Worksheet 7 – Solution

1. The number of accidents per week at a hazardous intersection varies with mean 2.2 and standard deviation 1.4. This distribution is discrete and so it is not normal.

   (a) Let $\bar{X}$ be the mean number of accidents per week at the intersection during a year (52 weeks). What is the *approximate* distribution of $\bar{X}$ according to the Central Limit Theorem (CLT)?
   According to the CLT, $\bar{X}$ will follow approximately a normal $N(2.2,\ 1.4/\sqrt{52})$ distribution.

   (b) What is (approximately) the probability that $\bar{X}$ is less than 2?

$$\begin{aligned}\mathbb{P}(\bar{X} \le 2) &\approx \mathbb{P}\left(Z \le \frac{2-2.2}{0.19}\right)\\ &= \mathbb{P}(Z \le -1.05) = 0.1469\end{aligned}$$

   (c) What is (approximately) the probability that there are on average more than 5 accidengts per week?

   The CLT tells us that the approximate distribution of the sum of iid random variable is approximately normal $N(2.2,\ \frac{1.4}{\sqrt{52}})$.

$$\begin{aligned}\mathbb{P}(\bar{X} > 5) &\approx \mathbb{P}\left(Z > \frac{5-2.2}{0.19}\right)\\ &= \mathbb{P}(Z > 14.4) = 0\end{aligned}$$

2. In a regular season, the Lakers play 82 games with probability to win $\mathbb{P}(win) = 0.7$.

   (a) What is (approximately) the distribution of the sample proportion?
   We have a fixed number of games, each game is assumed to be independent from the rest, the probability of a win is the same among all games. Also:

$$np = 82 \cdot 0.7 = 57.4 > 10 \quad \text{and } n(1-p) = 83 \cdot 0.3 = 24.6 > 10$$

   So, according to the CLT

$$\hat{p} \sim \mathcal{N}\left(0.7, \sqrt{\frac{0.7\,(1-0.7)}{82}}\right)$$

   (b) What is (approximately) the probability that the Lakers win more than 80% of the games during a regular season (i.e. 82 games in total)?

$$\begin{aligned}\mathbb{P}(\hat{p} > 0.8) &= \mathbb{P}\left(Z > \frac{0.8-0.7}{0.05}\right)\\ &= \mathbb{P}\left(Z > 2\right)\\ &= 1 - \mathbb{P}(Z \le 2) = 1 - 0.9772 = 0.0228.\end{aligned}$$

3. The level of nitrogen oxide (NOX) in the exhaust of a particular car model varies with mean 1.4g/mi and standard deviation 0.3g/mi. A company has 125 cars of this model in its fleet. If $\bar{X}$ is the mean NOX emission level for these cars, what is the level L such that the probability $\bar{X}$ is greater than L is only 0.01?

   We are practically looking for the top 1% of cars in terms of mean NOX emission level, that is

   $$\mathbb{P}(\bar{X} \geq L) = 0.01$$

   According to the CLT, the distribution of $\bar{X}$ is approximately normal $N(1.4, 0.3/\sqrt{125})$. So,

   $$\begin{aligned} \mathbb{P}(\bar{X} \geq L) &= 0.01 \\ \mathbb{P}(\bar{X} \leq L) &= 0.99 \\ \mathbb{P}(Z \geq \frac{L - 1.4}{0.3/\sqrt{125}}) &= 0.99 \end{aligned}$$

   The z-score that corresponds to 0.99 is 2.33. Therefore,

   $$\frac{L - 1.4}{0.027} = 2.33$$

   $$L = 1.46$$