# LU Factorization

- To further improve the efficiency of solving linear systems
- Factorizations of matrix $\mathbf{A}$ : $LU$ and $QR$
- LU Factorization Methods:
  - Using basic Gaussian Elimination (GE)
  - Factorization of Tridiagonal Matrix
  - Using Gaussian Elimination with pivoting
  - Direct LU Factorization
  - Factorizing Symmetrix Matrices (Cholesky Decomposition)
- Applications
- Analysis



# LU Decomposition

- A matrix $A$ can be decomposed into a lower triangular matrix $L$ and upper triangular matrix $U$ so that

$$\mathbf{A} = \mathbf{LU}$$

- $LU$ decomposition is performed once; can be used to solve multiple right hand sides.
- Similar to Gaussian elimination, care must be taken to avoid roundoff errors (partial or full pivoting)
- Special Cases: Banded matrices, Symmetric matrices.



# LU Decomposition: Motivation

*Flow around left loop*

$$20(i_1 - i_2) + 10(i_1 - i_3) = V_1$$

*Flow around upper right loop*

$$25i_2 + 10(i_2 - i_3) + 20(i_2 - i_1) = V_2$$

*Flow around lower right loop*

$$30i_3 + 10(i_3 - i_2) + 10(i_3 - i_1) = V_3$$

In Figure 3.1 we had $V_1 = 0$, $V_2 = 0$, and $V_3 = 200$.

The equations simplify to

$$\begin{aligned} 30i_1 - 20i_2 - 10i_3 &= V_1 \\ -20i_1 + 55i_2 - 10i_3 &= V_2 \\ -10i_1 - 10i_2 + 50i_3 &= V_3 \end{aligned}$$

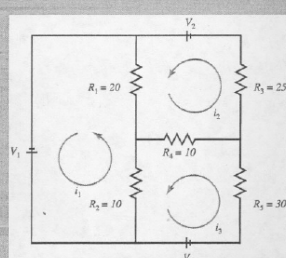


FIGURE 4.1 Simple electrical circuit.



# LU Decomposition

Forward pass of Gaussian elimination results in the upper triangular matrix

$$U = \begin{bmatrix} 1 & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & 1 & u_{23} & \cdots & u_{2n} \\ 0 & 0 & 1 & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

We can determine a matrix $L$ such that

$$LU = A \text{ , and}$$

$$L = \begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & 1 \end{bmatrix}$$

# LU Decomposition via Basic Gaussian Elimination

$$\mathbf{A} = \begin{bmatrix} 4 & 12 & 8 & 4 \\ 1 & 7 & 18 & 9 \\ 2 & 9 & 20 & 20 \\ 3 & 11 & 15 & 14 \end{bmatrix}.$$

Initialize

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{U} = \begin{bmatrix} 4 & 12 & 8 & 4 \\ 1 & 7 & 18 & 9 \\ 2 & 9 & 20 & 20 \\ 3 & 11 & 15 & 14 \end{bmatrix}$$

Step 1: $m(2,1) = -u(2,1)/u(1,1) = -1/4$; $m(3,1) = -u(3,1)/u(1,1) = -2/$
$m(4,1) = -u(4,1)/u(1,1) = -3/4$;

Now

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 1/2 & 0 & 1 & 0 \\ 3/4 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{U} = \begin{bmatrix} 4 & 12 & 8 & 4 \\ 0 & 4 & 16 & 8 \\ 0 & 3 & 16 & 18 \\ 0 & 2 & 9 & 11 \end{bmatrix}$$

Step 2: $m(3,2) = -u(3,2)/u(2,2) = -3/4$; $m(4,2) = -u(4,2)/u(2,2) = -2/$

Now

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 1/2 & 3/4 & 1 & 0 \\ 3/4 & 1/2 & 0 & 1 \end{bmatrix} \quad \mathbf{U} = \begin{bmatrix} 4 & 12 & 8 & 4 \\ 0 & 4 & 16 & 8 \\ 0 & 0 & 4 & 12 \\ 0 & 2 & 1 & 7 \end{bmatrix}$$

Step 3: $m(4,3) = -u(4,3)/u(3,3) = -1/4$;

