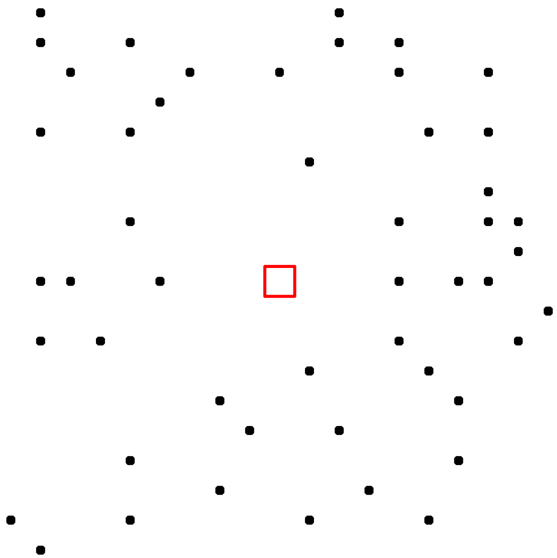
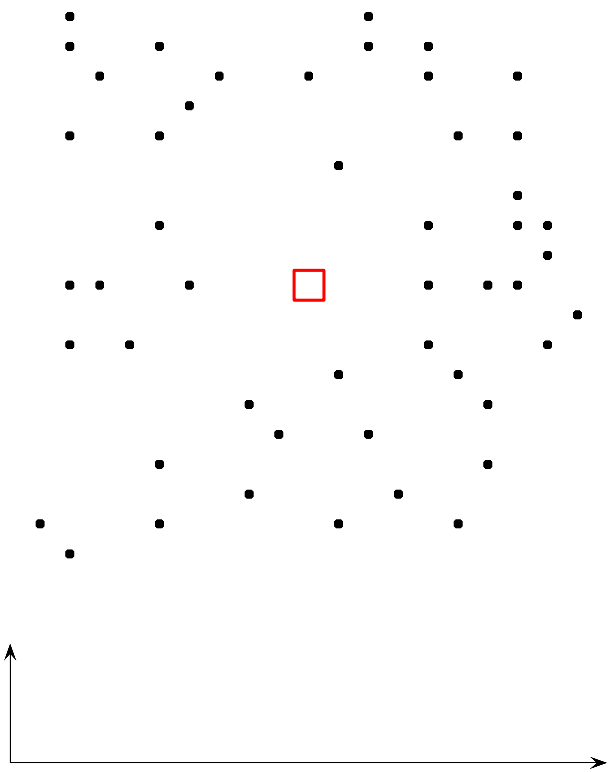
The Inventory Routing Problem (IRP)

- One depot
- A set of customers



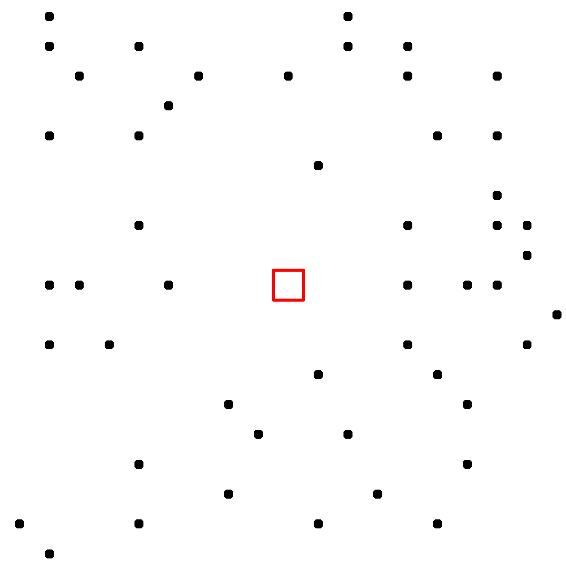
The Inventory Routing Problem (IRP)

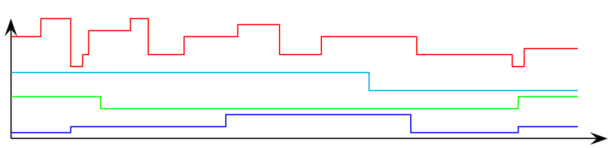
- One depot
- A set of customers
- A time horizon



The Inventory Routing Problem (IRP)

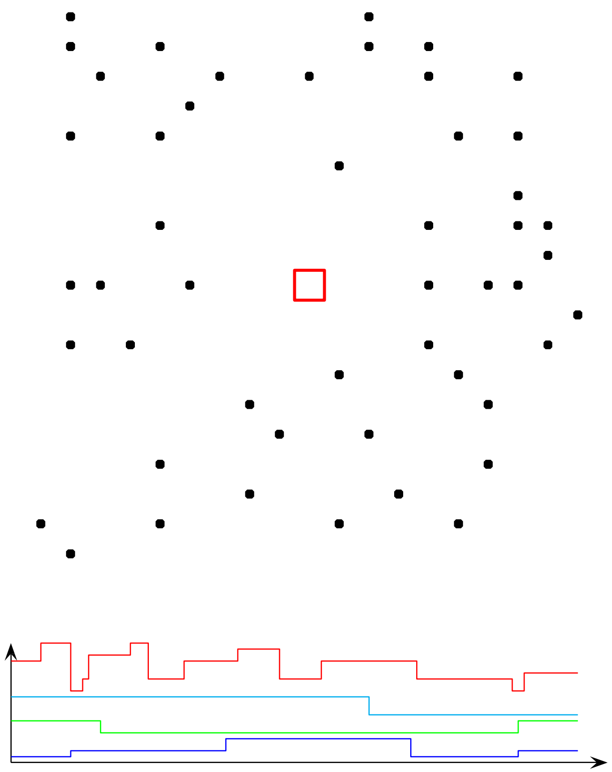
- One depot
- A set of customers
- A time horizon
- A known demand for each customer and each period





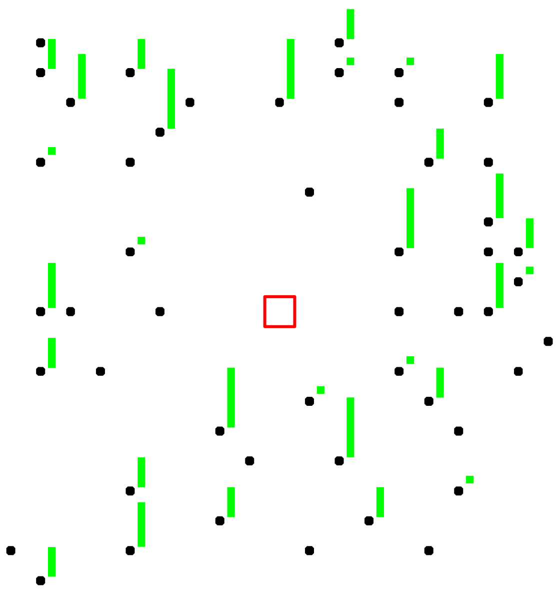
The Inventory Routing Problem (IRP)

