Lecture 8: Congestion Minimization

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1 Integer Programming and Linear Programming

Integer (linear) programming (IP) In an IP problem, we want to find an *integral* point that maximizes/minimizes a linear objective function under a few linear constraints.

$$
\begin{aligned}
\max_{x \in \mathbb{Z}^n} \quad & c_1 x_1 + \cdots + c_n x_n && \text{(objective function)} \\
\text{subject to} \quad & a_{11} x_1 + \cdots + a_{1n} x_n \le b_1 \\
& \vdots \qquad\qquad \vdots \qquad\qquad \vdots && \text{(constraints)} \\
& a_{m1} x_1 + \cdots + a_{mn} x_n \le b_m \\
& x_1, \dots, x_n \ge 0
\end{aligned}
$$

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. In the matrix form, IP can be written as follows.

$$
\begin{aligned}
\max_{x \in \mathbb{Z}^n} \quad & c^\intercal x \\
\text{subject to} \quad & Ax \le b \\
& x \ge 0
\end{aligned}
$$

If x is further restricted to $\{0,1\}^n$, then it is called a 0/1 IP problem.

Remark 1. 0/1 IP is NP-complete.

Example 2. Let $G = (V, E)$ be an undirected graph. A vertex cover S of G is a subset of vertices such that every edge has at least one endpoint from S. We can find a minimum vertex cover by solving the following 0/1 IP.

$$
\begin{aligned}
\min \quad & \sum_{v \in V} x_v \\
\text{subject to} \quad & x_u + x_v \ge 1, \quad \forall uv \in E \\
& x_v \in \{0,1\}, \quad \forall v \in V
\end{aligned}
$$

Each x_v indicates whether v is inside the vertex cover or not. In particular, under these constraints the set $S = \{v \in V : x_v = 1\}$ is indeed a vertex cover. The objective function $\sum_{v \in V} x_v$ equals the size of S, which we want to minimize. Since finding a minimum vertex cover is NP-complete, this shows that 0/1 IP is NP-complete.

Linear programming (LP) In an LP problem, we want to find a point in $\mathbb{R}^n$ that maximizes/minimizes a linear objective function under a few linear constraints. Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. In the matrix form, LP can be written as follows.

$$
\begin{aligned}
\max_{x \in \mathbb{R}^n} \quad & c^\intercal x \\
\text{subject to} \quad & Ax \le b \\
& x \ge 0
\end{aligned}
$$

Remark 3. LP can be solved in polynomial time. Examples of algorithms for LP include the simplex algorithm and the ellipsoid algorithm.

2 Congestion Minimization Problem

In the congestion minimization problem, we are given a directed graph $G = (V, E)$ and k source-sink pairs $(s_1, t_1), \ldots, (s_k, t_k)$. Our goal is to find k paths $p_1, \ldots, p_k$ where each p_i is from s_i to t_i, such that the congestion of these paths is minimized. Here, the congestion of an edge e is defined by

$$\operatorname{cong}(e) = |\{i \in [k] : e \in p_i\}| = \text{number of paths using } e,$$

and the congestion of paths $p_1, \ldots, p_k$ is defined by

$$\text{congestion} = \max_{e \in E} \operatorname{cong}(e).$$

Let $\mathcal{P}_i$ denote the set of all paths from s_i to t_i for every $i \in [k]$. For each path $p \in \bigcup_{i=1}^{k} \mathcal{P}_i$, let x_p be an indicator variable such that $x_p = 1$ if the path p is used and $x_p = 0$ otherwise. Let $W \in \mathbb{N}^+$ denote the congestion of paths that we want to minimize. The congestion minimization problem can be solved by the following IP.

$$\begin{aligned}
\min \quad & W \qquad\qquad \text{(IP for Congestion Minimization)} \\
\text{subject to} \quad & \sum_{p \in \mathcal{P}_i} x_p = 1, \quad \forall i \in [k] \\
& \sum_{i=1}^{k} \sum_{p \in \mathcal{P}_i : e \in p} x_p \le W, \quad \forall e \in E \\
& x_p \in \{0, 1\}, \quad \forall p \in \bigcup_{i=1}^{k} \mathcal{P}_i \\
& W \in \mathbb{N}^+
\end{aligned}$$

Solving this IP is computationally hard. We adopt the following strategy, which is a common approach in the design of approximation algorithm.

