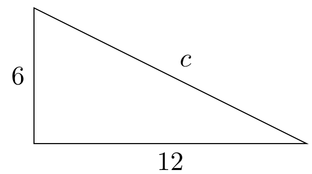
Group Practice Problems #1 - The Pythagorean Theorem: Sketch the right triangle that satisfies the given conditions. Then calculate the length of the remaining side. Simplify the radicals, if possible.

- One leg has length 12 and the other leg has length 6

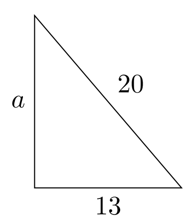


From the given information, we are able to label the lengths of both legs of the right triangle. We will also label the hypotenuse c. To solve for c, we will use the Pythagorean Theorem:

$$\begin{aligned} a^2 + b^2 &= c^2 && \text{Pythagorean Theorem} \\ 6^2 + 12^2 &= c^2 && \text{Substitute} \\ 36 + 144 &= c^2 \\ 180 &= c^2 \\ c &= \sqrt{180} && \text{Take the square root of both sides} \\ & && \text{We can ignore the negative case because this is a distance} \end{aligned}$$

Notice that $180 = 36 \cdot 5$, so that $\sqrt{180} = \sqrt{36} \cdot \sqrt{5} = 6\sqrt{5}$. Therefore, we have that the length of the hypotenuse is $6\sqrt{5}$ units. ($6\sqrt{5} \approx 13.42$)

- One leg has length 13 and the hypotenuse has length 20



From the given information, we are able to label one leg and the hypotenuse of the right triangle. We will label the remaining leg a. To solve for a, we will use the Pythagorean Theorem:

$$\begin{aligned} a^2 + b^2 &= c^2 && \text{Pythagorean Theorem} \\ a^2 + 13^2 &= 20^2 && \text{Substitute} \\ a^2 + 169 &= 400 \\ a^2 &= 231 \\ a &= \sqrt{231} && \text{Take the square root of both sides} \\ & && \text{We can ignore the negative case because this is a distance} \end{aligned}$$

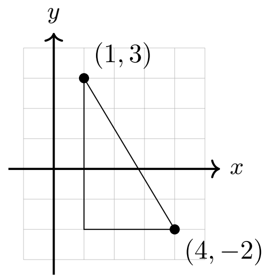
Notice that 231 does not have any square factors, so $\sqrt{231}$ does not simplify. Therefore, the length of the remaining leg is $\sqrt{231}$ units. ($\sqrt{231} \approx 15.20$)

Comments and Observations:

- We typically use the symbols a and b to represent the lengths of the legs, and we typically use c to represent the length of the hypotenuse. However, any symbol is acceptable. In particular, many students use x to represent the length of the unknown side out of habit. You should get comfortable with using all sorts of symbols as variables.

Group Practice Problems #2 - The Distance Formula: Plot each pair of points and draw the line segment connecting them. Then compute the distance between the points. Simplify the radicals, if possible.

- $(1, 3)$ and $(4, -2)$

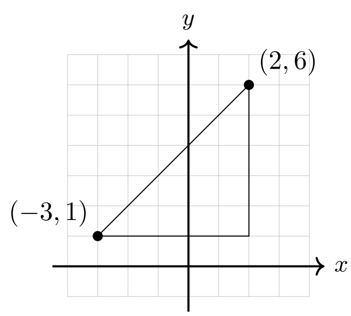


We calculate the distance between the points using the distance formula. We will call the point $(1, 3)$ the first point and $(4, -2)$ the second point.