Now

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 1/2 & 3/4 & 1 & 0 \\ 3/4 & 1/2 & 1/4 & 1 \end{bmatrix} \quad \mathbf{U} = \begin{bmatrix} 4 & 12 & 8 & 4 \\ 0 & 4 & 16 & 8 \\ 0 & 0 & 4 & 12 \\ 0 & 2 & 0 & 4 \end{bmatrix}$$

Multiply $\mathbf{L}$ by $\mathbf{U}$ to verify the result:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 1/2 & 3/4 & 1 & 0 \\ 3/4 & 1/2 & 1/4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & 12 & 8 & 4 \\ 0 & 4 & 16 & 8 \\ 0 & 0 & 4 & 12 \\ 0 & 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 12 & 8 & 4 \\ 1 & 7 & 18 & 9 \\ 2 & 9 & 20 & 20 \\ 3 & 11 & 15 & 14 \end{bmatrix}$$



# LU Decomposition via Basic Gaussian Elimination: Algorithm

```
Input
    A                                n-by-n matrix to be factored
    n                                dimension of A
Initialize
    L = I                            n-by-n identity matrix
    U = A
Compute
For k = 1 to n-1
    For i = k + 1 to n               each row of matrix U after the k^th row
        m(i,k) = -U(i, k)/U(k,k)
        For j = k to n               transform row i
            U(i, j) = U(i, j) + m(i,k)*U(k, j)
        End
        L(i,k) = -m(i,k)             update L matrix
    End
End
Return
    L                                lower triangular matrix
    U                                upper triangular matrix
```



# Banded Matrices

- Matrices that have non-zero elements close to the main diagonal
- Example: matrix with a bandwidth of 3 (or half band of 1)

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} & 0 \\ 0 & 0 & a_{43} & a_{44} & a_{45} \\ 0 & 0 & 0 & a_{54} & a_{55} \end{bmatrix}$$

- Efficiency: Reduced pivoting needed, as elements below bands are zero.
- Iterative techniques are generally more efficient for sparse matrices, such as banded systems.



# LU Decomposition of Tridiagonal Matrix

$$\mathbf{M} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix},$$

which can be represented by the vectors

$$\mathbf{d} = [2, 2, 2, 2];\ \mathbf{a} = [-1, -1, -1, 0];\ \mathbf{b} = [0, -1, -1, -1],$$

For this example, $n = 4$.

*Begin computation*

$$dd_1 = d_1 = 2$$

For $i = 2$

$$bb_2 = b_2/dd_1 = -1/2$$
$$dd_2 = d_2 - bb_2 a_1 = 2 - (-1/2)(-1) = 3/2$$

For $i = 3$

$$bb_3 = b_3/dd_2 = -1/(3/2) = -2/3$$
$$dd_3 = d_3 - bb_3 a_2 = 2 - (-2/3)(-1) = 4/3$$

For $i = 4$

$$bb_4 = b_4/dd_3 = -1/(4/3) = -3/4$$
$$dd_4 = d_4 - bb_4 a_3 = 2 - (-3/4)(-1) = 5/4$$

In general, the factorization is

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ bb_2 & 1 & 0 & 0 \\ 0 & bb_3 & 1 & 0 \\ 0 & 0 & bb_4 & 1 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} dd_1 & a_1 & 0 & 0 \\ 0 & dd_2 & a_2 & 0 \\ 0 & 0 & dd_3 & a_3 \\ 0 & 0 & 0 & dd_4 \end{bmatrix},$$

which, for this example, is

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}.$$

# LU Decomposition of Tridiagonal Matrix: Algorithm

*Input*

| | |
|---|---|
| a | *upper diagonal of M (matrix to be factored) with $a_n = 0$* |
| d | *diagonal of matrix M* |
| b | *lower diagonal of matrix M, with $b_1 = 0$* |
| n | *number of components in a, d, and b* |

*Initialize*

$bb_1 = 0$

$dd_1 = d_1$

*Compute*

For i = 2 to n

$$bb_i = \frac{b_i}{dd_{i-1}}$$

$$dd_i = d_i - bb_i a_{i-1}$$

End

*Return*

| | |
|---|---|
| bb | *lower diagonal of L (main diagonal is [1, 1, ..., 1])* |
| dd | *main diagonal of U (upper diagonal is a)* |



