Deterministic Dynamic Programming III: Practical and Computational Aspects

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Outline

Recap: Theory I

Paradigm:

- Single-agent infinite-horizon decision-making problem
  - Many neoclassical dynamic general equilibrium problems fall into this class
  - Competitive equilibrium $\equiv$ Pareto allocation problem
  - FWT #1 and FWT #2
- This is amenable to dynamic programming techniques

Recap: Theory II

Key lessons from last set of lectures:

- Infinite sequence problem — Bellman Principle of Optimality — equivalent to recursive Bellman equation problem
- Under regularity condition on payoff and constraint sets — exists a unique solution of a “value function” — to Bellman equation
- Also exists a strategy that supports the optimal value
- Focus on a special class of strategies: Markovian and stationary; existence of a Markovian-stationary policy mapping that generates this strategies in this class.

Theory to Practice I

This set of lectures

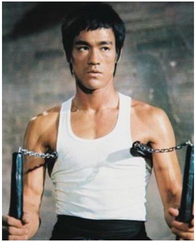
- How do we implement and analyze the solution(s)?
- ... When there is no analytical means of doing so?
- Banach Fixed-point Theorem gives a starting point for developing an algorithm.
- Algorithm is a set of instruction for implementing solution method (e.g. on computer).
- Analyses of approximate optimal solutions.

Non-analytic optimal growth (OGM) example I

- Make things more concrete ...
- Revisit our old friend, the Ramsey-Cass-Koopmans optimal growth problem
- We will apply techniques for characterizing:
  1. existence and uniqueness of solution v to Bellman operator: $w \mapsto T(w)$
  2. existence, and, under more primitive assumptions, uniqueness, of supporting stationary strategy
- But today ... This example has no analytical solution!
- Requires a new skillset ...

Non-analytic optimal growth (OGM) example II

Need some computational skills for characterizing equilibrium dynamics ...



Nunchuck skills and such.

Non-analytic optimal growth (OGM) example III

Idea:

- In the simple Solow-Swan or analytical optimal growth examples we could characterize equilibrium path by hand!
- It boiled down to an analytical mapping, e.g. $k_{+} = g(k)$. This g was the stationary policy function that induces the equilibrium/optimal Markovian-stationary strategy (trajectory of optimal actions).
- But now, in the optimal growth problem, the solution to the Bellman equation v, and its supporting stationary decision rule (a version of "g"), cannot be obtained by hand in general.

Non-analytic optimal growth (OGM) example IV

- Need for method to obtain approximately-optimal v and supporting g, say, $\hat{v}$ and $\hat{g}$ ...
- Then given $\hat{g}$, we can indirectly characterize the equilibrium/optimal strategies via simulation:
    - Impulse response (i.e. sample trajectory)
    - Limiting/asymptotic statistics (if $\hat{g}$ is a stochastic map)

Non-analytic optimal growth (OGM) example V

- Before we get to the computational aspects ... let's:
  - setup the optimal growth problem again; and
  - characterize/describe properties of an optimal solution as much as possible analytically.

Non-analytic OGM: Model setup I

- A single good (the neoclassical banana) - can be consumed or invested;
- Capital investment flow in period t, x_t;
- Capital depreciates at rate $\delta \in (0, 1]$;
- Capital stock accumulation:

$$k_{t+1} = (1 - \delta)k_t + x_t$$

  with initial stock $k_0 > 0$;

Non-analytic OGM: Model setup II

- Production function, $F : \mathbb{R}_+ \to \mathbb{R}_+$:

$$y_t = F(k_t);$$

- Total available resources at time t is

$$f(k_t) := (1-\delta)k_t + F(k_t);$$

  resource constraint: $f(k_t) \geq c_t + k_{t+1}$.
- Feasibility: $0 \leq k_{t+1} \leq f(k_t)$ for all $t \in \mathbb{N}$.
- Utility function: $U : \mathbb{R}_+ \to \mathbb{R}$, where for each $c_t \in \mathbb{R}_+$, $U(c_t) \in \mathbb{R}$; and
- Subjective discount factor $\beta \in [0, 1)$.

