Problem 21

Prove the following lemma:

Lemma:
Let $p(x) = a_0x^k + a_1x^{k-1} + \ldots + a_k$ and $g(n)$ be any discrete function. Then:

$$P(E)\left(b^n g(n)\right) = b^n P(E)g(n)$$

Attempt:

We know from the lecture notes that $P(E)$ is defined as:

$$P(E) = a_0E^k + a_1E^{k-1} + \ldots a_kI$$

and that:

$$P(E)b^n = \left(a_0b^k + a_1b^{k-1} + \ldots + a_k\right)b^n = p(b)b^n$$

Then we know the following will occur:

$$\begin{aligned} P(E)(b^n g(n)) &= \left(a_0E^k + a_1E^{k-1} + \ldots a_kI\right)b^n g(n) \\ &= a_0E^kb^ng(n) + a_1E^{k-1}b^ng(n) + \ldots + a_kIb^ng(n) \\ &= a_0b^{n+k}g(n+k) + a_1b^{n+k-1}g(n+k-1) + \ldots + a_kb^ng(n) \\ &= b^n(a_0b^kg(n+k) + a_1b^{k-1}g(n+k-1) + \ldots + a_kg(n)) \end{aligned}$$

from our definition of $p(x)$ we know that the right side of the final line above (the part in the parenthesis), is given by:

$$P(bE)g(n) = a_0b^kE^kg(n+k) + a_1b^{k-1}e^{k-1}g(n+k-1) + \ldots + a_kg(n)$$

(just plug it into the given definition of $p(x)$). So this gives us:

$$P(E)(b^ng(n)) = b^nP(bE)g(n)$$

Problem 22

Prove parts (ii) and (iii) of the following lemma:

Lemma:
For a fixed $k \in \mathbb{Z}^+$ and $x \in \mathbb{R}$, the following statements hold:

1. $\Delta x^{(k)} = kx^{(k-1)}$
2. $\Delta^n x^{(k)} = k(k-1)\cdots(k-n+1)x^{(k-n)}$
3. $\Delta^k x^{(k)} = k!$

Attempt:

We know that $x^{(k)}$ is defined as follows:

$$x^{(k)} = x(x-1)(x-2)\cdots(x-k+1),\ k \in \mathbb{Z}^+$$

$$(x+1)^{(k)} = (x+1)(x)(x-1)\cdots(x-k+2),\ k \in \mathbb{Z}^+$$

This can be similarly defined as:

$$x^{(k)} = \frac{x!}{(x-k)!} = x(x-1)(x-2)\cdots(x-k+1) \tag{1}$$

and as:

$$x^{(k)} = \prod_{j=1}^{k} x-j+1 = x(x-1)(x-2)\cdots(x-k+1) \tag{2}$$

$$(x+1)^{(k)} = \prod_{j=1}^{k-1} x-j+1 = (x+1)(x)(x-2)\cdots(x-k+2)$$

and from the lecture notes that:

$$\begin{aligned}
\Delta x^{(k)} &= (x+1)^{(k)} - x^{(k)} \\
&= (x+1)(x)(x-1)\cdots(x-k+2) - x(x-1)\cdots(x-k+2)(x-k+1) \\
&= (x)(x-1)\cdots(x-k+2)[x+1-(x-k+1)] \\
&= x(x-1)\cdots(x-k+2)k \\
&= kx^{(k-1)}
\end{aligned}$$

(i) If we express the above as $\Delta^1 x^{(k)}$, then we can do the following:

$$\begin{aligned}
\Delta\left[\Delta x^{(k)}\right] &= \Delta\left[k x^{(k-1)}\right] \\
&= k\Delta x^{(k-1)} \\
&= k\left[(x+1)^{(k-1)} - x^{(k-1)}\right] \\
&= k\Big[(x+1)(x)(x-1)\cdots(x-(k-1)+2) \\
&\quad - x(x-1)\cdots(x-(k-1)+2)(x-(k-1)+1)\Big] \\
&= k\Big[(x)(x-1)\cdots(x-(k-1)+2)\Big]\Big((x+1)-(x-(k-1)+1)\Big)
\end{aligned}$$

