Ramsey-Cass-Koopmans social planner's problem (special case)

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Outline

1. Outline

Tools

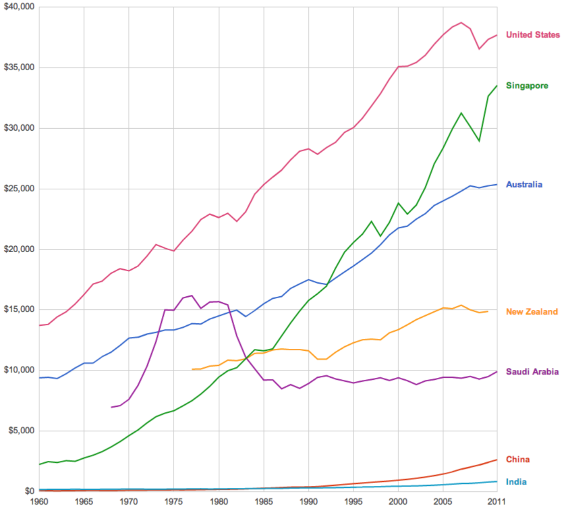
- Batteries not included:
  1. Optimization (Math A for Economists)
  2. High school calculus
  3. High school (elementary) algebra

2. Looking Ahead

1. Developing basic toolkit:
    - Dynamic Programming — important for single agent dynamic decision problems
    - Getting your hands dirty — implementation of DP and function approximation methods
2. Why you're doing this stuff now? What's the point? The *economics* beyond this neoclassical (frictionless-markets) RCK starting point:
    - Distortionary taxes in RCK
    - New Keynesian market frictions and role of monetary policy
    - Search models of labor (also, marriage markets, money and finance)
    - Heterogenous agent models in incomplete markets settings

3. A Workhorse Dynamic Problem

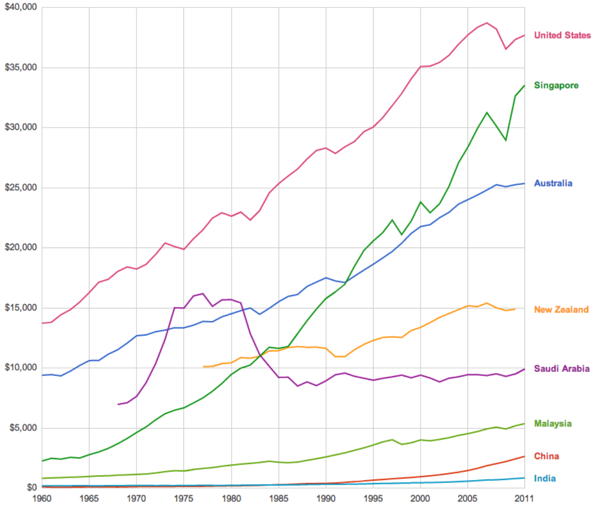
Economic Motivation



Real GDP per person as total of expenditure flows (in constant 2000 US dollars).

3. A Workhorse Dynamic Problem

Economic Motivation



Consider: Singapore vs. Malaysia.

3. A Workhorse Dynamic Problem

Economic Motivation

We are familiar with these questions in economic growth:

- Why do seemingly similar countries end up with different living standards after some time?
- Why the different economic performances?
    - e.g. why did Singapore take off post-1965 and grow faster than Malaysia?
    - e.g. why did South Korea take of and Phillipines did not quite do so, although they "looked similar" initially in 1960?
    - why did colonies overtake their colonial master? e.g. U.S. and Great Britain.

3. A Workhorse Dynamic Problem

Approach

- To understand *some* of these phenomena, we need a *dynamic* model.
- A model that can account explicitly for changes and growth over time:
  - We'll see a key dynamic variable is *capital.*
  - When capital (stock) varies over time as a result of changes in investment (flow).
  - Capital as a sufficient summary of "where" the economy is at each point in time: i.e. a *state* variable.
  - We have learned Solow model.
  - Ramsey-Cass-Koopmans foundation:
    - We'd like a deeper understanding of consumption/saving behaviour through choice problems.
    - But how do we describe this problem? Does a solution exist? What are its properties? How to find it?

3. A Workhorse Dynamic Problem

Key Takeaways

- Many dynamic decision problems are difficult to solve
- Typical: Infinite horizon decision processes.
- In previous Macro course, you learned to characterize solution(s) to an *infinite-horizon Lagrange problem*. ...

  ... How did you actually solve it? [You didn't. You'd tip-toed around this issue and merely *characterized* its solution in terms of FoCs.]

