Operational Possibilistic Theories and Generalized Tomography of Physical Theories

Honour's Thesis

June 2, 2025

Aaron Lengyel

University of Calgary

Supervised by
Dr. Carlo Maria Scandolo, Samuel Steakley

Operational Possibilistic Theories and Generalized Tomography of Physical Theories

Aaron Lengyel

June 2, 2025

Abstract

Probabilities are one of the cornerstones of applied mathematics, finding utility across statistics, biology, and physics. Whereas probability theory can serve as a useful tool to model our uncertainty in a complex, but fundamentally deterministic, system, in physics probabilities take on an entirely new level of importance in the quantum world. This is because, in the quantum picture, the universe is fundamentally nondeterministic. However, not all nondeterministic systems are probabilistic, since probabilities require an appropriate way to measure and quantify nondeterminism, typically done through the use of measure theory. We attempt to explore the indispensability of probabilities in understanding quantum nondeterminism by replacing probabilities with an alternative: possibilities. In possibility theory, outcomes can be labeled with two distinct values, **possible** and **impossible**. The focus of this research is to understand what occurs to generalizationw of quantum theory, called operational probabilistic theories, when the resulting outcomes can only be labeled as possible or impossible.

Contents

I Introduction

Probabilities are one of the cornerstones of applied mathematics, finding utility across statistics, biology, and physics. Whereas probability theory can serve as a useful tool to model our uncertainty in a complex, but fundamentally deterministic, system, in physics probabilities take on an entirely new level of importance in the quantum world. This is because, in the quantum picture, the universe is fundamentally nondeterministic. However, not all nondeterministic systems are probabilistic, since probabilities require an appropriate way to measure and quantify nondeterminism, typically done through the use of measure theory. We attempt to explore the indispensability of probabilities in understanding quantum nondeterminism by replacing probabilities with an alternative: possibilities. In possibility theory, outcomes can be labeled with two distinct values, **possible** and **impossible**. The focus of this research is to understand what occurs to a generalization of quantum theory, called operational probabilistic theories, when the resulting outcomes can only be labeled as possible or impossible.

I.1 Thesis Roadmap

The thesis is structured in the following 4 parts:

1. Introduction (Section I)
2. Background Notions (Section II)
3. Results (Section III)
   - General P-Theories (Section III.1)
   - Operational Possibilistic Theories (Section III.2)
   - Concrete Examples (Section III.3)
4. Conclusion and Next Steps (Section IV)

In section II, We stary by discussing the interpretations of probability and their applications in both classical and quantum theory. We discuss two perspectives, the frequentist and Bayesian, as well as their pitfalls. This motivates the development of alternative theories, namely possibilistic theory, which is the focus of this thesis. The remaining background will cover Category theory, quantum information, operational probabilistic theories, and basic lattice theory. The standard texts in Category theory are [Rie16; Lei16] and most examples and definitions are standard. The main proofs will appear in II.5 since these results are important and are also not written out explicitly in the literature. For quantum information we closely follow various standard texts on the subject [Cho22; Kok15; M A10; Wil16].

The results are built off the theory of operational probabilistic theories. These are a way of viewing physical theories through the lens of process theory, which is based heavily on category theory [Tul19]. The main sources on operational probabilistic theories are [CDP10; Sca19]. To understand possibilities we need to work with the details of the possibility (Boolean) lattice. Thus this section builds on basic notions of lattice theory found in [Nat; FJN95] to ideally present enough ideas to build possibilistic theories. The concept of a rig, or semiring, and their semimodules will also be discussed. This follows from [CL22].

In the final section, Section III.2, we cover the results. Here we build a concrete notion of a generalization of operational probabilistic theories on any arbitrary rig P, called a P-theory. This is built from the concept of process theories discussed in section II.7. Then we define a notion of tomographical indistinguishability and show that this forms a monoidal congruence as in definition II.15. This allows us to classify general theories by the existence of a functor via a main theorem (III.2). We can then understand possibilistic theories as process theories built over the Boolean lattice, and we analyze a particular example called the category of finite sets and relations **FinRel**.

II Background Notions

We begin by discussing probability in quantum theory and perspective on probability.

II.1 Probabilities in the Quantum Picture

Classically, probabilities arise from probability spaces in measure theory. Let Ω be a set of outcomes, and Σ a σ-algebra over X. This makes (Ω, Σ) a measurable space. If we imbue this space with a measure P such that $P(\Omega) = 1$ then this makes (Ω, Σ, P) a measure space called a probability space.

An experiment or test is associated with a random variable, this is a measurable function $X : \Omega \to \mathbb{R}$ from the measurable space (Ω, Σ) to the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ of borel sets of $\mathbb{R}$. This induces the pushforward measure of P by X as the measure:

$$\begin{aligned} P_X : \mathcal{B}(\mathbb{R}) &\to [0,1] \\ B &\mapsto P(X^{-1}(B)) = P(\{\omega \ : \ X(\omega) \in B\}) \end{aligned}$$

the latter set often abreviated as $P(X(\omega) \in B)$.

Quantum Mechanics on the other hand begins with the definition of a few basic postulates governing the status of observation, probabilities, and the description of states [Kok15; Cho22]. The most important of these is the Born Rule. A complete description of these postulates and the Born Rule is handled in section

II.6.2.

In the quantum picture, physical states are described by rays [Kok15] $|\psi\rangle$ in a Hilbert space $\mathcal{H}$. Given an outcome x we can associate a projector Π_x (II.23) such that the probability of x is given by:

$$p(x) = \langle\psi|\Pi_x|\psi\rangle$$

and the post-measurement quantum state is:

$$|\psi\rangle \mapsto \frac{\Pi_x |\psi\rangle}{p(x)}$$

For a noisy quantum system, we can have mixed states $\rho \in \mathrm{Herm}_{\geq 0}(\mathcal{H})$ where $\mathrm{Herm}_{\geq 0}(\mathcal{H})$ is the set of bounded positive semidefinite self-adjoint operators on $\mathcal{H}$, called a density operators. Then the probability of an outcome x is:

$$p(x) = \mathrm{tr}(E_x\rho)$$

where E_x is a POVM (see II.25) associated to the outcome x.

In this view, a physicist working in quantum mechanics can describe experiments and measurement outcomes purely in terms of linear operators and inner products. The proper treatment of projective and POVM measurement will be handled in the background section II.6.

II.2 Philosophical Perspectives on Probability

The notion of probability has two main interpretations, that of the frequentist and the Bayesian. We discuss both.

II.2.1 Frequentist Approach

The frequentist believes that probabilities can be determined empirically by counting the frequency of which an event occurs over an infinite period of time, or an infinite period of samples.

Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of independent and identically distributed (I.I.D.) samples with expectation valued μ. Let $\overline{X}_n = \frac{1}{n}\sum_{i=0}^{n} X_i$ denote the sample mean over n samples.

Theorem II.1 (Weak Law of Large Numbers)**.** $X_i \to \mu$ in probability, i.e. for any $\varepsilon > 0$

$$\lim_{n\to\infty} P(|\overline{X}_n - \mu| < \varepsilon) = 1$$

Theorem II.2 (Strong Law of Large Numbers)**.** $\overline{X}_n \to \mu$ almost surely, i.e. up to some outcome set N, where $P(N) = 0$.

Using these theorems we can estimate the probability of an event A occuring by estimating the expected value of the random variable:

$$I_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Then

$$E[I_A] = \int_X dP_{I_A} = \int_A dP = P(A)$$

While at first glance this perspective may appear intuitive, it has a few issues. First lets speak of its strengths. The frequentist model provides us with an objective way to estimate probabilities in the real world. To witness its drawbacks, suppose you are placing bets on the outcome of a professional sports game. As such there are only two possible outcomes, a win or a loss. However, due to the nature of sports games it is not possible to play the same game multiple times with all variables remaining constant, and so we only have one possible sample which we must use to estimate the probability.

Now suppose you wanted to estimate the probability of victory by looking at previous game outcomes. This can be done, however the assumption of independent and identically distributed events is very strong. Previous game performance, such as previous losses, may encourage teams to play with more intensity, increasing their chances of winning. Additionally, it is not uncommon for sports teams to acquire new players or to lose current players due to injury, affecting the range of effective strategies they can employ, violating the assumption of identically distributed samples.

Furthermore, when determining what outcomes to measure we have to ensure that our outcomes are not contained in a set of probability zero. This requires us to already have information on the probabilities before estimating, leading to a circular justification.

II.2.2 Bayesian Approach

On the other hand, the Bayesian approach replaces the notion of objective probabilities with “degrees of belief“ or credences.

Definition II.1 (Credences)**.** Let $\{A_j\}_{j\in J}$ represent a set of outcomes representing our current knowledge of a closed system. We define $P(A_j)$ to be our **degree of belief** in witnessing outcome A_j such that $\sum_{j\in J} P(A_j) = 1$ and $P(A_j) \in [0, 1]$ for all j, making $P(A_j)$ a set of probabilities.

Furthermore, we have an inductive scheme for updating our degrees of belief rationally.

Theorem II.3 (Bayes' Theorem)**.** Suppose we have been a witness to an event B. For each A_i, our new degree of belief in the occurrence of A_i having witnessed B is $P(A_i|B)$. This can be determined through the formula:

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{\sum_{j\in J} P(B|A_j)P(A_j)}$$

The probabilities $P(A_j)$ are called the **prior** probabilities and the probability $P(A_i|B)$ is called the **posterior**.

Clearly replacing the probabilities from the frequentist perspective with credences invokes subjectivity into our models of reality. Suppose we have two rational agents with a shared knowledge set $\{A_j\}_{j\in J}$. Let P_1 and P_2 denote the credences of agent 1 and agent 2 respectively. If P_1 and P_2 do not agree for all A_j, that is, agent 1 and agent 2 have different prior beliefs, then upon witnessing B and applying Bayes' Theorem, both agents will arrive at different posteriors. This can be a useful as it forces researchers to be upfront with their prior beliefs, however it makes it difficult to make objective statements about reality, a quality that is valued in the natural sciences.

We can also wonder if we can define a notion of degrees of belief that do not follow Bayes' Theorem. This is not possible due to the existence of Dutch Books. In the world of gambling, the house has a Dutch Book if it is guaranteed to win despite how much players bet.

Suppose that we had a knowledge set containing the outcomes A, B with degrees of belief $P(A) = P(B) = 0.2$ and $P(A \text{ and } B) = 0.1$. By Bayes' Theorem, $P(A|B) = 0.5$. Suppose instead we assigned the posterior a different credence $P(A|B) = 0.6$. Then there exists the following Dutch Book which can take advantage of our credences. Suppose we bet our credence in terms of dollars so we bet 2 cents on the occurence of outcome B.

Outcome	Wager	Winning	Net Outcome
B	\$0.02	\$0.1	\$0.08
not A given B	\$0.50	\$1.00	\$0.50
A given B	\$0.60	\$1.00	\$0.40

If B does not occur, we lose 2 cents. If B occurs and then A occurs we earn 8 cents on the bet, however if A then occurs we lose 50 cents and gain 40 cents from the event A given B, totalling in a loss: $\$0.08 - \$0.50 + \$0.40 = -\$0.02$. Likewise, if B occurs and A does not we gain $\$0.08 + \$0.50 - \$0.60 = -\$0.02$. In all cases we lose 2 cents from all events. However, had we believed $P(A|B) = 0.5$ according to Bayes' Theorem and bet accordingly we would have broken even on the bet.

So as you can see, by allowing probabilities to be represented as degrees of belief we must accept subjectivity in our measurements and also must take care to ensure that our beliefs are properly updated to avoid the risk of having our degrees of belief taken advantage of.

II.3 Possibilities

This motivates the the search for alternative theories for uncertainty. One such approach is the approach of possibilities. In this approach we assign two quantities: "possible' and "impossible", to outcomes. The distinction between "possible" and "impossible" is equivalent to the distinction between truth and falsehood. This makes the Boolean lattice $\mathbb{B} = (\{\perp, \top\}, \wedge, \vee)$ of interest for possibilistic reasoning (II.18).

In the meantime we will further refine the definitions of probabilistic theories so that we can speak about both probabilistic and possibilistic theories within the same framework.

In this section, we cover the necessary materials and prerequisites of the project.

II.4 Categories and Functors

For Category theory the textbooks [Lei16; Rie16] are recommended.

II.4.1 Categories

Definition II.2 (Category)**.** A **category** $\underline{\mathbb{C}}$ is a mathematical object consisting of a collection of objects, $\underline{\mathbb{C}}_0$, and a collection of arrows (also called morphisms) between them, $\underline{\mathbb{C}}_1$ such that:

C1: Every arrow f has two associated objects in $\underline{\mathbb{C}}_0$: its domain, $D_0(f)$, and codomain, $D_1(f)$. Each arrow is notated as $f : D_0(f) \to D_1(f)$.

C2: For every object A in $\underline{\mathbb{C}}_0$ there exists an identity arrow $1_A : A \to A$.

C3: The collection of arrows $\underline{\mathbb{C}}_1$ is closed under a composition operation $\circ$ such that: For all objects A, B, C and arrows $f : A \to B$ and $g : B \to C$, we can form the composite arrow $g \circ f : A \to C$ such that:

Comp1: (Identity) For all objects A, B, any arrow $f : A \to B$ satisfies:

$$1_B \circ f = f = f \circ 1_A$$

Comp2: (Associativity) For all objects A, B, C, D with arrows $f : A \to B, g : B \to C, h : C \to D$:

$$h \circ (g \circ f) = (h \circ g) \circ f$$

When speaking about particular morphisms it can be easier to reference a wider collection:

Definition II.3. Let $\underline{\mathbb{X}}$ be a category. For any two objects A, B, $\underline{\mathbb{X}}(A, B)$ is the collection of all arrows from A to B.

Categories can further be described by the properties of this collection (called an enrichment).

Definition II.4 (Locally Small)**.** A category $\underline{\mathbb{X}}$ is called locally small if for every pair of objects A, B, the collection $\underline{\mathbb{X}}(A, B)$ is a set (called the Homset). This is also often notated $\mathrm{Hom}_{\underline{\mathbb{X}}}(A, B)$.

For the entirety of this paper, we will be considering locally small categories.

Definition II.5 (Isomorphisms)**.** A morphism/arrow $f : A \to B$ in a category $\underline{\mathbb{C}}$ is an **isomorphism** if there exists an inverse arrow $g : B \to A$ such that $f \circ g = 1_B$ and $g \circ f = 1_A$. We write $f^{-1} = g$.

