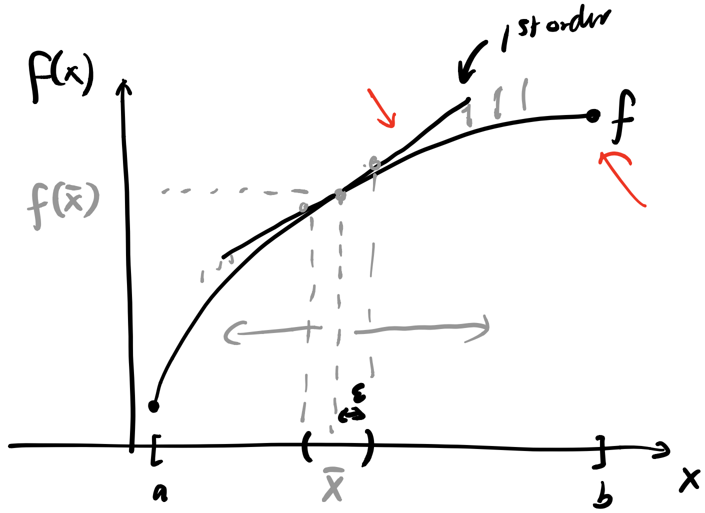
Taylor series expansion – univariate case



$$f(x) = \sum_{k=0}^{\infty} \frac{f^{k}(\bar{x})}{k!}(x-\bar{x})^{k}$$

$$f(x) \approx \underbrace{f(\bar{x})}_{k=0} + \underbrace{f'(\bar{x})(x-\bar{x})}_{k=1} + \cdots$$

1st order Taylor series

$$f(x) - f(\bar{x}) \simeq f'(\bar{x})(x-\bar{x})$$

$$= f'(\bar{x})\frac{(x-\bar{x})}{\bar{x}} \cdot \bar{x}$$

% deviation
from $\bar{x}$

Note: For small ($<\varepsilon$) deviations,

$$\frac{x-\bar{x}}{\bar{x}} \approx \ln\left(\frac{x}{\bar{x}}\right)$$

Multivariate Taylor series

$$f(x,y) \approx f_x(\bar{x},\bar{y})\,\bar{x}\ln\left(x/\bar{x}\right) + f_y(\bar{x},\bar{y})\,\bar{y}\ln\left(y/\bar{y}\right) + \ldots$$

Example:

Say we have a FONC with nonlinear terms/functions:

RBC example – FONC wrt k_{t+1}: ②

A_{t+1}

nonlinear in (c_{t+1}, k_{t+1})

$$\overbrace{u'(c_t)} = \mathbb{E}_t\left\{\beta \overbrace{u'(c_{t+1}) A_{t+1} \tilde{f}'(k_{t+1})}\right\}$$

① nonlinear in c_t

where $\tilde{f}(k)$ is total resource function.

TSE of ①

$$\left.\frac{\partial u'(c_t)}{\partial c_t}\right|_{c_t=\bar{c}} \cdot \underbrace{(c_t - \bar{c})}$$

$$= u''(\bar{c}) \cdot \overbrace{\bar{c} \ln\left(c_t/\bar{c}\right)}$$

TSE of 2

Let

$$h(c_{t+1}, k_{t+1}, A_{t+1}) \equiv \beta u'(c_{t+1}) \tilde{f}'(k_{t+1}) \times A_{t+1}$$

TSE:

$$h(c_{t+1}, k_{t+1}, A_{t+1}) - h(\bar{c}, \bar{k}, \bar{A})$$

$$\simeq h_c(\bar{c}, \bar{k}, \bar{A}) \cdot (c_{t+1} - \bar{c})$$

$$+ h_k(\bar{c}, \bar{k}, \bar{A}) \cdot (k_{t+1} - \bar{k})$$

$$+ h_A(\bar{c}, \bar{k}, \bar{A})(A_{t+1} - \bar{A})$$

$$= \beta \tilde{f}'(\bar{k}) u''(\bar{c}) \cdot \bar{c} \ln\left(\frac{c_{t+1}}{c}\right)$$

$$+ \beta u'(\bar{c}) \tilde{f}''(\bar{k}) \bar{k} \ln\left(\frac{k_{t+1}}{k}\right)$$

$$+ \beta u'(\bar{c}) \tilde{f}'(\bar{k}) \bar{A} \ln\left(\frac{A_{t+1}}{\bar{A}}\right)$$

