Probability Rules
Conditional & Joint probability

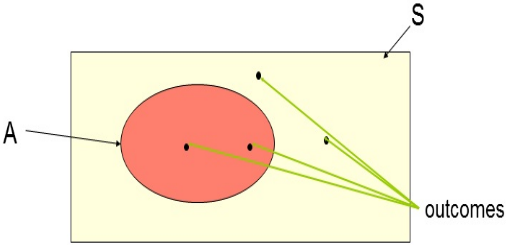
Lecture 2

01/09/2013

Modeling Random Phenomena

Probability Models

- Outcomes: possible results of an experiment
- Sample space/ Universe/ Population: set of all possible outcomes (denoted by S or Ω).
- Event: set of outcomes (e.g. A in the picture)

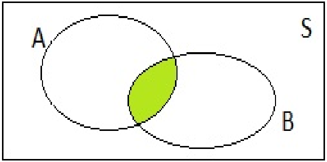
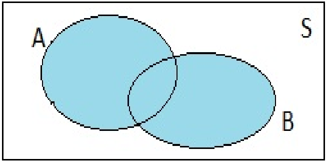
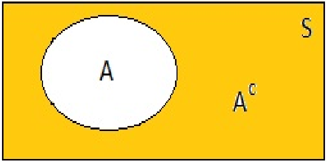


Probability

Assigning probabilities

- If all outcomes are *equally-likely*, i.e. have the same probability to occur,

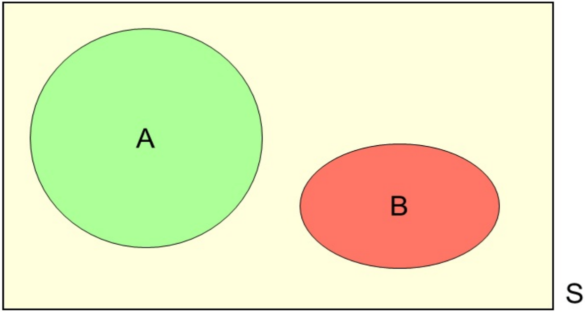
$$\mathbb{P}(\text{Event}) = \frac{\#\text{ outcomes in the event}}{\text{Total }\#\text{ of outcomes}}.$$

Sets	Notation	Interpretation	
A and B	$A \cap B$	outcomes in both A and B	
A or B	$A \cup B$	outcomes in A, B or both	
A complement	A^c	outcomes not in A	

OR Rule

- If A and B are mutually exclusive,

$$\mathbb{P}(A \text{ or } B) = \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$$



(*) A and B have no elements in common.

Conditional Probability

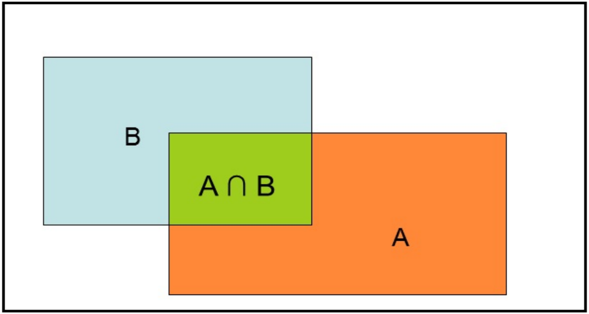
Roll a *fair* die and somebody tells us that *the outcome of the die is an odd number*.

1. What is the probability that the outcome is a five *given that the outcome of the die is an odd number*?
2. How about the probability of getting a 6 *given that the outcome of the die is an odd number*?

Conditional Probability

Conditional probability of event B given that event A has occurred:

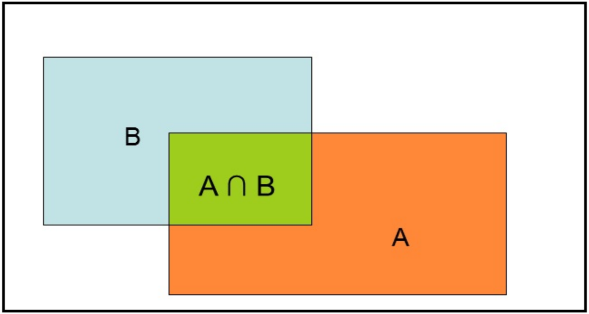
$$\mathbb{P}(B|A) = \textit{the probability of B given A}$$
$$= \frac{\text{\# of outcomes in } A \text{ and } B}{\text{\# of outcomes in } A}$$



