Expected Value & Variance

Lecture 5

01/16/2013

Expected Value

$$\mathbb{E}(X) = \sum_{all\ k} k\ \mathbb{P}(X = k)$$

Interpretation

1. A probability-weighted average.
2. Long-run average of X.
3. The fair value of a gamble.
4. The balance point for a probability histogram/bargraph.

How to compute $\mathbb{E}(X)$

k	2	5
$\mathbb{P}(X = k)$	1/3	2/3

$$\begin{aligned}\mathbb{E}(X) &= 2 \cdot \mathbb{P}(X = 2) + 5 \cdot \mathbb{P}(X = 5) \\ &= 2\ (1/3) + 5\ (2/3) \\ &= 2/3 + 10/3 = 12/3 = 4\end{aligned}$$

It is a *weighted* average of the values of X and the *weights* are the probabilities that correspond to these values.

A Bet

You pay \$100 for a bet.

If you win you get \$210 and if you lose you get \$0.

- X = your profit/loss

$$X = \$110, \text{ if you win}$$
$$X = -\$100, \text{ if you lose}$$

- Assume that the probabilities to win/lose are

$$\mathbb{P}(\text{ win }) = 0.55$$
$$\mathbb{P}(\text{ lose }) = 0.45$$

Is this a fair bet?

A Bet

- The pdf of X is

k	\$110	$-$\$100
$\mathbb{P}(X = k)$	0.55	0.45

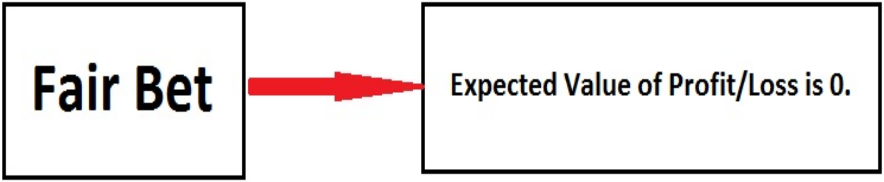
- To check whether the bet is fair or not, compute the expectation of your profit/loss:

$$\begin{aligned} \mathbb{E}(X) &= 110\,\mathbb{P}(X = 110) - 100\,\mathbb{P}(X = -100) \\ &= \$110\,(0.55) - \$100\,(0.45) \\ &= \$15.5 \end{aligned}$$

Is it fair?

It is **not a fair bet**, since the expectation is not 0!

Fair Bet



A Fair Game

- A clicker question has 5 possible answers. If you choose the correct answer, you get 1 point. How many points should I deduct if you choose a wrong answer for the grade to be *fair*?
  (Each answer is equally likely to be selected.)

(a) 0 points

(b) 1 point

(c) 0.25 points

(d) 0.5 points

A Fair Game

- Since it is equally likely to choose any of the 5 possible answers, consider X = points you get.

k	1	$-x$
$\mathbb{P}(X=k)$	$\frac{1}{5}$	$\frac{4}{5}$

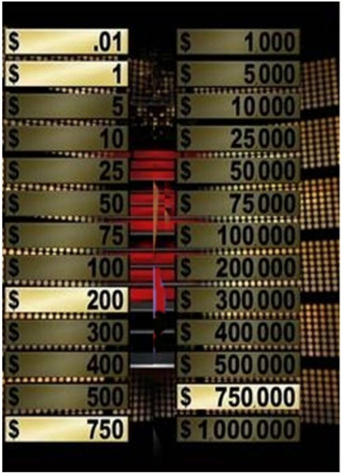
  Thus, $E(X) = \frac{1}{5} \cdot 1 - \frac{4}{5} \cdot x$.

