Group Practice Problems #1 - Chart of Values: Complete the following chart of values, except only complete the portions of the chart that correspond to the appropriate inverse trigonometric function. (For example, the sine function should only have values on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.)

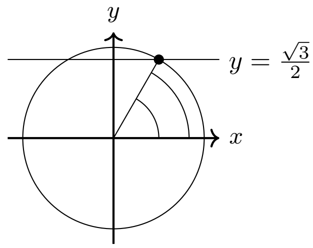
θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin(\theta)$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	—	—	—	—
$\cos(\theta)$	—	—	—	—	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$\tan(\theta)$	U	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	1	—	—	—	—

Comments and Observations:

- This can be thought of as an extended memorization exercise, but it's actually not too hard to see the patterns that make this less complicated. Everything is symmetric around its own zero values, with negative signs being introduced for the reflected portion.
- This exercise is less about having a chart to look up values as it is getting more comfortable with angles beyond the first quadrant. All of these values can be represented using a unit circle diagram, and that is a better conceptual approach to the problem than simply memorizing values.

Group Practice Problems #2 - Evaluating Inverse Trigonometric Functions: Determine the values of the following expressions. For each problem, draw a unit circle diagram and use it to identify the quadrant of the angle. Then state the corresponding reference angle.

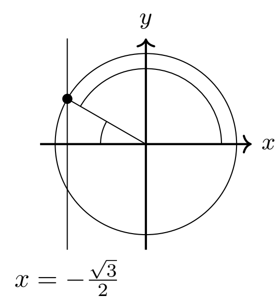
- $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$



The angle is in the first quadrant and the reference angle is $\frac{\pi}{3}$. Therefore, we have

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}.$$

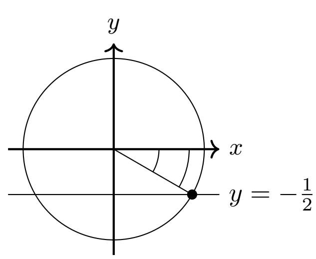
- $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$



The angle is in the second quadrant and the reference angle is $\frac{\pi}{6}$. Therefore, we have

$$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}.$$

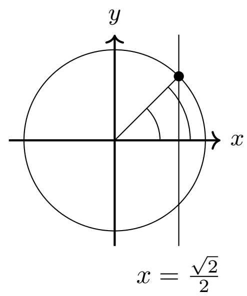
- $\sin^{-1}\left(-\frac{1}{2}\right)$



The angle is in the fourth quadrant and the reference angle is $\frac{\pi}{6}$. Therefore, we have

$$\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}.$$

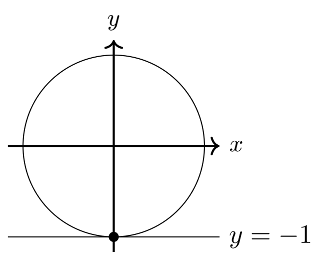
- $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$



The angle is in the first quadrant and the reference angle is $\frac{\pi}{4}$. Therefore, we have

$$\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}.$$

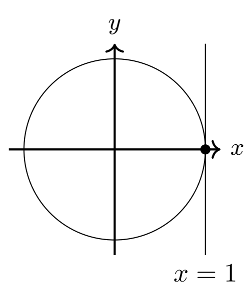
- $\sin^{-1}(-1)$



The intersection is along the negative y-axis and does not lie in any quadrant. This allows us to determine the angle from the geometry.

$$\sin^{-1}(-1) = -\frac{\pi}{2}.$$

- $\cos^{-1}(1)$

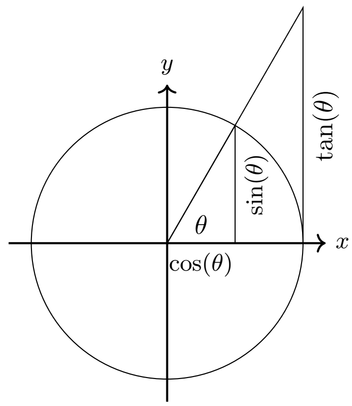


The intersection is along the positive x-axis and does not lie in any quadrant. This allows us to determine the angle from the geometry.

$$\cos^{-1}(1) = 0.$$

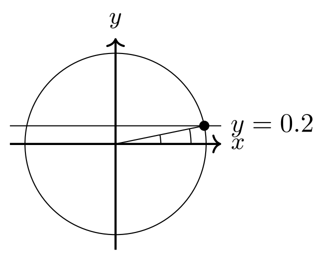
Comments and Observations:

- We can also get these values by reading them off of the chart from the first problem. However, it is going to be helpful to be able to see the geometry of the unit circle when we solve trigonometric equations. Focusing on just the inverse trigonometric functions will lead to missing half of the solutions of those equations.
- There is a way to put the tangent function into a unit circle diagram using similar triangles and the relationship $\frac{\sin(\theta)}{\cos(\theta)} = \frac{\tan(\theta)}{1}$. This diagram isn't essential to anything that follows. It's just an interesting geometric side note for visualizing the tangent function.



