Deterministic Dynamic Programming

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Outline

1. **Infinite-horizon Decisions**
   - Histories, Strategies, Value
2. **Recursive representation**
   - Bellman Functional Equation
   - Principle of Optimality
3. **Fixed Point of Bellman Functional**
   - What is a functional operator?
   - Existence and Uniqueness of v

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2. Infinite-horizon Decisions I

Notation:

- Time: $t \in \mathbb{N}$
- State space: $X \subset \mathbb{R}^n$, $n \geq 1$
- State (vector): $x_t \in X$
- Action space: $A \subset \mathbb{R}^k$, $k \geq 1$
- Control (vector): $u_t \in A$
- Feasible control set at x_t: $\Gamma(x_t)$
- Controllable transition law: $x_{t+1} = f(x_t, u_t)$, with $x_0 \in X$ given.
- Payoff criterion: $\sum_{t \in \mathbb{N}} \beta^t U(x_t, u_t)$, with $U : X \times A \rightarrow \mathbb{R}$, and $\beta \in (0, 1)$.

2. Infinite-horizon Decisions II

Planning problem:

$$\text{(P1)} \qquad v(x_0) = \sup_{\{u_t \in \Gamma(x_t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(u_t, x_t)$$

s.t.

$$x_{t+1} = f(x_t, u_t)$$

$$x_0 \text{ given.}$$

Value of the optimal problem is a function of the current state $v(x_t)$. We also call this the indirect utility.

2. Infinite-horizon Decisions III

Example

- Single firm's intertemporal profit maximization problem. Choosing optimal investment path.
- Single firm's intertemporal profit maximization problem. Choosing optimal pricing path.
- Optimal resource depletion problem
- Optimal growth/savings problem

2. Infinite-horizon Decisions IV

- We said it was impossible to exactly solve (PI) in a direct way. Why?
- The infinite sequence solution to (P1) can be found recursively as the solution to the "Bellman (functional) equation". Bellman Principle of Optimality (BPO).
  1. Need a well-defined valued function v consistent with (P1), which also satisfies BPO. Existence? Unique?
  2. Properties of solution σ that sustains v? Exists? Unique?
- Goal:
  1. Characterize "solution" v satisfying BPO generally. Regularity Conditions.
  2. Then describe conditions for σ to exist. Require more conditions to ensure unique σ.

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2. Infinite-horizon Decisions I

Histories, Strategies, Value

Deconstructing the infinite-sequence problem (P1). ...

A few notations:

- A t-history: $h^t = \{x_0, u_0, ..., x_{t-1}, u_{t-1}, x_t\}$.
- Set of all possible t-histories: Let $H_0 = X$. Then $H_t = \{h^t | t = 1, 2, ...\}$.
- Period t state under history h^t: $x_t[h^t]$.
- A strategy $\sigma = \{\sigma_t(h^t)\}_{t=0}^{\infty}$ is a plan of action specified as a function of the revealed history, such that for each $t \in \mathbb{N}$, $\sigma_t : H_t \rightarrow A$ and the actions are feasible for each history: $\sigma_t(h^t) \in \Gamma(x_t[h^t])$.
- Set of all strategies: Σ.

2. Infinite-horizon Decisions II

Histories, Strategies, Value

Selections from strategies. Generating infinite sequence of states and actions $\{x_t(\sigma, x_0), u_t(\sigma, x_0)\}_{t \in \mathbb{N}}$ for a given strategy σ.

Fix a strategy $\sigma \in \Sigma$.