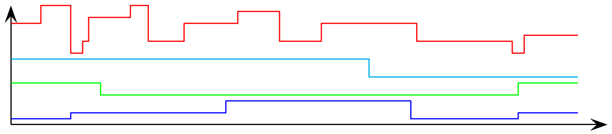
- One depot
- A set of customers
- A time horizon
- A known demand for each customer and each period
- Two different objectives



The Inventory Routing Problem (IRP)

- One depot
- A set of customers
- A time horizon
- A known demand for each customer and each period
- Two different objectives
    - Minimize Inventory cost





The Inventory Routing Problem (IRP)

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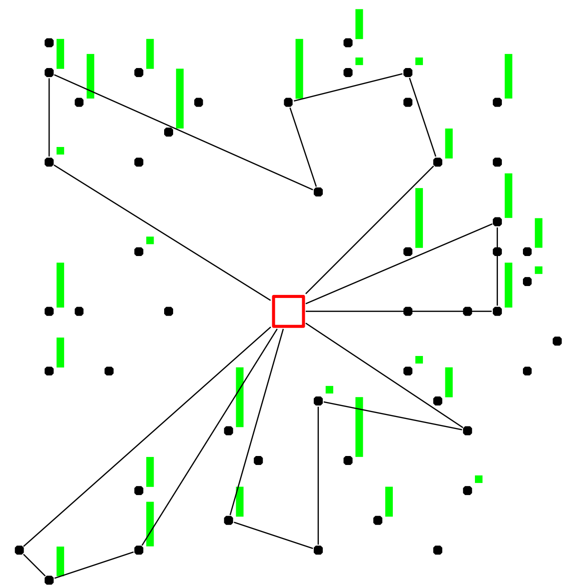
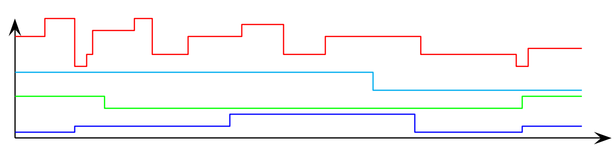
- A set of customers
- A time horizon
- A known demand for each customer and each period
- Two different objectives
  - Minimize Inventory cost
  - Minimize Routing cost





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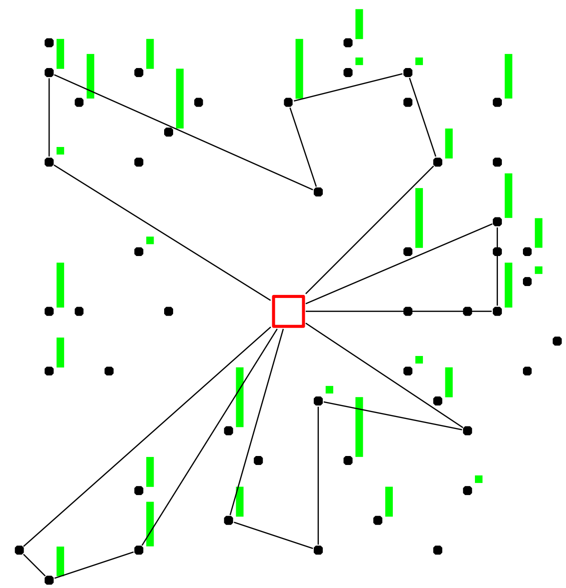
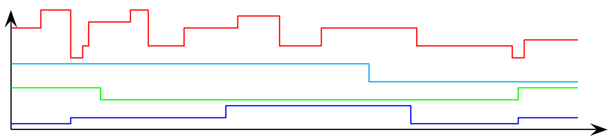




IRP is a real bi-objective optimization problem

Important Decisions (IRP)

For solving the IRP, we must make the following decisions:

Decisions

1. When deliver customers?
2. How much deliver?
3. With which routes?

All these decisions are linked together:

- increase delivery quantities → change routes or frequency
- change frequency → adapt delivery quantities
- ...

IRP variants

There exists a large number of variants

- planning horizon (finite/infinite)
- inventory costs and capacities
- production/demand rates (of single/multiple product)
- specific restrictions/regulations deterministic/stochastic
- demand/production
- initial inventory
- fleet (homogeneous/heterogeneous)

And different objectives

- usually a combination of inventory level and routing cost

The choice of policies

Standard policies:

DD Day-to-day delivery policy

If not enough in stock, deliver the missing demand

OU Order-up-to level policy

When you ship, ship the maximum (customer capacity)

ML Maximum level policy

Any quantity less than the maximum level (but which quantity?)

The frequency policy encoding

Solutions are modeled as a frequency f of the deliveries for each customer

$f = 1$	$\rightarrow$	DD policy
$f = 2$	$\rightarrow$	serve for the next two consecutive periods
$\cdots$	$\rightarrow$	...
$f = k$	$\rightarrow$	serve for the next k consecutive periods
$\cdots$	$\rightarrow$	...
$f = +\infty$	$\rightarrow$	OU policy

Evaluation of solutions

Each solution is measured with the two criteria

Inventory cost

Sum of all inventory levels at customers' at the end of each period. This can be computed in $O(np)$

Routing cost

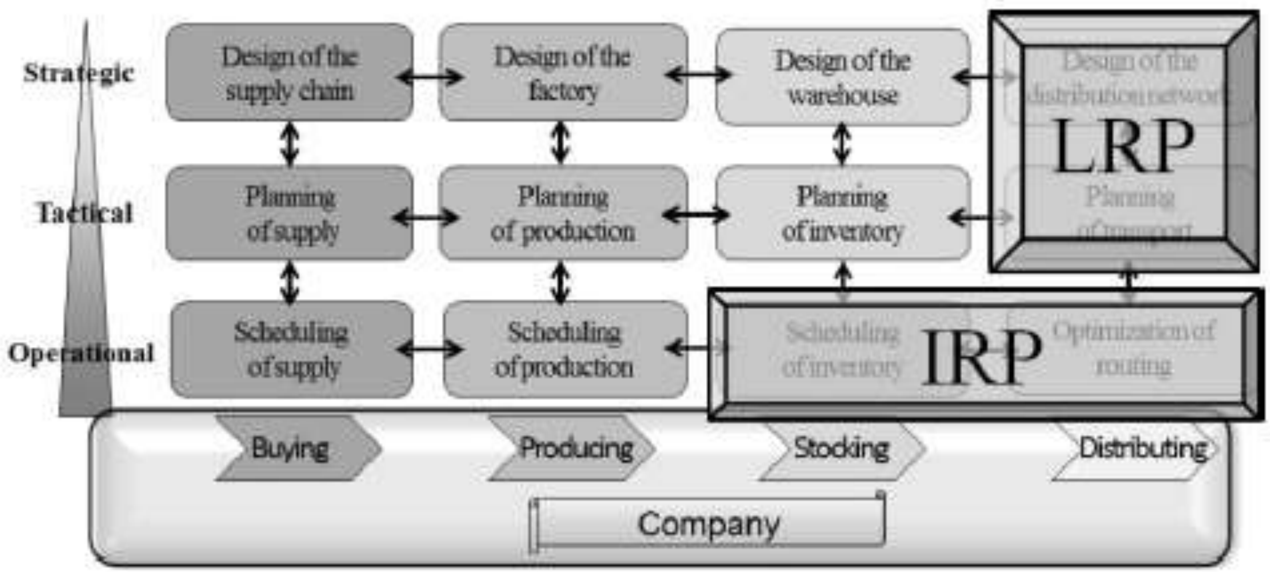
Sum of all distances run by the trucks at every period. Solve a VRP for each period. This is a NP-hard problem!!!

