Recursive Competitive Equilibrium: A Real Business Cycle application

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Outline

Motivation

- General equilibrium (with production) – time and uncertainty – application in *real business cycle* (RBC) model.
- Later applications: monetary business cycles and monetary policy.
- We are now putting together old tricks we have learned:
    1. Markovian stochastic processes
    2. Recursive methods for dynamic optimization
    3. Theory of complete markets

Plan of attack

1. Pareto problem for the planning economy under a benevolent social planner.
2. Competitive equilibrium allocation via:
   - Market for state/history contingent securities (either Arrow-Debreu or one-period Arrow securities),
   - Households supply labor service to Firm I and initial capital endowment to Firm II;
   - Firm II produces new capital stock and rents it to Firm I; and
   - Firm I produces output that can be sold as consumption (to household) and/or investment good (to Firm II).

   Implication: FWT # 1.
3. Markov process for shocks. This allows for characterization of *recursive competitive equilibrium*.

We'll start with 3., directly. Read 1-2 on your own (LS, Ch.12)

A stochastic growth model

We define the following objects, along with some assumptions, for the model:

- Production technology is subject to stochastic shocks.
- Stochastic shock, $s_t \in S$. Assume S finite.
- History of events leading up to $t \geq 0$ is $h^t = (s_t, s_{t-1}, ..., s_0)$.
- Unconditional probability of history h^t is $\pi_t(h^t)$.
- Conditional probability $\pi_\tau(h^\tau | h^t)$.
- Assume $\pi_0(s_0) = 1$ for initial state, $s_0 \in S$.
- Goods are differentiated by history/state, so the commodity space is represented by S.
- Identical households and firms (so we can drop the i's from individual decision rules).

Preferences

The representative (average) household orders consumption and leisure streams according to the following criterion:

$$\mathbb{E}\left\{\sum_{t=0}^{\infty}\beta^{t}u\left(c_{t}\left(h^{t}\right),l_{t}\left(h^{t}\right)\right)\Big|s_{0}\right\}=\sum_{t=0}^{\infty}\sum_{h^{t}}\beta^{t}u\left(c_{t}\left(h^{t}\right),l_{t}\left(h^{t}\right)\right)\pi_{t}\left(h^{t}\right)$$

with $\beta\in(0,1)$ and $u_{i}\left(c,l\right)>0$ and $u_{ii}\left(c,l\right)<0$, and assuming Inada conditions $\lim_{i\to 0}u_{i}\left(c,l\right)=+\infty$ for $i=c,l$.

New! Time constraint:

$$1 = l_t\left(h^t\right) + n_t\left(h^t\right) \tag{1}$$

where $l_t\left(h^t\right)$ is leisure and $n_t\left(h^t\right)$ is labor. Now we model endogenous labor supply (c.f. Brock and Mirman model.)

Households are assumed to be endowed with initial capital stock k_0.

Production technology

- $A_t(h^t)$ is the stochastic level of productivity.
- $F : \mathbb{R}_+ \times [0,1] \to \mathbb{R}_+$ is aggregate production function, s.t.:
  1. Constant returns to scale (homogeneous of degree 1):

     $$F(k,n) = nF\left(\hat{k},1\right) := nf\left(\hat{k}\right)$$

     where $\hat{k} = k/n$.
  2. $F \in C^2(\mathbb{R}_+ \times [0,1])$ s.t.:

     $$F_i(k,n) > 0, \qquad F_{ii}(k,n) < 0.$$

  3. Inada conditions: $\lim_{i\to 0} F_i(k,n) = \infty$ and $\lim_{i\to\infty} F_i(k,n) = 0$,

for $i = k, n$.

