Numerical Solutions of Quasi-One-Dimensional Nozzle Flows

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Abstract

This report deals with the quasi-one-dimensional flow through a convergent-divergent nozzle. First, consider the subsonic-supersonic isentropic nozzle flow, the governing equation is of non-conservation form. Next, apply the conservation form of governing equation to the same problem. Finally, study a case of shock capturing with the addition of artificial viscosity. We get numerical solutions by employing MacCormack Scheme.

Key Words: Quasi 1D Nozzle Flow;MacCormack Scheme;Shock Capturing

1 Subsonic-Supersonic Isentropic Nozzle Flow

1.1 Formulation

The partial differential equation form of the continuity equation suitable for unsteady,quasi-one-dimensional flow is

$$\frac{\partial(\rho A)}{\partial t} + \rho A\frac{\partial V}{\partial x} + \rho V\frac{\partial A}{\partial x} + VA\frac{\partial \rho}{\partial x} = 0 \tag{1}$$

The momentum equation appropriate for quasi-one-dimensional flow, written in nonconservation form is

$$\rho\frac{\partial V}{\partial t} + \rho V\frac{\partial V}{\partial x} = -R(\rho\frac{\partial T}{\partial x} + \frac{\partial \rho}{\partial x}) \tag{2}$$

The nonconservation form of the energy equation is

$$\rho c_v\frac{\partial T}{\partial t} + \rho V c_v\frac{\partial T}{\partial x} = -\rho RT(\frac{\partial V}{\partial x} + V\frac{\partial(lnA)}{\partial x}) \tag{3}$$

For nozzle flows, the flow-field variables are frequently expressed in terms of nondimensional variables,hence the particular form of the equations that will be most appropriate as well as convenient for the time-marching solution of quasi-one-dimensional nozzle flow is as follows:

$$Continuity: \frac{\partial \rho}{\partial t} = -\rho\frac{\partial V}{\partial x} - \rho V\frac{\partial(lnA)}{\partial x} - V\frac{\partial \rho}{\partial x} \tag{4}$$

$$Momentum: \frac{\partial V}{\partial t} = -V\frac{\partial V}{\partial x} - \frac{1}{\gamma}(\frac{\partial T}{\partial x} + \frac{T}{\rho}\frac{\partial \rho}{\partial x}) \tag{5}$$

$$Energy: \frac{\partial T}{\partial t} = -V\frac{\partial T}{\partial x} - (\gamma-1)T(\frac{\partial V}{\partial x} + V\frac{\partial(lnA)}{\partial x}) \tag{6}$$

In terms of the nondimensional velocity and temperature, the Mach number is given by

$$M = \frac{V}{\sqrt{T}} \tag{7}$$

1.2 MacCormack Scheme

At a given grid point at a given time step, the stability constraint exists on this system is written as

$$(\Delta t)_i^t = C\frac{\Delta x}{a_i^t + V_i^t} \tag{8}$$

Calculate $(\Delta t)_i^t$ at all grid points(i=1,,,n), and then choose the minimum value for use, that is

$$\Delta t = minimum(\Delta t_1^t, \Delta t_2^t,,, \Delta t_i^t,,, \Delta t_N^t) \tag{9}$$

First, consider the predictor step. Set up the spatial derivative as forward differences(i=1,,,n-1)

$$(\frac{\partial\rho}{\partial t})_i^t = -\rho_i^t\frac{V_{i+1}^t - V_i^t}{\Delta x} - \rho_i^t V_i^t\frac{lnA_{i+1} - lnA_i}{\Delta x} - V_i^t\frac{\rho_{i+1}^t - \rho_i^t}{\Delta x} \tag{10}$$

$$(\frac{\partial V}{\partial t})_i^t = -V_i^t\frac{V_{i+1}^t - V_i^t}{\Delta x} - \frac{1}{\gamma}(\frac{T_{i+1}^t - T_i^t}{\Delta x} + \frac{T_i^t}{\rho_i^t}\frac{\rho_{i+1}^t - \rho_i^t}{\Delta x}) \tag{11}$$

$$(\frac{\partial T}{\partial t})_i^t = -V_i^t\frac{T_{i+1}^t - T_i^t}{\Delta x} - (\gamma - 1)T_i^t(\frac{V_{i+1}^t - V_i^t}{\Delta x} + V_i^t\frac{lnA_{i+1} - lnA_i}{\Delta x}) \tag{12}$$

