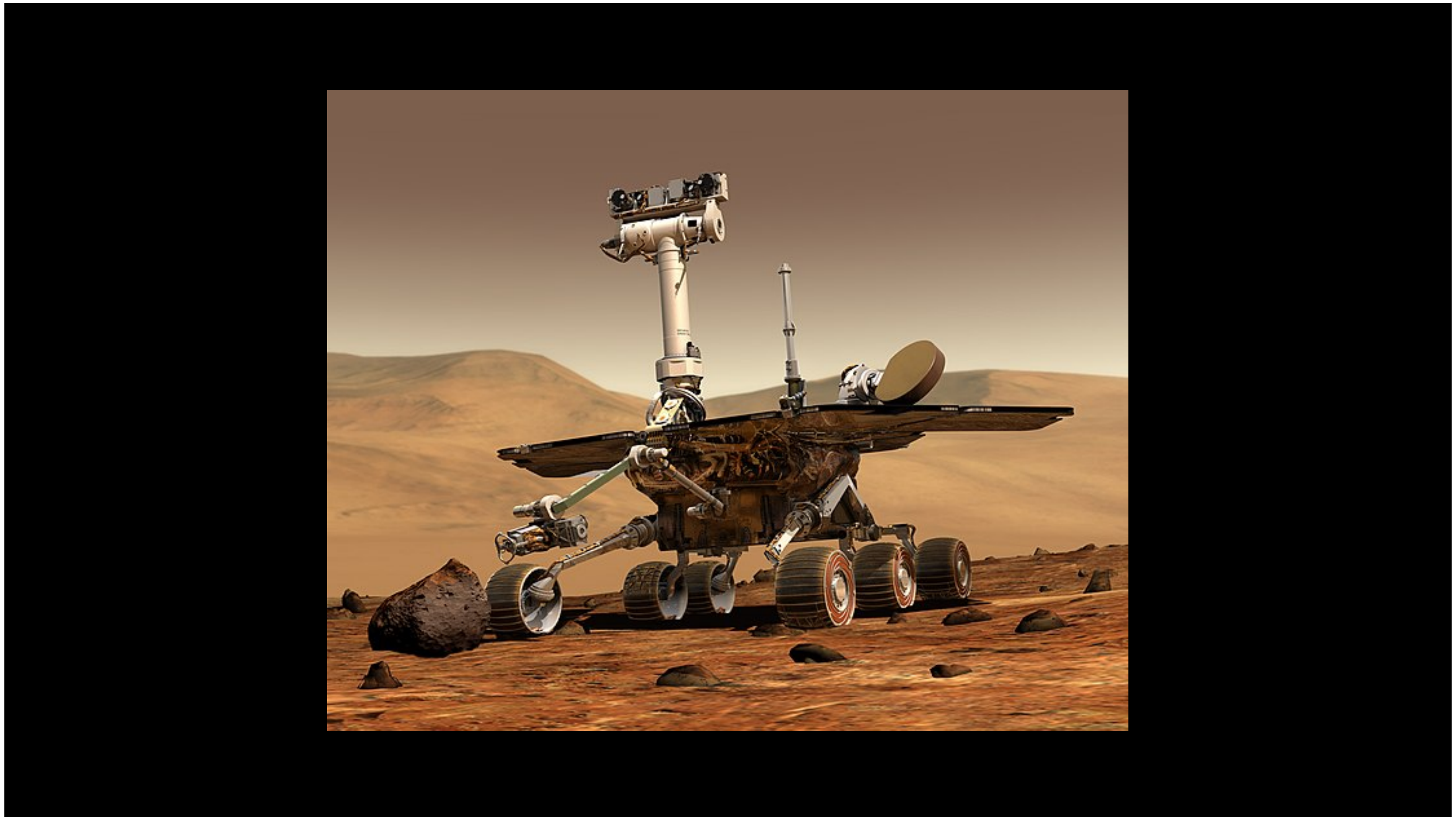
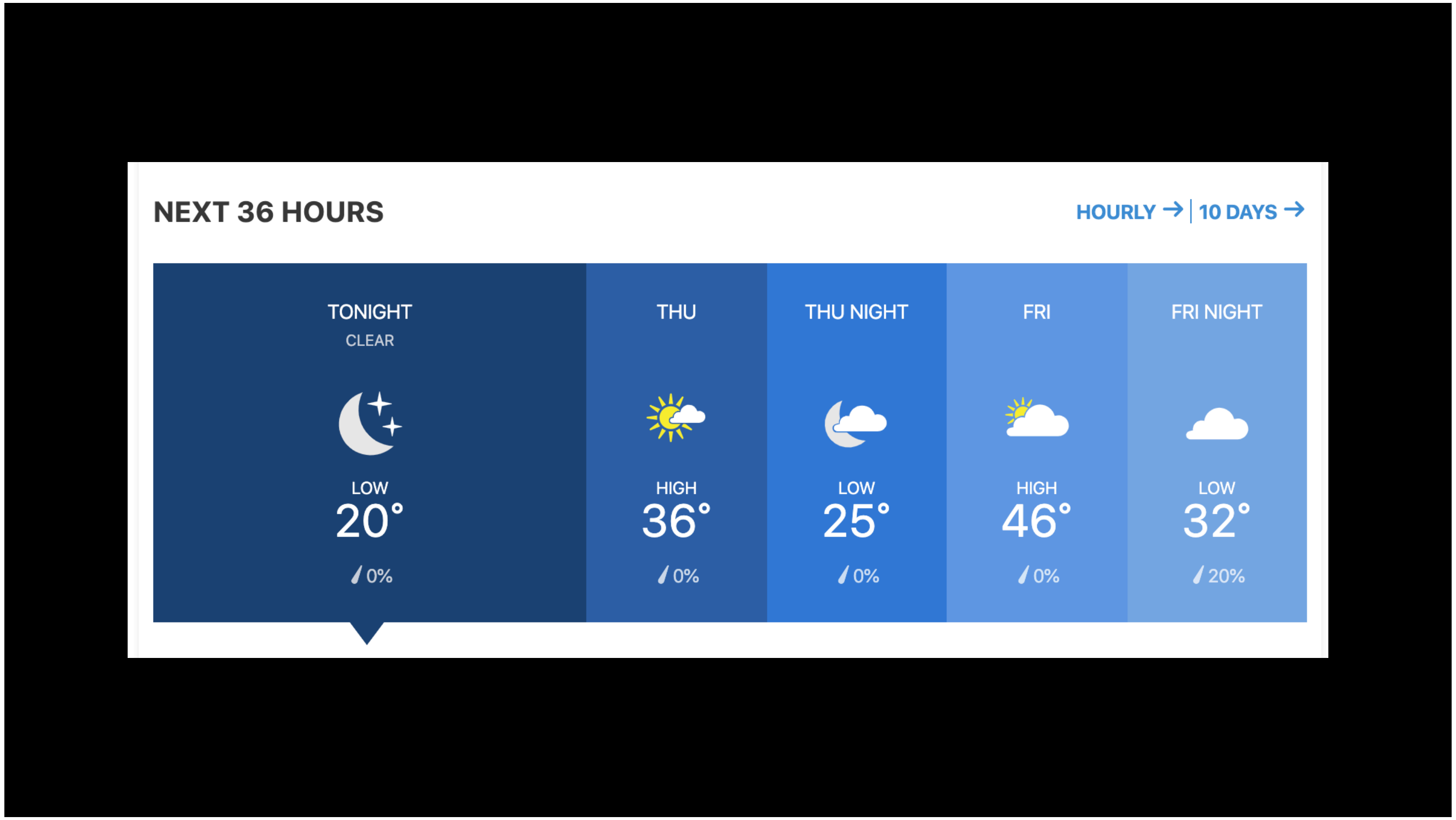
Introduction to

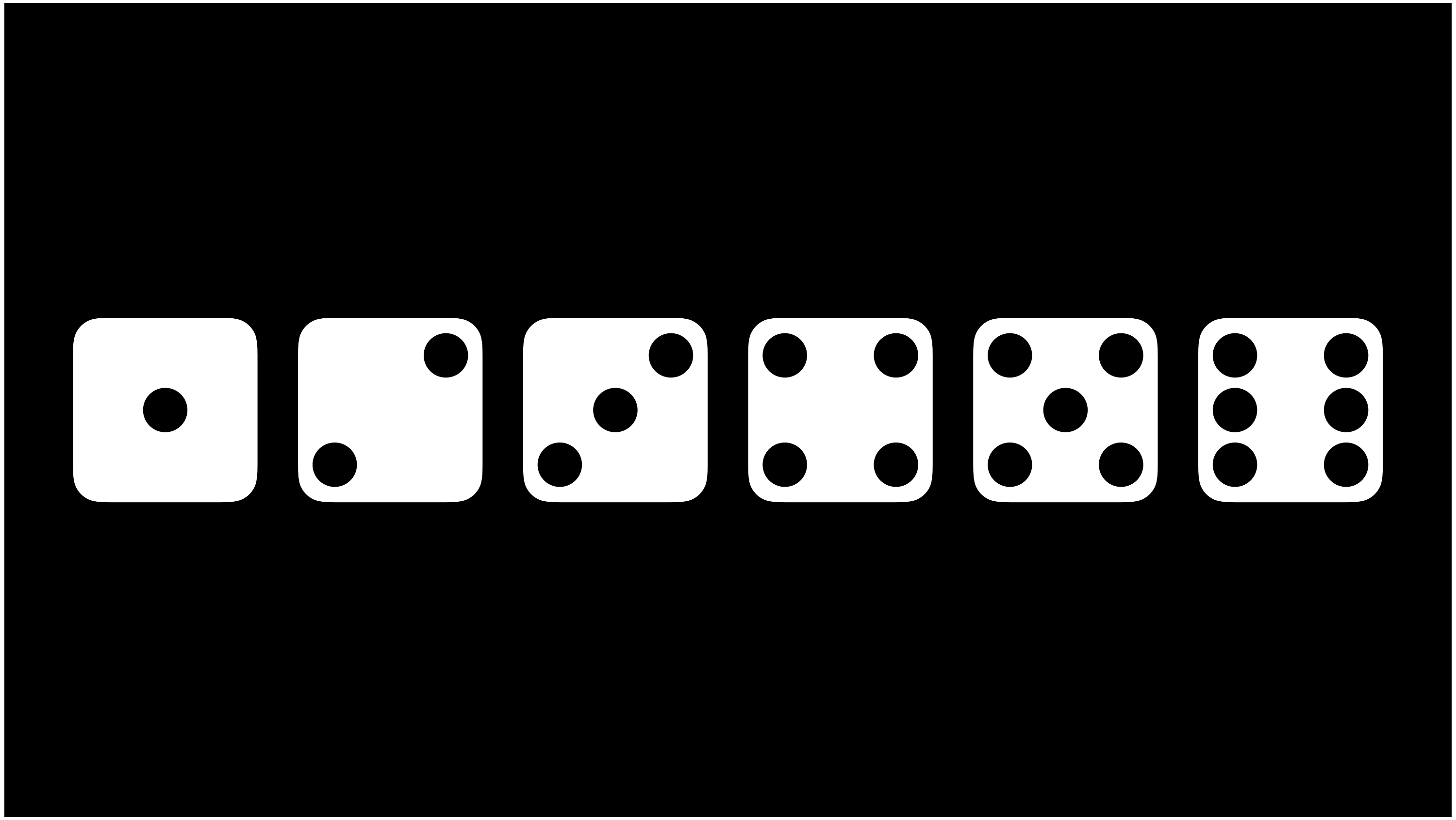
# Artificial Intelligence

with Python

# Uncertainty




NEXT 36 HOURS
HOURLY → | 10 DAYS →
TONIGHT
CLEAR
LOW
20°
0%
THU
HIGH
36°
0%
THU NIGHT
LOW
25°
0%
FRI
HIGH
46°
0%
FRI NIGHT
LOW
32°
20%



