Graph Labelings and Tournament Scheduling

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Thank you to anyone who supported me in any way.

Dediciation

My parents.

Abstract

During my research, I studied and became familiar with *distance magic* and *distance antimagic* labelings and their relation to tournament scheduling. Roughly speaking, the relation is as follows. Let the vertices on the graph represent teams in a tournament, and let an edge between two vertices a and b represent that team a will play team b in the tournament. Further, suppose we can rank the teams based on previous games, say, the preceding season. These integer rankings become labels for the vertices. Of particular interest were handicap tournaments, that is, tournaments designed to give each team an equal chance of winning.

Contents

List of Figures

List of Tables

1 Introduction

Labelings of graphs were introduced in the late 1960's, and have evolved due to pure mathematical curiosity as well as being the source of solutions for many pragmatic problems. For a survey of well known results pertaining to all types of graph labeling, we refer to [7]. One specific application is tournament scheduling. At first thought, one may not realize that there are actually multiple types of tournaments, each with their own characteristics. Some are deemed to be more fair than others. What "fair" actually is, though, is debatable. One may want the highest ranked team to have the best chance of winning. Or, one may like every team to have an equal chance of winning. Depending on the situation, a specific type of tournament might be saught for.

Let us take a look at some of the different types of tournaments. To do this, we will use the teams from the 2014 football season of the Northern Sun Intercollegiate Conference (NSIC) to generate some meaningful examples. The NSIC has 16 teams with 2 divisions, North and South, each with 8 teams. See Table 1 [12]. In the regular season each team played the other 7 teams in their division, and 4 teams in the other division, for a total of 11 regular season conference games. This seems reasonable, but can be problematic.

For example, consider a possible 2015 season for the top two teams in the North division, Minnesota Duluth (UMD) and Northern State. Each team plays every other team in their division, so this part of the schedule is more or less the same. However, it is quite possible that UMD plays the top four teams from the South: Minnesota State, Sioux Falls, Concordia-St. Paul, and Augustana, while Northern State plays the bottom four teams from the South: Wayne State, Upper Iowa, Southwest Minnesota State, and Winona State. This somewhat extreme, but very possible case, results in an obviously unfair schedule. Thus, if care is not taken, poor schedules can easily be written.

2014 Football Standings							
School	**Div**	**DPct.**	**Conf**	**CPct.**	**Overall**	**Pct.**	**Streak**
							North
Minnesota Duluth	7-0	1	11-0	1	13-1	0.929	L1
Northern State	6-1	0.857	8-3	0.727	8-3	0.727	W4
St. Cloud State	5-2	0.714	6-5	0.545	6-5	0.545	W3
U-Mary	3-4	0.429	5-6	0.455	5-6	0.455	L3
MSU Moorhead	3-4	0.429	4-7	0.364	4-7	0.364	W3
Bemidji State	3-4	0.429	3-8	0.273	3-8	0.273	L4
Minot State	1-6	0.143	1-10	0.091	1-10	0.091	L3
Minnesota Crookston	0-7	0	0-11	0	0-11	0	L11
							South
Minnesota State	7-0	1	11-0	1	14-1	0.933	L1
Sioux Falls	6-1	0.857	10-1	0.909	11-1	0.917	W3
Concordia-St. Paul	4-3	0.571	5-6	0.455	5-6	0.455	W4
Augustana	3-4	0.429	6-5	0.545	6-5	0.545	W1
Wayne State	3-4	0.429	5-6	0.455	5-6	0.455	L3
Upper Iowa	2-5	0.286	6-5	0.545	6-5	0.545	L2
Southwest Minnesota State	2-5	0.286	3-8	0.273	3-8	0.273	L1
Winona State	1-6	0.143	4-7	0.364	4-7	0.364	L4

Table 1: NSIC Teams

The divisional games actually compose a *round robin tournament*. A round robin tournament[1] (denoted RRT for convenience) is where each team plays every other team exactly once. So, an RRT with 8 teams consists of $56/2 = 28$ different games, since each team plays 7 games (a team cannot play themselves, and each game gets counted twice). Round robin tournaments are often considered fair in the sense that every team plays every other team, and can be represented by a *complete graph*. Also, if we rank the teams based on the previous season, we can measure the difficulty of the tournament for each team by looking at the strength of the opponents. For example, since

[1]formal definitions are presented in section 2

UMD finished first in the North division in 2014, they would be considered the *strongest* team going into 2015, and would be assigned a strength of 8. Similarly, since Minnesota Crookston finished last in 2014, they would be considered the *weakest* team going into 2015, and be assigned a strength of 1. Table 2 then sums this information up by adding up the strengths each team would face in a round robin tournament.

School	**Ranking**	**Strength**	**Total Strength of Opponents**
Minnesota Duluth	1	8	28
Northern State	2	7	29
St. Cloud State	3	6	30
U-Mary	4	5	31
MSU Moorhead	5	4	32
Bemidji State	6	3	33
Minot State	7	2	34
Minnesota Crookston	8	1	35

Table 2: Difficulty of an RRT for 8 Teams

Now suppose the NSIC decides they want to change their schedule, and simply use an RRT for their regular season, that is, each team plays all of the other 15 teams in the conference. The down side of this is that it takes more time. Assuming the usual one game per weekend, this would require a minimum of 15 weeks to complete, and a long season in terms of college football. You can imagine that the more teams there are, the less practical this is. The National Football League is home to 32 teams, and a full round robin tournament, 31 games per team, is not a feasible schedule. Is there any way to mimic an RRT without actually playing all the games? In fact, there is. A tournament of this type is called a *fair incomplete tournament* (or FIT). FIT's correspond to graphs with a *distance antimagic vertex labeling.*

For an example of this, consider the same 8 schools from the NSIC North. This can be done in three rounds, presented in Table 3. Table 4 shows the total stength of opponents faced in this FIT, notice the same increasing arithmetic pattern is encountered here as in the full RRT. In fact, a little

	Week 1	**Week 2**	**Week 3**
Game 1	UMD vs. St. Cloud St.	UMD vs. Crookston	UMD vs. Bemidji St.
Game 2	N. State vs. U-Mary	N. State vs. Minot St.	N. State vs. MSU M.
Game 3	MSU M. vs. Minot St.	St. Cloud St. vs. Bemidji St.	U-Mary vs. Minot St.
Game 4	Bemidji St vs. Crookston	U-Mary vs. MSU M.	Crookston vs. St. Cloud St.

Table 3: Fair Incomplete Tournament for 8 teams

further analysis shows that each team misses a total strength of 18.

School	**Ranking**	**Strength**	**Total Strength of Opponents**
Minnesota Duluth	1	8	10
Northern State	2	7	11
St. Cloud State	3	6	12
U-Mary	4	5	13
MSU Moorhead	5	4	14
Bemidji State	6	3	15
Minot State	7	2	16
Minnesota Crookston	8	1	17

Table 4: Difficulty of FIT for 8 Teams

Viewing tournaments in this fashion, i.e. by looking at the total strengths of a given team's opponents, we can see that RRT's and FIT's favor the stronger teams, since they face the weakest opponents. In an effort to even the playing field, an *equalized incomplete tournament* (EIT) accomplishes the goal of having each team face the same total strength of opponents. In the aforementioned FIT, each team missed a total strength of opponents of 18. So, the games we did not play in the FIT would in fact yield an EIT. This is not a coincidence. Equalized incomplete tournaments correspond to *distance magic vertex labeling* of graphs [4].

Let us play devil's advocate yet again, and try to find a reason why an EIT might not actually be deemed a "fair" tournament. Suppose you enter a weekend golf tournament that is open to the public. Fortunately, you've been very busy and productive in the summer finding and proving new results in your favorite area of mathematics, and therefore haven't had a lot of time to be on the golf course. So, you are a little rusty. You show up to the tournament to find out that the local pro, Chad Micheals[2] has also entered the tournament. Chad, as a professional golfer, is better than you at golf. You are both playing on the same course, and so, he has a better chance of winning the tournament than you do. This should help us see that in an equalized incomplete tournament the strongest team has the best chance of winning. Each team plays the same total strength of opponents, so the strongest team will likely do best (which is not bad in general).

What if you want each team to have not a fair, but *equal* chance of winning a tournament? After all, if you knew Chad Micheals was going to enter the golf tournament, you wouldn't have bothered entering in the first place because he would probably win. But what if you got to play the nine easiest holes on the course, while Chad had to play the nine most difficult holes on the course? This might be more appealing to you (and others) as you could actually stand a chance against Chad and win the tournament! A *handicap incomplete tournament* (HIT) is a tournament where the strongest teams play strong opponents and weaker teams play weak opponents, in the effort of giving each team an equal chance of winning the tournament. HIT's correspond to *handicap distance-antimagic labelings* of graphs.

School	**Ranking**	**Strength**	**Total Strength of Opponents**
Minnesota Duluth	1	8	17
Northern State	2	7	16
St. Cloud State	3	6	15
U-Mary	4	5	14
MSU Moorhead	5	4	13
Bemidji State	6	3	12
Minot State	7	2	11
Minnesota Crookston	8	1	10

Table 5: Difficulty of HIT for 8 Teams

[2]Chad Micheals is fictional

If we return back to our NSIC football teams, Table 5 shows how the total strength of opponents gets larger for stronger teams. This type of progression is typical for an HIT. Compare this to Table 4 for an interesting contrast. An example schedule for an HIT is shown below in Table 6.

	Week 1	**Week 2**	**Week 3**
Game 1	Crookston vs. Minot St.	Bemdiji St. vs. Crookston	Crookston vs. U-Mary
Game 2	Bemidji St. vs. MSU M.	Minot St. vs. MSU M.	Minot St. vs. St. Cloud St.
Game 3	U-Mary vs. St. Cloud St.	U-Mary vs. N. State	Bemidji St. vs. N. State
Game 4	N. State vs. UMD	UMD vs. St. Cloud St.	UMD vs. MSU M.

Table 6: Handicap Incomplete Tournament for 8 teams

It is worth mentioning that the aforementioned strategy of assigning strengths is a simplification of a more general idea. One may want to choose the strength based on number of wins, total number of points scored, or even use powers of 2 for assigning strengths. However, if we use distinct powers of 2, for example, it is impossible to have an EIT. The benefit of assigned strengths as an arithmetic progression with a common difference (the difference is usually 1) is that it often lends itself to a natural extension of a graph labeling. One can easily analyze the structure and define properties of different types of tournaments. We are then able to guarantee that the different types of tournaments actually exist.

2 Definitions

In this section we introduce formal definitions of the graphs and tournaments mentioned in the introduction, as well as other useful pieces for later sections. A *graph* $G = (V, E)$ consists of two sets. V, the *vertex set*, is a nonempty set of elements called *vertices*. E, the *edge set*, is a set of elements called edges. Each edge e in E is an unordered pair of vertices (u, v), called the end vertices of of e. It is possible to have a vertex joined to itself by an edge; such an edge is called a *loop*. If two or more edges of G have the same end vertices we say these are *parallel* edges. A graph is called *simple* if it has no loops and no parallel edges. We use the notation of [1] for all other standard graph theoretic terms unless specified otherwise.

We begin with the most familiar type of tournament, a complete Round Robin.

2.1 Round Robin Tournaments (RRT) and 1-Factors

A round robin tournament with an even number of teams, n, is a tournament of $n - 1$ rounds where each team plays the other $n - 1$ teams. A *round* is a collection of games where each team is matched with exactly one other team. This is often considered a fair tournament, and can be represented by a complete graph on n vertices. The vertices on the graph represent the teams, each labeled by its strength, and edges between vertices indicate the teams play each other in the tournament. This is shown in Figure 1, where each color represents a different round of the tournament. Each round is in fact a *perfect matching*, also known as *1-factor* of the graph.

Definition 2.1: Matching

Let G be a simple graph with vertex set V and edge set E. A set M of edges of G is called a *matching* in G if no two of the edges in M share a vertex.

Definition 2.2: Perfect Matching

If M is a matching in G such that every vertex of G is the end vertex of some edge in the matching M, then M is called a *perfect matching* or a *1-factor*. [1]

Definition 2.3: 1-Factorization

A *1-factorization* of a graph G is a partition of the edge set of G in 1-factors. [11]

In the context of tournament scheduling, it is more common to use the terminology 1-factor instead of perfect matching. We will say a graph G can be *1-factored* if a 1-factorization exists.

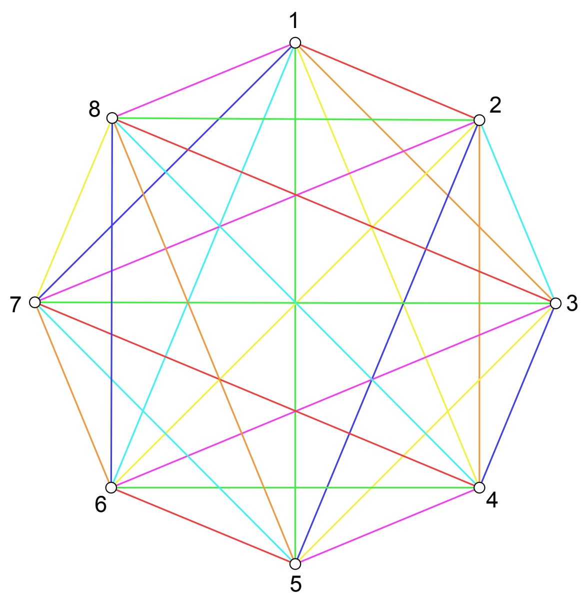


Figure 1: A complete graph on 8 teams representing an RRT

Round robin tournaments are fairly well known even among people with little or no mathematical background. Even though it is relatively simple to describe, RRT's are not necessarily trivial to design. Probably the most popular method known for RRT's is the *Kirkman tournament* . A Kirkman tournament provides a 1-factorization of K_{2n}, a complete graph on any even number of vertices. For details of this type of construction, as well as alternative solutions such as a *Steiner tournament* and other RRT properties, we refer the reader to [2].

In relation to 1-factorizations, we describe a graph that will be very important to us in Theorem 4.3. First we define the length of an edge. Place the vertices at uniform distance in a circle, starting with 1 at the top-center position, in a clock–wise fashion. Suppose we have an edge $[k \mid j]$, the *length* of this edge is the "circular distance" between the vertices k and j, i.e., the number of steps we need to take around the circle to get from k to j using the shorter of the two paths between them. Then for any subset $D \subseteq \{1, 2, \ldots, \lfloor n/2 \rfloor\}$, define $G_n(D)$ to be the graph with vertex set $\{1, 2, 3 \ldots, n\}$ and edge set consisting of all edges whose length is in D. This type of graph is sometimes referred to as a *circulant* graph. For example, Figure 2 is a graph on 6 vertices with all edges of length 1 and 2.

