Random Variables

Lecture 4

01/14/2013

Random Variables

- A *random variable* is a numerical variable whose value depends on the outcome of a *random phenomenon*.

- Each outcome is a number:

	1	2	3
$\mathbb{P}$	0.2	0.6	0.2

- Name the random variable **X**

  $\mathbb{P}(X = 1) = 0.2$

  $\mathbb{P}(X = 2) = 0.6, \ldots$

X	1	2	3
$\mathbb{P}$	0.2	0.6	0.2

Flip a Coin 3 times

- Outcomes

HHH	HHT	HTH	HTT
THH	THT	TTH	TTT

- X = number of Heads

  $\mathbb{P}(X = 3) = 1/8$

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  $\mathbb{P}(X = 2) = 3/8$

Flip a Coin 3 times

- Outcomes

HHH	HHT	HTH	HTT
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- X = number of Heads

  $\mathbb{P}(X = 3) = 1/8$

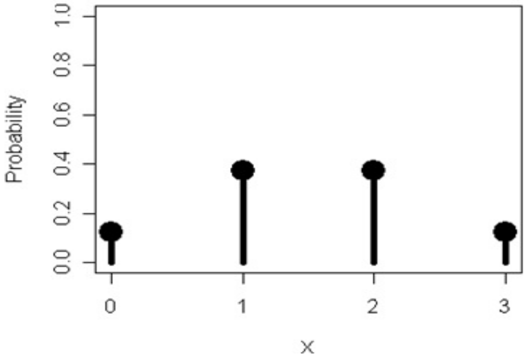
  $\mathbb{P}(X = 2) = 3/8$

  $\mathbb{P}(X = 1) = 3/8$

  $\mathbb{P}(X = 0) = 1/8$

Probability distribution function (pdf)

k	$X = 0$	$X = 1$	$X = 2$	$X = 3$
$\mathbb{P}(X = k)$	1/8	3/8	3/8	1/8



More generally

- X takes values $\{x_1, \ldots, x_k\}$

PDF

k	$X = x_1$	$\ldots$	$X = x_k$
$\mathbb{P}(X = k)$	p_1	$\ldots$	p_k

$p_1 = P(X = x_1)$,

$\ldots$

$p_k = P(X = x_k)$

Remarks

1. All probabilities add up to 1.

$$p_1 + \ldots + p_k = 1$$

2. For every k

$$0 \leq \mathbb{P}(X = k) \leq 1$$

3. All other numbers have probability 0.

Roll Two Dice

- Outcomes

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

- X = sum of two dice
- PDF

k	2	3	4	5	6	7	8	9	10	11	12
$\mathbb{P}(X = k)$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

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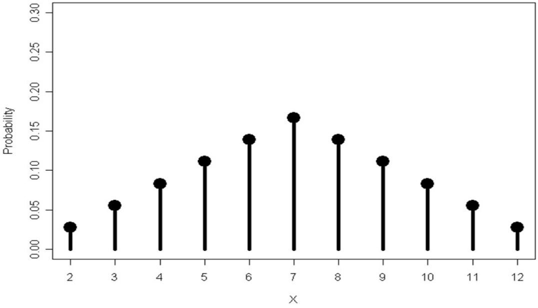

Turn on your clickers!

Two coins are tossed and the *total number of Tails is counted*. Which of the following would be a legitimate probability model for the total number of heads?

That is, which of the following satisfies the rules of probability?

(a)

k	0	1	2
$\mathbb{P}(X = k)$	1/3	2/3	1/3

(b)

k	0	1	2
$\mathbb{P}(X = k)$	1/16	5/8	5/16

(c)

k	0	1	2
$\mathbb{P}(X = k)$	1/2	3/4	-1/4

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Event $A = \{X = k\}$

- The events $\{X = k\}$ and $\{X = j\}$ are mutually exclusive.
- Simple OR Rule:

$$\mathbb{P}(X = k \text{ or } X = j) = \mathbb{P}(X = k) + \mathbb{P}(X = j)$$

Job interview

- 5 applicants are selected for an interview
- X = number of women

	$X = 0$	$X = 1$	$X = 2$	$X = 3$	$X = 4$	$X = 5$
$\mathbb{P}(X = x)$	0.0312	0.1563	0.3125	0.3125	0.1563	0.0312

- What is the probability that one or two women are selected for interview?