Conditional Probability

Conditional probability of event B given that event A has occurred:

$$\mathbb{P}(B|A) = \textit{the probability of } B \textit{ given } A$$
$$= \frac{\mathbb{P}(A \text{ and } B)}{\mathbb{P}(A)}$$

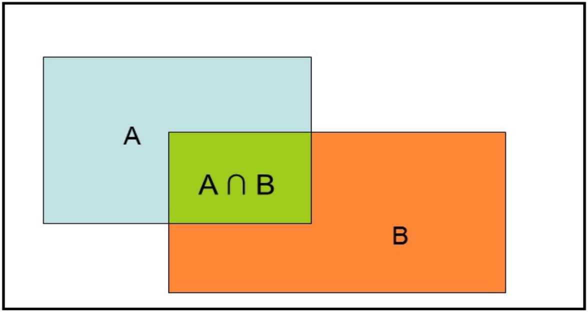


Conditional Probability

Conditional probability of event *B* given that event *A* has occurred:

$$\mathbb{P}(A|B) = \text{the probability of } A \text{ given } B$$

$$= \frac{\#\text{ of outcomes in } A \text{ and } B}{\#\text{ of outcomes in } B}$$



Conditional Probability

Conditional probability of event B given that event A has occurred:

$$\mathbb{P}(A|B) = \textit{the probability of A given B}$$
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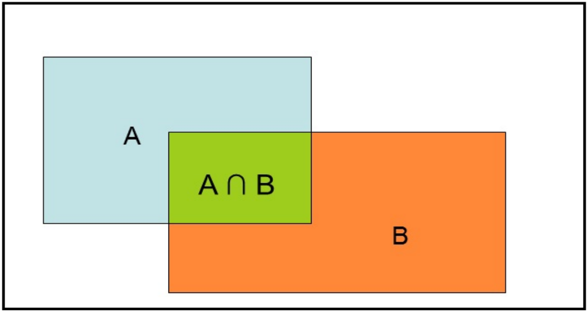
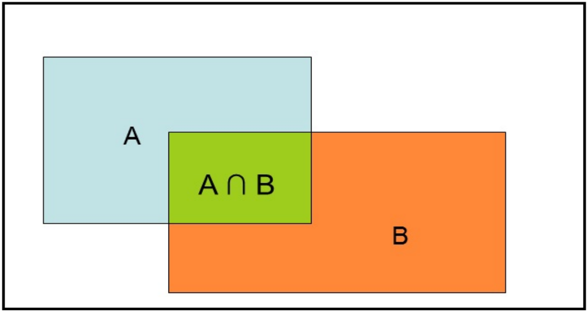


Joint Probability

- $\mathbb{P}(A \text{ and } B)$ = *joint* probability of A and B
  the probability that events A and B occur **together** ($A \cap B$).



Conditional Probability

What is the probability that the outcome of a fair die is five *given that the outcome of the die is an odd number*?

$$A = \{\text{roll a } 5\} = \{5\}$$
$$B = \{\text{roll an odd number}\} = \{1, 3, 5\}$$

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \text{ and } B)}{\mathbb{P}(B)} = \frac{\#\text{ of outcomes in } A \text{ and } B}{\#\text{ of outcomes in } B} = \frac{1}{3},$$

since A and $B = A \cap B = \{5\}$.

Another Example

If you know that a family has *two children* and one of them is a Boy, what is the probability that the other child is a Boy as well?

