SOLUTIONS: Deriving the spectral shape of the nanohertz stochastic gravitational-wave background

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Abstract

In this exercise you will learn about how a cosmic population of supermassive black-hole binaries produces a stochastic gravitational-wave signal in the nanohertz (PTA) band. You will proceed through the steps of building the stochastic signal to derive the shape of the gravitational-wave spectrum. You will also learn about the influence of different dynamical mechanisms on the orbital evolution of the binaries, and how these affect the shape of the spectrum at our lowest and most sensitive GW frequencies.

I. Introduction

The target source class of a pulsar-timing array (PTA) (Foster & Backer, 1990) is a cosmic population of supermassive black-hole binaries. These binaries (of typical mass $\sim 10^7 - 10^{10} M_\odot$) have a time-varying quadrupole moment and hence emit gravitational waves (GWs). Since we look for the influence of these GWs in the deviations of pulse times of arrival (TOA), a PTA's frequency sensitivity is dictated by our sampling of the pulsar's time series. The lowest frequency to which we are sensitive is typically the inverse of the total observation span, which is $\sim$ several nHz. This lowest frequency also sets the typical frequency resolution of our PTA detector. The highest frequency is set by how often we observe the pulsars to record a TOA (the Nyquist frequency frequency) – this is typically every few weeks, so our upper frequency is ~ 100 nHz.

In this frequency regime, where the binaries are at milliparsec to centiparsec separations, the orbital evolution due to GW emission is adiabatic and gradual. The binary orbit barely inspirals or evolves over our typical observation timespan[1] and hence the signal of each binary is a very simple and straightforward sinusoid. If a few binaries were exceptionally massive or nearby then they may be individually resolvable and able to be picked out by continuous-wave searches. If so, then we could potentially perform source characterization (e.g. determine sky location, mass, and other binary properties) on each of these binaries and look for electromagnetic counterparts, much like LIGO, except that our signals are much longer lived (PTAs = Gyrs, LIGO = seconds).

However we expect that the signals of most binaries will be drowned out by one other, such that what we actually detect will be the incoherent superposition of all of these sinusoidal signals, producing a stochastic background of gravitational radiation. "Stochastic" means that we can only probe the statistical properties of the signal, and won't recover phase information (i.e. we look for the power in the background and not the waveform itself). But those statistical properties provide us with powerful tests of the merger rate (Middleton et al., 2015; Arzoumanian et al., 2015) and angular distribution of the Universe's most massive galaxies (Taylor & Gair, 2013; Taylor et al., 2015; Mingarelli et al., 2013), the dynamics of galactic nuclei (Arzoumanian et al., 2015), and deviations of gravity from GR (Chamberlin & Siemens, 2012; Gair et al., 2015).

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[1] But the orbit will evolve between the time the GW passes by the pulsar and then passes by the Earth! This is important for building deterministic signal models for individually resolvable binaries, but will not be considered here.

In the following you will learn how to build the frequency spectrum of GWs from supermassive black-hole binaries, and learn how the spectral shape is influenced by astrophysical processes deep within galactic cores. Much of this material is explored in greater depth in Phinney (2001); Sesana et al. (2008) and references therein.

II. Key concepts

A stochastic background of GWs is usually described in terms of the energy density in GWs as a fraction of that required for closure density of the Universe. This is a common way for cosmologists to tabulate the composition of the Universe, and the only difference for GWs is that we describe the fractional energy density per logarithmic *observed* frequency interval:

$$\Omega_{\mathrm{GW}}(f) \equiv \frac{1}{\rho_{\mathrm{crit}}} \frac{\mathrm{d}\rho_{\mathrm{GW}}}{\mathrm{d}\ln f}, \tag{1}$$

where ρ_{GW} is the energy density in GWs, and ρ_{crit} is the energy density required for closure density of the Universe, defined as $\rho_{\mathrm{crit}} = 3H_0^2/(8\pi G)$.