Reduce inventory or transportation costs?

- Reasons for keeping low inventory
  - reduce inventory costs
- Reasons for keeping high inventory
  - reduce transportation costs

- Reasons for frequent transportation
  - reduce inventory costs
- Reasons for rare transportation
  - reduce transportation costs

Positioning of inventory and of location routing problems in supply chain management



Conventional Inventory

- Customer
  - monitors inventory levels
  - places orders
- Vendor
  - manufactures/purchases product
  - assembles order
  - loads vehicles
  - routes vehicles
  - makes deliveries

Problems with Conventional

- Large variation in demands on production and transportation facilities
- workload balancing
- utilization of resources
- unnecessary transportation costs
- urgent vs nonurgent orders
- setting priorities

Retailer Managed Inventory (RMI)

- In vehicle routing problems, the customers to be visited are given as well as the quantities to be delivered. The decisions to be taken concern the routes of the vehicles, i.e., the space traversed by the vehicles. We call these decisions over space. If the customers of a supplier, independently of each other and independently of the supplier, decide when and how much to order, the problems to be solved by the supplier over time are vehicle routing problems.
- In this kind of traditional distribution management, that we call Retailer Managed Inventory (RMI), the power of the supplier to optimize the distribution is strongly constrained by the decisions taken by the customers, even when the goal is the minimization of the transportation cost only.

Vendor Managed Inventory

- Customer
  - trusts the vendor to manage the inventory
- Vendor
  - monitors customers' inventory
    - customers call/fax/e-mail
    - remote telemetry units
    - set levels to trigger call-in
  - controls inventory replenishment & decides
    - when to deliver
    - how much to deliver
    - how to deliver

Vendor Managed Inventory

- VMI transfers inventory management (and possibly ownership) from the customer to the supplier
- VMI synchronizes the supply chain through the process of collaborative order fulfillment

Advantages of VMI

- Customer
  - less resources for inventory management
  - assurance that product will be available when required

- Vendor
  - more freedom in when & how to manufacture product and make deliveries
  - better coordination of inventory levels at different customers
  - better coordination of deliveries to decrease transportation cost

VMI Approach

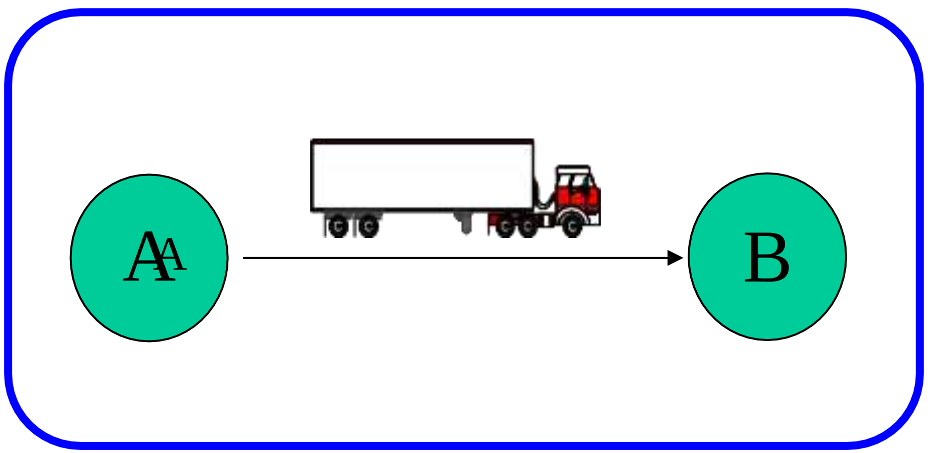
- Decide when to deliver to a customer
- Decide how much to deliver to a customer
- Decide on the delivery routes

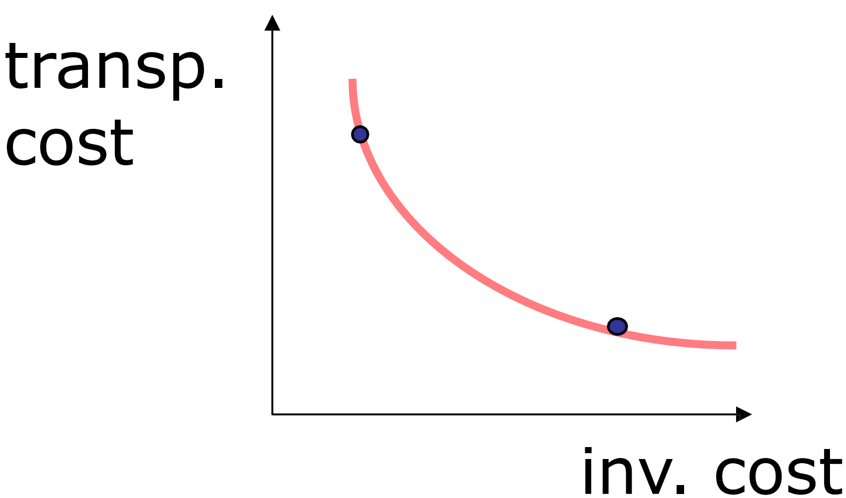
In IRPs, the decision space always includes timing and quantities and may include the routing too. In general, we can classify the IRPs according to the decision space as follows:

- 1. Decisions over time only: Single Link Shipping Problem.

In this case, the routes are given. The decisions concern the times and the quantities to deliver to the customers. In the Single Link Shipping Problem, a supplier serves one customer only and, thus, the route traversed by the vehicles is given, from the supplier to the customer and back. Similarly, in the Inventory Routing Problem with Direct Shipping, a supplier serves a set of customers with direct shipments to each separately. Again, the routes are fixed.