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} && \text{Distance formula} \\ &= \sqrt{(4-1)^2 + (-2-3)^2} && \text{Substitute} \\ &= \sqrt{3^2 + (-5)^2} \\ &= \sqrt{9 + 25} \\ &= \sqrt{34} \end{aligned}$$

Since 34 does not have any squared factors, $\sqrt{34}$ does not simplify. The distance between the points $(1, 3)$ and $(4, -2)$ is $\sqrt{34}$ units. ($\sqrt{34} \approx 5.83$)

- $(2, 6)$ and $(-3, 1)$



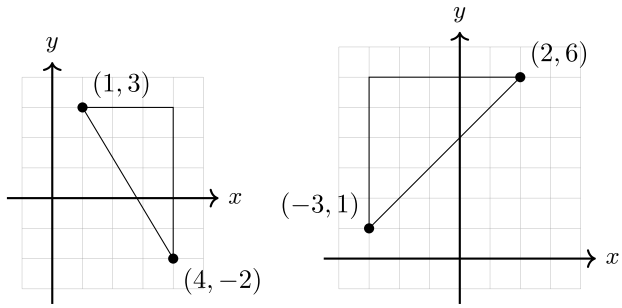
We calculate the distance between the points using the distance formula. We will call the point $(-3, 1)$ the first point and $(2, 6)$ the second point.

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} && \text{Distance formula} \\ &= \sqrt{(2-(-3))^2 + (6-1)^2} && \text{Substitute} \\ &= \sqrt{5^2 + 5^2} \\ &= \sqrt{25 + 25} \\ &= \sqrt{50} \end{aligned}$$

Notice that $50 = 25 \cdot 2$, so that $\sqrt{50} = \sqrt{25} \cdot \sqrt{2} = 5\sqrt{2}$. The distance between the points $(-3, 1)$ and $(2, 6)$ is $5\sqrt{2}$ units. ($5\sqrt{2} \approx 7.07$)

Comments and Observations:

- There are actually two right triangles that can be drawn in these diagrams, and both are considered correct.



- By looking at the right triangle, you should be able to visualize the values of the differences in the the distance formula. This helps you to catch errors if you make a substitution or arithmetic error.

- It makes no difference which point you consider to be the first point and which one you consider to be the second. The reason this is true is that the squared terms under the square root are unaffected by swapping their order (because that swap introduces a negative sign that is eliminated when squaring.

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

- The reason we typically write the ordering of the subtract as $x_2 - x_1$ and $y_2 - y_1$ instead of the other way around is because this gives us a "signed distance" that tells us which direction you must travel to get from point 1 to point 2. For example, in the first problem we got the values 3 and -5 under the square root, and we can see that to go from the point $(1, 3)$ to the point $(4, -2)$ we need to go right 3 and down 5. (This becomes more relevant when working with mathematical objects called vectors, which capture directional information. These come up in physics and in future math courses.)

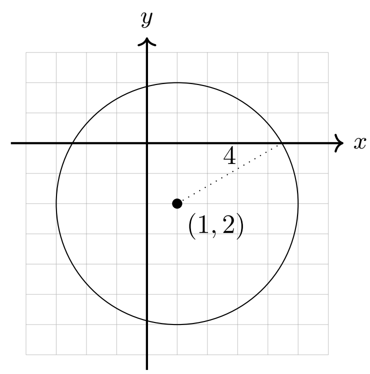
Group Practice Problems #3 - Equations and Graphs of Circles: Determine the equations of the following circles. Then draw a sketch of the graph. Be sure that the graph intersects the coordinate axes appropriately.

- The circle of radius 4 centered at the point $(1, -2)$.

To determine the equation of a circle, we need to identify the center and the radius. In this problem, these values are given to us directly, so that we can plug in the point $(1, -2)$ for the center and the value 4 for the radius.

$$\begin{aligned} (x-h)^2 + (y-k)^2 &= r^2 && \text{Standard form of a circle} \\ (x-1)^2 + (y-(-2))^2 &= 4^2 && \text{Substitute} \\ (x-1)^2 + (y+2)^2 &= 16 \end{aligned}$$

When sketching the graph, since the radius is larger than the individual coordinates, the circle will intersect both axes at two points.

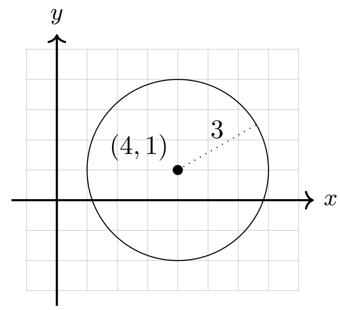


- The circle of diameter 6 centered at the point $(4, 1)$.