- For the exam to be fair, $E(X) = 0$. That is,

$$\frac{1}{5} \cdot 1 - \frac{4}{5} \cdot x = 0$$

$$x = \frac{1}{4} = 0.25 \text{ points}$$

Deal or No Deal



Banker's Offer = \$ 138,000

X = earnings if you continue

k	X = \$0.01	X = \$1	X = \$200	X = \$750	X = \$750,000
P(X=k)	1/5	1/5	1/5	1/5	1/5

$E(X) = (0.01)(1/5) + (1)(1/5) + (200)(1/5) + (750)(1/5) + (750{,}000)(1/5)$

$= \mathbf{150{,}190.20}$

Who wants to be a Millionaire?

Quit = \$100,000

Guess right X = \$250,000

Guess wrong X = \$32,000

Two cases:

- Choose at random among all 4 possible answers.
  $\mathbb{P}(right) = 1/4$

  $$\mathbb{E}(X) = (250,000)(1/4) + 32,000(3/4) = \$86,500$$

- Get the 50-50 option. So, you choose at random between 2 answers.
  $\mathbb{P}(right) = 1/2$

  $$\mathbb{E}(X) = (250,000)(1/2) + 32,000(1/2) = \$141,000$$

Changing the Random Variables

– Jane bought her car for \$16,500.

– Jane hopes to sell her car for S.

How much did the car *cost* to Jane? (Cost = Buy - Sell)

- Jane is going to sell it for \$15,000, if gas $>$ \$4/gal and \$12,000, if gas $<$ \$4/gal.
- We know that $\mathbb{P}(gas > \$4/gal) = 0.45$. So,

S	12,000	15,000
$\mathbb{P}$	0.55	0.45

$$\mathbb{E}(S) = 12,000(0.55) + 15,000(0.45) = 13,350$$

Changing the Random Variables

- X = cost = 16,500 - S

X	(16,500 - 12,000)	(16,500 - 15,000)
$\mathbb{P}$	0.55	0.45

$$\mathbb{E}(X) = (4,500)(0.55) + (1,500)(0.45) = 3,150$$

$$\mathbb{E}(X) = 16,500 - \mathbb{E}(S)$$

$$= 16,500 - 13,350 = 3,150$$

Adding a Constant

$$\mathbb{E}(X + \alpha) = \mathbb{E}(X) + \alpha$$

Functions of a Random Variable

k	10	12	15
$\mathbb{P}(X = k)$	0.4	0.5	0.1

$$\mathbb{E}(X) = 10(0.4) + 12(0.5) + 15(0.1) = 11.5$$

- Double the value of X

k	20	24	30
$\mathbb{P}(2\,X = 2\,k)$	0.4	0.5	0.1

$$\mathbb{E}(2\,X) = 20(0.4) + 24(0.5) + 30(0.1) = 23$$

$$\mathbb{E}(2\,X) = 2\,\mathbb{E}(X)$$

$$= 2(11.5) = 23$$

Multiplying by a Constant

$$\mathbb{E}(b\,X) = b\,\mathbb{E}(X)$$

Properties of $\mathbb{E}(X)$

1. $\mathbb{E}(X + \alpha) = \mathbb{E}(X) + \alpha$.
2. $E(b\, X) = b\, \mathbb{E}(X)$.
3. $\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y)$
4. If X and Y are independent, then

$$\mathbb{E}(XY) = \mathbb{E}(X)\, \mathbb{E}(Y)$$

Measuring the Spread

- Variance = how spread out the pdf is.
- Variance = average squared distance from the mean:
  - Mean: $\mathbb{E}(X)$
  - Distance from the mean: $X - \mathbb{E}(X)$
  - Squared distance from the mean: $(X - \mathbb{E}(X))^2$
  - Average squared distance from the mean:

$$Var(X) = \mathbb{E}(X - \mathbb{E}(X))^2$$

Another way to compute the Variance

$$Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$$

Remark!

$\mathbb{E}(X^2)$ is not the same as $(\mathbb{E}(X))^2$!!!!