- 2. Decisions over time and space The timing of the deliveries to each customer, the quantities to be delivered each time a delivery takes place and the routes traveled by the vehicles have to be decided at the same time.

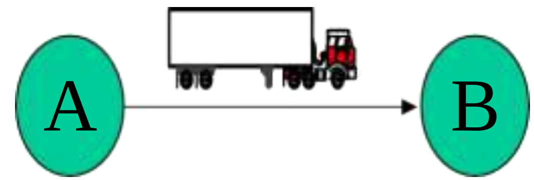




Determine shipping policies that optimize the trade-off between:

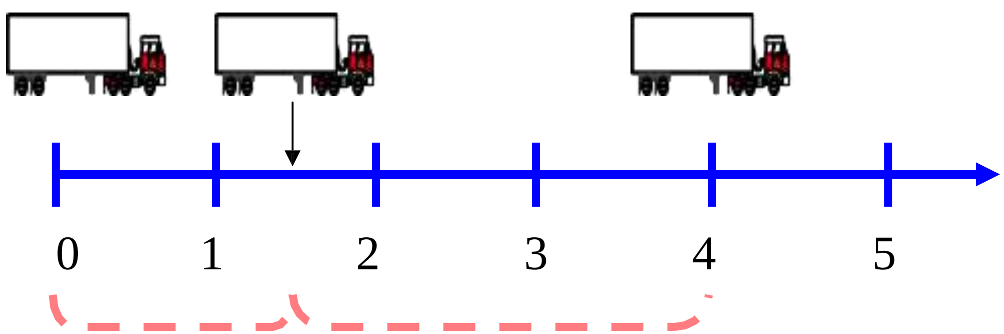
- Inventory cost
- Transportation cost

Problem Description

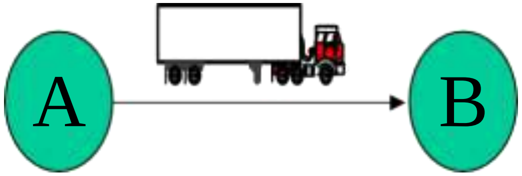


Fleet of vehicles:

- transportation capacity $r = 1$
- transportation cost c (A->B->A)



Goal



Determine shipping policies

that minimize

inventory cost + transportation cost

The continuous problem (capacitated EOQ)

Blumenfeld et al. (1985), Transportation Research B

- ✓ No minimum inter-shipment time
- ✓ Single frequency f
- ✓ Continuous time between shipments t= 1/f
- ✓ Single vehicle

$$\min hqt + \frac{c}{t}$$

$$vqt \le r$$

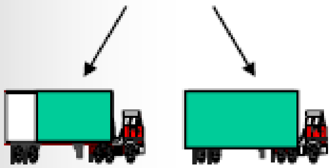
$$t \ge 0$$

EOQ= Economic Order Quantity

Daily volume vq
r= transportation capacity

Optimal solution

$$t^* = \min\left(\sqrt{\frac{c}{hq}}, \frac{r}{vq}\right)$$



Problem is a non-linear constrained optimization model

The continuous problem (capacitated EOQ)

Constraint (2) is the capacity constraint and constraint (3) defines the non-negativity of the decision variable of the problem.

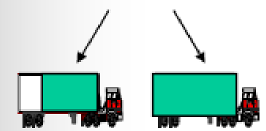
$$\min hqt + \frac{c}{t}$$

$$vqt \le r$$

$$t \ge 0$$

Optimal solution

$$t^* = \min\left(\sqrt{\frac{c}{hq}}, \frac{r}{vq}\right)$$



When the optimal delivery time is $\sqrt{\frac{c}{hq}} < \frac{r}{vq}$ then a vehicle is sent with partial load every t*

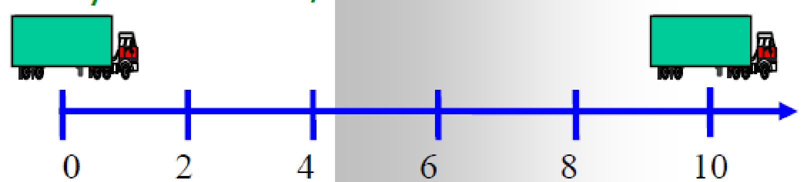
whereas when t*$= \frac{r}{vq}$ a full load vehicle is sent to the customer.

An example

transportation capacity r=50
fixed transportation cost c=100
daily volume vq=1x5=5

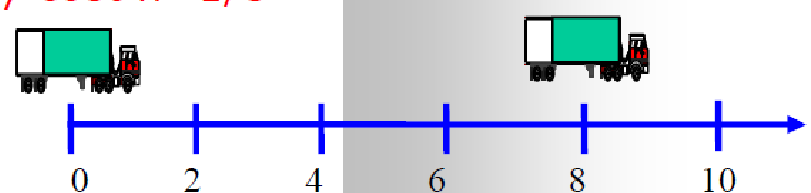
Case 1: Shipped product is "cheap"
daily unit inventory cost h=1/30

$$t^* = \frac{r}{vq} = 10$$



Case 2 : Shipped product is "expensive"
daily inventory cost h=1/3

$$t^* = \sqrt{\frac{c}{hq}} \cong 8$$



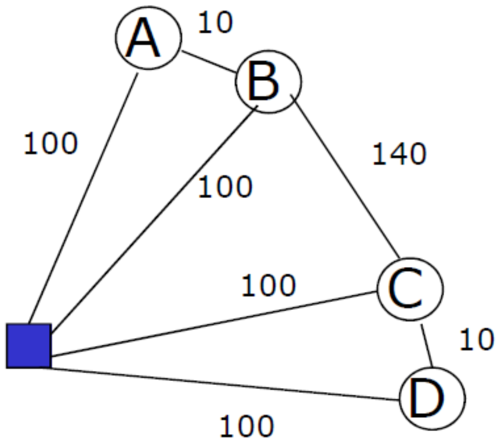
- Objective
  - Minimize distribution costs without causing any stock-outs over a finite horizon ***OR***
  - Maximize the expected total discounted value (rewards minus costs) over an infinite horizon

The optimal solution of any IRP depends on the objective function chosen.
The minimization of the transportation cost only is a suitable goal for a decision-maker who is responsible for the transportation only or for a situation where the inventory costs are not relevant when compared to the transportation costs. In this case, we may expect that an optimal solution prescribes infrequent transportation with highly loaded vehicles.
The minimization of the inventory costs is the goal in situations where the focus is on inventory management. In this case, we may expect frequent transportation. This is what happens when just-in-time is implemented.
The objective of minimizing the sum of the inventory and transportation costs is more suitable than the minimization of only one of the two cost components whenever a decision-maker is responsible for all the cost components. Tackling the transportation problem separately from the inventory management becomes in these cases a way to decompose a complex problem in simpler problems, but produces suboptimal solutions.