1. *Relaxation*: Solve an LP relaxation of the IP by removing the integral constraint, and obtain an optimal *fractional* solution;

2. *Rounding*: Transform the fractional solution we get into an *integral* solution of the original IP, and hope that it solves the IP approximately.

LP relaxation We relax the integral constraints $x_p \in \{0, 1\}$ to $x_p \in [0, 1]$ and $W \in \mathbb{N}^+$ to $W \ge 1$.

$$\begin{aligned}
\min \quad & W \qquad\qquad \text{(LP Relaxation)} \\
\text{subject to} \quad & \sum_{p \in \mathcal{P}_i} x_p = 1, \quad \forall i \in [k] \\
& \sum_{i=1}^{k} \sum_{p \in \mathcal{P}_i : e \in p} x_p \le W, \quad \forall e \in E \\
& 0 \le x_p \le 1, \quad \forall p \in \bigcup_{i=1}^{k} \mathcal{P}_i \\
& W \ge 1
\end{aligned}$$

Notice that there could be exponentially many variables in this LP relaxation since the number of paths from s_i to t_i can be exponentially large. We shall explain how to solve this LP relaxation efficiently later.

The following observation is important to us.

Fact 4. *Let $(x^{\text{IP}}_{\text{OPT}}, W^{\text{IP}}_{\text{OPT}})$ denote an optimal (integral) solution to the IP for congestion minimization, and let $(x^{\text{LP}}_{\text{OPT}}, W^{\text{LP}}_{\text{OPT}})$ denote an optimal (fractional) solution to the LP relaxation. Then we have*

$$W^{\text{IP}}_{\text{OPT}} \geq W^{\text{LP}}_{\text{OPT}}.$$

Randomized rounding Write $(x, W) = (x^{\text{LP}}_{\text{OPT}}, W^{\text{LP}}_{\text{OPT}})$ for simplicity. Our randomized rounding algorithm is given below.

Algorithm 1 Randomized rounding

Input: $(x, W) = (x^{\text{LP}}_{\text{OPT}}, W^{\text{LP}}_{\text{OPT}})$ an optimal (fractional) solution to the LP relaxation

for each $i \in [k]$ **do**

$(x_p)_{p \in \mathcal{P}_i}$ forms a distribution over $\mathcal{P}_i$ (since $\sum_{p \in \mathcal{P}_i} x_p = 1$), and we choose a path $p_i \in \mathcal{P}_i$ randomly from this distribution; that is

$$\Pr(\text{choose } p_i) = x_{p_i}, \quad \forall p_i \in \mathcal{P}_i$$

end for

return $p_1, \dots, p_k$

Lemma 5. *For every edge $e \in E$, we have*

$$\Pr\left(\text{cong}(e) \geq \frac{C \log n}{\log \log n} W\right) \leq \frac{1}{2n^2}$$

where $C > 0$ is a large absolute constant.

Given the lemma, we are able to bound the congestion of paths we get via randomized rounding. Recall that congestion $= \max_{e \in E} \text{cong}(e)$. We deduce that

$$\begin{aligned}
\Pr\left(\text{congestion} \geq \frac{C \log n}{\log \log n} W^{\text{IP}}_{\text{OPT}}\right) &\leq \Pr\left(\text{congestion} \geq \frac{C \log n}{\log \log n} W^{\text{LP}}_{\text{OPT}}\right) && \text{(Fact 4)} \\
&\leq \sum_{e \in E} \Pr\left(\text{cong}(e) \geq \frac{C \log n}{\log \log n} W^{\text{LP}}_{\text{OPT}}\right) && \text{(Union Bound)} \\
&\leq |E| \cdot \frac{1}{2n^2} && \text{(Lemma 5)} \\
&\leq \frac{1}{2}.
\end{aligned}$$

Therefore, with probability at least $1/2$, the congestion of paths we find is at most $\frac{C \log n}{\log \log n} W^{\text{IP}}_{\text{OPT}}$ where $W^{\text{IP}}_{\text{OPT}}$ is the minimum congestion. Namely, we can achieve an approximation ratio of $C \log n / \log \log n$ with high probability.

