LP: algebraic solution

- Consider (P) a LP in standard form

$$(P)\ \max\ x_0 = \underline{c}^T\,\underline{x}$$
$$A\underline{x} = \underline{b}$$
$$\underline{x} \geq \underline{0}$$
$$\underline{x} \in R^n$$

- Assume (P) is feasible and let B a feasible basis
- Optimality test

$$x_0 = \underline{c}^T\,\underline{x} = [\underline{c}_B^T \quad \underline{c}_N^T]\begin{bmatrix}\underline{x}_B \\ \underline{x}_N\end{bmatrix} = \underline{c}_B^T\,\underline{x}_B + \underline{c}_N^T\,\underline{x}_N \quad (1)$$

LP: algebraic solution

Example

$$\max\ x_0 = 2x_1 + x_2$$

$$\begin{aligned}
x_1 + x_2 + x_3 \quad &= 5 &\quad (e1)\\
-x_1 + x_2 \quad + x_4 \quad &= 0 &\quad (e2)\\
6x_1 + 2x_2 \quad + x_5 &= 21 &\quad (e3)
\end{aligned}$$

$$x_1 \geq 0\ x_2 \geq 0\ x_3 \geq 0\ x_4 \geq 0\ x_5 \geq 0$$

starting solution $\underline{x}_{B_2} = \begin{bmatrix} x_1 \\ x_2 \\ x_5 \end{bmatrix}$ objective function can be rewritten

$$x_0 = [2 \quad 1 \quad 0]\begin{bmatrix} x_1 \\ x_2 \\ x_5 \end{bmatrix} + [0 \quad 0]\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} \quad (e0)$$

LP: algebraic solution

The values of basic variables $\underline{x}_B = B^{-1}\underline{b} - B^{-1}N\,\underline{x}_N$ (2)

Substituting (2) in (1) $x_0 = \underline{c}_B^T B^{-1}\underline{b} - \left(\underline{c}_B^T B^{-1}N - \underline{c}_N^T\right)\underline{x}_N$ (3)

For a BFS $\underline{x}_N = \underline{0} \;\Rightarrow \underline{x}_B = B^{-1}\underline{b}$

$$\Rightarrow x_0 = \underline{c}_B^T B^{-1}\underline{b}$$

LP: algebraic solution

Example: rewriting the constraints equations

$$\underline{x}_{B_2} = \begin{bmatrix} x_1 \\ x_2 \\ x_5 \end{bmatrix} = B_2^{-1}\underline{b} - B_2^{-1}N\underline{x}_N = \begin{bmatrix} 1/2 & -1/2 & 0 \\ 1/2 & 1/2 & 0 \\ -4 & 2 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 0 \\ 21 \end{bmatrix} - \begin{bmatrix} 1/2 & -1/2 & 0 \\ 1/2 & 1/2 & 0 \\ -4 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$$

same result obtained from the linear system of equations

$$x_1 = 5/2 - 1/2\, x_3 + 1/2\, x_4 \qquad (e1)$$

$$x_2 = 5/2 - 1/2\, x_3 - 1/2\, x_4 \qquad (e2)$$

$$x_5 = 1 + 4x_3 - 2x_4 \qquad (e3)$$

LP: algebraic solution

Equations (2) and (3) in matrix form

$$\begin{bmatrix} x_0 \\ \underline{x}_B \end{bmatrix} = \begin{bmatrix} \underline{c}_B^T B^{-1} \underline{b} \\ B^{-1} \underline{b} \end{bmatrix} - \begin{bmatrix} \underline{c}_B^T B^{-1} N - \underline{c}_N^T \\ B^{-1} N \end{bmatrix} \underline{x}_N \quad (4)$$

$m+1$ equations (objective function and m constraints) in n variables

Example

$$x_0 = 15/2 - (3/2\,x_3 - 1/2\,x_4) \quad (e0)$$
$$x_1 = 5/2 - (1/2\,x_3 - 1/2\,x_4) \quad (e1)$$
$$x_2 = 5/2 - (1/2\,x_3 + 1/2\,x_4) \quad (e2)$$
$$x_5 = 1 - (-4x_3 + 2x_4) \quad (e3)$$

LP: algebraic solution

Rewrite(4) in algebraic form

Let:

- R set of indexes of non basic variables (i.e., columns of N)
- $x_{B_0} = x_0$ variable associated with objective value
- m basic variables $\underline{x}_B = \begin{bmatrix} x_{B_1} \\ \vdots \\ x_{B_m} \end{bmatrix}$
- Rhs of equations as a ($m+1$)-dim vector

$$\underline{y}_0 = \begin{bmatrix} \underline{c}_B^T B^{-1} \underline{b} \\ B^{-1} \underline{b} \end{bmatrix} = \begin{bmatrix} y_{00} \\ y_{10} \\ \vdots \\ y_{m0} \end{bmatrix}$$

values of the objective function

values of basic variables

LP: algebraic solution

– The coefficients of the equations (columns associated with non basic variables)

$$\underline{y}_j = \begin{bmatrix} \underline{c}_B^T B^{-1} \underline{a}_j - c_j \\ B^{-1} \underline{a}_j \end{bmatrix} = \begin{bmatrix} y_{0j} \\ y_{1j} \\ \vdots \\ y_{mj} \end{bmatrix} \qquad \forall j \in R$$

