Group Practice Problems #1 - Converting Between Degrees and Radians: Perform the following conversions. Show your conversion factor and the cancellation of units.

- Convert $\frac{7\pi}{10}$ radians to degrees.

The conversion factor for radians to degrees is $\frac{180 \text{ degrees}}{\pi \text{ radians}}$.

$$\begin{aligned}
\frac{7\pi}{10} \text{ radians} &= \frac{7\pi}{10} \text{ radians} \cdot \frac{180 \text{ degrees}}{\pi \text{ radians}} \\
&= \frac{7\cancel{\pi}}{\cancel{10}} \cancel{\text{radians}} \cdot \frac{18 \cdot \cancel{10} \text{ degrees}}{\cancel{\pi} \cancel{\text{radians}}} \\
&= 126 \text{ degrees} \\
&= 126^\circ
\end{aligned}$$

- Convert 15° to radians.

The conversion factor for radians to degrees is $\frac{\pi \text{ radians}}{180 \text{ degrees}}$.

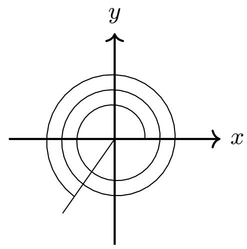
$$\begin{aligned}
15^\circ &= 15 \text{ degrees} \\
&= 15 \text{ degrees} \cdot \frac{\pi \text{ radians}}{180 \text{ degrees}} \\
&= \cancel{15} \cancel{\text{degrees}} \cdot \frac{\pi \text{ radians}}{\cancel{15} \cdot 12 \cancel{\text{degrees}}} \\
&= \frac{\pi}{12} \text{ radians}
\end{aligned}$$

Comments and Observations:

- There are equivalent forms of the conversion factors by thinking about the measure of the whole circle rather than half of a circle: $\frac{360 \text{ degrees}}{2\pi \text{ radians}}$ and $\frac{2\pi \text{ radians}}{360 \text{ degrees}}$. It is acceptable to use these if you prefer.
- Writing degrees as a word instead of with the symbol helps to make the degree cancellation easier to see. This step is not necessary, and in practice we typically don't bother showing the cancellation of the units because it always works out the same way. But it's good for practice.
- Radians is technically a dimensionless quantity, meaning that it actually adds nothing to write the units of radians on angles. The reason for this is that radians are the ratio of arc length (linear units) to the radius of the arc (linear units), leading to those units cancelling out. However, this is helpful to put the value into the context of being understood as an angle.

Group Practice Problems #2 - Coterminal Angles: Draw the angle in standard position, indicating the number of times the angle has wrapped around the origin. Then determine an angle coterminal to given angle in the interval $[0, 2\pi)$ or $[0°, 360°)$, depending on the units of the original angle.

- $955°$

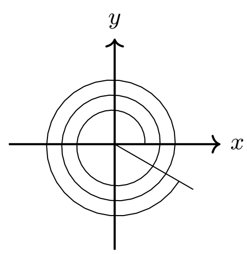


Since the angle wraps around the origin two full times, we need to subtract off two full rotations to get an angle in the desired interval.

$$955° - 2 \cdot 360° = 955° - 720° = 235°$$

An angle of $955°$ is coterminal with an angle of $235°$.

- $\frac{35\pi}{6}$

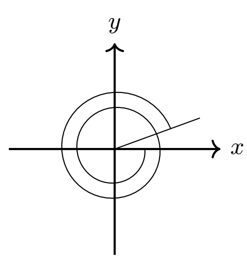


Since the angle wraps around the origin two full times, we need to subtract off two full rotations to get an angle in the desired interval.

$$\frac{35\pi}{6} - 2 \cdot 2\pi = \frac{35\pi}{6} - 4\pi = \frac{35\pi}{6} - \frac{24\pi}{6} = \frac{11\pi}{6}$$

An angle of $\frac{35\pi}{6}$ is coterminal with an angle of $\frac{11\pi}{6}$.

- $-700°$



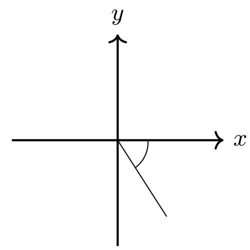
Since the angle wraps around the origin once in the negative direction, we need to add two full rotations to get an angle in the desired interval. (The first one unwinds the angle, and the second one moves the angle from a negative measure to a positive measure.)

$$-720° + 2 \cdot 360° = -700° + 720° = 20°$$

An angle of $-700°$ is coterminal with an angle of $20°$.

- -1 (radians)

In order to sketch this angle, we need to use some reference values. An right angle has a measure of $\frac{\pi}{2}$, which is approximately 1.57. This means that an angle of -1 radians will be less than a right angle in the negative direction.