3. A Workhorse Dynamic Problem

What we will learn

- Illustration of a method: "value-function iteration".
- Idea:
  - Transform intractable choice problem over infinite sequences ...
  - ... into a sequence of tractable finite-dimensional optimization problem (i.e. "recursive" problem).
- What is the catch?
  - Need to encode infinite continuation problems into a single object—"value function".
- This tutorial: glean the intuition by example.

3. A Workhorse Dynamic Problem

DP idea by example

1. A (hopefully) familiar dynamic decision model. A single decision maker's problem.
2. Original problem of choosing infinite sequences to maximize an infinite-horizon decision criterion.
3. Claim: (infinite-sequence optimization problem) $\equiv$ (recursive two-period, finite-dimensional optimization decision problems).

    - The latter is good, because we know how to deal with a sequence of finite-dimensioned optimization problems.
    - This claim motivates the following illustrative example.

3. A Workhorse Dynamic Problem

DP idea by example

Optimal savings problem:

- a single decision maker;
- planning horizon is indefinite—model time as $\{0, 1, 2, ...\} =: \mathbb{N} \ni t$
- planning problem begins at time $t = 0$
- Endowed with initial capital stock $k_0 \in X$
    - $X \subset \mathbb{R}_+$ is the set of possible capital stocks.
- Decision problem: optimal balance between consumption today and into the indefinite future.
- Dynamic link: A storage technology (capital) and a productive technology

3. A Workhorse Dynamic Problem

DP idea by example

At the beginning of any time t,

- given a fixed amount of capital stock k_t,
- the planner can access a technology described by the function $F : \mathbb{R}_+ \to \mathbb{R}_+$ to produce output,

$$y_t = F(k_t).$$

Assumption

The technology representation $F : \mathbb{R}_+ \to \mathbb{R}_+$*:*

1. *is continuous, strictly increasing and concave on* $\mathbb{R}_+$*, and is twice-continuously differentiable; and*
2. *satisfies the (Inada) conditions:* $\lim_{k \to 0} F_k(k) = \infty$*, and* $\lim_{k \to \infty} F_k(k) = 0$*.*

3. A Workhorse Dynamic Problem

Feasibility:

- Output y_t split between either current consumption, c_t, or it can be saved or invested, i_t. Feasibility constraint:

  $$F(k_t) = c_t + i_t.$$

  equiv. A national accounting identity for a closed economy.
- Feasibility is state contingent—what can be produced each period depends on the state variable k_t.
- Feasible set for consumption (or alternatively investment) at each k_t is

  $$\Gamma(k_t) := \{c_t \in \mathbb{R}_+ : c_t \leq F(k_t)\}.$$

- Feasibility correspondence:

  $$\Gamma = \bigcup_{k_t \in \mathbb{R}_+} \Gamma(k_t).$$

3. A Workhorse Dynamic Problem

Preferences:

- Preference ordering over consumption sequences: $(c_0, c_1, c_2, ...)$.
- Per-period utility function $U : \mathbb{R}_+ \to \mathbb{R}$.

Assumption

U *is*

- *a continuous function,*
- *strictly increasing and concave,*
- *twice continuously differentiable,*
- $U_c(0) \leq U_c$ *for some* $U_c < \infty$*, and,* $\lim_{c \to \infty} U_c(c) = 0$.

3. A Workhorse Dynamic Problem

Exercise

Show that $|U(c)| \leq K$*, where* $K < \infty$ *is some constant, for all feasible* $c \in \mathbb{R}_+$*. What is the expression for* K*?*

3. A Workhorse Dynamic Problem

Planning problem:

- The First and Second Welfare Theorems hold in this model.
- Focus on equivalent planner's problem.
- Planner's criterion function, evaluated at a particular consumption stream, $\{c_t\}_{t=0}^{\infty} := (c_0, c_1, c_2, ...)$, is

$$U(c_0) + \beta U(c_1) + \beta^2 U(c_1) + ... =: \sum_{t=0}^{\infty} \beta^t U(c_t),$$

  where $\beta \in (0, 1)$ is her *subjective discount factor*, or a model of her degree of impatience.

- Naturally, we can interpret $\beta^{-1} - 1$ as her subjective rate of time preference.

1. Outline

2. Looking Ahead

3. A Workhorse Dynamic Problem
   - Why use it? Economic Questions/Motivations.
   - Takeaways
   - DP idea by example

4. Solving Workhorse Dynamic Problem
   - Infinite-sequence problem
   - Toward a recursive problem
   - Love at first sight?

