LP: examples of the simplex method

Example: the production of paints (Product Mix)

$$\max Z = 3x_E + 2x_I$$
$$x_E + 2x_I \le 6$$
$$2x_E + x_I \le 8$$
$$-x_E + x_I \le 1$$
$$x_I \le 2$$
$$x_E \ge 0 \quad x_I \ge 0$$

In standard form

$$\max Z = 3x_E + 2x_I$$
$$x_E + 2x_I + s_1 = 6$$
$$2x_E + x_I + s_2 = 8$$
$$-x_E + x_I + s_3 = 1$$
$$x_I + s_4 = 2$$
$$x_E \ge 0, x_I \ge 0, s_1 \ge 0, s_2 \ge 0, s_3 \ge 0, s_4 \ge 0$$

n=6 m=4

Initialization with slack variables is possible

LP: examples of the simplex method

Equation for the starting BFS

$$\underline{x}_B = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 1 \\ 2 \end{bmatrix}$$

$$Z = 0 - (-3x_E - 2x_I)$$
$$s_1 = 6 - (x_E + 2x_I)$$
$$s_2 = 8 - (2x_E + x_I)$$
$$s_3 = 1 - (-x_E + x_I)$$
$$s_4 = 2 - (0x_E + x_I)$$

BFS not optimal

First iteration

x_E enters the basis

$$s_1 = 0 \Rightarrow x_E = 6$$
$$s_2 = 0 \Rightarrow x_E = 4$$

s_2 leaves the basis

$$Z = 12 - (3/2 s_2 - 1/2 x_I)$$
$$s_1 = 2 - (-1/2 s_2 + 3/2 x_I)$$
$$x_E = 4 - (1/2 s_2 + 1/2 x_I)$$
$$s_3 = 5 - (3/2 s_2 + 1/2 x_I)$$
$$s_4 = 1 - (0 s_2 + x_I)$$

BFS not optimal

Second iteration

x_I enters the basis

LP: examples of the simplex method

Equation for the second iteration

$$s_1 = 0 \Rightarrow x_I = 4/3$$
$$x_E = 0 \Rightarrow x_I = 8$$
$$s_3 = 0 \Rightarrow x_I = 10/3$$
$$s_4 = 0 \Rightarrow x_I = 2$$

s_1 leaves the basis

$$Z = 38/3 - (4/3\,s_2 + 1/3\,s_1)$$
$$x_I = 4/3 - (-1/3\,s_2 + 2/3\,s_1)$$
$$x_E = 10/3 - (2/3\,s_2 - 1/3\,s_1)$$
$$s_3 = 3 - (s_2 - s_1)$$
$$s_4 = 2/3 - (1/3\,s_2 - 2/3\,s_1)$$

BFS is optimal

The algorithm stops after 2 iterations

LP: examples of the simplex method

Simplex on the Tableau

Initial tableau

Entering

Pivot

Leaving

	$\boxed{x_E}$	x_I	s_1	s_2	s_3	s_4		
z	-3	-2	0	0	0	0	0	
s_1	1	2	1	0	0	0	6	$6/1$
$\boxed{s_2}$	$\boxed{2}$	1	0	1	0	0	8	$8/2$
s_3	-1	1	0	0	1	0	1	–
s_4	0	1	0	0	0	1	2	–

1st iteration: x_E enters s_2 leaves

LP: examples of the simplex method

Gauss-Jordan elimination for the first iteration

1. s_2 row divided by pivot
2. x_E becomes basic in place of s_2

3. Zero all coeff. in x_E column except the one in row x_E

	x_E	x_I	s_1	s_2	s_3	s_4	
z	-3	-2	0	0	0	0	0
s_1	1	2	1	0	0	0	6
x_E	1	1/2	0	1/2	0	0	4
s_3	-1	1	0	0	1	0	1
s_4	0	1	0	0	0	1	2

Example for objective row:

Multiply row x_E by -3

x_E | -3 -3/2 0 -3/2 0 0 | -12

and subtract it from row Z

Z | -3 -2 0 0 0 0 | 0

Obtaining the new row Z

Z | 0 -1/2 0 3/2 0 0 | 12

LP: examples of the simplex method

Gauss-Jordan elimination for the first iteration

Zero the coeff. in column x_E for row s_1 and s_3

	x_E	x_I	s_1	s_2	s_3	s_4	
z	0	$-1/2$	0	$3/2$	0	0	12
s_1	1	2	1	0	0	0	6
x_E	1	$1/2$	0	$1/2$	0	0	4
s_3	-1	1	0	0	1	0	1
s_4	0	1	0	0	0	1	2