# Probability

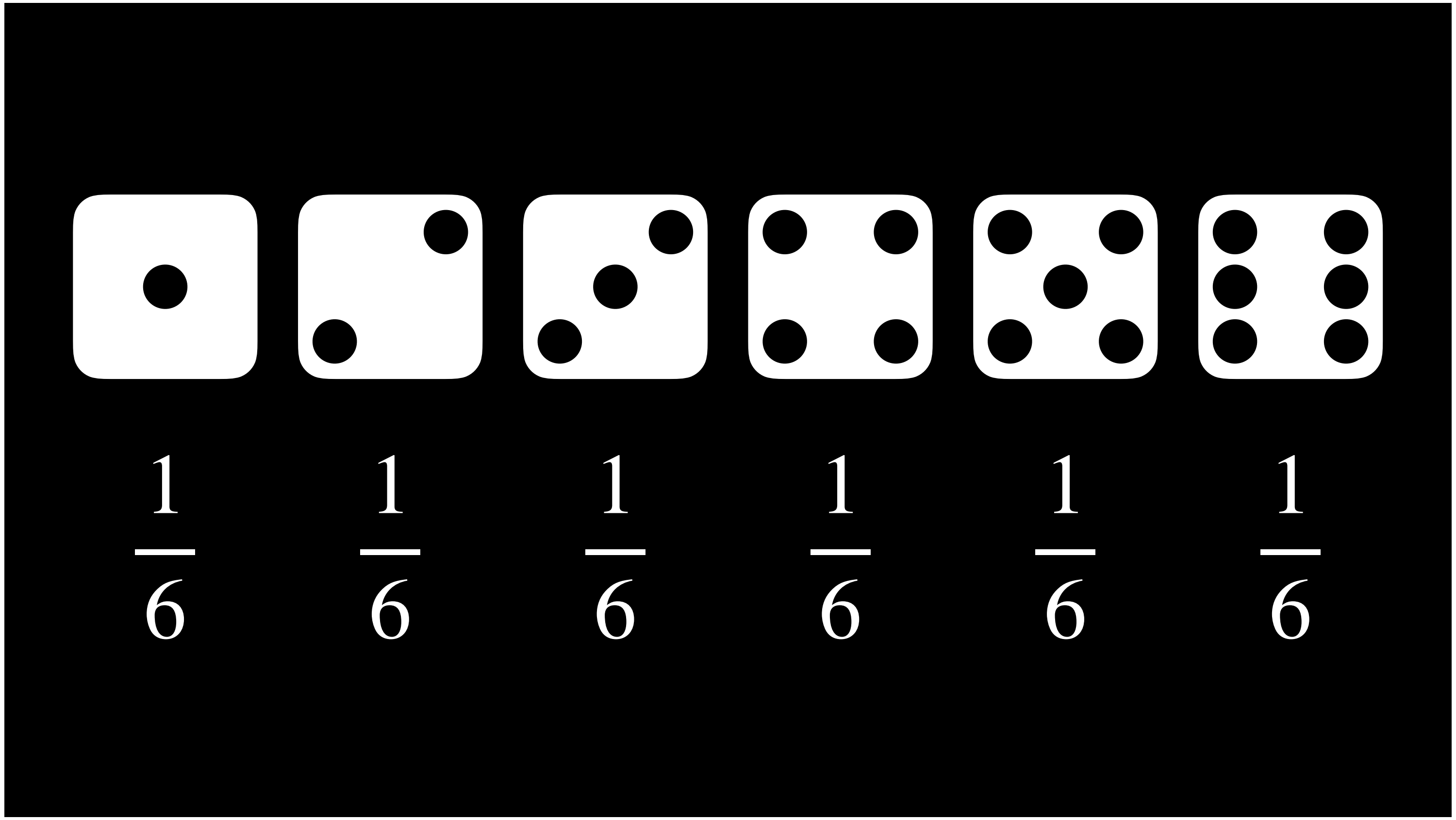
# Possible Worlds

$$\omega$$

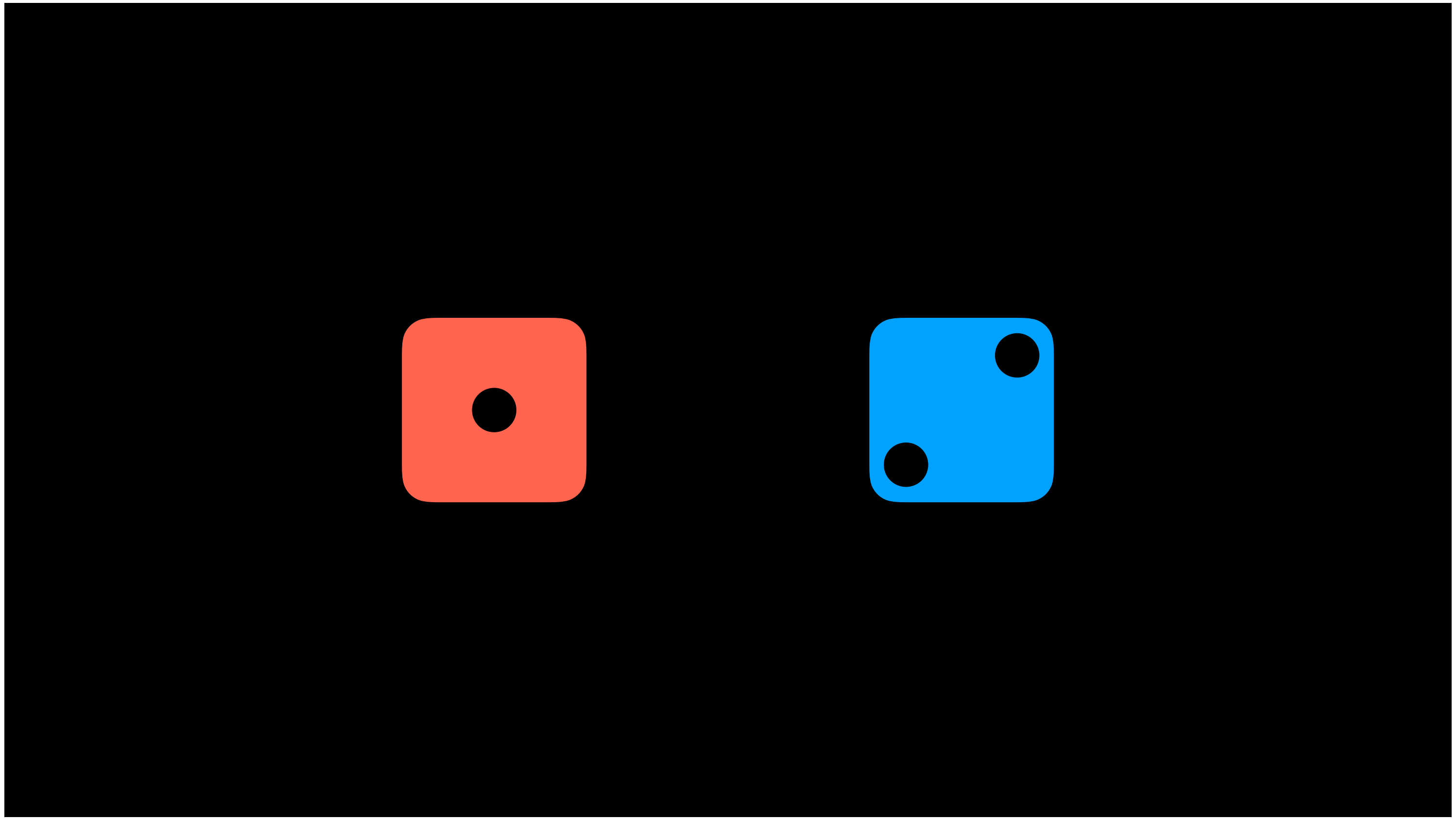
$$P(\omega)$$

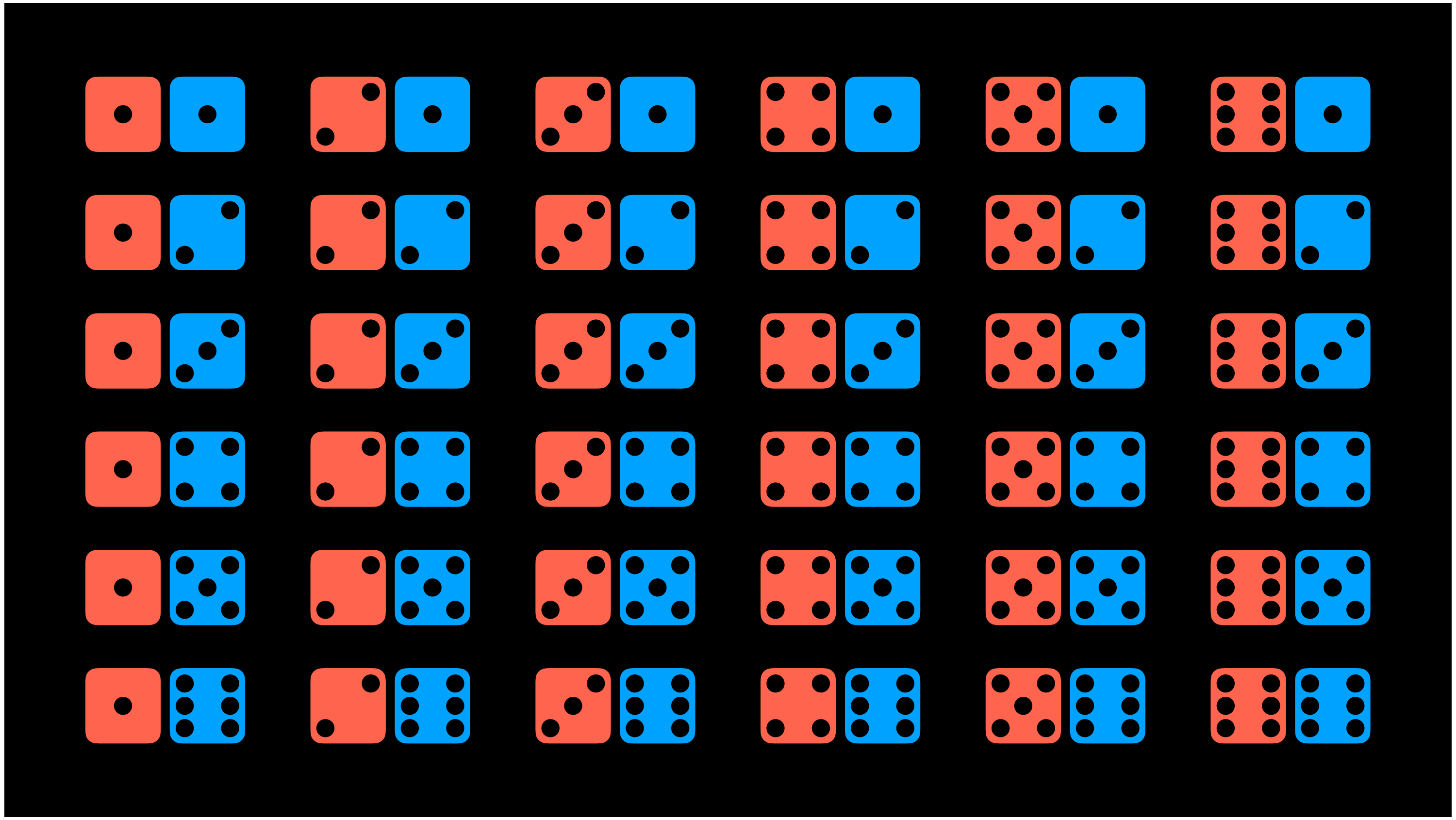
$$0 \le P(\omega) \le 1$$

$$\sum_{\omega \in \Omega} P(\omega) = 1$$


$\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$

$$P(\text{⚁}) = \frac{1}{6}$$





| | | | | | |
|---|---|---|---|---|---|
| 2 | 3 | 4 | 5 | 6 | 7 |
| 3 | 4 | 5 | 6 | 7 | 8 |
| 4 | 5 | 6 | 7 | 8 | 9 |
| 5 | 6 | 7 | 8 | 9 | 10 |
| 6 | 7 | 8 | 9 | 10 | 11 |
| 7 | 8 | 9 | 10 | 11 | 12 |

| | | | | | |
|---|---|---|---|---|---|
| 2 | 3 | 4 | 5 | 6 | 7 |
| 3 | 4 | 5 | 6 | 7 | 8 |
| 4 | 5 | 6 | 7 | 8 | 9 |
| 5 | 6 | 7 | 8 | 9 | 10 |
| 6 | 7 | 8 | 9 | 10 | 11 |
| 7 | 8 | 9 | 10 | 11 | 12 |

| | | | | | |
|---|---|---|---|---|---|
| 2 | 3 | 4 | 5 | 6 | 7 |
| 3 | 4 | 5 | 6 | 7 | 8 |
| 4 | 5 | 6 | 7 | 8 | 9 |
| 5 | 6 | 7 | 8 | 9 | 10 |
| 6 | 7 | 8 | 9 | 10 | 11 |
| 7 | 8 | 9 | 10 | 11 | 12 |

$$P(sum\ to\ 12) = \frac{1}{36}$$

$$P(sum\ to\ 7) = \frac{6}{36} = \frac{1}{6}$$

## unconditional probability

degree of belief in a proposition
in the absence of any other evidence

## conditional probability

degree of belief in a proposition given some evidence that has already been revealed conditional probability

