RCE-REE with Distortions: A Keynesian Example

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Outline

Motivation

New Keynesian model: A popular policy-relevant model.

- Monetary theory and policy:
  1. a version of the RBC model augmented with imperfect competition in product and/or factor markets
  2. PLUS pricing rigidity rationalize inflation and output policy trade-off with short-run Keynesian dynamics and long-run neoclassical features.
- Use this model to study, qualitatively, monetary policy and business cycle stabilization:
  1. interest-rate policy instrument
  2. monetary authority maximizing an average measure of private utility (social welfare) $\approx$ loss criterion in terms of inflation and output "gaps".
  3. Commitment (ex-ante optimal) benchmark vs. discretionary (ex-post optimal) policy.
- The NK model is not the only model of monetary business cycle and policy!

NK Microfoundations

- A cashless economy where there is a continuum of households defined on $[0, 1]$.
- Continuum of firms on $[0, 1]$, each producing a good differentiated by product type i.
- Household derives utility from consumption and leisure. Holds one-period Arrow securities. Rent labor services to firms. Own firms directly.
- Firms solve dynamic profit-max problems. [c.f. RBC firm.] Firm has (i) production unit – uses only labor, no capital; (ii) pricing department – sets pricing optimally with some probability.
- Monetary authority sets state-contingent interest rate policy. Simple rule/optimal rule.
- Solution by RCE-REE. Approximate with log-linearization approach.

Information

Let $h^t = \{s_0, ..., s_t\}$ be history of shocks up to and including date t.

This could be a list of technology and government spending shocks, for example.

Later, as usual, we generate h^t in a recursive, Markovian way.

Household problem

Each household buys a basket of a variety of the goods produced by each firm:

$$C_t = \left[\int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}} .$$

where $\varepsilon > 1$ is the constant elasticity of substitution between any two varieties i and j.

Note: Trick is to optimize the $C(i)$ statically, and Solve for decision function for C_t dynamically.

Total discounted expected utility criterion function:

$$\mathbb{E}\left\{\sum_{t=0}^{\infty}\beta^t[U(C_t(h^t),N_t(h^t))]\middle|s_0\right\}$$
$$=\sum_{t=0}^{\infty}\beta^t\int_{h^t}[U(C_t(h^t),N_t(h^t)]\rho(h^t)dh^t.$$

Assume $U(C,N)$ is separable in consumption and labor, is increasing, twice differentiable and concave in C and decreasing, twice-differentiable and convex in N.

The sequence of household budget constraint at each history h^t:

$$P_t(h^t)C_t(h^t) + \int_{s_{t+1}} Q_{t,t+1}(s_{t+1}|h^t)B_{t+1}(s_{t+1};h^t)ds_{t+1}$$
$$\leq W_t(h^t)N_t(h^t) + T_t(h^t) + B_t(h^t) + \int_0^1 \Pi_t(i)di. \quad (1)$$

Households choose the allocations $\{C_t(h^t), N_t(h^t), B_{t+1}(s_{t+1};h^t)\}_{t\geq 0}$ taking as given the prices $\{P_t(h^t), W_t(h^t), Q_{t,t+1}(s_{t+1}|h^t)\}_{t\geq 0}$ for every h^t and given initial asset holdings B_0.

Household stochastic intertemporal optimality

Now we drop explicit dependency on h^t ...

The optimality condition for the dynamic household decision problem is given by

$$Q_{t,t+1} = \beta \frac{P_t}{P_{t+1}} \frac{U_c(C_{t+1})}{U_c(C_t)} \rho(h^{t+1}|h^t). \tag{2}$$

which holds for every $t \geq 0$ and every h^t and h^{t+1} reachable from h^t. Note that $Q_{t,t}(h^t|h^t) = 1$.

Complete markets: If the household held some redundant asset (e.g. a riskless bond) that pays expected (nominal) return of $1 + i_t$ each period, the no-profitable-arbitrage condition implies that

$$1 + i_t = \frac{1}{\int_{s_{t+1}} Q_{t,t+1}(s_{t+1}|h^t) ds_{t+1}} := \frac{1}{\mathbb{E}_t Q_{t,t+1}}.$$

So we can take conditional expectations of (2) and then re-write it as

$$U_c(C_t) = \beta(1 + i_t)\mathbb{E}_t \left\{ U_c(C_{t+1}) \frac{P_t}{P_{t+1}} \right\}. \tag{3}$$

which holds for every $t \geq 0$ and every h^t.

Finally at the optimum (1) holds with equality for every $t \geq 0$ and every h^t.

Household intratemporal optimality

Household chooses labor supply optimally according to the familiar static first order condition:

$$-\frac{U_n(N_t)}{U_c(C_t)} = \frac{W_t}{P_t}. \tag{4}$$

Exercise

Show that given the choice of the consumption bundle each period (and in each h^t) C_t the optimal intratemporal allocation of expenditure between each variety of goods gives rise to the demand function for each variety i as

$$C_t(i) = \left(\frac{P_t(i)}{P_t} \right)^{-\varepsilon} C_t, \tag{5}$$

and the resulting aggregate price index for the bundle C_t is a CES aggregator of individual varieties' prices:

$$P_t = \left[\int_0^1 (P_t(i))^{1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}}.$$

Hint: The consumer must solve an expenditure minimization problem.

