Competitive Equilibrium with Complete Markets: Part I

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Outline

Motivation

Previously ...

Previously we look at a *planned* economy:

- Single optimizing planner.
- Characterized recursive optimal allocations as a DP problem.
- In reality, we have *decentralized* or *competitive economies*.

Osamu Tezuka's *Metropolis*

Motivation

And few lectures ahead ...

- Want to work toward the stochastic growth model as also basic *recursive competitive equilibrium* model.
- As a dynamic outcome where individuals and firms solve their *decentralized* optimal allocation problems independently.
- No planner.
- History (or State)-contingent, intertemporal relative prices, as the *allocative mechanism*.
- Resulting versions of first- and second fundamental welfare theorems. (Why?)

Motivation

General:

- What do we mean by "equilibrium"?
- Equilibrium as a " mapping" from the physical/primitive environment (preference, technology, information sets and market structure) to real allocations, s.t.
  1. agents optimize (they select their best actions),
  2. agents actions are consistent with each other's actions, and
  3. the allocations are feasible.
- An example of such an (economic) equilibrium concept is the Walrasian, or valuation equilibrium.
- Typically we want to show such equilibria exist and are unique.
- Inefficient (non-Paretian) equilibria when there exist externalities, incomplete information or incentive problems.

Motivation

Today ...

But, we take a small step first. Look at a model with no production.

1. Recall pretty Edgeworth box analysis of 2×2 pure-exchange economy?
2. **So what's new here?**
   - A model of *pure exchange infinite horizon* economy.
   - With *stochastic process for endowments*.
   - So, just infinite number of goods – indexed by time and history (current state, if Markov).
   - How to model intertemporal trading:
     - between decentralized/competitive agents,
     - in the presence of endowment risk?
     - and, what do people trade?

Today's Roadmap ...

But, we take a "small" step first. Look at a model with no production.

1. Set up a *pure-exchange-infinite-horizon* economy with *stochastic endowments*.
2. Benchmark: Characterizing social planner's (Pareto) allocation (PO).
3. Two market trading assumptions:
    - Arrow-Debreu time-0-trading economy (ADE) with Arrow-Debreu securities.
    - Radner sequential-trading economy (SME) with Arrow securities.
    - So, just infinite number of goods – indexed by time and history (current state, if Markov).
4. Allocative "equivalence" between ADE and SME and PO.
5. Finance: Asset pricing implications of the model.
6. Tractable specialization: Markovian endowments $\Rightarrow$ *Recursive competitive equilibrium*.
7. Some examples by hand.

Model Setup

- Stochastic event $s_t \in S = \{s_1, ..., s_n\}$ for $t \in \mathbb{N}$.
- Publicly observable history of events up to and including t: $h^t = (s_0, s_1, ..., s_t) \in S^t$.
- Unconditional probability of h^t given by probability measure $\pi_t\left(h^t\right)$.
- W.l.o.g., assume $\pi\left(s_0\right) = 1$.
- Probability of observing h^t *conditional* on realization of h^τ is $\pi\left(h^t | h^\tau\right)$, for any $t \geq \tau$.

Model Setup

- I agents indexed by $i = 1, ..., I$.
- Agent i's
  - Endowment: $y_t^i(h^t)$
  - history-dependent consumption plan, $c^i = \{c_t^i(h^t)\}_{t=0}^{\infty}$ for each $h^t \in S^t$
  - expected utility criterion:

$$U(c^i) = \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t^i(h^t)) \right\} = \sum_{t=0}^{\infty} \sum_{h^t} \beta^t u(c_t^i(h^t)) \pi_t(h^t)$$

  where

    - $u'(c) > 0, u''(c) < 0$
    - $\lim_{c \searrow 0} u'(c) = +\infty$

  to ensure $c_t > 0$ for all t

Remark. In the special case, $\pi_t\left(h^t\right)$ can be induced by a Markov process.

- So if $h^t \sim$ Markov, then, we can write

$$\pi_t\left(h^t\right) = \pi_t(s_t),$$

and recursively,

$$\pi_t\left(s_t\right) = \pi_t\left(s_t | s_{t-1}\right)\pi_t\left(s_{t-1}\right).$$

- For now we don't need to make this restriction, so we will keep track of all t-histories, h^t.

