Problem Set 7

1 Great Balls of (Relativistic) Fire

Based on Rybicki & Lightman 4.1.

In astronomy, it is frequently argued that a source of radiation that undergoes a fluctuation of duration Δt must have a physical diameter of order $D \lesssim c\Delta t$. This argument is based on the fact that even if all portions of the source undergo a disturbance at the same instant and for an infinitesimal period of time, the resulting signal at the observer will be smeared out over a time interval of $\Delta t_{min} \sim D/c$ because of the finite light travel time across the source. Suppose, however, that the source is an optically thick spherical shell of radius $R(t)$ that is expanding with relativistic velocity $\beta \sim 1, \gamma \gg 1$ and energized by a stationary point at its center. By consideration of relativistic beaming effects, show that if the observer sees a fluctuation from the shell of duration Δt at time t, the source may actually be of radius

$$R < 2\gamma^2 c\Delta t \tag{1}$$

rather than the much smaller limit given by the nonrelativistic considerations. In the rest frame of the shell surface, each surface element may be treated as an isotropic emitter.

This latter argument has been used to show that the active regions in quasars may be much larger than $c\Delta t \sim 1$ light month across, and thus avoids much energy being crammed into so small a volume.

2 The Blob

Based on Rybicki & Lightman 4.7.

An object emits a blob of material at speed v at an angle θ to the line-of-sight of a distant observer.

2.1 Show that the apparent transverse velocity inferred by the observer (i.e. the angular velocity on the sky times the distance to the object) is

$$v_{app} = \frac{v \sin\theta}{1 - (v/c)\cos\theta} \tag{2}$$

2.2 Show that v_{app} can exceed c; find the angle for which v_{app} is maximum, and show that this maximum is $v_{max} = \gamma v$.

2.3 Plot v_{app}/c versus θ for $\gamma = 10^2$. Does the viewing angle θ need to be especially small for superluminal motion to be perceived?

3 Powering Radio Lobes

The galaxy Cygnus A is one of the most powerful radio sources in the sky. Striking radio images (Figure 1) reveal a pair of immense lobes of emission, which sit about ~100 kpc outside the central galaxy. It is thought that these lobes are powered by a super-massive black hole (SMBH) at the galactic center. The SMBH somehow produces narrow, bipolar, relativistic jets, which propagate

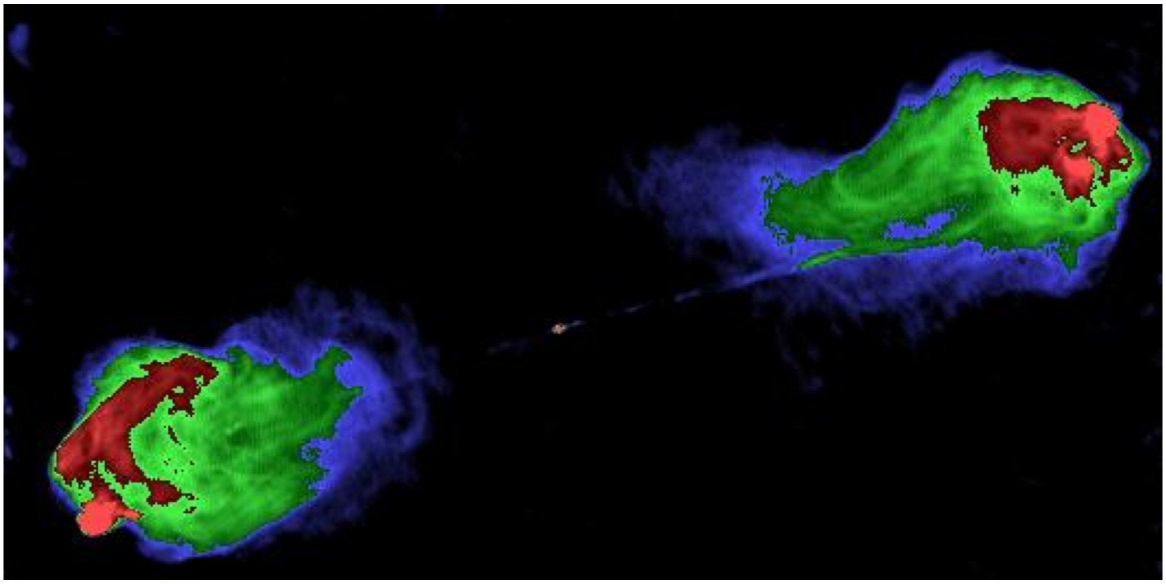

Figure 1: Radio image of Cygnus A, showing the extended lobes powered by narrow jets from the central supermassive black hole. The lobes are about 30 kpc in radius, and the emission is strongest at the front regions which are presumably interacting most strongly with the ambient medium.

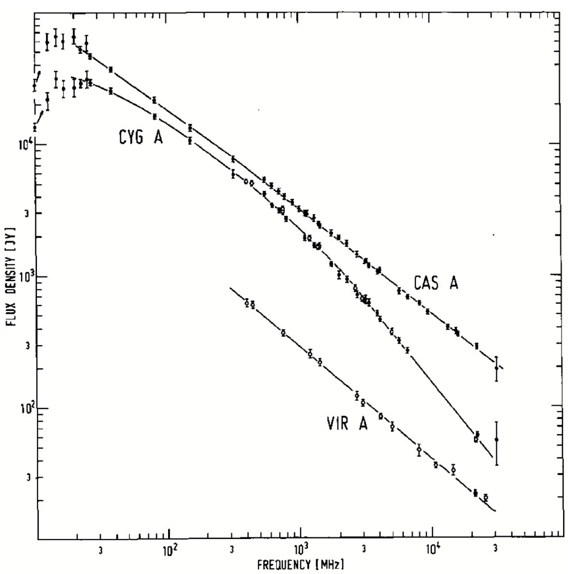

Figure 2: Radio spectrum of Cygnus A (and other objects) from Baars et al. (1977)

to the outskirts of the galaxy, interact with the ambient medium, and form shocks. The radio luminosity is the result of synchrotron emission from relativistic electrons produced in those shocks.