Job interview

	$X=0$	$X=1$	$X=2$	$X=3$	$X=4$	$X=5$
$\mathbb{P}(X=x)$	0.0312	0.1563	0.3125	0.3125	0.1563	0.0312

$$\begin{aligned}
&\mathbb{P}(\text{one or two women })\\
=\ &\mathbb{P}(X=1 \text{ or } X=2)\\
=\ &\mathbb{P}(X=1)+\mathbb{P}(X=2)=0.1563+0.3125=0.4688
\end{aligned}$$

Job interview

	$X=0$	$X=1$	$X=2$	$X=3$	$X=4$	$X=5$
$\mathbb{P}(X=x)$	0.0312	0.1563	0.3125	0.3125	0.1563	0.0312

- What is the probability that *at least 2* women are selected?

$$\begin{aligned}\mathbb{P}(X\geq 2) &= \mathbb{P}(X=2)+\mathbb{P}(X=3)+\mathbb{P}(X=4)+\mathbb{P}(X=5)\\ &= 0.3125+0.3125+0.1563+0.0312=0.8125.\end{aligned}$$

Turn on your clickers!

	$X=0$	$X=1$	$X=2$	$X=3$	$X=4$	$X=5$
$\mathbb{P}(X=x)$	0.0312	0.1563	0.3125	0.3125	0.1563	0.0312

- What is the probability that *at most* two women are selected for interview?

(a) 0.1875

(b) 0.5

(c) 0.8125

Turn on your clickers!

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$\mathbb{P}(X=x)$	0.0312	0.1563	0.3125	0.3125	0.1563	0.0312

- What is the probability that *at most* two women are selected for interview?

$$\begin{aligned}\mathbb{P}(X \leq 2) &= \mathbb{P}(X=0) + \mathbb{P}(X=1) + \mathbb{P}(X=2) \\ &= 0.0312 + 0.1563 + 0.3125 = 0.5\end{aligned}$$

Sigma Notation

A short hand for writing long sums

$$\sum_{k=1}^{5} k = 1 + 2 + 3 + 4 + 5$$

$$\begin{aligned}\sum_{k=1}^{5} k^2 &= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 \\ &= 1 + 4 + 9 + 16 + 25\end{aligned}$$

$$\begin{aligned}\sum_{0 \le k \le 3} (k-1)^2 &= (0-1)^2 + (1-1)^2 + (2-1)^2 + (3-1)^2 \\ &= 1 + 0 + 1 + 4\end{aligned}$$

Fundamental probability formula

- X = random variable
- A = a set of possible values of X (an event)

$$P(X \text{ takes a value in } A) = \sum_{\text{for all } k \text{ that are in } A} \mathbb{P}(X = k).$$

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k are the outcomes

probabilities from the pdf

Job Interview Example

Meaning	Probability	Σ notation	Compute
at least 3	$\mathbb{P}(X \geq 3)$	$\sum_{k \geq 3} \mathbb{P}(X = k)$	$\mathbb{P}(X = 3) + \mathbb{P}(X = 4) + \mathbb{P}(X = 5)$
at most 3	$\mathbb{P}(X \leq 3)$	$\sum_{k \leq 3} \mathbb{P}(X = k)$	$\mathbb{P}(X = 0) + \mathbb{P}(X = 1) + \mathbb{P}(X = 2) + \mathbb{P}(X = 3)$
less than 3	$\mathbb{P}(X < 3)$	$\sum_{k < 3} \mathbb{P}(X = k)$	$\mathbb{P}(X = 0) + \mathbb{P}(X = 1) + \mathbb{P}(X = 2)$
more than 3	$\mathbb{P}(X > 3)$	$\sum_{k > 3} \mathbb{P}(X = k)$	$\mathbb{P}(X = 4) + \mathbb{P}(X = 5)$

Turn on your clickers!