Example II.1. The category $\underline{\mathbb{1}}$ is the category with a single object and the identity on that object.

$$\begin{tikzcd} 0 \arrow[loop above, "1_0"] \end{tikzcd}$$

Example II.2. The Category $\underline{\mathbb{2}}$ is the category with 2 objects $0, 1$ and one morphism between them: $0 \to 1$.

$$\begin{tikzcd} 0 \arrow[loop above, "1_0"] \arrow[r, "f"] & 1 \arrow[loop above, "1_1"] \end{tikzcd}$$

Example II.3. Any partially ordered set $(P, \leq)$ forms a category $\underline{\mathbb{P}}$. The objects are the elements of P, and for any objects $A, B \in P$:

$$\underline{\mathbb{P}}(A, B) = \begin{cases} \{*\} & A \leq B \\ \emptyset & \text{otherwise} \end{cases}$$

In other words, a poset is a category where every homset is either the singleton set or empty.

Example II.4 (Set)**.** The category $\underline{\mathbb{Set}}$ is a small category with objects as sets and set functions as arrows.

Example II.5 (Opposite Category)**.** Let $\underline{\mathbb{X}}$ be a category. $\underline{\mathbb{X}}^{\mathrm{op}}$ is the category with the same objects $\underline{\mathbb{X}}_0^{\mathrm{op}} = \underline{\mathbb{X}}_0$, but the direction of the arrows are reversed. So if $f : A \to B$ is a morphism in $\underline{\mathbb{X}}$, $f : B \to A$ is a morphism in $\underline{\mathbb{X}}^{\mathrm{op}}$.

II.4.2 Functors

Definition II.6 (Functor)**.** A **functor** F is a map between categories $F : \underline{\mathbb{A}} \to \underline{\mathbb{B}}$ consisting of a map between objects $F_0 : \underline{\mathbb{A}}_0 \to \underline{\mathbb{B}}_0$ and a map between arrows $F_1 : \underline{\mathbb{A}}_1 \to \underline{\mathbb{B}}_1$ such that for objects A, B, C and arrows $f : A \to B, g : B \to C$ the following hold:

- $D_0(F_1(f)) = F_0(D_0(f))$ and $D_1(F_1(f)) = F_0(D_1(f))$
- $F_1(1_A) = 1_{F_0(A)}$
- $F_1(g \circ f) = F_1(g) \circ F_1(f)$

For notational convenience, if A is an object of the domain category $\underline{\mathbb{A}}$, $F_0(A)$ is denoted $F(A)$ or FA. If $f : A \to B$ is a morphism in the domain category $\underline{\mathbb{A}}$ $F_1(f)$ is denoted $F(f)$ or Ff.

In other words, a functor between two categories $\underline{\mathbb{A}}, \underline{\mathbb{B}}$ is a map $F : \underline{\mathbb{A}} \to \underline{\mathbb{B}}$, such that for any two objects A, B in $\underline{\mathbb{A}}_0$ and a morphism $f : A \to B$ in $\underline{\mathbb{A}}_1$, the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "F_0"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & F(A) \arrow[d, "F(f)"] \\ B \arrow[r, "F_0"] & F(B) \end{tikzcd}$$

Definition II.7 (Contravariance)**.** If $\underline{\mathbb{A}}, \underline{\mathbb{B}}$ are categories and $F : \underline{\mathbb{A}} \to \underline{\mathbb{B}}$ is a functor such that it sends a morphism $f : A \to B \in \underline{\mathbb{A}}_1$ to $F(f) : F(B) \to F(A) \in \underline{\mathbb{B}}_1$, then F is called a **contravariant functor**.

Regular functors as defined in II.6 are called**covariant** . If $F : \underline{\mathbb{A}} \to \underline{\mathbb{B}}$ is a contravariant functor, then it is covariant as a functor $F : \underline{\mathbb{A}}^{\mathrm{op}} \to \underline{\mathbb{B}}$ where $\underline{\mathbb{A}}^{\mathrm{op}}$ is the opposite category of $\underline{\mathbb{A}}$ (Example II.5).

Example II.6. If $\underline{\mathbb{X}}$ is a category, every object A can be understood as the image of a functor $O_A : \underline{\mathbb{1}} \to \underline{\mathbb{X}}$. Here $(O_A)_0 : \underline{\mathbb{1}}_0 \to \underline{\mathbb{X}}_0$ picks out the element and $(O_A)_1 : \underline{\mathbb{1}}_1 \to \underline{\mathbb{X}}_1$ preserves the identity:

$$\begin{tikzcd} 0 \arrow[r, "(O_A)_0"] \arrow[d, equal, "1_0"'] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[d, equal, "(O_A)_1(1_0)=1_A"] \\ 0 \arrow[r, "(O_A)_0"] & A \end{tikzcd}$$

Example II.7. Similarly, every morphism between objects $f : A \to B$ in $\underline{\mathbb{X}}$ can be understood as the image of a functor $M_f : \underline{\mathbb{2}} \to \underline{\mathbb{X}}$ where

$$\begin{aligned} (M_f)_0 : \underline{\mathbb{2}}_0 &\to \underline{\mathbb{X}}_0 \\ 0 &\mapsto A \\ 1 &\mapsto B \end{aligned}$$

picks out the domain and codomain, and $(M_f)_1 : \underline{\mathbb{2}}_1 \to \underline{\mathbb{X}}_1$ picks out the element f inside $\underline{\mathbb{X}}(A, B)$.

$$\begin{tikzcd}
0 \arrow[r, "M_f"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[d, "f"] \\
1 \arrow[r, "M_f"] & B
\end{tikzcd}$$

Example II.8 (The Category Cat)**.** The category $\underline{\mathbb{Cat}}$ is a category where the objects $\underline{\mathbb{A}}, \underline{\mathbb{B}}$ are categories, and the morphisms are functors $F : \underline{\mathbb{A}} \to \underline{\mathbb{B}}$.

Example II.9 (Diagrams)**.** Any diagram in a category can be described as a functor that "picks" out the objects. Let $\mathbf{J}$ be a small category for example:

$$\begin{tikzcd}
& 0 & \\
-1 \arrow[ur] & & 1 \arrow[ul]
\end{tikzcd}$$

Then a diagram $D : \mathbf{J} \to \underline{\mathbb{C}}$ of shape $\mathbf{J}$ would look something like:

$$\begin{tikzcd}
& Z & \\
X \arrow[ur, "f"'] & & Y \arrow[ul, "g"]
\end{tikzcd}$$

where $D(-1) = X, D(1) = Y, D(0) = Z$.

II.4.3 Natural Transformations

We are not simply restricted to talking about morphisms between categories.

Definition II.8 (Natural Transformations)**.** let $\underline{\mathbb{X}}, \underline{\mathbb{Y}}$ be categories and $F, G : \underline{\mathbb{X}} \to \underline{\mathbb{Y}}$ be functors. A **natural transformation** $\eta : F \implies G$ is a map between categories such that for every object $A \in \underline{\mathbb{X}}_0$ we have a morphism $\eta_A : F(A) \to G(A)$, called the component of η at A, and for every morphism $f : A \to B \in \underline{\mathbb{X}}_1$ we ask that $\eta_B \circ F(f) = G(f) \circ \eta_A$.

In other words, the diagram:

$$\begin{tikzcd}
F(A) \arrow[r, "\eta_A"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\circlearrowleft"] & G(A) \arrow[d, "G(f)"] \\
F(B) \arrow[r, "\eta_B"'] & G(B)
\end{tikzcd}$$

commutes.

If a natural transformation is invertible we call it a **natural isomorphism** .

Example II.10 (Functor Categories)**.** Let $\underline{\mathbb{A}}, \underline{\mathbb{B}}$ be categories. The object $\underline{\mathbb{Cat}}(\underline{\mathbb{A}}, \underline{\mathbb{B}})$ is a category where the objects are functors $F : \underline{\mathbb{A}} \to \underline{\mathbb{B}}$, and the morphisms are natural transformations.

II.4.4 Limits and Co-Limits

Definition II.9. Let $D : \mathbf{J} \to \underline{\mathbb{C}}$ be a diagram as in example II.9. A **cone** to D is an object N of $\underline{\mathbb{C}}$ with a family of maps: $\{\phi_X : N \to D(X)\}_{X \in \mathbf{J}_0}$ such that for any morphism $f : X \to Y$ we have $D(f) \circ \phi_X = \phi_Y$.

A **limit** of the diagram $F : \mathbf{J} \to \underline{\mathbb{C}}$ is a cone (L, φ) to D such that for any every cone (N, ϕ) to D there exists a unique morphism $\eta_N : N \to L$ such that $\varphi_X \circ \eta_N = \phi_X$ for all $X \in \mathbf{J}_0$. In other words, the diagram:

$$\begin{tikzcd}
& N \arrow[d, "\eta_N"] \arrow[ddl, "\phi_X"'] \arrow[ddr, "\phi_Y"] & \\
& L \arrow[dl, "\varphi_X"] \arrow[dr, "\varphi_Y"'] & \\
D(X) \arrow[rr, "D(f)"'] & & D(Y)
\end{tikzcd}$$

commutes.

The dual notion to a limit is a **colimit**.

The definition of a limit makes more sense with examples.

Example II.11 (Product)**.** Let X, Y be objects in a category $\underline{\mathbb{C}}$. The **product** of X and Y is an object P with projection maps π_X, π_Y such that for all objects Z with maps $f : Z \to X$ and $g : Z \to Y$.

$$\begin{tikzcd}
& Z \arrow[d, "f \times g"] \arrow[ddl, "f"'] \arrow[ddr, "g"] & \\
& P \arrow[dl, "\pi_X"] \arrow[dr, "\pi_Y"'] & \\
X & & Y
\end{tikzcd}$$

A product in the category of sets is the cartesian product.

Example II.12 (Coproducts)**.** Let X, Y be objects in a category $\underline{\mathbb{C}}$. A coproduct of X and Y is an object C together with two maps $\iota_X : X \to C$ and $\iota_Y : Y \to C$ such that for any object Z with maps $f : X \to Z$, $g : Y \to Z$ there exists a unique map $f + g : C \to Z$ such that:

$$\begin{tikzcd}
& Z & \\
& C \arrow[u, "f+g"] & \\
X \arrow[uur, "f"] \arrow[ur, "\iota_X"] & & Y \arrow[uul, "g"'] \arrow[ul, "\iota_Y"']
\end{tikzcd}$$

commutes.

A coproduct in the category of sets is the disjoint union. Note that product and coproducts are dual to each other. Products are limits while coproducts are colimits.

Example II.13 (Coequalizers)**.** Let X, Y be objects in a category $\underline{\mathbb{C}}$, and $f, g : X \to Y$. The coequalizer is an object $\text{coeq}(f, g) = Q$ with a map $q : Y \to Q$ such that for any map $h : Y \to Z$ where $h \circ f = h \circ g$, there exists a unique map $\tilde{h} : Q \to Z$ making the following diagram:

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[r, "h"] \arrow[d, "q"] & Z \\
& Q \arrow[ur, "\tilde{h}"'] &
\end{tikzcd}$$

commutes.

A coequalizer is a colimit, it will come in handy soon. A good example of a coequalizer is a quotient group.

II.4.5 Monoidal Categories

Definition II.10 (Monoidal Category)**.** A category $\underline{\mathbb{C}}$ is a **monoidal category** if there exists an object I (called the monoidal unit), a functor $\otimes : \underline{\mathbb{C}} \times \underline{\mathbb{C}} \to \underline{\mathbb{C}}$ and natural isomorphisms (II.4.3):

- (associator) $\alpha : - \otimes (- \otimes -) \implies (- \otimes -) \otimes -$
- (left unitor) $\lambda : I \otimes - \implies \mathbb{1}_{\underline{\mathbb{C}}}$
- (right unitor) $\rho : - \otimes I \implies \mathbb{1}_{\underline{\mathbb{C}}}$

such that

T1: For any two objects $A, B \in \mathbb{C}_0$ there exists $A \otimes B \in \underline{\mathbb{C}}_0$.

T2: If $f : A \to B$ and $g : C \to D$ are morphisms, then $f \otimes g : A \otimes C \to B \otimes D \in \underline{\mathbb{C}}_1$.

and for objects $A, B, C, D \in \underline{\mathbb{C}}_0$, the following diagrams commute:

$$\begin{tikzcd}
A \otimes (B \otimes (C \otimes D)) \arrow[rr, "\alpha_{A,B,C\otimes D}"] \arrow[d, "1_A \otimes \alpha_{B,C,D}"'] & & (A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha_{A\otimes B,C,D}"] \\
A \otimes ((B \otimes C) \otimes D) \arrow[dr, "\alpha_{A,B\otimes C,D}"'] \arrow[rr, phantom, "\circlearrowleft"] & & ((A \otimes B) \otimes C) \otimes D \\
& (A \otimes (B \otimes C)) \otimes D \arrow[ur, "\alpha_{A,B,C} \otimes 1_D"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes (I \otimes B) \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "1_A \otimes \lambda_B"'] & \circlearrowleft & (A \otimes I) \otimes B \arrow[dl, "\rho_A \otimes 1_B"] \\
& A \otimes B &
\end{tikzcd}$$

Definition II.11 (Strictness)**.** A monoidal category $(\underline{\mathbb{C}}, I, \otimes, \alpha, \lambda, \rho)$ is **strict** if the natural isomorphisms α, λ, ρ are identities. That is $A \otimes I = A = I \otimes A$ and $A \otimes (B \otimes C) = (A \otimes B) \otimes C$.

Definition II.12 (Symmetric Monoidal Category)**.** Let $(\underline{\mathbb{C}}, I, \otimes, \alpha, \lambda, \rho)$ be a monoidal category. $\underline{\mathbb{C}}$ is said to be **symmetric** if there exists a natural isomorphism $\sigma : -\otimes = \implies - \otimes -$, such that for objects $A, B, C \in \underline{\mathbb{C}}_0$ the following commute:

$$\begin{tikzcd}
A \otimes I \arrow[rr, "\sigma_{A,I}"] \arrow[dr, "\rho_A"'] & \circlearrowleft & I \otimes A \\
& A \arrow[ur, "{}_A"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\sigma_{A,B} \otimes 1_C"] \arrow[d, "\alpha_{A,B,C}"'] & (B \otimes A) \otimes C \arrow[d, "\alpha_{B,A,C}"] \\
A \otimes (B \otimes C) \arrow[d, "\sigma_{A,B\otimes C}"'] \arrow[r, phantom, "\circlearrowleft"] & B \otimes (A \otimes C) \arrow[d, "1_B \otimes \sigma_{A,C}"] \\
(B \otimes C) \otimes A \arrow[r, "\alpha_{B,C,A}"'] & B \otimes (C \otimes A)
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes B \arrow[rr, equal, "1_{A\otimes B}"] \arrow[dr, "\sigma_{A,B}"'] & \circlearrowleft & A \otimes B \\
& B \otimes A \arrow[ur, "\sigma_{B,A}"'] &
\end{tikzcd}$$

The use of a tensor product symbol ($\otimes$) is motivated by the following example:

Example II.14. Let R be a commutative ring. The category RMod is a category whose objects are R-modules, and morphisms are R-module homomorphisms. This is a symmetric monoidal category under the tensor product of R-modules $\otimes_R$ (see II.20) with identity element R viewed as an R-module.

