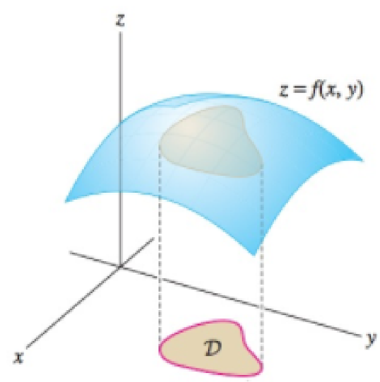
Lecture 20: Double Integrals over General Regions (§5.2)

In L19, we developed the idea of a double integral over a rectangular region. Next, we will develop a more general theory of integrals over a closed plane region using our established results as a starting point.

Consider a function $f(x, y)$ and a general plane region D. We enclosed this region D in a rectangle R and define a new function $F(x, y)$ on R by



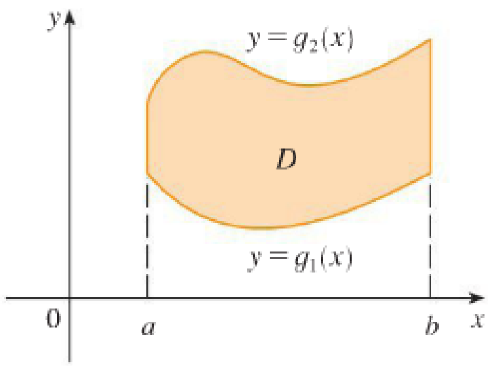
$$F(x,y) = \begin{cases} \quad & (x,y) \text{ in } D \\ \quad & (x,y) \text{ not in } D \end{cases}$$

If F is integrable over R, then we define the **double integral of *f* over *D*** by

$$\iint_D f(x,y)\,dA =$$

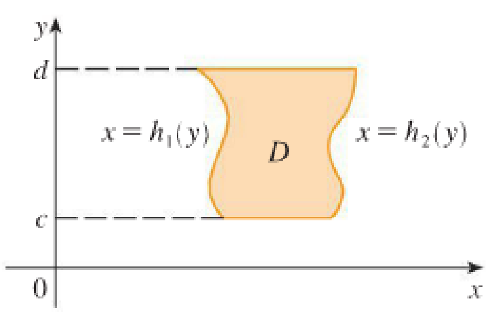
To evaluate the integral, we classify the region D into two types:

Type I: $D = \{(x, y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$



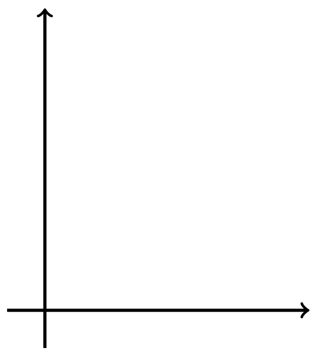
$$\iint_D f(x, y)\, dA =$$

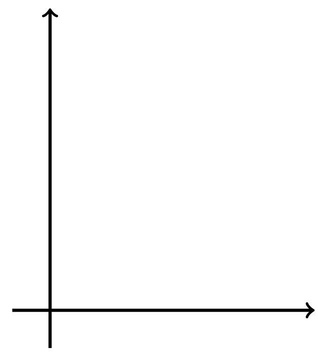
Type II: $D = \{(x, y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$



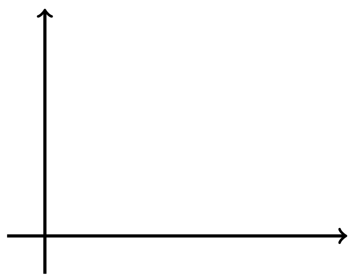
$$\iint_D f(x, y)\, dA =$$

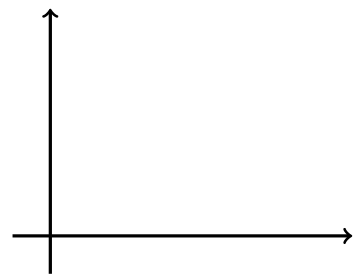
ex. Find $\displaystyle\iint_D 5\,dA$, where D is the triangle with vertices at $(0,0)$, $(0,4)$ and $(1,4)$.





<u>ex.</u> Find $\displaystyle\iint_D x\, dA$, where D is the region between $y = x$ and $y = x^2$ in the first quadrant.

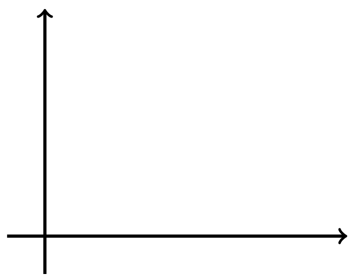


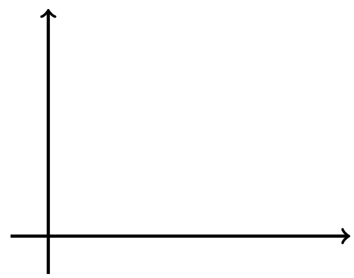


Changing the Order of Integration

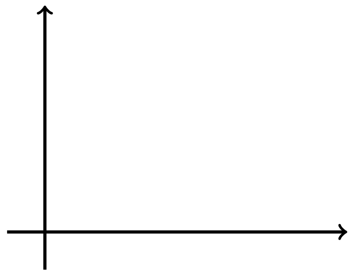
Sometimes changing the order of integration in the iterated integral may simplify the evaluation process. It is useful to use the limits in the original integral to first sketch the region D.

