NEW-KEYNESIAN BUSINESS CYCLES AND MONETARY POLICY

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Abstract

In this note, we study the simplest possible version of a sequential markets economy with production which is further augmented with imperfectly competitive firms. This provides a platform for allowing firms to change prices optimally as part of their profit maximization plan. By introducing a mechanism that creates staggered price changes, we can rationalize a Phillips curve for the economy. We also derive a Keynesian flavored IS curve from household optimization problems. We then use this simple platform to study the design of monetary policy, with a focus on commitment to an ex ante optimal policy versus an ex post or time-consistent optimal policy. We also discuss simple policy rules. Throughout we will assume that the monetary authority controls the short term nominal interest rate. ◄ ▷

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1. Introduction

In the last decade, many academic and central-bank macroeconomists working in the field of applied monetary theory and policy, have been utilizing a version of the real-business-cycle model augmented with imperfect competition in product and/or factor markets. Together with assumptions about pricing rigidity in these models, they are able to rationalize aggregate outcomes for inflation and output, amongst other things, that exhibit short-run Keynesian dynamics and long-run neoclassical features. Elsewhere, this has been termed the New neoclassical synthesis: Goodfriend and King (1997). The models have also been extended to versions with international or open economy dimensions. A small open economy example that has been widely cited is Galí and Monacelli (2005).

In this note, we will consider one such framework of the New-Keynesian (NK) approach which builds in the assumption of monopolistically competitive firms that get to set prices optimally. However, one version of the model due to Calvo (1983) assumes that firms get to do so only occasionally when they receive a random signal saying that they can adjust their prices. Alternatively, one can introduce an explicit cost of price adjustment in the style of Rotemberg (1982).

Having constructed one such model, we will consider the role of monetary policy in terms of stabilizing the business cycle, via an interest-rate policy instrument. For the simple model we have, we can show that the problem of a monetary authority maximizing an average measure of private utility (social welfare) can be approximated to the second order by a loss criterion in terms of inflation and output gaps from their respective policy targets.

We will study the role of commitment to an *ex ante* optimal policy rule and why such a policy cannot be *ex post* optimal, or is "time inconsistent". Hence we also study *ex post*, discretionary or time-consistent, optimal policy. We will also discuss policy design from the perspective of simple policy rules.

One should be cautious to note that this class of models, while tractable enough to address monetary policy issues, should not be taken as the only theory of monetary economies. Indeed, the basic model creates a role for government intervention via monetary policy, but does not require the explicit modelling of a medium of exchange (or money). We can append a money market equilibrium easily by introducing money in the utility function (MIU), or some transactions technology that requires some purchases of goods using cash – a cash-in-advance constraint (CIA) – or alternatively, a shopping-time cost that can be economized by holding money.[1] So to reiterate, these models essentially take a frictionless Arrow-Debreu or sequential markets economy and introduce additional market imperfections and frictions to allow for monetary policy intervention in the stabilization of relative prices.

[1] Even so, this approach may not be conceptually appealing. One rationalizes equilibrium allocations that include real money balance holdings only because one assumes people derive utility from holding money (MIU) or alternatively there is an assumed black-box friction (CIA or shopping-time) that then creates a role for money to relax the constraint or cost. There is a growing literature on search theoretic foundations of money, or "deep monetary theory" as its proponents would call it, but the focus there is mainly on providing a theoretical foundation of trading costs/frictions and the existence of money. There is little connection as yet, in this literature, to aggregate monetary policy analysis and design. An exception might be the recent work of Rocheteau and Wright (2005).

2. A simple version of the NK model

Consider a cashless economy where there is a continuum of households defined on the compact interval $[0, 1]$. We will assume specific function forms for various objects of interest in this economy. There is also a continuum of firms on $[0, 1]$ each producing a good differentiated by product type i.

2.1 Household problem

Each household buys a basket of a variety of the goods produced by each firm. Assume the aggregate bundle is given by the CES aggregator:

$$C_t = \left[\int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}}.$$

where $\varepsilon > 1$ is the constant elasticity of substitution between any two varieties i and j.

Let $h^t = \{s_0, ..., s_t\} \in H^t$ denote the history of shocks up to and including date t. For now we leave s undefined. This could be a list of technology and government spending shocks, for example.

Suppose agents order stochastic streams of consumption and labor according to the total discounted expected utility criterion function:

$$\mathbb{E} \left\{ \sum_{t=0}^{\infty} \beta^t [U(C_t(h^t), N_t(h^t)] \middle| s_0 \right\} = \sum_{t=0}^{\infty} \beta^t \int_{h^t} [U(C_t(h^t), N_t(h^t)] \rho(h^t) dh^t. \tag{1}$$

Assume $U(C, N)$ is separable in consumption and labor, is increasing, twice differentiable and concave in C and decreasing, twice-differentiable and convex in N.

Households are direct owners of these firms and thus receive profits $\Pi(i)$ from each firm i. They also earn nominal labor income, $W_t(h^t)N_t(h^t)$ from renting labor services to firms via a perfectly competitive market.[2] The households receive net transfers from the government given by $T_t(h^t)$. The household enters period t, under history h^t, with asset $B_t(h^t)$. The household can then spend its income and wealth by purchasing the consumption bundle $C_t(h^t)$ at price $P_t(h^t)$ and a portfolio of Arrow securities $B_{t+1}(s_{t+1}; h^t)$ with state contingent price $Q_{t,t+1}(s_{t+1}|h^t)$. So we write the sequence of budget constraint at each history h^t as

$$\begin{aligned} P_t(h^t)C_t(h^t) + \int_{s_{t+1}} Q_{t,t+1}(s_{t+1}|h^t) B_{t+1}(s_{t+1}; h^t) ds_{t+1} \\ \leq W_t(h^t)N_t(h^t) + T_t(h^t) + B_t(h^t) + \int_0^1 \Pi_t(i) di. \end{aligned} \tag{2}$$

Households choose the allocations $\{C_t(h^t), N_t(h^t), B_{t+1}(s_{t+1}; h^t)\}_{t\geq 0}$ taking as given the prices $\{P_t(h^t), W_t(h^t), Q_{t,t+1}(s_{t+1}|h^t)\}_{t\geq 0}$ for every h^t and given initial asset holdings B_0, to maximize (1) subject to (2).

[2] Alternative one could introduce imperfectly competitive labor markets or labor trades with search frictions.