– $\underline{y}_j$ are n-m vectors of ($m+1$)-dim where
  - a_j is the column of matrix N of the coefficients of the j-th non basic variable
  - c_j is the coefficient in c_N of the j-th non basic variable

Then (4) can be rewritten as

$$x_{B_i} = y_{i0} - \sum_{j \in R} y_{ij} x_j \quad \forall i = 0,1,\ldots,m \quad (5)$$

LP: algebraic solution

Example

$$x_0 = 15/2 - (3/2\, x_3 - 1/2\, x_4) \quad (e0)$$
$$x_1 = 5/2 - (1/2\, x_3 - 1/2\, x_4) \quad (e1)$$
$$x_2 = 5/2 - (1/2\, x_3 + 1/2\, x_4) \quad (e2)$$
$$x_5 = 1 - (-4x_3 + 2x_4) \quad (e3)$$

where

Value of objective

coeff. of x_3 ($j=3$)

coeff. of x_4 ($j=4$)

$x_{B_0} = x_0$	$y_{00} = 15/2$	$y_{03} = 3/2$	$y_{04} = -1/2$
$x_{B_1} = x_1$	$y_{10} = 5/2$	$y_{13} = 1/2$	$y_{14} = -1/2$
$x_{B_2} = x_2$	$y_{20} = 5/2$	$y_{23} = 1/2$	$y_{24} = 1/2$
$x_{B_3} = x_5$	$y_{30} = 1$	$y_{33} = -4$	$y_{34} = 2$

Value of basic variables

LP: optimality condition

- Equation (5) expresses the objective and constraints of problem (P) in standard form with respect to the (initial) basis B

$$x_{B_i} = y_{i0} - \sum_{j\in R} y_{ij} x_j \quad \forall i = 0,1,\ldots,m \qquad (5)$$

- Fixing in (5) $x_j=0\ \forall j\in R$ we obtain the objective value ($i=0$) and the basic solution ($i=1,\ldots,m$) for the current basis
- Let consider the objective equation ($i=0$)

$$x_{B_0} = y_{00} - \sum_{j\in R} y_{0j} x_j \quad (5a)$$

- Assume that in (5a) a coefficient of non basic variable x_k is $y_{0k}<0$
- How does change the objective value if x_k increases from zero?

LP: optimality condition

- Rewriting (5a) as function of x_k

Positive term added to y_{0k}

$$x_{B_0} = y_{00} - \overbrace{y_{0k}\,x_k}^{>0} > y_{00}$$

The objective increases (the solution improves)

y_{0k} is negative <0 $\quad$ >0 x_k becomes positive

Theorem (Optimality condition)

A basic solution of an LP problem is optimal if

1) $y_{i0} \geq 0 \quad i=1,\ldots,m$ (feasible)

2) $y_{0j} \geq 0 \;\forall j \in R$ (cannot be improved)

LP: change of basis

- If in (5a) the coefficient of a non basic variable x_k is $y_{0k}<0$ then the current basic solution is not optimal since the objective can be improved by increasing the value of x_k from zero
- The increase of x_k however could not in general be unbounded: increasing x_k also the values of the current basic variables in (5) change
- In the generic i-th equation (basic variable) when increasing x_k :

$$x_{B_i} = y_{i0} - \sum_{j \in R} y_{ij} x_j \quad i \neq 0 \quad \text{(5b)}$$

$$x_{B_i} = y_{i0} - y_{ik} x_k$$

- If $y_{ik}>0$ then increasing x_k the value of x_{B_i} decreases
- The maximum value for x_k is the one ensuring the solution feasibility:

$$x_{B_i} \geq 0 \qquad \forall i = 1,\dots, m$$

LP: change of basis

- The maximum value for x_k is the minimum such that for a basic variable i

$$x_k = \frac{y_{i0}}{y_{ik}} \Rightarrow x_{B_i} = 0$$

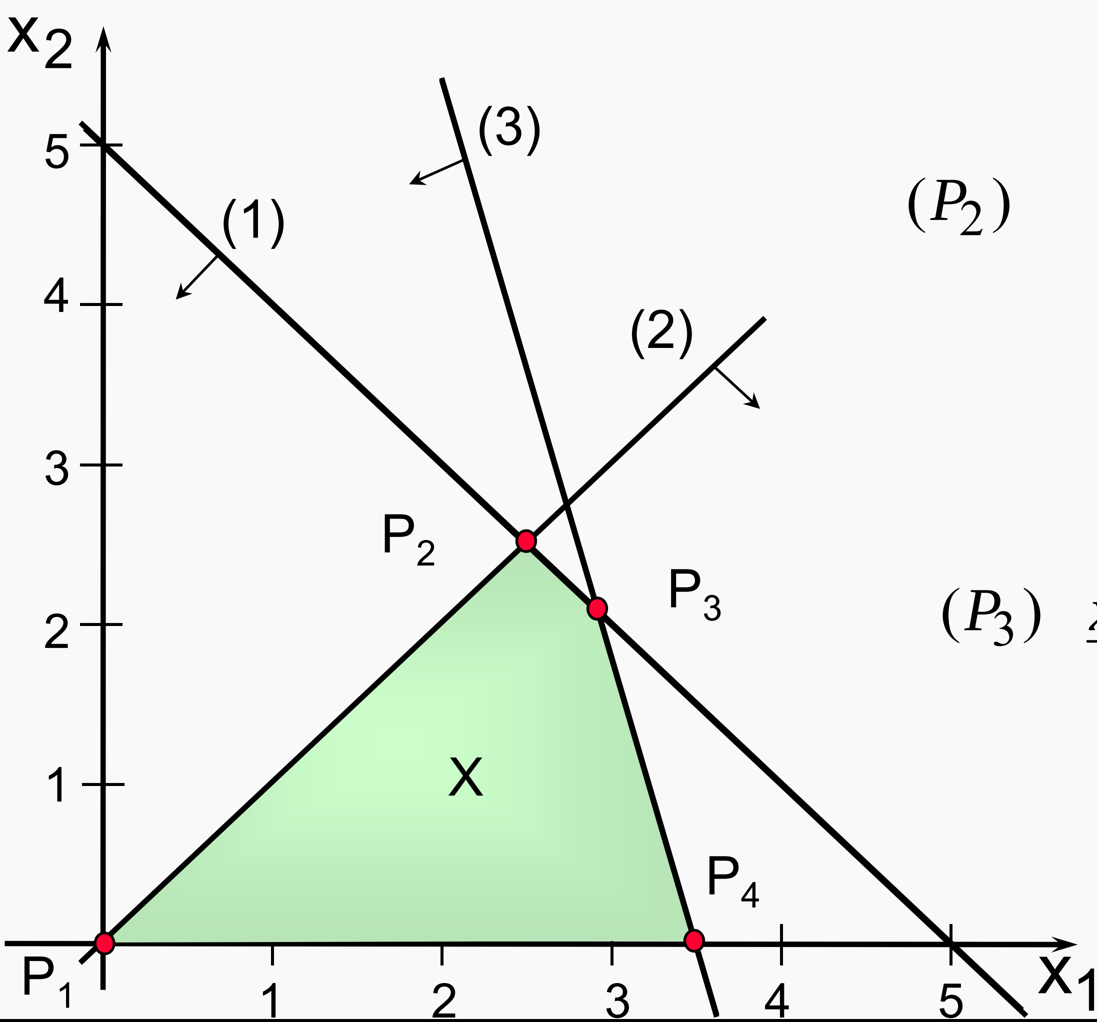
- Therefore x_k ***enters in the basis*** with such value and since the basis includes m variables, a variable must leave the current basis becoming non basic
- The leaving variable is the one in the current basis that first reaches zero due to the increase of x_k
- These computations are called *change of basis*
- In general, the value of the entering x_k is

$$\frac{y_{r0}}{y_{rk}} = \min_{\substack{i=1,\ldots,m \\ y_{ik}>0}} \left\{ \frac{y_{i0}}{y_{ik}} \right\}$$ being x_{B_r} the leaving variable

LP: change of basis – the geometry

Change of basis $\Rightarrow$ move to an adjacent vertex

quando cambio punto una variabile entra ed una esce, non mi è chiarissi cosa siano x12345. ma devi guardare quali sono le rette che intersecano il punto



P_2 and P_3 are adjacent

$$(P_2)\qquad \underline{x}_{B_2} = \begin{bmatrix} x_1 \\ x_2 \\ x_5 \end{bmatrix} \qquad \underline{x}_{N_2} = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$$

leaving

entering

$$(P_3)\qquad \underline{x}_{B_3} = \begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix} \qquad \underline{x}_{N_3} = \begin{bmatrix} x_3 \\ x_5 \end{bmatrix}$$

LP: change of basis

Pivoting: the algebraic computation for changing a basis

- Coefficient y_{rk} is called *pivot* and it is used to update the current basis when x_k enters the basis
- In (5) x_k replaces x_{B_r} in the new basis

devo scrivere la nuova variabile in funzione di

$$x_k = \frac{y_{r0}}{y_{rk}} - \sum_{j \in R-\{k\}} \frac{y_{rj}}{y_{rk}} x_j - \frac{1}{y_{rk}} x_{B_r} \quad (5'r)$$