1. At $t = 0$. Set $x_0(\sigma, x_0) = x_0$, so $h^0 = x_0$; and select $u_0(\sigma, x_0) = \sigma_0(h^0(\sigma, x_0))$. Then,

$$
\begin{aligned}
x_1(\sigma, x_0) =& f(x_0(\sigma, x_0), u_0(\sigma, x_0)) \\
h^1(\sigma, x_0) =& \{x_0(\sigma), u_0(\sigma, x_0), x_1(\sigma, x_0)\}
\end{aligned}
$$

2. Infinite-horizon Decisions III

Histories, Strategies, Value

2. At $t = 1$. Given recorded time-1 history $h^1(\sigma, x_0)$, planner picks action $u_1 = u_1(\sigma, x_0)$. Then,

$$u_1(\sigma, x_0) = \sigma_1(h^1(\sigma, x_0))$$
$$x_2(\sigma, x_0) = f(x_1(\sigma, x_0), u_1(\sigma, x_0))$$

3. So in general for $t \in \mathbb{N}$, we have the recursion under σ starting from x_0 as

$$h^t(\sigma, x_0) = \{x_0(\sigma), u_0(\sigma, x_0), ..., x_t(\sigma, x_0)\}$$
$$u_t(\sigma, x_0) = \sigma_t(h^t(\sigma, x_0))$$
$$x_{t+1}(\sigma, x_0) = f(x_t(\sigma, x_0), u_t(\sigma, x_0))$$

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2. Infinite-horizon Decisions IV

Histories, Strategies, Value

Assigning payoffs to σ-induced outcomes. Each strategy σ, starting from initial state x_0, generates a period-t return:

$$U_t(\sigma)(x_0) = U[x_t(\sigma, x_0), u_t(\sigma, x_0)].$$

Total discounted payoffs from x_0 under strategy σ as

$$W(\sigma)(x_0) = \sum_{t=0}^{\infty} \beta^t U_t(\sigma)(x_0).$$

By definition, the value function is the maximal of all total discounted returns, across all possible strategies:

$$v(x_0) = \sup_{\sigma \in \Sigma} W(\sigma)(x_0).$$

2. Infinite-horizon Decisions V

Histories, Strategies, Value

Brute force solution proposal?

- Construct all possible strategies σ from Σ, and
- Evaluate the discounted lifetime payoff of each strategy, $W(\sigma)(x_0)$.
- Pick the strategy (or strategies) that deliver the maximal value, $v(x_0) = \sup_{\sigma \in \Sigma} W(\sigma)(x_0)$.

... **No es posible!** ¿Por qué?

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3. Recursive representation I

Bellman Functional Equation

A typical assumption to ensure that v is well-defined would be the following:

Assumption

There is a real number $K < +\infty$ *such that* $|U(u_t, x_t)| \leq K$ *for all* $(u_t, x_t) \in A \times X$.

or

Assumption

$A \times X$ *is compact, and* U *is continuous on* $A \times X$.

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3. Recursive representation II

Bellman Functional Equation

Lemma

If U is bounded, then $v : X \to \mathbb{R}$ is bounded.

Lemma allows us to assign finite numbers when ordering or ranking alternative strategies.

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3. Recursive representation I

Principle of Optimality

Let $x \equiv x_t$ and $x' \equiv x_{t+1}$.

Theorem (Bellman principle of optimality)

For each $x \in X$, the value function $v : X \to \mathbb{R}$ of (P1) satisfies

$$v(x) = \sup_{u \in \Gamma(x)} \{U(x,u) + \beta v(x')\} \textit{ s.t. } x' = f(x,u) \tag{1}$$

3. Recursive representation II

Principle of Optimality

How to prove BPO. Part A:

- Define $W(x) = \sup_{u \in \Gamma(x)} \{U(x,u) + \beta v[f(x,u)]\}$, for any $x \in X$.
- Fix $\epsilon > 0$, then there is a strategy σ feasible from x s.t. $W[f(x,u);\sigma] \geq v[f(x,u)] - \epsilon$. (Why?)
- Then,

$$v(x) \geq U(x,u) + \beta v[(f(x,u);\sigma] - \beta\epsilon$$

3. Recursive representation III

Principle of Optimality

- Take ϵ to zero. Then,

$$v(x) \geq U(x,u) + \beta v[(f(x,u);\sigma].$$

- This holds for all $u \in \Gamma(x)$, so that $v(x) \geq W(x)$.