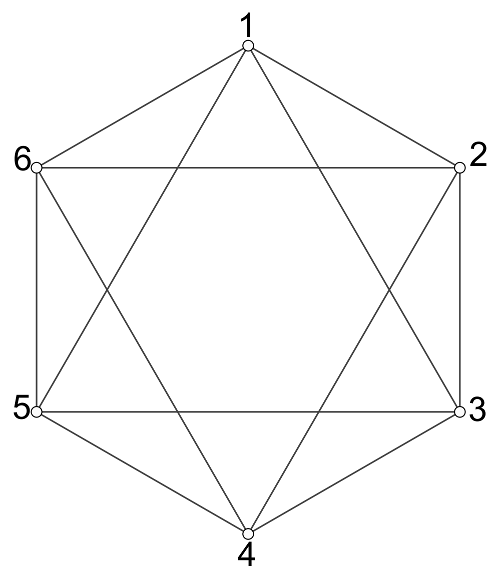


Figure 2: The graph $G_6(1, 2)$

There is a very specific requirement for when this special type of graph can be 1-factored. Lemma 2.1 is originally due to Stern and Lenz and we reference the constructive proof given in [11].

Lemma 2.1. *If D contains an element d where $\frac{n}{\gcd(d,n)}$ is even, then $G_n(D)$ has a 1-factorization.*

In other words, to see if $G_n(D)$ has a 1-factorization, all we need to do is find an edge length $d \in D$ so that $\frac{n}{\gcd(d,n)}$ is an even integer. Of course, $\gcd(d, n)$ denotes the greatest common divisor of the integers d and n.

2.2 Fair and Equalized Incomplete Tournaments (FIT and EIT)

Often times, a full RRT can take a long time. If there is not enough time to complete a full RRT, one may try to still conduct a similar tournament. If there is only time for g rounds of games, where $g < n$, then a special type of incomplete tournament may suffice. A *fair incomplete tournament* of n teams in g rounds, FIT(n, g), is a tournament where every team plays g other teams and the difficulty of the tournament for each team mimics that of a complete round robin tournament. Thus, the total *strength* of the opponents that each team misses is equal. The graph of an FIT admits a *distance antimagic vertex* labeling, with the additional property that the weights of each vertex forms a decreasing arithmetic progression with common difference equal to one.

The $n-g-1$ games that are not played in an FIT(n, g) form an *equalized incomplete tournament*, denoted EIT$(n, n-g-1)$. The total strength of opponents for each team is equal in an EIT. Graphs that admit a *distance magic vertex* labeling represent an EIT. The term distance magic labeling has evolved from previous terminology throughout the years. The concept was originally coined as a *sigma labeling* by Vilfred [5] in 1994, and then by Miller et. al. [6] using the name *1-vertex magic vertex labeling.* The definition of distance antimagic labeling nicely follows after the definition of distance magic labeling.

Definition 2.4: Distance Magic Labeling

A distance magic labeling of a graph G of order n is a bijection $f : V \to \{1, 2, ..., n\}$ with the property that there is a positive integer μ such that

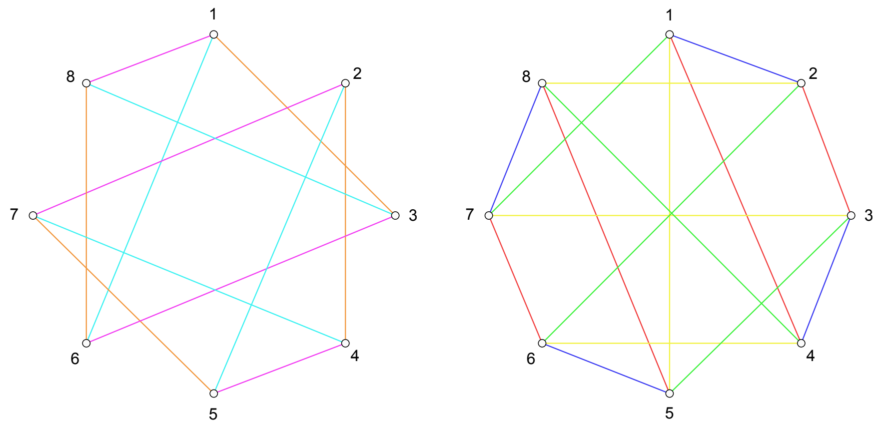
$$\sum_{y \in N(x)} f(y) = \mu \quad \text{for every} \quad x \in V.$$

The constant μ is called the magic constant of the labeling f, and $N(x)$ denotes the set of all vertices adjacent to v. The sum $\sum_{y \in N(x)} f(y)$ is called the weight of vertex x and is denoted $w(x)$. A graph that admits a distance magic labeling is called a distance magic graph. [5]

Definition 2.5: Distance d-Antimagic Labeling

A distance d-antimagic labeling of a graph $G(V, E)$ with n vertices is a bijection $\bar{f} : V \to \{1, 2, ..., n\}$ with the property that there exists an ordering of the vertices of G such that the weights $w(x_1), w(x_2), ..., w(x_n)$ forms an arithmetic progression with difference d. When $d = 1$, then $\bar{f}$ is called just distance antimagic labeling. A graph G is a distance d-antimagic graph if it allows a distance d-antimagic labeling, and a distance antimagic graph when $d = 1$. [4]

The *complement* $\overline{G}$ of G is defined to be the simple graph with the same vertex set as G where two vertices are adjacent in $\overline{G}$ precisely when they are not adjacent in G, see [1]. The complement of a distance magic graph is a distance antimagic graph and therefore the complement of an EIT is an FIT. An example of an FIT is shown in Figure 3a and is the associated graph with Table 4. That is, using team strength as vertex labels, the weights of the vertices match the total strength of opponents in the table. Similarly, an example EIT is shown in Figure 3b.



(a) an FIT(8,3) (b) an EIT(8,4)

Figure 3: Distance antimagic and distance magic graphs

2.3 Handicap Tournaments (HIT)

Round robin tournaments, often deemed the "fairest" of tournaments to the naked eye, actually favor the highest ranked team. This is because the total strength of its opponents is in fact the lowest. Therefore, since an FIT mimics an RRT, FIT's are also biased. Even if all teams play opponents with the same total strength, as in an EIT, then clearly the strongest team still has the highest chance of finishing with the most wins, for if not, said team was clearly not the strongest to begin with. Usually this is all ok, after all, one may typically want the strongest team to win the tournament. Handicap tournaments, however, are designed to offer teams a more balanced chance at winning the tournament. This might be very appealing to an audience and participants. It will be much more challenging to forecast the winner in a handicap tournament.

The term *handicap labeling* was originally introduced by Petr Kovář and Tereza Kovářová and previously referred to as *ordered distance antimagic labeling* by Froncek in [3]. We now formally define the handicap labeling.

Definition 2.6: Handicap Distance d-Antimagic Labeling

A handicap distance d-antimagic labeling of a graph $G(V, E)$ with n vertices is a bijection $\vec{f} : V \to \{1, 2, ...n\}$ with the property that $\vec{f}(x_i) = i$ and the sequence of the weights $w(x_1), w(x_2), ..., w(x_n)$ forms an increasing arithmetic progression with difference d. A graph G is a handicap distance d-antimagic graph if it allows a distance d-antimagic labeling, and handicap distance antimagic graph when $d = 1$. [4]

For convenience, we will use terms *handicap labeling* and *handicap graph* to refer to a handicap distance antimagic graph with $d = 1$ throughout the rest of this paper.

In a handicap tournament with n teams and g rounds, or HIT(n, g), we have $w(x_1) < w(x_2) < \cdots < w(x_n)$. Thus, roughly speaking, in a handicap tournament weaker teams play weaker opponents and stronger teams play stronger opponents. This is shown in Table 5 which corresponds to the graph shown in Figure 4. In a handicap labeling, for convenience we adopt the convention that $\vec{f}(x_i) = i$ for each vertex $x_i \in V$. Since $d = 1$, we can write the weight of a vertex as $w(i) = l + i$ for some integer l (for general d, $w(i) = l + d \cdot i$). In this example, we see from Table 5 that $l = 9$. Observe we have weights $w(1) = 10$ up to $w(8) = 17$.

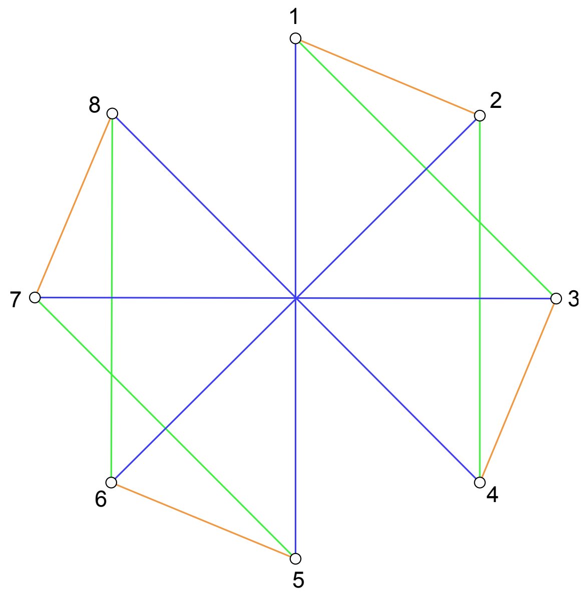


Figure 4: A handicap graph on 8 teams

Definition 2.7: Magic Squares

A *magic square* is a $p \times p$ array filled with the numbers $1, 2, \ldots, p^2$, each appearing once, such that the sum of each row, column, and the main and backward diagonal is equal to $p(p^2 + 1)/2$.

Seemingly unrelated, magic squares prove useful in the contruction of handicap graphs as in [4] and Theorem 4.4.

2.4 Bubble Graphs

Here we define a specific type of graph product that will be used in later sections. First, a formal definition.

Definition 2.8: Lexicographic Product

The lexicographic product $G[H]$ of graphs G and H with disjoint vertex sets $V(G)$ and $V(H)$ and edge sets X_1 and X_2 is the graph with vertex set $V(G) \times V(H)$ and $u = (u_1, u_2)$ is adjacent to $v = (v_1, v_2)$ whenever u_1 and v_1 are adjacent in G or $u_1 = v_1$ and u_2 is adjacent to v_2 in H.

In other words, $G[H]$ is obtained by replacing each vertex x_i in G with a copy of H, and linking these copies by edges of the complete bipartite graph $K_{t,t}$ where t is the number of vertices in H. Another way to think of this is that G is a "bubble graph" where the bubbles are the vertices. Each of these bubbles will house a certain number of smaller vertices. Often times, the bubble graph is less cluttered and easier to work with for constructing and analyzing properties of a graph. In the end, we will "pop" the bubbles or blow up the bubble graph (perform the product) to get $G[H]$ and achieve the desired result. Figure 5 gives an example of this where G is a bubble graph with three vertices and H is a graph on two vertices with no edges. Since H has two vertices, each edge in G then becomes a $K_{2,2}$ in the product $G[H]$. The lexicographic product is also called graph compositions.

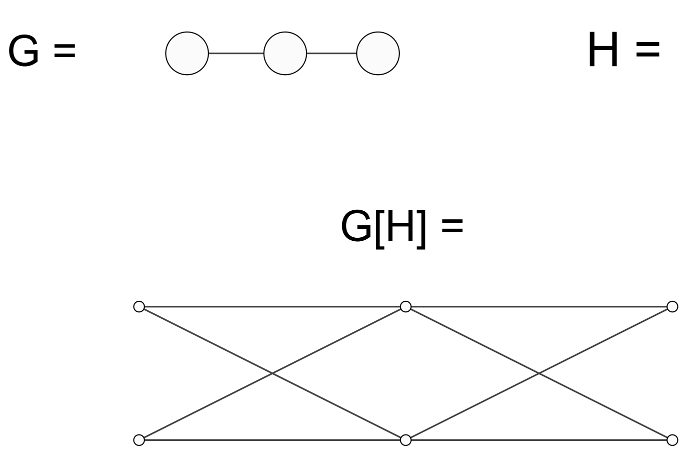


Figure 5: Example of lexicographic product

Since we are interested in tournaments, it makes sense to consider graphs that are *r-regular*. Thus in general the problem is: *For the r-regular graph on n vertices, what pairs (n,r) does there exist a handicap labeling?* While some simple numerical observations can show non-existence for certain cases of r and n, there is very little formal literature devoted to solving the problem, and solutions for many cases remain unseen. In the next sections, we discuss handicap labelings in more detail, and illustrate some new constructions and results developed this past summer. We begin with some basic observations.

3 Observations and Known Results

We will first illustrate a small numerical lemma. Recall that for a handicap graph, the weight of vertex labeled i is $w(i) = l + i$ for some integer constant l. When clear by context, we will often refer to a specific vertex by its label. The constant l is in fact determined by r and n.

Lemma 3.1. *Let G be a handicap graph that is r-regular on n vertices. Then $w(i) = l + i$ where $l = \frac{(r-1)(n+1)}{2}$.*

Proof. Consider the sum of the weights of all vertices. Since G is r-regular, every label is added r times, so we have

$$\sum_{i=1}^{n} w(i) = r \sum_{i=1}^{n} i$$

which can be rewritten as

$$\sum_{i=1}^{n} (l + i) = r \sum_{i=1}^{n} i\,.$$

By expanding and regrouping the first summation we get that

$$nl + \sum_{i=1}^{n} i = r \sum_{i=1}^{n} i$$

and then

$$nl = (r-1)\sum_{i=1}^{n} i = (r-1)\frac{n(n+1)}{2} \Rightarrow l = \frac{(r-1)(n+1)}{2}$$

as desired. $\square$

This can be used in conjunction with some other basic properties of graph theory to show there are many non-existence scenarios.

Theorem 3.1.

i. No 1-regular handicap graph exists.

ii. No 2-regular handicap graph exists.

iii. No handicap graph exists for n and r both even.

iv. No $(n-1)$-regular handicap graph exists.

v. No $(n-2t)$-regular handicap graph exists for all positive integers t.

vi. No handicap graph exists for $r \equiv 1 \pmod 4$ and $n \equiv 2 \pmod 4$.

vii. No $(n-3)$-regular handicap graph exists.

