Confidence Intervals for the Mean

Lecture 19

03/01/2013

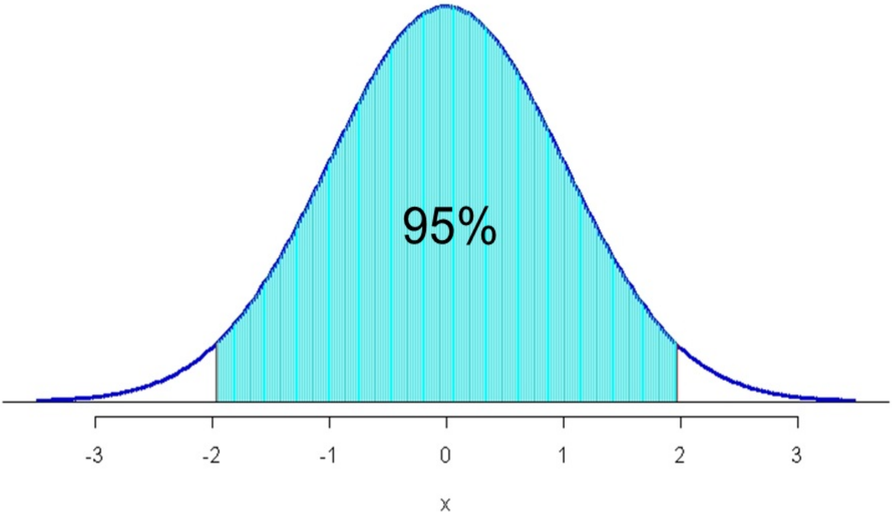
Central Limit Theorem

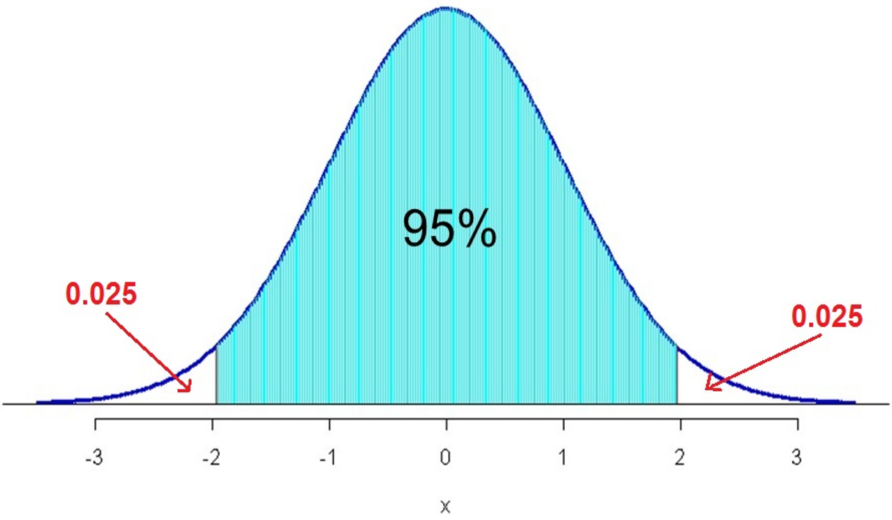
- $X_1, \ldots, X_n$: *simple random sample*
- Large sample size ($n > 30$)

$$\bar{x} \sim \mathcal{N}\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$$

Describe the Error

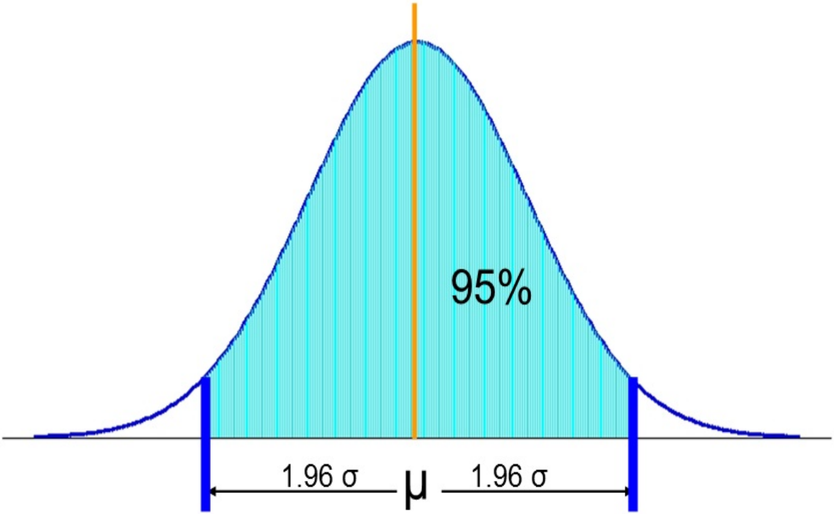
- How close is our estimate to the true value?
- How much uncertainty is there?
- Range of plausible values for μ.


95%
-3
-2
-1
0
1
2
3
x


95%
0.025
0.025
-3
-2
-1
0
1
2
3
x

95% Confidence Interval

- Standard Normal Probability
  - $\mathbb{P}(Z < -1.96) = 0.025$
  - $\mathbb{P}(-1.96 < Z < 1.96) = 0.95$
- Generally
  - Within 1.96 sigmas of the mean.


