COMPLETE MARKETS, GROWTH AND REAL BUSINESS CYCLES

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Abstract

The first part of this note follows closely Chapter 12 of Ljungqvist and Sargent (2004). The note is very preliminary and is to be used as a guide only. You are still advised to study the main text itself. I will provide an example in class to illustrate the general model. We will also consider two methods of computing the recursive competitive equilibrium in the real business cycle model. Suggestions and typo alerts are greatly appreciated. This is your chance to have your name branded into the posterior versions of these lecture notes. ◀ ▷

Contents

> Macroeconomics has progressed beyond the stage of searching for a theory to the stage of deriving the implications of theory. In this way, macroeconomics has become like the natural sciences. Unlike the natural sciences, though, macroeconomics involves people making decisions based on what they think will happen, and what will happen depends on what decisions they make. This means that the concept of equilibrium must be dynamic, and [...] this dynamism is at the core of modern macroeconomics.
>
> Edward C. Prescott (2006).

Part I

The first part of this note follows closely Chapter 12 of Ljungqvist and Sargent (2004). We consider a more general version of the stochastic growth model with endogenous labor supply. First we recall a version of the Pareto problem for the planning economy under a benevolent social planner. From this we can characterize the Pareto optimal allocations across time and states. Second, we consider decentralized, or competitive equilibrium, allocation by some market pricing mechanism. We decentralize the economy into markets with the following assumptions:

- There is a market for state/history contingent securities (either Arrow-Debreu or one-period Arrow securities),
- Households supply labor service to Firm I and initial capital endowment to Firm II;
- Firm II produces new capital stock and rents it to Firm I; and
- Firm I produces output that can be sold as consumption (to household) and/or investment good (to Firm II).

In this market economy with general history dependence, we show that competitive equilibrium allocations (under either securities markets assumption) are Pareto optimal, thus restating a version of the First Fundamental Welfare Theorem.

Third, we proceed to make the model more useful by assuming that the history of technology shocks are generated by a Markov process. This allows us to recast the general competitive equilibrium into one that is a *recursive competitive equilibrium*. Here we will distinguish between individual state variables (individual capital stock, k) and aggregate states (aggregate capital stock, K) in characterizing the equilibrium. However, the requirement of equilibrium also prescribes that individual choices must be consistent with the aggregate. Since we have identical agents in this model, all that is required in equilibrium is $k = K$. Further, since the model is stochastic and agents are solving their individual forward-looking optimization problems, equilibrium also requires that individual beliefs (governed by the subjective probabilities they assign to stochastic events *and* the transition law of the individual states) must be consistent with actual equilibrium beliefs (the objective probability distribution and transition law of the aggregate state). This is the bit that contributes to what is often called a "rational expectations equilibrium". These conditions, along with a pricing system that clears markets and the requirement that all agents are optimizing, define a recursive competitive equilibrium.

1. Endogenous aggregate state variables

In the previous pure endowment economies, we showed equivalence between allocations in the Arrow-Debreu economy and the sequential markets economy that trades one-period Arrow securities. We then assumed endowments were Markovian stochastic processes. Arrow securities can then be made functions of an aggregate current state vector, s_t. This allowed a recursive formulation of the household problems.

This equivalence is not so straightforward when part of the aggregate state variable is endogenous. We will look at the simple stochastic growth model and cast it in terms of decentralized competitive equilibrium. We will then show that the equivalence between allocations in the Arrow-Debreu economy and the sequential markets economy that trades one-period Arrow securities still arises in the competitive economies with production. However, the requirements here are a little bit more involved since we need to track the history of equilibrium interactions between agents' (i.e. households and firms) decisions.

The stochastic growth framework under decentralized competitive equilibrium provides a basic foundation for the modeling of aggregate variables of an economy. More recent variations on the theme have built in market imperfections, such as monopolistically competitive firms, pricing rigidity, information asymmetry, lack of contractual enforcement, distortionary taxation and so on, to allow roles for government intervention such as monetary and fiscal policy. We will sample a few of these economies in the topics to come.

2. Pareto optimality in the stochastic growth model

Here we consider the discrete state-space version of a stochastic growth model with endogenous labor supply. The only source of stochastic shocks are shifts to the production function.[1] We do not impose any restriction on the stochastic process at this stage, thus allowing for allocations to be history dependent in general. Later we will look at Markovian shocks which allow us to focus on recursive competitive equilibria where allocations are functions of only the current aggregate state of the economy.

We define the following objects, along with some assumptions, for the model:

- Production technology is subject to stochastic shocks.
- Stochastic event, $s_t \in S$.
- History of events leading up to $t \geq 0$ is $h^t = (s_t, s_{t-1}, ..., s_0)$.
- Unconditional probability of history h^t is $\pi_t\left(h^t\right)$.
- Conditional probability $\pi_\tau\left(h^\tau | h^t\right)$.
- Assume $\pi_0\left(s_0\right) = 1$ for any $s_0 \in S$.
- Goods are differentiated by history/state, so the commodity space is represented by S.
- Identical households and firms (so we can drop the i's from individual decision rules).

[1] We can introduce more shocks – e.g. on the preference side for example – to make the model quantitatively richer. We leave that as an exercise for the reader.

2.1 Planner's problem

First we set up the Pareto problem. To do so, we outline the components representing preferences, technology, and resource constraint. Then we characterize the Pareto optimal allocation.

2.1.1. Preferences

The representative (average) household orders consumption and leisure streams according to the following criterion:

$$\sum_{t=0}^{\infty}\sum_{h^t}\beta^t u\left(c_t\left(h^t\right),l_t\left(h^t\right)\right)\pi_t\left(h^t\right)$$

with $\beta \in (0,1)$ and $u_i(c,l) > 0$ and $u_{ii}(c,l) < 0$, and assuming Inada conditions $\lim_{i\to 0} u_i(c,l) = +\infty$ for $i = c, l$.

The household faces a time constraint, with total time endowment normalized to 1, without loss of generality:

$$1 = l_t\left(h^t\right) + n_t\left(h^t\right) \tag{1}$$

where $l_t\left(h^t\right)$ is leisure and $n_t\left(h^t\right)$ is labor. Households are assumed to be endowed with initial capital stock k_0.

2.1.2. Technology

Let the function $A : S^t \to \left[\underline{A}, \overline{A}\right] \subset \mathbb{R}$ such that for each $h^t \in S^t$, $A_t\left(h^t\right)$ is the stochastic level of productivity. Let $F : \mathbb{R}_+ \times [0,1] \to \mathbb{R}_+$ be the aggregate production function. Given capital stock and labor inputs (k,n), the output is $F(k,n)$.

ASSUMPTION 1. *F satisfies:*

1. *Constant returns to scale (homogeneous of degree 1):* $F(k,n) = nF\left(\hat{k},1\right) := nf\left(\hat{k}\right)$ *where* $\hat{k} = k/n$.
2. $F_i(k,n) > 0$ *and* $F_{ii}(k,n) < 0$
3. *Inada conditions:* $\lim_{i\to 0} F_i(k,n) = \infty$ *and* $\lim_{i\to\infty} F_i(k,n) = 0$,

for $i = k, n$.

