Problem Set 6

1 Pulsar Dispersion Measure

A radio astronomer uses the Green Bank Telescope to observe a pulsar in a band from 1 to 2 GHz, recording a frequency power spectrum about every 10 ms. The resulting (noisy) frequency spectrum is recorded as pulsar.dat (available on the course GitHub repo). It looks like noise, but I promise you that, for the right dispersion measure, there is a pulse in there.

1.1 De-disperse the measured power spectra to determine the DM of the pulsar. You may have to try many DMs to find the one that works. To help you, the beginning of the file is timed to the arrival time without dispersion delay. For bonus points, you could make your code not depend on this fact.

Just a warning: be careful about the difference between ω and ν. The reported frequencies in the data file are ν, of course.

1.2 For an assumed density of electrons in the interstellar medium of 0.03 cm^{-3}, calculate the distance to this pulsar. Does this seem reasonable (what scale would you use to judge "reasonable")?

1.3 For such an assumed electron density, is the radio astronomer safely observing above the plasma cut-off frequency?

1.4 Calculate the optical depth to Thomson scattering along the line-of-sight to this pulsar.

2 Faraday Rotation

Consider the propagation of light thorugh a magnetized plasma. The magnetic field is uniform: $\vec{B}_0 = B_0\hat{z}$. Light travels parallel to $\hat{z}$.

An electron in the plasma feels a force from the electromagnetic wave and a force from the externally imposed B-field. Its equation of motion reads

$$m\dot{\vec{v}} = -e\vec{E} - \frac{e}{c}\vec{v} \times \vec{B}_0 \tag{1}$$

where the electric field $\vec{E}$ can be decomp[osed into right-circularly polarized (RCP) and left-circularly-polarized (LCP) waves:

$$\vec{E} = E_0(\hat{x} \mp i\hat{y})e^{i(k_{\mp}z-\omega t)} \tag{2}$$

where it is understood that the real part of $\vec{E}$ should be taken. The upper sign $(-)$ corresponds to RCP waves, while the lower sign $(+)$ corresponds to LCP waves.

In the equation of motion above, we have neglected the Lorentz force from the wave's B-field, since it is small (by v/c) compared to the force from the wave's E-field.

2.1 Prove that the solution of the equation of motion reads

$$\vec{v} = \frac{-ie}{m(\omega \pm \omega_{cyc})}\vec{E} \tag{3}$$

where $\omega_{cyc} \equiv eB_0/mc$. This is the hardest part of the derivation for the dispersion relation of RCP and LCP waves.

2.2 Based on Rybicki & Lightman Problem 8.3.

The signal from a pulsed, polarized source is measured to have an arrival time delay that varies with frequency as $dt_p/d\omega = 1.1 \cdot 10^{-5}$ s^2, and a Faraday rotation that varies with frequency as $d\Delta\theta/d\omega = 1.9 \cdot 10^{-4}$ s. The measurements are made around the frequency $\omega = 10^8$ s^{-1}, and the source is at unknown distance from the earth. Find the mean magnetic field, $\langle B_\parallel \rangle$, in the interstellar space between the earth and the source:

$$\langle B_\parallel \rangle \equiv \frac{\int n B_\parallel \; ds}{\int n \; ds} \tag{4}$$

3 A Compton Monte Carlo

An intuitive and commonly applied method for modeling inverse Compton emission is to use a Monte Carlo approach, in which the radiation field is represented by discrete, individual packets of photons. The fate of each photon packet is determined by "rolling the dice", i.e., choosing random numbers drawn from the appropriate probability distributions. Here you will write your own simple Monte Carlo code, which could in principle be used to model real data. This is a numerical problem, so please provide a printout of your code, in addition to your plots.

Consider a supernova that produces optical photons, which then travel through a shell of shocked circumstellar gas, where they may be scattered by non-thermal, relativistic electrons. Assume that the supernova emits a luminosity of $L_s = 10^{43}$ erg/s. The luminosity carried by each Monte Carlo packet in our code is then $L_p = L_s/N_p$, where N_p is the number of photon packets we choose to use. We will assume for simplicity that all photons are emitted with the same frequency $h\nu_{in} = E_{in} = 1$ eV. These photons travel through a shocked shell of relativistic electrons with optical depth $\tau = 0.01$.

The Monte Carlo method proceeds by generating and scattering photon packets one by one. The energy of a scattered photon is given by applying two Doppler shifts — one into the rest frame, and one back out — which gives a final outgoing energy

$$E_{out} = E_{in}\gamma^2(1 - \beta\cos\theta_{in})(1 + \beta\cos\theta'_{out}) \tag{5}$$

where θ_{in} is the incoming angle (in the lab frame) and θ'_{out} is the outgoing angle (in the rest frame). We simulate a scattering by sampling the incoming and outgoing directions randomly. (Since we are considering $\tau \ll 1$, we will only need to scatter each packet once). Assume that $\cos\theta_{in}$ and $\cos\theta'_{out}$ are drawn uniformly (and randomly) from the range -1 to 1. This takes into account the dipole response pattern of Larmor radiation. After a random scattering, the packet has a new luminosity

$$L_{out} = L_p \frac{E_{out}}{E_{in}}(1 - e^{-\tau}) \tag{6}$$

where the term in parenthesis takes into account the fact that only a fraction $(1 - e^{-\tau})$ of the photons from the source are actually scattered. The packet can be "observed" by collecting it and binning its energy into a spectrum array. If we repeat this procedure with the N_p packets, we will gradually build up the observed spectrum. Because of the random nature of the algorithm, this spectrum will possess noise, and we will need a fairly large value for N_p to achieve a reasonable signal to noise (try 10^5 for testing, 10^6 for the final result).

3.1 Consider the case where we only have electrons of a single energy, $\gamma = 10$. Run the Monte Carlo procedure and plot the spectrum of scattered photons on a log-log plot. Your x-axis can be either eV or Hz, and y-axis should be a monochromatic luminosity, i.e., units of ergs s^{-1} eV^{-1} or ergs s^{-1} Hz^{-1}. Check that where the spectrum cuts off at high energy makes sense.

3.2 Now consider the more realistic case where the electrons have a power law distribution in energy, between the bounds $\gamma_{min} = 10$ and $\gamma_{max} = 100$. The probability, in any given scattering, that the electron has an energy γ is

$$P = A\gamma^{-p} \tag{7}$$

for $\gamma_{min} < \gamma < \gamma_{max}$, and 0 otherwise. A is a constant, and we'll take $p = 2.5$. For each scattering, we should then choose the γ of the electron by randomly sampling from this probability distribution (which can be easily done[1]). Run the Monte Carlo for this case and plot the spectrum due to inverse Compton scattering. Overplot the slope of the power law that you expect from analytic arguments.

By the way, your Monte Carlo code could probably already be used in a simple research paper. For example, you could model the x-ray emission from SN 2011fe and constrain the density of the circumstellar environment, as considered in Horesh et al. (2012). To improve the code, you would want to relax the assumption that all photons start with the same energy E_{in}. Instead, you can imagine randomly sampling the initial frequency of the photons from a real distribution (e.g., a blackbody). In addition, you could fairly easily generalize your code to treat multiple scatterings, or for the more realistic case where the scattering is not isotropic or inelastic. If you also kept track of the position of the photons, you could even consider the case where the scattering cloud is not spherically symmetric.

[1]http://mathworld.wolfram.com/RandomNumber.html