Deterministic Dynamic Programming II

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Outline

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2. **Recursive Problem**
   - Existence and Uniqueness of v
3. **Optimal strategies**
   - Stationary, Markovian strategies
   - Existence: Stationary strategies
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Infinite-horizon (sequence) problem I

Notation:

- Time: $t \in \mathbb{N}$
- State space: $X \subset \mathbb{R}^n$, $n \geq 1$
- State (vector): $x_t \in X$
- Action space: $A \subset \mathbb{R}^k$, $k \geq 1$
- Control (vector): $u_t \in A$
- Feasible control set at x_t: $\Gamma(x_t)$
- Controllable transition law: $x_{t+1} = f(x_t, u_t)$, with $x_0 \in X$ given.
- Payoff criterion: $\sum_{t \in \mathbb{N}} \beta^t U(x_t, u_t)$, with $U : X \times A \to \mathbb{R}$, and $\beta \in (0, 1)$.

Infinite-horizon (sequence) problem II

Planning problem:

$$\text{(P1)} \qquad v(x_0) = \sup_{\{u_t \in \Gamma(x_t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(u_t, x_t)$$

$$\text{s.t.}$$

$$x_{t+1} = f(x_t, u_t)$$

$$x_0 \text{ given.}$$

Value of the optimal problem is a function of the current state $v(x_t)$. We also call this the indirect utility.

Recursive Problem I

Theorem (Bellman principle of optimality)

For each $x \in X$, the value function $v : X \to \mathbb{R}$ of (P1) satisfies

$$v(x) = \sup_{u \in \Gamma(x)} \{U(x,u) + \beta v(x')\} \text{ s.t. } x' = f(x,u) \tag{1}$$

Recursive Problem II

We discussed ...

- Let

$$B(X) := \{w : X \to \mathbb{R} |\text{ w is bounded }\}.$$

- For any $w \in B(X)$, let $T(w)$ be a function(al) such that

$$T(w)(x) = \sup_{u \in \Gamma(x)} \{U(x,u) + \beta w(f(x,u))\}$$

  at any $x \in X$.
- Since U and w are bounded, then $T(w) \in B(X)$: Bellman operator T preserves boundedness.

Existence and Uniqueness I

How do we know v exists? ...

... If it does, is it unique?

Existence and Uniqueness II

It turns out, under some regularity conditions, we can apply the Banach fixed point theorem.

Goal:

- Show that the Bellman operator $w \mapsto T(w)$ is a contraction mapping on a complete metric space.
- Then these conditions suffices for the existence of a unique function v, such that $v = T(v)$.

Existence and Uniqueness III

Some conditions:

- Set of candidate value functions W is a *Complete metric space*
- Bellman operator $T : W \to W$ is a β-*contraction map*

... we need to understand what these things are.

Existence and Uniqueness IV

Recall the definition of a metric space, say (W, d)?

Definition

A sequence $\{w_n\}_{n\in\mathbb{N}}$ in any metric space (W, d) is Cauchy, if for every positive real number $\varepsilon > 0$, there is a positive integer N such that for all natural numbers $m, n > N$,

$$d(w_m, w_n) < \varepsilon.$$

Existence and Uniqueness V

- In our usage here, $W = B(X)$
  ... the space of bounded real-valued functions with domain X.

- What is d?
  ... Need to measure how "close" two functions $v, w \in W$ are.

- Choose d as sup-norm metric:

$$d_\infty(v, w) = \sup_{x \in X} |v(x) - w(x)|.$$

  ... least upper bound of the absolute distance between two functions evaluated at every $x \in X$.

  ... If $d_\infty(v, w) \to 0$,

  ... then the two functions are "the same" at every $x \in X$.

Existence and Uniqueness VI

Definition

A *complete* (or Cauchy) metric space if every Cauchy sequence in that space converges to a limit in the same space.

Let $B(X) := \{f : X \to \mathbb{R} | f \text{ bounded}\}$.

Lemma

Let $d_\infty(v, w) = \sup_{x \in X} |v(x) - w(x)|$ *for* $v, w \in B(X)$. *The metric space* $(B(X), d_\infty)$ *is complete.*

Existence and Uniqueness VII

Proof: (Sketch).

- Use construct of a Cauchy sequence in $B(X)$.
- Show pointwise convergence: Fix $x \in X$. Exists some $N_\varepsilon \in \mathbb{Z}_+$, $d_\infty(v_n(x) - v_m(x)) < \varepsilon$ for $n, m > N_\varepsilon$.
- What else do we know? v_n bounded. Cauchy-ness means limit of sequence v also bounded.
- Show uniform convergence: Cauchy-ness of $\{v_n\}$ also means, exists some $N_\varepsilon \in \mathbb{Z}_+$, $d_\infty(v_n(x) - v_m(x)) < \varepsilon$ for $n, m > N_\varepsilon$, for every $x \in X$.

Existence and Uniqueness VIII

Definition (β-contraction)

- Let (W, d) be a metric space and the map $T : W \to W$.
- Let $T(w) :=: Tw$ be the value of T at $w \in W$.
- T is a contraction with modulus $0 \leq \beta < 1$ if $d(Tw, Tv) \leq \beta d(w, v)$ for all $w, v \in W$.

Existence and Uniqueness IX

Theorem (Banach Fixed Point Theorem)

If (W, d) is a complete metric space and $T : W \to W$ is a β-contraction, then there is a fixed point for T and it is unique.