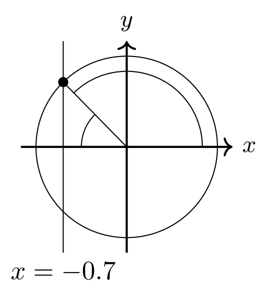
Group Practice Problems #3 - Approximating Inverse Trigonometric Functions: Use a calculator to determine the values of the following expressions. For each problem, draw a unit circle diagram and use it to identify the quadrant of the angle. Then calculate the corresponding reference angle. All of your calculations should be done in degrees.

- $\sin^{-1}(0.2)$



Using a calculator, we get that $\sin^{-1}(0.2) = 11.57°$. Since this is in the first quadrant, it is equal to the reference angle, and so the reference angle is also $11.57°$.

- $\cos^{-1}(-0.7)$



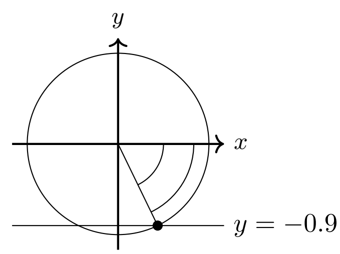
Using a calculator, we get that $\cos^{-1}(-0.7) = 134.43°$. Since this is in the second quadrant, the reference angle can be calculated by $180° - 134.43° = 45.57°$.

Comments and Observations:

- The part that is important in this problem is understanding how to use the geometry of unit circle to work back to the reference angle. The reference angle will be important for solving trigonometric equations.

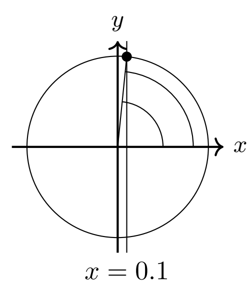
Group Practice Problems #4 - Approximating Inverse Trigonometric Functions: Use a calculator to determine the values of the following expressions. For each problem, draw a unit circle diagram and use it to identify the quadrant of the angle. Then calculate the corresponding reference angle. All of your calculations should be done in radians.

- $\sin^{-1}(-0.9)$



Using a calculator, we get that $\sin^{-1}(-0.9) = -1.12$ (radians). Since this is in the fourth quadrant, the reference angle is the absolute value of this, so that the reference angle is 1.12 (radians).

- $\cos^{-1}(0.1)$



Using a calculator, we get that $\cos^{-1}(0.1) = 1.47$ (radians). Since this is in the first quadrant, the reference angle is the same as this, so that the reference angle is 1.47 (radians).

Comments and Observations:

- It takes some time to adjust to thinking in radians when they are written as decimals and not as multiples of π. It is helpful to know the following approximations to help think about the quadrant of the angles.

$$90^\circ = \frac{\pi}{2} \approx 1.57, \qquad 180^\circ = \pi \approx 3.14, \qquad 270^\circ = \frac{3\pi}{2} \approx 4.71, \qquad 360^\circ = 2\pi \approx 6.28$$

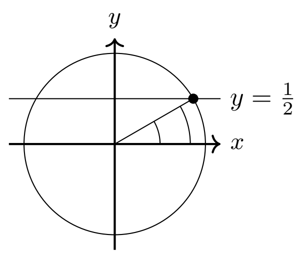
Group Practice Problems #5 - Compositions of inverse trigonometric functions with trigonometric functions: Determine the exact values of the following expressions.

- $\sin^{-1}\left(\cos\left(\frac{\pi}{3}\right)\right)$

Notice that $\cos(\frac{\pi}{3})$ is a known value from the basic chart, and we can substitute this value into the expression.

$$\sin^{-1}\left(\cos\left(\frac{\pi}{3}\right)\right) = \sin^{-1}\left(\frac{1}{2}\right)$$

We can calculate the inverse sine by looking at the unit circle diagram to identify the quadrant and the reference angle.