# LU Decomposition via GE with Pivoting

**Example 4.3 LU Factorization with Pivoting**

Consider the matrix **A** introduced in Example 3.6

$$\mathbf{A} = \begin{bmatrix} 2 & 6 & 10 \\ 1 & 3 & 3 \\ 3 & 14 & 28 \end{bmatrix}$$

*Initialize*

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{U} = \begin{bmatrix} 2 & 6 & 10 \\ 1 & 3 & 3 \\ 3 & 14 & 28 \end{bmatrix} \quad \mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$k = 1$

Interchange rows 1 and 3 in matrices **U** and **P**

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{U} = \begin{bmatrix} 3 & 14 & 28 \\ 1 & 3 & 3 \\ 2 & 6 & 10 \end{bmatrix} \quad \mathbf{P} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The first stage of elimination gives

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 2/3 & 0 & 1 \end{bmatrix} \quad \mathbf{U} = \begin{bmatrix} 3 & 14 & 28 \\ 0 & -5/3 & -19/3 \\ 0 & -10/3 & -26/3 \end{bmatrix} \quad \mathbf{P} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$k = 2$

Interchange rows 2 and 3 in matrices **U** and **P**, also rows 2 and 3, column 1 in **L**

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & 0 & 1 \end{bmatrix} \quad \mathbf{U} = \begin{bmatrix} 3 & 14 & 28 \\ 0 & -10/3 & -26/3 \\ 0 & -5/3 & -19/3 \end{bmatrix} \quad \mathbf{P} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The second stage of elimination gives

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & 1/2 & 1 \end{bmatrix} \quad \mathbf{U} = \begin{bmatrix} 3 & 14 & 28 \\ 0 & -10/3 & -26/3 \\ 0 & 0 & -2 \end{bmatrix} \quad \mathbf{P} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Verify result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & 1/2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 14 & 28 \\ 0 & -10/3 & -26/3 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 14 & 28 \\ 2 & 6 & 10 \\ 1 & 3 & 3 \end{bmatrix}$$



# LU Decomposition via GE with Pivoting: Algorithm

```
Input
   A                              matrix to be factored (n-by-n)
Initialize
   U = A
   L = I                          n-by-n identity matrix
   P = I
Compute
For k = 1 to n-1
   pivot = |U(k,k)|               pivot element
   p = k                          pivot row
   For i = k + 1 to n
      If (|U(i,k)| > pivot)
            pivot = |U(i,k)|      update pivot element
            p = i                 update pivot row
      End
   End
   If (p > k)
      t1 = u(k, :)                interchange rows k and p of matrix U
      U(k, :) = U(p, :)
      U(p, :) = t1
      t2 = P(k, :)                interchange rows k and p of matrix P
      P(k, :) = P(p, :)
      P(p, :) = t2
      For j = 1 to k-1            (if k > 1)
         t3 = L(k, j)             interchange columns 1 . . . k - 1 of rows k and p
         L(k, j) = L(p,j)         in matrix L
         L(p, j) = t3
      End
   End
   For i = k + 1 to n
      s = -U(i, k)/U(k,k)
      U(i, :) = U(i, :) + s*U(k, :)   update row i in U
      L(i, k) = -s
   End
End
Return
   L                              lower triangular matrix
   U                              upper triangular matrix
```



# Direct LU Decomposition

$$\begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & 1 & u_{23} & \cdots & u_{2n} \\ 0 & 0 & 1 & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

The above equation shows the coefficient matrix $A$ can be decomposed into $L$ and $U$; $L$ and $U$ can be determined by the computing the product and equating like terms.

### Determining $L$ and $U$

$$
\begin{aligned}
l_{i1} &= a_{i1}, i = 1, 2, \ldots, n \\
u_{1j} &= \frac{a_{1j}}{l_{11}}, j = 2, 3, \ldots, n \\
l_{ij} &= a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj}, \\
&\quad \text{for } j = 2, 3, \ldots, n-1 \text{ and } i = j, j+1, \ldots, n \\
u_{ji} &= \frac{a_{ji} - \sum_{k=1}^{j-1} l_{jk} u_{ki}}{l_{jj}}, \\
&\quad \text{for } j = 2, 3, \ldots, n-1 \text{ and } i = j+1, j+2, \ldots, n \\
l_{nn} &= a_{nn} - \sum_{k=1}^{n-1} l_{nk} u_{kn}
\end{aligned}
$$



