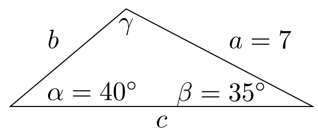
Group Practice Problems #1 - The Law of Sines: Solve the triangle. Assume that the sides and angles are labeled following the convention. There should only be one solution.

- $\alpha = 40°$, $a = 7$, $\beta = 35°$

We will begin by drawing a picture and labeling the sides and angles.



We can determine the third angle using the fact that the sum of the angles is 180°:

$$\gamma = 180° - \alpha - \beta = 180° - 40° - 35° = 105°$$

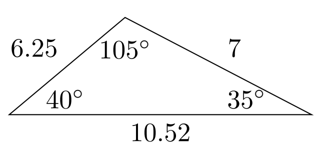
We will now apply the law of sines to determine the remaining two sides. We will first solve for b:

$$\begin{aligned} \frac{\sin(\alpha)}{a} &= \frac{\sin(\beta)}{b} && \text{Law of sines} \\ b &= a \cdot \frac{\sin(\beta)}{\sin(\alpha)} && \text{Solve for } b \\ b &= 7 \cdot \frac{\sin(35°)}{\sin(40°)} && \text{Substitute} \\ b &\approx 6.25 \end{aligned}$$

Then we will solve for c:

$$\begin{aligned} \frac{\sin(\alpha)}{a} &= \frac{\sin(\gamma)}{c} && \text{Law of sines} \\ c &= a \cdot \frac{\sin(\gamma)}{\sin(\alpha)} && \text{Solve for } c \\ c &= 7 \cdot \frac{\sin(105°)}{\sin(40°)} && \text{Substitute} \\ c &\approx 10.52 \end{aligned}$$

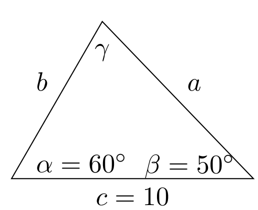
This gives us the fully-solved triangle.



a	b	c	α	β	γ
7	6.25	10.52	40°	35°	105°

- $\alpha = 60°$, $\beta = 50°$, $c = 10$

We will begin by drawing a picture and labeling the sides and angles.



We can determine the third angle using the fact that the sum of the angles is $180°$:

$$\gamma = 180° - \alpha - \beta = 180° - 60° - 50° = 70°$$

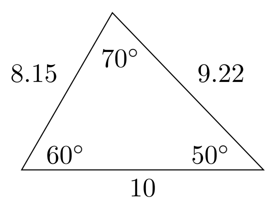
We will now apply the law of sines to determine the remaining two sides. We will first solve for a:

$$\begin{aligned}
\frac{\sin(\alpha)}{a} &= \frac{\sin(\gamma)}{c} && \text{Law of sines} \\
a &= c \cdot \frac{\sin(\alpha)}{\sin(\gamma)} && \text{Solve for } a \\
a &= 10 \cdot \frac{\sin(60°)}{\sin(70°)} && \text{Substitute} \\
a &\approx 9.22
\end{aligned}$$

Then we will solve for b:

$$\begin{aligned}
\frac{\sin(\beta)}{b} &= \frac{\sin(\gamma)}{c} && \text{Law of sines} \\
b &= c \cdot \frac{\sin(\beta)}{\sin(\gamma)} && \text{Solve for } b \\
b &= 10 \cdot \frac{\sin(50°)}{\sin(70°)} && \text{Substitute} \\
b &\approx 8.15
\end{aligned}$$

This results in the following triangle.



a	b	c	α	β	γ
9.22	8.15	10	60°	50°	70°

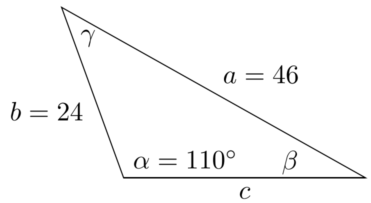
Comments and Observations:

- Sketching the triangle in advance can be helpful as an organizational tool even if you have the dimensions and angles wrong. (But it is good to at least try to get the angles approximately correct.)

Group Practice Problems #2 - The Ambiguous Case: Solve the triangle. Assume that the sides and angles are labeled following the convention. There may be zero, one, or two solutions.

- $a = 46$, $b = 24$, $\alpha = 110°$

We will begin by drawing a picture and labeling the sides and angles.