Proof (i – vi). Numbers one through six were proven by Petr Kovář et al. in [10]. The proofs are mostly numerical arguments based on $w(i) = l + i$ and using $l = \frac{(r-1)(n+1)}{2}$. We will use a similar strategy to prove *vii.* Number six relies on parity and the fact that the number of vertices of odd degree must be even. □

Proof (vii). By contradiction. Suppose G is handicap on n vertices, with $r = n - 3$. The complement, $\overline{G}$, is then 2-regular distance antimagic with common difference 2. To see this, observe that in G,

$$w(i) = l + i = \frac{(r-1)(n+1)}{2} + i = \frac{(n-4)(n+1)}{2} + i\,.$$

Let $\bar{w}(i)$ be the weight of vertex i in $\overline{G}$. We can compute $\bar{w}(i)$ by taking $w(i)$ and subtracting the total weight of i in the complete graph. In the complete graph, i is joined to every other vertex, so the weight would be the sum of the first n positive integers minus itself. So we have

$$\bar{w}(i) = \frac{n(n+1)}{2} - i - w(i) = \frac{n(n+1)}{2} - i - \left(\frac{(n-4)(n+1)}{2} + i\right)$$

$$= \frac{n(n+1)}{2} - \frac{(n-4)(n+1)}{2} - 2i = \frac{n(n+1) - (n-4)(n+1)}{2} - 2i$$

$$= \frac{n^2 + n - (n^2 - 3n - 4)}{2} - 2i = \frac{4n+4}{2} - 2i = 2n + 2 - 2i\,.$$

Thus, $\bar{w}(i) = 2n + 2 - 2i$, so as i ranges from 1 to n, the weight in the complement has a common difference of 2. Now, $\bar{w}(n) = 2n + 2 - 2n = 2$, which is impossible, since $\overline{G}$ is 2-regular, the minumum weight of any vertex is $1 + 2 = 3$. $\square$

This can also be done without venturing into the complement. An alternative proof is given below.

Alternative Proof (vii). Again by contradiction. Suppose G is handicap on n vertices, with $r = n - 3$. Thus, we can compute an upper bound for $w(n)$. Since n cannot be joined to itself, $w(n) \leq \sum_{i=3}^{n-1} i$ and

$$\sum_{i=3}^{n-1} i = \frac{(n-1)n}{2} - (1+2) = \frac{n^2 - n - 6}{2}\,.$$

Since we know

$$w(i) = \frac{(n-4)(n+1)}{2} + i \Rightarrow w(n) = \frac{(n-4)(n+1)}{2} + n$$
$$= \frac{n^2 - 3n - 4}{2} + n = \frac{n^2 - n - 4}{2}$$

we see that

$$w(n) \leq \sum_{i=3}^{n-1} i \iff \frac{n^2 - n - 4}{2} \leq \frac{n^2 - n - 6}{2}$$
$$\iff n^2 - n - 4 \leq n^2 - n - 6 \iff -4 \leq -6$$

which is a contradiction. $\square$

4 Results

Two strategies were investigated for constructing handicap graphs. One was to use a recursive type of a solution based on the methods used in [10]. That is, given a handicap graph on n vertices with a desired property, show that there exists a handicap graph on $n + k$ vertices with same property for some integer k. This seemed promising at the start, but posed more challenging than expected. Another strategy was to do a direct construction for certain cases of r and n. Our results use the latter.

If i is joined to k by an edge, we will use the notation $[i|k]$.Further, $[a, b|c, d]$ will denote the complete bipartite graph where a and b are both adjacent to c and d and vice-versa.

4.1 Case $n \equiv (0 \bmod 8)$

Theorem 4.1: n ≡ 0 (mod 8)

For n $\equiv$ 0 (mod 8) and $r \equiv 1, 3 \pmod 4$, there exists an r-regular handicap graph G on n vertices for all feasible values of r, that is, $3 \le r \le n - 5$.

Proof by Construction. First note that Thereom 3.1 proves non-existence for all other r values other than those claimed above. Since r is odd and at least 3, we can partition the edges at each vertex as follows: $2s$ black edges, 2 blue edges, and 1 red edge, for some nonnegative integer s. In other words we will have $2s$ 1-factors with edges colored black, a pair of 1-factors that are colored blue, and a single 1-factor colored red. The construction is complete in a three step process.

Step 1: The red edges will be used specifically to create the arithmetic progression required in the labeling by connecting $[1|4k+1]$, $[2|4k+2]$, $[3|4k+3]\ldots$, and $[4k|8k]$. This naturally partitions the vertex set into "lower" and "upper" sets.

Let $w_r(i)$ denote the weight of vertex i obtained from the red edges. We have that

$$w_r(i) = 4k + i \text{ for } i \in [1, 4k]$$

and

$$w_r(i) = -4k + i \text{ for } i \in [4k+1, 8k]\,.$$

Step 2: Now we construct the two blue edges to each vertex. For the lower vertices, the blue edges will be copies of $K_{2,2}$ as: $[1, 4k|2, 4k-1], [3, 4k-2|4, 4k-3], \ \ldots \ , [2k-1, 2k+2|2k, 2k+1]$, and the upper vertices will be done in a similar manner: $[4k+1, 8k|4k+2, 8k-1], [4k+3, 8k-2|4k+4, 8k-3], \ \ldots \ , [6k-1, 6k+2|6k, 6k+1]$. See Figures 6 and 7.

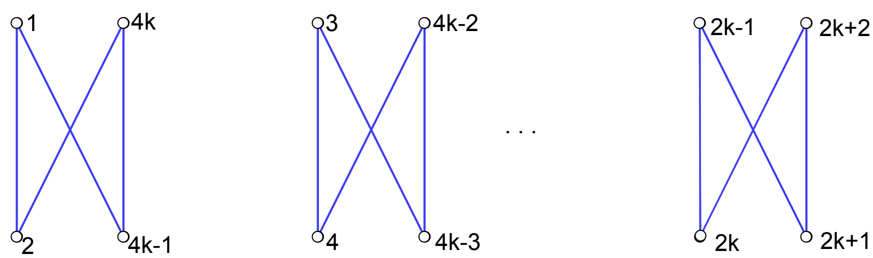


Figure 6: Lower Blue Edges

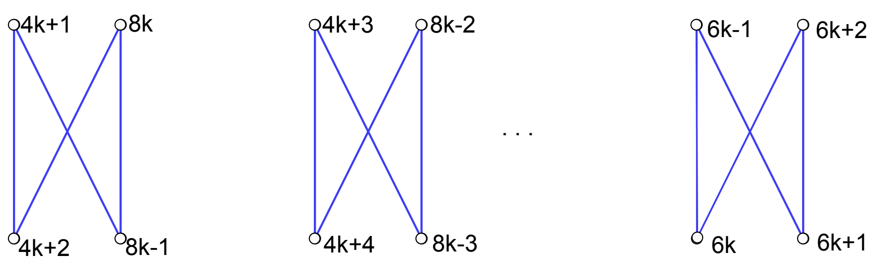


Figure 7: Upper Blue Edges

Let $w_b(i)$ denote the weight of vertex i obtained from the blue edges. Then

$$w_b(i) = 4k + 1 \text{ for } i \in [1, 4k]$$

and

$$w_b(i) = 12k + 1 \text{ for } i \in [4k+1, 8k]$$

so we have that

$$w_b(i) + w_r(i) = 4k + 1 + 4k + i = 8k + 1 + i \text{ for } i \in [1, 4k]$$

and

$$w_b(i) + w_r(i) = 12k + 1 - 4k + i = 8k + 1 + i \text{ for } i \in [4k + 1, 8k] \,.$$

Thus the weight of each vertex with the red and blue edges is $8k + 1 + i$ for each i, which is exactly what we want. The graph of red and blue edges is currently 3-regular and handicap with $l = 8k + 1$. Thus, if we can have the black edges contribute the same weight μ to each vertex, we will not be effecting the arithmetic progression of our weights, and therefore, still have a handicap graph with higher regularities.

Step 3:

Now our goal is to add $2s$ black edges such that the subgraph induced by the black edges is vertex magic. We need to be careful, though, to make sure that we are not trying to reuse any of the red or blue edges that are used in steps 1 and 2. To do this, we pair the vertices 1 with $8k$, 2 with $8k - 1$, ..., and $4k$ with $4k + 1$, so that the sum of these pairs is $8k + 1$. Each of these pairs can be thought of as a graph H with with no edges. Each pair becomes a vertex in our bubble graph B. In B, there will be an edge between two bubbles $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ if and only if there would be a red or blue edge (or both) between either x_1 or x_2 and y_1 or y_2. For clarity, we will color an edge red in B if it comes from step 1. Once all edges from step 1 are accounted for, we then add the edges from step 2 and of course color those blue. While the colors in the bubble graph are not important, it helps to see where the edges came from. What happens here is the red and blue edges create separate components of B, each of which is K_4.

To see this, take any bubble $J = (a, 8k + 1 - a)$. Since there is a red edge $[a|4k + a]$, we have $[J|K]$ where $K = (4k + 1 - a, 4k + a)$. We know the other half of the bubble K must have weight $4k + 1 - a$ since the sum inside each bubble is $8k + 1$. We also have the blue $K_{2,2}$ involving a, namely

$[a, 4k+1-a|a+1, 4k-a]$. Specifically, since there exists a blue edge $[a|a+1]$, we have $[J|L]$ where $L = (a+1, 8k-a)$. Similarly, $[J|M]$ where $M = (4k-1, 4k+1+a)$. Checking all other existing red and blue edges, we have a red edge $[4k-a|8k-a]$, and the blue $K_{2,2} = [4k+a, 8k+1-a|4k+1+a, 8k-a]$. Observe that any red or blue edges that would emerge from the four bubbles J, K, L, and M only result in edges between these four bubbles. See Figure 8.

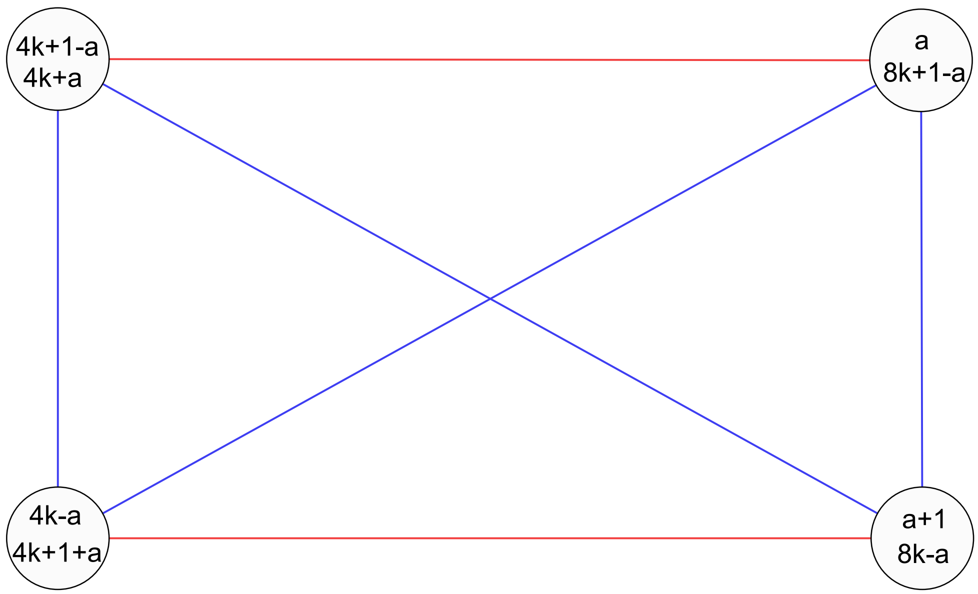


Figure 8: Bubble Structure

Since we have $\frac{n}{2}$ bubbles, $\overline{B} = K_{\frac{n}{2}} - \frac{n}{8}K_4$. This is in fact isomorphic to the complete multipartite graph $K_{\frac{n}{8}[4]}$, that is, a graph with $\frac{n}{8}$ partite sets of size 4. It is a well known result that the complete multipartite graph on an even number of vertices can be 1-factored [9]. Since this is such an important tool for our own results, we state it as a theorem below, followed by a short explanation.

Theorem 4.2: 1-factorization of complete multipartite graphs

Let K be a complete multipartite graph with n vertices. If n is even then K is class 1 and can be 1-factored. [9]

Let Δ be the largest vertex degree of a graph G. A graph G is *class 1* if Δ is the number of colors required to color each edge so that no two edges incident to the same vertex have the same color. Since $\overline{B}$ is a complete multipartite graph where each partite set is of size 4, Δ is the degree of every vertex in $\overline{B}$. Thus, the edges can be colored by exactly Δ different colors so that each edge incident to the same vertex is a different color. The subgraphs induced by each color then are edge disjoint 1-factors.

Now, the bubble graph B is 3-regular, and the complement $\overline{B}$ will be $(\frac{n}{2}-4)$-regular. $\overline{B}$ is the graph where we will pull our black edges from. Each black edge in $\overline{B}$ equates to a $K_{2,2}$ in the blown up graph $\overline{B}[H]$, therefore, each 1-factor induced on $\overline{B}$ will consist of a 2-regular distance magic graph we can add to the red and blue edges, as desired. If we use all available black edges, we can add $2(\frac{n}{2}-4) = n-8$ black edges to increase regularity, for a max regularity of $n-8+1+2 = n-5$. □

As it often is with graph theory, seeing pictures of a graph is often much more useful than the most explicit textual explanation. We offer the following not only as an example but as an aid in understanding the main ideas of the contruction for Thereom 4.1.

4.1.1 Example Construction of 5-regular Handicap Graph on $n = 32$ Vertices

Since $n = 32 = 8 \cdot 4$, we have in this example that $k = 4$.

