DECENTRALIZED (WALRASIAN) EQUILIBRIUM

Preference representation

$$\sum_{t=0}^{\infty} \beta^t u(c_t, n_t)$$

NEW

endogenous labor now.

where

$$u_c > 0$$

$$u_{cc} < 0$$

$$\lim_{c \to 0} u_c(c, \cdot) = \infty$$

$$u_n < 0$$

$$u_{nn} < 0$$

Production technology

$$Y_t = F(K_t, AN_t)$$

Agg. effective labor

Aggregate output

Agg. cap.

where

$$F(0, \cdot) = F(\cdot, 0) = F(0,0) = 0$$

$$F_1 > 0, \quad F_2 > 0, \quad F_{11} < 0, \quad F_{22} < 0$$

$$\lim_{K \to 0} F_1(K, 1) = \infty$$

$$\lim_{K \to \infty} F_1(K, 1) = 0$$

F is hom. deg. 1.

Two Theories of Competitive Equilibrium

M1 ◦ Arrow - Debreu with one-and-for all market for future contracts.

M2 ◦ Sequential (Competitive) Eqm with forward contracts.

Caveat:

- Note for now, we have a deterministic economy — no <u>risk</u>.

- So futures / forwards are perfect-foresight contracts / claims to future exchange.

- Later, we generalize this to risky environment. So then these contracts are

    - date, and

    - state / history -

  contingent.

M1. Arrow-Debreu (Date-0 trading)

Notation:

- q_t^0

  relative price of
  a claim to date-t consumption
  in units of date-0 consumption.

- w_t

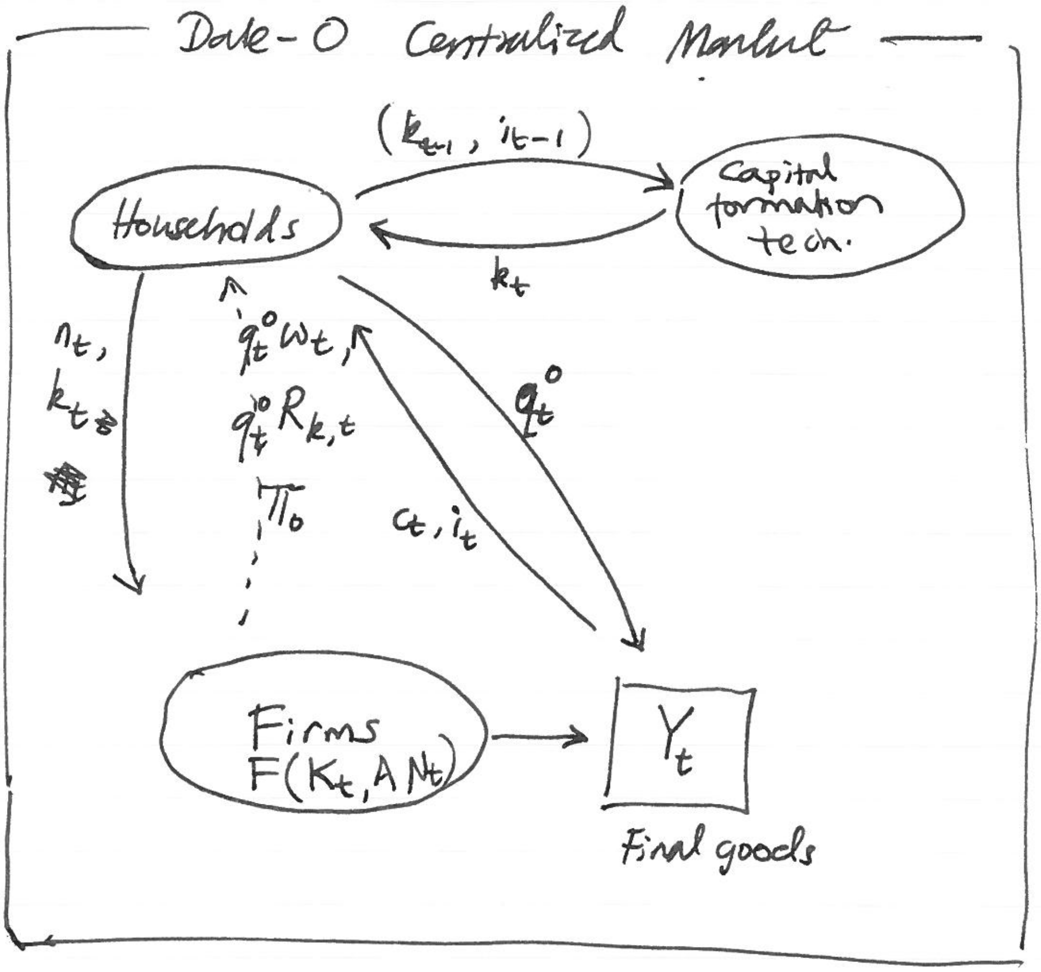
  Date t (real) wage rate
  in units of date-t consumption

- $R_{k,t}$

  ? (real) rental rate
  in units of date-t consumption

- Π_0 total discounted profits in
  units of date-0 consumption.

