LP: Duality Theory

Basic concepts

- **Dual problem** = a derived problem D associated with an original problem P (primal) whose solution is linked to the one of P
- In LP D is obtained by precise construction rules and the link between P and D is very strict
- We analyse the economic meaning of the link between P and D considering a Product Mix problem:
  - find the optimal production level in order to optimize the use of resources

(P) $\max\ \underline{c}^T \underline{x}$

$$A\underline{x} \le \underline{b}$$

$$\underline{x} \ge \underline{0}$$

Optimal solution

$$\underline{x}_B = B^{-1}\underline{b}$$

$$Z = \underline{c}_B{}^T B^{-1}\underline{b} \Rightarrow Z = \underline{y}^T\underline{b}$$

$$\underline{y}^T = \underline{c}_B{}^T B^{-1}$$

LP: Duality Theory

Basic concepts

- Optimal solution = the best way of valorising the resources
- Vector $\underline{y}$ = optimal unitary value of the resources (due to their transformation in the optimal level of the products)
- If we increase the availability of resource i of δ units

$$\underline{x}_B = B^{-1}\underline{b}$$

$$Z = \underline{c}_B{}^T B^{-1}\underline{b} \Rightarrow Z = \underline{y}^T \underline{b} \qquad b_i \leftarrow b_i + \delta \qquad Z = Z + y_i\delta$$

$$\underline{y}^T = \underline{c}_B{}^T B^{-1}$$

LP: Duality Theory

Basic concepts

- The optimality conditions for (P)

$$\underline{r} = \underline{c}_B{}^T B^{-1} N - \underline{c}_N{}^T \geq \underline{0}^T \quad \Rightarrow \quad \underline{y}^T N - \underline{c}_N{}^T \geq \underline{0}^T$$

$$\boxed{\sum_{i=1}^{m} a_{ij} y_i} \geq c_j \qquad \forall j \in R$$

Unitary value for (non basic) product j computed from the unitary value of resources y_i associated with the optimal solution

- The optimal resource values $\underline{y}$ are so that the *synthetic* values for any product not produced in the optimal mix is not lower by the true values of such products
- There is no convenience in changing the optimal production mix

LP: Duality Theory

Basic concepts

- A problem equivalent to (P):
  - Determine the minimum values $\underline{y}$ to assign to resources corresponding to the mix of production providing the maximum profit
- m variables (one for resource)
- n constraints (one for each product) stating that the synthetic product values are not lower to the real product values

(D)
$$\min \ \underline{b}^T \underline{y}$$
$$A^T \underline{y} \geq \underline{c}$$
$$\underline{y} \geq \underline{0}$$
$$\underline{y} \in \mathbf{R}^m$$

(D) is the dual of (P)

LP: Duality Theory

Definitions

A Dual problem can always be associated with a Primal problem

Primal Problem (P)

$$
\begin{aligned}
&\max \quad c_1 x_1 + \cdots + c_n x_n \\
&s.t. \\
&a_{11} x_1 + \cdots + a_{1n} x_n \le b_1 \\
&\vdots \\
&a_{m1} x_1 + \cdots + a_{mn} x_n \le b_m
\end{aligned}
$$

(n variables, m constraints)

x_j are the primal variables

Dual Problem (D)

$$
\begin{aligned}
&\min \quad b_1 y_1 + \cdots + b_m y_m \\
&s.t. \\
&a_{11} y_1 + \cdots + a_{m1} y_m \ge c_1 \\
&\vdots \\
&a_{1n} y_1 + \cdots + a_{mn} y_m \ge c_n
\end{aligned}
$$

(m variables, n constraints)

y_i are the dual variables

LP: Duality Theory

Symmetric form

- The pair (P) and (D) is the symmetric form of duality
- In matrix form

$$
\begin{array}{llcll}
(P) & \max\ \underline{c}^T \underline{x} & & (D) & \min\ \underline{b}^T \underline{y} \\
 & A\underline{x} \le \underline{b} & \Longleftrightarrow & & A^T \underline{y} \ge \underline{c} \\
 & \underline{x} \ge \underline{0} & & & \underline{y} \ge \underline{0} \\
 & \underline{x} \in \mathbb{R}^n & & & \underline{y} \in \mathbb{R}^m
\end{array}
$$

- (P) is a canonical form of maximization
- (D) is a canonical form of minimization
- Matrix A is transposed in the constraints of (D)

