Independence

Lecture 3

01/11/2013

Probability rules

1. Conditional probability of A *given* B

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \textbf{ and } B)}{\mathbb{P}(B)}$$

   Conditional probability of B *given* A

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \textbf{ and } B)}{\mathbb{P}(A)}.$$

2. Multiplication Rule

$$\mathbb{P}(A \textbf{ and } B) = \mathbb{P}(B) \cdot \mathbb{P}(A|B).$$

$$\mathbb{P}(A \textbf{ and } B) = \mathbb{P}(A) \cdot \mathbb{P}(B|A).$$

Probability rules

3. General Addition Rule (the General OR Rule)

$$\mathbb{P}(A \textbf{ or } B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \textbf{ and } B).$$

4. Total Probability

$$\mathbb{P}(A) = \mathbb{P}(A \textbf{ and } B) + \mathbb{P}(A \textbf{ and } B^c)$$

5. Complement Rule

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^c)$$

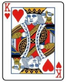
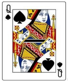
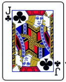
$$\mathbb{P}(A|B) = 1 - \mathbb{P}(A^c|B)$$

Example

Standard Deck of cards

- A standard deck of cards has 52 cards.
- 4 suits (Hearts ♥, Spades ♠, Diamonds ♦, Clubs ♣)
- Each suit has:

  Numbers from 1-10

  Faces (King , Queen , Jack )

- We draw a card at random.

  What is the probability that we draw a King or a Heart?

Example

- Use the general addition rule:

$$\mathbb{P}(\text{King}) = \frac{4}{52}$$

$$\mathbb{P}(\text{Hearts}) = \frac{13}{52}$$

$$\mathbb{P}(\text{King and Hearts}) = \mathbb{P}(\text{King of Hearts}) = \frac{1}{52}$$

$$\begin{aligned}\mathbb{P}(\text{King or Hearts}) &= \mathbb{P}(\text{King}) + \mathbb{P}(\text{Hearts}) - \mathbb{P}(\text{King and Hearts}) \\ &= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}\end{aligned}$$

Example

- Roll a fair die.
  What is the probability to roll an *even number* or a *3*?

$$\mathbb{P}(\text{Roll Even}) = \frac{3}{6}$$

$$\mathbb{P}(\text{Roll 3}) = \frac{1}{6}$$

- Rolling an $\{$Even number$\}$ and $\{$Rolling a 3$\}$ are two *mutually exclusive events*.

$$\begin{aligned}\mathbb{P}(\text{Even or 3}) &= \mathbb{P}(\text{Roll Even}) + \mathbb{P}(\text{Roll 3}) \\ &= \frac{1}{6} + \frac{3}{6} = \frac{4}{6} = \frac{2}{3}\end{aligned}$$

Independence

A and *B* are *independent* events if

$$\mathbb{P}(B|A) = \mathbb{P}(B), \quad \mathbb{P}(B|A^c) = \mathbb{P}(B),$$
$$\mathbb{P}(A|B) = \mathbb{P}(A), \quad \mathbb{P}(A|B^c) = \mathbb{P}(A).$$

- Joint Probability of two Independent events:

$$\mathbb{P}(A \text{ and } B) = \mathbb{P}(A) \cdot \mathbb{P}(B).$$

Is King independent of Hearts?

$$\mathbb{P}(\text{King}) = \frac{4}{52}$$

$$\mathbb{P}(\text{Hearts}) = \frac{13}{52}$$

$$\mathbb{P}(\text{King and Hearts}) = \mathbb{P}(\text{King of Hearts}) = \frac{1}{52}$$

$$\mathbb{P}(\text{King}) \cdot \mathbb{P}(\text{Hearts}) = \frac{4}{52} \cdot \frac{13}{52} = \frac{52}{52^2} = \frac{1}{52}$$

They are independent!

Is King independent of Hearts?

Another way to check for independence:

$$\mathbb{P}(\text{King}) = \frac{4}{52} = \frac{1}{13}$$

$$\begin{aligned}
\mathbb{P}(\text{King}|\text{Hearts}) &= \frac{\mathbb{P}(\text{King and Hearts})}{\mathbb{P}(\text{Hearts})} \\
&= \frac{\mathbb{P}(\text{King of Hearts})}{\mathbb{P}(\text{Hearts})} \\
&= \frac{\frac{1}{52}}{\frac{13}{52}} = \frac{1}{13}
\end{aligned}$$

$$\mathbb{P}(\text{King}|\text{Hearts}) = \mathbb{P}(\text{King})$$

They are independent!

Is Face independent of King?

$$\mathbb{P}(\text{King}) = \frac{4}{52}$$

$$\mathbb{P}(\text{Face}) = \frac{12}{52}$$

$$\mathbb{P}(\text{King and Face}) = \mathbb{P}(\text{King}) = \frac{4}{52}$$

$$\mathbb{P}(\text{King}) \cdot \mathbb{P}(\text{Face}) = \frac{4}{52} \cdot \frac{12}{52} = \frac{48}{52^2}$$

They are NOT independent!

