Competitive Equilibrium with Complete Markets: Part II

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Outline

What Next?

- Competitive/Decentralized equilibrium:
  - Arrow-Debreu time-0-trading economy (ADE) with Arrow-Debreu securities.
  - Radner sequential-trading economy (SME) with Arrow securities.
- Show ADE $\Leftrightarrow$ SME $\Leftrightarrow$ PO.

Motivation

Previously ...

Previously we look at a *planned* economy:

- Single optimizing planner.
- Characterized recursive optimal allocations as a DP problem.
- In reality, we have *decentralized* or *competitive economies*.

Osamu Tezuka's *Metropolis*

Motivation

And few lectures ahead ...

- Want to work toward the stochastic growth model as also basic *recursive competitive equilibrium* model.
- As a dynamic outcome where individuals and firms solve their *decentralized* optimal allocation problems independently.
- No planner.
- History (or State)-contingent, intertemporal relative prices, as the *allocative mechanism*.
- Resulting versions of first- and second fundamental welfare theorems. (Why?)

Model Setup

- Stochastic event $s_t \in S = \{s_1, ..., s_n\}$ for $t \in \mathbb{N}$.
- Publicly observable history of events up to and including t: $h^t = (s_0, s_1, ..., s_t) \in S^t$.
- Unconditional probability of h^t given by probability measure $\pi_t\left(h^t\right)$.
- W.l.o.g., assume $\pi\left(s_0\right) = 1$.
- Probability of observing h^t *conditional* on realization of h^τ is $\pi\left(h^t | h^\tau\right)$, for any $t \geq \tau$.

Model Setup

- I agents indexed by $i = 1, ..., I$.
- Agent i's
  - Endowment: $y_t^i\left(h^t\right)$
  - history-dependent consumption plan, $c^i = \left\{c_t^i\left(h^t\right)\right\}_{t=0}^{\infty}$ for each $h^t \in S^t$
  - expected utility criterion:

$$U\left(c^i\right) = \mathbb{E}_0\left\{\sum_{t=0}^{\infty} \beta^t u\left(c_t^i\left(h^t\right)\right)\right\} = \sum_{t=0}^{\infty} \sum_{h^t} \beta^t u\left(c_t^i\left(h^t\right)\right) \pi_t\left(h^t\right)$$

  where
  - $u'(c) > 0, u''(c) < 0$
  - $\lim_{c \searrow 0} u'(c) = +\infty$

  to ensure $c_t > 0$ for all t

Model Setup

A *feasible allocation* must satisfy

$$\sum_{i=1}^{I} c_t^i \left(h^t\right) \leq \sum_{i=1}^{I} y_t^i \left(h^t\right)$$

for all t and for all h^t.

Remember?

A first-order neccesary condition for Pareto optimum is

$$\beta^t u'\left(c_t^i\left(h^t\right)\right)\pi_t\left(h^t\right) = \frac{\theta_t\left(h^t\right)}{\lambda_i}$$

for all $i = 1, ..., I$, and for all $t \geq 0$ and all h^t.

Remember?

Consider two agents, $i \neq j$. The ratio of their marginal utilities at each period, for all possible histories, is

$$\frac{u'\left(c_t^i\left(h^t\right)\right)}{u'\left(c_t^j\left(h^t\right)\right)} = \frac{\lambda_j}{\lambda_i}$$

This implies:

$$c_t^i\left(h^t\right) = u'^{-1}\left[\frac{\lambda_j}{\lambda_i} u'\left(c_t^j\left(h^t\right)\right)\right]$$

Theorem

A Pareto optimal allocation is a function of the realized aggregate endowment and does not depend on

1. *the particular history* h^t *leading up to that outcome, nor*
2. *the realization of individual endowments,*

so that if $h^t \neq h^\tau$ *are such that* $\sum_j y_t^j\left(h^t\right) = \sum_j y_\tau^j\left(h^\tau\right)$ *then* $c_t^i\left(h^t\right) = c_\tau^i\left(h^\tau\right)$.

