Fourier Transforms and Discrete Fourier Transforms

Nicolás Quesada

August 29, 2016

The Fourier Transform (FT) of a function can be defined as

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-\mathrm{i}\omega t}. \tag{1}$$

We can approximate this integral by a Riemann sum

$$F(\omega) \approx \int_{-\infty}^{\infty} dt \left(\sum_{j=0}^{N-1} f(t_j)\delta(t-t_j)\Delta t \right) e^{-\mathrm{i}\omega t} = \Delta t \sum_{j=0}^{N-1} f(t_j)e^{-\mathrm{i}\omega t_j}. \tag{2}$$

where we use the following definitions

$$\Delta t = \frac{t_f - t_i}{N} \text{ and } t_j = t_i + j\Delta t. \tag{3}$$

Note however that $t_0 = t_i$ but $t_N = t_i/N + (1 - 1/N)t_f \neq t_f$. Nevertheless, when $N \gg 1$ one has $t_N \to t_f$. Note also that if N is even then $t_{N/2} = (t_f + t_i)/2$.

Now let us pick ω to be a multiple of some (at the moment) undetermined frequency $\Delta\omega$, $\omega = k\Delta\omega$ and write $t_j = t_i + j\Delta t$

$$F(k\Delta\omega) \approx \Delta t \sum_{j=0}^{N-1} f(t_j)e^{-\mathrm{i}k\Delta\omega(t_i + j\Delta t)} = \Delta t e^{-\mathrm{i}k\Delta\omega t_i} \sum_{j=0}^{N-1} f(t_j)e^{-\mathrm{i}kj\Delta\omega\Delta t}. \tag{4}$$

It is convenient to pick $\Delta\omega = 2\pi/(\Delta t N) = 2\pi/(t_f - t_i)$ to get

$$F(k\Delta\omega) \approx \Delta t e^{-\mathrm{i}k\Delta\omega t_i} \sum_{j=0}^{N-1} f(t_j)e^{-2\pi\mathrm{i}kj/N}. \tag{5}$$

If we define $x_j = f(t_j)$ then we can identify

$$X_k = \sum_{j=0}^{N-1} x_j e^{-2\pi\mathrm{i}kj/N} \equiv \mathcal{F}\left(\{x_j\}\right), \tag{6}$$

as the Discrete Fourier transform of the sequence x_j and write the FT $F(k\Delta\omega)$ in terms of the DFT

$$F(k\Delta\omega) \approx \Delta t e^{-\mathrm{i}k\Delta\omega t_i} X_k. \tag{7}$$

Typically one will pick $t_i = -t_f$ in which case the prefactor in Eq. (5) becomes

$$e^{-\mathrm{i}k\Delta\omega t_i} = e^{-\mathrm{i}k\frac{(2\pi)}{(-t_i-t_i)}t_i} = e^{\mathrm{i}k\pi} = (-1)^k. \tag{8}$$

Now let us assume that the function $f(t)$ is real and symmetric, $f(t) = f(-t)$. Then, one can easily show that the function $F(\omega)$ is real. Does the same hold true for the sequences x_j and X_k? Indeed, it does, if one defines a symmetric sequence to satisfy $x_j = x_{N-j}$ then one can, using the definition of DFT in Eq. (6), show that $X_k = X_k^*$. Now, how one should sample $f(t)$ in such away that the Fourier transform obtained by using the DFT satisfies the type of symmetries mentioned before? It turns out that by sampling as in Eq. (3) with $t_f = -t_i$ one gets the desired property since

$$\begin{aligned} x_{N-j} &= f(t_i + (N-j)\Delta t) = f(t_i + N\Delta t - j\Delta t) = f(t_i + t_f - t_i - j\Delta t) \\ &= f(t_f - j\Delta t) = f(-t_i - j\Delta t) = f(t_i + j\Delta t) = x_j. \end{aligned} \tag{9}$$

In the last two equalities we used the fact that $t_i = -t_f$ and that $f(t) = f(-t)$.

One final question is how to get rid of the annoying factor $(-1)^k$ in Eq. (7). We can rewrite it as

$$F(k\Delta\omega) \approx \Delta t e^{-\mathrm{i}k\Delta\omega t_i} X_k = \Delta t \sum_{j=0}^{N-1} f(t_j) e^{-2\pi\mathrm{i}kj/N} e^{-\mathrm{i}\pi k} = \Delta t \sum_{j=0}^{N-1} f(t_j) e^{-2\pi\mathrm{i}k(j+N/2)/N}. \tag{10}$$

Now note that the j indices are only defined modulo N, if we let $j \to j+N$ we get the same DFT since $e^{2\pi\mathrm{i}k(j+N)/N} = e^{2\pi\mathrm{i}kj/N}$, thus we can identify

$$x_{(j+N/2)\mathrm{mod}N} = f(t_j), \tag{11}$$

and write

$$F(k\Delta\omega) \approx \Delta t \mathcal{F}\left(\{x_{(j+N/2)\mathrm{mod}N}\}\right), \tag{12}$$

which gives directly the FT in terms of the DFT of the (re-arranged) sampled values.