On regular handicap graphs of even order

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Abstract

Let $G = (V, E)$ be a simple graph of order n. A bijection $f : V \to \{1, 2, \dots, n\}$ is a *handicap labeling* of G if there exists an integer ℓ such that $\sum_{u \in N(v)} f(u) = \ell + f(v)$ for all $v \in V$, where $N(v)$ is the set of all vertices adjacent to v. Any graph which admits a handicap labeling is a *handicap graph.*

We present an overview of results, which completely answer the question of existence of regular handicap graphs of even order.

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1 Introduction and definitions

Let $G = (V, E)$ be a simple graph of order n. A bijection $f : V \to \{1, 2, \ldots, n\}$ is a *handicap labeling* of G if there exists an integer ℓ such that $\sum_{u \in N(v)} f(u) = \ell + f(v)$ for all $v \in V$, where $N(v)$ is the set of all vertices adjacent to v. Any graph which admits a handicap labeling is a *handicap graph*.

Handicap labelings were introduced as a modification of a distance magic labeling, which is a bijection $f : V \to \{1, 2, \ldots, n\}$ with the property that the sum $\sum_{u \in N(v)} f(u)$ equals the same value for every $v \in V$. The motivation of both labelings lies in scheduling of incomplete tournaments with teams ordered linearly according to their strength. The label $f(i)$ represents the rank that decreases with the team strength. We identify vertex names with their labels, thus by i we understand the vertex labeled i. A distance magic labeling of a graph represents a schedule of an incomplete tournament in which all teams should have an equally strong set of opponents, while in a handicap tournament a certain advantage is given to weaker teams: the weaker the team, the bigger its advantage. This hopes to support attractive tournaments in which each game counts. An excellent up-to-date overview of recent results on labelings is the review by Gallian [6], a specialized survey on distance magic labelings and its application to tournaments is [1].

For any graph with given regularity and given order an easy counting argument shows (see [7]) the set of vertex weights is given by the following lemma, unlike vertex-magic total labelings, where for the same graph different weights using the same set of labels can be obtained.

Lemma 1.1 *In an r-regular handicap graph with n vertices the weight of every vertex is $w(i) = (r-1)(n+1)/2 + i$.*

Each vertex weight is an integer value obtained as a sum of integers, thus

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$n\backslash r$	$0 \pmod 4$	$1 \pmod 4$	$2 \pmod 4$	$3 \pmod 4$
$0 \pmod 4$	Th 1.2		Th 1.2	
$1 \pmod 4$		PP		PP
$2 \pmod 4$	Th 1.2	Th 1.3	Th 1.2	
$3 \pmod 4$		PP		PP

Table 1
Overview of cases r and n modulo 4.

from the previous lemma immediately follows.

Theorem 1.2 *There exists no r-regular handicap graph with n vertices if both r and n are even.*

By parity principle, no odd-regular graph with an odd number of vertices exists, thus an r-regular handicap graph with n vertices cannot have r and n of the same parity. Moreover, another counting argument shows the following, see e.g. [7] or [8].

Theorem 1.3 *There exists no r-regular handicap graph with n vertices if $r \equiv 1 \pmod 4$ and $n \equiv 2 \pmod 4$.*

Table 1 summarizes for which regularities r and orders n the existence of an r-regular handicap graph is excluded by parity principle (PP) or by a theorem. Since for each empty field at least one handicap graph with corresponding regularity and order was known, several authors started working on constructions, which would provide at least one handicap graph for any given regularity and order. This paper shows that the joined effort was successful.

Handicap graphs with extremely low or high regularity do not exist. Using Lemma 1.1 and Theorem 1.2, the following lemma was proved in [7] and in [8].

Lemma 1.4 *No nontrivial r-regular handicap graph with n vertices exists if $r = 1$, $r = 2$, $r = n - 1$ and $r = n - 2t$, where $t \in [1, \lfloor n/2 \rfloor]$.*

A similar but slightly more technical approach as in Lemma 1.4 can be used to prove the following lemma. See [7] and [8].

Lemma 1.5 *There is no $(n - 3)$-regular handicap graph of order n.*

The existence of 3-, 5-, $\ldots$, $(n - 11)$-regular handicap graphs is settled in [7]. A summary is given in Section 2. The existence of $(n - 3)$-, $(n - 5)$-, $\ldots$, 7-regular handicap graphs is settled in [5]. A summary is in Sections 3 and 4. There are a few small orders for which no 3-regular handicap graphs exist. In Section 5 we show that all these results form a complete classification of regular handicap graphs of even order.