Note that if F is homogeneous of degree 1, then $F(k,n) = nF\left(\hat{k},1\right)$ and the first derivatives are homogeneous of degree 0 functions:

$$F_k(k,n) = \frac{\partial nf(k/n)}{\partial k} = f'\left(\hat{k}\right),$$

$$F_n(k,n) = \frac{\partial nf(k/n)}{\partial n} = f\left(\hat{k}\right) - f'\left(\hat{k}\right)\hat{k}.$$

Let $\delta \in (0,1]$ be the depreciation rate of capital. Net capital stock at the end of period t under history h^t must equal new investment:

$$k_{t+1}\left(h^t\right) - (1-\delta)\,k_t\left(h^{t-1}\right) = x_t\left(h^t\right). \tag{2}$$

2.1.3. Resource constraint

Aggregate resource demand in terms of consumption and investment must be feasibly met by aggregate output:

$$c_t\left(h^t\right)+x_t\left(h^t\right)\leq A_t\left(h^t\right)F\left(k_t\left(h^{t-1}\right),n_t\left(h^t\right)\right).$$

2.1.4. Pareto problem

Consider centralized planner's problem. The planner controls allocation

$$\left\{c_t\left(h^t\right),l_t\left(h^t\right),x_t\left(h^t\right),n_t\left(h^t\right),k_{t+1}\left(h^t\right)\right\}_{t=0}^{\infty}.$$

Given k_0 and $A_t\left(h^t\right)$ the planner maximizes

$$\begin{aligned}L=\sum_{t=0}^{\infty}\sum_{h^t}\beta^t\pi_t\left(h^t\right)&\left\{u\left(c_t\left(h^t\right),1-n_t\left(h^t\right)\right)\right.\\&+\mu_t\left(h^t\right)\left[A_t\left(h^t\right)F\left(k_t\left(h^{t-1}\right),n_t\left(h^t\right)\right)+(1-\delta)k_t\left(h^{t-1}\right)\right.\\&\left.\left.-c_t\left(h^t\right)-k_{t+1}\left(h^t\right)\right]\right\}.\end{aligned}$$

The first-order conditions with respect to $c_t\left(h^t\right),n_t\left(h^t\right)$, and $k_{t+1}\left(h^t\right)$ are:

$$\begin{aligned}&u_c\left(c_t\left(h^t\right)\right)=\mu_t\left(h^t\right),\\&u_l\left(n_t\left(h^t\right)\right)=\mu_t\left(h^t\right)A_t\left(h^t\right)F_n\left(k_t\left(h^{t-1}\right),n_t\left(h^t\right)\right),\\&\pi_t\left(h^t\right)\mu_t\left(h^t\right)=\beta\sum_{h^{t+1}|h^t}\pi_{t+1}\left(h^{t+1}\right)\mu_{t+1}\left(h^{t+1}\right)\\&\qquad\cdot\left[A_{t+1}\left(h^{t+1}\right)F_k\left(k_{t+1}\left(h^t\right),n_{t+1}\left(h^{t+1}\right)\right)+(1-\delta)\right],\end{aligned}$$

for all $t\geq 0$ and all h^t. The first-order condition with respect to $\mu_t(h^t)$ yields the resource constraint

$$c_t\left(h^t\right)+x_t\left(h^t\right)\leq A_t\left(h^t\right)F\left(k_t\left(h^{t-1}\right),n_t\left(h^t\right)\right),\tag{3}$$

for all $t\geq 0$ and all h^t.

Simplify further to get rid of the dynamic Lagrange multipliers:

$$\frac{u_l\left(n_t\left(h^t\right)\right)}{u_c\left(c_t\left(h^t\right)\right)}=A_t\left(h^t\right)F_n\left(k_t\left(h^{t-1}\right),n_t\left(h^t\right)\right)\tag{4}$$

This static or intratemporal optimality condition tells us that the planner allocates resources optimally to the point where the marginal rate of substitution between consumption and leisure is equal to the opportunity cost of working as measured by the marginal product of labor. This condition characterizes the optimal allocation of labor/leisure.

The last optimality condition is quite familiar to us by now:

$$\begin{aligned}1=\beta\sum_{h^{t+1}|h^t}\pi_{t+1}\left(h^{t+1}|h^t\right)&\frac{u_c\left(c_{t+1}\left(h^{t+1}\right)\right)}{u_c\left(c_t\left(h^t\right)\right)}\\&\cdot\left[A_{t+1}\left(h^{t+1}\right)F_k\left(k_{t+1}\left(h^t\right),n_{t+1}\left(h^{t+1}\right)\right)+(1-\delta)\right]\end{aligned}\tag{5}$$

for all t and all h^t. This is an Euler equation giving us the price of a time t history h^t claim to a random one-period gross return on capital. It is also the intertemporal condition for optimal state-contingent consumption allocation.

Definition 1. *A Pareto optimal allocation or plan is the bounded sequence of functions* $\{c_t(h^t), n_t(h^t), k_{t+1}(h^t), l_t(h^t), x_t(h^t)\}_{t=0}^{\infty}$ *satisfying*

1. *feasibility and optimality conditions in equations (3), (4), (5),*
2. *the time constraint and capital accumulation equation in (1) and (2), respectively,*
3. *initial conditions* $k_0 > 0$ *and* s_0 *given, and a boundary condition*

$$\lim_{t\to\infty} \sum_{h^t} \pi_t(h^t|s_0)\beta^{t+1}k_{t+1}(h^t) = 0.$$

3. Competitive equilibrium in the stochastic growth model

Now we turn to markets and consider how real allocations are made when there is no benevolent planner. We consider the two formulations for a decentralized, competitive equilibrium for the stochastic growth model.

- Time-0 trading Arrow-Debreu economy, and
- Sequential markets with Arrow securities.

Either formulation of the competitive equilibrium will give identical allocations. Further, the allocations are one possible outcome of the social planner's Pareto problem.

3.1 Time-0 trading

Consider first the time-0 trading Arrow-Debreu economy. We assume the existence of a complete set of time- and history-contingent securities. Firms are perfectly competitive in the product and factor markets.

We follow the decentralization assumptions in Ljungqvist and Sargent (2004):

- Representative household buys consumption, supply labor and sells initial capital endowment to Firm II.
- Firm I hires labor from household and capital from Firm II. The delivery of labor service and its price is time and history contingent.
- Firm I produces consumption and investment goods using technology F. The delivery of consumption/investment good and its price is time and history contingent.
- Firm II buys k_0 from h/hold and investment good from Firm I.
- Firm II produces new capital stock to rent to Firm I. The delivery of capital service and its price is time and history contingent.

Thus there is a competitive labor market, product (consumption and investment good) market, a physical capital market, and a market for Arrow-Debreu contingent claims on all these resources.

3.1.1. Household

Households maximizes

$$\sum_{t=0}^{\infty} \sum_{h^t} \beta^t \pi_t\left(h^t\right) u\left(c_t\left(h^t\right), 1 - n_t\left(h^t\right)\right)$$

subject to the time-0 intertemporal budget constraint:

$$\sum_{t=0}^{\infty}\sum_{h^t} q_t^0\left(h^t\right) c_t\left(h^t\right) \leq \sum_{t=0}^{\infty}\sum_{h^t} w_t^0\left(h^t\right) n_t\left(h^t\right) + p_{k0}k_0$$

The first-order conditions with respect to $c_t(h^t)$ and $n_t(h^t)$ are:

$$\beta^t u_c\left(c_t\left(h^t\right), 1-n_t\left(h^t\right)\right)\pi_t\left(h^t\right) = \eta q_t^0\left(h^t\right) \tag{6}$$

$$\beta^t u_l\left(c_t\left(h^t\right), 1-n_t\left(h^t\right)\right)\pi_t\left(h^t\right) = \eta w_t^0\left(h^t\right). \tag{7}$$

for all t and all h^t, and where $\eta > 0$.

3.1.2. Type I Firm

Firm I enters into state-contingent contracts at $t = 0$ to rent capital $k_t^I\left(h^t\right)$ and labor $n_t\left(h^t\right)$ for each possible h^t and $t \geq 0$. Firm I maximizes time-0 value of expected profits (total revenue from sales in the product market net of capital and labor rental costs)

$$\sum_{t=0}^{\infty}\sum_{h^t} q_t^0\left(h^t\right)\left[c_t\left(h^t\right) + x_t\left(h^t\right)\right] - r_t^0\left(h^t\right) k_t^I\left(h^t\right) - w_t^0\left(h^t\right) n_t\left(h^t\right)$$

subject to

$$c_t\left(h^t\right) + x_t\left(h^t\right) \leq A_t\left(h^t\right) F\left(k_t^I\left(h^t\right), n_t\left(h^t\right)\right). \tag{8}$$

Note that $q_t^0\left(h^t\right)$ is the time-t, history h^t contingent Arrow-Debreu price of one unit of output (transformable into consumption or investment goods) in time-0 terms. Likewise $r_t^0(h^t)$ and $w_t^0(h^t)$ are the corresponding contingent prices for capital and labor services.