Firms' problem

- Each firm on $[0, 1]$ produces a differentiated good indexed by $i \in [0, 1]$.
- Monopolistically competitive.
- Let's break each firm into two departments:
  1. Production, and
  2. Pricing

Optimal production

Firm i hires labor $N_t(i)$ taking the competitive wage W_t as given and produces output using a linear technology:

$$Y_t(i) = A_t N_t(i),$$

where A is the stochastic technology level common to all firms.

Since firms have same technology, then common marginal cost MC_t. Each firm solves a cost minimization problem:

$$\min_{N_t(i)} W_t N_t(i) + MC_t[Y_t(i) - A_t N_t(i)]$$

with efficient production given by

$$W_t = MC_t A_t. \tag{6}$$

Note: No physical capital!

Optimal pricing

- In standard monopoly result: Set optimal price as markup over marginal cost. [Draw racy micro picture!]
- Calvo (1983, JME)*: Now random arrival of signal each period: "OK to reset optimal price". *Original Calvo model in continuous time – signal given by Poisson arrival process.
- So in dynamic context, firms solve a dynamic markup, taking into account stochastic demands, probability of "price stickiness", stochastic marginal costs, to maximize P.V. total expected profits.

Model of price stickiness:

- Constant probability $(1-\theta)$ per period that firms get to adjust their prices.
- By the law of large numbers, the fraction of firms $=$probability they do not change prices in one period, θ.
- Let P_t^* be the new aggregate price set in period t. Evolution of the aggregate price index is then,

$$P_t = \left[\theta P_{t-1}^{1-\varepsilon} + (1-\theta)(P_t^*)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}, \tag{7}$$

- Current aggregate CPI index depends on its previous value (by a fraction of firms stuck with the old price) and the new price set by current price-resetting firms.

Criticisms?

1. θ is not state-dependent. C.f. Dotsey, King and Wolman (QJE) state-dependent pricing.
2. What explains θ? What if firms' pricing decisions take into account other firms' decisions?
3. Some recent empirical challenges to model's predictions.

Now let's go back to firm i who gets to reset its price at time t. This firm knows that the probability it will be stuck with this new price for the next $k \geq 0$ periods ahead will be θ^k. Firm i chooses $P_t^*(i)$ to maximize its expected present discounted stream of profits

$$\mathbb{E}_t \sum_{k=0}^{\infty} \theta^k Q_{t,t+k} C_{t+k}(i)[P_t(i) - MC_{t+k}] \tag{8}$$

such that

$$C_{t+k}(i) = \left(\frac{P_t(i)}{P_{t+k}}\right)^{-\varepsilon} C_{t+k}.$$

A usual Keynesian model where output of firms are determined on the "short side" or in other words, demand determined.

Exercise

Show that the optimal pricing strategy of firm i is characterized by

$$P_t^*(i) = \frac{\varepsilon}{\varepsilon - 1} \frac{\mathbb{E}_t \sum_{k=0}^{\infty} \theta^k Q_{t,t+k} MC_{t+k} C_{t+k}(i)}{\mathbb{E}_t \sum_{k=0}^{\infty} \theta^k Q_{t,t+k} C_{t+k}(i)}. \tag{9}$$

Remark. If $\theta = 0$, prices are completely flexible, the first-order condition for optimal price setting collapses to

$$P_t^*(i) = \frac{\varepsilon}{\varepsilon - 1} MC_t.$$

This is the familiar (undergraduate microeconomics) static monopolist's markup of price over marginal cost, which depends on the own-price elasticity of demand.

1. Goods markets clearing. For each i,

$$Y_t(i) = A_t N_t(i) = C_t(i).$$

Integrating over all firms on $[0,1]$ we have $Y_t := \int_0^1 Y_t(i)di$, so that aggregate market clearing is

$$Y_t = A_t N_t = C_t \int_0^1 \left(\frac{P_t(i)}{P_t} \right)^{-\varepsilon} di \tag{10}$$

2. Labor market clearing implies real marginal cost as

$$mc_t := \frac{MC_t}{P_t} = -\frac{U_n(N_t)}{U_c(C_t)} \frac{1}{A_t}. \tag{11}$$

3. Symmetric pricing equilibrium where all firms that get to reset prices will chose the same price:

$$P_t^* = \frac{\varepsilon}{\varepsilon - 1} \frac{\mathbb{E}_t \sum_{k=0}^{\infty} (\beta\theta)^k \Gamma_{t,t+k} MC_{t+k} P_{t+k}^{\varepsilon} C_{t+k}}{\mathbb{E}_t \sum_{k=0}^{\infty} (\beta\theta)^k \Gamma_{t,t+k} P_{t+k}^{\varepsilon} C_{t+k}}. \tag{12}$$

where the demand function (5) and (3) are used,

$$\Gamma_{t,t+1} = \beta^{-1} Q_{t,t+1}.$$

Log-linear approximation of RCE-REE

We need to make some assumption about preferences. Assume that the per period utility function is given by

$$U(C,N) = \frac{C^{1-\sigma}}{1-\sigma} - \frac{N^{1+\gamma}}{1+\gamma} \tag{13}$$

where $\sigma, \gamma > 0$.