Example for row s_1 :

Multiply row x_E by 1 (unchanged)

x_E | 1 1/2 0 1/2 0 0 | 4

Subtract from row s_1

s_1 | 1 2 1 0 0 0 | 6

Obtaining the new row s_1

s_1 | 0 3/2 1 -1/2 0 0 | 2

…

	x_E	x_I	s_1	s_2	s_3	s_4	
z	0	$-1/2$	0	$3/2$	0	0	12
s_1	0	$3/2$	1	$-1/2$	0	0	2
x_E	1	$1/2$	0	$1/2$	0	0	4
s_3	-1	1	0	0	1	0	1
s_4	0	1	0	0	0	1	2

Update rows

LP: examples of the simplex method

Second iteration

Tableau after 1st iteration

Pivot

entering

	x_E	$\boxed{x_I}$	s_1	s_2	s_3	s_4	
z	0	$-1/2$	0	$3/2$	0	0	12
$\boxed{s_1}$	0	$\boxed{3/2}$	1	$-1/2$	0	0	2
x_E	1	$1/2$	0	$1/2$	0	0	4
s_3	0	$3/2$	0	$1/2$	1	0	5
s_4	0	1	0	0	0	1	2

$2 \cdot 2/3 = 4/3$

$4 \cdot 2 = 8$

$5 \cdot 2/3 = 10/3$

2

leaving

2nd iteration: x_I enters s_1 leaves

LP: examples of the simplex method

Tableau after 2nd iteration

Non negative coeff.

Optimal objective value

	x_E	x_I	s_1	s_2	s_3	s_4	
z	0	0	1/3	4/3	0	0	38/3
x_I	0	1	2/3	−1/3	0	0	4/3
x_E	1	0	−1/3	2/3	0	0	10/3
s_3	0	0	−1	1	1	0	3
s_4	0	0	−2/3	1/3	0	1	2/3

Optimal basis

Optimal solution

Optimal tableau

LP: examples of the simplex method

A second example

Standard form

$$\max x_0 = 2x_1 + x_2$$

$$\begin{aligned}
x_1 + x_2 + x_3 \quad &= 5 \\
-x_1 + x_2 \quad + x_4 \quad &= 0 \\
6x_1 + 2x_2 \quad + x_5 &= 21 \\
x_1 \ge 0 \, x_2 \ge 0 \, x_3 \ge 0 \, x_4 \ge 0 \, x_5 &\ge 0
\end{aligned}$$

Initialization with slack variables

$$\underline{x}_B = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 21 \end{bmatrix}$$

(Degenerate)

LP: examples of the simplex method

Initial tableau

	x_3	x_4	x_5	$\boxed{x_1}$ (entering)	x_2		
x_0	0	0	0	-2	-1	0	
x_3	1	0	0	1	1	5	$5/1$
x_4	0	1	0	-1	1	0	–
leaving $\boxed{x_5}$	0	0	1	$\boxed{6}$	2	21	$21/6 = 7/2$

Pivot

1st iteration: x_1 enters x_5 leaves

LP: examples of the simplex method

Tableau after 1st iteration

Pivot

New objective value

Leaving

Entering

	x_3	x_4	x_5	x_1	$\boxed{x_2}$		
x_0	0	0	1/3	0	−1/3	$\boxed{7}$	
$\boxed{x_3}$	1	0	−1/6	0	$\boxed{2/3}$	3/2	$3/2 \cdot 3/2 = 9/4 (= 2{,}25)$
x_4	0	1	1/6	0	4/3	7/2	$7/2 \cdot 3/4 = 21/8 (= 2{,}62)$
x_1	0	0	1/6	1	1/3	7/2	$7/2 \cdot 3 = 21/2 (= 10{,}5)$

2nd iteration: x_2 enters x_3 leaves

LP: examples of the simplex method

Tableau after 2nd iteration

non negative coeff.