2.1.1. Optimal intertemporal decision

Now that we are clear of the history dependence (or later in the Markovian shock case, state dependence) of the pricing and allocation functions, I will dispense with the h^t notation throughout.[3]

The optimality condition for the dynamic household decision problem is given by

$$Q_{t,t+1} = \beta \frac{P_t}{P_{t+1}} \frac{U_c(C_{t+1})}{U_c(C_t)} \rho(h^{t+1}|h^t). \tag{3}$$

which holds for every $t \geq 0$ and every h^t and h^{t+1} reachable from h^t, and $\rho(h^{t+1}|h^t)$ is the probability of h^{t+1} conditional on h^t. Note that $Q_{t,t}(h^t|h^t) = 1$.

We saw previously that if the household held some other redundant asset (e.g. a riskless bond) that pays an expected (nominal) return of $1 + i_t$ each period, the no-arbitrage condition implies that

$$1 + i_t = \frac{1}{\int_{s_{t+1}} Q_{t,t+1}(s_{t+1}|h^t) ds_{t+1}} := \frac{1}{\mathbb{E}_t Q_{t,t+1}}.$$

So we can take conditional expectations of (3) and then re-write it as

$$U_c(C_t) = \beta(1 + i_t)\mathbb{E}_t \left\{ U_c(C_{t+1}) \frac{P_t}{P_{t+1}} \right\}. \tag{4}$$

which holds for every $t \geq 0$ and every h^t. Finally at the optimum (2) holds with equality for every $t \geq 0$ and every h^t.

2.1.2. Optimal intratemporal decisions

First, the household choose labor supply optimally according to the familiar static first order condition:

$$-\frac{U_n(N_t)}{U_c(C_t)} = \frac{W_t}{P_t}. \tag{5}$$

Second, within each period the household optimally allocates its expenditure of the variety of consumptions for a given budget for C_t.

Exercise 1. *Show that given the choice of the consumption bundle each period C_t the optimal intratemporal allocation of expenditure between each variety of goods gives rise to the demand function for each variety i as*

$$C_t(i) = \left(\frac{P_t(i)}{P_t} \right)^{-\varepsilon} C_t, \tag{6}$$

and the resulting aggregate price index for the bundle C_t is a CES aggregator of individual varieties' prices:

$$P_t = \left[\int_0^1 (P_t(i))^{1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}}.$$

Hint: The consumer must solve an expenditure minimization problem.

[3] However, bear in mind that the first order conditions characterizing optimal decision processes by agents would be for each period $t \geq 0$ and each probable history h^t.

2.2 Firms' optimal production and pricing

Each firm on $[0, 1]$ produces a differentiated good indexed by $i \in [0, 1]$. Since they are monopolistically competitive in the differentiated product markets, they can set an optimal price which is some markup over their marginal cost of hiring labor. In the absence of price-change frictions, these firms will just set the usual static monopolistic markup of price over the marginal cost. However, we will introduce a stochastic process that determines when firms get to reset their prices, and when they do, they do so in a dynamic optimal fashion.

2.2.1. Optimal production

Firm i hires labor $N_t(i)$ taking the competitive wage W_t as given and produces output using a linear technology:

$$Y_t(i) = A_t N_t(i),$$

where A is the stochastic technology level common to all firms. Since firms are competitive in the factor market, its marginal cost will be the common marginal cost MC_t so that each firm solves a cost minimization problem:

$$\min_{N_t(i)} W_t N_t(i) + MC_t[Y_t(i) - A_t N_t(i)]$$

with efficient production given by

$$W_t = MC_t A_t. \tag{7}$$

Notice that we have conveniently dropped capital as a production input in this model. This model can be readily generalized to include capital accumulation as in the real business cycle models.

2.2.2. Optimal pricing

The price-setting firm i would like to set prices optimally as some markup over its current marginal cost, depending on the elasticity of demand and marginal cost each period. However, following Calvo (1983), we will assume that there is a random arrival process for a signal for firms to change prices each period. This is a simple an exogenous way to introduce the fact that not all firms adjust their prices frequently. The resulting optimal pricing strategy will not depend on time or the history of price changes by all firms (i.e. other firms' pricing strategies).

Suppose each period, there is a constant probability $(1 - \theta)$ that firms get to adjust their prices. By the law of large numbers, the fraction of firms not able to change their prices in one period is also the probability they do not change prices in one period, θ. Let P_t^* be the new aggregate price set in period t. By the law of large numbers, the evolution of the aggregate price index is then,

$$P_t = \left[\theta P_{t-1}^{1-\varepsilon} + (1-\theta)(P_t^*)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}, \tag{8}$$

which says that the current aggregate CPI index depends on its previous value (by a fraction of firms stuck with the old price) and the new price set by current price-resetting firms.

Now let's go back to firm i who gets to reset its price at time t. This firm knows that the probability it will be stuck with this new price for the next $k \geq 0$ periods ahead will be θ^k. Firm i chooses $P_t^*(i)$ to maximize its expected present discounted stream of profits

$$\mathbb{E}_t \sum_{k=0}^{\infty} \theta^k Q_{t,t+k} C_{t+k}(i)[P_t(i) - MC_{t+k}] \tag{9}$$

such that

$$C_{t+k}(i) = \left(\frac{P_t(i)}{P_{t+k}}\right)^{-\varepsilon} C_{t+k}.$$

Notice this is the usual Keynesian model where output of firms are determined on the "short side" or in other words, demand determined.

EXERCISE 2. *Show that the optimal pricing strategy of firm i is characterized by*

$$P_t^*(i) = \frac{\varepsilon}{\varepsilon - 1} \frac{\mathbb{E}_t \sum_{k=0}^{\infty} \theta^k Q_{t,t+k} MC_{t+k} C_{t+k}(i)}{\mathbb{E}_t \sum_{k=0}^{\infty} \theta^k Q_{t,t+k} C_{t+k}(i)}. \tag{10}$$

Notice that if $\theta = 0$, which says that the probability of no price change is zero (or prices are completely flexible), the first-order condition for optimal price setting collapses to

$$P_t^*(i) = \frac{\varepsilon}{\varepsilon - 1} MC_t.$$

This is the familiar (undergraduate microeconomics) static monopolist's markup of price over marginal cost, which depends on the own-price elasticity of demand.

