Mathematical Symobls

A guide to the most common symbols
used in higher mathematics

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Abstract

The following is a comprehensive list and explanation of every common symbol and use of notation in higher mathematics.

This is meant to be a definitive resource, however there will be mistakes.

Contents

Set Theory

Set Builder Notation

Set builder notation is a common notation used for defining sets. In set builder notation you are defining a set to be a collection of elements where a given property is true:

$$S = \{x \text{ such that } x \text{ is even}\}$$

This is an example set of even numbers.

such that (| or :)

The | and : symbols are used in set builder notation to mean *such that* or *such as*. Returning to our previous example we can rewrite our set S as:

$$S = \{x : x \text{ is even}\} = \{x \mid x \text{ is even}\}$$

element of ($\in$)

This epsilon looking symbol is used to denote the fact that a certain symbol represents an element of some set. It is usually written as element$\in$set. For example,

$$x \in \{0, 1, 2, 3\}$$

this means that x can be either 0,1,2 or 3.

We can exclude a mathematical object by being in a set by doing the not in sign $\notin$.

contains ($\ni$)

This is the reverse of the set element sign and is used to indicate that a particular set has a specific object. For example,

$$\{0, 1, 2, 3\} \ni 2$$

This is read as: the set contains 2.

Like with the set element symbol, we can cross this one out to mean *does not contain* $\not\ni$

$$\{0, 1, 2, 3\} \not\ni 4$$

subset of ($\subset$)

The subset symbol denotes a subset

$$\{0, 1\} \subset \{0, 1, 2, 3\}$$

superset of ($\supset$)

$$\mathbb{N} \supset \{0, 1, 2, 3\}$$

Just like with subsets, we put an underline if the two sets may be equal $\supseteq$

complement ($\complement$)

The complement of a set is every element that is not in that set.

$$A^{\complement} = \{x : x \notin A\}$$

For example:

$$\{0, 1, 2, 3\}^{\complement} = \{4, 5, .., \infty\}$$

Other notations look like

union ($\cup$)

The union of two sets A, B is a new set where any element is in A or in B.

$$A \cup B = \{x : x \in A \vee x \in B\}$$

intersection ($\cap$)

The intersection of two sets A, B is every element in A that is also in B.

$$A \cap B = \{x : x \in A \wedge x \in B\}$$

Algebra

Prime Numbers ($\mathbb{P}$)

The prime numbers are numbers which can not be factored into the product of other numbers. The prime numbers don't include one as one can be factored into itself multiple times, it can also be factored into every other prime.

$$\mathbb{P} = \{2, 3, 5, 7, 11, 13...\infty\}$$

Natural Numbers ($\mathbb{N}$)

The natural numbers is the set of counting numbers. These are the numbers we learn in grade school when we first learn basic arithmetic and counting. 1 apple, 2 apples, 3 apples, etc.

$$\mathbb{N} = \{0, 1, 2.., \infty\}$$

The natural numbers come in two flavours, one with zero $\mathbb{N}_0$ and one without zero $\mathbb{N}_1$. The reason for ommiting zero is a philosophical debate that is beyond the scope of this presentation, so we won't be talking about it.

The natural numbers are closed under addition and multiplication. That is to say that if we have two natural numbers and we add or multiply them together, the result is another natural number:

$$\begin{aligned} a + b \in \mathbb{N}, &\quad \forall a, b \in \mathbb{N} \\ ab \in \mathbb{N}, &\quad \forall a, b \in \mathbb{N} \end{aligned}$$

Integer Numbers ($\mathbb{Z}$)

The integers are natural numbers but we include negative values:

$$\mathbb{Z} = \{-\infty, .., -1, 0, 1, .., \infty\}$$

The Z stands for the german word *Zahlen*, which means an integer number.