Model Setup

A *feasible allocation* must satisfy

$$\sum_{i=1}^{I} c_t^i\left(h^t\right) \leq \sum_{i=1}^{I} y_t^i\left(h^t\right) \tag{1}$$

for all t and for all h^t.

What does this say?

What Next?

- Benchmark: Characterizing social planner's (Pareto) allocation (PO).
- Competitive/Decentralized equilibrium:
  - Arrow-Debreu time-0-trading economy (ADE) with Arrow-Debreu securities.
  - Radner sequential-trading economy (SME) with Arrow securities.
- Allocative "equivalence" between ADE and SME and PO.

Benchmark: Pareto allocation

- Pareto weights $\lambda_i \geq 0$ for all $i = 1, ..., I$ agents.
- The planner allocates plans $c^i = \left\{c_t^i\left(h^t\right)\right\}_{t=0}^{\infty}$ for all $i = 1, ..., I$, by solving the following problem:

$$\max_{\left(c^i\right)_{i=1}^I} W = \sum_{i=1}^{I} \lambda_i U\left(c^i\right)$$

subject to

$$\sum_{i=1}^{I} c_t^i\left(h^t\right) \leq \sum_{i=1}^{I} y_t^i\left(h^t\right)$$

for all t and for all h^t.

- Recall:

$$U\left(c^{i}\right)=\sum_{t=0}^{\infty}\sum_{h^{t}}\beta^{t}u\left(c_{t}^{i}\left(h^{t}\right)\right)\pi_{t}\left(h^{t}\right).$$

- So we can re-write this problem out as

$$\max_{\left(c^{i}\right)_{i=1}^{I}} W=\sum_{t=0}^{\infty}\sum_{h^{t}}\sum_{i=1}^{I}\lambda_{i}\beta^{t}u\left(c_{t}^{i}\left(h^{t}\right)\right)\pi_{t}\left(h^{t}\right)$$

subject to

$$\sum_{i=1}^{I}c_{t}^{i}\left(h^{t}\right)\leq\sum_{i=1}^{I}y_{t}^{i}\left(h^{t}\right)$$

for all t and for all h^{t}.

The Lagrangian for the planner's problem is

$$L = \sum_{t=0}^{\infty} \sum_{h^t} \Bigg\{ \sum_{i=1}^{I} \lambda_i \beta^t u\left(c_t^i\left(h^t\right)\right) \pi_t\left(h^t\right) \\ + \theta_t\left(h^t\right) \sum_{i=1}^{I} \left[y_t^i\left(h^t\right) - c_t^i\left(h^t\right)\right] \Bigg\}$$

Note:

- Langrange multiplier is time and history dependent.
- Langrange multiplier independent of i. Why?
- $\theta_t\left(h^t\right) \geq 0$ for all h^t and all t. Why?

A first-order neccesary condition for a maximum is

$$\beta^t u'\left(c_t^i\left(h^t\right)\right)\pi_t\left(h^t\right)=\frac{\theta_t\left(h^t\right)}{\lambda_i}$$

for all $i=1,...,I$, and for all $t\geq 0$ and all h^t.

Intuition: Following each history h^t, at any t, Pareto optimality requires,

- For each agent i of importance λ^i in the planner's criterion,
- the P.V. of the marginal utility of h^t-consumption, $c_t^i\left(h^t\right)$, which occurs with probability $\pi_t\left(h^t\right)$,
- is equated with the P.V. marginal social cost (following history h^t) of providing $c_t^i\left(h^t\right)$.

Big Love. Consider two agents, $i \neq j$. The ratio of their marginal utilities at each period, for all possible histories, is

$$\frac{u'\left(c_t^i\left(h^t\right)\right)}{u'\left(c_t^j\left(h^t\right)\right)} = \frac{\lambda_j}{\lambda_i}$$

i.e. The planner's MRS of consumption between i-van and j-elena is a function of Her relative love for each individual.

This implies:

$$c_t^i\left(h^t\right) = u'^{-1}\left[\frac{\lambda_j}{\lambda_i} u'\left(c_t^j\left(h^t\right)\right)\right] \tag{2}$$

where u'^{-1} is the inverse function of u'.