Optimal objective value

	x_3	x_4	x_5	x_1	x_2	
x_0	$1/2$	0	$1/4$	0	0	$31/4$
x_2	$3/2$	0	$-1/4$	0	1	$9/4$
x_4	-2	1	$1/2$	0	0	$1/2$
x_1	$-1/2$	0	$1/4$	1	0	$11/4$

Optimal basis

Optimal solution

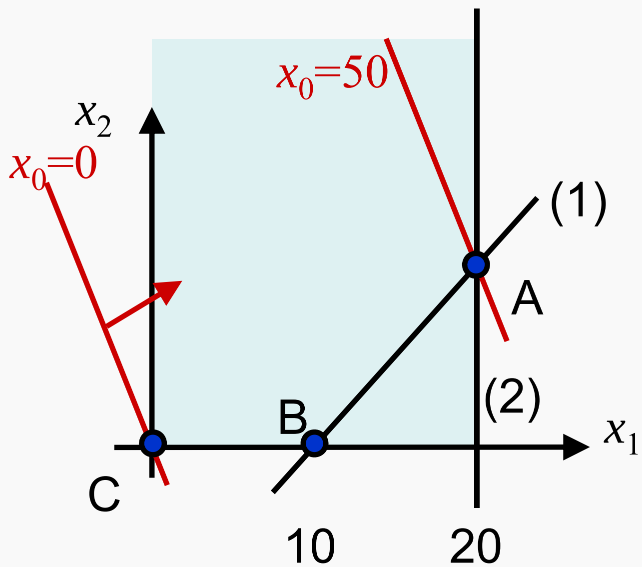
Optimal tableau

LP: the simplex in tabular form – special cases

Unbounded solution

Example

$$\max\ x_0 = 2x_1 + x_2$$
$$x_1 - x_2 \le 10$$
$$2x_1 \le 40$$
$$x_1 \ge 0\ x_2 \ge 0$$



in standard form

$$\max\ x_0 = 2x_1 + x_2$$
$$x_1 - x_2 + x_3 = 10$$
$$2x_1 + x_4 = 40$$
$$x_1 \ge 0\ x_2 \ge 0\ x_3 \ge 0\ x_4 \ge 0$$

Initializing with slacks (vertex C)

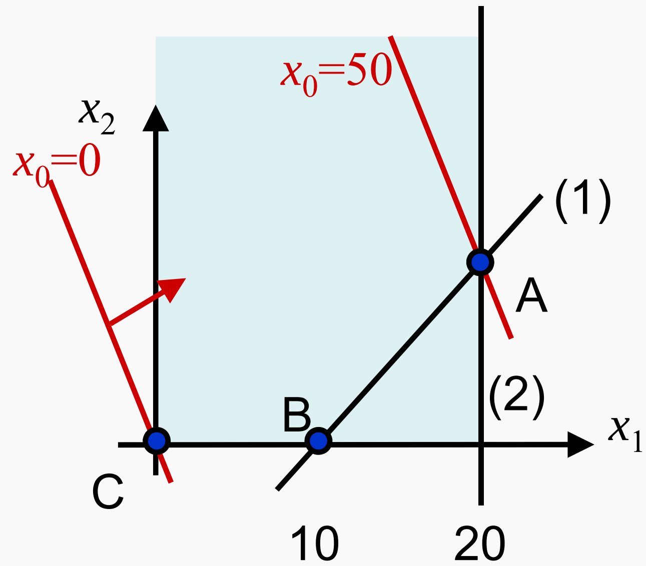
$$\underline{x}_B = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10 \\ 40 \end{bmatrix}$$

LP: the simplex in tabular form – special cases

Unbounded solution

- Initial Tableau

	x_3	x_4	x_1	x_2	
x_0	0	0	-2	-1	0
x_3	1	0	1	-1	10
x_4	0	1	2	0	40



x_1 enters x_3 leaves
From C to B along axes x_1
Tableau after first iteration

	x_3	x_4	x_1	x_2	
x_0	2	0	0	-3	20
x_1	1	0	1	-1	10
x_4	-2	1	0	2	20

x_2 enters x_4 leaves
From B to A along constraint (1)

