Euler's case as the greatest mathematician of the 18th century

Leonhard Euler (1707-1783) was a Swiss mathematician who made pivotal contributions to a wide range of subjects including calculus, mechanics and number theory. It is near impossible to compare mathematicians of different eras due to the accumulating nature of mathematics and the everchanging landscape of societal education. Thus, this analysis shall draw comparisons between 18th century mathematicians with a focus on Euler and argue for his case as the greatest mathematician of that era.

Euler's contributions to mathematics and a wide variety of other disciplines are plentiful and the extensive list of his work cannot all be covered in this introductory analysis. Instead, this study aims to use specific parts of his work as supporting evidence in relation to a set of criteria. This set consists of three criterion that have been selected to serve as metrics for comparison between Euler and two other celebrated 18th century mathematicians, Laplace (1749-1827) and Lagrange (1736-1813). First, the immediate impact of Euler's work is assessed, followed by its longevity and relevance in modern developments. Then, the influence of the mathematician in question is judged.

There is no doubt that Euler's work had an instant impact on both the theoretical and practical advancements of his time. He built upon many theoretical questions posed by his predecessors such as Fermat's Last Theorem and the Basel problem. The former was a claim found in the margins of Pierre de Fermat's copy of Diophantus' *Arithmetica* by his son posthumously. The theorem states that no three natural numbers x, y and z satisfy the equation $x^n + y^n = z^n$ for any integer n greater than 2. Fermat proved it for n = 4 in the 17th century, and Euler managed to prove it for n = 3 using Fermat's method of infinite descent. This is one of mathematics' most famous theorems, and it will appear later as an example of Euler's vast influence.

Euler's solution to the Basel problem is even more impressive given that it had stumped many of the best mathematicians of the time, including members of the Bernoulli family, Leibniz and de Moivre. First proposed by Pietro Mengoli in 1644 and brought to the attention of the wider mathematics community by Jacob Bernoulli in 1699, the problem involves finding the infinite sum of squares of reciprocals of the natural numbers. Euler not only found the value of the infinite series; he extended his method to provide a general solution for the sum of all even powered reciprocals of the natural numbers. The most remarkable aspect of his solution was the appearance of the value π. It was completely unexpected that the ratio of the circumference of any circle to its diameter would form part of the result of the infinite sum (Sangwin, 2001).

The examples above illustrate that Euler built upon the work of his predecessors and solved problems that had defeated many of his contemporaries. Of course, there are a few others that made progress on the mathematical challenges of Euler's time. For example, Jacob and Johann Bernoulli developed Leibniz's infinitesimal calculus in the early 18th century. Euler was younger than both the Bernoulli brothers, and he was even tutored by Johann at the University of Basel. It was inevitable that he, together with Lagrange, would build upon the Bernoullis' work and develop the calculus of variations in the 1750s, although it would not be called as such till 1766. However, the importance of Euler's 1755 publication *Institutiones Calculi Differentialis* would only be recognized later, perhaps due to its radical representation of the calculus compared with previous textbooks. In assessing its initial impact, around 80% of all copies remained unsold 6 years after its publication. Nevertheless, it is known that Lagrange had the greatest respect for

Euler, and Laplace himself said: “Read Euler: he is our master in everything” (O’Connor and Robertson).

From Laplace’s quote, it is clear he held Euler in high esteem. Famously referred to as the “French Newton”, it is convenient to start the discussion on immediate practical impact with his major works on physics and astronomy. Laplace’s five-volume *Méchanique celeste* which was published between 1799-1825 offered a solution to the mechanical problem presented by the movements of the heavenly bodies. He applied Sir Isaac Newton’s gravitational theory to the solar system and “successfully accounted for all observed deviations of the planets from their theoretical orbits” (Whitrow, 2021). Furthermore, the Laplace transform which is named after its inventor appears in many branches of mathematics and engineering. Clearly, the man who placed Euler on such a high pedestal is himself a great astronomer and mathematician.