Summing up all i and invoking the resource constraint we have

$$\sum_{i=1}^{I} c_t^i\left(h^t\right) = \sum_{i=1}^{I} u'^{-1}\left[\frac{\lambda_j}{\lambda_i} u'\left(c_t^j\left(h^t\right)\right)\right] = \sum_{i=1}^{I} y_t^i\left(h^t\right) \tag{3}$$

The LHS is one equation in $c_t^j\left(h^t\right)$ and thus $c_t^j\left(h^t\right)$ depends only on *current realization* of *aggregate endowment* (RHS), for all $j = 1, ..., I$, for all t, for all h^t.

We summarize our result as follows.

Theorem

A Pareto optimal allocation is a function of the realized aggregate endowment and does not depend on

1. *the particular history* h^t *leading up to that outcome, nor*
2. *the realization of individual endowments,*

so that if $h^t \neq h^\tau$ *are such that* $\sum_j y_t^j \left(h^t\right) = \sum_j y_\tau^j \left(h^\tau\right)$ *then* $c_t^i \left(h^t\right) = c_\tau^i \left(h^\tau\right)$.

Remark. Agent i's share of aggregate endowment depends on on time-invariant Pareto weight, λ_i. This may no longer be true in economies with incentive problems due to lack of enforcement or incomplete information.

Example

Let $u(c) = \ln(c)$ and $I = 2$. Then for all $i, j \in \{1, 2\}$ and $i \neq j$, the Pareto allocation is characterized by

$$\frac{u'(c_t^2(h^t))}{u'(c_t^1(h^t))} = \frac{c_t^1(h^t)}{c_t^2(h^t)} = \frac{\lambda_1}{\lambda_2} \Rightarrow c_t^1(h^t) = \frac{\lambda_1 c_t^2(h^t)}{\lambda_2}.$$

at each $t \in \mathbb{N}$, following each h^t.

Interior solution: By feasibility constraint,

$$\frac{\lambda_1 c_t^2(h^t)}{\lambda_2} + c_2(h^t) = \sum_{i=1}^{2} y_t^i(h^t).$$

at each $t \in \mathbb{N}$, following each h^t.

Example (Cont'd)

So then

$$c_2(h^t) = \left(\frac{\lambda_2}{\lambda_1 + \lambda_2} \right) \sum_{i=1}^{2} y_t^i(h^t).$$

and

$$c_1(h^t) = \left(\frac{\lambda_1}{\lambda_1 + \lambda_2} \right) \sum_{i=1}^{2} y_t^i(h^t).$$

For example, if planner cares equally about both agents, $\lambda_1 = \lambda_2$, then the Pareto allocation at each $t \in \mathbb{N}$, following each h^t, is to split the realized total endowment equally between them.

What Next?

- Competitive/Decentralized equilibrium:
  - Arrow-Debreu time-0-trading economy (ADE) with Arrow-Debreu securities.
  - Radner sequential-trading economy (SME) with Arrow securities.
- Allocative "equivalence" between ADE and SME and PO.

Market Structure 1: Arrow-Debreu economy

- All trades happen at the beginning of time, time 0, after s_0 is known.
- At $t = 0$ agents exchange date and history-contingent claims on $c_t(h^t)$ at price $q_t^0\left(h^t\right)$. ("Forward contracts".)
- (Assume) a complete set of securities for all possible histories h^t.
- After time 0 trades agreed to at $t = 0$ are executed as events unfold (no ex-post deviation). No more trades occur after $t = 0$.

Remark. This implies that each agent faces only one budget constraint accounting for all trades across time and probable histories.