Corollary (First fundamental welfare theorem)

The competitive equilibrium is a particular Pareto optimal allocation, where $\mu_i = \lambda_i^{-1}$ for all $i = 1, ..., I$, is unique (up to a multiplication by a positive scalar). Furthermore, the shadow prices for the planner $\theta_t \left(h^t\right)$ are equal to Arrow-Debreu equilibrium prices $q_t^0 \left(h^t\right)$.

What Next?

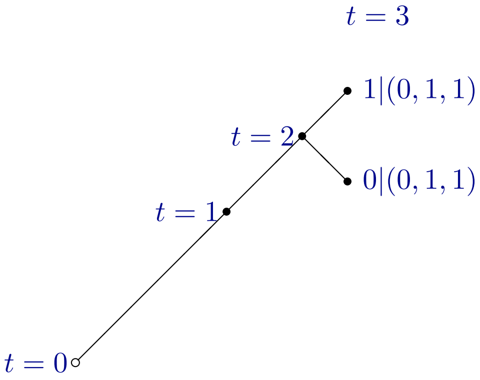
- We have already charactered Pareto-optimal allocation (PO).
- We have studied one assumption for a decentralized market economy: ADE
- Next we study alternative SME. Note along the way, connections btw SME and ADE via asset pricing relationships.
- W.t.s. FWT: Allocative “equivalence” between ADE and SME and PO.

Sequential Markets Economy

Market trading structure – assumptions:

- Trade occurs at each $t \in \mathbb{N}$.
- Trade in one-period complete Arrow securities.
- At each t reached with history h^t, traders meet to trade for history h^{t+1}-contingent goods deliverable in $t+1$.

Example



$S = \{0, 1\}$. At $t = 2$, trades occurs for only $t = 3$ goods at states that can be reached from the realized $t = 2$ history, $h^2 = (0, 1, 1)$.

Preliminaries: Two new items ...

Now markets exist *sequentially*.

At start of each period t, after each h^t, traders need to keep track of what is *feasibly tradable*.

This depend on:

1. Wealth as a state variable.
2. A restriction that prevents forever-borrowing schemes by agents.

Remark. These two things not needed in ADE. Why?

Relevant state variable

- Need to find an appropriate *individual* state variable.
- This state variable tracks available opportunity set;
- to provide the right choices of consumption so that there will be enough resources left for future trades on contingent claims.
- State variable is the present value (in terms of history h^t and date t) of expected current and future net claims – i.e. current wealth of the consumer.

ADE again! Agent i's wealth at time-t is just the time-t *expected value of all her current and future net claims* conditional on time-t, history h^t:

$$\begin{aligned}\Omega_t^i\left(h^t\right) &= \sum_{\tau=t}^{\infty}\sum_{h^\tau|h^t} q_\tau^t\left(h^\tau\right) d_\tau(h^\tau) \\ &= \sum_{\tau=t}^{\infty}\sum_{h^\tau|h^t} q_\tau^t\left(h^\tau\right)\left[c_\tau^i\left(h^\tau\right) - y_\tau^i\left(h^\tau\right)\right]\end{aligned}$$

But,

$$q_\tau^t\left(h^\tau\right) = \beta^{\tau-t}\frac{u'\left(c_\tau^i\left(h^\tau\right)\right)}{u'\left(c_t^i\left(h^t\right)\right)}\pi_\tau\left(h^\tau|h^t\right)$$

Note $\Omega_t^i\left(h^t\right)$ has the same expression as the value of a tail asset!

Since, aggregate endowment must equal aggregate consumption (following any h^t), then

$$\sum_{i=1}^{I} \Omega_t^i \left(h^t\right) = 0.$$

for all t and all h^t.

Remarks:

- The cross-sectional distribution of tail wealth across all agents i sums to zero, since all contingent debt sellers are balanced out by buyers.
- When we move from this ADE to the SME, we can relate this tail wealth of each i, $\Omega_t^i \left(h^t\right)$, to the time t history h^t individual asset of the sequential markets world.

SBC and Debt limits

- In ADE, households face a single intertemporal budget constraint that ensures intertemporal solvency.
- In the sequential markets setting, there will be a sequence of budget constraints, indexed by t and h^t.
- Need to ensure sequential asset trades are not open to "Ponzi schemes" – i.e. consumers cannot forever be consuming more than their endowments.
- We will consider the weakest possible restrictions called "natural debt limits" .