To determine the equation of a circle, we need to identify the center and the radius. In this problem, we are given the diameter and the center, but we need the radius and the center. Since the radius of a circle is half of the diameter, we know that the radius is 3. The center is given to us explicitly as $(4, 1)$. We can plug these values into the formula.

$$\begin{aligned} (x-h)^2 + (y-k)^2 &= r^2 && \text{Standard form of a circle} \\ (x-4)^2 + (y-1)^2 &= 3^2 && \text{Substitute} \\ (x-4)^2 + (y-1)^2 &= 9 \end{aligned}$$

When sketching the graph, notice that the radius is smaller than the size of the x-coordinate, which means that the circle is not large enough to reach the y-axis. But it will intersect the x-axis because the size of the y-coordinate is smaller than the radius.



- The circle whose diameter is the line segment from the point $(-2, 1)$ to the point $(4, 3)$.

To determine the equation of a circle, we need to identify the center and the radius. In this problem, we are only given a line segment that represent the diameter of the circle. The center of the circle is located at the midpoint of the diameter, which can be calculated using the midpoint formula.

$$\begin{aligned}
(h, k) &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) && \text{Midpoint formula} \\
&= \left(\frac{-2 + 4}{2}, \frac{1 + 3}{2}\right) && \text{Substitute} \\
&= (-1, 2)
\end{aligned}$$

The radius is half of the diameter, and we can calculate the diameter using the distance between the two points. We will start by calculating the diameter:

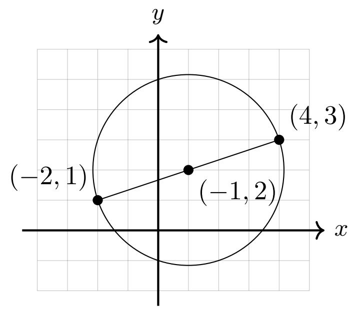
$$\begin{aligned}
d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} && \text{Distance formula} \\
&= \sqrt{(4 - (-2))^2 + (3 - 1)^2} && \text{Substitute} \\
&= \sqrt{6^2 + 2^2} \\
&= \sqrt{40} \\
&= 2\sqrt{10} && \text{Simplify the radical}
\end{aligned}$$

Since the radius is half of this, we get that $r = \sqrt{10}$.

We can now plug these values into the formula for a circle.

$$\begin{aligned}
(x - h)^2 + (y - k)^2 &= r^2 && \text{Standard form of a circle} \\
(x - (-1))^2 + (y - 2)^2 &= \left(\sqrt{10}\right)^2 && \text{Substitute} \\
(x + 1)^2 + (y - 2)^2 &= 10
\end{aligned}$$

When sketching the graph of the circle, we can intuitively see that the diameter makes it appear that the circle will intersect both coordinate axes in two points.



Comments and Observations:

- When starting off, some students make the error of trying to plug in the coordinates of the center into the x and y rather than the h and k. It is important to keep the meaning of the symbols in mind. In the standard equation of a circle, the x and y values represent an arbitrary point on the circle whose coordinates are (x, y), and there are many points that satisfy the equation.

- No matter what information is given to you, to get the equation of the circle, you always need to identify the center and the radius. It is up to you to figure out how to get that from the given information. A sketch can be helpful if you're stuck on this part.

- For the last problem, a carefully-drawn sketch can also be used to identify the midpoint if you did not remember the midpoint formula.

Group Practice Problems #4 - Circles and the Coordinate Axes: Determine the x-intercepts and y-intercepts of the following circles. Then sketch a graph of the circle.

- The circle whose equation is $(x-1)^2 + (y+2)^2 = 9$.

To determine the x-intercepts, we set $y = 0$ and solve for x.