Inventory Routing Problem

- Extensions
  - Operating modes
  - Delivery time windows
  - Delivery times (fixed plus variable part)

A first example



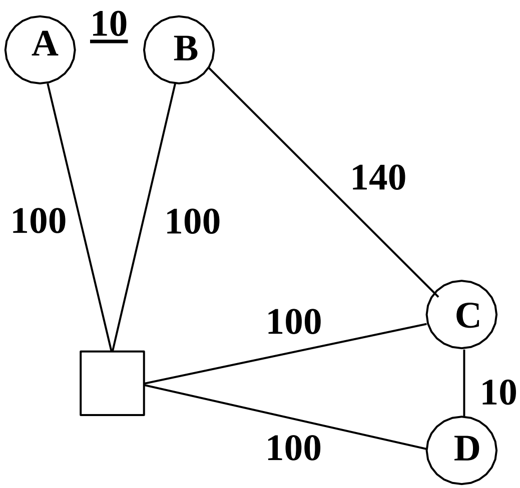
	capacity	consumption
A	5000	1000
B	3000	3000
C	2000	2000
D	4000	1500

Initial inventory of i =capacity
The supplier has unlimited availability
Capacity of vehicles 5000
? Periodic policy of minimum cost

Bell et al (1983), Interfaces

Time is assumed to be discrete, for example structured in days.
A product has to be distributed from a supplier to a set of four customers with capacitated vehicles.
Decisions have to be taken, for each day, on which customers to serve, how much to deliver and the routes to travel.
There is no limitation on the availability of product at the supplier.

The objective is to find a periodic distribution policy, i.e., a plan on whom to serve, how much to deliver and the routes traveled by the vehicles, to be repeated regularly, that minimizes the total transportation cost. The policy must be such that a stock out is never caused at any of the customers, that the maximum inventory level at the customers is not exceeded and the vehicle capacity is satisfied. The periodicity of the policy implies that the inventory levels at the end of the period must be equal to the initial levels.



	A	B	C	D
Daily Use	1000	3000	2000	1500
Max. Delv.	5000	3000	2000	4000

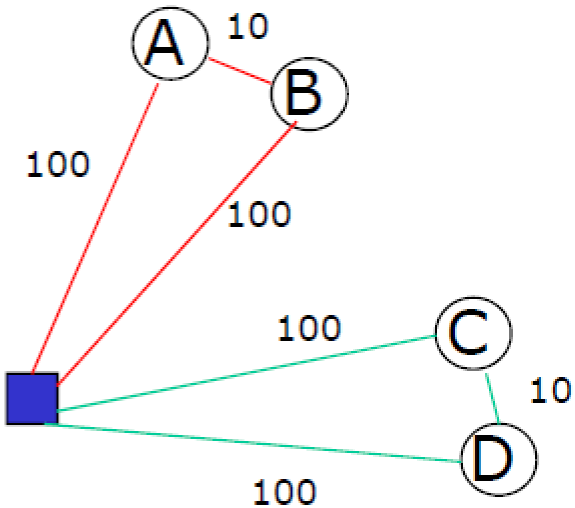
Truck capacity is 5000

There are 14 possible customer combinations

A-B-C-D	A-B
A-B-C	A-C
A-B-D	A-D
A-C-D	B-C
A	B-D
B	C-D
C	D

There are an infinite number of possible delivery volumes

A natural solution



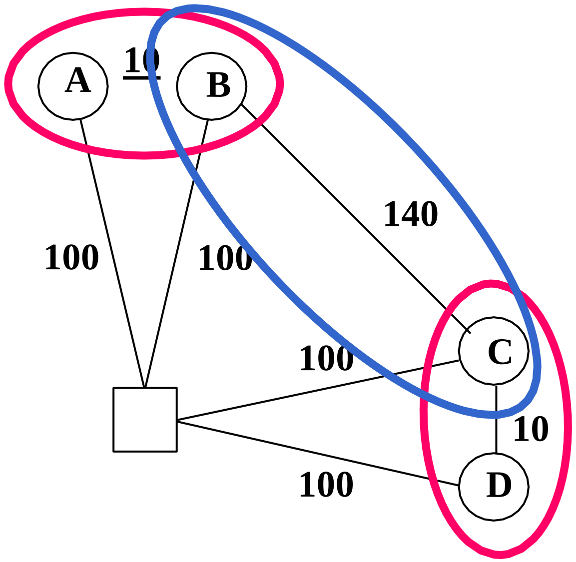
	capacity	consumption
A	5000	1000
B	3000	3000
C	2000	2000
D	4000	1500

Initial inventory of i =capacity
The supplier has unlimited availability
2 vehicles
Capacity of vehicles 5000

Repeat the red and the green tours every day
Daily cost: 420

A better solution

380 miles per day



	A	B	C	D
Daily Use	1000	3000	2000	1500
Max. Delv.	5000	3000	2000	4000

The "natural" solution

Daily schedule

trip 1: deliver 1000 to A & 3000 to B

trip 2: deliver 2000 to C & 1500 to D

420 miles per day

A better solution

Day 1 schedule

trip 1: deliver 3000 to B & 2000 to C

Day 2 schedule

trip 1: deliver 2000 to A & 3000 to B

trip 2: deliver 2000 to C & 3000 to D

Questo esempio mostra quale sia effettivamente l'impatto delle decisioni relative a quanto e quando servire sui costi di trasporto.

Shipping times

1. *Continuous*. A shipment can be performed at any time (starting from 0).

2. *Continuous with a minimum intershipment time*.

A shipment can be performed at any time (starting from 0), but the time between any pair of consecutive shipments (intershipment time) cannot be lower than a given minimum intershipment time, due to shipment/receiving set-up time requirements.

3. *Discrete Shipments* can be performed only at multiples of a minimum intershipment time. Since the minimum intershipment time can be normalized to 1 without loss of generality, shipments are performed at discrete times.

The planning horizon over which an optimal policy is looked at may be:

1. *Infinite.* In this case, the IRP aims at determining a long-term distribution plan that may be useful, for example, to determine the fleet of vehicles, the number of drivers, and the organization of the distribution area in regions. More detailed operational problems may be solved at a later stage.
2. *Finite.* The length of the planning horizon depends on the specific situation tackled. A short horizon is more operational than a long horizon.

