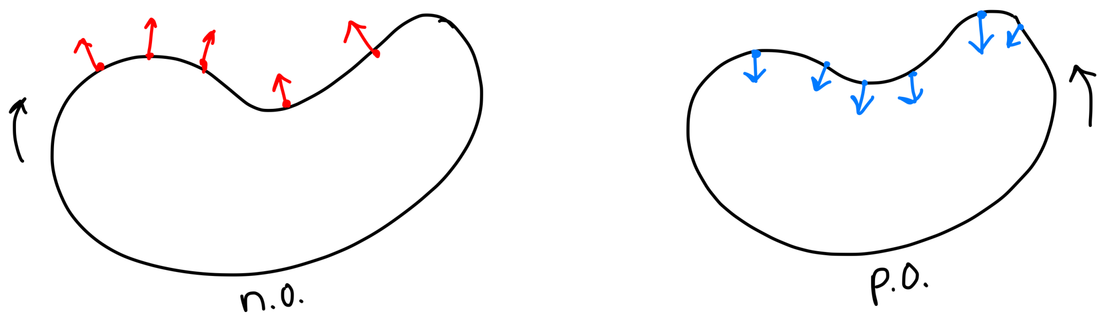


$\square$

Definition 12.9 (Positively oriented vs. negatively oriented)
A simple closed plane curve $\gamma\colon [a,b] \to \mathbb{R}^2$ is positively oriented if the interior is always on one's left as one traverses γ:

$$\forall \varepsilon > 0, \forall t \in [a,b], \forall S \in (0,\varepsilon),$$

$\gamma(t) + Sn_s(t)$ is in the interior and negatively oriented if the exterior is always on one's left and $\gamma(t) + Sn_s(t)$ is in the exterior.



Remarks 12.10

i. γ is either positively oriented or negatively oriented, as $\deg f_{\gamma(t)+Sn_s(t)}, t(s) \in \underbrace{[a,b] \times (0,\varepsilon)}_{\text{path-connected}}$ is constant, hence 0 (n.o.) or ± 1 (p.o.)

ii.
$$\text{p.o.} \iff i = 1$$
$$\text{n.o.} \iff i = -1$$

13 Feb 2, 2022

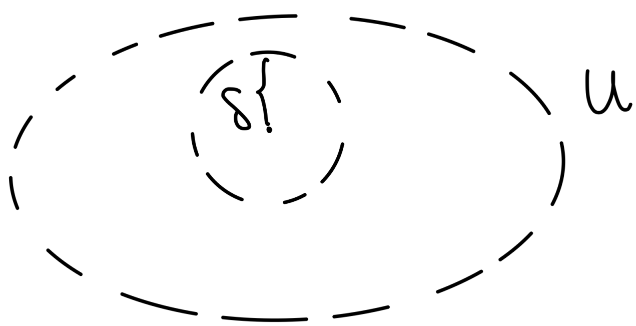
13.1 Jordan's Theorem (Cont'd)

Definition 13.1 (Piecewise regular curve, closed, simple, positively vs. negatively oriented)
A piecewise regular curve is a continuous function $\boldsymbol{\gamma}\colon [a,b] \to \mathbb{R}^n$ with partition $a = t_0 < t_1 < \cdots < t_m = b$ such that $\boldsymbol{\gamma}\colon := \boldsymbol{\gamma}|_{[t_{i-1},t_i]}\colon [t_{i-1}, t_i] \to \mathbb{R}^n$ is a regular curve for each $i = 1, \ldots, m$. Such a curve is called closed if $\boldsymbol{\gamma}(a) = \boldsymbol{\gamma}(b)$, simple if $\boldsymbol{\gamma}$ is one-to-one on $[a, b)$.
When $n = 2$, such a curve is called positively oriented if $\vec{n}_s(t)$ points toward the interior of C for all $t \in [a, b]$ corresponding to smooth points, and negatively oriented if $\vec{n}_s(t)$ points toward the exterior of C.

Remark 13.2 Jordan's theorem is true for piecewise regular simple closed plane curves.

13.2 Green's Theorem

Recall 13.3 $U \subset \mathbb{R}^n$ is open $\iff \forall x \in U, \exists \delta > 0, \forall y \in \mathbb{R}^n, |y - x| < \delta \implies y \in U$.

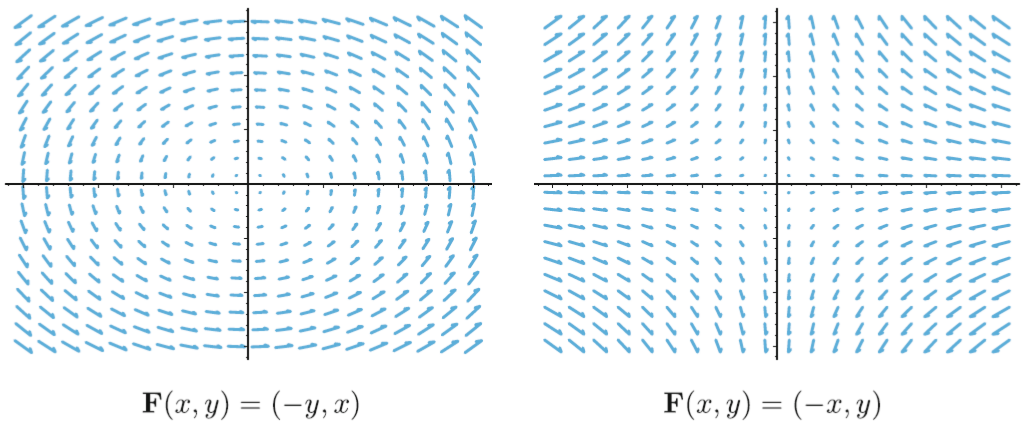


Definition 13.4 (Vector field)
A vector field on an open subset $U \subset \mathbb{R}^n$ is a smooth function $\vec{F}\colon U \to \mathbb{R}^n$, where smooth means

$$\vec{F} = (F_1, \cdots, F_n), F_1, \cdots, F_n \text{ smooth}$$

Remark 13.5 A vector field assigns to each point in U a vector in $\mathbb{R}^n$.

Example 13.6



An "oriented curve" will mean a regular curve $\boldsymbol{\gamma}\colon I \to \mathbb{R}^n$ with its trace.

Definition 13.7 (line integral)
Let C be an oriented curve parametrized as $\boldsymbol{\gamma}\colon [a,b] \to \mathbb{R}^n$, $\mathbf{F}$ be a vector field whose domain contains C. The line integral of $\mathbf{F}$ along C is defined as:

$$\int_C \mathbf{F} \cdot d\boldsymbol{\gamma} := \int_a^b \langle \mathbf{F}(\boldsymbol{\gamma}(t)), \boldsymbol{\gamma}'(t) \rangle \, dt$$

when C is simple closed, then the line integral is also denoted by

$$\oint_C \mathbf{F} \cdot d\boldsymbol{\gamma}$$

Remark 13.8 $\mathbf{F}$ force field $\implies \int_C \mathbf{F} \cdot d\boldsymbol{\gamma}$ total work along C.

Proposition 13.9
The line integral is unchanged by any orientation-preserving reparametrization, and multiplied by -1 by any orientation-reversing reparametrization.

Proof. Homework. $\square$

Remark 13.10 This shows that the line integral is well-defined for an equivalence class of oriented curves modulo orientation-preserving reparametrization. We will work with such a class, instead of an oriented curve itself.

Example 13.11
$\mathbf{F}(x, y) = (-y, x)$.
$C_1 :=$ the counterclockwise circle of radius 3 centered at the origin.
C_1 can be parametrized by $\boldsymbol{\gamma}_1(t) = (3\cos t, 3\sin t), t \in [0, 2\pi]$

$$\begin{aligned}
\oint_{C_1} \mathbf{F} \cdot d\boldsymbol{\gamma}_1 &= \int_0^{2\pi} \langle \mathbf{F}(\boldsymbol{\gamma}_1(t)), \boldsymbol{\gamma}'(t) \rangle \, dt \\
&= \int_0^{2\pi} \langle (-3\sin t, 3\cos t), (-3\sin t, 3\cos t) \rangle \, dt \\
&= \int_0^{2\pi} 9\sin^2 t + 9\cos^2 t dt \\
&= 9 \int_0^{2\pi} dt = 18\pi
\end{aligned}$$

$C_2 :=$ the graph of the parabola $y = x^2$ from $(-1, 1)$ to $(1, 1)$.
$\boldsymbol{\gamma}_2(t) = (t, t^2), \quad t \in [-1, 1]$.

$$\begin{aligned}
\int_{C_2} \mathbf{F}(t) \cdot d\boldsymbol{\gamma}_2 &= \int_{-1}^{1} \langle \mathbf{F}(\boldsymbol{\gamma}_2(t)), \boldsymbol{\gamma}'(t) \rangle \, dt \\
&= \int_{-1}^{1} \Big\langle (-t^2, t), (1, 2t) \Big\rangle \, dt \\
&= \int_{-1}^{1} (-t^2 + 2t^2) dt \\
&= \left[\frac{t^3}{3} \right]_{-1}^{1} = \frac{2}{3}
\end{aligned}$$

Remark 13.12 The line integral can be defined for a piecewise-regular curve $(C, \boldsymbol{\gamma})$ with smooth pieces $(C_1, \boldsymbol{\gamma}_i)$:

$$\int_C \mathbf{F} \cdot d\boldsymbol{\gamma} := \sum_i \int_{C_i} \mathbf{F} \cdot d\boldsymbol{\gamma}_i$$

Theorem 13.13 (Green's Theorem)
Let C be a positively oriented piecewise-regular simple closed plane curve parametrized by $\boldsymbol{\gamma}\colon [a, b] \to \mathbb{R}^2$, D be the interior of C. Let $\mathbf{F}$ be a vector field whose domain contains $C \cup D$. Then,

$$\oint_C \mathbf{F} \cdot d\boldsymbol{\gamma} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy,$$

where $\mathbf{F} = (P, Q)$.

Proof. Read Tapp for the proof. $\square$

Remark 13.14 Green's Theorem is a special case of the generalized Stokes Theorem:

$$\int_{\partial D} F = \int_D dF,$$

where D is a "region" in $\mathbb{R}^n$ with boundary ∂D, F is a "function" on D, d is a "derivative".

Corollary 13.15
Let $(C, \boldsymbol{\gamma}), \mathbf{F}$ as in Green's Theorem. Write $\boldsymbol{\gamma}(t) = (x(t), y(t))$. Then,

$$\begin{aligned} Area(D) &= \int_a^b x(t)y'(t)\,dt \\ &= -\int_a^b x'(t)y(t)\,dt \end{aligned}$$

Proof. Apply Green's Theorem to:

$$\mathbf{F}_1(x,y) = (0,x),\ \mathbf{F}_2(x,y) = (-y,0).$$

For instance,

$$\int_C \mathbf{F}_1 \cdot d\boldsymbol{\gamma} = \iint_D \left(\frac{\partial x}{\partial x} - \frac{\partial 0}{\partial y} \right) dxdy$$

$$\begin{aligned} \text{L.H.S.} &= \int_a^b \langle (0, x(t)), (x'(t), y'(t)) \rangle \, dt \\ &= \int_a^b x(t)y'(t)\,dt \\ \text{R.H.S.} &= \iint_D dxdy = Area(D) \end{aligned}$$

$\square$

14 Feb 4, 2022

14.1 Isoperimetric Inequality

Theorem 14.1 (Isoperimetric Inequality)
Let C be a simple closed plane curve, ℓ be the arc length of C, A be the area of the interior of C. Then

$$\ell^2 \geq 4\pi A$$

Moreover,

$$\text{“ = ”} \iff C \text{ is a circle.}$$

Remark 14.2 Theorem says among all simple closed plane curves with fixed perimeter, the circle bounds the largest area.