- Then substituting x_k in the other equation (5)

sostituisco poi la funzione calcolata prima

$$x_{B_i} = y_{i0} - y_{ik} \frac{y_{r0}}{y_{rk}} - \sum_{j \in R-\{k\}} \left(y_{ij} - y_{ik} \frac{y_{rj}}{y_{rk}} \right) x_j + \frac{y_{rj}}{y_{rk}} x_{B_r} \quad (5'i)$$

$$\forall i = 0,1,\ldots,m \quad i \neq r$$

LP: change of basis

- Equations (5’r) and (5’i) are the new (5) after the change
- Fixing to zero the new non basic variables

$$x_j = 0 \quad \forall j \in R - \{k\} \qquad x_{B_r} = 0$$

we obtain the new solution (BFS)

$$x_k = \frac{y_{r0}}{y_{rk}} \qquad x_{B_i} = y_{i0} - y_{ik}\frac{y_{r0}}{y_{rk}} \qquad \forall i = 0, \ldots, m \quad i \neq r$$

LP: change of basis

Example: change of basis from vertex P_2 to vertex P_3 (x_4 entering)

$$(P_2) \qquad \underline{x}_{B_2} = \begin{bmatrix} x_1 \\ x_2 \\ x_5 \end{bmatrix} \qquad \underline{x}_{N_2} = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$$

$$\begin{aligned} x_1 &= 5/2 - (1/2\,x_3 - 1/2\,x_4) && (e1) \\ x_2 &= 5/2 - (1/2\,x_3 + 1/2\,x_4) && (e2) \\ x_5 &= 1 - (-4x_3 + 2x_4) && (e3) \end{aligned}$$

$$\begin{aligned} x_2 &= 5/2 - 1/2\,x_4 && (e2) \\ x_5 &= 1 - 2x_4 && (e3) \end{aligned}$$

$$\begin{aligned} x_2 &= 0 \Rightarrow x_4 = 5 && (e2) \\ x_5 &= 0 \Rightarrow x_4 = 1/2 && (e3) \end{aligned}$$

$$x_4 = 1/2 - (-2x_3 + 1/2\,x_5) \qquad (e3')$$

$$\begin{aligned} x_1 &= 11/4 - (-1/2\,x_3 + 1/4\,x_5) && (e1') \\ x_2 &= 9/4 - (3/2\,x_3 - 1/4\,x_5) && (e2') \end{aligned}$$

LP: change of basis

The equations for the new basis

$$x_1 = 11/4 - (-1/2\, x_3 + 1/4\, x_5) \quad (e1')$$
$$x_2 = 9/4 - (3/2\, x_3 - 1/4\, x_5) \quad (e2')$$
$$x_4 = 1/2 - (-2 x_3 + 1/2\, x_5) \quad (e3')$$

$$(P_3) \quad \underline{x}_{B_3} = \begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 11/4 \\ 9/4 \\ 1/2 \end{bmatrix} \qquad \underline{x}_{N_3} = \begin{bmatrix} x_3 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 1 & 1 & \boxed{0} \\ -1 & 1 & \boxed{0} \\ 6 & 2 & \boxed{1} \end{bmatrix} \qquad A = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & \boxed{0} & \boxed{0} \\ -1 & 1 & 0 & \boxed{1} & \boxed{0} \\ 6 & 2 & 0 & \boxed{0} & \boxed{1} \end{bmatrix} \end{array} \qquad B_3 = \begin{bmatrix} 1 & 1 & \boxed{0} \\ -1 & 1 & \boxed{1} \\ 6 & 2 & \boxed{0} \end{bmatrix}$$

LP: change of basis

- Updating (5) for i=0 (objective) we obtain the improved solution

$$x_{B_0} = y_{00} - \underbrace{y_{0k}}_{<0} \overbrace{\frac{y_{r0}}{\underbrace{y_{rk}}_{>0}}}^{>0} > y_{00} \qquad \text{(5') for i=0}$$

Example: substituting x_4 in $(e0)$

$$x_4 = 1/2 - (-2x_3 + 1/2\,x_5) \qquad (e3')$$

$$x_0 = 15/2 - (3/2\,x_3 - 1/2\,x_4) \quad (e0) \Longrightarrow x_0 = 31/4 - (1/2\,x_3 + 1/4\,x_5) \quad (e0')$$

The objective increased from 7.5 to 7.75

This solution is **optimal** … why?

il simplesso poi di fatto è solo un cambiare la base con un metodo sensato dietro

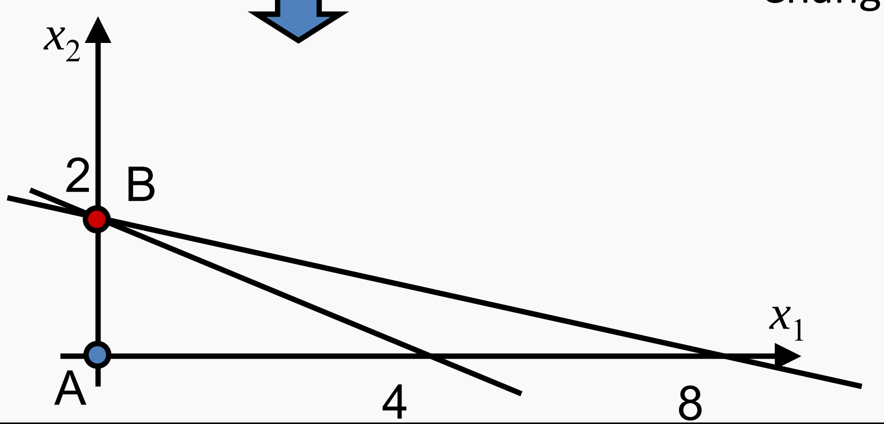
Degenerate solutions

- For an optimal degenerate solution optimality condition (2) may not hold
- However, if a finite optimal solution exists, then a basis exists satisfying the optimality conditions