$$P(a \mid b)$$

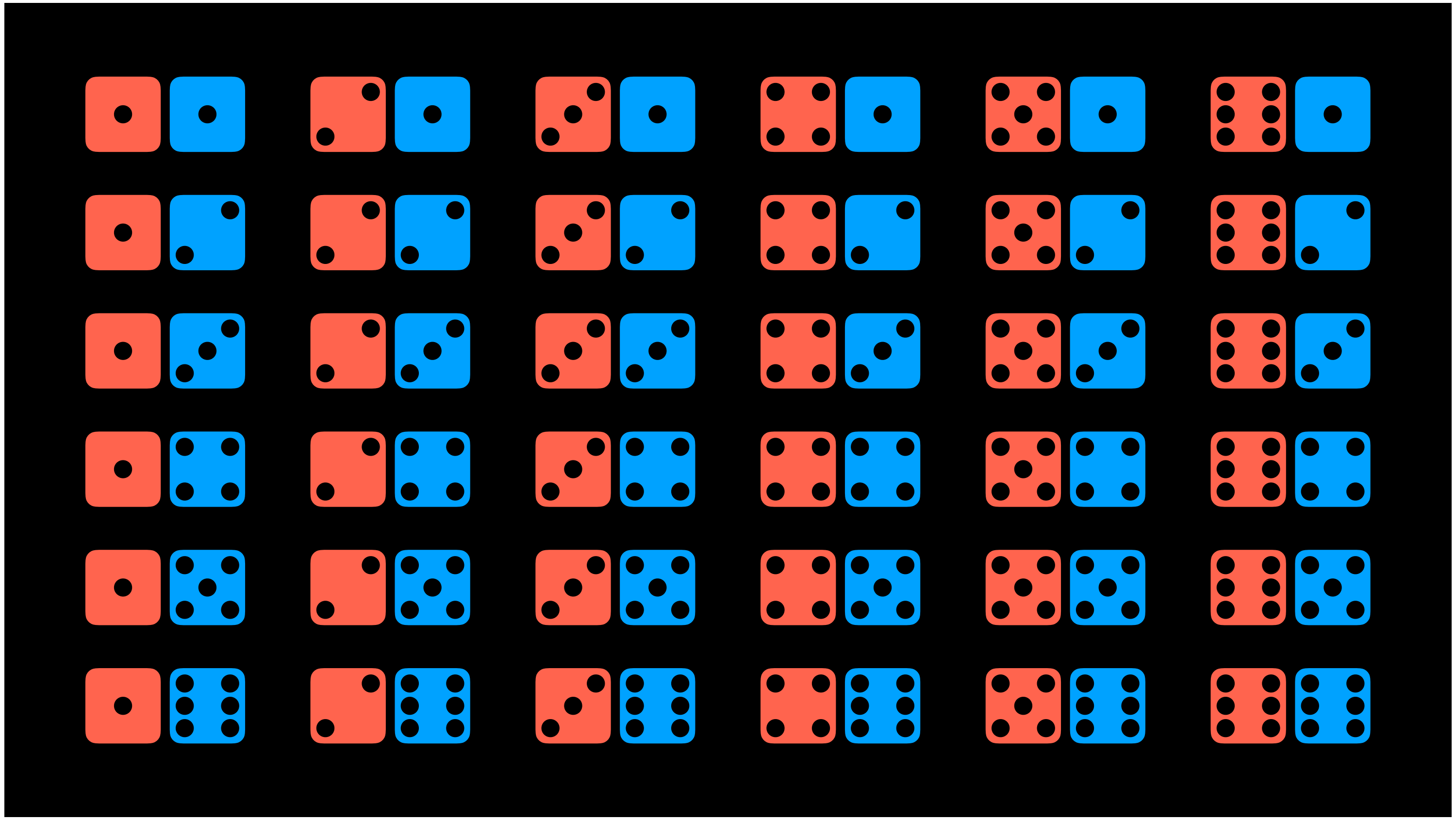
$$P(rain\ today \mid rain\ yesterday)$$

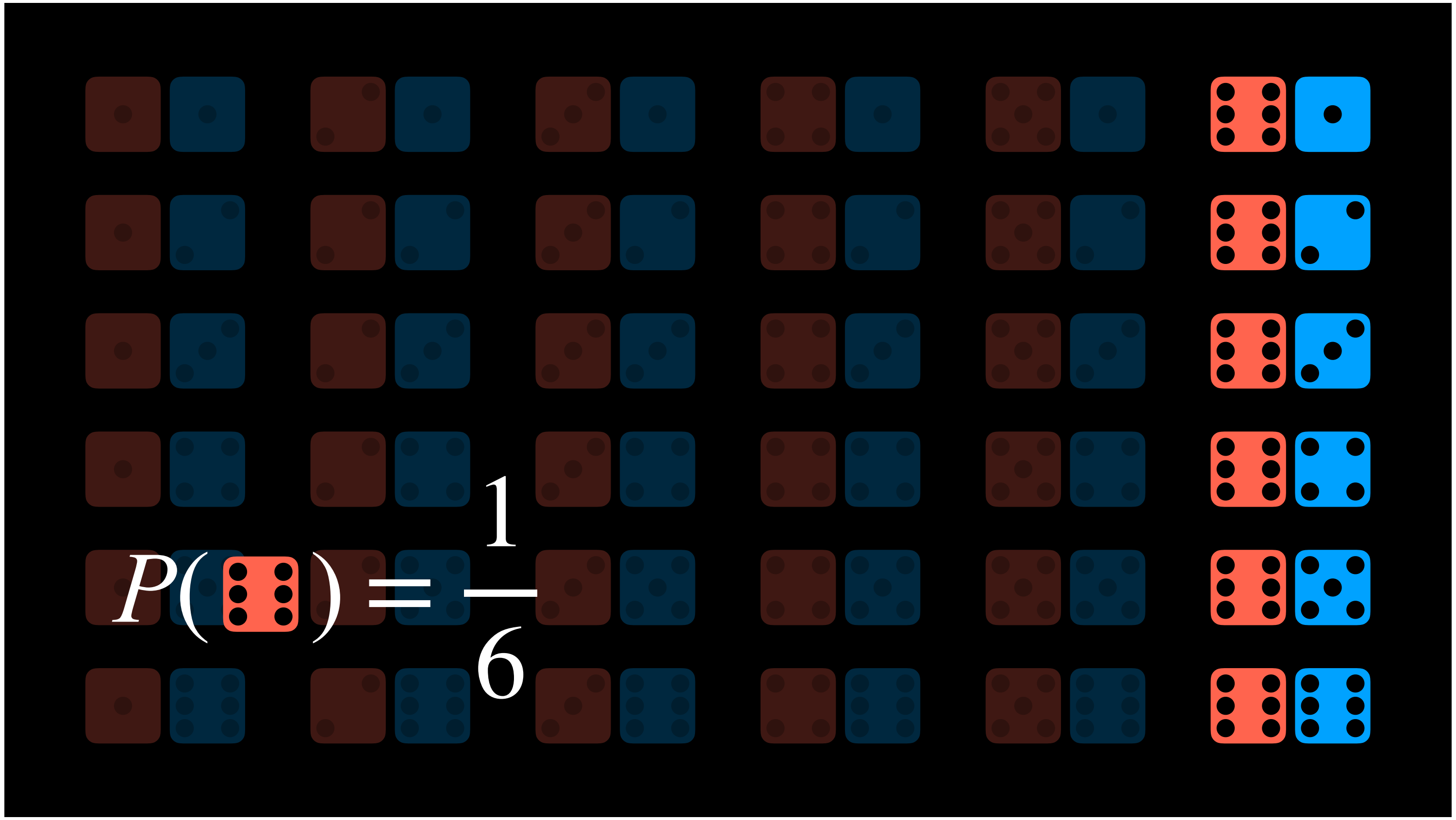
$$P(route\ change \mid traffic\ conditions)$$

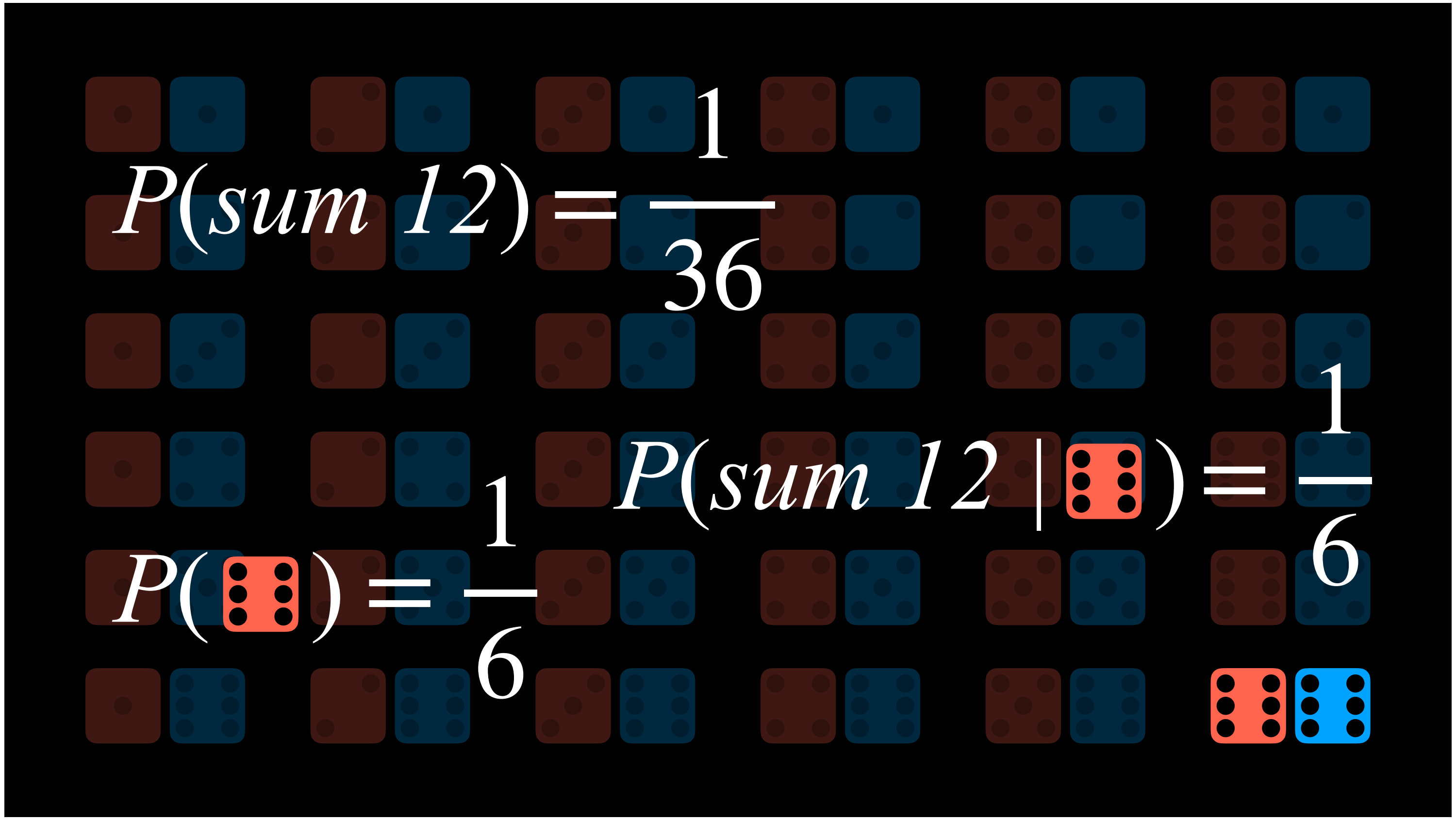
$$P(disease \mid test\ results)$$

$$P(a \mid b) = \frac{P(a \land b)}{P(b)}$$

$P(sum\ 12 \mid$  $)$




$P(\ ) = \frac{1}{6}$


$P(sum\ 12) = \frac{1}{36}$
$P(sum\ 12 \mid ⚅) = \frac{1}{6}$
$P(⚅) = \frac{1}{6}$

$$P(a \mid b) = \frac{P(a \wedge b)}{P(b)}$$

$$P(a \wedge b) = P(b)P(a \mid b)$$

$$P(a \wedge b) = P(a)P(b \mid a)$$

## random variable

a variable in probability theory with a domain of possible values it can take on

random variable

$Roll$

$\{1, 2, 3, 4, 5, 6\}$

random variable

*Weather*

$\{sun, cloud, rain, wind, snow\}$

random variable

*Traffic*

$\{none, light, heavy\}$

random variable

$Flight$

$\{on\ time, delayed, cancelled\}$

## probability distribution

$$P(\mathit{Flight} = \mathit{on\ time}) = 0.6$$
$$P(\mathit{Flight} = \mathit{delayed}) = 0.3$$
$$P(\mathit{Flight} = \mathit{cancelled}) = 0.1$$

probability distribution

$$\mathbf{P}(Flight) = \langle 0.6, 0.3, 0.1 \rangle$$

## independence

the knowledge that one event occurs does not affect the probability of the other event

**independence**

$$P(a \wedge b) = P(a)P(b \mid a)$$

independence

$$P(a \wedge b) = P(a)P(b)$$

## independence

$$P(⚅⚅) = P(⚅)P(⚅)$$

$$= \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

## independence

$$P(\text{⚅⚃}) \neq P(\text{⚅})P(\text{⚃})$$

$$= \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

**independence**

$$P(\text{⚅⚃}) \neq P(\text{⚅})P(\text{⚃} \mid \text{⚅})$$

$$= \frac{1}{6} \cdot 0 = 0$$

# Bayes' Rule

$$P(a \wedge b) = P(b)\ P(a \mid b)$$

$$P(a \wedge b) = P(a)\ P(b \mid a)$$

$$P(a)\ P(b \mid a) = P(b)\ P(a \mid b)$$

## Bayes' Rule

$$P(b \mid a) = \frac{P(b)\ P(a \mid b)}{P(a)}$$

## Bayes' Rule

$$P(b \mid a) = \frac{P(a \mid b)\, P(b)}{P(a)}$$



Given clouds in the morning,
what's the probability of rain in the afternoon?