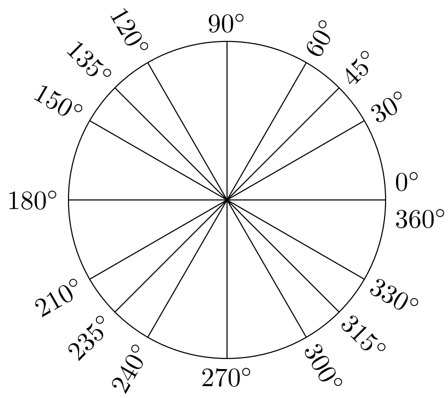


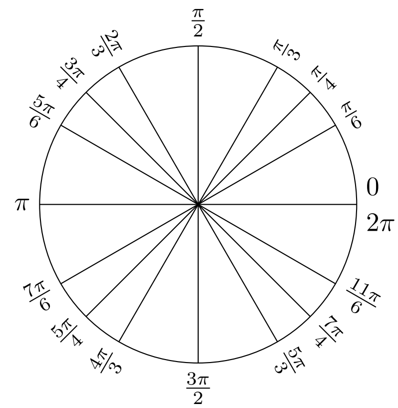
To get an angle in the desired interval, we need to add one full positive rotation. This gives an angle of $-1 + 2\pi$, which cannot be simplified any further and is approximately equal to 5.28.

Comments and Observations:

- The last problem is a reminder that radian measures do not always have to have a π in them. The angle measures can be any real number at all, and the only reason we see a lot of π values in the radian measure is because we tend to fractions of a complete circle in problems. When we get to working with inverse trigonometric functions and solving trigonometric equations, we'll see many more decimal representations of angles measured in radians.

Group Practice Problems #3 - Unit Circle: Sketch two unit circles and label all of the common angles that are included in the standard diagram. Use degrees in one diagram and radians in the other.



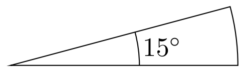


Comments and Observations:

- Ideally, these diagrams would be constructed from the 30° and 45° angles and the $\frac{\pi}{6}$ and $\frac{\pi}{4}$ angles. It is a good opportunity to practice thinking about fractions when creating the radian diagram.
- Over time, these values (both degrees and radians, and also the connection between the two) should basically just be memorized.

Group Practice Problems #4 - Circular Arcs: Draw a diagram that matches the description and answer the question.

- Determine the arc length of a 15° arc of a 3-inch radius circle.



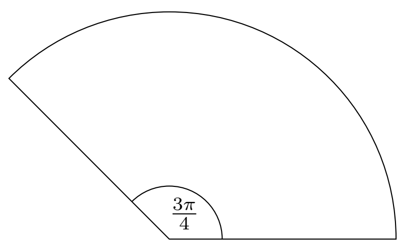
In order to use the arc length formula, we need to convert the angle measure from degrees to radians.

$$15^\circ = 15^\circ \cdot \frac{\pi \text{ radians}}{180^\circ} = \frac{\pi}{12}$$

Then we plug the values into the arc length formula.

$$s = r\theta = (3 \text{ inches}) \cdot \frac{\pi}{12} = \frac{\pi}{4} \text{ inches} \approx 0.79 \text{ inches}$$

- Determine the arc length of a $\frac{3\pi}{4}$ radian arc of a 2-meter diameter circle.



Since the angle is already given in radians, we just need to plug the values into the arc length formula.

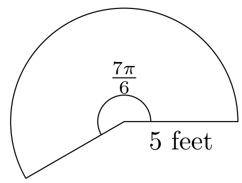
$$s = r\theta = (2 \text{ meters}) \cdot \frac{3\pi}{4} = \frac{3\pi}{2} \text{ meters} \approx 4.71 \text{ meters}$$

Comments and Observations:

- The primary source of error for these problems is the failure to convert from degrees to radians. Alternatively, there is a formula for arc length when the angle is given in degrees, but it is a bit clumsy to have to remember two different formulas.
- When the problem has units, it's important to use those units in your final answer. They were included in the calculations as well, and it's a good practice to track those units through the calculation, but if you're doing everything else right, the units should work themselves out naturally.

Group Practice Problems #5 - Areas of Sectors: Draw a diagram that matches the description and answer the question.

- Determine the area of a $\frac{7\pi}{6}$ radian sector of a circle with a 5-foot radius.

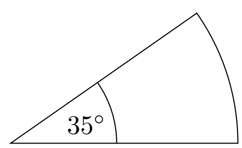


We are given the radius and the measure of the central angle, so to calculate the area, we simply plug these values into the formula.

$$A = \frac{\theta}{2}r^2 = \frac{7\pi}{12} \cdot (5\text{ ft})^2 = \frac{175\pi}{12}\text{ ft}^2 \approx 45.81\text{ ft}^2$$

- An 18-inch (diameter) pizza is cut into wedges. The central angle of one of the wedges is measured to be 35°. Determine the area of the pizza. (The central angle of the slice of pizza is measured at the "tip" of the pizza. Assume that the slices have straight sides and that the tip is the center of the pizza.)