Step 1: We start with the red edges by connecting $[1 \mid 4k+1], [2 \mid 4k+2], [3 \mid 4k+3] \ldots$, and $[4k \mid 8k]$, since $k = 4$ we have $[1 \mid 17], [2 \mid 18], [3 \mid 19] \ldots$, and $[16 \mid 32]$. See Figure 9. At the end of step 1 we have the following weights:

$$w_r(i) = 4 \cdot 4 + i \text{ for } i \in [1, 16] \text{ (lower vertices) and}$$

$$w_r(i) = i - 4 \cdot 4 \text{ for } i \in [17, 32] \text{ (upper vertices).}$$

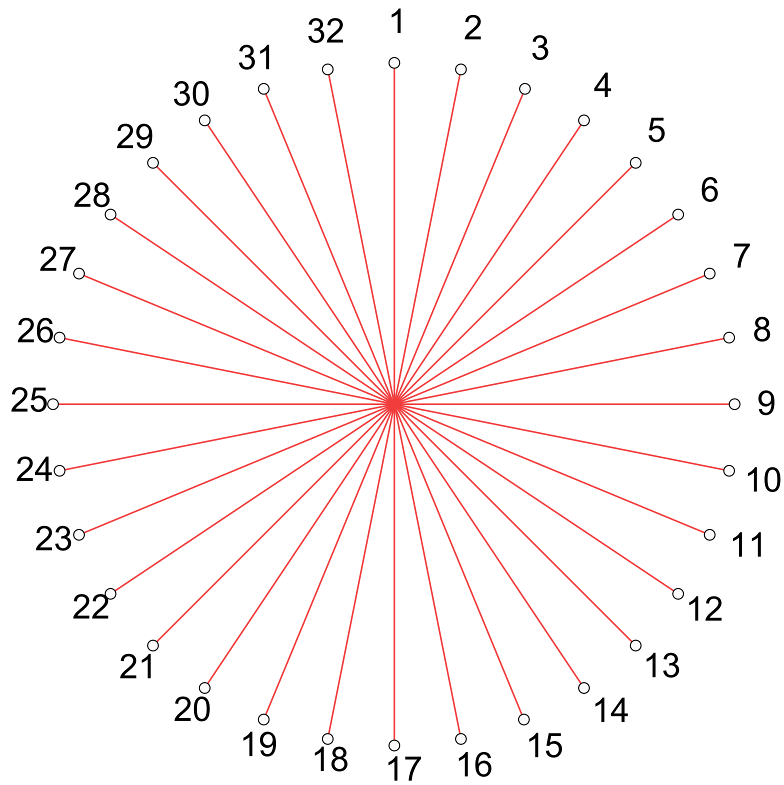


Figure 9: Step 1 on 32 vertices

Step 2: We now add the blue $K_{2,2}$'s. For the lower vertices: $[1, 4k \mid 2, 4k-1], [3, 4k-2 \mid 4, 4k-3], \ldots, [2k+1, 2k+2 \mid 2k, 2k+1]$, that is, $[1, 16 \mid 2, 15], [3, 14 \mid 4, 13], \ldots, [7, 10 \mid 8, 9]$. Thus we are adding a weight of 17 to each lower vertex.

For the upper vertices: $[4k+1, 8k \mid 4k+2, 8k-1], [4k+3, 8k-2 \mid 4k+4, 8k-3], \ldots, [6k-1, 6k+2 \mid 6k, 6k+1]$, that is, $[17, 32 \mid 18, 31], [19, 30 \mid 20, 29], \ldots, [23, 26 \mid 24, 25]$. Thus we are adding a weight of 49 to each upper vertex. See Figure 10 (notice the graph is not connected). We now have the following weights:

$$w_b(i) + w_r(i) = 4 \cdot 4 + 1 + 4 \cdot 4 + i \text{ for lower vertices and}$$

$$w_b(i) + w_r(i) = 12 \cdot 4 + 1 + i - 4 \cdot 4 \text{ for upper vertices.}$$

Thus, we currently have a 3-regular handicap graph with $l = 33$.

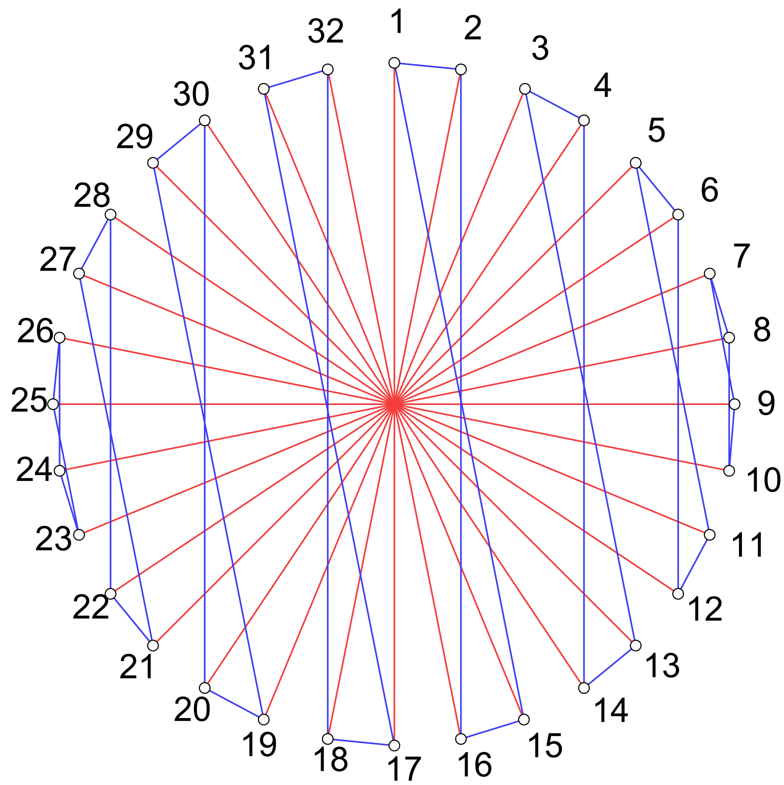


Figure 10: Step 2 on 32 vertices

Step 3: Now we take a look at which black edges are available to use. First we construct the bubble graph B, drawing red or blue edges between bubbles for edges already used in step 1 or 2. This is shown in Figure 11.

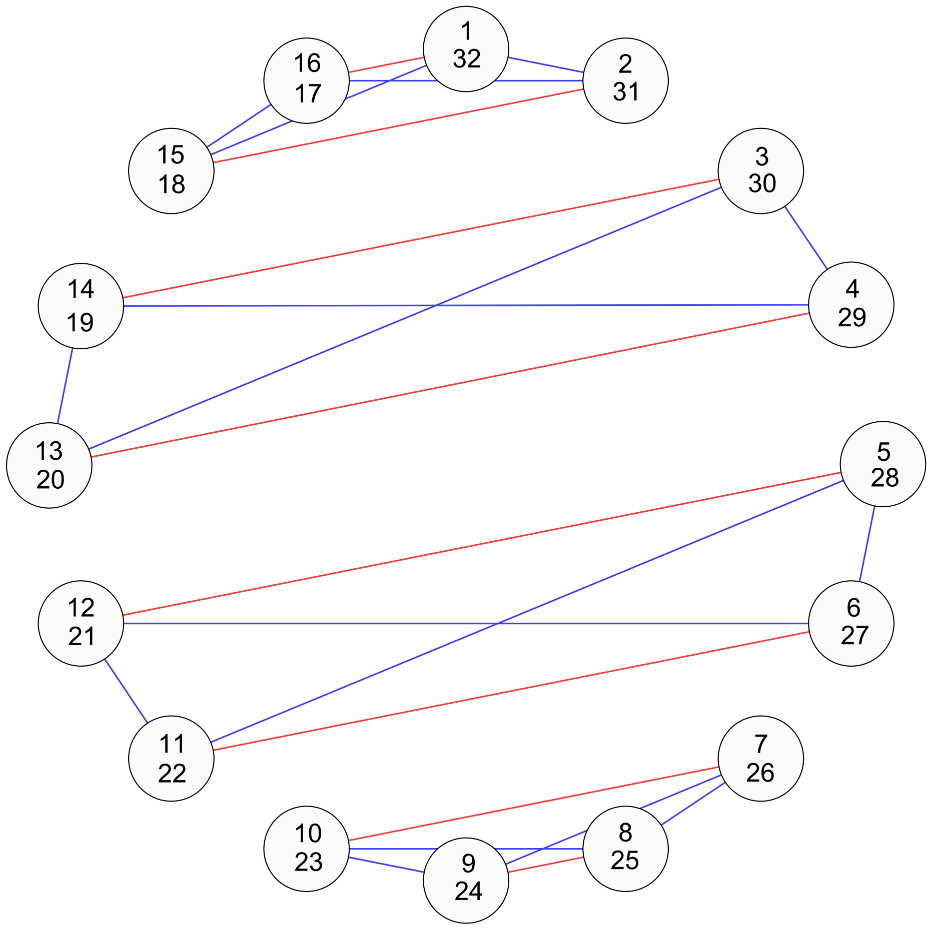


Figure 11: Bubble Graph on 32 vertices

Figure 11 makes it more evident that $\overline{B}$ is a complete multipartite graph. Let us refer to each bubble by the minimum of the two labels it contains. Consider the bubbles 1, 2, 15, and 16. In the complement, these will form one partite set, and each will have a black edge to bubbles 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, and 14. Further, since 3, 4, 13, and 14 are all connected in B, these will form another partite set in $\overline{B}$, and be joined to 1, 2, 5, 6, 7, 8, 9, 10, 11, 12, 15, and 16. We continue in this fashion to see $\overline{B}$ is $K_{4,4,4,4}$, or $K_{4[4]}$. $\overline{B}$ is shown in Figure 12.

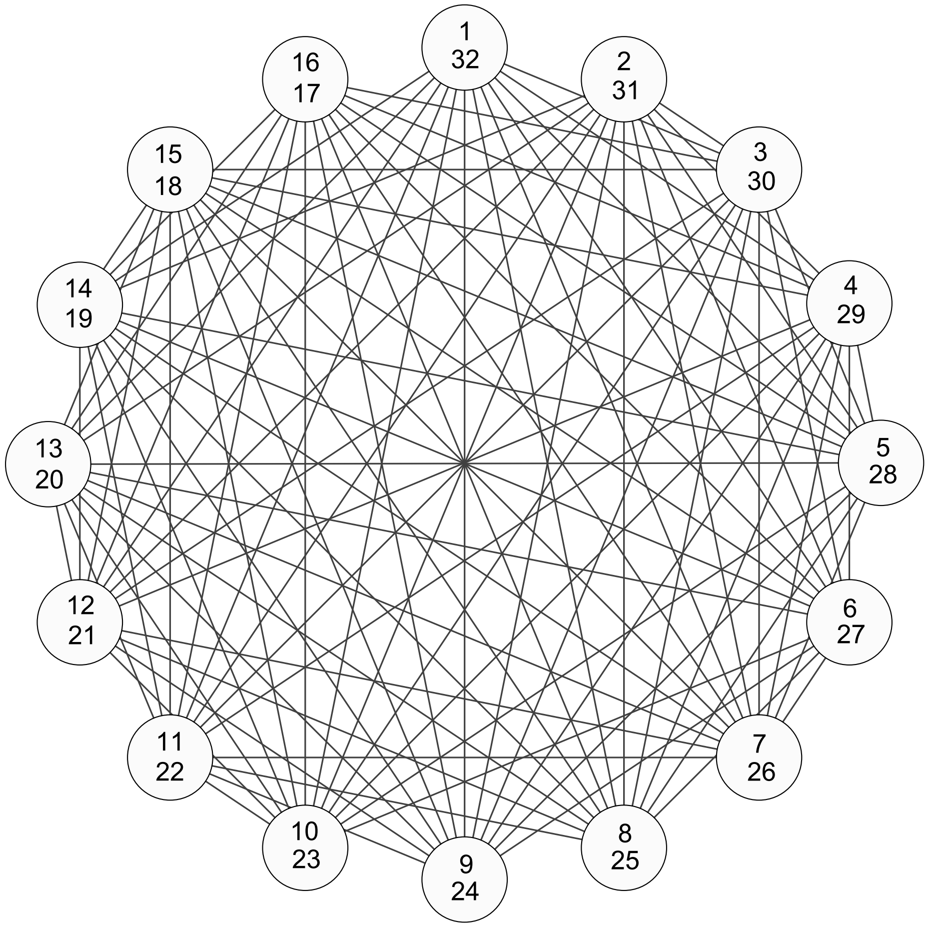


Figure 12: Complement of Bubble Graph on 32 vertices

Lastly we can choose our favorite 1-factor from a 1-factorization of $\overline{B}$ to increase regularity by two in our handicap graph. No matter what we choose, we increase the weight of each vertex by 33 (the sum of the vertices in each bubble), resulting in a 5-regular handicap graph on 32 vertices with $w(i) = 66 + i$ for all i. Our final graph is shown in Figure 13.

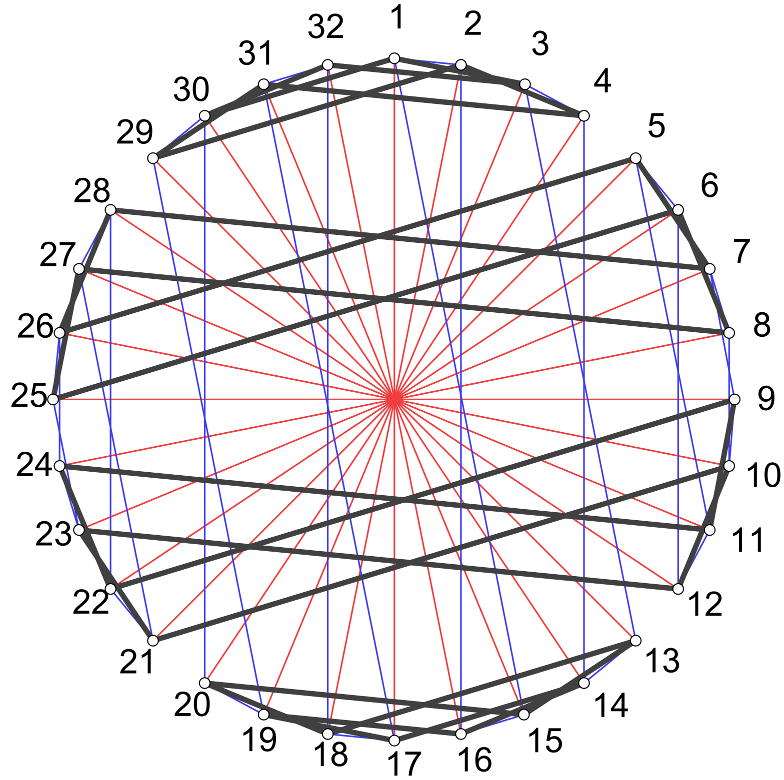


Figure 13: Step 3 on 32 vertices (increasing regularity)

4.2 Case $n \equiv (4 \bmod 8)$

Theorem 4.3: n ≡ 4 (mod 8)

For n $\equiv$ 4 (mod 8) and $r \equiv 1, 3 \pmod 4$, there exists an r-regular handicap graph G for $7 \leq r \leq n-5$.

Proof by Construction. Similar to the proof for the case of $n \equiv 0 \pmod 8$, we will split the edges up into three colors. Suppose $r = 2s+7$. We will have 1 red edge, 6 blue edges, and $2s$ black edges. Since we have 6 blue edges we will break that stage up into three parts.

