Artificial Intelligence

First Order Logic

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Agenda - First Order Logic

1. Motivations
2. Syntax
3. Semantics
4. Reasoning Techniques
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   - Herbrand Theorem
   - Semi-decidability of First Order Logic
5. Intended interpretations
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Agenda - First Order Logic

Problem

We want to model and reason about a scenario with n persons. In particular, we want to say when two or more persons know each other.

Solution in Propositional Logic - Language definition

1. Solution 1: for any two <u>distinct</u> persons p_1 and p_2, introduce a propositional variable which is true when p_1 knows p_2,
2. Solution 2: for any two persons p_1 and p_2, introduce a propositional variable which is true when p_1 knows p_2,
3. Solution 3: for any two <u>distinct</u> persons p_1 and p_2, introduce a propositional variable which is true when p_1 and p_2 know each other,
4. Solution 4: for any two persons p_1 and p_2, introduce a propositional variable which is true when p_1 and p_2 know each other.

1. Soln 1 & 3: we do not model (and thus we cannot reason about) whether a person “knows himself”,
2. Soln 3 & 4: we assume that the knowledge relation is symmetric. Thus, we cannot model (and thus reason about) scenarios in which Alice knows Bob but Bob does not know Alice.

Assume we adopt Solution 1.

Solution in Propositional Logic - problem formulation

$K(p_1, p_2)$ is a propositional variable modeling that p_1 knows p_2.

Assume we have Alice (a), Bob (b) and Charles (c) and we know that Alice knows Bob and that Bob knows Charles.

Q: Does Bob know Alice?

Bob knows Alice if the known facts entail $K(b, a)$. $K(b, a)$ is not entailed by $K(a, b), K(b, c)$.

Q: Is it the case that Bob does not know Alice?

Bob does not know Alice if the known facts entail $\neg K(b, a)$. $\neg K(b, a)$ is not entailed by $K(a, b), K(b, c)$.

Thus, it is possible both that Bob knows Alice and that Bob does not know Alice.

Solution in Propositional Logic - problem formulation

$K(p_1, p_2)$ is a propositional variable modeling that p_1 knows p_2.

Assume we have Alice (a), Bob (b) and Charles (c) and we know that Alice knows Bob and that Bob knows Charles.

Q: Assume we also know that the knowledge relation is symmetric, does Bob know Alice?

Then, we have to introduce the additional axioms:

$$\begin{array}{ll} K(a,b) \rightarrow K(b,a) & K(a,c) \rightarrow K(c,a) \\ K(b,a) \rightarrow K(a,b) & K(b,c) \rightarrow K(c,b) \\ K(c,a) \rightarrow K(a,c) & K(c,b) \rightarrow K(b,c) \end{array}$$

The above facts and $K(a,b)$ entail $K(b,a)$ and now the answer is yes.

For n people, there are $n(n-1)$ additional axioms.

Solution in Propositional Logic - problem formulation

$K(p_1, p_2)$ is a propositional variable modeling that p_1 knows p_2.

Assume we have Alice (a), Bob (b) and Charles (c) and we know that Alice knows Bob and that Bob knows Charles.

Q: If the knowledge relation is transitive, does Alice know Charles?

The answer is yes, but modeling transitivity of the knowledge relation needs 6 additional axioms like the following:

$$K(a, b) \rightarrow (K(b, c) \rightarrow K(a, c)).$$

In general, with n people, $n(n-1)(n-2) \simeq n^3$ axioms.

Solution in Propositional Logic - problem formulation

$K(p_1, p_2)$ is a propositional variable modeling that p_1 knows p_2.

Assume we have Alice (a), Bob (b) and Charles (c) and possibly many others, and we do not know how many. We know that Alice knows Bob and that Bob knows Charles.

Q: If the knowledge relation is transitive, does Charles know Alice?

The problem cannot be modeled in propositional logic. Propositional logic can still be used to solve the problem once we restrict/ground the set of people to Alice, Bob and Charles and a few others (thanks to Herbrand's theorem).

Q: Assuming every one knows someone. Does there exist a person who knows everybody?

We cannot restrict to a "few people" and the problem cannot be reduced to propositional logic.

Motivations: From propositional to first order logic

Propositional Logic:

1. Great for modeling and reasoning about *finitely many facts*, i.e., scenarios with finitely many objects and relations among objects.
2. Even in finite scenarios, cannot be used to succinctly assert general relations holding among the objects.
3. Cannot be used to model scenarios with infinitely many objects.

From propositional to first order logic

First Order Logic extends propositional logic language introducing:

1. *Terms*, representing objects in the scenario, like
    1. *Constants*, like a, b and c, for Alice, Bob and Charles, or 1, 2, 3, ...
    2. *Variables*, like x, y, z ranging over objects, and
    3. *Function symbols* modeling functions, like $age(a)$ or $+$, which, when applied to a term, return another term.
2. *Quantifiers* to express that something holds for one ($\exists$) or for all ($\forall$) objects.
3. *Propositional Variables*, each one modeling a particular fact which is either true or false.
4. *Predicate Symbols*, each one modeling a particular n-ary relation among the objects.

Agenda - First Order Logic

Syntax

Elements in the Language

- Constants, denoted with a, b, c, ...
- Function Symbols, denoted with f, g, h, ... with arguments
- Variables, denoted with x, y, z, ...
- Propositional Variables, denoted with capital letters
- Predicate Symbols, denoted with P, Q, R, ... with arguments
- Propositional Connectives ($\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$)
- Quantifiers ($\forall$, $\exists$)

Propositional connectives and quantifiers are also called *logical constants*, because they have a fixed interpretation.

Note:

1. Constants can be seen and will be treated as special cases of 0-ary function symbols,
2. Propositional Variables can be seen and will be treated as as special cases of 0-ary predicate symbols.

Terms

1. A variable is a term
2. If f is a function symbol with arity $n \geq 0$ and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is a term
3. Nothing else is a term

Ground Term

A term is *ground* iff it does not contain variables in it.