3 main ingredients for the proof

i. (Corollary of) Green's Theorem:
Let C be a positively oriented piecewise-regular simple closed plane curve, parametrized by $\boldsymbol{\gamma}(t) = (x(t), y(t)), t \in [a, b]$, D be the interior of C. Then

$$Area(D) = \int_a^b x(t)y'(t)\,dt = -\int_a^b x'(t)y(t)\,dt$$

ii. Schwartz inequality:
$\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then

$$\langle \mathbf{x}, \mathbf{y} \rangle \leq |\mathbf{x}| \cdot |\mathbf{y}|$$

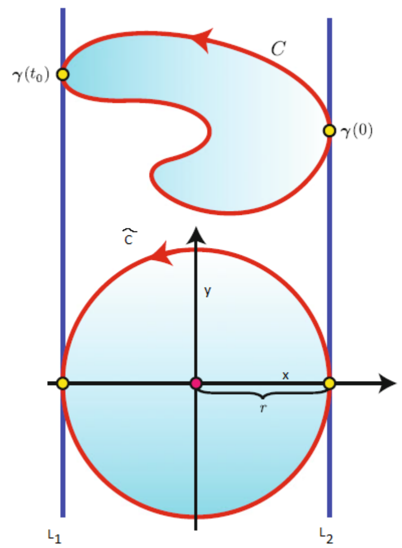
and,

$$\begin{aligned} \text{“ = ”} &\iff \mathbf{x}, \mathbf{y} \text{ point toward the same direction} \\ &\underset{\text{if } \mathbf{x} \neq \mathbf{0}}{\iff} \exists c \geq 0, \mathbf{y} = c\mathbf{x} \end{aligned}$$

iii. AM-GM inequality:
$a, b \geq 0$, then

$$\sqrt{ab} \leq \frac{a+b}{2}, \text{ and “ =''} \iff a = b$$

Proof of Isoperimetric Inequality. Let C be a simple closed plane curve. Let L_1, L_2 be two parallel tangent lines to C so that C is between L_1, L_2:



Set $r :=$ (the distance between L_1, L_2) $\times \frac{1}{2}$
Let $\tilde{C}$ be a circle tangent to L_1, L_2. Then $\tilde{C}$ has radius r.
Choose a coordinate system $\{x, y\}$ for $\mathbb{R}^2$ so that $\tilde{C}$ has center $(0,0)$, $L_1 = \{x = -r\}$, $L_2 = \{x = r\}$.
Let $\boldsymbol{\gamma}\colon [0,\ell] \to \mathbb{R}^2$ be a positively oriented unit-speed parametrization of C. May assume $\boldsymbol{\gamma}(0) =$ the tangent point with L_2. Let $t_0 \in [0,\ell]$ such that $\boldsymbol{\gamma}(t_0) =$ the tangent point with L_1.
Let: $\mathbf{B}\colon [0,\ell] \to \mathbb{R}^2$ be the parametrization of $\tilde{C}$ given by $\mathbf{B}(t) = (x(t), \tilde{y}(t))$, where

$$\tilde{y}(t) := \begin{cases} \sqrt{r^2 - x(t)^2} & \text{if } t \in [0, t_0] \\ -\sqrt{r^2 - x(t)^2} & \text{if } t \in [t_0, \ell] \end{cases}$$

Note: $\mathbf{B}$ might not be regular, nor simple, but no issue when computing the area. By Green's Theorem,

$$A = \int_0^\ell x(t)y'(t)\,dt$$
$$\pi r^2 = -\int_0^\ell x'(t)\tilde{y}(t)\,dt$$

Then,

$$\begin{aligned} A + \pi r^2 &= \int_0^\ell \Big(x(t)y'(t) - x'(t)\tilde{y}(t)\Big)dt \\ &= \int_0^\ell \langle (x'(t), y'(t)), (-\tilde{y}(t), x(t)) \rangle\,dt \\ &\le \int_0^\ell \underbrace{|(x'(t), y'(t))|}_{\substack{=1 \text{ unit-speed} \\ \text{by Schwartz Inequality}}} \cdot \underbrace{|(-\tilde{y}(t), x(t))|}_{=r} dt \\ &= \int_0^\ell r dt = \ell r \end{aligned}$$

By AM-GM inequality,

$$\sqrt{A \cdot \pi r^2} \leq \frac{A + \pi r^2}{2} \leq \frac{\ell r}{2}$$
$$A \cdot \pi r^2 \leq \frac{\ell^2 r^2}{4}$$
$$4\pi A \leq \ell^2$$

The first statement is proved!
Suppose $4\pi A = \ell^2$.
Then the Schwartz, AM-GM inequalities are equalities:

1. $\exists c \geq 0, (-\tilde{y}(t), x(t)) = c(x'(t), y'(t))$
2. $A = \pi r^2$.

From (i.),

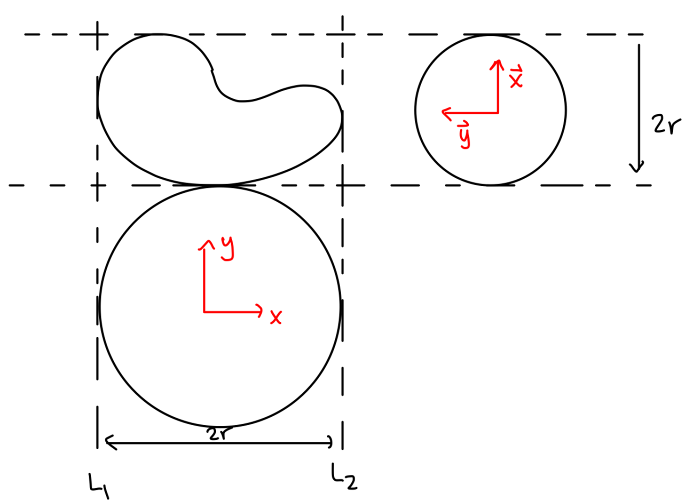
$$\underbrace{|\tilde{y}(t), x(t))|}_{=r} = c \cdot \underbrace{|(x'(t), y'(t))|}_{=1}$$

$$\implies c = r$$
$$\implies (-\tilde{y}(t), x(t)) = r(x'(t), y'(t))$$
$$\implies x(t) = ry'(t) \tag{1}$$

From (ii),

$$r = \sqrt{\frac{A}{\pi}}$$

This shows r does not depend on the directions of L_1, L_2.



Now we repeat the process for two parallel tangent lines perpendicular to L_1, L_2.
Let $\{\overline{x}, \overline{y}\}$ be the corresponding coordinate system. Then $\overline{x}(t) = r\overline{y}'(t)$. On the other hand,

$$\begin{cases} \overline{x}(t) = y(t) + d \\ \overline{y}(t) = -x(t) = \ell \end{cases} \quad \text{for } d, \ell \in \mathbb{R}$$

Then

$$y(t) + d = -rx'(t) \tag{2}$$

Then

$$\begin{aligned} &x(t)^2 + (y(t)+d)^2 \\ &= (ry'(t))^2 + (-rx'(t))^2 && \text{(by 1 and 2)} \\ &= r^2(x'(t)^2 + y'(t)^2) = r^2 && \text{(unit-speed)} \end{aligned}$$

Therefore C is a circle of radius r. □

14.2 The Derivative of Functions from $\mathbb{R}^m$ to $\mathbb{R}^n$

Definition 14.3 (Partial derivatives, C^r on U, smooth on U)
Let $f\colon U \subset \mathbb{R}^m \to \mathbb{R}^n$ be a function, when $U \subset \mathbb{R}^m$ is an open subset.
The partial derivative of f with respect to x at $p \in U$ is defined as

$$\frac{\partial f}{\partial x_i}(p) = f_{x_i}(p) \coloneqq \lim_{h \to 0} \frac{f(p + he_i) - f(p)}{h},$$

where $e_i = (0, \cdots, 0, 1, 0, \cdots, 0) \in \mathbb{R}^n$.
A second order partial derivative is a partial derivative of a partial derivative, and so on.
For instance,

$$\frac{\partial^2 f}{\partial x_j \partial x_i} = f_{x_i, x_j} \coloneqq \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)$$

f is called C^r on U if all r-th partial derivatives exist and are continuous on U. f is smooth on U if $\forall r \in \mathbb{Z} > 0$, f is C^r on U.

15 Feb 7, 2022

15.1 The Derivative of Functions from $\mathbb{R}^m$ to $\mathbb{R}^n$ (Cont'd)

Recall 15.1 Given $f\colon U \subset \mathbb{R}^m \to \mathbb{R}^n$,
f is C^r on $U \iff$ All r-th partial derivatives exist and are continuous on U.
f is smooth on $U \iff \forall r \in \mathbb{Z}_{>0}, f$ is C^r on U.

Remarks 15.2

i. If f is C_2, then $\forall i, j, f_{x_i,x_j} = f_{x_j,x_i}$.
If f is smooth, then mixed partials of any order commute.

ii. f is smooth $\iff \forall i, f_i$ is smooth, where $f = (f_1, \cdots, f_n)$.

Example 15.3
Let $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the smooth function defined by

$$f(x, y) = (\sin(xy), e^{x+5y}, x^2y^3)$$

Then,

$$f_x = (y\cos(xy), e^{x+5y}, 3xy^3)$$
$$f_y = (x\cos(xy), 5e^{x+5y}, 3x^2y^2)$$

and

$$f_{xx} = (-y^2\sin(xy), e^{x+5y}, 2y^3)$$
$$f_{xy} = (f_x)_y = (\cos(xy) - xy\sin(xy), 5e^{x+5y}, 6xy^2) = f_{yx}$$
$$f_{yy} = (-x^2\sin(xy), 25e^{x+5y}, 6x^2y)$$

Proposition 15.4
If $g\colon V \subset \mathbb{R}^\ell \to \mathbb{R}^m, f\colon U \subset \mathbb{R}^m \to \mathbb{R}^n$ are smooth and $g(V) \subset U$, then $f \circ g\colon V \subset \mathbb{R}^\ell \to \mathbb{R}^n$ is smooth.

Example 15.5
We have

$$f(x, y) = (\sin(xy), e^{x+5y}, x^2y^3)$$

so f is the composition of the smooth functions:

$$(x, y) \longmapsto (xy, x + 5y, x^2y^3)$$

and

$$(u, v, w) \longmapsto (\sin(u), e^v, w)$$

Definition 15.6 (Differentiable)
Let $f\colon U \subset \mathbb{R}^m \to \mathbb{R}^n, p \in U. f$ is differentiable at $p \in U$ if there exists an $n \times m$ matrix M such that

$$f(p+v) = f(p) + M \cdot v + E(v),$$

where

$$\lim_{v \to 0} \frac{E(v)}{|v|} = 0.$$

In this case, M is called the derivative of f at p, and denoted by $f'(p)$.

Remarks 15.7

i. $f'(p)$ is uniquely determined if it exists.
ii. $f(p+v) = f(p) + f'(p) \cdot v + E(v)$ is the first order Taylor approximation of f at p.
iii. Smooth $\implies$ differentiable $\implies$ $f_{x_i}, \cdots, f_{x_m}$ all exist.

Proposition 15.8
Suppose $f\colon U \subset \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at $p \in U$. Then

$$f'(p) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_1}{\partial x_m}(p) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(p) & \cdots & \frac{\partial f_n}{\partial x_m}(p) \end{pmatrix}$$

where $f'(p)$ is the Jacobian matrix of f at p.

Example 15.9
Let $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$f(x,y) = (x^2y^3, x + y^4, y + 1)$$

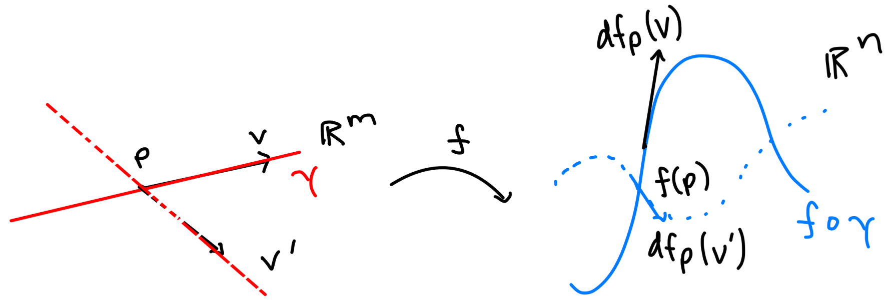
Then,

$$f' = \begin{pmatrix} \frac{\partial}{\partial x}(x^2y^3) & \frac{\partial}{\partial y}(x^2y^3) \\ \frac{\partial}{\partial x}(x+y^4) & \frac{\partial}{\partial y}(x+y^4) \\ \frac{\partial}{\partial x}(y+1) & \frac{\partial}{\partial y}(y+1) \end{pmatrix} = \begin{pmatrix} 2xy^3 & 3x^2y^2 \\ 1 & 4y^3 \\ 0 & 1 \end{pmatrix}$$

Definition 15.10 (Directional derivative)
Let $f\colon U \subset \mathbb{R}^m \to \mathbb{R}^n, p \in U, v \in \mathbb{R}^m$. The directional derivative of f in the direction of v at p is defined as

$$df_p(v) := \lim_{t \to 0} \frac{f(p+tv) - f(p)}{t} = (f \circ \gamma)'(0),$$

where $\gamma(t) = p + tv$.



Remarks 15.11

i. $\dfrac{\partial f}{\partial x_i}(p) = df_p(e_i)$

ii. differentiable at $p \implies \forall v \in \mathbb{R}^m, df_p(v)$ exists

Proposition 15.12
Suppose $f\colon U \subset \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at $p \in U$. Then $\forall v \in \mathbb{R}^m, df_p(v) = f'(p) \cdot v$.