2.3 Market clearing and competitive equilibrium

In a competitive equilibrium, goods markets have to clear for each variety i of goods so that

$$Y_t(i) = A_t N_t(i) = C_t(i).$$

Given competitive labor markets, each firm will hire the same units of labor $N_t(i) = N_t$. Integrating over all firms on $[0, 1]$ we have $Y_t := \int_0^1 Y_t(i)di$, so that aggregate market clearing is

$$Y_t = A_t N_t = C_t \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} di \tag{11}$$

where we can denote

$$d_t = \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} di$$

as the dispersion of relative prices across producers.

Labor market clearing requires that (5) equals (7) such that we get a measure of real marginal cost as

$$mc_t := \frac{MC_t}{P_t} = -\frac{U_n(N_t)}{U_c(C_t)} \frac{1}{A_t}. \tag{12}$$

We will assume a symmetric pricing equilibrium where all firms that get to reset prices will chose the same price so that the first-order condition becomes

$$P_t^* = \frac{\varepsilon}{\varepsilon - 1}\frac{\mathbb{E}_t \sum_{k=0}^{\infty}(\beta\theta)^k \Gamma_{t,t+k} MC_{t+k} P_{t+k}^{\varepsilon} C_{t+k}}{\mathbb{E}_t \sum_{k=0}^{\infty}(\beta\theta)^k \Gamma_{t,t+k} P_{t+k}^{\varepsilon} C_{t+k}}. \tag{13}$$

where the demand function (6) and (4) are used. Specifically,

$$\Gamma_{t,t+1} = \frac{P_t}{P_{t+1}}\frac{U_c(C_{t+1})}{U_c(C_t)}\rho(h^{t+1}|h^t).$$

2.4 Log-linear approximations

In the steady state $P_t = P_{t-1} = P_t^* = P_{ss}$ and $Q_{t,t+1} = \beta$ or $\Gamma_{t,t+1} = 1$. Goods market clearing becomes:

$$\hat{Y}_t = \hat{N}_t + \hat{A}_t = \hat{C}_t. \tag{14}$$

Assume that the per period utility function is given by

$$U(C, N) = \frac{C^{1-\sigma}}{1-\sigma} - \frac{N^{1+\gamma}}{1+\gamma} \tag{15}$$

where $\sigma, \gamma > 0$.

Exercise 3. *Show that the labor market clearing condition now can be approximated as*

$$\hat{mc}_t = (\gamma + \sigma)\hat{Y}_t - (1+\gamma)\hat{A}_t. \tag{16}$$

Next we log-linearize the firms' optimal pricing decisions.

Exercise 4. *Show that the log-linearized approximation of (13) is given by*

$$\hat{P}_t^* = (1-\beta\theta)\mathbb{E}_t \sum_{k=0}^{\infty}(\beta\theta)^k[\hat{mc}_{t+k} + \hat{P}_{t+k}] \tag{17}$$

$$= (1-\beta\theta)[\hat{mc}_t + \hat{P}_t] + \beta\theta\mathbb{E}_t\hat{P}_{t+1}^*. \tag{18}$$

Next show that the log-linearized version of (8) is

$$\hat{P}_t = \theta\hat{P}_{t-1} + (1-\theta)\hat{P}_t^*. \tag{19}$$

Subtracting $\hat{P}_{t-1}$ on both sides of (19) we get CPI inflation as

$$\pi_t := \hat{P}_t - \hat{P}_{t-1} = (1-\theta)(\hat{P}_t^* - \hat{P}_{t-1}) \tag{20}$$

Combining (19) and (20) we obtain the New-Keynesian Phillips curve:

$$\pi_t = \beta\mathbb{E}_t\pi_{t+1} + \frac{(1-\theta)(1-\beta\theta)}{\theta}\hat{mc}_t.$$

Given expectations of one-period ahead inflation, this equation gives the locus of inflation and real marginal cost pairs such that current inflation is increasing in real marginal cost. The intuition is that, holding inflation expectation constant, an increase in demand results in a rise in output, which increases labor demand, and that results in increased real wage and this increase real marginal cost. This results in a rise in current inflation. The dependency of current inflation on expected future inflation arises from the fact that price-setting firms are forward looking in their price setting strategy, so they need to forecast future prices.

Finally, notice that the more sticky prices are (higher θ), the smaller is elasticity term on real marginal cost, so that the less sensitive is inflation to variations in real marginal cost. That is, variations in current aggregate demand does not affect current inflation much.

The household's optimal consumption plan is approximately given by

$$\hat{C}_t = \mathbb{E}_t \hat{C}_{t+1} - \frac{1}{\sigma}\hat{r}_t$$

where $\hat{r}_t = \hat{i}_t - \mathbb{E}_t \pi_{t+1}$. We can use goods market clearing to re-write this as

$$\hat{Y}_t = \mathbb{E}_t \hat{Y}_{t+1} - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t \pi_{t+1}).$$

We can interpret this Euler equation in terms of output as a forward-looking Keynesian IS curve. Aggregate demand depends on expected future output conditions and the ex ante real interest rate.

2.4.1. Natural level of output, interest and output gaps

For policy purposes, we are interested in a model where output is is defined in terms of deviation from some potential output. In this model, potential output is also termed the "natural level of output". There is nothing natural about it, really. This natural output is defined as the level of output that would prevail if prices were completely flexible. This implies that $\hat{mc}_t = 0$ in all periods and states of the world. From (16) we have the natural output as

$$\hat{Y}_t^n = \left(\frac{1+\gamma}{\sigma+\gamma}\right)\hat{A}_t.$$

Notice that under flexible prices, this level of output is independent of the interest rate, hence monetary policy. It only depends only on the level of technology, which indeed makes this model a special case of the RBC model without capital.

Now define output gap as the deviation of output from the flexible price output:

$$x_t = \hat{Y}_t - \hat{Y}_t^n.$$

Using this in (16) again, we get

$$\hat{mc}_t = (\gamma + \sigma)x_t.$$

So aggregate supply, or the Phillips curve, can be written as

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t. \tag{21}$$

where

$$\kappa = \frac{(\gamma+\sigma)(1-\theta)(1-\beta\theta)}{\theta}.$$

So now we have a clear relationship linking output gap to inflation. The higher is output realtive to potential, the higher is current inflation, all else constant.

