# Deriving the Envelop condition ("Benveniste - Scheinkman formula")

$$v(k) = \max_{0 \le k_+ \le f(k)} \{ \underbrace{u[f(k) - k_+] + \beta v(k_+)}_{W(k,k_+)} \} \quad (1)$$

FOC wrt $k_+$:

$$u'[f(k) - k_+](-1) + \beta v'(k_+) = 0 \quad (2)$$

An optimal solution function $g$ s.t. $k_+ = g(k)$ satisfies (1).

Evaluate objective function $W$ at $k_+ = g(k)$, given fixed $k$:

$$v(k) = W[k, g(k)]$$

by definition of value function $v$ in (1)

$$= u[f(k) - \underbrace{g(k)}_{\text{optimizer}}] + \beta v[\underbrace{g(k)}_{\text{optimizer}}] \quad (3)$$

**Benveniste - Scheinkman Theorem** $X \ni k$ is a convex set. $v: X \to \mathbb{R}$ is concave. Let $k_0 \in \text{int}(X)$, and $D$ is a neighborhood of $k_0$. If $\exists\, W: D \to \mathbb{R}$ which is concave and differentiable with $W(k_0) = v(k_0)$ and $W(k) \le v(k)$ $\forall\, k \in D$, then $v'(k_0)$ exists and $v'(k_0) = W'(k_0)$.

Using the B-S formula on (3):

$$v'(k) = W'[k, g(k)] \tag{4}$$

$$\begin{matrix} v'(k) & = & W'[k, g(k)] \\ \equiv & & \equiv \\ \dfrac{\partial v(k)}{\partial k} & & \dfrac{\partial W[k, g(k)]}{\partial k} \end{matrix}$$

Note that RHS of (4) is just

$$\begin{aligned} W'[k, g(k)] &= u'[f(k) - g(k)]\, f'(k) \\ &\quad - u'[f(k) - g(k)]\, g'(k) \\ &\quad + \beta v'[g(k)] \cdot g'(k) \\ &= u'[f(k) - g(k)]\, f'(k) \\ &\quad + \{\beta v'[g(k)] - u'[f(k) - g(k)]\}\, g'(k) \end{aligned} \tag{5}$$

Marginal (optimal) value of $k$

From FOC (2), evaluated at $k_+ = g(k)$, we have

$$u'[f(k) - g(k)] = \beta v'[g(k)] \tag{2'}$$

Plug (2') into (5), Roberto es tu tío ...

$$\boxed{v'(k) = W'[k, g(k)] = u'(c^*)\, f'(k)} \tag{6}$$

where $c^* = f(k) - g(k)$. This is B-S!