Problem 26

Show that the Casoratian $W(n)$ given by:

$$W(n) = \begin{vmatrix} x_1(n) & x_2(n) & \cdots & x_k(n) \\ x_1(n+1) & x_2(n+1) & \cdots & x_k(n+1) \\ \vdots & & & \vdots \\ x_1(n+k-1) & x_2(n+k-1) & \cdots & x_k(n+k-1) \end{vmatrix} \tag{1}$$

may be given by the formula:

$$W(n) = \begin{vmatrix} x_1(n) & x_2(n) & \cdots & x_k(n) \\ \Delta x_1(n) & \Delta x_2(n) & \cdots & \Delta x_k(n) \\ \vdots & & & \vdots \\ \Delta^{k-1} x_1(n) & \Delta^{k-1} x_2(n) & \cdots & \Delta^{k-1} x_k(n) \end{vmatrix}$$

Attempt:

We know by definition that, $\Delta x_i(n) = x_i(n+1) - x_i(n)$, and that $\Delta x_i(n+k-1) = x_i(n+k) - x_i(n+k-1)$

Problem 27

Consider the second order difference equation

$$u(n+2) - \frac{n+3}{n+2}u(n+1) + \frac{2}{n+2}u(n) = 0$$

(a) show that $u_1(n) = \dfrac{2^n}{n!}$ is a solution of the equation.

(b) Use the above formula to find another solution $u_2(n)$ of the equation.

Attempt:

We know that $x^{(k)}$ is defined as follows:

$$x^{(k)} = x(x-1)(x-2)\cdots(x-k+1),\ k \in \mathbb{Z}^+$$

$$(x+1)^{(k)} = (x+1)(x)(x-1)\cdots(x-k+2),\ k \in \mathbb{Z}^+$$

This can be similarly defined as:

$$x^{(k)} = \frac{x!}{(x-k)!} = x(x-1)(x-2)\cdots(x-k+1) \tag{2}$$

and as:

$$x^{(k)} = \prod_{j=1}^{k} x - j + 1 = x(x-1)(x-2)\cdots(x-k+1) \tag{3}$$

$$(x+1)^{(k)} = \prod_{j=1}^{k-1} x - j + 1 = (x+1)(x)(x-2)\cdots(x-k+2)$$

and from the lecture notes that:

$$\begin{aligned}
\Delta x^{(k)} &= (x+1)^{(k)} - x^{(k)} \\
&= (x+1)(x)(x-1)\cdots(x-k+2) - x(x-1)\cdots(x-k+2)(x-k+1) \\
&= (x)(x-1)\cdots(x-k+2)[x+1-(x-k+1)] \\
&= x(x-1)\cdots(x-k+2)k \\
&= kx^{(k-1)}
\end{aligned}$$

(i) If we express the above as $\Delta^1 x^{(k)}$, then we can do the following:

$$\begin{aligned}
\Delta\left[\Delta x^{(k)}\right] &= \Delta\left[k x^{(k-1)}\right] \\
&= k\Delta x^{(k-1)} \\
&= k\left[(x+1)^{(k-1)} - x^{(k-1)}\right] \\
&= k\Big[(x+1)(x)(x-1)\cdots(x-(k-1)+2) \\
&\quad - x(x-1)\cdots(x-(k-1)+2)(x-(k-1)+1)\Big] \\
&= k\Big[(x)(x-1)\cdots(x-(k-1)+2)\Big]\Big((x+1)-(x-(k-1)+1)\Big)
\end{aligned}$$