Re-write Firm I's problem as maximizing

$$\sum_{t=0}^{\infty}\sum_{h^t} n_t\left(h^t\right)\left\{q_t^0\left(h^t\right) A_t\left(h^t\right) f\left(\hat{k}_t\left(h^t\right)\right) - r_t^0\left(h^t\right)\hat{k}_t^I\left(h^t\right) - w_t^0\left(h^t\right)\right\}$$

Optimality with respect to $\hat{k}_t^I\left(h^t\right)$ is

$$r_t^0\left(h^t\right) = q_t^0\left(h^t\right) A_t\left(h^t\right) f'\left(\hat{k}_t^I\left(h^t\right)\right)$$

Optimal labor demand decision by Firm I each period is characterized by

$$\left\{q_t^0\left(h^t\right) A_t\left(h^t\right)\left[f\left(\hat{k}_t\left(h^t\right)\right) - f'\left(\hat{k}_t^I\left(h^t\right)\right)\hat{k}_t^I\left(h^t\right)\right] - w_t^0\left(h^t\right)\right\}$$

$$\begin{cases} < 0, n_t^*\left(h^t\right) = 0 \\ = 0, n_t^*\left(h^t\right) \in [0,\infty) \\ > 0, n_t^*\left(h^t\right) = \infty \end{cases}$$

Inada conditions on $F(k,n)$ imply that $k_t^I\left(h^t\right), n_t^*\left(h^t\right) \in (0,\infty)$. So the stuff in curly brackets must equal zero giving us the optimal labor demand equation:

$$\begin{aligned} w_t^0\left(h^t\right) &= q_t^0\left(h^t\right) A_t\left(h^t\right)\left[f\left(\hat{k}_t\left(h^t\right)\right) - f'\left(\hat{k}_t^I\left(h^t\right)\right)\hat{k}_t^I\left(h^t\right)\right] \\ &= q_t^0\left(h^t\right) A_t\left(h^t\right) F_n\left(k_t^I\left(h^t\right), n_t\left(h^t\right)\right) \end{aligned} \tag{9}$$

and capital demand

$$r_t^0\left(h^t\right) = q_t^0\left(h^t\right) A_t\left(h^t\right) F_k\left(k_t^I\left(h^t\right), n_t\left(h^t\right)\right). \tag{10}$$

3.1.3. Type II Firm

Firm II buys initial capital, k_0, from household and buys investment good from Firm I, and produces in new capital, which is rented to Firm I. Firm II maximizes

$$\sum_{t=0}^{\infty}\sum_{h^t}\left\{r_t^0\left(h^t\right) k_t^{II}\left(h^{t-1}\right) - q_t^0\left(h^t\right) x_t\left(h^t\right)\right\} - p_{k0}k_0^{II}$$

subject to

$$k_{t+1}^{II}\left(h^t\right) = (1-\delta) k_t^{II}\left(h^{t-1}\right) + x_t\left(h^t\right).$$

Note the subtle difference in history dependence in terms of the capital stock between both firms. From Firm II's perspective, it takes one period to assemble new capital before it becomes input for Firm I production. So it rents $k_t = k_t^{II}(h^{t-1})$, which is predetermined at time t, to firm I. In contrast, Firm I chooses how much time-t capital to rent contingent on history h^t, so that Firm I hires $k_t = k_t^I\left(h^t\right)$.

We can show that equilibrium prices satisfy

$$p_{k0} = r_0^0\left(s_0\right) + q_0^0\left(s_0\right)(1-\delta) \tag{11}$$

$$q_t^0\left(h^t\right) = \sum_{h^{t+1}|h^t}\left[r_{t+1}^0\left(h^{t+1}\right) + q_{t+1}^0\left(h^{t+1}\right)(1-\delta)\right] \tag{12}$$

since k_0^{II} and $k_{t+1}^{II}\left(h^t\right)$ are strictly positive and finite in equilibrium.

3.1.4. Equilibrium prices and quantities

The labor market clears in equilibrium. This is pinned down by substituting (9) into (7):

$$\beta^t u_l\left(c_t\left(h^t\right), 1-n_t\left(h^t\right)\right)\pi_t\left(h^t\right) = \eta q_t^0\left(h^t\right) A_t\left(h^t\right) F_n\left(k_t^I\left(h^t\right), n_t\left(h^t\right)\right) \tag{13}$$

and the capital market clears: substitute (12), (10) into (6):

$$\begin{aligned}
&\beta^t u_c\left(c_t\left(h^t\right), 1-n_t\left(h^t\right)\right)\pi_t\left(h^t\right) \\
&= \eta\sum_{h^{t+1}|h^t}\left[r_{t+1}^0\left(h^{t+1}\right) + q_{t+1}^0\left(h^{t+1}\right)(1-\delta)\right] \\
&\qquad\qquad = \eta\sum_{h^{t+1}|h^t} q_{t+1}^0\left(h^{t+1}\right)\left[A_{t+1}\left(h^{t+1}\right) F_k\left(h^{t+1}\right) + (1-\delta)\right]
\end{aligned} \tag{14}$$

Recall from (6) we have

$$q_t^0\left(h^t\right) = \frac{\beta^t u_c\left(c_t\left(h^t\right), 1-n_t\left(h^t\right)\right)\pi_t\left(h^t\right)}{\eta}$$

and we can do the same for $q_{t+1}^0\left(h^{t+1}\right)$.

Using these expressions in (13) and (14) we have, respectively

$$\frac{u_l\left(c_t\left(h^t\right), 1-n_t\left(h^t\right)\right)\pi_t\left(h^t\right)}{u_c\left(c_t\left(h^t\right), 1-n_t\left(h^t\right)\right)\pi_t\left(h^t\right)} = A_t\left(h^t\right) F_n\left(k_t^I\left(h^t\right), n_t\left(h^t\right)\right)$$

which is identical to the the planner's optimal labor allocation condition (4); and

$$\beta^t u_c\left(h^t\right)\pi_t\left(h^t\right) = \sum_{h^{t+1}|h^t} \beta^t u_c\left(h^{t+1}\right)\pi_{t+1}\left(h^{t+1}\right) \cdot \left[A_{t+1}\left(h^{t+1}\right)F_k\left(h^{t+1}\right) + (1-\delta)\right]$$

which is equal to the planner's optimal consumption allocation condition (5). Finally the product market also clears so that (8) holds with equality. By Walras law, we also know that the Arrow-Debreu securities market has to clear.

Definition 2. *An Arrow-Debreu competitive equilibrium in this economy is a sequence of pricing functions* $\{q_t^0(h^t), w_t^0(h^t), r_t^0(h^t)\}_{t\geq 0}$ *supporting the sequence of allocation functions* $\{c_t(h^t), n_t(h^t), k_{t+1}(h^t), x_t(h^t), l_t(h^t)\}_{t\geq 0}$ *such that*

1. *Households optimize: (6) and (7),*
2. *Firms maximize profits: (9)-(10), and (11)-(12), and*
3. *Markets clear: (13), (14) and (8).*

for all h^t *and all* $t \geq 0$.

3.1.5. A first fundamental welfare theorem

Again, without boring ourselves silly, we note that the Arrow-Debreu allocations will be identical to the Pareto optimal allocations chosen by the planner. This is another manifestation of the first fundamental welfare theorem in the case of a production economy with perfectly competitive markets without externalities or informational asymmetries.