Euler also produced some notable works in this field. He devoted substantial effort to develop lunar theories, but only managed a partial solution as predicting the Moon’s motion was incredibly difficult before the emergence of computers due to its nature as a three-body-problem (Verdun, 2018). Even so, his first lunar theory publication in 1753 assisted the British Admiralty in calculating lunar tables to determine longitude at sea. In addition, Euler made some preliminary calculations on the behavior of the planet Uranus which later astronomers used to discover the next planet Neptune. Furthermore, it is not too much to say that his paper titled *Discovery of a New Principle of Mechanics* published in 1752 completely transformed the mechanics of rigid bodies. It was in this paper that he first presented the famous Euler equations.

It is simple enough to infer that both Laplace and Euler contributed much to the subjects of astronomy and physics. Conversely, trying to determine who was the greater mathematician is no trivial task. In this matter, the fact that a scientist as renowned as Laplace would refer to Euler as "…our master in everything" speaks for itself.

Moving on to the second criterion, longevity and relevance in modern developments. This can be described as the importance and impact of a mathematician's work in modern times. As an example, consider why no Greek mathematics text older than Euclid's *Elements* (c.300 BC) has survived. The *Elements* was so highly revered that it superseded all older mathematical texts. Furthermore, it served as a set of axioms to guide the logical exploration of scientist and mathematicians well into the 19th century, which demonstrates the importance of Euclid's work (Lambek, 2017). 18th century mathematicians are assessed in a similar manner with regards to their work on calculus, arguably the most important branch of modern mathematics.

Although the discovery of calculus was made by Newton and Leibniz independently, it was Leibniz's version which was built upon by the Bernoulli family as its rules and notation were easier to apply. Leibnizian calculus was initially concerned with "geometric quantities as embodied in the curve" (Bos, 1974). It has since evolved into many branches of mathematics such as stochastic calculus, which has applications in finance through models such as the Vasicek model. This model has become more applicable recently due to England's Monetary Policy Committee considering adding negative interest rates into their toolkit to support the economy given the events of 2020 (Tenreyro, 2021)**.** A huge leap must have taken place so calculus could advance from a tool to study geometric quantities into its use today. In particular, the 18th century saw a shift of interest from calculus as a study of curves to a means of expressing relations between quantities in general. Euler and Lagrange both significantly progressed this area of the calculus.

While the concept of a function was introduced before Euler, he was the one who extended it to involve more than one variable. This allowed the study of problems involving multiple degrees of freedom as opposed to only one degree of freedom (curves). Thus, it is not too much to say that Euler was the key figure in the separation of analysis from geometry. In 1908, Ball wrote that the material in the first volume of Euler's *Analysis Infinitorum* (1748) is "found in modern textbooks on algebra, theory of equations, and trigonometry". Furthermore, Euler integrated Leibniz's differential calculus and Newtonian laws of motion in his publication *Mechanica* (1738), which was important for its application and formulation of first order ordinary differential equations and second order differential equations. Along with the works of the Bernoulli family, Barrow-Green states that Euler's *Mechanica* established differential equations as a key part of the calculus (2021).

On the other hand, Lagrange is usually credited with introducing the derivative which replaced the differential as the fundamental concept of calculus. This is an important achievement, as the differential had many logical inconsistencies unbefitting of an underlying concept. Note that the notion of derivatives was only possible due to the emergence of the function as defined by Euler. Lagrange also made sizable contributions to the calculus of variations, and even used it to generalize results Euler had obtained pertaining to mechanics. His publication in 1788 called *Mécanique analytique* brought the field of mechanics into a new era of mathematical analysis and is important for its use of differential equations. The method of using Lagrange multipliers to solve minimization problems subject to constraints was physically interpreted in this book (Carpinteri and Paggi, 2014). In modern times, the method has applications in machine learning where it can be used in conjunction with iterative methods such as Newton-Raphson when attempting to minimize a cost function that does not have any minima.

The rise of analysis as a separate area of study was possibly the most crucial development for modern mathematics. It is difficult to suggest Euler's contribution in this field was clearly superior to Lagrange, as the latter rose to prominence after Euler and generalized much of his work. Indeed, there are contrasting opinions on this issue. Boyer praises Euler's work as having a similar impact on modern analytic geometry and trigonometry as Euclid's *Elements* did on ancient geometry (2021). Meanwhile, there are others such as Ball that believe Lagrange was the greatest mathematician of the 18th century (2012). To resolve this debate, one must consider the third and last criterion, the influence of the mathematicians in question.

