Problem Set 5 (part 1)

1 Multiple Multipoles

1.1 Write down the electric field a distance r away from a monopole of charge q.

1.2 Someone moves another monopole of charge q next to the original monopole. The two charges are separated by a distance b. Derive, to order-of-magnitude, the factor by which this dipole electric field is reduced from the monopole field.

1.3 Someone moves two more charges, $-q$ and q, into position to form a square. The edge of the square has length b. Going around the square, the charges are $-q$, q, $-q$, and q. Derive, to order-of-magnitude, the factor by which this quadrupole electric field is reduced from the dipole field.

1.4 Draw a picture of a pure electric octopole, that is, a charge distribution for which the electric field decreases as $1/r^5$ (and no less gradually). For bonus points, draw a picture of an electric hexadecapole (electric field dies no less gradually than $1/r^6$).

1.5 Now imagine the dipole and quadrupole configurations rotating about their centers-of-charge with frequency ν. We have a rotating barbell and a rotating square, respectively. (Think CO and H_2).
To order-of-magnitude, what is the maximum distance from each object inside of which the electric fields are nearly perfectly in phase with the rotation? This is the boundary of the near zone, inside of which the electric field geometry rotates with frequency ν as if it were a rigid body.

1.6 Electromagnetic waves are emitted at the boundary of the near zone, into the far (radiation) zone.
Write down (one line of argument suffices) the factor by which the power carried by waves emitted by the rotating quadrupole is smaller than the power carried by waves emitted by the rotating dipole.

2 Rotating Magnetic Dipole

Based on Rybicki & Lightman Problem 3.1.
A pulsar is convetionally believed to be a rotating neutron star. Such a astar is likely to have a strong magnetic field, B_0, since it traps lines of force during its collapse. If the magnetic axis of the neutron star does not line up with the rotation axis, there will be magnetic dipole radiation from the time-changing magnetic dipole, $m(t)$. Assume that the mass and radius of the neutron star ar M and R, respectively; that the angle between the magnetic and rotation axes is α; and that the rotational angular velocity is ω.

2.1 Find an expression for the radiated power P in terms of ω, R, B_0, and α.

2.2 Assuming that the rotational energy of the pulsar is the ultimate source of the radiated power, find an expression for the slow-down timescale, τ, of the pulsar.

2.3 For $M = 1M_\odot$, $R = 10^6$ cm, $B_0 = 10^{12}$ gauss, $\alpha = 90°$, find P and τ for $\omega = 10^4$ s^{-1}, 10^3 s^{-1}, 10^2 s^{-1}. The highest rate $\omega = 10^4$ s^{-1} is believed to be typical of newly formed pulsars.
Hint: The Larmor power formula gives the power radiated by an accelerating electric dipole moment. This problem is similar except that it concerns an accelerating magnetic dipole moment.

Problem Set 5 (part 2)

3 21 (Flavors of) Temperatures for 21cm Radiation

This problem revisits the famous hyperfine transition in neutral hydrogen. Here we try to understand what sets the excitation (spin) temperature, $T_{\rm ex}$. (In a previous problem set, we merely contented ourselves with the statement that $T_{\rm ex} \gg T_*$. While we're at it, we learn more astronomer's jargon.
In general, the excitation temperature of a transition is influenced by two factors: (1) the radiation field, and (2) collisions with surrounding species.
The radiation field can either excite the atom through photon absorption, or de-excite through stimulated emission. Measure the strength of the ambient radiation field at 21 cm by $\bar{J}$, the mean (i.e., angle-averaged) intensity integrated over the hyperfine line ν profile (recall Rybicki & Lightman chapter 1).
As for collisions, consider here exciting and de-exciting collisions with fellow neutral hydrogen atoms; Purcell and Field (1957, hereafter PF) conclude that collisions between a given electronic-ground-state H atom and other electronic-ground-state H atoms are most important in the predominantly neutral HI clouds of the ISM. (Electrons are ∼42 times faster and tend to dominate the excitation dynamics in other situations, but assume here that there are too few of them in these cold clouds.) Denote the collisional excitation rate coefficient by q_{12}, and the collision de-excitation rate coefficient by q_{21} Assume for this problem that both hyperfine-excited and hyperfine- ground atoms can excite or de-excite the hyperfine level in an atom. [1]