Definition II.13 (Monoidal Functors)**.** A functor $F : \underline{\mathbb{A}} \to \underline{\mathbb{B}}$ between two monoidal categories is **strong monoidal** if and only if $F(A \otimes B) \cong F(A) \otimes F(B)$, it is **strict monoidal** if and only if $F(A \otimes B) = F(A) \otimes F(B)$.

In future, if we say that a functor is monoidal without clarifying whether it is strict or strong monoidal we allow it to be both.

Monoidal categories are useful because they allow us to use a graphical language of string diagrams (which we will call circuit diagrams) [Sel11]. Examples of these diagrams can be found in Section II.7.

II.5 Congruences on Categories

Definition II.14 (Categorical Congruences)**.** A **congruence** R on a category $\underline{\mathbb{C}}$ is a family of equivalence relations: For objects $A, B, C \in \underline{\mathbb{C}}_0$, $R_{A,B}$ is an equivalence relation on $\underline{\mathbb{C}}(A,B)$, such that if $f \sim_{R_{A,B}} g$ in $\underline{\mathbb{C}}(A,B)$ and $h \sim_{R_{B,C}} k$ in $\underline{\mathbb{C}}(B,C)$, then $h \circ f \sim_{R_{A,C}} k \circ g$ in $\underline{\mathbb{C}}(A,C)$.

Example II.15.

II.5.1 Quotient Categories

Proposition II.1 (Quotient Category)**.** Let R be a congruence on a category $\underline{\mathbb{C}}$. There exists a category $\underline{\mathbb{C}}/R$, unique up to unique isomorphism, called the **quotient category**, and functor $[\cdot]_R$, called the **quotient functor**. Such that for any functor $F : \underline{\mathbb{C}} \to \underline{\mathbb{D}}$ such that if $f \sim g$ in $\underline{\mathbb{C}}$ then $F(f) = F(g)$ in $\underline{\mathbb{D}}$ there is a unique factorization of F through the quotient.

In other words, there exists a functor $\tilde{F} : {}^{\underline{\mathbb{C}}}\!/_{R} \to \underline{\mathbb{D}}$ such that the diagram

$$\begin{tikzcd}
\underline{\mathbb{C}} \arrow[rr, "F"] \arrow[dr, "{[\cdot]_R}"'] & \circlearrowleft & \underline{\mathbb{D}} \\
& {}^{\underline{\mathbb{C}}}\!/_{R} \arrow[ur, "\tilde{F}"'] &
\end{tikzcd}$$

commutes.

Proof. We can define the quotient as a coequalizer for the functors: $[f]_{f,g\in R}, [g]_{f,g\in R} : \coprod_{(f,g)\in R} \underline{2} \to \underline{\mathbb{C}}$, where $[f]_{f,g\in R}$ picks out all the arrows which appear on the left of a relation and $[g]_{f,g\in R}$ picks out all the arrows which appear on the right. This is a generalization of the functor in Example II.7.

Hence, there is a functor $[\cdot]_R : \underline{\mathbb{C}} \to \mathrm{coeq}([f]_{f,g\in R}, [g]_{f,g\in R})$ such that $[\cdot]_R \circ [f]_{f,g\in R} = [\cdot]_R \circ [g]_{f,g\in R}$.

$$\begin{tikzcd}
\coprod_{(f,g)\in R} \underline{2} \arrow[rr, shift left, "{[f]_{f,g\in R}}"] \arrow[rr, shift right, "{[g]_{f,g\in R}}"'] \arrow[dr] & \circlearrowleft & \underline{\mathbb{C}} \arrow[dl, "{[\cdot]_R}"] \\
& {}^{\underline{\mathbb{C}}}\!/_{R} = \mathrm{coeq}([f]_{f,g\in R}, [g]_{f,g\in R}) &
\end{tikzcd}$$

If $h, k : A \to B$ are arrows in $\underline{\mathbb{C}}$ such that $h \sim k$, then $[f]_{f,g\in R}$ picks out the left arrow h and $[g]_{f,g\in R}$ picks out the right arrow k. Hence, under $[\cdot]_R$ both arrows are equalized, $[f] = [g] : A \to B$. So the objects in $\underline{\mathbb{C}}/R$ are the objects in $\underline{\mathbb{C}}$ and the morphisms are equivalence classes, that is ${}^{\underline{\mathbb{C}}(A,B)}\!/_{R} = {}^{\underline{\mathbb{C}}}\!/_{R}(A,B)$.

By the universal property of the coequalizer, any functor $F : \underline{\mathbb{C}} \to \underline{\mathbb{D}}$ that sends the pair $f \sim g$ to $F(f) = F(g)$ equalizes the functor pair and so must factor through ${}^{\underline{\mathbb{C}}}\!/_{R}$ as desired. ♡

II.5.2 Congruences on Monoidal Categories

Definition II.15 (Monoidal Congruence)**.** A monoidal congruence R on a monoidal category $\underline{\mathbb{C}}$ is a categorical congruence on $\underline{\mathbb{C}}$ such that if $f \sim g$ and $h \sim k$ under R, then $f \otimes h \sim g \otimes k$ in R.

II.5.3 Monoidal Quotient Categories

Theorem II.4 (Monoidal Quotient Category)**.** Let R be a congruence on a category $\underline{\mathbb{C}}$. There exists a monoidal category $\underline{\mathbb{C}}/R$, unique up to unique isomorphism, called the **quotient category**, and functor $[\cdot]_R$, called the **quotient functor**. Such that for any functor $F : \underline{\mathbb{C}} \to \underline{\mathbb{D}}$ such that if $f \sim g$ in $\underline{\mathbb{C}}$ then $F(f) = F(g)$ in $\underline{\mathbb{D}}$ there is a unique factorization of F through the quotient.

In other words, there exists a monoidal functor $\tilde{F} : {}^{\underline{\mathbb{C}}}\!/_{R} \to \underline{\mathbb{D}}$ such that the diagram

$$\begin{tikzcd}
\underline{\mathbb{C}} \arrow[rr, "F"] \arrow[dr, "{[\cdot]_R}"'] & \circlearrowleft & \underline{\mathbb{D}} \\
& {}^{\underline{\mathbb{C}}}\!/_{R} \arrow[ur, "\tilde{F}"'] &
\end{tikzcd}$$

commutes

Proof. Since R is a also congruence on a category we can take $\underline{\mathbb{C}}/R$. It suffices to show that $\underline{\mathbb{C}}/R$ is a quotient category with a monoidal product on equivalence classes. That is, we need to construct a functor $\boxtimes : {}^{\underline{\mathbb{C}}}\!/_{R} \times {}^{\underline{\mathbb{C}}}\!/_{R} \to {}^{\underline{\mathbb{C}}}\!/_{R}$ to serve as our monoidal product.

We invoke a few theorems from [Lei16]. Since $\underline{\mathbb{Cat}}$ is what's known as cartesian closed, the taking products of colimits is also a colimit ${}^{\underline{\mathbb{C}}}\!/_{R} \times {}^{\underline{\mathbb{C}}}\!/_{R}$ [Lei16, p. 163] [Ste]. The details of this proof would require introducing techniques and details irrelevant to the rest of this thesis, as a result we omit them. However it follows that ${}^{\underline{\mathbb{C}}}\!/_{R} \times {}^{\underline{\mathbb{C}}}\!/_{R}$ is a coequalizer.

Take the tensor functor $\otimes : \underline{\mathbb{C}} \times \underline{\mathbb{C}} \to \underline{\mathbb{C}}$ and compose it with the functor $[-]_R : \underline{\mathbb{C}} \to {}^{\underline{\mathbb{C}}}\!/_{R}$. Since R is monoidal congruence, for $f \sim g$ and $h \sim k$, $[f \otimes h] = [g \otimes k]$. That is, $f \otimes h \sim g \otimes k$ and $[-]_R \circ \otimes(f,h) = [-]_R \circ \otimes(g,k)$. Then by the universal property of the quotient there exists a functor $\widetilde{[-]_R \circ \otimes} : {}^{\underline{\mathbb{C}}}\!/_{R} \times {}^{\underline{\mathbb{C}}}\!/_{R} \to {}^{\underline{\mathbb{C}}}\!/_{R}$ such that:

$$\begin{tikzcd}
\underline{\mathbb{C}} \times \underline{\mathbb{C}} \arrow[r, "{[-]_R \circ \otimes}"] \arrow[d, "{[]_{R\times R}}"'] & {}^{\underline{\mathbb{C}}}\!/_{R} \\
{}^{\underline{\mathbb{C}}}\!/_{R} \times {}^{\underline{\mathbb{C}}}\!/_{R} \arrow[ur, "\widetilde{[-]_R \circ \otimes}"'] &
\end{tikzcd}$$

define $\boxtimes = \widetilde{[-]_R \circ \otimes}$, and it's clear that for two equivalence classes $[f] : A \to B, [g] : C \to D$, $[f] \boxtimes [g] = [f \otimes g] : A \otimes B \to$

$C \otimes D$.

Now we need to check coherence. Since the natural isomorphisms α, λ, ρ have components that belong to their own respective equivalence classes $[\alpha_{A,B,C}], [\lambda_A], [\rho_A]$ for objects A, B, C, we can just assign them to be the new natural isomorphisms with components $\tilde{\alpha}_{A,B,C} = [\alpha_{A,B,C}], \lambda_A = [\lambda_A], \rho_A = [\rho_A]$. Since $[-]_R$ is a functor it preserves all commutative diagrams hence:

$$\begin{tikzcd}
A \otimes (B \otimes (C \otimes D)) \arrow[rr, "{[\alpha_{A,B,C\otimes D}]}"] \arrow[d, "{[1_A \otimes \alpha_{B,C,D}]}"'] & & (A \otimes B) \otimes (C \otimes D) \arrow[d, "{[\alpha_{A\otimes B,C,D}]}"] \\
A \otimes ((B \otimes C) \otimes D) \arrow[dr, "{[\alpha_{A,B\otimes C,D}]}"'] & \circlearrowleft & ((A \otimes B) \otimes C) \otimes D \\
& (A \otimes (B \otimes C)) \otimes D \arrow[ur, "{[\alpha_{A,B,C} \otimes 1_D]}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes (I \otimes B) \arrow[rr, "{[\alpha_{A,I,B}]}"] \arrow[dr, "{[1_A \otimes \lambda_B]}"'] & \circlearrowleft & (A \otimes I) \otimes B \arrow[dl, "{[\rho_A \otimes 1_B]}"] \\
& A \otimes B &
\end{tikzcd}$$

commute in $\underline{\mathbb{C}}/R$.

$\heartsuit$

Corollary II.4.1. Let R be a monoidal congruence on a monoidal category $\underline{\mathbb{C}}$.

- If $\underline{\mathbb{C}}$ is strict, so is ${}^{\underline{\mathbb{C}}}\!/_{R}$
- If $\underline{\mathbb{C}}$ is symmetric, so is ${}^{\underline{\mathbb{C}}}\!/_{R}$

Proof. Repeat the construction as in Theorem II.4.

$\heartsuit$

II.6 Quantum Information Theory

Now we outline the basics of quantum information theory, of interests since it will be brought as examples throughout the thesis.

II.6.1 Hilbert Spaces

Definition II.16 (Inner Product Spaces)**.** Let V be a vector space over a field $K = \mathbb{R}, \mathbb{C}$. An **inner product** on V is a sesquilinear form: $\langle \cdot, \cdot \rangle : V \times V \mapsto K$. Such that for all $u, v, w \in V$ and $\alpha, \beta \in K$:

1. $\langle v, u \rangle = \overline{\langle u, v \rangle}$
2. $\langle v, \alpha w + \beta u, w \rangle = \alpha \langle v, w \rangle + \beta \langle u, w \rangle$
3. $\langle v, v \rangle \geq 0$

A vector space V and inner product $\langle \cdot, \cdot \rangle$ form an **inner product space** $(V, \langle \cdot, \cdot \rangle)$.

Definition II.17 (Normed Spaces)**.** Let V be a vector space over a field $K = \mathbb{R}, \mathbb{C}$. A **norm** on V is a function $||\cdot|| : V \to \mathbb{R}$ such that for all $v, u \in V$ and $\alpha \in K$:

1. $||v|| = 0 \iff v = 0$
2. $||v|| \geq 0$
3. $||\alpha v|| = |\alpha| \cdot ||v||$
4. $||v + u|| \leq ||v|| + ||u||$

A vector space V equipped with a norm $||\cdot||$ is a **normed vector space** $(V, ||\cdot||)$.

Definition II.18 (Hilbert Space)**.** A Hilbert space $\mathcal{H}$ is a complete normed inner product space.

Definition II.19 (Dual space)**.** Let V be a vector space over a field K, the dual space V^* is the vector space of linear functionals $f : V \to K$, also called covectors. Every vector $v \in V$ has a dual $v^* \in V^*$, if V is finite then v^* is the conjugate transpose $v^\dagger$. If V is an inner product space, if $v, w \in V$ then $w^*(v) = \langle v, w \rangle$.

Definition II.20 (Tensor Product)**.** Let V, U, W be vector spaces over a field K. The **tensor product** of V and U, $V \otimes U$ is the unique object $V \otimes U$, such that for any bilinear map $T : V \times U \to W$ there is a unique factorization of T as a linear map $\tilde{T} : V \otimes U \to W$.

$$\begin{tikzcd}
V \times U \arrow[rr, "T"] \arrow[dr, "\otimes"'] & \circlearrowleft & W \\
& V \otimes U \arrow[ur, dashed, "\tilde{T}"'] &
\end{tikzcd}$$

In other words the tensor product characterize multilinear transformations in linear terms.