# Direct LU Decomposition: Example

**Example 4.4 Doolittle Form of LU Factorization**

To find the LU factorization for $\mathbf{A} = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 20 & 32 \\ 5 & 32 & 64 \end{bmatrix}$

by Doolittle's method, we write the desired product as

$$\begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 20 & 32 \\ 5 & 32 & 64 \end{bmatrix}.$$

We begin by solving for the first row of **U** and the first column of **L**:

$$\begin{aligned} &(1)\, u_{11} = a_{11} = 1, \quad (1)\, u_{12} = a_{12} = 4, \quad (1)\, u_{13} = a_{13} = 5; \\ &\ell_{21}\, u_{11} = a_{21} = 4, \quad \ell_{31}\, u_{11} = a_{31} = 5. \end{aligned}$$

Next, using these values, we find the second row of **U** and the second column of **L**:

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & \ell_{32} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 & 5 \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 20 & 32 \\ 5 & 32 & 64 \end{bmatrix},$$

$$(4)(4) + u_{22} = 20 \quad \Rightarrow \quad u_{22} = 20 - 16 = 4;$$

$$\begin{aligned} (4)(5) + u_{23} = 32 \quad &\Rightarrow \quad u_{23} = 32 - 20 = 12; \\ (5)(4) + \ell_{32}\, u_{22} = 32 \quad &\Rightarrow \quad \ell_{32} = (32 - 20)/4 = 3. \end{aligned}$$

Finally, the only remaining unknown in matrix **U** is determined:

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 & 5 \\ 0 & 4 & 12 \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 20 & 32 \\ 5 & 32 & 64 \end{bmatrix},$$

$$(5)(5) + (3)(12) + u_{33} = 64 \quad \Rightarrow \quad u_{33} = 64 - 25 - 36 = 3.$$

The factorization is

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & 3 & 1 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 4 & 12 \\ 0 & 0 & 3 \end{bmatrix}.$$



# Direct LU Decomposition: Algorithm

```
Input
    A           n-by-n matrix to be factored
    n           dimension of A
Initialize
    U = 0(n)    initialize U to n-by-n zero matrix
    L = I(n)    initialize L to identity matrix
Compute
For k = 1 to n
    U(k, k) = A(k, k) - L(k, 1:k - 1)*U(1:k - 1, k)
    For j = k + 1 to n
            U(k, j) = A(k, j) - L(k, 1:k - 1)*U(1:k - 1), j)
            L(j, k) = (A(j, k) - L(j, 1:k - 1)*U(1:k - 1, k))/U(k, k)
    End
End
```



# Symmetric Matrices

- Symmetric square matrices - common in engineering, for example stiffness matrix (stiffness properties of structures).
- For a symmetric matrix $A$

  $$A = A^T$$

  where $A^T$ is the transpose of $A$. Hence
- For a symmetric matrix $A$

  $$A = LL^T$$

# Symmetric Matrices: LU Decomposition

- We can use Cholesky Decomposition to compute the LU decomposition of $A$

$$
\begin{aligned}
l_{11} &= \sqrt{a_{11}} \\
l_{ii} &= \sqrt{a_{ii} - \sum_{k=1}^{i-1} l_{ik}^2}, \text{ for } i = 2, 3, \ldots, n \\
l_{ij} &= \frac{a_{ij} - \sum_{k=1}^{j-1} l_{ik} l_{jk}}{l_{jj}}, \text{ for } j = 2, 3, \ldots, i-1, \text{ and } j < 1
\end{aligned}
$$



# LU (Cholesky) Decomposition of Symmetric Matrices

**Example 4.5 Cholesky LU Factorization**

The Cholesky form of LU factorization for $\mathbf{A} = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 20 & 32 \\ 5 & 32 & 64 \end{bmatrix}$ requires the diagonal of the $\mathbf{L}$ and $\mathbf{U}$ matrices to be equal, so we write the desired product as

$$\begin{bmatrix} x_{11} & 0 & 0 \\ \ell_{21} & x_{22} & 0 \\ \ell_{31} & \ell_{32} & x_{33} \end{bmatrix} \cdot \begin{bmatrix} x_{11} & u_{12} & u_{13} \\ 0 & x_{22} & u_{23} \\ 0 & 0 & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 20 & 32 \\ 5 & 32 & 64 \end{bmatrix}$$

