Problem 11

For the following systems determine if the equilibrium point $x^* = 0$ is semi-asymptotically stable from the left or the right.

(i) $x(n+1) = \big(x(n)\big)^3 + \big(x(n)\big)^2 + x(n)$

(ii) $x(n+1) = \big(x(n)\big)^3 - \big(x(n)\big)^2 + x(n)$

Attempt:

(i) To check the semi-asymptotic stability, we need to know three things:

- what are the equilibrium points, x^*.
- Do the equilibrium points satisfy the definition of semi-stable: An equilibrium point x^* of $x(n+1) = f(x(n))$ is semistable from the right if given $\varepsilon > 0$ there exists a $\delta > 0$ such that if $x(0) > x^*$, $x(0) - x^* < \delta$, then $x(n) - x^* < \varepsilon$.
- If the equilibrium points are semi-stable, do they satisfy the definition of semi-asymptotic stability:
  If $\lim_{n\to\infty} x(n) = x^*$, whenever $x(0) - x^* < \eta$ $\{(x^* - x(0)) < n\}$, then x^* is said to be semi-asymptotically stable from the right. If $f'(x^*) = 1$ and
  - $f''(x^*) < 0$, then x^* is semiasympotically stable from the right.
  - $f''(x^*) > 0$, then x^* is semiasympotically stable from the left.

Since I am lazy, and I can't figure out how to do the *actual* analysis (I tried, but couldn't get the definitions to work), we are just going to use the last 2 conditions.

$$f'(0) = 3(0^2) + 2(0) + 1 = 1$$
$$f''(0) = 6(0) + 2 = 2$$

So this is semi-asymptotically stable from the left.

(ii) Similarly,

$$f'(0) = 3(0^2) - 2(0) + 1 = 1$$
$$f''(0) = 6(0) - 2 = -2$$

So this is asymptotically stable from the right.

Problem 12

Consider the logistic equation:

$$x(n+1) = \mu x(n)(1 - x(n))$$

(i) Let $\mu = 3.4$ and $x(0) = 0.45$. Show that we get a 2-periodic point (compute the periodic orbit).

(ii) Let $\mu = 3.5$ and $x(0) = 0.5$. Show that we get a 4-periodic point. (compute the periodic orbit). Provide the code you used.

Attempt:

1. we get the values, $0.451965325993535, 0.842155078316515$

Listing 1: 2 values

```
%% Problem 12
%% Initial conditions
mu = 3.4;
x(1) = .45; %x(0)

%% loop
n=50;
for i=1:n+1
        x(i+1)=logisticeq(3.4,x(i));
end
x= reshape(x,2,(n+2)/2)'

%% Functions
function output = logisticeq(mu,x)
    output = mu*x*(1-x);
end
```

2. we get the values:

$$0.500884210307218, 0.874997263602464, 0.382819683017324, 0.826940706591439$$

Listing 2: 2 values

```
%% Problem 12b
%% Initial conditions
mu = 3.5;
x(1) = .5; %x(0)

```

```
%% loop
rows = 50;
cols = 4;
n=rows*cols-1;
for i=1:n
        x(i+1)=logisticeq(mu,x(i));
end
x=reshape(x,cols,rows)'

%% Functions
function output = logisticeq(mu,x)
    output = mu*x*(1-x);
end
function output = logisticeqsq(mu,x)
    a = mu*x*(1-x);
    output = mu*a*(1-a);
end
function output = logisticeqcube(mu,x)
    one = mu*x*(1-x);
    two = mu*one*(1-one);
    output =mu*two*(1-two);
end
function output = logisticeqquad(mu,x)
    one = mu*x*(1-x);
    two = mu*one*(1-one);
    three=mu*two*(1-two);
    output =mu*three*(1-three);
end
```

Problem 13

Consider the tent map:

$$\text{Let} \quad f(x) = \begin{cases} 2x, & \text{for } 0 \leq x \leq \frac{1}{2} \\ 2(1-x), & \text{for } \frac{1}{2} < x \leq 1. \end{cases}$$

The tent map iteration is given by:

$$x(n+1) = f(x(n))$$

Let $y(n+1) = g(y(n))$, with $g(y) = f^2(y) = f(f(y))$

(a) Obtain an explicit form for the function $g(x)$.

(b) Find the fixed points of g.

(c) Determine the two-cycles of f.