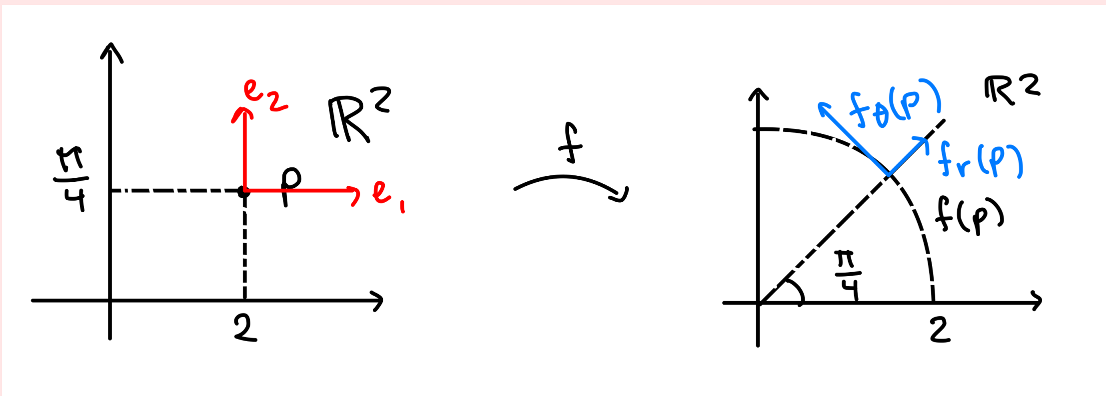
Remark 15.13 Proposition says:

- $df_p\colon \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation represented by $f'(p)$.
- df_p is also called the derivative of f at p and will be identified with the matrix: $df_p = f'(p)$

Example 15.14

Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2, (r,\theta) \mapsto (r\cos\theta, r\sin\theta), p = (2, \frac{\pi}{4}) \in \mathbb{R}^2$. Then,

$$
\begin{aligned}
df_p &= \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}\Bigg|_{(r,\theta)=(2,\frac{\pi}{4})} \\
&= \begin{pmatrix} \frac{\sqrt{2}}{2} & -\sqrt{2} \\ \underbrace{\frac{\sqrt{2}}{2}}_{f_r(p)} & \underbrace{\sqrt{2}}_{f_\theta(p)} \end{pmatrix}
\end{aligned}
$$



Proposition 15.15 (Chain Rule)

If $g\colon V \subset \mathbb{R}^\ell \to \mathbb{R}^m, f\colon U \subset \mathbb{R}^m \to \mathbb{R}^n$ are differentiable and $g(V) \subset U$, then $\forall q \in V$,

$$(f \circ g)'(q) = \underbrace{f'(g(q))}_{n\times m} \cdot \underbrace{g'(q)}_{m\times\ell}$$

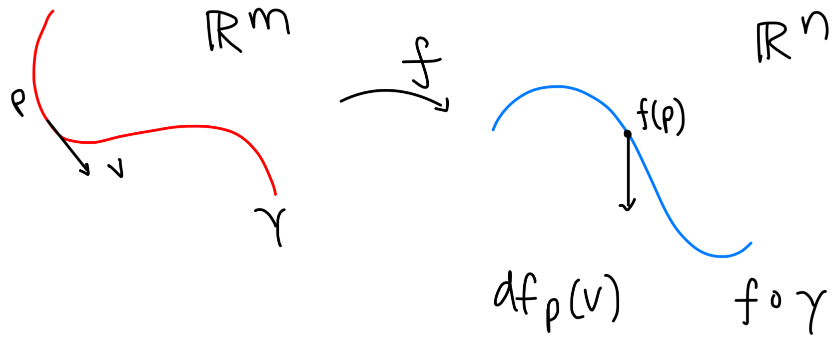
or,

$$d(f\circ g)_q = (df)_{g(q)} \circ (dg)_q$$

Corollary 15.16

Let $f\colon U \subset \mathbb{R}^m \to \mathbb{R}^n$ be a differentiable function, $\gamma\colon I \to U \subset \mathbb{R}^m$ be a regular curve such that $0 \in I, \gamma(0) = p \in U, \gamma'(0) = v \in \mathbb{R}^m$. Then

$$df_p(v) = (f\circ\gamma)'(0).$$



Proof. Apply the Chain rule to f and $g = \gamma$. $\square$

16 Feb 9, 2022

16.1 The Derivative of Functions from $\mathbb{R}^m$ to $\mathbb{R}^n$ (Cont'd)

Recall 16.1 $f\colon U \subset \mathbb{R}^m \to \mathbb{R}^n$ smooth, $P \in U$, $df_p\colon \mathbb{R}^m \to \mathbb{R}^n$ is represented by

$$f'(p) = \left(\frac{\partial f_i}{\partial x_j}(p)\right)$$

Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be smooth. Suppose f is a smooth bijection such that $f^{-1}\colon \mathbb{R}^n \to \mathbb{R}^n$ is smooth. Then $\forall p \in \mathbb{R}^n, df_p$ is an invertible linear transformation. Indeed, by Chain rule,

$$I_n = d(f \circ f^{-1})_{f(p)} = (df)_p \circ (df^{-1})_{f(p)}$$

$$I_n = d(f^{-1} \circ f)_p = (df^{-1})_{f(p)} \circ (df)_p$$

Therefore, $(df)_p$ is invertible.

Recall 16.2 A neighborhood of $p \in \mathbb{R}^n$ is an open subset $U \subset \mathbb{R}^n$ such that $p \in U$.

Theorem 16.3 (Inverse Function Theorem)
Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be smooth on a neighborhood of $p \in \mathbb{R}^n$. Suppose df_p is an invertible linear transformation. Then

$$\exists U \subset \mathbb{R}^n\colon \text{ neighborhood of } p$$

$$\exists V \subset \mathbb{R}^n\colon \text{ neighborhood of } f(p)$$

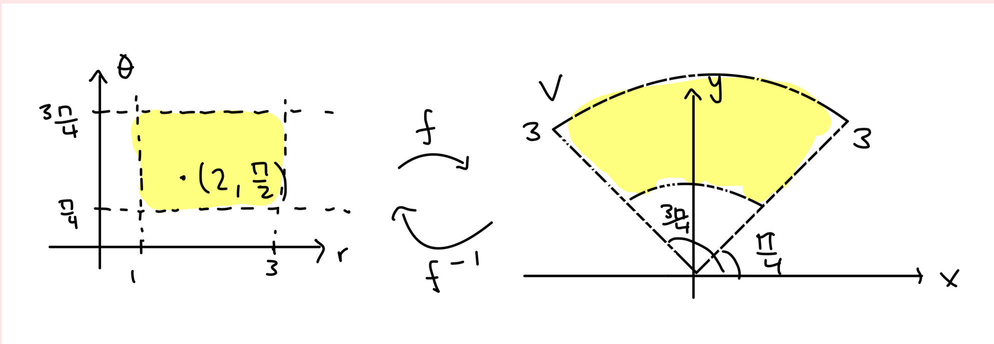
such that $f\colon U \to V$ is a smooth bijection with $f^{-1}\colon V \to U$ smooth.

Example 16.4
Suppose $f(r,\theta) = (r\cos\theta, r\sin\theta)$, $(r,\theta) \in \mathbb{R}^2$. Then,

$$df_{(r,\theta)} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

So,

$$\det df_{(r,\theta)} = r\cos^2\theta + r\sin^2\theta = r \neq 0$$
$$\iff df_{(r,\theta)} \text{ is invertible.}$$



where

$$f^{-1}(x,y) = \left(\sqrt{x^2+y^2}, \cos^{-1}\frac{x}{\sqrt{x^2+y^2}}\right).$$

f is not a bijection on $\{(r,\theta) \in \mathbb{R}^2 \mid r \neq 0\}$, and

$$f(r,\theta) = f(r, \theta + 2\pi m), \quad m \in \mathbb{Z}$$

Definition 16.5 (Smooth (on a set))
Let $X \subset \mathbb{R}^m$ be a subset (not necessarily open). $f\colon X \to \mathbb{R}^n$ is called <u>smooth</u> if $\forall p \in X, \exists U\colon$ neighborhood of p in $\mathbb{R}^m$, $\exists \tilde{f}\colon U \to \mathbb{R}^n$ smooth such that

$$f\mid_{U\cap X} = \tilde{f}\mid_{U\cap X}.$$

Example 16.6
Consider $S^2 \coloneqq \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$.
Define

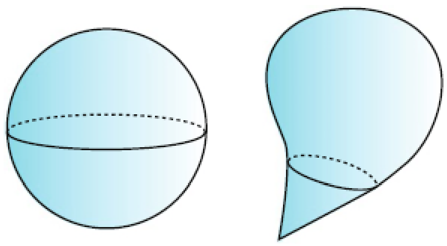
$$f(x,y,z) = x^2+y^2+z^2,$$
$$g(x,y,z) = \frac{1}{x^2+y^2+z^2}.$$

Then $f, g\colon S^2 \to \mathbb{R}$ are both smooth. As for g, g is smooth on $\mathbb{R}^3 - \{0,0,0)\}$, and $S^2 \subseteq \mathbb{R}^3 - \{(0,0,0)\}$.

16.2 Diffeomorphisms

Definition 16.7 (Diffeomorphic/Diffeomorphism)
$X \subset \mathbb{R}^m, Y \subset \mathbb{R}^n$ are called diffeomorphic if $\exists f\colon X \to Y$ smooth bijection such that $f^{-1}\colon Y \to X$ is smooth. In this case, f is called a diffeomorphism between X, Y.

Remark 16.8 If we require "continuous" instead of "smooth" in the above definition, then we get the definition of homeomorphism.

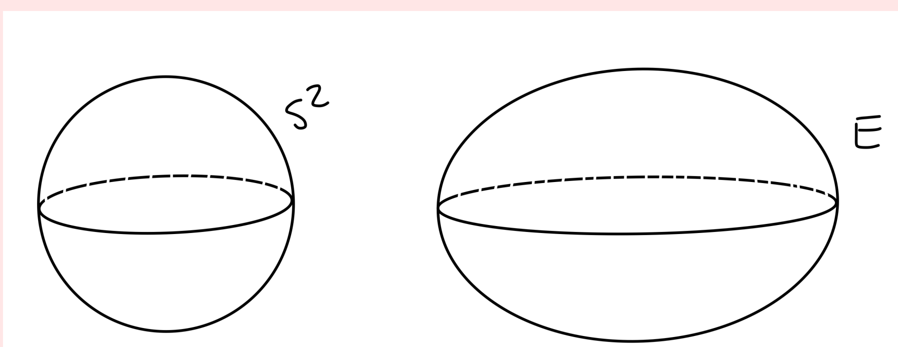


Homeomorphic
but not diffeomorphic

Example 16.9
Consider S^2, and

$$E = \left\{(x, y, z) \in \mathbb{R}^3 \middle| \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = c\right\}$$

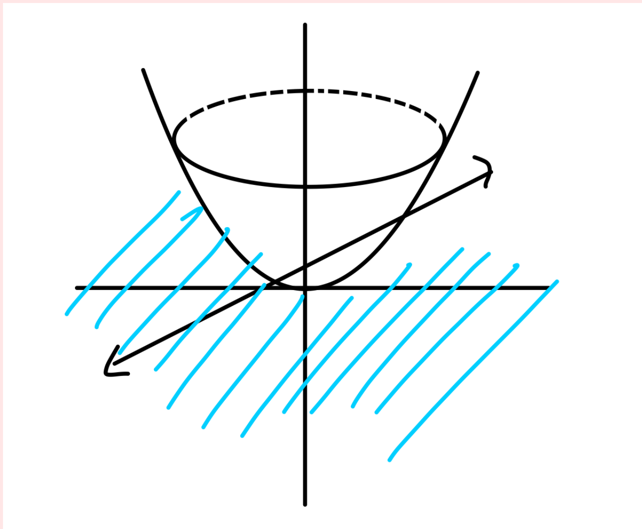
where $a, b, c > 0$ (fixed).



S^2, E are diffeomorphic:

$$f\colon S^2 \to E, (x, y, z) \longmapsto (ax, by, cz),$$
$$f^{-1}\colon E \to S^2, (x, y, z) \longmapsto \left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right),$$

Example 16.10
Consider $P \coloneqq \{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 + y^2\}$



Show $P, \mathbb{R}^2$ are diffeomorphic.
Define

$$f\colon P \to \mathbb{R}^2, (x, y, z) \longmapsto (x, y)$$
$$g\colon \mathbb{R}^2 \to P, (x, y) \longmapsto (x, y, x^2 + y^2)$$

Now f, g are both smooth. Also f is a bijection with $f = g^{-1}$. Indeed,

$$g \circ f(x, y, z) = g(x, y) = (x, y, x^2 + y^2) = (x, y, z)$$

So $g \circ f = id_p$. Similarly, $f \circ g = id_{\mathbb{R}^2}$. Therefore, $f\colon P \to \mathbb{R}^2$ is a smooth bijection with smooth inverse g. So f is a smooth diffeomorphism.