- 80% of rainy afternoons start with cloudy mornings.
- 40% of days have cloudy mornings.
- 10% of days have rainy afternoons.

$$P(rain \mid clouds) = \frac{P(clouds \mid rain)P(rain)}{P(clouds)}$$

$$= \frac{(.8)(.1)}{.4}$$

$$= 0.2$$

Knowing

$$P(cloudy\ morning \mid rainy\ afternoon)$$

we can calculate

$$P(rainy\ afternoon \mid cloudy\ morning)$$

Knowing

$$P(\textit{visible effect} \mid \textit{unknown cause})$$

we can calculate

$$P(\textit{unknown cause} \mid \textit{visible effect})$$

Knowing

$$P(\textit{medical test result} \mid \textit{disease})$$

we can calculate

$$P(\textit{disease} \mid \textit{medical test result})$$

Knowing

$$P(\textit{blurry text} \mid \textit{counterfeit bill})$$

we can calculate

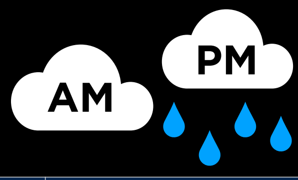
$$P(\textit{counterfeit bill} \mid \textit{blurry text})$$

# Joint Probability



| $C = cloud$ | $C = \neg cloud$ |
|---|---|
| 0.4 | 0.6 |

| $R = rain$ | $R = \neg rain$ |
|---|---|
| 0.1 | 0.9 |




| | $R = rain$ | $R = \neg rain$ |
|---|---|---|
| $C = cloud$ | 0.08 | 0.32 |
| $C = \neg cloud$ | 0.02 | 0.58 |

$\mathbf{P}(\mathrm{C} \mid \mathrm{rain})$

$$\mathbf{P}(\mathrm{C} \mid \mathrm{rain}) = \frac{\mathbf{P}(\mathrm{C}, \mathrm{rain})}{P(\mathrm{rain})} = \alpha\mathbf{P}(\mathrm{C}, \mathrm{rain})$$

$$= \alpha\langle 0.08, 0.02\rangle = \langle 0.8, 0.2\rangle$$

| | R = *rain* | R = ¬*rain* |
|---|---|---|
| C = *cloud* | 0.08 | 0.32 |
| C = ¬*cloud* | 0.02 | 0.58 |

# Probability Rules

## Negation

$$P(\neg a) = 1 - P(a)$$

## Inclusion-Exclusion

$$P(a \vee b) = P(a) + P(b) - P(a \wedge b)$$

# Marginalization

$$P(a) = P(a, b) + P(a, \neg b)$$

# Marginalization

$$P(X = x_i) = \sum_j P(X = x_i, Y = y_j)$$

# Marginalization

| | R = *rain* | R = ¬*rain* |
|---|---|---|
| C = *cloud* | 0.08 | 0.32 |
| C = ¬*cloud* | 0.02 | 0.58 |

$$P(\mathrm{C} = cloud)$$
$$= P(\mathrm{C} = cloud, R = rain) + P(\mathrm{C} = cloud, R = \neg rain)$$
$$= 0.08 + 0.32$$
$$= 0.40$$

# Conditioning

$$P(a) = P(a \mid b)P(b) + P(a \mid \neg b)P(\neg b)$$

## Conditioning

$$P(X = x_i) = \sum_j P(X = x_i \mid Y = y_j) P(Y = y_j)$$

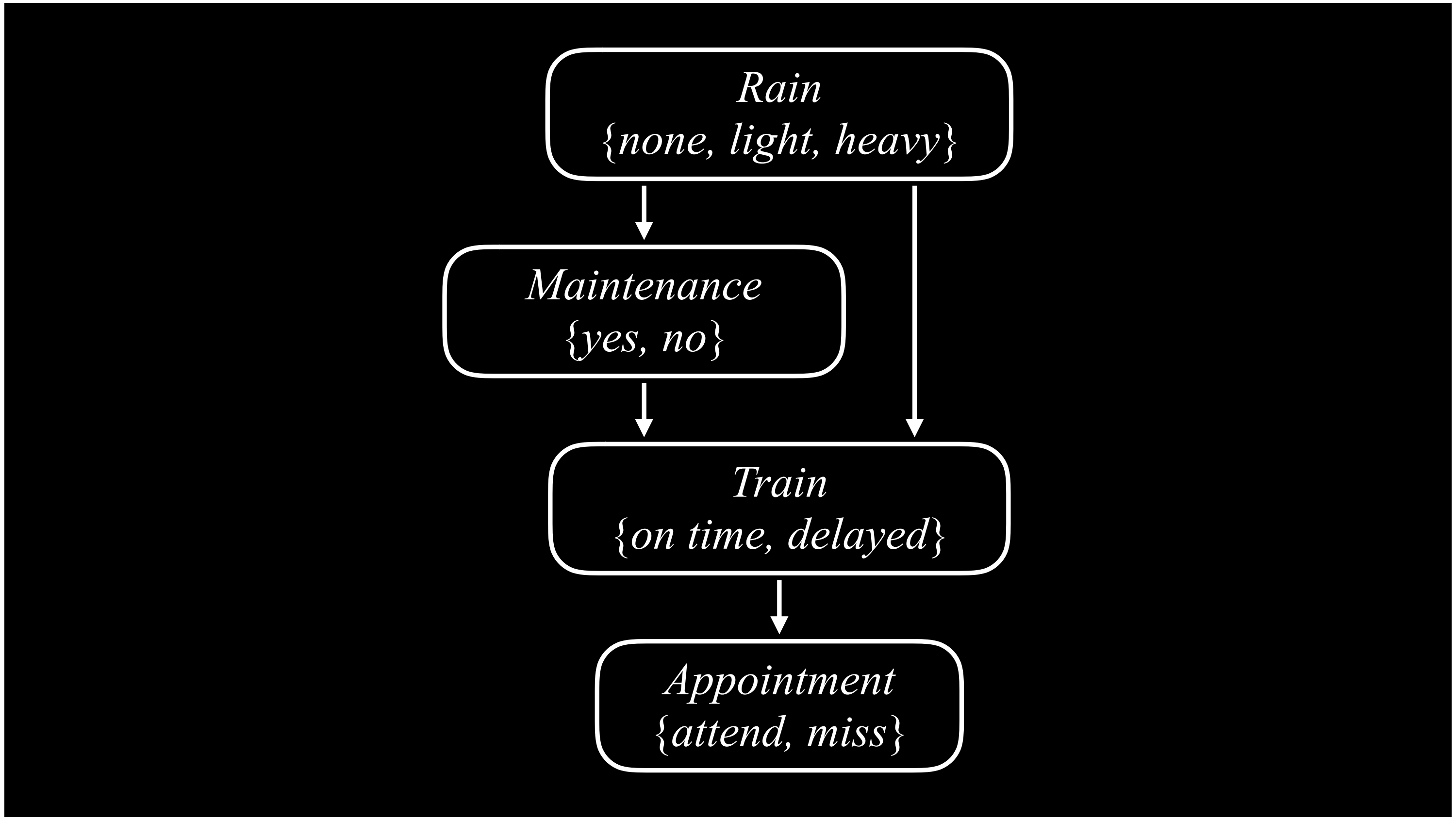
# Bayesian Networks

## Bayesian network

data structure that represents the dependencies among random variables

# Bayesian network

- directed graph
- each node represents a random variable
- arrow from $X$ to $Y$ means $X$ is a parent of $Y$
- each node $X$ has probability distribution $\mathbf{P}(X \mid Parents(X))$

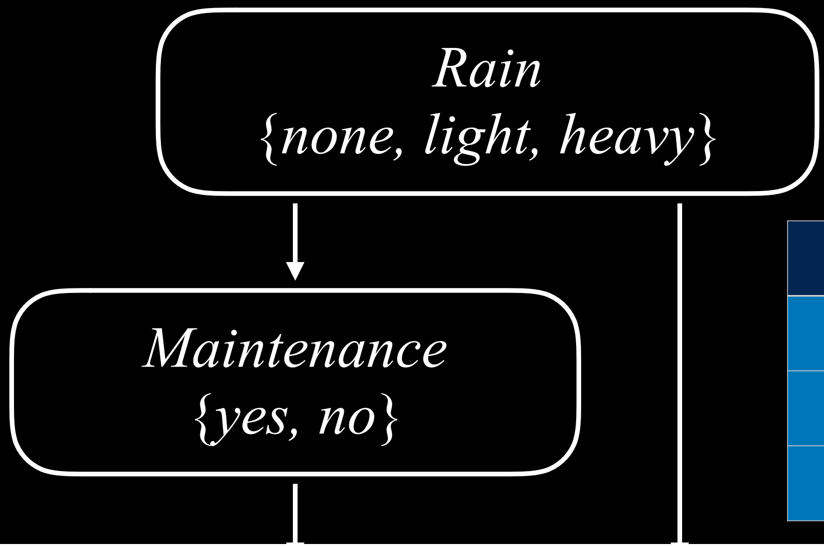


Rain
{none, light, heavy}
Maintenance
{yes, no}
Train
{on time, delayed}
Appointment
{attend, miss}

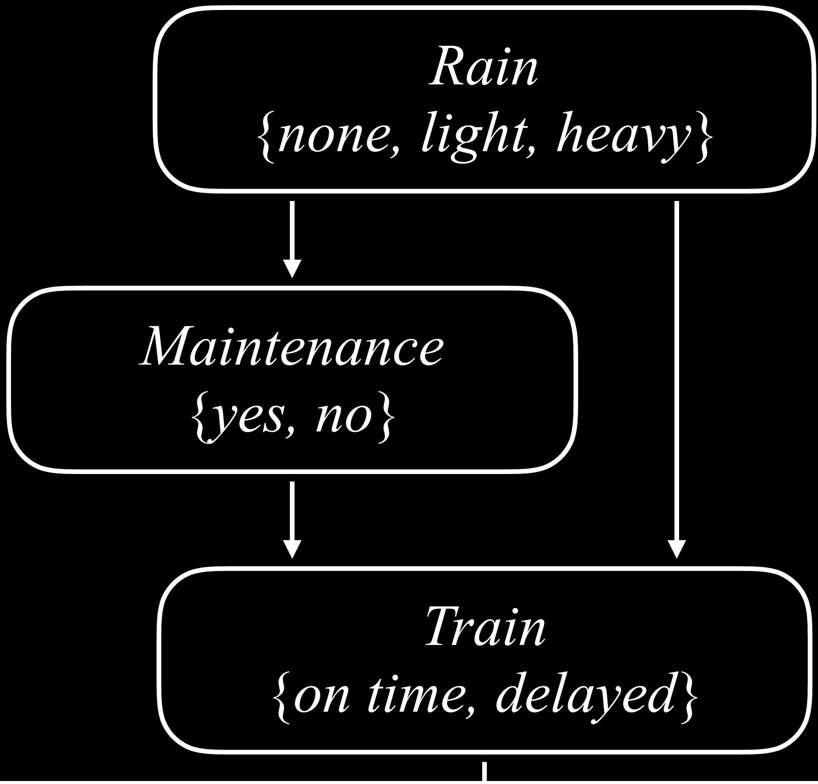
*Rain*

{*none*, *light*, *heavy*}

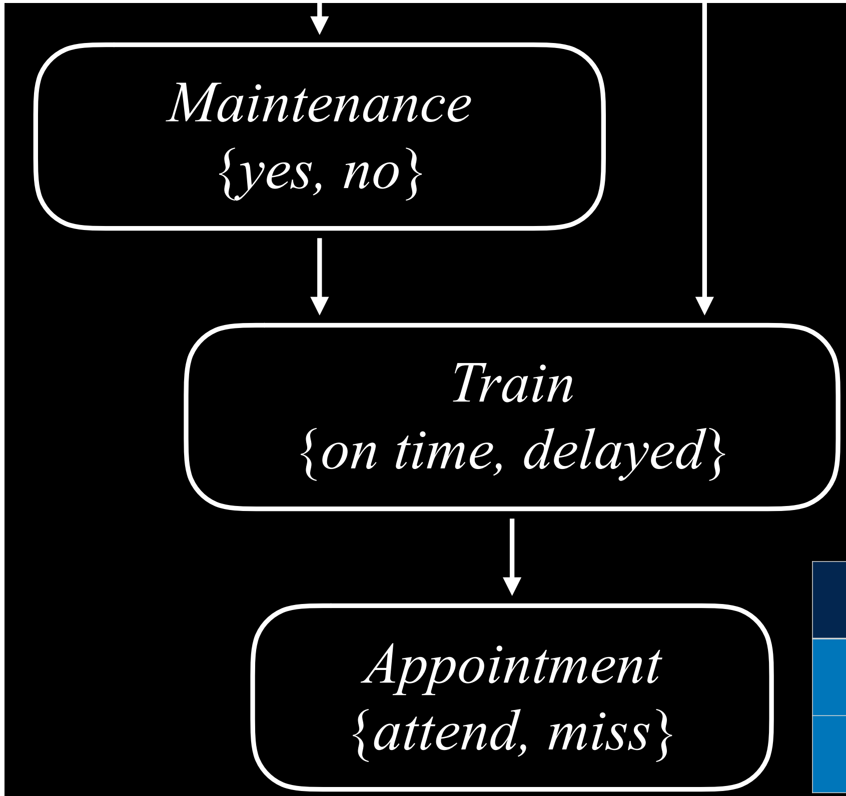
| *none* | *light* | *heavy* |
|---|---|---|
| 0.7 | 0.2 | 0.1 |