- Sample space = set of possible pairs of children

$$\Omega = \{BB, BG, GB, GG\}.$$

- The probability that both children are Boys given that at least one is a Boy is

$$\mathbb{P}(\text{ both Boys } | \text{at least one boy}) = \frac{\mathbb{P}(\text{ both Boys } \textbf{and} \text{ at least one boy})}{\mathbb{P}(\text{at least one boy})}.$$

Example

$$\mathbb{P}(\text{both Boys \textbf{and} at least one boy}) = \mathbb{P}(\text{both Boys})$$
$$= \frac{\#\text{ of pairs both are Boys}}{\text{total }\#\text{ of pairs}} = \frac{1}{4}$$

$$\mathbb{P}(\text{at least one Boy}) = \frac{\#\text{ of pairs with at least one Boy}}{\text{total }\#\text{ of pairs}} = \frac{3}{4}$$

$$\mathbb{P}(\text{ both Boys }|\text{at least one boy}) = \frac{1/4}{3/4} = \frac{1}{3}.$$

$\mathbb{P}(A|B) \neq \mathbb{P}(B|A)$

- $\mathbb{P}(A|B)$ = conditional probability of A given B.
- $\mathbb{P}(B|A)$ = conditional probability of B given A.
- Example:

$$A = \{\text{roll a 5}\}$$

$$B = \{\text{roll an odd number}\}$$

  - $\mathbb{P}(A|B) = \mathbb{P}(\text{roll a 5}|\text{roll an odd}) = 1/3$.
  - $\mathbb{P}(B|A) = \mathbb{P}(\text{roll an odd}|\text{roll a 5}) = 1$.

Probability Rules

Probability rules

1. Conditional probability of A *given* B

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \textbf{ and } B)}{\mathbb{P}(B)}$$

Conditional probability of B *given* A

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \textbf{ and } B)}{\mathbb{P}(A)}.$$

2. Multiplication Rule

$$\mathbb{P}(A \textbf{ and } B) = \mathbb{P}(B) \cdot \mathbb{P}(A|B).$$

$$\mathbb{P}(A \textbf{ and } B) = \mathbb{P}(A) \cdot \mathbb{P}(B|A).$$

Probability rules

3. General Addition Rule (the General OR Rule)

$$\mathbb{P}(A \textbf{ or } B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \textbf{ and } B).$$

4. Total Probability

$$\mathbb{P}(A) = \mathbb{P}(A \textbf{ and } B) + \mathbb{P}(A \textbf{ and } B^c)$$

5. Complement Rule

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^c)$$

$$\mathbb{P}(A|B) = 1 - \mathbb{P}(A^c|B)$$

Example

Employed and unemployed men and women in a particular CA county.

Contingency Table:

	Employed	Unemployed	
Male	0.511	0.044	
Female	0.156	0.289	

- Joint Probabilities:

$$\mathbb{P}(\text{Male } \textbf{and} \text{ Employed}) = 0.511$$

$$\mathbb{P}(\text{Female } \textbf{and} \text{ Employed}) = 0.156$$

Example: *Unemployment*

Contingency Table:

	Employed	Unemployed	
Male	0.511	0.044	0.555
Female	0.156	0.289	0.445
	0.667	0.333	

- Marginal Probabilities:

  $\mathbb{P}(\text{Male}) = 0.555 \qquad \mathbb{P}(\text{Female}) = 0.445$

  $\mathbb{P}(\text{Employed}) = 0.667 \qquad \mathbb{P}(\text{Unemployed}) = 0.333$

Example: *Unemployment*

Contingency Table:

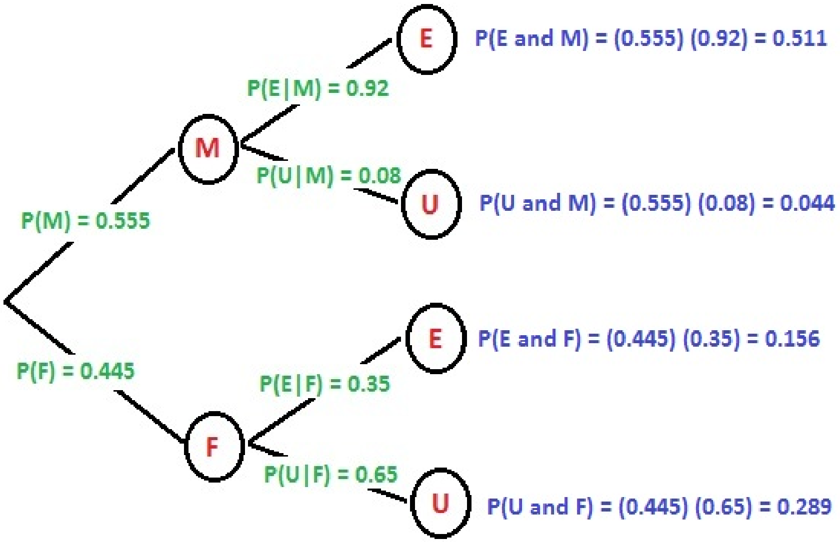
	Employed	Unemployed	
Male	0.511	0.044	0.555
Female	0.156	0.289	0.445
	0.667	0.333	

