Binomial Distribution (cont'd)

Lecture 7

01/25/2013

Counting Sequences

n Factorial

$$n! = n\,(n-1)\,(n-2)\;\ldots\;(3)(2)1$$
$$0! = 1$$

- Examples:

$$4! = 4\,(3)\,(2)\,1 = 24$$
$$5! = 5\,(4)\,(3)\,(2)\,1 = (5)\,4! = 120$$
$$10! = 10\,(9)\,(8)\,(7)\,(6)\,5! = 3,628,800$$

Combinations

n choose k

$$ {}_nC_k = \frac{n!}{k!\,(n-k)!} $$

- $n = 6,\ k = 4$

$$ {}_6C_4 = \frac{6!}{4!\,2!} = \frac{6(5)(4)(3)(2)1}{(4)(3)(2)(1) \times (2)1} = \frac{720}{48} = 15 $$

- $n = 20,\ k = 4$

$$ \begin{aligned} {}_{20}C_4 &= \frac{20!}{4!\,(20-4)!} = \frac{20\,(19)\,(18)\,(17)\,16!}{4(3)(2)1 \times 16!} \\ &= \frac{116,280}{24} = 4,845 \end{aligned} $$

Example

We have a pool of

- 15 true-false (T/F) questions and
- 20 multiple-choice questions.

and we want to make an exam with 10 questions exactly 4 of which are T/F. Ignoring the order of the questions, how many exams with **exactly 4 T/F questions** can be constructed?

Example

- The total number of ways to choose 4 T/F questions is

$$_{15}C_4 = \frac{15!}{4!\,11!} = 1,365$$

- The total number of ways to choose 6 multiple choice questions is

$$_{20}C_6 = \frac{20!}{6!\,14!} = 38,760$$

- The total number of exams with 4 T/F and 6 multiple choice questions are

$$_{15}C_4 \times {}_{20}C_6 = 1,365 \times 38,760 = 52,907,400$$

Another Example

We have a pool of *15 true-false questions* and *20 multiple-choice questions*. Among exams with 10 questions, what is the probability to pick an exam with exactly 4 T/F questions (and 6 multiple choice)?

- Recall the classical definition of probability:

$$\begin{aligned}\mathbb{P}(\text{4 T/F Questions}) &= \frac{\text{\# of exams with 4 T/F Questions}}{\text{\# of exams with 10 Questions}} \\ &= \frac{{}_{15}C_4 \times {}_{20}C_6}{{}_{35}C_{10}} \\ &= \frac{52,907,400}{183,579,396} = 0.2882\end{aligned}$$

Turn on your clickers!

20 people are running for 4 different offices in a labor union. Mary, John, Pat and Tom are among them.

What is the **probability** that Mary, John, Pat and Tom are elected officers?

(a) $\frac{1}{{}_{20}C_4} = 0.0002$

(b) ${}_{20}C_4 = 4{,}845$

(c) $4! = 24$

(d) $1/4! = 0.0417$

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Probability of Flipping 6 of 10

$$\mathbb{P}(\text{6 Heads in 10 Tries}) = {}_{10}C_6 \left(\frac{1}{2}\right)^{10}$$

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# of orderings

Probability of each one

Probability of Flipping 6 of 10

$$\mathbb{P}(\text{6 Heads in 10 Tries}) = \underbrace{{}_{10}C_6}_{\text{\# of orderings}} \underbrace{\left(\frac{1}{2}\right)^{10}}_{\text{Probability of each one}}$$

$$\frac{10!}{6!\,4!}\left(\frac{1}{1024}\right) = \frac{10(9)(8)(7)}{4(3)(2)1}\left(\frac{1}{1024}\right) = \frac{210}{1024} = 0.2051$$

Other Probabilities

Flip a Biased Coin 5 times

$$\mathbb{P}(H) = 0.2 \text{ and } \mathbb{P}(T) = 0.8$$

$$\mathbb{P}(HTTHT) = 0.2(0.8)(0.8)(0.2)(0.8) = (0.2)^2(0.8)^3 = 0.02048$$
$$\mathbb{P}(TTHTH) = 0.8(0.8)(0.2)(0.8)(0.2) = (0.2)^2(0.8)^3 = 0.02048$$
$$\mathbb{P}(TTTHH) = 0.8(0.8)(0.8)(0.2)(0.2) = (0.2)^2(0.8)^3 = 0.02048$$

$$\mathbb{P}(\text{ 2 Heads }) =_5 C_2(0.2)^2(0.8)^3 = 10(0.02048) = 0.2048$$

Binomial Experiment

1. n trials
2. Trials are independent
3. Two possible outcomes
   - "Success", 1
   - "Failure", 0
4. $\mathbb{P}($ "Success" $) = p$
   $\mathbb{P}($ "Failure" $) = 1 - p$
   p is constant for all trials.