The angle is in the first quadrant and the reference angle is $\frac{\pi}{6}$. Therefore, we have

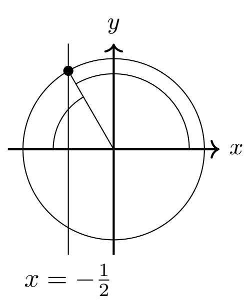
$$\sin^{-1}\left(\cos\left(\frac{\pi}{3}\right)\right) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}.$$

- $\cos^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right)$

We can calculate $\sin(-\frac{\pi}{6})$ by noting that the angle is in the fourth quadrant (making the sine function negative) with reference angle $\frac{\pi}{6}$.

$$\cos^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right) = \cos^{-1}\left(-\sin\left(\frac{\pi}{6}\right)\right) = \cos^{-1}\left(-\frac{1}{2}\right)$$

We can calculate the cosine by looking at the unit circle diagram to identify the quadrant and the reference angle.



The angle is in the second quadrant and the reference angle is $\frac{\pi}{3}$. Therefore, we have

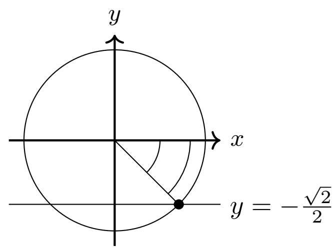
$$\cos^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right) = \cos^{-1}\left(-\frac{1}{2}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

- $\sin^{-1}\left(\sin\left(\frac{7\pi}{4}\right)\right)$

We can calculate $\sin(-\frac{\pi}{6})$ by noting that the angle is in the fourth quadrant (making the sine function negative) with reference angle $\frac{\pi}{4}$.

$$\sin^{-1}\left(\sin\left(\frac{7\pi}{4}\right)\right) = \sin^{-1}\left(-\sin\left(\frac{\pi}{4}\right)\right) = \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$$

We can calculate the inverse sine by looking at the unit circle diagram to identify the quadrant and the reference angle.



The angle is in the fourth quadrant and the reference angle is $\frac{\pi}{4}$. Therefore, we have

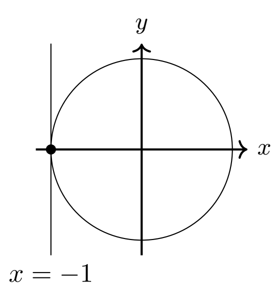
$$\sin^{-1}\left(\sin\left(\frac{7\pi}{4}\right)\right) = \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$$

- $\cos^{-1}\left(\cos\left(-9\pi\right)\right)$

To calculate $\cos(-9\pi)$, we first note that -9π is coterminal with π. This gives us the following:

$$\cos^{-1}\left(\cos\left(-9\pi\right)\right) = \cos^{-1}\left(\cos\left(\pi\right)\right) = \cos^{-1}\left(-1\right)$$

To evaluate the inverse cosine, we examine the unit circle diagram.



We can see that the intersection is on the negative x-axis, so that

$$\cos^{-1}\left(\cos\left(-9\pi\right)\right) = \cos^{-1}\left(-1\right) = \pi.$$

Comments and Observations:

- The last two examples emphasize the fact that the inverse trigonometric function does not always "cancel out" the trig function. In other words, it's not always the case that $\sin^{-1}(\sin(\theta)) = \theta$ or that $\cos^{-1}(\cos(\theta)) = \theta$. You need to work through the evaluation of the inner function and then think about the range of the inverse trigonometric function to get the correct result.

Group Practice Problems #6 - Compositions of trigonometric functions with inverse trigonometric functions using Algebraic Methods: Use an algebraic method to determine the exact values of the following expressions.

- $\sin\left(\cos^{-1}\left(\frac{2}{9}\right)\right)$

We will let $\theta = \cos^{-1}\left(\frac{2}{9}\right)$ so that $\cos(\theta) = \frac{2}{9}$. Then we will use the Pythagorean identity to determine $\sin(\theta)$.

$$\begin{aligned}
\sin^2(\theta) + \cos^2(\theta) &= 1 && \text{Pythagorean identity} \\
\sin^2(\theta) + \left(\frac{2}{9}\right)^2 &= 1 && \text{Substitute} \\
\sin^2(\theta) + \frac{4}{81} &= 1 \\
\sin^2(\theta) &= \frac{77}{81} && \text{Subtract } \frac{4}{81} \text{ from both sides} \\
\sin(\theta) &= \pm\frac{\sqrt{77}}{9} && \text{Take the square root of both sides}
\end{aligned}$$

Since θ is defined by an inverse cosine function, we know that θ is in either the first or second quadrant. Since we are taking the inverse cosine of a positive value, we know that the angle is in the first quadrant. Therefore, the sine function is also positive and we have

$$\sin\left(\cos^{-1}\left(\frac{2}{9}\right)\right) = \sin(\theta) = \frac{\sqrt{77}}{9}$$

- $\cot\left(\sin^{-1}\left(\frac{5}{11}\right)\right)$

We will let $\theta = \sin^{-1}\left(\frac{5}{11}\right)$ so that $\sin(\theta) = \frac{5}{11}$. Then we will use the Pythagorean identity to determine $\cos(\theta)$.