Attempt:

(a) we know that $g(x) = f(f(x))$, so we must put $2x$ into each, and $2(1-x)$ into each, and adjust the initial intervals. This gives us:

$$g(x) = \begin{cases} 2(2(x)), & 0 \leq x < \frac{1}{4} \\ 2(1-(2x)) & \frac{1}{4} \leq x < \frac{1}{2} \\ 2(1-2(1-x))) & \frac{1}{2} \leq x < \frac{3}{4} \\ 2(2(1-x)) & \frac{3}{4} \leq x \leq 1 \end{cases}$$

(b) Our fixed points are:

$$\begin{cases} 4x^* - x^* = 0, & 0 \leq x < \frac{1}{4} \\ 2 - 4x^* - x^* = 0 & \frac{1}{4} \leq x < \frac{1}{2} \\ 4x^* - 2 - x^* = 0 & \frac{1}{2} \leq x < \frac{3}{4} \\ 4 - 4x^* - x^* = 0 & \frac{3}{4} \leq x \leq 1 \end{cases}$$

$$\begin{cases} x^* = 0, & 0 \leq x < \frac{1}{4} \\ x^* = \frac{2}{5} & \frac{1}{4} \leq x < \frac{1}{2} \\ x^* = \frac{2}{3} & \frac{1}{2} \leq x < \frac{3}{4} \\ x^* = \frac{4}{5} & \frac{3}{4} \leq x \leq 1 \end{cases}$$

(c) The two cycles are .8, .4.

Listing 3: 2 values

```
%% Problem 13c
clear all
%% Initial conditions
x(1) = .9; %x(0)

%% loop
rows = 8;
cols = 2;
n=rows*cols-1;
for i=1:n
        x(i+1)=prob13c(x(i));
end
x=reshape(x,cols,rows)'

%% Functions
function output = prob13c(x)
    if x<=.5
        output = 2*x;
    else
        output = 2*(1-x);
    end
end
```

Problem 14

Recall that the tent map problem:

$$x(n+1) = f\big(x(n)\big), \text{ with } f(x) \begin{cases} 2x, & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2(1-x), & \text{if } \frac{1}{2} < x \leq 1 \end{cases}$$

What is the stability of the two cycles?

Attempt:

We can apply the following Theorem:

Theorem 1.21:
Let $O(b) = \{b = x(0), x(1), \dots, x(k-1)\}$ be a k-cycle of a continuously differentiable function f. Then the following statements hold:

(i) The k-cycle $O(b)$ is asymptotically stable if:

$$\left|f'\big(x(0)\big)f'\big(x(1)\big)\dots f'\big(x(k-1)\big)\right| < 1$$

(ii) The k-cycle $O(b)$ is asymptotically unstable if:

$$\left|f'\big(x(0)\big)f'\big(x(1)\big)\dots f'\big(x(k-1)\big)\right| > 1$$

Then

$$|f'(.4)f'(.8)| = |2 * (-2)| = 4 > 1$$

So this is asymptotically unstable.

Problem 15

Consider the function $B(x)$ (Baker's function) given by:

$$B(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2x - 1, & \text{if } \frac{1}{2} < x \leq 1 \end{cases}$$

Sketch the graph of B^2 and find the two-cycles of B
Attempt:

(a) we know that $g(x) = f(f(x))$, so we must put $2x$ into each, and $2x - 1$ into each, and adjust the initial intervals. This gives us:

$$g(x) = \begin{cases} 2(2(x)), & 0 \leq x < \frac{1}{4} \\ 2(2x) - 1 & \frac{1}{4} \leq x < \frac{1}{2} \\ 2(2x - 1) & \frac{1}{2} \leq x < \frac{3}{4} \\ 2(2x - 1) - 1 & \frac{3}{4} \leq x \leq 1 \end{cases}$$

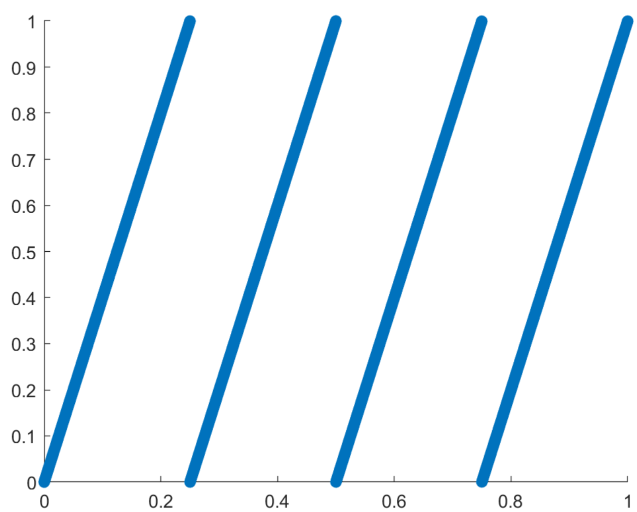

Figure 1: plot of B^2

(b) Our fixed points are:

$$\begin{cases} 4x^* - x^* = 0, & 0 \leq x < \frac{1}{4} \\ 4x^* - 1 - x^* = 0 & \frac{1}{4} \leq x < \frac{1}{2} \\ 4x^* - 2 - x^* = 0 & \frac{1}{2} \leq x < \frac{3}{4} \\ 4x^* - 3 - x^* = 0 & \frac{3}{4} \leq x \leq 1 \end{cases}$$

$$\begin{cases} x^* = 0, & 0 \leq x < \frac{1}{4} \\ x^* = \frac{1}{3} & \frac{1}{4} \leq x < \frac{1}{2} \\ x^* = \frac{2}{3} & \frac{1}{2} \leq x < \frac{3}{4} \\ x^* = 1 & \frac{3}{4} \leq x \leq 1 \end{cases}$$

(c) Since $x^* = 0, 1$ are solutions of $B(x)$, they cannot be solutions, therefore the 2-cycle is given by $\frac{1}{3}, \frac{2}{3}$
We can see these plotted here:

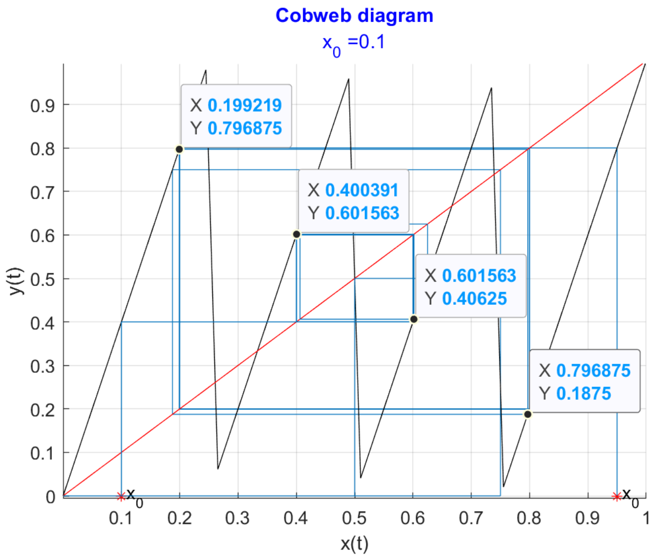

Figure 2: An image of the cycles (code included at the end)

Problem 16

what is the stability of the two cycle for $\mu = 1 + \sqrt{6}$? Justify your answer.
Attempt:

$F_\mu(x) = \mu x(1-x)$, and so

$$\begin{aligned} [F_\mu^2(x(0))]' &= F_\mu'(x(0))F_\mu'(x(1)) \\ &= \mu(1-2x(0))\mu(1-2x(1)) \\ &= \mu^2(1-2x(0))(1-2x(1)) \end{aligned}$$

If:

$$x(0) = \frac{\left[(1+\mu) - \sqrt{(\mu-3)(\mu+1)}\right]}{2\mu}$$

$$x(1) = \frac{\left[(1+\mu) + \sqrt{(\mu-3)(\mu+1)}\right]}{2\mu}$$

We can write this as:

$$\mu^2\left(1 - 2\frac{\left[(1+\mu) - \sqrt{(\mu-3)(\mu+1)}\right]}{2\mu}\right)\left(1 - 2\frac{\left[(1+\mu) + \sqrt{(\mu-3)(\mu+1)}\right]}{2\mu}\right)$$

simplifying further:

$$\mu^2\left(\frac{\mu - \left[(1+\mu) - \sqrt{(\mu-3)(\mu+1)}\right]}{\mu}\right)\left(\frac{\mu - \left[(1+\mu) + \sqrt{(\mu-3)(\mu+1)}\right]}{\mu}\right)$$

$$\left(\mu - \left[(1+\mu) - \sqrt{(\mu-3)(\mu+1)}\right]\right)\left(\mu - \left[(1+\mu) + \sqrt{(\mu-3)(\mu+1)}\right]\right)$$

$$\left(\left[-1 + \sqrt{(\mu-3)(\mu+1)}\right]\right)\left(\left[-1 - \sqrt{(\mu-3)(\mu+1)}\right]\right)$$