- Conditional Probabilities:

$$\mathbb{P}(\text{Employed} \mid \text{Male}) = \frac{\mathbb{P}(\text{Employed } \textbf{and} \text{ Male})}{\mathbb{P}(\text{Male})} = \frac{0.511}{0.555} = 0.921$$

$$\mathbb{P}(\text{Male} \mid \text{Employed}) = \frac{\mathbb{P}(\text{Employed } \textbf{and} \text{ Male})}{\mathbb{P}(\text{Employed})} = \frac{0.511}{0.667} = 0.766$$

Tree Diagram



Two types of Questions

1. $\mathbb{P}(A \text{ and } B)$
   - Deduce conditional probability $\mathbb{P}(B|A)$.
   - We compute the joint: $\mathbb{P}(A \text{ and } B) = \mathbb{P}(B|A)\mathbb{P}(A)$
2. $\mathbb{P}(B|A)$
   - We know the joint probability = $\mathbb{P}(A \text{ and } B)$
   - We compute the conditional: $\mathbb{P}(B|A) = \frac{\mathbb{P}(A \text{ and } B)}{\mathbb{P}(A)}$.

Example: *Drug Test*

- 1% of students use cocain.
- Drug test is 99% accurate.
- Pat (a student) tests positive for cocaine.

What is the probability that Pat uses cocaine *given* she tests positive?

Example: *Drug Test*

- 1% of students use cocain
- Drug test is 99% accurate

- *What is the probability that Pat uses cocaine, given she tests positive?*
  - We know that:

$$\mathbb{P}(\text{Pat uses}) = 0.01$$
$$\mathbb{P}(\text{Test + } | \text{ Pat uses}) = 0.99$$
$$\mathbb{P}(\text{Test + } | \text{ Pat DOESN'T use}) = 1 - 0.99 = 0.01$$

  - But, we need to compute

$$\mathbb{P}(\text{Pat uses} | \text{ Test +}) = ???$$

Example: *Drug Test*

Let's start with the definition

$$\mathbb{P}(\text{Pat uses}|\text{ Test }+) = \frac{\mathbb{P}(\text{Pat uses } \textbf{and} \text{ Test }+)}{\mathbb{P}(\text{Test }+)}.$$

- Use the multiplication rule:

$$\begin{aligned}\mathbb{P}(\text{Pat uses } \textbf{and} \text{ Test }+) &= \mathbb{P}(\text{Pat uses})\;\; \mathbb{P}(\text{Test }+ \mid \text{Pat uses}) \\ &= (0.01)\,(0.99) = 0.0099\end{aligned}$$

Example: *Drug Test*

1. Use the total probability rule:

$$\mathbb{P}(\textit{Test}+) = \mathbb{P}(\text{Test + } \textbf{ and } \text{ Pat uses}) + \mathbb{P}(\text{Test + } \textbf{ and } \text{ Pat DOESN'T use})$$

2. Use the multiplication rule:

$$\mathbb{P}(\text{Test + } \textbf{ and } \text{ Pat DOESN'T use}) = \mathbb{P}(\text{Pat DOESN'T use})\,\mathbb{P}(\text{Test + } | \text{ Pat DOESN'T use})$$

3. Use the complement rule:

$$\mathbb{P}(\text{Pat DOESN'T use}) = 1 - \mathbb{P}(\text{Pat uses}) = 1 - 0.01 = 0.99$$

Example: *Drug Test*

Finally,

$$\mathbb{P}(\textit{Test}+) = 0.0099 + (0.99)(0.01) = 0.0198$$

$$\mathbb{P}(\text{Pat uses}|\text{ Test }+) = \frac{\mathbb{P}(\text{Pat uses } \textbf{and} \text{ Test }+)}{\mathbb{P}(\text{Test }+)} = \frac{0.0099}{0.0198} = \frac{1}{2}$$