In the steady state $P_t = P_{t-1} = P_t^* = P_{ss}$ and $Q_{t,t+1} = \beta$ or $\Gamma_{t,t+1} = 1$. Goods market clearing becomes:

$$\hat{Y}_t = \hat{N}_t + \hat{A}_t = \hat{C}_t. \tag{14}$$

Exercise

Show that the labor market clearing condition now can be approximated as

$$\hat{mc}_t = (\gamma + \sigma)\hat{Y}_t - (1+\gamma)\hat{A}_t. \tag{15}$$

Intuition from this approximate equilibrium condition?

Supply side. Next we log-linearize the firms' optimal pricing decisions.

Exercise

Show that the log-linearized approximation of (12) is given by

$$\hat{P}_t^* = (1-\beta\theta)\mathbb{E}_t \sum_{k=0}^{\infty} [\hat{mc}_{t+k} + \hat{P}_{t+k}]$$
$$= (1-\beta\theta)[\hat{mc}_t + \hat{P}_t] + \beta\theta\mathbb{E}_t \hat{P}_{t+1}^*. \quad (16)$$

Intuition of this last equation?

Exercise

Next show that the log-linearized version of (7) is

$$\hat{P}_t = \theta \hat{P}_{t-1} + (1-\theta)\hat{P}_t^*. \tag{17}$$

Subtracting $\hat{P}_{t-1}$ on both sides of (17) we get CPI inflation as

$$\pi_t := \hat{P}_t - \hat{P}_{t-1} = (1-\theta)(\hat{P}^*_t - \hat{P}_{t-1}) \tag{18}$$

Combining (17) and (18) we obtain the New-Keynesian Phillips curve:

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \frac{(1-\theta)(1-\beta\theta)}{\theta} \hat{mc}_t.$$

Demand side. The household's optimal consumption plan is approximately given by

$$\hat{C}_t = \mathbb{E}_t \hat{C}_{t+1} - \frac{1}{\sigma}\hat{r}_t$$

where $\hat{r}_t = \hat{i}_t - \mathbb{E}_t \pi_{t+1}$. We can use goods market clearing to re-write this as

$$\hat{Y}_t = \mathbb{E}_t \hat{Y}_{t+1} - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t \pi_{t+1}).$$

We can interpret this Euler equation in terms of output as a forward-looking Keynesian IS curve. Aggregate demand depends on expected future output conditions and the ex ante real interest rate.

Deviations from "potential" ...

Natural output level is defined as the level of output that would prevail if prices were completely flexible. This implies that $\hat{mc}_t = 0$ in all periods and states of the world. From (15) we have the natural output as

$$\hat{Y}_t^n = \left(\frac{1+\gamma}{\sigma+\gamma}\right)\hat{A}_t.$$

Notice that under flexible prices, this level of output is independent of the interest rate, hence monetary policy. It only depends only on the level of technology, which indeed makes this model a special case of the RBC model without capital.

Now define output gap as the deviation of output from the flexible price output:

$$x_t = \hat{Y}_t - \hat{Y}_t^n.$$

Using this in (15) again, we get

$$\hat{mc}_t = (\gamma + \sigma) x_t.$$

So aggregate supply, or the Phillips curve, can be written as

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t. \tag{19}$$

where

$$\kappa = \frac{(\gamma + \sigma)(1 - \theta)(1 - \beta\theta)}{\theta}.$$

Exercise

Show also that the IS curve can be re-written in output gap terms as

$$x_t = \mathbb{E}_t x_{t+1} - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t \pi_{t+1} - r_t^n). \tag{20}$$

where that natural real interest rate is

$$r_t^n := \sigma \mathbb{E}[\hat{Y}_{t+1}^n - \hat{Y}_t^n] = \frac{\sigma(1+\gamma)}{\sigma+\gamma}\mathbb{E}_t[\hat{A}_{t+1} - \hat{A}_t].$$

Definition

Given the stochastic process $\{\hat{A}_t\}_{t\geq 0}$ and a monetary policy plan $\{\hat{i}_t\}_{t\geq 0}$, a RCE-REE in this approximate economy is the set of bounded stochastic processes $\{x_t, \pi_t\}_{t\geq 0}$ satisfying:

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t. \tag{19}$$

$$x_t = \mathbb{E}_t x_{t+1} - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t \pi_{t+1} - r_t^n). \tag{20}$$

$$r_t^n := \sigma \mathbb{E}[\hat{Y}_{t+1}^n - \hat{Y}_t^n] = \frac{\sigma(1+\gamma)}{\sigma+\gamma} \mathbb{E}_t[\hat{A}_{t+1} - \hat{A}_t].$$

Remarks:

1. So a log-linear approximation of the RCE-REE conditions of the closed-economy NK model yields a 2-equation expectational difference equation system in (x, π).
2. Given specification of monetary policy, pinning down process for i, we find REE by finding stable solution to this system.
3. Note: Clarida, Galí, and Gertler (2001, AER P & P) (CGG):
    - small open economy version with complete exchange rate pass through, yields *qualitatively* the same equations!
    - "Openness" measured by share of imports in consumption bundle affect only slope of the IS curve w.r.t. ex-ante real interest rate.
4. Breaking the CGG "isomorphism" between the closed vs. open economy models.
    - Monacelli (2007, JMCB) allow for incomplete imports price pass-through, assuming imported monopoly.
    - Alonso-Carrera and Kam (2015, MD) show that this break can be obtained in a simpler model with incomplete international securities market.