Definition II.21 (Bras and Kets)**.** Let $\mathcal{H}$ be a Hilbert space. A vector $\psi \in \mathcal{H}$ is written with a ket $|\psi\rangle$, and a covector $\xi^\dagger \in \mathcal{H}^*$ is written with a bra $\langle \xi|$. The inner product in $\mathcal{H}$ can be rewritten by placing a bra and ket together and removing one of the pipes in the middle: $\langle \psi, \xi \rangle = \langle \psi | \xi \rangle$ in the quantum information theory convention. The tensor product is denoted by concatenation: $|\psi\xi\rangle = |\psi\rangle \otimes |\xi\rangle$.

The operator $|\psi\rangle \langle \xi|$ corresponds to a linear transformation that takes a vector $|\varphi\rangle$ and returns the vector $\langle \xi|\varphi\rangle \cdot |\psi\rangle$. Linear operators $|\psi\rangle \langle \psi|$ are called projectors and they project on to the linear subspace $\text{span}\{|\psi\rangle\} \leq \mathcal{H}$.

Definition II.22 (Ray)**.** Let $\mathcal{H}$ be a complex Hilbert space. By a **ray** inside of $\mathcal{H}$ we mean an equivalence class of nonzero vectors such that $|\psi\rangle \sim |\xi\rangle \iff |\psi\rangle = \alpha |\xi\rangle$ for some $\alpha \in \mathbb{C} \setminus \{0\}$.

II.6.2 Quantum Postulates and the Born Rule

The mathematical formulation of standard quantum mechanics is built on a few postulates. In the proceeding section, and the rest, we will assume all Hilbert spaces are finite dimensional.

Postulate 1. A physical system labeled by A, is described by a Hilbert space $\mathcal{H}_A$, and a physical state of this system at a fixed time t is a ray $|\psi(t)\rangle_A$ with norm 1. That is a state is an equivalence class of normalized (norm 1) vectors $[|\psi\rangle]$ such that $|\xi\rangle \in [|\psi\rangle] \iff |\psi\rangle = e^{i\phi}|\xi\rangle$ for $\phi \in [0, 2\pi)$.

Given two distinct physical systems $\mathcal{H}_A$ and $\mathcal{H}_B$, we can form the composite system $\mathcal{H}_A \otimes \mathcal{H}_B$ under the tensor product of vector spaces. Physical states are vector $|\psi\rangle_{AB}$.

Postulate 2. Every physical observable $\mathcal{A}$ in a physical system $\mathcal{H}$ corresponds to a bounded Hermitian operator A.

Postulate 3. If A is an observable in a physical system $\mathcal{H}$, the pointwise spectrum of A corresponds to the measurable outcomes of the observable.

Postulate 4 (Born Rule)**.** Let A be an observable in a physical system $\mathcal{H}$. The probability of observing outcome a when measuring in a system in state $|\psi\rangle$ is:

$$P(a) = \langle\psi|\Pi_a|\psi\rangle = \sum_{n=1} |\langle a_n|\psi\rangle\ |^2$$

where Π_a is the projection onto the eigenspace of a with orthonormal basis $\{|a_n\rangle\}$.

In quantum mechanics measuring a physical state changes it. This is expressed in the following postulate.

Postulate 5 (Post Measurement State)**.** Let A be an observable in a physical system $\mathcal{H}$. If $\mathcal{H}$ is prepared in the state $|\psi\rangle$ and outcome $a \in \sigma_p(A)$ is measured, the post-measurement quantum state is determined by:

$$|\psi\rangle \mapsto \frac{\Pi_a|\psi\rangle}{\sqrt{P(a)}}$$

where Π_a is the projector onto the eigenspace of a.

More details on Postulates 4 and 5 will be expanded on in II.6.3.

Postulate 6. The time evolution of a state vector from a fixed time t_0 to a time t is given by a Unitary transformation via the Schrodinger equation:

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = U(t;t_0)|\psi(t_0)\rangle$$

II.6.3 PVMs

As discussed above, physical observables $\mathcal{A}$ correspond to Hermitian operators on a Hilbert space $\mathcal{H}$. It is a standard result in functional analysis that Hermitian operators can be decomposed as the sum of rank 1 projectors:

$$A = \sum_{a\in\mathcal{A}} a\Pi_a$$

This motivates a constructive approach to measuring observable quantities. Let $\{a\}_{a\in\mathcal{A}}$ be a set of outcomes described by some observable quantity. We can measure this observable in our physical system $\mathcal{H}$ so long as we can construct a

Definition II.23 (Projective Valued Measures)**.** Let $\mathcal{A}$ be a finite set of observable outcomes, then $(\mathcal{A}, \mathcal{P}(\mathcal{A}))$ is a measurable space where $\mathcal{P}(\mathcal{A})$ is the power set of $\mathcal{A}$. Let $\mathcal{H}$ be a quantum system. By a **projective valued measure** we define a function from subsets of $\mathcal{A}$ to projection operators on $\mathcal{H}$.

$$\begin{aligned}\pi : \mathcal{P}(\mathcal{A}) &\to \mathrm{Herm}_{\geq 0}(\mathcal{H})\\ E &\mapsto \Pi_E\end{aligned}$$

where Π_E is a projective operator onto a subspace of $\mathcal{H}$ such that for any partition $\{E_i\}_{i=1}^n$ of $\mathcal{A}$, $\sum_i^n \Pi_{E_i} = I_{\mathcal{H}}$. In particular we can decompose the system into orthogonal spaces $\mathcal{H} = \bigoplus_i^n \mathrm{Im}(\Pi_{E_i})$. If $\mathcal{H}$ is prepared in state $|\psi\rangle$, we can define a measure:

$$\begin{aligned}P_\psi : \mathcal{P}(\mathcal{A}) &\to [0,1]\\ E &\mapsto \langle\psi|\Pi_E|\psi\rangle\end{aligned}$$

making $(\mathcal{A}, \mathcal{P}(\mathcal{A}), P_\psi)$ into a probability space. We can then describe the operator associated to $\mathcal{A}$ as:

$$A = \sum_{a\in\mathcal{A}} P_\psi(a)\Pi_{\{a\}}$$

II.6.4 Mixed States

It is often the case in the lab where we may not have complete knowledge of what state the physical system $\mathcal{H}$ is in. In this case we can associate a probability distribution over the possible states. Such a distribution is called a **statistical ensemble**. Thus, a quantum state can no longer be associated with a particular state vector $|\psi\rangle \in \mathcal{H}$ but instead as a distribution of vectors.

Definition II.24 (Density Operator)**.** Let $\{|i\rangle\}_{i\in I}$ be a set of possible states, and $p(i)$ the probability of witnessing state $|i\rangle$. The **density operator** corresponding to this ensemble is:

$$\rho = \sum_{i\in I} p(i)|i\rangle\langle i| \tag{1}$$

The state can now be represented with this density operator, which we call a **mixed state**.

The reason this is called the density operator is because for a measurement selecting for an outcome i the probability is given through the Hilbert-Schmidt inner product with the density operator, so it acts in some ways like a probability density function:

$$p(i) = \operatorname{tr}(E_i \rho) \tag{2}$$

Proposition II.2. The following are true for a density operator ρ on a Hilbert space $\mathcal{H}$:

D1: $\operatorname{tr}\rho = 1$

D2: $\rho^\dagger = \rho$

D3: ρ is positive semi-definite

The density operator has trace 1.

II.6.5 POVMs

In order to make sense of the scenario in eq. 2 we need to construct a notion of measurability that can be applied to a density operator.

Definition II.25 (Positive Operator Valued Measure)**.** Let $M = (\Omega, \Sigma)$ be a measurable space and $\mathcal{H}$ a Hilbert space. A **postive operator value measure** on M is a function into bounded positive semi-definite Hermitian operators on $\mathcal{H}$:

$$\begin{aligned} \pi : \Sigma &\to \mathrm{Herm}_{\geq 0}(\mathcal{H}) \\ E &\mapsto \Delta_E \end{aligned}$$

such that $\Delta_\emptyset = 0$, $\Delta_\Omega = \mathcal{I}_\mathcal{H}$, $\Delta_{\bigcup E_i} = \sum_i E_i$, where $\mathcal{I}_\mathcal{H}$ is the identity operator on the Hilbert space and $\{E_i\}$ is a collection of disjoint subsets of Ω.

For any quantum state ρ we can define a probability measure:

$$\begin{aligned} P_\rho : \Sigma &\to [0,1] \\ E &\mapsto \operatorname{tr}(\Delta_E \rho) \end{aligned}$$

making (Ω, Σ, P_ρ) a probability space.

II.6.6 Channels, Operations, and Instruments

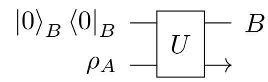
Definition II.26 (Quantum Operators)**.** A **quantum operator**, $\mathcal{E}$, is a transformation that sends quantum states to quantum states in a complete sense [M A10]. This means that when tensored with respect to the identity channel of some reference system R, $\mathcal{E} \otimes \mathcal{I}_R$ sends any quantum state ρ_{AR} to something that can be identified as a quantum state $\rho_{BR} = \mathcal{E} \otimes \mathcal{I}_R(\rho_{AR})$.

In a closed quantum system quantum operators are called unitary channels. They are represented by unitary matrices U and a unitary channel $\mathcal{U}$ sends states from one system $\mathcal{H}_A$ to $\mathcal{H}_B$ via:

$$\rho_B = mathcalU(\rho_A) = U \rho_A U^\dagger$$

where U is a unitary matrix.

Example II.16. For example, suppose we had the following quantum circuit from a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ to $\mathcal{H}_B$:



the corresponding quantum operation looks like:

$$\mathcal{E}(\rho_A) = \operatorname{tr}_A(U(\rho_B \otimes |0\rangle \langle 0|)U^\dagger)$$

With this we can construct a symmetric strict monoidal category **Quant** where the objects are vector spaces of positive semi-definite operators on a Hilbert space $\mathcal{H}_A$, and the maps correspond to quantum operators. This construction will motivate the next section.

II.7 Operational Theories

Operational Theories seek to develop a description of physical theories using similar postulates to the description of states, measurements, and operators as in quantum mechanics. To do this we need to abstract the experimental process.

II.7.1 Systems and Events

An experimenter inside of a physical theory

1. A set of physical systems S
   - Some physical description of any experimental configuration
2. A set of physical events E
   - Things that happen to systems changing their physical state
3. A set of physical tests T
   - Collections of mutually exclusive events that can occur on a system

Before we begin by defining these, let us look at a motivating example:

Example II.17 (Quantum Systems)**.** In the theory of finite dimensional quantum mechanics, every physical system is described by a complex Hilbert space $\mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_C, \ldots$ as in postulate 1. Hence, the set of systems is a monoid generated by the set of complex Hilbert spaces with prime dimension: $S = \langle \mathcal{H} \ : \ \dim \mathcal{H} \text{ is prime} \rangle$.

In this thesis, general systems will be labeled by roman letters $A, B, C, \ldots$, and events with caligraphic letters $\mathcal{E}, \mathcal{F}, \mathcal{T}$.

Definition II.27 (System)**.** A **system** is a physical description of the experimental configuration confined to some amount of degrees of freedom.

It is typical for theories to have their limitations, that is there may be a physical configuration of states that can't be reasonably measured or are simply not of interest to the experimenter. For example, if you are an experimenter working in quantum photonics, physical descriptions of sound waves will not have a proper place or description. We can cover all of these cases in a catch-all called the **trivial system**.

Definition II.28 (Trivial System)**.** The **trivial system**, I, represents any physical description not of interest to the theory itself.

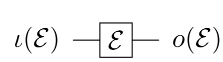
In order to remain coherent with the case of quantum mechanics as in postulate 1, systems should be able to be freely composed together to form joint systems.

Definition II.29 (Combination of Systems)**.** Given two systems A, B, we can form the **combined system** AB, with special care to note that $A(BC) = (AB)C$ for all systems A, B, C. That is, the hierarchy in which we compose systems does not matter. Furthermore, combining a system A with the trivial system doesn't change the system, $IA = AI = A$.

Definition II.30. The set of systems S of an operational theory is a monoid closed under composition of systems.

Definition II.31 (Events)**.** An **event** in an operational theory is a physical process that occurs on a system to create a new system. We can represent an event $\mathcal{E}$ as a black box which take in an input system $\iota(\mathcal{E})$ and performs some physical process to return an output system $o(\mathcal{E})$:

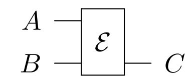
$\mathcal{E} : \iota(\mathcal{E}) \to o(\mathcal{E}) \quad :=$



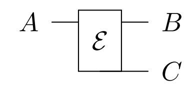
This notation comes from the intuition of treating events as physical devices in our experiment.

Events that take combined systems as input can be thought of as devices with multiple input wires, and events which have combined systems as outputs can be thought of as devices with multiple output wires:

$\mathcal{E} : AB \to C \quad :=$

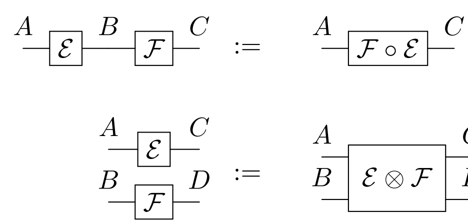


$\mathcal{E} : A \to BC \quad :=$



Definition II.32 (Sequential and Parallel composition of Events)**.** If $\mathcal{E}, \mathcal{F}$ are two events such that the input of $\mathcal{E}$ matches the input of $\mathcal{F}$, $o(\mathcal{E}) = \iota(\mathcal{F})$, then we can compose these two events together by connecting the wires. This is called sequential composition and we denote it $\mathcal{F} \circ \mathcal{E}$ inspired by the notation for composition of functions.

Furthermore, if $\mathcal{E}, \mathcal{F}$ are arbitrary events, we can consider the situation of having these two events occur locally on each of their associated systems as one larger event combining both systems. This kind of operation is always available and we observe these as independent events. We denote the parallel composition via a tensor product: $\mathcal{E} \otimes \mathcal{F} : \iota(\mathcal{E})\iota(\mathcal{F}) \to o(\mathcal{E})o(\mathcal{F})$ inspired by the tensor product in Hilbert spaces.

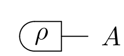


Definition II.33. An event $\rho : I \to A$ from the trivial system into a system A is called a **preparation event**, and an event $e : B \to I$ from a system B to the trivial system is called an **observation event**. Inspired by quantum theory, we denote preparation events as $|\rho)_A$ and observation events with $(E|_B$. In the case when the base system is equal or known we will omit the subscript.

