Problem Set 3

1 Hyperfine Emission from Neutral Hydrogen

Neutral hydrogen in the electronic ground state can be in one of two hyperfine states. Denote the number density of atoms in the ground hyperfine level (singlet state) as n_0, and the number density of atoms in the excited hyperfine level (triplet state) as n_1. DEFINE the excitation temperature, T_{ex}, of the transition as

$$\frac{n_1}{n_0} = \frac{g_1}{g_0} e^{-h\nu/kT_{\mathrm{ex}}}. \tag{1}$$

Here, $h\nu = hc/\lambda$ is the mean energy difference between the levels, and $g_0 = 1$ and $g_1 = 3$ are the statistical weights of the levels. The excitation temperature is merely another way of expressing the ratio of ground state to excited state populations. By definition, if a gas is in local thermodynamic equilibrium (LTE) at some gas kinetic temperature T, then $T_{\mathrm{ex}} = T$; the level populations are distributed in Boltzmann fashion at the local temperature T. Some people refer to the excitation temperature for the $\lambda = 21$ cm transition as the *spin temperature*. But use of the term "excitation temperature" is general to any line transition; it is simply a measure of how excited an atom is.

1.1
Define $T_* = h\nu/k$ and compute its value. It is likely that $T_{\mathrm{ex}} \gg T_*$. For the remainder of this problem, work in the $T_*/T_{\mathrm{ex}} \ll 1$ limit.

1.2
Write down the absorption coefficient, α_ν (units of per length), for this transition. Express your answer in terms of $\Delta\nu$ (we will discuss next week that $\Delta\nu$ is determined by $\phi(\nu)$, the line profile function), A_{21} (Einstein A coefficient), λ, whatever densities you need, and T_*/T_{ex}. Do not forget the correction for stimulated emission.

1.3
Write down the volume emissivity, j_ν (units of erg s^{-1} cm^{-3} Hz^{-1} sr^{-1}), for this transition. Use whatever quantities defined above that you need.

1.4
Write down the source function, S_ν, for this transition.

1.5
Write down the specific intensity, I_ν, of a cloud of HI that is optically thin along the line-of-sight (l-o-s). Take the l-o-s dimension of the cloud to be L, and give the answer only to leading order in $\tau \ll 1$, where τ is the optical depth at an arbitrary wavelength.
Does your answer depend on T_{ex}? If someone gives you a spectrum of the 21-cm line that appears in emission and tells you that the line was emitted from an optically thin cloud, what physical quantities relating to that cloud can you infer from the spectrum?

1.6
Write down the optical depth of the cloud. Does your answer depend on T_{ex}?

1.7

For what column densities is a cloud of HI marginally optically thick? Use a gas density of $n = 1$ cm^{-3}, a gas temperature of $T = 100$ K, an excitation temperature $T_{\text{ex}} = T$, and let's take $\Delta\nu$ to be determined by Doppler shifting from random atomic motion (i.e. $\Delta\nu \sim \nu_0 \frac{v}{c}$ for v being the typical velocity of atoms at temperature T).

2 Hyperfine ^{3}He$^+$

Observations of the hyperfine transition in ^{3}He$^+$ are used to probe the ^{3}He$^+/H$ abundance in the galaxy. This abundance reflects the primordial yield from big bang nucleosynthesis and galactic chemical evolution.

2.1

Estimate, using the scaling relations presented in class and whatever facts you remember, the wavelength of the ground-state electronic (i.e. Ly-α equivalent) transition of ^{3}He$^+$.

2.2

Estimate, using the scaling relations presented in class and whatever facts you remember, the wavelength of the ground-state hyperfine transition of ^{3}He$^+$. Compare to the true answer of 3.46 cm.

2.3

Estimate A_{21} for the ground-state electronic (Ly-α-like) and hyperfine (21cm-like) transitions of ^{3}He$^+$.

3 Rotating Magnetic Dipole

Rybicki & Lightman Problem 3.1.

A pulsar is convetionally believed to be a rotating neutron star. Such a star is likely to have a strong magnetic field, B_0, since it traps lines of force during its collapse. If the magnetic axis of the neutron star does not line up with the rotation axis, there will be magnetic dipole radiation from the time-changing magnetic dipole, $m(t)$. Assume that the mass and radius of the neutron star ar M and R, respectively; that the angle between the magnetic and rotation axes is α; and that the rotational angular velocity is ω.

3.1

Find an expression for the radiated power P in terms of ω, R, B_0, and α.

3.2

Assuming that the rotational energy of the pulsar is the ultimate source of the radiated power, find an expression for the slow-down timescale, τ, of the pulsar.

3.3

For $M = 1M_\odot$, $R = 10^6$ cm, $B_0 = 10^{12}$ gauss, $\alpha = 90°$, find P and τ for $\omega = 10^4$ s^{-1}, 10^3 s^{-1}, 10^2 s^{-1}. The highest rate $\omega = 10^4$ s^{-1} is believed to be typical of newly formed pulsars.

Hint: The Larmor power formula gives the power radiated by an accelerating electric dipole moment. This problem is similar except that it concerns an accelerating magnetic dipole moment.