Problem 21

Prove the following lemma:

Lemma:
Let $p(x) = a_0x^k + a_1x^{k-1} + \ldots + a_k$ and $g(n)$ be any discrete function. Then:

$$P(E)\left(b^n g(n)\right) = b^n P(E)g(n)$$

Attempt:

We know from the lecture notes that $P(E)$ is defined as:

$$P(E) = a_0E^k + a_1E^{k-1} + \ldots a_kI$$

and that:

$$P(E)b^n = \left(a_0b^k + a_1b^{k-1} + \ldots + a_k\right)b^n = p(b)b^n$$

Then we know the following will occur:

$$\begin{aligned}
P(E)(b^ng(n)) &= \left(a_0E^k + a_1E^{k-1} + \ldots a_kI\right)b^ng(n) \\
&= a_0E^kb^ng(n) + a_1E^{k-1}b^ng(n) + \ldots + a_kIb^ng(n) \\
&= a_0b^{n+k}g(n+k) + a_1b^{n+k-1}g(n+k-1) + \ldots + a_kb^ng(n) \\
&= b^n(a_0b^kg(n+k) + a_1b^{k-1}g(n+k-1) + \ldots + a_kg(n))
\end{aligned}$$

from our definition of $p(x)$ we know that the right side of the final line above (the part in the parenthesis), is given by:

$$P(bE)g(n) = a_0b^kE^kg(n+k) + a_1b^{k-1}e^{k-1}g(n+k-1) + \ldots + a_kg(n)$$

(just plug it into the given definition of $p(x)$). So this gives us:

$$P(E)(b^ng(n)) = b^nP(bE)g(n)$$

Problem 22

Prove parts (ii) and (iii) of the following lemma:

Lemma:
For a fixed $k \in \mathbb{Z}^+$ and $x \in \mathbb{R}$, the following statements hold:

1. $\Delta x^{(k)} = kx^{(k-1)}$
2. $\Delta^n x^{(k)} = k(k-1)\cdots(k-n+1)x^{(k-n)}$
3. $\Delta^k x^{(k)} = k!$

Attempt:

We know that $x^{(k)}$ is defined as follows:

$$x^{(k)} = x(x-1)(x-2)\cdots(x-k+1),\ k \in \mathbb{Z}^+$$

and from the lecture notes that:

$$\begin{aligned}
\Delta x^{(k)} &= (x+1)^{(k)} - x^{(k)} \\
&= (x+1)(x)(x-1)\dots(x-k+2) - x(x-1)\dots(x-k+2)(x-k+1) \\
&= (x)(x-1)\dots(x-k+2)[x+1-(x-k+1)] \\
&= x(x-1)\dots(x-k+2)k \\
&= kx^{(k-1)}
\end{aligned}$$

(i) If we express the above as $\Delta^1 x^{(k)}$, then we can do the following:

$$\begin{aligned}
\Delta\left[\Delta x^{(k)}\right] &= \Delta\left[kx^{(k-1)}\right] \\
&= k\Delta x^{(k-1)} \\
&= k\left[(x+1)^{(k-1)} - x^{(k-1)}\right] \\
&= k
\end{aligned}$$

Problem 23

Prove the Quotient rule:

$$\Delta\left[\frac{x(n)}{y(n)}\right] = \frac{y(n)\Delta x(n) - x(n)\Delta y(n)}{y(n)Ey(n)}$$

Attempt:

(a) we know that $g(x) = f(f(x))$, so we must put $2x$ into each, and $2(1-x)$ into each, and adjust the initial intervals. This gives us:

$$g(x) = \begin{cases} 2(2(x)), & 0 \leq x < \frac{1}{4} \\ 2(1-(2x)) & \frac{1}{4} \leq x < \frac{1}{2} \\ 2(1-2(1-x))) & \frac{1}{2} \leq x < \frac{3}{4} \\ 2(2(1-x)) & \frac{3}{4} \leq x \leq 1 \end{cases}$$

(b) Our fixed points are:

$$\begin{cases} 4x^* - x^* = 0, & 0 \leq x < \frac{1}{4} \\ 2 - 4x^* - x^* = 0 & \frac{1}{4} \leq x < \frac{1}{2} \\ 4x^* - 2 - x^* = 0 & \frac{1}{2} \leq x < \frac{3}{4} \\ 4 - 4x^* - x^* = 0 & \frac{3}{4} \leq x \leq 1 \end{cases}$$

$$\begin{cases} x^* = 0, & 0 \leq x < \frac{1}{4} \\ x^* = \frac{2}{5} & \frac{1}{4} \leq x < \frac{1}{2} \\ x^* = \frac{2}{3} & \frac{1}{2} \leq x < \frac{3}{4} \\ x^* = \frac{4}{5} & \frac{3}{4} \leq x \leq 1 \end{cases}$$

(c) The two cycles are .8, .4.