Sources of FWT failure

- Why are fiscal-monetary policy makers are needed in this world?
- We appeal to the failure of FWT here.
- There are two sources of distortions to an otherwise neoclassical economy here:
  1. Monopolistic distortion
  2. Sticky price distortion
- That is, the RCE-REE allocation will not be a Pareto optimal allocation.

Distortion 1: monopolistic deadweight loss

- Absent intervention, output is inefficiently lower that under perfectly competitive markets, even in the long run with flexible prices.
- WLOG, suppose at steady state $A_{ss} = 1$. Let $\mu := \varepsilon/(\varepsilon - 1)$. Then, from labor-market clearing condition:

$$mc_{ss} := \frac{MC_{ss}}{P_{ss}} = \frac{1}{\mu_{ss}} = \frac{W_{ss}}{P_{ss}}.$$

- From the household optimal labor supply,

$$\frac{N^{\gamma}}{C^{-\sigma}} = \frac{W_{ss}}{P_{ss}}.$$

- Further, $Y_{ss} = C_{ss} = N_{ss}$, so that

$$Y_{ss}^{\gamma+\sigma} = \frac{1}{\mu_{ss}} \Rightarrow Y_{ss} = \left(\frac{1}{\mu_{ss}}\right)^{\frac{1}{\gamma+\sigma}}.$$

- The efficient level of output by firms, Y^*, would be the case $\mu_{ss} = 1$. Since for monopolistically competitive firms $\mu_{ss} > 1$, it means that $Y_{ss} < Y^* = 1$.

We can get rid of this monopolistic distortion, in theory, easily.

- Use fiscal policy.
- The optimal FP in this case is to set a lump sum subsidy to employment, $T(h^t) = T > 0$, such that

$$mc_{ss}^* = mc_{ss} + T = \frac{W_{ss}}{P_{ss}} + T = 1 \Rightarrow Y_{ss} = Y^* = 1.$$

Distortion 2: sticky relative price distortion

- Calvo pricing model introduces staggered price setting via θ.
- Consider the term

$$d_t := \int_0^1 \left(\frac{P_t(i)}{P_t} \right)^{-\varepsilon} di$$

defined as the dispersion of relative prices across producers.

- This can be written as

$$d_t = (1-\theta) \left(\frac{1-\theta \pi_t^{\varepsilon-1}}{1-\theta} \right)^{\frac{-\varepsilon}{1-\varepsilon}} + \theta \pi_t^{\varepsilon} d_{t-1}.$$

- In a deterministic steady state, we have

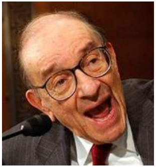
$$d_{ss} = \frac{1-\theta}{1-\theta \pi_{ss}^{\theta}} \left(\frac{1-\theta \pi_{ss}^{\varepsilon-1}}{1-\theta} \right)^{\frac{-\varepsilon}{1-\varepsilon}} \begin{cases} = 1, & \text{if } \pi_{ss} = 1 \text{ or } \theta = 0 \\ > 1, & \text{otherwise} \end{cases}.$$

- The relative price distortion across producers who face identical marginal costs is minimized when either there is zero inflation at steady state, or when prices are perfectly flexible.

Specifying monetary policy

We now want to study the behavior of the RCE-REE of the NK model under alternative monetary-policy settings:

1. MP follows simple policy rule. We don't need a FOMC. Just a monkey who knows how to use a calculator.



Pan troglodytes vs. Sophisticus Alere Fartacus

2. Model FOMC decision making as an optimal planning problem, with commitment.
3. Model FOMC decision making as an optimal planning problem, without commitment.

Recall ... with a new addition ...

Definition

Given the stochastic process $\{\hat{A}_t, u_t\}_{t\geq 0}$ and a monetary policy plan $\{\hat{i}_t\}_{t\geq 0}$, a RCE-REE in this approximate economy is the set of bounded stochastic processes $\{x_t, \pi_t\}_{t\geq 0}$ satisfying:

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t + u_t. \tag{19}$$

$$x_t = \mathbb{E}_t x_{t+1} - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t \pi_{t+1} - r_t^n). \tag{20}$$

$$r_t^n := \sigma \mathbb{E}[\hat{Y}_{t+1}^n - \hat{Y}_t^n] = \frac{\sigma(1+\gamma)}{\sigma+\gamma} \mathbb{E}_t[\hat{A}_{t+1} - \hat{A}_t].$$

Simple policy function

First we close the RCE-REE with a description of a monetary policy plan $\{\hat{i}_t\}_{t\geq 0}$ induced by simple "Taylor rule":

$$\hat{i}_t = \phi_\pi \pi_t; \qquad \phi_\pi > 0. \tag{21}$$

WLOG, we can set $u_t = 0$, for all t.

Private agents know central bank follows this simple strategy for policy.

The (approximate) equilibrium is then given by (19), (20) and (21).

Substituting out $\hat{i}_t$ the approximate REE is characterized by

$$\begin{pmatrix} \pi_t \\ x_t \end{pmatrix} = \frac{1}{\sigma + \kappa\phi_\pi} \begin{pmatrix} \sigma\beta + \kappa & \sigma\kappa \\ 1 - \beta\phi_\pi & \sigma \end{pmatrix} \begin{pmatrix} \mathbb{E}_t\pi_{t+1} \\ \mathbb{E}_t x_{t+1} \end{pmatrix} + \frac{1}{\sigma + \kappa\phi_\pi} \begin{pmatrix} \kappa \\ 1 \end{pmatrix} r_t^n. \qquad (✠)$$

Proposition

A sequence of allocation and relative pricing functions, $\{x_t, \pi_t\}$*, satisfying the conditions for the approximate RCE-REE, is unique (stable) if and only if the eigenvalues of*

$$\mathbf{F} = \frac{1}{\sigma + \kappa\phi_\pi} \begin{pmatrix} \sigma\beta + \kappa & \sigma\kappa \\ 1 - \beta\phi_\pi & \sigma \end{pmatrix}$$

lie inside the unit circle.