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17.1 Diffeomorphisms (Cont'd)

Recall 17.1 $X \subset \mathbb{R}^m, Y \subset \mathbb{R}^n$

$$\begin{aligned} f\colon X \to \mathbb{R}^n \text{ is smooth } \iff & \forall p \in X, \exists U \subseteq \mathbb{R}^m \text{ nbd of } P, \\ & \exists \tilde{f}\colon U \to \mathbb{R}^n \\ & \text{s.t. } f = \tilde{f} \text{ on } X \cap U. \end{aligned}$$

$f\colon X \to Y$ is a diffeomorphism $\iff$ f is a smooth bijection with f^{-1} smooth.

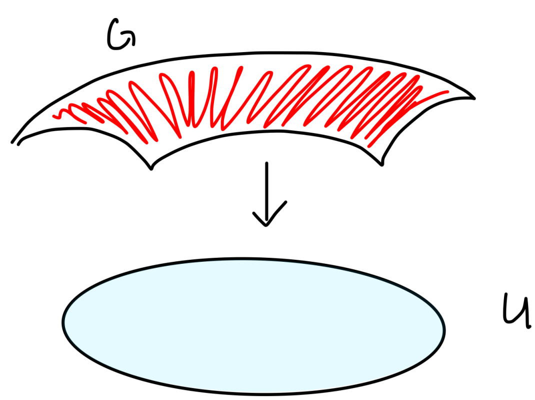
Lemma 17.2
Let $f\colon U \subset \mathbb{R}^2 \to \mathbb{R}$ be smooth,

$$G \coloneqq \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in U, z = f(x, y)\}$$

Then G is diffeomorphic to U by

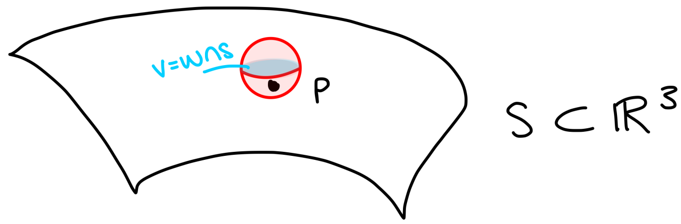
$$\sigma\colon G \to U, (x, y, z) \mapsto (x, y),$$

$$\sigma^{-1}\colon U \to G, (x, y) \mapsto (x, y, f(x, y))$$

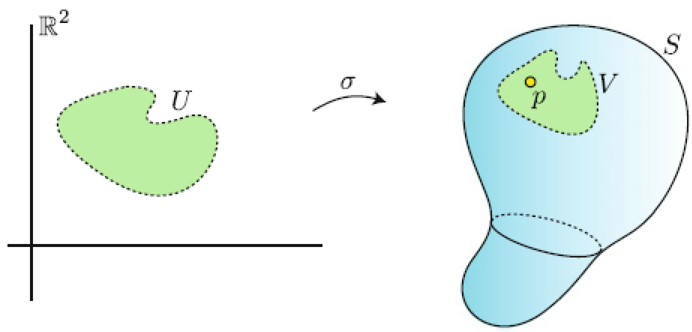


17.2 Regular Surfaces

Recall 17.3 Consider $S \subset \mathbb{R}^n$. $V \subset S$ is open if $\exists W \in \mathbb{R}^n$ open, $V = S \cap W$. A neighborhood (or nbd) of $p \in S$ in S is an open subset of S that contains p.



Definition 17.4 (Regular surface, surface patch)
$S \subset \mathbb{R}^3$ is called a regular surface if $\forall p \in S, \exists V \subseteq S$ neighborhood of $p, \exists U \subseteq \mathbb{R}^2$ open, $\exists \sigma \colon U \to V$ diffeomorphism.
σ is called a surface patch or coordinate chart. A collection of surface patches that cover every point of S is called an atlas for S.



A surface patch $\sigma : U \to V$ that covers the point $p \in S$

Remark 17.5 An analogue of a regular curve is a parametrized surface, which will be defined later.

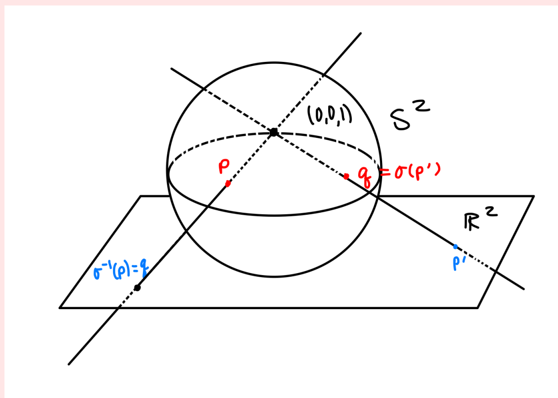
Example 17.6
Let $U \subset \mathbb{R}^2 \subset \mathbb{R}^3$ be an open subset. Then U is a regular surface. Indeed, $id \colon U \to U$ is a surface patch that covers U.

Example 17.7
Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$
Show S^2 is a regular surface.
We will define 2 surface patches that cover S^2:

$$\sigma \colon \mathbb{R}^2 \to S^2 - \{(0,0,1)\},$$

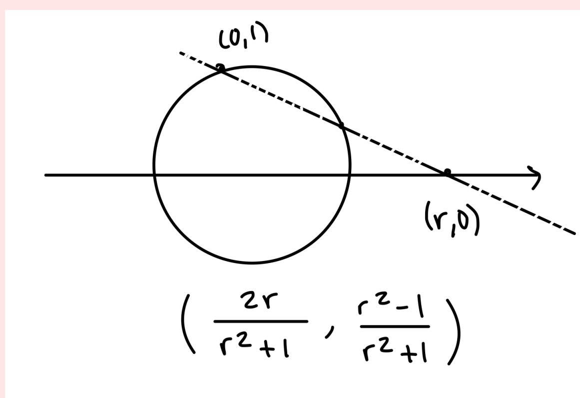
$$\tau \colon \mathbb{R}^2 \to S^2 - \{(0,0,-1)\}$$

We use the stereographic projection:



Where,

$$\sigma(x, y) = \left(\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}\right)$$



Then, $\sigma^{-1} \colon S^2 - \{(0,0,1) \to \mathbb{R}^2$ is defined by

$$\sigma^{-1}(x, y, z) = \left(\frac{2x}{1 - z}, \frac{2y}{1 - z}\right)$$

σ, σ^{-1} are both smooth, so σ is a diffeomorphism. Therefore, σ is a surface patch. Similar for τ.

Example 17.8

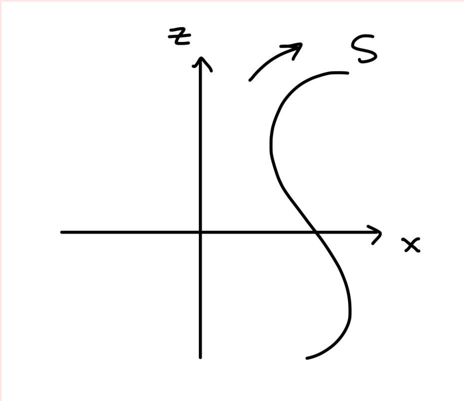
Let $\gamma(t) = (x(t), 0, z(t)), t \in (a, b)$ be a regular curve in the xz planes. Define $c :=$ the trace of γ. Suppose:

i. $\forall t \in (a, b), x(t) > 0$

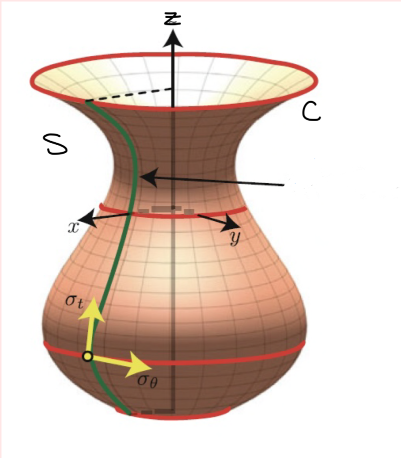
ii. $\forall t \in (a, b), z'(t) > 0$

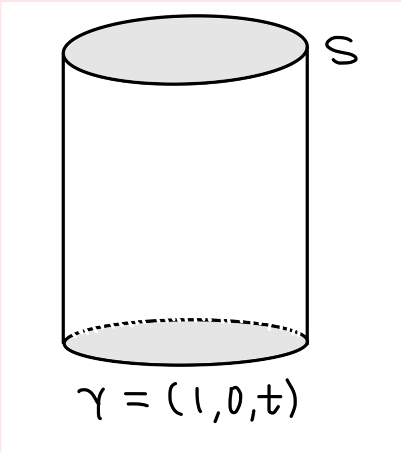
(i) $\implies$ c does not intersect the z-axis

(ii) $\implies$ c has no self-intersection

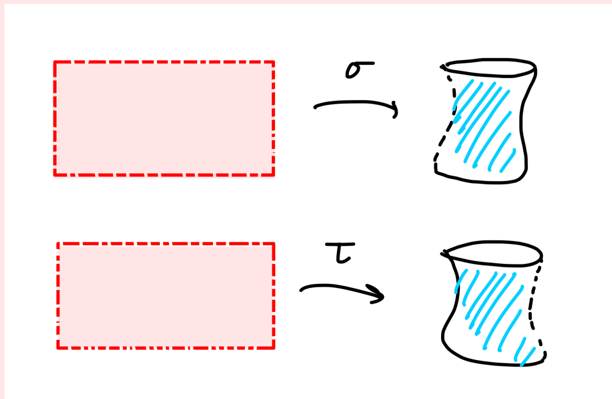


Let S be the surface of revolution resulting from revolving C about the z-axis.





We show S is a regular surface.
Idea:



Define $\sigma\colon (a,b)\times(-\pi,\pi)\to S$ by

$$\begin{aligned}\sigma(t,\theta) &= (x(t)\cos\theta, x(t)\sin\theta, z(t))\\ &= R_\theta(\gamma(t)),\end{aligned}$$

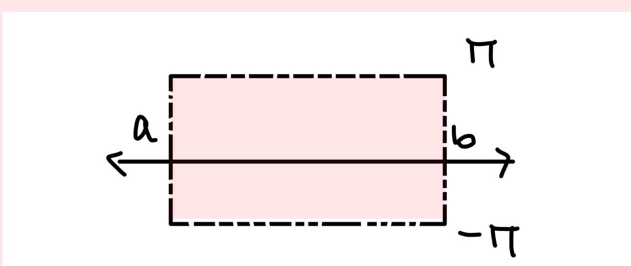
where,

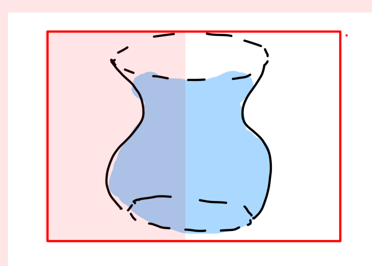
$$R_\theta = \begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}$$

which is the rotation by θ about the z-axis.
We need to show σ is a surface patch.

$$U := (a,b)\times(-\pi,\pi)\subset\mathbb{R}^2$$

$$V := \sigma(U) = S - \underbrace{\{(x,y,z)\mid y=0, x\le 0\}}_{\text{closed}} \subset S \text{ open}$$





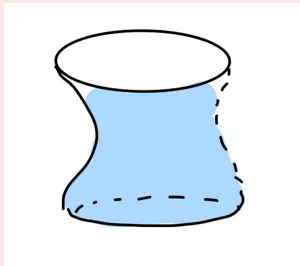
Enough to show $\sigma\colon U\to V$ is a diffeomorphism.
σ is a smooth bijection. As for σ^{-1}, the smoothness can be checked locally, for instance,

$$\{(x,y,z)\in V\mid x>0\},$$

then

$$\sigma^{-1}(x,y,z) = (t(z), \tan^{-1}(y/x)),$$

where $t(z)$ is the inverse of $z(t)$. Therefore σ is a surface patch. Similarly, we define $\tau\colon (a,b)\times(0,2\pi)\to S$ by $\tau(t,\theta) = (x(t)\cos\theta, x(t)\sin\theta, z(t))$. τ covers



Therefore, S is a regular surface covered by σ, τ.