Is Face independent of King?

Another way to check for independence:

$$\mathbb{P}(\text{King}) = \frac{4}{52} = \frac{1}{13}$$

$$\begin{aligned}\mathbb{P}(\text{King}|\text{Face}) &= \frac{\mathbb{P}(\text{King and Face})}{\mathbb{P}(\text{Face})} \\ &= \frac{\mathbb{P}(\text{King})}{\mathbb{P}(\text{Face})} \\ &= \frac{\frac{4}{52}}{\frac{12}{52}} = \frac{4}{12} = \frac{1}{3}\end{aligned}$$

$$\mathbb{P}(\text{King}|\text{Face}) \neq \mathbb{P}(\text{King})$$

They are NOT independent!

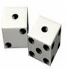
Mutually Exclusive vs. Independent Events

Independent Events	***Mutually Exclusive Events***
$\mathbb{P}(A \text{ and } B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$	$\mathbb{P}(A \text{ and } B) = 0$
$\mathbb{P}(A \vert B) = \mathbb{P}(A)$	$\mathbb{P}(A \vert B) = 0$
$\mathbb{P}(A \text{ or } B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A) \cdot \mathbb{P}(B)$	$\mathbb{P}(A \text{ or } B) = \mathbb{P}(A) + \mathbb{P}(B)$

Rules of Thumb

- Not $\longrightarrow$ 1 – Probability
- OR & Mutually Exclusive $\longrightarrow$ Add
- AND & Independence $\longrightarrow$ Multiply

Successive Events

- $\mathbb{P}(\text{Heads then Tails}) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$
- $\mathbb{P}(\text{Roll}$  $) = \left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = \frac{1}{36}$
- $\mathbb{P}(\text{Roll 1 and then an even number}) = \left(\frac{1}{6}\right)\left(\frac{1}{2}\right) = \frac{1}{12}$

Repeated Events

- A = Flip Heads 6 times (HHHHHH)

$$\mathbb{P}(A) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$
$$= \left(\frac{1}{2}\right)^6 = \frac{1}{64}$$

- B = Get HTHTTH

$$\mathbb{P}(B) = \frac{1}{64}$$

Sampling without Replacement

- A jar contains 6 blue and 6 red marbles.
- You pull out 3.
- $\mathbb{P}(\text{1st is blue}) = \frac{6}{12} = 0.5$
- $\mathbb{P}(\text{2nd is blue} \mid \text{1st is blue}) = \frac{5}{11} = 0.45$
- $\mathbb{P}(\text{3rd is blue}|\text{1st two were blue}) = \frac{4}{10} = 0.4$

$$\begin{aligned}\mathbb{P}(\text{1st two are blue}) &= \mathbb{P}(\text{1st is blue}) \cdot \mathbb{P}(\text{2nd is blue} \mid \text{1st is blue}) \\ &= (0.5)(0.45) = 0.225\end{aligned}$$

Sampling without Replacement

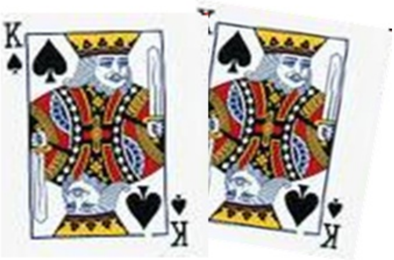
- Draw two cards from a standard deck of cards:
- A = { First card is a King }
  B = { Second card is a King }

$$\mathbb{P}(B|A) = \frac{3}{51}$$

$$\begin{aligned}
\mathbb{P}(\text{Draw two Kings}) &= \mathbb{P}(A \text{ and } B) \\
&= \mathbb{P}(A) \cdot \mathbb{P}(B|A) \\
&= \frac{4}{52} \cdot \frac{3}{51} = 0.0045
\end{aligned}$$

Sampling without Replacement

- Draw two cards from a standard deck of cards:
- Only one can be K♠

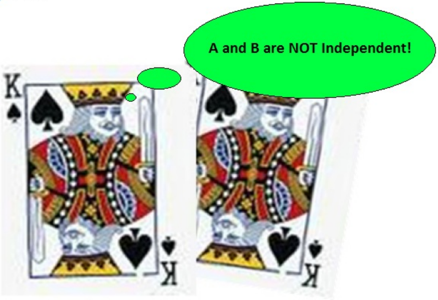


A = First is K♠

B = Second is K♠

Sampling without Replacement

- Draw two cards from a standard deck of cards:
- Only one can be K♠



A = First is K♠

B = Second is K♠

Winning the Lottery

- Choose 3 numbers from 1 to 47

$$\mathbb{P}(\text{Winning}) = \left(\frac{1}{47}\right)^3 = \frac{1}{103,823} = 0.0000096$$

- Only with independence
- Independence requires replacement

Practice Problems will be posted on Gauchospace.
Sections start on Monday.

Have a good weekend!