Non-analytic OGM: problem statement I

The planner's infinite-sequence problem:

$$\max_{\{c_t, k_{t+1}\}_{t \in \mathbb{N}}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

subject to

$$\begin{aligned} k_0 &= k \text{ given}, \\ f(k_t) &\geq c_t + k_{t+1}, \\ 0 &\leq k_{t+1} \leq f(k_t), \\ c_t, k_t &\in \mathbb{R}_+. \end{aligned}$$

for all $t \in \mathbb{N}$.

Non-analytic OGM: problem statement II

Impose more information—optimal program $\Rightarrow$ no waste. So ...

$$\max_{\{c_t, k_{t+1}\}_{t \in \mathbb{N}}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

subject to

$$\begin{aligned} k_0 &= k \text{ given}, \\ f(k_t) &= c_t + k_{t+1}, \\ 0 \leq k_{t+1} &\leq f(k_t). \end{aligned}$$

for all $t \in \mathbb{N}$.

Non-analytic OGM: problem statement III

Notation:

- $k := k_t$
- $k' := k_{t+1}$

Non-analytic OGM: problem statement IV

By the BPO we can re-cast this problem recursively as a DP problem $\{X, A, \Gamma, U, \kappa, \beta\}$:

- State space, $X = \mathbb{R}_+$. State variable $k \in X$.
- Action space, $A = \mathbb{R}_+$.
- State transition function $\kappa : X \times A \to X$ such that $k' = \kappa(k, c) := f(k) - c$.
- Feasible action correspondence $\Gamma : X \to 2^A$ such that $\Gamma(k) = [0, f(k)]$.
- Per period payoff from action $c \in \Gamma(k)$ given state k, $U(k, c) := U(c)$. (Special case)

Non-analytic OGM: problem statement V

The recursive planning problem:

$$v(k) = \max_{k' \in \Gamma(k)} \{U(f(k) - k') + \beta v(k')\}$$

Non-analytic OGM: Characterizations I

Things to look out for:

- Existence of a unique solution to the Bellman equation?
- When do (stationary) optimal strategies exist?
- Is an optimal strategy unique here?
- Characterizing optimal strategy.
  - What are the dynamic properties – i.e. the trajectory of $\{c_t, k_t\}_{t\in\mathbb{N}}$ under the optimal strategy?
  - What is the behavior of the transitional path (short run)?
  - The steady state (long run)?

Non-analytic OGM: Characterizations II

Additional structure:

- $X = A$ is a compact set.
- $U : X \to \mathbb{R}$ continuous
- $f : X \to \mathbb{R}_+$ continuous and nondecreasing

Non-analytic OGM: Characterizations III

Stationary strategies. Slight abuse of notation:

- Stationary strategies generically denoted by π.
- Now we'll also make it synonymous with stationary decision rule $\pi : X \to A$, which generates such a strategy.
- Used interchangeable in notes.

Theorem (Existence)

There exists a stationary optimal strategy $\pi : X \to A$ for the optimal growth model given by $\{X, A, \Gamma, U, \kappa, \beta\}$, such that

$$\begin{aligned} v(\pi)(k) &= \max_{k' \in \Gamma(k)} \{U(f(k) - k') + \beta v(\pi)[k']\} \\ &= U(\pi(k)) + \beta v[\kappa(k, \pi(k))]\}. \end{aligned}$$

Non-analytic OGM: Characterizations IV

Theorem

$v : X \to \mathbb{R}$ is a nondecreasing function on X.

Theorem

If additionally, $U : X \to \mathbb{R}$ is strictly increasing, then $v : X \to \mathbb{R}$ is strictly increasing.

Non-analytic OGM: Characterizations V

More regularization:

- $U : X \to \mathbb{R}$ is strictly concave.

Then ...

Proposition

Given the above assumptions, the optimal savings level $\pi(k) := f(k) - c(k)$ under the optimal strategy π, where $k' = \pi(k)$, is nondecreasing on X.

Non-analytic OGM: Characterizations VI

Even more discipline:

- $f : X \to \mathbb{R}_+$ is concave.

Theorem

Under the assumptions above, the value function v is (weakly) concave on X.