Example

Consider consumer 1 whose endowments are such that $y_H > y_L$ and

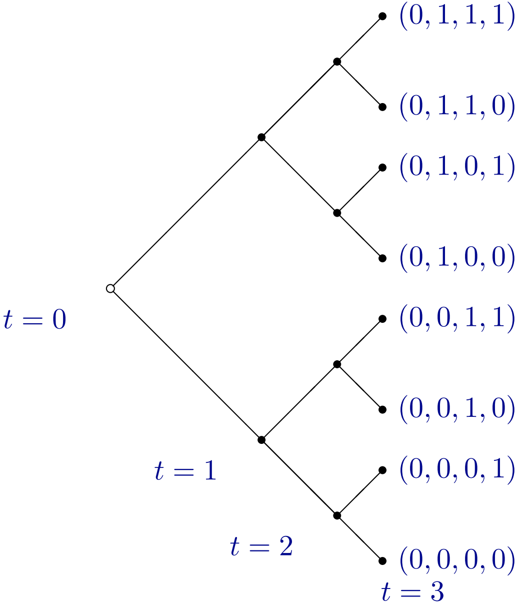
$$y_t(s_t) = \begin{cases} y_H = 1, & \text{if } s_t = 0 \\ \\ y_L = 0, & \text{if } s_t = 1 \end{cases}.$$

Suppose $s_0 = 0$ known.

Example (Cont'd)

Consider consumer 1 whose endowments are such that $y_H > y_L$ and

$$y_t^1(s_t) = \begin{cases} y_H = 1, & \text{if } s_t = 0 \\ y_L = 0, & \text{if } s_t = 1 \end{cases}.$$


$(0,1,1,1)$
$(0,1,1,0)$
$(0,1,0,1)$
$(0,1,0,0)$
$t = 0$
$(0,0,1,1)$
$(0,0,1,0)$
$(0,0,0,1)$
$t = 1$
$t = 2$
$(0,0,0,0)$
$t = 3$

Example (Cont'd)

So for a realized history $h^3 = (0, 1, 1, 1)$, Consumer 1 at $t = 0$ would have executed a subset of the financial contracts s.t.

- He transfers some of his endowment to another consumer at $t = 0$,
- He receives delivery of some endowment from others at $t = 1, 2, 3$.

In the Arrow-Debreu economy, each agent $i \in \{1, ..., I\}$ solves

$$\max_{\left\{c_t^i(h^t)\right\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{h^t} \beta^t u\left(c_t^i\left(h^t\right)\right) \pi_t\left(h^t\right)$$

subject to

$$\sum_{t=0}^{\infty} \sum_{h^t} q_t^0\left(h^t\right) c_t^i\left(h^t\right) \leq \sum_{t=0}^{\infty} \sum_{h^t} q_t^0\left(h^t\right) y_t^i\left(h^t\right) \tag{4}$$

where $q_t^0\left(h^t\right)$ is the relative price, determined at time 0, of time t, history h^t consumption good.

What is the price relative to? We have let one good be the numeraire good (more on this later).

The optimal choice of each agent i is characterized by the (interior) first-order conditions:

$$\frac{\partial U\left(c^i\right)}{\partial c_t^i\left(h^t\right)} = \beta^t u'\left(c_t^i\left(h^t\right)\right)\pi_t\left(h^t\right) = \mu_i q_t^0\left(h^t\right) \tag{5}$$

$$\sum_{t=0}^{\infty}\sum_{h^t} q_t^0\left(h^t\right) c_t^i\left(h^t\right) = \sum_{t=0}^{\infty}\sum_{h^t} q_t^0\left(h^t\right) y_t^i\left(h^t\right) \tag{6}$$

for each $t \geq 0$ and $h^t \in S^t$.

Remark. Notice that μ^i is agent i's time-independent Lagrange multiplier on their time-0 intertemporal budget constraint.

Definition (Arrow-Debreu competitive equilibrium)

A competitive equilibrium is an allocation $\left\{c_t^i\left(h^t\right)\right\}_{t=0}^{\infty}$ and a price system $\left\{q_t^0\left(h^t\right)\right\}_{t=0}^{\infty}$ such that

- Agents optimize, so (5) and (6) hold, and
- Markets clear (feasible allocation):

$$\sum_{i=1}^{I} c_t^i\left(h^t\right) = \sum_{i=1}^{I} y_t^i\left(h^t\right)$$

for all $i = 1, ..., I$, and for all histories h^t.