EXERCISE 5. *Show also that the IS curve can be re-written in output gap terms as*

$$x_t = \mathbb{E}_t x_{t+1} - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t \pi_{t+1} - r_t^n). \tag{22}$$

where that natural real interest rate is

$$r_t^n := \sigma \mathbb{E}[\hat{Y}_{t+1}^n - \hat{Y}_t^n] = \frac{\sigma(1+\gamma)}{\sigma+\gamma} \mathbb{E}_t[\hat{A}_{t+1} - \hat{A}_t].$$

DEFINITION 1. *Given the stochastic process $\{\hat{A}_t\}_{t\geq 0}$ and a monetary policy plan $\{\hat{i}_t\}_{t\geq 0}$, a rational expectations decentralized equilibrium (REE) in this approximate economy is the set of bounded stochastic processes $\{x_t, \pi_t\}_{t\geq 0}$ satisfying (21) and (22).*

3. Departure from fundamental welfare theorems

At this point, one might ask what role the policy maker has in terms of intervening in this particular economy. There are two sources of distortions to an otherwise neoclassical economy here.

3.1 Monopolistic distortion

First, there is the presence of market power by each firm producing a differentiated good. Thus, output is inefficiently lower that under perfectly competitive markets, even in the long run with flexible prices. This distortion is independent of monetary policy and will have to be corrected by lump-sum fiscal policy transfers.

Without loss of generality suppose at steady state $A_{ss} = 1$. Let $\mu = \varepsilon/(\varepsilon - 1)$. From our example previously, the deterministic steady state of labor market clearing (12) given our separable utility function (15) gives the steady state real marginal cost as

$$mc_{ss} := \frac{MC_{ss}}{P_{ss}} = \frac{1}{\mu_{ss}} = \frac{W_{ss}}{P_{ss}}.$$

From the household optimal labor supply, we have

$$\frac{N^\gamma}{C^{-\sigma}} = \frac{W_{ss}}{P_{ss}}.$$

Further, $Y_{ss} = C_{ss} = N_{ss}$, so that

$$Y_{ss}^{\gamma+\sigma} = \frac{1}{\mu_{ss}} \Rightarrow Y_{ss} = \left(\frac{1}{\mu_{ss}}\right)^{\frac{1}{\gamma+\sigma}}.$$

The efficient level of output by firms, Y^*, would be the case where all firms are setting price equal to marginal cost, so that $\mu_{ss} = 1$. Since for monopolistically competitive firms $\mu_{ss} > 1$, it means that $Y_{ss} < Y^* = 1$.

3.2 Sticky price distortion

Second, because the Calvo pricing model introduces price stickiness via staggered price setting, there exists relative price dispersion d_t between firms that adjust prices and firms that do not. This distortion prevails in the long run if inflation is not zero. That is, in the long run, if inflation is non-zero, the relative price of firms stuck with an old price relative to the aggregate (including the prices of firms that adjust) declines, even though firms face the same marginal cost of production. So non-adjusting firms are relatively worse off in the long run with non-zero inflation. The Calvo staggered pricing assumption also implies that in the short run, fluctuations in the price markup around its constant desired level of results in welfare losses.

Recall, in relating the demand of variety i to the aggregate demand and supply, we had the goods market clearing condition (11). Consider the term

$$d_t = \int_0^1 \left(\frac{P_t(i)}{P_t} \right)^{-\varepsilon} di$$

defined as the dispersion of relative prices across producers. In Appendix A we show that this can be written as

$$d_t = (1-\theta) \left(\frac{1-\theta\pi_t^{\varepsilon-1}}{1-\theta} \right)^{\frac{-\varepsilon}{1-\varepsilon}} + \theta\pi_t^{\varepsilon} d_{t-1}.$$

In a deterministic steady state, we have

$$d_{ss} = \frac{1-\theta}{1-\theta\pi_{ss}^{\varepsilon}} \left(\frac{1-\theta\pi_{ss}^{\varepsilon-1}}{1-\theta} \right)^{\frac{-\varepsilon}{1-\varepsilon}} \begin{cases} = 1, & \text{if } \pi_{ss} = 1 \text{ or } \theta = 0 \\ > 1, & \text{otherwise} \end{cases}.$$

In other words, the relative price distortion across producers who face identical marginal costs is minimized when either there is zero inflation at steady state, or when prices are perfectly flexible.

4. Specifying the interest rate process

To close the model we need to specify how the process $\{\hat{A}_t\}_{t\geq 0}$ is generated and also how the nominal interest rate $\{\hat{i}_t\}_{t\geq 0}$ is determined. For now we take as given $\{\hat{A}_t\}_{t\geq 0}$. We consider three kinds of monetary policy design that are common in this literature. First, there is the assumption that the monetary authority implements policy according to some simple rule. Second, one can assume the monetary authority commits to some optimal time-0 contingent plan that solves a dynamic programming problem of maximizing some utilitarian social welfare, or approximately, minimizing a loss criterion. Third, one can show in this model that the optimal time-0 contingent plan is not time consistent. Given expectations of the private sector, the monetary authority can re-optimize and construct another time-t optimal plan which ends up making the time-0 plan no longer optimal ex post. A time consistent or ex post optimal policy then requires an equilibrium concept that specifies that the monetary authority has no further incentive to deviate from that decision rule and that private agents will play the strategy where they anticipate correctly that they will have no incentive to deviate. We will characterize the notion of equilibrium under such ex post optimal policy as a Markov perfect equilibrium. But first, we take a little detour in the next section to make an important observation.

4.1 Creating a nontrivial monetary problem

If we reconsider the approximate equilibrium conditions again for the model:

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t, \tag{21}$$

and

$$x_t = \mathbb{E}_t x_{t+1} - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t \pi_{t+1} - r_t^n), \tag{22}$$

where

$$r_t^n := \sigma \mathbb{E}[\hat{Y}_{t+1}^n - \hat{Y}_t^n] = \frac{\sigma(1+\gamma)}{\sigma+\gamma}\mathbb{E}_t[\hat{A}_{t+1} - \hat{A}_t],$$

the short run monetary policy solution is quite trivial. To overcome the effect of the sole productivity shock on r_t^n and hence output gap x_t and inflation π_t, all the monetary authority needs to do is raise $\hat{i}_t$ by the same proportion as the increase in r_t^n to completely neutralize the shock. There is no inflation-output-gap trade-off when the model has only productivity shocks acting on the natural rate of interest. To make the problem interesting, researchers have added another shock to the Phillips curve equation, u_t, often interpreted as a cost push shock. This can be derived by assuming perturbations to the nominal marginal cost term in the firms' profit function in the optimal pricing decision stage. For our purposes, we will just tack on u_t to the end of the Phillips curve:

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t + u_t, \tag{21'}$$

so that the monetary policy design problem become non-trivial.