Listing 1: 2 values

```matlab
%% Problem 13c
clear all
%% Initial conditions
x(1) = .9; %x(0)

%% loop
rows = 8;
cols = 2;
n=rows*cols-1;
for i=1:n
```

```
        x(i+1)=prob13c(x(i));
end
x=reshape(x,cols,rows)'

%% Functions
function output = prob13c(x)
    if x<=.5
        output = 2*x;
    else
        output = 2*(1-x);
    end
end
```

Problem 24

Prove the following lemma:

Lemma:
Let $X_1(n)$ and $X_2(n)$ be two solutions of:

$$x(k+n) + P_1(n)x(n+k-1) + \cdots + p_k(n)x(n) = 0 \tag{1}$$

Then the following statements are true:

1. $x(n) = x_1(n) + x_2(n)$ is a solution of (1)
2. $\tilde{x}(n) = ax_1(n)$ is a solution of (1) for any constant a

Attempt:

We can apply the following Theorem:

Theorem 1.21:
Let $O(b) = \{b = x(0), x(1), \ldots, x(k-1)\}$ be a k-cycle of a continuously differentiable function f. Then the following statements hold:

(i) The k-cycle $O(b)$ is asymptotically stable if:

$$\left|f'\big(x(0)\big)f'\big(x(1)\big)\ldots f'\big(x(k-1)\big)\right| < 1$$

(ii) The k-cycle $O(b)$ is asymptotically unstable if:

$$\left|f'\big(x(0)\big)f'\big(x(1)\big)\ldots f'\big(x(k-1)\big)\right| > 1$$

Then

$$|f'(.4)f'(.8)| = |2 * (-2)| = 4 > 1$$

So this is asymptotically unstable.

Problem 25

Find the Casoratian of the following functions:

1. $(-2)^n, 2^n, 3$
2. $0, 2^n, 3$
3. $2^n, 3, 2^{n+2}$

Attempt:

(a) we know that $g(x) = f(f(x))$, so we must put $2x$ into each, and $2x - 1$ into each, and adjust the initial intervals. This gives us:

$$g(x) = \begin{cases} 2(2(x)), & 0 \leq x < \frac{1}{4} \\ 2(2x) - 1 & \frac{1}{4} \leq x < \frac{1}{2} \\ 2(2x-1) & \frac{1}{2} \leq x < \frac{3}{4} \\ 2(2x-1) - 1 & \frac{3}{4} \leq x \leq 1 \end{cases}$$

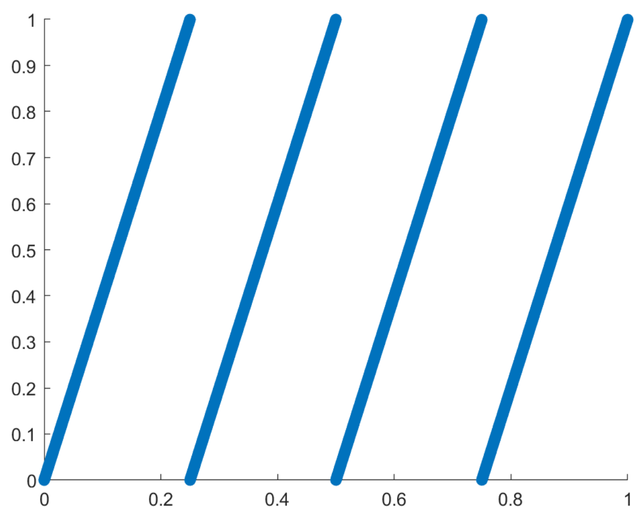

Figure 1: plot of B^2

(b) Our fixed points are:

$$\begin{cases} 4x^* - x^* = 0, & 0 \leq x < \frac{1}{4} \\ 4x^* - 1 - x^* = 0 & \frac{1}{4} \leq x < \frac{1}{2} \\ 4x^* - 2 - x^* = 0 & \frac{1}{2} \leq x < \frac{3}{4} \\ 4x^* - 3 - x^* = 0 & \frac{3}{4} \leq x \leq 1 \end{cases}$$

$$\begin{cases} x^* = 0, & 0 \le x < \frac{1}{4} \\ x^* = \frac{1}{3} & \frac{1}{4} \le x < \frac{1}{2} \\ x^* = \frac{2}{3} & \frac{1}{2} \le x < \frac{3}{4} \\ x^* = 1 & \frac{3}{4} \le x \le 1 \end{cases}$$

