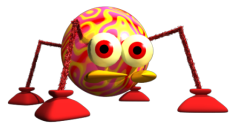

MAT 485 Project 1

Cal Poly Pomona

A Two-Mass Oscillator

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1 Introduction

In this paper we will model the two-mass oscillator system. Consider the diagram shown in Figure 1.

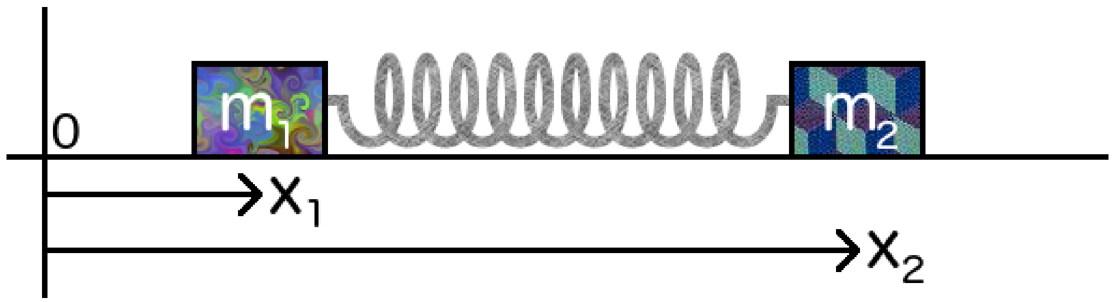


Figure 1: The two-mass oscillator system

In this system there are two masses attached horizontally by a spring. We will explore the scenario in which the spring's force is governed by Hooke's Law. We will use a coordinate system relative to an arbitrary fixed origin. The first object has mass m_1 and is positioned at x_1. For the second mass, m_2 and x_2 are defined similarly. The spring constant is determined by the parameter k, and the unstretched length of the spring is L.
The displacement by which the spring is stretch is given by $x_2 - x_1 - L$. Therefore, by Hooke's Law, we can determine the acceleration of each mass,

$$m_1 \frac{d^2 x_1}{dt^2} = k(x_2 - x_1 - L),$$
$$m_2 \frac{d^2 x_2}{dt^2} = -k(x_2 - x_1 - L).$$

From here, the following can be shown:

$$\frac{d^2}{dt^2}(m_1 x_1 + m_2 x_2) = 0, \tag{1}$$
$$\frac{d^2 z}{dt^2} = -k\left(\frac{1}{m_1} + \frac{1}{m_2}\right) z, \text{ where } z = x_2 - x_1 - L, \tag{2}$$

In the following sections we will assume these initial conditions:

$$x_1(0) = \alpha, \quad \frac{dx_1}{dt}(0) = \gamma$$

$$x_2(0) = \beta, \quad \frac{dx_2}{dt}(0) = \delta.$$

2 Analytical Work

We will now solve equation (1) analytically by integrating twice.

$$\begin{aligned}
&\frac{d^2}{dt^2}(m_1x_1 + m_2x_2) = 0 \\
\implies &\frac{d}{dt}(m_1x_1 + m_2x_2) = C \\
\implies &m_1x_1 + m_2x_2 = Ct + D,
\end{aligned}$$

for constants of integration C and D. Now we impose the initial conditions. We have the system of equations,

$$\begin{aligned}
m_1\alpha + m_2\beta &= D, \\
m_1\gamma + m_2\delta &= C,
\end{aligned}$$

and thus,

$$m_1x_1 + m_2x_2 = (m_1\gamma + m_2\delta)t + m_1\alpha + m_2\beta. \tag{3}$$

Now, we will solve equation (2) using an eigenvalue approach. If we do this then we will find that the roots of the characteristic polynomial are:

$$r = \pm\sqrt{k\left(\frac{1}{m_1} + \frac{1}{m_2}\right)}i$$

We will let $\omega = \sqrt{k\left(\frac{1}{m_1} + \frac{1}{m_2}\right)}$. This implies that the the general solution to equation (2) is:

$$z(t) = c_1\cos(\omega t) + c_2\sin(\omega t)$$

We will now impose the initial conditions in order to find c_1 and c_2. We will also back-substitute x_1 and x_2 into the equation, which results in:

$$x_2 - x_1 - L = (\beta - \alpha - L)\cos(\omega t) + \frac{\delta - \gamma}{\omega}\sin(\omega t) \tag{4}$$

3 Numerical Work

We used the MATLAB function `fsolve`, a nonlinear implicit equation solver, together with the analytically derived equations (3) and (4), to explore the system. We used two sets of initial conditions.

3.1 Volumeless Entity: The Spring Singularity

In this section we will explore what happens when we choose a set of initial conditions that maximizes the displacement of each individual mass. When this happens the two curves can be seen touching at a single point along the center of mass line. This is not physically possible because it implies that the spring has popped out of existence or has phased into the two masses. Even though it is not possible it is still interesting to look at and makes a really pretty picture. It turns out that it is a little bit difficult to find the correct constants for this to happen because we need the total maximum amplitude of each curve to be L, which would mean solving this equation for each constant:

$$\sqrt{(\beta - \alpha - L)^2 + \left(\frac{\delta - \gamma}{\omega}\right)^2} = L$$

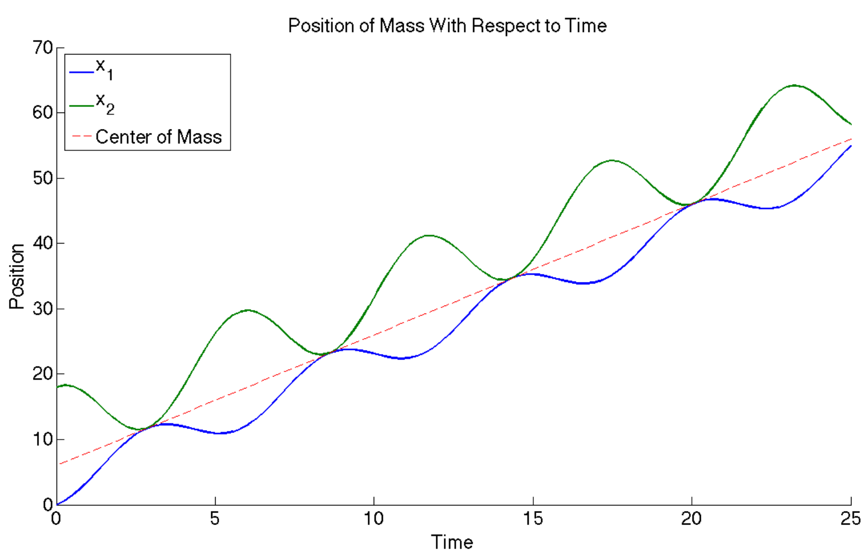
This is incredibly difficult, but we can impose some requirements that makes this a simpler task such as making one of the terms under the square root 0:

$$\delta - \gamma = 0 \implies \beta - \alpha = 2L$$

If we do this using the parameters

L	α	β	γ	δ	m_1	m_2	k
9	0	18	2	2	10	5	4

we then get a plot that looks like:



4 Discussion