Example

a is a ground term. $f(g(a, b))$ is a ground term. $f(g(x, b))$ is not a ground term.

Atomic Formula

An atomic formula (or atom) is an expression of the form $P(t_1, \ldots, t_n)$ where P is a predictive symbol with $n \geq 0$ arguments and $t_1, \ldots, t_n$ are terms.

Example

If h and f are 1-ary function symbols, and P and A are binary predicative symbols, then $P(a, b)$ and $A(h(a), f(x))$ are atomic formulas.

A Literal

A *Literal* is an atomic formula or the negation of an atomic formula.

Example

$P(a, b)$ and $\neg A(h(a), f(x))$ are literals, while $\neg\neg A(h(a), f(x))$ is not.

Well Formed Formulas (wffs)

1. An atomic formula is a wff
2. If w, w_1, w_2 are wffs then, $\neg w$, $(w_1 \wedge w_2)$, $(w_1 \vee w_2)$, $(w_1 \to w_2)$, and $(w_1 \leftrightarrow w_2)$ are also wffs
3. If w is a wff and x is a variable, then $\forall x.w$ and $\exists x.w$ are also wff
4. Nothing else is a wff

In the wff $\mathcal{Q}x.w$, $\mathcal{Q} \in \{\forall, \exists\}$, w is the *matrix (or scope)* of $\mathcal{Q}x$.

Note: Sometimes,

1. "$\overline{P(t_1, \ldots, t_n)}$" is used in place of "$\neg P(t_1, \ldots, t_n)$",
2. "$(w_1 \supset w_2)$" is used in place of "$(w_1 \to w_2)$",
3. "$(w_1 \equiv w_2)$" is used in place of "$(w_1 \leftrightarrow w_2)$".

Closed wffs

An occurrence of a variable x is *bounded* iff it occurs:

- immediately to the right of a quantifier, or
- in the scope of a quantification of the type $\mathcal{Q}x$.

If an occurrence is not bounded then it is *free*.

A wff is *closed* if it does not contain occurrences of free variables.

Example

- In $\forall x.P(x,y)$, y is free: the wff is not closed.
- y is also free in wff $\forall x.(P(x,y) \rightarrow \exists y.Q(y))$, and thus the wff is not closed.
- $\forall x.\forall y.(P(x,y) \rightarrow \exists y.Q(y))$ is closed.

Assumption 1: Closed formulas

Closed wffs

From here on, we assume that wffs are closed.

The formal treatment of not closed wffs introduces complications in the definition of the semantics. Further, for the satisfiability and validity problems (formal definitions will be given), we can always "reduce" to closed wffs.

Fact: For a wff w and a constant a not occurring in w,

1. w is satisfiable iff $\exists x.w$ is satisfiable iff $w[a/x]$ is satisfiable.
2. w is valid iff $\forall x.w$ is valid iff $w[a/x]$ is valid,

where $w[a/x]$ is the wff obtained by replacing all free occurrences of x in w with a.)

Assumption 2: Distinct quantified variables

Distinct quantified variables

From here on, we assume that wffs do not contain variables bounded by two distinct quantifiers.

As in the previous case, such assumption simplifies the presentation of the semantics. Further, renaming bounded variables does not change the "meaning" of the wff. It is thus possible, for any wff, to construct an "equivalent" satisfying the assumption.

Example

wffs on the same line are "equivalent":

$\forall x.w(x)$	$\forall y.w(y)$	
$\exists x.w_1(x) \vee \forall x.w_2(x)$	$\exists x.w_1(x) \vee \forall y.w_2(y)$	$\exists y.w_1(y) \vee \forall x.w_2(x)$
$\exists x.(w_1(x) \vee \forall x.w_2(x))$	$\exists x.(w_1(x) \vee \forall y.w_2(y))$	$\exists y.(w_1(y) \vee \forall x.w_2(x))$

Problem

We want to model and reason about a scenario with n persons. In particular, we want to say when two or more persons know each other.

Solution in First Order Logic - language definition

K is now a binary predicate symbol.

$K(p_1, p_2)$ is now an atomic formula saying that (the object represented by) p_1 is in the K (knowledge) relation (the object represented by) p_2.

Comparing to the 4 solutions in the propositional case:

1. the K relation takes any two terms, even the same term twice, and we can write e.g., $K(a, a)$;
2. the K relation is not ensured to be reflexive or symmetric or transitive: if we want it to be (or not to be) reflexive or symmetric or transitive we have to state it.

Solution in First Order Logic - problem formulation

$K(p_1, p_2)$ is now an atomic formula saying that (the object represented by) p_1 is in the K (knowledge) relation (the object represented by) p_2.

Assume we have Alice (a), Bob (b) and Charles (c) and we know that Alice knows Bob and that Bob knows Charles.

Q: Does Bob know Alice?

Bob knows Alice if the known facts entail $K(b, a)$. $K(b, a)$ is not entailed by $K(a, b), K(b, c)$.

Q: Is it the case that Bob does not know Alice?

Bob does not know Alice if the known facts entail $\neg K(b, a)$. $\neg K(b, a)$ is not entailed by $K(a, b), K(b, c)$.

Thus, it is possible both that Bob knows Alice and that Bob does not know Alice.

Solution in First Order Logic - problem formulation

$K(p_1, p_2)$ is now an atomic formula saying that (the object represented by) p_1 is in the K (knowledge) relation (the object represented by) p_2.

Assume we have Alice (a), Bob (b) and Charles (c) and we know that Alice knows Bob and that Bob knows Charles.

Q: Assume we also know that the knowledge relation is symmetric, does Bob know Alice?

Then, we have to introduce one additional axiom:

$$\forall x.\forall y.(K(x, y) \rightarrow K(y, x)).$$

The above fact and $K(a, b)$ entail $K(b, a)$ and now the answer is yes.

For n people, there is still one additional axiom.

Solution in First Order Logic - problem formulation

$K(p_1, p_2)$ is now an atomic formula saying that (the object represented by) p_1 is in the K (knowledge) relation (the object represented by) p_2.