Definition (Natural debt limit)

Let the Arrow-Debreu price in terms of the time t history h^t numeraire good be $q_\tau^t\left(h^\tau\right)$ for $\tau \geq t$. The value of the *tail* of i's endowment sequence at time t given history h^t,

$$A_t^i\left(h^t\right) = \sum_{\tau=t}^{\infty} \sum_{s^\tau | h^t} q_\tau^t\left(h^\tau\right) y_\tau^i\left(s^\tau\right),$$

is the *natural debt limit* at time t and history h^t.

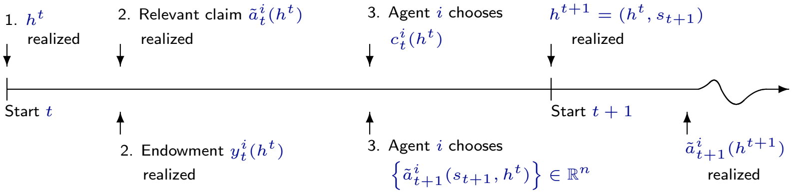
Remarks

$$A_t^i\left(h^t\right) = \sum_{\tau=t}^{\infty} \sum_{s^\tau | h^t} q_\tau^t\left(h^\tau\right) y_\tau^i\left(s^\tau\right)$$

- The maximal amount that i can repay his debt starting from time t is thus the tail value of his endowment starting out from time t given history h^t.
- Alternatively, this says the worst i can do is to consume zero forever from time t to repay existing debt at time t history h^t.
- At each time t, i will face one such borrowing constraint for each possible realization h^{t+1} the next period.

Sequential trades: Timing and Actions

Markets are open for trade in one period-ahead state-contingent claims every period.



Suppose the pricing kernel $\widetilde{Q}_t\left(s_{t+1}|h^t\right)$ exists.

$\widetilde{Q}_t\left(s_{t+1}|h^t\right)$: price of one unit of time $t+1$ consumption, contingent on realization of s_{t+1} in $t+1$, given t-history h^t.

Agent i's sequence of budget constraints:

$$\widetilde{c}_t^i\left(h^t\right)+\sum_{s_{t+1}}\widetilde{a}_{t+1}^i\left(s_{t+1},h^t\right)\widetilde{Q}_t\left(s_{t+1}|h^t\right)\leq y_t^i\left(h^t\right)+\widetilde{a}_t^i\left(h^t\right)$$

for $t\geq 0$, at each h^t.

Note at time t, given h^t, i chooses

- Current consumption: $\widetilde{c}_t^i\left(h^t\right)$, and
- Quantities of all possible n number of state-contingent claims next period:

$$\left(\widetilde{a}_{t+1}^i\left(s_{t+1}, h^t\right)\right) \in \mathbb{R}^n$$

No-Ponzi borrowing constraint:

$$-\widetilde{a}_{t+1}^{i}\left(s_{t+1}\right) \leq A_{t+1}^{i}\left(h^{t+1}\right) = \sum_{\tau=t+1}^{\infty} \sum_{h^{\tau}|h^{t+1}} q_{\tau}^{t+1}\left(h^{\tau}\right) y_{\tau}^{i}\left(h^{\tau}\right).$$

Huh? ...

- Amount of debt i brings into all possible $s_{t+1} \in S$,
- *Must* be repayable, in the worst case,
- by expected discounted (real) value of tail endowments, remaining from $t+1$ onward.
- Worst case: consume nothing forever from $t+1$ on ...! Merd!

