Useful Identities in Quantum Optics

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Notation

Latin letters with caret $(\hat{a}, \hat{x}, \hat{y}, \hat{z})$ denote linear operators.
Greek letters (α, λ) denote $\mathbb{C}$ numbers.
The hermitian adjoint of $\hat{a}$ is denoted $\hat{a}^\dagger$.
The commutator of two operators $\hat{x}$ and $\hat{y}$ is defined as: $[\hat{x}, \hat{y}] := \hat{x}\hat{y} - \hat{y}\hat{x}$

Leibniz's law

$$[\hat{x}, \hat{y}\hat{z}] = [\hat{x}, \hat{y}]\hat{z} + \hat{y}[\hat{x}, \hat{z}]$$
$$[\hat{x}\hat{y}, \hat{z}] = [\hat{x}, \hat{z}]\hat{y} + \hat{x}[\hat{y}, \hat{z}]$$

Jacobi Identity

$$[\hat{x}, [\hat{y}, \hat{z}]] + [\hat{y}, [\hat{z}, \hat{x}]] + [\hat{z}, [\hat{x}, \hat{y}]] = 0$$

Baker-Campbell-Hausdorff formula

If $[\hat{x}, [\hat{x}, \hat{y}]] = [\hat{y}, [\hat{x}, \hat{y}]] = 0$ then:

$$e^{\hat{x}+\hat{y}} = \exp\left(-\frac{1}{2}[\hat{x}, \hat{y}]\right) e^{\hat{x}} e^{\hat{y}} = \exp\left(\frac{1}{2}[\hat{x}, \hat{y}]\right) e^{\hat{y}} e^{\hat{x}}$$

Hadamard lemma

$$e^{i\lambda\hat{x}}\hat{y}e^{-i\lambda\hat{x}} = \hat{y} + i\lambda[\hat{x}, \hat{y}] + \frac{(i\lambda)^2}{2!}[\hat{x}, [\hat{x}, \hat{y}]] + \ldots$$

Identities for Bosonic Operators

The bosonic operators $\hat{a}$ and $\hat{a}^\dagger$ satisfy $[\hat{a}, \hat{a}^\dagger] = \hat{1}$

$$[\hat{a}, f(\hat{a}, \hat{a}^\dagger)] = \frac{\partial f}{\partial \hat{a}^\dagger}$$
$$[\hat{a}^\dagger, f(\hat{a}, \hat{a}^\dagger)] = -\frac{\partial f}{\partial \hat{a}}$$
$$e^{-\alpha\hat{a}^\dagger\hat{a}} f(\hat{a}, \hat{a}^\dagger) e^{\alpha\hat{a}^\dagger\hat{a}} = f(\hat{a}e^{\alpha}, \hat{a}^\dagger e^{-\alpha})$$

References

[1] J.J. Sakurai, *Modern Quantum Mechanics*, Addison Wesley Longman

[2] M.O. Scully, M.S. Zubairy *Quantum Optics*, Cambridge University Press.