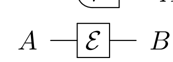
Events from the trivial system to itself $\mu : I \to I$ will be called scalars. They can be represented as closed circuits $(E|\rho)$.

Here we make note of a few notations inspired by quantum theory that we will use throughout the thesis:

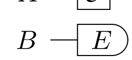
$|\rho)_A : I \to A \quad :=$



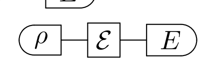
$\mathcal{E} : A \to B \quad :=$



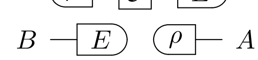
$(e|_B : B \to I \quad :=$



$(e|_B \mathcal{E} |\rho)_A : I \to I \quad :=$



$|\rho)_A (e|_B : B \to A \quad :=$

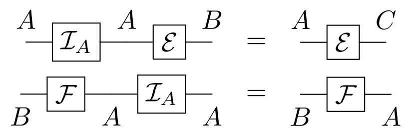


Definition II.34 (Tests)**.** A physical **test** between two systems A, B is simply a collection of events $\{\mathcal{E}_x\}_{x \in X}$ where each $\mathcal{E} : A \to B$. If $|X| = 1$, then we call the test **deterministic**. In the case of a deterministic test we omit the brackets.

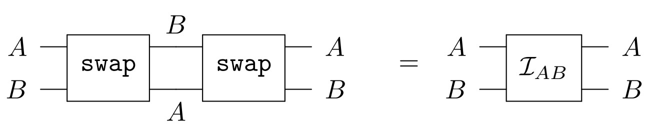
Composition of tests can be done in parallel and sequentially, however we must respect that a test is a collection of events so we must make sure that these compositions do not take us out of either test. In effect, we take sequential and parallel composition componentwise:

$$\{\mathcal{E}_i\}_{i\in X} \circ \{\mathcal{F}_j\}_{j\in Y} = \{\mathcal{E}_i \circ \mathcal{F}_j\}_{(i,j)\in X\times Y}$$
$$\{\mathcal{E}_i\}_{i\in X} \otimes \{\mathcal{F}_j\}_{j\in Y} = \{\mathcal{E}_i \otimes \mathcal{F}_j\}_{(i,j)\in X\times Y}$$

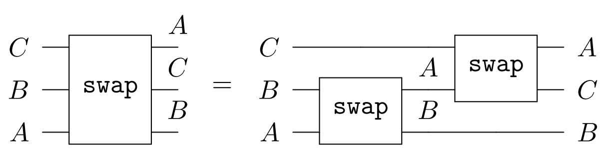
Definition II.35 (Identity Events)**.** Every system A has a deterministic test $\mathcal{I}_A : A \to A$ which corresponds to doing nothing on that event. This means that for every event $\mathcal{E} : A \to B$ and $\mathcal{F} : B \to A$: $\mathcal{E} \circ \mathcal{I}_A = \mathcal{E}$ and $\mathcal{I}_A \circ \mathcal{F} = \mathcal{F}$:



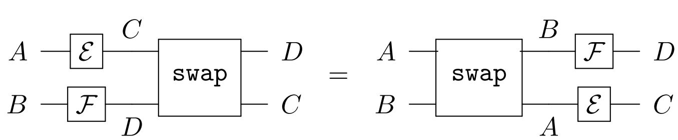
Definition II.36 (Swap)**.** For systems A, B we also specify the existence of a deterministic test $\texttt{swap} : AB \to BA$ that swaps compositions. Clearly swapping two systems should yield the original system back, so we require:



The way we swap combined systems is by swapping each component individually:



Furthermore, this test swaps events that have been composed in parallel:



We now have enough to define an operational theory generally.

Definition II.37. An operational theory, $\mathbf{T}$, is a theory specifying systems, events, and tests, including the swap and identity tests, with combination of systems and sequential and parallel composition of events.

Theorem II.5. An operational theory of processes $\mathbf{T}$ forms a symmetric strict monoidal category $\underline{\mathbb{I}} = \mathbf{Proc}(\mathbf{T})$.

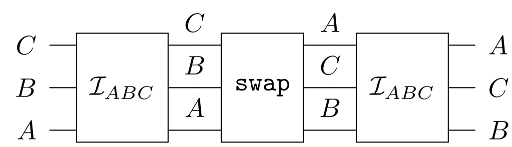
Proof. By construction we simply take the objects to be labels for systems, and morphisms to be events between systems, and the identity events as identities. Composition is sequential composition of events. This forms a category in the sense of definition II.2.

Now we must make this into a monoidal category. The trivial system I acts as our monoidal unit and the monoidal operation $\otimes : \underline{\mathbb{I}} \times \underline{\mathbb{I}} \to \underline{\mathbb{I}}$ that acts accordingly: For objects A, B $A \otimes B = AB$, and for morphisms $\mathcal{E} : A \to B$ and $\mathcal{F} : C \to D$, gives us their parallel composition $\mathcal{E} \otimes \mathcal{F}$. Since composition of systems is associative the natural transformations α, λ, ρ correspond to identities. This generates a strict monoidal category. We can then define a natural isomorphism $\sigma : - \otimes - \implies - \otimes -$ whose components $\sigma_{A,B}$ correspond to the event $\texttt{swap} : AB \to BA$, we need to check if this gives us the appropriate diagrams as in definition II.12:

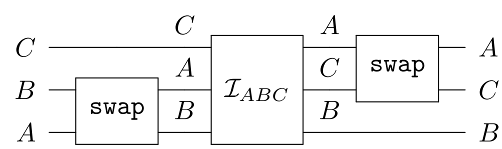
Now by the equality $A \otimes I = A = I \otimes A$ we get our first diagram:

$$\begin{tikzcd}
A \otimes I \arrow[rr, "\texttt{swap}"] \arrow[dr, equal, "\rho_A"'] & \circlearrowleft & I \otimes A \arrow[dl, equal, "A"] \\
& A &
\end{tikzcd}$$

Because the circuit:



is the same as the circuit:



this gives us the commutative diagram:

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[rr, "\sigma_{A,B} \otimes 1_C"] \arrow[d, equal, "\alpha_{A,B,C}"'] & & (B \otimes A) \otimes C \arrow[d, equal, "\alpha_{B,A,C}"] \\
A \otimes (B \otimes C) \arrow[d, "\sigma_{A,B\otimes C}"'] & \circlearrowleft & B \otimes (A \otimes C) \arrow[d, "1_B \otimes \sigma_{A,C}"] \\
(B \otimes C) \otimes A \arrow[rr, equal, "\alpha_{B,C,A}"'] & & B \otimes (C \otimes A)
\end{tikzcd}$$

Finally, by the definition of the swap tests we have:

$$\begin{tikzcd}
A \otimes B \arrow[rr, equal, "1_{A\otimes B}"] \arrow[dr, "\sigma_{A,B}"'] & \circlearrowleft & A \otimes B \\
& B \otimes A \arrow[ur, "\sigma_{B,A}"'] &
\end{tikzcd}$$

♡

II.8 Rigs and Semimodules

When dealing with possibilities we will need to work with a special structure called a semiring and semimodules. The main theory of semimodules is covered in [CL22].

Definition II.38 (Rig)**.** A **rig** (also **semiring**), is a quintuple $(R, +, \cdot, 0, 1)$ where R is a set and both $(R, +, 0)$ and $(R, \cdot, 1)$ are monoids. Furthermore, $+$ and $\cdot$ distribute over each other, furthermore for any $r \in R$, $0 \cdot r = 0$. In other words, a rig is a "ring without negatives".

We woll continue assuming R is a commutative rig. Just as rings have modules, rigs have semimodules:

Definition II.39 (Semimodule)**.** A semimodule over a rig $R = (R, +, \cdot, 0, 1)$ is a commutative monoid $(M, +, \vec{0})$ and an action $\bullet : R \times M \to M$ such that for $r, s \in R$ and $\vec{x}, \vec{y} \in M$:

M1: $r \bullet (\vec{x} +_M \vec{y}) = r \bullet \vec{x} +_M r \bullet \vec{y}$

M2: $(r +_R s) \bullet \vec{x} = r \bullet \vec{x} +_M s \bullet \vec{y}$

M3: $(r \cdot s) \bullet \vec{x} = r \bullet (s \bullet \vec{x})$

M4: $0 \bullet x = \vec{0}$

M5: $1 \bullet \vec{x} = \vec{x}$

M6: $r \bullet \vec{0} = \vec{0}$

Definition II.40 (R-semimodule homomorphism)**.** Let R be a rig and M, N R-semimodules. A map $f : M \to N$ is an **R-semimodule homomorphism** if and only if for any $x, y \in M$:

H1: $f(x +_N y) = f(x) +_M f(y)$

H2: $f(r \bullet x) = r \bullet f(x)$

H3: $f(1_N) = 1_M$

H4: $f(0_N) = 0_M$

For a rig R, R-semimodules and R-semimodule homomorphisms form a category R**SMod**.

Proposition II.3. If R is a rig and N an R-semimodule, and define the set $M = \{f_n : R \to N \mid n \in N\}$ where for each $n \in N$, $f_n \in M$ is an R-module homomorphism that sends $r \in R$ to $f_n(r) = r \bullet n$. Then M is an R-semimodule and $N \cong M$ as R-semimodules.

Proof. We consider what f_n does to the multiplicative identity in R for $n \in N$. By definition $f_n(1) = 1 \bullet n = n$. To get an R-semimodule out of M we need to show it has a commutative monoid structure. Let $n, m \in N$ and let $r \in R$, then the map $f_{n+m}(r) = r \bullet (n + m) = r \bullet n + r \bullet m$. Hence $f_{n+m} = f_n + f_m$ and so M is closed under this addition since N is additively closed. We can define an action of R on M as follows:

$$\begin{aligned} \bullet : R \times M &\to M \\ (r, f_n) &\mapsto r \bullet f_n : R \to N \\ &\qquad\qquad r' \mapsto r' \bullet (r \bullet n) \end{aligned}$$

clearly $r \bullet f_n = f_{r \bullet n}$ as R-semimodule homomorphisms.

$$\begin{aligned} \phi : N &\to M & \psi : M &\to N \\ n &\mapsto f_n & f_n &\mapsto f_n(1) \end{aligned}$$

Clearly for some $n \in N$, then $\psi(\phi(n)) = \psi(f_n) = f_n(1) = n$ and for some $f_n \in M$ $\phi(\psi(f_n)) = \phi(f_n(1)) = \phi(n) = f_n$. Hence $\psi = \phi$ and ϕ gives us a bijection. This bijection can be upgraded to an isomorphism because $\phi(n + m) = f_{n+m} = f_n + f_m = \phi(n) + \phi(m)$, and $\phi(r \bullet n) = f_{r \bullet n} = r \bullet f_n = r \bullet \phi(n)$.

♡

II.9 Lattice Theory

We introduce the theory of lattices [Nat] which will be tangentially useful.

Definition II.41 (Lattice)**.** A **lattice** $\mathcal{L}$ is an algebra $(L, \wedge, \vee)$ such that for $x, y, z \in L$:

L1 $x \wedge x = x$ and $x \vee x = x$

L2 $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$

L3 $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$

L4 $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$

If L is only defined with one such operation $(L, \vee)$ then it is a **semilattice**.

Definition II.42 (Distributive Lattices)**.** A lattice $\mathcal{L} = (L, \wedge, \vee)$ is distributive if for all $x, y, z \in L$: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Definition II.43 (Bounded Lattice)**.** A lattice $\mathcal{L} = (L, \wedge, \vee, 0, 1)$ is bounded if $0, 1 \in L$ and for all $x \in L$:

- $x \wedge 1 = x$ and $x \vee 1 = 1$
- $x \vee 0 = 0$ and $x \vee 0 = x$

Proposition II.4. Every bounded distributive lattice is a rig.

Proof. Let $(L, \wedge, \vee, 0, 1)$ be a bounded distributive lattice. Clearly $(L, \vee, 0)$ and $(L, \wedge, 1)$ define commutative monoids, and both $\vee$ and $\wedge$ distribute over each other. Finally, for any $x \in L$, $0 \vee x = 0 = x \vee 0$.

♡

II.9.1 The Boolean Lattice $\mathbb{B}$

Example II.18 (Boolean Lattice)**.** The boolean lattice $\mathbb{B} = (\{\bot, \top\}, \wedge, \vee, \bot, \top)$, is a lattice with meet and join operations corresponding to logical conjunction and logical disjunction:

- $\top \wedge \top = \top$
- $\top \vee \top = \top$
- $\top \wedge \bot = \bot$
- $\top \vee \bot = \top$
- $\bot \vee \bot = \bot$
- $\bot \wedge \bot = \bot$

Clearly $\mathbb{B}$ is a bounded distributive lattice. Hence by proposition II.4 it's also a rig, so we can consider its semimodules.

III The Results

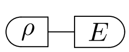
We proceed now to the results inspired by the theory of Operational Probabilistic Theories (OPTs) as covered in [CDP10; Sca19; Tul19]. The historical treatment of operational probabilistic theories typically takes an operational theory $\underline{\mathbb{I}}$ as in §II.7, and assigns probabilities to outcomes like so:

Definition III.1 (Operational Probabilistic Theory)**.** An operational probabilistic theory is an operational theory $\underline{\mathbb{I}}$ such that for any test $\{\mathcal{P}_x\}_{x\in X} : I \to I$ we associate a probability $p_x \in [0,1]$ to each $\mathcal{P}_x$ for $x \in X$, such that $\sum_{x\in X} p_x = 1$. Furthermore, for scalars $\mu, \lambda : I \to I$, $P(\mu \otimes \lambda) = P(\mu \circ \lambda) = P(\mu)P(\lambda)$.

It is then a natural construction to create equivalence classes of preparation and observation events based on these probabilities:

$$|\rho)_A \implies \hat{\rho} : \underline{\mathbb{I}}(A, I) \to [0,1]$$
$$(E|_A \mapsto (E|\rho)$$

where $(E|\rho)$ is the probability associated to the event:

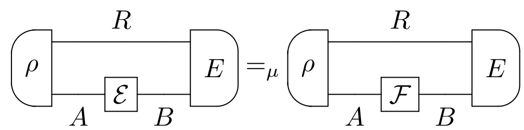


Definition III.2 (Tomography)**.** We say that two preparation events $\rho, \sigma : I \to A$ are tomographically indistinguishable ($\rho \sim \sigma$) if and only if $\hat{\rho}(e) = \hat{\sigma}(e)$ for all observation events $E : A \to I$. We call the equivalence classes $|\rho)$ the states of A and we can take the set of all states of A, $\mathsf{St}(A)$.