95%
1.96 σ
μ
1.96 σ

2.5% Percentile

$$\mathbb{P}(Z < -1.96) = 0.025$$

From $Z = \dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}}$ we get $\bar{x} = \mu + Z \cdot \dfrac{\sigma}{\sqrt{n}}$

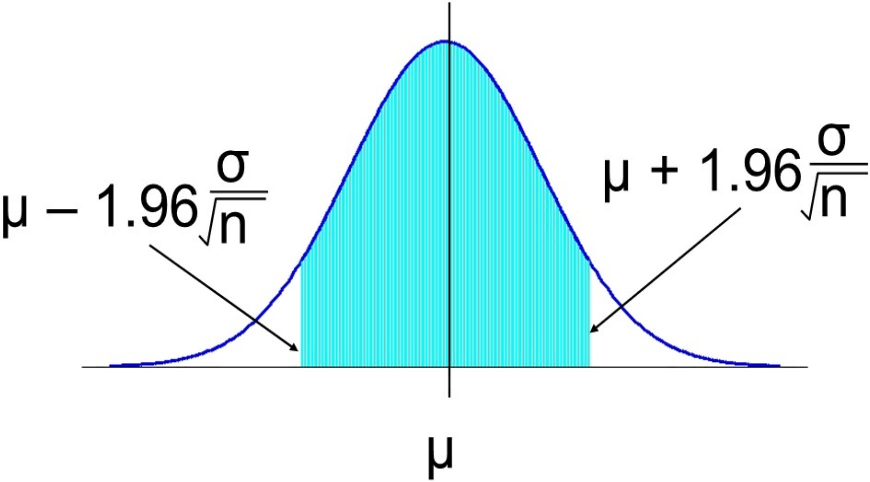
Therefore,

$$\mathbb{P}\left(\bar{x} < \mu - 1.96\frac{\sigma}{\sqrt{n}}\right) = 0.025$$

and

$$\mathbb{P}\left(\bar{x} > \mu + 1.96\frac{\sigma}{\sqrt{n}}\right) = 0.025$$

Sampling Distribution



68-95-99.7% Confidence intervals for μ

$$\left(\bar{x} - \frac{\sigma}{\sqrt{n}}, \bar{x} + \frac{\sigma}{\sqrt{n}}\right)$$

is an approximate *68% confidence interval for μ*,

$$\left(\bar{x} - 1.96\,\frac{\sigma}{\sqrt{n}}, \bar{x} + 1.96\,\frac{\sigma}{\sqrt{n}}\right)$$

is an approximate *95% confidence interval for μ*.

Interpretation

- A confidence interval is *random*.
- If you collect different samples of size n, you will observe a different value for $\bar{x}$ each time and consequently you will obtain a different confidence interval each time.
- The CLT for $\bar{x}$ guarantees that
  about 95% of these intervals will include the true average μ.

Significance Level: α

- Each confidence level corresponds to a significance level

$$\alpha = 1 \; - \; \text{confidence level}$$

- *Example*: A 68% CI corresponds to significance level equal to

$$\alpha = 1 - 0.68 = 0.32$$

CI for significance level α

$$\bar{x} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}},$$

where α is the significance level and $z_{\alpha/2}$ is the z-score that corresponds to a probability equal to $\alpha/2$.

Margin of Error

CI for significance level α

$$\bar{x} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}},$$

where α is the significance level and $z_{\alpha/2}$ is the z-score that corresponds to a probability equal to $\alpha/2$.

- The Margin of Error is

$$E = z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$

Example: *Vehicle's age*

In a survey of 30 adults, the *average age of a person's primary vehicle* is 5.6 years.

Assuming that the population standard deviation is 0.8 years, find the *best point estimate of the population mean and a 90% confidence interval for the population mea*n.

Example: *Vehicle's age*

- The best point estimate of the population mean is $\bar{x} = 5.6$ years.
- Since we care about a 90% confidence interval

$$\alpha = 1 - 0.9 = 0.1$$

- The following the z-score:

$$z_{\alpha/2} = z_{0.1/2} = z_{0.05} = -1.65$$

$$\begin{aligned} 5.6 - 1.65\frac{0.8}{\sqrt{30}} < \quad & \mu \quad < 5.6 + 1.65\frac{0.8}{\sqrt{30}} \\ 5.6 - 0.24 < \quad & \mu \quad 5.6 + 0.24 \\ 5.36 < \quad & \mu \quad < 5.84 \end{aligned}$$

Steps to Compute a Confidence Interval for μ with known σ

- Step 1: Compute (if necessary) the sample mean of the sample. Sometimes this might be given to you.
- Step 2: Find $\alpha/2$. If you are asked to find a 95% confidence interval, then $\alpha = 1 - 0.95 = 0.05$ and $\alpha/2 = 0.025$.
- Step 3: Find $z_{\alpha/2}$, that is the corresponding z-score from the normal distribution table.
- Step 4: Substitute in the formula

$$\bar{x} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}.$$