We can further describe the stochastic background signal in terms of its *characteristic strain*, $h_c(f)$, which modifies the usual definition of a GW strain-amplitude by incorporating the beneficial influence of an inspiraling signal on the signal-to-noise ratio in our detector, i.e. the more GW cycles we record in our time series, the better the signal is sampled, and thus we get better detection prospects. So, $h_c(f) \propto \sqrt{N_{\mathrm{cycles}}} \times h \propto \sqrt{f} h$. The energy density is then written as,

$$\frac{1}{\rho_{\mathrm{crit}}} \frac{\mathrm{d}\rho_{\mathrm{GW}}}{\mathrm{d}\ln f} = \frac{\pi}{4} f^2 h_c(f)^2 \tag{2}$$

We now focus on a background signal being produced by a cosmological population of GW events, where the event distribution is characterized by a comoving number density per redshift $\mathrm{d}n/\mathrm{d}z$, with each event producing GW energy per logarithmic rest-frame frequency interval $\mathrm{d}E/\mathrm{d}\ln f_r$. The total energy density in GWs per logarithmic observed frequency interval, $\mathrm{d}\rho/\mathrm{d}\ln f$ is given by integrating over all of these events.

Write down an expression for $\mathrm{d}\rho/\mathrm{d}\ln f$ in terms of $\mathrm{d}n/\mathrm{d}z$ and $\mathrm{d}E/\mathrm{d}\ln f_r$. In practice you would also need to redshift the energy carried by the gravitational waves, but that is not necessary for these scaling relations.

$$\frac{\mathrm{d}\rho}{\mathrm{d}\ln f} = \int \mathrm{d}z \frac{\mathrm{d}n}{\mathrm{d}z} \frac{1}{(1+z)} \frac{\mathrm{d}E}{\mathrm{d}\ln f_r}$$

III. Binary radiation

The GW events we mostly care about are continuous GWs from supermassive black-hole binaries (SMBHBs). These binaries are the natural by-product of galaxy formation, which is a "bottom up" hierarchical process, where smaller dark matter halos and galaxies merge to form larger ones over cosmic time. The history of large elliptical galaxies is one of repeated cataclysmic merger events. Since most massive galaxies are thought to harbor supermassive black-holes in their center, this means that when the galaxies merge, so do the black holes. Dynamical friction causes the black holes to sink within the common merger remnant, eventually finding each other at the center of the galactic potential well. The binary may get stuck (or "stalled") at a separation of a parsec, or dynamics in the nuclei may quickly evolve it down to milliparsec separations where GW emission becomes the dominant driving mechanism of the orbit.

In this stage of the orbit, we can model the GW emission to the leading Newtonian quadrupole order where the masses only enter through the combined quantity known as the "chirp mass" $\mathcal{M} = (m_1 m_2)^{3/5}/(m_1 + m_2)^{1/5}$. If we have circular binaries then the radiation is emitted at a frequency equal to twice the orbital frequency, $f = 2 f_{\mathrm{orb}}$. The energy radiated per logarithmic frequency interval is given by (Thorne, 1987)

$$\frac{\mathrm{d}E}{\mathrm{d}\ln f_r} = \frac{\mathrm{d}t_r}{\mathrm{d}\ln f_r} \pi^2 d_L(z)^2 f_r^2 h^2 \tag{3}$$

where $\mathrm{d}t_r/\mathrm{d}\ln f_r$ describes the frequency evolution of the binary according to whichever mechanism (e.g. dynamical, or GW emission) dominates that evolution. The quantity $d_L(z)$ is the luminosity distance of the binary. Crucially, regardless of the mechanism that is driving the orbit, the strain-amplitude of the GW emission has the same form:

$$h = \frac{8\pi^{2/3}}{10^{1/2}} \frac{\mathcal{M}^{5/3}}{d_L(z)} f_r^{2/3} \tag{4}$$

IV. Binary orbital evolution

Since the comoving number density of merging binaries has no frequency dependence, the final component we need to compute the strain spectrum is the orbital frequency evolution of the binary. For GWs this evolution is described by (Peters, 1964)

$$\frac{\mathrm{d}f_{\mathrm{orb}}}{\mathrm{d}t} = \frac{48}{5\pi\mathcal{M}^2} (2\pi\mathcal{M} f_{\mathrm{orb}})^{11/3} \frac{1 + (73/24)e^2 + (37/96)e^4}{(1-e^2)^{7/2}} \tag{5}$$

where e is the binary eccentricity.