4.2 Simple policy rule

We will turn off the shock u_t for the moment since we do not need it here. As a first step to understanding the role of monetary policy in the model, consider the monetary authority following a fixed decision rule given by the reaction function

$$\hat{i}_t = \phi_\pi \pi_t; \qquad \phi_\pi > 0. \tag{23}$$

Since the private agents know that the central bank follows this simple strategy for policy, the equilibrium is then characterized approximately by (21), (22) and (23). Substituting out $\hat{i}_t$ the approximate REE is characterized by

$$\begin{pmatrix} \pi_t \\ x_t \end{pmatrix} = \frac{1}{\sigma + \kappa\phi_\pi}\begin{pmatrix} \sigma\beta + \kappa & \sigma\kappa \\ 1 - \beta\phi_\pi & \sigma \end{pmatrix}\begin{pmatrix} \mathbb{E}_t \pi_{t+1} \\ \mathbb{E}_t x_{t+1} \end{pmatrix} + \frac{1}{\sigma + \kappa\phi_\pi}\begin{pmatrix} \kappa \\ 1 \end{pmatrix} r_t^n.$$

Following Blanchard and Kahn (1980), it can be shown that a unique (stable) REE solution $\{x_t, \pi_t\}$ exists if and only if the eigenvalues of

$$\mathbf{F} = \frac{1}{\sigma + \kappa\phi_\pi}\begin{pmatrix} \sigma\beta + \kappa & \sigma\kappa \\ 1 - \beta\phi_\pi & \sigma \end{pmatrix}$$

lie inside the unit circle.[4] Let f_{ii} be the (i,i)-th element of $\mathbf{F}$. The characteristic polynomial of $\mathbf{F}$ is

$$\begin{aligned} P(\lambda) &= \det(\mathbf{F} - \lambda \mathbf{I}_2) \\ &= \lambda^2 - (f_{11} + f_{22})\lambda + (f_{11}f_{22} - f_{12}f_{21}) \end{aligned}$$

[4]Intuitively, for backward looking difference equations we solve backwards and stability then requires a convergent sequence which then requires the matrix on the lag vector to have stable eigenvalues. For forward looking expectational difference equations, we solve forward, and this requires the forward sequences to be convergent so that the transversality condition is satisfied, and the solutions are stable (and therefore bounded) stochastic processes.

$$= \lambda^2 - trace(\mathbf{F})\lambda + \det(\mathbf{F}).$$

The (at most two distinct) eigenvalues solve $P(\lambda) = 0$. We can apply the following theorem in determining the conditions for the eigenvalues to be inside the unit circle.

THEOREM 1. *Let* $\mathbf{A}$ *be a* $n \times n$ *matrix with eigenvalues* $\lambda_1, ..., \lambda_n$. *Then*

1. $\lambda_1 + \lambda_2 + ... + \lambda_n = trace(\mathbf{A})$, *and*
2. $\lambda_1 \cdot \lambda_2 \cdot ... \cdot \lambda_n = \det(\mathbf{A})$.

Proof: See Simon and Blume (1994), Theorem 23.9. □

For the eigenvalues λ_1, λ_2 to be inside the unit circle, we need the following conditions.

PROPOSITION 1. $\mathbf{F}$ *is a stable matrix, or* $|\lambda_i| < 1$, *if and only if*

1. $|\det(\mathbf{F})| < 1$, *and*
2. $|-trace(\mathbf{F})| - \det(\mathbf{F}) < 1$.

Now we can apply this result to our model to check for stability of an REE solution. Since

$$|\det(\mathbf{F})| = \left| \frac{\sigma\beta}{\sigma + \kappa\phi_\pi} \right|$$

the first condition $|\det(\mathbf{F})| < 1$ requires $\beta - 1 < \kappa\phi_\pi/\sigma$. This holds for any $\phi_\pi \geq 0$. The second condition $|\text{trace}(\mathbf{F})| - |\det(\mathbf{F})| < 1$ requires that

$$\frac{\sigma(\beta+1)+\kappa}{\sigma + \kappa\phi_\pi} < \frac{\sigma(\beta+1)+\kappa\phi_\pi}{\sigma + \kappa\phi_\pi}$$

which holds if and only if $\phi_\pi \geq 1$.

PROPOSITION 2. *Given the stochastic process* $\{\hat{A}_t\}_{t\geq 0}$, *there exists a unique (stable) a rational expectations decentralized equilibrium (REE)* $\{x_t, \pi_t\}_{t\geq 0}$ *satisfying (21) and (22) under a monetary policy plan* $\{\hat{i}_t\}_{t\geq 0}$ *defined by*

$$\hat{i}_t = \phi_\pi \pi_t$$

if and only if $\phi_\pi \geq 1$.

In other words the approximate REE is stable only when the central bank, equipped with the simple policy is able to respond by more than one-for-one to a change in the inflation rate. For example, if inflation rises by one percent, the central bank has to raise the nominal interest rate by $\phi_\pi > 1$ percent, to achieve a stable or unique REE. If not, the equilibrium either has multiple or explosive REE.

EXERCISE 6. *Characterize the condition for a unique REE in the linearized model when the monetary authority follows the simple Taylor-type feedback rule:*

$$\hat{i}_t = \phi_\pi \pi_t + \phi_x x_t.$$

Explain the intuition behind this condition.

4.3 Optimal policy

Since we are dealing with the REE of the log-linear approximate model, we will be looking for approximately optimal monetary policy decision rules that are linear. Thus we will approximate the monetary authority's objective up to a second-order or quadratic accuracy. In this simple closed-economy model, it turns out the objective of a benevolent monetary authority – i.e. to maximize the ex ante average household's expected lifetime utility – can be approximated by the objective of minimizing an expected discounted total quadratic loss function which depends on stochastic paths of output gap and inflation. Thus we have a welfare-theoretic foundation of a central-bank objective of stabilizing output gap and inflation.

Suppose the monetary authority seeks to maximize the expected total discounted utility of average households. For each given history h^∞ let the welfare criterion be

$$W(s_0)\Big|_{h^\infty \ni s_0} = \sum_{t=0}^{\infty} \beta^t U(C_t(h^t), N_t(h^t)) = \sum_{t=0}^{\infty} \beta^t \left\{ \frac{[C_t(h^t)]^{1-\sigma}}{1-\sigma} - \frac{[N_t(h^t)]^{1+\gamma}}{1+\gamma} \right\}.$$

In the background, we will assume that fiscal policy uses lump-sum transfers to offset the monopolistic distortion on the steady state output. That is we want the short run dynamics to be defined as fluctuations relative to a flexible price equilibrium with a first-best allocation, which nests the deterministic steady state with the efficient level of output $Y^* = 1$. To do so, we need the following precise assumption.