A family has 4 children.

$$X = \text{number of girls}$$

What is the probability that this family has *at most three* girls, if

	$X = 0$	$X = 1$	$X = 2$	$X = 3$	$X = 4$
$\mathbb{P}(X = x)$	0.0625	0.25	0.375	0.25	0.0625

(a) 0.3125 , (b) 0.0625, (c) 0.6875, (d) 0.9375

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Example

	$X=0$	$X=1$	$X=2$	$X=3$	$X=4$
$\mathbb{P}(X=x)$	0.0625	0.25	0.375	0.25	0.0625

We want to compute the following probability:

$$
\begin{aligned}
& \mathbb{P}(\text{ at most 3 girls }) = \mathbb{P}(X \leq 3) \\
= \quad & \mathbb{P}(X=0) + \mathbb{P}(X=1) + \mathbb{P}(X=2) + \mathbb{P}(X=3) \\
= \quad & 0.0625 + 0.25 + 0.375 + 0.25 = \mathbf{0.9375}.
\end{aligned}
$$

Example

	$X = 0$	$X = 1$	$X = 2$	$X = 3$	$X = 4$
$\mathbb{P}(X = x)$	0.0625	0.25	0.375	0.25	0.0625

Another way to compute this probability is through the complement rule:

$$\begin{aligned} \mathbb{P}(X \leq 3) &= 1 - \mathbb{P}(X > 3) \\ &= 1 - \mathbb{P}(X = 4) \\ &= 1 - 0.0625 = \mathbf{0.9375}. \end{aligned}$$

Expected Value

$$\mathbb{E}(X) = \sum_{\text{all } k} k \, \mathbb{P}(X = k)$$

Interpretation

1. A probability-weighted average.
2. Long-run average of X.
3. The fair value of a gamble.
4. The balance point for a probability histogram/bargraph.

How to compute $\mathbb{E}(X)$

k	2	5
$\mathbb{P}(X = k)$	1/3	2/3

$$
\begin{aligned}
\mathbb{E}(X) &= 2 \cdot \mathbb{P}(X = 2) + 5 \cdot \mathbb{P}(X = 5) \\
&= 2\,(1/3) + 5\,(2/3) \\
&= 2/3 + 10/3 = 12/3 = 4
\end{aligned}
$$

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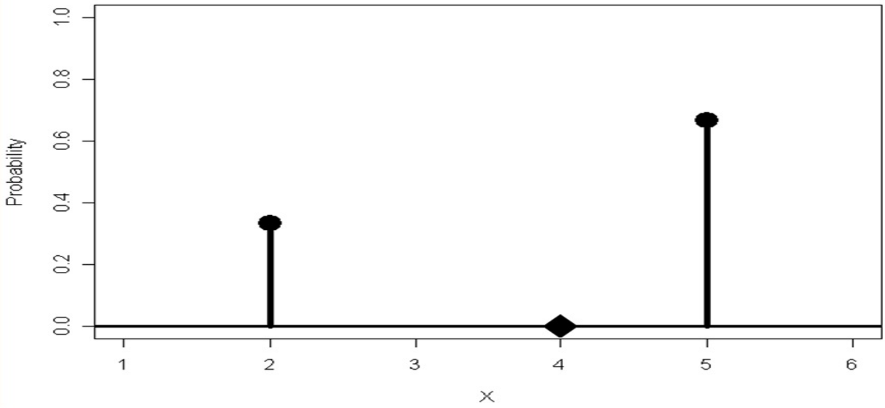

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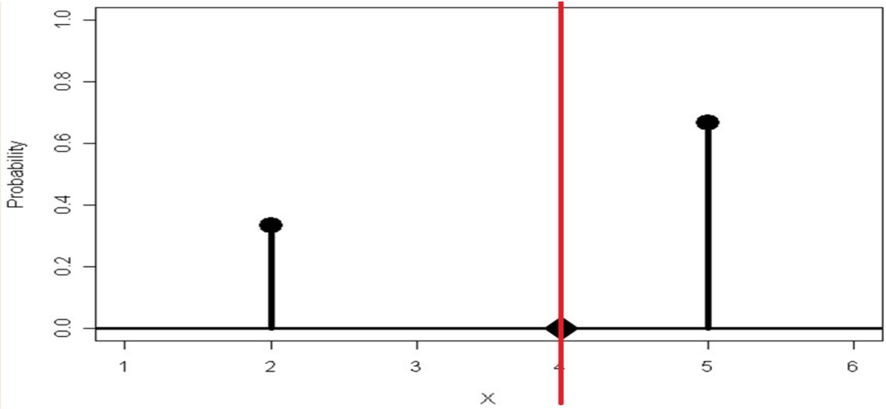

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