$$\max\ x_0 = 3x_1 + 9x_2$$
$$x_1 + 4x_2 \le 8 \qquad (e1)$$
$$x_1 + 2x_2 \le 4 \qquad (e2)$$
$$x_1 \ge 0\ x_2 \ge 0$$

In standard form the BFS associated with A

$$x_0 = 0 - (-3x_1 - 9x_2) \quad (e0)$$
$$x_3 = 8 - (x_1 + 4x_2) \quad (e1)$$
$$x_4 = 4 - (x_1 + 2x_2) \quad (e2)$$



Change in vertex B when entering x_2

$$x_3 = 0 \Rightarrow x_2 = 2 \qquad (e1)$$
$$x_4 = 0 \Rightarrow x_2 = 2 \qquad (e2)$$

Two equivalent leaving variables: x_3 or x_4

Degenerate solutions

- Vertex B is *overdetermined*

Change of basis when leaving variable is x_3

$$x_0 = 18 - (-3/4\, x_1 + 9/4\, x_3) \qquad (e0')$$
$$x_2 = 2 - (1/4\, x_1 + 1/4\, x_3) \qquad (e1')$$
$$x_4 = 0 - (1/2\, x_1 - 1/2\, x_3) \qquad (e2')$$

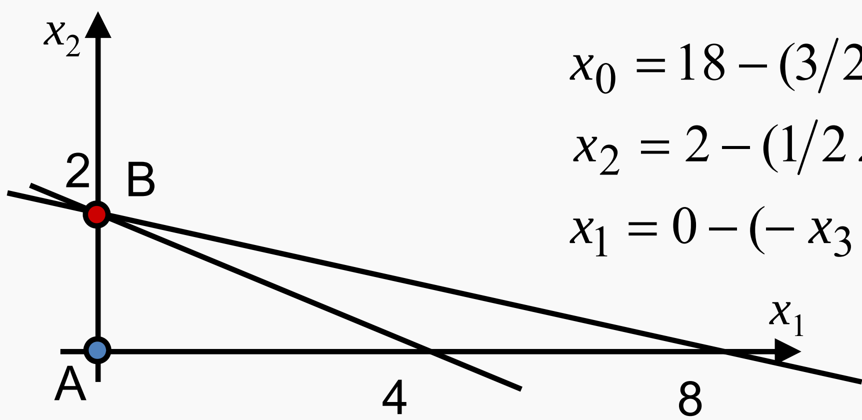
The solution is improved (18) but it is not optimal

x_1 can enter the basis

$$x_2 = 0 \Rightarrow x_1 = 8 \qquad (e1')$$
$$x_4 = 0 \Rightarrow x_1 = 0 \qquad (e2')$$

x_1 enters with value zero replacing x_4

The new basis is still vertex B



$$x_0 = 18 - (3/2\, x_3 + 3/2\, x_4) \qquad (e0'')$$
$$x_2 = 2 - (1/2\, x_3 - 1/2\, x_4) \qquad (e1'')$$
$$x_1 = 0 - (-\, x_3 + 2\, x_4) \qquad (e2'')$$

The objective is 18 but the solution is optimal

The last pivoting was not necessary

Degenerate solutions

- In case of degenerate solution the computation may be misled: some not necessary pivoting may occur
- This phenomenon is called **cycling**

quando il simplesso ripetutamente cambia base ma rimane sempre nello stesso vertice pensando di star migliorando la situazione.

- Theoretically the computation may enter an infinite loop
- However this is quite rare and strategies exist to prevent it

LP: choosing the entering variable

- If the current BSF is not optimal there could be more alternative entering variables
- The choice of the entering variable may affect the computation time but not prevent to find the optimal solution
- Two possible selection criteria for choosing the entering non basic x_k

a) steepest ascent (Dantzig) method $$y_{0k} = \min_{\substack{j \in R \\ y_{0j} < 0}} y_{0j}$$

è meglio fare questo che il max poiche dal punto di vista computazionale è molto migliore

b) largest increase: compute the actual objective increase and choose the maximum

$$k = \arg \max_{\substack{j \in R \\ y_{0j} < 0}} \left(- y_{0j} \frac{y_{r0}}{y_{rj}} \right)$$

scelgo la variabile che comporta il mx di questa funzione (penso sia quello negativo piu grande)

r is the index of the leaving variable due to the entering x_j

LP: choosing the leaving variable

Unbounded solution

Two possibilities when entering a non basic variable:

a) $y_{rk}>0$ for r (already considered; geometric interpretation?)