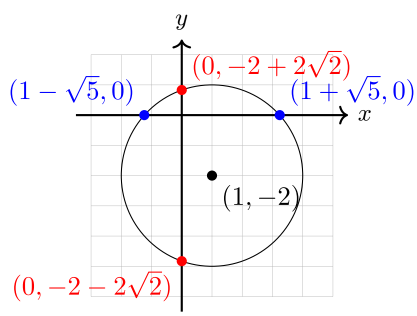
$$\begin{aligned}
(x-1)^2 + (y+2)^2 &= 9 && \text{Equation of the circle} \\
(x-1)^2 + (0+2)^2 &= 9 && \text{Substitute } y = 0 \\
(x-1)^2 + 4 &= 9 && \\
(x-1)^2 &= 5 && \text{Subtract 4 from both sides} \\
x - 1 &= \pm\sqrt{5} && \text{Take the square root of both sides} \\
x &= 1 \pm \sqrt{5} && \text{Add 1 to both sides}
\end{aligned}$$

So the x-intercepts are located at $(1+\sqrt{5}, 0)$ and $(1-\sqrt{5}, 0)$. These points are approximately located at $(3.24, 0)$ and $(-1.24, 0)$.

To determine the y-intercepts, we set $x = 0$ and solve for y.

$$\begin{aligned}
(x-1)^2 + (y+2)^2 &= 9 && \text{Equation of the circle} \\
(0-1)^2 + (y+2)^2 &= 9 && \text{Substitute } x = 0 \\
1 + (y+2)^2 &= 9 && \\
(y+2)^2 &= 8 && \text{Subtract 1 from both sides} \\
y + 2 &= \pm 2\sqrt{2} && \text{Take the square root of both sides} \\
y &= -2 \pm 2\sqrt{2} && \text{Subtract 2 from both sides}
\end{aligned}$$

So the y-intercepts are located at $(0, -2+\sqrt{2})$ and $(0, -2-\sqrt{2})$. These points are approximately located at $(0, 0.83)$ and $(0, -4.83)$.



- The circle of diameter 8 centered at the point $(5, 2)$.

We first need to determine the equation of the circle. Since the diameter is 8, we know that the radius is 4. This means that the equation of the circle is $(x-5)^2 + (y-2)^2 = 16$.

To determine the x-intercepts, we set $y = 0$ and solve for x.

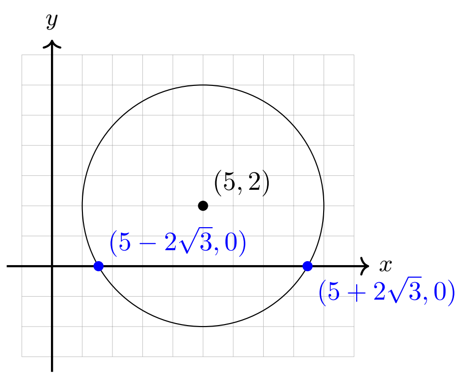
$$
\begin{aligned}
(x-5)^2 + (y-2)^2 &= 16 && \text{Equation of the circle} \\
(x-5)^2 + (0-2)^2 &= 16 && \text{Substitute } y = 0 \\
(x-5)^2 + 4 &= 16 && \\
(x-5)^2 &= 12 && \text{Subtract 4 from both sides} \\
x - 5 &= \pm 2\sqrt{3} && \text{Take the square root of both sides} \\
x &= 5 \pm 2\sqrt{3} && \text{Add 5 to both sides}
\end{aligned}
$$

So the x-intercepts are located at $(5 + 2\sqrt{3}, 0)$ and $(5 - 2\sqrt{3}, 0)$. These points are approximately located at $(8.46, 0)$ and $(1.54, 0)$.

To determine the y-intercepts, we set $x = 0$ and solve for y.

$$
\begin{aligned}
(x-5)^2 + (y-2)^2 &= 16 && \text{Equation of the circle} \\
(0-5)^2 + (y-2)^2 &= 16 && \text{Substitute } x = 0 \\
25 + (y-2)^2 &= 16 && \\
(y-2)^2 &= -9 && \text{Subtract 25 from both sides}
\end{aligned}
$$

Since the square root of a negative number is not a real number, there are no solutions to the equation, and so there are no y-intercepts.



Comments and Observations:

- Finding the x-intercepts is the same as finding the intersection of the circle with the line $y = 0$. And finding the y-intercepts is the same as finding the intersection with the line $x = 0$. Thinking about those lines is helpful for keeping track of which variable you're supposed to set equal to zero.
- Many students struggle in situations where there is no intercept. Some students simply ignore the negative sign and take the square root, while others recognize that they can't take a square root but don't know how to interpret that result. Drawing a careful sketch of the circle can help guide your thinking towards the correct conclusion.