Assume we have Alice (a), Bob (b) and Charles (c) and we know that Alice knows Bob and that Bob knows Charles.

Q: If the knowledge relation is transitive, does Alice know Charles?

The answer is yes, and modeling transitivity of the knowledge relation needs one additional axiom:

$$\forall x.\forall y.\forall z.(K(x, y) \rightarrow (K(y, z) \rightarrow K(x, z))).$$

Even with n people, just 1 axiom.

Solution in First Order Logic - problem formulation

$K(p_1, p_2)$ is now an atomic formula saying that (the object represented by) p_1 is in the K (knowledge) relation (the object represented by) p_2.

Assume we have Alice (a), Bob (b) and Charles (c) and possibly *many* others, and we do not know how many. We know that Alice knows Bob and that Bob knows Charles.

Q: If the knowledge relation is transitive, does Charles know Alice?

We have to check whether $K(c, a)$ or $\neg K(c, a)$ is entailed by the previous formulation.

Answer: it is possible both $K(c, a)$ and $\neg K(c, a)$, and this will be determined through reduction to propositional logic (Herbrand Domain is finite)

Solution in First Order Logic - problem formulation

$K(p_1, p_2)$ is now an atomic formula saying that (the object represented by) p_1 is in the K (knowledge) relation (the object represented by) p_2.

Q: Assuming every one knows someone. Does there exist a person who knows everybody?
Introduce the axiom

$$\forall x.\exists y.K(x, y)$$

and check whether it entails:

$$\exists x.\forall y.K(x, y)$$

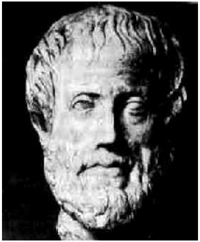
Answer: No, it is possible but not certain. The problem cannot be reduced to propositional logic (Herbrand Domain is not finite).

Agenda - First Order Logic

Dire di ciò che è che non è, o di ciò che non è che è ,è falso, mentre dire di ciò che è che è o di ciò che non è che non è , è vero.

To say of what is that it is not, or of what is not that it is, is false, while to say of what is that it is end of what is not that it is not, is true.

Aristotele



Semantics - Intuitions

Goal: Define the *truth* or *falsity* of wffs on the basis of the meaning given to the elements in the language.

To ascribe meaning to all sentences of a first-order language, the following information is needed.

1. A non empty set D, called *domain*,
2. For every n-ary function symbol, an n-ary function from D^n to D as its interpretation (that is, a function $D^n \mapsto D$),
3. For every n-ary predicate symbol, an n-ary relation on D as its interpretation (that is, a subset of D^n).

Semantics

Interpretation

An *interpretation* is a couple: $I = \langle D, g \rangle$ where D is a non-empty set called the *domain* and g is a function such that

- $g(f) : D^n \to D$ for any function symbol f with n arguments
- $g(P) \subseteq D^n$ for any predicative symbol P with n arguments

Note: When $n = 0$, $D^n = D^0$ is a set with one element. Thus,

1. if f is a 0-ary function symbol (i.e., is a constant), $g(f)$ is an element of D, and
2. if P is a 0-ary predicate symbol (i.e., is a propositional variable), $g(P)$ is either the empty set (meaning that P is false), or D^0 (meaning that P is true).

Assumption 3: Names for objects

Names for objects in the domain

Given an interpretation $\langle D, g \rangle$, for every element $d \in D$, we assume to

1. extend the language by including d in the set of constants, and
2. interpret d as itself, i.e., $g(d) = d$.

Such assumption guarantees that for any object in D, we have at least one name for it.

If for a $d \in D$, there is already a ground term (e.g., a constant) in the language whose interpretation is d, there is no need to further extend the language with a new additional name for d.

Interpretation of a ground term

The interpretation of a ground term $f(t_1, \ldots, t_n)$ given the interpretation $I = \langle D, g \rangle$, is the element of D corresponding to

$$g(f)(g(t_1), \ldots, g(t_n))$$

i.e., the result of the application of the function $g(f) : D^n \mapsto D$ to the interpretation $g(t_1), \ldots, g(t_n)$ of $t_1, \ldots, t_n$.

Example

Given the constants a and b the function symbol f,

1. if $D = \mathbb{N}$, $g(a) = 1$, $g(b) = 2$ and $g(f) = +$,

$$g(f(a,b)) = 3, \qquad g(f(f(7,a),b)) = +(+(7,1),2) = 10,$$

2. if $D = \{0,1,2,3\}$, $g(a) = 1$, $g(b) = 2$ and, for ground terms t_1 and t_2, $g(f(t_1,t_2)) = (t_1 + t_2) mod 4$, then

$$g(f(a,b)) = 3 \qquad g(f(f(3,a),b)) = +(+(3,1),2) mod 4 = 2.$$

Satisfiability of a wff wrt an interpretation

Satisfiability of a wff wrt an interpretation

A closed wff *w is satisfied by an interpretation $I = \langle D, g \rangle$* ($\models_I w$), iff:

- if w is $P(t_1, \ldots, t_n)$ then $\models_I w$ iff $\langle g(t_1), \ldots, g(t_n) \rangle \in g(P)$;
- if w is $\neg w_1$ then $\models_I w$ iff $\not\models_I w_1$;
- if w is $(w_1 \wedge w_2)$ then $\models_I w$ iff $\models_I w_1$ and $\models_I w_2$;
- if w is $(w_1 \vee w_2)$ then $\models_I w$ iff $\models_I w_1$ or $\models_I w_2$;
- if w is $(w_1 \rightarrow w_2)$ then $\models_I w$ iff $\not\models_I w_1$ or $\models_I w_2$;
- if w is $(w_1 \leftrightarrow w_2)$ then $\models_I w$ iff $\models_I (w_1 \rightarrow w_2)$ and $\models_I (w_2 \rightarrow w_1)$;
- if w is $\forall x.w_1$ then $\models_I w$ iff for each $d \in D$, $\models_I w_1[d/x]$;
- if w is $\exists x.w_1$ then $\models_I w$ iff there exist $d \in D$ such that $\models_I w_1[d/x]$.