(c) Since $x^* = 0, 1$ are solutions of $B(x)$, they cannot be solutions, therefore the 2-cycle is given by $\frac{1}{3}, \frac{2}{3}$
We can see these plotted here:

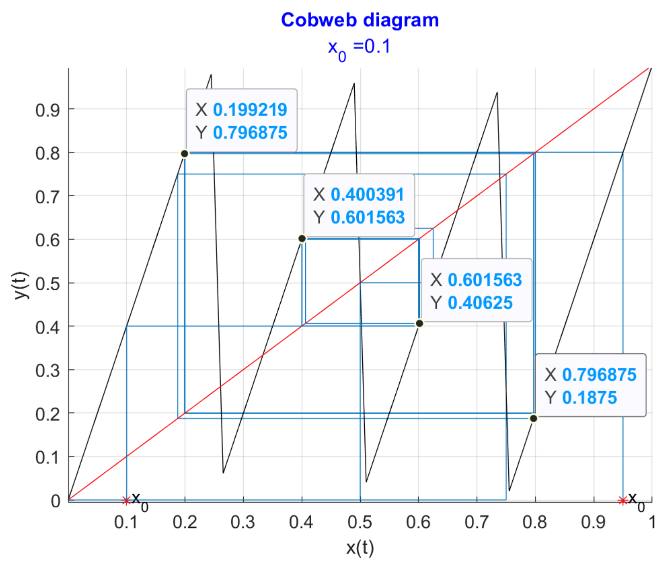

Figure 2: An image of the cycles (code included at the end)

Problem 16

what is the stability of the two cycle for $\mu = 1 + \sqrt{6}$? Justify your answer.
Attempt:

$F_\mu(x) = \mu x(1-x)$, and so

$$\begin{aligned} [F_\mu^2(x(0))]' &= F_\mu'(x(0))F_\mu'(x(1)) \\ &= \mu(1-2x(0))\mu(1-2x(1)) \\ &= \mu^2(1-2x(0))(1-2x(1)) \end{aligned}$$

If:

$$x(0) = \frac{\left[(1+\mu) - \sqrt{(\mu-3)(\mu+1)}\right]}{2\mu}$$

$$x(1) = \frac{\left[(1+\mu) + \sqrt{(\mu-3)(\mu+1)}\right]}{2\mu}$$

We can write this as:

$$\mu^2 \left(1 - 2\frac{\left[(1+\mu) - \sqrt{(\mu-3)(\mu+1)}\right]}{2\mu}\right) \left(1 - 2\frac{\left[(1+\mu) + \sqrt{(\mu-3)(\mu+1)}\right]}{2\mu}\right)$$

simplifying further:

$$\mu^2 \left(\frac{\mu - \left[(1+\mu) - \sqrt{(\mu-3)(\mu+1)}\right]}{\mu}\right) \left(\frac{\mu - \left[(1+\mu) + \sqrt{(\mu-3)(\mu+1)}\right]}{\mu}\right)$$

$$\left(\mu - \left[(1+\mu) - \sqrt{(\mu-3)(\mu+1)}\right]\right) \left(\mu - \left[(1+\mu) + \sqrt{(\mu-3)(\mu+1)}\right]\right)$$

$$\left(\left[-1 + \sqrt{(\mu-3)(\mu+1)}\right]\right) \left(\left[-1 - \sqrt{(\mu-3)(\mu+1)}\right]\right)$$

$$([1 - (\mu-3)(\mu+1)])$$

$$\left(\left[1 - \mu^2 + 2\mu + 3\right]\right)$$

$$\left(\left[4 - (1+\sqrt{6})^2 + 2(1+\sqrt{6})\right]\right)$$

$$\left(\left[4 - 1 - 2\sqrt{6} - 6 + 2 + 2\sqrt{6}) = -1\right]\right)$$

To assess the stability let's first try (**Thm 1.16**) and check the first derivative of our function:

$$\begin{aligned} F''(x) &= [F'(x(0))F'(x(1))]' \\ &= F''(x(0))F'(x(1)) + F'(x(0))F''(x(1)) \\ &= -2\mu(F'(x(1)) + F'(x(0))) \end{aligned}$$

$$\begin{aligned} F'''(x) &= [F'(x(0))F'(x(1))]'' \\ &= [F''(x(0))F'(x(1)) + F'(x(0))F''(x(1))]' \\ &= [-2\mu F'(x(1)) - 2\mu F'(x(0))]' \\ &= 4\mu^2 + 4\mu^2 \end{aligned}$$

$$Sf = -f'''(x^*) - \frac{3}{2}(f''(x^*))^2 = -8\mu^2 - \frac{3}{2}(-2\mu(-1))^2$$

$$= -8\mu^2 - 3\mu^2 = -11(7 + 2\sqrt{6}) < 0$$

so we know that this is asymptotically stable.

Problem 17

Iterate $x(n+1) = F_\mu\big(x(u)\big)$ for a large number of iterations. Plot the last portion of the sequence $\{x(n)\}$ as a function of μ to get a bifurcation diagram of F_μ

Attempt:

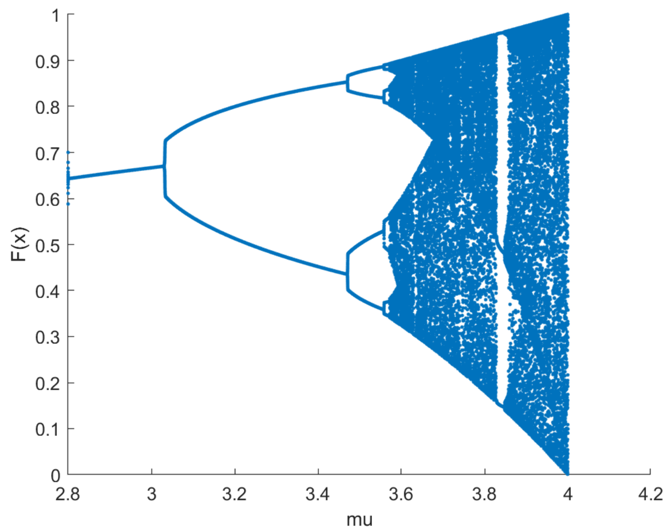

Figure 3: Bifurcation diagram

Listing 2: bifurcation

```
%% Problem 17
clear all
%% Initial conditions
x(1) =.7; %x(0)

%% loop
rows = 70000;
cols = 1;
n=rows*cols-1;
xplot=linspace(2.8,4,n+1);
for i=1:n
        x(i+1)=logistic(xplot(i),x(i));
end
x=reshape(x,cols,rows)'

scatter(xplot,x,'.')
xlabel('mu')
ylabel('F(x)')
```

```
%% Functions
function output = logistic(mu,x)

    output = mu*x-mu*x*x;
end
```

Problem 18

Show that:

$$E^k x(n) = \sum_{i=0}^{k} \binom{k}{i} \Delta^{k-i} x(n)$$

Attempt:

From before we know that $\Delta = E - I$, so using the previous proof:

$$\begin{aligned}\Delta^k x(n) &= (E-I)^k x(n) \\ &= \sum_{i=0}^{k} (-1)^i \binom{k}{i} E^{k-i} x(n) \\ &= \sum_{i=0}^{k} (-1)^i \binom{k}{i} x(n+k-i)\end{aligned}$$

Since $E = \Delta + I$, and $E^k = (\Delta + I)^k$, and $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$:

$$\begin{aligned}E^k &= (\Delta + I)^k \\ &= \sum_{i=0}^{k} \binom{k}{i} \Delta^{k-i} (I)^i\end{aligned}$$

