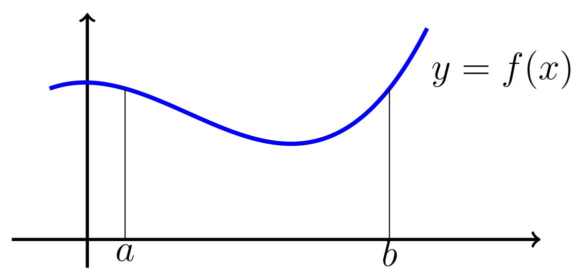
Lecture 19: Double Integrals over Rectangles (§5.1)

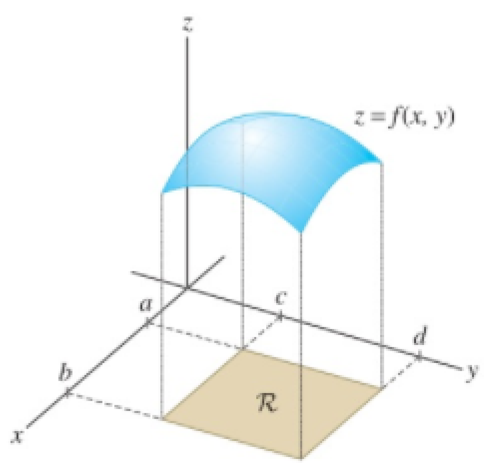
In Calculus 1, given a continuous function $f(x) \geq 0$ on an interval $[a, b]$, we define the area under the curve as a definite integral

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$



We wish to use a similar technique to motivate the definition of a definite integral of a function of two variables:

Consider a continuous function $f(x, y) \geq 0$ on a rectangular region $R = \{(x, y)\,|\,a \leq x \leq b,\ c \leq y \leq d\} = [a, b]\times[c, d]$.



Goal: Find the volume of the solid that lies above R and under the surface $z = f(x, y)$.

We divide the region R into small rectangle R_{ij} and we simplify the process by

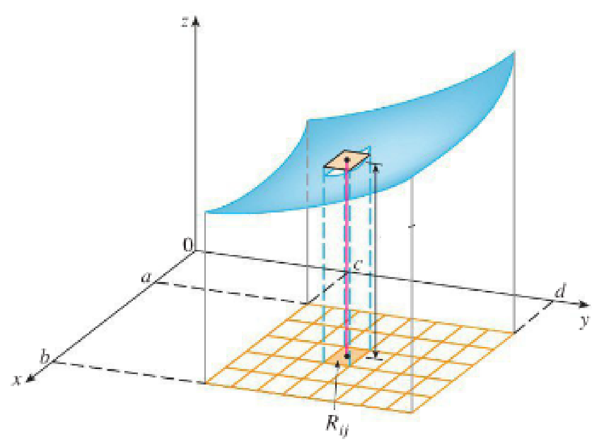
1. partitioning the intervals equally,

$$a = x_0 < x_1 < \cdots < x_i < \cdots < x_m = b,\ \ \Delta x = \frac{b-a}{m}$$

$$c = y_0 < y_1 < \cdots < y_j < \cdots < y_n = d,\ \ \Delta y = \frac{d-c}{n}$$

and

2. choosing the upper right-hand corner point of the sub-rectangle R_{ij} as the sample point.



Then volume under the surface can be approximated by

$$V \approx$$

Next, let both m and n go to infinity and we define the volume of the solid under the surface $f(x, y)$ and above the rectangle R is

$$V =$$

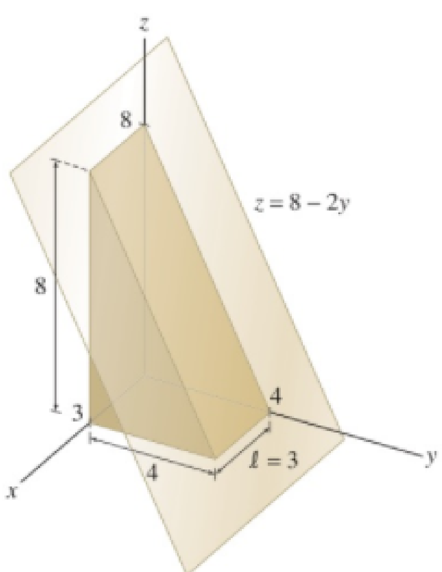
if the limit exists.

<u>Def.</u> The **double integral** of $f(x, y)$ over a rectangle R is defined as

$$\iint\limits_R f(x, y)\, dA =$$

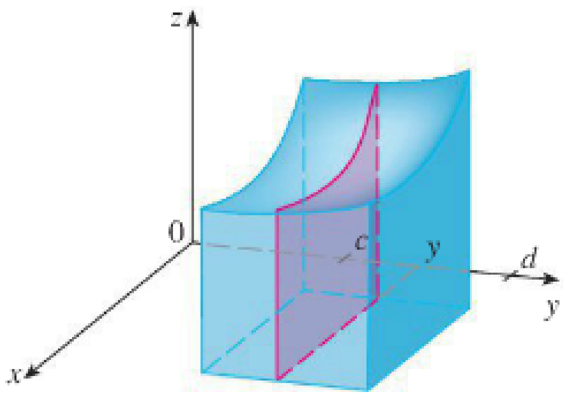
If the limit exists, we say that f is **integrable** over R.