Example

You flip a biased coin 100 times

1. $n = 100$
2. Tosses are Independent
3. "Heads" = "success"
   "Tails" = "failure"
4. $p = \mathbb{P}(\text{ "Heads" }) = 0.44$ (this is given)
   $1 - p = \mathbb{P}(\text{ "Tails" }) = 0.56$

$X =$ # of Heads

Binomial Probability

$$\mathbb{P}(X = k) =_n C_k \, p^k \, (1 - p)^{n-k}$$

- What is the probability that 40 Heads come up in the 100 tosses?

$$\begin{aligned}
&\mathbb{P}(\text{ 40 Heads }) = \mathbb{P}(X = 40) \\
&=_{100} C_{40} \, (0.44)^{40} \, (0.56)^{60} \\
&= (1.3746 \times 10^{28}) \, (5.47151 \times 10^{-15}) \, (7.78541 \times 10^{-16}) \\
&= 0.0586
\end{aligned}$$

Turn on your clickers!

Using the information from the previous example,

$$\mathbb{P}(X = 0) + \mathbb{P}(X = 1) + \mathbb{P}(X = 2) + \ldots + \mathbb{P}(X = 100) = ??$$

(a) 0

(b) 1

(c) 0.357

(d) 0.643

Binomial Distribution

- Binomial Experiment
  1. n trials
  2. Trials are independent
  3. Two possible outcomes
  4. $\mathbb{P}($ "Success" $) = p$

- Binomial Probability

$$\begin{aligned}\mathbb{P}(X = k) &= {}_nC_k \, p^k \, (1-p)^{n-k} \\ &= \frac{n!}{k! \, (n-k)!} \, p^k \, (1-p)^{n-k}\end{aligned}$$

Example: *The sock drawer*

A drawer contains 6 red and 4 black socks. What is the probability that you select 2 red socks when you pick 5 out of the drawer (With replacement)?

1. There are 5 trials.
2. Each draw is independent of the other, since we choose a sock each time with replacement.
3. Since we are interested in selecting *red* socks, we call this a **success** and selecting a *black* sock is a **failure**.
4. For each trial, the probability of success is $p = 6/10 = 0.6$ is constant.

This is a *Binomial experiment*.

Example: *The sock drawer*

A drawer contains 6 red and 4 black socks. What is the probability that you select 2 red socks when you pick 5 out of the drawer (With replacement)?

X : # of Red socks (i.e. successes) in 5 trials.

$$
\begin{aligned}
\mathbb{P}(X = 2) &= {}_5C_2 \cdot \left(\frac{6}{10}\right)^2 \cdot \left(1 - \frac{6}{10}\right)^{5-2} \\
&= 10 \cdot (0.6)^2 \cdot (0.4)^3 \\
&= 0.2304.
\end{aligned}
$$

Example: *The sock drawer*

A drawer contains 6 red and 4 black socks. What is the probability that you select *at least 4 red socks* when you pick 5 out of the drawer (With replacement)?

$$\begin{aligned}
\mathbb{P}(X \geq 4) &= \mathbb{P}(X = 4) + \mathbb{P}(X = 5) \\
&=_5 C_4 \cdot (0.6)^4 \cdot (0.4)^1 +_5 C_5 \cdot (0.6)^5 \cdot (0.4)^0 \\
&= 5 \cdot (0.6)^4 \cdot (0.4)^1 + (0.6)^5 \cdot (0.4)^0 \\
&= 0.2592 + 0.0778 = 0.3369
\end{aligned}$$

“At least one...”

I play the *pick-3* every day for 300 days. What is the probability I win at least once?

$$\mathbb{P}(X \geq 1) = \mathbb{P}(X = 1) + \mathbb{P}(X = 2) + \mathbb{P}(X = 3) + \dots$$

Trick

$$\begin{aligned} \mathbb{P}(X \geq 1) &= 1 - \mathbb{P}(X = 0) \\ &= 1 -_3 C_0 \, (0.001)^0 \, (0.999)^{300} \\ &= 0.2593 \end{aligned}$$

Take the shorter road

Alexandra is going to work 7 days a week and on any given day she is late with probability 0.643 . What is the probability that she is late no more than 5 days next week?

$$\begin{aligned}\mathbb{P}(X \leq 5) = \mathbb{P}(X = 0) + \mathbb{P}(X = 1) + \mathbb{P}(X = 2)\\ + \mathbb{P}(X = 3) + \mathbb{P}(X = 4) + \mathbb{P}(X = 5)\end{aligned}$$

$$\begin{aligned}\mathbb{P}(X \leq 5) &= 1 - \mathbb{P}(X = 6) - \mathbb{P}(X = 7)\\ &= 1 \ - \ {}_7C_6 \ (0.643)^6 (0.357)^1 \ - \ {}_7C_7 \ (0.643)^7 (0.357)^0\\ &= 0.7779\end{aligned}$$

Turn on your clickers!

You pick 3 students from a class of 7 boys and 8 girls. If

$$X = \text{number of boys we selected}$$

Is X a Binomial random variable?

(a) Yes!

(b) No, because we have sampling without replacement and the probability of choosing a boy from trial to trial changes, depending on whether we chose a boy or girl on the previous trials.