If $\models_I w$, we also say that *w is true in I* and that I is a *model* of w.
If $\not\models_I w$, we also say that *w is false in I* and that I is an *counter-model* of w.

Exercise

Determine what are the truth values of the following closed wff:

$$\exists x.\exists y.P(f(x,y),a) \tag{1}$$

$$\forall x.\forall y.(P(x,y) \rightarrow P(y,x)) \tag{2}$$

$$\forall x.\forall y.\forall z.(((P(x,y) \wedge P(y,z)) \rightarrow P(x,z)) \tag{3}$$

given the interpretation $I = \langle D, g \rangle$ in which:

- $D = \mathbb{N}$, $g(P) = \leq$, $g(f) = \cdot$, *and* $g(a) = 2$

Exercise

Determine what are the truth values of the following closed wff:

$$\exists x . \exists y . P(f(x, y), a) \tag{1}$$

$$\forall x . \forall y . (P(x, y) \rightarrow P(y, x)) \tag{2}$$

$$\forall x . \forall y . \forall z . (((P(x, y) \wedge P(y, z)) \rightarrow P(x, z)) \tag{3}$$

given the interpretation $I = \langle D, g \rangle$ *in which:*

- $D = \mathbb{N}$, $g(P) = \leq$, $g(f) = \cdot$, *and* $g(a) = 2$
- $D = \mathbb{Z}$, $g(P) =$ '=', $g(f) = +$, *and* $g(a) = 0$

Exercise

Determine what are the truth values of the following closed wff:

$$\exists x.\exists y.P(f(x,y),a) \quad (1)$$

$$\forall x.\forall y.(P(x,y) \rightarrow P(y,x)) \quad (2)$$

$$\forall x.\forall y.\forall z.(((P(x,y) \wedge P(y,z)) \rightarrow P(x,z)) \quad (3)$$

given the interpretation $I = \langle D, g \rangle$ in which:

- $D = \mathbb{N}$, $g(P) = \leq$, $g(f) = \cdot$, *and* $g(a) = 2$
- $D = \mathbb{Z}$, $g(P) =$ '=', $g(f) = +$, *and* $g(a) = 0$
- $D = \mathfrak{P}(\mathbb{Z})$ *(set of all sets of integers)*, $g(P) = \subseteq$, $g(f) = \cap$, *and* $g(a) = \emptyset$

Exercise

Given the following statement (in which the disjunction must be understood exclusively):

A number is even or odd.

Find:

1. *first-order language to express it,*
2. *a wff to represent it,*
3. *a model and a counter-model.*

Solution

Language: 3 predicative letters with argument: Num, $Even$, Odd.

wff: $\forall x.(Num(x) \rightarrow \neg(Even(x) \leftrightarrow Odd(x)))$

Model: $I = \langle D, g \rangle$ such that $g(Num) = \emptyset$.

Counter-model: $I = \langle D, g \rangle$ such that $g(Num) \neq \emptyset$ and $g(Even) = g(Odd) = D$.

Exercise

Given the following statement:

Every person has a father and a mother.

Find:

1. *first-order language to express it,*
2. *a wff to represent it,*
3. *a model and an counter-model.*

Solution

Language: *a predicative letter with one argument, Person, and two predicative letters with two arguments, Father and Mother.*

wff: $\forall x.(Person(x) \rightarrow (\exists y.Father(y,x) \wedge \exists z.Mother(z,x)))$

Model: *Any* $I = \langle D, g \rangle$ *such that* $g(Person) = \emptyset$.

Counter-model: *Any* $I = \langle D, g \rangle$ *such that* $g(Person) \neq \emptyset$ *and* $g(Father) = g(Mother) = \emptyset$.

Satisfiability

Satisfiability

A closed wff w is *satisfiable* iff it has a model, i.e. there exists an intepretation I such that $\models_I w$.

Example

From the previous exercise,

$$\forall x.(Person(x) \rightarrow (\exists y.Father(y,x) \wedge \exists z.Mother(z,x))) \tag{4}$$

has a model and a counter-model. Thus, (4) is satisfiable, and also

$$\neg\forall x.(Person(x) \rightarrow (\exists y.Father(y,x) \wedge \exists z.Mother(z,x)))$$

is satisfiable.

Validity

Validity

A closed wff w is *valid* ($\models w$) iff it is true for each intepretation I.

Example

The following formula is valid:

$$\forall x.P(x) \rightarrow \exists y.P(y).$$

Indeed, assume that there exists an interpretation $I = \langle D, g \rangle$ falsifying it. Then, it must satisfy $\forall x.P(x)$ (and thus for each $d \in D$, $\models_I P(d)$) and falsify $\exists y.P(y)$ (and thus, for each $d \in D$, $\not\models_I P(d)$), which is not possible (D is not empty by definition).

Logical equivalence

Logical equivalence

Two wffs w_1 and w_2 with free variables $x_1, \dots, x_n$ ($n \geq 0$) are *logically equivalent* iff $\models \forall x_1 . \dots . \forall x_n .(w_1 \leftrightarrow w_2)$.

Fact

If w_1 and w_2 are logically equivalent then they have the same "meaning", and we can substitute w_1 with w_2 in any wff and get another logically equivalent wff.

Example

The following three formulas are logically equivalent ($\circ \in \{\wedge, \vee\}$):

$$\forall x.\exists y.(P(x,z) \circ Q(y,z))$$
$$\exists y.\forall x.(P(x,z) \circ Q(y,z))$$
$$\forall x.P(x,z) \circ \exists y.Q(y,z).$$

Logical consequence

Logical consequence, entailment

A closed wff w is a *logical consequence* of a set Γ of closed wffs ($\Gamma \models w$, or Γ *entails* w) iff every model of Γ is also a model of w.