Hence, the predicted values of ρ, V and T (i=1,,,n-1) is

$$\rho_i^{t+\Delta t} = \rho_i^t + (\frac{\partial\rho}{\partial t})_i^t\Delta t \tag{13}$$

$$V_i^{t+\Delta t} = V_i^t + (\frac{\partial V}{\partial t})_i^t\Delta t \tag{14}$$

$$T_i^{t+\Delta t} = T_i^t + (\frac{\partial T}{\partial t})_i^t\Delta t \tag{15}$$

Next is the corrector step, replace the spatial derivatives with rearward differences, using the predicted quantities.

$$(\frac{\partial\rho}{\partial t})_i^{t+\Delta t} = -\rho_i^{t+\Delta t}\frac{V_i^{t+\Delta t} - V_{i-1}^{t+\Delta t}}{\Delta x} - \rho_i^{t+\Delta t}V_i^{t+\Delta t}\frac{lnA_{i+1} - lnA_i}{\Delta x} - V_i^{t+\Delta t}\frac{\rho_{i+1}^{t+\Delta t} - rho_i^{t+\Delta t}}{\Delta x} \tag{16}$$

$$(\frac{\partial V}{\partial t})_i^{t+\Delta t} = -V_i^{t+\Delta t}\frac{V_i^{t+\Delta t} - V_{i-1}^{t+\Delta t}}{\Delta x} - \frac{1}{\gamma}(\frac{T_i^{t+\Delta t} - T_{i-1}^{t+\Delta t}}{\Delta x} + \frac{T_i^{t+\Delta t}}{\rho_i^{t+\Delta t}}\frac{\rho_i^{t+\Delta t} - \rho_{i-1}^{t+\Delta t}}{\Delta x}) \tag{17}$$

$$(\frac{\partial T}{\partial t})_i^{t+\Delta t} = -V_i^{t+\Delta t}\frac{T_i^{t+\Delta t} - T_{i-1}^{t+\Delta t}}{\Delta x} - (\gamma - 1)T_i^{t+\Delta t}(\frac{V_i^t - V_{i-1}^{t+\Delta t}}{\Delta x} + V_i^{t+\Delta t}\frac{lnA_i - lnA_{i-1}}{\Delta x}) \tag{18}$$

The average time derivatives are given by

$$(\frac{\partial\rho}{\partial t})_{av} = 0.5((\frac{\partial\rho}{\partial t})_i^t + (\frac{\partial\rho}{\partial t})_i^{t+\Delta t}) \tag{19}$$

$$(\frac{\partial V}{\partial t})_{av} = 0.5((\frac{\partial V}{\partial t})_i^t + (\frac{\partial V}{\partial t})_i^{t+\Delta t}) \tag{20}$$

$$(\frac{\partial T}{\partial t})_{av} = 0.5((\frac{\partial T}{\partial t})_i^t + (\frac{\partial T}{\partial t})_i^{t+\Delta t}) \tag{21}$$

Finally, for the corrected values of the flow-field variables at time $t + \Delta t$

$$\rho_i^{t+\Delta t} = \rho_i^t + (\frac{\partial \rho}{\partial t})_{av}^t \Delta t \tag{22}$$

$$V_i^{t+\Delta t} = V_i^t + (\frac{\partial V}{\partial t})_{av}^t \Delta t \tag{23}$$

$$T_i^{t+\Delta t} = T_i^t + (\frac{\partial T}{\partial t})_{av}^t \Delta t \tag{24}$$

1.3 Boundary Conditions

Nozzle Shape and Initial Conditions:

The nozzle shape $S(x) = 1 + 2.2(x - 1.5)^2 \; 0 <= x <= 3$ as ρ and T decrease and V increases as the flow expands through the nozzle, for simplicity, assume linear variations of the flow-field variables as a function of x. At time $t = 0$

$$\rho = 1 - 0.3146x \tag{25}$$

$$T = 1 - 0.2134x \tag{26}$$

$$V = (0.1 + 1.09)T^{\frac{1}{2}} \tag{27}$$

Subsonic Inflow Boundary:

$$V_1 = 2V_2 - V_3 \tag{28}$$

$$\rho_1 = 1 \tag{29}$$

$$T_1 = 1 \tag{30}$$

Supersonic Outflow Boundary:

Allow all flow-field variables to float and use linear extrapolation based on the flow-field values at the internal points,

$$V_N = 2V_{N-1} - V_{N-2} \tag{31}$$

$$\rho_N = 2\rho_{N-1} - \rho_{N-2} \tag{32}$$

$$T_N = 2T_{N-1} - T_{N-2} \tag{33}$$

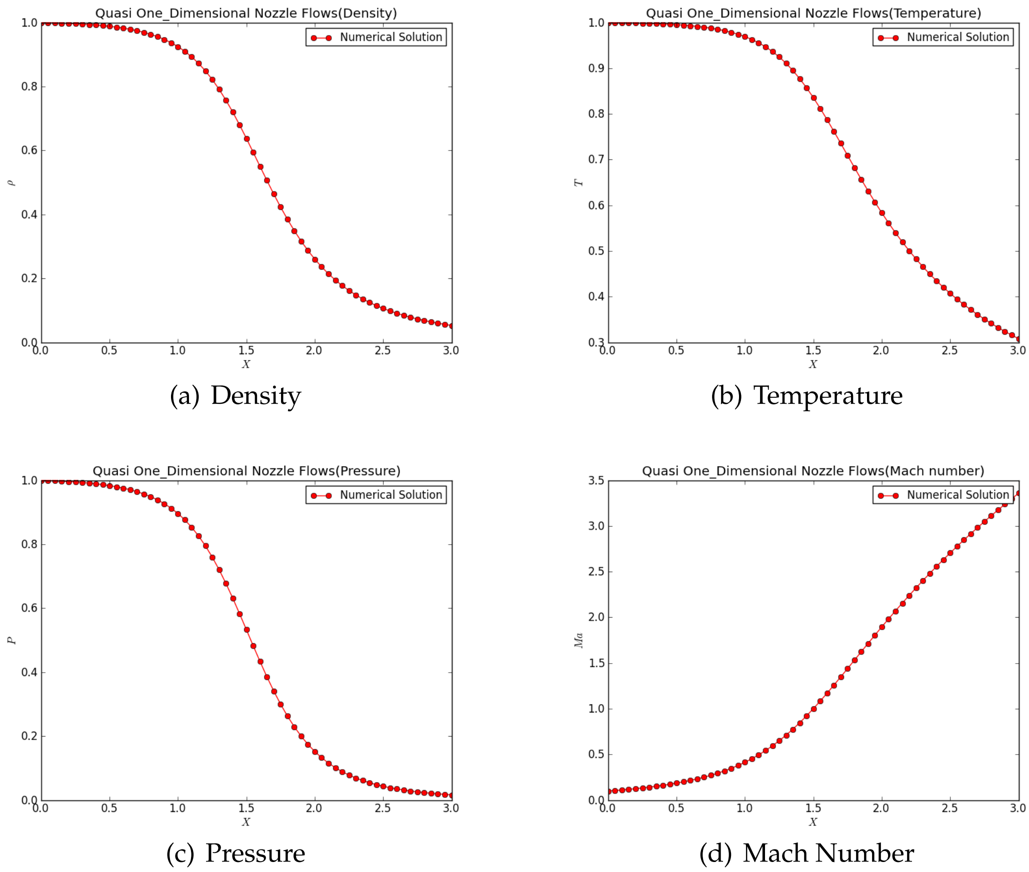


Figure 1: Subsonic-Supersonic Flow(Governing Equation in Non-Conservation Form)

1.4 Results

2 Subsonic-Supersonic Isentropic Nozzle Flow: Governing Equations in Conservation Form

2.1 Formulation

The nondimensional conservation form of the continuity, momentum, and energy equations for quasi-one-dimensional flow can be expressed in a generic form. Define the elements of the solutions vector U, and the flux vector F, and the source term J as follows:

$$U_1 = \rho A \tag{34}$$

$$U_2 = \rho A V \tag{35}$$

$$U_3 = \rho(\frac{e}{\gamma - 1} + \frac{\gamma}{2}V^2)A \tag{36}$$

$$F_1 = \rho AV \tag{37}$$

$$F_2 = \rho AV^2 + \frac{1}{\gamma}pA \tag{38}$$

$$F_3 = \rho(\frac{e}{\gamma - 1} + \frac{\gamma}{2}V^2)VA + pAV \tag{39}$$

$$J_2 = \frac{1}{\gamma}p\frac{\partial A}{\partial x} \tag{40}$$

Thus

$$\frac{\partial U_1}{\partial t} = -\frac{\partial F_1}{\partial x} \tag{41}$$

$$\frac{\partial U_2}{\partial t} = -\frac{\partial F_2}{\partial x} + J_2 \tag{42}$$

$$\frac{\partial U_3}{\partial t} = -\frac{\partial F_3}{\partial x} \tag{43}$$

To obtain the primitive variables ($\rho, V, T, etc.$), decode the elements U_1, U_2, U_3 as follows:

$$\rho = \frac{U_1}{A} \tag{44}$$

$$V = \frac{U_2}{U_1} \tag{45}$$

$$T = e = (\gamma - 1)(\frac{U_3}{U_1} - \frac{\gamma}{2}V^2) \tag{46}$$

$$p = \rho T \tag{47}$$

Pure form of the flux terms:

$$F_1 = U_2 \tag{48}$$

$$F_2 = \frac{U_2^2}{U_1} + \frac{\gamma - 1}{\gamma}(U_3 - \frac{\gamma}{2}\frac{U_2^2}{U_1}) \tag{49}$$

$$F_3 = \gamma\frac{U_2 U_3}{U_1} - \frac{\gamma(\gamma - 1)}{2}\frac{U_2^3}{U_1^2} \tag{50}$$

$$J_2 = \frac{\gamma - 1}{\gamma - 1}(U_3 - \frac{\gamma}{2}\frac{U_2^2}{U_1})\frac{\partial(lnA)}{\partial x} \tag{51}$$

2.2 B.C and I.C.

B.C.:

$$U_{1_{(i=1)}} = S_{i=1} \tag{52}$$

$$U_{2_{(i=1)}} = 2U_{2_{(i=2)}} U_{2_{(i=3)}} \tag{53}$$

$$U_3 = U_1\left(\frac{T}{\gamma - 1} + \frac{\gamma}{2}V^2\right) \tag{54}$$

$$(U_1)_N = 2(U_1)_{N-1} - (U_1)_{N-2} \tag{55}$$

$$(U_2)_N = 2(U_2)_{N-1} - (U_2)_{N-2} \tag{56}$$

$$(U_3)_N = 2(U_3)_{N-1} - (U_3)_{N-2} \tag{57}$$

I.C.:
For ($0 <= x <= 0.5$):

$$\rho = 1.0; T = 1.0 \tag{58}$$

For ($0.5 <= x <= 1.5$):

$$\rho = 1.0 - 0.366(x - 0.5); T = 1.0 - 0.167(x - 0.5) \tag{59}$$

For ($1.5 <= x <= 3.5$):

$$\rho = 0.634 - 0.3879(x - 1.5); T = 0.833 - 0.3507(x - 1.5) \tag{60}$$

$$V = \frac{0.59}{\rho A} \tag{61}$$

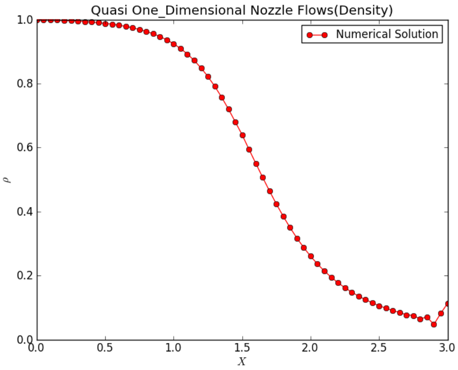
The shape of the nozzle is the same as discussed in previous case.