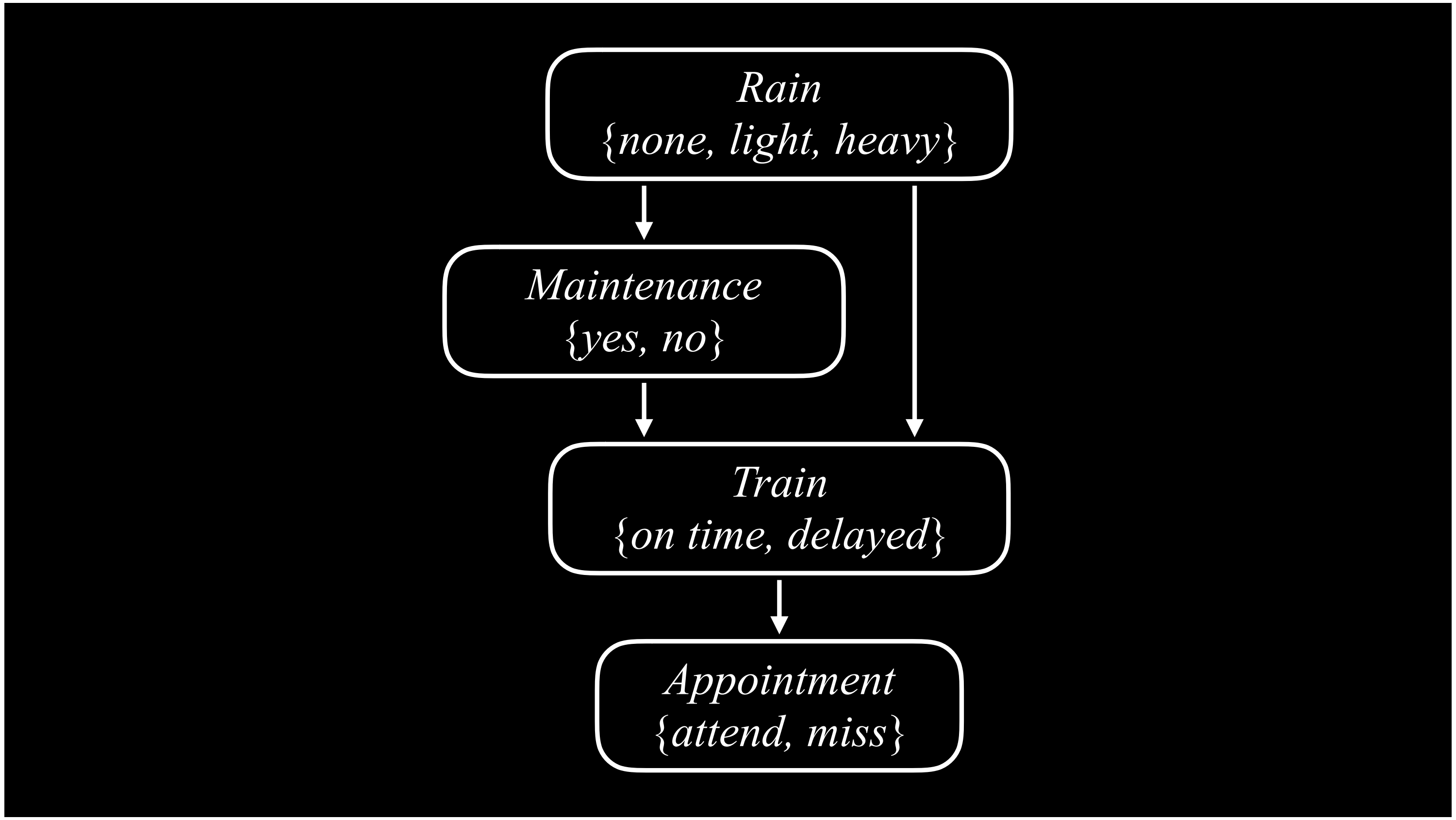

| ***R*** | *yes* | *no* |
|---|---|---|
| *none* | 0.4 | 0.6 |
| *light* | 0.2 | 0.8 |
| *heavy* | 0.1 | 0.9 |



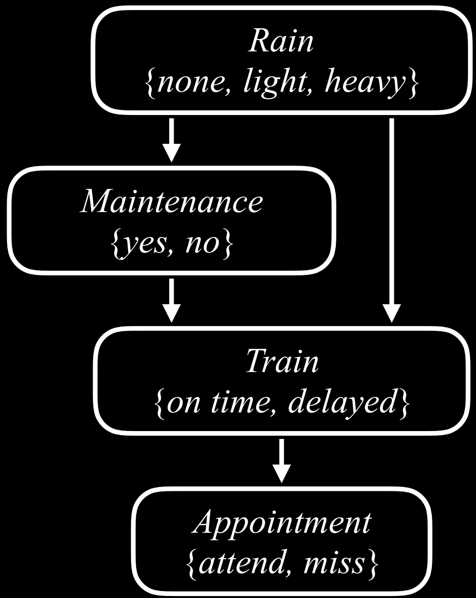

| ***R*** | ***M*** | *on time* | *delayed* |
|---|---|---|---|
| *none* | *yes* | 0.8 | 0.2 |
| *none* | *no* | 0.9 | 0.1 |
| *light* | *yes* | 0.6 | 0.4 |
| *light* | *no* | 0.7 | 0.3 |
| *heavy* | *yes* | 0.4 | 0.6 |
| *heavy* | *no* | 0.5 | 0.5 |




| ***T*** | *attend* | *miss* |
|---|---|---|
| *on time* | 0.9 | 0.1 |
| *delayed* | 0.6 | 0.4 |


Rain
{none, light, heavy}
Maintenance
{yes, no}
Train
{on time, delayed}
Appointment
{attend, miss}

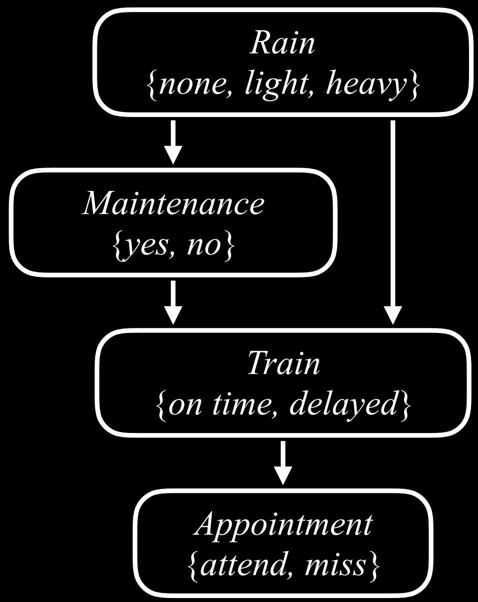
## Computing Joint Probabilities




$$P(light)$$

$$P(light)$$

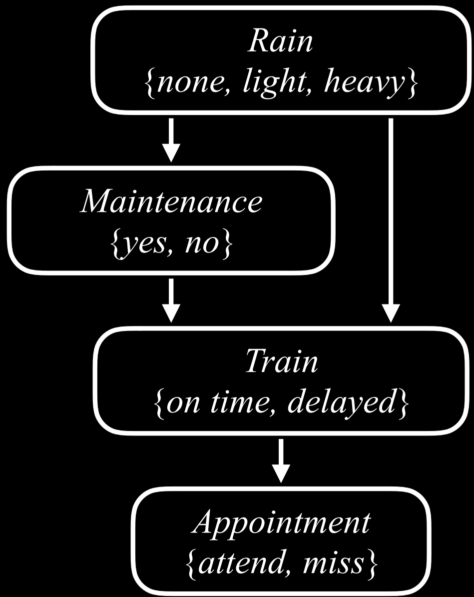
# Computing Joint Probabilities




$$P(light, no)$$

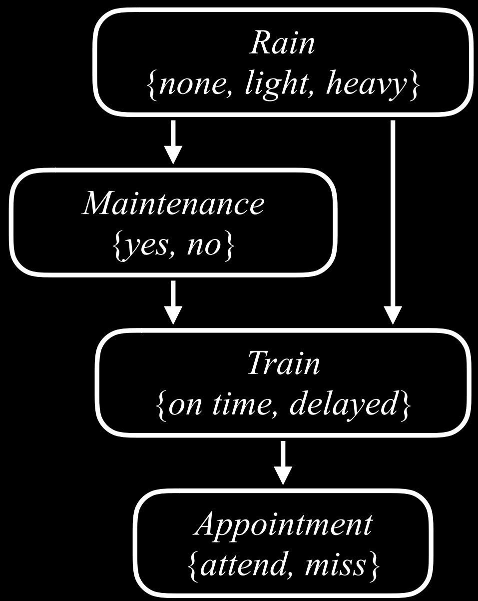
$$P(light)\, P(no \mid light)$$

# Computing Joint Probabilities




$$P(light, no, delayed)$$

$$P(light)\ P(no \mid light)\ P(delayed \mid light, no)$$




## Computing Joint Probabilities

$$P(light, no, delayed, miss)$$

$$P(light)\ P(no \mid light)\ P(delayed \mid light, no)\ P(miss \mid delayed)$$

# Inference

# Inference

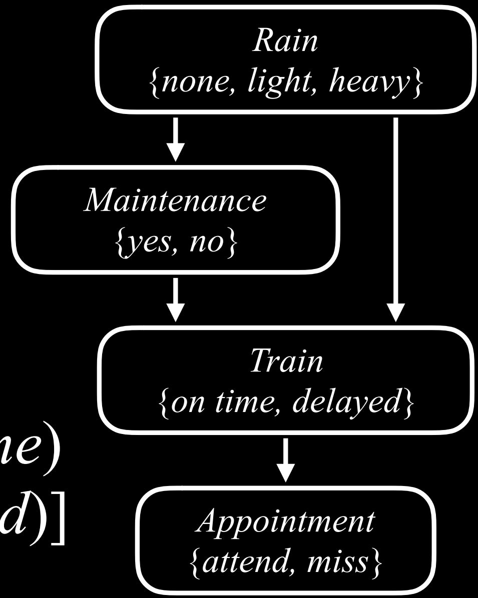
- Query $\mathbf{X}$: variable for which to compute distribution
- Evidence variables $\mathbf{E}$: observed variables for event $\mathbf{e}$
- Hidden variables $\mathbf{Y}$: non-evidence, non-query variable.

- Goal: Calculate $\mathbf{P}(X \mid \mathbf{e})$

$P(\text{Appointment} \mid \textit{light, no})$

$= \alpha\, P(\text{Appointment}, \textit{light, no})$

$= \alpha\, [P(\text{Appointment}, \textit{light, no, on time})$
$\quad + P(\text{Appointment}, \textit{light, no, delayed})]$



