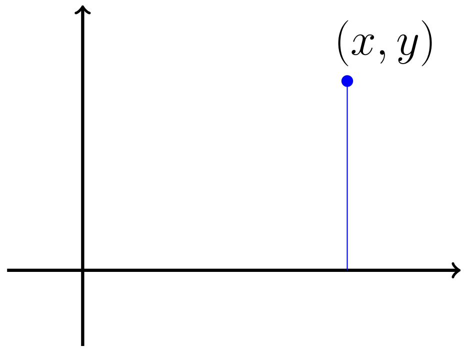
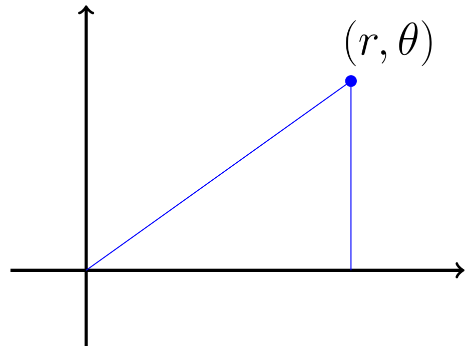
Lecture 21: Change of Coordinates (§1.3&2.7)

Recall polar coordinates in Calc 2:





The rectangular coordinates of a point (x, y) and its **polar coordinates** (r, θ) are related by the equations

Polar to rectangular:

$$\boxed{x = r\cos\theta \qquad y = r\sin\theta}$$

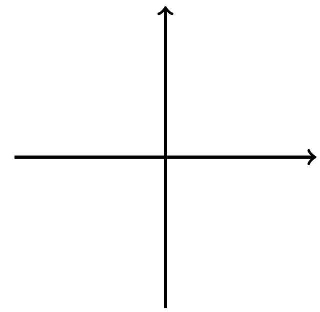
Rectangular to polar:

$$\boxed{r = \sqrt{x^2 + y^2} \qquad \tan\theta = \frac{y}{x}}$$

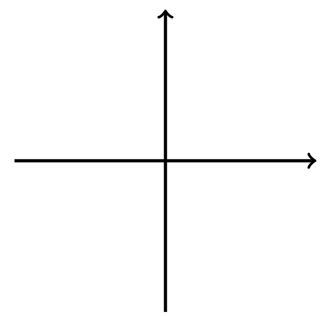
NOTE: We restrict $r \geq 0$ for the convenience of multiple integration in the later lectures.

ex. Convert to an equation in rectangular coordinates.

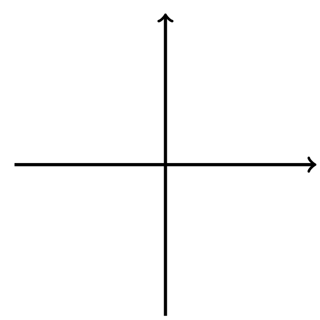
1. $r = 2$



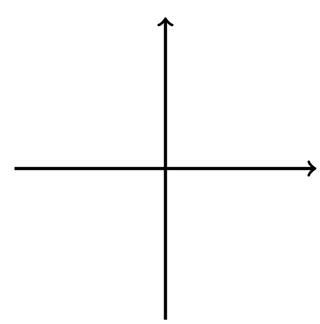
2. $r = 2\cos\theta$



3. $r = 2\sin\theta$

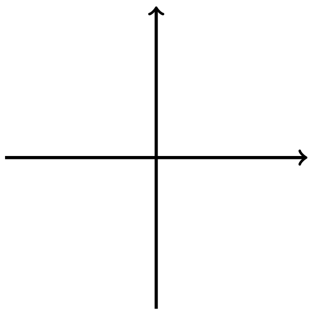


4. $r = 2\csc\theta$

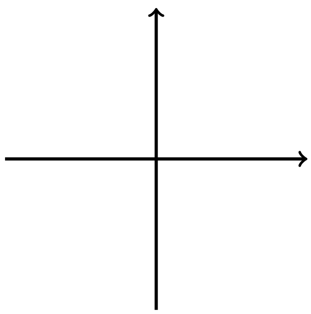


ex. Convert to an equation in polar coordinates.