Step 1: For $n = 8k+4$, we will do the red edges as follows: $[1 \mid 2k+2]$, $[2 \mid 2k+3]$, $[3 \mid 2k+4]$, $\ldots$, and $[2k+1 \mid 4k+2]$ (uses the lower vertices), followed by $[4k+3 \mid 6k+4], [4k+4 \mid 6k+5], \ldots,$ and $[6k+3 \mid 8k+4]$ (uses the upper vertices). Let $w_r(i)$ denote the weight of vertex i obtained from the red edges. We have that

$$w_r(i) = 2k+1+i \text{ for } i \in [1, 2k+1] \cup [4k+3, 6k+3]$$

and

$$w_r(i) = i-(2k+1) \text{ for } i \in [2k+2, 4k+2] \cup [6k+4, 8k+4].$$

Step 2.1: In the first stage of adding blue edges, we construct multiple copies of $K_{2,2}$ that include exactly half of the vertices. Namely $[1, 6k+3 \mid 2, 6k+2], [2, 6k+2 \mid 3, 6k+1], \ldots, [2k, 4k+4 \mid 2k+1, 4k+3], [2k+1, 4k+3 \mid 1, 6k+3]$. We will call the set of vertices used A, so $A = \{1, 2, \ldots, 2k+1, 4k+3, 4k+4, \ldots, 6k+3\}$.

Step 2.2: Similar to the first stage, we add $K_{2,2}$'s to the other half of the vertices, specifically $[2k+2, 8k+4 \mid 2k+3, 8k+3], \ldots, [4k+1, 6k+5 \mid 4k+2, 6k+4], [4k+2, 6k+4 \mid 2k+2, 8k+4]$. We name the set of vertices used here B, so $B = \{2k+2, 2k+3, \ldots, 4k+2, 6k+4, 6k+5, \ldots, 8k+4\}$.

Step 2.3: The graph induced by the blue edges is currently 4-regular. To add the last two edges we intertwine the $K_{2,2}$'s already created. For each new $K_{2,2}$ one partite set comes from A and one partite set comes from B. For example, we take the first partite set from step 2.1, and connect it to the second partite set from step 2.2. In general, connect $[1, 6k+3 | 2k+3, 8k+3], [2, 6k+2 | 2k+5, 8k+1], \ldots, [2k+1, 4k+3 | 2k+2, 8k+4]$. Let $w_b(i)$ denote the weight of vertex i obtained from the blue edges. We have that

$$w_b(i) = 22k+14 \text{ for } i \in [1, 2k+1] \cup [4k+3, 6k+3]$$

and

$$w_b(i) = 26k+16 \text{ for } i \in [2k+2, 4k+2] \cup [6k+4, 8k+4].$$

Then for

$$i \in [1, 2k+1] \cup [4k+3, 6k+3]$$

$$w_b(i) + w_r(i) = 22k+14+2k+i = 24k+15+i$$

and for

$$i \in [2k+2, 4k+2] \cup [6k+4, 8k+4]$$

$$w_b(i) + w_r(i) = 26k + 16 + i - (2k+1) = 24k + 15 + i\,.$$

So we have a 7-regular handicap graph, with $l = 24k + 15$. Again, if we can have the black edges contribute the same weight μ to each vertex, we will not effecting the arithmetic progression of our weights, and therefore, still have a handicap graph with higher regularities.

Step 3: Recall that $r = 2s + 7$. Our goal now is to show that we can add $2s$ black edges such that the graph induced by the black edges is distance magic. Pair the vertices 1 with $8k+4$, 2 with $8k+3$, $\ldots$, and $4k+2$ with $4k+3$, so that sum of these pairs is $8k+5$. Each pair can be thought of as a graph H with no edges and becomes a vertex in our bubble graph B. In B, there will be an edge between two bubbles $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ if and only if there would be a red or blue edge (or both) between either x_1 or x_2 and y_1 or y_2.

To more easily understand the structure of the bubble graph, we look at the edge lengths. We refer to each bubble by the minimum of the two labels it contains. Place the bubbles at uniform distance in a circle, starting with 1 at the top-center position, in a clock–wise fashion.

In step 1, we define our red edges, all of which are length $2k+1$. In step 2.1, we see blue edges come in a couple different lengths, namely 1 and $2k$. In step 2.2, we see blue edges also come in length 1 and $2k$. In step 2.3, we have blue edges of lengths 1 and $2k$ as well. Thus, in B, the edges are all of length 1, $2k$, and $2k+1$. Since $n = 8k+4$ we have exacly $n' = 4k+2$ bubbles. For any given bubble, there are 2 bubbles at length 1 away (one clockwise and one counter-clockwise), 2 bubbles at length $2k$ away, and exactly one bubble at length $2k+1$ away. If all edges of lengths 1, $2k$, and $2k+1$ are used in B, B is 5-regular. Thus, $\overline{B}$ will have all edges of the lengths that are not present in B, so $\overline{B}$ is isomorphic to a circulant graph $G(\{3, \ldots, 2k-1\}, n')$.

Recall Lemma 2.1, which says that $\overline{B}$ can 1-factored if there exists an edge length d of $\overline{B}$ so that $n'/\gcd(d, n')$ is an even integer. This can be done as follows. Let d be an odd edge length in the edge set of $\overline{B}$. Such a d exists since 3 will always be an edge length used in $\overline{B}$. Recall that $n' = 4k+2$. Let $n'' = n'/2 = 2k+1$, thus n'' is odd. Now, since d is odd and n' is even we have that

$$\gcd(d, n') = \gcd(d, n'/2) = \gcd(d, n'')$$

is an odd integer since both d and n'' are odd. Now, since the $\gcd(d, n'')$ divides both n'' and d, $n''/\gcd(d, n'')$ is an integer. Thus,

$$\frac{n''}{\gcd(d, n'')} = \frac{n''}{\gcd(d, n')} \Rightarrow \frac{2n''}{\gcd(d, n')} = \frac{n'}{\gcd(d, n')}$$

is an even integer. And so, by Lemma 2.1, $\overline{B}$ can be 1-factored.

Each black edge in $\overline{B}$ equates to a $K_{2,2}$ in the blown up graph $\overline{B}[H]$. Therefore, each 1-factor in $\overline{B}$ will contribute a 2-regular distance magic graph to the red and blue edges. We can add $2(\frac{n}{2} - 6) = n - 12$ black edges to increase regularity, if desired, for a max of $n - 12 + 1 + 6 = n - 5$. $\square$

Again, the reader my find it useful to see an example of the construction for Theorem 4.3, so we present one here.

4.2.1 Example Construction of 7-regular Handicap Graph on $n = 28$ Vertices

In this example, $n = 28 = 8(3) + 4$, so $k = 3$. The resulting graph is just 7-regular, but with 28 vertices it is somewhat dense for the human eye to digest. Thus at the end of the example we offer an alternative view of the graph by seperating red and blue edges. This more clearly indicates what the structure of these graphs look like.

Step 1: We start with the red edges by connecting $[1 \mid 2k+2], [2 \mid 2k+3], [3 \mid 2k+4], \ldots,$ and $[2k+1 \mid 4k+2]$, followed by $[4k+3 \mid 6k+4], [4k+4 \mid 6k+5], \ldots,$ and $[6k+3 \mid 8k+4]$. So for the lower vertices we have $[1 \mid 8], [2 \mid 9], [3 \mid 10], \dots,$ and $[7 \mid 14]$. For the upper vertices we have $[15 \mid 22], [16 \mid 23], \dots,$ and $[21 \mid 28]$. This is shown in Figures 14 and 20. Let $w_r(i)$ denote the weight of vertex i obtained from the red edges. We have that

$$w_r(i) = 7 + i \text{ for } i \in [1, 7] \cup [15, 21]$$

and

$$w_r(i) = i - 7 \text{ for } i \in [8, 14] \cup [22, 28]\,.$$

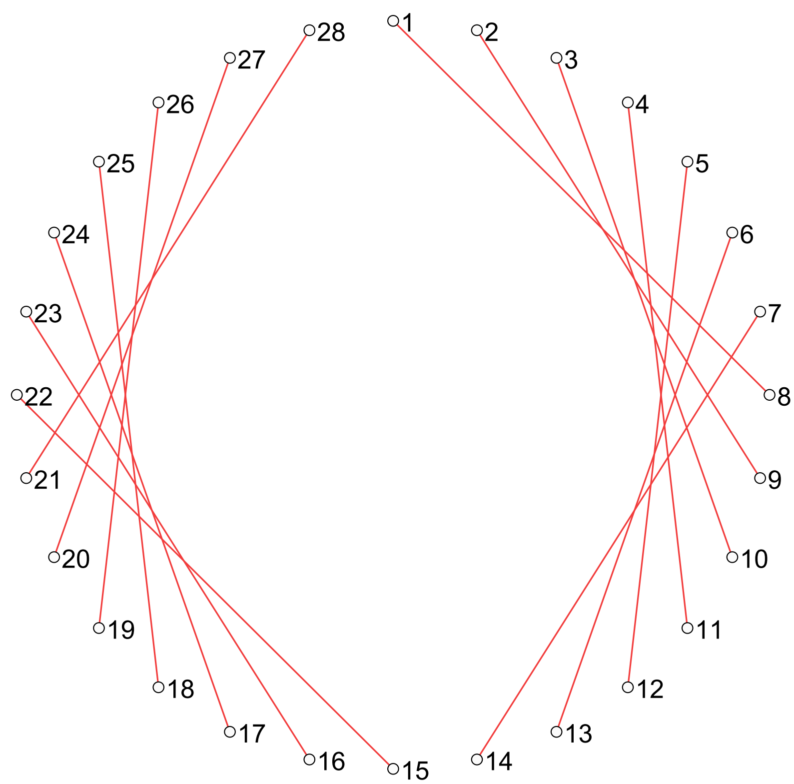


Figure 14: Step 1 on 28 vertices

Step 2.1: We now add the first set of blue $K_{2,2}$'s. Namely $[1, 21 \mid 2, 20], [2, 20 \mid 3, 19], \dots, [6, 16 \mid 7, 15], [7, 15 \mid 1, 21]$. See Figure 15. Thus, in this example $A = \{1, 2, \dots, 7, 15, 16, \dots, 21\}$.

Step 2.2: We now add the second set of blue $K_{2,2}$'s, $[8, 28 \mid 9, 27], \dots, [13, 23 \mid 14, 22], [14, 22 \mid 8, 28]$. See Figure 16. In this example $B = \{8, 9, \dots, 14, 22, 23, \dots, 28\}$.

Steps 2.1 and 2.2 are shown in the alternative view in Figure 21.

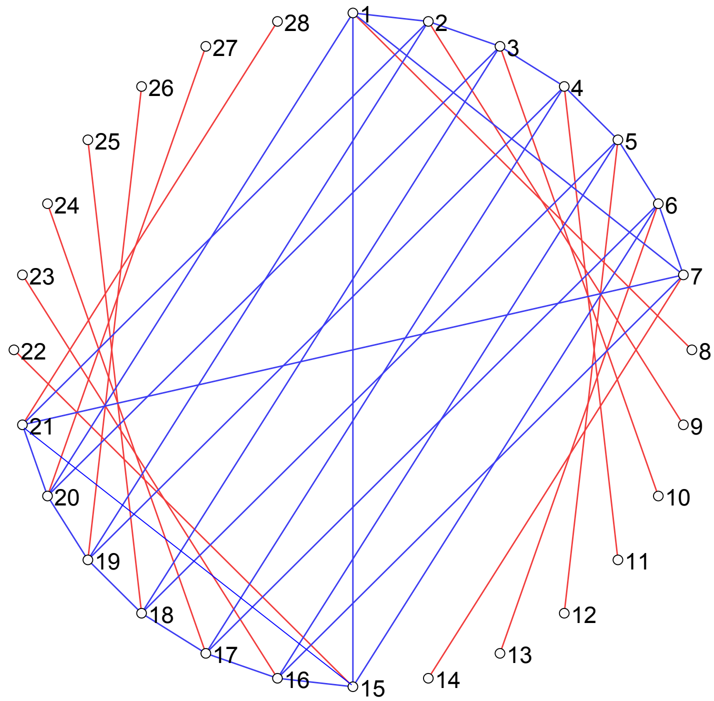


Figure 15: Step 2.1 on 28 vertices

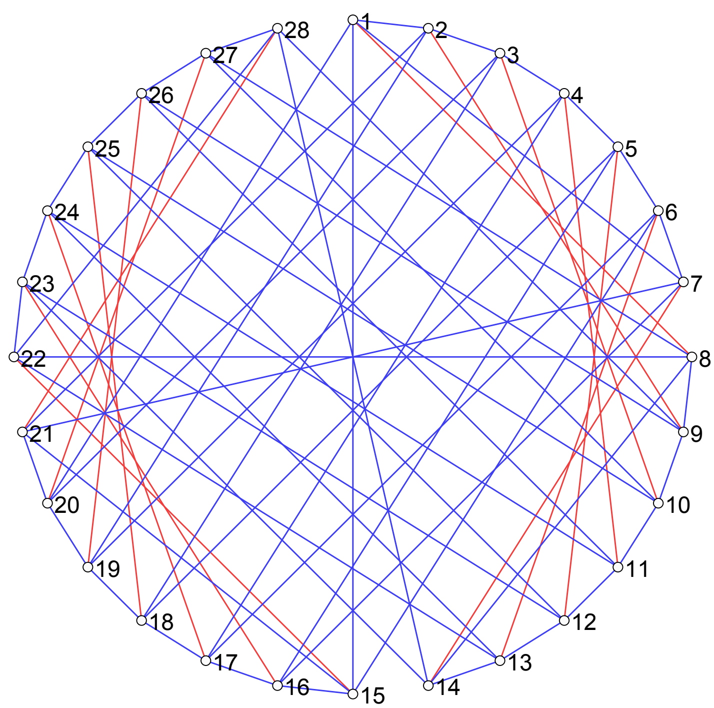


Figure 16: Step 2.2 on 28 vertices

Step 2.3: In this step, each new $K_{2,2}$ has one partite set that comes from A and one partite set from B. For example, the first will be $[1, 21 \mid 9, 27]$. This can be seen in Figure 22. In a similar fashion we complete the process, adding $[2, 20 \mid 14, 22], \ldots, [7, 15 \mid 8, 28]$. The completion of this process can be seen in Figures 17 and 23. Let $w_b(i)$ denote the weight obtained from the blue edges for vertex i. Then

$$w_b(i) = 22(3) + 14 \text{ for } i \in [1, 7] \cup [15, 21]$$

and

$$w_b(i) = 26(3) + 16 \text{ for } i \in [8, 14] \cup [22, 28]$$

so we have that

$$\text{for } i \in [1, 7] \cup [15, 21]$$

$$w_b(i) + w_r(i) = 22(3) + 14 + 2(3) + i = 24(3) + 15 + i$$

and

$$\text{for } i \in [8, 14] \cup [22, 28]$$

$$w_b(i) + w_r(i) = 26(3) + 16 + i - (2(3) + 1) = 24(3) + 15 + i\,.$$

Thus we have a 7-regular handicap graph with $l = 24(3) + 15$. For completeness, we will illustrate the process of step 3 even though we are not going to add any black edges to this example.