b) $y_{ik}\leq 0 \ \forall i=1,\ldots,m \ \Rightarrow$ unbounded solution

Increasing x_k no current basic variable decreases to zero

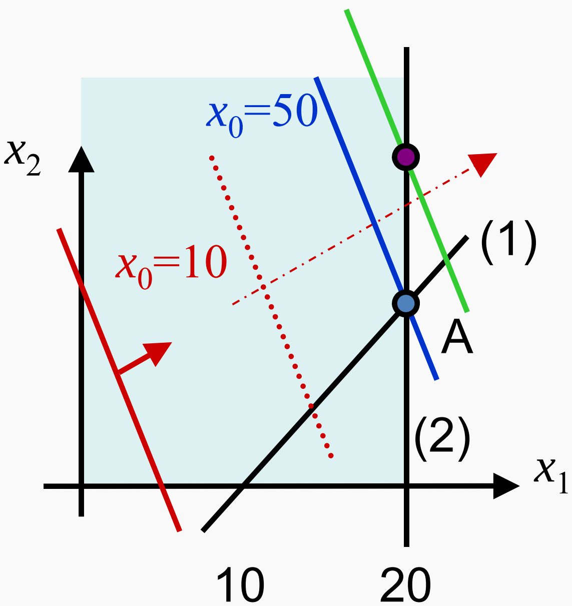
$$x_{B_i} = y_{i0} \overbrace{- \underbrace{y_{ik}}_{<0} \underbrace{x_k}_{>0}}^{>0} \geq 0 \quad \text{always!}$$

prendo la negativa, se ce ne sono piu di una prendo quella piu negativa

LP: unbounded solution

Example

$$\max\ x_0 = 2x_1 + x_2$$
$$x_1 - x_2 \le 10 \qquad (e1)$$
$$2x_1 \le 40 \qquad (e2)$$
$$x_1 \ge 0\ x_2 \ge 0$$



Standard form and starting basis associated with A

$$x_0 = 50 - (-x_3 + 3/2\, x_4) \qquad (e0)$$
$$x_2 = 10 - (-x_3 + 1/2\, x_4) \qquad (e1)$$
$$x_1 = 20 - (0x_3 + 1/2\, x_4) \qquad (e2)$$

Not optimal solution: x_3 enters the basis

No variable between x_1 and x_2 leaves the basis

The objective can increase to infinity along constraint (2)

LP: The simplex algorithm

- Designed by G. Dantzig in 1947
- Simplex is the polytope with the minimum number of vertices for a given dimension
- The simplex algorithm includes 5 steps

1. **Initialization**

   Determine a starting BFS

2. **Optimality test**

   if $y_{0j} \geq 0 \ \forall j \in R$ the current BFS is optimal then stop

   Otherwise go to 3

3. **Select the entering variable**

   Choose a non basic variable x_k such that $y_{0k} < 0$ (e.g., by steepest ascent) and go to 4

LP: The simplex algorithm

4. **Select the leaving variable**

   Choose variable x_{B_r} such that

$$\frac{y_{r0}}{y_{rk}} = \min_{\substack{i=1,\dots,M \\ y_{ik}>0}} \left\{ \frac{y_{i0}}{y_{ik}} \right\}$$

   If $y_{ik} \leq 0\ \forall i=1,\dots m$ the problem has unbounded solution then stop

5. **Pivoting**

   Solve the equations

$$x_{B_i} = y_{i0} - \sum_{j \in R} y_{ij} x_j \quad \forall i = 0,1,\dots,m$$

   obtaining x_k and x_{B_i} $i \neq r$, as a function of x_j, $j \in R$-$\{k\}$ and x_{B_r}

   Go to 2

LP: finding an initial solution

- Three possibilities to initialize the Simplex method
  - Slack variables
  - Two-Phase Method
  - Big-M Method
- Initialize means solve the *feasibility problem*

- **Slack variables**

  All constraints are inequality ≤, then slack variables are used to convert the problem in standard form

$$
\begin{aligned}
\max \quad & x_0 = \underline{c}^T \underline{x} \\
& A\underline{x} \le \underline{b} \\
& \underline{x} \ge \underline{0} \\
& \underline{x} \in R^n
\end{aligned}
\qquad \Longrightarrow \qquad
\begin{aligned}
\max \quad & x_0 = \underline{c}^T \underline{x} \\
& A\underline{x} + I\underline{s} = \underline{b} \\
& \underline{x} \ge \underline{0} \quad \underline{s} \ge \underline{0} \\
& \underline{x} \in R^n \quad \underline{s} \in R^m
\end{aligned}
$$

LP: finding an initial solution

- Each slack variable is associated with a single constraints
- The extended variable vector includes $n+m$ and the constraints matrix has now m rows and $n+m$ columns

$$A\underline{x} + I\underline{s} = \underline{b} \Rightarrow [A|I]\begin{bmatrix}\underline{x}\\ \underline{s}\end{bmatrix} = \underline{b}$$