ASSUMPTION 1. *There exists a constant subsidy to employment $T(h^t) = T > 0$ such that*

$$mc_{ss}^* = mc_{ss} + T = \frac{W_{ss}}{P_{ss}} + T = 1 \Rightarrow Y_{ss} = Y^* = 1.$$

Without going through a detailed proof, we make the following claim.[5]

PROPOSITION 3. *To a second-order approximation around the efficient steady state with zero inflation,*

$$W(s_0)\Big|_{h^\infty \ni s_0} \approx -\frac{U_c(Y_{ss})Y_{ss}}{2} \sum_{t=0}^{\infty} \beta^t \left\{ (\sigma+\gamma)[x_t(h^t)]^2 + \frac{\theta\varepsilon}{(1-\theta\beta)(1-\theta)}[\pi_t(h^t)]^2 \right\}.$$

From this point, we will re-write the utilitarian welfare criterion as the approximate quadratic loss function in terms of output gap and inflation:[6]

$$W(s_0) \approx -\frac{1}{2}\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \omega[x_t(h^t)]^2 + [\pi_t(h^t)]^2 \right\}. \tag{24}$$

[5]See Walsh (2003), Woodford (2003) or Galí and Monacelli (2005) for the approximation.

[6]It should be noted that without Assumption 1, the approximated criterion function will become

$$W(s_0) \approx -\frac{1}{2}\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \omega[x_t(h^t) - \overline{x}]^2 + [\pi_t(h^t)]^2 \right\}.$$

where $\overline{x} = Y^* - Y_{ss}|_{T=0}$ is the difference between the efficient steady-state level of output and the steady state output when there is monopolistic distortion. Thus in the absence of a first-best fiscal policy, the monetary authority attempts to compensate by targeting an output gap that is non-zero, which would imply an *average inflation bias* in the steady state.

where

$$\omega = \frac{U_c(Y_{ss})Y_{ss}(\sigma+\gamma)(1-\theta\beta)(1-\theta)}{\theta\varepsilon}.$$

4.3.1. Commitment to ex ante optimal policy

We will characterize the benchmark optimal policy plan under the assumption of a central bank that commits to a time-0 strategy or contingent plan for monetary policy which minimizes the expected total discounted losses (24) subject to the decentralized equilibrium conditions defined by (22) and (21'). Note that these constraints are the forward looking IS and Phillips equations which characterize the optimal responses of the private sector (household and firms in a competitive equilibrium) to given policy plans.

The Lagrangian for the monetary authority under commitment to this plan is

$$L = \mathbb{E}_0\left\{ \sum_{t=0}^{\infty} \beta^t \left[\frac{1}{2}\ (\pi_t^2 + \omega x_t^2) + \phi_t(\pi_t - \beta\mathbb{E}_t\pi_{t+1} - \kappa x_t - u_t) \right.\right.$$
$$\left.\left. \psi_t\left(x_t - \mathbb{E}_t x_{t+1} + \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t\pi_{t+1} - r_t^n)\right)\right]\right\}$$

where there is one pair of $(\phi_t(h^t), \psi_t(h^t))$ for every $t \geq 0$ and h^t.

The optimal time-0 plan $\{\pi_t(h^t), x_t(h^t), \hat{i}_t(h^t)\}$ is characterized by the set of first-order conditions:

$$\begin{aligned} \pi_t + \phi_t - \phi_{t-1} =& 0, \\ \omega x_t - \kappa\phi_t =& 0 \\ \mathbb{E}_0\psi_t =& 0. \end{aligned}$$

for all $t \geq 0$ and all histories h^t. Notice that the last first-order condition implies that the IS curve is not binding at all for any history or time period – it does not really factor in the central banks constrained optimization problem. Simplifying the first-order conditions, the "optimal commitment policy" is to set inflation such that

$$\pi_t^C = \begin{cases} = -\frac{\omega}{\kappa}(x_t - x_{t-1}), & \text{for } t > 0 \\ \\ = -\frac{\omega}{\kappa}x_0, & \text{for } t = 0 \end{cases}, \tag{25}$$

for every h^t.

The policy under the assumption of perfect commitment to the time-0 optimal plan has the following feature. At time 0, the monetary authority implements $\pi_0 = -\frac{\omega}{\kappa}x_0$ and promises to set $\pi_t(h^t) = -\frac{\omega}{\kappa}(x_t(h^t|h^{t-1}) - x_{t-1}(h^{t-1}))$ for all $t > 0$. When $t \geq 1$, the central bank *by assumption* implements the promised plan for every possible history, which is to lower inflation by ω/κ percent in any state of the world if output growth increases by one percent.

One implication of the assumption that the central bank commits to implementing its time-0 optimal plan is that this ex-ante optimal policy is history dependent. For all $t \geq 1$, if we substitute (25) into (21') we have a second-order difference equation in output gap:

$$x_t(1+\beta+\kappa^2/\omega) = \beta\mathbb{E}_t x_{t+1} + x_{t-1} - \frac{\kappa}{\omega}u_t.$$

Now assume that $u_t \sim$ i.i.d.$(0, \sigma_u^2)$. Let $x_{-1} = 0$. The stable solution to this is

$$x_t = a x_{t-1} - \frac{a\kappa}{\omega}u_t$$

where

$$a = \frac{1+\beta+\frac{\kappa^2}{\omega}}{2\beta}\left[1 - \sqrt{1 - \frac{4\beta}{\left(1+\beta+\frac{\kappa^2}{\omega}\right)^2}}\right] < 1.$$

We can then show by backward subsitution that:

$$x_t = -\frac{a\kappa}{\omega}\sum_{i=0}^{\infty} a^i u_{t-i}.$$

So the policy under commitment to the optimal time-0 plan is to contract output gap in response to not just a contemporaneous cost-push shock, but also to all prior shocks from the infinite past. An interpretation of this history dependent policy is that the central bank ties its hands to past promises (which is reflected in the dependence on the history of shocks) and does not re-optimize at any future stage. The central bank is just like Ulysses in Homer's classic tale, *Odyssey*, who instructs the crew of his ship to tie him to the mast of the ship, lest they be tempted by the Sirens and the ship would run aground. Implicit in this ex-ante optimal policy is the assumption that the central bank possesses a commitment device which makes it adhere to the the time-0 optimal plan. This is evident if we back out the optimal policy with commitment in terms of the interest rate instrument. Suppose we have an AR(1) process for the markup shock:

$$u_t = \zeta u_{t-1} + z_t,$$

where $\zeta \in (0,1)$ and $z_t \sim$ i.i.d.$(0, \sigma_z^2)$. The optimal commitment policy, for all $t > 1$, in terms of an intrument rule representation can be derived from combining the optimal policy conditions earlier and our IS curve. This rule turns out to be:

$$\hat{i}_t = r_t^n + \left[a\sigma + \frac{(1-a)\omega}{\kappa}\right]x_t - a\sigma x_{t-1} + \left[a\zeta + \frac{a(1-\zeta)\sigma\kappa}{\omega}\right]u_t.$$