18 Feb 14, 2022

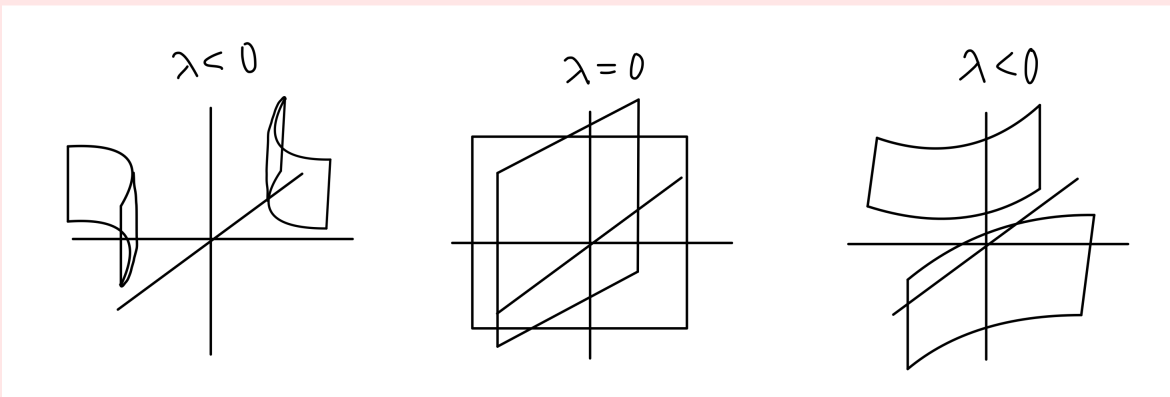
18.1 Regular Surfaces (Cont'd)

Recall 18.1 $f\colon U \subset \mathbb{R}^3 \to \mathbb{R}, \lambda \in \mathbb{R}$

$$f^{-1} = \{p \in U \mid f(p) = \lambda\}$$

Example 18.2
Suppose $f(x, y, z) = xy, \lambda \in \mathbb{R}$ Then,

$$f^{-1}(\lambda) = \{(x, y, z) \in \mathbb{R}^3 \mid xy = \lambda\} \subset \mathbb{R}^3$$



For which λ is $f^{-1}(\lambda)$ a regular surface?

Definition 18.3 (Regular value)
Let $f\colon U \subset \mathbb{R}^3 \to \mathbb{R}$ be smooth, $\lambda \in \mathbb{R}$. λ is called a regular value of f if

$$\forall p \in f^{-1}(\lambda), \quad df_p = \begin{pmatrix} \frac{\partial f}{\partial x}(p) & \frac{\partial f}{\partial y}(p) & \frac{\partial f}{\partial z}(p) \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$$

That is, one of

$$\frac{\partial f}{\partial x}(p), \frac{\partial f}{\partial y}(p), \frac{\partial f}{\partial z}(p) \neq 0$$

Theorem 18.4 (Regular Value Theorem)
Under the same setting, if λ is a regular value, then f^{-1} is a regular surface.

Remark 18.5 Theorem shows the existence of some surface patches that cover $f^{-1}(\lambda)$, but does not give an explicit construction of such patches.

Example 18.6
$f(x,y,z) = x^2 + y^2 - z^2 \quad f\colon \mathbb{R}^3 \to \mathbb{R}$ is smooth
Therefore,

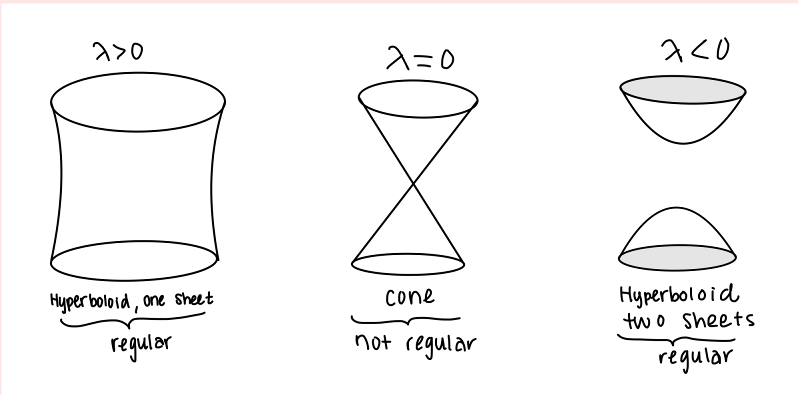
$$df_{(x,y,z)} = \begin{pmatrix} 2x & 2y & 2z \end{pmatrix},$$

so,

$$df_{(x,y,z)} = 0 \iff (x,y,z) = (0,0,0)$$

so,

$$f^{-1}(\lambda) = \{(x,y,z) \in \mathbb{R}^3 \mid \underbrace{x^2 + y^2 - z^2 = \lambda}_{x^2+y^2=z^2+\lambda}\}$$



Proof of Regular Value Theorem (Sketch). $f\colon U \subset \mathbb{R}^3 \to \mathbb{R}$ is smooth, $\lambda \in \mathbb{R}$ a regular value.
$S \coloneqq f^{-1}(\lambda), p \in S$.
We know

$$df_p = \begin{pmatrix} \frac{\partial f}{\partial x}(p) & \frac{\partial f}{\partial y}(p) & \frac{\partial f}{\partial z}(p) \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$$

We may assume

$$\frac{\partial f}{\partial z}(p) \neq 0.$$

Define

$$g\colon U \subset \mathbb{R}^3 \to \mathbb{R}^3 \text{ by } g(x,y,z) = \big(x, y, f(x,y,z)\big)$$

Then g is smooth, and

$$dg_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial f}{\partial x}(p) & \frac{\partial f}{\partial y}(p) & \frac{\partial f}{\partial z}(p) \end{pmatrix}$$

So $\det dg_p = \frac{\partial f}{\partial z}(p) \neq 0$. So dg_p is invertible.
By the inverse function theorem,

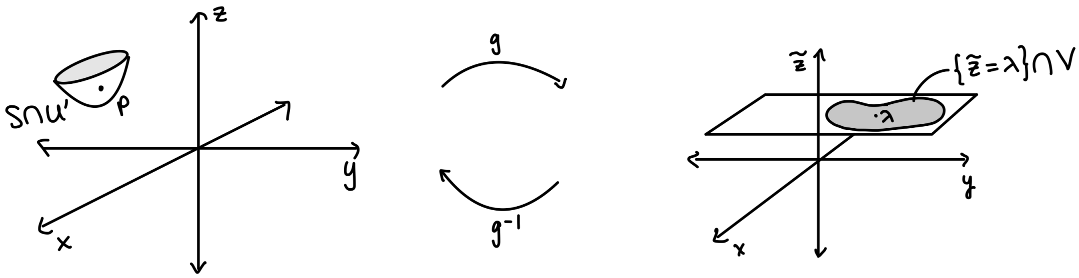
$$p \in \exists U' \subset U \text{ nbd}, \quad g(p) \in \exists V \subset \mathbb{R}^3 \text{ nbd}$$

such that

$$g\colon U' \to V \text{ is a diffeomorphism}$$

Then g restricts to a diffeomorphism:

$$S \cap U' = \{f(x, y, z) = \lambda\} \cap U' \longmapsto \{\tilde{z} = \lambda\} \cap V$$



$\{\tilde{z} = \lambda\} \cap V$ can be identified with an open subset of $\mathbb{R}^2$, and $y^{-1} \colon \{\tilde{z} = \lambda\} \cap V \to S \cap U'$ is a surface patch containing p. □

18.2 Parametrized Surfaces

Definition 18.7 (Parametrized surface)
A parametrized surface is a smooth function $f \colon U \subset \mathbb{R}^2 \to \mathbb{R}^3$ such that $\forall p \in U, \operatorname{rank} df_p = 2$.

Remarks 18.8

i. A parametrized surface is a function, but a regular surface is a set.
ii. A regular curve is a smooth function $\gamma \colon I \subset \mathbb{R} \to \mathbb{R}^n$ such that $\forall p \in I, \operatorname{rank} d\gamma_p = 1 \implies \operatorname{rank} \gamma'(p) \iff \gamma'(p) \neq 0$
iii. $\sigma(U)$ is not a regular surface in general, as it might have a self-intersection.

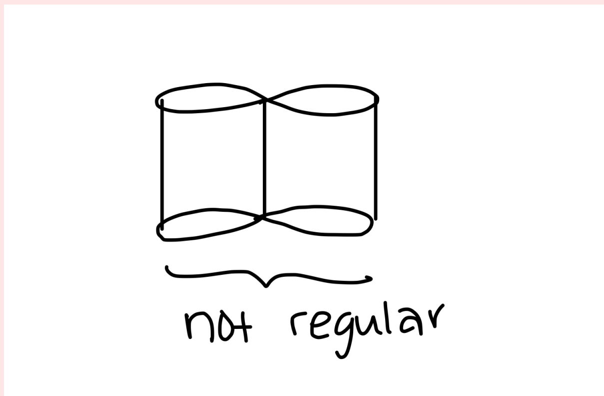
Example 18.9
$\sigma(x, y) = (\cos x, \sin 2x, y)$
Then $\sigma\colon \mathbb{R}^2 \to \mathbb{R}^3$ is smooth. So

$$d\sigma_{(x,y)} = \underbrace{\begin{pmatrix} -\sin x & 0 \\ 2\cos 2x & 0 \\ 0 & 1 \end{pmatrix}}_{\text{first column can never be 0}}$$

$$\implies \operatorname{rank} d\sigma_{(x,y)} = 2$$

$$\implies \sigma \text{ is a parametrized surface}$$

So, $\sigma(\mathbb{R}^2)$ is



which is not regular.

Lemma 18.10
A surface patch $\sigma\colon U \subset \mathbb{R}^2 \to V \subset S \subset \mathbb{R}^3$ is a parametrized surface.

Proof. Let $p \in U, q := \sigma(p)$.
σ^{-1} locally extends to a smooth function on $\mathbb{R}^3$.

$$I_2 = d(\sigma^{-1} \circ \sigma)_p = d\sigma_q^{-1} \circ d\sigma_p$$
$$\implies \ker(d\sigma_p) = 0$$
$$\underset{\text{rank-nullity}}{\implies} \operatorname{rank}(d\sigma_p) = 0$$

□

Roughly speaking, a surface patch is equivalent to the image of a parametrized surface that is one-to-one on the domain. Strictly speaking, this is not true. In addition, $\sigma\colon U \to \sigma(U)$ has to be a homeomorphism. Read §3.10 in Tapp for the details.

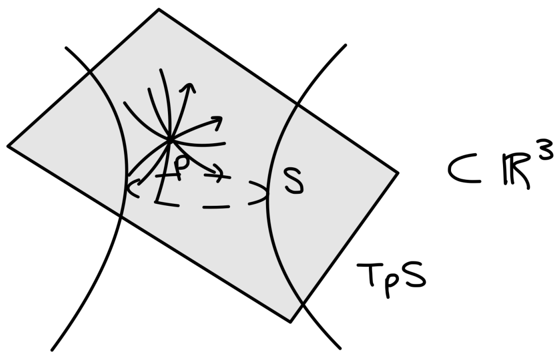
19 Feb 16, 2022

19.1 Tangent Planes

Definition 19.1 (Tangent space, tangent vector)
Let $S \subset \mathbb{R}^n$ be a subset, $p \in S$. The tangent space to S at p is defined as follows:

$$T_pS \coloneqq \{\gamma'(0) \mid \gamma\colon (-\varepsilon, \varepsilon) \to S \subset \mathbb{R}^n \text{ curve s.t. } \gamma(0) = p\}$$

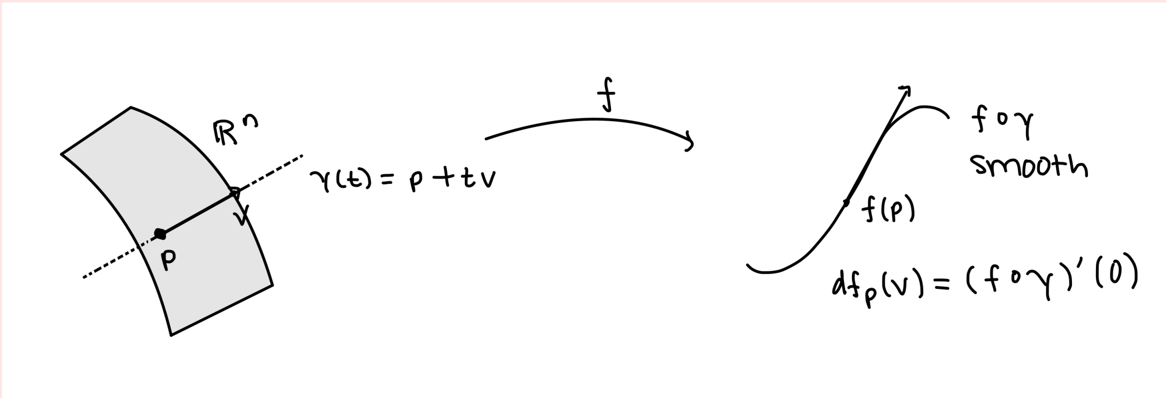
A vector in T_pS is called a tangent vector to S at p.