From a solution point of view, when we have such economies, it is easier to solve the planner's problem first for the allocations and then we can back out the implied relative prices in the competitive equilibrium using the optimality conditions we have just derived. We summarize this solution strategy below:

1. Solve for allocations under PO problem.
2. *Given allocations*, work out relative prices from household and firms' first-order conditions:
   - $q_t^0\left(h^t\right)$ from (6),
   - $w_t^0\left(h^t\right)$ from (7),
   - $r_t^0\left(h^t\right)$ from (10), and p_{k0} from (11).
3. To determine absolute prices, pin down a numeraire. For example, set $q_0^0\left(s_0\right) = 1$ which is equivalent to $\eta = u_c\left(s_0\right)$.

3.1.6. Implied wealth dynamics

We can also track the implied household wealth (net of stream of labor income) over time. Convert all relative prices to time t history h^t values. For $\tau \geq t$:

$$q_\tau^t\left(h^\tau\right) := \frac{q_\tau^0\left(h^\tau\right)}{q_t^0\left(h^t\right)} = \beta^{\tau-t}\frac{u_c\left(c_\tau\left(h^\tau\right)\right)}{u_c\left(c_t\left(h^t\right)\right)}\pi_\tau\left(h^\tau|h^t\right).$$

$$w_\tau^t\left(h^\tau\right) := \frac{w_\tau^0\left(h^\tau\right)}{q_t^0\left(h^t\right)}$$

$$r_\tau^t\left(h^\tau\right) := \frac{r_\tau^0\left(h^\tau\right)}{q_t^0\left(h^t\right)}$$

We leave the reader here with a parting section gift of verifying that household wealth (value of all its time$-t$ and future net claims) in terms of date t, history h^t numeraire, is

$$\Upsilon_t\left(h^t\right) := \sum_{\tau=t}^{\infty}\sum_{h^\tau|h^t}\left\{q_\tau^t\left(h^\tau\right)c_\tau\left(h^\tau\right) - w_\tau^t\left(h^\tau\right)n_\tau\left(h^\tau\right)\right\}$$

$$\vdots$$

$$= \left[r_t^t\left(h^t\right) + (1-\delta)\right]k_t\left(h^{t-1}\right) \quad (15)$$

Economically, the household wealth at time t under history h^t equals the rental income from current capital service and the value of remaining undepreciated capital.

3.2 Sequential trading economy

As before in the pure exchange economy, we can show that the economy with sequential trading in Arrow securities are dynamically consistent with the time-0 trading assumption in the Arrow-Debreu world. Define "tilde" variables as sequential markets variables.

We will use the following notation. At time t history h^t, guess existence of

- wage rate $\widetilde{w}_t\left(h^t\right)$
- rental rate $\widetilde{r}_t\left(h^t\right)$
- Arrow security prices (pricing kernel) $\widetilde{Q}_t\left(s_{t+1}|h^t\right)$.

Let $\widetilde{a}_t\left(h^t\right)$ be claim to time t consumption that households brings into time t if history h^t is realized.

3.2.1. Household

Now the household faces a sequence of budget constraint, one for each h^t, and $t \geq 0$:

$$\widetilde{c}_t\left(h^t\right) + \sum_{s_{t+1}}\widetilde{a}_{t+1}\left(s_{t+1},h^t\right)\widetilde{Q}_t\left(s_{t+1}|h^t\right) \leq \widetilde{w}_t\left(h^t\right)\widetilde{n}_t\left(h^t\right) + \widetilde{a}_t\left(h^t\right).$$

where $\left\{\widetilde{a}_{t+1}\left(s_{t+1},h^t\right)\right\}_{s_{t+1}}$ is a **vector** of claims on time $t+1$ consumption. The dimension of $\left\{\widetilde{a}_{t+1}\left(s_{t+1},h^t\right)\right\}_{s_{t+1}}$ is the same as the number of possible s_{t+1} reachable, given history h^t.

We also need to rule out Ponzi schemes so that intertemporally, household budgets constraints are bounded. One way to do this is to use the *natural debt limit* which constrains debt levels state-by-state:

$$-\widetilde{a}_{t+1}\left(s_{t+1}\right) \leq A_{t+1}\left(h^{t+1}\right) = \sum_{\tau=t+1}^{\infty}\sum_{h^\tau|h^{t+1}}\widetilde{w}_\tau\left(h^\tau\right)\widetilde{n}_\tau\left(h^\tau\right).$$

Another way is to assume an arbitrary constant $A \geq 0$ such that

$$-\widetilde{a}_{t+1}\left(s_{t+1}\right) \leq A.$$

for all s_{t+1} given h^t. Here we can just set $A = 0$. Why?

The household maximizes the Lagrangian:

$$L = \sum_{t=0}^{\infty} \sum_{h^t} \left\{ \beta^t u\left(\widetilde{c}_t\left(h^t\right), 1 - \widetilde{n}_t\left(h^t\right)\right) \pi_t\left(h^t\right) \right.$$
$$+ \eta_t\left(h^t\right) \left[\widetilde{w}_\tau\left(h^\tau\right) \widetilde{n}_\tau\left(h^\tau\right) + \widetilde{a}_t\left(h^t\right) - \widetilde{c}_t\left(h^t\right) \right.$$
$$\left. \left. - \sum_{s_{t+1}} \widetilde{a}_{t+1}\left(s_{t+1}, h^t\right) \widetilde{Q}_t\left(s_{t+1} | h^t\right) \right] + \nu_t\left(h^t; s_{t+1}\right) \widetilde{a}^i_{t+1}\left(s_{t+1}\right) \right\}$$

Note that the Inada conditions (utility and production functions) imply that $\widetilde{c}_t\left(h^t\right) \in (0, \infty)$ and $\widetilde{n}_t\left(h^t\right) \in (0, \infty)$. So the conjecture $-\widetilde{a}^i_{t+1}\left(s_{t+1}\right) \leq A = 0$ is not binding so that $\nu_t\left(h^t; s_{t+1}\right) = 0$ almost surely.

Optimality for the household is characterized by

$$\widetilde{w}_\tau\left(h^\tau\right) = \frac{u_l\left(\widetilde{c}_t\left(h^t\right), 1 - \widetilde{n}_t\left(h^t\right)\right)}{u_c\left(\widetilde{c}_t\left(h^t\right), 1 - \widetilde{n}_t\left(h^t\right)\right)}$$

$$\widetilde{Q}_t\left(s_{t+1} | h^t\right) = \beta \frac{u_c\left(\widetilde{c}_{t+1}\left(h^{t+1}\right), 1 - \widetilde{n}_{t+1}\left(h^{t+1}\right)\right)}{u_c\left(\widetilde{c}_t\left(h^t\right), 1 - \widetilde{n}_t\left(h^t\right)\right)} \pi\left(h^{t+1} | h^t\right)$$

for all t, h^t and s_{t+1}.

