

# An identity involving the imaginary error function $\operatorname{erfi}(x)$

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The imaginary error function  $\operatorname{erfi}(x)$  is costumarily defined as  $\operatorname{erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt e^{t^2}$ . It is related to the error function by  $\operatorname{erfi}(x) = \operatorname{erf}(ix)/i$ , and can be also defined as

$$\frac{1}{\pi} \int_{-\infty}^{\infty} dy \frac{e^{ay^2+2by}}{y} = \frac{2}{\pi} \int_0^{\infty} dy e^{ay^2} \frac{\sinh(2by)}{y} = \operatorname{erfi}\left(\frac{b}{\sqrt{-a}}\right). \quad (1)$$

for  $a < 0$ . In this note we investigate a property of a function related to the imaginary error function.

We can define two functions

$$\phi_0(x) \equiv \frac{\exp(-x^2)}{\sqrt[4]{\pi/2}}, \quad \phi_1(x) \equiv \sqrt{3}\phi_0(x) \operatorname{erfi}\left(\sqrt{\frac{2}{3}}x\right), \quad (2)$$

whose  $\mathcal{L}^2$ -orthogonality is seen from their respective parities.

We here show that  $\phi_1$  is, like  $\phi_0$ ,  $\mathcal{L}^2$ -normalized; to this end consider the integral

$$I = \int_{-\infty}^{\infty} dx \phi_1^2(x) = 3\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} dx \exp(-2x^2) \operatorname{erfi}^2\left(\sqrt{\frac{2}{3}}x\right). \quad (3)$$

Note that this integral is, to the best of our knowledge, not currently tabulated in standard computer algebra systems (CAS) like Mathematica or Maple or the Wolfram Functions Site. We use Eq. (1) to write

$$I = \frac{3}{\pi^2} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} dx \exp(-2x^2) \int_{-\infty}^{\infty} \frac{du}{u} \exp\left(-\frac{3}{2}u^2 + 2ux\right) \int_{-\infty}^{\infty} \frac{dv}{v} \exp\left(-\frac{3}{2}v^2 + 2vx\right) \quad (4a)$$

$$= \frac{3}{\pi^2} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{du}{u} \int_{-\infty}^{\infty} \frac{dv}{v} \exp\left(-\frac{3}{2}u^2 - \frac{3}{2}v^2\right) \int_{-\infty}^{\infty} dx \exp(-2x^2 + 2ux + 2vx) \quad (4b)$$

$$= \frac{3}{\pi^2} \int_{-\infty}^{\infty} \frac{du}{u} \exp(-u^2) \int_{-\infty}^{\infty} \frac{dv}{v} \exp(-v^2 + uv), \quad (4c)$$

where we have liberally switched the order of the integrals. We now again use Eq. (1) to find

$$I = \frac{3}{\pi^2} \int_{-\infty}^{\infty} \frac{du}{u} \exp(-u^2) \underbrace{\left(\pi \operatorname{erfi}\left(\frac{u}{2}\right)\right)}_{=\pi/6} = \frac{6}{\pi} \int_0^{\infty} \frac{du}{u} \exp(-u^2) \operatorname{erfi}\left(\frac{u}{2}\right) = 1. \quad (5)$$

The last integral can be evaluated using any of the aforementioned CAS. The last equation shows that the set  $\{\phi_0, \phi_1\}$  forms an (incomplete) orthonormal set in  $\mathcal{L}^2$ .