3.1 Write down the equation of global (not detailed!) balance for this transition. That is, write down the statement that the rate of excitations (from all possible channels) per volume per time equals the rate of de-excitations (from all possible channels) per volume per time.
Use only the following variables: n_1 and n_2 are the number densities of atoms in the ground and excited states, respectively, $n = n_1 + n_2$, $T_{\rm K}$ is the kinetic temperature of the atoms that move according to a Maxwellian, q_{12}, any Einstein coefficients you want, $\bar{J}$,and the statistical weights g_1 and g_2 of the ground and excited states, respectively.
What you have written down is an equation for the excitation temperature ($T_{\rm ex} \leftrightarrow n_1/n_2$) in terms of the radiation field and the rate of collisions. Regard the latter two as given throughout this problem.

3.2 DEFINE a "radiation temperature," $T_{\rm R}$, from $\bar{J}$ as

$$\bar{J}_\nu \equiv B_\nu(T_{\rm R}) \tag{1}$$

[1] In truth, excitations and de-excitations proceed, as PF describe, by "spin-exchange" collisions, in which an electron with a certain spin in one H atom swaps places with an electron having a different spin coming from the colliding H atom. A given atom can swap its way up to the hyperfine excited state, or swap its way down to the hyperfine ground state. Here we follow PF and place the relative probabilities of undergoing a swap-up versus a swap-down into q_{12} and q_{21}.

Note that we are NOT saying the ambient radiation field is Planckian. We are merely DEFINING a number $T_{\rm R}$ by using Planck's function, B_ν, where $\nu = 1420$ MHz, the frequency of the 21 cm line.
Re-write your equation in the previous section to solve for $T_{\rm ex}$ in terms of the following variables: $T_{\rm K}$, $T_{\rm R}$, $T \equiv h\nu/k$ (recall last problem set), and the dimensionless variable

$$z \equiv \frac{g_1 n q_{12} T_*}{g_2 A_{21} T_{\rm K}} \tag{2}$$

where $A_{21} = 2.85 \cdot 10^{-15}$ s^{-1} is the Einstein decay coefficient. Use the very likely condition that $T_{\rm K}$, $T_{\rm R} \gg T_*$ to rid your equation of all exponentials.
Verify that if $z \gg 1$, $T_{\rm ex} \approx T_{\rm K}$ (collisions beat radiation; the transition is in LTE at $T_{\rm K}$), but that if $z \ll 1$, $T_{\rm ex} \approx T_{\rm R}$ (radiation beats collisions; the transition is not in LTE at $T_{\rm K}$).

3.3 To order of magnitude (actually much better than that), what fraction of HI is in the excited hyperfine state? Recall that $g_1 = 1$ and $g_2 = 3$ and use the very likely condition that $T_{\rm K}$, $T_{\rm R} \gg T_*$.

3.4 Estimate the value for z for an HI cloud at $T_{\rm K} = 100K$, $n = 1\text{cm}^3$. Use Table 1 and equation (9) of PF; note that PF's collision frequency $\nu = n\langle\sigma v\rangle$ is not the same as our line frequency ν ; call PF's $\nu = \nu_{\rm PF}$; then $nq_{12} = 3\nu_{\rm PF}/8$. (For those interested, the 3/8 can be understood easily; skim the first 3 pages of PF and use your answer from the previous section).
Based on your answer, would you expect collisions or radiation to be more important in determining the degree of excitation?

3.5 "Critical densities," n_{crit}, for exciting the line by collisions are defined by setting the rate of spontaneous decays equal to the rate of collisional de-excitations. Show that such a procedure gives

$$n_{crit} = \frac{A_{21}}{q_{21}} \tag{3}$$

and solve for its value for this line at $T_{\rm K} = 100$K.
One can define n_{crit} for any line transition at any temperature; it is a crude gauge of the density of colliders required for collisions to be important in setting the level populations.

3.6 Suppose radio observations are made that spatially resolve emission from a uniform HI cloud that is optically thick to its own 21 cm line radiation. Prove that the observed specific intensity, I_ν, equals

$$I_\nu = \frac{2kT_{\rm ex}}{\lambda^2}. \tag{4}$$