

## Problem Set 3 (part 1)

### 1 Blackbody Flux

Derive the blackbody flux formula

$$F = \sigma T^4 \quad (1)$$

from the Planck function for specific intensity,

$$B_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1} \quad (2)$$

You may use the integral  $\int_0^\infty \frac{dx}{x^5(e^{1/x}-1)} = \pi^4/15$ . Those who know what they have to do can stop reading here. Those who need a bit of help should read on.

Imagine a perfectly flat, blackbody patch at the center of a sphere of radius  $R$ . The patch has tiny surface area  $dA$  and is lying flat at  $r = 0$  in the  $\theta = \pi/2$  plane (in spherical coordinates, where  $r$ ,  $\theta$ ,  $\phi$  are the radius, polar angle, and azimuth). Only one side of the patch is warm; the other side is at absolute zero. Assume that the warm side radiates as a perfect thermal blackbody.

A thermal blackbody emits radiation isotropically — i.e., with no preference for direction. In other words, imagine each point on the patch beaming out a tiny, uniform, hemispherical dome of rays (like an umbrella). It is a hemispherical dome and not a full sphere because only one side of the patch is warm. Note that the Planck blackbody function has no variable inside it that specifies direction of emission; the direction of emission doesn't matter for purely thermal emission. The only variables that need to be specified for the Planck function are  $T$  and  $\nu$ .

Despite the isotropic nature of the emission, because the patch is geometrically flat, has definite area, and emits only on one side of itself, a detector glued to the inside of the sphere at  $r = R$  detects different amounts of radiation depending on where it is placed. For example, if the detector is glued on the half of the sphere that can't see the bright side of the patch, the detector detects nothing. If the detector is glued to the sphere directly above the patch, the detector picks up the most number of photons, because it sees the full face-on area of the patch (namely,  $dA$ ). If the detector is glued at an angle to the pole, then it sees less than area  $dA$ . If it is glued in the plane of the patch, it sees nothing but an infinitesimally thin line segment.

This problem asks you to place detectors everywhere on the inside of the sphere and sum up the radiation collected; this operation is equivalent to “integrating the Planck function over all solid angles into which the radiation is beamed.”

Calculate the total luminosity (in units of energy/time) emitted by the patch of area  $dA$ . You must integrate the Planck function over the entire emitting area (which is maximally  $dA$  but in general will be less, depending on the viewing geometry), over all solid angles into which the radiation is beamed, and over all frequencies. Think about placing tiny detectors all over the inside of the sphere and asking how much energy/time each of the detectors receives.

Then divide the luminosity by  $dA$  to calculate the total flux (in units of energy/time/area) emitted by the patch. You should recover the usual blackbody flux formula,  $\sigma T^4$ . By definition,  $\sigma T^4$  is the total amount of energy radiated per time per unit area of a blackbody surface, radiated into all solid angles and over all frequencies.

## 2 Flat Disks

This problem forms the foundation for understanding the spectral energy distributions of circumstellar disks, e.g., those surrounding pre-main-sequence stars and AGN.

Consider a perfectly flat, blackbody disk encircling a blackbody star. The star has radius  $R_*$  and effective temperature  $T_*$ . The disk begins at a stellocentric radius far from the star, i.e. the inner disk radius  $r_i \gg R_*$ . The disk extends to infinity.

Calculate the temperature of the disk,  $T(r)$ , as a function of radius,  $r$ . Make whatever approximations you deem necessary in light of the fact that  $r_i \gg R_*$ . Be sure you get the scaling of  $T$  with  $r$  correctly. It is less important that you get the numerical coefficient correctly.

This disk is PERFECTLY flat. Do not consider individual particles in the disk.

Also, neglect radial transport of energy. Each annulus is independent of neighboring annuli. This is a fine approximation for thin disks since they transport more radiation vertically than radially (the temperature gradient is steeper vertically than radially).

## Problem Set 3 (part 2)

### 3 Practice with $j_\nu$ , $\alpha_\nu$ , $S_\nu$ , $I_\nu$

**3.1** A plane-parallel slab of uniformly dense gas is known to be in LTE (local thermodynamic equilibrium) at a uniform temperature  $T$ . Its thickness normal to its surface is  $s$ . Its absorption coefficient is  $\alpha_{\nu,\text{gas}}$ . Write down the specific intensity,  $I_\nu$ , viewed normal to the slab, in terms of the variables given.

**3.2** The same slab is now filled uniformly with non-emissive dust having absorption coefficient  $\alpha_{\nu,\text{dust}}$ . The dust is non-emissive, so its emissivity  $j_{\nu,\text{dust}} = 0$ . Write down  $I_\nu$  viewed normal to the slab, in terms of all variables given so far.

**3.3** The slab of gas and dust is further mixed with a third component: an emissive, non-absorptive uniform medium having emissivity  $j_{\nu,\text{med}}$  and absorption coefficient  $\alpha_{\nu,\text{med}} = 0$ . Write down  $I_\nu$  viewed normal to the slab, in terms of all variables given.

A physical realization of this problem might be an HII region surrounding an ionizing O star. The material in LTE would be the fully ionized plasma, emitting thermal bremsstrahlung radiation. The dust would be dust. The emissive, non-absorptive medium would be the same ionized plasma emitting recombination (line) radiation. For the assumptions stated in the problem to be valid, we would have to evaluate  $\nu$  at, say, an optical recombination line like  $\text{H}\alpha$ .

## 4 Detailed Balance

Suppose an atom has 2 energy states  $E_1$  and  $E_2$ . This atom occasionally collides with an  $e^-$ , whereupon it may change states. Let us say that during a collision, there is a probability of  $P_{12}$  that the atom transitions from state 1 to state 2, and  $P_{21}$  for the reverse (from 2 to 1).

**4.1** If these are the only transitions, what do you expect the relative populations to be of atoms in state 1 versus state 2 after many collisions? Write some code that verifies this by simulating it (and please submit your code with your homework, along with the output of the code). Pick  $P_{12}$  and  $P_{21}$  to be something illustrative. Does this system satisfy detailed balance?

**4.2** Modify your code above to add a third energy state,  $E_3$ , along with the relevant transition probabilities to/from all other states. Now what are the relative populations in each state (both numerically for chosen values of  $P_{13}$ ,  $P_{23}$ ,  $\dots$ , and as an analytic function of these probabilities). Does this system satisfy detailed balance?