LP: Duality Theory

Symmetric form

Transformation rules

(P)		(D)	
max	$\Rightarrow$	min	constraints $\geq$
min	$\Rightarrow$	max	constraints $\leq$

Notes:

- The type of constraints of (D) is linked to the optimization sense of (P)
- Transformation rules are symmetric
- The dual of (D) is (P)

LP: Duality Theory

General form - Equality constraints

Problem (P) in standard form

$$(P)\ \max\ \underline{c}^T \underline{x}$$
$$A\underline{x} = \underline{b}$$
$$\underline{x} \geq \underline{0}$$
$$\underline{x} \in \mathbb{R}^n$$

writing $A\underline{x} = \underline{b}$ as $\begin{cases} A\underline{x} \leq \underline{b} \\ A\underline{x} \geq \underline{b} \ \Rightarrow \ -A\underline{x} \leq -\underline{b} \end{cases}$

$$(P)\ \max\ \underline{c}^T \underline{x}$$
$$A\underline{x} \leq \underline{b} \qquad (1)$$
$$-A\underline{x} \leq -\underline{b} \qquad (2)$$
$$\underline{x} \geq \underline{0}$$
$$\underline{x} \in \mathbb{R}^n$$

$\Longleftrightarrow$

$$(D)\ \min\ (\underline{b}^T \underline{u} - \underline{b}^T \underline{v})$$
$$A^T \underline{u} - A^T \underline{v} \geq \underline{c}$$
$$\underline{u} \geq \underline{0} \quad \underline{v} \geq \underline{0}$$
$$\underline{u}, \underline{v} \in \mathbb{R}^m$$

LP: Duality Theory

General form - Equality constraints

Introducing m dual variables $\underline{y}$ and substituting

$$\underline{y} = \underline{u} - \underline{v}$$

the final form of (D)

$$(D) \quad \min \underline{b}^T \underline{y}$$

$$A^T \underline{y} \geq \underline{c}$$

$$\underline{y} \quad free$$

$$\underline{y} \in \mathbb{R}^m$$

$\underline{y}$ are not constrained variables

LP: Duality Theory

General form

Transformation rules

(P)		(D)	
max	$\Rightarrow$	min	constraints $\geq$
min	$\Rightarrow$	max	constraints $\leq$
Constraint =	$\Rightarrow$	free variable	
Free variable	$\Rightarrow$	Constraint =	

The transformations are reversible: the dual of the dual is the primal

LP: Duality Theory

Why duality is important:

- Solutions of (P) and (D) are linked (solving (D) is equivalent to solving (P))
- Dual solutions has an economic interpretation relevant for the sensitivity analysis
- Solution algorithms exploiting duality: Dual Simplex, Primal-Dual Algorithm (alternative to the Primal Simplex)
- Often may be convenient solving the dual instead of primal:
  - Primal simplex use the inverse of the basis matrix $m \times m$
  - The computation burden is mainly due to the number of constraints
  - Solving the dual may be convenient if is has a smaller number of constraints

Duality: fundamental results

A pair of problems (P) and (D)

$$(P)\ \max\ \underline{c}^T\underline{x} \qquad\qquad (D)\ \min\ \underline{b}^T\underline{y}$$

$$A\underline{x} \le \underline{b} \qquad\qquad A^T\underline{y} \ge \underline{c}$$

$$\underline{x} \ge \underline{0} \qquad\qquad \underline{y} \ge \underline{0}$$

Weak Duality Theorem

Let $\underline{x}$ and $\underline{y}$ feasible solution respectively for (P) and (D), then

$$\underline{c}^T\underline{x} \le \underline{b}^T\underline{y}$$

Duality: fundamental results

Demonstration

Being $\underline{y}$ feasible for (D) then $A^T \underline{y} \geq \underline{c}$

Being $\underline{x}$ feasible for (P) then $A\underline{x} \leq \underline{b}$

multiplying

$$\underline{x} \geq \underline{0} \;\Rightarrow\; (A^T \underline{y})^T \underline{x} \geq \underline{c}^T \underline{x}$$

$$\underline{y} \geq \underline{0} \;\Rightarrow\; (A\underline{x})^T \underline{y} \leq \underline{b}^T \underline{y}$$

linking the two expressions

$$\underline{c}^T \underline{x} \leq (A^T \underline{y})^T \underline{x} = \underline{x}^T A^T \underline{y} = (A\underline{x})^T \underline{y} \leq \underline{b}^T \underline{y}$$

then

$$\underline{c}^T \underline{x} \leq \underline{b}^T \underline{y}$$

(qed)