The characteristic polynomial of $\mathbf{F}$ is

$$P(\lambda) = \lambda^2 - \mathrm{trace}(\mathbf{F})\lambda + \det(\mathbf{F}).$$

The (at most two distinct) eigenvalues, λ, solve $P(\lambda) = 0$.

Theorem

Let $\mathbf{A}$ *be a* $n \times n$ *matrix with eigenvalues* $\lambda_1, ..., \lambda_n$. *Then*

1. $\lambda_1 + \lambda_2 + ... + \lambda_n = trace(\mathbf{A})$, *and*
2. $\lambda_1 \cdot \lambda_2 \cdot ... \cdot \lambda_n = \det(\mathbf{A})$.

Proposition

$\mathbf{F}$ *is a stable matrix, or* $|\lambda_i| < 1$, *if and only if*

1. $|\det(\mathbf{F})| < 1$, *and*
2. $|-trace(\mathbf{F})| - \det(\mathbf{F}) < 1$.

Application: checking for stability of RCE-REE Now we can apply this result to our model to check for stability of an REE solution.

1. Since

$$|\det(\mathbf{F})| = \left| \frac{\sigma\beta}{\sigma + \kappa\phi_\pi} \right|$$

the first condition $|\det(\mathbf{F})| < 1$ requires $\beta - 1 < \kappa\phi_\pi/\sigma$. This holds for any $\phi_\pi \geq 0$.

2. The second condition $|\text{trace}(\mathbf{F})| - |\det(\mathbf{F})| < 1$ requires that

$$\frac{\sigma(\beta+1)+\kappa}{\sigma+\kappa\phi_\pi} < \frac{\sigma(\beta+1)+\kappa\phi_\pi}{\sigma+\kappa\phi_\pi}$$

which holds if and only if $\phi_\pi \geq 1$.

Proposition

Given the stochastic process $\{\hat{A}_t\}_{t\geq 0}$, *there exists a unique (stable) a rational expectations decentralized equilibrium (REE)* $\{x_t, \pi_t\}_{t\geq 0}$ *satisfying (19) and (20) under a monetary policy plan* $\{\hat{i}_t\}_{t\geq 0}$ *defined by*

$$\hat{i}_t = \phi_\pi \pi_t$$

if and only if $\phi_\pi \geq 1$.

Remark.

- The approximate REE is stable only when the central bank, equipped with the simple policy is able to respond by more than one-for-one to a change in the inflation rate.
- E.g.: if inflation rises by one percent, the central bank has to raise the nominal interest rate by $\phi_\pi > 1$ percent.
- If not, the equilibrium either has multiple or explosive REE.
- Note, stability condition(s) for the linearized RCE-REE, is independent of the "noise statistics" in (✠), i.e. the last term:

$$+\frac{1}{\sigma + \kappa\phi_\pi}\begin{pmatrix} \kappa \\ 1 \end{pmatrix} r_t^n.$$

Why?

What if the Taylor rule includes response to contemporaneous output gap too?

Exercise

Characterize the condition for a unique REE in the linearized model when the monetary authority follows the simple Taylor-type feedback rule:

$$\hat{i}_t = \phi_\pi \pi_t + \phi_x x_t.$$

Explain the intuition behind this condition.

Suppose the monetary authority seeks to maximize the expected total discounted utility of average households. For each given history h^∞ let the welfare criterion be

$$\begin{aligned} W(s_0)\Big|_{h^\infty \ni s_0} &= \sum_{t=0}^{\infty} \beta^t U(C_t(h^t), N_t(h^t)) \\ &= \sum_{t=0}^{\infty} \beta^t \left\{ \frac{[C_t(h^t)]^{1-\sigma}}{1-\sigma} - \frac{[N_t(h^t)]^{1+\gamma}}{1+\gamma} \right\}. \end{aligned}$$

Assume that fiscal policy uses lump-sum transfers to offset the monopolistic distortion on the steady state output.

That is we want the short run dynamics to be defined as fluctuations relative to a flexible price equilibrium with a first-best allocation, which nests the deterministic steady state with the efficient level of output $Y^* = 1$.

Proposition

To a second-order approximation around the efficient steady state with zero inflation,

$$W(s_0)\Big|_{h^\infty \ni s_0} \approx -\frac{U_c(Y_{ss})Y_{ss}}{2}\sum_{t=0}^{\infty}\beta^t\Bigg\{(\sigma+\gamma)[x_t(h^t)]^2 + \frac{\theta\varepsilon}{(1-\theta\beta)(1-\theta)}[\pi_t(h^t)]^2\Bigg\}.$$

Or in short-hand:

$$W(s_0) \approx -\frac{1}{2}\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left\{\omega[x_t(h^t)-\overline{x}]^2 + [\pi_t(h^t)]^2\right\}.$$

where $\overline{x} = Y^* - Y_{ss}|_{T=0}$, and,

$$\omega = \frac{U_c(Y_{ss})Y_{ss}(\sigma+\gamma)(1-\theta\beta)(1-\theta)}{\theta\varepsilon}.$$

Benchmark optimal policy plan under the commitment to a time-0 contingent plan for monetary policy which

1. minimizes the expected total discounted losses, subject to,
2. the decentralized equilibrium conditions.

