Problem Set 3

1 Detailed Balance

Suppose an atom has 2 energy states E_1 and E_2. This atom occasionally collides with an e^-, whereupon it may change states. Let us say that during a collision, there is a probability of P_{12} that the atom transitions from state 1 to state 2, and P_{21} for the reverse (from 2 to 1).

1.1

If these are the only transitions, what do you expect the relative populations to be of atoms in state 1 versus state 2 after many collisions? Write some code that verifies this by simulating it (and please submit your code with your homework, along with the output of the code). Pick P_{12} and P_{21} to be something illustrative. What are the relative populations in each state? Does this system satisfy detailed balance? Does it matter if $P_{12} = P_{21}$? Why/why not?

2 Hyperfine Emission from Neutral Hydrogen

Neutral hydrogen in the electronic ground state can be in one of two hyperfine states. Denote the number density of atoms in the ground hyperfine level(singlet state) as n_0, and the number density of atoms in the excited hyperfine level (triplet state) as n_1. DEFINE the excitation temperature, $T_{\rm ex}$, of the transition as

$$\frac{n_1}{n_0} = \frac{g_1}{g_0} e^{-h\nu/kT_{\rm ex}}. \tag{1}$$

Here, $h\nu = hc/\lambda$ is the mean energy difference between the levels, and $g_0 = 1$ and $g_1 = 3$ are the statistical weights of the levels. The excitation temperature is merely another way of expressing the ratio of ground state toexcited state populations. By definition, if a gas is in local thermodynamic equilibrium (LTE) at some gas kinetic temperature T, then $T_{\rm ex} = T$; the level populations are distributed in Boltzmann fashion at the local temperature T. Some people refer to the excitation temperature for the $\lambda = 21$ cm transition as the *spin temperature*. But use of the term "excitation temperature" is general to any line transition; it is simply a measure of how excited an atom is.

2.1

Define $T_* = h\nu/k$ and compute its value. It is likely that $T_{\rm ex} \gg T_*$. For the remainder of this problem, work in the $T_*/T_{\rm ex} \ll 1$ limit.

2.2

Write down the absorption coefficient, α_ν (units of per length), for this transition. Express your answer in terms of $\phi(\nu)$(the line profile function), A_{21} (Einstein A coefficient), λ, whatever densities you need, and $T_*/T_{\rm ex}$. Do not forget the correction for stimulated emission.

2.3

Write down the volume emissivity, j_ν (units of erg s^{-1} cm^{-3} Hz^{-1} sr^{-1}), for this transition. Use whatever quantities defined above that you need.

2.4

Write down the source function, S_ν, for this transition.

2.5

Write down the specific intensity, I_ν, of a cloud of HI that is optically thin along the line-of-sight (l-o-s). Take the l-o-s dimension of the cloud to be L, and give the answer only to leading order in $\tau \ll 1$, where τ is the optical depth at an arbitrary wavelength.
Does your answer depend on $T_{\rm ex}$? If someone gives you a spectrum of the 21-cm line that appears in emission and tells you that the line was emitted from an optically thin cloud, what physical quantities can you infer from the spectrum?

2.6

Write down the optical depth of the cloud. Does your answer depend on $T_{\rm ex}$?

2.7

How large would L have to be for the cloud to be marginally optically thick? Use a gas density of $n = 1$ cm^{-3}, a gas temperature of $T = 100$ K, and an excitation temperature $T_{\rm ex} = T$. Assume the line is only thermally broadened.

3 Hyperfine ^{3}He$^+$

Observations of the hyperfine transition in ^{3}He$^+$ are used to probe the ^{3}He$^+/H$ abundance in the galaxy. This abundance reflects the primordial yield from big bang nucleosynthesis and galactic chemical evolution.

3.1

Estimate, using the scaling relations presented in class and whatever facts you remember, the wavelength of the ground-state electronic (i.e. Ly-α equivalent) transition of ^{3}He$^+$.

3.2

Estimate, using the scaling relations presented in class and whatever facts you remember, the wavelength of the ground-state hyperfine transition of ^{3}He$^+$. Compare to the true answer of 3.46 cm.

3.3

Estimate A_{21} for the ground-state electronic (Ly-α-like) transition of ^{3}He$^+$.

4 Rotating Magnetic Dipole

Based on Rybicki & Lightman Problem 3.1.
A pulsar is convetionally believed to be a rotating neutron star. Such a star is likely to have a strong magnetic field, B_0, since it traps lines of force during its collapse. If the magnetic axis of the neutron star does not line up with the rotation axis, there will be magnetic dipole radiation from the time-changing magnetic dipole, $m(t)$. Assume that the mass and radius of the neutron star ar M and R, respectively; that the angle between the magnetic and rotation axes is α; and that the rotational angular velocity is ω.

4.1

Find an expression for the radiated power P in terms of ω, R, B_0, and α.

4.2

Assuming that the rotational energy of the pulsar is the ultimate source of the radiated power, find an expression for the slow-down timescale, τ, of the pulsar.

4.3

For $M = 1M_{\odot}$, $R = 10^6$ cm, $B_0 = 10^{12}$ gauss, $\alpha = 90°$, find P and τ for $\omega = 10^4$ s^{-1}, 10^3 s^{-1}, 10^2 s^{-1}. The highest rate $\omega = 10^4$ s^{-1} is believed to be typical of newly formed pulsars.
Hint: The Larmor power formula gives the power radiated by an accelerating electric dipole moment. This problem is similar except that it concerns an accelerating magnetic dipole moment.