Duality: fundamental results

Consequence:

The solution of one problem gives a bound for the other

$\min W = \underline{b}^T \underline{y}$	$\underline{b}^T \underline{y}$	Upper Bound for Z
$\max Z = \underline{c}^T \underline{x}$	$\underline{c}^T \underline{x}$	Lower Bound for W

Corollary (1)

If (P) is unbounded $\Rightarrow$ (D) is unfeasible

If (D) is unbounded $\Rightarrow$ (P) is unfeasible

Duality: fundamental results

Demonstration (1)

Assuming (D) has optimal finite solution then

$$W^* = \underline{b}^T \underline{y}^*$$

is upper bound for Z and (P) is bounded

Assuming (P) unbounded but that a feasible solution $\underline{y}^\circ$ exists for (D) then for the weak duality

$$\underline{c}^T \underline{x} \leq \underline{b}^T \underline{y}^\circ$$

contradicting the assumption that (P) is unbounded (qed)

Duality: fundamental results

Corollary (2)

If $\underline{x}^\circ$ is a feasible solution for (P) and $\underline{y}^\circ$ is a feasible solution for (D) such that

$$\underline{c}^T \underline{x}^\circ = \underline{b}^T \underline{y}^\circ$$

then they are optimal solution respectively for (P) and (D)

Demonstration (2)

Assume by absurd that $\underline{y}^\circ$ is not optimal for (D) then it must exist a different solution $\underline{y}$ such that

$\underline{b}^T \underline{y} < \underline{b}^T \underline{y}^\circ$ then $\underline{b}^T \underline{y} < \underline{b}^T \underline{y}^\circ = \underline{c}^T \underline{x}^\circ \quad \Rightarrow \underline{c}^T \underline{x}^\circ > \underline{b}^T \underline{y}$

contradicting $\underline{c}^T \underline{x} \leq \underline{b}^T \underline{y}^\circ$ (qed)

Duality: fundamental results

Strong Duality Theorem

If (P) has optimal feasible bounded solution $\underline{x}^*$ then (D) has optimal feasible bounded solution $\underline{y}^*$ and

$$\underline{c}^T \underline{x}^* = \underline{b}^T \underline{y}^*$$

Demonstration

Let (P) be $\max \quad \underline{c}^T \underline{x}$ in standard form Redefine

$$A\underline{x} \le \underline{b} \qquad A\underline{x} + I\underline{s} = \underline{b}$$

$$\underline{x} \ge \underline{0} \qquad \underline{x} \leftarrow \begin{bmatrix} \underline{x} \\ \underline{s} \end{bmatrix} \qquad A \leftarrow \begin{bmatrix} A & I \end{bmatrix}$$

Assume (P) has bounded optimum $\underline{x}$*

then the optimal basis is $\underline{x}^* = \begin{bmatrix} \underline{x}_B^* \\ \underline{x}_N^* \end{bmatrix} = \begin{bmatrix} B^{-1}\underline{b} \\ \underline{0} \end{bmatrix}$

Duality: fundamental results

Demonstration (cont.)

The optimal objective function is

$$Z^* = \underline{c}_B^{\,T} B^{-1} \underline{b} - (\underline{c}_B^{\,T} B^{-1} N - \underline{c}_N^{\,T}) \underline{x}^*_N$$

since it is optimal then $\underline{c}_B^{\,T} B^{-1} N - \underline{c}_N^{\,T} \geq \underline{0}^T$

Since the variable vector $\underline{x}$ has now $n+m$ elements, the non basic variables are $n+m-m$ then these are n relationships associated with them

Similarly we can consider the relationships associated with the basic variables that are trivially true

$$\underline{c}_B^{\,T} B^{-1} B - \underline{c}_B^{\,T} \geq \underline{0}^T$$

Duality: fundamental results

Demonstration (cont.)