3. Recursive representation IV

Principle of Optimality

Part B:

- Since there is an $\epsilon > 0$ s.t.

$$v(x) - \epsilon \leq W(x;\sigma) = U(x,u) + \beta W[f(x,u);\sigma],$$

  and, since $v[f(x,u)] \geq W[f(x,u);\sigma]$, then

$$\begin{aligned} v(x) - \epsilon &\leq U(x,u) + \beta W[f(x,u);\sigma] \\ &\leq \sup_{u \in \Gamma(x)} \{U(x,u) + \beta v(f(x,u))\} = W(x). \end{aligned}$$

- Take ϵ (arbitrary) to zero, so then, $v(x) \leq W(x)$.

3. Recursive representation V

Principle of Optimality

Combining Part A and Part B:

- We have $v(x) = W(x)$.
- Since by definition,

$$W(x) := \sup_{u \in \Gamma(x)} \{U(x,u) + \beta v[f(x,u)]\}$$

...

- Then, $v(x) = \sup_{u \in \Gamma(x)} \{U(x,u) + \beta v[f(x,u)]\}$.

3. Recursive representation VI

Principle of Optimality

Checkpoint!

Remember our heuristic example, where *we thought we could* re-write the infinite-sequence problem as this?

....

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4. Fixed Point of Bellman Functional I

Checkpoint!

But, also recall we said, if we could rewrite the infinite sequence problem recursively ...

... as the BPO shows we can:

$$v(x) = \sup_{u \in \Gamma(x)} \{U(x,u) + \beta v[f(x,u)]\}$$

... or, equivalently,

$$v(x) = \sup_{x' \in \Gamma(x)} \{U[x, f^{-1}(x';x)] + \beta v[x']\}$$

... **we still have to find v!**

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4. Fixed Point of Bellman Functional I

What is a functional operator?

Why Bellman "operator"?

First order of business:

- Let

$$B(X) := \{w : X \to \mathbb{R}| \text{ w is bounded } \}.$$

4. Fixed Point of Bellman Functional II

What is a functional operator?

- For any $w \in B(X)$, let $T(w)$ be a function(al) such that

$$T(w)(x) = \sup_{u \in \Gamma(x)} \{U(x,u) + \beta w(f(x,u))\}$$

  at any $x \in X$.

- Since U and w are bounded, then $T(w) \in B(X)$.

4. Fixed Point of Bellman Functional III

What is a functional operator?

Remarks:

- So $T : B(X) \to B(X)$ is a self-map.
- Here, T maps the set of bounded functions into itself.
- Here, T operates on functions,

  ... a.k.a. "Bellman operator",

  ... "Bellman functional equation",

  ... or "Bellman equation".

4. Fixed Point of Bellman Functional IV

What is a functional operator?

- By analogy, recall the Solow-Swan equilibrium map, $g : X \rightarrow X$? That was an operator that takes a point into another point in an interval $X \subseteq \mathbb{R}_+$.
- Here, a "point" in $B(X)$ is a function!

  The fixed-point solution of the map $T : B(X) \rightarrow B(X)$, is v, the value function!
- Recall, v in economic terms is an indirect utility. ...

  ... By construction it encodes the "best" (total) payoff.

  Then supporting v must be some optimal strategy σ^* ... ?

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4. Fixed Point of Bellman Functional I

Existence and Uniqueness of v

How do we know v exists? ...

... If it does, is it unique?

4. Fixed Point of Bellman Functional II

Existence and Uniqueness of v

It turns out, under some regularity conditions, we can apply the Banach fixed point theorem.

Goal

- Show that the Bellman operator $w \mapsto T(w)$ is a contraction mapping on a complete metric space.
- Then these conditions suffices for the existence of a unique function v, such that $v = T(v)$.