Notice that (5) implies

$$\frac{u'\left(c_t^i\left(h^t\right)\right)}{u'\left(c_t^j\left(h^t\right)\right)} = \frac{\mu_i}{\mu_j} \Rightarrow c_t^i\left(h^t\right) = u'^{-1}\left[\frac{\mu_i}{\mu_j} u'\left(c_t^j\left(h^t\right)\right)\right] \tag{7}$$

for all $i, j = 1, ..., I$. Using this and the binding feasibility constraint

$$\sum_{i=1}^{I} c_t^i\left(h^t\right) = \sum_{i=1}^{I} y_t^i\left(h^t\right)$$

we have

$$\sum_{i=1}^{I} u'^{-1}\left[\frac{\mu_i}{\mu_j} u'\left(c_t^j\left(h^t\right)\right)\right] = \sum_{i=1}^{I} y_t^i\left(h^t\right) \tag{8}$$

which looks just like (3) in the Pareto problem.

Theorem

The Arrow-Debreu competitive equilibrium allocation is a function of the realized aggregate endowment and does not depend on

1. *the particular history* h^t *leading up to that outcome, nor*
2. *the realization of individual endowments*

so that if $h^t \neq h^\tau$ *and* $\sum_j y_t^j\left(h^t\right) = \sum_j y_\tau^j\left(h^\tau\right)$ *then* $c_t^i\left(h^t\right) = c_\tau^i\left(h^\tau\right)$.

Remark. This result echoes the allocation delivered under a Pareto planner.

Corollary (First fundamental welfare theorem)

The competitive equilibrium is a particular Pareto optimal allocation, where $\mu_i = \lambda_i^{-1}$ *for all* $i = 1, ..., I$*, is unique (up to a multiplication by a positive scalar). Furthermore, the shadow prices for the planner* $\theta_t\left(h^t\right)$ *are equal to Arrow-Debreu equilibrium price* $q_t^0\left(h^t\right)$.

Note the equilibrium price system $\{q_t^0(h^t)\}_{t=0}^{\infty}$, s.t.

$$q_t^0(h^t) = \mu_i^{-1} \frac{\partial U(c^i)}{\partial c_t^i(h^t)} = \frac{\beta^t}{\mu_i} u'(c_t^i(h^t)) \pi_t(h^t)$$

is a function of equilibrium allocations $\{c_t^i(h^t)\}_{t=0}^{\infty}$ for all $i = 1, ..., I$; and has arbitrary units.

Set numeraire as $q_0^0(s_0) = 1$, so the price system is relative to time 0 goods.

This also implies $\mu_i = u'(c_0^i(s_0))$. Then,

$$q_t^0(h^t) = \beta^t \frac{u'(c_t^i(h^t))}{u'(c_0^i(s_0))} \pi_t(h^t).$$

Asset Pricing Implications

Pricing redundant assets. Suppose the claim on consumption contingent on the realization of time t and history h^t is given by $d_t\left(h^t\right)$ such that

$$d_t\left(h^t\right) = \begin{cases} 0 & \text{if } \widetilde{h}^t \neq h^t \\ 1 & \text{if } \widetilde{h}^t = h^t \end{cases} .$$

So holder of $d_t\left(h^t\right)$ has right to claim delivery of one unit of consumption if at time t the history were to be $h^t = \tilde{h}^t$. The price of that unit of consumption is just $q_t^0\left(h^t\right)$.

Therefore, the price of any asset at time 0 that can synthesize this stream of contingent claims has to be

$$p_0^0\left(s_0\right)=\sum_{t=0}^{\infty}\sum_{h^t} q_t^0\left(h^t\right) d_t\left(h^t\right).$$

Otherwise, there exists profitable arbitrage!

Pricing tail assets. Suppose we have an investor who wishes to know what the price of an asset that entitles her to a stream of dividends starting from some future date $\tau \geq t \geq 1$ is.