Now our "log-linearized" Euler eqn is

$$u''(\bar{c})\cdot \bar{c}\,\ln\left(c_t/\bar{c}\right)$$

$$= \beta\, \mathbb{E}_t \Big\{ \tilde{f}'(\bar{k})\, u''(\bar{c})\cdot \bar{c}\,\ln\left(\frac{c_{t+1}}{\bar{c}}\right) + u'(\bar{c})\,\tilde{f}''(\bar{k})\,\bar{k}\,\ln\left(\frac{k_{t+1}}{\bar{k}}\right) + u'(\bar{c})\,\tilde{f}'(\bar{k})\,\bar{A}\,\ln\left(A_{t+1}/\bar{A}\right) \Big\}$$

Divide both sides by $u''(\bar{c})\,\bar{c}$,

$$\ln\left(c_t/\bar{c}\right) = \beta\, \mathbb{E}_t \Big\{ \tilde{f}'(\bar{k})\,\ln\left(\frac{c_{t+1}}{\bar{c}}\right) \left[\frac{u'(\bar{c})}{u''(\bar{c})\,\bar{c}}\right] \Big[\tilde{f}''(\bar{k})\,\bar{k}\,\ln\left(\frac{k_{t+1}}{\bar{k}}\right) + \tilde{f}'(\bar{k})\,\bar{A}\,\ln\left(A_{t+1}/\bar{A}\right) \Big] \Big\} \quad (\star)$$

- If $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ then

$$\sigma = \frac{u''(\bar{c})\,\bar{c}}{u'(\bar{c})}$$

- Also if $\tilde{f}(k) = Ak^{\alpha}$

then $\tilde{f}'(k) = \alpha A k^{\alpha-1}$

$$\tilde{f}''(k) = \alpha(\alpha-1)Ak^{\alpha-2}$$

$$= \frac{(\alpha-1)\tilde{f}'(k)}{k}$$

- Let $\hat{c}_t = \ln\left(c_t/\bar{c}\right)$

$$\hat{k}_t = \ln\left(k_{t+1}/\bar{k}\right)$$

(*) can be rewritten as

$$\hat{c}_t = \beta \mathbb{E}_t \left\{ \left(\alpha A \bar{k}^{\alpha-1} \right) \left[\hat{c}_{t+1} - \frac{1}{\sigma} \left(\hat{a}_{t+1} + (1-\alpha) \hat{k}_{t+1} \right) \right] \right\}$$

Note also in RCE:

$$\underbrace{MPK(\bar{k})}_{\alpha A \bar{k}^{\alpha-1}} = 1/\beta$$

$$\Rightarrow \hat{c}_t = \cancel{\beta} \mathbb{E}_t \left\{ \left(\overset{1/\beta}{\overset{\parallel}{\cancel{\alpha A \bar{k}^{\alpha-1}}}} \right) \left[\hat{c}_{t+1} + \frac{1}{\sigma} \hat{a}_{t+1} - \frac{1}{\sigma} (1-\alpha) \hat{k}_{t+1} \right] \right\}$$

$$\Rightarrow \boxed{\hat{c}_t = \mathbb{E}_t\left\{ \hat{c}_{t+1} + \overbrace{\tfrac{1}{\sigma}}^{a_1}\hat{a}_{t+1} - \overbrace{\tfrac{1}{\sigma}(1-\alpha)}^{a_2}\hat{k}_{t+1} \right\}}$$

Loglinear R.C.

$$\hat{a}_t + \alpha \hat{k}_t = a_2 \hat{c}_t + a_3 \hat{k}_{t+1}$$

Note: $\mathbb{E}_t \hat{a}_{t+1} = \rho \hat{a}_t$

$$\begin{pmatrix} 1 & a_2 \\ 0 & a_3 \end{pmatrix} \begin{pmatrix} \mathbb{E}_t c_{t+1} \\ k_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -a_2 & \alpha \end{pmatrix} \begin{pmatrix} \hat{c}_t \\ \hat{k}_t \end{pmatrix} + \begin{pmatrix} -a_1\rho \\ 1 \end{pmatrix} \hat{a}_t$$

Guess a solution function

$$\hat{c}_t = R\hat{k}_t + S\hat{a}_t$$

$$\hat{k}_{t+1} = P\hat{k}_t + Q\hat{a}_t$$

Find unknown coefficient

(R, S, P, Q)