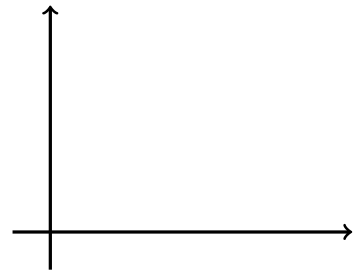
ex. Evaluate: $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\, dx\, dy$





ex. Evaluate: $\displaystyle\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy$

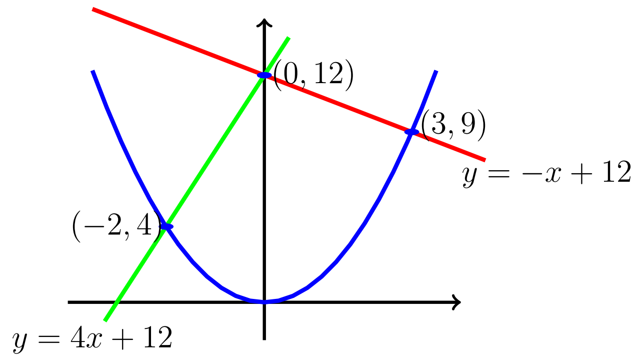




Area of Regions by Double Integrals

Def. The area of the region D is $A(D) =$

ex. Set up a double integral that represents the area of the region bounded by $y = x^2$, $y = -x + 12$, and $y = 4x + 12$.

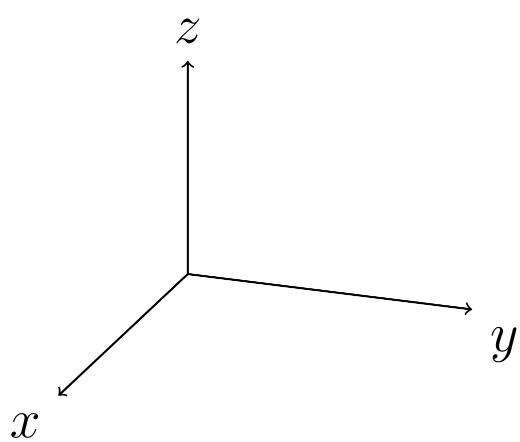


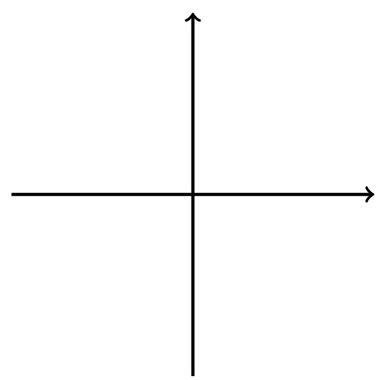
Region Between Two Surfaces

The techniques developed in this lecture can also be used to determine the volume between two continuous surfaces $z_1 = f(x, y)$ and $z_2 = g(x, y)$ with $g(x, y) \leq f(x, y)$ on a region D in the xy-plane.

$$V =$$

<u>ex.</u> Set up an integral for the volume of the solid between the surfaces $z = x^2 + y^2$ and $z = 9$.





Now You Try It (NYTI):

1. Find the volume below $z = 1 - y$ above the region $-1 \leq x \leq 1$, $0 \leq y \leq 1 - x^2$. 4/5

2. Find the volume in the first octant bounded by $y^2 = 4 - x$ and $y = 2z$. 2

3. Evaluate (you may reverse the order of integration):

(a) $\displaystyle\int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy$ $1 - \cos(1)$

(b) $\displaystyle\int_0^1 \int_{x^2}^1 \sqrt{y} \sin y\, dy\, dx$ $\sin(1) - \cos(1)$

4. Sketch the region of integration and change the order of integration.

(a) $\displaystyle\int_0^{\pi/2}\int_0^{\cos x} f(x,y)\,dy\,dx$

$$\int_0^1\int_0^{\cos^{-1} y} f(x,y)\,dx\,dy$$

(b) $\displaystyle\int_{-2}^{2}\int_0^{\sqrt{4-y^2}} f(x,y)\,dx\,dy$

$$\int_0^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)\,dy\,dx$$

(c) $\displaystyle\int_1^{10}\int_0^{\ln(y)} f(x,y)\,dx\,dy$

$$\int_0^{\ln(10)}\int_{e^x}^{10} f(x,y)\,dy\,dx$$