It remains to prove Lemma 5. We shall use the following version of Chernoff bounds which is suitable when the deviation is large.

Lemma 6 (Chernoff Bounds)**.** *Let $X_1, \dots, X_n$ be* independent *random variables taking values in $[0, 1]$. Let $X = \sum_{i=1}^n X_i$ be their sum with mean $\mu = \mathbb{E}X = \sum_{i=1}^n \mathbb{E}X_i$. Then we have*

$$\forall \delta \geq 0: \quad \Pr\left(X \geq (1+\delta)\mu\right) \leq \left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^\mu \leq \left(\frac{e}{1+\delta}\right)^{(1+\delta)\mu};$$

$$\textit{Equivalently}, \quad \forall a \geq \mu: \quad \Pr\left(X \geq a\right) \leq \left(\frac{e\mu}{a}\right)^a.$$

Proof of Lemma 5. Let X_i be an indicator random variable such that $X_i = 1$ if $e \in p_i$ and $X_i = 0$ otherwise. Note that $X_1, \dots, X_k$ are jointly independent. Then we have

$$X = \sum_{i=1}^{k} X_i = \mathrm{cong}(e).$$

Notice that

$$\mathbb{E}X_i = \Pr(e \in p_i) = \sum_{p_i \in \mathcal{P}_i : e \in p_i} \Pr(\text{choose } p_i) = \sum_{p_i \in \mathcal{P}_i : e \in p_i} x_{p_i}.$$

Therefore, by the constraint of LP we have

$$\mathbb{E}X = \sum_{i=1}^{k} \mathbb{E}X_i = \sum_{i=1}^{k} \sum_{p_i \in \mathcal{P}_i : e \in p_i} x_{p_i} \le W$$

We deduce from the Chernoff bound (Lemma 6) that

$$\begin{aligned}
\Pr\left(X \ge \frac{C \log n}{\log\log n} W\right) &\le \left(\frac{e\mathbb{E}X}{\frac{C \log n}{\log\log n} W}\right)^{\frac{C \log n}{\log\log n} W} && \text{(Chernoff Bound (Lemma 6))} \\
&\le \left(\frac{\log\log n}{\log n}\right)^{\frac{C \log n}{\log\log n}} && (\mathbb{E}X \le W,\ e \le C,\ W \ge 1) \\
&\le \left(\frac{1}{\sqrt{\log n}}\right)^{\frac{C \log n}{\log\log n}} && (\log\log n \le \sqrt{\log n} \text{ when } n \text{ is sufficiently large}) \\
&= e^{-\frac{1}{2}\log\log n \cdot \frac{C \log n}{\log\log n}} && \le \frac{1}{2n^2},
\end{aligned}$$

where $C > 0$ is a large constant independent of n. □

3 Solving LP Relaxation

We give a compact representation of the original LP which contains polynomially many variables and constraints, and hence can be solved in polynomial time. For every vertex v, let $E_{\text{in}}(v)$ be the set of edges directed into v, and $E_{\text{out}}(v)$ the set of edges directed out of v.

$$\begin{aligned}
\min \quad & W && \text{(Compact Representation)} \\
\text{subject to} \quad & \sum_{e \in E_{\text{in}}(v)} x_e^i = \sum_{e \in E_{\text{out}}(v)} x_e^i, \quad \forall i \in [k],\ \forall v \in V \setminus \{s_i, t_i\} \\
& \sum_{e \in E_{\text{out}}(s_i)} x_e^i = 1, \quad \forall i \in [k] \\
& \sum_{e \in E_{\text{in}}(t_i)} x_e^i = 1, \quad \forall i \in [k] \\
& \sum_{i=1}^{k} x_e^i \le W, \quad \forall e \in E \\
& 0 \le x_e^i \le 1, \quad \forall i \in [k],\ \forall e \in E \\
& W \ge 1
\end{aligned}$$

Fact 7. *The two LPs are equivalent.*

See Michael Dinitz's note for details.