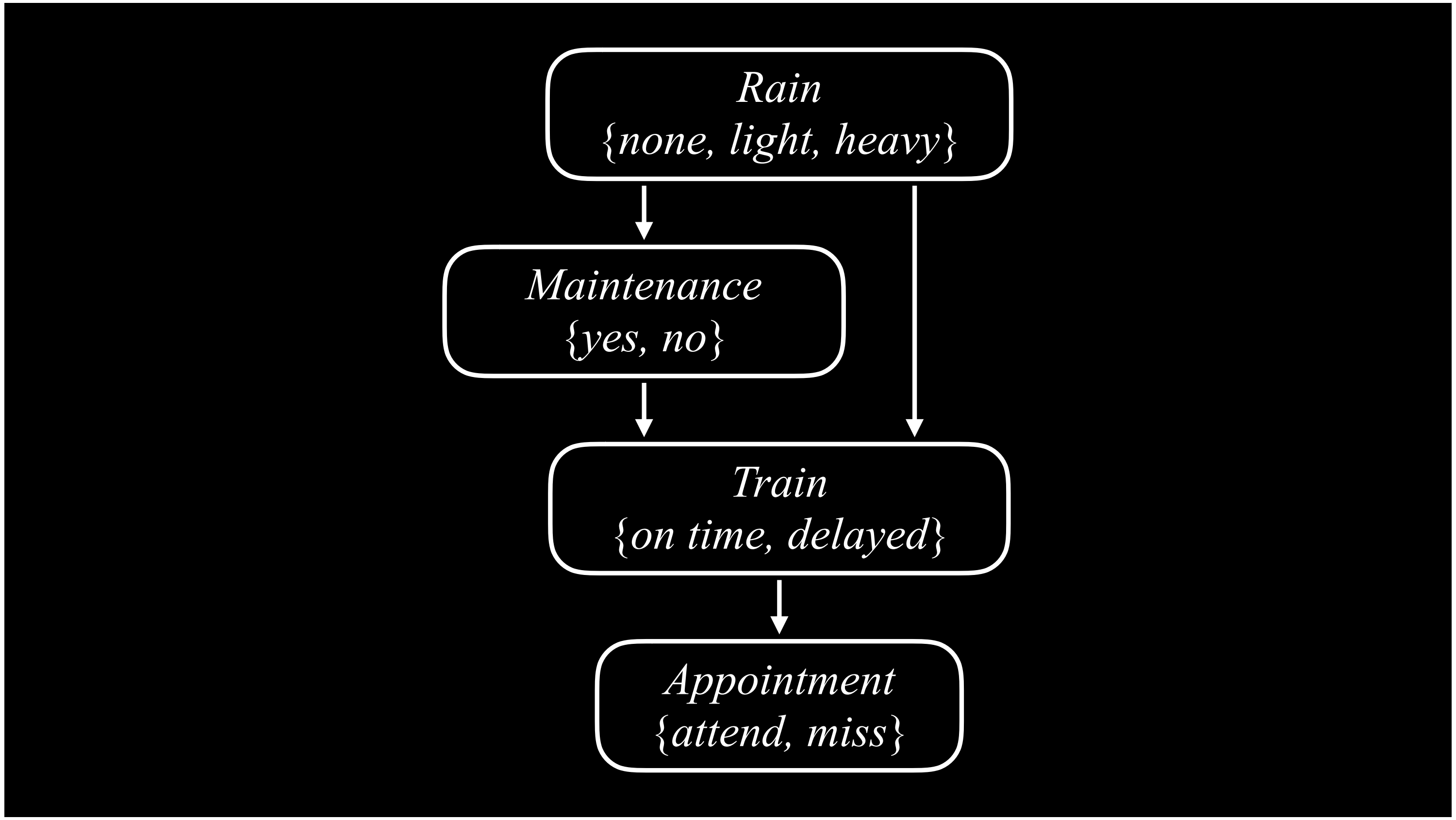
# Inference by Enumeration

$$\mathbf{P}(X \mid \mathbf{e}) = \alpha\, \mathbf{P}(X, \mathbf{e}) = \alpha \sum_{\mathbf{y}} \mathbf{P}(X, \mathbf{e}, \mathbf{y})$$

$X$ is the query variable.
$\mathbf{e}$ is the evidence.
$\mathbf{y}$ ranges over values of hidden variables.
$\alpha$ normalizes the result.

# Approximate Inference

# Sampling

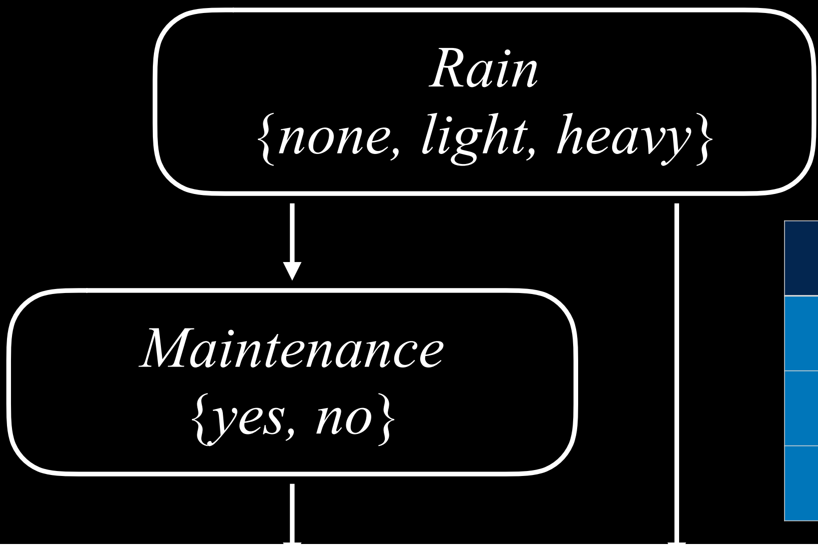

Rain
{none, light, heavy}
Maintenance
{yes, no}
Train
{on time, delayed}
Appointment
{attend, miss}

| R = *none* |
|---|
| |
| |
| |

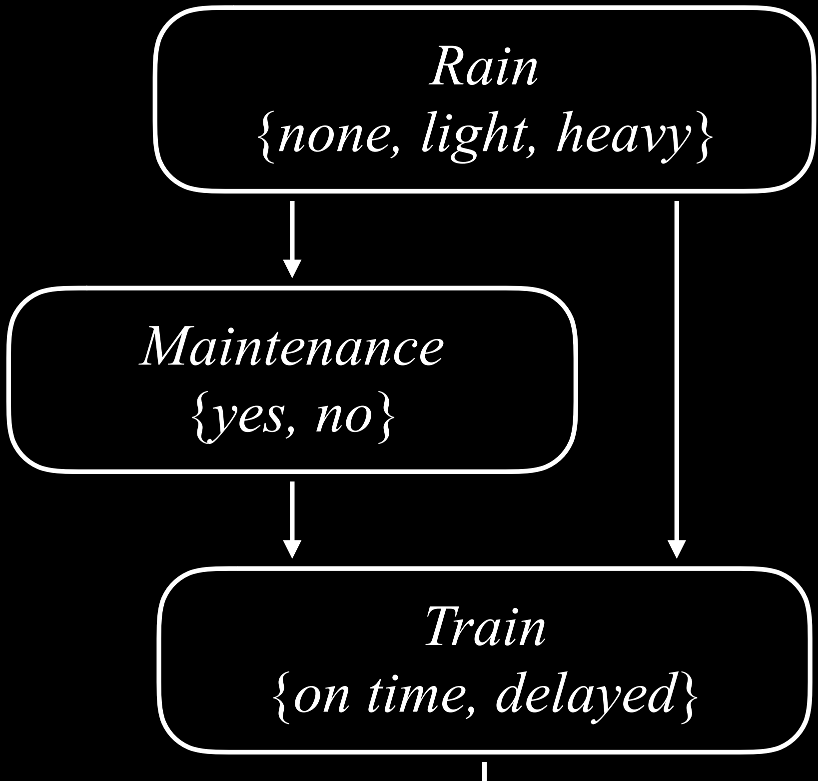
*Rain*
{*none, light, heavy*}

| *none* | *light* | *heavy* |
|---|---|---|
| 0.7 | 0.2 | 0.1 |

| |
|---|
| R = *none* |
| M = *yes* |
| |
| |

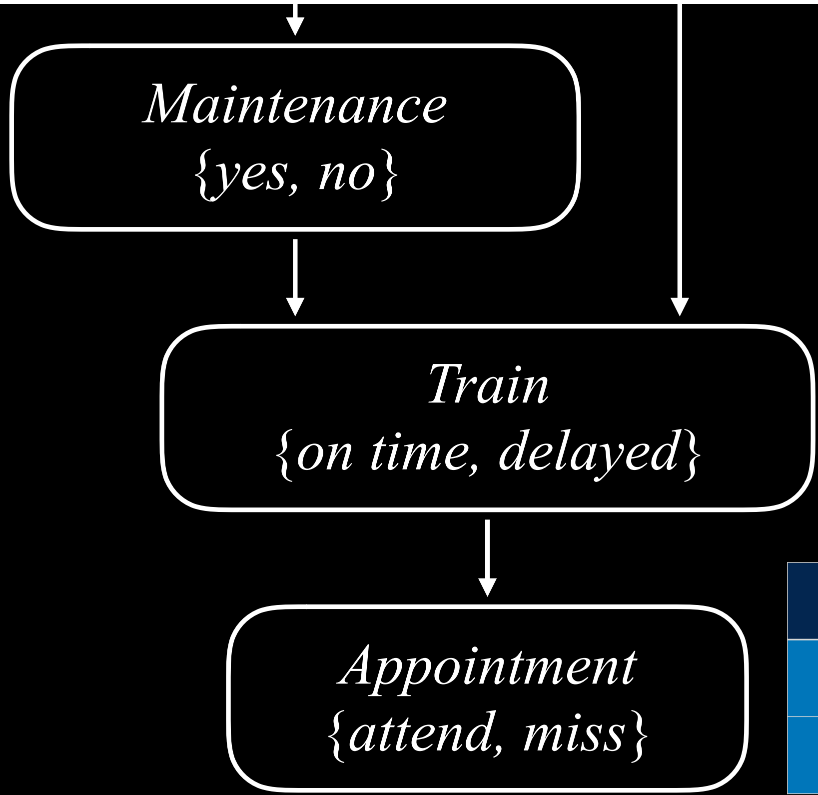



| ***R*** | *yes* | *no* |
|---|---|---|
| *none* | 0.4 | 0.6 |
| *light* | 0.2 | 0.8 |
| *heavy* | 0.1 | 0.9 |




| |
|---|
| R = *none* |
| M = *yes* |
| T = *on time* |
| |

| ***R*** | ***M*** | *on time* | *delayed* |
|---|---|---|---|
| *none* | *yes* | 0.8 | 0.2 |
| *none* | *no* | 0.9 | 0.1 |
| *light* | *yes* | 0.6 | 0.4 |
| *light* | *no* | 0.7 | 0.3 |
| *heavy* | *yes* | 0.4 | 0.6 |
| *heavy* | *no* | 0.5 | 0.5 |




| |
|---|
| R = *none* |
| M = *yes* |
| T = *on time* |
| A = *attend* |

| ***T*** | *attend* | *miss* |
|---|---|---|
| *on time* | 0.9 | 0.1 |
| *delayed* | 0.6 | 0.4 |

| |
|---|
| R = *none* |
| M = *yes* |
| T = *on time* |
| A = *attend* |

| | | | |
|---|---|---|---|
| R = *light* | R = *light* | R = *none* | R = *none* |
| M = *no* | M = *yes* | M = *no* | M = *yes* |
| T = *on time* | T = *delayed* | T = *on time* | T = *on time* |
| A = *miss* | A = *attend* | A = *attend* | A = *attend* |

| | | | |
|---|---|---|---|
| R = *none* | R = *none* | R = *heavy* | R = *light* |
| M = *yes* | M = *yes* | M = *no* | M = *no* |
| T = *on time* | T = *on time* | T = *delayed* | T = *on time* |
| A = *attend* | A = *attend* | A = *miss* | A = *attend* |

$$\mathrm{P}(\mathit{Train} = \mathit{on\ time})\ ?$$

| | | | |
|---|---|---|---|
| R = *light* | R = *light* | R = *none* | R = *none* |
| M = *no* | M = *yes* | M = *no* | M = *yes* |
| T = *on time* | T = *delayed* | T = *on time* | T = *on time* |
| A = *miss* | A = *attend* | A = *attend* | A = *attend* |

| | | | |
|---|---|---|---|
| R = *none* | R = *none* | R = *heavy* | R = *light* |
| M = *yes* | M = *yes* | M = *no* | M = *no* |
| T = *on time* | T = *on time* | T = *delayed* | T = *on time* |
| A = *attend* | A = *attend* | A = *miss* | A = *attend* |

| | | | |
|---|---|---|---|
| R = *light* | M = *no* | T = *on time* | A = *miss* |
| R = *light* | M = *yes* | T = *delayed* | A = *attend* |
| R = *none* | M = *no* | T = *on time* | A = *attend* |
| R = *none* | M = *yes* | T = *on time* | A = *attend* |
| R = *none* | M = *yes* | T = *on time* | A = *attend* |
| R = *none* | M = *yes* | T = *on time* | A = *attend* |
| R = *heavy* | M = *no* | T = *delayed* | A = *miss* |
| R = *light* | M = *no* | T = *on time* | A = *attend* |

$$P(\text{Rain} = \textit{light} \mid \text{Train} = \textit{on time})\ ?$$

| | | | |
|---|---|---|---|
| R = *light* | R = *light* | R = *none* | R = *none* |
| M = *no* | M = *yes* | M = *no* | M = *yes* |
| T = *on time* | T = *delayed* | T = *on time* | T = *on time* |
| A = *miss* | A = *attend* | A = *attend* | A = *attend* |

| | | | |
|---|---|---|---|
| R = *none* | R = *none* | R = *heavy* | R = *light* |
| M = *yes* | M = *yes* | M = *no* | M = *no* |
| T = *on time* | T = *on time* | T = *delayed* | T = *on time* |
| A = *attend* | A = *attend* | A = *miss* | A = *attend* |

| | | | |
|---|---|---|---|
| R = *light* | R = *light* | R = *none* | R = *none* |
| M = *no* | M = *yes* | M = *no* | M = *yes* |
| T = *on time* | T = *delayed* | T = *on time* | T = *on time* |
| A = *miss* | A = *attend* | A = *attend* | A = *attend* |

| | | | |
|---|---|---|---|
| R = *none* | R = *none* | R = *heavy* | R = *light* |
| M = *yes* | M = *yes* | M = *no* | M = *no* |
| T = *on time* | T = *on time* | T = *delayed* | T = *on time* |
| A = *attend* | A = *attend* | A = *miss* | A = *attend* |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| R = *light* | R = *light* | R = *none* | R = *none* | R = *none* | R = *none* | R = *heavy* | R = *light* |
| M = *no* | M = *yes* | M = *no* | M = *yes* | M = *yes* | M = *yes* | M = *no* | M = *no* |
| T = *on time* | T = *delayed* | T = *on time* | T = *on time* | T = *on time* | T = *on time* | T = *delayed* | T = *on time* |
| A = *miss* | A = *attend* | A = *attend* | A = *attend* | A = *attend* | A = *attend* | A = *miss* | A = *attend* |

