Group Practice Problems #1 - The Values of the Other Trigonometric Functions: For each angle θ, determine $\tan(\theta)$, $\cot(\theta)$, $\sec(\theta)$, and $\csc(\theta)$.

- $\theta = 30^\circ$

Notice that $\sin(30^\circ) = \frac{1}{2}$ and $\cos(30^\circ) = \frac{\sqrt{3}}{2}$, which allows us to calculate the other trigonometric functions:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}$$
$$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{1}{1/\sqrt{3}} = \sqrt{3}$$
$$\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}}$$
$$\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{1}{1/2} = 2$$

- $\theta = \frac{3\pi}{4}$

Notice that θ is in the second quadrant, which means the sine function will be positive and the cosine function will be negative. Also notice that the reference angle is $\frac{\pi}{4}$. This means that $\sin(\frac{3\pi}{4}) = \sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$ and $\cos(\frac{3\pi}{4}) = -\cos(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}$. We can calculate the other trigonometric functions from this information.

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{\sqrt{2}/2}{-\sqrt{2}/2} = -1$$
$$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{1}{-1} = -1$$
$$\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{1}{-\sqrt{2}/2} = -\frac{2}{\sqrt{2}}$$
$$\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{1}{\sqrt{2}/2} = \frac{2}{\sqrt{2}}$$

- $\theta = 225^\circ$

Notice that θ is in the third quadrant, which means the sine and cosine functions will both be negative. Also notice that the reference angle is 45°. This means that $\sin(225^\circ) = -\sin(45^\circ) = -\frac{\sqrt{2}}{2}$ and $\cos(225^\circ) = -\cos(45^\circ) = -\frac{\sqrt{2}}{2}$. We can calculate the other trigonometric functions from this information.

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{-\sqrt{2}/2}{-\sqrt{2}/2} = 1$$
$$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{1}{-1} = 1$$
$$\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{1}{-\sqrt{2}/2} = -\frac{2}{\sqrt{2}}$$
$$\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{1}{-\sqrt{2}/2} = -\frac{2}{\sqrt{2}}$$

- $\theta = \frac{7\pi}{6}$

Notice that θ is in the third quadrant, which means the sine and cosine functions will both be negative. Also notice that the reference angle is $\frac{\pi}{6}$. This means that $\sin(\frac{7\pi}{6}) = -\sin(\frac{\pi}{6}) = \frac{1}{2}$ and $\cos(\frac{7\pi}{6}) = -\cos(\frac{\pi}{6}) = -\frac{\sqrt{3}}{2}$. We can calculate the other trigonometric functions from this information.

$$\begin{aligned}
\tan(\theta) &= \frac{\sin(\theta)}{\cos(\theta)} = \frac{-1/2}{-\sqrt{3}/2} = \frac{1}{\sqrt{3}} \\
\cot(\theta) &= \frac{1}{\tan(\theta)} = \frac{1}{1/\sqrt{3}} = \sqrt{3} \\
\sec(\theta) &= \frac{1}{\cos(\theta)} = \frac{1}{-\sqrt{3}/2} = -\frac{2}{\sqrt{3}} \\
\csc(\theta) &= \frac{1}{\sin(\theta)} = \frac{1}{-1/2} = -2
\end{aligned}$$

Comments and Observations:

- There are some that prefer to write everything with a rationalized denominator. I see no specific harm in that, but I also generally see little benefit to that. Radicals in the denominator are not particularly terrible in a context where you may be taking reciprocals of the fractions. (But you should at least know how to do it!)

Group Practice Problems #2 - Determining the Values of Trigonometric Functions with Algebra: Use algebraic techniques to solve the following problems.

- Suppose that $\sec(\theta) = \frac{7}{3}$ and that $\frac{3\pi}{2} < \theta < 2\pi$. Determine $\cot(\theta)$.

We will start by substituting the known quantity into the appropriate Pythagorean Identity:

$$\begin{aligned}
\tan^2(\theta) + 1 &= \sec^2(\theta) && \text{The Pythagorean Identity} \\
\tan^2(\theta) + 1 &= \left(\frac{7}{3}\right)^2 && \text{Substitute} \\
\tan^2(\theta) + 1 &= \frac{49}{9} && \\
\tan^2(\theta) &= \frac{40}{9} && \text{Subtract 1 from both sides} \\
\tan(\theta) &= \pm\frac{\sqrt{40}}{3} && \text{Take the square root of both sides}
\end{aligned}$$

Since we know that θ is the fourth quadrant, we know that $\tan(\theta)$ will be negative. This means that $\tan(\theta) = -\frac{\sqrt{40}}{3}$. Therefore, $\cot(\theta) = \frac{1}{\tan(\theta)} = -\frac{3}{\sqrt{40}}$.