This is called tomography. We can then construct vector space $\mathsf{St}_{\mathbb{R}}(A)$ associated with each system A [CDP10].

Tomography looks exactly the same for observation events, that is $E, F : A \to I$ are tomographically indistinguishable if $(E|\rho) = (F|\rho)$ for all $\rho : I \to A$. We call an equivalence class of observation events an effect, and they live in the set $\mathsf{Eff}(A)$ and similarly have an associated vector space $\mathsf{Eff}_{\mathbb{R}}(A)$.

Tomography on general events looks different however. This generalizes the notion of a quantum operator as in II.6.6. We say two events $\mathcal{E}, \mathcal{F} : A \to B$ are tomographically indistinguishable if for any reference system R, and preparation $|\rho)_{AR}$ and obsersvation $(E|_{BR}$:



$\forall R \in \underline{\mathbb{I}}_0$
$\forall \rho \in \underline{\mathbb{I}}(I, A)$
$\forall E \in \underline{\mathbb{I}}(A, I)$

Equivalence classes of events are called transformations and live in the set $\mathsf{Trans}(A, B)$ and correspond to a set of linear transformations $\mathsf{Trans}_{\mathbb{R}}(A, B)$.

However, it is never shown that tomography generates a valid monoidal congruence on a category. As a result, taking equivalence classes is unfounded unless proven. We will create a generalization of operational probabilistic theories over an arbitrary rig P, called P-theories, and show that the definition of tomography as taken in OPTs is in fact a monoidal congruence relation.

We will then prove a theorem that lets us realize general P-theories as a category with a special kind of functor into P-semimodules. This will allow us to classify what we mean by an operational possibilistic theory. Then we will talk more about operational possibilistic theories and give a concrete construction of one in the category of finite sets and relations, **FinRel**.

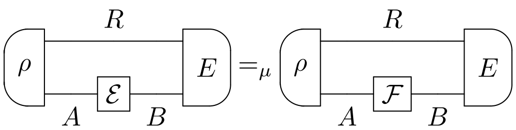
III.1 Generalized P-theories

Definition III.3 (Operataional P-theory)**.** Let P be a commutative rig and $\underline{\mathbb{I}}$ an operational theory in the sense of II.7 (symmetric strict monoidal category). An operational P-theory is a pair $(\underline{\mathbb{I}}, \mu)$, where $\mu : \underline{\mathbb{I}}(I, I) \to P^{\times}$ is a monoid homomorphism into the multiplicative monoid of P, so that $\mu(\lambda \circ \nu) = \mu(\lambda \otimes \nu) = \mu(\lambda) \cdot_P \mu(\nu)$ for scalars λ, ν in $\underline{\mathbb{I}}$. Such that for tests $\{(E_i|_B\}_{i\in I}$, $\{\mathcal{T}_j : A \to B\}_{j\in J}$, and $\{|\rho_k)_A\}_{k\in K}$

$$\sum_{i,j,k} \mu\left((E_i|_B \mathcal{T}_j |\rho_k)_A\right) = 1_P$$

The monoid homomorphism μ gives rise to an internal congruence relation on the events, which we will call μ-Tomography.

Definition III.4 (General μ-Tomography)**.** Let $(\underline{\mathbb{I}}, \mu)$ be an operational P-theory. Two events $\mathcal{E}, \mathcal{F} \in \underline{\mathbb{I}}(A, B)$ are said to be μ-*Tomographically Equivalent* if and only if for any reference system R and preparation and observation events $|\rho)_{AR}, (E|_{BR}$, we have $\mu\left((E|_{BS}(\mathcal{E}\otimes\mathcal{I}_S)|\rho_{AS})\right) = \mu\left((E|_{BS}(\otimes\mathcal{I}_S)|\rho_{AS})\right)$. Diagrammatically we will write:

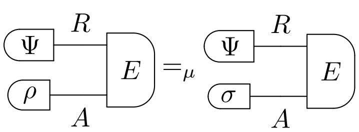


$\forall R \in \underline{\mathbb{I}}_0$
$\forall \rho \in \underline{\mathbb{I}}(I, A)$
$\forall E \in \underline{\mathbb{I}}(A, I)$

to indicate these circuits are mapped to the same scalar under μ, and we write $\mathcal{E} \sim_\mu \mathcal{F}$

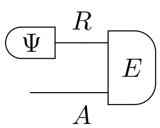
Let us look at what μ-tomographic equivalence looks like on preparation and observation events.

Definition III.5 (Preparation μ-Tomography)**.** Let $|\rho)_A, |\sigma)_A$ be μ-tomographically indistinguishable preparation events. Hence for any reference system R, and preparation event $|\Psi)_{IR} = |\Psi)_R$ and observation event $(E|_{AR}$:

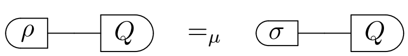


$\forall R \in \underline{\mathbb{I}}_0$
$\forall \Psi \in \underline{\mathbb{I}}(I, R)$
$\forall E \in \underline{\mathbb{I}}(R\otimes A, I)$

Now since $(E|_{AR}|\Psi)_R$ is an observation event on A as witnessed by the circuit:



Let $(Q|_A = (E|_{AR}|\psi)_R$, we have that

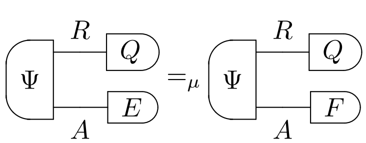


$\forall Q \in \underline{\mathbb{I}}(I, A)$

Likewise, given any observation event $(Q|_A \in \underline{\mathbb{I}}(A, I)$ if $(Q|_A$ is not already able to be written in terms of some $(E|_{AR}$ and $|\Psi)_R$ as above, we can choose $R = I$, and the scalar $|\Psi)_I = \mathcal{I}_I : I \to I$, then $(Q|_A = (Q|_{AI}\mathcal{I}_A \otimes \mathcal{I}_I$, which is in the correct form.

So this definition is equivalent to the definition in definition III.2 of asking that two preparation events are equivalent when they agree on observations.

Definition III.6 (Observation μ-Tomography)**.** We can repeat this to say two observation events $(E|, (F| : A \to I$ are tomographically equivalent when:



$\forall R \in \underline{\mathbb{I}}_0$
$\forall \Psi \in \underline{\mathbb{I}}(I, A\otimes R)$
$\forall Q \in \underline{\mathbb{I}}(R, I)$

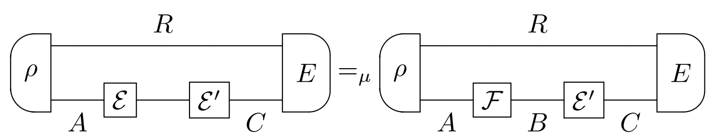
which through a dual argument to the preparation events is equivalent to asking that any two observations agree on preparations.

It should be clear that this relation $\sim_\mu$ induces an equivalence relation on homsets of $\underline{\mathbb{I}}$. This begs the question of whether or not this is a proper congruence relation.

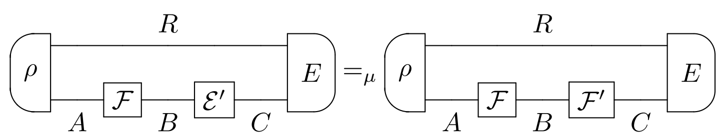
Theorem III.1. *If $(\underline{\mathbb{I}}, \mu)$ is an operationa P-theory, then μ-tomography is a monoidal congruence on the category $\underline{\mathbb{I}}$.*

Proof. Working in an operational P-theory $(\underline{\mathbb{I}}, \mu)$, we will proceed in full generality for arbitrary events, as the cases for preparation and observational tomography arise as a special case.

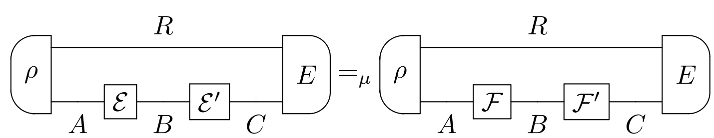
Let $\mathcal{E}, \mathcal{F}$ be μ-tomographically indistinguishable events from A to B, and let $\mathcal{E}', \mathcal{F}'$ be μ-tomographically indistinguishable events from B to C. Let R be any arbitrary reference system, and $|\rho)_{AR}$ any preparation event and $(E|_{BR}$ any observation event. Then since $\mathcal{E} \sim_\mu \mathcal{F}$ and $(E|_{BR}(\mathcal{E}' \otimes \mathcal{I}_R)$ is an observation event on $A \otimes B$, it follows that:



now since $\mathcal{F} \otimes \mathcal{I}_R|\rho)_{AR}$ is a preparation event on $A \otimes R$ and $\mathcal{E}' \sim_\mu \mathcal{F}'$ it follows that:

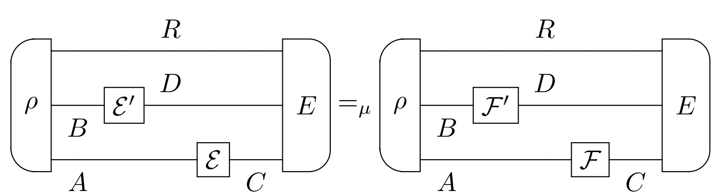


hence



Thus $\mathcal{E}' \circ \mathcal{E} \sim \mu\mathcal{F}' \circ \mathcal{F}$, showing that $\sim_\mu$ is at least a categorical congruence. Now we must check that it plays nice with the tensor product.

Now suppose that we had indistinguishable events $\mathcal{E} \sim_\mu \mathcal{F}$ between systems A am C and $\mathcal{E}' \sim_\mu \mathcal{F}'$ between systems B an D. Since $\mathcal{E} \sim \mu\mathcal{F}$, we have that $\mathcal{E} \otimes \mathcal{I}_B \sim_\mu \mathcal{F} \otimes \mathcal{I}_B$, by definition of tomographic equivalence, likewise $\mathcal{I}_A \otimes \mathcal{E}' \sim \mathcal{I}_A \otimes \mathcal{F}'$. Using the fact that the $\mathcal{E} \otimes \mathcal{E}' = (\mathcal{E} \otimes \mathcal{I}_B) \circ (\mathcal{I}_A \otimes \mathcal{E}')$ and $\mathcal{F} \otimes \mathcal{F}' = (\mathcal{F} \otimes \mathcal{I}_B) \circ (\mathcal{I}_A \otimes \mathcal{F}')$ we now can use the fact that we are composing indistinguishable maps to get that $(\mathcal{E} \otimes \mathcal{I}_B) \circ (\mathcal{I}_A \otimes \mathcal{E}') \sim (\mathcal{F} \otimes \mathcal{I}_B) \circ (\mathcal{I}_A \otimes \mathcal{F}')$, which is exactly the statement $\mathcal{E} \otimes \mathcal{E} \sim \mathcal{F} \otimes \mathcal{F}'$. Diagrammatically:



♡

Since μ-tomography defines a proper monoidal congruence we can justify working exclusively in the monoidal category ${}^{\underline{\mathbb{I}}}\!/\!{\sim_\mu}$. We can now directly generalize our notion of states, effects, and transformations for an operational P-theory as morphisms in the quotient category:

$$\mathsf{St}(A) := {}^{\underline{\mathbb{I}}(I,A)}\!/\!{\sim_\mu} = {}^{\underline{\mathbb{I}}}\!/\!{\sim_\mu}(I,A)$$
$$\mathsf{Eff}(A) := {}^{\underline{\mathbb{I}}(A,I)}\!/\!{\sim_\mu} = {}^{\underline{\mathbb{I}}}\!/\!{\sim_\mu}(A,I)$$
$$\mathsf{Trans}(A,B) := {}^{\underline{\mathbb{I}}(A,B)}\!/\!{\sim_\mu} = {}^{\underline{\mathbb{I}}}\!/\!{\sim_\mu}(A,B)$$

We can now see that probabilistic theories arise as a special case

Example III.1 (Operational Probabilistic Theories)**.** An **operational probabilistic theory** is an operational $\mathbb{R}_+$-theory where $\mu = P$ is called the probability map and assigns positive scalars inside the interval [0,1].

Example III.2 (Operational Probabilistic Theories)**.** An **operational possibilistic theory** is an operational $\mathbb{B}$-theory where μ is called the possibility map and written P_+.

Definition III.7 (Morphisms of P-Theories)**.** Let $(\underline{\mathbb{I}},\mu)$ and $(\underline{\mathbb{E}},\lambda)$ be two operational P-theories. A morphism of P-theories $(F,\phi) : (\underline{\mathbb{I}},\mu) \to (\underline{\mathbb{E}},\lambda)$ is comprised of a monoidal functor $F : {}^{\underline{\mathbb{I}}}\!/\!{\sim_\mu} \to {}^{\underline{\mathbb{E}}}\!/\!{\sim_\lambda}$ and a monoid homomorphism

$$\phi : \underline{\mathbb{I}}(I_{\underline{\mathbb{I}}}, I_{\underline{\mathbb{I}}}) \to \underline{\mathbb{E}}(I_{\underline{\mathbb{E}}}, I_{\underline{\mathbb{E}}})$$

that sends $\nu : I_{\underline{\mathbb{I}}} \to I_{\underline{\mathbb{I}}}$ to $F\nu : I_{\underline{\mathbb{E}}} \to I_{\underline{\mathbb{E}}}$. That is for states $|\rho)_A$, transformations $\mathcal{T} : A \to B$ and effects $(E|_B$ in ${}^{\underline{\mathbb{I}}}\!/\!{\sim_\mu}$: This induces an endomoprhism on P, sending

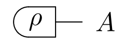
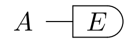
$$\mu(E|_B\mathcal{T}|\rho)_A \mapsto \lambda(F(E)|_{F(B)}F(\mathcal{T})|F(\rho)_{F(A)}$$

Given two morphisms $(F,\phi) : (\underline{\mathbb{I}},\mu) \to (\underline{\mathbb{E}},\lambda), (G,\varphi) : (\underline{\mathbb{E}},\mu) \to (\underline{\mathbb{V}},\nu)$ we can compose them by composing $G\circ F$ as functors and $\varphi\circ\phi$ as monoid homomorphisms. $(G\circ F, \varphi\circ\phi)$ is thus a P-theory homomorphisms such that for states $|\rho)_A$, transformations $\mathcal{T} : A \to B$ and effects $(E|_B$ in ${}^{\underline{\mathbb{I}}}\!/\!{\sim_\mu}$: $\varphi\circ\phi$ sends $\mu(E|_B\mathcal{T}|\rho)_A$ to the scalar $\nu((GF(E)|_{GF(B)}GF(\mathcal{T})|GF(\rho))_{GF(A)})$.