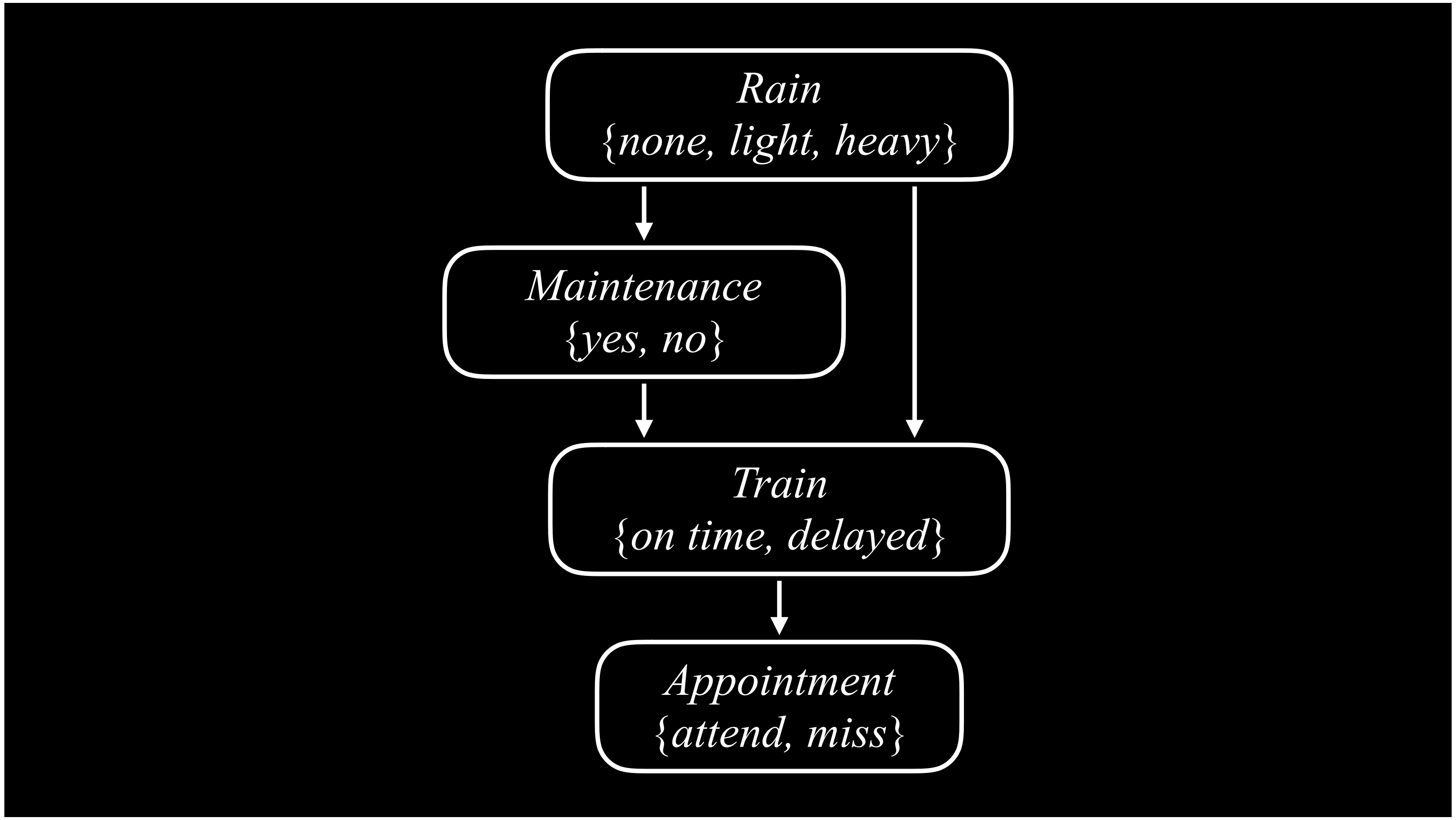
# Rejection Sampling

# Likelihood Weighting

# Likelihood Weighting

- Start by fixing the values for evidence variables.
- Sample the non-evidence variables using conditional probabilities in the Bayesian Network.
- Weight each sample by its **likelihood**: the probability of all of the evidence.

$P(\text{Rain} = \textit{light} \mid \text{Train} = \textit{on time})$ ?



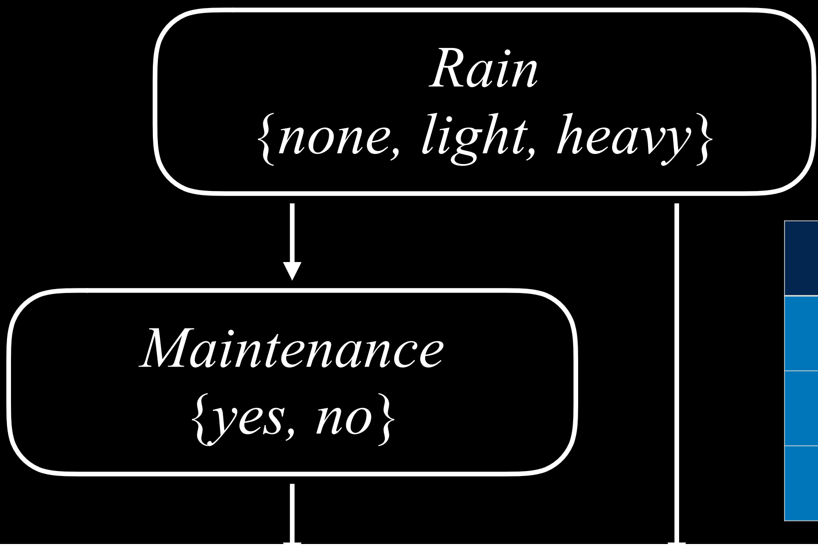
Rain
{none, light, heavy}
Maintenance
{yes, no}
Train
{on time, delayed}
Appointment
{attend, miss}

| |
|---|
| R = *light* |
| |
| T = *on time* |
| |

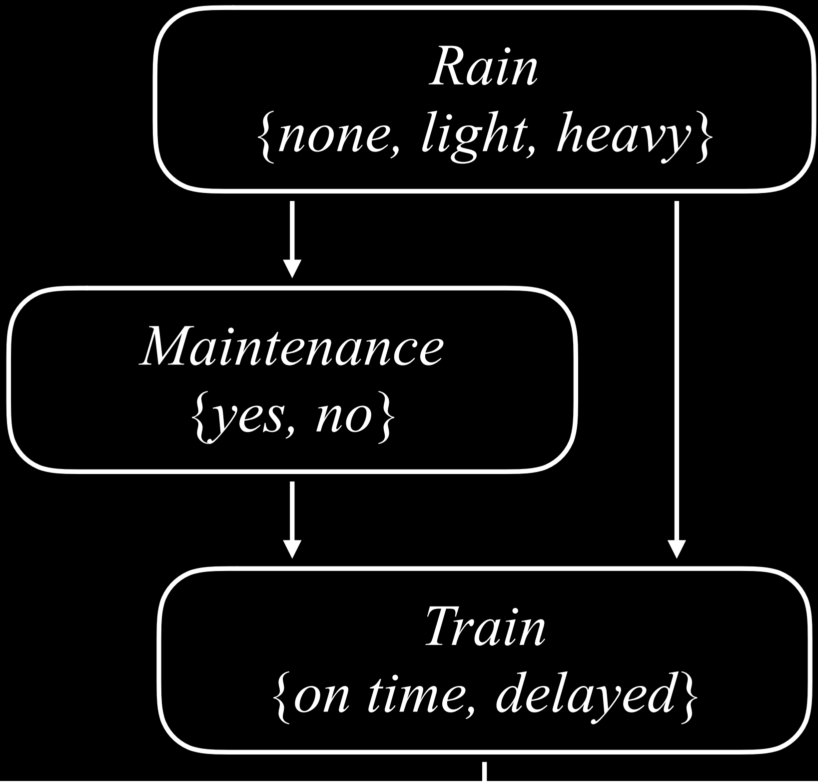
*Rain*
{*none, light, heavy*}

| *none* | *light* | *heavy* |
|---|---|---|
| 0.7 | 0.2 | 0.1 |

| |
|---|
| R = *light* |
| M = *yes* |
| T = *on time* |
| |

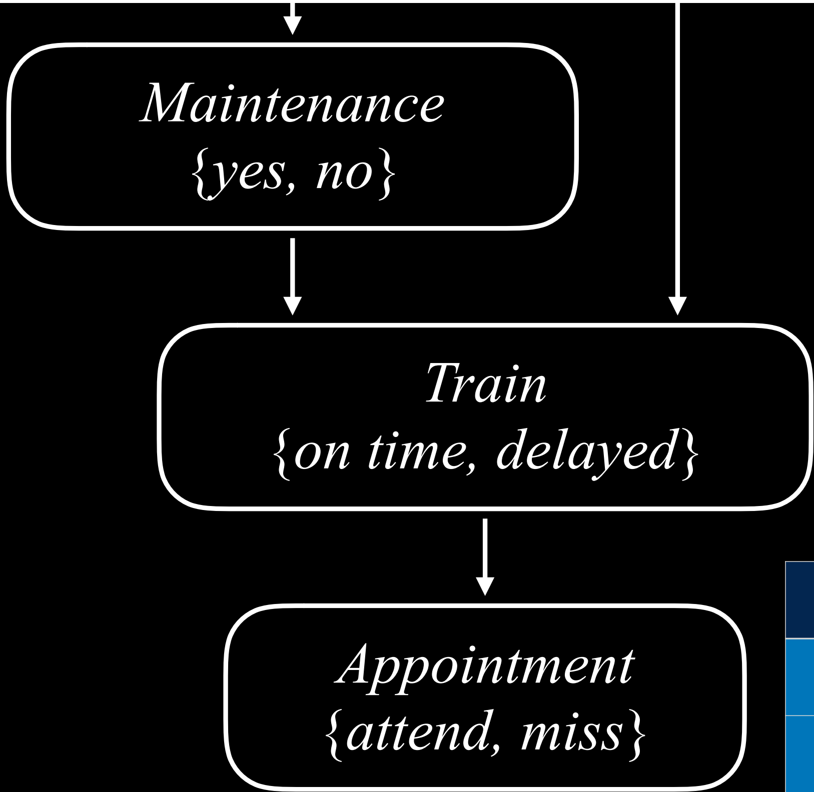



| ***R*** | *yes* | *no* |
|---|---|---|
| *none* | 0.4 | 0.6 |
| *light* | 0.2 | 0.8 |
| *heavy* | 0.1 | 0.9 |



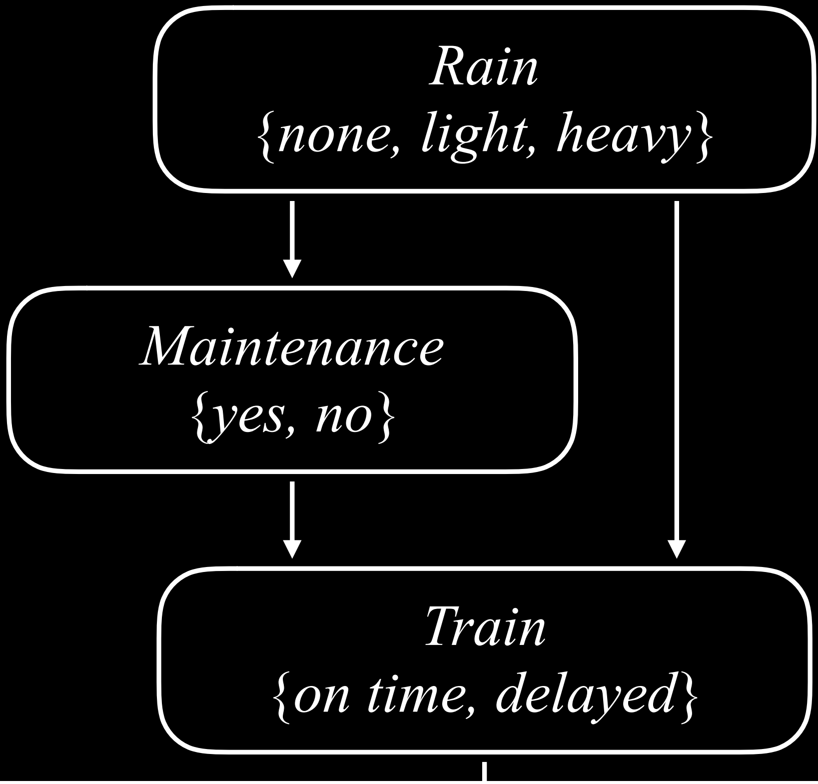

| |
|---|
| R = *light* |
| M = *yes* |
| T = *on time* |
| |

| ***R*** | ***M*** | *on time* | *delayed* |
|---|---|---|---|
| *none* | *yes* | 0.8 | 0.2 |
| *none* | *no* | 0.9 | 0.1 |
| *light* | *yes* | 0.6 | 0.4 |
| *light* | *no* | 0.7 | 0.3 |
| *heavy* | *yes* | 0.4 | 0.6 |
| *heavy* | *no* | 0.5 | 0.5 |