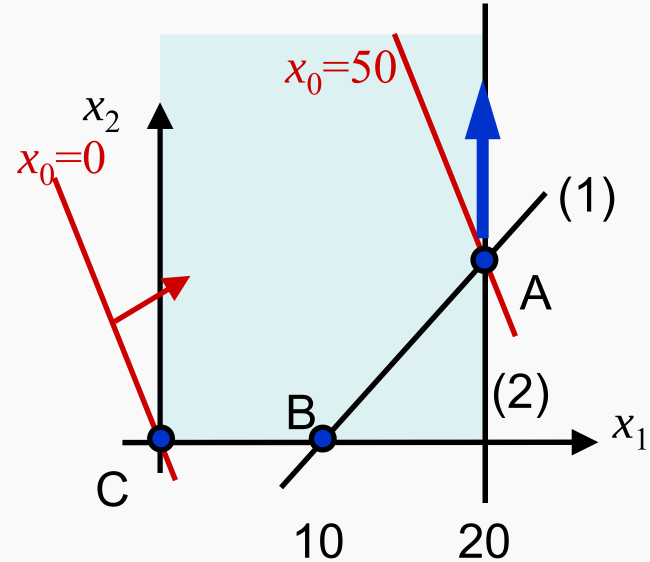
LP: the simplex in tabular form – special cases

Unbounded solution

Tableau after second iteration

	x_3	x_4	x_1	x_2	
x_0	-1	$3/2$	0	0	50
x_1	0	$1/2$	1	0	20
x_2	-1	$1/2$	0	1	10

x_3 enters moving solution along (2) without reaching any other constraint



Solution grows to infinity (unbounded solution)

Simplex stops

LP: the simplex in tabular form – special cases

Unbounded solution

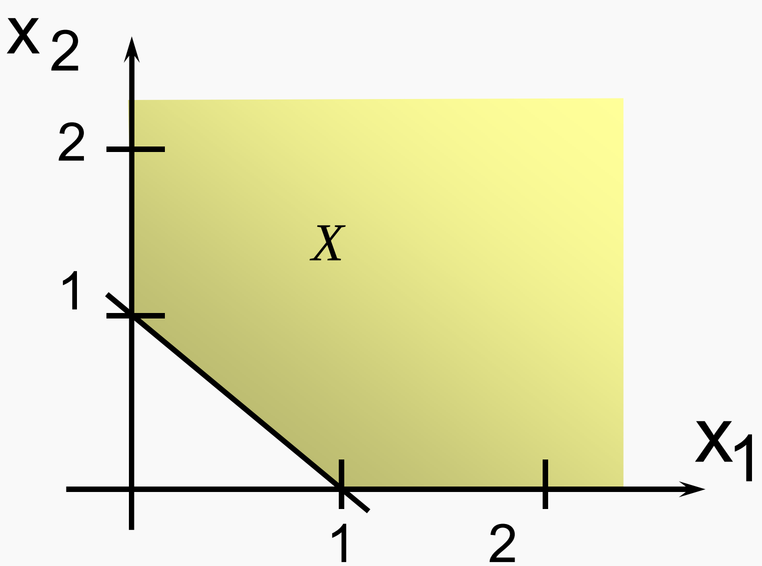
A second example

$$\max x_0 = x_1$$
$$x_1 + x_2 \geq 1$$
$$x_1 \geq 0 \; x_2 \geq 0$$

in standard form

$$\max x_0 = x_1$$
$$x_1 + x_2 - x_3 = 1$$
$$x_1 \geq 0 \; x_2 \geq 0 \; x_3 \geq 0$$

n=3, m=1



Polyhedron X is open

We cannot initialize with slacks.
Why?

LP: the simplex in tabular form – special cases

Unbounded solution

- Initialize the tableau with the BFS for vertex x_1=1 x_2=0

$$\underline{x}_B = [x_1] = [1] \quad \underline{x}_N = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \underline{0}$$

	x_1	x_2	x_3	
x_0	-1	0	0	0
x_1	1	1	-1	1

Initial tableau ….
But something is wrong (what?)

