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Randomized Nyström Preconditioning with RPChloesky

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Declaration

I hereby declare that this work is fully my own.

Signed: Aaron Dinesh Date: December 4, 2024

0.1 Question 1

Using the result from L6S76, show that the approximation $\hat{A}^{(k)}$ returned after k steps of RPChloesky satisfies

$$\mathbb{E}\left[\|A - \hat{A}^{(k)}\|_2\right] \leq 3 \cdot \mathrm{sr}_p(A) \cdot \lambda_p$$

for $k \geq (p-1)(\frac{1}{2} + \log(\frac{\eta^{-1}}{2}))$ with $\mathrm{sr}_p(A)$ defined in [1].

Using the definition of $\mathrm{sr}_p(A)$ from [1] we can see that:

$$\begin{aligned}
\mathrm{sr}_p(A) \cdot \lambda_p &= \left[\lambda_p^{-1} \sum_{j>p}^{n} \lambda_j\right] \lambda_p \\
&= \sum_{j>p}^{n} \lambda_j \\
&= \mathrm{trace}(A - \mathcal{T}_{p-1}(A))
\end{aligned}$$

Where $\mathcal{T}_r(A)$ denotes the best rank-r approximation of A. We then also note that:

$$\mathbb{E}\left[\|A - \hat{A}^{(k)}\|\right] \leq \mathbb{E}\left[\mathrm{trace}(A - \hat{A}^{(k)})\right]$$

Then we can use the theorem from L6S76 to begin our proof:

$$\begin{aligned}
\mathbb{E}\left[\|A - \hat{A}^{(k)}\|\right] &\leq \mathbb{E}\left[\mathrm{trace}(A - \hat{A}^{(k)})\|\right] \\
&\leq (1+\epsilon)\mathrm{trace}(A - \mathcal{T}_{p-1}(A))
\end{aligned}$$

To complete the proof we let $\epsilon = 2$, and then the equation above becomes:

$$\mathbb{E}\left[\|A - \hat{A}^{(k)}\|\right] \leq 3 \cdot \mathrm{trace}(A - \mathcal{T}_{p-1}(A))$$

According to the theorem in L6S76, for the above bound to hold we need to choose:

$$\begin{aligned}
k &\geq \frac{r}{\epsilon} + r\log(\frac{1}{\epsilon\eta}) \\
&= \frac{p-1}{2} + (p-1)\log(\frac{\eta^{-1}}{2}) \\
&= (p-1)(\frac{1}{2} + \log(\frac{\eta^{-1}}{2}))
\end{aligned}$$

where $\eta = \mathrm{trace}(A - \mathcal{T}_{p-1}(A))/\mathrm{trace}(A)$

Bibliography

[1] Z. Frangella, J. A. Tropp, and M. Udell, "Randomized nyström preconditioning," 2021. [Online]. Available: https://arxiv.org/abs/2110.02820