The Lagrangian for the monetary authority under commitment to this plan is

$$L = \frac{1}{2}\mathbb{E}_0\Bigg\{ \sum_{t=0}^{\infty} \beta^t \Bigg[(\pi_t^2 + \omega x_t^2) + \phi_t(\pi_t - \beta \mathbb{E}_t \pi_{t+1} - \kappa x_t - u_t) \\ \psi_t \left(x_t - \mathbb{E}_t x_{t+1} + \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t \pi_{t+1} - r_t^n) \right) \Bigg] \Bigg\}$$

where there is one pair of $(\phi_t(h^t), \psi_t(h^t))$ for every $t \geq 0$ and h^t.

The optimal time-0 plan $\{\pi_t(h^t), x_t(h^t), \hat{i}_t(h^t)\}$ is characterized by the set of first-order conditions:

$$\begin{aligned} \pi_t + \phi_t - \phi_{t-1} =& 0, \\ \omega x_t - \kappa\phi_t =& 0 \\ \mathbb{E}_0 \psi_t =& 0. \end{aligned}$$

for all $t \geq 0$ and all histories h^t.

Simplifying the first-order conditions, the "optimal commitment policy" is to set inflation such that

$$\pi_t^C = \begin{cases} = -\frac{\omega}{\kappa}(x_t - x_{t-1}), & \text{for } t > 0 \\ \\ = -\frac{\omega}{\kappa}x_0, & \text{for } t = 0 \end{cases}, \tag{22}$$

for every h^t.

Features of ex-ante optimal plan

1. At time 0, the monetary authority implements $\pi_0 = -\frac{\omega}{\kappa} x_0$ and promises to set $\pi_t(h^t) = -\frac{\omega}{\kappa}(x_t(h^t|h^{t-1}) - x_{t-1}(h^{t-1}))$ for all $t > 0$.
2. When $t \geq 1$, the central bank *by assumption* implements the promised plan for every possible history, which is to lower inflation by ω/κ percent in any state of the world if output growth increases by one percent.
3. ex-ante optimal policy is history dependent.

Note: For all $t \geq 1$, if we substitute (22) into the Phillips curve constraint, we have a second-order difference equation in output gap:

$$x_t(1+\beta+\kappa^2/\omega) = \beta\mathbb{E}_t x_{t+1} + x_{t-1} - \frac{\kappa}{\omega}u_t.$$

Example (History dependence of commitment plan)

Now assume that $u_t \sim$ i.i.d.$(0, \sigma_u^2)$. Let $x_{-1} = 0$. The stable solution to this is

$$x_t = a x_{t-1} - \frac{a\kappa}{\omega} u_t$$

where

$$a = \frac{1 + \beta + \frac{\kappa^2}{\omega}}{2\beta} \left[1 - \sqrt{1 - \frac{4\beta}{\left(1 + \beta + \frac{\kappa^2}{\omega}\right)^2}} \right] < 1.$$

Example (cont'd)

We can then show by backward subsitution that:

$$x_t = -\frac{a\kappa}{\omega}\sum_{i=0}^{\infty} a^i u_{t-i}.$$

So the policy under commitment to the optimal time-0 plan is to contract output gap in response to not just a contemporaneous cost-push shock, but also to all prior shocks from the infinite past.

Intuition. The monetary authority ties its hands to past promises and does not re-optimize at any future stage, even it may be "profitable" to do so.

Ulysses and the Sirens from Homer's *Odyssey*

Example (cont'd)

Optimal policy rule with commitment:

$$\hat{i}_t = r_t^n + \left[a\sigma + \frac{(1-a)\omega}{\kappa} \right] x_t - a\sigma x_{t-1} + \left[a\rho + \frac{a(1-\rho)\sigma\kappa}{\omega} \right] u_t .$$

The optimal interest rate decision rule neutralizes variations in the natural interest rate completely, and responds to cost push shocks, and contemporaneous and *lagged* output gap.

Optimal discretionary policy

If monetary authority re-evaluates its optimal plan in every period, then at any time $\tau \geq 0$ the central bank's time-τ optimal contingent plan is characterized by the time-τ version of (22):

$$\pi_t^D = \begin{cases} = -\frac{\omega}{\kappa}(x_t - x_{t-1}), & \text{for } t > \tau \\ \\ = -\frac{\omega}{\kappa}x_t, & \text{for } t = \tau \end{cases}. \tag{23}$$

So in every period $t > 0$, by re-optimizing,

$$\pi_t^D = -\frac{\omega}{\kappa}x_t. \tag{24}$$

But this policy rule is different to that formulated before under the assumption of commitment, which was

$$\pi_t^C = -\frac{\omega}{\kappa}(x_t - x_{t-1}).$$

- At the beginning of every period $t > 0$, the central bank has the incentive to deviate from the time-0 optimal contingent plan and follow its time-t optimal action (25); and
- taking as given the optimal plans (thus expectations) of private households and firms.
- But if private agents are forward looking and rational, they would anticipate such an incentive of the central bank to deviate from its time$-t$ optimal plan, and encode this into their individual optimal plans.
- This leads us to require a notion of equilibrium where all decision makers (players) play a Nash equilibrium in the continuation game, following any state of the economy.