The term on the left in square brackets is identically $x^{(k+2)}$ since:

$$x^{(k-2)} = x(x-1)\cdots(x-(k-2)+1) = x(x-1)\cdots(x-k+3)$$

and the term on the right simplifies to:

$$\Big((x+1)-(x-(k-1)+1)\Big) = k-1$$

giving us:

$$\Delta^2 x^{(k)} = k(k-1)x^{(k-2)}$$

Now let's assume that this pattern holds up to some $(n-1)$, then:

$$\Delta^{n-1}x^{(k)} = k(k-1)\cdots((k-(n-1)-1))x^{(k-(n-1)}$$

then:

$$\begin{aligned}
\Delta\left[\Delta^{n-1}x^{(k)}\right] &= \Delta\left[k(k-1)\cdots(k-(n-1)-1)x^{(k-(n-1))}\right] \\
&= \Delta\left[k(k-1)\cdots(k-n)x^{(k-n+1)}\right]
\end{aligned}$$

We can express the quantity on the left inside the brackets as:

$$k(k-1)\cdots(k-n) = \frac{k!}{(k-n-1)!} \tag{4}$$

and since it is a scalar quantity that does not depend on x and is thus not effected by the Δ operator, we can move it outside. (since the Δ operator is linear)

$$\begin{aligned}
\Delta\left[\Delta^{n-1}x^{(k)}\right] &= \frac{k!}{(k-n-1)!}\Delta x^{(k-n+1)} \\
&= \frac{k!}{(k-n-1)!}\left[(x+1)^{(k-n+1)} - x^{(k-n+1)}\right]
\end{aligned}$$

We know that the structure of the operator as before in $\Delta^2 x^{(k)}$, the bracketed part will have the form from (3), where $(x+1)^{(k)}$ shifts both indicies down by one:

$$\prod_{j=0}^{k-n} x-j+1-\prod_{j=1}^{k-n+1} x-j+1$$

which simplifies to:

$$\left[\prod_{j=1}^{k-n} x-j+1\right]\left[(x+1)-(x-(k-n+1)+1)\right]=\left[\prod_{j=1}^{k-n} x-j+1\right](k-n+1)$$

going back to what we had, we can substitute into the bracketed part:

$$\begin{aligned}\Delta\left[\Delta^{n-1}x^{(k)}\right]&=\frac{k!}{(k-n-1)!}\left[(x+1)^{(k-n+1)}-x^{(k-n+1)}\right]\\&=\frac{k!}{(k-n-1)!}\left[\prod_{j=1}^{k-n} x-j+1\right](k-n+1)\end{aligned}$$

Simplifying further using (3) again we then get:

$$\Delta^n x^{(k)}=\frac{k!}{(k-n-1)!}x^{(k-n)}(k-n+1) \tag{5}$$

$$\Delta^n x^{(k)}=k(k-1)\cdots(k-n)(k-n+1)x^{(k-n)}$$

(ii) Using (5), if $n=k$:

$$\Delta^k x^{(k)}=\frac{k!}{(k-k-1)!}x^{(k-k)}(k-k+1)$$

and using (4)

$$\begin{aligned}\Delta^k x^{(k)}&=\frac{k!}{(-1)!}\frac{x!}{(x-(k-k))!}(0+1)\\&=\frac{k!}{(-1)!}(1)(1)\\&=\frac{k!}{(-1)!}\end{aligned}$$

The term in brackets is zero, since the indicies are backwards. For the term on the left, we know that negative interger factorials have the form:

$$n!=\frac{(-1)^{-n-1}}{(-n-1)!}$$

and that $-1!=1$, so this gives us:

$$\Delta x^{(k)}=k!$$

Problem 28

Write the general solution of the difference equation:

$$(E-3)^2(E^2+4)x(n) = 0$$

Attempt:

since $\Delta = E - I$,

$$\begin{aligned}
\Delta\left[\frac{x(n)}{y(n)}\right] &= (E-I)\left[\frac{x(n)}{y(n)}\right] \\
&= E\left[\frac{x(n)}{y(n)}\right] - I\left[\frac{x(n)}{y(n)}\right] \\
&= \left[\frac{x(n+1)}{y(n+1)}\right] - \left[\frac{x(n)}{y(n)}\right] \\
&= \left[\frac{x(n+1)y(n) - x(n)y(n+1)}{y(n)y(n+1)}\right] \\
&= \left[\frac{x(n+1)y(n) + [y(n)x(n) - y(n)x(n)] - x(n)y(n+1)}{y(n)y(n+1)}\right] \\
&= \left[\frac{x(n+1)y(n) - y(n)x(n) - x(n)y(n+1) + y(n)x(n)]}{y(n)y(n+1)}\right] \\
&= \left[\frac{y(n)[x(n+1) - x(n)] - x(n)[y(n+1) - y(n)]}{y(n)y(n+1)}\right] \\
&= \left[\frac{y(n)\Delta x(n) - x(n)\Delta y(n)}{y(n)Ey(n)}\right]
\end{aligned}$$

Problem 29

Find a homogenous difference equation whose solution is:

$$y(n) = 2^{n-1} - 5^{n+1}$$

Attempt:

1. if $X_1(n)$ and $X_2(n)$ are solutions to **??**, then:

$$X_1(k+n) + p_1(n)X_1(n+k-1) + \cdots + p_k(n)X_1(n) = 0$$

$$X_2(k+n) + p_1(n)X_2(n+k-1) + \cdots + p_k(n)X_2(n) = 0$$

So if we plug in $x(n) = X_1(n) + X_2(n)$ for (**??**)

$$\begin{aligned} x(n) =& X_1(k+n) + X_2(k+n) + p_1(n)X_1(n+k-1) + p_1(n)X_2(n+k-1) \\ &+ \cdots + p_k(n)X_1(n) + p_k(n)X_2(n) \\ &= X_1(k+n) + p_1(n)X_1(n+k-1) + \cdots + p_k(n)X_1(n) \\ &\quad + X_2(k+n) + p_1(n)X_2(n+k-1) + \cdots + p_k(n)X_2(n) \\ &= 0 + 0 \end{aligned}$$

But this is just equal to the two given solutions meaning the sum of each (when we split them) is zero, so this is also a solution.

2. Doing something similar:

$$\begin{aligned} \tilde{x}(n) &= aX_1(k+n) + p_1(n)aX_1(n+k-1) + \cdots + p_k(n)aX_1(n) \\ &= a(X_1(k+n) + p_1(n)X_1(n+k-1) + \cdots + p_k(n)X_1(n)) \\ &= a(0) \\ &= 0 \end{aligned}$$

So this is also a solution.

Problem 30

Find a partic

Attempt:

(a) let $n = 1$

$$\begin{vmatrix} 1 & 1 & 3 \\ -2 & 2 & 3 \\ 4 & 4 & 3 \end{vmatrix} = 1 \begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} - (-2) \begin{vmatrix} 1 & 3 \\ 4 & 3 \end{vmatrix} + 4 \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix}$$

$$= 6 - 12 + 6 - 24 + 12 - 24 = -12 - 6 - 18 = -36$$

since this $\neq 0$ these are linearly independent

(b)

$$\begin{vmatrix} 0 & 1 & 3 \\ 0 & 2 & 3 \\ 0 & 4 & 3 \end{vmatrix} = 0 \begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} - 0 \begin{vmatrix} 1 & 3 \\ 4 & 3 \end{vmatrix} + 0 \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = 0$$

So this is linearly dependent.

(c)

$$\begin{vmatrix} 1 & 3 & 4 \\ 2 & 3 & 8 \\ 4 & 3 & 16 \end{vmatrix} = 1 \begin{vmatrix} 3 & 8 \\ 3 & 16 \end{vmatrix} - 2 \begin{vmatrix} 3 & 4 \\ 3 & 16 \end{vmatrix} + 4 \begin{vmatrix} 3 & 4 \\ 3 & 8 \end{vmatrix}$$

$$= 48 - 24 + -96 + 24 + 96 - 48 = 0$$

So this is linearly dependent