and proceed to solve for the unknowns in a systematic manner. The first stage of calculations gives

$$
\begin{aligned}
x_{11}\, x_{11} &= a_{11} = 1 &\Rightarrow\quad& x_{11} = 1; \\
x_{11}\, u_{12} &= a_{12} = 4 &\Rightarrow\quad& u_{12} = 4/1 = 4; \\
x_{11}\, u_{13} &= a_{13} = 5 &\Rightarrow\quad& u_{13} = 5/1 = 5; \\
\ell_{21}\, x_{11} &= a_{21} = 4 &\Rightarrow\quad& \ell_{21} = 4/1 = 4; \\
\ell_{31}\, x_{11} &= a_{31} = 5 &\Rightarrow\quad& \ell_{31} = 5/1 = 5.
\end{aligned}
$$

Note that if $\mathbf{A}$ is symmetric, $a_{1j} = a_{j1}$, and it is automatic that $\ell_{j1} = u_{1j}$.

Next, from the values computed in the first stage, the product is

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & x_{22} & 0 \\ 5 & \ell_{32} & x_{33} \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 & 5 \\ 0 & x_{22} & u_{23} \\ 0 & 0 & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 20 & 32 \\ 5 & 32 & 64 \end{bmatrix}$$

we thus compute

$$
\begin{aligned}
(4)(4) + x_{22}\, x_{22} &= 20 &\Rightarrow\quad& x_{22} = (20 - 16)^{1/2} = 2 \\
(4)(5) + x_{22}\, u_{23} &= 32 &\Rightarrow\quad& u_{23} = (32 - 20)/2 = 6 \\
(5)(4) + \ell_{32}\, x_{22} &= 32 &\Rightarrow\quad& \ell_{32} = (32 - 20)/2 = 6
\end{aligned}
$$

Finally, we find the last unknown:

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 6 & x_{33} \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 20 & 32 \\ 5 & 32 & 64 \end{bmatrix}$$

$$(5)(5) + (6)(6) + x_{33}\, x_{33} = 64 \Rightarrow x_{33} = (64 - 25 - 36)^{1/2} = \sqrt{3}$$

The LU factorization, with $\mathbf{L}$ and $\mathbf{U}$ as follows, satisfies $\mathbf{L} = \mathbf{U}^T$

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 6 & \sqrt{3} \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & \sqrt{3} \end{bmatrix}.$$



# LU (Cholesky) Decomposition: Algorithm

```
Input
    A                              symmetric n-by-n matrix
    n                              number of rows or columns in A
Initialize
    L = I_n                        n-by-n identity matrix
    If (A ≠ A^T) Error             A must be symmetric
Compute
For k = 1 to n
    x = L(k, 1:k − 1)              columns 1 to k-1 of the k^th row of L
    L(k,k) = √(A(k,k) − x·x^T)
    For j = k + 1 to n
        y = L(j, 1:k − 1)          columns 1 to k-1 of the j^th row of L
                       L(j,k) = (A(j,k) − y·x^T)/L(k,k)
    End
End
U = L^T
Return
    L                              computed lower triangular matrix
    U                              u is defined to be L^T
```



# Applications of LU Decomposition:Solving Linear systems

$$
\begin{aligned}
[A]\vec{X} &= [L][U][\vec{X}] = C \\
&= [L]\vec{e} = C
\end{aligned}
$$

where $\vec{e} = [U]\vec{X}$

- Solve for $\vec{e}$ using $L$ (forward substitution)
- Solve for $\vec{X}$ using $U$ (back substitution)

# Solving Linear Systems

### Forward Substitution

Solve for $\vec{e}$,

$$
\begin{aligned}
e_1 &= \frac{C_1}{l_{11}} \\
e_i &= \frac{C_i - \sum_{j=1}^{n} l_{ij} e_j}{l_{ii}}, \text{ for } i = 2, 3, \ldots n
\end{aligned}
$$

### Back Substitution

Solve for $\vec{X}$

$$
\begin{aligned}
X_n &= e_n \\
X_i &= e_i - \sum_{j=i+1}^{n} u_{ij} X_j, \text{ for } i = n-1, n-2, \ldots 1
\end{aligned}
$$