Time 0 price of an asset that pays dividend stream $\{d_\tau(h^\tau)\}_{\tau \geq t}$, if h^t is realized:

$$p_t^0(h^t) = \sum_{\tau \geq t} \sum_{h^\tau | h^t} q_\tau^0(h^\tau) d_\tau(h^\tau)$$

We can write the state price deflator in time $t \leq \tau$ terms, for all agents i, as

$$\begin{aligned} q_\tau^t\left(h^\tau\right) &= \frac{q_\tau^0\left(h^\tau\right)}{q_t^0\left(h^t\right)} = \frac{\beta^\tau u'\left(c_\tau^i\left(h^\tau\right)\right)\pi_\tau\left(h^\tau\right)}{\beta^t u'\left(c_t^i\left(h^t\right)\right)\pi_t\left(h^t\right)} \\ &= \beta^{\tau-t}\frac{u'\left(c_\tau^i\left(h^\tau\right)\right)}{u'\left(c_t^i\left(h^t\right)\right)}\pi_t\left(h^\tau | h^t\right). \end{aligned} \quad (9)$$

This equation says that the price of time τ delivery, history h^τ-contingent consumption, relative to the price of time $t \leq \tau$ delivery, history h^t-contingent consumption is equal to the marginal rate of substitution of consumption across τ and t, taking into account the probability that state h^τ is realized, given realization of h^t.

So the price at t of the tail asset is

$$p_t^t\left(h^t\right) = \sum_{\tau \geq t} \sum_{h^\tau | h^t} q_\tau^t\left(h^\tau\right) d_\tau\left(h^\tau\right)$$

where numeraire is time t, history h^t, consumption good, or $q_t^t\left(h^t\right) = 1$. This says that an asset (e.g. equity) bought at time t entitles buyer/owner to dividends from $t \geq 1$ onward.

Pricing one-period returns. The one-period version of the state price deflator (9) is

$$q_{\tau+1}^{\tau}\left(h^{\tau+1}\right) = \beta \frac{u'\left(c_{t+1}^{i}\left(h^{\tau+1}\right)\right)}{u'\left(c_{\tau}^{i}\left(h^{\tau}\right)\right)} \pi_{\tau+1}\left(h^{\tau+1}|h^{\tau}\right).$$

So the price, at time τ history h^{τ}, of a claim to a random payoff $\omega\left(s_{\tau+1}\right)$ will be

$$\begin{aligned} p_{\tau}^{\tau}\left(h^{\tau}\right) &= \sum_{h^{\tau+1}} q_{\tau+1}^{\tau}\left(h^{\tau+1}\right)\omega\left(s_{\tau+1}\right) \\ &= \sum_{h^{\tau+1}} \left[\beta \frac{u'\left(c_{\tau+1}^{i}\left(h^{\tau+1}\right)\right)}{u'\left(c_{\tau}^{i}\left(h^{\tau}\right)\right)}\omega\left(s_{\tau+1}\right)\right]\pi_{\tau+1}\left(h^{\tau+1}|h^{\tau}\right) \\ &:= \mathbb{E}_{\tau}\left[\beta \frac{u'\left(c_{\tau+1}^{i}\right)}{u'\left(c_{\tau}^{i}\right)}\omega\left(s_{\tau+1}\right)\right] \qquad (10) \end{aligned}$$

for all $i \in \{1, ..., I\}$.

Dividing through on both sides, we get

$$1 = \mathbb{E}_\tau \left[\beta \frac{u'\left(c^i_{\tau+1}\right)}{u'\left(c^i_\tau\right)} \frac{\omega\left(s_{\tau+1}\right)}{p^\tau_\tau\left(h^\tau\right)} \right] := \mathbb{E}_\tau \left[m_{\tau+1} R_{\tau+1} \right].$$

$R_{\tau+1} := \omega\left(s_{\tau+1}\right) / p^\tau_\tau(h^\tau)$ is the *one-period gross return on the asset.* and $m_{\tau+1} = \beta u'\left(c^i_{\tau+1}\right) / u'\left(c^i_\tau\right)$ is called a *stochastic discount factor* on this return. Both are random variables whose expected values are with respect to the distribution of $s_{\tau+1}$ conditional on h^τ. This is just a stochastic Euler equation relating optimal risky intertemporal consumption allocations to asset returns!

What Next?

- Competitive/Decentralized equilibrium:
  - Arrow-Debreu time-0-trading economy (ADE) with Arrow-Debreu securities.
  - Radner sequential-trading economy (SME) with Arrow securities.
- Show PO $\Leftrightarrow$ ADE $\Leftarrow$ SME $\Leftrightarrow$ PO.