Example

(7) is a logical consequence of (5) and (6), where:

$$\forall x.(Happy(x) \rightarrow (Hippie(x) \vee \neg Poor(x))) \tag{5}$$

$$\exists x.(Happy(x) \wedge Poor(x)) \tag{6}$$

$$\exists x.Hippie(x) \tag{7}$$

Each model $I = \langle D, g \rangle$ of (5) and (6) must satisfy:

1. foreach $d \in D$ if $d \in g(Happy)$ then $d \in g(Hippie)$ or $d \notin g(Poor)$;
2. there exists $c \in D$ such that $c \in g(Happy) \cap g(Poor)$.

Let $c \in D$ such that $c \in g(Happy) \cap g(Poor)$. Since $c \in g(Happy)$, by the 1st item, we conclude that $c \in g(Hippie)$ or $c \notin g(Poor)$. But since we know that $c \in g(Poor)$, then necessarily $c \in g(Hippie)$ and then $\exists x.Hippie(x)$ is true in I. So (7) is true in every model of (5) and (6).

Consistency

Consistency

A closed wff w is *inconsistent* or *contradictory* iff w entails any formula.

Example

The following formula is contradictory:

$$\forall x.P(x) \land \exists y.\neg P(y).$$

Indeed, w has no model.

Satisfiability, Validity, Logical consequence, Consistency

1. A wff w is valid if and only if $\neg w$ is <u>unsatisfiable</u>,
2. Two wffs w_1 and w_2 with free variables $x_1, \dots, x_n$ ($n \geq 0$) are logically equivalent if and only if $\neg\forall x_1 . \dots . \forall x_n.(w_1 \leftrightarrow w_2)$ is <u>unsatisfiable</u>,
3. If $w, w_1, \dots, w_n$ are closed wffs, $w_1, \dots, w_n \models w$ iff $(w_1 \wedge \dots \wedge w_n) \models w$ iff $\models (w_1 \wedge \dots \wedge w_n) \rightarrow w$ iff $(w_1 \wedge \dots \wedge w_n \wedge \neg w)$ is <u>unsatisfiable</u>.
4. A wff w is consistent iff w is <u>satisfiable</u>.

<u>Validity, logical equivalence, logical consequence and consistency can be reduced to satisfiability.</u>

Agenda - First Order Logic

Normal forms

Classes of wffs can be reduced to other wffs by actual procedures that preserve some properties in semantics (for example the logical equivalence or satisfiability)

- Prenex Normal Form
- Conjunctive and Disjunctive Normal Form
- Skolem Normal Form

Prenex Normal Form

Negation Normal Form

A wff is in *negation normal form* if the negation operator $\neg$ is only applied to atomic formulas and the only other allowed Boolean operators are conjunction $\wedge$ and disjunction $\vee$.

Theorem

Every wff has a logical equivalent in negation normal form.

Demostration.

Following equivalences can be used as rules for re-writing:

$(w_1 \leftrightarrow w_2) \leftrightarrow ((w_1 \rightarrow w_2) \wedge (w_2 \rightarrow w_1))$	$\neg\neg w \leftrightarrow w$
$\neg(w_1 \wedge w_2) \leftrightarrow (\neg w_1 \vee \neg w_2)$	$\neg(w_1 \vee w_2) \leftrightarrow (\neg w_1 \wedge \neg w_2)$
$(w_1 \rightarrow w_2) \leftrightarrow (\neg w_1 \vee w_2)$	
$\neg\forall x.w \leftrightarrow \exists x.\neg w$	$\neg\exists x.w \leftrightarrow \forall x.\neg w$

$\square$

Prenex Normal Form

Prenex Normal Form

The wffs of the form $\mathcal{Q}_1 x_1 \dots \mathcal{Q}_n x_n.w$ where $\mathcal{Q}_i \in \{\forall, \exists\}$ and w is wff that contains no quantifiers are called wff in *prenex normal form*.

Theorem

Every wff has a logical equivalent in prenex normal form.

Proof.

Following equivalences can be used as rules for re-writing wffs in negation normal form:

$\forall x.(w_1 \wedge w_2) \leftrightarrow (w_1 \wedge \forall x.w_2)^1$	$\forall x.(w_1 \wedge w_2) \leftrightarrow (\forall x.w_1 \wedge w_2)^2$
$\forall x.(w_1 \vee w_2) \leftrightarrow (w_1 \vee \forall x.w_2)^1$	$\forall x.(w_1 \vee w_2) \leftrightarrow (\forall x.w_1 \vee w_2)^2$
$\exists x.(w_1 \vee w_2) \leftrightarrow (w_1 \vee \exists x.w_2)^1$	$\exists x.(w_1 \vee w_2) \leftrightarrow (\exists x.w_1 \vee w_2)^2$
$\exists x.(w_1 \wedge w_2) \leftrightarrow (w_1 \wedge \exists x.w_2)^1$	$\exists x.(w_1 \wedge w_2) \leftrightarrow (\exists x.w_1 \wedge w_2)^2$
$\forall x.(w_1 \wedge w_2) \leftrightarrow (\forall x.w_1 \wedge \forall x.w_2)$	$\exists x.(w_1 \vee w_2) \leftrightarrow (\exists x.w_1 \vee \exists x.w_2)$
$\forall x.w[x/y] \leftrightarrow \forall y.w$	$\forall z.w[z/x] \leftrightarrow \forall y.w[y/x]$

[1] If x is not free in w_1

[2] If x is not free in w_2

□

Prenex normal form

Example

Given ($\circ \in \{\wedge, \vee\}$, $Q_1, Q_2 \in \{\forall, \exists\}$)

$$Q_1 x.P(x,z) \circ Q_2 y.Q(y,z),$$

the following two wffs in prenex normal form are equivalent:

$$Q_1 x.Q_2 y.(P(x,z) \circ Q(y,z))$$
$$Q_2 y.Q_1 x.(P(x,z) \circ Q(y,z))$$

Q: What if $Q_1 = Q_2$?

Q: What if $Q_1 \neq Q_2$?