- Suppose that $\tan(\theta) = 4$ and that $\pi < \theta < \frac{3\pi}{2}$. Determine $\csc(\theta)$.

Since $\tan(\theta) = 4$, we have that $\cot(\theta) = \frac{1}{4}$. We will substitute this quantity into the appropriate Pythagorean Identity.

$$\begin{aligned}
1 + \cot(\theta) &= \csc^2(\theta) && \text{The Pythagorean Identity} \\
1 + \left(\frac{1}{4}\right)^2 &= \csc^2(\theta) && \text{Substitute} \\
1 + \frac{1}{16} &= \csc^2(\theta) && \\
\csc^2(\theta) &= \frac{17}{16} && \\
\csc(\theta) &= \pm\frac{\sqrt{17}}{4} && \text{Take the square root of both sides}
\end{aligned}$$

Since we know that θ is the third quadrant, we know that $\csc(\theta)$ will be negative. This means that $\csc(\theta) = -\frac{\sqrt{17}}{4}$.

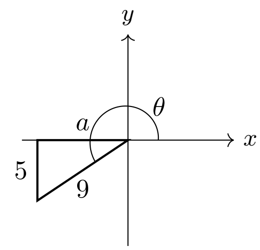
Comments and Observations:

- There are often multiple paths to the final answer. As long as the algebra is valid, it doesn't matter which path you choose.

Group Practice Problems #3 - Determining the Values of Trigonometric Functions with Geometry: Use geometric techniques to solve the following problems.

- Suppose that $\csc(\theta) = -\frac{9}{5}$ and that $\pi < \theta < \frac{3\pi}{2}$. Determine $\cot(\theta)$.

Notice that $\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{\text{hypotenuse}}{\text{opposite}}$ and θ is in the third quadrant, so we can draw a triangle in the appropriate position to reflect this relationship.



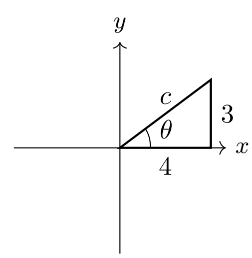
We will use the Pythagorean Theorem to determine the length of the remaining side.

$$\begin{aligned}
a^2 + b^2 &= c^2 && \text{The Pythagorean Theorem}\\
a^2 + 5^2 &= 9^2 && \text{Substitute}\\
a^2 + 25 &= 81 &&\\
a^2 &= 56 && \text{Subtract 25 from both sides}\\
a &= \pm\sqrt{56} = \pm 2\sqrt{14} && \text{Take the square root of both sides}
\end{aligned}$$

We will pick the value of a to be negative to match the geometry of the diagram. Then since $\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\text{adjacent}}{\text{opposite}}$, we can see that $\cot(\theta) = -\frac{2\sqrt{14}}{5}$.

- Suppose that $\cot(\theta) = \frac{4}{3}$ and that $0 < \theta < \frac{\pi}{2}$. Determine $\sec(\theta)$.

Notice that $\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\text{adjacent}}{\text{opposite}}$ and θ is in the first quadrant, so we can draw a triangle in the appropriate position to reflect this relationship.



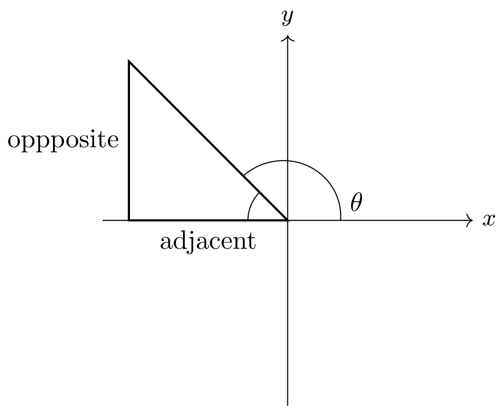
We will use the Pythagorean Theorem to determine the length of the remaining side.

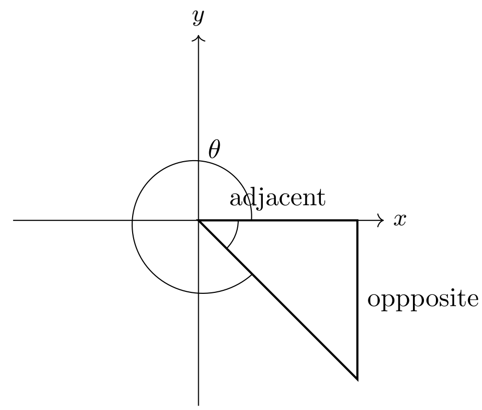
$$\begin{aligned}
a^2 + b^2 &= c^2 && \text{The Pythagorean Theorem}\\
3^2 + 4^2 &= c^2 && \text{Substitute}\\
9 + 16 &= c^2 &&\\
c^2 &= 25 &&\\
c &= \pm 5 && \text{Take the square root of both sides}
\end{aligned}$$

Since c is the distance from the origin, we will take it to be positive. Then since $\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{\text{hypotenuse}}{\text{adjacent}}$, we can see that $\sec(\theta) = \frac{5}{4}$.

