CSE 632: Analysis of Algorithms II: Randomized Algorithms **Spring 2024**

Lecture 10: Max-Cut

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Max-cut problem: Given a weighted graph $G = (V, E)$ with edges weights $w(e) > 0$ for each $e \in E$, find a cut $(S, V \setminus S)$ which maximizes

$$w(S, V \setminus S) := \sum_{\substack{e=uv\in E \\ u\in S,\, v\in V\setminus S}} w(e).$$

1 Simple and LP-Based Algorithms

Lemma 1. *A random cut $(S, V \setminus S)$ satisfies*

$$\mathbb{E}[w(S, V \setminus S)] = \frac{1}{2}W \geq \frac{1}{2}\mathsf{OPT},$$

where $W = \sum_{e\in E} w(e)$ is the total weight of all edges, and OPT is the weight of a maximum cut.

Proof. Define $X_v = 1$ if $v \in S$ and $X_v = 0$ otherwise; thus $\{X_v : v \in V\}$ are i.i.d. Bernoulli random variables with mean $1/2$. By the linearity of expectation, we deduce that

$$\mathbb{E}[w(S, V \setminus S)] = \sum_{e=uv\in E} w(e) \Pr(X_u \neq X_v) = \sum_{e=uv\in E} \frac{w(e)}{2} = \frac{W}{2},$$

as claimed. □

Lemma 1 gives a simple randomized approximation algorithm for max-cut with approximation ratio $1/2$ in expectation.

Alternatively, we could try to use an LP-based algorithm. For each vertex $v \in V$ let x_v be the indicator variable for $v \in S$, and for each edge $e = uv \in E$ let y_e be the indicator variable for $x_u \neq x_v$. Then the max-cut problem is equivalent to the following IP.

$$\begin{aligned} \max \quad & \sum_{e\in E} w(e) y_e && \text{(IP for max-cut)} \\ \text{subject to} \quad & y_e \leq x_u + x_v \leq 2 - y_e, \quad \forall e = uv \in E \\ & x_v \in \{0,1\}, \quad \forall v \in V \\ & y_e \in \{0,1\}, \quad \forall e \in E \end{aligned}$$

Using the relaxation-rounding approach, we solve the LP relaxation where we replace $x_v, y_e \in \{0,1\}$ by $x_v, y_e \in [0,1]$, and obtain an optimal LP solution $(\hat{x}^*, \hat{y}^*)$. Then we apply randomized rounding: For each $v \in V$ independently, we set $x_v = 1$ with probability $\hat{x}^*_v$ and $x_v = 0$ otherwise (i.e., x_v is an independent Bernoulli random variable with mean $\hat{x}^*_v$).

However, this LP relaxation is actually trivial and a poor approximation of the original IP. In particular, it has a simple optimal solution $\hat{x}^*_v = \frac{1}{2}$ for all $v \in V$ and $\hat{y}^*_e = 1$ for all $e \in E$. It is straightforward to check that this solution is indeed feasible and achieves the optimal value $\sum_{e\in E} w(e)\hat{y}^*_e = \sum_{e\in E} w(e) = W$. Furthermore, applying randomized rounding to this optimal LP solution recovers the simple algorithm from Lemma 1.

2 Semidefinite Programming Approach

We represent the max-cut problem as an equivalent Integer Quadratic Programming (IQP). Let $x_v = +1$ if $v \in S$ and $x_v = -1$ otherwise (instead of using $\{0,1\}$). Observe that $x_u \neq x_v$ if and only if $x_u x_v = -1$. So, if y_e is the indicator variable for $x_u \neq x_v$ where $e = uv \in E$, then

$$y_e = \frac{1 - x_u x_v}{2}.$$

We then obtain an IQP for max-cut.

$$\begin{aligned} \max \quad & \sum_{e=uv \in E} w(e) \frac{1 - x_u x_v}{2} \\ \text{subject to} \quad & x_v \in \{\pm 1\}, \quad \forall v \in V \end{aligned} \tag{IP for max-cut}$$

IQP is NP-hard. Our plan is still to use the relaxation-rounding approach.