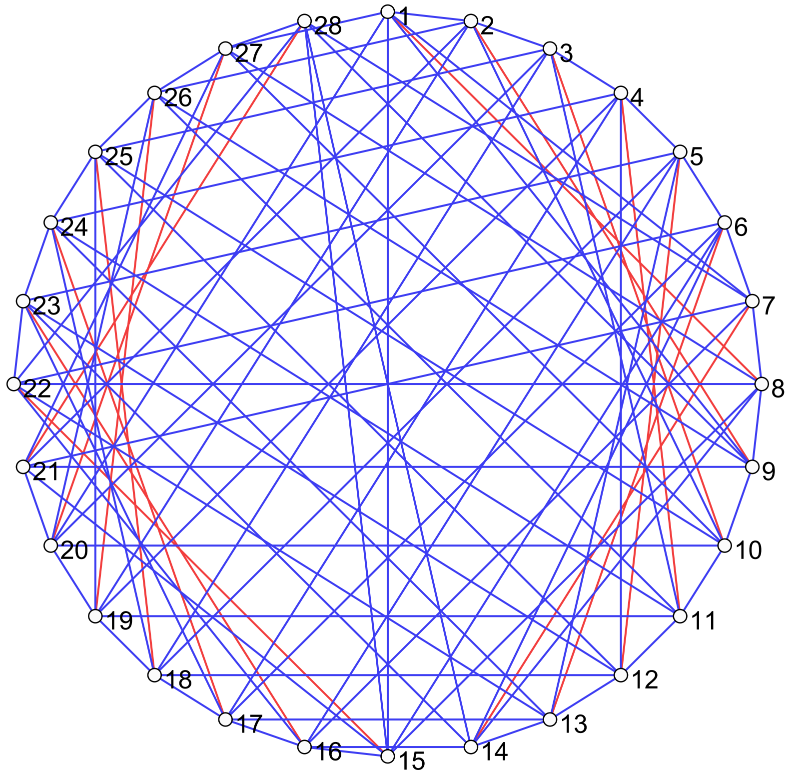


Figure 17: Step 2.3 on 28 vertices

Step 3: Now we took a look at which black edges are available to use. First we construct the bubble graph B by pairing the vertices to form bubbles so that sum of each pair is 29. Then we draw red or blue edges between bubbles for edges already used in step 1 or 2. The beautiful structure of this graph is shown in Figure 18.

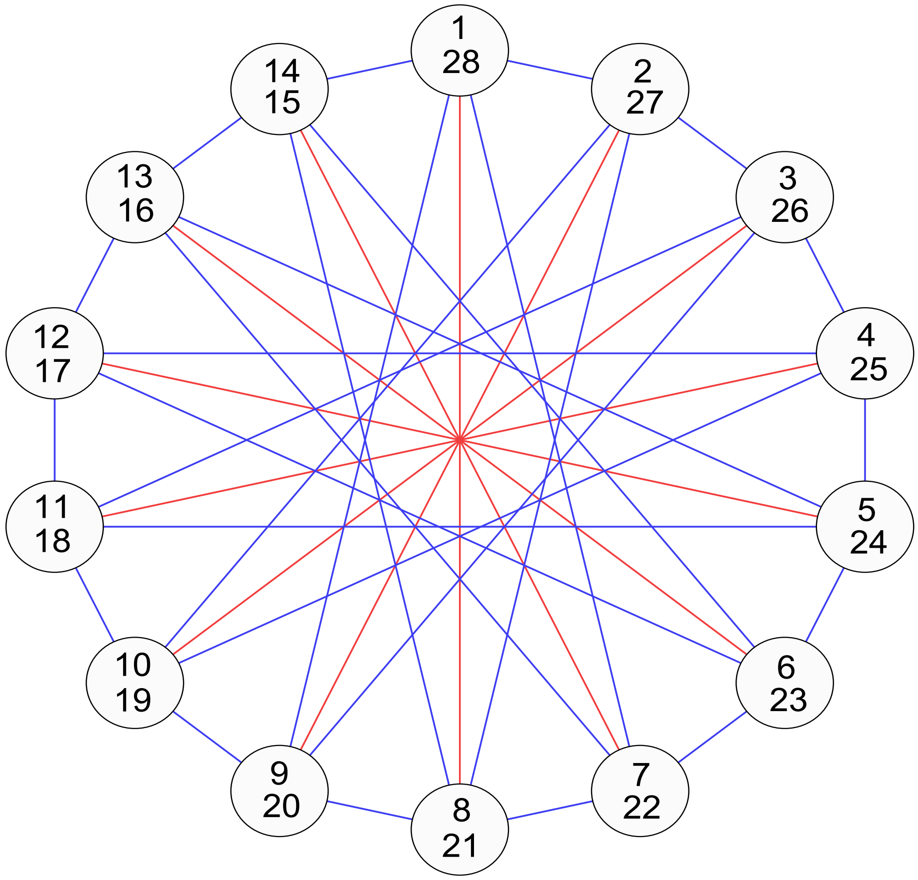


Figure 18: Bubble Graph on 28 vertices

We then would take the complement of this to get $\overline{B}$, shown in Figure 19. This is where we would pull black edges from to increase regularity. $\overline{B}$ is 8-regular, and since each black edge equates to a $K_{2,2}$ in the blown up graph, we can have a handicap graph that has max regularity equal to $8(2)+1+6 = 23$, i.e. $n-5 = 28-5 = 23$, if desired.

