EXAMPLE OF LU FACTORIZATION

Assume the matrix is

$$\mathbf{A} = \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix}.$$

Then $\mathbf{E}_1$ with

$$\mathbf{E}_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix},$$

takes $\mathbf{A}$ to

$$\begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 10 & 4 & -9 \\ 0 & -16 & -11 & 18 \end{bmatrix}. \qquad \mathbf{E}_1^{-1}\mathbf{E}_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 0 & 1 \end{bmatrix}$$

changes $\mathbf{A}$ to

$$\begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -3 & 2 \end{bmatrix}. \qquad \mathbf{L} = \mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\mathbf{E}_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{bmatrix}$$

changes $\mathbf{A}$ to

$$\mathbf{U} = \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

For these $\mathbf{L}$ and $\mathbf{U}$, $\mathbf{A} = \mathbf{LU}$.

Now let

$$\mathbf{b} = \begin{bmatrix} -9 \\ 5 \\ 7 \\ 11 \end{bmatrix}.$$

To solve $\mathbf{Ax} = \mathbf{b}$, first solve $\mathbf{Ly} = \mathbf{b}$ and then $\mathbf{Ux} = \mathbf{y}$.

The equations for $\mathbf{y}$ are

$$\begin{aligned} y_1 &= -9, \\ y_2 &= 5 + y_1 = 5 - 9 = -4, \\ y_3 &= 7 - 2y_1 + 5y_2 = 7 + 18 - 20 = 5, \\ y_4 &= 11 + 3y_1 - 8y_2 - 3y_3 = 11 - 27 + 32 - 15 = 1. \end{aligned}$$

The equations for **x** are

$$
\begin{aligned}
-x_4 &= y_4 = 1, \\
x_4 &= -1, \\
-x_3 &= -x_4 + y_3 = 1 + 5 = 6, \\
x_3 &= -6, \\
-2x_2 &= x_3 - 2x_4 + y_2 = -6 + 2 - 4 = -8, \\
x_2 &= 4, \\
3x_1 &= 7x_2 + 2x_3 - 2x_4 + y_1 = 28 - 12 + 2 - 9 = 9, \\
x_1 &= 3.
\end{aligned}
$$

Once the factorization $\mathbf{A} = \mathbf{LU}$ has been found, then several equations of the form $\mathbf{Ax} = \mathbf{b}$ can easily be solve for different values of $\mathbf{b}$.