Fourier Transforms and Discrete Fourier Transforms

Nicolás Quesada

August 23, 2016

The Fourier Transform (FT) of a function can be defined as

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-\mathrm{i}\omega t}. \tag{1}$$

One would like to approximate the value of the FT. To this end let us replace the function $f(t)$ by a bunch of unit impulse functions that "mimic" its behaviour

$$\tilde{f}(t) \approx \sum_{j=0}^{N-1} f(t_j)\delta(t - t_j)\Delta t, \tag{2}$$

where

$$\Delta t = \frac{t_f - t_i}{N} \text{ and } t_j = t_i + j\Delta t. \tag{3}$$

With these definitions one has

$$\int_{t_j - \Delta t/2}^{t_j + \Delta t/2} dt \tilde{f}(t) = f(t_j)\Delta t \approx \int_{t_j - \Delta t/2}^{t_j + \Delta t/2} dt f(t). \tag{4}$$

Note however that $t_0 = x_i$ but $t_N = t_i/N + (1 - 1/N)t_f \neq t_f$. Nevertheless, when $N \gg 1$ one has $t_N \to t_f$. Note also that if N is even then $t_{N/2} = (t_f + t_i)/2$. Having $\tilde{f}$ we can use it to approximate f and to calculate its Fourier transform

$$F(\omega) \approx \int_{-\infty}^{\infty} dt \left(\sum_{j=0}^{N-1} f(t_j)\delta(t - t_j)\Delta t \right) e^{-\mathrm{i}\omega t} = \Delta t \sum_{j=0}^{N-1} f(t_j) e^{-\mathrm{i}\omega t_j}. \tag{5}$$

Now let us pick ω to be a multiple of some (at the moment) undetermined frequency ω_0, $\omega = k\omega_0$ and write $t_j = t_i + j\Delta t$

$$F(k\omega_0) \approx \Delta t \sum_{j=0}^{N-1} f(t_j) e^{-\mathrm{i}k\omega_0 (t_i + j\Delta t)} = \Delta t e^{-\mathrm{i}k\omega_0 t_i} \sum_{j=0}^{N-1} f(t_j) e^{-\mathrm{i}k\omega_0 j \Delta t}. \tag{6}$$

It is convenient to pick $\omega_0 = 2\pi/(\Delta t N) = 2\pi/(t_f - t_i)$ to get

$$F(k\omega_0) \approx \Delta t e^{-\mathrm{i}k\omega_0 t_i} \sum_{j=0}^{N-1} f(t_j) e^{-2\pi \mathrm{i} kj/N}. \tag{7}$$

If we define $x_j = f(t_j)$ then we can identify

$$X_k = \sum_{j=0}^{N-1} x_j e^{-2\pi \mathrm{i} kj/N} \equiv \mathcal{F}\left(\{x_j\}\right), \tag{8}$$

as the Discrete Fourier transform of the sequence x_j and write the FT $F(k\omega_0)$ in terms of the DFT

$$F(k\omega_0) \approx \Delta t e^{-\mathrm{i}k\omega_0 t_i} X_k. \tag{9}$$

Typically one will pick $t_i = -t_f$ in which case the prefactor in Eq. (7) becomes

$$e^{-\mathrm{i}k\omega_0 t_i} = e^{-\mathrm{i}k\frac{(2\pi)}{(-t_i - t_i)}t_i} = e^{\mathrm{i}k\pi} = (-1)^k. \tag{10}$$

Now let us assume that the function $f(t)$ is real and symmetric, $f(t) = f(-t)$. Then, one can easily show that the function $F(\omega)$ is real. Does the same hold true for the sequences x_j and X_k? Indeed, it does, if one defines a symmetric sequence to satisfy $x_j = x_{N-j}$ then one can, using the definition of DFT in Eq. (8), show that $X_k = X_k^*$. Now, how one should sample $f(t)$ in such away that the Fourier transform obtained by using the DFT satisfies the type of symmetries mentioned before? It turns out that by sampling as in Eq. (3) with $t_f = -t_i$ one gets the desired property since

$$\begin{aligned} x_{N-j} &= f(t_i + (N-j)\Delta t) = f(t_i + N\Delta t - j\Delta t) = f(t_i + t_f - t_i - j\Delta t) \\ &= f(t_f - j\Delta t) = f(-t_i - j\Delta t) = f(t_i + j\Delta t) = x_j. \end{aligned} \tag{11}$$

In the last two equalities we used the fact that $t_i = -t_f$ and that $f(t) = f(-t)$.

One final question is how to get rid of the annoying factor $(-1)^k$ in Eq. (9). We can rewrite it as

$$F(k\omega_0) \approx \Delta t e^{-\mathrm{i}k\omega_0 t_i} X_k = \Delta t \sum_{j=0}^{N-1} f(t_j) e^{-2\pi \mathrm{i} kj/N} e^{-\mathrm{i}\pi k} = \Delta t \sum_{j=0}^{N-1} f(t_j) e^{-2\pi \mathrm{i} k(j+N/2)/N}. \tag{12}$$

Now note that the j indices are only defined modulo N, if we let $j \to j + N$ we get the same DFT since $e^{2\pi \mathrm{i} k(j+N)/N} = e^{2\pi \mathrm{i} kj/N}$, thus we can identify

$$x_{(j+N/2) \bmod N} = f(t_j), \tag{13}$$

and write

$$F(k\omega_0) \approx \Delta t \mathcal{F}\left(\{x_{(j+N/2) \bmod N}\}\right), \tag{14}$$

which gives directly the FT in terms of the DFT of the (re-arranged) sampled values.