- The initial basis can be $B=I$
- The basic variables in the initial BFS are the slack variables

$$A'\underline{x}' = \underline{b} \Rightarrow A' = [A|I] \quad \underline{x}' = \begin{bmatrix}\underline{x}\\ \underline{s}\end{bmatrix} \quad \begin{bmatrix}\underline{x}_B\\ \underline{x}_N\end{bmatrix} = \begin{bmatrix}\underline{s}\\ \underline{x}\end{bmatrix} = \begin{bmatrix}\underline{b}\\ \underline{0}\end{bmatrix}$$

- In general, whenever A include an identical $m \times m$ matrix I it is possible to select $B=I$ as initial basis and using as basic variables the ones associated with the columns of I

LP: finding an initial solution

Example

$$\max x_0 = 2x_1 + x_2$$

$$\begin{aligned} x_1 + x_2 + x_3 \quad &= 5 \\ -x_1 + x_2 \quad + x_4 \quad &= 0 \\ 6x_1 + 2x_2 \quad + x_5 &= 21 \end{aligned}$$

$$x_1 \geq 0 \; x_2 \geq 0 \; x_3 \geq 0 \; x_4 \geq 0 \; x_5 \geq 0$$

slack

$$A = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left[\begin{array}{cc|ccc} 1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 6 & 2 & 0 & 0 & 1 \end{array}\right] \end{array}$$

I

The initial BFS is

$$\begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \underline{b} = \begin{bmatrix} 5 \\ 0 \\ 21 \end{bmatrix}$$

Corresponding to vertex P_1

It is a degenerate solution

LP: finding an initial solution

- **Two-phase method**
  - General method whose first phase allows to determine the feasibility of a set of constraints
  - It builds an auxiliary *feasibility problem*

$$\text{(P)} \quad \max \ x_0 = \underline{c}^T \underline{x}$$
$$A\underline{x} = \underline{b}$$
$$\underline{x} \geq \underline{0}$$

I Phase (definition and solution of the auxiliary problem)

$$\text{(A)} \quad \min \ z = \underline{1}^T \underline{y} = \sum_{i=1}^{m} y_i$$
$$A\underline{x} + I\underline{y} = \underline{b}$$
$$\underline{x} \geq \underline{0} \quad \underline{y} \geq \underline{0}$$

where $\underline{1} = \begin{array}{c} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ \vdots \\ m \end{array}$

LP: finding an initial solution

- **Two-phase method**

I Phase (definition and solution of the auxiliary problem)

  - Auxiliary variable y_i allow to define an initial BFS for (A)
  - Let $\underline{z}$ be the new vector of variables

$$\underline{z} = \begin{bmatrix} \underline{y} \\ \underline{x} \end{bmatrix} \begin{matrix} m \\ n \end{matrix} \qquad \text{Initial BFS} \quad \begin{bmatrix} \underline{z}_B \\ \underline{z}_N \end{bmatrix} = \begin{bmatrix} \underline{y} \\ \underline{x} \end{bmatrix} = \begin{bmatrix} \underline{b} \\ \underline{0} \end{bmatrix}$$

  - (A) is solved by simplex method

LP: finding an initial solution

- **Two-phase method**

II Phase (Initialization of the original problem)

– If in the optimal solution for (A) $z=0$, then no y_i, $i=1,\ldots,m$, in the optimal basis for (A) and (P) is feasible

– The optimal solution for (A) $\underline{z}^* = \begin{bmatrix} \underline{y}^* \\ \underline{x}^* \end{bmatrix}$ with $\underline{y}^* = \underline{0}$

$$A\underline{x}^* + I\,\underline{y}^* = \underline{b} \Rightarrow A\underline{x}^* = \underline{b}$$

then $\underline{x}^*$ is feasible for (P) and is the initial BFS for (P)

– Instead if $z>0$, at least one y_i > 0 then (P) is not feasible (No vector $\underline{x}$ can satisfy the constraints of (P))

LP: finding an initial solution

- **Big-M method**
  - M (Big-M) usually denotes a constant coefficient that is significantly larger than any other coefficients of a problem
  - We need again auxiliary variables y_i, $i=1,\ldots,m$, included in the constraints as for the Two-phase method
  - The objective function is modified to penalize the auxiliary variables

$$
\text{(P')} \quad \max \underline{c}^T \underline{x} - M \underline{1}^T \underline{y} = \sum_{i=1}^{n} c_i x_i - M \sum_{j=1}^{m} y_j
$$

$$
A\underline{x} + I\underline{y} = \underline{b}
$$

$$
\underline{x} \geq \underline{0} \quad \underline{y} \geq \underline{0}
$$

$$
M >> |c_i|, |b_j|, |a_{ij}| \forall i, j
$$

LP: finding an initial solution

- **Big-M method**
  - The auxiliary variables y_i, $i=1,\dots,m$, are used to initialize the simplex as in the Two-phase method
  - The (P') is solved
  - The big-M forces the auxiliary variables to leave the basis
  - If in the optimal solution to (P') all y_i are non basic then

$$\underline{z}^* = \begin{bmatrix} \underline{y}^* \\ \underline{x}^* \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{x}^* \end{bmatrix} \Rightarrow A\underline{x}^* + I\,\underline{y}^* = \underline{b} \Rightarrow A\underline{x}^* = \underline{b}$$

  and $\underline{x}^*$ is the optimal solution to(P)
  - If instead at least one y_i is basic in the optimal solution to (P'), problem (P) is not feasible
  - Big-M method initializes and solves (if feasible) the original problem