Typical examples of structured policies, often inspired by practical relevance, are the following:

1. Zero Inventory Ordering (ZIO) Any customer is replenished if and only if its inventory level is down to zero.
2. Periodic. A period P has to be found. Any operation performed at time t, 0 < t<P, is repeated at times t + kP, k=1; 2; ...: 3.
3. Frequency-based. These are periodic policies in which shipments are performed on the basis of one or several frequencies. For each frequency, the intershipment time is constant. In the single frequency policy, the intershipment time can be continuous or discrete. If more frequencies are allowed, then each frequency has an associated integer intershipment time and the period P of the policy is the minimum common multiplier of the intershipment times.
4. Full load. Shipments are performed using full load vehicles only.
5. Direct shipping. Any customer is served independently by direct shipments from the supplier only. Routes that visit more than one customer are not allowed.
6. Order-up-to level Any customer has defined a maximum inventory level. Every time a customer is served, the delivered quantity is such that the maximum inventory level at the customer is reached.

Typical examples of structured policies, often inspired by practical relevance, are the following:

7. Maximum level This class of policies generalizes the order-up-to level policies. Any customer has defined a maximum inventory level. Every time a customer is served, the delivered quantity is such that the inventory level at the customer is not greater than the maximum level.

8. Fixed partition The set of customers is partitioned into a number of sets such that each set is served separately and independently of the other sets. In other words, any route visits customers of the same set. The partition is typically based on the geographical location of the customers.

9. Partition-based. This class of policies generalizes the fixed partition policies. Customers are partitioned into sets, as in the fixed partition policy. A route may visit customers of a set only, but also customers of specific combinations of two or more sets

Example

The following examples show that in the worst case, the solution obtained when the objective is to minimize the transportation cost only or the inventory cost only can be infinitely worse than the one obtained by minimizing the sum of the costs.

Example 1 Consider the case with one supplier and one customer. The shipping times are discrete. A set I of products has to be shipped from the supplier to the customer. Each product $i \in I$ has a production and consumption rate q_i, a unit volume v_i and unit inventory cost h_i. Let ϵ be such that $\frac{1}{\epsilon}$ is an integer number. Suppose that the total volume per time unit is $v = \sum_{i \in I} v_i q_i = \epsilon$ and the associated inventory cost is $h = \sum_{i \in I} h_i q_i = \frac{1}{\epsilon}$. The transportation is performed by vehicles having capacity $C = 1$. The transportation cost per trip, i.e., the cost to go from the supplier to the customer and return to the supplier, is $c = 100$.

The Inventory costs

Let us first compute the average total cost per time unit of a single frequency policy, i.e., a policy where a shipment takes place every τ times.

Let us focus on any time interval of length τ .
As the production rate q_i of any product $i \in \mathrm{I}$ is constant over time, the total quantity produced every τ times is $q_i\tau$.

The total inventory level at the supplier over τ is $\frac{(q_i\tau)\tau}{2}$.
The average inventory level per time unit is $\frac{(q_i\tau)\tau}{2\tau} = \frac{(q_i\tau)}{2}$

Since the unit inventory cost h_i is charged for any product $i \in \mathrm{I}$;
the average inventory cost at the supplier is $\frac{\sum_{i\epsilon I}(h_i q_i \tau)}{2}$ per time unit.

Since the average inventory cost at the customer is identical,
the total average inventory cost is $\sum_{i\epsilon I}(h_i q_i \tau)$

Transportation costs

Let us now compute the transportation cost.
As the total volume shipped every τ time is $v\tau$;
$y = \left\lceil \frac{v\tau}{C} \right\rceil$ so y is the number of vehicles needed every τ time.
Therefore, the average transportation cost per time unit is $\frac{c}{\tau}\left\lceil \frac{v\tau}{C} \right\rceil$.
Hence, the average total cost per time unit is $z^{SF}(\tau) = \sum_{i\epsilon I}(h_i q_i \tau) + \frac{c}{\tau}\left\lceil \frac{v\tau}{C} \right\rceil$

$$h = \sum_{i \in I} h_i q_i = \frac{1}{\epsilon}. \qquad v = \sum_{i \in I} v_i q_i = \epsilon$$

If the transportation cost only is taken into account,
one fully loaded vehicle is sent every $\tau = \frac{1}{\epsilon}$ time with average total cost per time unit $z^{SF}\left(\frac{1}{\epsilon}\right) = \sum_{i\epsilon I}(h_i q_i \tau) + \frac{c}{\tau}\left\lceil \frac{v\tau}{C} \right\rceil = h\tau + cv = \frac{1}{\epsilon^2} + 100\epsilon$

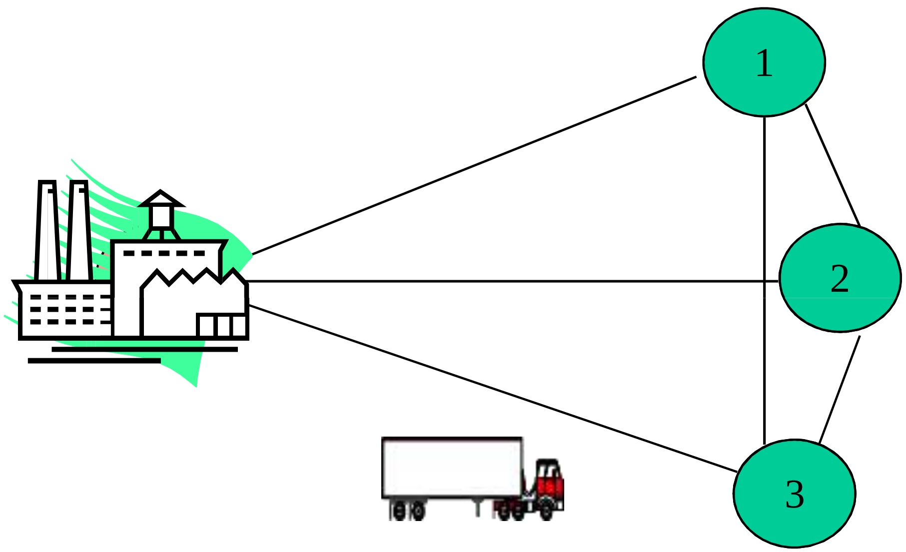
A different solution is obtained by sending one vehicle per time unit, with a resulting average total cost per time unit
$z^{SF}(1) = \sum_{i\epsilon I}(h_i q_i \tau) + \frac{c}{\tau}\left\lceil \frac{v\tau}{C} \right\rceil = \frac{1}{\epsilon} + 100$

IRP costs

$$\frac{z^{SF}\left(\frac{1}{\epsilon}\right)}{z^{SF}(1)} = \frac{\frac{1}{\epsilon^2} + 100\epsilon}{\frac{1}{\epsilon} + 100} \rightarrow \infty \quad \epsilon \rightarrow 0.$$

The above example shows that whenever the inventory costs are relevant but are ignored in the optimization, a very poor solution is likely to be found and implemented.