Remarks 19.2

i. $0 \in T_pS$, as one can consider $\gamma(t) = p$ (constant)
ii. Another possible definition of the tangent space is: $p + T_pS$ (tangent space placed at p).
iii. If $p \in U \subset S$ neighborhood, then $T_pU = T_pS$ as one can shrink the domain of γ.

Example 19.3
Let $p \in \mathbb{R}^n$. Then $T_p\mathbb{R}^n = \mathbb{R}^n$.



If $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is smooth, then $df_p\colon T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^m$.
If $p \in S \subset \mathbb{R}^n$, then $T_pS \subset T_p\mathbb{R}^n = \mathbb{R}^n$.

Proposition 19.4
Let $S \subset \mathbb{R}^3$ be a regular surface, $p \in S$. If

$$\sigma\colon U \subset \mathbb{R}^2 \to V \subset S$$
$$q \mapsto p$$

is a surface patch, then

$$T_pS = \operatorname{Im} d\sigma_p = \operatorname{span}(\sigma_x(p), \sigma_y(p)) \subset T_p\mathbb{R}^3$$

If $p \in S \subset \mathbb{R}^n$, then

$$T_pS \subset T_p\mathbb{R}^n = \mathbb{R}^n.$$

In particular, $T_pS \subset T_p\mathbb{R}^3 = \mathbb{R}^3$ is a 2-dimensional linear subspace.

Remark 19.5 $d\sigma_p\colon T_q\mathbb{R}^2 \to T_pS \subset T_p\mathbb{R}^3$ is a linear isomorphism.

Proof.

$$\begin{aligned} T_pS &= \{\gamma'(0) \mid \gamma \text{ is a curve on } V \text{ s.t. } \gamma(0) = p\} \\ &= \{(\sigma \circ B)'(0) \mid B \text{ curve on } U \text{ s.t } B(0) = q\} \\ &\to \sigma \text{ is a diffeomorphism, so it identifies regular curves} \\ &\quad \text{on } U \text{ and those on } V \\ &= \operatorname{Im} d\sigma_q \end{aligned}$$

□

Example 19.6
$S \coloneqq \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$
So,

$$p = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1\right) \in S$$

Compute T_pS.
Recall:

$$\sigma(\theta, t) = (\cos\theta, \sin\theta, t)$$

gives a surface patch for S.

$$\sigma\left(\frac{\pi}{4}, 1\right) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1\right) = p$$

Set $q = \left(\frac{\pi}{4}, 1\right)$. So,

$$\begin{aligned} d\sigma_q &= \begin{pmatrix} -\sin\theta & 0 \\ \cos\theta & 0 \\ 0 & 1 \end{pmatrix}_q \\ &= \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

$$\implies T_pS = \operatorname{Im} d\sigma_q = \operatorname{Span}\left(\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$$

Then,

$$n = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

So,

$$T_pS = \{(x, y, z) \in \mathbb{R}^3 \mid x + y = 0\}$$

Example 19.7
$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2\}$, $p = (0, 0, 0) \in S$.
Then, $T_pS = S$ (homework) is not a subspace. Hence, S is not a regular surface.

Definition 19.8 (Derivative (domain is a surface))
Let $S \subset \mathbb{R}^n$ be a subset, $f: S \to \mathbb{R}^m$ be smooth, $p \in S$. The derivative of f at p is defined as

$$df_p: T_pS \to T_{f(p)}\mathbb{R}^m, \quad v \mapsto (f \circ \gamma)'(0),$$

where γ is any curve on S such that $\gamma(0) = p, \gamma'(0) = v$.

Lemma 19.9
df_p is well-defined.

Proof. We need to check:

i. $f \circ \gamma(-\varepsilon, \varepsilon) \to \mathbb{R}^m$ is smooth

ii. $df_p(v)$ is independent of γ.

f locally extends to a smooth function $\tilde{f}$ on a neighborhood of p in $\mathbb{R}^n$.
i.) $f \circ \gamma = \tilde{f} \circ \gamma$ is the composition of smooth functions whose domains are open in some vector space, so smooth.
ii.) $df_p(v) = (f \circ \gamma)'(0) = (\tilde{f} \circ \gamma)'(0) = \tilde{f}' \circ \gamma(0) \cdot \gamma'(0) = \tilde{f}'(p) \cdot v$
$\implies$ This does not depend on γ. $\square$

20 Feb 18, 2022

20.1 Tangent Planes (Cont'd)

Recall 20.1 $S \subset \mathbb{R}^n$ a subset, $p \in S$,

$$T_pS = \{\gamma'(0) \mid \gamma \text{ is a curve on } S \text{ s.t. } \gamma(0) = p\}$$

$f\colon S \to \mathbb{R}^n$ smooth,

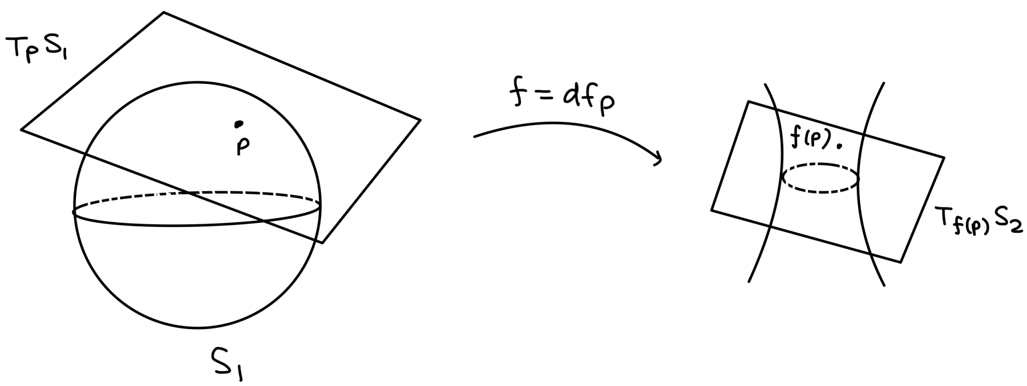
$$\begin{aligned} df_p\colon T_pS &\to T_{f(p)}\mathbb{R}^n \\ v = \gamma'(0) &\mapsto (f \circ \gamma)'(0) \end{aligned}$$

Proposition 20.2
Let $S \subset \mathbb{R}^3$ be a regular surface, $f\colon S \to \mathbb{R}^m$ be smooth, $p \in S$. Then $df_p\colon T_pS \to T_{f(p)}\mathbb{R}^m$ is a linear transformation.

Proof. f extends locally to a smooth function $\tilde{f}$ on a neighborhood of p in $\mathbb{R}^3$.
$\forall v \in T_pS$,

$$df + p(v) = (f \circ \gamma)'(0) = (\tilde{f} \circ \gamma)'(0) = d\tilde{f}_p(v)$$

$\implies$ df_p is the restriction of the linear transformation $d\tilde{f}_p\colon T_p\mathbb{R}^3 \to T_{f(p)}\mathbb{R}^m$ to the subspace $T_pS \subset T_p\mathbb{R}^3 = \mathbb{R}^3$. ∎



Theorem 20.3 (Inverse Function Theorem for Regular Surfaces)
Let $S_1, S_2 \subset \mathbb{R}^3$ be regular surfaces, $f\colon S_1 \to S_2$ be smooth. Suppose $\exists p \in S_1, df_p\colon T_pS_1 \to T_{f(p)}S_2$ is invertible, then f is locally a diffeomorphism at p:

$$p \in \exists V_1 \subset S_1 \text{ nbd}$$

$$f(p) \in \exists V_2 \subset S_2 \text{ nbd}$$

such that $f\colon V_1 \to V_2$ is a diffeomorphism.

Proof. We take surface patches at $p \in S_1, f(p) \in S_2$,

$$\sigma_1 \colon U_1 \to V_1, \quad \sigma_2 \colon U_2 \to V_2$$

We may assume $f(V_1) \subset V_2$ by shrinking U_1, V_1 if necessary.

$$\begin{tikzcd} V_1 \subset S_1 \arrow[r, "f"] & V_2 \subset S_2 \\ U_1 \subset \mathbb{R}^2 \arrow[u, "\sigma_1"] \arrow[r, "\psi"'] & U_2 \subset \mathbb{R}^2 \arrow[u, "\sigma_2"'] \end{tikzcd}$$

$$\text{Define } \psi \coloneqq \sigma_2^{-1} \circ f \circ \sigma_1$$

Then ψ is smooth, as $\sigma_1, f, \sigma_2^{-1}$ are all smooth. Set $q \coloneqq \sigma^{-1}(p) \in U_1$. Then

$$\begin{tikzcd} T_pS_1 \arrow[r, "df_p"] & T_{f(p)}S_2 \\ T_q\mathbb{R}^2 \arrow[u, "d(\sigma_1)_q"] \arrow[r, "d\psi_q"'] & T_{\psi(q)}\mathbb{R}^2 \arrow[u, "d(\sigma_2)_{\psi(q)}"'] \end{tikzcd} \text{ by the chain rule}$$

Here, $(d\sigma_1)_{q_1}, (d\sigma_2)_{\psi(q)}$ are invertible, and $(df)_p$ is invertible by assumption.

$$\implies d\psi_q \colon T_q\mathbb{R}^2 \to T_{\psi(q)}\mathbb{R}^2 \text{ is invertible}$$

and by Inverse Function theorem,

$$\implies \psi \colon U_1 \to U_2 \text{ is a diffeomorphism}$$

by shrinking U_1 if necessary.

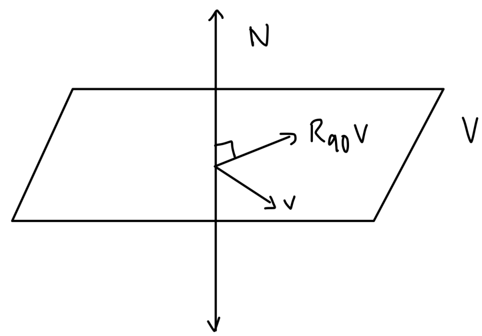
$$\implies f = \sigma_2 \circ \psi \circ (\sigma_1)^{-1}$$

is the composition of diffeomorphisms $\sigma_2, \psi, \sigma_1^{-1}$, hence f is a diffeomorphism. □

20.2 Orientable Surfaces

An orientation for a regular surface allows us to define the notions of "clockwise" and "counterclockwise" in a constant manner. First, we define an orientation for a 2-dimensional vector space $V \subset \mathbb{R}^3$.

Definition 20.4 (External definition for orientation)
An orientation for $\mathcal{V}$ is a choice of a unit normal vector N to $\mathcal{V}$.



Remarks 20.5

i. There are exactly 2 orientations for $\mathcal{V}$.

ii. There is no normal orientation for $\mathcal{V}$.

iii. Given an orientation N for $\mathcal{V}$, $\forall v \in \mathcal{V}$, $R_{90}v := N \times v$.

Definition 20.6 (Internal definition for orientation)
An equivalence relation on the set of ordered basis for $\mathcal{V}$ is defined as follows:

$$\{v_1, v_2\} \sim \{v_1', v_2'\}$$

$$\iff \begin{cases} v_1' = a_1v_1 + a_2v_2 \\ v_2' = b_1v_1 + b_2v_2 \end{cases} \quad \text{then} \;\; \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} > 0$$

An <u>orientation</u> for $\mathcal{V}$ is an equivalence class of ordered basis for $\mathcal{V}$.