How to compute $Var(X)$

k	1	2	3
$\mathbb{P}(X = k)$	0.2	0.6	0.2

1. $\mathbb{E}(X) = 1(0.2) + 2(0.6) + 3(0.2) = 2$.
2. $\mathbb{E}(X^2) = \sum k^2 \, \mathbb{P}(X = k)$

k^2	1	4	9
$\mathbb{P}(X = k)$	0.2	0.6	0.2

$$\mathbb{E}(X^2) = 1(0.2) + 4(0.6) + 9(0.2) = 4.4$$

3. Compute the Variance:

$$\begin{aligned} Var(X) &= \mathbb{E}(X^2) - (\mathbb{E}(X))^2 \\ &= 4.4 - 2^2 = 0.4 \end{aligned}$$

Standard Deviation

- Standard Deviation = $\sqrt{\text{Variance}}$

$$\sigma = \sqrt{Var(X)}$$

Properties

- $Var(\alpha X) = \alpha^2 \; Var(X)$

  For example,

  $$Var(3 \cdot X) = 3^2 \cdot Var(X) = 9 \cdot Var(X)$$

- $Var(\alpha + X) = Var(X)$

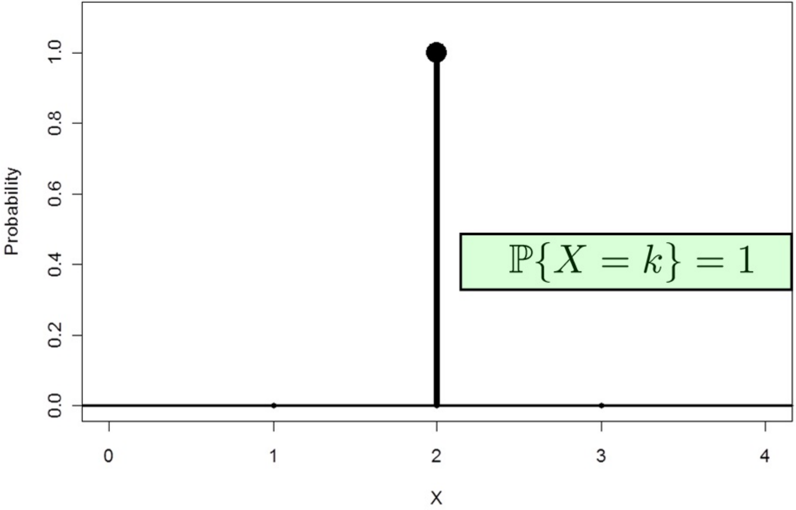
  For example,

  $$Var(3 + X) = Var(X)$$

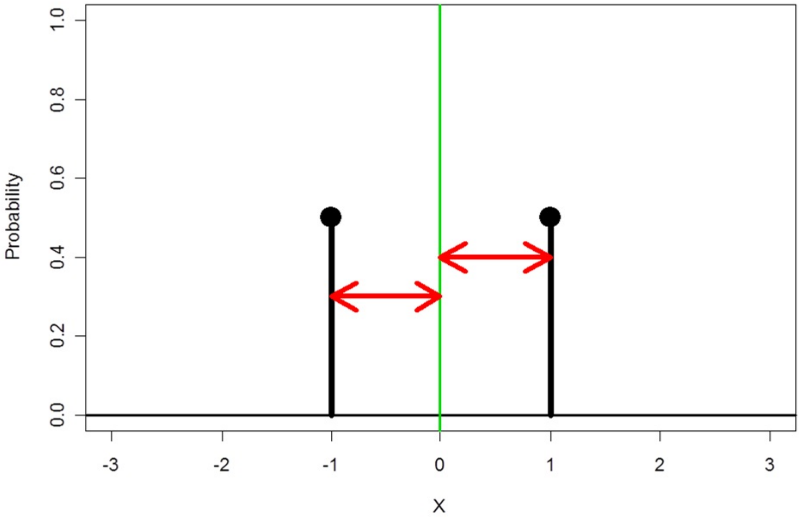
- If X and Y are **independent**

  $$Var(X + Y) = Var(X) + Var(Y)$$

Variance = 0



Var(X) = 1



Var(X) = 4 !!