2.3 Results

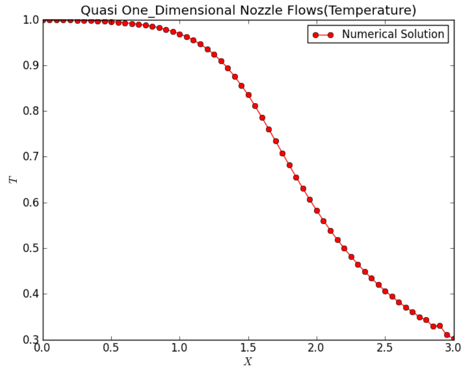
3 Shock Capture

In this section, we will use the conservation form of the governing equations to numerically capture normal shock waves within the nozzle. When we practice the art of shock capturing, the smoothing and stabilization of the solution by the addition of some type of numerical dissipation is absolutely necessary. Add artificial viscosity as follows:

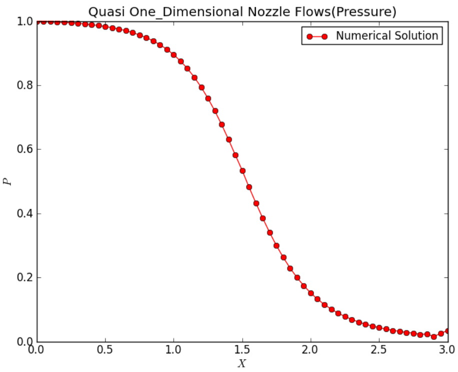
$$S_i^t = \frac{C_x|(p)_{i+1}^t - 2(p)_i^t + (p)_{i-1}^t|}{(p)_{i+1}^t + 2(p)_i^t + (p)_{i-1}^t}(U_{i+1}^t - 2U_i^t + U_{i-1}^t) \tag{62}$$



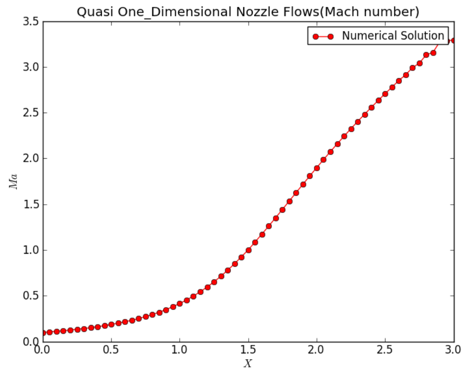
(a) Density



(b) Temperature



(c) Pressure



(d) Mach Number

Figure 2: Subsonic-Supersonic Flow(Governing Equation in Conservation Form)

3.1 B.C. and I.C.

B.C.:

$$(U_1)_N = 2(U_1)_{N-1} - (U_1)_{N-2} \tag{63}$$

$$(U_2)_N = 2(U_2)_{N-1} - (U_2)_{N-2} \tag{64}$$

$$V_N = \frac{(U_2)_N}{(U_1)_N} \tag{65}$$

$$(U_3)_N = \frac{0.6784S}{\gamma - 1} + \frac{\gamma}{2}(U_2)_N V_N \tag{66}$$

I.C.:
For $(0 <= x <= 0.5)$:

$$\rho = 1.0; T = 1.0 \tag{67}$$

For ($0.5 <= x <= 1.5$):

$$\rho = 1.0 - 0.366(x - 0.5); T = 1.0 - 0.167(x - 0.5) \tag{68}$$

For ($1.5 <= x <= 2.1$):

$$\rho = 0.634 - 0.702(x - 1.5); T = 0.833 - 0.4908(x - 1.5) \tag{69}$$

For ($2.1 <= x <= 3.0$):

$$\rho = 0.5892 + 0.10228(x - 2.1); T = 0.93968 - 0.0622(x - 2.1) \tag{70}$$

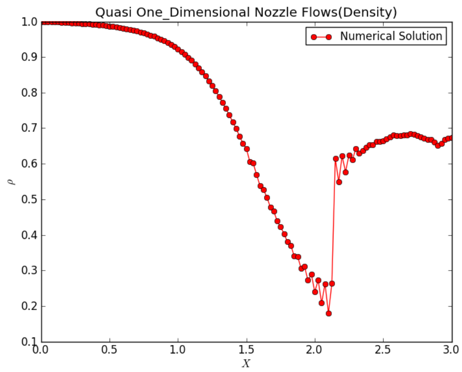
3.2 Results

Compare the shock capturing numerical results through the nozzle, the addition of artificial viscosity(**Right Column**) eliminated the oscillations that were encountered with no artificial viscosity(**Left Column**).