Walrasian Market.



Ownership structure?

- H/hold owns
  - capital <u>stock</u>
  - labor
  - firms

- Budget constraint per period t:

$$c_t + i_t = \underbrace{R_{ht} k_t}_{\substack{\text{income from}\\\text{renting capital}\\\text{out}}} + \underbrace{w_t n_t}_{\substack{\text{income from}\\\text{renting}\\\text{labor out}}} + \underbrace{\Pi_t}_{\substack{\text{date-}t\\\text{profit}\\\text{flow.}}}$$

PV:

$$\sum_{t=0}^{\infty} q_t^0 \left(c_t + i_t \right) = \sum_{t=0}^{\infty} q_t^0 \left[R_{ht} k_t + w_t n_t \right] + \overset{?}{\Pi_0}$$

- And...

$$k_{t+1} = (1-\delta) k_t + i_t \quad , \quad k_0 \text{ given.}$$

Household's Problem : Write this out!

Ex. Show that h/hold's optimal plan, given $\{q_t^0, w_t, R_{k,t}\}_{t\geq 0}$ is:

$$c_t: \quad \beta^t u_c(c_t, n_t) = \mu q_t^0$$

$$n_t: \quad -\beta^t u_n(c_t, n_t) = \mu q_t^0 w_t$$

$$k_{t+1} \text{ (or } i_t\text{)}: \quad q_t^0 = q_{t+1}^0 (R_{k,t+1} + 1 - \delta)$$

for every date $t \geq 0$.

Firm's Problem

$$\Pi_0 = \max_{\{K_t, N_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} q_t^0 \left[F(K_t, AN_t) - R_{kt} K_t - w_t N_t \right]$$

Ex. FOC:

$$F_K(K_t, AN_t) = R_{kt}$$

$$AF_N(K_t, AN_t) = w_t$$

Show $\Pi_0 = 0$ since F is hom. dg. 1.

Definition: An (Arrow-Debreu) competitive equilibrium with a date-0 once-and-for all trade is a sequence of allocations $\{C_t, I_t, N_t, K_{t+1}\}_{t\geq 0}$ prices $\{q_t^0, w_t, R_{k,t}\}_{t\geq 0}$ s.t. given:

(1) prices, $\{c_t, i_t, n_t, k_{t+1}\}_{t\geq 0}$ solve the h/hold problem;

(2) prices, $\{K_t, N_t\}_{t\geq 0}$ solve the firm problem; and

(3) markets clear:

$$C_t = c_t$$

$$K_t = k_t$$

$$I_t = i_t$$

$$N_t = n_t$$

so that

$$K_{t+1} = (1-\delta)K_t + I_t$$

and

$$C_t + I_t = F(K_t, AN_t).$$

M2. Sequential (Competitive) Equilibrium

Assume: H/hold own capital ...

H/hold Sequential B.C.:

$$c_t + i_t = R_{kt} k_t + w_t n_t + \Pi_t$$

Firm solves static problem (why?):

$$\Pi_t = \max_{K_t, N_t} \{ F(K_t, AN_t) - R_{kt} K_t - w_t N_t \}$$

Definition:

A sequential CE is a $\underbrace{\text{sequence of quantities}}_{\text{"an allocation"}}$

$x = \{c_t, i_t, n_t, k_{t+1}\}_{t \geq 0}$ to each h/hold, aggregate quantities $X = \{C_t, I_t, N_t, K_{t+1}\}_{t \geq 0}$ and prices $\{w_t, R_{kt}\}$ s.t.

(1) given prices, x solves h/hold problem;

(2) " X solves firm's problem,

(3) markets clear at each date ...

(see previous)

(1) H/hold problem has the FOC's

$$u_c(c_t, n_t) = \beta u_c(c_{t+1}, n_{t+1})\left[R_{kt+1} + (1-\delta)\right]$$

$$-\frac{u_n(c_t, n_t)}{u_c(c_t, n_t)} = w_t$$

and the TVC$_\infty$:

$$\lim_{t \to \infty} \beta^t u_c(c_t, n_t) k_{t+1} = 0$$

Ex. Show that the allocations in M1 and M2 are identical.

M2'. Alternative Ownership

- H/hold own firms by trading shares in equity market.

Choose: $\{ c_t , \underbrace{\psi_{t+1}}_{\text{shares}} , n_t \}$

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t , n_t)$$

s.t.

$$c_t + \psi_{t+1} \underbrace{(V_t - D_t)}_{\substack{\text{ex-dividend} \\ \text{market value} \\ \text{of firm.}}} = \psi_t V_t + w_t n_t$$

- Continuum of firms on $[0,1]$. Each firm chooses $\{ N_t , I_t \}_{t \geq 0}$ to

$$V_t = \max_{\{I_{t+j} , N_{t+j}\}} \sum_{j=0}^{\infty} M_{t,t+j} D_{t+j}$$

discount factor/ pricing kernel

s.t.

$$D_{t+j} = F(K_{t+j} , AN_{t+j}) - w_{t+j} N_{t+j} - I_{t+j}$$

$$K_{t+j+1} = (1-\delta) K_{t+j} + I_{t+j}$$

$$M_{t,t} = 1.$$

Definition : A SCE in M2' is an allocation $x' = \{ c_t, n_t\ \psi_{t+1} \}$, $X' = \{ K_{t+1}, N_t, I_t \}$ and prices $\{ V_t , w_t , M_{t,t+j} \}$ s.t.

