Problem Set 7

1 Saha and the Redshift of Recombination

Based on Rybicki & Lightman 9.4.
The thermal de Broglie wavelength of electrons at temperature T is defined by $\lambda = h/(2\pi mkT)^{1/2}$. The degree of degeneracy of the electrions can be measured by the number of electrons in a cube λ on a side:

$$\xi \equiv n_e \lambda^3 \approx 4.1 \cdot 10^{-16}\ n_e T^{-\frac{3}{2}} \tag{1}$$

For many cases of physical interest, the electrons are very non-degenerate, with the quantity $\gamma \equiv \ln \xi^{-1}$ being of order 10 to 30. We want to investigate the consequences of the Boltzmann and Saha equations of γ being large and only weakly dependent on temperature. For the present purposes, assume that the partition functions are independent of temperature and of order unity.

1.1 Show that the value of temperature at which the stage of ionization passes from j to $j+1$ (i.e. when the $j+1$th electron is knocked off) is given approximately by

$$kT \sim \frac{\chi}{\gamma} \tag{2}$$

where χ is the ionization potential between stages j and $j+1$. Therefore, this temperature is much smaller than the ionization potential expressed in temperature units.

1.2 The rapidity with which the ionization stage changes is measured by the temperature range ΔT over which the ration of populations N_j/N_{j+1} changes substantially. Show that

$$\frac{\Delta T}{T} \sim \left[\frac{d\mathrm{log}(N_{j+1}/N_j)}{d\mathrm{logT}}\right]^{-1} \sim \gamma^{-1} \tag{3}$$

Therefore, ΔT is much smaller than T itself, and the change occurs rapidly.

1.3 Using the Boltzmann equation and the result from §1.1, show that when γ is large, an atom or ion stays mostly in its ground state before being ionized.

2 Blowing Strömgren Bubbles

Consider a lone O star emitting η Lyman limit photons per second. It sits inside hydrogen gas of infinite extent and of number density n. The star ionizes an HII regiona.k.a. a "Strömgren sphere," after Bengt Strömgren, who understood that such spheres have sharp boundariesin which hydrogen is nearly completely ionized.
Every Lyman limit photon goes towards ionizing a neutral hydrogen atom. That is, every photon emitted by the star goes towards maintaining the Strömgren bubble. Put yet another way, no Lyman limit photon emitted by the star travels past the radius of the Strömgren sphere.
The rate at which Lyman limit photons are emitted by the central star equals the rate of radiative recombinations in the ionized gas. The sphere is nearly completely ionized[1]. These facts of photo-ionization equilibrium determine the approximate radius of the Strömgren sphere.
The temperature inside the sphere is about 10000 K. (A class on the interstellar medium can show you why.)

[1]The sphere cannot be 100% ionized because then there would be no neutrals to absorb the Lyman limit photons that are continuously streaming out of the star.

2.1 Derive a symbolic expression for the radius of the Strömgren sphere using the above variables and whatever variables were introduced in lecture.

2.2 What is the timescale, t_{rec}, over which a free proton radiatively recombines in the sphere? That is, how long would a free proton have to wait before undergoing a radiative recombination? Give both a symbolic expression, and a numerical evaluation for $n = 1$ cm^{-3}.

2.3 If the star were initially "off," and the gas surrounding it initially neutral, what is the timescale for the Strömgren sphere to develop after the star were turned "on"? That is, how long does the star take to blow an ionized bubble? Think simply and to order-of-magnitude; you should get the same answer as 2.2.

3 Time to Relax in the Strömgren Sphere

It is often assumed that velocity distributions of particles are Maxwellian. The validity of this assumption rests on the ability of particles to collide elastically with one another and share their kinetic energy. For a Maxwellian to be appropriate, the timescale for a collision must be short compared to other timescales of interest. This problem tests these assumptions for the case of the Strömgren sphere of nearly completely ionized hydrogen of problem 2.

3.1 Establishing the electron (kinetic) temperature: what is the timescale, t_e, for free electrons in the Strömgren sphere to collide with one another? Consider collisions occurring at relative velocities typical of those in an electron gas at temperature T_e. Work only to order-of-magnitude and express your answer in terms of n, T_e, and other fundamental constants.

3.2 Establishing the proton (kinetic) temperature: repeat 3.1, but for protons, and consider collisions at relative velocities typical of those in a proton gas of temperature T_p. Call the proton relaxation time tp.

3.3 Establishing a common (kinetic) temperature: suppose that initially, $T_e > T_p$. What is the timescale over which electrons and protons equilibrate to a common kinetic temperature? This is not merely the timescale for a proton to collide with an electron. You must consider also the amount of energy exchanged between an electron and proton during each encounter. Estimate, to order-of-magnitude, the time it takes a cold proton to acquire the same kinetic energy as a hot electron. Call this time t_{ep}. Again, express your answer symbolically.
Hint: you might find it helpful to switch the charge on the electron and consider head-on collisions between the positive electron and positive proton.

3.4 Numerically evaluate t_e/t_{rec}, t_p/t_{rec}, and t_{ep}/t_{rec}, for $T_e \sim T_p \sim 10^4$ K (but $T_e \neq T_p$ so that $t_{ep} \neq 0$). Is assuming a Maxwellian distribution of velocities at a common temperature for both electrons and ions a good approximation in Strömgren spheres?