4. Solving Workhorse Dynamic Problem

Infinite-sequence problem

- Denote $X := [0, \overline{k}]$.
- The interior of the set X is denoted by $\text{int}(X) := \{k \in X : 0 < k < \overline{k}\}$.
- Claim: Given the strict concavity and Inada assumptions on F and U, ...

  ... and an initial $k_0 \in \text{int}(X)$, ...

  ... our solution in terms of optimal path of k_t, $t \geq 1$, will be interior — i.e. $k_t \in \text{int}(X)$ for all $t \in \mathbb{N}$.

4. Solving Workhorse Dynamic Problem

Infinite-sequence problem

Planner's infinite sequence problem, beginning from her initial given state k_0, is:

$$v(k_0) = \max_{\{k_{t+1} \in \Gamma(k_t)\}_{t \in \mathbb{N}}} \left\{ \sum_{t=0}^{\infty} \beta^t U[F(k_t) - k_{t+1}] \right. \\ \left. : k_0 \text{ given} \right\}$$

The left hand side of this problem gives us the value of the optimal problem that begins from a given k_0.

What is an economic name for this object?

4. Solving Workhorse Dynamic Problem

Toward a recursive problem

- Let's specialize the example further:
  - $U(c) = \ln(c)$, and
  - $F(k) = k^\alpha$, where $\alpha \in (0,1)$.
- Why these functional forms?
  - This renders the example soluble by hand.
- Now, this sequence problem is:

$$v(k_0) = \max_{\{k_{t+1} \in \Gamma(k_t)\}_{t \in \mathbb{N}}} \left\{ \sum_{t=0}^{\infty} \beta^t \ln(k_t^\alpha - k_{t+1}) : k_0 \text{ given} \right\}.$$

4. Solving Workhorse Dynamic Problem

Toward a recursive problem

Re-write the above sequence problem in two bits:

$$v(k_0) = \max_{k_1, \{k_{t+1}\}_{t=1}^{\infty}} \left\{ \ln(k_0^{\alpha} - k_1) + \beta \left[\sum_{t=0}^{\infty} \beta^t \ln(k_{t+1}^{\alpha} - k_{t+2}) \right] : \begin{array}{l} \text{(i) } k_0 \text{ given }, \\ \text{(ii) } k_{t+1} \in \Gamma(k_t), \text{ all } t \in \mathbb{N} \end{array} \right\}.$$

4. Solving Workhorse Dynamic Problem

Toward a recursive problem

Remarks:

- The second term on the right in square brackets, heuristically, looks like another infinite sequence program — one that has a value beginning from a given k_1.
- Suppose that indeed k_1 was chosen "consistently" as part of the optimal program beginning from k_0.
- Then the appropriate value function for evaluating the continuation program, beginning at time $t = 1$ at state k_1 would have to be $v(k_1)$, right?

$$v(k_1) = \max_{\{k_{t+1} \in \Gamma(k_t)\}_{t=1}^{\infty}} \left\{ \sum_{t=1}^{\infty} \beta^t \ln(k_t^{\alpha} - k_{t+1}) : k_1 \text{ given} \right\}.$$

4. Solving Workhorse Dynamic Problem

Toward a recursive problem

- If this intuition is correct, then we can re-write the above program beginning at $t = 0$, as

$$v(k_0) = \max_{k_1 \in \Gamma(k_0)} \left\{ \ln(k_0^\alpha - k_1) + \beta v(k_1) : k_0 \text{ given} \right\}.$$

4. Solving Workhorse Dynamic Problem

Toward a recursive problem

- Inductively, write this also for any period $t \in \mathbb{N}$.

  ... Let's dispense with the hoity-toity $t-$ and $(t+1)-$notation.

  .... Let $k := k_t$ and $k_+ := k_{t+1}$.

- Then, from the infinite sequence problem, we could (heuristically) rewrite it as the following *R*ecursive *F*unctional mapping:

$$v(k) = \max_{k_+ \in \Gamma(k)} \left\{ \ln(k^\alpha - k_+) + \beta v(k_+) : k \text{ given} \right\}.$$

Oh Joy!

4. Solving Workhorse Dynamic Problem

Love at first sight?

- At first glance ...
- This seems an easy problem to solve, given that we no longer have to choose infinite list $\{k_{t+1}\}_{t \in \mathbb{N}}$.
- We now just choose a finite object k_+ (i.e. k_{t+1}) as some function, say g_t of the current decision state k (i.e. k_t).
- $k_{t+1} = g_t(k_t)$ is just a solution to the right-hand-side finite-dimensional maximization problem.
- Easy, no?

4. Solving Workhorse Dynamic Problem

Love at first sight?

But we still have a **problem** to deal with. ...

What is the function v?

... We actually don't know what v looks like.