Note that if F is homogeneous of degree 1, then

$$F(k,n) = nF\left(\hat{k},1\right)$$

and the first derivatives are homogeneous of degree 0 functions:

$$F_k(k,n) = \frac{\partial nf(k/n)}{\partial k} = f'\left(\hat{k}\right),$$

$$F_n(k,n) = \frac{\partial nf(k/n)}{\partial n} = f\left(\hat{k}\right) - f'\left(\hat{k}\right)\hat{k}.$$

Let $\delta \in (0,1]$ be the depreciation rate of capital. Net capital stock at the end of period t under history h^t must equal new investment:

$$k_{t+1}\left(h^t\right) - (1-\delta)\,k_t\left(h^{t-1}\right) = x_t\left(h^t\right). \tag{2}$$

Aggregate resource constraint

Aggregate consumption and investment demand must be feasibly met by aggregate supply of output:

$$c_t\left(h^t\right)+x_t\left(h^t\right)\leq A_t\left(h^t\right)F\left(k_t\left(h^{t-1}\right),n_t\left(h^t\right)\right).$$

Remarks

Using the general history-dependent setup before, we can conclude similar results, as in the pure-exchange economy we studied last time:

1. Equivalence in allocation in PO, ADE and SME setups. Reason? FWT's hold.
2. Relative prices pin down the correct intertemporal (capital/consumption), and intratemporal (labor) trade-offs.
3. Convenient result:
   - Calculate Pareto allocation.
   - By FWT # 1, PO allocation = ADE allocation = SME allocation.
   - Reconstruct ADE prices or SME prices via allocations, and FOC's.

Problem: How do we solve for PO allocation if there is general history dependence?

Markov equilibrium in RBC models

One way out, for tractability of our solution, is to assume forcing processes are Markov.

Assume:

- $s \in S$ and s_t is a Markov chain $(\pi(s'|s), \pi_0)$ with $\pi_0(s_0) = 1$ for any $s_0 \in S$.
- $A_t(h^t)$ is generated by a measurable, time-independent function $\mathcal{A}$ of own lag, and current shock:

$$\left(A_{t-1}\left(h^{t-1}\right), s_t\right) \mapsto \mathcal{A}\left(A_{t-1}\left(h^{t-1}\right), s_t\right) = A_t\left(h^t\right).$$

Example (TFP with multiplicative shocks)

$$A_t\left(h^t\right) = A_{t-1}(h^{t-1})s_t = A_{-1}\prod_{\tau=0}^{t} s_\tau,$$

with A_{-1} given.

So next-period technology depends on last period technology and current shock s.

Dropping the time subscripts, we write

$$A' = \mathcal{A}\left(A, s\right) = As.$$

Example (continued)

Note:

$$A_t\left(h^t\right) = A_{t-1}(h^{t-1})s_t = A_{-1}\prod_{\tau=0}^{t} s_\tau,$$

with A_{-1} given.

Then:

$$\begin{aligned}\ln(A_t\left(h^t\right)) &= \ln(A_{t-1}(h^{t-1})) + \ln(s_t),\\ &= \ln(A_{-1}) + \sum_{\tau=0}^{t}\ln(s_\tau).\end{aligned}$$

The likelihood of a t-history, h^t can now be recursively computed:

$$\begin{aligned}\pi_t\left(h^t\right) &= \pi\left(s_t|s_{t-1}\right)\pi_{t-1}\left(h^{t-1}\right)\\ &= \pi\left(s_t|s_{t-1}\right)\pi\left(s_{t-1}|s_{t-2}\right)\cdots\pi\left(s_1|s_0\right)\pi_0\left(s_0\right)\end{aligned}$$

So then, we can dedude

$$\pi_t(s_t) = \pi\left(s_t|s_{t-1}\right)\pi_{t-1}(s_{t-1}).$$

Notation

Let the aggregate beginning of period capital stock be $K := K_t(h^{t-1})$. To summarize current position of the economy, define aggregate state as

$$X := (K, A, s)$$

We will next look at the Pareto problem and then the SME, in a recursive setup.

To distinguish the notatation:

Planner	**RCE**
C	c
K	k
N	n

Recursive Pareto Problem

The planner's problem now can be written down recursively as the Bellman equation

$$v(K, A, s) = \max_{C,N,K'} \left\{ u(C, 1-N) + \beta \sum_{s'} \pi(s'|s)\, v(K', A', s') \right\}$$

subject to

$$\begin{aligned} K' + C &\leq AsF(K, N) + (1-\delta) K \\ A' &= As. \end{aligned}$$

with K_0 given.