We relax the integral constraint $x_v \in \{\pm 1\}$ to allowing x_v to be any unit vector in $\mathbb{R}^n$. The product $x_u x_v$ in the objective function then becomes the dot product $x_u \cdot x_v$ between two unit vectors. Recall that

$$x_u \cdot x_v = \|x_u\| \|x_v\| \cos\theta_{uv} = \cos\theta_{uv}$$

where θ_{uv} is the angle between x_u and x_v, and $\cos\theta_{uv}$ is called the *cosine similarity* between the two vectors. We thus obtain a SemiDefinite Programming (SDP) relaxation.

$$\begin{aligned} \max \quad & \sum_{e=uv \in E} w(e) \frac{1 - x_u \cdot x_v}{2} \\ \text{subject to} \quad & x_v \in \mathbb{R}^n, \ \|x_v\| = 1, \quad \forall v \in V \end{aligned} \tag{SDP relaxation}$$

SDP can be solved in polynomial time (using ellipsoid method, interior point method, etc.).

Suppose we solve the SDP relaxation and obtain an (approximately) optimal SDP solution $x = (x_v)_{v \in V}$. We need to round x to find a cut $(S, V \setminus S)$. Our plan is to choose a random hyperplane H through the origin, which divides the whole space $\mathbb{R}^n$ into two sides. Each vertex $v \in V$ is classified into S or $V \setminus S$ depending on which side the vector x_v lies in.

More precisely, we choose a random hyperplane H through the origin by selecting a random unit vector $r \in \mathbb{R}^n$ as the normal vector to the hyperplane H. For any unit vector $x \in \mathbb{R}^n$, let θ be the angle between x and r, and we have:

- $r \cdot x = 0 \Leftrightarrow \theta = \frac{\pi}{2} \Leftrightarrow x$ is on H;
- $r \cdot x > 0 \Leftrightarrow \theta \in [0, \frac{\pi}{2}) \Leftrightarrow x$ lies on the same side of H as r;
- $r \cdot x < 0 \Leftrightarrow \theta \in (\frac{\pi}{2}, \pi] \Leftrightarrow x$ lies on the opposite side of H as r.

We then obtain our randomized rounding procedure.

Algorithm 1 Randomized rounding for SDP relaxation

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Choose a random unit vector r ∈ ℝ^n
    return S ← {v ∈ V : r · x_v > 0}
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Lemma 2. *It holds*

$$\mathbb{E}[w(S, V \setminus S)] \geq \alpha \mathsf{OPT}$$

where $\alpha \approx 0.878$ is a constant.

Proof. By the linearity of expectation, we have

$$\begin{aligned}\mathbb{E}[w(S, V \setminus S)] &= \sum_{e \in E} w(e) \Pr\left(e \in E(S, V \setminus S)\right) \\ &= \sum_{e = uv \in E} w(e) \Pr\left(H \text{ splits } x_u \text{ and } x_v\right).\end{aligned}$$

Consider the plane P containing x_u and x_v. Let θ_{uv} be the angle between x_u and x_v; recall $x_u \cdot x_v = \cos\theta_{uv}$. The projection of a random hyperplane H through the origin on P is a random line L through the origin. Therefore, we obtain

$$\begin{aligned}\Pr\left(H \text{ splits } x_u \text{ and } x_v\right) &= \Pr\left(L \text{ splits } x_u \text{ and } x_v \text{ on } P\right) \\ &= \frac{\theta_{uv}}{\pi} \\ &= \frac{\arccos(x_u \cdot x_v)}{\pi}.\end{aligned}$$

Therefore, we deduce that

$$\begin{aligned}\mathbb{E}[w(S, V \setminus S)] &= \sum_{e = uv \in E} w(e) \frac{\arccos(x_u \cdot x_v)}{\pi} \\ &\geq \alpha \sum_{e = uv \in E} w(e) \frac{1 - x_u \cdot x_v}{2} \\ &\geq \alpha \mathsf{OPT},\end{aligned}$$

where α is chosen to be the best (maximum) constant such that

$$\frac{1}{\pi} \arccos t \geq \alpha \frac{1 - t}{2}, \quad \forall t \in [-1, 1].$$

More precisely,

$$\alpha = \min_{t \in [-1,1]} \frac{2}{\pi} \frac{\arccos t}{1 - t} = \frac{2}{\pi} \min_{\theta \in [0, \pi]} \frac{\theta}{1 - \cos\theta} \approx 0.878.$$

This shows the lemma. □

By Lemma 2, we obtain an SDP-based randomized approximation algorithm which finds a cut of weight at least $0.878\mathsf{OPT}$ in expectation.