

# Fourier Transforms and Discrete Fourier Transforms

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The Fourier Transform (FT) of a function can be defined as

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}. \quad (1)$$

One would like to approximate the value of the FT. To this end let us replace the function  $f(t)$  by a bunch of unit impulse functions that “mimic” its behaviour

$$\tilde{f}(t) \approx \sum_{j=0}^{N-1} f(t_j) \delta(t - t_j) \Delta t, \quad (2)$$

where

$$\Delta t = \frac{t_f - t_i}{N} \text{ and } t_j = t_i + j\Delta t. \quad (3)$$

With these definitions one has

$$\int_{t_j - \Delta t/2}^{t_j + \Delta t/2} dt \tilde{f}(t) = f(t_j) \Delta t \approx \int_{t_j - \Delta t/2}^{t_j + \Delta t/2} dt f(t). \quad (4)$$

Note however that  $t_0 = t_i$  but  $t_N = t_i/N + (1 - 1/N)t_f \neq t_f$ . Nevertheless, when  $N \gg 1$  one has  $t_N \rightarrow t_f$ . Note also that if  $N$  is even then  $t_{N/2} = (t_f + t_i)/2$ . Having  $\tilde{f}$  we can use it to approximate  $f$  and to calculate its Fourier transform

$$F(\omega) \approx \int_{-\infty}^{\infty} dt \left( \sum_{j=0}^{N-1} f(t_j) \delta(t - t_j) \Delta t \right) e^{-i\omega t} = \Delta t \sum_{j=0}^{N-1} f(t_j) e^{-i\omega t_j}. \quad (5)$$

Now let us pick  $\omega$  to be a multiple of some (at the moment) undetermined frequency  $\Delta\omega$ ,  $\omega = k\Delta\omega$  and write  $t_j = t_i + j\Delta t$

$$F(k\Delta\omega) \approx \Delta t \sum_{j=0}^{N-1} f(t_j) e^{-ik\Delta\omega(t_i + j\Delta t)} = \Delta t e^{-ik\Delta\omega t_i} \sum_{j=0}^{N-1} f(t_j) e^{-ik\Delta\omega j\Delta t}. \quad (6)$$

It is convenient to pick  $\Delta\omega = 2\pi/(\Delta t N) = 2\pi/(t_f - t_i)$  to get

$$F(k\Delta\omega) \approx \Delta t e^{-ik\Delta\omega t_i} \sum_{j=0}^{N-1} f(t_j) e^{-2\pi i k j / N}. \quad (7)$$

If we define  $x_j = f(t_j)$  then we can identify

$$X_k = \sum_{j=0}^{N-1} x_j e^{-2\pi i k j / N} \equiv \mathcal{F}(\{x_j\}), \quad (8)$$

as the Discrete Fourier transform of the sequence  $x_j$  and write the FT  $F(k\Delta\omega)$  in terms of the DFT

$$F(k\Delta\omega) \approx \Delta t e^{-ik\Delta\omega t_i} X_k. \quad (9)$$

Typically one will pick  $t_i = -t_f$  in which case the prefactor in Eq. (7) becomes

$$e^{-ik\Delta\omega t_i} = e^{-ik \frac{(2\pi)}{(-t_i - t_i)} t_i} = e^{ik\pi} = (-1)^k. \quad (10)$$

Now let us assume that the function  $f(t)$  is real and symmetric,  $f(t) = f(-t)$ . Then, one can easily show that the function  $F(\omega)$  is real. Does the same hold true for the sequences  $x_j$  and  $X_k$ ? Indeed, it does, if one defines a symmetric sequence to satisfy  $x_j = x_{N-j}$  then one can, using the definition of DFT in Eq. (8), show that  $X_k = X_k^*$ . Now, how one should sample  $f(t)$  in such away that the Fourier transform obtained by using the DFT satisfies the type of symmetries mentioned before? It turns out that by sampling as in Eq. (3) with  $t_f = -t_i$  one gets the desired property since

$$\begin{aligned} x_{N-j} &= f(t_i + (N-j)\Delta t) = f(t_i + N\Delta t - j\Delta t) = f(t_i + t_f - t_i - j\Delta t) \\ &= f(t_f - j\Delta t) = f(-t_i - j\Delta t) = f(t_i + j\Delta t) = x_j. \end{aligned} \quad (11)$$

In the last two equalities we used the fact that  $t_i = -t_f$  and that  $f(t) = f(-t)$ .

One final question is how to get rid of the annoying factor  $(-1)^k$  in Eq. (9). We can rewrite it as

$$F(k\Delta\omega) \approx \Delta t e^{-ik\Delta\omega t_i} X_k = \Delta t \sum_{j=0}^{N-1} f(t_j) e^{-2\pi i k j / N} e^{-i\pi k} = \Delta t \sum_{j=0}^{N-1} f(t_j) e^{-2\pi i k (j+N/2) / N}. \quad (12)$$

Now note that the  $j$  indices are only defined modulo  $N$ , if we let  $j \rightarrow j + N$  we get the same DFT since  $e^{2\pi i k (j+N) / N} = e^{2\pi i k j / N}$ , thus we can identify

$$x_{(j+N/2) \bmod N} = f(t_j), \quad (13)$$

and write

$$F(k\Delta\omega) \approx \Delta t \mathcal{F}(\{x_{(j+N/2) \bmod N}\}), \quad (14)$$

which gives directly the FT in terms of the DFT of the (re-arranged) sampled values.