Then we have the system of n+m inequalities

$$\left.\begin{array}{l} \underline{c}_B^{\ T} B^{-1} N - \underline{c}_N^{\ T} \geq \underline{0}^T \\ \underline{c}_B^{\ T} B^{-1} B - \underline{c}_B^{\ T} \geq \underline{0}^T \end{array}\right. \Rightarrow \underline{c}_B^{\ T} B^{-1} A - \underline{c}^T \geq \underline{0}^T$$

Substituting $A \leftarrow \begin{bmatrix} A & I \end{bmatrix}$ and splitting accordingly

$$\underline{c}_B^{\ T} B^{-1} A - \underline{c}^T \geq \underline{0}^T \qquad \text{then fixing} \quad \underline{y}^T = \underline{c}_B^T B^{-1}$$

$$\underline{c}_B^{\ T} B^{-1} I - \underline{0}^T \geq \underline{0}^T \qquad \text{we obtain} \quad \underline{y}^T A - \underline{c}^T \geq \underline{0}^T$$

$$\underline{y}^T \geq \underline{0}^T \quad \text{feasible solution for (D)}$$

The objective is $\underline{y}^T \underline{b} = \underline{c}_B^T B^{-1} \underline{b} = \underline{c}^T \underline{x}^*$

then by corollary (2) $\underline{y}$ is optimal for (D) (qed)

Duality: primal-dual relationships

Primal – Dual relationships

- The results are independent from the type of pair (P) and (D) considered
- Strong duality states the equivalence in solving the two problems
- Weak and Strong duality theorems highlight the link between (P) and (D) solutions:
  - The optimal basis B for (P) produces an optimal feasible solution for (D)
  - A feasible basis B for (P) produces a feasible solution for (D) if and only if B is optimal

Duality: primal-dual relationships

Consider a pair P-D and a basis B feasible for (P)

(P) $$\max \underline{c}_B^T \underline{x}_B + \underline{c}_N^T \underline{x}_N$$
$$B\underline{x}_B + N\underline{x}_N = \underline{b}$$
$$\underline{x}_B \geq \underline{0} \quad \underline{x}_N \geq \underline{0}$$

(D) $$\min \underline{b}^T \underline{y}$$
$$\underline{y}^T B \geq \underline{c}_B^T$$
$$\underline{y}^T N \geq \underline{c}_N^T$$
$$\underline{y} \ \mathit{free}$$

Since B is feasible

$$\underline{\overline{x}} \in X \Rightarrow B^{-1}\underline{b} \geq \underline{0}$$

Generating from B a solution $\underline{y}$

$$\underline{\overline{y}}^T = \underline{c}_B^T B^{-1}$$

Substituting in (D)

$$\underline{\overline{y}}^T \in Y \Rightarrow \begin{cases} a) & (\underline{c}_B^T B^{-1})B \geq \underline{c}_B^T \\ b) & (\underline{c}_B^T B^{-1})N \geq \underline{c}_N^T \end{cases}$$

Duality: primal-dual relationships

Consider a pair P-D and a basis B feasible for (P)

We obtain

$$\underline{\bar{y}}^T \in Y \Rightarrow \begin{cases} a) & (\underline{c}_B^T B^{-1})B \geq \underline{c}_B^T \\ b) & (\underline{c}_B^T B^{-1})N \geq \underline{c}_N^T \end{cases}$$

$$a) \quad \underline{c}_B^T \geq \underline{c}_B^T \quad \text{(always true)}$$

$$b) \quad \underline{c}_B^T B^{-1} N - \underline{c}_N^T \geq \underline{0}$$

$$\boxed{\underline{c}_B^T B^{-1} \underline{a}_j - c_j \geq 0 \qquad j \in R}$$

Optimality conditions on reduced costs of (P)

Basis B is optimal and the pair

$$(\underline{\bar{x}}, \underline{\bar{y}})$$

includes feasible solution for both (P) and (D) and then

$$\underline{c}^T \underline{\bar{x}} = \underline{b}^T \underline{\bar{y}}$$

Duality: primal-dual relationships

Note that

- Only the optimal basis for (P) generates a feasible (and optimal) solution for (D)
- The basis for (P) at a generic simplex iteration provides the *simplex multipliers* vector

$$\underline{\pi} = \underline{c}_B^T B^{-1}$$

that is not a feasible solution to (D).