The optimal interest rate decision rule neutralizes variations in the natural interest rate completely, and responds to cost push shocks, and contemporaneous and *lagged* output gap. This history dependence in the rule reflects the commitment to the ex ante plan, to past promises. We will see later that in the absence of commitment, the policy rule will be much more aggressive and we lose the policy gradualism outcome.

4.3.2. Ex post optimal policy

Now consider removing the assumption that the central bank can and does commit to the optimal time-0 plan. Previously, by assumption the central bank can commit. Now if there is the option for the central bank to re-evaluate its optimal plan in every period, then at any time $\tau \geq 0$ the central bank's time-τ

optimal contingent plan is characterized by the time-τ version of (25):

$$\pi_t^D = \begin{cases} = -\frac{\omega}{\kappa}(x_t - x_{t-1}), & \text{for } t > \tau \\ \\ = -\frac{\omega}{\kappa}x_t, & \text{for } t = \tau \end{cases}. \tag{26}$$

So in every period $t > 0$,

$$\pi_t^D = -\frac{\omega}{\kappa}x_t, \tag{27}$$

but this policy rule is different to that formulated before under the assumption of commitment, which was

$$\pi_t^C = -\frac{\omega}{\kappa}(x_t - x_{t-1}).$$

So now we have concluded that at the beginning of every period $t > 0$, if the central bank re-optimizes it has the incentive to deviate from the time-0 optimal contingent plan and follow its time-t optimal action (27), given the optimal plans (thus expectations) of private households and firms. But if private agents are forward looking and rational, they would anticipate such an incentive of the central bank to deviate from its time$-t$ optimal plan, and encode this into their individual optimal plans. This leads us to require a notion of equilibrium where all decision makers (players) are optimizing in response to each other, following any state of the economy.

First, we will make the following simplifying structure on the model's exogenous processes so that the economy can be described recursively.

Assumption 2. *The history of shocks $h^t = (s_0, ..., s_t) = (u_0, A_0, ..., u_t, A_t)$ is Markov, so that $\rho(h^t) = \rho((u_t, A_t)|h^{t-1})\rho(h^{t-1})$.*

Second, we can model this as a dynamic game between the central bank that sequentially optimizes by choosing $\{\pi_t, x_t, \hat{i}_t\}$ and the private sector (households and firms) that react optimally (in the competitive equilibrium) in their consumption, production and pricing plans to a given policy plan, by choosing

$$\{\mathbb{E}_t\pi_{t+1}(\hat{i}_{t+1}; s_{t+1}), \mathbb{E}_t x_{t+1}(\hat{i}_{t+1}; s_{t+1})\}.$$

We focus our attention to a simpler (but more restrictive) solution concept called a Markov perfect equilibrium.[7]

Definition 2. *A Markov perfect equilibrium in this game is such that beginning from any current state $s_t = (u_t, A_t)$ of the system,*

1. *the policy $\Phi = \{\pi_t, x_t, \hat{i}_t\}$ characterized by (27) is optimal for the central bank, and*
2. *private sector expectations conditioned on the central bank strategy*

$$\{\mathbb{E}_t\pi_{t+1}(\Phi; s_{t+1}), \mathbb{E}_t x_{t+1}(\Phi; s_{t+1})\}$$

satisfy the REE defined by (22) and (21'),

for all $t \geq 0$.

[7]Ljungqvist and Sargent (2004) in Chapter 22 use the more general solution concept of subgame perfection. See also Mailath and Samuelson (2006) for more detailed and precise expositions on repeated and dynamics games.

Notice that we ignore strategic behavior *between* private agents – i.e. between households and firms, or between firms themselves, by assuming a competitive equilibrium definition for characterizing their optimal actions. This assumption is tantamount to saying that private agents are too small to influence the aggregate dynamics by their own actions. On the contrary, the government is a large player.

4.3.3. An anatomy of the Markov equilibrium

First, we consider the decision problem of the central bank that cannot commit to the *time* $-$ 0 optimal plan, again. This is Part 1 of Definition 2. The central bank at any period $t \geq 0$ seeks to solve

$$\begin{aligned} V(s_t) &= \min_{x_t, \pi_t, \hat{i}_t} \frac{1}{2}(\omega x_t^2 + \pi_t^2) + \mathbb{E}_t \sum_{\tau=1}^{\infty} \beta^\tau (\omega x_{t+\tau}^2 + \pi_{t+\tau}^2) \\ &= \min_{x_t, \pi_t, \hat{i}_t} \frac{1}{2}(\omega x_t^2 + \pi_t^2) + \beta \mathbb{E}_t V(s_{t+1}) \end{aligned}$$

subject to

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t + u_t, \tag{21'}$$

$$x_t = \mathbb{E}_t x_{t+1} - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t \pi_{t+1} - r_t^n), \tag{22}$$

where

$$r_t^n := \sigma \mathbb{E}[\hat{Y}_{t+1}^n - \hat{Y}_t^n] = \frac{\sigma(1+\gamma)}{\sigma+\gamma} \mathbb{E}_t[\hat{A}_{t+1} - \hat{A}_t].$$

So we see that indeed the first-order condition for the RHS problem is (27). A re-optimizing central bank here ignores its continuation value since its future self or the subsequent central bank will again undo whatever plan it has put in place from the current period. Now, such an incentive to undo past promises or plan, is what we called *time inconsistency* in the time-0 optimal plan.

Second, such time-inconsistent plans by successive central banks, must be anticipated by private agents. This is Part 2 of Defintion 2. Thus, private sector expectations are formed "correctly" such that

$$\mathbb{E}_t \pi_{t+1} = -\frac{\omega}{\kappa} \mathbb{E}_t x_{t+1}. \tag{28}$$

and (22) and (21') are satisfied. These three equations literally say that the private agents are internalizing all future ex-post optimal policy of the central bank given by (27) in their expectations formation, and they incorporate these expectations into their optimal consumption, production, and pricing competitive equilibrium strategies as summarized by the IS and Phillips curves.