One of the key algebraic properties between the natural numbers and the integers is that the integers are closed under addition and it's inverse operation subtraction. If you notice, the sum of two natural numbers is always a natural number, $2 + 2 = 4$, but that isn't the case with subtraction. Take $1 - 2$ for instance. In the integers you can always add or subtract two integers and get an integer back. The integers are also closed under multiplication:

$$\begin{aligned} a \pm b \in \mathbb{Z}, &\quad \forall a, b \in \mathbb{Z} \\ ab \in \mathbb{Z}, &\quad \forall a, b \in \mathbb{Z} \end{aligned}$$

An important variant, the integers modulo n, is written like $\mathbb{Z}_n$ and is the set:

$$\mathbb{Z}_n = \{0, 1, 2, .., n - 1\}$$

This is the set of all integers $\mod n$

Rational Numbers ($\mathbb{Q}$)

The rational numbers is the set of all integer ratios:

$$\mathbb{Q} = \{-\infty, .., -\tfrac{2}{1}, -\tfrac{3}{2}, -\tfrac{1}{1}, 0, \tfrac{1}{1}, \tfrac{3}{2}, \tfrac{2}{1}, .., \infty\}$$

Algebraic Numbers ($\mathbb{A}$)

The algebraic numbers are all the numbers that are solutions to algebraic systems, mainly polynomial equations:

$$\mathbb{A} = \{x \mid \sum_{i=0}^{n} a_i x^i, \quad \forall a \in \mathbb{R}\}$$

Real Numbers ($\mathbb{R}$)

This is the most common set of numbers that you will encounter. Essentially the real numbers, often called the **reals**, is the set of rational and irrational numbers. Any number you can think of is contained inside the real numbers.

$$\mathbb{R} = \{-\infty, .., -e, -\frac{3}{2}, 0, 1, \pi, .., \infty\}$$

A corrollary to the real numbers is the irrational numbers: The irrational numbers is the set difference of the real numbers and the rational numbers $\mathbb{R} - \mathbb{Q}$:

$$\mathbb{R} - \mathbb{Q} = \{-\infty, .., -\pi, -e, -\Phi, -\phi, 0, \phi, \Phi, e, \pi, ..., \infty\}$$

Imaginary Numbers ($\mathbb{I}$)

The imaginary numbers are simply every real number multiplied by the imaginary number $i = \sqrt{-1}$

$$\mathbb{I} = \{-\infty i, .., -2i, -i, 0, i, 2i, .., \infty i\}$$

Complex Numbers ($\mathbb{C}$)

The complex numbers are essentially linear combinations of the real and imaginary numbers:

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

This contains all real numbers and all imaginary numbers and forms the largest number set.

$$\mathbb{C} = \{..., 34(\cos\frac{\pi}{72} + i\sin\frac{\pi}{72}), -(5+6i), -i, -1, 0, 1, i, e^{i\frac{\pi}{3}}, 3+5i, ...\infty\}$$

Complex numbers prove useful when forming an algebraic framework for dealing with 2 dimensional rotations. In the complex plane, multiplication by i translates essentially to a rotation by 180 degrees:

$$i(1 + i) = e^{i\frac{\pi}{2}} e^{i\frac{\pi}{4}} = -1 + i = e^{\frac{-\pi}{4}}$$

Quaternions ($\mathbb{H}$)

Quaternions are similar to the complex numbers except they have a few more imaginary values j and k which are related to 1 and i through the following identities:

$$\begin{aligned} &ij = k && jk = i && ki = j \\ &ji = -k && kj = -i && ik = -j \quad -1 = i^2 = j^2 = k^2 \\ &ij = -ji && kj = -jk && ki = -ik \end{aligned}$$

Quaternions are essentially linear combinations of $1, i, j, k$:

It can also be written as the linear combination of $c + dj$:

$$\mathbb{H} = \{c + dj : c, d \in \mathbb{C}\} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$$

It's important to note that the quaternions **do not** form an associative algebra over the complex numbers, they are a vector space over the real numbers.

Quaternions are useful because they allow us to describe rotations in 3D space. This can easily be seen when we describe the quaternions as being $c + dj$ with a complex c and d. Both c and d describe changes in one angel, and since we have two angles we are now able to describe rotations in 3D space.

Take the quaternion $1 + i + j + k$, this can be factored into $c, d = (1 + i)$, $(1 + i) + (1 + i)j$.