Agent i chooses $\{c_t^i(h^t), \tilde{a}_{t+1}^i(s_{t+1}, h^t)\}_{t=0}^{\infty}$ to:

$$\max \sum_{\tau=0}^{\infty} \sum_{h^t} \Bigg\{ \beta^t u\left(\widetilde{c}_t^i\left(h^t\right)\right) \pi_t\left(h^t\right)$$

$$+ \qquad \eta_t^i\left(h^t\right) \Bigg[y_t^i\left(h^t\right) + \widetilde{a}_t^i\left(h^t\right)$$

$$- \widetilde{c}_t^i\left(h^t\right) - \sum_{s_{t+1}} \widetilde{a}_{t+1}^i\left(s_{t+1}, h^t\right) \widetilde{Q}_t\left(s_{t+1}|h^t\right) \Bigg]$$

$$+ \nu_t^i\left(h^t; s_{t+1}\right) \left[\widetilde{a}_{t+1}^i\left(s_{t+1}, h^t\right) + A_{t+1}^i\left(h^{t+1}\right)\right] \Bigg\}$$

for given initial wealth $\widetilde{a}_0^i\left(h^0\right)$.

Remarks: Following each h^t,

- There are n no-Ponzi constraints to consider. Why?
- So there are n Lagrange multipliers $\nu_t^i\left(h^t; s_{t+1}\right)$, one for each possible $s_{t+1} \in S$.
- For each s_{t+1} need to calculate upper bound on negative assets:

$$A_{t+1}^i\left(h^{t+1}\right) = \sum_{\tau=t+1}^{\infty} \sum_{h^\tau | h^{t+1}} q_\tau^{t+1}\left(h^\tau\right) y_\tau^i\left(h^\tau\right).$$

Optimal decision by agents i:
No-Ponzi constraints not binding. Why? So $\nu_t^i\left(h^t; s_{t+1}\right)=0$ for all t, all h^t.

Then necessary (and sufficient) condition for optimal consumption-asset-accumulation strategy is

$$\widetilde{Q}_t\left(s_{t+1}|h^t\right)=\beta\frac{u'\left(\widetilde{c}_{t+1}^i\left(h^{t+1}\right)\right)}{u'\left(\widetilde{c}_t^i\left(h^t\right)\right)}\pi_t\left(h^{t+1}|h^t\right)$$

for all s_{t+1}, $t\geq 0$ and h^t.

Crickey! This is a familiar looking one-period pricing kernel we encountered in the Arrow-Debreu economy!

Definition

A **distribution of wealth** is a vector $\overrightarrow{\widetilde{a}}_t\left(h^t\right) = \left\{\widetilde{a}_t^i\left(h^t\right)\right\}_{i=1}^I$ satisfying $\sum_i \widetilde{a}_t^i\left(h^t\right) = 0$.

Definition

A **sequential trading competitive equilibrium** is an initial distribution of wealth $\overrightarrow{\widetilde{a}}_0\left(s_0\right)$, an allocation (sequence of allocations for all agents) $\left\{\widetilde{c}^i\right\}_{i=1}^I$ and pricing kernels $\widetilde{Q}_t\left(s_{t+1}|h^t\right)$ such that

1. for all i, given $\widetilde{a}_0^i\left(s_0\right)$ and $\widetilde{Q}_t\left(s_{t+1}|h^t\right)$, the consumption allocation $\widetilde{c}^i = \left\{\widetilde{c}_t^i\right\}_{t=0}^\infty$ solves agent i's optimization problem;
2. for all realizations of $\left\{h^t\right\}_{t=0}^\infty$ the agent's consumption allocation and implied asset portfolios $\left\{\widetilde{c}_t^i\left(h^t\right), \left\{\widetilde{a}_{t+1}^i\left(s_{t+1}, h^t\right)\right\}_{s_{t+1}}\right\}_{t\in\mathbb{N}}$ satisfy $\sum_i \widetilde{c}_t^i\left(h^t\right) = \sum_i y_t^i\left(h^t\right)$ and $\sum_i \widetilde{a}_{t+1}^i\left(s_{t+1}, h^t\right) = 0$ for all s_{t+1}.

Theorem

The time-0 trading arrangement in the Arrow-Debreu equilibrium with complete markets has the same allocations as the sequential trading arrangement with one-period complete Arrow securities,

$$\{c^i\}_{i=1}^I = \{\tilde{c}^i\}_{i=1}^I ,$$

for an appropriate initial distribution of wealth in the sequential markets equilibrium, $\{\tilde{a}_0^i(s_0)\}_{i=1}^I$.