$$([1 - (\mu-3)(\mu+1)])$$

$$\left(\left[1 - \mu^2 + 2\mu + 3\right]\right)$$

$$\left(\left[4 - (1+\sqrt{6})^2 + 2(1+\sqrt{6})\right]\right)$$

$$\left(\left[4 - 1 - 2\sqrt{6} - 6 + 2 + 2\sqrt{6}) = -1\right]\right)$$

To assess the stability let's first try (**Thm 1.16**) and check the first derivative of our function:

$$\begin{aligned} F''(x) &= [F'(x(0))F'(x(1))]' \\ &= F''(x(0))F'(x(1)) + F'(x(0))F''(x(1)) \\ &= -2\mu(F'(x(1)) + F'(x(0))) \end{aligned}$$

$$\begin{aligned} F'''(x) &= [F'(x(0))F'(x(1))]'' \\ &= [F''(x(0))F'(x(1)) + F'(x(0))F''(x(1))]' \\ &= [-2\mu F'(x(1)) - 2\mu F'(x(0))]' \\ &= 4\mu^2 + 4\mu^2 \end{aligned}$$

$$Sf = -f'''(x^*) - \frac{3}{2}(f''(x^*))^2 = -8\mu^2 - \frac{3}{2}(-2\mu(-1))^2$$

$$= -8\mu^2 - 3\mu^2 = -11(7 + 2\sqrt{6}) < 0$$

so we know that this is asymptotically stable.

Problem 17

Iterate $x(n+1) = F_\mu\big(x(u)\big)$ for a large number of iterations. Plot the last portion of the sequence $\{x(n)\}$ as a function of μ to get a bifurcation diagram of F_μ

Attempt:

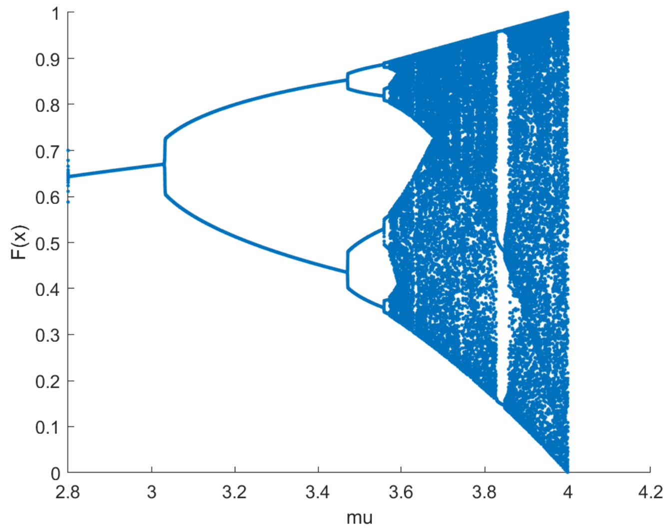

Figure 3: Bifurcation diagram

Listing 4: bifurcation

```
%% Problem 17
clear all
%% Initial conditions
x(1) =.7; %x(0)

%% loop
rows = 70000;
cols = 1;
n=rows*cols-1;
xplot=linspace(2.8,4,n+1);
for i=1:n
        x(i+1)=logistic(xplot(i),x(i));
end
x=reshape(x,cols,rows)'

scatter(xplot,x,'.')
xlabel('mu')
ylabel('F(x)')
```

```
%% Functions
function output = logistic(mu,x)

    output = mu*x-mu*x*x;
end
```

Problem 18

Show that:

$$E^k x(n) = \sum_{i=0}^{k} \binom{k}{i} \Delta^{k-i} x(n)$$

Attempt:

From before we know that $\Delta = E - I$, so using the previous proof:

$$\begin{aligned}
\Delta^k x(n) &= (E - I)^k x(n) \\
&= \sum_{i=0}^{k} (-1)^i \binom{k}{i} E^{k-i} x(n) \\
&= \sum_{i=0}^{k} (-1)^i \binom{k}{i} x(n + k - i)
\end{aligned}$$

Since $E = \Delta + I$, and $E^k = (\Delta + I)^k$, and $(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$:

$$\begin{aligned}
E^k &= (\Delta + I)^k \\
&= \sum_{i=0}^{k} \binom{k}{i} \Delta^{k-i} (I)^i
\end{aligned}$$