3.2.2. *Type I Firm*

Each period, Firm I solves a static problem:

$$\max \left\{ A_t\left(h^t\right) F\left(\widetilde{k}^I_t\left(h^t\right), \widetilde{n}_t\left(h^t\right)\right) - \widetilde{r}_t\left(h^t\right) \widetilde{k}^I_t\left(h^t\right) - \widetilde{w}_t\left(h^t\right) \widetilde{n}_t\left(h^t\right) \right\}$$

with optimality given by:

$$\widetilde{r}_t\left(h^t\right) = A_t\left(h^t\right) F_k\left(\widetilde{k}^I_t\left(h^t\right), \widetilde{n}_t\left(h^t\right)\right)$$

$$\widetilde{w}_t\left(h^t\right) = A_t\left(h^t\right) F_n\left(\widetilde{k}^I_t\left(h^t\right), \widetilde{n}_t\left(h^t\right)\right).$$

Firm I makes zero profit, and is willing to produce whatever quantity the market demands:

$$A_t\left(h^t\right) F\left(\widetilde{k}^I_t\left(h^t\right), \widetilde{n}_t\left(h^t\right)\right) = \widetilde{c}_t\left(h^t\right) + \widetilde{x}_t\left(h^t\right).$$

3.2.3. *Type II Firm*

At each $t \geq 0$, Firm II chooses $\widetilde{k}^{II}_{t+1}\left(h^t\right)$ to solve

$$\max \left\{ \widetilde{k}^{II}_{t+1}\left(h^t\right) \sum_{s_{t+1}} \widetilde{Q}_t\left(s_{t+1} | h^t\right) \left[\widetilde{r}_{t+1}\left(h^{t+1}\right) + (1 - \delta)\right] - \widetilde{k}^{II}_{t+1}\left(h^t\right) \right\}.$$

where the term in curly parentheses is the expected $t+1$ rental revenue plus undepreciated (liquidation value) capital net of capital stock stored at end of t to be rented to Firm I in $t+1$.

The zero profit first-order condition is

$$\sum_{s_{t+1}} \widetilde{Q}_t\left(s_{t+1}|h^t\right)\left[\widetilde{r}_{t+1}\left(h^{t+1}\right)+(1-\delta)\right]-1=0. \tag{16}$$

The firm breaks even at any $\widetilde{k}_{t+1}^{II}\left(h^t\right)$ satisfying this condition.

3.2.4. Equilibrium prices and quantities

We can show that in terms of allocations, the sequential markets economy and the Arrow-Debreu economy are equivalent. That is,

$$\left\{\widetilde{c}_t\left(h^t\right),\widetilde{l}_t\left(h^t\right),\widetilde{x}_t\left(h^t\right),\widetilde{n}_t\left(h^t\right),\widetilde{k}_{t+1}\left(h^t\right)\right\}_{t=0}^{\infty}= \\ \left\{c_t\left(h^t\right),l_t\left(h^t\right),x_t\left(h^t\right),n_t\left(h^t\right),k_{t+1}\left(h^t\right)\right\}_{t=0}^{\infty}.$$

Here is an outline of the proof:

1. Guess price

$$\widetilde{Q}_t\left(s_{t+1}|h^t\right)=q_{t+1}^t\left(h^{t+1}\right)$$
$$\widetilde{w}_t\left(h^t\right)=w_t^t\left(h^t\right)$$
$$\widetilde{r}_t\left(h^t\right)=r_t^t\left(h^t\right)$$

2. Guess portfolio $\widetilde{a}_{t+1}\left(s_{t+1}\right)=\Upsilon_{t+1}\left(h^{t+1}\right)$ for all s_{t+1}.
3. Show asset portfolios are affordable given chosen allocations, where

$$\widetilde{a}_0=\left[r_0^0\left(s_0\right)+(1-\delta)\right]k_0=p_{k0}k_0.$$

Definition 3. *An sequential-markets competitive equilibrium in this economy is a sequence of pricing functions $\{Q(s_{t+1}|h^t),\widetilde{w}_t^0(h^t),\widetilde{r}_t^0(h^t)\}_{t\geq 0}$ supporting the sequence of allocation functions $\{c_t(h^t),n_t(h^t),k_{t+1}(h^t),x_t(h^t),l_t(h^t)\}_{t\geq 0}$ such that*

1. *Households optimize,*
2. *Firms maximize profits, and*
3. *Markets clear.*

for all h^t and all $t\geq 0$.

3.2.5. Why conjecture $A=0$ is correct

It turns out that Firm II is entirely owned by its creditor, the household. Firm II finances $\widetilde{k}_{t+1}^{II}\left(h^t\right)$ in time t, by issuing one-period securities that promise to pay

$$\left[\widetilde{r}_{t+1}\left(h^{t+1}\right)+(1-\delta)\right]\widetilde{k}_{t+1}^{II}\left(h^t\right)$$

for all s_{t+1}, t and h^t, to break even, according to (16).

Taking expectations at time t, history h^t (value of these payouts in time t consumption good):

$$\widetilde{k}_{t+1}^{II}\left(h^t\right)\sum_{s_{t+1}}\widetilde{Q}_t\left(s_{t+1}|h^t\right)\left[\widetilde{r}_{t+1}\left(h^{t+1}\right)+(1-\delta)\right]$$

The firm breaks even by issuing these claims to households – i.e. households are owners of Firm II.

Recall (15) implied wealth of Arrow-Debreu households:

$$\Upsilon_t\left(h^t\right)=\left[r_t^t\left(h^t\right)+(1-\delta)\right]k_t\left(h^{t-1}\right)=\widetilde{a}_{t+1}\left(s_{t+1}\right)$$

equals the sequential market economy households' desired next period beginning wealth (positive), which is the value of Firm II at t. That is,

$$-\widetilde{a}_{t+1}\left(s_{t+1}\right)<A=0.$$

4. A recursive stochastic growth model

We have so far taken history to be generated by some unspecified stochastic process. That resulted in allocations and prices being completely history dependent. So we had to describe optimization as a Lagrangian problem – a problem that we cannot hope to solve. If we force history to be induced by a Markov process, optimization can be recursively computed as a Bellman equation problem.

Now we specialize the exogenous forcing process for technology, $A_t\left(h^t\right)$. Let $s\in S$ and s_t is a Markov process $\left(\pi\left(s'|s\right),\pi_0\left(s_0\right)\right)$ with $\pi_0\left(s_0\right)=1$ for any $s_0\in S$.

The likelihood of a t-history, h^t can now be recursively computed:

$$\begin{aligned}\pi_t\left(h^t\right)&=\pi\left(s_t|s_{t-1}\right)\pi_{t-1}\left(h^{t-1}\right)\\&=\pi\left(s_t|s_{t-1}\right)\pi\left(s_{t-1}|s_{t-2}\right)\cdots\pi\left(s_1|s_0\right)\pi_0\left(s_0\right)\end{aligned}$$

by exploiting the transition probability function or the Markov matrix in the case of finite states for the shock. Note that now

$$\frac{\pi_t\left(h^{t+1}\right)}{\pi_t\left(h^t\right)}=\pi(s_{t+1}|s_t).$$

Assume $A_t\left(h^t\right)=\mathcal{A}\left(A_{t-1}\left(h^{t-1}\right),s_t\right)$, a time-invariant function $\mathcal{A}$ measurable with respect to s_t. For example,

$$A_t\left(h^t\right)=A_{t-1}(h^{t-1})s_t=A_{-1}s_0s_1\cdots s_t,$$

with A_{-1} given. So current technology $A_t\left(h^t\right):=A'$ depends on last period technology and current shock s. Dropping the time subscripts, we write

$$A'=\mathcal{A}\left(A,s\right)=As.$$

Let the aggregate beginning of period capital stock be $K:=K_t(h^{t-1})$. To summarize current position of the economy, define aggregate state as

$$X:=(K,A,s)$$

We will next look at the Pareto problem and then the sequential markets competitive equilibrium for the stochastic growth model with recursive technology shocks. To distinguish the notatation we use capital letters for the planners variables and lower case for the competitive equilibrium:

Planner	**RCE**
C	c
K	k
N	n

We'll show they are the same in the end.

4.1 Recursive planning problem

The planner's problem now can be written down recursively as the Bellman equation

$$v(K,A,s) = \max\left\{u(C,1-N) + \beta\sum_{s'}\pi\left(s'|s\right)v\left(K',A',s'\right)\right\}$$

subject to

$$\begin{aligned} K' + C &\leq AsF(K,N) + (1-\delta)K \\ A' &= As. \end{aligned}$$

The optimal policy functions solving the Bellman equation are of the form

$$\begin{aligned} C &= C(K,A,s) \\ N &= N(K,A,s) \\ K' &= K(K,A,s) \end{aligned}$$

Note that the transition law for technology shocks together with the optimal policy function for capital accumulation,

$$A' = As \tag{17}$$

and

$$K' = K(K,A,s), \tag{18}$$

along with the Markov kernel for the shock $\pi(s'|s)$, induce a transition density function $\Pi(X'|X)$ on the aggregate state $X = (K,A,s)$. $\Pi(X'|X)$ gives us the frequency of observing the next-period aggregate state X' given a known current state X.