Proof.

First we show "ADE $\Rightarrow$ SME".

- Take Arrow-Debreu equilibrium $q_t^0\left(h^t\right)$ as given.
- Suppose $\exists\ \widetilde{Q}_t\left(s_{t+1}|h^t\right)$ satisfying recursion

$$\begin{aligned} q_{t+1}^0\left(h^{t+1}\right) &= \widetilde{Q}_t\left(s_{t+1}|h^t\right) q_t^0\left(h^t\right) \\ \Leftrightarrow &\quad \widetilde{Q}_t\left(s_{t+1}|h^t\right) = \frac{q_{t+1}^0\left(h^{t+1}\right)}{q_t^0\left(h^t\right)} = q_{t+1}^t\left(h^{t+1}\right). \end{aligned}$$

- To show guess is true, take Arrow-Debreu equilibrium first-order conditions from two succesive periods and write:

$$\beta \frac{u'\left(c_{t+1}^i\left(h^{t+1}\right)\right)}{u'\left(c_t^i\left(h^t\right)\right)} \pi_t\left(h^{t+1}|h^t\right) = \frac{q_{t+1}^0\left(h^{t+1}\right)}{q_t^0\left(h^t\right)}$$

Proof (cont'd).

- But then if guess is true, it must be that

$$\beta \frac{u'\left(c_{t+1}^{i}\left(h^{t+1}\right)\right)}{u'\left(c_{t}^{i}\left(h^{t}\right)\right)} \pi_{t}\left(h^{t+1} | h^{t}\right) = \widetilde{Q}_{t}\left(s_{t+1} | h^{t}\right) = \beta \frac{u'\left(\widetilde{c}_{t+1}^{i}\left(h^{t+1}\right)\right)}{u'\left(\widetilde{c}_{t}^{i}\left(h^{t}\right)\right)} \pi_{t}\left(h^{t+1} | h^{t}\right).$$

- So then, Arrow-Debreu equilibrium is equivalent to the sequential markets equilibrium in terms of allocations,

$$\left\{c^{i}\right\}_{i=1}^{I} = \left\{\widetilde{c}^{i}\right\}_{i=1}^{I}.$$

Proof (cont'd).

Next we show "ADE $\Leftarrow$ SME":

- Pick $\left\{\widetilde{a}_0^i\left(s_0\right)\right\}_{i=1}^I$ s.t. SBCs in SME consistent with IBC for ADE. Guess that $\left\{\widetilde{a}_0^i\left(s_0\right)\right\}_{i=1}^I = \mathbf{0}_{I\times 1}$.
- Why? In Arrow-Debreu equilibrium, at time 0, agents bring in only their endowments, $y_0\left(s_0\right)$.
- At $t \geq 0$ and history h^t, i chooses asset portfolio, $\widetilde{a}_{t+1}^i\left(s_{t+1}, h^t\right) = \Omega_{t+1}^i\left(h^{t+1}\right)$ for all s_{t+1}.
- The expected value in date t terms is

$$\sum_{s_{t+1}} \widetilde{a}_{t+1}^i\left(s_{t+1}, h^t\right) \quad \widetilde{Q}_t\left(s_{t+1}|h^t\right) = \sum_{s_{t+1}} \Omega_{t+1}^i\left(h^{t+1}\right) q_{t+1}^t\left(h^{t+1}\right)$$

$$= \sum_{\tau=t+1}^{\infty} \sum_{h^\tau|h^t} q_\tau^t\left(h^\tau\right)\left[c_\tau^i\left(h^\tau\right) - y_\tau^i\left(h^\tau\right)\right]$$

just the tail value of wealth!

Proof (cont'd).

- Show that i can afford this portfolio strategy. Use SME SBCs. At time 0 given $\widetilde{a}_0^i(s_0) = 0$,

$$\widetilde{c}_0^i(s_0) + \sum_{t=1}^{\infty} \sum_{h^t} q_t^0(s_t) \left[c_t^i(h^t) - y_t^i(h^t) \right] = y_t^i(s_0) + 0$$

- But this is the same as IBC in the ADE.
- So $\widetilde{c}_0^i(s_0) = c_0^i(s_0)$.