# Solving Linear Systems: Example

**Example 4.6 Solving an Electrical Circuit for Several Voltages**

Consider again the electric circuit of Figure 3.1 with resistances as shown, but with $V_1 = 0$, $V_2 = 80$, and $V_3 = 0$. The system to be solved is $\mathbf{A}\,\mathbf{x} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 30 & -20 & -10 \\ -20 & 55 & -10 \\ -10 & -10 & 50 \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} 0 \\ 80 \\ 0 \end{bmatrix}.$$

$\mathbf{A} = \mathbf{L}\,\mathbf{U}$, with

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ -1/3 & -2/5 & 1 \end{bmatrix} \text{ and } \mathbf{U} = \begin{bmatrix} 30 & -20 & -10 \\ 0 & 125/3 & -50/3 \\ 0 & 0 & 40 \end{bmatrix}.$$

We first solve $\mathbf{L}\,\mathbf{y} = \mathbf{b}$, i.e.,

$$\begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ -1/3 & -2/5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 80 \\ 0 \end{bmatrix},$$

and find, by forward substitution, that

$$y_1 = 0, \; y_2 = 80, \text{ and } y_3 = (2/5)(80) = 32.$$

Next, we solve $\mathbf{U}\,\mathbf{x} = \mathbf{y}$, or

$$\begin{bmatrix} 30 & -20 & -10 \\ 0 & 125/3 & -50/3 \\ 0 & 0 & 40 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 80 \\ 32 \end{bmatrix},$$

and find, by backward substitution, that

$$
\begin{aligned}
x_3 &= 32/40 = 4/5, \\
x_2 &= (80 + 40/3)(3/125) = 56/25,
\end{aligned}
$$

and

$$x_1 = (1/30)[-20(56/25) - 10(4/5)] = 44/25.$$

The resulting electrical currents in each of the three loops are

$$i_1 = 1.76, \; i_2 = 2.24, \text{ and } i_3 = 0.80.$$



# Solving Linear Systems: Algorithm

```
Input
    L          lower triangular matrix (with 1's on diagonal)
    U          upper triangular matrix
    B          right-hand side matrix (n-by-m)
    n          number of rows or columns in L or U, number of rows in B
    m          number of columns in B

Solve L Z = B using forward substitution
For j = 1 to m
    Z(1,j) = B(1,j)
    For i = 2 to n
        Z(i, j) = B(i, j) - L(i, :)*Z(:, j)          i^th row of L dot j^th col of Z
    End
End

Solve U X = Z using back substitution
For j = 1 to m
    X(n, :) = Z(n, :)/U(i, i)
    For i = n - 1 to 1
        X(i, j) = (Z(i, j) - U(i, :)*X((:, j))/U(i, i)   i^th row of U dot j^th col of Z
    End
End

Return
    X                 matrix of solutions
```



# Application 2: Tridiagonal Systems

```
Input
    a          above diagonal of matrix U; (with a_n = 0)
    d          diagonal of matrix U
    b          below diagonal of matrix L (with b_1 = 0)
    r          right-hand side of linear system
    n          number of components in each vector
Solve L z = b using forward substitution
z(1) = r(1)
For k = 2 to n
    z(k) = r(k) - b(k) z(k - 1)
End
Solve U x = z using back substitution
x(n) = z(n)/d(n)
For k = n - 1 to 1
    x(k) = (z(k) - a(k) x(k + 1))/d(k)
End
Return
    x          vector of solution
```

**Example 4.7 Solving a Tridiagonal System Using LU Factorization**

To illustrate the use of the preceding algorithm, we solve the linear system, $\mathbf{M}\mathbf{x} = \mathbf{r}$, with $\mathbf{r} = (4\ -3\ 9\ -10)$, and the LU factorization of $\mathbf{M}$ given by the vectors $\mathbf{a} = (-1\ -1\ -1\ 0)$, $\mathbf{d} = (2\ 3/2\ 4/3\ 5/4)$, $\mathbf{b} = (0\ -1/2\ -2/3\ -3/4)$ (see Example 4.2).