We will start by solving for β:

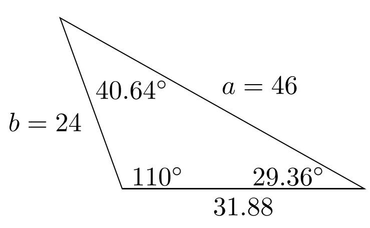
$$
\begin{aligned}
\frac{\sin(\alpha)}{a} &= \frac{\sin(\beta)}{b} && \text{Law of sines} \\
\sin(\beta) &= \frac{b}{a}\sin(\alpha) && \text{Solve for } \sin(\beta) \\
\sin(\beta) &= \frac{24}{46}\sin(110°) && \text{Substitute} \\
\sin(\beta) &= 0.4903
\end{aligned}
$$

Based on the unit circle diagram, there will be solutions in both the first and second quadrants. Using a calculator, we get $\sin^{-1}(.4903) \approx 29.36°$. This means that the two possible solutions are $\beta \approx 29.36°$ and $\beta \approx 180° - 29.36° = 150.64°$.

If $\beta \approx 29.36°$, then $\gamma \approx 180° - \alpha - \beta = 180° - 110° - 29.36° = 40.64°$. With this value, we can solve for c:

$$
\begin{aligned}
\frac{\sin(\alpha)}{a} &= \frac{\sin(\gamma)}{c} && \text{Law of sines} \\
c &= a \cdot \frac{\sin(\gamma)}{\sin(\alpha)} && \text{Solve for } c \\
c &= 46 \cdot \frac{\sin(40.64°)}{\sin(110°)} && \text{Substitute} \\
c &= 31.88
\end{aligned}
$$

This gives the following triangle:

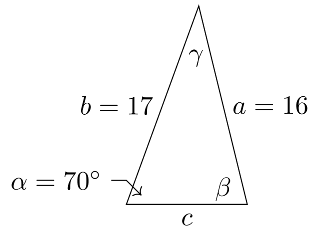


a	b	c	α	β	γ
46	24	31.88	110°	29.36°	40.64°

If $\beta \approx 150.64°$, then we would have $\gamma \approx 180° - \alpha - \beta = 180° - 110° - 150.64° = -80.64°$, which is not valid. Therefore, there is only one solution.

- $a = 16$, $b = 17$, $\alpha = 70°$

We will begin by drawing a picture and labeling the sides and angles.



We will start by solving for β:

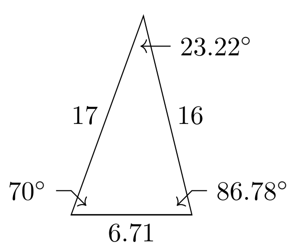
$$
\begin{aligned}
\frac{\sin(\alpha)}{a} &= \frac{\sin(\beta)}{b} && \text{Law of sines} \\
\sin(\beta) &= \frac{b}{a}\sin(\alpha) && \text{Solve for } \sin(\beta) \\
\sin(\beta) &= \frac{17}{16}\sin(70^\circ) && \text{Substitute} \\
\sin(\beta) &= 0.9984
\end{aligned}
$$

Based on the unit circle diagram, there will be solutions in both the first and second quadrants. Using a calculator, we get $\sin^{-1}(.9984) \approx 86.78^\circ$. This means that the two possible solutions are $\beta \approx 86.78^\circ$ and $\beta \approx 180^\circ - 86.78^\circ = 93.22^\circ$.

If $\beta \approx 86.78^\circ$, then $\gamma \approx 180^\circ - \alpha - \beta = 180^\circ - 70^\circ - 86.78^\circ = 23.22^\circ$. With this value, we can solve for c:

$$
\begin{aligned}
\frac{\sin(\alpha)}{a} &= \frac{\sin(\gamma)}{c} && \text{Law of sines} \\
c &= a \cdot \frac{\sin(\gamma)}{\sin(\alpha)} && \text{Solve for } c \\
c &= 16 \cdot \frac{\sin(23.22^\circ)}{\sin(70^\circ)} && \text{Substitute} \\
c &= 6.71
\end{aligned}
$$

This gives the following triangle:

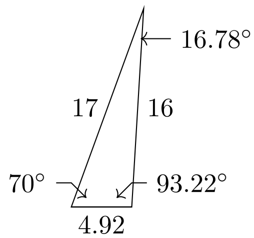


a	b	c	α	β	γ
16	17	6.71	70°	86.78°	23.22°

If $\beta \approx 93.22^\circ$, then $\gamma \approx 180^\circ - \alpha - \beta = 180^\circ - 70^\circ - 93.22^\circ = 16.78^\circ$. With this value, we can solve for c:

$$
\begin{aligned}
\frac{\sin(\alpha)}{a} &= \frac{\sin(\gamma)}{c} && \text{Law of sines} \\
c &= a \cdot \frac{\sin(\gamma)}{\sin(\alpha)} && \text{Solve for } c \\
c &= 16 \cdot \frac{\sin(16.78^\circ)}{\sin(70^\circ)} && \text{Substitute} \\
c &= 4.92
\end{aligned}
$$

This gives the following triangle:



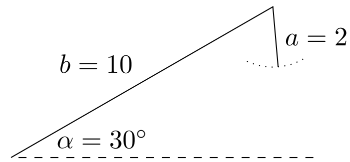
a	b	c	α	β	γ
16	17	6.71	70°	93.22°	16.78°

Comments and Observations:

- When sketching the initial triangle, it's only important to ensure the labels are placed correctly. You won't know in advance how many triangles there might be.
- The final triangle diagrams are not perfect. The main feature to keep an eye on is that obtuse angles are drawn as obtuse, otherwise it creates significant cognitive dissonance.

Group Practice Problems #3 - Studying the Ambiguous Case: In order to study the ambiguous case more deeply, we're going to focus on a sequence of problems where we are changing only a single variable in the problem. We will keep the angle and one of the side lengths fixed, and run through a set of different values for the second side. Write a sentence that explains how your work shows you the number of solutions. (You may find it helpful to try to sketch an accurate picture.)

- $a = 2$, $b = 10$, $\alpha = 30°$

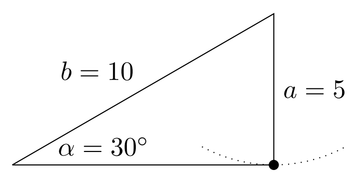


We will start by solving for β:

$$\begin{aligned}
\frac{\sin(\alpha)}{a} &= \frac{\sin(\beta)}{b} && \text{Law of sines}\\
\sin(\beta) &= \frac{b}{a}\sin(\alpha) && \text{Solve for } \sin(\beta)\\
\sin(\beta) &= \frac{10}{2}\sin(30°) && \text{Substitute}\\
\sin(\beta) &= 2.5
\end{aligned}$$

There are no solutions to this equation. From the diagram, we can see that side a is too short to complete the triangle, and so no triangles can be formed.

- $a = 5$, $b = 10$, $\alpha = 30°$

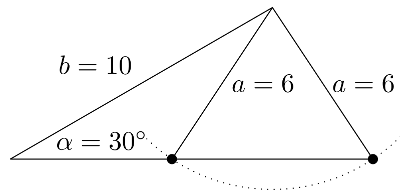


We will start by solving for β:

$$\begin{aligned}
\frac{\sin(\alpha)}{a} &= \frac{\sin(\beta)}{b} && \text{Law of sines}\\
\sin(\beta) &= \frac{b}{a}\sin(\alpha) && \text{Solve for } \sin(\beta)\\
\sin(\beta) &= \frac{10}{5}\sin(30°) && \text{Substitute}\\
\sin(\beta) &= 1
\end{aligned}$$

Based on the unit circle diagram, there is exactly one solution to this equation. That solution is $\beta = 90°$. From the diagram, we can see that side a is just barely long enough to reach the third side, and so only one triangle can be formed.

- $a = 6$, $b = 10$, $\alpha = 30°$



We will start by solving for β:

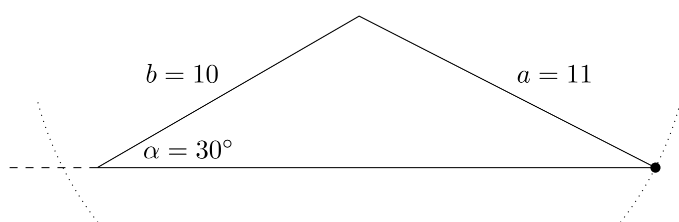
$$
\begin{aligned}
\frac{\sin(\alpha)}{a} &= \frac{\sin(\beta)}{b} && \text{Law of sines} \\
\sin(\beta) &= \frac{b}{a}\sin(\alpha) && \text{Solve for } \sin(\beta) \\
\sin(\beta) &= \frac{10}{6}\sin(30°) && \text{Substitute} \\
\sin(\beta) &= \frac{5}{6}
\end{aligned}
$$