Policy functions

The optimal policy functions solving the Bellman equation are of the form

$$\begin{aligned} C &= C\left(K, A, s\right) \\ N &= N\left(K, A, s\right) \\ K' &= K\left(K, A, s\right) \end{aligned}$$

Markov kernel under Pareto plan

Note that the transition law for technology shocks together with the optimal policy function for capital accumulation,

$$A' = As \tag{3}$$

and

$$K' = K(K, A, s), \tag{4}$$

along with the Markov kernel for the shock $\pi(s'|s)$, induce a transition density function $\Pi(X'|X)$ on the aggregate state $X = (K, A, s)$. $\Pi(X'|X)$ gives us the frequency of observing the next-period aggregate state X' given a known current state X.

Example

Recall Brock-Mirman exercise in lecture slide.

1. Under Pareto planner's optimal plan, law of motion of capital was log-linear in own state and shock:

$$\ln(k_{t+1}) = \ln(\alpha\beta) + \alpha \ln(k_t) + \ln(\theta_t).$$

2. Shock was iid normal.
3. Recursively generated stochastic path for k_t will be conditionally log-normal.

Characterizing Pareto plan

Define derivatives *at the optimum* as

$$u_i(X) := u_i(C(K, A, s), 1 - N(K, A, s)) \text{ for } i = C, l$$

$$F_j(X) := F_j(K, N(K, A, s)) \text{ for } j = K, N$$

Proposition

Pareto optimal allocation is a sequence of state-contingent functions $\{C_t(X_t), K_{t+1}(X_t), N_t(X_t)\}_{t=0}^{\infty}$ *satisfying:*

1. *Intra-temporal optimality:*

$$\frac{u_l(X)}{u_C(X)} = AsF_N(X)$$

2. *Intertemporal, and across state, optimality:*

$$1 = \beta \sum_{X'} \Pi(X'|X) \frac{u_C(X')}{u_C(X)} \left[A's'F_K(X') + (1-\delta)\right]$$

3. *Feasibility at every* X*:*

$$K' + C = AsF(K, N) + (1-\delta)K.$$

Example (ejemplo con números)

Suppose $A_t = 1$ for all t. Then $X = (K, s)$. Let

$$u(C, 1-N) = \frac{C^{1-\sigma}}{1-\sigma} + \eta(1-N); \qquad \sigma > 0, \eta > 0.$$

Let

$$F(K, N) = sK^{\alpha}N^{1-\alpha}; \qquad \alpha \in (0,1).$$

Assume $S = \{s_1, s_2\} \ni s_t \sim$ Markov-(P, π_0), with

$$P = \begin{pmatrix} p_{11} & 1-p_{11} \\ p_{21} & 1-p_{21} \end{pmatrix}$$

and $p, q \in (0,1)$.

Example (cont'd)

Pareto planner's Bellman equation:

$$v\left(K, s_i\right) = \max_{C,N,K'} \left\{ \frac{C^{1-\sigma}}{1-\sigma} + \eta(1-N) + \beta \sum_{s'} P_{ij} v\left(K', s_j\right) \right\}$$

subject to

$$K' + C \leq s_i F\left(K, N\right) + \left(1-\delta\right) K$$

with K_0 given.

Example (cont'd)

A Pareto optimum is given by

1. a value function $v : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ and
2. optimal decision functions

$$\begin{aligned} C_t &= C(K_t, s_t), \\ N_t &= N(K_t, s_t), \\ K_{t+1} &= K(K_t, s_t) \end{aligned}$$

satisfying the Bellman equation above.

Example (cont'd)

So Pareto allocation, *at each* $(K_t, s_t) = (K_t, s_i)$, $i = 1, 2$, satisfies:

- optimal labor allocation:

$$\eta[C(K_t, s_i)]^\sigma = s_i(1-\alpha)\left(\frac{K_t}{N(K_t, s_i)}\right)^\alpha$$

Example (cont'd)

- optimal consumption trade-offs:

$$[C(K_t, s_i)]^{-\sigma} =$$
$$\beta\left\{p_{i1}[C(K_{t+1}, s_1)]^{-\sigma}\left[s_1\alpha\left(\frac{K_{t+1}}{N(K_{t+1}, s_1)}\right)^{\alpha-1} + (1-\delta)\right]\right.$$
$$\left. + p_{i2}[C(K_{t+1}, s_2)]^{-\sigma}\left[s_2\alpha\left(\frac{K_{t+1}}{N(K_{t+1}, s_2)}\right)^{\alpha-1} + (1-\delta)\right]\right\}$$

Example (cont'd)

- and feasibility:

$$
\begin{aligned}
& K(K_t, s_i) + C(K_t, s_i) \\
& \qquad = s_i F\left(K_t, N(K_t, s_i)\right) + (1-\delta) K_t
\end{aligned}
$$

Recursive SME

Now we consider the decentralized economy with sequential markets.