Then:

$$E^k x(n) = \sum_{i=0}^{k} \binom{k}{i} \Delta^{k-i} (I)^i x(n)$$

$$E^k x(n) = \sum_{i=0}^{k} \binom{k}{i} \Delta^{k-i} (I) x(n)$$

$$E^k x(n) = \sum_{i=0}^{k} \binom{k}{i} \Delta^{k-i} x(n)$$

Problem 19

Show that E and Δ are linear operators. That is, show that for all constants a and b:

$$\Delta[ax(n) + by(n)] = a\Delta x(n) + b\Delta y(n)$$

and

$$E[ax(n) + by(n)] = aEx(n) + bEy(n)$$

Attempt:

$$\begin{aligned} a\Delta[x(n)] + b\Delta[y(n)] &= a[x(n+1) - x(n)] + b[y(n+1) - y(n)] \\ &= [ax(n+1) - ax(n)] + [by(n+1) - by(n)] \\ &= [ax(n+1) + by(n+1)] - [ax(n) + by(n)] \\ &= \Delta[ax(n) + by(n)] \end{aligned}$$

$$\begin{aligned} E[ax(n) + by(n)] &= [ax(n+1) + by(n+1)] \\ &= a[x(n+1)] + b[y(n+1)] \\ &= aE[x(n)] + bE[y(n)] \end{aligned}$$

Problem 20

Prove that the following statements hold:

(i) $\displaystyle\sum_{i=n_0}^{n-1} \Delta x(n) = x(n) - x(n_0)$

(ii) $\displaystyle\Delta\left(\sum_{k=n_0}^{n-1} x(k)\right) = x(n)$

Attempt:

(i)

$$\begin{aligned} \sum_{i=n_0}^{n-1} \Delta x(n) &= \sum_{i=n_0}^{n-1} x(i+1) - x(i) \\ &= \sum_{i=n_0}^{n-1} -x(i) + x(i+1) \\ &= [-x(n_0) + x(n_0+1)] + [-x(n_0+1) + x(n_0+2)] + \dots \\ &+ [-x(n-2) + x(n-2+1)] + [-x(n-1) + x(n-1+1)] \end{aligned}$$

But it is easy to observe that the terms between cancel:

$$
\begin{aligned}
&= -x(n_0) + [x(n_0+1) - x(n_0+1)] + [x(n_0+2) - x(n_0+2)] \dots \\
&[x(n-2) - x(n-2)] + [x(n-2+1) + -x(n-1)] + x(n-1+1) \\
&= -x(n_0) + x(n)
\end{aligned}
$$

So we get: $\sum_{i=n_0}^{n-1} \Delta x(n) = x(n) - x(n_0)$

(ii)

$$
\begin{aligned}
\Delta\left(\sum_{k=n_0}^{n-1} x(k)\right) &= \\
&= \left(\sum_{k=n_0}^{n} x(k)\right) - \left(\sum_{k=n_0}^{n-1} x(k)\right) \\
&= x(n) + \left(\sum_{k=n_0}^{n-1} x(k)\right) - \left(\sum_{k=n_0}^{n-1} x(k)\right)
\end{aligned}
$$

so $\Delta\left(\sum_{k=n_0}^{n-1} x(k)\right) = x(n)$

Listing 5: cobweb

```
%% Initializing variables
clear all
clc
% Initial Conditions
a=2;
b=1;
x0 = -3 ;
fp = (a-1)/b;

%% Cobweb
for i =-7:7
cobweb(@fiveb,i,-10,10,100)
hold off
end
%% Functions
function output = nm(x)
    output = .5.*(x+3./x);
end
```

```
function output = fiveb(x)
    output=2*x/(1+x);
end

function cobweb(f,initialpoint,intervalstart,intervalend,iterations)
x=linspace(intervalstart,intervalend,iterations);
y=f(x);
hold on
grid on
%axis([intervalstart intervalend intervalstart intervalend ])
plot(initialpoint,0,'r*')
text(initialpoint,0,' x_0')
title(['Cobweb diagram'],['x_0 =', num2str(initialpoint)],'Color','blue');
plot(x,y,'k');
plot(x,x,'r');
xlabel('x(t)');
ylabel('y(t)');
x(1)=initialpoint;

line( [x(1),x(1)], [0,f(x(1))] );
line([x(1),f(x(1))],[f(x(1)),f(x(1))]);
for i=1:iterations
    x(i+1)= f(x(i));
    line([x(i+1),x(i+1)],[x(i+1),f(x(i+1))]);
    line([x(i+1),f(x(i+1))],[f(x(i+1)),f(x(i+1))]);
end

end
```