The term on the left in square brackets is identically $x^{(k+2)}$ since:

$$x^{(k-2)} = x(x-1)\cdots(x-(k-2)+1) = x(x-1)\cdots(x-k+3)$$

and the term on the right simplifies to:

$$\Big((x+1)-(x-(k-1)+1)\Big) = k-1$$

giving us:

$$\Delta^2 x^{(k)} = k(k-1)x^{(k-2)}$$

Now let's assume that this pattern holds up to some $(n-1)$, then:

$$\Delta^{n-1}x^{(k)} = k(k-1)\cdots((k-(n-1)-1))x^{(k-(n-1)}$$

then:

$$\begin{aligned}
\Delta\left[\Delta^{n-1}x^{(k)}\right] &= \Delta\left[k(k-1)\cdots(k-(n-1)-1)x^{(k-(n-1))}\right] \\
&= \Delta\left[k(k-1)\cdots(k-n)x^{(k-n+1)}\right]
\end{aligned}$$

We can express the quantity on the left inside the brackets as:

$$k(k-1)\cdots(k-n) = \frac{k!}{(k-n-1)!} \tag{3}$$

and since it is a scalar quantity that does not depend on x and is thus not effected by the Δ operator, we can move it outside. (since the Δ operator is linear)

$$\begin{aligned}
\Delta\left[\Delta^{n-1}x^{(k)}\right] &= \frac{k!}{(k-n-1)!}\Delta x^{(k-n+1)} \\
&= \frac{k!}{(k-n-1)!}\left[(x+1)^{(k-n+1)} - x^{(k-n+1)}\right]
\end{aligned}$$

We know that the structure of the operator as before in $\Delta^2 x^{(k)}$, the bracketed part will have the form from (2), where $(x+1)^{(k)}$ shifts both indicies down by one:

$$\prod_{j=0}^{k-n} x - j + 1 - \prod_{j=1}^{k-n+1} x - j + 1$$

which simplifies to:

$$\left[\prod_{j=1}^{k-n} x - j + 1\right] \left[(x+1) - (x - (k-n+1) + 1)\right] = \left[\prod_{j=1}^{k-n} x - j + 1\right](k-n+1)$$

going back to what we had, we can substitute into the bracketed part:

$$\begin{aligned}\Delta\left[\Delta^{n-1} x^{(k)}\right] &= \frac{k!}{(k-n-1)!}\left[(x+1)^{(k-n+1)} - x^{(k-n+1)}\right] \\ &= \frac{k!}{(k-n-1)!}\left[\prod_{j=1}^{k-n} x - j + 1\right](k-n+1)\end{aligned}$$

Simplifying further using (2) again we then get:

$$\Delta^n x^{(k)} = \frac{k!}{(k-n-1)!} x^{(k-n)}(k-n+1) \tag{4}$$

$$\Delta^n x^{(k)} = k(k-1)\cdots(k-n)(k-n+1)x^{(k-n)}$$

(ii) Using (4), if $n = k$:

$$\Delta^k x^{(k)} = \frac{k!}{(k-k-1)!} x^{(k-k)}(k-k+1)$$

and using (3)

$$\begin{aligned}\Delta^k x^{(k)} &= \frac{k!}{(-1)!}\frac{x!}{(x-(k-k))!}(0+1) \\ &= \frac{k!}{(-1)!}(1)(1) \\ &= \frac{k!}{(-1)!}\end{aligned}$$

The term in brackets is zero, since the indicies are backwards. For the term on the left, we know that negative interger factorials have the form:

$$n! = \frac{(-1)^{-n-1}}{(-n-1)!}$$

and that $-1! = 1$, so this gives us:

$$\Delta x^{(k)} = k!$$

Problem 23

Prove the Quotient rule:

$$\Delta\left[\frac{x(n)}{y(n)}\right]=\frac{y(n)\Delta x(n)-x(n)\Delta y(n)}{y(n)Ey(n)}$$