Conjunctive and Disjunctive Normal Form

Abbreviations:

- $\bigwedge_{i=1}^{n} w_i$ is short for $(w_1 \wedge (w_2 \wedge (\ldots \wedge w_n)\ldots))$
- $\bigvee_{i=1}^{n} w_i$ is short for $(w_1 \vee (w_2 \vee (\ldots \vee w_n)\ldots))$

Theorem

For every wff without quantifiers, a logically equivalent wff can be built in:

Conjunctive Normal form (CNF): $\bigwedge_{i=1}^{m} \bigvee_{j=1}^{n(i)} k_{i,j}$, *OR*

Disjunctive Normal Form (DNF): $\bigvee_{i=1}^{p} \bigwedge_{j=1}^{q(i)} l_{i,j}$

where $k_{i,j}$ and $l_{i,j}$ are literals.

Wffs in Prenex and in CNF or DNF

Theorem

For every wff, there exists a logically equivalent wff in prenex normal form and whose matrix is in Conjunctive Normal Form (CNF) or in Disjunctive Normal Form (DNF).

Procedure.

1. First convert the formula in prenex normal form, and
2. Then convert the matrix in CNF or in DNF.

□

Skolem Normal Form

Skolem Normal Form

A closed, prenex wff without existential quantifiers is in *Skolem Normal Form*.

Theorem

For every wff w it is possible to construct an equisatisfiable one $sko(w)$ in Skolem Normal Form.

NB: The construction of the Skolem normal form of a wff may lead to different wffs (all of which are equisatisfiable) and so the notation $sko(w)$ is actually improper.
NB: In some textbooks, a wff is in Skolem Normal Form if it also satisfies the additional requirement to have the matrix in CNF.

Procedure.

The main steps to construct $sko(w)$ are:

1. transformation of w in prenex normal form, and
2. substitution of each occurrence of an existentially quantified variable x in the matrix with $f(x_1, \ldots, x_n)$ $(n \geq 0)$ where
   1. f is a newly introduced n-ary function symbol.
   2. $x_1, \ldots, x_n$ are the n universally quantified variables occurring before x in the prefix, and
3. removal of the existential quantifiers and their bounded variables from the prefix.

$\square$

Exercise

Determine a Skolem normal form of the wff:

$$\forall x.(P(x) \to \neg\forall z.\exists y.(R(x,z) \land Q(x,y)))$$

Solution

First we consider the wff in prenex normal form:

$$\forall x.\exists z.\forall y.(\neg P(x) \lor \neg R(x,z) \lor \neg Q(x,y))$$

Then we calculate the Skolem normal form:

$$\forall x.\forall y.(\neg P(x) \lor \neg R(x,f(x)) \lor \neg Q(x,y))$$

Exercise

Determine a Skolem normal form of the wff:

$$\forall x.P(x) \rightarrow \neg\forall z.\exists y.(R(z) \wedge Q(y))$$

Solution

First we consider the wff in prenex normal form:

$$\exists x.\exists z.\forall y.(\neg P(x) \vee \neg R(z) \vee \neg Q(y))$$

Then we calculate the Skolem normal form:

$$\forall y.(\neg P(a) \vee \neg R(b) \vee \neg Q(y))$$

Exercise

Determine a Skolem normal form of the wff ($\circ \in \{\wedge, \vee\}$):

$$\forall x.P(x) \circ \exists y.Q(y)$$

Solution

Two logically equivalent wffs in prenex normal form, and two different Skolem normal forms:

$$\begin{array}{ccc} \forall x.\exists y.(P(x) \circ Q(y)) & \qquad & \exists y.\forall x.(P(x) \circ Q(y)) \\ \Downarrow & & \Downarrow \\ \forall x.(P(x) \circ Q(f(x))) & & \forall x.(P(x) \circ Q(a)) \end{array}$$

Equivalent, but with different properties (see Herbrand Theorem). The one of the right is better.

Herbrand Interpretation

Herbrand Interpretation

Let Γ be a set of wffs in Skolem normal form.
$I_H = \langle D_H, g_H \rangle$ is a *Herbrand interpretation* of Γ if

1. D_H is the set of ground terms that can be constructed using the n-ary ($n \geq 0$) function symbols in Γ (if Γ does not contain constants, one is created), and
2. g_H is such that, for each ground term $f(t_1, \ldots, t_n)$ ($n \geq 0$) in D_H,

$$g(f(t_1, \ldots, t_n)) = f(t_1, \ldots, t_n).$$

D_H is called the *Herbrand Domain* of Γ.

Example

The Herbrand Domain of $\forall x.(P(x) \circ Q(a))$ is $D_H = \{a\}$.
The Herbrand Domain of $\forall x.(P(x) \circ Q(f(x)))$ is

$$D_H = \{c, f(c), f(f(c)), f(f(f(c))), \ldots\} = \{f^n(c) : n \geq 0\}.$$

Ground instances of a set of wffs

Ground instances of a set of wffs

Let Γ be a set of wffs in Skolem normal form. We define the *set $GI(\Gamma)$ of ground instances of Γ* as the set of ground wffs obtained by

1. substituting the variables in the matrix of a formula in Γ with a term in the Herbrand Domain of Γ, and
2. deleting the prefix.

Example

The set of ground instances of $\forall x.(P(x) \circ Q(a))$ is $\{(P(a) \circ Q(a))\}$.
The set of ground instances of $\Gamma = \{\forall x.(P(x) \circ Q(f(x)))\}$ is

$$GI(\Gamma) = \{(P(f^n(c)) \circ Q(f^{n+1}(c))) : n \geq 0\}.$$

Herbrand and Compactness Theorems

Theorem (Herbrand)

The following facts are equivalent:

1. Γ *has a model.*
2. Γ *has a Herbrand model.*
3. *The set of ground instances of* Γ *has a model.*
4. *The set of ground instances of* Γ *has a Herbrand model.*

Theorem (Compactness)

Let Γ *be an arbitrary set of wff (with or without free variables).* Γ *has a model iff every finite subset of* Γ *has a model.*

Semi-decidability of First Order Logic

Γ, w closed wffs (if Γ and/or w are not closed, consider their universal closure).