Lemma 20.7
External def $\iff$ internal def
More precisely,

$$\{\text{eq. classes}\} \to \{\text{unit normals}\}$$
$$\{v_1, v_2\} \mapsto \frac{v_1 \times v_2}{|v_1 \times v_2|}$$

Proof. Let $\{v_1, v_2\}, \{v_1', v_2'\}$ be ordered basis for $\mathbb{R}^2$. Then, we can write

$$\begin{cases} v_1' = a_1v_1 + a_2v_2 \\ v_2' = b_1v_1 + b_2v_2 \end{cases}$$

Then,

$$\begin{aligned} v_1' \times v_2' &= (a_1v_1 + a_2v_2) \times (b_1v_1 + b_2v_2) \\ &= (a_1b_2 - a_2b_1)v_1 \times v_2 \\ &= \det(A)v_1 \times v_2 \end{aligned}$$

where $A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$. Then,

$$\frac{v_1' \times v_2'}{|v_1' \times v_2'|} = \begin{cases} \dfrac{v_1 \times v_2}{|v_1 \times v_2|} & \text{if } \det(A) > 0 \\ -\dfrac{v_1 \times v_2}{|v_1 \times v_2|} & \text{if } \det(a) < 0. \end{cases}$$

$\square$

21 Feb 23, 2022

21.1 Orientable Surfaces (Cont'd)

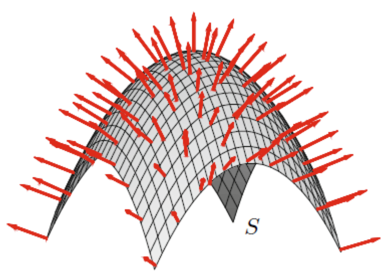
Recall 21.1 Suppose $V \subset \mathbb{R}^3$ is a 2-dimensional subspace.

i. orientation = the choice of a unit normal vector to V.

ii. Orientation = an equivalence class of $\{v_1, v_2\}$, an ordered basis for V, where $\{v_1, v_2\} \sim \{v_1', v_2'\} \iff$ determinant of the change of basis > 0

Last time: (i.) $\iff$ (ii.)

Definition 21.2 (Vector field on S, normal field on S)
Let $S \subset \mathbb{R}^3$ be a regular surface. A vector field on S is a smooth function $f\colon S \to \mathbb{R}^3$. A normal field on S is a vector field $N\colon S \to \mathbb{R}^3$ such that $\forall p \in S, N(p)$ is a unit normal vector to $T_pS \subset T_p\mathbb{R}^3 = \mathbb{R}^3$.



Definition 21.3 (Orientable, oriented surface)
A regular surface is called orientable if a normal field exists on it. An orientation for an orientable surface is the choice of a normal field in it. An oriented surface is an orientable surface with a given orientation.

Remark 21.4 (The trace of) a regular plane curve is always orientable by n_s:

Lemma 21.5
A regular surface is always locally orientable.

Proof. Let $\sigma\colon U \subset \mathbb{R}^2 \to V \subset S$ be a surface patch. We need to show V is orientable.
Take $p \in V, q = \sigma^{-1}(p) \in U$.
Then

$$d\sigma_q = \begin{pmatrix} \sigma_u(q) & \sigma_v(q) \end{pmatrix}, \text{ where } \{u, v\} \text{ is the coordinate system for } U \subset \mathbb{R}^2$$

And

$$N(p) := \frac{\sigma_u(q) \times \sigma_v(q)}{|\sigma_u(q) \times \sigma_v(q)|} \text{ is a normal field on } V$$

$\square$

Remark 21.6 A regular surface is orientable if one can extend a local orientation to a global one in a constant manner.

Example 21.7
Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be smooth,

$$G = \{(x, y, z) \in \mathbb{R}^3 \mid z = f(x, y)\}$$

G is a regular surface covered by a single surface patch:

$$\sigma\colon \mathbb{R}^2 \to G, \quad (x, y) \mapsto \big(x, y, f(x, y)\big)$$

Then G is orientable:

$$\begin{aligned} N = \frac{\sigma_x \times \sigma_y}{|\sigma_x \times \sigma_y|} &= \frac{(1, 0, f_x) \times (0, 1, f_y)}{|(1, 0, f_x) \times (0, 1, f_y)|} \\ &= \frac{(-f_x, -f_y, 1)}{\sqrt{f_x^2 + f_y^2 + 1}} \end{aligned}$$

Therefore, by N, G is orientable.

Example 21.8
Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be smooth, $\lambda \in \mathbb{R}$ be a regular value, $S = f^{-1}(\lambda)$.
We need to show S is orientable.

$$\forall p \in S, \quad df_p = \begin{pmatrix} \frac{\partial f}{\partial x}(p) & \frac{\partial f}{\partial y}(p) & \frac{\partial f}{\partial z}(p) \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$$

So,

$$\nabla f_p = df_p^T = \left(\frac{\partial f}{\partial x}(p), \frac{\partial f}{\partial y}(p), \frac{\partial f}{\partial z}(p) \right) \neq (0,0,0)$$

We have $f(x,y,z) = x^2 + y^2 + z^2$. So

$$\nabla f_p = (2x, 2y, 2z) = 2p$$

$$N(p) = \frac{\nabla f_p}{|\nabla f_p|} = \frac{p}{|p|}$$

We need to show $N(p) = \dfrac{\nabla f_p}{|\nabla f_p|}$ is an orientation for S. Let γ be a curve on S such that $\gamma(0) = p$.

$$\begin{aligned} \langle N(p), \gamma'(0) \rangle &= \left\langle \frac{\nabla f_p}{|\nabla f_p|}, \gamma'(0) \right\rangle \\ &= \frac{1}{|\nabla f_p|} f'(p)\gamma'(0) \\ &= \frac{1}{|\nabla f_p|} (f \circ \gamma)'(0) = 0 \end{aligned}$$

And notice $f \circ \gamma(p) = \lambda = \text{constant}$

Recall 21.9 $S \subset \mathbb{R}^n$ is <u>connected</u> $\iff$ IVT holds for any continuous function on $S \iff S$ is path-connected

Example 21.10

$$\left.\begin{array}{l} [a,b] \\ [a,b) \\ (a,b] \\ (a,b) \end{array}\right\} \text{connected}$$

Proposition 21.11
A connected regular surface $S \subset \mathbb{R}^3$ has exactly 2 orientations for S.

Proof. Choose an orientation $N\colon S \to \mathbb{R}^3$. Then $-N\colon S \to \mathbb{R}^3$ is also an orientation for S. We need to show if M is an orientation for S, then $M = N$ or $-N$.

Define $f\colon S \to \mathbb{R}$ by

$$f(p) = \langle M(p), N(p) \rangle = \pm 1$$

Then f is continuous on S, S is connected.

$$\underset{\text{IVT}}{\Longrightarrow} f \text{ should be constant}$$

$$\Longrightarrow M = \begin{cases} N & \text{if } \forall p, f(p) = 1 \\ -N & \text{if } \forall p, f(p) = -1 \end{cases}$$

□

Definition 21.12 (Orientation-preserving)
Let $(V, \{v_1, v_2\}), (w, \{w_1, w_2\})$ be oriented 2-dimensional vector spaces. A linear isomorphism $L\colon V \to W$ is called orientation-preserving if

$$\{L(v_1), L(v_2)\} \sim \{w_1, w_2\}$$

Remarks 21.13

i. This is well defined:

$$\{v_1, v_2\} \sim \{v_1', v_2'\} \implies \{L(v_1), L(v_2)\} \sim \{L(v_1'), L(v_2')\}$$

because L is linear, and the change-of-basis matrices are the same.

ii. A linear isomorphism L is either orientation-preserving or orientation-reserving.

22 Feb 25, 2022

22.1 Midterm 2

23 Feb 28, 2022

23.1 Orientable Surfaces (Cont'd)

Recall 23.1 V, W are 2-dimensional oriented vector spaces $\{v_1, v_2\}, \{w_1, w_2\}$ A linear isomorphism $L\colon V \to W$ is orientation-preserving $\iff \{L(v_1), L(v_2)\} \sim \{w_1, w_2\}$.

Example 23.2
Consider A: 2×2 invertible matrix

$$L_A\colon (\mathbb{R}^2, \{e_1, e_2\}) \to (\mathbb{R}^2, \{e_1, e_2\})$$

is orientation preserving

$$\begin{aligned} &\iff \{Ae_2, Ae_2\} \sim \{e_1, e_2\} \\ &\iff \det A > 0, \end{aligned}$$

Definition 23.3 (Orientation-preserving diffeomorphism)
Let $S_1, S_2 \subset \mathbb{R}^3$ be oriented regular surfaces. A diffeomorphism $f\colon S_1 \to S_2$ is called orientation preserving $\iff \forall p \in S_1, df_p\colon T_pS_1 \to T_{f(p)}S_2$ is orientation-preserving.

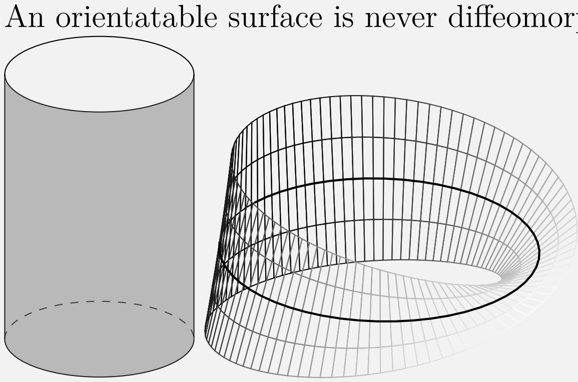
Proposition 23.4
Let $S_1, S_2 \subset \mathbb{R}^3$ be connected oriented surfaces, $f\colon S_1 \to S_2$ be a diffeomorphism. Then,

i. f is either orientation-preserving or orientation-reserving.
ii. f is orientation preserving $\iff \exists p \in S_1, df_p$ is orientation preserving.

(ii) follows from (i). For (i), we need the following:

Lemma 23.5
Let $S_1, S_2 \subset \mathbb{R}^3$ be regular surfaces, $f\colon S_1 \to S_2$ be a diffeomorphism. Then, an orientation on S_1 (if exists) induces an orientation for S_2.

Remark 23.6 This shows:
An orientatable surface is never diffeomorphic to a non-orientable one.



Proof of Lemma (Sketch). $\sigma_1 \colon \mathbb{R}^2 \to V_1 \subset S_1$ a surface patch, $V_2 \coloneqq f(V_1), \sigma_2 = f \circ \sigma_1$

$$\implies \sigma_2 \colon U \subset \mathbb{R}^2 \to V_2 \subset S_2 \text{ is a surface patch}$$

$$\begin{tikzcd}
U \subset \underset{p\in}{\mathbb{R}^2} \arrow[rr, "\sigma_1"] \arrow[drr, "\sigma_2"'] & & q_1 \in V_1 \subset S_1 \arrow[d, "f"] \\
& & q_2 \in V_2 \subset S_2
\end{tikzcd}$$

$$N_1(q_1) = \frac{\sigma_{1,x}(p) \times \sigma_{1,y}(p)}{|\sigma_{1,x}(p) \times \sigma_{2,y}(p)|}, \text{ for } V_1 \text{ induces}$$

$$N_2(q_2) = \frac{\sigma_{2,x}(p) \times \sigma_{2,y}(p)}{|\sigma_{2,x}(p) \times \sigma_{2,y}(p)|} \text{ for } V_2$$

An orientation for S_1 is equivalent to an info on how to patch $N_1(q)$ in a consistent manner. This allows us to patch $N_2(q_2)$ to obtain an orientation for S_2. $\square$

Proof of Proposition 23.4. (ii) $S_1, S_2 \subset \mathbb{R}^3$ connected oriented surfaces, $f \colon S_1 \to S_2$ is a diffeomorphism. f is orientation preserving $\iff$ the orientation for S_2 equals the orientation induced by S_1. Note that connected orientable surfaces have exactly 2 orientations. $\square$

Example 23.7

Consider $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ with the orientation $N(p) = p$.
Let $A \in \mathcal{O}(3), L_A \colon \mathbb{R}^3 \to \mathbb{R}^3$ (this is also a diffeomorphism). L_A restricts to a diffeomorphism $f \colon S^2 \to S^2$, because it preserves the length.
Fix $p \in S^2$. Enough to show df_p is orientation preserving $\iff \det(A) = 1$. Let $\{v_1, v_2\}$ be an ordered basis for T_pS^2 such that

$$\frac{v_2 \times v_2}{|v_1 \times v_2|} = N(p)$$

We need to show

$$\frac{df_p(v_1) \times df_p(v_2)}{|df_p(v_1) \times df_p(v_2)|} = N(f(p)) \iff \det(A) = 1$$

$$\text{L.H.S.} \iff \{df_p(v_1), df_p(v_2), \underbrace{N(f(p))}_{=f(p)=A\cdot N(p)}\}$$

$$\iff \{Av_1, Av_2, A \cdot N(p)\} \text{ is right-handed}$$

$$\iff \det(A) = 1$$

Remark 23.8 The Mobius strip is non-orientable.