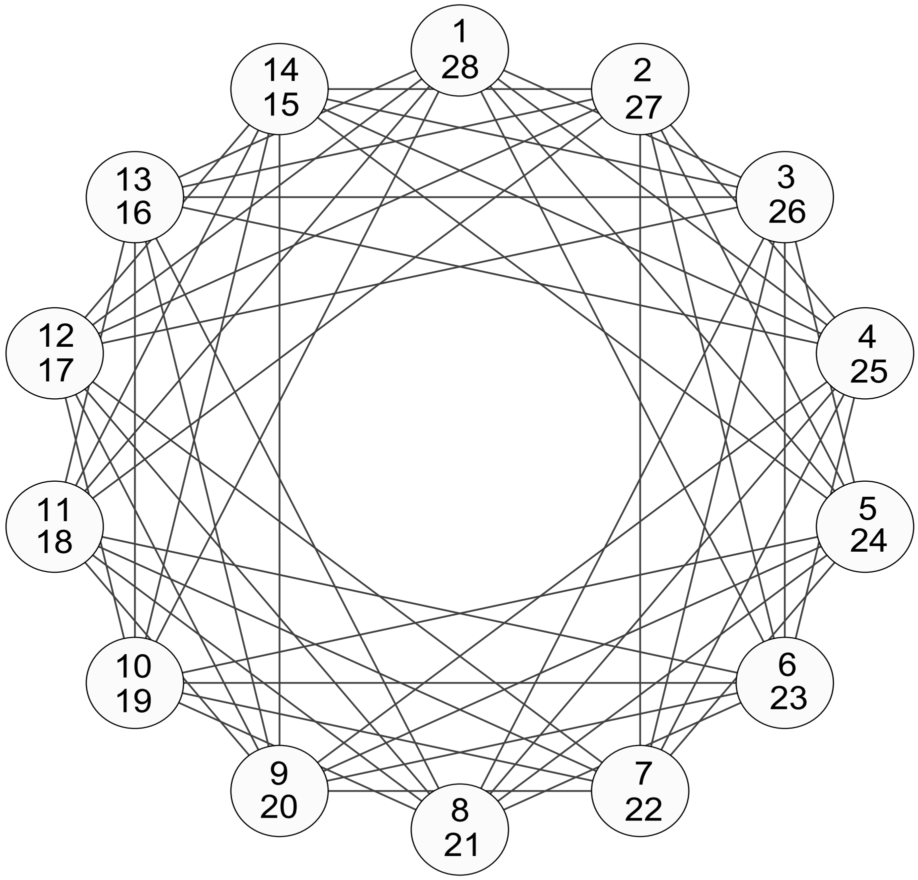


Figure 19: Complement of Bubble Graph on 28 vertices

Figure 20: Construction of Step 1 on 28 vertices

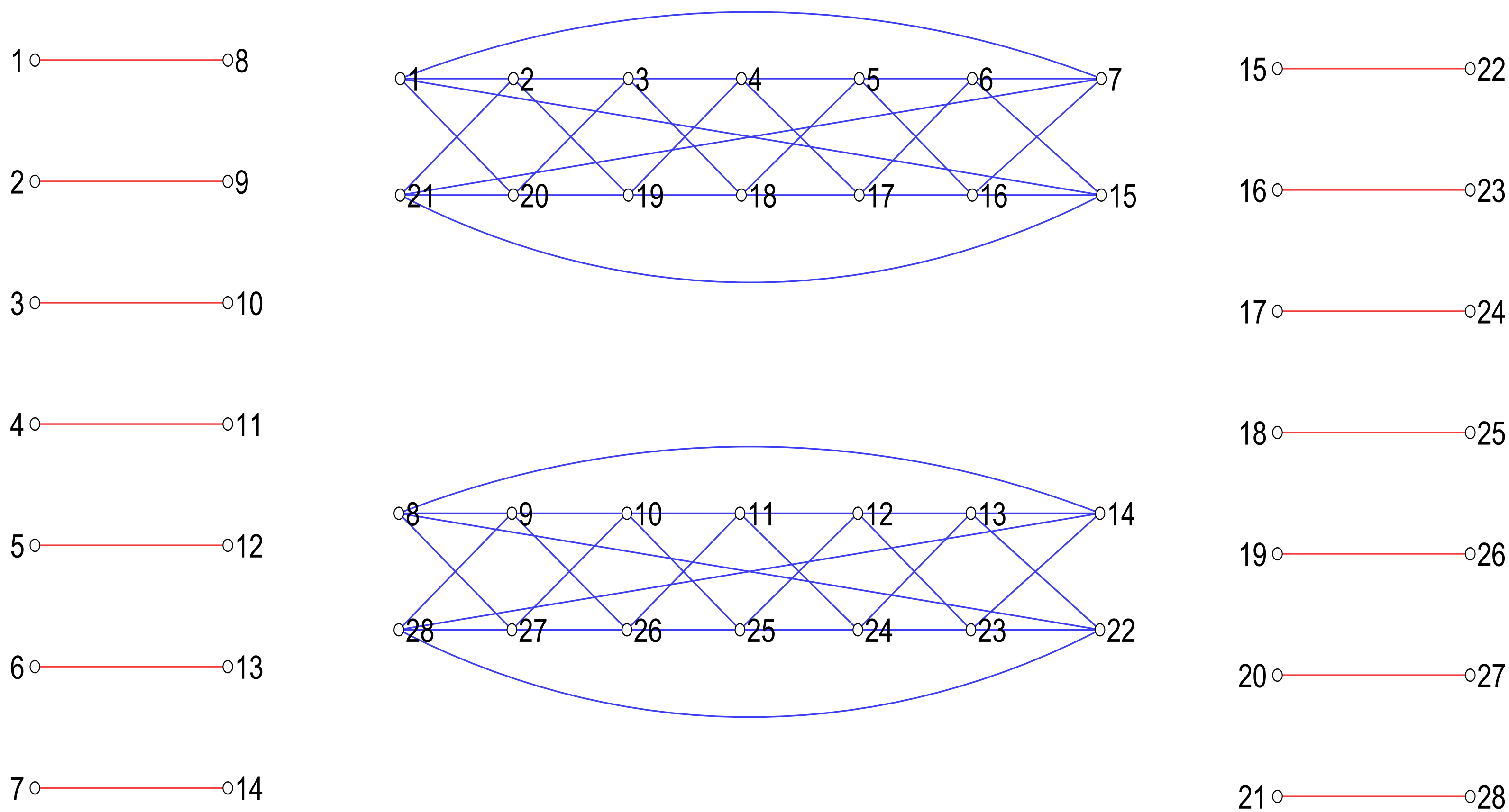


Figure 21: Construction of Step 2.1 and 2.2 on 28 vertices

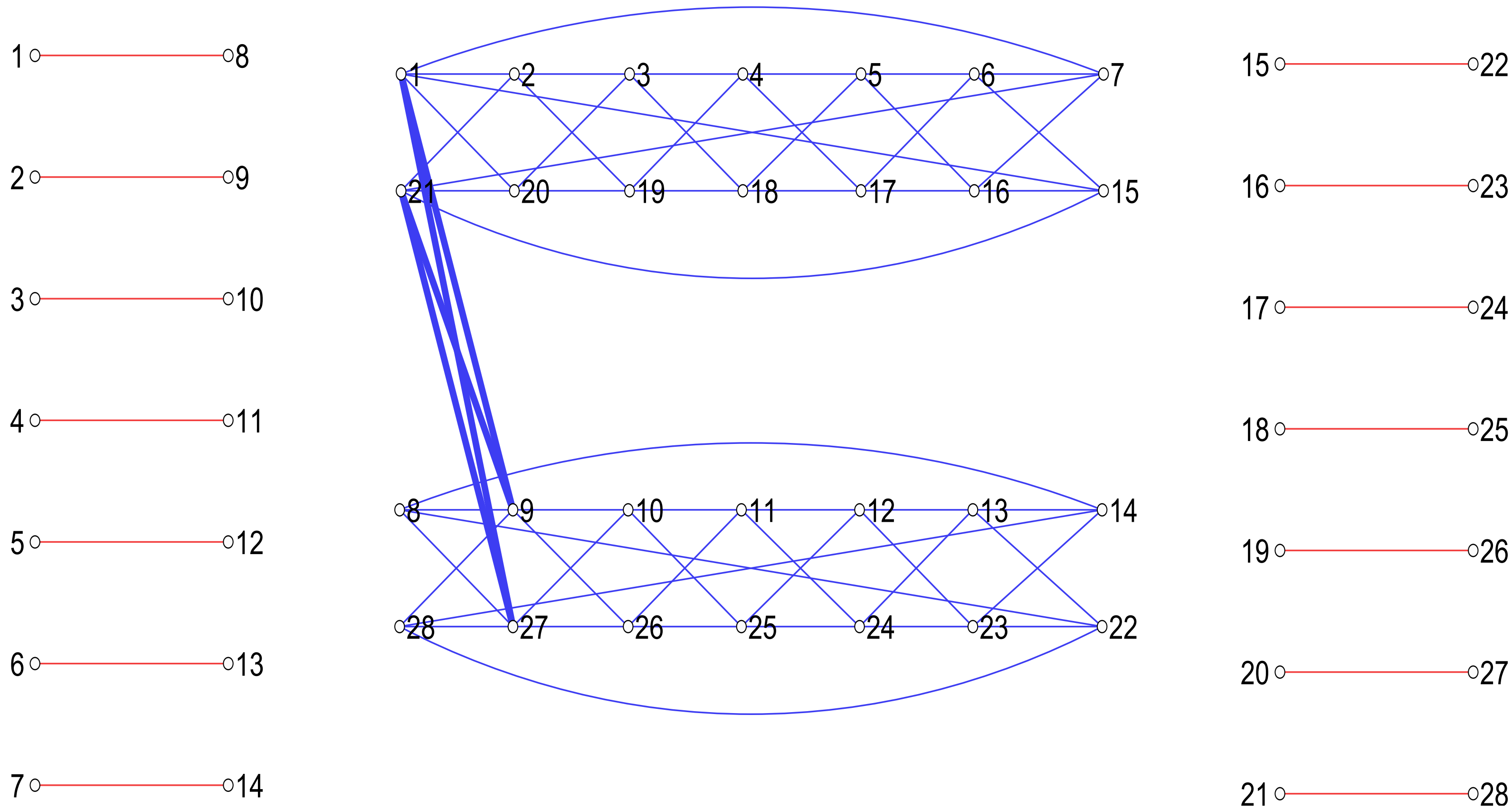


Figure 22: Construction of Step 2.3 on 28 vertices, adding first $K_{2,2}$

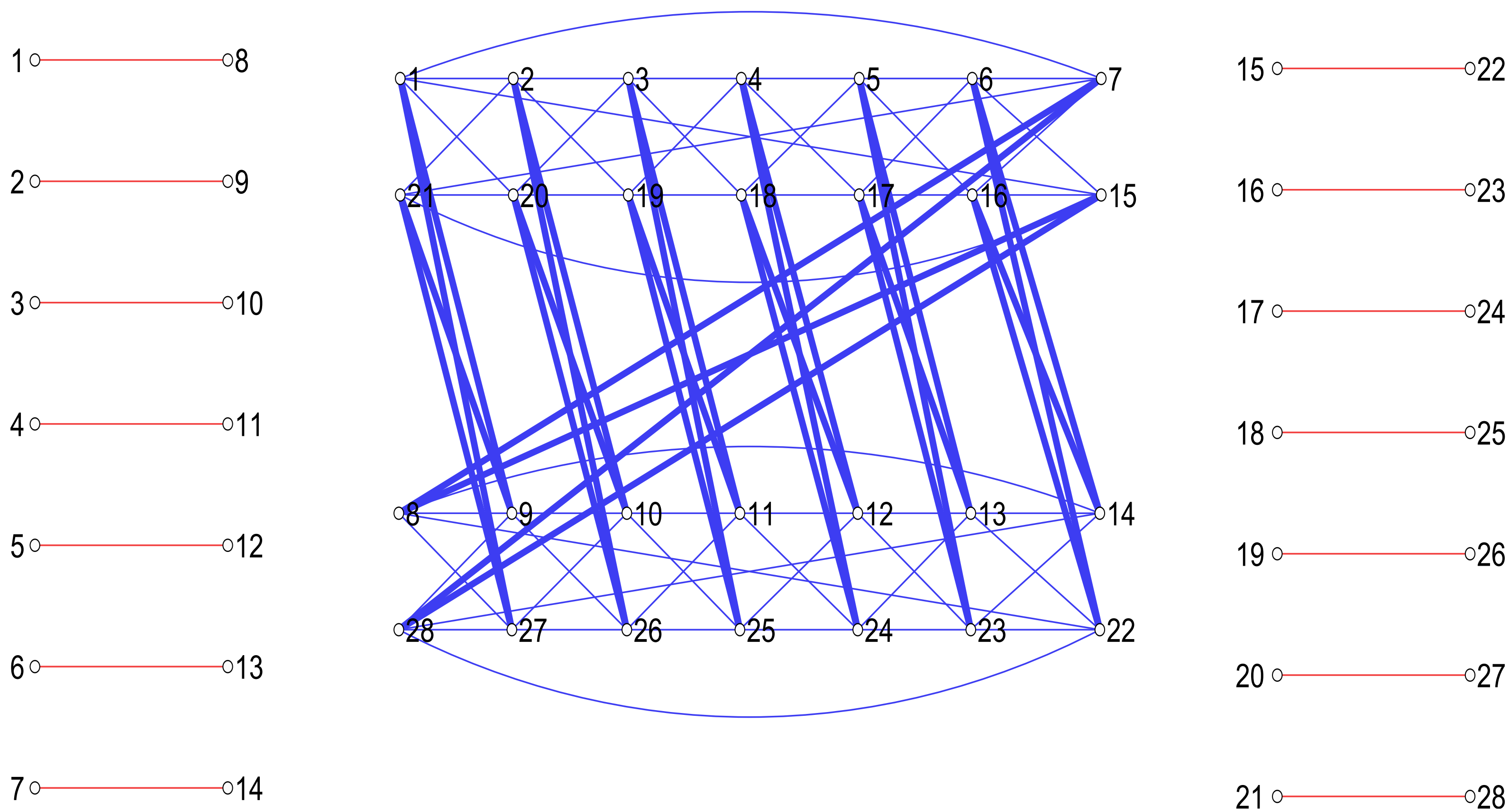


Figure 23: Construction of Step 2.3 on 28 vertices

4.3 Case $n \equiv (2 \bmod 4)$

We now move to a different type of result. After communications with P. Kovář we became aware that results for the case where $n \equiv (2 \bmod 4)$ were incomplete. Specifically, the dense graph with $r = n - 7$ was unsolved. What follows is a solution for this unsolved case, also done by construction.

Theorem 4.4: n ≡ (2 mod 4) (part 1)

For a handicap graph G with $n \equiv 2 \pmod 4$ and $r = n - 7$, there exists a handicap graph G' on $n + 16$ vertices and $r = n + 16 - 7$.

Proof by Construction. The idea for this construction is that the complement of a handicap graph is a distance antimagic graph where the weights $w(1), w(2), \ldots, w(n)$ form a decreasing arithmetic progression with common difference 2 (see the proof for Theorem 3.1). What we do here is construct the complement of the desired handicap graph, that is, a 6-regular distance antimagic graph with common difference 2 on $n+16$ vertices. G is handicap, so $w(i) = l + i$ and

$$l = \frac{(r-1)(n+1)}{2} = \frac{(n-8)(n+1)}{2} = \frac{n^2 - 7n - 8}{2}\,.$$

In a complete graph, vertex i has a weight of

$$\sum_{k=1}^{n} k - i = \frac{n(n+1)}{2} - i = \frac{n^2+n}{2} - i\,.$$

Let $w^c(i)$ be the weight of vertex i in $\overline{G}$. Since G is $(n-7)$-regular, $\overline{G}$ is 6-regular and

$$w^c(i) = \frac{n^2+n}{2} - i - (l+i) = \frac{n^2+n}{2} - \frac{n^2-7n-8}{2} - 2i = \frac{8n+8}{2} - 2i = 4n + 4 - 2i\,.$$

Now, consider an additional component with vertex set H (Table 7) constructed from a 4×4 magic square (see Definition 2.4). Each vertex is adjacent to every other vertex in its own row and column. Here, 16 is adjacent to 3, 2, 13, 5, 9, and 4. Vertex with label 10 is adjacent to 5, 11, 8, 3, 6, and 15. The associated graph is shown in Figure 24. Letting σ equal the row and column sum, we see that $\sigma = 34$.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Table 7: The vertex labels of H as a magic square

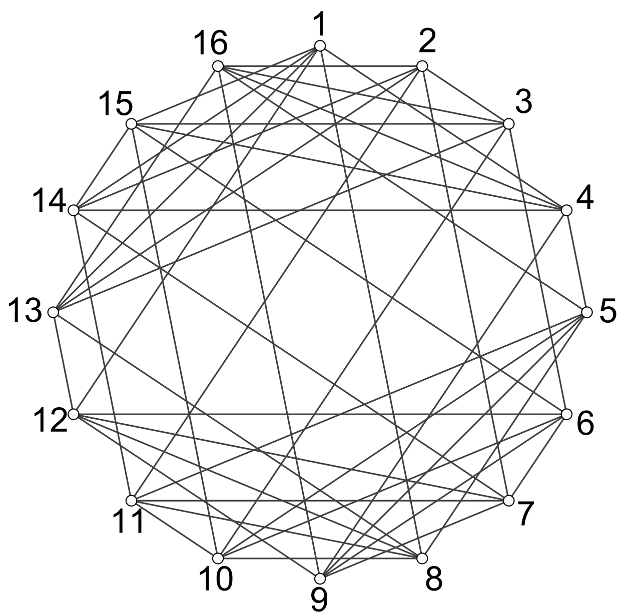


Figure 24: The Graph represented by Table 7

The weight of a vertex in a magic square is easily computed by adding the row sum and column sum, which is 2σ, and making sure to subtract the vertex label twice (since its label was counted once in the row sum and once in the column sum).

Now, to construct G', we let j denote the label of a vertex in G'.

For

$$i \in H, i = 1, 2, \ldots, 8, \text{ let } j = i$$

$$i \in H, i = 9, 10, \ldots, 16, \text{ let } j = i + n$$

$$i \in \overline{G}, \text{ let } j = i + 8 .$$

16+n	3	2	13+n
5	10+n	11+n	8
9+n	6	7	12+n
4	15+n	14+n	1

Table 8: Modified magic square, showing $\sigma_j = 34 + 2n$

Since each column/row has exactly two of 9, 10,. . . , 16, the new magic constant is $\sigma_j = 34 + 2n$. The modified magic square can be seen in Table 8. Thus for $i \in H, i = 1, 2, \ldots, 8$,

$$w^c(i) = w^c(j) = 2(34 + 2n) - 2j = 68 + 4n - 2j .$$

Now, for $i \in H, i = 9, 10, \ldots, 16$, only one row neighbor and one column neighbor are increased by n, so we have for $i \in H, i = 9, 10, \ldots, 16$,

$$w^c(i) = w^c(j - n) = 2(34 + n) - 2i = 2(34 + n) - 2(j - n)$$

$$= 68 + 4n - 2j .$$

Lastly, since we increased each vertex in $\overline{G}$ by 8, and $\overline{G}$ is 6-regular, we added a total weight of $6 \cdot 8 = 48$ to each vertex in $\overline{G}$. From this we have for $i \in \overline{G}$,

$$w^c(i) = w^c(j-8) = (4n+4)+6\cdot 8-2i = 4n+52-2(j-8) = 68+4n-2j .$$

Thus, $\overline{G'} = H \cup \overline{G}$ has the desired property that the weight of each vertex is $4n + 68 - 2j$, so $\overline{G'}$ is distance antimagic with common difference 2. Therefore, G' is a handicap graph on $n + 16$ vertices and it is connected. □

Using the above construction, we can confirm the following theorem.

Theorem 4.5: $n \equiv 2 \pmod 4$ (part 2)

For $n \equiv 2 \pmod 4$ and $r = n - 7$, there exists an r-regular handicap graph G on n vertices for $n \geq 14$.

Proof. The first case to consider is $n = 10$ since n must be large enough to allow the desired regularity. A brute force search found that no 3-regular handicap graph on 10 vertices exists. The next four cases are $n = 14, 18, 22$, and 26, all of which exist. These graphs are shown in the figures that follow. In the appendix we have provided tables detailing the calculation of the weights, showing that the graphs are indeed handicap. Since the construction of Theorem 4.4 adds 16 vertices to create another handicap graph, and $n = 14, 18, 22$, and 26 are four consecutive cases modulo 4, we can iterate the construction to achieve an $(n-7)$-regular handicap graph for all $n \equiv 2 \pmod 4$ for $n \geq 14$.