Define derivatives at the optimum as

$$u_i(X) := u_i(C(K,A,s), 1-N(K,A,s)) \text{ for } i = c,n$$

$$F_j(X) := F_j(K, N(K,A,s)) \text{ for } j = k,n$$

Then Pareto optimality of the allocations $\{C, K', N\}$ is characterized by

$$\frac{u_l(X)}{u_c(X)} = AsF_n(X)$$

$$1 = \beta\sum_{X'}\Pi\left(X'|X\right)\frac{u_c(X')}{u_c(X)}\left[A's'F_k\left(X'\right) + (1-\delta)\right]$$

along with

$$K' + C = AsF(K,N) + (1-\delta)K.$$

4.2 Recursive competitive equilibrium with sequential markets

Now we consider the decentralized economy with sequential markets. In describing the decision problem of households as price takers, we now denote K as the aggregate capital stock determined in the aggregate equilibrium, but individual capital k is chosen by individual household/firms. We will impose a notion of a "consistent" – in the sense of beliefs and actions – equilibrium later, so that $k = K$. In doing so, we make explicit the idea of a *rational expectations equilibrium* or REE.

Because agents are "small" – viz. they take as given prices $r(X)$, $w(X)$, $Q(X'|X)$. In deciding their individual optimal actions (consuming, producing, selling and buying), agents need to forecast the evolution of (K, A, s) in order to track prices.

For simplicity, assume that all agents know the true law of motion for technology shock,

$$A' = As.$$

but they assign subjective probabilities to the shocks s, denoted as $\hat{\pi}(s'|s)$, and have subjective views on how the aggregate capital stock K evolves. We will denote their *perceived law of motion* (PLM) for the aggregate capital stock as

$$K' = G(X) = G(K, A, s),$$

This subjective beliefs system $(G, \hat{\pi})$, in turn, will induce a perceived transition density for the aggregate state vector, $\hat{\Pi}(X'|X)$. When we define a REE, we will impose consistency of beliefs on G and $\hat{\pi}$. Next we turn to describing the recursive decentralized economy.

4.2.1. Household

Individually, households hold the belief system $(G, \hat{\pi})$ or more compactly $\hat{\Pi}(X'|X)$. Households need to form a view of the evolution of the aggregate state $X = (K, A, s)$. They also track their own individual state a, which is the wealth they carry into each current period.

Let $V(a, X)$ be the value function for the household at current state (a, X) which is the *subjectively* maximal indirect expected total discounted utility. We say "subjectively" because the household, off-equilibrium, calculates its expected continuation value using the subjective probabilities defined by $\hat{\Pi}(X'|X)$. The household recursive problem is now given by the Bellman equation:

$$V(a, X) = \max \left\{ u(c, 1-n) + \beta \sum_{X'} V\left(\overline{a}\left(X'\right), X'\right) \hat{\Pi}\left(X'|X\right) \right\}$$

subject to

$$c + \sum_{X'} Q\left(X'|X\right) \overline{a}\left(X'\right) \leq w(X)\, n + a$$

Note that the price of a unit of consumption is normalized to unity so that now $w(X)$ denotes the real wage rate. $Q(X'|X)$ is the price of a unit of Arrow security that pays off in terms of consumption if next-period state X' occurs given current state X.

Household's (subjectively) optimal policy functions are

$$c = \sigma^c(a, X)$$

$$n = \sigma^n(a, X)$$

$$\overline{a}\left(X'\right) = \sigma^a\left(a, X; X'\right)$$

Let the marginal utilities of consumption and leisure *under the subjectively optimal household strategies*, respectively, be denoted by

$$\overline{u}_c(a, X) := u_c(\sigma^c(a, X), 1 - \sigma^n(a, X)),$$

and

$$\overline{u}_l(a, X) := u_l(\sigma^c(a, X), 1 - \sigma^n(a, X)).$$

(Subjective) optimality is then characterized by

$$\frac{\overline{u}_l(a, X)}{\overline{u}_c(a, X)} = w(X)$$

$$Q(X'|X) = \beta\hat{\Pi}(X'|X)\frac{\overline{u}_c(a', X')}{\overline{u}_c(a, X)}$$

The first equation is the optimal leisure-consumption trade-off which, at the margin, must equal the opportunity cost of not working (the real wage). The second one is household's perceived stochastic Euler equation, which governs the subjectively optimal consumption plan between any period and state.

4.2.2. Type I Firm

The static profit maximization problem for Firm I has the following zero profit conditions:

$$r(X) = AsF_k(k, n),$$

$$w(X) = AsF_n(k, n).$$

Firm I produces output up to the point where the marginal product of capital (labor) equals the market real rental rate on capital (labor).

4.2.3. Type II Firm

The Type II firm now solves its profit maximization problem with the optimality condition (zero profit condition) given by

$$1 = \sum_{X'} Q(X'|X)\left[r(X') + (1 - \delta)\right].$$

which again says that Firm II will rent any amount of capital to firm I as long as the marginal cost of a unit of investment good it purchases (equal to 1), is equal to the marginal return on its rental sales. The marginal return is made up of the state contingent rental return plus undepreciated capital return, and both discounted using the Arrow pricing kernel (stochastic discount factor), since the contracts between Firm II and Firm I are written in those terms.

4.3 Recursive competitive equilibrium

We now want to impose *equilibrium conditions* on

- arbitrary prices $r(X)$, $w(X)$, $Q(X'|X)$ and
- beliefs $(G(X), \hat{\pi})$ or $\hat{\Pi}(X'|X)$.

That is, (i) we want firms and households as to behave as price takers, so that $k = K$, *after* solving their optimum problems, and (ii) a rational expectations competitive equilibrium.

State-contingent claim issued by Firm II must match the demand by household:

$$\overline{a}\left(X'\right) = \left[r\left(X'\right) + (1-\delta)\right] K'$$

and so beginning-of-period assets in the household budget constraint must satisfy

$$a = \left[r\left(X\right) + (1-\delta)\right] K$$

Substitute these into the household sequential budget constraint:

$$c + \sum_{X'} Q\left(X'|X\right) \left[r\left(X'\right) + (1-\delta)\right] K' = w\left(X\right) n + \left[r\left(X\right) + (1-\delta)\right] K.$$

Using Firm II's first-order condition we further get

$$c + K' = w\left(X\right) n + \left[r\left(X\right) + (1-\delta)\right] K.$$

and also using Firm I's first-order condition we further reduce the household budget constraint to get

$$K' = AsF\left(K, \sigma^n\left(a, X\right)\right) + (1-\delta) K - \sigma^c\left(a, X\right)$$

where Euler's theorem on the homogeneous of degree one function $F(\cdot)$ was used.

Given individual optimality condition, impose equilibrium conditions which requires individual choices to be consistent with the aggregate:

$$\begin{aligned} &k = K, \\ &n = \sigma^n\left(a, X\right) = N, \\ &c = \sigma^c\left(a, X\right) = C, \\ &a = \left[r\left(X\right) + (1-\delta)\right] K. \end{aligned}$$

Then we can write solely in terms of aggregate state X:

$$\begin{aligned} K' = {} & AsF\left(K, \sigma^n\left(\left[r\left(X\right) + (1-\delta)\right] K, X\right)\right) \\ & + (1-\delta) K - \sigma^c\left(\left[r\left(X\right) + (1-\delta)\right] K, X\right) := \mathcal{G}[G(X)]. \end{aligned} \tag{19}$$

Note that after substituting out firm decisions, the RHS contains household's (subjectively) optimal decisions, under the beliefs induced by the PLM $\left(G\left(X\right), \hat{\pi}\right)$, or $\hat{\Pi}\left(X'|X\right)$. This is the optimal evolution of the aggregate capital stock, or the *actual law of motion* (ALM) for K. What we have here is a mapping from PLM to ALM for K.