If the diameter is 18 inches, then the radius is 9 inches.



Before we can plug this into the formula, we need to convert 35° degrees to radians.

$$35^\circ = 35^\circ \cdot \frac{\pi\text{ radians}}{180^\circ} = \frac{7\pi}{36}\text{ radians}$$

And now we can plug these values into the formula for the area of a sector.

$$A = \frac{\theta}{2}r^2 = \frac{7\pi}{72} \cdot (9\text{ in})^2 = \frac{63\pi}{8}\text{ in}^2 \approx 24.74\text{ in}^2$$

Comments and Observations:

- These problems are similar to the arc length problems. It's mostly just about making sure the angle is in radians and plugging in values into the formula correctly.

Group Practice Problems #6 - Circular motion: Solve the problem.

- A bicycle with a 24-inch diameter wheel is traveling at 15 miles per hour. Determine the angular speed of the wheels in radians per minute. How many revolutions per minute do the wheels make?

We first need to convert the linear speed of miles per hour to inches per minute to get all of the units to be the same as the desired result.

$$\begin{aligned} 15\,\frac{\text{miles}}{\text{hour}} &= 15\,\frac{\text{miles}}{\text{hour}} \cdot \frac{1 \text{ hour}}{60 \text{ minutes}} \cdot \frac{5280 \text{ feet}}{1 \text{ mile}} \\ &= 1320\,\frac{\text{feet}}{\text{minute}} \end{aligned}$$

To determine the angular speed, we use the relationship $v = r\omega$, where v is the linear speed, r is the radius, and ω is the angular speed. Notice that a 24-inch diameter wheel has a 1-foot radius.

$$\begin{aligned} v &= r\omega \\ 1320\,\frac{\text{feet}}{\text{minute}} &= 1 \text{ foot} \cdot \omega \\ \omega &= 1320\,\frac{1}{\text{minute}} \end{aligned}$$

This shows us that the wheel rotates at a speed of 1320 radians per minute.
To convert this quantity to revolutions per minute, we note that one revolution is equivalent to 2π radians and use that as a conversion factor.

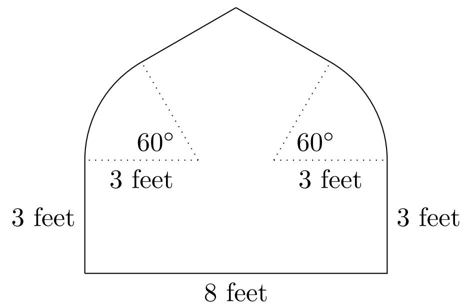
$$\begin{aligned} 1320\,\frac{\text{radians}}{\text{minute}} &= 1320\,\frac{\text{radians}}{\text{minute}} \cdot \frac{1 \text{ revolution}}{2\pi \text{ radians}} \\ &= \frac{660}{\pi}\,\frac{\text{revolutions}}{\text{minute}} \\ &\approx 210.08\,\frac{\text{revolutions}}{\text{minute}} \end{aligned}$$

The wheel is rotating at a rate of approximately 210.08 revolutions per minute.

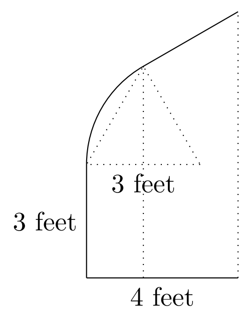
Comments and Observations:

- Many students are confused by the introduction of radians into the units. This is allowed because radians are a pure unit because it is a fraction of two linear units. This means that we can insert and remove radians in the units at any time. We will only want to do this when the context of the problem suggests that making that change is helpful for understanding the meaning of the value.

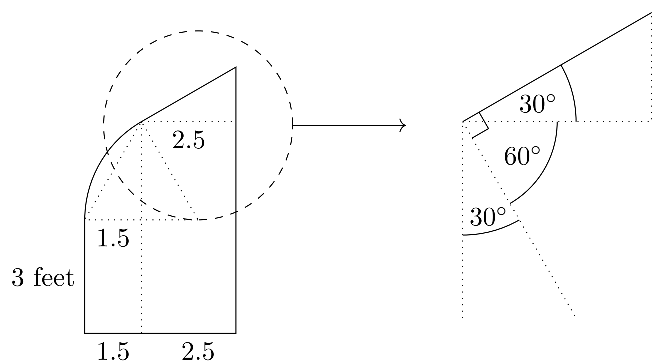
Group Practice Problems #7 - Challenge Problem: Determine the perimeter and area of the following figure. Give your answer as a decimal approximation to two decimal places. (Hint: Connecting the points on the 60° angle creates an equilateral triangle that can be used to help determine the length of the line segments that create the peak.)