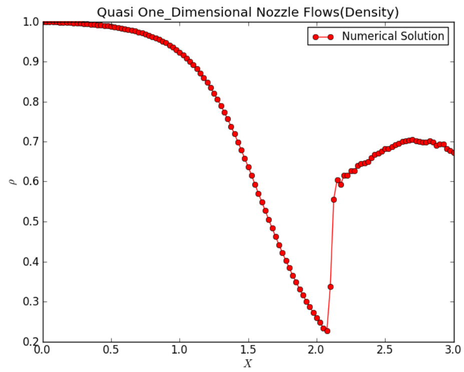
References

[1] Anderson,J.D., ”Computational Fluid Dynamics: The Basics with Application”, 2002.

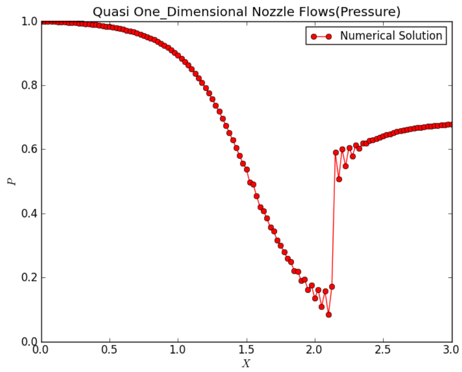
[2] Jaan Kiusalaas, ”Numerical Methods in Engineering with Python”, 2005.



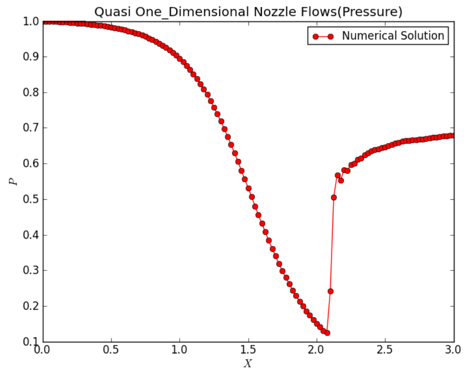
(a) Density



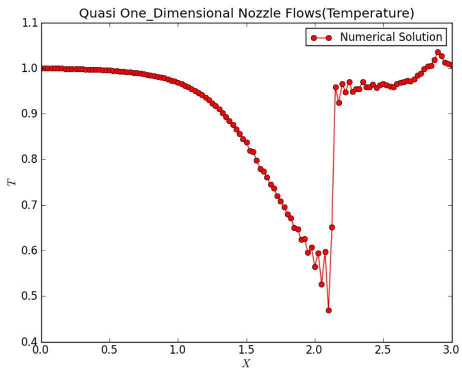
(b) Density



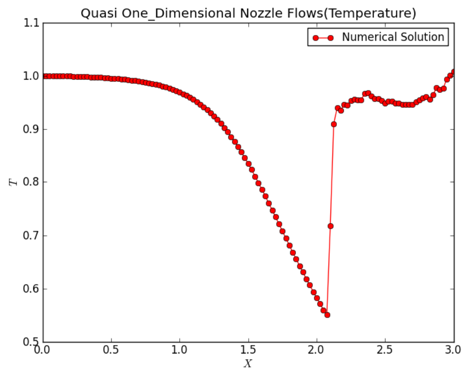
(c) Pressure



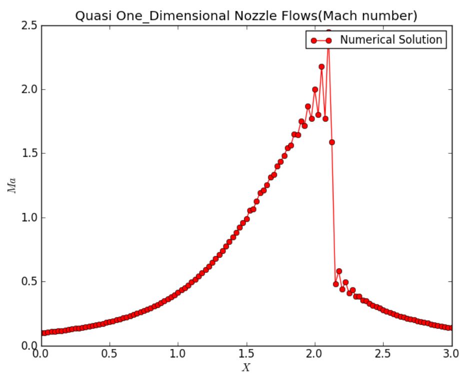
(d) Pressure



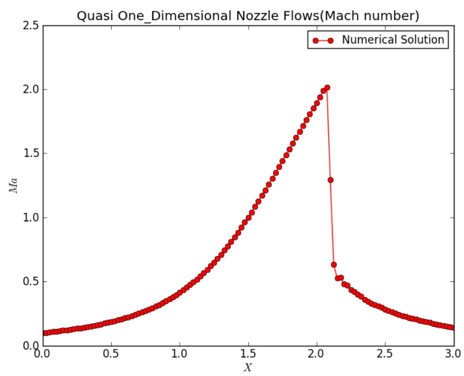
(e) Temperature



(f) Temperature



(g) Mach Number



(h) Mach Number

Figure 3: Shock Capturing