Then:

$$E^k x(n) = \sum_{i=0}^{k} \binom{k}{i} \Delta^{k-i} (I)^i x(n)$$

$$E^k x(n) = \sum_{i=0}^{k} \binom{k}{i} \Delta^{k-i} (I) x(n)$$

$$E^k x(n) = \sum_{i=0}^{k} \binom{k}{i} \Delta^{k-i} x(n)$$

Problem 19

Show that E and Δ are linear operators. That is, show that for all constants a and b:

$$\Delta[ax(n) + by(n)] = a\Delta x(n) + b\Delta y(n)$$
$$\text{and}$$
$$E[ax(n) + by(n)] = aEx(n) + bEy(n)$$

Attempt:

$$\begin{aligned} a\Delta[x(n)] + b\Delta[y(n)] &= a[x(n+1) - x(n)] + b[y(n+1) - y(n)] \\ &= [ax(n+1) - ax(n)] + [by(n+1) - by(n)] \\ &= [ax(n+1) + by(n+1)] - [ax(n) + by(n)] \\ &= \Delta[ax(n) + by(n)] \end{aligned}$$

$$\begin{aligned} E[ax(n) + by(n)] &= [ax(n+1) + by(n+1)] \\ &= a[x(n+1)] + b[y(n+1)] \\ &= aE[x(n)] + bE[y(n)] \end{aligned}$$

Problem 20

Prove that the following statements hold:

(i) $\displaystyle\sum_{i=n_0}^{n-1} \Delta x(n) = x(n) - x(n_0)$

(ii) $\displaystyle\Delta\left(\sum_{k=n_0}^{n-1} x(k)\right) = x(n)$

Attempt:

(i)

$$\begin{aligned} \sum_{i=n_0}^{n-1} \Delta x(n) &= \sum_{i=n_0}^{n-1} x(i+1) - x(i) \\ &= \sum_{i=n_0}^{n-1} -x(i) + x(i+1) \\ &= [-x(n_0) + x(n_0+1)] + [-x(n_0+1) + x(n_0+2)] + \dots \\ &+ [-x(n-2) + x(n-2+1)] + [-x(n-1) + x(n-1+1)] \end{aligned}$$

But it is easy to observe that the terms between cancel:

$$\begin{aligned}
&= -x(n_0) + [x(n_0+1) - x(n_0+1)] + [x(n_0+2) - x(n_0+2)] \dots \\
&[x(n-2) - x(n-2)] + [x(n-2+1) + -x(n-1)] + x(n-1+1) \\
&= -x(n_0) + x(n)
\end{aligned}$$

So we get: $\displaystyle\sum_{i=n_0}^{n-1} \Delta x(n) = x(n) - x(n_0)$

(ii)

$$\begin{aligned}
\Delta\left(\sum_{k=n_0}^{n-1} x(k)\right) &= \\
&= \left(\sum_{k=n_0}^{n} x(k)\right) - \left(\sum_{k=n_0}^{n-1} x(k)\right) \\
&= x(n) + \left(\sum_{k=n_0}^{n-1} x(k)\right) - \left(\sum_{k=n_0}^{n-1} x(k)\right)
\end{aligned}$$

so $\displaystyle\Delta\left(\sum_{k=n_0}^{n-1} x(k)\right) = x(n)$

Listing 3: cobweb

```
%% Initializing variables
clear all
clc
% Initial Conditions
a=2;
b=1;
x0 = -3 ;
fp = (a-1)/b;

%% Cobweb
for i =-7:7
cobweb(@fiveb,i,-10,10,100)
hold off
end
%% Functions
function output = nm(x)
    output = .5.*(x+3./x);
end
```

```

function output = fiveb(x)
    output=2*x/(1+x);
end

function cobweb(f,initialpoint,intervalstart,intervalend,iterations)
x=linspace(intervalstart,intervalend,iterations);
y=f(x);
hold on
grid on
%axis([intervalstart intervalend intervalstart intervalend ])
plot(initialpoint,0,'r*')
text(initialpoint,0,' x_0')
title(['Cobweb diagram'],['x_0 =', num2str(initialpoint)],'Color','blue');
plot(x,y,'k');
plot(x,x,'r');
xlabel('x(t)');
ylabel('y(t)');
x(1)=initialpoint;

line( [x(1),x(1)], [0,f(x(1))] );
line([x(1),f(x(1))],[f(x(1)),f(x(1))]);
for i=1:iterations
    x(i+1)= f(x(i));
    line([x(i+1),x(i+1)],[x(i+1),f(x(i+1))]);
    line([x(i+1),f(x(i+1))],[f(x(i+1)),f(x(i+1))]);
end

end
```