Problem Set 2

1 Blackbody Flux

Derive the blackbody flux formula

$$F = \sigma T^4 \tag{1}$$

from the Planck function for specific intensity,

$$B_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1} \tag{2}$$

You may use the integral $\int_0^\infty \frac{dx}{x^5(e^{1/x}-1)} = \pi^4/15$. Those who know what they have to do can stop reading here. Those who need a bit of help should read on.

Imagine a perfectly flat, blackbody patch at the center of a sphere of radius R. The patch has tiny surface area dA and is lying flat at $r = 0$ in the $\theta = \pi/2$ plane (in spherical coordinates, where r, θ, ϕ are the radius, polar angle, and azimuth). Only one side of the patch is warm; the other side is at absolute zero. Assume that the warm side radiates as a perfect thermal blackbody.

A thermal blackbody emits radiation isotropically — i.e., with no preference for direction. In other words, imagine each point on the patch beaming out a tiny, uniform, hemispherical dome of rays (like an umbrella). It is a hemispherical dome and not a full sphere because only one side of the patch is warm. Note that the Planck blackbody function has no variable inside it that specifies direction of emission; the direction of emission doesn't matter for purely thermal emission. The only variables that need to be specified for the Planck function are T and ν.

Despite the isotropic nature of the emission, because the patch is geometrically flat, has definite area, and emits only on one side of itself, a detector glued to the inside of the sphere at $r = R$ detects different amounts of radiation depending on where it is placed. For example, if the detector is glued on the half of the sphere that cant see the bright side of the patch, the detector detects nothing. If the detector is glued to the sphere directly above the patch, the detector picks up the most number of photons, because it sees the full face-on area of the patch (namely, dA). If the detector is glued at an angle to the pole, then it sees less than area dA. If it is glued in the plane of the patch, it sees nothing but a infinitesimally thin line segment.

This problem asks you to place detectors everywhere on the inside of the sphere and sum up the radiation collected; this operation is equivalent to "integrating the Planck function over all solid angles into which the radiation is beamed."

Calculate the total luminosity (in units of energy/time) emitted by the patch of area dA. You must integrate the Planck function over the entire emitting area (which is maximally dA but in general will be less, depending on the viewing geometry), over all solid angles into which the radiation is beamed, and over all frequencies. Think about placing tiny detectors all over the inside of the sphere and asking how much energy/time each of the detectors receives.

Then divide the luminosity by dA to calculate the total flux (in units of energy/time/area) emitted by the patch. You should recover the usual blackbody flux formula, σT^4. By definition, σT^4 is the total amount of energy radiated per time per unit area of a blackbody surface, radiated into all solid angles and over all frequencies.

2 Flat Disks

This problem forms the foundation for understanding the spectral energy distributions of circumstellar disks, e.g., those surrounding pre-main-sequence stars and AGN.
Consider a perfectly flat, blackbody disk encircling a blackbody star. The star has radius R_* and effective temperature T_*. The disk begins at a stellocentric radius far from the star, i.e. the inner disk radisu $r_i \gg R_*$. The disk extends to infinity.
Calculate the temperature of the disk, $T(r)$, as a function of radius, r. Make whatever approximations you deem necessary in light of the fact that $r_i \gg R_*$. Be sure you get the scaling of T with r correctly. It is less important that you get the numerical coefficient correctly.
This disk is PERFECTLY flat. Do not consider individual particles in the disk.
Also, neglect radial transport of energy. Each annulus is independent of neighboring annuli. This is a fine approximation for thin disks since they transport more radiation vertically than radially (the temperature gradient is steeper vertically than radially).

3 Practice with j_ν, α_ν, S_ν, I_ν

3.1

A plane-parallel slab of uniformly dense gas is known to be in LTE (local thermodynamic equilibrium) at a uniform temperature T. Its thickness normal to its surface is s. Its absorption coefficient is $\alpha_{\nu,\mathrm{gas}}$. Write down the specific intensity, I_ν, viewed normal to the slab, in terms of the variables given.

3.2

The same slab is now filled uniformly with non-emissive dust having absorption coefficient $\alpha_{\nu,\mathrm{dust}}$. The dust is non-emissive, so its emissivity $j_{\nu,\mathrm{dust}} = 0$. Write down I_ν viewed normal to the slab, in terms of all variables given so far.

3.3

The slab of gas and dust is further mixed with a third component: an emissive, non-absorptive uniform medium having emissivity $j_{\nu,\mathrm{med}}$ and absorption coefficient $\alpha_{\nu,\mathrm{med}} = 0$. Write down I_ν viewed normal to the slab, in terms of all variables given.
A physical realization of this problem might be an HII region surrounding an ionizing O star. The material in LTE would be the fully ionized plasma, emitting thermal bremsstrahlung radiation. The dust would be dust. The emissive, non-absorptive medium would be the same ionized plasma emitting recombination (line) radiation. For the assumptions stated in the problem to be valid, we would have to evaluate ν at, say, an optical recombination line like Hα.

4 Bremsstrahlung Spectrum

In deriving the spectrum of bremsstrahlung emission, we made an assumption that, in an individual collision of an electron moving at velocity v with an ion of charge Ze we could relate the impact parameter b to a unique frequency of emission, ν. The relationship we derived was

$$\nu \sim \frac{v}{4b} \tag{3}$$

In this problem, we are going to numerically examine the validity of that assumption.

4.1

Write a python program that simulates a collision event. For this simulation, assume an ion of charge e is located at the origin and does not move. Inject an electron beginning at something like $\vec{x} = (-500a_0, 100a_0)$ (Bohr radii) moving with velocity $3 \cdot 10^7$ cm/s in the $\hat{x}$ direction. Simulate a time interval of 0.2 ps, with many thousands of time steps. At each time step, use the position of the electron to compute the force vector acting on the electron. Assuming that force vector acts for time Δt corresponding to the time resolution of your simulation, compute what the velocity will be in the next time step of your simulation. Similarly, use the current velocity to predict the position of the electron in the next step of your simulation. Plot the trajectory of your electron, and decide what the resulting impact parameter b was for the point of closest approach.

4.2

Now plot, as a function of time, the acceleration along the $\hat{x}$ and $\hat{y}$ directions. In the style of the lecture on thermal bremsstrahlung, estimate the time interval corresponding to an oscillation period in the EM wave that results from accelerating the electron. Deduce a frequency of emitted radiation.

4.3

Use the Fourier transform of your acceleration profile versus time to deduce the power spectrum of the radiation. How should you combine emission from accelerations in the $\hat{x}$ and $\hat{y}$ directions? At what frequency does the power spectrum peak? To what degree is emission concentrated at a single frequency?

4.4

How well does your result motivate the approximation we made in lecture? For whatever deficiencies you see in our approximation, discuss how that might affect our estimate of the emissivity j_ν of free-free emission.