Central Limit Theorem

① SAMPLE MEAN

$X_1, X_2, \dots, X_n$ — independent, identically distributed, $n > 30$

common mean μ
common s.d. σ

Then,

$$\bar{X} \sim \text{Normal}\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$$

② SAMPLE PROPORTION

$X_1, X_2, \dots, X_n$ — Binomial experiment (Recall 4 assumptions), $np > 10$, $n(1-p) > 10$

$\overset{?}{\text{Bin}}(n,p)$

$\Downarrow$

$E(X_i) = n \cdot p$

$Var(X_i) = np(1-p)$

Then,

$$\hat{p} \sim \text{Normal}\left(\cancel{n}p, \sqrt{\frac{p(1-p)}{\cancel{n}n}}\right)$$

Example ①

$X =$ student's score. (# of correct answers).

? PMF OF X

1) This is a Binomial experiment. Why?

(1) There are 30 trials/questions.

(2) Each question is answered independently of the other (independent trials)

(3) There are two possible outcomes.

- success : answer correctly
- failure : answer wrong..

(4) Fixed probability of success among trials.

$p = P(\text{answer correctly}) = \frac{1}{4} = 0.25$

since multiple-choice with 4 possible answers.

$\Downarrow$

$$\boxed{X \sim \text{Binomial}(30, 0.25).}$$

$$P_X(x) = {}_{30}C_x (0.25)^x (1-0.25)^{30-x}.$$

2) 81 students

Average score of 81 students:

$$\overline{X} = \frac{X_1 + \cdots + X_{81}}{81}.$$

$X_1, X_2, \ldots, X_{81} \sim \text{Binomial}(30, 0.25).$

<u>Assumptions</u> : 81 students are independent.
81 students are identically distributed since they are all in the same class.
81 > 30 !

Thus, $\bar{X} \sim \text{Normal}\left(\mu, \frac{\sigma}{\sqrt{81}}\right)$.

What is μ and what σ?

→ Since $X_i \sim \text{Binomial}(30, 0.25) \Rightarrow$

$\mu = E(X_i) = 30(0.25) = 7.5$
$\sigma = \sqrt{Var(X_i)} = \sqrt{(30)(0.25)(0.75)} = \sqrt{5.625}$
$= \underline{\underline{2.37}}$

$\Rightarrow \quad \bar{X} \sim \text{Normal}\left(\underline{7.5}, \frac{2.37}{\sqrt{81}} = \underline{0.26}\right)$

$P(\bar{X} > 8)$?

i/ Standardize $\frac{8 - 7.5}{0.26} = 1.92$

ii/ Look at the standard Normal table

iii) Since it's "greater than" . . . $P(Z > 1.92) = 1 - 0.9783$
$= \boxed{0.0217}$

Example 2

$n = 100$ $\qquad\qquad$ $\hat{p} = ?$

$p = 0.2$

$$\hat{p} \sim \text{Normal}\left(0.2,\ \sqrt{\underbrace{\frac{0.2(0.8)}{100}}_{0.04}}\right)$$

$P(\hat{p} < .30)$

Since $\hat{p}$ is Normal we need to find z-score for 0.30:

$$z = \frac{0.30 - 0.20}{0.04} = \underline{\underline{2.5}}$$

$$P(\hat{p} < .3) = P(z < 2.5) = 0.9938$$