<u>ex.</u> Evaluate $\iint\limits_R (8 - 2y)\, dA$, where $R = [0, 3] \times [0, 4]$.



Iterated Integrals

To calculate the volume of the solid given in the definition of double integral, we can take the solid, slice it into thin sections perpendicular to the y-axis, and add the volumes of the each slice.



Therefore, $V =$

$$\boxed{V = \iint\limits_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy}$$

The right-hand side of the equality is an **iterated integral** and we work <u>from inside out</u>.

ex. Evaluate the iterated integral

$$\int_0^4 \int_0^3 \frac{1}{\sqrt{3x+4y}}\, dx\, dy$$

Try this: $\displaystyle \int_0^3 \int_0^4 \frac{1}{\sqrt{3x+4y}}\, dy\, dx = \frac{34}{9}$

Fubini's Theorem

If f is continuous on the rectangle $R = [a, b] \times [c, d]$, then

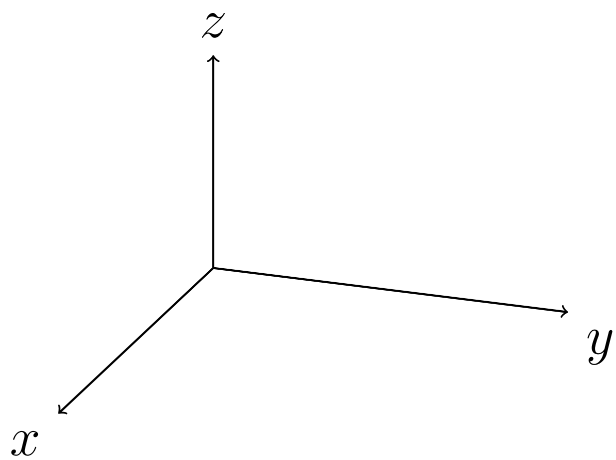
$$\iint\limits_R f(x, y)\, dA =$$

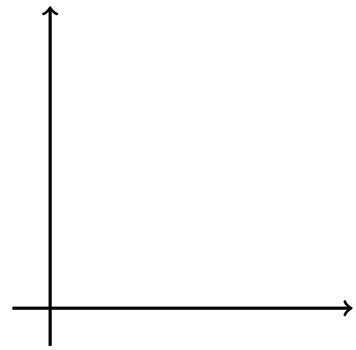
While the order of integration of a double integral may not affect its value, the ease of integration may be dependent on the order of integration.

<u>**ex.**</u> Evaluate: $\displaystyle\int_0^1 \int_0^1 \frac{y}{(xy+1)^2}\, dy\, dx$

<u>ex.</u> Evaluate $\displaystyle\int_0^1 \int_0^1 \frac{y}{(xy+1)^2}\,dy\,dx$ by changing the order of integration.

ex. Find the volume of the solid in the first octant enclosed by $z = 4 - x^2$ and $y = 2$.





<u>Special case:</u> If $f(x,y) = g(x)h(y)$ on $R = [a,b] \times [c,d]$, then

$$\iint\limits_R g(x)h(y)\,dA =$$

<u>**ex.**</u> Evaluate: $\displaystyle\int_0^1 \int_0^1 ye^{x+y}\,dy\,dx$

Now You Try It (NYTI):

1. Evaluate $\displaystyle\iint\limits_R (4-2y)\,dA,\ \ R=[0,1]\times[0,1]$ by interpreting it as the volume of the solid. 3

2. Given $\displaystyle\int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,dy\,dx=\frac{1}{2}$ and $\displaystyle\int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,dx\,dy=-\frac{1}{2}.$

Do the answers contradict Fubini's Theorem and why? no

3. Evaluate

(a) $\displaystyle\int_0^1 \int_0^2 y e^{x-y}\, dx\, dy$ $(e^2-1)(1-2e^{-1})$

(b) $\displaystyle\int_0^1 \int_0^1 xy\sqrt{x^2+y^2}\, dy\, dx$ $\frac{2}{15}(2\sqrt{2}-1)$

(c) $\displaystyle\iint_R x\sin(x+y)\,dA,\;\; R=[0,\pi/6]\times[0,\pi/3]$ $\qquad \frac{\sqrt{3}-1}{2}-\frac{\pi}{12}$

(d) $\displaystyle\iint_R \frac{x}{1+xy}\,dA,\;\; R=[0,1]\times[0,1]$ $\qquad 2\ln(2)-1$