Considering that the concept of the calculus of variations dates back thousands of years, it was only in the 18th century when it arguably saw its most sudden and rapid development (Roubicek, 2014). Perhaps a reason for this was the nature of collaboration between Euler and Lagrange. A 19-year-old Lagrange wrote to Euler in 1755 describing a new and less geometric dependent algorithm on the subject, which Euler had devoted "an important treatise 11 years earlier" (Knorr et all. 2020). Instead of rejecting or claiming those ideas as his own, Euler demonstrated humility and worked together with Lagrange, spending the next few years revising the calculus of variations using the newly introduced methods. He also put forward Lagrange as a candidate for election to the Berlin Academy. The dynamics between these two men suggests Euler had a hugely important influence on Lagrange's career. Stedall states "Lagrange always retained a respectful distance from Euler, whom he regarded as his elder and superior" (2012).

Lagrange himself became an influential mentor to Sophie Germain (1776-1831). Unlike previous female mathematicians, Germain is the first one known to have educated herself to at least the undergraduate level, capable of submitting written work to Lagrange (Laubenbacher and Pengelley, 2010). While facing great adversity from cultural expectations of women at the time, Germain contributed original works in elasticity and number theory, the former of which won her a French Academy prize. Germain was the one to make the next step in Fermat's Last Theorem after Euler had solved the case for n=3. She proved the theorem for a class of numbers for all n less than 100, and her method would be later extended by Legendre to include all n less than 197. The theorem was only recently completely solved by Andrew Wiles, right before the start of the 21st century. An aspect worth highlighting in Germain's story is her focus on books on analysis by Lagrange during the period of her self-education. Had Euler not extended courtesy to Lagrange back in 1755, there is a chance this work would not have existed. Euler's influence on Lagrange might even have had an impact on Lagrange's acceptance and mentoring of Germain. These points are contentious but give an example of the potential reach of Euler's seminal role in mathematical development, even after his death in 1783.

Euler's sphere of influence extends to many of the mathematical notation used today. Already seen previously was his notion of a function, for which he introduced f and parentheses. He also popularized the symbols π and e for pi and the base of the natural logarithm, respectively. Among others are the symbols for the sum and imaginary numbers, along with the format in which to label the sides of a triangle and their opposite angles. It is natural to question why Euler's notation has not been superseded or replaced by proposals from different mathematicians of the 18th century and beyond. The information already presented in this study suggests that Euler was regarded as the leading figure of mathematics in Europe by his peers, hence aside from their practicality, the respect he commanded along with his numerous publications may have made widespread adoption of his notation straightforward.

Euler's publications greatly influenced academics and non-academics alike. One of his projects that enjoyed immense success with the mass public is titled *Lettres à une Princesse d'Allemagne* which translates in English to *Letters to a German Princess*. Written originally to Princess Charlotte (1745-1808), this collection of epistles was translated into almost every major European language due to its concise and clear explanations of difficult concepts relating to physics and philosophy. This suggests Euler was a talented teacher who was able to express ideas “without recourse to formulas and equations” with “his gentle mode of persuasion” (Stén, 2021). His letters to Princess Charlotte are especially applicable in the current climate where pure mathematics is taught separately from other subjects and nearly all doctoral students are required to teach. Referring to Euler’s work would allow scholars to gain another perspective on the motivations behind mathematics, improve the way they convey information and provide a look into the mind of one the most brilliant mathematicians ever.

Overall, Euler built upon the works on his predecessors, solved problems his contemporaries could not and established analysis as a new branch of mathematics. He introduced and popularized many modern notations, and material from his work appeared in mathematics textbooks up till at least the 20th century. Due to his influence, other great mathematicians such as Lagrange and Laplace thrived in an academic environment that gave rise to collaboration. He was also a talented teacher whose work would serve as a good example for modern day scholars. These points place Euler as the premier mathematician of the 18th century.

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