We begin by solving $\mathbf{L}\,\mathbf{z} = \mathbf{b}$ using forward substitution

$$
\begin{aligned}
z(1) &= r(1) && = 4 \\
z(2) &= r(2) - b(2)\,z(1) = -3 - (-1/2)(4) && = -1 \\
z(3) &= r(3) - b(3)\,z(2) = 9 - (-2/3)(-1) && = 25/3 \\
z(4) &= r(4) - b(4)\,z(3) = -10 - (-3/4)(25/5) && = -15/4
\end{aligned}
$$

We then solve $\mathbf{U}\,\mathbf{x} = \mathbf{z}$ using back substitution

$$
\begin{aligned}
x(1) &= z(4)/(4) && = -3 \\
x(3) &= (z(3) - a(3)\,x(4))/d(3) = (25/3 - (-1)(-3))(3/4) && = 4 \\
x(2) &= (z(2) - a(2)\,x(3))/d(2) = (-1 - (-1)(4))(2/3) && = 2 \\
x(1) &= (z(1) - a(1)\,x(2))/d(1) = (4 - (-1)2))(1/2) && = 3
\end{aligned}
$$

The solution vector is

$$\mathbf{x} = (3\ 2\ 4\ -3)$$

# Application 3: Matrix Inverse

**Example 4.8 Inverse of a Matrix**

Using the LU factorization of $\mathbf{A}$ and the algorithm to solve $\mathbf{L\,U\,x} = \mathbf{B}$ given in section 4.3.1, we can construct $\mathbf{X} = \mathbf{A}^{-1}$ by taking $\mathbf{B}$ to be the identity matrix. To illustrate the process, we find the inverse of $\mathbf{A}$, using its L U factorization, where

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 5 \\ 0 & 0 & 8 \end{bmatrix}.$$

First we solve $\mathbf{L\,Y} = \mathbf{I}$ for $\mathbf{Y}$, that is,

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The first column of $\mathbf{Y}$ is the solution vector for the system

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_{11} \\ y_{21} \\ y_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The second and third columns of $\mathbf{Y}$ are found in a similar manner, using the second and third columns of $\mathbf{I}$. Each column of $\mathbf{Y}$ is found by forward substitution, using the corresponding column of $\mathbf{I}$. The solution that is obtained is

$$\mathbf{Y} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}.$$

Finally, we solve $\mathbf{U\,X} = \mathbf{Y}$ for $\mathbf{X}$; then $\mathbf{A}^{-1} = \mathbf{X}$. We have

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 5 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}.$$

The solution is easily found for each column of $\mathbf{X}$, using back substitution and the corresponding column of $\mathbf{Y}$. The solution is

$$\mathbf{X} = \mathbf{A}^{-1} = \begin{bmatrix} 1/8 & -5/8 & 3/8 \\ -1/8 & -3/8 & 5/8 \\ 3/8 & 1/8 & 1/8 \end{bmatrix}.$$



# Analysis: LU Decomposition

- Why does LU decomposition work?
- Consider the first elimination step:

$$\begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & a_{32}' & a_{33}' \end{bmatrix}$$

- Define

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & 0 & 1 \end{bmatrix}, M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & m_{32} & 1 \end{bmatrix}$$

- Hence

$$M_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & 0 & 1 \end{bmatrix}, M_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -m_{32} & 1 \end{bmatrix},$$



# Analysis: LU Decomposition

- Gradual tranformation of I towards LU:

$$\begin{aligned} I.A &= A \\ I.M_1^{-1}.M_1.A &= A \\ I.M_1^{-1}.M_2^{-1}.M_2.M_1.A &= A \end{aligned}$$

- $M_2.M_1.A$ is $U$ and $I.M_1^{-1}.M_2^{-1}$ is $L$, given by

$$\begin{bmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & -m_{32} & 1 \end{bmatrix}$$



# Analysis:LU Decomposition with Pivoting

- Decomposition results in the factorization of $PA$, where $P$ is the permutation matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- Assume no pivoting in column 1, pivoting in column 2:

$$\begin{aligned} I.A &= A \\ I.M_1^{-1}.M_1.A &= A \\ I.M_1^{-1}.P.P.M_1.A &= A \\ I.M_1^{-1}.P.M_2^{-1}.M_2.P.M_1.A &= A \end{aligned}$$

- $M_2.P.M_1.A$ is upper triangular, but $I.M_1^{-1}.P.M_2^{-1}$ is not lower triangular.
- Premultiply both sides by $P$:

$$PM_1^{-1}.P.M_2^{-1}.M_2.P.M_1.A = P.A$$