LP: the simplex in tabular form – special cases

Unbounded solution

Objective must be a function of non basic variables x_2 and x_3 only

We need a pivoting

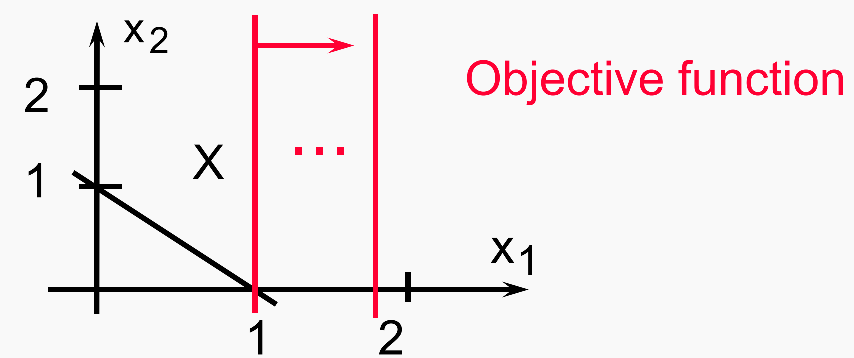
	x_1	x_2	x_3	
x_0	-1	0	0	0
x_1	1	1	-1	1

Subtracting row x_1 to objective row

	x_1	x_2	x_3	
x_0	0	1	-1	1
x_1	1	1	-1	1

Correct objective value

x_3 enters without violating the constraint



Increasing x_3 the solution moves on axes x_1 (following an *extreme direction*)

LP: extreme directions

How many extreme directions are there in a polyhedron?

If $\underline{d}$ is a direction for a polyhedron $X = \{A\underline{x} = \underline{b},\ \underline{x} \geq \underline{0}\}$

then $A(\underline{x} + \lambda \underline{d}) = \underline{b} \Rightarrow A\underline{d} = \underline{0}$

Rewriting as a function of basis B

$$A\underline{d} = \underline{0} \Rightarrow [B|N]\begin{bmatrix} \underline{d}_B \\ \underline{d}_N \end{bmatrix} = \underline{0} \Rightarrow B\underline{d}_B + N\underline{d}_N = \underline{0}$$

Obtaining $\underline{d}_B = -B^{-1}N\underline{d}_N$

Then arbitrarily fixing d_N we compute an extreme direction as

$$\underline{d} = \begin{bmatrix} \underline{d}_B \\ \underline{d}_N \end{bmatrix} = \begin{bmatrix} -B^{-1}N\underline{d}_N \\ \underline{d}_N \end{bmatrix}$$

LP: extreme directions

A possible choice

$$\underline{d}_N = \underline{e}_j = \begin{array}{c@{}c} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} & \begin{array}{c} 1 \\ \\ j \\ \\ n-m \end{array} \end{array} \qquad \underline{e}_j \in \mathbf{R}^{n-m}$$

Obtaining $\underline{d} = \begin{bmatrix} -B^{-1}\underline{a}_j \\ \underline{e}_j \end{bmatrix}$

where a_j is the j-th column of N

For each basis B distinct n-m vectors a_j can be chosen

Then the maximum number of extreme direction is

$$(n-m)\binom{n}{m}$$

LP: extreme directions

- Computing the extreme directions for the example (basis matrix formed by x_1 column)

$$A = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \end{bmatrix} \qquad N = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

$$\underline{d}_1 = \begin{bmatrix} -B^{-1}\underline{a}_j \\ \underline{e}_j \end{bmatrix} = \begin{bmatrix} -1 \cdot 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} (j=1) \qquad \underline{d}_2 = \begin{bmatrix} -B^{-1}\underline{a}_j \\ \underline{e}_j \end{bmatrix} = \begin{bmatrix} -1 \cdot -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} (j=2)$$

For both directions $A\underline{d}=\underline{0}$ (e.g., for $\underline{d}_2$)

$$A\underline{d}_2 = \underline{0} \Rightarrow \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1 - 1 = 0 \quad (n=3, m=1)$$

LP: finite optimality condition

Theorem

Given the LP problem

$$\max x_0 = \underline{c}^T \underline{x}$$
$$A\underline{x} = \underline{b}$$
$$\underline{x} \geq \underline{0}$$

Let d_j, j=1,...,D the extreme directions of the non empty polyhedron X={$A\underline{x}$=$\underline{b}$, $\underline{x}$ ≥0}

An finite optimal solution exists if and only if

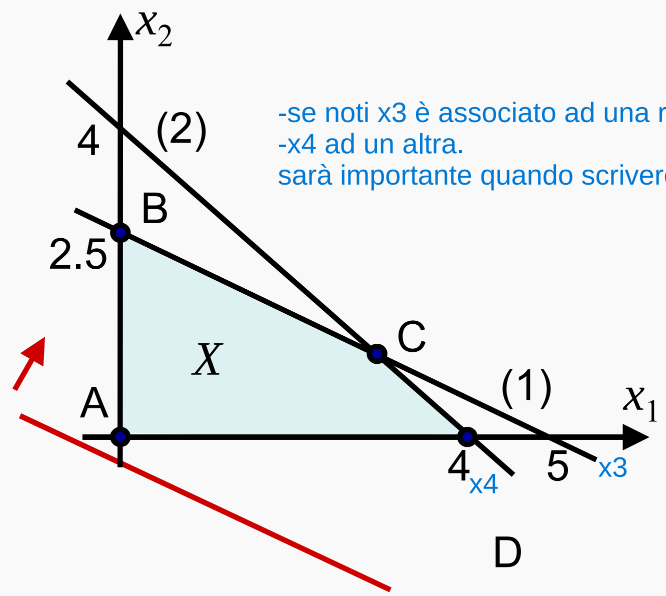
$$\underline{c}^T \underline{d}_j \leq 0 \qquad \forall j = 1,\ldots,D$$