Proof.

- For all $t > 0$, we can write $\widetilde{a}_t^i\left(h^t\right) = \Omega_t^i\left(h^t\right)$, and the time t, h^t-BC is

$$\widetilde{c}_t^i\left(h^t\right) + \sum_{s_{t+1}} \widetilde{a}_{t+1}^i\left(s_{t+1}, h^t\right)\widetilde{Q}_t\left(s_{t+1}|h^t\right) = y_t^i\left(h^t\right) + \widetilde{a}_t^i\left(h^t\right)$$

$$\Rightarrow \sum_{s_{t+1}} \widetilde{a}_{t+1}^i\left(s_{t+1}, h^t\right)\widetilde{Q}_t\left(s_{t+1}|h^t\right) = \Omega_t^i\left(h^t\right) - \left[\widetilde{c}_t^i\left(h^t\right) - y_t^i\left(h^t\right)\right]$$

$$\Rightarrow \sum_{s_{t+1}} \Omega_{t+1}^i\left(h^{t+1}\right) q_{t+1}^t\left(h^{t+1}\right) = \Omega_t^i\left(h^t\right) - \left[\widetilde{c}_t^i\left(h^t\right) - y_t^i\left(h^t\right)\right]$$

$$\Rightarrow \sum_{\tau=t+1}^{\infty} \sum_{h^\tau|h^t} q_\tau^t\left(h^\tau\right)\left[c_\tau^i\left(h^\tau\right) - y_\tau^i\left(h^\tau\right)\right]$$

$$= \Omega_t^i\left(h^t\right) - \left[\widetilde{c}_t^i\left(h^t\right) - y_t^i\left(h^t\right)\right]$$

It then follows that $\widetilde{c}_t^i\left(h^t\right) = c_t^i\left(h^t\right)$ for all t and h^t. □

Notes

The equivalence between Arrow-Debreu equilibrium and Arrow's sequential markets equilibrium follows from two key factors:

- Agents are v.N-M expected utility maximizers – their once-and-for-all time 0 choices are *time consistent*. Past actions affect future payoffs but future actions do not affect past payoffs.
- Under complete markets, the budget sets defined by the two formulations are equivalent, and thus Arrow-Debreu equilibrium prices of contingent claims are equal to Arrow's spot prices weighted by the price in period 1 of the appropriate Arrow security.

Recursive (Markov) Competitive Equilibrium

- The assumptions about the state variables so far in the Arrow-Debreu equilibrium and sequential markets equilibrium economies are too general to be useful – at each t, state variables are made up of the entire history leading up to t, i.e. $h^t := (s_0, s_1, ..., s_t)$.
- For practical purposes, we need to discipline the evolution of the state further – e.g. to be Markovian – so only a few state variables suffice to describe the position of the economy at each time period.
- We'll look at a *recursive competitive equilibrium* formulation of the sequential markets equilibrium and Arrow-Debreu equilibrium.

Endowments with Markov property

Consider the state space, S. So exogenous event is $s \in S$. Let s be governed by a Markov chain:

- $\pi_0(s) = \Pr(s_0 = s)$.
- $\pi(s'|s) = \Pr(s_{t+1} = h'|s_t = s)$.

The Markov chain induces a sequence of probability measures on histories h^t. The probability of realizing history h^t is

$$\pi_t(h^t) = \pi(s_t|s_{t-1})\pi(s_{t-1}|s_{t-2})\cdots\pi(s_1|s_0)\pi_0(s_0).$$

where it is assumed $\pi_0(s_0) = 1$.

The Markov property says that

$$\pi_t\left(h^t|h^k\right) = \pi\left(s_t|s_{t-1}\right)\pi\left(s_{t-1}|s_{t-2}\right)\cdots\pi\left(s_{k+1}|s_k\right)$$

where $\pi_t\left(h^t|h^k\right)$ depends only on state s_k at $k<t$ and the history prior to k is redundant.