Write down Eq. (5) for a circular binary with $e = 0$. For circular systems the GW frequency is $f = 2f_{\rm orb}$. Rearrange the equation you wrote down to be explicitly in terms of the gravitational wave frequency. Finally, write down $\mathrm{d}t/\mathrm{d}\ln f$.

$$\begin{aligned}
\frac{\mathrm{d}f_{\rm orb}}{\mathrm{d}t} &= \frac{48}{5\pi\mathcal{M}^2}(2\pi\mathcal{M}f_{\rm orb})^{11/3}, \\
\frac{\mathrm{d}f_r}{\mathrm{d}t} &= \frac{96}{5\pi\mathcal{M}^2}(\pi\mathcal{M}f_r)^{11/3}, \\
\frac{\mathrm{d}t}{\mathrm{d}\ln f_r} &= \frac{5\pi\mathcal{M}^2}{96(\pi\mathcal{M})^{11/3}}f_r^{-8/3}.
\end{aligned}$$

Plug Eq. (4) and your expression for $\mathrm{d}t/\mathrm{d}\ln f$ into Eq. (3) to get an expression for $\mathrm{d}E/\mathrm{d}\ln f$.

$$\frac{\mathrm{d}E}{\mathrm{d}\ln f_r} \propto f_r^{-8/3} \times f_r^2 \times f_r^{4/3} \propto f_r^{2/3}$$

Plug your expression for $\mathrm{d}E/\mathrm{d}\ln f$ into the first equation your wrote down for $\mathrm{d}\rho/\mathrm{d}\ln f$.

$$\frac{\mathrm{d}\rho}{\mathrm{d}\ln f} \propto f^{2/3}$$

Finally, rearrange Eq. (2) using your expression for $\mathrm{d}\rho/\mathrm{d}\ln f$, and deduce a simple scaling relation for the characteristic strain spectrum of the background, h_c, as a function of observed GW frequency, f. Don't worry about normalization or precise equations. Simply work out how h_c scales with GW frequency.

$$\frac{\mathrm{d}\rho}{\mathrm{d}\ln f} \propto f^2 h_c^2 \propto f^{2/3},$$
$$h_c(f) \propto f^{-2/3}$$

V. Extensions

The scaling relation you just derived assumes that the orbital evolution of the binary is always driven by the emission of GWs. However, when the binaries first sink within the common merger remnant to find each other, dynamical friction is only efficient down to an orbital separation of a few parsecs. It will unlikely be sufficient to drive the binary to milliparsec separations where GW emission becomes the dominant evolving mechanism. Therefore, if we can't get the binaries to evolve swiftly from $\sim$ pc to $\sim$ mpc then the black holes won't merge within a Hubble time, causing the centers of massive galaxies to be populated by "stalled binaries". This is known as the "final parsec problem".

There are several mechanisms which have been proposed to mitigate this problem. The first is for many stars from within the galactic core to slingshot off the SMBHB, carrying away energy from the binary orbit and thereby driving the orbit to smaller separations (Quinlan, 1996). The second is for a gaseous disk to form around, and accrete onto, the black-hole binary – the binary exerts a torque on the disk which causes energy to be drained from the binary orbit (Ivanov et al., 1999). Finally, it is possible that the binaries retain some eccentricity from earlier in their evolution or are driven to higher eccentricity by the aforementioned stellar-slingshots/disk-accretion. If the binaries do have some eccentricity once they evolve into the sensitivity band of PTAs then the rate of orbital evolution will be much higher than for circular systems [see Eq. (5)].

Depending on the density of stars in the galactic core or the accretion rate of gas onto the black-holes, dynamical mechanisms may dominate the orbital evolution at $\sim$ pc to $\sim$ mpc separations, i.e. before GW emission dominates. If so then $\mathrm{d}f/\mathrm{d}t$ will not be equal to Eq. (5) at low orbital frequency (large orbital separations). For stellar slingshots the orbital evolution scales as $\mathrm{d}f/\mathrm{d}t \propto f^{1/3}$, and for gaseous accretion the scaling is $\mathrm{d}f/\mathrm{d}t \propto f^{4/3}$. For a recent review and application of these galactic environmental influences on the GW strain spectrum of supermassive black-hole binaries, see Arzoumanian et al. (2015).

Using these scalings for the orbital evolution (but remembering to keep the strain-amplitude h the same) derive new scaling relationships for the characteristic strain, $h_c(f)$ of the background signal.