Duality: economic interpretation

Consider problem (P) $\max \ \underline{c}^T \underline{x}$ a (non degenerate) solution

$$A\underline{x} \leq \underline{b}$$
$$\underline{x} \geq \underline{0}$$

$$\underline{x}^* = \begin{bmatrix} \underline{x}_B^* \\ \underline{x}_N^* \end{bmatrix} = \begin{bmatrix} B^{-1}\underline{b} \\ \underline{0} \end{bmatrix} \qquad \underline{x}_B^* > \underline{0}$$

A small variation $\Delta\underline{b}>0$ of $\underline{b}$ does not change the optimal basis B

$$\underline{x}^* = \begin{bmatrix} \underline{x}_B^* + \Delta\underline{x}_B^* \\ \underline{x}_N^* \end{bmatrix} = \begin{bmatrix} B^{-1}(\underline{b} + \Delta\underline{b}) \\ \underline{0} \end{bmatrix}$$

The objective changes

$$x_0 = \underline{c}_B^T B^{-1}(\underline{b} + \Delta\underline{b}) \Rightarrow \Delta x_0 = \underline{c}_B^T B^{-1}\Delta\underline{b} \Rightarrow \Delta x_0 = \underline{y}^{*T} \Delta\underline{b}$$

$\underline{y}$* are the marginal prices (values) of resources (***shadow prices***)

Duality: economic interpretation

The optimal tableau for the paint production example

The reduced costs for slack variables = shadow prices

$$r_k = \underline{c}_B^T B^{-1} \underline{a}_k - c_k = \underline{c}_B^T B^{-1} \underline{e}_k = \underline{y}^{*T} \underline{e}_k = y_k^*$$

	x_1	x_2	x_3	x_4	x_5	
x_0	0	0	1/3	4/3	0	38/3
x_2	0	1	2/3	−1/3	0	4/3
x_1	1	0	−1/3	2/3	0	10/3
x_5	0	0	−2/3	1/3	1	2/3

$$\Rightarrow \underline{y}^{*T} = [1/3 \quad 4/3 \quad 0]$$

why $y_3^*=0$?

- it is worth increasing only the scarse resources
- For abundant resources $y_i^*=0$

Duality: complementary slackness

Consider a pair (P) (D) in canonical form and convert them into standard form

(P) $\max \underline{c}^T \underline{x}$

$A\underline{x} \leq \underline{b}$

$\underline{x} \geq \underline{0}$

$\rightarrow$

$A\underline{x} + I\underline{s} = \underline{b}$

$\underline{x} \geq \underline{0}$ n var.

$\underline{s} \geq \underline{0}$ m slack var.

(D) $\min \underline{b}^T \underline{y}$

$A^T \underline{y} \geq \underline{c}$

$\underline{y} \geq \underline{0}$

$\rightarrow$

$A^T \underline{y} - I\underline{v} = \underline{c}$

$\underline{y} \geq \underline{0}$ m var.

$\underline{v} \geq \underline{0}$ n surplus var.

- Primal constraints $\Leftrightarrow$ slack and dual variables
- Dual constraints $\Leftrightarrow$ surplus and primal variables

Duality: complementary slackness

A pair of solutions optimal respectively for (P) and (D) are said *complementary*

Complementary Slackness Theorem

Given a pair $\underline{x}$ and $\underline{y}$ respectively feasible solution for (P) and (D), $\underline{x}$ and $\underline{y}$ are optimal for (P) and (D) if and only if

$$s_i \cdot y_i = (b_i - \underline{a}^i \underline{x}) \cdot y_i = 0 \qquad i = 1, \dots, m$$

$$v_j \cdot x_j = (\underline{a}_j^T \underline{y} - c_j) \cdot x_j = 0 \qquad j = 1, \dots, n$$

where $\underline{a}^i$ i-th row of A

$\underline{a}_j$ j-th column of A

(Demonstration by exercise)

Duality: complementary slackness

Consequences of CS Theorem :

a. $x_j > 0 \Rightarrow \underline{a}_j^T \underline{y} = c_j$ (dual constr. saturated: v_j=0)

b. $\underline{a}_j^T \underline{y} > c_j \Rightarrow x_j = 0$ (dual constr. non saturated: v_j>0)

c. $y_i > 0 \Rightarrow \underline{a}^i \underline{x} = b_i$ (primal constr. saturated: s_i=0)

d. $\underline{a}^i \underline{x} < b_i \Rightarrow y_i = 0$ (primal constr. non saturated: s_i>0)

Pairs P-D: economic interpretation

Product Mix problem

Determine the optimal production levels for a set of products in order to maximize the profit from their sale, respecting the limited availability of the necessary production resources