(1) h/holds opt.

(2) firms opt.

(3) Market clear :

$$\psi_t = 1$$

total shares owned by h/holds

$$C_t = c_t$$

$$N_t = n_t$$

$$\cancel{C_t}\ c_t + I_t = F(K_t , AN_t)$$

Note:

(1) h/hold optimize:

φ_{t+1} : $$u_c(c_t, n_t)(V_t - D_t) = \beta u_c(c_{t+1}, n_{t+1}) V_{t+1}$$

$$\Rightarrow \quad V_t = D_t + \frac{\beta u_c(c_{t+1}, n_{t+1})}{u_c(c_t, n_t)} V_{t+1}$$

Solving fwd: in eqm:

$$V_t = \sum_{j=0}^{\infty} \underbrace{\frac{\beta^j u_c(c_{t+j}, n_{t+j})}{u_c(c_t, n_t)}}_{\substack{\text{c.f.} \\ \text{equilibrium} \\ \text{pricing kernel}}} \cdot D_{t+j}$$

Interpretation in words?

and h/hold TVC_∞ implies:

$$\lim_{j \to \infty} \beta^j u_c(c_{t+j}, n_{t+j}) V_{t+j} = 0.$$

Interpretation in words?

No-bubbles eqm

(2) Firms optimize :

~~Lagrange~~ Lagrange Multiplier on capital acc. eqn.

$Q_{t,t+j} M_{t,t+j}$

$I_t : \quad Q_t = 1$

$K_{t+1} : \quad Q_t = M_{t,t+1} \left[F_k (K_{t+1}, AN_{t+1}) - (1-\delta) Q_{t+1} \right]$

Tobin's q.

marginal value of the firm following an additional unit increase in capital

$\equiv$

shadow price of capital.

Why MZ' ?

- Allows us to explicitly model asset pricing behavior from an eqm theory perspective.

Note:

Since $Q_t = 1$, we have:

$$M_{t,t+1}\left[F_k\left(K_{t+1}, AN_{t+1}\right) - (1-\delta)\right] = 1.$$

$$\Leftrightarrow \quad u_c(c_t, n_t) = \beta u_c(c_{t+1}, n_{t+1})\left[F_k(K_{t+1}, AN_{t+1}) - (1-\delta)\right]$$

Allocations in M1, M2 and M2' are identical.

Social Planner

$$\max_{\{c_t, n_t, i_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t, n_t)$$

s.t.

$$c_t + i_t = F(k_t, An_t)$$

$$k_{t+1} = (1-\delta) k_t + i_t .$$

Claim FOC's identical to M1, M2. or M2'.

SPP		CE
		$\{C_t, N_t, I_t\}$
$\{c_t, n_t, i_t\}$	$\Longleftrightarrow$	M1
	$\Longleftrightarrow$	M2
	$\Longleftrightarrow$	M2'

Prescott - Mehra (1980)

- Recursive Competitive Equilibrium

- Observation: from M2

$$F_K(K, AN) = R_k(K)$$

$$F_N(K, AN) = W(K).$$

- Intuition:

  - If in SME or ADE prices are function of only current state of capital K, then CE can be characterized recursively as a function of K alone: $K_+ = G(K)$.

- H/hold problem

$$V(k, K) = \max_{c, n, k_+} u(c, n) + \beta V(k', K')$$

fixed

s.t.

$$c + k_+ - (1-\delta)k = k \cdot R_k(K) + W(K) \cdot n$$

$$K_+^{*} = G(K)$$

"perceived law of motion"

Definition

A P.M RCE in this economy is a value function $V(k,K)$, decision functions $c = G^c(k,K)$, $n = G^n(k,K)$, $k_+ = G^k(k,K)$, the l.o.m. $K_+ = G(K)$ and pricing fns $R_k(K)$ and $W(K)$ s.t.

(1) given price functions and $K \mapsto G(K)$, $V(k,K)$ satisfies h/hold Bellman eqn,

(2) pricing fns satisfy

$$W(K) = F_N(K, AN)$$

$$R_k(K) = F_K(K, AN)$$

(3) Markets clear:

- $k = K$, $k_+ = K^+$
- $G^k(K,K) = G(K)$
- $c(K,K) = C$
- $n(K,K) = N$
- $K_+ - (1-\delta)K + C = F(K,N)$.

Lesson:

- Infinite sequence problems:

$$\begin{array}{c} \mathrm{SPP} \\ \Updownarrow \\ \mathrm{M1} \Leftrightarrow \mathrm{M2} \Leftrightarrow \mathrm{M2'} \\ \Updownarrow \\ \mathrm{RCE} \end{array}$$

$\boxed{\text{tractable!}}$