Arbitrary decentralization as:

- Household
- Firm I
- Firm II

Notation

Relative prices:

- *real rental rate on capital stock,* $r(X)$,
- *real rental rate on labor,* $w(X)$,
- *one-period Arrow security price,* $Q(X'|X)$.

Remarks

- Agents are "small" – viz. they take as given prices.
- In deciding their individual optimal actions (consuming, producing, selling and buying), agents need to forecast the evolution of $X = (K, A, s)$ in order to track prices.
- For simplicity, assume that all agents know:

$$A' = \mathcal{A}(s, A) := As.$$

but form subjective probabilities of shocks s, denoted as $\hat{\pi}(s'|s)$.

- So they have subjective views on stochastic evolution of K: $K' = G(X)$.

Remarks

- Idea:
  1. Observe current $X = (K, A, s)$
  2. Form beliefs $(G, \hat{\pi})$, induce belief on continuation states:

$$s' \sim \hat{\pi}(s'|s),$$
$$K' = G(X).$$

They know l.o.m. for A:

$$(s, A) \mapsto \mathcal{A}(s, A) = A'.$$

  3. Induced perceived Markov kernel on aggregate state:

$$X' \sim G \circ \mathcal{A} \circ \hat{\pi}(X'|X) \equiv \hat{\Pi}(X'|X).$$

Remarks

- Subjective beliefs system $(G, \hat{\pi})$, induces a *perceived transition density* for the aggregate state vector, $\hat{\Pi}(X'|X)$.
- So where does rational expectations come in?
- Rational expectations equilibrium (REE): impose consistency of beliefs on G and $\hat{\pi}$.
- Idea of REE: a fixed point in beliefs – subjective beliefs/probabilities coincide with actual equilibrium or objective probabilities.
- Beauty of RE: no need to model evolution of expectations, or how people learn. [Contra: literature in A.I., stochastic control, and psychology – people take time to learn about the "model" of the environment they play in.]

The Royal Swedish Academy of Sciences awarded the Bank of Sweden Prize in Economic Sciences in Memory of Alfred Nobel, 1995, to Professor Robert E. Lucas, Jr., University of Chicago, "for having developed and applied the hypothesis of rational expectations, and thereby having transformed macroeconomic analysis and deepened our understanding of economic policy".

Households

- Belief system $\hat{\Pi}(X'|X)$.
- They also track their own individual state a, which is the wealth they carry into each current period.
- $V(a, X)$, household value function at current state (a, X) which is the *subjectively* maximal indirect expected total discounted utility.
- We say "subjectively" because the household, off-equilibrium, calculates its expected continuation value using the subjective probabilities defined by $\hat{\Pi}(X'|X)$.

Households

- The household recursive problem is now given by the Bellman equation:

$$V(a, X) = \max \left\{ u(c, 1-n) + \beta \sum_{X'} V\left(\bar{a}(X'), X'\right) \hat{\Pi}(X'|X) \right\}$$

subject to

$$c + \sum_{X'} Q(X'|X)\, \bar{a}(X') \leq w(X)\, n + a$$

Household's (subjectively) optimal policy functions are

$$c = \sigma^c\left(a, X\right)$$

$$n = \sigma^n\left(a, X\right)$$

$$\overline{a}\left(X'\right) = \sigma^a\left(a, X; X'\right)$$

Let the marginal utilities of consumption and leisure *under the subjectively optimal household strategies*, respectively, be denoted by

$$\overline{u}_c\,(a, X) := u_c\left(\sigma^c\,(a, X)\,, 1 - \sigma^n\,(a, X)\right),$$

and

$$\overline{u}_l\,(a, X) := u_l\left(\sigma^c\,(a, X)\,, 1 - \sigma^n\,(a, X)\right).$$

(Subjective) optimality is then characterized by

$$\frac{\overline{u}_l\left(a, X\right)}{\overline{u}_c\left(a, X\right)} = w\left(X\right)$$

$$Q\left(X'|X\right) = \beta\hat{\Pi}\left(X'|X\right)\frac{\overline{u}_c\left(a', X'\right)}{\overline{u}_c\left(a, X\right)}$$

$$c + \sum_{X'} Q\left(X'|X\right)\overline{a}\left(X'\right) \leq w\left(X\right)n + a$$

Meaning?