$$\begin{aligned}
\sin^2(\theta) + \cos^2(\theta) &= 1 && \text{Pythagorean identity} \\
\left(\frac{5}{11}\right)^2 + \cos^2(\theta) &= 1 && \text{Substitute} \\
\frac{25}{121} + \cos^2(\theta) &= 1 \\
\cos^2(\theta) &= \frac{96}{121} && \text{Subtract } \frac{25}{121} \text{ from both sides} \\
\cos(\theta) &= \pm\frac{4\sqrt{6}}{11} && \text{Take the square root of both sides}
\end{aligned}$$

Since θ is defined by an inverse sine function, we know that θ is in either the first or fourth quadrant. Since we are taking the inverse sine of a positive value, we know that the angle is in the first quadrant. Therefore, the cosine function is also positive. We can then get the result using identities.

$$\cot\left(\sin^{-1}\left(\frac{5}{11}\right)\right) = \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)} = \frac{4\sqrt{6}/11}{5/11} = \frac{4\sqrt{6}}{5}$$

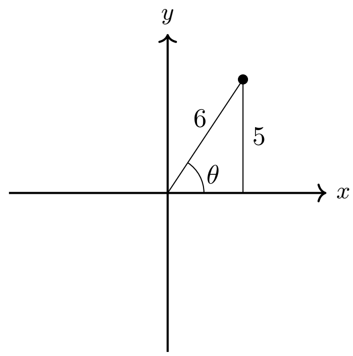
Comments and Observations:

- In both problems, we didn't need to identify specifically that the angle was in the first quadrant. In the first problem, knowing that θ is in either the first or second quadrant is enough to know that the sine function will be positive. And in the second problem, knowing that θ is in either the first or fourth quadrant would be enough to know that the cosine function will be positive.

Group Practice Problems #7 - Compositions of trigonometric functions with inverse trigonometric functions using Geometric Methods: Use an algebraic method to determine the exact values of the following expressions.

- $\sec\left(\sin^{-1}\left(\frac{5}{6}\right)\right)$

We will let $\theta = \sin^{-1}(\frac{5}{6})$ so that $\sin(\theta) = \frac{5}{6}$. Using $\sin(\theta) = \frac{y}{r}$, we can draw a triangle in the plane that matches this situation.



We can use the Pythagorean Theorem to determine the x-coordinate.

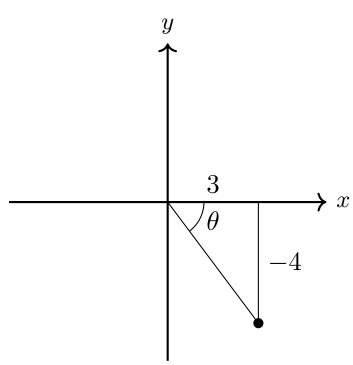
$a^2 + b^2 = c^2$	The Pythagorean Theorem
$a^2 + 5^2 = 6^2$	Substitute
$a^2 + 25 = 36$	
$a^2 = 11$	Subtract 25 from both sides
$a = \pm\sqrt{11}$	Take the square root of both sides

To be consistent with our diagram, we will choose the positive case, so that the x-coordinate is $\sqrt{11}$. Then we have that

$$\sec\left(\sin^{-1}\left(\frac{5}{6}\right)\right) = \sec(\theta) = \frac{r}{x} = \frac{6}{\sqrt{11}}.$$

- $\cos\left(\tan^{-1}\left(-\frac{4}{3}\right)\right)$

We will let $\theta = \tan^{-1}(-\frac{4}{3})$ so that $\tan(\theta) = -\frac{4}{3}$. Using $\tan(\theta) = \frac{y}{x}$, we can draw a triangle in the plane that matches this situation. Notice that since the inverse tangent function gives an angle in either the first or fourth quadrant, we should draw this angle in the fourth quadrant to match the sign.



We can use the distance formula to determine the length of the hypotenuse.

$$\begin{aligned}
r &= \sqrt{x^2+y^2} && \text{The distance formula} \\
&= \sqrt{3^2+(-4)^2} && \text{Substitute} \\
&= \sqrt{9+16} \\
&= \sqrt{25} \\
&= 5
\end{aligned}$$

Therefore, we have

$$\cos\left(\tan^{-1}\left(-\frac{4}{3}\right)\right) = \cos(\theta) = \frac{3}{5}$$

Comments and Observations:

- You can also do these calculations using the opposite, adjacent, and hypotenuse relationships for the trigonometric functions, and then just making sure you have the correct sign at the end of the problem.