Non-analytic OGM: Characterizations VII

Theorem (Unique optimal strategy)

Under the assumptions above, the correspondence $G^* : X \to P(A)$ *defined by*

$$G^*(k) = \left\{ k' \middle| \max_{k' \in \Gamma(k)} \{U(f(k) - k') + \beta v(k')\}, k \in X \right\}.$$

is a singleton set (a set of only one maximizer k'*) for each state* $k \in X$*. Therefore* G^* *admits a unique optimal strategy* π*. Furthermore,* π *is a continuous function on* X*.*

... these standard results obtain from the Theorem of the Maximum. (Recall in Math A.)

Non-analytic OGM: Characterizations VIII

- If we layer more assumptions on primitives U and f, we can say even more about the properties of the optimal strategy π.
- Assume further:
  1. $U \in C^1((0,\infty))$ and $\lim_{c \searrow 0} U'(c) = \infty$.
  2. $f \in C^1((0,\infty))$ and $\lim_{k \searrow 0} f'(k) = 1/\beta$.

Non-analytic OGM: Characterizations IX

Theorem (Interior optimum)

Under Assumptions above, the solution $k' = \pi(k)$ *is such that* $\pi(k) \in (0, f(k))$ *for all* $k \in X$.

Non-analytic OGM: Characterizations X

Implication ... first-order condition for optimality in this model always holds with equality.

Theorem (Euler equation)

Under assumption above, for each $k \in (0, f(k))$, π satisfies

$$U_c[f(k) - \pi(k)] = \beta U_c[f(k') - \pi(k')] f_k(\pi(k))$$

such that $k' = \pi(k)$.

Non-analytic OGM: Characterizations XI

Relation to first-order perturbation method studied last semester:

- "Log-linearization" perturbation methods assumes that solution $\pi : X \to X$ is always interior, so Euler condition is a strict equality condition.
- Method requires that U is continuously twice-differentiable; f is at least continuously once-differentiable, so that you can apply the Taylor approximation theorem
- Approximate solution $\hat{\pi}$ must be of some linear class of stationary-Markovian decision rules satisfying "linearized" FOC, and ...
- The linear optimal control rule $\hat{\pi}$ must be one which is stabilizing (in the linearized system) so that it implies boundedness of the equilibrium dynamics (implies TVC holds).

Non-analytic OGM: Characterizations XII

From Euler equation we can show that the sequence of consumption decisions from any initial state k are monotone. Define $c(k) = f(k) - \pi(k)$.

Theorem

Under Assumptions above, c is increasing on X. That is, for $k > \tilde{k}$, $c(k) > c(\tilde{k})$.

Non-analytic OGM: Characterizations XIII

Theorem

Given any initial condition $k \in X$, the sequence of states $\{k_{t+1}(k)\}_{t\in\mathbb{N}}$ under the optimal policy function $\pi : X \to P(A)$, and the sequence of consumption levels $\{c_t(k)\}_{t\in\mathbb{N}}$ converge to k_{ss} and c_{ss} respectively. Furthermore, k_{ss} and c_{ss} are unique.

Algorithm and Implementation I

- We had taken the description of the optimal solution of the RCK problem as far as possible.
- In this example, we were able to describe the properties of v and π with features that increase with additional regularity assumptions on primitives, U and f.
- To actually *solve* for the optimal outcomes, we need to resort to numerical approximation and computation.
- First we need to setup a strategy for approximation and computation — i.e. develop an algorithm.

Algorithm and Implementation II

Let $\epsilon > 0$. Algorithm:

1. Pick some initial guess for $v_n : X \to \mathbb{R}$, where, $n = 0$.
2. Solve and evaluate the Bellman operator:

$$v_{n+1}(k) = \max_{k' \in \Gamma(k)} \{U[f(k) - k'] + \beta v_n(k')\},$$

   for every $k \in X$. Also store:

$$\pi_n(k) = \text{arg} \max_{k' \in \Gamma(k)} \{U[f(k) - k'] + \beta v_n(k')\}.$$

3. Calculate distance between consecutive updates: $d(v_{n+1}, v_n)$.
4. While $d(v_{n+1}, v_n) \geq \epsilon$, repeat Steps 2-4.

Algorithm and Implementation III

Implementation issues:

- How to represent state space $X = [\underline{k}, \overline{k}]$ (infinite set) on computer (finite storage)?
- How to represent v_n, $n = 0, 1, ...$, each a function, an element of an infinite dimensional space?
- How to compute the "$\max$" operator in Step 2?
- How to represent and store π_n, also a function?
- What is the appropriate metric d?