Attempt:

since $\Delta = E - I$,

$$\begin{aligned}
\Delta\left[\frac{x(n)}{y(n)}\right] &= (E-I)\left[\frac{x(n)}{y(n)}\right]\\
&= E\left[\frac{x(n)}{y(n)}\right]-I\left[\frac{x(n)}{y(n)}\right]\\
&= \left[\frac{x(n+1)}{y(n+1)}\right]-\left[\frac{x(n)}{y(n)}\right]\\
&= \left[\frac{x(n+1)y(n)-x(n)y(n+1)}{y(n)y(n+1)}\right]\\
&= \left[\frac{x(n+1)y(n)+[y(n)x(n)-y(n)x(n)]-x(n)y(n+1)}{y(n)y(n+1)}\right]\\
&= \left[\frac{x(n+1)y(n)-y(n)x(n)-x(n)y(n+1)+y(n)x(n)]}{y(n)y(n+1)}\right]\\
&= \left[\frac{y(n)[x(n+1)-x(n)]-x(n)[y(n+1)-y(n)]}{y(n)y(n+1)}\right]\\
&= \left[\frac{y(n)\Delta x(n)-x(n)\Delta y(n)}{y(n)Ey(n)}\right]
\end{aligned}$$

Problem 24

Prove the following lemma:

Lemma:
Let $X_1(n)$ and $X_2(n)$ be two solutions of:

$$x(k+n) + p_1(n)x(n+k-1) + \cdots + p_k(n)x(n) = 0 \tag{5}$$

Then the following statements are true:

1. $x(n) = X_1(n) + X_2(n)$ is a solution of (5)

2. $\tilde{x}(n) = aX_1(n)$ is a solution of (5) for any constant a

Attempt:

1. if $X_1(n)$ and $X_2(n)$ are solutions to 5, then:

$$X_1(k+n) + p_1(n)X_1(n+k-1) + \cdots + p_k(n)X_1(n) = 0$$

$$X_2(k+n) + p_1(n)X_2(n+k-1) + \cdots + p_k(n)X_2(n) = 0$$

So if we plug in $x(n) = X_1(n) + X_2(n)$ for (5)

$$\begin{aligned} x(n) =& X_1(k+n) + X_2(k+n) + p_1(n)X_1(n+k-1) + p_1(n)X_2(n+k-1) \\ &+ \cdots + p_k(n)X_1(n) + p_k(n)X_2(n) \\ =& X_1(k+n) + p_1(n)X_1(n+k-1) + \cdots + p_k(n)X_1(n) \\ &+ X_2(k+n) + p_1(n)X_2(n+k-1) + \cdots + p_k(n)X_2(n) \\ =& 0 + 0 \end{aligned}$$

But this is just equal to the two given solutions meaning the sum of each (when we split them) is zero, so this is also a solution.

2. Doing something similar:

$$\begin{aligned} \tilde{x}(n) &= aX_1(k+n) + p_1(n)aX_1(n+k-1) + \cdots + p_k(n)aX_1(n) \\ &= a(X_1(k+n) + p_1(n)X_1(n+k-1) + \cdots + p_k(n)X_1(n)) \\ &= a(0) \\ &= 0 \end{aligned}$$

So this is also a solution.

Problem 25

Find the Casoratian of the following functions:

1. $(-2)^n, 2^n, 3$
2. $0, 2^n, 3$
3. $2^n, 3, 2^{n+2}$

Attempt:

(a) let $n = 1$

$$\begin{vmatrix} 1 & 1 & 3 \\ -2 & 2 & 3 \\ 4 & 4 & 3 \end{vmatrix} = 1\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 3 \\ 4 & 3 \end{vmatrix} + 4\begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix}$$

$$= 6 - 12 + 6 - 24 + 12 - 24 = -12 - 6 - 18 = -36$$

since this $\neq 0$ these are linearly independent

(b)

$$\begin{vmatrix} 0 & 1 & 3 \\ 0 & 2 & 3 \\ 0 & 4 & 3 \end{vmatrix} = 0\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} - 0\begin{vmatrix} 1 & 3 \\ 4 & 3 \end{vmatrix} + 0\begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = 0$$

So this is linearly dependent.

(c)

$$\begin{vmatrix} 1 & 3 & 4 \\ 2 & 3 & 8 \\ 4 & 3 & 16 \end{vmatrix} = 1\begin{vmatrix} 3 & 8 \\ 3 & 16 \end{vmatrix} - 2\begin{vmatrix} 3 & 4 \\ 3 & 16 \end{vmatrix} + 4\begin{vmatrix} 3 & 4 \\ 3 & 8 \end{vmatrix}$$

$$= 48 - 24 + -96 + 24 + 96 - 48 = 0$$

So this is linearly dependent