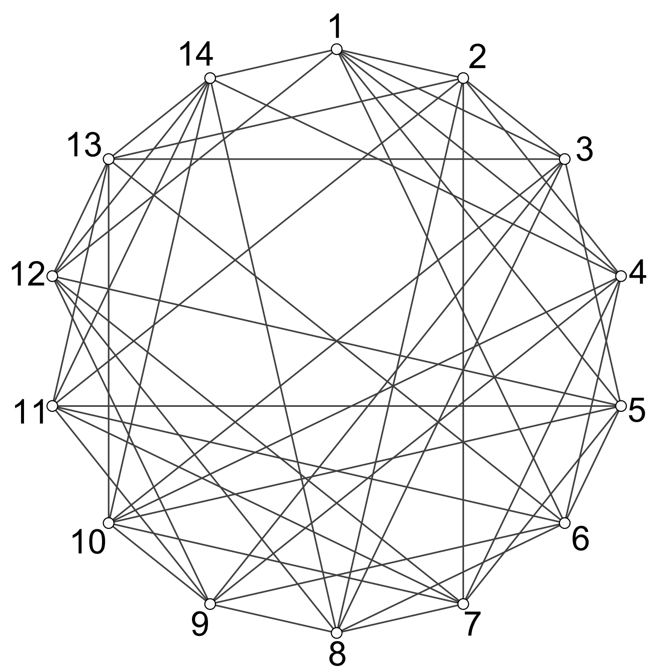


Figure 25: 7-regular handicap graph on 14 vertices

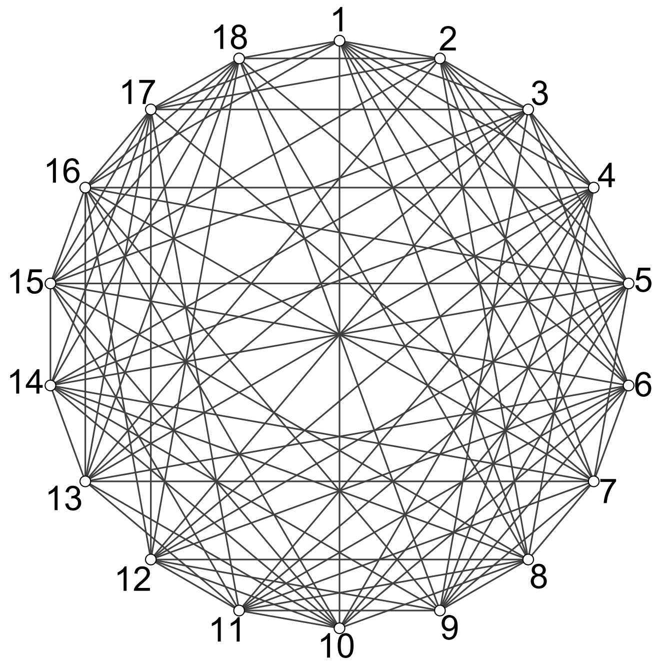


Figure 26: 11-regular handicap graph on 18 vertices

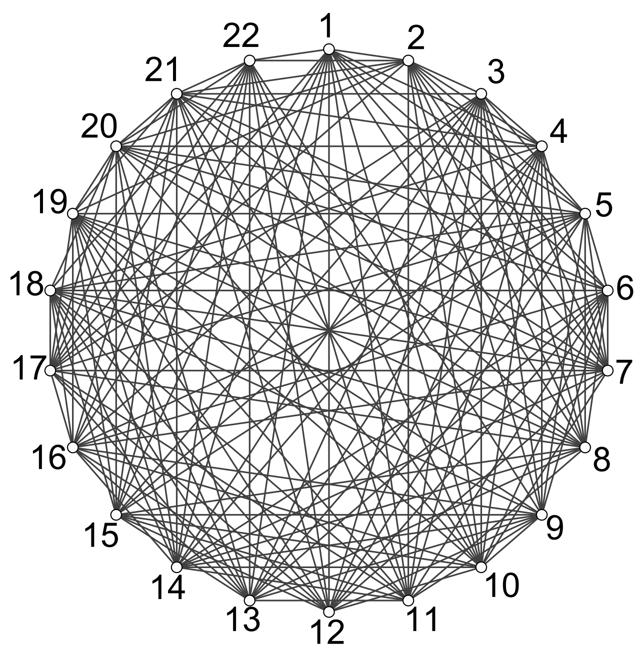


Figure 27: 15-regular handicap graph on 22 vertices

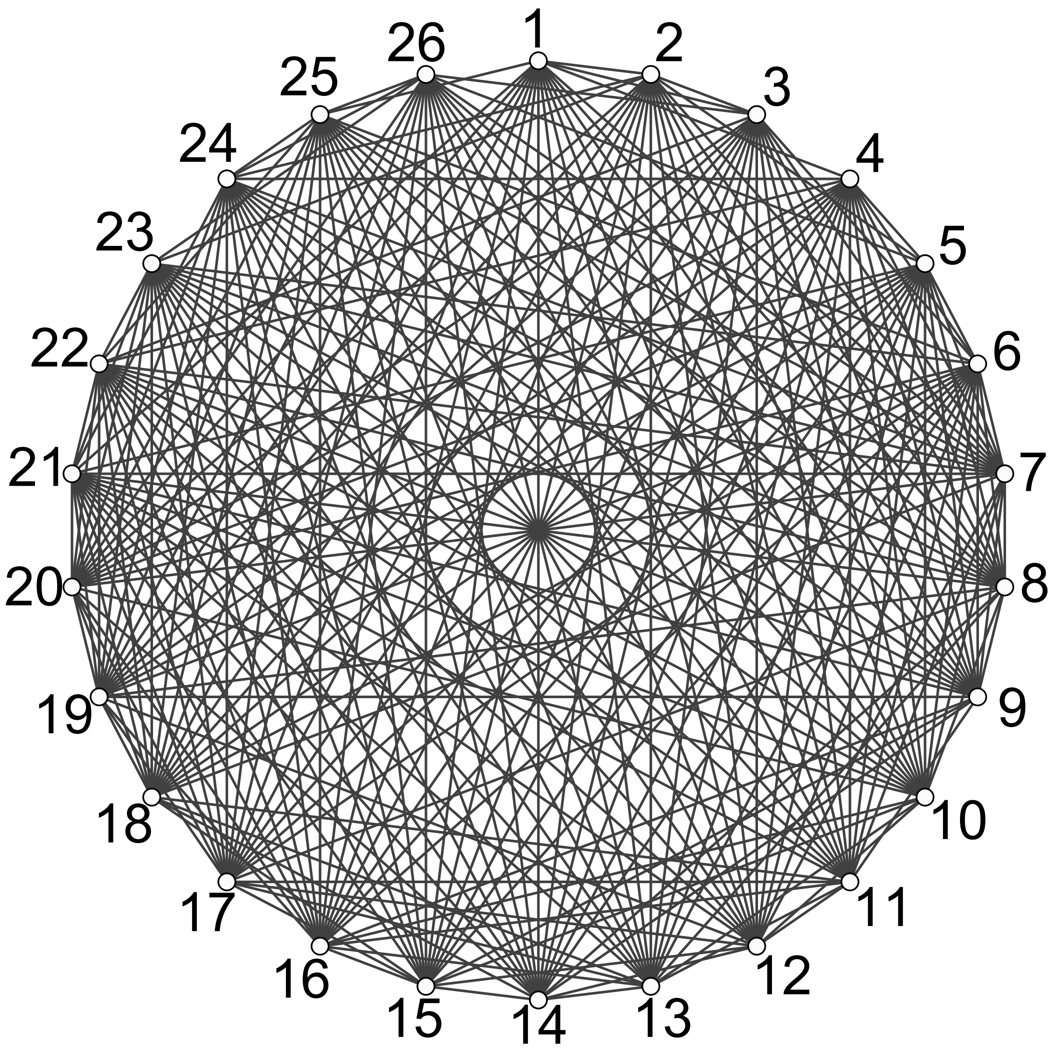


Figure 28: 19-regular handicap graph on 26 vertices

□

5 Conclusion and Future Work

We take a moment to summarize the results in Table 9.

n\r	0 (mod 4)	1 (mod 4)	2 (mod 4)	3 (mod 4)
0 (mod 4)	DNE	Thm 4.1 & Thm 4.2	DNE	Thm 4.1 & Thm 4.2
1 (mod 4)	?	DNE	?	DNE
2 (mod 4)	DNE	DNE	DNE	Thm 4.4 & Kovář
3 (mod 4)	?	DNE	?	DNE

Table 9: Overview of cases modulo 4

For $n \equiv 2 \pmod 4$ and $r \equiv 1{,}3 \pmod 4$ our results in combination with the soon to be published results of Kovář will solve that case completely. He covers most feasible values of r except the most dense scenario, $r = n - 7$, and our theorem handles that specific case. It's also worth mentioning that for $n \equiv (0 \bmod 4)$ and r odd our results are split into two cases. Theorem 4.1 covers all feasible values of r with $n \equiv (0 \bmod 8)$, while Theorem 4.3 covers all feasible values of r for $n \equiv (4 \bmod 8)$, *except* $r = 3$, or 5. These last two cases, however, are solved in [10]. We summarize these results in the theorem that follows.

Theorem 5.1: Froncek, Kovář, Shepanik

An r-regular handicap graph G on n vertices exists when $n \geq 8$ and

1. $n \equiv 0 \pmod 4$ if and only if $3 \leq r \leq n - 5$ and r is odd
2. $n \equiv 2 \pmod 4$ if and only if $3 \leq r \leq n - 7$ and $r \equiv 3 \pmod 4$

except when $r = 3$ and $n = 10, 12, 14, 18, 22,$ and 26.

What remains open here are cases for n odd and r even. Kovář has found specific examples of handicap graphs that fit each of the four missing cases. However, we have not had enough time to find any general results in this area. Lastly, we see that over the years different types of tournaments, and therefore different types of labelings, have evolved due to different goals of scheduling. While handicap tournaments and handicap labelings may be the most recent venture, I highly doubt it will be the last type of tournament defined. New goals may arise and give birth to a new type of tournament. This new genre of tournament is most certianly best studied as a graph labeling problem and such a labeling may not even exist yet.

Appendix

Tables of Weights for Select Handicap Graphs

7-Regular Handicap Graph on 14 Vertices

Vertex	Neighbors							Weights
1	2	3	4	5	6	12	14	**46**
2	1	3	4	7	8	11	13	**47**
3	1	2	5	8	9	10	13	**48**
4	1	2	6	7	9	10	14	**49**
5	1	3	6	7	10	11	12	**50**
6	1	4	5	8	9	11	13	**51**
7	2	4	5	8	10	11	12	**52**
8	2	3	6	7	9	12	14	**53**
9	3	4	6	8	10	11	12	**54**
10	3	4	5	7	9	13	14	**55**
11	2	5	6	7	9	13	14	**56**
12	1	5	7	8	9	13	14	**57**
13	2	3	6	10	11	12	14	**58**
14	1	4	8	10	11	12	13	**59**

Table 10

11-Regular Handicap Graph on 18 Vertices

Vertex	Neighbors											Weights
1	2	3	4	5	6	7	8	10	16	17	18	**96**
2	1	3	4	5	6	7	8	13	15	17	18	**97**
3	1	2	4	5	6	9	12	13	14	15	17	**98**
4	1	2	3	8	9	10	11	12	13	14	16	**99**
5	1	2	3	7	9	10	11	12	14	15	16	**100**
6	1	2	3	7	9	10	11	12	13	15	18	**101**
7	1	2	5	6	8	9	11	13	14	16	17	**102**
8	1	2	4	7	9	10	11	12	14	15	18	**103**
9	3	4	5	6	7	8	11	12	14	16	18	**104**
10	1	4	5	6	8	11	12	13	14	15	16	**105**
11	4	5	6	7	8	9	10	12	13	15	17	**106**
12	3	4	5	6	8	9	10	11	16	17	18	**107**
13	2	3	4	6	7	10	11	14	16	17	18	**108**
14	3	4	5	7	8	9	10	13	15	17	18	**109**
15	2	3	5	6	8	10	11	14	16	17	18	**110**
16	1	4	5	7	9	10	12	13	15	17	18	**111**
17	1	2	3	7	11	12	13	14	15	16	18	**112**
18	1	2	6	8	9	12	13	14	15	16	17	**113**

Table 11

15-Regular Handicap Graph on 22 Vertices

Vertices	Neighbors															Weights
1	2	4	5	6	7	8	10	11	12	13	14	15	16	17	22	**162**
2	1	3	4	5	6	7	8	9	10	11	17	19	20	21	22	**163**
3	2	4	5	7	8	9	10	11	12	13	14	15	16	17	21	**164**
4	1	2	3	7	8	9	10	11	12	13	14	15	18	20	22	**165**
5	1	2	3	6	7	8	9	12	13	15	16	17	18	19	20	**166**
6	1	2	5	7	8	9	10	11	12	13	14	16	18	20	21	**167**
7	1	2	3	4	5	6	10	12	14	16	17	18	19	20	21	**168**
8	1	2	3	4	5	6	11	13	14	15	17	18	19	20	21	**169**
9	2	3	4	5	6	10	11	12	13	14	15	16	18	19	22	**170**
10	1	2	3	4	6	7	9	12	14	15	16	19	20	21	22	**171**
11	1	2	3	4	6	8	9	13	14	15	17	18	19	21	22	**172**
12	1	3	4	5	6	7	9	10	15	16	17	18	19	21	22	**173**
13	1	3	4	5	6	8	9	11	15	16	17	18	19	20	22	**174**
14	1	3	4	6	7	8	9	10	11	16	18	19	20	21	22	**175**
15	1	3	4	5	8	9	10	11	12	13	18	19	20	21	22	**176**
16	1	3	5	6	7	9	10	12	13	14	17	18	19	21	22	**177**
17	1	2	3	5	7	8	11	12	13	16	18	19	20	21	22	**178**
18	4	5	6	7	8	9	11	12	13	14	15	16	17	20	22	**179**
19	2	5	7	8	9	10	11	12	13	14	15	16	17	20	21	**180**
20	2	4	5	6	7	8	10	13	14	15	17	18	19	21	22	**181**
21	2	3	6	7	8	10	11	12	14	15	16	17	19	20	22	**182**
22	1	2	4	9	10	11	12	13	14	15	16	17	18	20	21	**183**

Table 12

Handicap Graph on 26 Vertices

Vertex	Neighbors																			Weights
1	2	3	4	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	25	**244**
2	1	3	5	7	8	9	10	11	12	13	14	15	16	17	18	19	20	23	24	**245**
3	1	2	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	22	26	**246**
4	1	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	22	24	**247**
5	2	4	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	26	**248**
6	3	4	5	7	8	9	10	11	12	13	14	15	16	17	18	19	20	23	25	**249**
7	1	2	3	4	5	6	8	9	10	11	12	18	20	21	22	23	24	25	26	**250**
8	1	2	3	4	5	6	7	9	10	13	14	17	19	21	22	23	24	25	26	**251**
9	1	2	3	4	5	6	7	8	11	14	15	16	19	21	22	23	24	25	26	**252**
10	1	2	3	4	5	6	7	8	12	13	15	16	20	21	22	23	24	25	26	**253**
11	1	2	3	4	5	6	7	9	12	13	16	17	18	21	22	23	24	25	26	**254**
12	1	2	3	4	5	6	7	10	11	14	15	17	19	21	22	23	24	25	26	**255**
13	1	2	3	4	5	6	8	10	11	14	16	17	18	21	22	23	24	25	26	**256**
14	1	2	3	4	5	6	8	9	12	13	15	18	20	21	22	23	24	25	26	**257**
15	1	2	3	4	5	6	9	10	12	14	16	17	18	21	22	23	24	25	26	**258**
16	1	2	3	4	5	6	9	10	11	13	15	19	20	21	22	23	24	25	26	**259**
17	1	2	3	4	5	6	8	11	12	13	15	19	20	21	22	23	24	25	26	**260**
18	1	2	3	4	5	6	7	11	13	14	15	19	20	21	22	23	24	25	26	**261**
19	1	2	3	4	5	6	8	9	12	16	17	18	20	21	22	23	24	25	26	**262**
20	1	2	3	4	5	6	7	10	14	16	17	18	19	21	22	23	24	25	26	**263**
21	1	5	7	8	9	10	11	12	13	14	15	16	17	18	19	20	22	23	24	**264**
22	3	4	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	23	25	**265**
23	2	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	26	**266**
24	2	4	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	25	26	**267**
25	1	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	22	24	26	**268**
26	3	5	7	8	9	10	11	12	13	14	15	16	17	18	19	20	23	24	25	**269**

Table 13

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