The Operational P-theories of some commutative rig P form a category called P**-Theory**, whose objects are operational P-theories and morphisms are morphisms of P-theories. In particular we have the category **OPT** of operational probabilistic theories and the category $\mathbf{OPT}^+$ of operational possibilistic theories.

III.1.1 Main Theorem

Let P be a commutative rig, and let $(\underline{\mathbb{I}},\mu)$ be a an operational P-theory. We can associate to each preparation $|\rho)_A$ in $\underline{\mathbb{I}}$, a map into P


$$\implies \hat{\rho} : \underline{\mathbb{I}}(A,I) \to P$$

$$\mapsto \mu(e\circ\rho)$$

We can repeat this on events $(E| : A \to I$ to obtain a map $\hat{E} : \underline{\mathbb{I}}(I,A) \to \mathbb{B}$. Note that $\hat{\rho}(E) = \hat{E}(\rho)$ for all preparations $|\rho)_A$ and observations $(E|_A$.

More importantly is the fact that in the quotient category ${}^{\underline{\mathbb{I}}}\!/\!{\sim_\mu}$ each map $\hat{\rho}$ is bijective on the set of states, likewise for effects. To demarcate the set of functions into P from the set of states themselves we will put a hat on top $\hat{\mathsf{St}}(A)$ ($\hat{\mathsf{Eff}}(A)$ for effects). We will proceed thinking in terms of states but the following constructions can be carried out for effects.

Let ρ,σ be states of a system A, since these are P-valued functions we can define an addition operation on them, defining:

$$\hat{\rho}+\hat{\sigma} : \mathsf{Eff}(A) \to P$$
$$E \mapsto \hat{\rho}(E) + \hat{\sigma}(E)$$

where the addition is addition in P. Clearly the set $\hat{\mathsf{St}}(A)$ is not necessarily closed under this addition operation note that $\hat{\mathsf{St}}(A) \subseteq P^{\mathsf{Eff}(A)}$, where $P^{\mathsf{Eff}(A)}$ is the set of functions from events into P. This has the following P-semimodule structure:

$$\alpha : P \times P^{\mathsf{Eff}(A)} \to P^{\mathsf{Eff}(A)}$$
$$(p,f) \mapsto \alpha(p,f) : \mathsf{Eff}(A) \to P$$
$$E \mapsto p\cdot f(E)$$

Hence even though we don't have closure in $\hat{\mathsf{St}}(A)$ we can still justify taking P-linear combinations in $\hat{\mathsf{St}}(A)$. This is clearly a P-semimodule, which we will call $\mathsf{St}_P(A) = \operatorname{span}\hat{\mathsf{St}}(A)$.

Likewise events on A have an associated P-semimodule $\mathsf{Eff}_P(A) = \operatorname{span}\hat{\mathsf{Eff}}(A)$. It should be clear by definition that effects are contained in the dual of the semimodule of states: $\mathsf{Eff}_P(A) \leq \mathsf{St}^*_P(A)$.

Now let $\mathcal{T} : A \to B$ be a transformation in ${}^{\underline{\mathbb{I}}}\!/\!{\sim_\mu}$. Clearly $\mathcal{T}$ can be extended to a set function on $\hat{\mathsf{St}}(A)$ in the following way:

$$\widehat{\mathcal{T}} : \hat{\mathsf{St}}(A) \to \hat{\mathsf{St}}(B)$$
$$\hat{\rho} \mapsto \widehat{\mathcal{T}\rho}$$

This induces a P-semimodule homomorphism $\widehat{\mathcal{T}} : \mathsf{St}_P(A) \to \mathsf{St}_P(B)$. Thus we have an associate P-semimodule $\mathsf{Trans}_P(A,B)$ of transformations.

If we have two transformations $\mathcal{E} : A \to B$ and $\mathcal{F} : B \to C$, we can compose them in the obvious way:

$$\begin{aligned} \widehat{\mathcal{F} \circ \mathcal{E}} : \hat{\mathsf{St}}(A) &\to \hat{\mathsf{St}}(C) \\ \hat{\rho} &\mapsto \widehat{\mathcal{F}}(\widehat{\mathcal{E}\rho}) \end{aligned}$$

We may now wonder if we have preservation of tensor products. Unfortunately in general this is not true. Let A, B be systems, any pair of states $|\rho)_A, |\sigma)_B$ can be combined to form a state $|\rho)_A \otimes |\sigma)_B$ in $A \otimes B$. However, in general, we cannot assume that every state in $A \otimes B$ is generated by tensor products of states in A and B. This means that $\mathsf{St}_P(A) \otimes_P \mathsf{St}_P(A) \leq \mathsf{St}_P(A \otimes B)$ where the $\otimes_P$ is the tensor product of P-semimodules [Kat04, p. 289]. When these two semimodules are isomorphic for all systems A and B we say that our operational P-theory satisfies local μ-tomography.

Definition III.8 (Local μ-Tomography)**.** An operational P-theory $(\underline{\mathbb{I}}, \mu)$ satisfies local μ-tomography if and only if $\mathsf{St}_P(A \otimes B) \cong \mathsf{St}_P(A) \otimes_P \mathsf{St}_P(B)$.

We will return to local tomography later, but first the construction we've just undergone shows us how to construct a P-semimodule for each system and how to construct P-semimodule homomorphisms from events. This correspondence is captured through the following theorem.

Theorem III.2 (Representation Theorem)**.** Let $\underline{\mathbb{I}}$ be a symmetric strict monoidal category and P a commutative rig. The following are equivalent:

- There exists a monoid homomorphism μ as in III.3 that makes $(\underline{\mathbb{I}}, \mu)$ an operational P-theory
- There exists a functor $\underline{\mathbb{I}} \to P\mathbf{Mod}$ that sends I to P

Proof. Let $(\underline{\mathbb{I}}, \mu)$ characterize an operational P-theory. We will construct the functor $\underline{\mathbb{I}} \to P\mathbf{Mod}$. Define $(\hat{-}) : \underline{\mathbb{I}}/\sim_\mu \to P\mathbf{Mod}$ as the functor that sends every system A to $\mathsf{St}_P(A)$ and every event $\mathcal{T} : A \to B$ to the corresponding P-semimodule homomorphism $\widehat{\mathcal{T}} : \mathsf{St}_P(A) \to \mathsf{St}_P(B)$. From the construction we did earlier we know that this is well-defined, meaning it preserves morphisms. Furthermore, since every element in $\mathsf{St}_P(A)$ is in bijective correspondence with a morphism: $P \to \mathsf{St}_P(A)$ (proposition II.3), $(\hat{-})$ sends a state $|\rho)_A$ to a morphism $(\hat{-})\rho : P \to \mathsf{St}_P(A)$ which we identify with the element $\hat{\rho} \in \mathsf{St}_P(A)$. Likewise events are sent into the dual $\mathsf{Eff}_P(A) \leq \mathsf{St}^*_P(A)$. Moreover, $(\hat{-})$ sends I into P. Let $\lambda : I \to I$ be a scalar in our quotient category. Then $\hat{\lambda}$ is just $\mu(\lambda)$. Hence $(\hat{-})$ acts like μ locally on $\underline{\mathbb{I}}/\sim_\mu(I, I)$. For this reason we motivate the notation $\hat{\mu} = (\hat{-})$.

Now we can extend this to a full functor $\underline{\mathbb{I}} \to P\mathbf{SMod}$ by composing with the quotient functor:

$$\begin{tikzcd} \underline{\mathbb{I}} \arrow[rr, "\hat{\mu}\circ{[-]_{\sim_\mu}}"] \arrow[dr, "{[-]_{\sim_\mu}}"'] & & P\mathbf{SMod} \\ & \underline{\mathbb{I}}/\sim_\mu \arrow[ur, "\hat{\mu}"'] \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

Now suppose $\underline{\mathbb{I}}$ is a symmetric strict monoidal category with a functor $F : \underline{\mathbb{I}} \to P\mathbf{SMod}$ such that it sends I to P. Since F maps endomorphisms on I to endomorphisms on P in a way that preserves composition, this induces a monoid homomorphism $\mu_F : \underline{\mathbb{I}}(I, I) \to P$. Now we simply define our tests to be collections $\{(E_i|_B\}_{i \in I}$, $\{\mathcal{T}_j : A \to B\}_{j \in J}$, and $\{|\rho_k)_A\}_{k \in K}$ such that:

$$\sum_{i,j,k} \mu\left((E_i|_B \mathcal{T}_j |\rho_k)_A\right) = 1_P$$

as before. Giving us an operational P-theory.

♡

Indeed, it turns out that our module homomorphism μ can be extended to a complete functor over $\underline{\mathbb{I}}$, and in fact any operational P-theory can be characterized by this functor. Let P be a commutative rig. We can reintroduce our notion of an operational theory under the representation theorem.

An **operational P-theory** is a pair $(\underline{\mathbb{I}}, \mu)$ where $\underline{\mathbb{I}}$ is an operational theory and $\mu : \underline{\mathbb{I}} \to P\mathbf{SMod}$ is a functor that sends $\mu(I) \to P$, and each system to its corresponding P-semimodule of states $\mathsf{St}_P(A)$, and it sends tests $\{(E_i|_B\}_{i \in I}$, $\{\mathcal{T}_j : A \to B\}_{j \in J}$, and $\{|\rho_k)_A\}_{k \in K}$ to the identity when composed:

$$\sum_{i,j,k} \mu\left((E_i|_B \mathcal{T}_j |\rho_k)_A\right) = 1_P$$

We also get the following corollaries:

Corollary III.2.1 (μ-Tomography)**.** Two events $\mathcal{E}, \mathcal{F}$ in an operational P-theory $(\underline{\mathbb{I}}, \mu)$ are μ-tomographically equivalent ($\mathcal{E} \sim_\mu \mathcal{F}$) if and only if $\mu(\mathcal{E}) = \mu(\mathcal{F})$.

Corollary III.2.2 (Local Tomography)**.** An operational P-theory $(\underline{\mathbb{I}}, \mu)$ satisfies local tomography if and only if μ is a monoidal functor.

III.2 Operational Possibilistic Theories

An operational possibilistic theory is an operational theory where we assign one of the quantities "possible" or "impossible" to outcomes. From the representation theorem we can define any operational possibilistic theory as a pair $(\underline{\mathbb{I}}, P_+)$ where $\underline{\mathbb{I}}$ is a symmetric strict monoidal category and $P_+ : \underline{\mathbb{T}} \to \mathbb{B}\mathbf{SMod}$ a functor that sends I to $\mathbb{B}$.

A test in some operational possibilistic theory therefore corresponds to sets $\{(E_i|_B\}_{i\in I}$, $\{\mathcal{T}_j : A \to B\}_{j\in J}$, and $\{|\rho_k)_A\}_{k\in K}$ such that:

$$\bigvee_{i,j,k} P_+\left((E_i|_B \mathcal{T}_j|\rho_k)_A\right) = \top$$

One particular statement of note is that for all systems A, $\hat{\mathsf{St}}(A)$, is a $\mathbb{B}$-semimodule under the action:

$$\begin{aligned}\alpha : \mathbb{B} \times \hat{\mathsf{St}}(A) &\to \hat{\mathsf{St}}(A)\\ (\top, \hat{\rho}) &\mapsto \hat{\rho}\\ (\bot, \hat{\rho}) &\mapsto \bot\end{aligned}$$

Hence we can say $\hat{\mathsf{St}}(A) = \mathsf{St}_{\mathbb{B}}(A)$.

Possibilistic theories are related to probabilistic ones through the following functor:

Proposition III.1 (Possibilistic Collapse Functor)**.** There exists a functor $\mathrm{coll} : \mathbf{OPT} \to \mathbf{OPT}^+$ called the collapse functor, which sends an operational probabilistic theory $(\underline{\mathbb{T}}, P)$ to the possibilistic theory $(\underline{\mathbb{T}}, P_+)$ where

$$\begin{aligned}P_+ : \underline{\mathbb{T}}(I, I) &\to \mathbb{B}\\ \lambda &\mapsto \begin{cases}\top & P(\lambda) > 0\\ \bot & P(\lambda) = 0\end{cases}\end{aligned}$$

Proof. Let $(\underline{\mathbb{T}}, P)$ and $(\underline{\mathbb{E}}, Q)$ be two operational probabilistic theories with a functor $(F, \varphi) : (\underline{\mathbb{T}}, P) \to (\underline{\mathbb{E}}, Q)$.

In the quotient category $\underline{\mathbb{T}}/\sim_P$ define the module homomorphism

$$\begin{aligned}P_+ : \underline{\mathbb{T}}(I_{\underline{\mathbb{T}}}, I_{\underline{\mathbb{T}}}) &\to \mathbb{B}\\ \lambda &\mapsto \begin{cases}\top & P(\lambda) > 0\\ \bot & P(\lambda) = 0\end{cases}\end{aligned}$$

then clearly if $\{\mathcal{P}_x : I \to I\}_{x\in X}$ is a test on the trivial system $\bigvee_{x\in X} P_+(\mathcal{P}_i) = \top$. Then this defines an operational possibilistic theory. Likewise we can do the same construction from $\underline{\mathbb{E}}/\sim_Q$.

Since F is a monoidal functor on the quotient categories and φ is a monoid homomorphism $\varphi : \underline{\mathbb{T}}(I_{\underline{\mathbb{T}}}, I_{\underline{\mathbb{T}}}) \to \underline{\mathbb{E}}(I_{\underline{\mathbb{E}}}, I_{\underline{\mathbb{E}}})$, it follows that the pair (F, φ) defines a valid morphism between the possibilistic theories $(\underline{\mathbb{T}}, P_+)$ and $(\underline{\mathbb{E}}, Q_+)$. Since it acts as the identity on morphisms it is well-defined and preserves composition and identity morphisms, thus making coll a proper functor.

♡

Let us look at a concrete example of an operataional possibilistic theory.

III.3 FinRel

Example III.3 (The Category **FinRel**)**.** The category **FinRel** is the category of relations on finite sets. It is a category whose objects are sets: $A, B, C, \ldots$ and morphisms R are relations $R : A \to B$. Composition of two relations $R : A \to B$ and $S : B \to C$ is given by a new relation:

$$S \circ R = \{(a, c) \mid \exists b \in B \quad aRb \text{ and } bSc\} \subseteq A \times C$$

This is a strict symmetric monoidal category with the product $\otimes = \times$ the cartesian product on sets, and $\{*\}$ acting as the monoidal unit [CF16; Tul19]. Since every non-empty finite set A is simply the disjoint union of a $|A|$ copies of the singleton set $\{*\}$, we have that this monoidal category is generated by a single object $\{*\}$.