LP: the simplex in tabular form

The Tableau

- Table of coefficients of objective function and constraints
- It corresponds to equations

$$x_{B_i} + \sum_{j \in R} y_{ij} x_j = y_{i0} \quad \forall i = 0, 1, \dots, m$$

LP: the simplex in tabular form

$$x_{B_i} + \sum_{j\in R} y_{ij}\, x_j = y_{i0} \qquad \forall\, i = 0,1,\dots,m$$

Basic variables

Objective coeff.

Non basic variables

Objective value (current solution)

$$\begin{array}{c|ccccc:ccccc|c}
 & x_{B_1} & \cdots & x_{B_r} & \cdots & x_{B_m} & \cdots & x_j & \cdots & x_k & \cdots & \\
\hline
x_0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & y_{0j} & \cdots & y_{0k} & \cdots & y_{00} \\
\hline
x_{B_1} & 1 & \cdots & 0 & \cdots & 0 & \cdots & y_{1j} & \cdots & y_{1k} & \cdots & y_{10} \\
\vdots & \vdots & \ddots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
x_{B_r} & 0 & \cdots & 1 & \cdots & 0 & \cdots & y_{rj} & \cdots & y_{rk} & \cdots & y_{r0} \\
\vdots & \vdots & & \vdots & \ddots & \vdots & & \vdots & & \vdots & & \vdots \\
x_{B_m} & 0 & \cdots & 0 & \cdots & 1 & \cdots & y_{mj} & \cdots & y_{mk} & \cdots & y_{m0} \\
\hline
\end{array}$$

Basic variables coeff.

Non basic variables coeff.

Basic variables values (current BFS)

LP: the simplex in tabular form

The Simplex Algorithm on the Tableau

1. **Initialization**
   – Build the starting tableau for an initial BFS

2. **Optimality test**
   – If in the objective row (x_0) there is no negative coefficient the current solution is optimal and the algorithm stop
   – Otherwise go to 3

3. **Select the entering variable**
   – Choose a non basic variable x_k such that y_{0k}<0 (e.g., the smallest coefficient) and go to 4

LP: the simplex in tabular form

The Simplex Algorithm on the Tableau

4. **Select the leaving variable**

   – If $y_{ik}<0\ \forall i=1,\ldots,m$, in the column of x_k the problem is unbounded and the algorithm stops

   – Otherwise compute $\dfrac{y_{i0}}{y_{ik}} \qquad i=1,\ldots,m$

   – Select the *r*-th row associated with the smallest ratio (y_{rk} is the *pivot*)

LP: the simplex in tabular form

The Simplex Algorithm on the Tableau

5 Pivoting

- Apply to the tableau the *Gauss-Jordan elimination method*:
  - Substitute x_k in the basis to x_{B_r} by diving r-th by the pivot
  - Subtract the new r-th row to any other row i multiplied by the coefficient in column k in order to obtain a zero coefficient (final column k should have all zero coefficient but that in row r equal to 1)

- Substitute in the first column variable x_k to the one that left the basis
- Go to 2

Simplex method: key ideas

Iterative procedure whose steps has a precise geometrical meaning

1. Start from an initial solution corresponding to a polyhedron vertex
2. Test the current solution optimality verifying if moving from the current vertex along a polyhedron edge can improve the objective
3. If no such improvement direction exists then the current basis is optimal and stop
4. Otherwise select a improving direction and move along the polyhedron edge
5. Determine the change of basis finding the first constraint reached moving along the edge
6. If no such constraint exists then the direction is an extreme one, the problem has no bounded solution and stop
7. Otherwise compute the new current basis associated with the reached vertex and iterate at 2

LP: internal point method (general concepts)

- In 1984 N. Karmakar, researcher at AT&T, developed a solution method alternative to simplex
- His method starts from a feasible solution and moves towards the optimal one through points internal to the feasibility polyhedron
- The method is much more complicated than simplex
- Karmakar's Algorithm was the first polynomial time algorithm for LP
- Simplex has an exponential complexity, than it is theoretically worse than interior point algorithms
- However, simplex implementations usually behave effectively in most of the cases
- From 1984 both internal point algorithms and Simplex have been extensively studied so that their performances greatly improved
- Simplex is used to solve problem with hundred thousand (even million) variables and constraints