$\Gamma \models w$ **iff**

Semi-decidability of First Order Logic

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Semi-decidability of First Order Logic

Γ, w closed wffs (if Γ and/or w are not closed, consider their universal closure).

$\Gamma \models w$ **iff**

$\Gamma \cup \{\neg w\}$ is unsat **iff**

$sko(\Gamma) \cup sko(\{\neg w\})$ is unsat **iff** (for the Herbrand Thm)

$GI(sko(\Gamma) \cup sko(\{\neg w\}))$ is unsat **iff**

Semi-decidability of First Order Logic

Γ, w closed wffs (if Γ and/or w are not closed, consider their universal closure).

$\Gamma \models w$ **iff**

$\Gamma \cup \{\neg w\}$ is unsat **iff** nega il risultato

$sko(\Gamma) \cup sko(\{\neg w\})$ is unsat **iff** (for the Herbrand Thm)

$GI(sko(\Gamma) \cup sko(\{\neg w\}))$ is unsat **iff** (for Compactness Thm)

there is a finite subset of $GI(sko(\Gamma) \cup sko(\{\neg w\}))$ that is unsat.

Decidability of First Order Logic if D_H is finite

Since

1. If $\Gamma \cup \{\neg w\}$ is a set of wffs with finite Herbrand domain D_H, then the set of ground instances $GI(sko(\Gamma) \cup sko(\{\neg w\}))$ is finite.
2. If a set of wffs is unsatisiable then any superset is also unsatisfiable.

Fact: Let $\Gamma \cup \{\neg w\}$ be a set of wffs with finite Herbrand domain. $\Gamma \models w$ **iff** the conjunction of the wffs in $GI(sko(\Gamma) \cup sko(\{\neg w\}))$ is unsatisfiable.

Corollary

The Satisfiability is decidable for the following two classes of wffs:

- *prenex wff form $\exists\vec{x}.\forall\vec{y}.w$ without function symbols,*
- *wff without function symbols and whose predicate symbols are unary.*

Exercise

Consider the wffs

$$\forall x.(Happy(x) \rightarrow (Hippie(x) \vee \neg Poor(x))) \tag{8}$$

$$\exists x.(Happy(x) \wedge Poor(x)) \tag{9}$$

$$\exists x.Hippie(x) \tag{10}$$

and show that $\{(8), (9)\} \models (10)$ *using Herbrand Theorem.*

Solution

$\{(8),(9)\} \models (10)$ *iff* $\{(8),(9),\neg(10)\}$ *is unsatisfiable, iff (Skolem Theorem) the following set of wffs is unsatisfiable:*

$$\begin{aligned} \forall x.(Happy(x) \rightarrow (Hippie(x) \vee \neg Poor(x))) \\ (Happy(a) \wedge Poor(a)) \\ \forall x.\neg Hippie(x) \end{aligned}$$

iff (Herbrand's theorem) the following set of propositional wffs is unsatisfiable:

$$\begin{aligned} (Happy(a) \rightarrow (Hippie(a) \vee \neg Poor(a))) \\ (Happy(a) \wedge Poor(a)) \\ \neg Hippie(a) \end{aligned}$$

It is easy to show that such a set of wffs is unsatisfiable and this allows us to conclude that $\{(5),(6)\} \models (7)$.

Exercise

Show whether the statement "Each person knows somebody" follows from "there exists a person who knows everybody", i.e.,

$$\exists x.\forall y.K(x, y) \models \forall s.\exists t.K(s, t).$$

Solution

$$\exists x.\forall y.K(x,y) \models \forall s.\exists t.K(s,t).$$

iff

$$\exists x.\forall y.K(x,y), \exists s.\forall t.\neg K(s,t)$$

is unsatisfiable, iff (Skolem Theorem) the following set of wffs is unsatisfiable:

$$\forall y.K(a,y), \forall t.\neg K(b,t)$$

iff (Herbrand's theorem $D_H = \{a, b\}$) the following set of propositional wffs is unsatisfiable:

$$K(a,a), K(a,b), \neg K(b,a), \neg K(b,b).$$

It is easy to show that such a set of wffs is satisfiable and this allows us to conclude that

$$\exists x.\forall y.K(x,y) \not\models \forall s.\exists t.K(s,t).$$

Exercise

We know that:

$$\exists x.\forall y.K(x,y) \not\models \forall s.\exists t.K(s,t).$$

what if we assume the K relation is reflexive?

Exercise

We know that:

$$\exists x.\forall y.K(x,y) \not\models \forall s.\exists t.K(s,t).$$

what if we assume the K relation is symmetric?

Exercise

We know that:

$$\exists x . \forall y . K(x, y) \not\models \forall s . \exists t . K(s, t).$$

what if we assume the K relation is transitive?

Exercise

Show whether the statement "For each person there is someone who knows him" follows from "there exists a person who knows everybody", i.e.,

$$\exists x.\forall y.K(x, y) \models \forall s.\exists t.K(t, s).$$

Solution

$$\exists x.\forall y.K(x,y) \models \forall s.\exists t.K(t,s).$$

iff

$$\exists x.\forall y.K(x,y), \exists s.\forall t.\neg K(t,s)$$

is unsatisfiable, iff (Skolem Theorem) the following set of wffs is unsatisfiable:

$$\forall y.K(a,y), \forall t.\neg K(t,b)$$

iff (Herbrand's theorem $D_H = \{a, b\}$) the following set of propositional wffs is unsatisfiable:

$$K(a,a), K(a,b), \neg K(a,b), \neg K(b,b).$$

It is easy to show that such a set of wffs is unsatisfiable and this allows us to conclude that

$$\exists x.\forall y.K(x,y) \models \forall s.\exists t.K(t,s).$$

Exercise

We know that:

$$\exists x.\forall y.K(x,y) \models \forall s.\exists t.K(t,s).$$

what if we assume the K relation is reflexive or symmetric or transitive?