23.2 Isometries and the First Fundamental Form

Consider the inner product, $\langle -, - \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ symmetric bilinear.
Consider the norm, "$\Longleftrightarrow$" $|\cdot|: \mathbb{R}^n \to \mathbb{R}$, the norm.
$|x| = \sqrt{\langle x, x \rangle}$
$\langle x, y \rangle = \frac{1}{2}(\langle x + y, x + y \rangle - \langle x, x \rangle - \langle y, y \rangle) = \frac{1}{2}(|x + y|^2 - |x|^2 - |y|^2)$

24 Mar 2, 2022

24.1 Isometries and the First Fundamental Form (Cont'd)

Notation: $S \subset \mathbb{R}^3$ regular surface, $p \in S$.

$$\langle -, - \rangle_p \colon T_pS \times T_pS \to \mathbb{R}, (x, y) \mapsto \langle x, y \rangle_{\mathbb{R}^2}$$

$$| \cdot |_p \colon T_pS \to \mathbb{R}, x \mapsto |x|_{p^3}$$

Definition 24.1 (First fundamental form)
The first fundamental form on a regular surface $S \subset \mathbb{R}^3$ is the assignment to each $p \in S$,

$$\text{the inner product } \langle -, - \rangle_p \colon T_pS \times T_pS \to \mathbb{R}$$

$$\text{or the norm } | \cdot |_p \colon T_pS \to \mathbb{R}$$

Example 24.2
Let $S \subset \mathbb{R}^3$ be a regular surface.

1. $\gamma \colon [a, b] \to S$ regular curve on S, then

$$\text{Arc length } = \int_a^b |\gamma'(t)| dt$$

2. $\gamma_1, \gamma_2 \colon (-\epsilon, \epsilon) \to S$ regular curves on S intersection at $p = \gamma_1(0) = \gamma_2(0)$.

Definition 24.3 (Isometry)
A diffeomorphism $f \colon S_1 \to S_2$ between regular surfaces is called an isometry if df preserves the first fundamental form

$$\forall p \in S_1, \forall x, y \in T_pS_1, \langle x, y \rangle_p = \langle df_p(x), df_p(y) \rangle_{f(p)}$$

Example 24.4
Suppose $S \coloneqq \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$

$$\sigma\colon (-\pi, \pi) \times \mathbb{R} \to S - \{(x, y, z) \in \mathbb{R}^3 \mid x \leq 0\}$$

$$(\theta, t) \mapsto (\cos\theta, \sin\theta, t)$$

is a surface patch.
Show σ is an isometry.

$$p = (\theta, t) \in (-\pi, \pi) \times \mathbb{R},$$

$$d\sigma_p = \underbrace{\begin{pmatrix} -\sin\theta & 0 \\ \cos\theta & 0 \\ 0 & 1 \end{pmatrix}}_{\text{orthonormal}} = M$$

Which is orthonormal

$$\implies M^T M = \begin{pmatrix} \langle v_1, v_1\rangle_{\mathbb{R}^3} & \langle v_1, v_2\rangle_{\mathbb{R}^3} \\ \langle v_2, v_1\rangle_{\mathbb{R}^3} & \langle v_2, v_2\rangle_{\mathbb{R}^3} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

$\forall x, y \in T_pS$, $\langle Mx, My\rangle_{\mathbb{R}^3}$

$$\begin{aligned} \langle d\sigma_p(x), d\sigma_p(y)\rangle_{f(p)} &= (Mx)^T My \\ &= x^T M^T My = x^T y \\ &= \langle x, y\rangle_p \end{aligned}$$

Lemma 24.5
Let $S \subset \mathbb{R}^3$ be a regular surface, $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a rigid motion. Then $S' = f(S) \subset \mathbb{R}^3$ is a regular surface and $f\colon S \to S'$ is an isometry.

Proof (Sketch). S' is a regular surface, because the composition of f and a surface patch for S is a surface patch for S'.
If $f = T_p \circ L_A$, where $p_0 \in \mathbb{R}^3, A \in O(3)$ then $\forall p, df_p = L_A$ preserves the lengths, because $A \in O(3)$. □

Example 24.6
$S \subset \mathbb{R}^3$ regular surface,

$$f\colon \mathbb{R}^3 \to \mathbb{R}^3, \quad x \mapsto 2x,$$

$$S' = f(S) \subset \mathbb{R}^3$$

Then $f\colon S \to S'$ is not an isometry.

$$|df_p(x)|_{f(p)} = 2|x|_p$$

Definition 24.7 (Intrinsic)
A measurement for regular surfaces is called intrinsic if it is preserved by isometries.

Remark 24.8 An intrinsic measurement comes from the surface only, and ignores the container $\mathbb{R}^3$.

Proposition 24.9
On a regular surface,

i. The arc length of a regular curve;
ii. The angle between two regular curves of an intersection point

are both intrinsic.

Proof.

i. $S \subset \mathbb{R}^3$ be a regular surface. $\gamma\colon [a,b] \to S$ be a regular curve on S.
We want to show if $S \to S'$ is an isometry, where $S' \subset \mathbb{R}^3$ is a regular surface,

$$\text{Arc length of } \gamma = \text{Arc length of } f \circ \gamma$$

So,

$$|(f\circ\gamma)'(t)|_{f(\gamma(t))} = |df_{\gamma(t)}(\gamma'(t))|_{f(\gamma(t))} = |\gamma'(t)|_{\gamma(t)}$$

So,

$$\text{Arc length of } f\circ\gamma = \int_a^b |(f\circ\gamma)'(t)|_{f(\gamma(t))}dt = \int_a^b |\gamma'(t)|_{\gamma(t)}dt = \text{ Arc length of } \gamma$$

ii. Similar.

$\square$

Example 24.10
For a regular surface $S \subset \mathbb{R}^3$, S itself, the set of points on S, is extrinsic.

Example 24.11
The tangent planes are also extrinsic.

25 Mar 4, 2022

25.1 The First Fundamental Form in Local Coordinates

Remark 25.1 The correct definitions for a regular surface, the tangent plane, the first fundamental form should be intrinsic. But our correct definitions are not.

Definition 25.2 (Symmetric bilinear form)
Let V be a vector space. A symmetric bilinear form is a pairing

$$\langle , \rangle : V \times V \to \mathbb{R}$$

such that

i. $\forall x, y, z \in V$,

$$\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$$

$$\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$$

ii. $\forall x, y \in V, \forall c \in \mathbb{R}$

$$\langle cx, y\rangle = \langle x, cy\rangle = c\,\langle x, y\rangle$$

iii. $\forall x, y \in V$,

$$\langle x, y\rangle = \langle y, x\rangle$$

Definition 25.3 (Quadratic form)
The quadratic form associated with $\langle , \rangle$ is defined as $q(x) = \langle x, x\rangle$.

Remark 25.4 $\langle x, y\rangle = \dfrac{1}{2}(q(x + y) - q(x) - q(y))$

Example 25.5

$$\langle , \rangle_{\mathbb{R}^n} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R},$$

$$| \cdot |^2_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}$$

Example 25.6

$$\begin{aligned}\langle,\rangle : \mathbb{R}^2 \times \mathbb{R}^2 &\to \mathbb{R}\\ (x,y) &\mapsto x_1y_1 + 2x_1y_2 + 2x_2y_1 + 3x_2y_2\\ &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\\ q\colon \mathbb{R}^2 &\to \mathbb{R}\\ x &\mapsto x_1^2 + 4x_1x_2 + 3x_2^2\end{aligned}$$

Example 25.7
$S \subset \mathbb{R}^3$ regular surface, $p \in S$

$$\begin{aligned}\langle,\rangle : T_pS \times T_pS &\to \mathbb{R}\\ |\cdot|_p^2 \colon T_pS &\to \mathbb{R}\end{aligned}$$

Definition 25.8 (Symmetric matrix)
Let V be an n-dimensional vector space with symmetric bilinear form $\langle,\rangle$, $\{v_1, v_2, \dots, v_n\}$ be a basis for V.
The symmetric matrix

$$\begin{pmatrix} \langle v_1, v_1\rangle & \cdots & \langle v_1, v_n\rangle \\ \vdots & \ddots & \vdots \\ \langle v_n, v_1\rangle & \cdots & \langle v_n, v_n\rangle \end{pmatrix}$$

the matrix representation of $\langle,\rangle$ with respect to $\{v_1, v_2, \dots, v_n\}$.

Example 25.9
$n = 2$, $x, y \in V$. Write

$$x = x_1v_1 + x_2v_2, \quad y = y_1v_1 + y_2v_2$$

$$\begin{aligned}\langle x, y\rangle &= \langle x_1v_1 + x_2v_2, y_1v_1 + y_2v_2\rangle\\ &= x_1y_1\langle v_1, v_2\rangle + (x_1y_1 + x_1y_1)\langle v_1, v_2\rangle + x_2y_2\langle v_2, v_2\rangle\\ &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} \langle v_1, v_1\rangle & \langle v_1, v_2\rangle \\ \langle v_2, v_1\rangle & \langle v_2, v_2\rangle \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\end{aligned}$$

$$q(x) = x_1^2\langle v_1, v_1\rangle + 2x_1x_2\langle v_1, v_2\rangle + x_2^2\langle v_1, v_2\rangle$$

Goal: Pull back the first fundamental form via a surface patch, and describe the matrix representation.

Let $S \subset \mathbb{R}^3$ be a regular surface,

$$\sigma\colon U \subset \mathbb{R}^2 \to V \subset S$$

be a surface patch. Fix $p \in V, q = \sigma(q) \in U$, $\{\sigma_u(q), \sigma_v(q)\}$ is a basis for T_pS, and

$$\begin{pmatrix} |\sigma_u(q)|_p^2 & \langle \sigma_u(q), \sigma_v(q)\rangle_p \\ \langle \sigma_v(q), \sigma_u(q)\rangle_p & |\sigma_v(q)|_p^2 \end{pmatrix} \text{ represented } \langle , \rangle_p$$

$$\begin{cases} E(q) = \langle \sigma_u(q), \sigma_u(q)\rangle_p = |\sigma_u(q)|_p^2 \\ F(q) = \langle \sigma_u(q), \sigma_v(q)\rangle_p \\ G(q) = \langle \sigma_v(q), \sigma_v(q)\rangle = |\sigma(v)|_p^2 \end{cases}$$

$x = (a, b) \in T_pU = \mathbb{R}^2$

$$\begin{aligned} |d\sigma_q(x)|_p^2 &= |a\sigma_u(q) + b\sigma_v(q)|_p^2 \\ &= a^2E(q) + 2abF(q) + b^2G(q) \\ &= E(q)(du_q(x))^2 + 2F(q)du_q(x)dv_q(x) + G(q)dv_q(x) \end{aligned}$$

where

$$\begin{aligned} u &\colon \mathbb{R}^2 \to \mathbb{R}, (u, v) \mapsto u \\ v &\colon \mathbb{R}^2 \to \mathbb{R}, (u, v) \mapsto v \end{aligned}$$

Definition 25.10 (First fundamental form with respect to σ)
The first fundamental form with respect to σ is

$$\mathcal{F}_1 = E(du)^2 + 2Fdudv + G(dv)^2$$

Remark 25.11 $\mathcal{F}_1$ is the pull back of the first fundamental form via σ:

$$\begin{aligned} q \in U &\mapsto (\mathcal{F}_1)_q \\ T_qU &\to \mathbb{R} \\ x &\mapsto |d\sigma_q(x)|_{\sigma(q)}^2 \end{aligned}$$

Example 25.12

Consider the spherical coordinates for S^2

$$\begin{aligned} \sigma \colon (0, 2\pi) \times (0, \pi) &\to S^2 \\ (\theta, \phi) &\mapsto \cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi \end{aligned}$$

Describe $\mathcal{F}_1$ with respect to the surface patch.

$$d\sigma = \begin{pmatrix} -\sin\theta\cos\phi & \cos\theta\cos\phi \\ \cos\theta\cos\phi & \sin\theta\cos\phi \\ 0 & -\sin\phi \end{pmatrix}$$

$$\begin{cases} E = |\sigma_u|^2 = \sin^2\phi \\ F = \langle \sigma_u, \sigma_v \rangle = 0 \\ G = |\sigma_v|^2 = 1 \end{cases} \implies \mathcal{F}_1 = \sin\phi (d\theta)^2 + (d\phi)^2$$

Lemma 25.13

Let $S \subset \mathbb{R}^3$ be a regular surface, $\sigma \colon U \subset \mathbb{R}^2 \to V \subset S$ be a surface patch.
Let $\sigma \colon [a, b] \to U$ be a regular curve on U.

$$\begin{tikzcd} {[a,b]} \arrow[r, "\gamma"] \arrow[dr, "\sigma\circ\gamma"'] & U \subset \mathbb{R}^2 \arrow[d, "\sigma"] \\ & V \subset S \end{tikzcd}$$

Then the arc length of $\sigma \circ \gamma$ is equal to

$$\int_a^b \sqrt{\mathcal{F}_1} = \int_a^b \sqrt{(\mathcal{F}_1)_{\gamma(t)}(\gamma'(t))}$$

Proof.

$$|(\sigma \circ \gamma)'(t)| = |d\sigma(\gamma'(t))| = \sqrt{(\mathcal{F}_1)_{\gamma(t)}(\gamma'(t))}$$

□