Definition 4. *A consistent set of beliefs, or rational expectations equilibrium is a fixed-point of the mapping from perceived G and price system to an actual $\mathcal{G}$ and $\hat{\pi} = \pi$, so that $(G, \hat{\pi}) = (\mathcal{G}, \pi)$.*

In our case, REE or PLM = ALM, requires

$$\begin{aligned} G\left(X\right) = {} & AsF\left(K, \sigma^n\left(\left[r\left(X\right) + (1-\delta)\right] K, X\right)\right) \\ & + (1-\delta) K - \sigma^c\left(\left[r\left(X\right) + (1-\delta)\right] K, X\right) := \mathcal{G}[G(X)]. \end{aligned} \tag{20}$$

Note LHS is implicitly a function of PLM $G(X)$.

DEFINITION 5. *A RCE with Arrow securities is*

- *a price system* $r(X), w(X), Q(X'|X)$,
- *a perceived law of motion* $K' = G(X)$ *and induced transition density* $\hat{\Pi}(X'|X)$, *and*
- *a household value function* $V(a, X)$ *and decision rules*

$$\sigma^c(a, X), \sigma^n(a, X), \sigma^a\left(a, X; X'\right)$$

such that

1. *Given the price system and beliefs* $\hat{\Pi}(X'|X)$, *the decision rules and value function solve the household's optimum problem;*
2. *For all* X, $r(X), w(X)$ *solve Firm I's optimum problem given 1;*
3. $Q(X'|X)$ *and* $r(X)$ *solve Firm II's problem given 1 and 2;*
4. *Markets clear,* $G(X), r(X), \sigma^c(a, X), \sigma^n(a, X)$ *satisfy ALM = PLM; and*
5. $\hat{\pi} = \pi$.

REMARK 1. *Requirements 4 and 5 impose rational expectations, so that* $\hat{\Pi}(X'|X) = \Pi(X'|X)$.

Part II

5. An example recursive economy

We will use the generic notation $\mathbb{E}$ now for the expectations operator so that our model can admit either finite-state-space technology shocks or continuous-state-space shocks.[2] How the shocks are specified will be a matter of choice for the researcher.

Now we will illustrate a more condensed example where we side-step Firm II altogether and assume that the household owns capital stock in every period and rents it to Firm I. We will impose the requirement of rational expectations from the outset so households will be forecasting based on the equilibrium, "correct" or model-consistent belief system.

5.1 Exogenous forcing process

Suppose the level of technology follows a continuous-state first-order Markov process:

$$A_t = A_{t-1}^{\rho} \exp\{s_t\}, \tag{21}$$

and $s_t \sim$ i.i.d.$(0, \sigma^2)$.

5.2 Preferences

Let the aggregate state be $X := (K, A)$ and the individual state k. Suppose that the per-period utility for the household is

$$u(c, 1-n) = \epsilon_c \ln(c) + (1-\epsilon_c)\ln(1-n).$$

where $\epsilon_c \in (0, 1)$.

The household budget constraint is given by:

$$c + x = w(K, A)n + r(K, A)k,$$

which has that the household can purchase consumption or investment goods using labor or capital income. With the investment good, the household can produce more capital to rent to Firm I:

$$k' = x + (1-\delta)k.$$

So the household Bellman equation is now

$$V(k, K, A) = \max_{c,k',n} \left\{ [\epsilon_c \ln(c) + (1-\epsilon_c)\ln(1-n)] + \beta \mathbb{E}[V(k', K', A')|A] \right\} \tag{22}$$

subject to

$$k' = [r(K, A) + (1-\delta)]k + w(K, A)n - c \tag{23}$$

We now suppress the dependency of prices and allocations on states and re-introduce the time subscripts. An optimal plan by the household with respect to n and k' satisfies

$$w_t = \left(\frac{1-\epsilon_c}{\epsilon_c} \right) \frac{c_t}{1-n_t}, \tag{24}$$

[2] With a Markovian economy, the expectations of (a function of) next period's state conditional on the current realized state is conventionally written as $\mathbb{E}_t[f(x_{t+1})] = \mathbb{E}[f(x_{t+1})|x_t]$.

and

$$\frac{1}{c_t} = \beta \mathbb{E}_t \left\{ \frac{(r_{t+1} + 1 - \delta)}{c_{t+1}} \right\}, \tag{25}$$

for all states and dates $t \geq 0$. Or we can define $R_t = r_t + 1 - \delta$, so that the Euler equation is re-written as

$$\frac{1}{c_t} = \beta \mathbb{E}_t \left\{ \frac{R_{t+1}}{c_{t+1}} \right\}. \tag{26}$$

5.3 Technology

Firm I operates the following technology:

$$F(k, n) = Ak^{\alpha} n^{1-\alpha}$$

where $\alpha \in (0, 1)$. Firm I hires labor and capital services from the household up to the point that it maximizes profits:

$$w_t = (1 - \alpha) A_t \left(\frac{k_t}{n_t} \right)^{\alpha}, \tag{27}$$

and

$$r_t = \alpha A_t \left(\frac{k_t}{n_t} \right)^{\alpha - 1} \Rightarrow R_t = \alpha A_t \left(\frac{k_t}{n_t} \right)^{\alpha - 1} + (1 - \delta). \tag{28}$$

5.4 Recursive competitive equilibrium

A recursive competitive equilibrium with rational expectations is then a sequence of pricing functions $\{w_t(K_t, A_t), r_t(K_t, A_t)\}_{t \geq 0}$ such that

1. Households solve their Bellman equation problem: (24) and (26), and
2. Firms maximize profit, (27) and (28), and
3. Markets clear: $k = K$, $n = N$, $c = C$, $x = X$ and

$$K_{t+1} = (1 - \delta) K_t + AK^{\alpha} N^{1-\alpha} - C_t.$$

5.5 Computing a sample equilibrium

In this section we show how to compute a solution to the recursive competitive equilibrium using a simple log-linear approximation technique.[3] This method is commonly used in the early real business cycle literature and also more recently in the monetary policy models with New Keynesian features (more on this later). The idea is to attack the optimality conditions (describing the competitive equilibrium) themselves and impose a stability condition (much stronger assumption than the transversality condition) on the equilibrium sequences.

This simple model has six parameters $(\beta, \epsilon_c, \delta, \alpha, \rho, \sigma^2)$. At a deterministic steady state where $A_{ss} = 1$ we have

$$r_{ss} = \beta^{-1} - 1 + \delta$$
$$\delta = X_{ss}/K_{ss}.$$

[3]You could also have used the non-linear discretized state space method we considered earlier in this course, but I leave it here as an exercise for the enthusiast.

Recall we defined $R_t = r_t + 1 - \delta$ so that now $R_{ss} = r_{ss} + 1 - \delta$.

So we can calibrate the values of β and δ from long run data on the real return on capital and investment-capital ratio. We also know from data the long run capital-labor ratio, so we can calibrate the real return on capital and labor as

$$\begin{aligned} r_{ss} &= \alpha A_{ss} \left(\frac{K_{ss}}{N_{ss}} \right)^{\alpha-1} \\ w_{ss} &= (1-\alpha) A_{ss} \left(\frac{K_{ss}}{N_{ss}} \right)^{\alpha}. \end{aligned}$$

We can also calculate the long run capital-output ratio from data, so then we can find

$$\frac{X_{ss}}{Y_{ss}} = \delta \frac{K_{ss}}{Y_{ss}}$$

and from the accounting identity we have

$$\frac{C_{ss}}{Y_{ss}} = 1 - \frac{X_{ss}}{Y_{ss}}.$$

It turns out that we don't need to parameterize ϵ_c since it does not appear in our linearized system below. A typical estimate of the share of capital in output is $\alpha = 0.36$. So now we are ready to approximate and then solve the approximate model.