Existence and Uniqueness X

Proof (Sketch):

- Show existence. There is at least one v such that $v = T(v)$.
  - Use definition of a β-contraction and apply triangle inequality.
  - Arrive at $\{T^n w\}$ as a Cauchy sequence, $w \in W$ and $T^n w := T[T^{n-1}(\cdots T(w))]$.
  - So there exists a limit in W, $v = \lim_{n\to\infty} T^n w$.
  - Show that $v = T(v)$. (Use triangle inequality again.)
- Show that limit $v \in W$ is unique.
  - Suppose not. Then there is another fixed point s.t. $\tilde{v} = T(\tilde{v})$.
  - But property of T as β-contraction results in contradiction.

Existence and Uniqueness XI

Workflow:

- Show that the DP problem, described by the Bellman operator $T : W \to W$, is a β-contraction.
- Check that W is a complete metric space.
- Then there is a unique value function that solves the Bellman equation!

... how do we check that $T : W \to W$, is a β-contraction?

Existence and Uniqueness XII

Lemma (Blackwell's sufficient conditions for a contraction)

Let $M : B(X) \to B(X)$ *be any map satisfying*

1. *Monotonicity: For any* $v, w \in B(X)$ *such that* $w \geq v \Rightarrow Mw \geq Mv$.
2. *Discounting: There exists a* $0 \leq \beta < 1$ *such that* $M(w + c) = Mw + \beta c$, *for all* $w \in B(X)$ *and* $c \in \mathbb{R}$. *(Define* $(f + c)(x) = f(x) + c$.*)*

Then M *is a contraction with modulus* β.

Fixed-point result for Bellman equation

Theorem

$v : X \to \mathbb{R}$ is the unique fixed point of the Bellman operator $T : B(X) \to B(X)$, such that if $w \in B(X)$ is any function satisfying

$$w(x) = \sup_{u \in \Gamma(x)} \{U(x, u) + \beta w(f(x, u))\}$$

at any $x \in X$, then it must be that $w = v$.

Optimal strategies I

Theorem

If $U : X \times A \to \mathbb{R}$ is bounded, then a strategy σ is optimal if and only if w_σ satisfies the Bellman equation

$$w_\sigma(x) = \sup_{u \in \Gamma(x)} \{U(x,u) + \beta w_\sigma(f(x,u))\}$$

at each $x \in X$.

Optimal strategies II

Remarks:

- If our Bellman operator defines a mapping T which is a contraction on $B(X)$, we can apply the Banach fixed-point theorem.

  ... i.e. there is a unique bounded value function solving the Bellman equation.

- The last theorem says that a strategy is optimal if, and only if, it induces (or supports) a total payoff that satisfies the Bellman equation.

Stationary strategies I

Simpler strategies? Recursivity of Bellman equation suggests that we can restrict attention to:

- Markov strategies; and
- Stationary strategies

Stationary strategies II

Definition

A Markovian strategy π for the DP problem $\{X, A, U, f, \Gamma, \beta\}$ is a strategy such that $\pi = \{\pi_t\}_{t\in\mathbb{N}}$ and $\pi_t = \pi_t(x_t[h^t])$, where for each t, $\pi_t : X \to A$ such that $\pi_t(x_t) \in \Gamma(x_t)$.

Note:

- $\pi_t = \pi_t(x_t[h^t])$ is the requirement that each period's action is conditioned on the history h^t only insofar as it affects the current state.
- Further, this action has to be in the set of feasible actions determined by the current state.

Stationary strategies III

Definition

A Markovian strategy $\pi = \{\pi_t\}_{t\in\mathbb{N}}$ with the further property that $\pi_t(x) = \pi_\tau(x) = \pi(x)$ for all $x \in X$ and all $t, \tau \in \mathbb{N}$, and $t \neq \tau$, is called a stationary strategy.

Note the further restriction that decision functions for each period that are *time-invariant* functions of the current state only.

Stationary strategies: existence I

Additional regularity assumptions ...

Assumption

U is continuous on $X \times A$.

Assumption

f is continuous on $X \times A$.

Assumption

Γ is a continuous, compact valued correspondence on X.

Stationary strategies: existence II

Existence of π^*

There exists a strategy π^* from the class of "stationary strategies" that is optimal – viz. this strategy satisfies the Bellman Principle of Optimality as stated in Theorem 3.1.

Stationary strategies: existence III

With these additional assumptions plus Assumption that U is bounded on $X \times A$:

1. Existence of a unique *continuous* and *bounded* value function that satisfies the Bellman Principle of Optimality;
2. Existence of a well-defined feasible action correspondence admitting a stationary optimal strategy that satisfies the Bellman Principle of Optimality; and
3. This stationary strategy delivers a total discounted payoff that is equal to the value function, and is indeed an optimal strategy.

Stationary strategies: existence IV

Theorem

If the stationary dynamic programming problem $\{X, A, \Gamma, f, U, \beta\}$ *satisfies Assumptions made on* (U, f, Γ), *then there exists a stationary optimal policy* π^*.

Furthermore the value function $v = W(\pi^*)$ *is bounded and continuous on* X, *and satisfies for each* $x \in X$,

$$\begin{aligned} v(x) &= \max_{u \in \Gamma(x)} \{U(x, u) + \beta v(f(x, u))\} \\ &= U(x, \pi^*(x)) + \beta W(\pi^*)(f(x, \pi^*(x))). \end{aligned}$$

Discussion I

- We began with a heuristic example that we could solve by hand.
- We brute-force solved it using the method of "value function iteration".
- It turns out what we were doing there was constructing a Cauchy sequence of value functions $\{v_n\}_n$.
- While in that example, the per-period payoff function U was unbounded (i.e. log), we had implicitly made the domain X compact, and therefore bounded.

Discussion II

- More generally, even with the same Ramsey optimal growth example (see Chapter 2) but without the specific functional restrictions on U, f and $\delta = 1$, we can now be assured that there is a unique value function solving the Bellman equation.
- Moreover, under regularity conditions studied today, there exists a Markovian and stationary solution that can be recursively used to construct the optimal strategy beginning from some initial state.