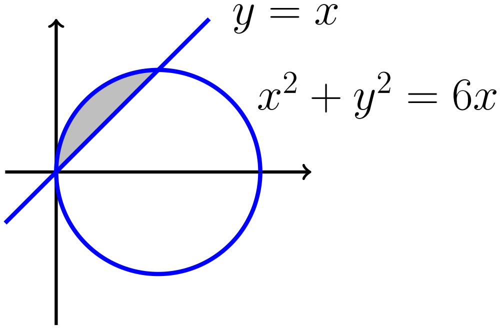
1. $x^2 + y^2 = 5$



2. $y = \sqrt{4x - x^2}$



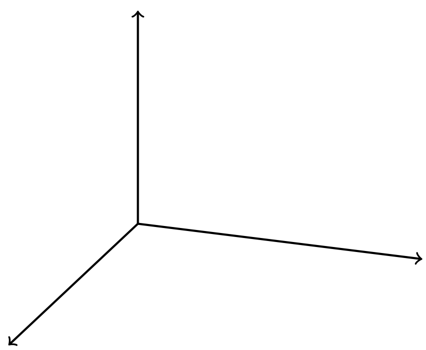
ex. Describe the shaded region in polar coordinates.



Next, we introduce two generalizations of polar coordinates to $\mathbb{R}^3$, cylindrical and spherical coordinates.

Cylindrical Coordinates (r, θ, z)

In $\mathbb{R}^2$, a point with rectangular coordinates (x, y) can be easily converted to (r, θ) in polar coordinates and vice versa. We use the same concept in $\mathbb{R}^3$ to do the conversion between rectangular and cylindrical coordinates.



Cylindrical to rectangular:

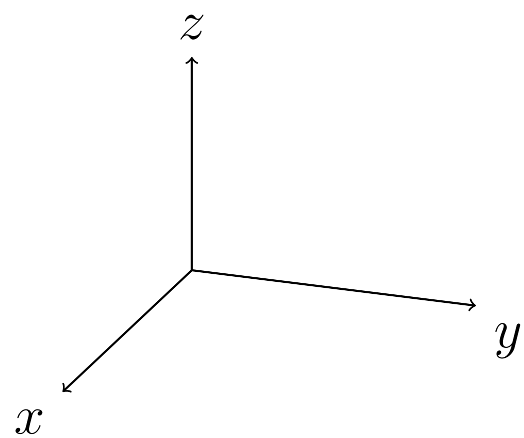
$$\boxed{x = r\cos\theta \qquad y = r\sin\theta \qquad z = z}$$

Rectangular to cylindrical:

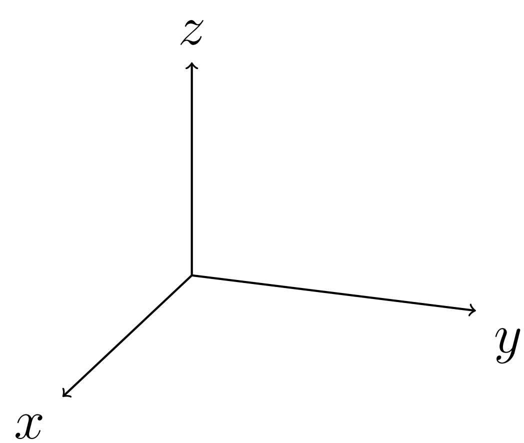
$$\boxed{r = \sqrt{x^2 + y^2} \qquad \tan\theta = \frac{y}{x} \qquad z = z}$$

ex. Describe the solid E in cylindrical coordinates.