Since this is a strictly symmetric monoidal category, this category is also an operational theory. We now describe the functor $P_+ : \mathbf{FinRel} \to \mathbb{B}\mathbf{SMod}$. It turns out that P_+ is an isomorphism [CF16].

III.3.1 States and Effects

The preparation event $|\rho)_A : \{*\} \to A$ is a subset $\rho \subseteq \{*\} \times A = A$, where $*\rho a$ is simply the relation $a \in A$. So each preparation picks out certain elements in A. Likewise for observations: if $(E|_A : A \to \{*\}$, then $E \subseteq A \times \{*\} = A$. Since every relation $R = \{(a, b) \ : \ aRb \quad a \in A \ \ b \in B\}$ has a converse relation $R^* = \{(b, a) \ : \ bRa \quad a \in A \ \ b \in B\}$ then every preparation event $|\rho)_A$ has an associated observation event which we will denote with by dagger$(\rho|_A = |\rho)^\dagger$. The same goes for observations. Let us examine what occurs when we compose a preparation and observation event. Let $|\rho)_A$ and $(E|_A$ be our preparation and observation events. Then these are both subsets of A. By definition their composition is the relation:

$$(E|\rho) = \{(*, *) \ : \ \exists a \in A \quad * \rho a \text{ and } aE*\}$$

This means that $(E|\rho)$ is nonempty (and contains a single element) if there exists at least one element in the intersection of ρ and E (as subsets of A). If they do not share any elements then $(E|\rho) = \emptyset$.

Therefore we define a monoid homomorphism $P_+ : \mathbf{FinRel}(*, *) \to \mathbb{B}$ $P_+(\mathcal{I}_{\{*\}}) = \top$ and $P_+(0_\emptyset) = \bot$, where $0_{\{*\}} = \emptyset$ represents the empty set inclusion relation. From the representation theorem, this will generalize to the functor $P_+ : \mathbf{FinRel} \to \mathbb{B}\mathbf{SMod}$.

We now perform P_+-tomography and go into the category $\mathbf{FinRel}/\sim_{P_+}$.

Let A be a set. Now since every element $a \in A$ is a singleton subset $\{a\} \subset A$, every a has an associated preparation ${}_A|a)$ and observation $(a|_A = {}_A|a)^\dagger$. Let us work with observations. Let $a \in A$, and $|\rho)_A : \{*\} \to A$ be a preparation in A with ρ its associated subset of A. Then $(a|\rho) = \top$ if and only if $a \in \rho$. Since events correspond to subsets, then two preparations $|\rho)_A, |\sigma)_A$ would be P_+-tomographically indistinguishable if $(E|\rho) = (E|\sigma)$ for all observations E. However since each element induces an observation, this means that $\rho \sim_{P_+} \sigma$ if $|\rho)_A$ and $|\sigma)_A$ correspond to the same subsets. This means that the states of A are exactly all the subsets of A. The same reasoning holds for the effects of A. Thus $\mathsf{St}(A) = \mathcal{P}(A) = \mathsf{Eff}(A)$, the power set of A.

Now consider the functions $\hat{\rho}, \hat{\sigma} : \mathsf{Eff}(A) \to \mathbb{B}$ associated to the states, that is the set $\hat{\mathsf{St}}(A)$. They can be added as follows:

$$\begin{aligned} \hat{\rho} \vee \hat{\sigma} : \mathsf{Eff}(A) &\to \mathbb{B} \\ E &\mapsto \hat{\rho}(E) \vee \hat{\sigma}(E) \end{aligned}$$

Let $(E|_A$ be an effect on A, then $\hat{\rho} \vee \hat{\sigma}(E) = \hat{\rho}(E) \vee \hat{\sigma}(E) = \top$ if and only if E shares an element with either ρ or σ. Moreover, if $a \in A$, then $\hat{\rho} \vee \hat{\sigma}(a) = \hat{\rho}(a) \vee \hat{\sigma}(a) = \top$ if and only if a lies in either ρ or σ. Therefore the operation $\vee$ in $\mathbb{B}$ corresponds to set union: $\hat{\rho} \vee \hat{\sigma} = \widehat{\rho \cup \sigma}$.

Furthermore, let's examine if multiplication of states is well defined. Let $|\rho)_A, |\sigma)_A$, and define their multiplication as follows:

$$\begin{aligned} \hat{\rho} \wedge \hat{\sigma} : \mathsf{Eff}(A) &\to \mathbb{B} \\ E &\mapsto \hat{\rho}(E) \wedge \hat{\sigma}(E) \end{aligned}$$

Then using a similar analysis, if $a \in A$, $\hat{\rho} \wedge \hat{\sigma}(a) = \top$ if and only if a lies in both ρ and σ. Hence the operation $\wedge$ in $\mathbb{B}$ corresponds to set intersection: $\hat{\rho} \wedge \hat{\sigma} = \widehat{\rho \cap \sigma}$.

other, we can form a distributive bounded lattice Since $\emptyset$ and A are included in the states of A: let $0 = \hat{\emptyset}$ and $1 = \hat{A}$. This is motivated from the fact that $\hat{\emptyset}(E) = \perp$ and $\hat{A}(E) = \top$ for all observations $(E|_A$. Then $\mathsf{St}(A) = \{\mathcal{P}(A), 0, 1, \vee, \wedge\}$ is also a bounded distributive lattice.

In particular, every subset $S \subseteq A$ is generated by unions of elements in A, $S = \bigcup_{a \in S}\{a\}$. Let $|\rho)_A$ be an arbitrary state of A. Then $\hat{\rho}$ can be written as a $\mathbb{B}$-linear combination $\hat{\rho} = \hat{a} \vee \hat{b} \vee \ldots \vee \hat{z}$ where $a, b, \ldots, z \in A$. Hence the elements of A (or really their singletons) form a basis for $\operatorname{span} \mathsf{St}(A) = \mathsf{St}_{\mathbb{B}}(A)$. Since there are only $|A|$ elements, $\mathsf{St}_{\mathbb{B}}(A) = \mathbb{B}^{|A|}$. Therefore, the states of A can be represented by binary strings of $|A|$-bits, with each bit representing the inclusion of a particular element. The state of $\emptyset$ is the binary string with all bits turned off and the state of A is the binary string with all bits turned on. The same representation can be done to events.

Let $|\rho)_A$ be a state, then this can be represented by some binary number. An effect $(E|_A$ is also a binary number that represents a particular subset we want to observe. By because composing $(E|\rho)$ together is equivalent to choosing elements in ρ from elements in E, the corresponding bit operation corresponds to flipping off bits of $\hat{\rho}$ that aren't bits of $\hat{E}$, and then seeing if at least one bit remains on. Hence $(E|_A$ behaves like what is known in computer science circles as a bit mask [Wik].

III.3.2 Transformations

Transformations map states into states, this means that a transformation $\mathcal{T} : A \to B$ maps subsets of A into subsets of B. This induces a $\mathbb{B}$-semimodule transformation $P_+(\mathcal{T}) = \widehat{\mathcal{T}} : \mathbb{B}^{|A|} \to \mathbb{B}^{|B|}$, which maps bits to bits.

Let $\mathcal{T} : A \to B$ be a transformation (and hence relation) between sets A and B, and $|\rho)_A$ a state in A. Then $|\mathcal{T}\rho)_A$ should be a state with corresponding subset:

$$\mathcal{T}\rho = \{b \in B \ : \ \exists a \in A \ \ a \in \rho \text{ and } a\mathcal{T}b\}$$

This is better observed if we consider what it does to the state $1_A = |A)_A$:

$$\mathcal{T}\rho = \{b \in B \ : \ \exists a \in A \ \ a\mathcal{T}b\}$$

so a transformation associated to $\mathcal{T}$ is going to be a function that selects certain $b \in B$ based on some $a \in A$. This amounts to flipping off bits in $\hat{A}$, ($\hat{\rho}$ for a general state).

Hence $\mathcal{T}$ corresponds to a matrix that sends $\mathsf{St}_{\mathbb{B}}(A)$ to $\mathsf{St}_{\mathbb{B}}(B)$, which we will denote $\hat{\mathcal{T}} : \mathsf{St}_{\mathbb{B}}(A) \to \mathsf{St}_{\mathbb{B}}(B)$.

IV Conclusion

In conclusion, we took the notion of operational probabilistic theories and made them generalize to any arbitrary rig P, which we call a P-theory. We then combined proved that the notion of tomography taken from the literature is indeed a monoidal congruence. We then prove a theorem that lets us associate any P-theory with a pair of a process theory with a particular functor that describes physical systems as P-semimodules. This theorem lets us identify when a functor is a P-theory. We then took a look inside a particular example of a $\mathbb{B}$-theory called an operational possibilistic theory, **FinRel**, and identified it's representations in terms of $\mathbb{B}$-semimodules, showing that the possibilistic theory corresponding to finite relations of sets gives us a category of digital circuits.

IV.1 Next Steps

We highlight some interesting questions and new pathways that the results of this research have opened up:

- In Category Theory in general, more or less 'desirable' properties often arise from a notion called adjunction. This is when two functors map between categories in a nice way. Is there a rigorous notion of adjunction between probabilistic and possibilistic theories?
- Let P be a commutative Rig. Do the properties of this rig have any impact on the the sorts of states and effects a P-theory can express. For example, are there rigs that guarantee every P-theory also satisfies the local tomography principle?
- Can Bayesian inference be described in terms of endomorphisms on an operational probabilistic theory?
  - Since each probabilistic theory can be described in terms of a functor, endomorphisms should correspond to some kind of natural transformation
  - Does this natural transformation satisfy some universal property (i.e. any kind of inference on a probabilistic theory must be in some way Bayesian) or is there a more general kind of inference hidden away as a limit of probabilistic functors in some theory?
  - Can we generalize Bayesian inference to arbitrary P-theories?
  - Is there a natural analog to Bayesian inference in the possibilistic case?
- What are possible operational possibilistic descriptions of quantum theory
- Other kinds of 'relationship categories' (such as the category of linear relations $\mathbf{FinRel}_k$) have been explored for topics such as control theory [BE15]. Can the theory of operational possibilistic theories be applied to understand control systems?
- Is there a relation between operational possibilistic theories and decision problems in computability? Can algorithms be formalized as closed circuits in some kind of operational possibilistic theory?
- In the general literature of operational probabilistic theories there are many kinds of classifications of OPTs based on various properties they can satisfy. These include subjects like causality and purification. What are the generalized P-theory analogs to these and how do results in OPTs generalize to possibilities and general P-theories.

References

[BE15] J. C. Baez and J. Erbele. *Categories in Control*. 2015. arXiv: 1405.6881 [math.CT]. URL: https://arxiv.org/abs/1405.6881.

[CDP10] G. Chiribella, G. M. D'Ariano, and P. Perinotti. "Probabilistic theories with purification". In: *Physical Review A* 81.6 (June 2010). ISSN: 1094-1622. DOI: 10.1103/physreva.81.062348. URL: http://dx.doi.org/10.1103/PhysRevA.81.062348.

[CF16] B. Coya and B. Fong. *Corelations are the prop for extraspecial commutative Frobenius monoids*. 2016. arXiv: 1601.02307 [math.CT]. URL: https://arxiv.org/abs/1601.02307.

[Cho22] M. Choi. *A Quantum Computation Workbook*. Springer Cham, 2022.

[CL22] I. Chajda and H. Länger. "Semimodules over commutative semirings and modules over unitary commutative rings". In: *LMA* 70.7 (2022), pp. 1329–1344. DOI: 10.1080/03081087.2020.1760192. eprint: https://doi.org/10.1080/03081087.2020.1760192. URL: https://doi.org/10.1080/03081087.2020.1760192.

[FJN95] R.S. Freese, J. Ježek, and J.B. Nation. *Free Lattices*. Advances in the Mathematical Sciences. American Mathematical Society, 1995. ISBN: 9780821803899. URL: https://books.google.ca/books?id=YdPxBwAAQBAJ.

[Kat04] Y. Katsov. "On Flat Semimodules over Semirings". In: *algebra universalis* 51.2 (Aug. 2004), pp. 287–299. ISSN: 1420-8911. DOI: 10.1007/s00012-004-1865-1. URL: https://doi.org/10.1007/s00012-004-1865-1.

[Kok15] P. Kok. *Advanced Quantum Mechanics*. 2015. URL: https://www.pieter-kok.staff.shef.ac.uk/docs/PHY472.pdf.

[Lei16] T. Leinster. *Basic Category Theory*. 2016. arXiv: 1612.09375 [math.CT]. URL: https://arxiv.org/abs/1612.09375.

[M A10] I. L. Chuang M. A. Nielsen. *Quantum Computation and Quantum Information*. Cambridge University Press, 2010, 2010.

[Nat] J. B. Nation. *Notes on Lattice Theory*. URL: https://math.hawaii.edu/~jb/lattice2017.pdf.

[Rie16] E. Riehl. *Category Theory in Context*. Courier Dover Publications, 2016.

[Sca19] C. M. Scandolo. *Information-theoretic foundations of thermodynamics in general probabilistic theories*. 2019. arXiv: 1901.08054 [quant-ph]. URL: https://arxiv.org/abs/1901.08054.

[Sel11] P. Selinger. "A Survey of Graphical Languages for Monoidal Categories". In: *New Structures for Physics*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2011, pp. 289–355. ISBN: 978-3-642-12821-9. DOI: 10.1007/978-3-642-12821-9_4. URL: https://doi.org/10.1007/978-3-642-12821-9_4.

[Ste] S. Steakley. "Not yet published". personal correspondence.

[Tul19] S. Tull. *Categorical Operational Physics*. 2019. arXiv: 1902.00343 [quant-ph]. URL: https://arxiv.org/abs/1902.00343.

[Wik] Wikipedia. *Mask (computing)*. URL: https://en.wikipedia.org/wiki/Mask_(computing).

[Wil16] M. Wilde. "From Classical to Quantum Shannon Theory". In: *Quantum Information Theory*. Cambridge University Press, Nov. 2016, pp. xi–xii. DOI: 10.1017/9781316809976.001. URL: http://dx.doi.org/10.1017/9781316809976.001.