Such optimum corresponds to an extreme point of X

LP: alternative optimal solutions

Example

$$\max\ x_0 = 2x_1 + 4x_2$$
$$x_1 + 2x_2 \le 5$$
$$x_1 + x_2 \le 4$$
$$x_1 \ge 0 \quad x_2 \ge 0$$



-se noti x3 è associato ad una retta.
-x4 ad un altra.
sarà importante quando scriveremo la base

in standard form

$$\max\ x_0 = 2x_1 + 4x_2$$
$$x_1 + 2x_2 + x_3 = 5$$
$$x_1 + x_2 + x_4 = 4$$
$$x_1, x_2, x_3, x_4 \ge 0$$

	x_3	x_4	x_1	x_2	
x_0	0	0	-2	-4	0
x_3	1	0	1	2	5
x_4	0	1	1	1	4

x_2 enters
x_3 leaves

Optimal tableau

	x_3	x_4	x_1	x_2	
x_0	2	0	0	0	10
x_2	$1/2$	0	$1/2$	1	$5/2$
x_4	$-1/2$	1	$1/2$	0	$3/2$

LP: alternative optimal solutions

There are alternative optimal solutions: pivoting to let x_1 enter the basis

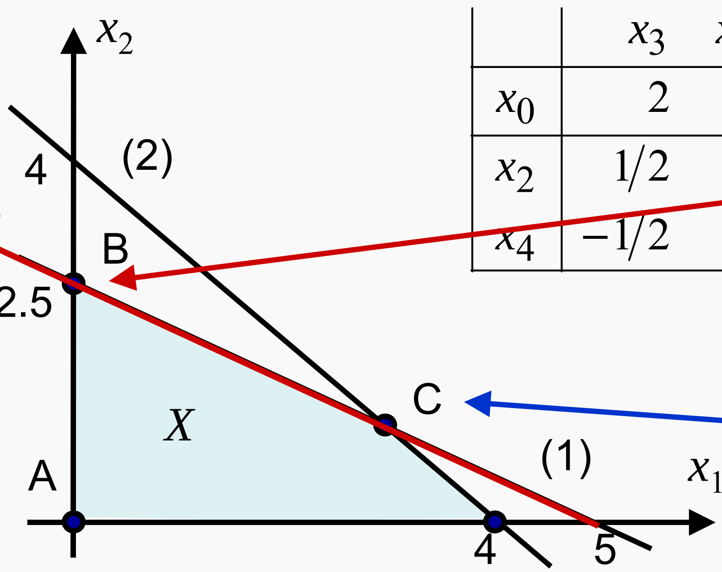
	x_3	x_4	x_1	x_2	
x_0	2	0	0	0	10
x_2	1/2	0	1/2	1	5/2
x_4	−1/2	1	1/2	0	3/2

x_1 enters
x_4 leaves

	x_3	x_4	x_1	x_2	
x_0	$\boxed{2 \quad 0}$		0	0	10
x_2	1	−1	0	1	1
x_1	−1	2	1	0	3

Alternative optimum

quindi quelle nella base sono x2 e x4 poiche x1 è chiaramente nulla e anche x3 poiche x2 è 2.5 che verifica la sua retta. invece l'altra retta avra bisogno di x4 per essere verificata



	x_3	x_4	x_1	x_2	
x_0	2	0	0	0	10
x_2	1/2	0	1/2	1	5/2
x_4	−1/2	1	1/2	0	3/2

All points on BC segment are optimal

	x_3	x_4	x_1	x_2	
x_0	2	0	0	0	10
x_2	1	−1	0	1	1
x_1	−1	2	1	0	3

LP: initialization of simplex on tableau

Example – Two-phase Method

$$\min x_0 = 4x_1 + x_2$$
$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

Not a canonical form

Converting in standard form

$$\max\ x_0 = -4x_1 - x_2$$
$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 - x_3 = 6$$
$$x_1 + 2x_2 + x_4 = 4$$
$$x_1, x_2, x_3, x_4 \geq 0$$