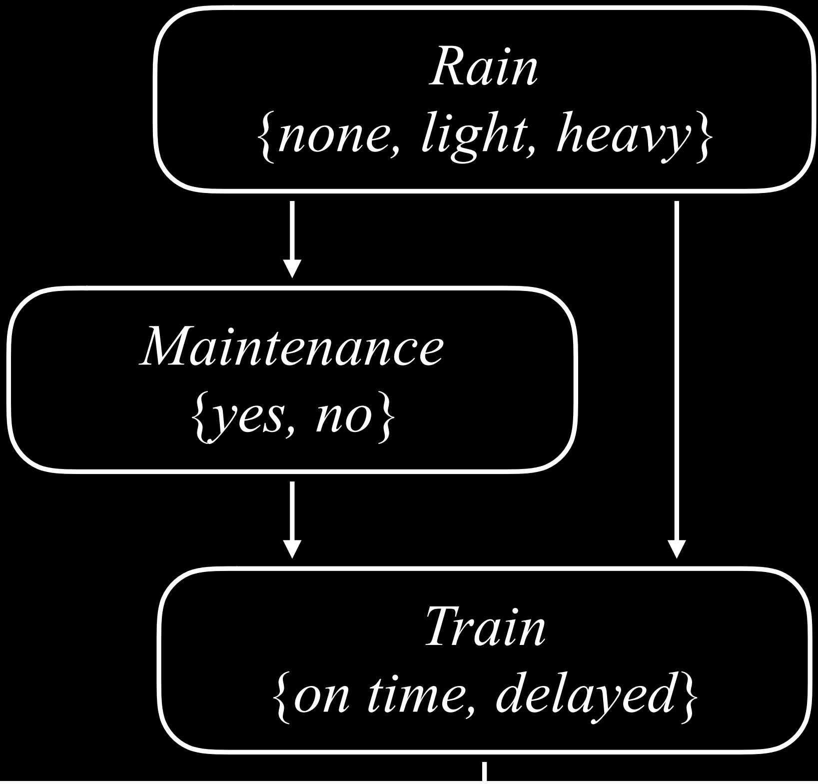

| |
|---|
| R = *light* |
| M = *yes* |
| T = *on time* |
| A = *attend* |

| ***T*** | *attend* | *miss* |
|---|---|---|
| *on time* | 0.9 | 0.1 |
| *delayed* | 0.6 | 0.4 |




| |
|---|
| R = *light* |
| M = *yes* |
| T = *on time* |
| A = *attend* |

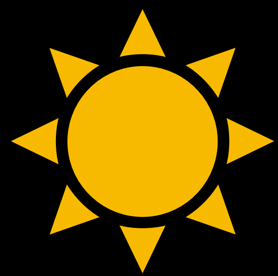
| ***R*** | ***M*** | *on time* | *delayed* |
|---|---|---|---|
| *none* | *yes* | 0.8 | 0.2 |
| *none* | *no* | 0.9 | 0.1 |
| *light* | *yes* | 0.6 | 0.4 |
| *light* | *no* | 0.7 | 0.3 |
| *heavy* | *yes* | 0.4 | 0.6 |
| *heavy* | *no* | 0.5 | 0.5 |




| |
|---|
| R = *light* |
| M = *yes* |
| T = *on time* |
| A = *attend* |

| ***R*** | ***M*** | *on time* | *delayed* |
|---|---|---|---|
| *none* | *yes* | 0.8 | 0.2 |
| *none* | *no* | 0.9 | 0.1 |
| *light* | *yes* | 0.6 | 0.4 |
| *light* | *no* | 0.7 | 0.3 |
| *heavy* | *yes* | 0.4 | 0.6 |
| *heavy* | *no* | 0.5 | 0.5 |

# Uncertainty over Time

$X_t$: Weather at time t

## Markov assumption

the assumption that the current state depends on only a finite fixed number of previous states

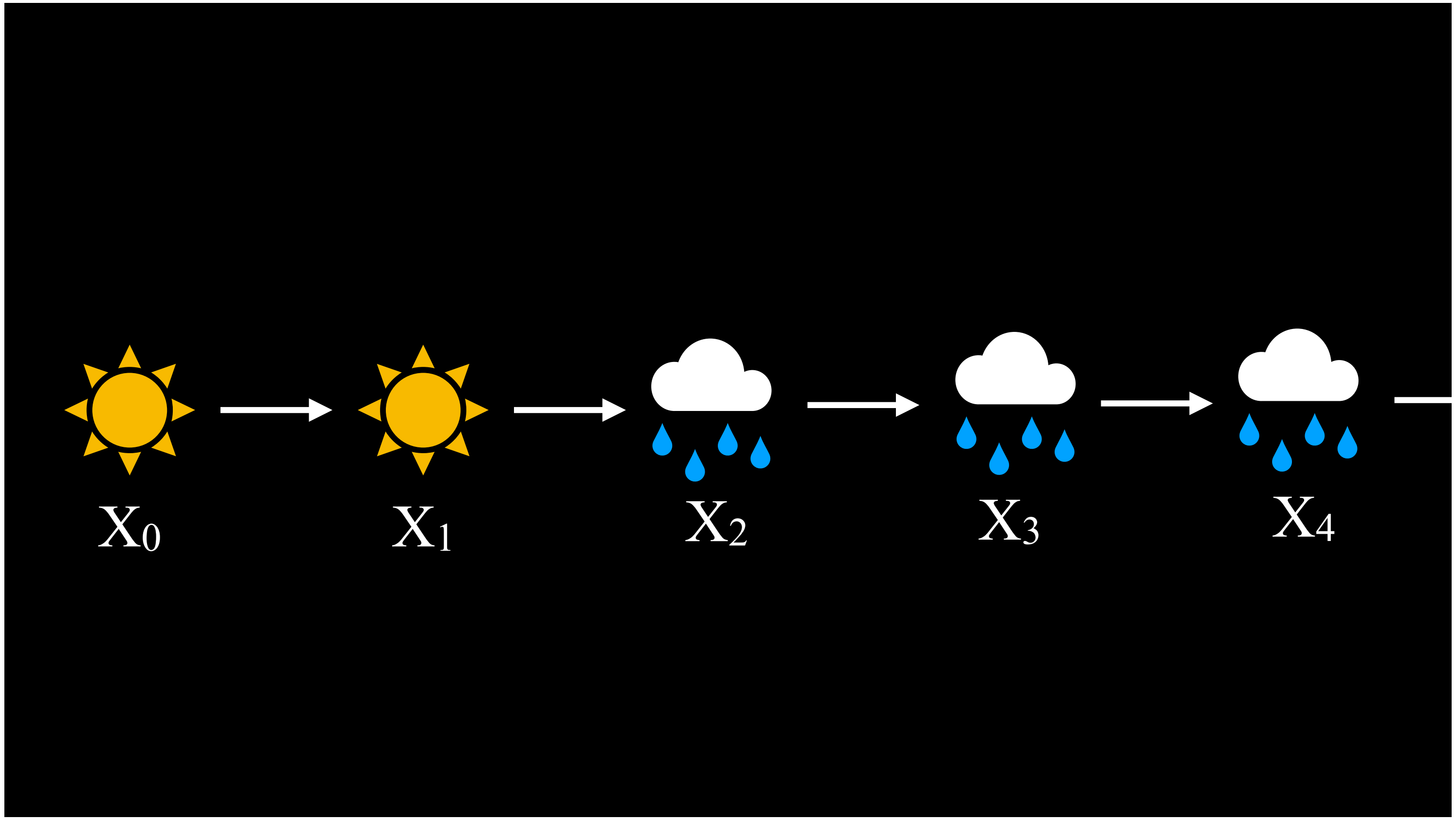
# Markov Chain

## Markov chain

a sequence of random variables where the distribution of each variable follows the Markov assumption

# Transition Model

| | | Tomorrow ($X_{t+1}$) | |
|---|---|---|---|
| | |  |  |
| Today ($X_t$) |  | 0.8 | 0.2 |
| |  | 0.3 | 0.7 |



$X_0$
$X_1$
$X_2$
$X_3$
$X_4$

# Sensor Models

| Hidden State | Observation |
|---|---|
| robot's position | robot's sensor data |
| words spoken | audio waveforms |
| user engagement | website or app analytics |
| weather | umbrella |

# Hidden Markov Models

# Hidden Markov Model

a Markov model for a system with hidden states that generate some observed event

# Sensor Model

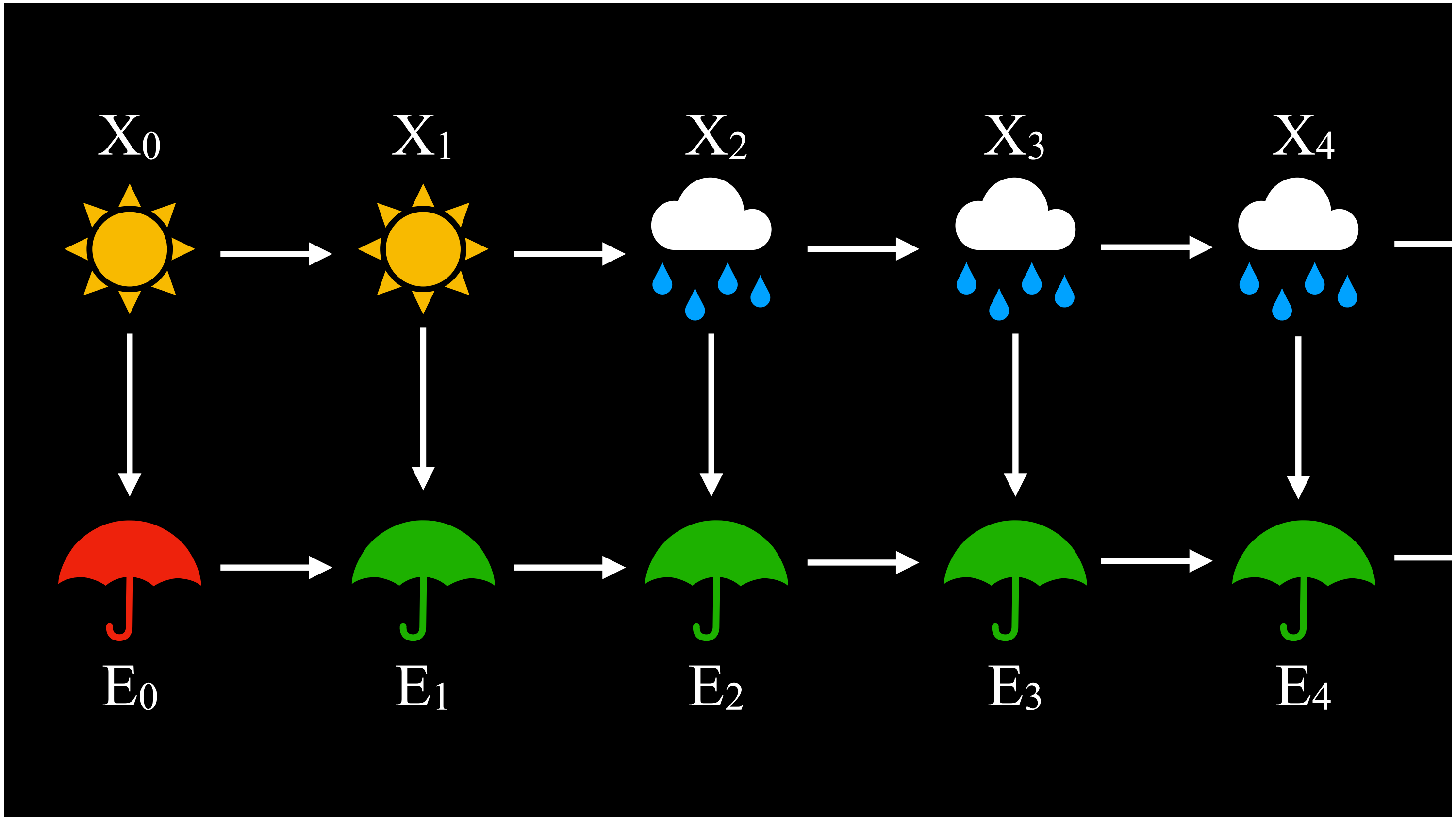
Observation ($E_t$)

State ($X_t$)

| |  |  |
|---|---|---|
|  | 0.2 | 0.8 |
|  | 0.9 | 0.1 |

## sensor Markov assumption

the assumption that the evidence variable
depends only the corresponding state


$X_0$
$X_1$
$X_2$
$X_3$
$X_4$
$E_0$
$E_1$
$E_2$
$E_3$
$E_4$

| Task | Definition |
| --- | --- |
| filtering | given observations from start until now, calculate distribution for **current** state |
| prediction | given observations from start until now, calculate distribution for a **future** state |
| smoothing | given observations from start until now, calculate distribution for **past** state |
| most likely explanation | given observations from start until now, calculate most likely **sequence** of states |

# Uncertainty

Introduction to

**Artificial Intelligence**

with Python