An inventory routing problem



n customers
H time horizon
1 vehicle

r_{0t} Production at t

B_0 Inventory of 0 at t

r_{st} Demand of s at t

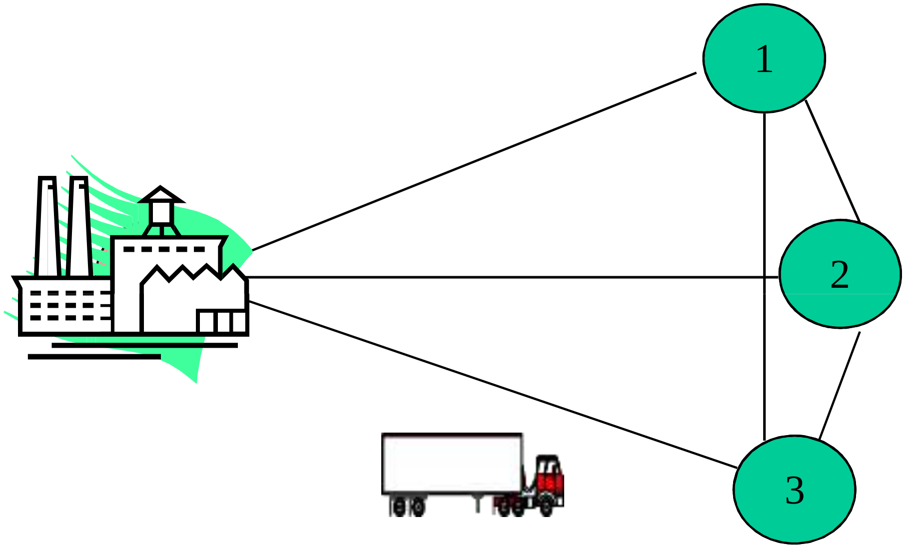
U_s Capacity of s

I_{st} Inventory of s at t

How much to deliver to s at time t

to minimize
routing costs + inventory costs

Replenishment policies

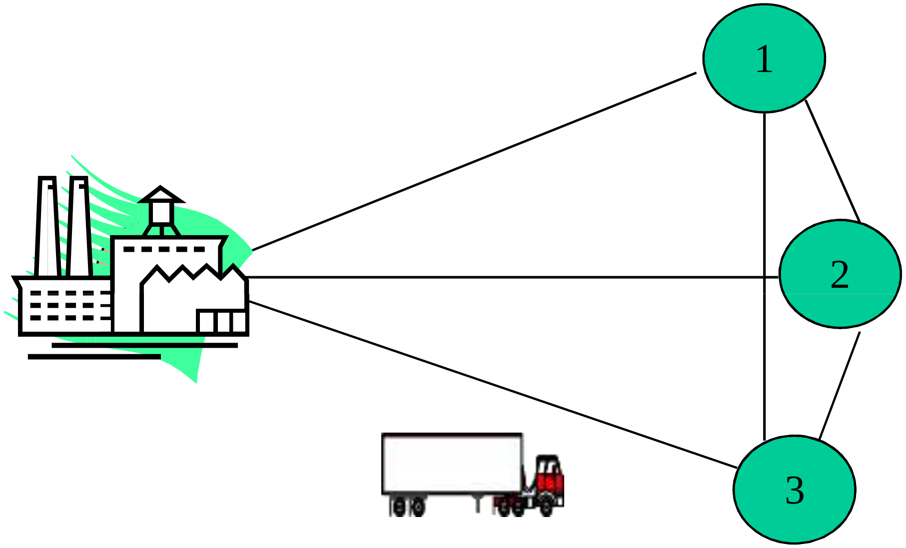


Order-Up-to Level (OU)

Maximum Level (ML)

Constraints on the quantities to deliver

Replenishment policies



Order-Up-to Level (OU)

Maximum Level (ML)

→ Constraints on the quantities to deliver

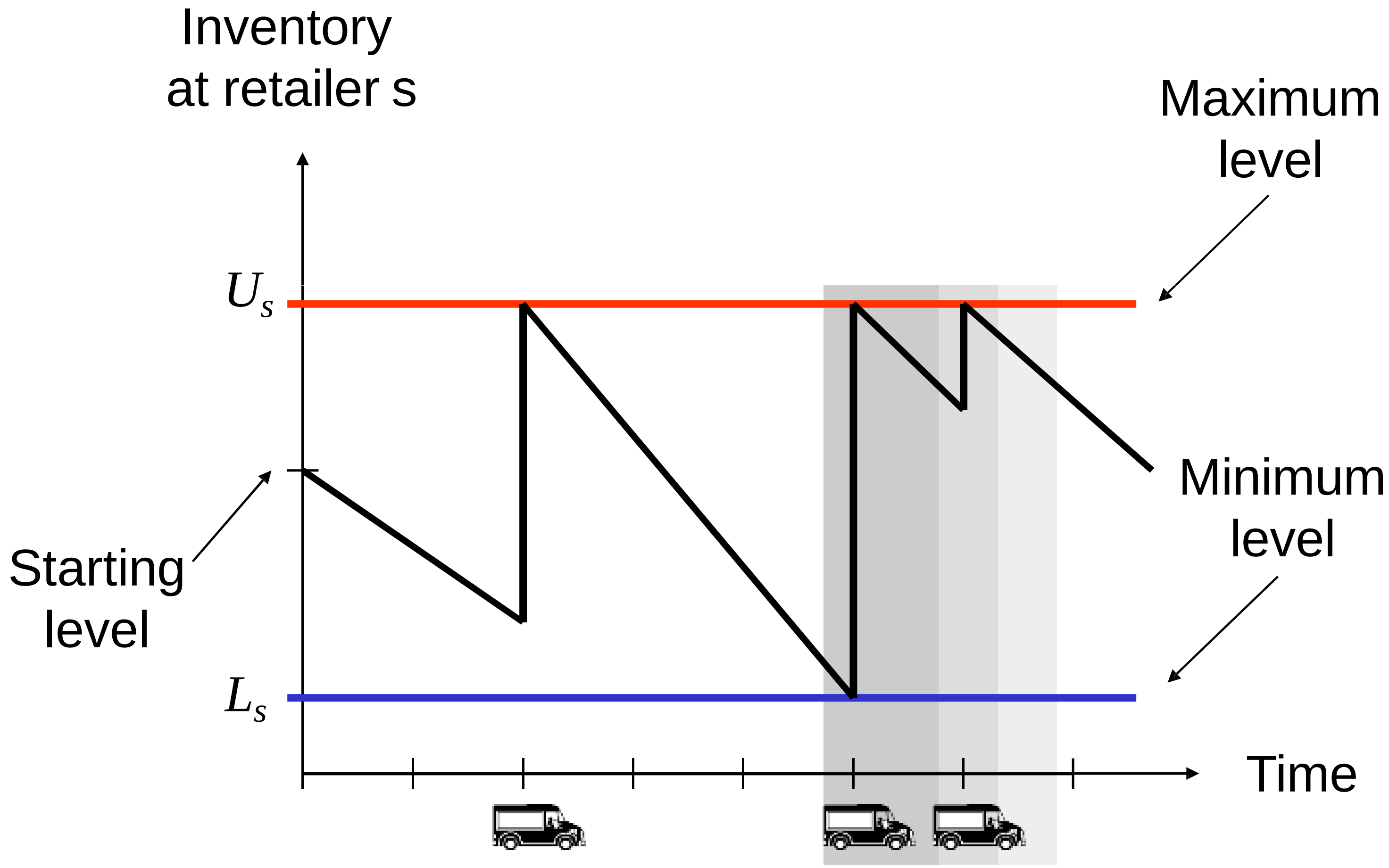
The deterministic order-up-to level policy