Semi-decidability of First Order Logic if D_H is infinite

If the Herbrand domain D_H is infinite, then the set $GI(sko(\Gamma) \cup sko(\{\neg w\}))$ is infinite.
In such case case, we can enumerate all its finite subsets and check whether each is (un)satisfiable:

- If $\Gamma \models w$ then sooner or later the procedure ends by finding a finite subset of $GI(sko(\Gamma \cup \{\neg w\}))$ which is unsatisfiable and thus we return a positive response.
- If $\Gamma \not\models w$ then the procedure does not terminate.

Exercise

Show whether the statement "there exists a person who knows everybody" follows from "everybody knows everybody", i.e.,

$$\forall x.\forall y.K(x, y) \models \exists s.\forall t.K(s, t).$$

Solution

$$\forall x.\forall y.K(x,y) \models \exists s.\forall t.K(s,t).$$

iff

$$\forall x.\forall y.K(x,y), \forall s.\exists t.\neg K(s,t)$$

is unsatisfiable, iff (Skolem Theorem) the following set of wffs is unsatisfiable:

$$\forall x.\forall y.K(x,y), \forall s.\neg K(s,f(s)).$$

Herbrand's domain $D_H = \{f^n(a) : n \geq 0\}$*). Consider the subset of ground instances obtained by substituting the variables with* $\{a, f(a)\}$ *of* D_H*. Then,*

$$K(a,a), K(a,f(a)), K(f(a),a), K(f(a),f(a)), \neg K(a,f(a)), \neg K(f(a),f(f(a))).$$

It is easy to show that such a set of wffs is unsatisfiable and this allows us to conclude that

$$\forall x.\forall y.K(x,y) \models \exists s.\forall t.K(s,t).$$

Exercise

Show whether the statement "there exists a person who knows everybody" follows from "everybody is known by someone", i.e.,

$$\forall x.\exists y.K(y, x) \models \exists s.\forall t.K(s, t).$$

Solution

$$\forall x.\exists y.K(y,x) \models \exists s.\forall t.K(s,t).$$

iff

$$\forall x.\exists y.K(y,x), \forall s.\exists t.\neg K(s,t)$$

is unsatisfiable, iff (Skolem Theorem) the following set of wffs is unsatisfiable:

$$\forall x.K(g(x),x), \forall s.\neg K(s,f(s)).$$

Herbrand's domain D_H is infinite and each finite subset of the ground instances you may try is satisfiable.

Agenda - First Order Logic

Motivations

An interpretation is a pair $\langle D, g \rangle$ in which

1. D is an *arbitrary* non empty set, and
2. g interprets all the n-ary function and predicate symbols as *arbitrary* functions from $D^n \mapsto D$ and subsets of D^n respectively.

but

Motivations

An interpretation is a pair $\langle D, g \rangle$ in which

1. D is an *arbitrary* non empty set, and
2. g interprets all the n-ary function and predicate symbols as *arbitrary* functions from $D^n \mapsto D$ and subsets of D^n respectively.

but it may be the case that we do want to restrict our attention to a subset of all the possible interpretations, usually called intended interpretations.

Example

Consider for example the atomic wff $<(\times(3, x), a)$, that we can write as

$$3x < a.$$

We may want to consider as D either $\mathbb{N}$ or $\mathbb{Z}$ or $\mathbb{R}$, and have the standard interpretation for the symbols 3, $\times$ and $<$.

Analogously, for the equality symbol $=$ that we want to be interpreted as identity: for any two ground terms t_1 and t_2,

$$t_1 = t_2$$

is true in an interpretation $\langle D, g\rangle$ iff $g(t_1) = g(t_2)$.

Intended Interpretations

Satisfiability, Validity and Entailment

Let Σ be the set of intended interpretations.

1. A wff w is *satisfiable wrt* Σ iff for some $I \in \Sigma$, $\models_I w$.
2. A wff w is *valid wrt* Σ iff for each interpretation $I \in \Sigma$, $\models_I w$.
3. A wff w is *entailed by* Γ *wrt* Σ iff for each interpretation $I \in \Sigma$, $\models_I w$ whenever I is a model of the wffs in Γ.

Note: whenever Σ is a single interpretation, the notion of satisfiability and validity collapse and it is common to speak just about satisfiability.

Note: Herbrand Theorem holds whenever the set of all possible interpretations is considered. Whenever we consider a subset Σ, Hebrand Theorem no longer necessarily holds.

Finite, single intended interpretation

Assume

1. that there is only one intended interpretation $\langle D, g \rangle$,
2. that D consists of n objects $a_1, \ldots, a_n$,
3. that $a_1, \ldots, a_n$ are also ground terms in the language and that each a_i is interpreted as itself (i.e., $g(a_i) = a_i$), $1 \leq i \leq n$.

.

Such assumptions are satisfied in finite scenarios with finitely many objects, each with a name.

Fact: In the above 3 hypotheses,

1. $\forall x.w$ is equivalent to

$$(w[a_1/x] \wedge \ldots \wedge w[a_n/x]),$$

2. $\exists x.w$ is equivalent to

$$(w[a_1/x] \vee \ldots \vee w[a_n/x]).$$

Thus, in such assumptions, for any wff there is an equivalent, propositional one. Still, first order logic provides a much easier, more compact representation language.

Agenda - First Order Logic

1. Motivations
2. Syntax
3. Semantics
4. Reasoning Techniques
   - Normal forms
   - Herbrand Theorem
   - Semi-decidability of First Order Logic
5. Intended interpretations
6. Beyond first order classical logic

Beyond first order classical logic

- Logics:
  1. Many-Sorted Logic
  2. Second and Higher Order Logics
  3. Modal and Temporal First order Logics
- Other reasoning techniques in FOL:
  1. Techniques for reduction to normal forms
  2. Resolution theorem provers
  3. Interactive theorem provers
- Reasoning Tasks:
  1. Minimal model computation