Based on the unit circle diagram, there are two solutions to this equation. Using a calculator, we get that $\sin^{-1}(\frac{5}{6}) \approx 56.44°$, so that the two solutions are $\beta \approx 56.44°$ and $\beta \approx 180° - 56.44° = 123.56°$. We need to determine whether these are valid angles by attempting to calculate the third angle of the triangle.

If $\beta \approx 56.44°$ then $\gamma = 180° - \alpha - \beta \approx 180° - 30° - 56.44° = 93.56°$. And if $\beta \approx 123.56°$ then $\gamma = 180° - \alpha - \beta \approx 180° - 30° - 123.56° = 26.44°$. In both cases, we get a valid triangle.

These two solutions represent the two ways that the leg can "swing down" to hit the third side and create triangles.

- $a = 11$, $b = 10$, $\alpha = 30°$



We will start by solving for β:

$$
\begin{aligned}
\frac{\sin(\alpha)}{a} &= \frac{\sin(\beta)}{b} && \text{Law of sines} \\
\sin(\beta) &= \frac{b}{a}\sin(\alpha) && \text{Solve for } \sin(\beta) \\
\sin(\beta) &= \frac{10}{11}\sin(30°) && \text{Substitute} \\
\sin(\beta) &= \frac{5}{11}
\end{aligned}
$$

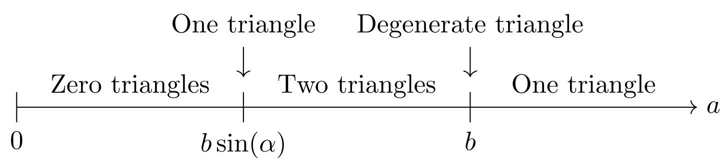
Based on the unit circle diagram, there are two solutions to this equation. Using a calculator, we get that $\sin^{-1}(\frac{5}{6}) \approx 27.04°$, so that the two solutions are $\beta \approx 27.04°$ and $\beta \approx 180° - 27.04° = 152.96°$. We need to determine whether these are valid angles by attempting to calculate the third angle of the triangle.

If $\beta \approx 27.04°$ then $\gamma = 180° - \alpha - \beta \approx 180° - 30° - 27.04° = 122.96°$. And if $\beta \approx 152.96°$ then $\gamma = 180° - \alpha - \beta \approx 180° - 30° - 152.96° = -2.96°$. In the second case, we get an invalid triangle, which means that there is only one solution.

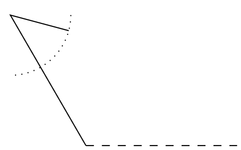
The reason the second angle becomes invalid is because it creates a triangle on the wrong side of the angle α.

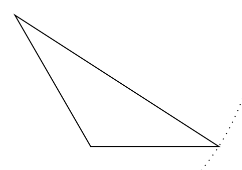
Comments and Observations:

- We can actually build out the entire logic of when you will get different numbers of solutions for a given acute angle from these diagrams.
  - If $\frac{b}{a}\sin(\alpha) > 1$, then there are no solutions. Equivalently, if $a < b\sin(\alpha)$ then there are no solutions.
  - If $\frac{b}{a}\sin(\alpha) = 1$, then there is exactly one solution and the result is a right triangle. Equivalently, if $a = b\sin(\alpha)$ then there is exactly one solution.
  - If $\frac{b}{a}\sin(\alpha) < 1$ and $a < b$, then there are two solutions, one of which gives an acute triangle and the other an obtuse triangle. Equivalently, if $b\sin(\alpha) < a < b$ then there are two solutions.
  - If $a > b$, then there is only one solution.



- If α is an obtuse angle, you get zero solutions or one solution, depending on whether the given leg (side a) is long enough to reach the other leg.

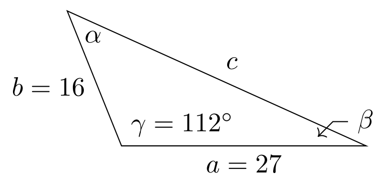

Too short – No triangle


Long enough – One triangle

Group Practice Problems #4 - The Law of Cosines: Solve the triangle. Assume that the sides and angles are labeled following the convention.