(P) $\max \ \underline{c}^T \underline{x}$

$$A\underline{x} \le \underline{b}$$

$$\underline{x} \ge \underline{0}$$

x_j	production level for j-th product
c_j	unitary profit for j-th product
b_i	availability of i-th resource
a_{ij}	quantity of i-th resource need for producing a unit of j-th product

i-th constraint of (P)

$$\underline{a}^i \underline{x} \le b_i$$

the total usage of the i-th resource cannot exceed its availability

Pairs P-D: economic interpretation

Dual of Product Mix problem

Having chosen to sell the productive resources, determine their minimum unit price convenient for selling, so that their sale is at least as convenient than selling the products produced with the resources

(D) $\min \ \underline{b}^T \underline{y}$

$A^T \underline{y} \geq \underline{c}$

$\underline{y} \geq \underline{0}$

y_i	unit price of i-th resource
c_j	unit profit for j-th product
b_j	availability of i-th resource
a_{ij}	quantity of i-th resource need for producing a unit of j-th product

j-th constraint of (D)

$\underline{a}_j^T \underline{y} \geq c_j$

the unit value for j-th product, computed from the price of the resources need for its production, must be not lower than the actual unitary price of such product

Pairs P-D: economic interpretation

Complementary slackness

- Primal:
  - The value of a resource (<u>optimal dual variable</u>) is positive if the resource is scarce, i.e., when the associated slack is zero

$$s_i = b_i - \underline{a}^i \underline{x} \Rightarrow s_i y_i = 0$$

- Dual
  - The production level of a product (optimal primal variable) is positive if the unitary profit from its sale equals the profit from the sale of the resources needed fot its production at their optimal (minimum) price, i.e., when the surplus of revenue from the resource sale is zero

$$v_j = \underline{a}_j^T \underline{y} - c_j \Rightarrow x_j v_j = 0$$

Pairs P-D: economic interpretation

Diet problem (blending)

Determine the cheapest balanced diet buying n different foods. A diet is balanced if it meets the yearly minimum levels of calories and other nutritional elements (e.g., protein, calcium, iron, vitamins). Determine the quantity that must be purchased for each food, minimizing the total cost by meeting the minimum nutritional levels

$$
\text{(P)}\quad \min\ \underline{c}^T \underline{x} \qquad A\underline{x} \geq \underline{b} \qquad \underline{x} \geq \underline{0}
$$

x_j	quantity of j-th food to purchase
c_j	unit cost for j-th food
b_i	minimum level for i-th nutrient
a_{ij}	quantity of i-th nutrient in a unit of j-th food

i-th constraint of (P)

$$\underline{a}^i \underline{x} \geq b_i$$

the total quantity of i-th nutritional element provided by purchased foods must be at least the minimum level for the balanced diet

Pairs P-D: economic interpretation

Dual of Diet problem

If we want to individually buy m nutrients (e.g., in pills) to get the balanced diet, determine the maximum price for individual elements so that their purchase is competitive with that of foods containing them.

(D) $\max \ \underline{b}^T \underline{y}$

$A^T \underline{y} \leq \underline{c}$

$\underline{y} \geq \underline{0}$

y_i	maximum price of i-th nutrient
c_j	unit cost of j-th food
b_i	minimum level for i-th nutrient
a_{ij}	quantity of i-th nutrient in a unit of j-th food

j-th constraint of (D)

$\underline{a}_j^T \underline{y} \leq c_j$

the unit price for j-th "synthetic" food (i.e., the sum of the prices of the quantities of nutrients provided by a unit of j-th food) must not exceed the actual price of j-th food

Solution methods for LP

Primal Simplex

It determines a sequence of solutions feasible for (P) trying to reach dual feasibility (i.e., primal optimality).

– At each iteration = feasible sub-optimal solutions
– Termination = optimal solution

Dual Simplex

It determines a sequence of solutions feasible for (D) trying to reach primal feasibility

– At each iteration = *dual-feasible* solutions (super-optimal; optimal but not feasible for (P)
– Termination = feasible solution for (P)

Solution methods for LP

Primal-Dual Method

It determines a sequence of solution pair ($\underline{x}$, $\underline{y}$) respectively feasible for (P) and (D) trying to satisfy the complementary slackness conditions

Internal point methods

- Ellipsoid method (Khachiyan, 1979) (first polynomial algorithm for LP; no competitive implementations)
- Karmakar algorithm (1984)