Homework Assignment

Due: 20/10/2014

1. The heat equation $T_t = \alpha T_{xx}$ governs the time-dependent temperature distribution in a homogeneous constant property solid under conditions where the temperature varies only in one space dimension. Physically, this may be nearly realized in a long thin rod or very large (infinite) wall of finite thickness. Consider a large wall of thickness L whose initial temperature is given by $T(x,0) = c\sin(\pi x / L)$, and boundary condition $T(0,t) = T(L,t) = 0$, the exact solution is $T(x,t) = c\exp\left(-\frac{\alpha\pi^2 t}{L^2}\right)\sin(\pi x / L)$.

   Let c = 100 °C, L = 1 m, α = 0.02 m^2/h. We will consider two explicit methods: (i) Simple explicit method. Stability requires that $\alpha\Delta t / \Delta x^2 \le 0.5$ for this method. (ii) Alternating direction explicit (ADE) method suggested by Barakat and Clark (1966). In this algorithm, the equation for p_j^{n+1} can be solved explicitly starting from the boundary at x = 0, whereas the equation for q_j^{n+1} should be solved starting at the boundary at x = L. The scheme is unconditionally stable. Develop computer programs to solve the problem using methods (i) and (ii) and make the following comparisons:

   (a) For Δx = 0.1 and Δt = 0.1, compare the results from (i), (ii) and the exact solution for t = 10 h in a graph.
   (b) Repeat the above comparison after refining the space grid, Δx = 0.066667. Is the reduction in error as suggested by $O(\Delta x^2)$?
   (c) Demonstrate that method (i) does become unstable as $\alpha\Delta t / \Delta x^2 > 0.5$. One suggestion is to plot the centerline temperature vs. time for $\alpha\Delta t / \Delta x^2 = 0.6$ for 10-20 hours of problem time.
   (d) For method (ii), Δx = 0.1, choose Δt such that $\alpha\Delta t / \Delta x^2 = 1$ and compare the results and the exact solution for t = 10 h. Increase $\alpha\Delta t / \Delta x^2$ to 2, 3, etc, and show the discrepancy becomes noticeable.

2. The heat equation with a source term is $T_t = \alpha T_{xx} + S(x) \quad 0 \le x \le L_x$. The initial and boundary conditions are $T(x,0) = 0 \quad T(0,t) = 0 \quad T(L_x,t) = T_{steady}(L_x)$. Take α =1, L_x =15, and $S(x) = -(x^2 - 4x + 2)e^{-x}$. The exact steady solution is $T_{steady}(x) = x^2 e^{-x}$.

   (a) Verify the exact solution is $T_{steady}(x)$. Plot $T_{steady}(x)$.
   (b) Using explicit Euler for time advancement and the second-order central difference scheme for the spatial derivative, solve the equation to steady state on a uniform grid. Plot the exact and numerical steady solutions for N_x =10, 20.
   (c) Repeat your calculations using the non-uniform grid $x_j = L_x[1 - \cos(\frac{\pi j}{2N_x})]$, where $J = 0, \cdots, N_x$, and an appropriate finite difference scheme for non-uniform grid.
   (d) Transform the differential equation to a new coordinate system using the transform $\xi = \cos^{-1}(1 - x / L_x)$. Solve the resulting equation to the steady state and plot the exact and numerical steady solutions for N_x =10, 20.

(e) Repeat (c) using the Crank-Nicolson method for time advancement. Show that you can take fewer time steps to reach steady state.

For each method, find the maximum time step required for stable solutions. Also, for each method with N_x =20, plot the transient solutions at two intermediate times, e.g., t = 2 and t = 10. Compare and discuss all results obtained in terms of accuracy and stability. Compare the number of time steps required for each method to reach steady state.

3. Solve the steady-state 2D heat conduction equation in the unit square, $0 < x < 1$ and $0 < y < 1$, using mesh $\Delta x = 0.05$ and $\Delta y = 0.05$. Compare the center temperature with the exact solution. Use boundary conditions: $T = 0$ at $x = 0$, $x = 1$; $\frac{\partial T}{\partial y} = 0$ at $y = 0$; $T = \sin(\pi x)$ at $y = 1$.

4. Use MacCormack method to solve the linearized Burgers equation $u_t + cu_x = \mu u_{xx}$, $c = 0.5$, $\mu = 0.02$ for initial condition $u(x,0) = 0$, $0 \le x \le 1$, and boundary condition with $u(0,t) = 100$ and $u(1,t) = 0$ (t >0) with $\Delta x = 0.02$ and $\Delta t = 0.01$. Find the steady-state solution and compare with the exact solution $u(x) = 100\dfrac{1-\exp[25*(x-1)]}{1-\exp(-25)}$.

   MacCormack method:
   Predictor:
$$\overline{u}_j^{n+1} = u_j^n - v\left(u_{j+1}^n - u_j^n\right) + r\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$
   Corrector:
$$u_j^{n+1} = \frac{1}{2}\left[u_j^n + \overline{u}_j^{n+1} - v\left(\overline{u}_j^{n+1} - \overline{u}_{j-1}^{n+1}\right) + r\left(\overline{u}_{j+1}^{n+1} - 2\overline{u}_j^{n+1} + \overline{u}_{j-1}^{n+1}\right)\right]$$