Problem Set 4 (part 1)

1 Eddington Limit

Based on Rybicki & Lightman, Problem 1.4

1.1 Derive the condition for an optically thin cloud to be ejected by radiation pressure from a luminous object of mass M and luminosity L:

$$M/L < \kappa/4\pi Gc \tag{1}$$

Remember that κ is the mass absorption coefficient (of the cloud).

1.2 Calculate the terminal velocity v of the cloud, based on radiation and gravity alone, if it starts at rest at a distance R from the object. Show that

$$v^2 = \frac{2GM}{R}\left(\frac{\kappa L}{4\pi GMc} - 1\right). \tag{2}$$

1.3 One can estimate the (minimum) value of κ for hydrogen from Thomson scattering off free elections when the hydrogen is ionized. The Thomson cross-section is $\sigma_T = 6.6 \cdot 10^{-25}$ cm^2. The mass scattering coefficient is therefore $> \sigma_T/m_{\mathrm{H}}$, where m_{H} is the mass of a hydrogen atom. Show that the maximum luminosity that a central mass M can have and not spontaneously eject hydrogen by radiation pressure is

$$\begin{aligned} L_{\mathrm{EDD}} &= 4\pi GMcm_{\mathrm{H}}/\sigma_T \\ &= 1.25 \cdot 10^{38} \text{ erg s}^{-1}(M/M_\odot), \end{aligned} \tag{3}$$

where $M_\odot$ is the mass of the Sun. This is called the *Eddington Limit*.

2 Hyperfine Emission from Neutral Hydrogen

Neutral hydrogen in the electronic ground state can be in one of two hyperfine states. Denote the number density of atoms in the ground hyperfine level (singlet state) as n_0, and the number density of atoms in the excited hyperfine level (triplet state) as n_1. DEFINE the excitation temperature, T_{ex}, of the transition as

$$\frac{n_1}{n_0} = \frac{g_1}{g_0} e^{-h\nu/kT_{\mathrm{ex}}}. \tag{4}$$

Here, $h\nu = hc/\lambda$ is the mean energy difference between the levels, and $g_0 = 1$ and $g_1 = 3$ are the statistical weights of the levels. The excitation temperature is merely another way of expressing the ratio of ground state to excited state populations. By definition, if a gas is in local thermodynamic equilibrium (LTE) at some gas kinetic temperature T, then $T_{\mathrm{ex}} = T$; the level populations are distributed in Boltzmann fashion at the local temperature T. Some people refer to the excitation temperature for the $\lambda = 21$ cm transition as the *spin temperature*. But use of the term "excitation temperature" is general to any line transition; it is simply a measure of how excited an atom is.

2.1 Define $T_* = h\nu/k$ and compute its value. It is likely that $T_{\mathrm{ex}} \gg T_*$. For the remainder of this problem, work in the $T_*/T_{\mathrm{ex}} \ll 1$ limit.

2.2 Write down the absorption coefficient, α_ν (units of per length), for this transition. Express your answer in terms of $\phi(\nu)$(the line profile function), A_{21} (Einstein A coefficient), λ, whatever densities you need, and $T_*/T_{\rm ex}$. Do not forget the correction for stimulated emission.

2.3 Write down the volume emissivity, j_ν (units of erg s^{-1} cm^{-3} Hz^{-1} sr^{-1}), for this transition. Use whatever quantities defined above that you need.

2.4 Write down the source function, S_ν, for this transition.

2.5 Write down the specific intensity, I_ν, of a cloud of HI that is optically thin along the line-of-sight (l-o-s). Take the l-o-s dimension of the cloud to be L, and give the answer only to leading order in $\tau \ll 1$, where τ is the optical depth at an arbitrary wavelength.
Does your answer depend on $T_{\rm ex}$? If someone gives you a spectrum of the 21-cm line that appears in emission and tells you that the line was emitted from an optically thin cloud, what physical quantities can you infer from the spectrum?

2.6 Write down the optical depth of the cloud. Does your answer depend on $T_{\rm ex}$?

2.7 How large would L have to be for the cloud to be marginally optically thick? Use a gas density of $n = 1$ cm^{-3}, a gas temperature of $T = 100$ K, and an excitation temperature $T_{\rm ex} = T$. Assume the line is only thermally broadened.