Markov perfect equilibrium

To use this solution concept for the dynamic game,

Assumption

The history of shocks $h^t = (s_0, ..., s_t) = (u_0, A_0, ..., u_t, A_t)$ *is Markov, so that* $\rho(h^t) = \rho((u_t, A_t)|h^{t-1})\rho(h^{t-1})$.

- Generally, even if shocks are Markovian, Subgame Perfection Equilibria are still history dependent.
- So in applied settings, restrict attention to Stationary Markov Perfect Equilibrium (MPE), where by assumption best-responses are time-invariant functions of only the current state.

So we have:

1. a dynamic game between the central bank and the private sector.
2. CB strategy $\{\pi_t, x_t, \hat{i}_t\}$
3. private sector (households and firms) strategy

$$\{\mathbb{E}_t \pi_{t+1}(\hat{i}_{t+1}; s_{t+1}), \mathbb{E}_t x_{t+1}(\hat{i}_{t+1}; s_{t+1})\}.$$

4. We focus our attention on Markov perfect equilibria.

Definition

A Markov perfect equilibrium in this game is such that beginning from any current state $s_t = (u_t, A_t)$ of the system,

1. the policy $\Phi = \{\pi_t, x_t, \hat{i}_t\}$ characterized by (25) is optimal for the central bank, and
2. private sector expectations conditioned on the central bank strategy

$$\{\mathbb{E}_t \pi_{t+1}(\Phi; s_{t+1}), \mathbb{E}_t x_{t+1}(\Phi; s_{t+1})\}$$

satisfy the RCE-REE conditions (IS and Phillips curves).

for all $t \geq 0$.

Step 1. (best-response of CB) The central bank at any period $t \geq 0$ seeks to solve

$$V(s_t) = \min_{x_t, \pi_t, \hat{i}_t} \frac{1}{2}(\omega x_t^2 + \pi_t^2) + \mathbb{E}_t \sum_{\tau=1}^{\infty} \beta^{\tau} (\omega x_{t+\tau}^2 + \pi_{t+\tau}^2)$$

$$= \min_{x_t, \pi_t, \hat{i}_t} \frac{1}{2}(\omega x_t^2 + \pi_t^2) + \beta \mathbb{E}_t V(s_{t+1})$$

subject to

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t + u_t,$$

$$x_t = \mathbb{E}_t x_{t+1} - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t \pi_{t+1} - r_t^n),$$

FONC for a maximum on the RHS is:

$$\pi_t^D = -\frac{\omega}{\kappa} x_t, \tag{25}$$

which is what we worked out before, if we said the CB re-optimizes.

Step 2. (best response of agents) Such time-inconsistent plans by successive central banks, must be anticipated by private agents. Private sector expectations are formed rationally such that

$$\mathbb{E}_t \pi_{t+1} = -\frac{\omega}{\kappa} \mathbb{E}_t x_{t+1}. \tag{26}$$

and the RCE-REE conditions are satisfied.

Step 3. (finding solution to MPE) Substitute (26) into IS curve to get

$$\left(1+\frac{\kappa^2}{\omega}\right)\pi_t = \beta \mathbb{E}_t \pi_{t+1} + u_t.$$

Exercise

Show that in the Markov perfect equilibrium inflation is

$$\pi_t = \frac{1}{\beta}\mathbb{E}_t \sum_{\tau=1}^{\infty} \left(\frac{\beta}{1+\frac{\kappa^2}{\omega}}\right)^{\tau} u_{t+\tau-1}. \tag{27}$$

Example

Suppose $u_t \sim \text{i.i.d.}(0, \sigma_u^2)$. Then the Markov equilibrium inflation process is given by

$$\pi_t = \left(\frac{\omega}{\omega + \kappa^2}\right) u_t.$$

Using (25) the output gap in the Markov equilibrium is then given by

$$x_t = -\left(\frac{\kappa}{\omega + \kappa^2}\right) u_t.$$

Remark. In this example with uncorrelated cost-push shocks,

1. the Markov equilibrium under ex post optimal monetary policy delivers inflation that rises less than one-for-one to a cost-push shock.
2. Also monetary policy partially neutralizes the cost push shock by allowing output gap to fall by less than one-for-one to a cost push shock.
3. Recall we claimed that the interest rate policy can perfectly offset the effect a technology shock on the natural interest rate.
4. Indeed, if we substitute these results into the IS curve, we can back out the implicit ex post optimal interest rate policy rule as

$$\hat{i}_t = r_t^n + \sigma \left(\frac{\kappa}{\omega + \kappa^2} \right) u_t.$$

   Thus, a rise in r_t^n is matched by a one-for-one rise in $\hat{i}_t$ which has no effect then on x_t or π_t. However, a rise in u_t is not completely neutralized by $\hat{i}_t$.
5. Effect of ω?

Numerical experiments

Questions to ask:

1. What are the gains from commitment, or losses from discretionary policy for the central bank?
2. Comparison of dynamics (response) of output and inflation, and interest rate.
3. Illustration of commitment vs. discretion in terms on business cycle moments.