Comments and Observations:

- Technically, in this approach we're working with the reference angle so that the angle must be drawn in the appropriate position. If the triangle is in either the second or fourth quadrant, the relative positions for the lengths of the legs (opposite and adjacent) will need to be understood relative to the angle formed with x-axis.





Group Practice Problems #4 - Simplifying Expressions: Rewrite the expression as an expression involving a single trigonometric function.

- $\cos(t)\csc(t)$

$$\begin{aligned}
\cos(t)\csc(t) &= \cos(t)\cdot\frac{1}{\sin(t)} && \text{Reciprocal identity}\\
&= \frac{\cos(t)}{\sin(t)} && \text{Fraction arithmetic}\\
&= \cot(t) && \text{Definition of cotangent}
\end{aligned}$$

- $\frac{\tan(t)}{\sec(t)-\cos(t)}$

$$\begin{aligned}
\frac{\tan(t)}{\sec(t)-\cos(t)} &= \frac{\tan(t)}{\frac{1}{\cos(t)}-\cos(t)} && \text{Reciprocal identity}\\
&= \frac{\tan(t)}{\frac{1-\cos^2(t)}{\cos(t)}} && \text{Common denominator}\\
&= \tan(t)\cdot\frac{\cos(t)}{1-\cos^2(t)} && \text{Fraction arithmetic}\\
&= \frac{\sin(t)}{\cos(t)}\cdot\frac{\cos(t)}{1-\cos^2(t)} && \text{Definition of tangent}\\
&= \frac{\sin(t)}{\cos(t)}\cdot\frac{\cos(t)}{\sin^2(t)} && \text{Pythagorean Identity}\\
&= \frac{1}{\sin(t)} && \text{Fraction arithmetic}\\
&= \csc(t) && \text{Reciprocal identity}
\end{aligned}$$

Comments and Observations:

- These calculations are mostly just algebra practice. It's not really necessary to write out every single algebraic step, but you should definitely be able to clearly verbalize what's happening at every step.

Group Practice Problems #5 - Proving Identities: Prove the following mathematical identities.

- $1 + \cot(x) = \cos(x)\left(\sec(x) + \csc(x)\right)$

$$\begin{aligned}
\cos(x)(\sec(x) + \csc(x)) &= \cos(x)\left(\frac{1}{\cos(x)} + \frac{1}{\sin(x)}\right) && \text{Reciprocal identity} \\
&= 1 + \frac{\cos(x)}{\sin(x)} && \text{Distributive property} \\
&= 1 + \cot(x) && \text{Definition of cotangent}
\end{aligned}$$

- $\frac{(1+\cos(\alpha))(1-\cos(\alpha))}{\sin(\alpha)} = \sin(\alpha)$

$$\begin{aligned}
\frac{(1+\cos(\alpha))(1-\cos(\alpha))}{\sin(\alpha)} &= \frac{1-\cos^2(\alpha)}{\sin(\alpha)} && \text{Expand the numerator} \\
&= \frac{\sin^2(\alpha)}{\sin(\alpha)} && \text{Pythagorean Theorem} \\
&= \sin(\alpha) && \text{Simplify}
\end{aligned}$$

- $\frac{\sin^4(t)-\cos^4(t)}{\sin(t)-\cos(t)} = \sin(t) + \cos(t)$

$$\begin{aligned}
\frac{\sin^4(t)-\cos^4(t)}{\sin(t)-\cos(t)} &= \frac{(\sin^2(t)+\cos^2(t))(\sin^2(t)-\cos^2(t))}{\sin(t)-\cos(t)} && \text{Factor} \\
&= \frac{\sin^2(t)-\cos^2(t)}{\sin(t)-\cos(t)} && \text{Pythagorean Identity} \\
&= \frac{(\sin(t)+\cos(t))(\sin(t)-\cos(t))}{\sin(t)-\cos(t)} && \text{Factor} \\
&= \sin(t)+\cos(t) && \text{Simplify}
\end{aligned}$$

Comments and Observations:

- This is more algebra practice.