Early existence results were based on constructions of complements of unions of Cartesian products of complete graphs $c(K_a \square K_b)$ and properties of magic rectangles and magic rectangle sets by the first author [2], [3]. Because the spectrum of admissible pairs (n, r) was very sparse, we do not include them in our further discussion.

Handicap graphs with odd order were studied by the first author in [4].

2 Constructions based on induction

In this section, we describe an inductive construction of 3- and 5-regular handicap graphs from [7]. Next we show that we can add edge-disjoint distance magic graphs with the same set of vertices to obtain handicap graphs of regularity up to $n - 11$.

The construction of 3-regular handicap graphs is based on the following lemma.

Lemma 2.1 *Let G be a 3-regular handicap graph with n vertices. Then a 3-regular handicap graph on $n + 8$ vertices exists.*

The proof is constructive. To any given 3-regular handicap graph G with n vertices we add a vertex disjoint copy of the graph H (Figure 1), while every label of G is increased by 4. Taking for G handicap graphs on 8, 20, 30, and 34 vertices, we cover by induction all even orders except 4, 6, 10, 12, 14, 18, 22, and 26 for which no handicap graph exists. This was verified by an exhaustive computer search.

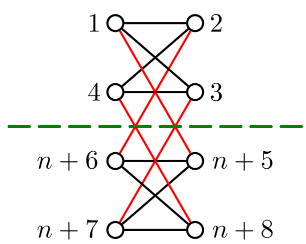

Fig. 1. Graph H.

Similarly, using the following lemma it was shown in [7], that a 5-regular handicap graphs exists for all even orders $n \geq 12$.

Lemma 2.2 *Let G be a 5-regular handicap graph with n vertices. Then a 5-regular handicap graph on $n + 12$ vertices exists.*

Finally, by taking the handicap graphs found above and constructing edge-disjoint distance magic graphs on the same set of vertices, the following was shown in [7].

Theorem 2.3 *There exists an r-regular handicap graph of even order n for all $n \geq 28$ and $3 \leq r \leq n - 11$ except when both $n \equiv 2 \pmod 4$ and $r \equiv 1 \pmod 4$.*

This leaves the existence of handicap graphs open for regularities $n - 7$ and $n - 9$ and finitely many small orders.

3 Direct constructions

A different approach is used in [5] and [8] to obtain graphs with higher maximum degrees.

Theorem 3.1 *There exists an r-regular handicap graph of even order $n \equiv 0 \pmod 8$ for all $n \geq 8$ and all odd r satisfying $3 \leq r \leq n - 5$.*

Proof. The construction has three steps. An example of a graph constructed in this manner is shown in Figure 2. The edges are colored red, blue, and black, to correspond with steps one, two, and three, respectively.

Let $n = 8k$. In the first step we find a one-factor, joining (by red edges) vertices whose labels differ by $4k$. In the second step, we group vertices into pairs such that in $4k$ pairs we have vertices whose labels add up to $4k + 1$; in the remaining $4k$ pairs the labels add up to $12k + 1$. Then we connect (by blue edges) the pair labeled $[1|4k]$ with $[2|4k-1]$, $[3|4k-2]$ with $[4|4k-3]$,..., $[2k-1|2k+2]$ with $[2k|2k+1]$, and similarly in the upper part, $[4k+1|8k]$ with $[4k+2|8k-1]$, $[4k+3|8k-2]$ with $[4k+4|8k-3]$,..., $[6k-1|6k+2]$ with $[6k|6k+1]$. This gives a 3-regular handicap graph G^3. Next we amalgamate vertices labeled i and $8k+1-i$ for $i = 1, 2, \ldots, 4k$ into a single vertex to obtain a cubic graph B (multiple edges are replaced by a single edge) and find its $(4k-4)$-regular complement $\bar{B}$. Then we find a one-factorization of $\bar{B}$. To increase the degree of G^3 by two to obtain G^5, we select a one-factor F of $\bar{B}$ and add to G^3 (black) edges of $2kK_{2,2}$, where each $K_{2,2}$ corresponds to one edge of F. It should be obvious that continuing this way we obtain graphs G^{2j+1} with all required regularities. $\square$

A similar but more sophisticated method is used to prove the following.