Example

$\pi_3\left(h^3|h^2\right) = \pi_3\left(\left(s_0,s_1,s_2,s_3\right)|\left(s_0,s_1,s_2\right)\right) = \pi\left(s_3|s_2\right).$

Then, for each $i=1,...,I$, we can write endowments as

$$y_t^i\left(h^t\right) = y^i\left(s_t\right)$$

Since s_t is a Markov process, $y_t^i\left(s_t\right)$ will also be a Markov process.

Equilibrium inherits Markov property

Theorem

Given $y_t^i\left(s^t\right)$ *a Markov process, the Arrow-Debreu equilibrium price of date-*τ *history* h^τ *consumption goods in terms of date* t, $0 \leq t \leq \tau$, *history* h^t *goods is* **not history dependent:** $q_\tau^t\left(h^\tau\right) = q_j^k\left(\widetilde{h}^k\right)$ *for* $j, k \geq 0$ *such that* $\tau - t = k - j$ *and* $\left(s_t, s_{t+1}, ..., s_\tau\right) = \left(\widetilde{s}_j, \widetilde{s}_{j+1}, ..., \widetilde{s}_k\right)$.

Remark. Natural debt limits and household wealth are also history independent: $A_t^i\left(s^t\right) = A^i\left(s_t\right)$ and $\Omega_t^i\left(s^t\right) = \Omega^i\left(s_t\right)$.

- Each agent enters every period with wealth independent of past endowment realizations.
- Past trades have fully insured away all idiosyncratic endowment risks.
- So an agent enters the current period with current-state contingent wealth just sufficient to fund a trading scheme that insures against future idiosyncratic risks.
- The pricing kernel $Q\left(s_t | s_{t-1}\right)$ thus provides the correct signal, along with market clearing, to coordinate trade in time $t-1$ such that all idiosyncratic risks are eliminated.
- However, if there are aggregate risks, they would still have to be borne by all agents.

Now $y_t^i(s_t)$ Markov.

Agent i's competitive equilibrium sequence problem, given $Q(s'|s)$, is now recursive:

$$v^i(a,s) = \max_{c,\widehat{a}(s')} \left\{ u(c) + \beta \sum_{s'} v^i\left(\widehat{a}(s'), s'\right) \pi(s'|s) \right\}$$

subject to

$$c + \sum_{s'} \widehat{a}(s') Q(s'|s) \leq y^i(s) + a$$

$$c \geq 0$$

$$-\widehat{a}(s') \leq A^i(s') \text{ for all } s' \in S.$$

Let the optimal decision rules associated with the fixed-point solution of the Bellman equation be

$$c = h^i(a, s)$$

$$\widehat{a}(s') = g^i(a, s, s')$$

for each $i = \{1, ..., I\}$.

We can show that this optimal solution depends on the price kernel $Q(s'|s)$.

Evaluate the first-order condition for the RHS of each Bellman equation and apply the Benveniste-Scheinkman formula to get

$$Q(s_{t+1}|s_t) = \beta \frac{u'\left(c^i_{t+1}\right)}{u'\left(c^i_t\right)} \pi(s_{t+1}|s_t)$$

where $c = h^i(a, s)$ and $\widehat{a}(s') = g^i(a, s, s')$.

Definition

A recursive competitive equililibrium is an initial distribution of wealth $\{a_0^i\}_{i=1}^I$ a pricing kernel Q$(s'|s)$, sets of value functions $\{v^i(a,s)\}_{i=1}^I$ and decision rules $\{h^i(a,s), g^i(a,s,s')\}_{i=1}^I$ such that

1. for all i, given a_0^i and the pricing kernel, the decision rules solve the household i's problem;
2. for all histories $\{s_t\}_{t=0}^\infty$, the consumption and assets $\{\{c_t^i, \{\widehat{a}_{t+1}^i(s')\}_{s'}\}_i\}_{t=0}^\infty$ implied by the decision rules satisfy $\sum_i c_t^i = \sum_i y^i(s_t)$ and $\sum_i \widehat{a}_{t+1}^i(s') = 0$ for all t and s'.