If we multiply the left side by i we will get:

$$i(1 + i + j + k) = i + i^2 + ij + ik = -1 + i - j + k$$

Our result is equivalent to $(-1 + i) + (-1 + i)j$. Which is equivalent to rotating the real components of c and d by $-\pi$.

If we were to multiply by i on the right we will get:

$$(1 + i + j + k)i = i + i^2 + ji + ki = -1 + i + j - k$$

Our result is equivalent to $(-1 + i) + (1 - i)j$ which is equivalent to rotating the real part of c and the imaginary part of d by $-\pi$

Octonions ($\mathbb{O}$)

Octonions are

Boolean Domain ($\mathbb{B}$)

The boolean domain is the the domain of True and False. Written usually as 1s and 0s respectively. It's mainly used in computer science.

$$\mathbb{B} = \{0, 1\}$$

maps to ($\mapsto$)

proportional to ($\propto$)

logic

identity ($=$)

In logic the equals sign means that two statements are the same.

$$P = Q$$

So this is read P **is** Q.

negation ($\neg$)

The negation operator simply reverses a statement's truth value. Pronounced not. For example, if proposition P is **true**, then $\neg P$ is **false**.

possibly ($\Diamond$)

necessarily ($\Box$)

logical equivalence ($\equiv$)

The logical equivalence operator indicates that two statements are the same conceptually.

$$P \equiv Q$$

definition ($:=$)

if then ($\rightarrow$)

This is one of the most basic logical statements, an *if then* statement basically has the form if P is true, then Q must be true also. The syntax is an arrow pointing from P to Q.

$$P \rightarrow Q$$

Keep in mind that this isn't always true in reverse. There's what's called the converse conditional, essentially it's the same except you would say if Q then P, and the syntax is a left arrow.

$$P \leftarrow Q$$

implies ($\implies$)

implied by ($\impliedby$)

if and only if ($\iff$)

An if and only if statement, also called the biconditional, is a relationship between two statements where both the conditional and it's converse are true. In english this is like saying, if P then Q, and if Q then P.

$$P \iff Q \equiv (P \rightarrow Q) \wedge (P \leftarrow Q)$$

The syntax is a two sided arrow pointing to both statements.

logically equivalent to ($:\iff$)

tautology ($\top$)

Also called Verum, a tautology represents something that is always true.

$$\top$$

It is equivalent to the statement if P then P for all P:

$$\top P \equiv \forall P(P \rightarrow P)$$

contradiction ($\perp$)

Also called Falsum, a contradiction is the opposite of a tautology.

logical conjunction ($\wedge$)

The logical conjuction has the same semantic meaning as the english word *and*.

$$P \wedge Q$$

Means P and Q

logical or ($\vee$)

xor ($\oplus$)

The XOR operator can be used as a closed addition operator over the boolean domain:

$$0 \oplus 0 = 0 = 1 \oplus 1$$
$$1 \oplus 0 = 1 = 0 \oplus 1$$

for all ($\forall$)

This quantifier states that what ever statement we make is true for every possible value that some variable can take. The syntax is an upside down capital A.

For example, let's write out the following statement:
For all n where n is an even integer, n divided by 2 is an integer.

$$n/2 \in \mathbb{Z},\ \forall n \in 2\mathbb{Z}$$

existence ($\exists$)

The existence quantifier, denoted by this backwards capital E, declares that the statements that follows is true for at least one value. There is a modified version that has an exclamation mark behind the E.

For example take the statement:
There exists a single real number x such that x + y = y for all real numbers y.

$$\exists! x \in \mathbb{R} : x + y = y,\ \forall y \in \mathbb{R}$$

proves ($\vdash$)

models ($\models$)

Because ($\because$)

Therefore ($\therefore$)

Q.E.D. ($\blacksquare$)

An abbreviation of a lating phrase *Quod Erat Demonstratum*, meaning "which was to be demonstrated". It is a fancy notation to demonstrate the end of a mathematical proof, usually denoted by a black square at the bottom of a proof.# Algebra