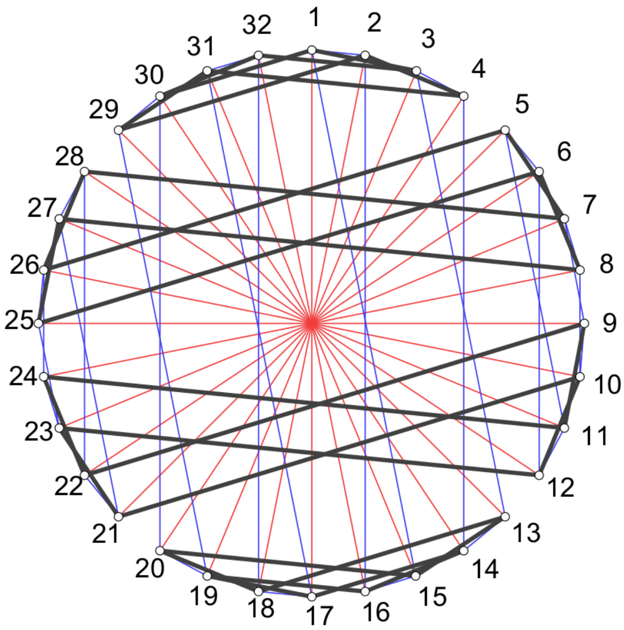

Fig. 2. 5-regular handicap graph on 32 vertices.

Theorem 3.2 *There exists an r-regular handicap graph of even order n for all $n \geq 12$, $n \equiv 4 \pmod 8$, and all odd r satisfying $7 \leq r \leq n - 5$.*

4 Constructions based on complements of graphs

A complementary approach is used in [5] and [8] for $n \equiv 2 \pmod 4$ and $r = n - 7$. The idea for this construction is that the complement of a handicap graph is a distance antimagic graph where the weights $w(1), w(2), \dots, w(n)$ form a decreasing arithmetic progression with common difference 2. The construction is inductive as those in Section 2 and uses the following lemma.

Lemma 4.1 *If G is an $(n - 7)$-regular handicap graph of order n, then there exists an $(n + 16 - 7)$-regular graph G' of order $n + 16$.*

For $n \in \{14, 18, 22, 26\}$, handicap graphs of order n and regularity $(n - 7)$ were found computationally and non-existence for order 10 was proved by brute force. Thus the following.

Theorem 4.2 *An $(n - 7)$-regular handicap graph of order $n \equiv 2 \pmod 4$ exists if and only if $n \geq 14$.*

$n \backslash r$	$0 \pmod 4$	$1 \pmod 4$	$2 \pmod 4$	$3 \pmod 4$
$0 \pmod 8$	Th 1.2	Th 2.3 & 3.1	Th 1.2	Th 2.3 & 3.1
$1 \pmod 8$		PP		PP
$2 \pmod 8$	Th 1.2	Th 1.3	Th 1.2	Th 2.3 & 4.2
$3 \pmod 8$		PP		PP
$4 \pmod 8$	Th 1.2	Th 2.3 & 3.2	Th 1.2	Th 2.3 & 3.2
$5 \pmod 8$		PP		PP
$6 \pmod 8$	Th 1.2	Th 1.3	Th 1.2	Th 2.3 & 4.2
$7 \pmod 8$		PP		PP

Table 2
Overview of cases r modulo 4 and n modulo 8.

5 Conclusion

Now we summarize all results. The necessary conditions show that regular graphs with extremely low or high regularity cannot be handicap. Also, parity conditions exclude the regularity and order of a handicap graph to be of the same parity. Finally, it was shown by brute force that there are no 3-regular handicap graphs of orders 10, 12, 14, 18, 22, and 26. For all remaining pairs (n, r), handicap graphs exist. In Section 2 there are constructions for graphs with regularity 3, 5, up to $n-11$ and in Sections 3 and 4 we cover regularities up to $n-5$ for $n \equiv 0 \pmod 4$ and $n-7$ for $n \equiv 2 \pmod 4$, respectively, by providing constructions of such handicap graphs. Notice that for $n \equiv 2 \pmod 4$ handicap graphs of regularity $n-9$ are excluded by Theorem 1.3.

Thus, we can conclude that the (obvious) necessary conditions for a regular handicap graph are sufficient with only six exceptional 3-regular graphs.

Theorem 5.1 *An r-regular handicap graph G on n vertices exists when $n \geq 8$ and*

(i) $n \equiv 0 \pmod 4$ *if and only if* $3 \leq r \leq n-5$ *and* r *is odd*

(ii) $n \equiv 2 \pmod 4$ *if and only if* $3 \leq r \leq n-7$ *and* $r \equiv 3 \pmod 4$,

except when $r = 3$ *and* $n \in \{10, 12, 14, 18, 22, 26\}$.

Proof. The proof follows from Theorems 1.2, 1.3, 2.3, 3.1, 3.2, and 4.2. □

A natural follow-up is to examine handicap graphs of odd orders. By Theorem 1.2 such handicap graphs have to have even regularity. Some construction can be found in [4]. For all feasible orders (with few exceptional small cases) at least one regular handicap graph was found.

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