- $a = 27$, $b = 16$, $\gamma = 112°$

We will begin by drawing a picture and labeling the sides and angles.



We will use the law of cosines to determine the length of the third side of the triangle.

$c^2 = a^2 + b^2 - 2ab\cos(\gamma)$	Law of cosines
$c^2 = 27^2 + 16^2 - 2 \cdot 27 \cdot 16\cos(112°)$	Substitute
$c^2 \approx 729 + 256 + 323.6601$	
$c^2 \approx 1308.6601$	
$c \approx 36.18$	Take the square root of both sides

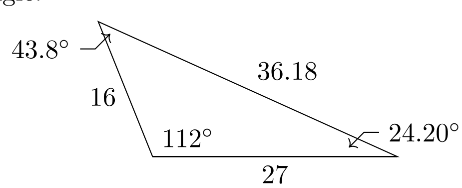
We can also use the alternate forms of the law of cosines to determine one of the remaining angles.

$\cos(\beta) = \dfrac{a^2 + c^2 - b^2}{2ac}$	Law of cosines
$\cos(\beta) \approx \dfrac{27^2 + 36.18^2 - 16^2}{2 \cdot 27 \cdot 36.18}$	Substitute
$\cos(\beta) \approx \dfrac{729 + 1308.9924 - 256}{1953.72}$	
$\cos(\beta) \approx 0.9121$	
$\beta \approx \cos^{-1}(0.8169) \approx 24.20°$	Take the inverse cosine of both sides

To get the last angle, we simply use that the sum of the angles is 180°.

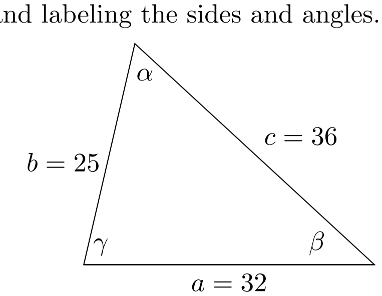
$$\alpha = 180° - \beta - \gamma \approx 180° - 24.20° - 112° = 43.8°$$

This results in the following triangle:



- $a = 32$, $b = 25$, $c = 36$

We will begin by drawing a picture and labeling the sides and angles.



We will use the law of cosines to determine α and β:

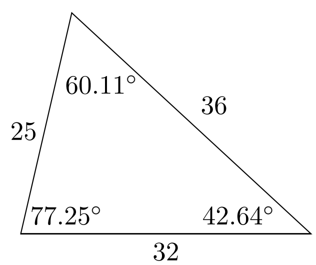
$$
\begin{aligned}
\cos(\alpha) &= \frac{b^2+c^2-a^2}{2bc} && \text{Law of cosines}\\
\cos(\alpha) &= \frac{25^2+36^2-32^2}{2\cdot 25\cdot 36} && \text{Substitute}\\
\cos(\alpha) &= \frac{625+1296-1024}{1800} && \\
\cos(\alpha) &\approx 0.4983 && \\
\alpha &\approx \cos^{-1}(0.4983) \approx 60.11^\circ && \text{Take the inverse cosine of both sides}
\end{aligned}
$$

$$
\begin{aligned}
\cos(\beta) &= \frac{a^2+c^2-b^2}{2ac} && \text{Law of cosines}\\
\cos(\beta) &= \frac{32^2+36^2-25^2}{2\cdot 32\cdot 36} && \text{Substitute}\\
\cos(\beta) & \frac{1024+1296-625}{2304} && \\
\cos(\beta) &\approx 0.7357 && \\
\beta &\approx \cos^{-1}(0.7357) \approx 42.64^\circ && \text{Take the inverse cosine of both sides}
\end{aligned}
$$

We can then calculate the third angle using the fact that the sum of the angles is 180°.

$$\gamma = 180^\circ - \alpha - \beta \approx 180^\circ - 60.11^\circ - 42.64^\circ = 77.25^\circ$$

This gives the following triangle:



Comments and Observations:

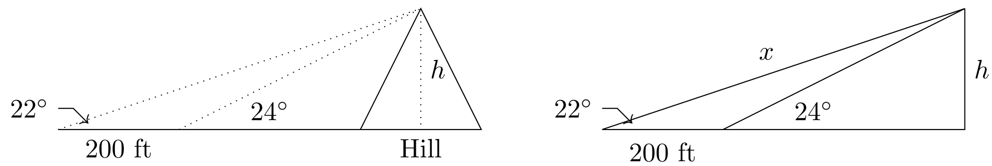
- In the first problem, some rounding error is introduced when squaring out the third side. It is more accurate to use the c^2 value from earlier in the calculation instead of squaring the term after taking the square root. The error for this problem is on the order of about 0.01 degrees. If you need to be more precise, you can keep additional decimals earlier in the calculation.

