# An identity involving the imaginary error function erfi($x$)

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The imaginary error function erfi($x$) is costumarily defined as $\mathrm{erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt\, e^{t^2}$. It is related to the error function by erfi($x$)= erf(i$x$)/i, and can be also defined as

$$\frac{1}{\pi} \fint_{-\infty}^{\infty} dy\, \frac{e^{ay^2+2by}}{y} = \frac{2}{\pi} \int_0^{\infty} dy\, e^{ay^2} \frac{\sinh(2by)}{y} = \mathrm{erfi}\left(\frac{b}{\sqrt{-a}}\right). \tag{1}$$

for $a < 0$. In this note we investigate an interesting property of the imaginary error function.

We can define two functions

$$\phi_0(x) \equiv \frac{\exp\left(-x^2\right)}{\sqrt[4]{\pi/2}}, \quad \phi_1(x) \equiv \sqrt{3}\phi_0\left(x\right) \mathrm{erfi}\left(\sqrt{\frac{2}{3}}x\right), \tag{2}$$

whose $\mathcal{L}^2$-orthogonality is seen from their respective parities.

We here show that $\phi_1$ is also normalized; to this end consider the integral

$$I = \int_{-\infty}^{\infty} dx\ \phi^2\left(x\right) = 3\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} dx \exp\left(-2x^2\right) \mathrm{erfi}^2\left(\sqrt{\frac{2}{3}}x\right). \tag{3}$$

Note that this integral is, to the best of our not knowledge, not currently tabulated in standard computer algebra systems (CAS) like Mathematica or Maple or the Wolfram Functions Site. We use Eq. (1) to write

$$I = \frac{3}{\pi^2}\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} dx \exp\left(-2x^2\right) \fint_{-\infty}^{\infty} \frac{du}{u} \exp\left(-\frac{3}{2}u^2 + 2ux\right) \fint_{-\infty}^{\infty} \frac{dv}{v} \exp\left(-\frac{3}{2}v^2 + 2vx\right) \tag{4a}$$

$$= \frac{3}{\pi^2}\sqrt{\frac{2}{\pi}} \fint_{-\infty}^{\infty} \frac{du}{u} \fint_{-\infty}^{\infty} \frac{dv}{v} \exp\left(-\frac{3}{2}u^2 - \frac{3}{2}v^2\right) \int_{-\infty}^{\infty} dx \exp\left(-2x^2 + 2ux + 2vx\right) \tag{4b}$$

$$= \frac{3}{\pi^2} \fint_{-\infty}^{\infty} \frac{du}{u} \exp\left(-u^2\right) \fint_{-\infty}^{\infty} \frac{dv}{v} \exp\left(-v^2 + uv\right), \tag{4c}$$

where we have liberally switched the order of the integrals. We now again use Eq. (1) to find

$$I = \frac{3}{\pi^2} \fint_{-\infty}^{\infty} \frac{du}{u} \exp\left(-u^2\right) \left(\pi \mathrm{erfi}\left(\frac{u}{2}\right)\right) = \frac{6}{\pi} \underbrace{\int_0^{\infty} \frac{du}{u} \exp\left(-u^2\right) \mathrm{erfi}\left(\frac{u}{2}\right)}_{=\pi/6} = 1. \tag{5}$$

The last integral can be evaluated using any of the aforementioned CAS. The last equation shows that the set $\{\phi_0, \phi_1\}$ forms an (incomplete) ortho*normal* set in $\mathcal{L}^2$.