$$E = \{(x, y, z) \mid x^2 + y^2 \leq 5,\ 0 \leq z \leq 7\}$$



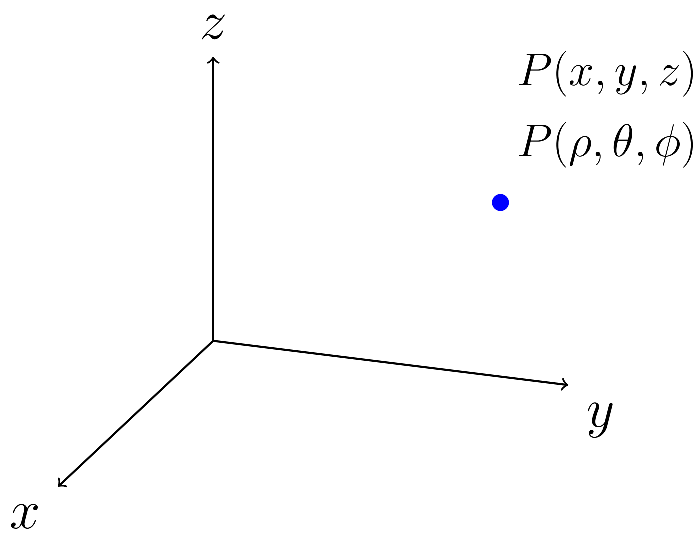
ex. Find an equation in cylindrical coordinates for the surface $z = \sqrt{4x^2 + 4y^2}$.



Spherical Coordinates

In $\mathbb{R}^3$, a point P in the **spherical coordinate** system is represented by the ordered triple (ρ, θ, ϕ), where

- ρ is the distance between P and the origin $(\rho \geq 0)$
- θ is the same angle used to describe the location in cylindrical coordinates $(0 \leq \theta \leq 2\pi)$
- ϕ is the angle formed by the positive z-axis and line segment OP $(0 \leq \phi \leq \pi)$



Spherical to rectangular:

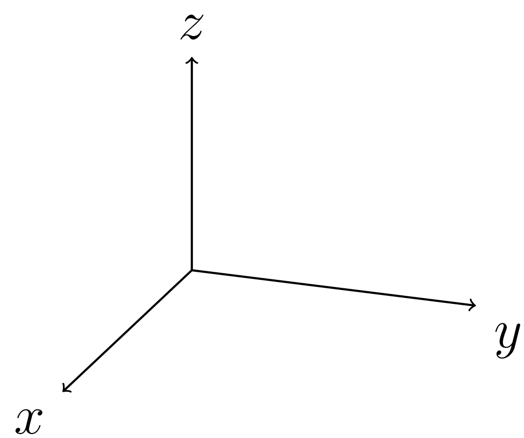
$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi$$

Rectangular to spherical:

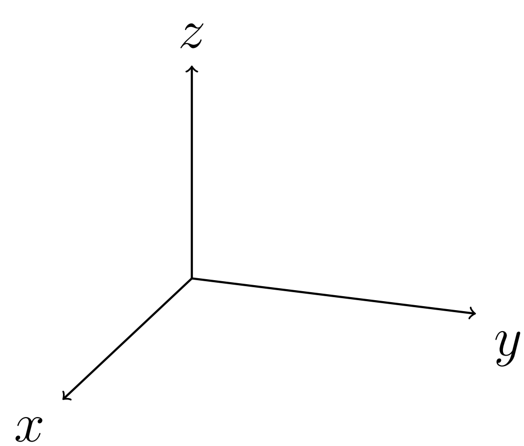
$$\rho = \sqrt{x^2 + y^2 + z^2} \qquad \tan\theta = \frac{y}{x} \qquad \cos\phi = \frac{z}{\rho}$$

ex. Describe the solid E in spherical coordinates.

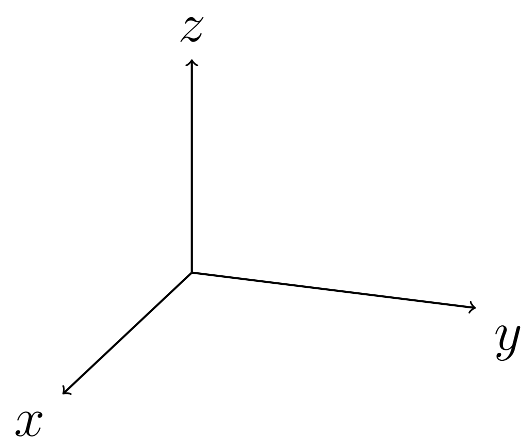
$$E = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 9,\ x,\ y,\ z \geq 0\}$$