Under (approx.) RCE-REE with optimal policy we find stable solutions satisfying:

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t + u_t. \tag{19}$$

$$x_t = \mathbb{E}_t x_{t+1} - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t \pi_{t+1} - r_t^n). \tag{20}$$

and, under commitment, optimal policy trade-off is:

$$\pi_t = -\frac{\omega}{\kappa}(x_t - x_{t-1}).$$

or under discretion, optimal policy trade-off is:

$$\pi_t = -\frac{\omega}{\kappa} x_t.$$

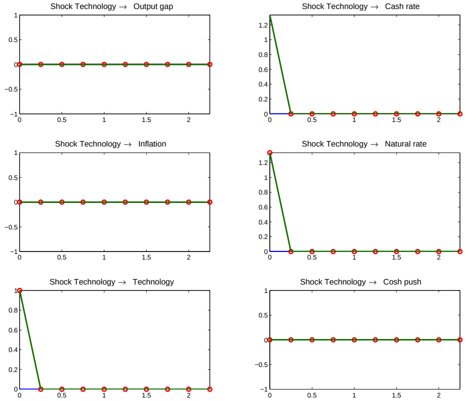
Using Harald Uhlig's setup, this system can be written down as

$$
\begin{aligned}
\mathbf{0} &= \mathbf{A}\mathbf{x}_t + \mathbf{B}\mathbf{x}_{t-1} + \mathbf{C}\mathbf{y}_t + \mathbf{D}\mathbf{z}_t \\
\mathbf{0} &= \mathbb{E}_t\mathbf{F}\mathbf{x}_{t+1} + \mathbf{G}\mathbf{x}_t + \mathbf{H}\mathbf{x}_{t-1} \\
&\qquad + \mathbf{J}\mathbb{E}_t\mathbf{y}_{t+1} + \mathbf{K}\mathbf{y}_t + \mathbf{L}\mathbb{E}_t\mathbf{z}_{t+1} + \mathbf{M}\mathbf{z}_t \\
\mathbf{z}_{t+1} &= \mathbf{N}\mathbf{z}_t + \varepsilon_{t+1}, \qquad \varepsilon_{t+1} \sim (\mathbf{0}, \boldsymbol{\Sigma})
\end{aligned}
$$

I let:

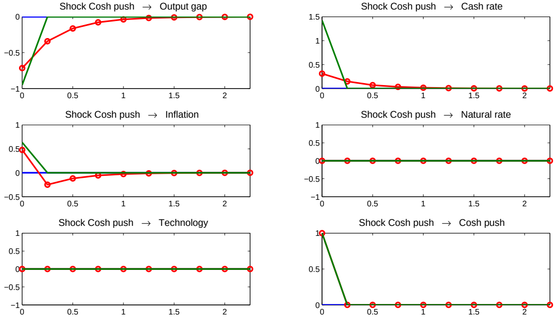
- Endogenous state variables, $\mathbf{x}$: (x, i)
- Endogenous other variables $\mathbf{y}$: (π, r^n)
- Exogenous state variables $\mathbf{z}$: (a, u).

Experiment 1: No persistence in shocks



Impulse response functions to TFP shock, a. Circles-red: commitment; Solid-green: Discretion

Experiment 1: No persistence in shocks

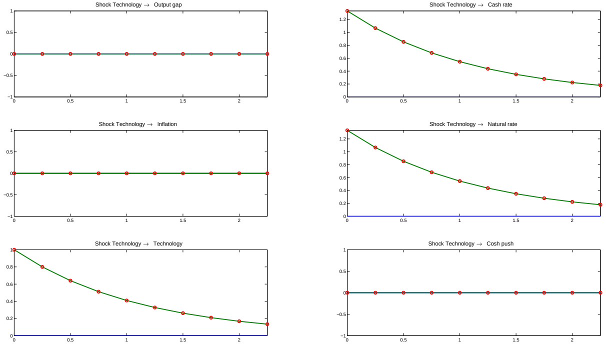


Impulse response functions to cost shock, u. Circles-red: commitment; Solid-green: Discretion

Table: Standard deviation (%)

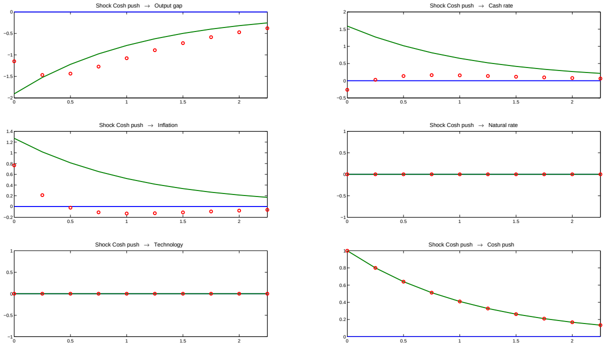
Variables	Commitment	Discretion
Output gap	0.5862	0.7327
Cash rate	0.9505	1.4304
Inflation	0.4428	0.4885
Natural rate	0.9155	0.9155
Technology	0.6866	0.6866
Cost push	0.7715	0.7715

Experiment 2: Persistent shocks



$\rho_a = 0.8$. Impulse response functions to TFP shock, a. Circles-red: commitment; Solid-green: Discretion

Experiment 2: Persistent shocks



$\rho_u = 0.8$. Impulse response functions to cost shock, u. Circles-red: commitment; Solid-green: Discretion