Problem Set 5

1 Blowing Strömgren Bubbles

Consider a lone O star emitting η Lyman limit photons per second. It sits inside hydrogen gas of infinite extent and of number density n. The star ionizes an HII region–a.k.a. a "Strömgren sphere," after Bengt Strömgren, who understood that such spheres have sharp boundariesin which hydrogen is nearly completely ionized.

Every Lyman limit photon goes towards ionizing a neutral hydrogen atom. That is, every photon emitted by the star goes towards maintaining the Strömgren bubble. Put yet another way, no Lyman limit photon emitted by the star travels past the radius of the Strömgren sphere.

The rate at which Lyman limit photons are emitted by the central star equals the rate of radiative recombinations in the ionized gas. The sphere is nearly completely ionized[1]. These facts of photo-ionization equilibrium determine the approximate radius of the Strömgren sphere.

The temperature inside the sphere is about 10000 K. (A class on the interstellar medium can show you why.)

1.1 OPTIONAL

Derive a symbolic expression for the radius of the Strömgren sphere using the above variables and whatever variables were introduced in lecture.

1.2 What is the timescale, t_{rec}, over which a free proton radiatively recombines in the sphere? That is, how long would a free proton have to wait before undergoing a radiative recombination? Give both a symbolic expression, and a numerical evaluation for $n = 1$ cm^{-3}.

1.3 If the star were initially "off," and the gas surrounding it initially neutral, what is the timescale for the Strömgren sphere to develop after the star were turned "on"? That is, how long does the star take to blow an ionized bubble? Think simply and to order-of-magnitude; you should get the same answer as 1.2.

2 Time to Relax in the Strömgren Sphere

It is often assumed that velocity distributions of particles are Maxwellian. The validity of this assumption rests on the ability of particles to collide elastically with one another and share their kinetic energy. For a Maxwellian to be appropriate, the timescale for a collision must be short compared to other timescales of interest. This problem tests these assumptions for the case of the Strömgren sphere of nearly completely ionized hydrogen of problem 1.

2.1 Establishing the electron (kinetic) temperature: what is the timescale, t_e, for free electrons in the Strömgren sphere to collide with one another? Consider collisions occurring at relative velocities typical of those in an electron gas at temperature T_e. Work only to order-of-magnitude and express your answer in terms of n, T_e, and other fundamental constants.

I think it is fair to say that this problem is done most easily in cgs units.

[1] The sphere cannot be 100% ionized because then there would be no neutrals to absorb the Lyman limit photons that are continuously streaming out of the star.

2.2 Establishing the proton (kinetic) temperature: repeat 2.1, but for protons, and consider collisions at relative velocities typical of those in a proton gas of temperature T_p. Call the proton relaxation time tp.

2.3 Establishing a common (kinetic) temperature: suppose that initially, $T_e > T_p$. What is the timescale over which electrons and protons equilibrate to a common kinetic temperature? This is not merely the timescale for a proton to collide with an electron. You must consider also the amount of energy exchanged between an electron and proton during each encounter. Estimate, to order-of-magnitude, the time it takes a cold proton to acquire the same kinetic energy as a hot electron. Call this time t_{ep}. Again, express your answer symbolically.

Hint: you might find it helpful to switch the charge on the electron and consider head-on collisions between the positive electron and positive proton.

2.4 Numerically evaluate t_e/t_{rec}, t_p/t_{rec}, and t_{ep}/t_{rec}, for $T_e \sim T_p \sim 10^4$ K (but $T_e \neq T_p$ so that $t_{ep} \neq 0$). Is assuming a Maxwellian distribution of velocities at a common temperature for both electrons and ions a good approximation in Strömgren spheres?

3 Pulsar Dispersion Measure

A radio astronomer uses the Green Bank Telescope to observe a pulsar in a band from 1 to 2 GHz, recording a frequency power spectrum about every 10 ms. The resulting (noisy) frequency spectrum is recorded as pulsar.dat (available on the course GitHub repo). It looks like noise, but I promise you that, for the right dispersion measure, there is a pulse in there.

3.1 De-disperse the measured power spectra to determine the DM of the pulsar. You may have to try many DMs to find the one that works. To help you, the beginning of the file is timed to the arrival time without dispersion delay. For bonus points, you could make your code not depend on this fact.

Just a warning: be careful about the difference between ω and ν. The reported frequencies in the data file are ν, of course.

3.2 For an assumed density of electrons in the interstellar medium of 0.03 cm^{-3}, calculate the distance to this pulsar. Does this seem reasonable (what scale would you use to judge "reasonable")?

3.3 For such an assumed electron density, is the radio astronomer safely observing above the plasma cut-off frequency?

3.4 In an earlier problem set (the Eddington limit), we made use of the Thomson cross-section, $\sigma_T = 6.65 \cdot 10^{-25}$ cm^2, for a photon scattering off an electron. We will discuss Thomson scattering in more detail later, but for now, calculate the optical depth to Thomson scattering along the line-of-sight to this pulsar.

4 Faraday Rotation

Consider the propagation of light thorugh a magnetized plasma. The magnetic field is uniform: $\vec{B}_0 = B_0\hat{z}$. Light travels parallel to $\hat{z}$.

An electron in the plasma feels a force from the electromagnetic wave and a force from the externally imposed B-field. Its equation of motion reads

$$m\dot{\vec{v}} = -e\vec{E} - \frac{e}{c}\vec{v} \times \vec{B}_0 \tag{1}$$

where the electric field $\vec{E}$ can be decomp[osed into right-circularly polarized (RCP) and left-circularly-polarized (LCP) waves:

$$\vec{E} = E_0(\hat{x} \mp i\hat{y})e^{i(k_\mp z - \omega t)} \tag{2}$$

where it is understood that the real part of $\vec{E}$ should be taken. The upper sign $(-)$ corresponds to RCP waves, while the lower sign $(+)$ corresponds to LCP waves.

In the equation of motion above, we have neglected the Lorentz force from the wave's B-field, since it is small (by v/c) compared to the force from the wave's E-field.

4.1 Prove that the solution of the equation of motion reads

$$\vec{v} = \frac{-ie}{m(\omega \pm \omega_{cyc})}\vec{E} \tag{3}$$

where $\omega_{cyc} \equiv eB_0/mc$. This is the hardest part of the derivation for the dispersion relation of RCP and LCP waves.

4.2 Based on Rybicki & Lightman Problem 8.3.

The signal from a pulsed, polarized source is measured to have an arrival time delay that varies with frequency as $dt_p/d\omega = 1.1 \cdot 10^{-5}$ s^2, and a Faraday rotation that varies with frequency as $d\Delta\theta/d\omega = 1.9 \cdot 10^{-4}$ s. The measurements are made around the frequency $\omega = 10^8$ s^{-1}, and the source is at unknown distance from the earth. Find the mean magnetic field, $\langle B_\parallel \rangle$, in the interstellar space between the earth and the source:

$$\langle B_\parallel \rangle \equiv \frac{\int nB_\parallel \, ds}{\int n \, ds} \tag{4}$$