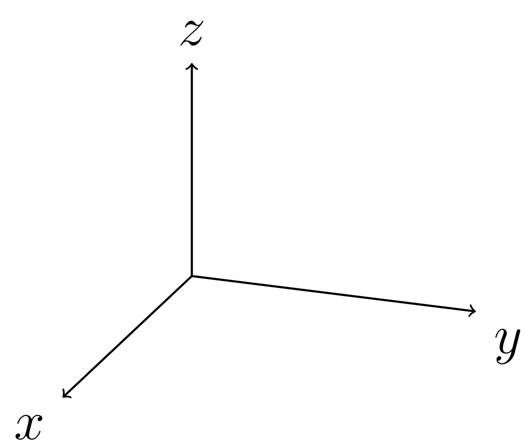
ex. Describe the surface $z = \sqrt{x^2 + y^2}$ in spherical coordinates.



ex. Identify the surface $r = 4\sin\theta$ in cylindrical coordinates.

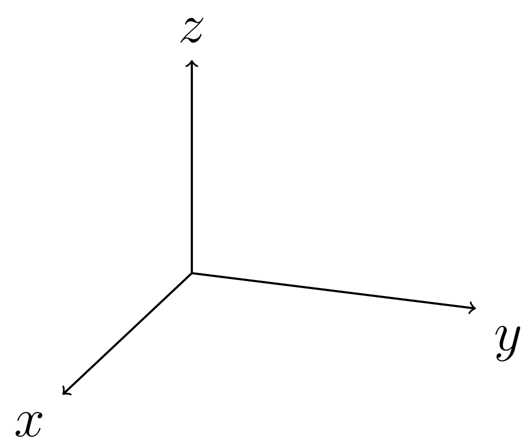


ex. Identify the surface $\rho = 4\cos\phi$ in spherical coordinates.



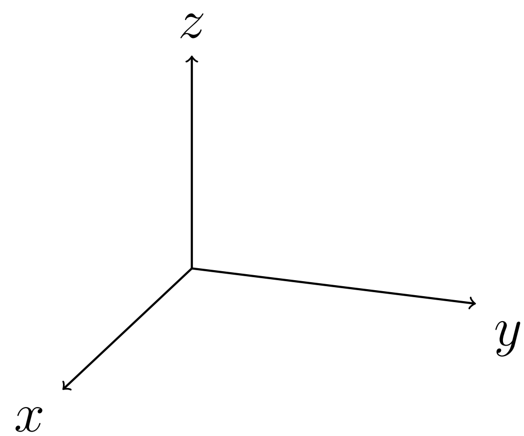
ex. Sketch the solid described by the given inequalities:

$$r^2 \le z \le 8 - r^2$$



ex. Sketch the solid described by the given inequalities:

$$\rho \leq 1,\ 0 \leq \phi \leq \frac{\pi}{4},\ 0 \leq \theta \leq \frac{\pi}{2}$$



Now You Try It (NYTI):

1. (1) Convert the point $\left(2, \dfrac{3\pi}{4}, -1\right)$ in cylindrical coordinates to rectangular coordinates. $(-\sqrt{2}, \sqrt{2}, -1)$

(2) Convert the point $(-3\sqrt{3}, -3, 5)$ in rectangular coordinates to cylindrical coordinates. $(6, 7\pi/6, 5)$

2. (1) Convert the point $\left(4, \dfrac{\pi}{3}, \dfrac{\pi}{4}\right)$ in spherical coordinates to rectangular coordinates. $(\sqrt{2}, \sqrt{6}, 2\sqrt{2})$

(2) Convert the point $(2, -2\sqrt{3}, 3)$ in rectangular coordinates to spherical coordinates. $\left(5, \frac{5\pi}{3}, \cos^{-1}\left(\frac{3}{5}\right)\right)$

3. Identify the surface in cylindrical coordinates.

(1) $r^2 + z^2 = 4$

a sphere centered at the origin with radius 2

(2) $r = 4\cos\theta$

a circular cylinder with radius 2

4. Identify the surface in spherical coordinates.

(1) $\rho\cos\phi = 1$

a horizontal plane $z = 1$

(2) $\rho = \cos\phi$

a sphere of radius $\frac{1}{2}$ centered at $\left(0, 0, \frac{1}{2}\right)$