Figure 2 shows the observed radio spectrum of Cyg A, which resembles what we might expect from synchrotron emission — broad-band, roughly power-law, clearly non-thermal. The observed flux increases down to a frequency of at least 10 MHz, where the value is $F_\nu \approx 10^4 Jy$. In this region, one can reasonably fit a power law to the spectrum and write the specific luminosity

$$L_\nu = 4\pi d^2 F_\nu = 5 \times 10^{36} \left(\frac{\nu}{10\ \text{MHz}}\right)^{-0.8} \text{erg s}^{-1}\text{Hz}^{-1} \tag{3}$$

where we have taken the distance to Cyg A to be $d \approx 230$ Mpc. Integrating over equation 3 over the observed range 10^7 to 10^{11} Hz gives a luminosity of $L \sim 10^{-45}$erg s^{-1} (about 1000 times the energy radiated by a supernova at peak).

What are the energetics and magnetic field strengths involved in these luminous radio lobes, and can a SMBH really provide the necessary oomph? The total energy[1] in the lobes can be written

$$E = (U_e + U_B) \times 2V \tag{4}$$

where U_e,U_B are the electron and magnetic field energy densities, respectively, and V is the volume of a lobe[2] As it turns out, we do not have enough information to directly determine these energy densities, but we can follow the famous arguments of Geoffery Burbidge to derive a lower limit. The implied energies, we will find, are massive — the equivalent of a billion supernova explosions or more. As usual, well take the electron number density to be described by a power law distribution in Lorentz factor

$$n\ d\gamma = C\gamma^p\ d\gamma \tag{5}$$

where C is some constant, and the power law cuts off at some lower value, $\gamma_{\min}$.

3.1 Infer the value of p from the observations.

3.2 Integrate equation 5 to get an expression for the energy density of electrons, U_e, in terms of C and $\gamma_{\min}$.

What should we use for $\gamma_{\min}$ and C? It looks like the observed spectrum might be peaking at $v_m \approx 10$ MHz. We can therefore associate[3] 10 MHz with the critical frequency of electrons with Lorentz factor $\gamma_{\min}$. To determine the constant C, we can use the observed specific luminosity at the frequency $\nu_m = 10$ MHz. From synchrotron theory, we know that the specific luminosity (ergs s^{-1} Hz^{-1}) from a power-law distribution of electrons is

$$L_\nu \approx \frac{2C}{3}\frac{U_B \sigma_T c}{\nu_{cyc}}\left(\frac{\nu}{\nu_{cyc}}\right)^{\frac{1+p}{2}} \times V \tag{6}$$

3.3 Show that the energy density of electrons can be written

$$U_e = A\frac{L_m \nu_m^{\frac{1}{2}}}{V} B^{-\frac{3}{2}} \tag{7}$$

where L_m is the observed specific luminosity at ν_m, and A is some combination of numerical factors and fundamental constants.

To proceed, we need to know the magnetic field strength. Unfortunately, there is no easy way to measure this directly. What is typically done instead is to make a minimum energy argument. The magnetic energy density increases with B, whereas equation 7 shows that U_e grows as B decreases. Thus, there must be some value of B that minimizes the total energy. This value will at least give us a lower limit to the necessary energetics.

[1] We are going to ignore the energy in protons, since there is no easy way to measure them directly. But presumably there is as much or more energy in protons than electrons

[2] The lobes each have a radius of around 30 kpc. We have included a factor of 2 in the equation since there are two of them.

[3] This may or may not be correct; the data are getting a little sketchy around 10 MHz, owing to the growing opacity of the ionosphere at these frequencies. It could also be that the turnover is real, but due to synchrotron self-absorption, not the minimum Lorentz factor. If the ν_m is, in fact, smaller than 10 MHz, this would imply an even smaller $\gamma_{\min}$, and hence even more energy. Thus, what follows is at least a lower limit to the energy.

3.4 Show that the total energy is minimized when $U_B = \frac{3}{4}U_e$ — that is, when U_e and U_b are roughly equal. For this reason, the minimum energy argument is often also called an equipartition argument.

3.5 Adopting the equipartition argument above, what is (numerically) the magnetic field strength in the emitting regions? What is the minimum Lorentz factor, γ_{min}?

3.6 What is the total energy in the radio lobes? How does this compare to the typical energy of a supernova?

3.7 How much mass mass would a black hole have to eat in order to create this energy? Assume that the rest mass energy of the accreted material is processed into jet energy with 10% efficiency. Does the SMBH hypothesis hold together? — i.e., is your estimate for the swallowed mass consistent with SMBH masses?

Comment: We have provided no physical justification for why this system should generate magnetic fields with $U_B \approx U_e$. We have only shown that equipartiation is the optimal configuration for radiating efficiently. However, if the magnetic field is produced by turbulent motions in the plasma, which also play a role in accelerating the high energy particles, perhaps there is some rationale for thinking that we may reach something close to equipartition. This conjecture can now be tested by detailed particle-in-cell simulations, which follow the motions of individual particles in a plasma while simultaneously solving Maxwell's equations to determine the fields they generate. In the absence of any better information, people often simply assume an equipartition B-field in order to carry out synchrotron analyses. Note that we have also neglected here the energy in relativistic protons, which could exceed that in electrons and magnetic fields.