I Phase (Definition and solution of the auxiliary problem)

$$\min z = y_1 + y_2$$
$$3x_1 + x_2 \qquad\qquad + y_1 = 3$$
$$4x_1 + 3x_2 - x_3 \quad + y_2 = 6$$
$$x_1 + 2x_2 \qquad\qquad + x_4 \quad = 4$$
$$x_1, x_2, x_3, x_4, y_1, y_2 \geq 0$$

Only two auxiliary variables

Not in standard form

$$\Rightarrow \max z = -y_1 - y_2$$

LP: initialization of simplex on tableau

Tableau for I phase

We must eliminate the basic variables from the objective row

	x_1	x_2	x_3	x_4	y_1	y_2	
z	0	0	0	0	1	1	0
y_1	3	1	0	0	1	0	3
y_2	4	3	-1	0	0	1	6
x_4	1	2	0	1	0	0	4

Initial tableau

	x_1	x_2	x_3	x_4	y_1	y_2	
z	-7	-4	1	0	0	0	-9
y_1	3	1	0	0	1	0	3
y_2	4	3	-1	0	0	1	6
x_4	1	2	0	1	0	0	4

Il tableau finale

i numeri della riga z per le variabili nella base DEVONO essere 0

	x_1	x_2	x_3	x_4	y_1	y_2	
z	0	0	0	0	1	1	0
x_1	1	0	$1/5$	0	$3/5$	$-1/5$	$3/5$
x_2	0	1	$-3/5$	0	$-4/5$	$3/5$	$6/5$
x_4	0	0	1	1	1	-1	1

y_1 and y_2 out of optimal basis

Values for initializing the original tableau

LP: initialization of simplex on tableau

II Phase (Initializing and solving the original problem)

Eliminate the basic variables from the objective row

	x_1	x_2	x_3	x_4	
x_0	$\boxed{4}$	$\boxed{1}$	0	0	0
x_1	1	0	$1/5$	0	$3/5$
x_2	0	1	$-3/5$	0	$6/5$
x_4	0	0	1	1	1

Initial tableau for the original problem

	x_1	x_2	x_3	x_4	
x_0	0	0	$-1/5$	0	$-18/5$
x_1	1	0	$1/5$	0	$3/5$
x_2	0	1	$-3/5$	0	$6/5$
x_4	0	0	1	1	1

Solve for exercise …

LP: initialization of simplex on tableau

The Big-M method

$$\max x_0 = 3x_1 + 2x_2$$
$$2x_1 + x_2 \le 2$$
$$3x_1 + 4x_2 \ge 12$$
$$x_1, x_2 \ge 0$$

In standard form

$$\max x_0 = 3x_1 + 2x_2$$
$$2x_1 + x_2 + x_3 = 2$$
$$3x_1 + 4x_2 - x_4 = 12$$
$$x_1, x_2, x_3, x_4 \ge 0$$

x_4 is a surplus

The modified problem

$$\max x_0 = 3x_1 + 2x_2 - My$$
$$2x_1 + x_2 + x_3 = 2$$
$$3x_1 + 4x_2 - x_4 + y = 12$$
$$x_1, x_2, x_3, x_4, y \ge 0$$

	x_1	x_2	x_3	x_4	y	
x_0	-3	-2	0	0	M	0
x_3	2	1	1	0	0	2
y	3	4	0	-1	1	12

LP: initialization of simplex on tableau

Eliminating y from the objective function row

Initial tableau

	x_1	x_2	x_3	x_4	y	
x_0	$-3-3M$	$\boxed{-2-4M}$	0	M	0	$-12M$
x_3	2	1	1	0	0	2
y	3	4	0	-1	1	12

x_2 enters x_3 leaves

Tableau after the first iteration

	x_1	x_2	x_3	x_4	y	
x_0	$\boxed{1+5M}$	0	$2+4M$	M	0	$4-4M$
x_2	2	1	1	0	0	2
y	-5	0	-4	-1	1	4

The tableau is optimal (M > 0)

y is in the optimal basis

The problem is unfeasible