Group Practice Problems #5 - Circles and Lines: Determine the points of intersection (if any) of the following circles and lines. Then sketch a graph of the circle and line, and label the points of intersection.

- The circle whose equation is $(x-2)^2+(y-5)^2=40$ and the line $y=x+1$.

We start by substituting $y=x+1$ into the equation $(x-2)^2+(y-5)^2=40$ to solve for x.

$$\begin{aligned}
(x-2)^2+(y-5)^2 &= 40 && \text{Equation of the circle}\\
(x-2)^2+((x+1)-5)^2 &= 40 && \text{Substitute } y=x+1\\
(x-2)^2+(x-4)^2 &= 40\\
(x^2-4x+4)+(x^2-8x+16) &= 40\\
2x^2-12x+20 &= 40\\
2x^2-12x-20 &= 0 && \text{Subtract 4 from both sides}\\
x^2-6x-10 &= 0 && \text{Divide both sides by 2}
\end{aligned}$$

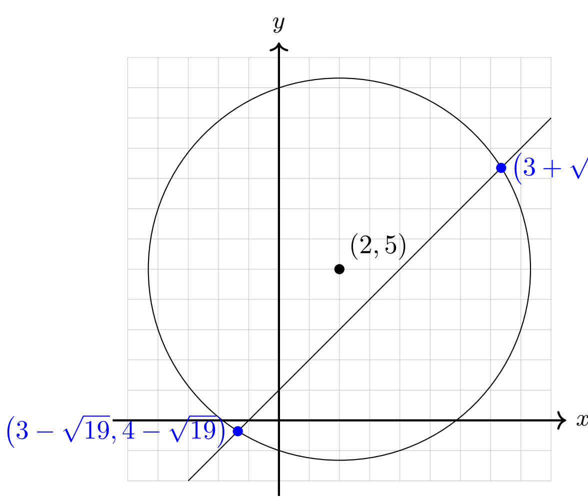
Since this does not easily factor, we will use the quadratic formula.

$$\begin{aligned}
x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} && \text{The quadratic formula}\\
&= \frac{-(-6)\pm\sqrt{(-6)^2-4\cdot 1\cdot(-10)}}{2\cdot 1} && \text{Substitute}\\
&= \frac{6\pm\sqrt{36+40}}{2}\\
&= \frac{6\pm\sqrt{76}}{2}\\
&= \frac{6\pm 2\sqrt{19}}{2} && \text{Simplify the radical}\\
&= 3\pm\sqrt{19} && \text{Factor a 2 out of the numerator and cancel}
\end{aligned}$$

We then plug these values into the equation $y=x+1$ to get the corresponding y-coordinates.

$$x=3+\sqrt{19} \implies y=(3+\sqrt{19})+1=4+\sqrt{19}$$
$$x=3-\sqrt{19} \implies y=(3-\sqrt{19})+1=4-\sqrt{19}$$

Therefore, the points of intersection are $\left(3+\sqrt{19},4+\sqrt{19}\right)$ and $\left(3-\sqrt{19},4-\sqrt{19}\right)$. These points are approximately located at $(7.35,8.35)$ and $(-1.36,-0.36)$.



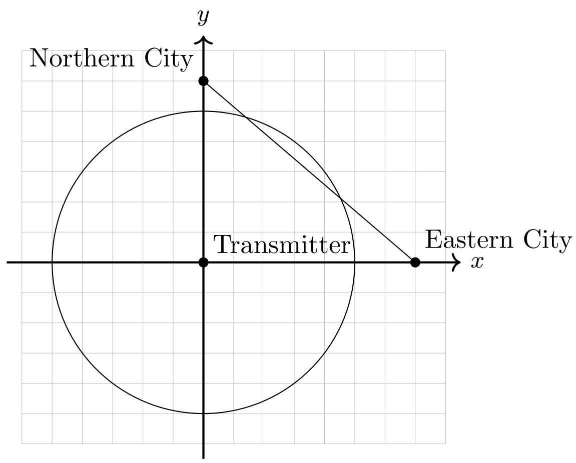
Comments and Observations:

- This problem follows a general approach to solving simultaneous equations. The goal is to try to isolate one variable so that the expression can be substituted into the other equation, resulting in an equation with just a single variable.
- Many students freeze up when they reach the quadratic equation. The general methods are to attempt to factor it, and if it doesn't factor easily to use the quadratic formula. There are some students that jump straight to the quadratic formula every time. This isn't wrong, but it is more of a handicap than a strength to fail to recognize various factoring patterns (or failing to recognize that a quadratic doesn't factor).