Define $\hat{x}_t = \ln(X_t/X_{ss})$ as the percentage deviation of the original variable in levels from its steady state value. Using a first-order Taylor series approximation of the true nonlinear dynamical system around the latter's deterministic steady state, we have the following approximate linear system of expectational difference equations:

$$\begin{aligned} \hat{a}_t &= \rho \hat{a}_{t-1} + s_t \\ \hat{k}_{t+1} &\approx (1-\delta)\hat{k}_t + \delta \hat{x}_t, \\ \mathbb{E}_t \hat{c}_{t+1} &\approx \hat{c}_t + \mathbb{E}_t \hat{r}_{t+1} \\ \hat{y}_t &= \hat{a}_t + (1-\alpha)\hat{n}_t + \alpha \hat{k}_t \\ \hat{y}_t &\approx \omega_c \hat{c}_t + \omega_x \hat{x}_t \\ R_{ss}\hat{r}_t &= \alpha \frac{Y_{ss}}{K_{ss}} (\hat{y}_t - \hat{k}_t) \\ \hat{w}_t &= \hat{y}_t - \hat{n}_t \\ \omega_l \hat{w}_t &= \hat{n}_t + \omega_l \hat{c}_t \end{aligned}$$

where $\omega_c = C_{ss}/Y_{ss}$, $\omega_x = X_{ss}/Y_{ss}$ and $\omega_l = (1-N_{ss})/N_{ss}$.

The system above can be reduced so that we only need to solve a two-dimensional system given by the first-order form:

$$\begin{bmatrix} \hat{k}_{t+1} \\ \mathbb{E}_t \hat{c}_{t+1} \end{bmatrix} = M \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix} + P \hat{a}_t.$$

We leave as an exercise to the reader to figure out the parameters in M and P.

5.5.1. Method of undetermined coefficients

To find the approximated REE solution we can use the method of undetermined coefficients. This example here in fact turns out to be a scalar case of the more general version in Uhlig (1999). We guess a linear optimal decision rule of the form[4]

$$\hat{c}_t = d_1 \hat{k}_t + d_2 \hat{a}_t.$$

If this is correct, then

$$\mathbb{E}_t \hat{c}_{t+1} = d_1 \hat{k}_{t+1} + d_2 \mathbb{E}_t \hat{a}_{t+1} = d_1 \hat{k}_{t+1} + d_2 \rho \hat{a}_t.$$

The decision rule coefficients (d_1, d_2) are as yet undetermined. We will solve for them using the stability conditions implied by an optimal plan. Under this rule, the expectational difference equation system can be written as

$$\begin{bmatrix} \hat{k}_{t+1} \\ d_1 \hat{k}_{t+1} + d_2 \rho \hat{a}_t \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ d_1 \hat{k}_t + d_2 \hat{a}_t \end{bmatrix} + \begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix} \hat{a}_t$$

This controlled system under the conjectured decision rule, provides two expressions for the dynamics of $\hat{k}_{t+1}$ as a function of $(\hat{k}_t, \hat{a}_t)$ which must be valid for all realizations of $(\hat{k}_t, \hat{a}_t)$. To see this clearly, we expand the system individually as:

$$\hat{k}_{t+1} = (m_{11} + m_{12} d_1) \hat{k}_t + (p_{11} + m_{12} d_2) \hat{a}_t$$

and

$$d_1 \hat{k}_{t+1} = (m_{21} + m_{22} d_1) \hat{k}_t + [p_{21} + (m_{22} - \rho) d_2] \hat{a}_t.$$

A REE for this approximate linear system implies that these two equations must be the same. That is, they characterize the same optimal contingent capital accumulation plan under the REE. Therefore, equating these two equation to eliminate the $\hat{k}_{t+1}$ terms we have

$$d_1 (m_{11} + m_{12} d_1) \hat{k}_t + d_1 (p_{11} + m_{12} d_2) \hat{a}_t = (m_{21} + m_{22} d_1) \hat{k}_t + [p_{21} + (m_{22} - \rho) d_2] \hat{a}_t.$$

Equating the coefficients, we then must have

$$m_{12} d_1^2 + (m_{11} - m_{22}) d_1 - m_{21} = 0$$

and

$$d_2 = \frac{-d_1 p_{11} + p_{21}}{d_1 m_{12} - (m_{22} - \rho)}.$$

The first is the characteristic equation which is a quadratic in the root d_1. To ensure that the dynamics of $\hat{k}_t$ is stable (so that it is also bounded) for any initial condition, we need to pick the d_1 that would ensure $|(m_{11} + m_{12} d_1)| < 1$ so that the path of capital is given by a stable REE (i.e. will satisfy the transversality

[4]We could also simultaneous guess

$$\hat{k}_{t+1} = d_3 \hat{k}_t + d_4 \hat{a}_t,$$

which is what the next problem set asks you to do.

condition). Given the values of $(m_{11}, m_{12}, m_{21}, m_{22})$ we pick the root

$$d_1 = \frac{-(m_{11} - m_{22}) \pm \sqrt{(m_{11} - m_{22})^2 + 4m_{12}m_{21}}}{2m_{12}}$$

such that $|(m_{11} + m_{12}d_1)| < 1$.

Exercise 1. *Show that the two possible distinct eigenvalues, $\lambda := (m_{11} + m_{12}d_1)$, occur as a reciprocal pair – viz. one eigenvalue is inversely related to the other so that there can be at most only one stable REE. We say that, in this case, the REE is unique and saddle-path stable.*

Once we know d_1 we can solve for d_2 using the previous condition. Finally, we can solve for the optimal capital accumulation decision rule using what we derived before:

$$\hat{k}_{t+1} = (m_{11} + m_{12}d_1)\hat{k}_t + (p_{11} + m_{12}d_2)\hat{a}_t,$$

and the consumption decision rule as

$$\hat{c}_t = d_1\hat{k}_t + d_2\hat{a}_t,$$

given

$$\hat{a}_t = \rho\hat{a}_{t-1} + s_t.$$

This is just a linear stochastic difference equation system (in the "state-space form") which we can then use to compute impulse responses and covariances of all other variables in the system, since they are just static identities. For an empiricist familiar with Vector Autoregressions (VARs), this (log-linear) approximate solution to the RCE-REE of the model implies a VAR with theory-implied restrictions on the VAR coefficient matrix and covariance structure.[5] The formulas for doing this was discussed in previous topics and in Ljungqvist and Sargent (2004), chapter 2. We leave the computation as an exercise for the reader.

Finally, what if we have larger models with more than one Euler equation? The technique discussed in Uhlig (1999) is a generalization of this simple hands on example.

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[5]What this means, from a philosophy-of-doing-science view, is that if we are to have any useful interpretation of the behavior underlying time-series observations/outcomes of quantities and prices in reality, we need to have a theory to restrict what would otherwise be a free-form statistical relation between the variables in the data. Here, a particular theory gives us a strong identification scheme to separate out what is a supply shock and what is a demand shock (not present in our example here). This then, allows us to identify demand versus supply curve shifts, which are unobservables in purely price-quantity variations in the data. An ad-hoc VAR without theory will not be able to answer such a question. That is, the statistical time-series approach to modelling is often constrained to forecasting exercises, but any attempts to use these models for interpretation or policy analysis would be a foolish exercise. Hence, the Nobel laureate Tom Sargent once warned: "Beware of the econometrician bearing free parameters".

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