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$$\frac{\mathrm{d}E}{\mathrm{d}\ln f_r} \propto f_r^{2/3} \times f_r^2 \times f_r^{4/3} \propto f_r^4,$$

$$h_c(f) \propto f$$

GAS

$$\frac{\mathrm{d}E}{\mathrm{d}\ln f_r} \propto f_r^{-1/3} \times f_r^2 \times f_r^{4/3} \propto f_r^3,$$

$$h_c(f) \propto f^{1/2}$$

So if the evolution is driven by dynamical mechanisms at low orbital frequencies and GW emission at high orbital frequencies, use your scaling relations to sketch a quick diagram (or make a plot if your like) of what the characteristic strain spectrum looks like from $\sim$ nHz to $\sim$100 nHz. [HINT: the behavior is easier to see if you plot $\log_{10} h_c$ against $\log_{10} f$.]

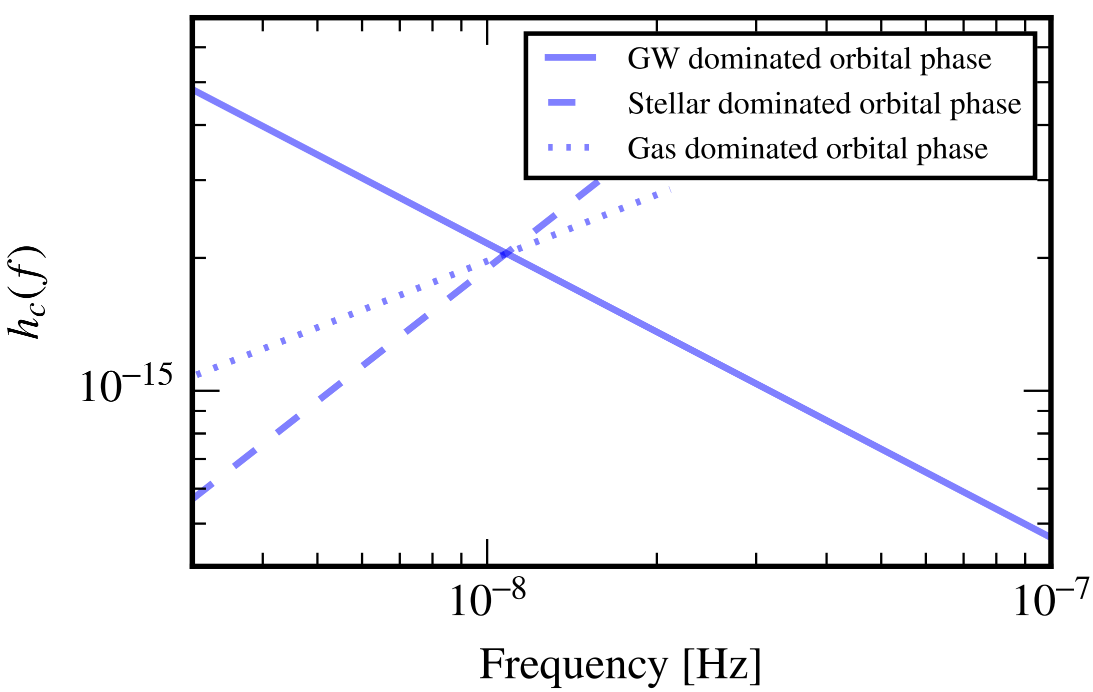

Figure 1: *Characteristic strain spectrum sketch showing the slope of the spectrum in the stellar, gas, and GW dominated stages of the orbital evolution. The actual spectrum will have a smooth transition in these stages, and thus a smooth turnover. The position of the turnover will depend on the strength of the coupling to the galactic environment, e.g. the density of stars in the galactic core, the accretion rate of a gaseous disk onto the black hole binaries, etc.*

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Glossary

adiabatic When a GW orbital inspiral is described as adiabatic, this means that the orbital period is much shorter than the timescale on which energy is radiated from the orbit. 1

closure density This is the total density of all constituent quantities in the Universe such that the spatial geometry of the Universe would be flat i.e. $k = 0$. 2

comoving This can be thought of as a cosmological quantity with the expansion of the Universe factored out, e.g. if there is no evolution in a set of objects then the comoving number density will remain fixed with redshift. 2, 3

Hubble time This is the age of the Universe if expansion had been linear. Since expansion was not linear this is only a common proxy. 5

Nyquist frequency Half of the sampling rate of a discrete signal processing system. 1

strain The strain induced by a gravitational wave is the differential change in proper separations caused by the passage of the wave as it stretches and squeezes spacetime. 2, 3, 4, 5, 6