Firm I

The static profit maximization problem for Firm I has the following zero profit conditions:

$$r(X) = AsF_k(k, n),$$

$$w(X) = AsF_n(k, n).$$

Firm I produces output up to the point where the marginal product of capital (labor) equals the market real rental rate on capital (labor).

Firm II

The Type II firm optimality condition (zero profit condition):

$$1 = \sum_{X'} Q\left(X'|X\right)\left[r\left(X'\right) + (1-\delta)\right].$$

Firm II will rent any amount of capital to Firm I as long as:

- the marginal cost of a unit of investment good it purchases (equal to 1),
- is equal to the expected discounted return on its rental sales.
- Discounted using the Arrow pricing kernel (stochastic discount factor), since the contracts between Firm II and Firm I are written in those terms.

Recursive competitive equilibrium

Idea. We now want to impose *equilibrium conditions* on

- arbitrary prices $r(X)$, $w(X)$, $Q(X'|X)$;
- arbitrary beliefs $(G(X), \hat{\pi})$ or $\hat{\Pi}(X'|X)$.

That is, we want equilibrium where:

- firms and households are price takers in solving their optimum problems,
- market clearing holds, and
- subjective beliefs coincide with objective probabilities.

Step 1. Given $\{w(X_t), r(X_t), Q_{t,t+1}(X_{t+1}|X_t)\}_{t\geq 0}$, solve household problem. Obtain subjectively optimal decision rules:

$$c = \sigma^c(a, X)$$

$$n = \sigma^n(a, X)$$

$$\bar{a}(X') = \sigma^a(a, X; X')$$

Step 2. Solve Firm I and Firm II profit maximization problems.
Profit maximization:

$$r(X) = AsF_k(k, n),$$

$$w(X) = AsF_n(k, n).$$

and

$$1 = \sum_{X'} Q(X'|X)\left[r(X') + (1-\delta)\right].$$

Step 2. Market clearing (Arrow securities market):

- Household asset holding equals Firm II's issue of state-contingent debt:

$$\bar{a}\left(X'\right) = \left[r\left(X'\right) + (1-\delta)\right] K'$$

- and so beginning-of-period assets in the household budget constraint must satisfy

$$a = \left[r\left(X\right) + (1-\delta)\right] K$$

Step 3a. Market clearing (labor and goods):

- Substitute asset market clearing into the household sequential budget constraint:

$$c + \sum_{X'} Q\left(X'|X\right)\left[r\left(X'\right) + (1-\delta)\right] K' \\ = w\left(X\right) n + \left[r\left(X\right) + (1-\delta)\right] K.$$

- using Firm I and II first-order conditions, household subjective optimal decisions, reduce the household budget constraint:

$$K' = AsF\left(K, \sigma^{n}\left(\left[r\left(X\right) + (1-\delta)\right] K, X\right)\right) \\ + (1-\delta) K - \sigma^{c}\left(\left[r\left(X\right) + (1-\delta)\right] K, X\right) \quad (\dagger)$$

Remark.

- Note that RHS (†) depends on subjectively optimal decision functions $\sigma(a, X)$ induced by perceived law of motion (PLM), $(G, \hat{\pi})$:

$$\begin{aligned} K' = {} & AsF\left(K, \sigma^{n}\left(\left[r\left(X\right) + \left(1-\delta\right)\right]K, X\right)\right) \\ & + \left(1-\delta\right)K - \sigma^{c}\left(\left[r\left(X\right) + \left(1-\delta\right)\right]K, X\right) \\ := {} & \mathcal{G}[G(X)]. \end{aligned}$$

- But RHS, for given PLM $(G, \hat{\pi})$, produces actual l.o.m. (ALM). Another way to see it: the RCE with subjective beliefs (PLM) is a mapping $G(X) \mapsto \mathcal{G}[G(X)]$.
- But under PLM $(G, \hat{\pi})$, we said

$$K' = G(X).$$

  So a RCE plus REE requires that PLM = ALM, or the REE l.o.m. $G^{*}(X)$ is the fixed point of the mapping of subjective beliefs and price system into an actual outcome, $G(X) \mapsto \mathcal{G}[G(X)]$.