j-step ahead pricing kernel

- Since a complete set of markets exists for all j periods ahead contingent claims,
- a consumer i, at the end of period t, can always buy $z^i_{t,j}(s_{t+j})$ units of contingent consumption claims, for $j \geq 1$.
- Recall that agent i's sequential budget constraint is

$$c^i_t + \sum_{s_{t+1}} Q_1(s_{t+1}|s_t)a^i_{t+1}(s_{t+1}) \leq y^i(s_t) + a^i_t.$$

- The agent's next period wealth depends on next period state s_{t+1} and the composition of the asset portfolio:

$$a^i_{t+1}(s_{t+1}) = z^i_{t,1}(s_{t+1}) + \sum_{j=2}^{\infty} \sum_{s_{t+j}} Q_{j-1}(s_{t+j}|s_{t+1}) z^i_{t,j}(s_{t+j}).$$

  So the outcome s_{t+1} will determine which element of the n-dimensional vector

$$\begin{bmatrix} z^i_{t,1}(s_{t+1} = s_1) \\ \vdots \\ z^i_{t,1}(s_{t+1} = s_n) \end{bmatrix}$$

  pays off at time $t + 1$.

- But this is only one component of time $t + 1$ wealth.

$$a^i_{t+1}(s_{t+1}) = z^i_{t,1}(s_{t+1}) + \sum_{j=2}^{\infty} \sum_{s_{t+j}} Q_{j-1}(s_{t+j}|s_{t+1}) z^i_{t,j}(s_{t+j}).$$

- The second term on the RHS, conditional of outcome s_{t+1} when time $t+1$ arrives, is the expected (capital gains or losses) from holding longer term claims on $t+1+j$, $j \geq 1$, consumption.
- Together they make up next-period – i.e. $t+1$ – wealth *if* $s_{t+1} \in S$ is to be realized.
- Using this fact, the sequence of budget constraints becomes

$$c^i_t + \sum_{j=1}^{\infty} \sum_{s_{t+j}} Q_j(s_{t+j}|s_t) z^i_{t,j}(s_{t+j}) \leq y^i(s_t) + a^i_t.$$

Note that the first-order condition for a optimal plan by agent i will imply that

$$Q_j(s_{t+j}|s_t) = \beta \sum_{s_{t+1} \in S} \frac{u'[c^i_{t+1}(s_{t+1})]}{u'[c^i_t(s_t)]} \pi(s_{t+1}|s_t) Q_{j-1}(s_{t+j}|s_{t+1}).$$

Since agent's optimal RCE strategy also requires satisfying,

$$Q_1(s_{t+1}|s_t) := Q(s_{t+1}|s_t) = \beta \frac{u'\left(c^i_{t+1}\right)}{u'\left(c^i_t\right)} \pi(s_{t+1}|s_t)$$

we then have

$$Q_j(s_{t+j}|s_t) = \sum_{s_{t+1} \in S} Q_1(s_{t+1}|s_t) Q_{j-1}(s_{t+j}|s_{t+1}),$$

a recursive formula for computing j-step ahead pricing kernels for $j = 2, 3, \ldots$.

Arbitrage-free pricing and redundant assets

- Suppose, apart from purchasing $z_{t,j}(s_{t+j})$ units of j-step ahead complete Arrow securities, Sam also trades an ex-dividend stock called a Lucas tree.
- A unit of this stock allows Sam to have the right to a unit of fruit or dividend $d(s_{t+1})$ from this Lucas tree, if state s_{t+1} occurs.
- Sam can buy N_t units of this stock. The ex-divident price is $p(s_t)$.
- So Sam can obtain $N_t[p(s_{t+1}) + d(s_{t+1})]$ units of consumption in $t+1$ if s_{t+1} occurs then.

See notes and LS, Ch.8 for details....

In equilibrium, two arbitrage-free pricing conditions

$$p(s_t) = \sum_{s_{t+1}} Q_1(s_{t+1}|s_t)[p(s_{t+1}) + d(s_{t+1})],$$

$$Q_j(s_{t+j}|s_t) = \sum_{s_{t+1}} Q_{j-1}(s_{t+j}|s_{t+1})Q_1(s_{t+1}|s_t), \qquad j = 2, 3, \ldots.$$

for all $t \in \mathbb{N}$.

Meaning?