Third, given the characterization of the central bank and private sector best-responses in the Markov equilibrium, we can solve for the equilibrium contingent stochastic paths for inflation, output and interest rate. Substitute (27) into (21') to get:

$$\left(1 + \frac{\kappa^2}{\omega}\right) \pi_t = \beta \mathbb{E}_t \pi_{t+1} + u_t.$$

Note that the coefficient on the LHS is strictly positive. We can iterate this relation forward (and using the law of iterated expectations) as the next exercise shows, to derive the equilibrium inflation process.

Exercise 7. *Show that in the Markov perfect equilibrium inflation is*

$$\pi_t = \lim_{\tau \uparrow +\infty} \left\{ \left(\frac{\beta}{1+\frac{\kappa^2}{\omega}} \right)^{\tau} \mathbb{E}_t \pi_{t+\tau} \right\} + \frac{1}{\beta} \mathbb{E}_t \sum_{\tau=1}^{\infty} \left(\frac{\beta}{1+\frac{\kappa^2}{\omega}} \right)^{\tau} u_{t+\tau-1}$$
$$= \frac{1}{\beta} \mathbb{E}_t \sum_{\tau=1}^{\infty} \left(\frac{\beta}{1+\frac{\kappa^2}{\omega}} \right)^{\tau} u_{t+\tau-1}. \tag{29}$$

Example 1. *Suppose $u_t \sim i.i.d.(0, \sigma_u^2)$. Then the Markov equilibrium inflation process is given by*

$$\pi_t = \left(\frac{\omega}{\omega + \kappa^2} \right) u_t.$$

Using (27) the output gap in the Markov equilibrium is then given by

$$x_t = - \left(\frac{\kappa}{\omega + \kappa^2} \right) u_t.$$

In this example with uncorrelated cost-push shocks, the Markov equilibrium under ex post optimal monetary policy delivers inflation that rises less than one-for-one to a cost-push shock. Also monetary policy partially neutralizes the cost push shock by allowing output gap to fall by less than one-for-one to a cost push shock. Recall we claimed that the interest rate policy can perfectly offset the effect a technology shock on the natural interest rate. Indeed, if we substitute these results into (22) we can back out the implicit ex post optimal interest rate policy rule as

$$\hat{i}_t = r_t^n + \sigma \left(\frac{\kappa}{\omega + \kappa^2} \right) u_t.$$

Thus, a rise in r_t^n is matched by a one-for-one rise in $\hat{i}_t$ which has no effect then on x_t or π_t. However, a rise in u_t is not completely neutralized by $\hat{i}_t$. Notice that the larger is ω – i.e. the more concern the central bank has for output gap relative to inflation – the less aggressive is the response of the cash rate to cost-push shocks. This reflects the output-gap-inflation trade-off that is faced by the central bank with respect to u_t.

Exercise 8. *Derive the corresponding Markov perfect equilibrium processes for $\{\pi_t, x_t, \hat{i}_t\}$ when the cost push shock is a Markov process:*

$$u_t = \zeta u_{t-1} + z_t,$$

where $\zeta \in (0,1)$ and $z_t \sim i.i.d.(0, \sigma_z^2)$.

5. Numerical solution and policy simulation

We will consider some computational exercise of the results we have seen thus far to illustrate the effect on the simple model's business cycle properties. We will do this in the Thursday computer lab series.

Questions to ask:

1. What are the gains from commitment, or losses from discretionary policy for the central bank?
2. Comparison of dynamics (response) of output and inflation, and interest rate.
3. Illustration of commitment vs. discretion in terms on business cycle moments.

Appendix A

Effect of inflation on relative price dispersion across industries

Recall, in relating the demand of variety i to the aggregate demand and supply, we had the goods market clearing condtion (11). Consider the term

$$d_t = \int_0^1 \left(\frac{P_t(i)}{P_t} \right)^{-\varepsilon} di$$

defined as the dispersion of relative prices across producers. Since θ is the probability that the price remains the same in one period, and in a symmetric pricing equilibrium, we can re-write the dispersion as a recursive expression:

$$\begin{aligned}
d_t &= (1-\theta) \left(\frac{P_t^*}{P_t} \right)^{-\varepsilon} + \theta(1-\theta) \left(\frac{P_{t-1}^*}{P_t} \right)^{-\varepsilon} \theta^2 (1-\theta) \left(\frac{P_{t-2}^*}{P_t} \right)^{-\varepsilon} + ... \\
&= (1-\theta) \sum_{k=0}^{\infty} \theta^k \left(\frac{P_{t-k}^*}{P_t} \right)^{-\varepsilon} \\
&= (1-\theta) \left(\frac{P_t^*}{P_t} \right)^{-\varepsilon} + \theta \left(\frac{P_{t-1}}{P_t} \right)^{-\varepsilon} \left[(1-\theta) \sum_{k=0}^{\infty} \theta^k \left(\frac{P_{t-k-1}^*}{P_{t-1}} \right)^{-\varepsilon} \right] \\
&= (1-\theta) \left(\frac{P_t^*}{P_t} \right)^{-\varepsilon} + \theta \pi_t^{\varepsilon} d_{t-1}.
\end{aligned}$$

This describes the evolution of the dispersion of relative producer prices as a function of past dispersion, the new price level relative to the aggregate price, and the aggregate inflation rate.

From (8) we have

$$P_t = \left[\theta P_{t-1}^{1-\varepsilon} + (1-\theta)(P_t^*)^{1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}}$$

and dividing through by $P_t^{1-\varepsilon}$ we have

$$1 = \theta \pi_t^{\varepsilon-1} + (1-\theta) \left(\frac{P_t^*}{P_t} \right)^{1-\varepsilon}$$

and re-arranging we have

$$\left(\frac{P_t^*}{P_t} \right) = \left(\frac{1-\theta\pi_t^{\varepsilon-1}}{1-\theta} \right)^{\frac{1}{1-\varepsilon}}.$$

Combining this with the evolution of the dispersion term we get

$$d_t = (1-\theta) \left(\frac{1-\theta\pi_t^{\varepsilon-1}}{1-\theta} \right)^{\frac{-\varepsilon}{1-\varepsilon}} + \theta \pi_t^{\varepsilon} d_{t-1}.$$

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