Group Practice Problems #6 - Challenge Problem: Draw a diagram and create a mathematical model of the problem. Then answer the question.

- A small radio transmitter broadcasts in a 50 mile radius. If you drive along a straight line from a city 60 miles north of the transmitter to a second city 70 miles to the east of the transmitter, during how much of the drive (in miles) will you pick up a signal from the transmitter?

We will set up a diagram to represent this problem. The transmitter will be located at the origin. This means that the range of the transmitter is a 50-mile radius circle centered at the origin. This can be represented by the equation $x^2 + y^2 = 50^2$. The road being traveled can be represented by a line that passes through the points $(0, 60)$ (for the city to the north) and $(70, 0)$ (for the city to the east). We need to find the two points of intersection between the circle and the line, and then find the distance between those points.



Before we can get the intersection of the circle and the line, we need to get an equation for the line. We will use the slope-intercept form because we can see that the y-intercept is $(0, 60)$. To get the slope, we use "rise-over-run" to get a slope of $\frac{-60}{70} = -\frac{6}{7}$. This means that the equation of the line is $y = -\frac{6}{7}x + 60$.

We will substitute this into the equation of the circle and solve for x.

$$\begin{aligned} x^2 + y^2 &= 50^2 && \text{Equation of the circle} \\ x^2 + \left(-\frac{6}{7}x + 60\right)^2 &= 2500 && \text{Substitute the equation of the line for } y \\ x^2 + \frac{36}{49}x^2 - \frac{720}{7}x + 3600 &= 2500 \\ 49x^2 + 36x^2 - 5040x + 176400 &= 122500 \\ 85x^2 - 5040x + 53900 &= 0 && \text{Subtract 122500 from both sides} \\ 17x^2 - 1008 + 10780 &= 0 && \text{Divide both sides by 5} \end{aligned}$$

Using Wolfram Alpha, we find that the solutions to this equation are $x = 14$ and $x = \frac{770}{17} \approx 45.29$. We use these values to get the corresponding y-coordinates.

$$\begin{aligned} x = 14 &\implies y = -\frac{6}{7}x + 60 = -\frac{6}{7} \cdot (14) + 60 = 48 \\ x \approx 45.29 &\implies y = -\frac{6}{7}x + 60 \approx \frac{6}{7} \cdot (45.29) + 60 = 12.18 \end{aligned}$$

We will now find the distance between these two points.

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} && \text{Distance formula} \\ &\approx \sqrt{(45.29 - 14)^2 + (12.18 - 48)^2} && \text{Substitute} \\ &= \sqrt{31.29^2 + (-35.82)^2} \\ &= \sqrt{2262.1365} \\ &\approx 47.56 \end{aligned}$$

Therefore, signal will be picked up from the transmitter for approximately 47.56 miles of the drive between the two cities.

Comments and Observations:

- Drawing a careful diagram is an important skill for solving word problems. It's very difficult to make progress on word problems without understanding the words that describe the situation. Drawing a diagram can help to organize the information and put everything into context.
- It is not expected that students would be able to identify that the quadratic equation factors. It's not until after using computational technology (or a long calculation with the quadratic formula) that one will recognize that exact solutions . For a problem in which the numbers are sufficiently large as to make it unreasonable to continue to work analytically, the use of technology to find the solutions is reasonable. And the use of decimal approximations from that point forward are also reasonable. However, it is essential to indicate where the use of technology is employed.
- In a write-up of a word problem such as this one, it's helpful to take the time to describe how the diagram is being created. This helps the reader to follow along with you as you set up the problem, rather than just starting with a context-less picture or equations.
- It also helps to describe the calculation that is going to be executed so that the reader has a roadmap to follow as they read through the calculations.