- When taking the inverse cosine, we will always get the correct angle without worry about an ambiguous case because the other potential angles would be in the third or fourth quadrant, and will not have a meaningful interpretation as an angle of a triangle. If you were to use the law of sines, you would need to make sure that the longest side is across from the largest angle.

Group Practice Problems #5 - Word Problems: Solve the word problem. Draw an accurate sketch of the problem and explain your steps.

- To measure the height of a hill, a woman measures the angle of elevation to the top of the hill to be 24°. She then moves back 200 feet and measures the angle of elevation to be 22°. Find the height of the hill.

We will start by sketching a diagram and isolating the relevant measurements.



We will start with the triangle on the left. Notice that the supplementary angle to the 24°angle is 156°, which makes the angle at the top $180° - 156° - 22° = 2°$. We can now apply the law of sines to determine the length of the side opposite the obtuse angle.

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} \qquad \text{Law of sines}$$

$$\frac{\sin(2°)}{200} = \frac{\sin(156°)}{x} \qquad \text{Substitute}$$

$$x = 200 \cdot \frac{\sin(156°)}{\sin(2°)} \qquad \text{Solve for } x$$

$$x \approx 2330.90$$

We can now use this as the hypotenuse of the right triangle to calculate the h:

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \text{Definition of sine}$$

$$\sin(22°) = \frac{h}{x} \qquad \text{Substitute}$$

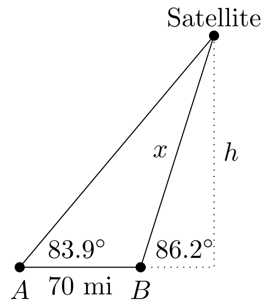
$$h = x\sin(22°) \qquad \text{Solve for } h$$

$$h \approx (2330.90) \cdot (0.3746) \approx 873.16 \qquad \text{Substitute}$$

The hill is approximately 873 feet tall.

- The path of a satellite orbiting the earth causes it to pass directly over two tracking stations A and B, which are 70 miles apart. When the satellite is on the same side of both stations, the angles of elevation at A and B are measured to be 86.2° and 83.9°, respectively. How far is the satellite from station A and how high is the satellite above the ground? (For this problem, we are ignoring the earth's curvature.)

We will start with a diagram. Note that x is the distance from station B to the satellite.



Using the small right triangle, we have the relationship $\sin(86.2°) = \frac{h}{x}$, so that $h = x\sin(86.2°)$.

We can also see that the supplemental angle to the 86.2°angle is 93.8°, so that the angle at the top of the triangle on the left is $180° - 83.9° - 93.8° = 2.3°$. We can now use the law of sines to determine x.

$$\begin{aligned}
\frac{\sin(\alpha)}{a} &= \frac{\sin(\beta)}{b} && \text{Law of sines} \\
\frac{\sin(2.3°)}{70} &= \frac{\sin(83.9°)}{x} && \text{Substitute} \\
x &= 70 \cdot \frac{\sin(83.9°)}{\sin(2.3°)} && \text{Solve for } x \\
x &\approx 1734.38
\end{aligned}$$

This value can then be plugged into the formula for h:

$$h = x\sin(86.2°) \approx (1734.38) \cdot (0.9978) \approx 1730.56$$

The satellite is about 1731 miles above the earth.

Comments and Observations:

- Read the text of the problem carefully! For the second problem, it states that the satellite is on the same side of both stations, but many students draw the picture with the satellite between the stations. Some students also calculate the distance from one of the tracking stations to the satellite rather than calculating the height of the satellite above the ground. There are no tricks to avoid this other than taking your time and reading the words carefully.
- We could have used the same approach for both problems, but picked different solutions to demonstrate that there are multiple ways to approach these problems. In fact, it's possible to solve both of these problems without using anything from this section! All you need is the definition of the tangent function.