Step 3b. REE and RCE

Definition

A consistent set of beliefs, or *rational expectations equilibrium* is a fixed-point of the mapping from perceived G and price system to an actual $\mathcal{G}$ and $\hat{\pi} = \pi$, so that $(G, \hat{\pi}) = (\mathcal{G}, \pi)$.

In our case, REE or PLM = ALM, requires

$$G(X) = AsF(K, \sigma^n([r(X) + (1-\delta)]K, X)) \\ + (1-\delta)K - \sigma^c([r(X) + (1-\delta)]K, X) := \mathcal{G}[G(X)].$$

Definition

A recursive competitive equilibrium with rational expectations is a sequence of pricing functions $\{w(X_t), r(X_t), Q_{t,t+1}(X_{t+1}|X_t)\}_{t\geq 0}$ and decision functions $\{\sigma(a_t, X_t)\}_{t=0}^{\infty}$ such that

1. $V(a, X)$ and $\sigma(a_t, X_t)$ solve household Bellman equation problem given prices,
2. Firms I and II maximize profit, and
3. Markets clear with REE: $k = K$, $\sigma^n(a, X) = n = N$, $\sigma^c(a, X) = c = C$, $\sigma^a(a, X; X') = [r(X') + (1 - \delta)]K'$ with

$$K' = G^*(X) = \mathcal{G}[G^*(X)],$$

and $\hat{\pi} = \pi$.

Remark.
So in a RCE with rational expectations, beliefs $(G, \hat{\pi})$ coincide with REE model (G^*, π).

So then,

$$
\begin{aligned}
G \circ \mathcal{A} \circ \hat{\pi}(X'|X) &\equiv \hat{\Pi}\left(X'|X\right) \\
&= \Pi\left(X'|X\right) \equiv G^* \circ \mathcal{A} \circ \pi(X'|X).
\end{aligned}
$$

and $\Pi\left(X'|X\right)$ is the *Markov kernel describing the RCE under REE*.

RCE cookbook steps

1. Setup decentralized problem.
2. Solve it. RCE contingent prices and allocations described by optimality, market clearing and REE conditions.
3. Solution to RCE is represented by a dynamic stochastic system in terms of the RCE Markov kernel: $\Pi(X'|X)$.
4. Study RCE behavior? Simulate outcomes $\{X_t\}_{t=0}^{\infty}$ given X_0:
   - $X_1 \sim \Pi(X_1|X_0)$,
   - $X_2 \sim \Pi(X_2|X_1)$,

     $\vdots$

   - $X_{t+1} \sim \Pi(X_{t+1}|X_t)$.

Solving RCE via planner's problem

1. Solve recursive planning problem. Find planner's optimal decision functions:

$$\begin{aligned} C &= C(K, A, s) \\ N &= N(K, A, s) \\ K' &= K(K, A, s) \end{aligned}$$

These fully describe Pareto optimal state-contingent allocations.

2. Invoke FWT # 1: Pareto allocations (in this model) are also RCE allocations. (What if models have frictions?)

3. Back out state-contingent RCE prices via RCE conditions:

$$\begin{aligned} Q(X'|X) &= \beta\hat{\Pi}(X'|X)\frac{\overline{u}_c(a', X')}{\overline{u}_c(a, X)} \\ r(X) &= AsF_k(k, n), \\ w(X) &= AsF_n(k, n). \end{aligned}$$

Recall ...

We've already figured out how to solve this model (optimal stochastic growth planning problem). Now if we apply FWT#1, we can also construct RCE pricing functions and outcomes.

1. Translate this theory and notion of RCE with REE into action.
2. Demonstrate how an approximate solution to RCE (with REE) of a model implies a dynamic stochastic system in form of an equilibrium Markov kernel or Markov operator, Π.
3. With Π, we can do many things:
    - Study stochastic dynamics of model (mostly via Monte Carlo simulation).
    - "Comparative statics"-like analysis via impulse response analysis.
    - Take model to the data using calibration, maximum-likelihood estimation, or Bayesian estimation methods.