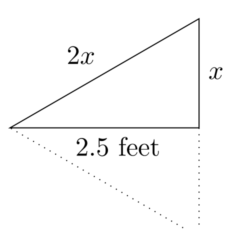
We will start by analyzing the peak in the middle. By symmetry, we will work with just the portion on the left. We will start by drawing the equilateral triangle formed by the 60°angle. From the top of the triangle angle, we will draw both a vertical line.



Notice that since this cuts the equilateral triangle in half, we can use this information to calculate some additional lengths. The dashed circle indicates a zoomed-in view of the indicated portion of the diagram. (The units have been suppressed in some places for readability.)



Notice that the uppermost triangle is half of an equilateral triangle. This means that the horizontal portion of the triangle is half the length of the hypotenuse.



We can now apply the Pythagorean Theorem to this triangle to determine x.

$$
\begin{aligned}
a^2+b^2 &= c^2 && \text{The Pythagorean Theorem}\\
(2.5)^2+x^2 &= (2x)^2 && \text{Substitute}\\
6.25+x^2 &= 4x^2 && \\
3x^2 &= 6.25 && \text{Subtract } x^2 \text{ from both sides and rearrange}\\
x^2 &= \frac{6.25}{3} && \text{Divide both sides by 3}\\
x &= \sqrt{\frac{6.25}{3}} && \text{Take the square root of both sides (rejecting the negative value)}\\
x &\approx 1.443 &&
\end{aligned}
$$

Therefore, the length of the diagonal segment at the peak of the diagram is $2x \approx 2.886$ feet in length.

To calculate the length of the circular arc, we use the arc length formula. Notice that $60^\circ = \frac{\pi}{3}$ radians.

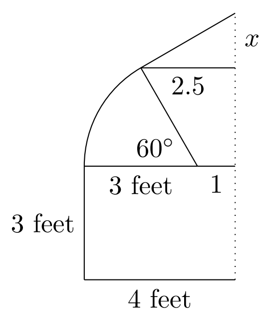
$$s = r\theta = 3 \cdot \frac{\pi}{3} = \pi \approx 3.142$$

The remainder of the half-diagram consists of line segments that are length 3 feet and 4 feet. This shows us that the total length of the perimeter for half of the diagram (in feet) is approximately

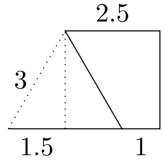
$$2.886 + 3.142 + 3 + 4 = 13.028.$$

Therefore, the total perimeter is approximately 26.056 feet.

To determine the area of the half-diagram, we will break it into the four regions shown.



We will focus our attention on the trapezoidal region first. The height of this region is same as the height of the equilateral triangle formed by the arc.



To determine the height, we use the Pythagorean Theorem on the right triangle at the left side of the diagram.

$$
\begin{aligned}
a^2+b^2 &= c^2 && \text{The Pythagorean Theorem}\\
(1.5)^2+b^2 &= 3^2 && \text{Substitute}\\
2.25+b^2 &= 9 && \\
b^2 &= 6.75 && \text{Subtract 2.25 from both sides}\\
b &= \sqrt{6.75} && \text{Take the square root of both sides (rejecting the negative value)}\\
b &\approx 2.60 &&
\end{aligned}
$$

Therefore, the area of the trapezoidal portion is $\frac{b_1+b_2}{2} \cdot h \approx \frac{1+2.5}{2} \cdot (2.60) = 4.55$ square feet.

The area of the sector can be determined using the appropriate area formula. Note that $60^\circ = \frac{\pi}{3}$ radians. (The units of the calculation is square feet.)

$$A_{\text{sector}} = \frac{1}{2}\theta r^2 = \frac{1}{2} \cdot \frac{\pi}{3} \cdot 3^2 = \frac{3\pi}{2} \approx 4.71$$

The area of the upper triangle can be determined from the value of x determined previously.

$$A_{\text{triangle}} = \frac{1}{2}bh \approx \frac{1}{2} \cdot (2.5) \cdot (1.443) = 1.80$$

Lastly, the area of the rectangle is 12 square feet. This shows us that the total area of the half diagram is $A = A_{\text{trapezoid}} + A_{\text{sector}} + A_{\text{triangle}} + A_{\text{rectangle}} \approx 4.55 + 4.71 + 1.80 + 12 = 23.06$ square feet. Therefore, the total area of the diagram is approximately 46.12 square feet.

Comments and Observations:

- This challenge problem secretly draws in an idea that will be important in the development of the unit circle, namely the relationship between the sides of a 30-60-90 right triangle. The "true" relationship is hidden because we've squared out the numerical quantities, which masks the deeper connection. The two most common representations of the 30-60-90 triangle are with a general length for the hypotenuse and a hypotenuse of length 1.