- Each retailer defines a minimum and a maximum level of the inventory

- The supplier determines the set of delivery time instants

- Every time a retailer is visited, the shipping quantity is such that the maximum level of the inventory is reached at the retailer

The deterministic order-up-to level policy



Problem formulation

Decisions:

- For each retailer *s*:
  the set of delivery time instants

- For each delivery time instant *t*:
  the route followed by the vehicle

Objective function:

Min Inv. Supplier + Inv. Retailers +Routing

Decision variables

x_{st} : quantity (≥ 0) shipped to retailer s at time t

$$z_{it} = \begin{cases} 1 & \text{if node } i \in M' \text{ (retailer or supplier)} \\ & \text{is visited at time } t \\ 0 & \text{otherwise} \end{cases}$$

$$y_{ij}^t = \begin{cases} 1 & \text{if the arc } (i,j) \text{ is traveled at time } t \\ 0 & \text{otherwise} \end{cases}$$

$$i \in M \quad j \in M, j < i \quad t \in T$$

$$y_{i0}^t \in \{0,1,2\} \quad i \in M \quad t \in T$$

Constraints

- *Inventory definition at the supplier*

$$B_t = B_{t-1} + r_{0t-1} - \sum x_{st-1} \quad t \in T'$$

- *Stock-out constraints at the supplier*

$$B_t \geq \sum_{s \in M} x_{st} \quad t \in T$$

r_{0t} Production at t

B_0 Inventory of 0 at t

r_{st} Demand of s at t

U_s Capacity of s

I_{st} Inventory of s at t

Constraints

- *Inventory definition at the retailers*

$$I_{st} = I_{st-1} + x_{st-1} - r_{st-1} \qquad s \in M, t \in T'$$

- *Stock-out constraints at the retailers*

$$I_{st} \geq 0 \qquad s \in M, t \in T'$$

- *Capacity constraints*

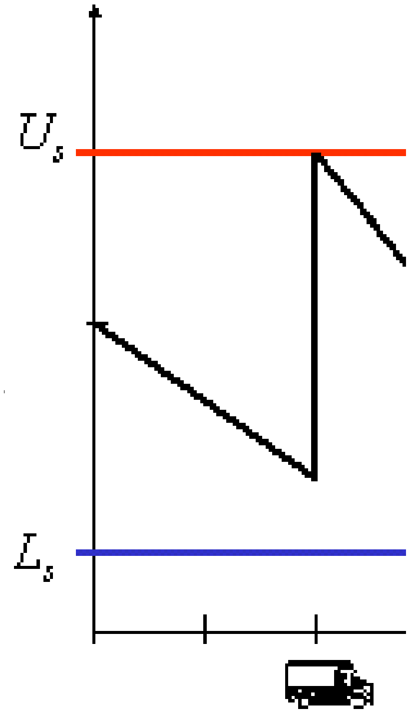
$$\sum_{s \in M} x_{st} \leq C \qquad t \in T$$

Constraints

- *Order-up-to level constraints*

The quantity shipped to s at time t is

$$\nearrow U_s - I_{st}$$
$$\searrow 0$$



$$x_{st} \geq U_s z_{st} - I_{st} \qquad s \in M \qquad t \in T$$

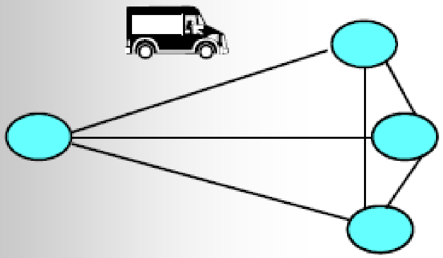
$$x_{st} \leq U_s - I_{st} \qquad s \in M \qquad t \in T$$

$$x_{st} \leq U_s z_{st} \qquad s \in M \qquad t \in T$$

Constraints

- *Routing constraints*

$$\sum_{s \in M} x_{st} \leq C z_{0t} \quad t \in T$$



$$\sum_{j \in M', j<i} y_{ij}^{t} + \sum_{j \in M', j>i} y_{ji}^{t} = 2 z_{it} \quad i \in M', t \in T$$

Objective function

$$\min \sum_{t\in T'} h_0 B_t + \sum_{s\in M} \sum_{t\in T'} h_s I_{st} + \sum_{i\in M'} \sum_{j\in M', j<i} \sum_{t\in T} c_{ij} y_{ij}^t$$

Inventory cost supplier

Inventory cost retailers

Routing cost

Valid inequalities

Service inequalities

$$I_{st} \geq (1-z_{st})r_{st} \quad s \in M, t \in T$$

$$I_{st} \geq r_{st}$$

$$I_{st} \geq 0$$

Extended service inequalities

$$I_{st-k} \geq \left(\sum_{j=0}^{k} r_{st-j}\right)\left(1-\sum_{j=0}^{k} z_{st-j}\right) \quad s \in M', t \in T, k = 0,\dots,t$$

If s is not served there must be sufficient inventory

Valid inequalities

Order-up inequalities

$$I_{st} \geq U_s z_{st-k} - \sum_{j=t-k}^{t-1} r_{sj} \quad s \in M, t \in T, k = 1,\dots,t-1$$

Inventory is related to the order-up to level replenishment policy

Visits inequalities

$$\sum_{s \in M} z_{st} \leq z_{ot} \quad s \in M, t \in T$$

The supplier has to be visited at time t if at least one retailer is served at time t

- Subtours elimination constraints