Group Practice Problems #1 - Basic Applications of the Identities: Simplify the following expressions. Describe your steps and state the identities that you have used.

- $\frac{\tan(\theta)}{\cos(-\theta)}\left(1-\sin^2(\theta)\right)$

$$\begin{aligned}
\frac{\tan(\theta)}{\cos(-\theta)}\left(1-\sin^2(\theta)\right) &= \frac{\tan(\theta)}{\cos(\theta)}\left(1-\sin^2(\theta)\right) && \text{Negative angle identity for cosine}\\
&= \frac{\tan(\theta)}{\cos(\theta)}\cdot\cos^2(\theta) && \text{Pythagorean Identity}\\
&= \tan(\theta)\cdot\cos(\theta)\\
&= \frac{\sin(\theta)}{\cos(\theta)}\cdot\cos(\theta) && \text{Definition of the tangent function}\\
&= \sin(\theta)
\end{aligned}$$

- $-\frac{\sin(\theta)}{\cos^2(\theta)}\tan(-\theta)\left(\csc(\theta)+1\right)\left(\frac{1}{\sin(\theta)}-1\right)$

$$\begin{aligned}
&-\frac{\sin(\theta)}{\cos^2(\theta)}\tan(-\theta)\left(\csc(\theta)+1\right)\left(\frac{1}{\sin(\theta)}-1\right)\\
&\quad= -\frac{\sin(\theta)}{\cos^2(\theta)}\cdot\left(-\tan(\theta)\right)\left(\csc(\theta)+1\right)\left(\frac{1}{\sin(\theta)}-1\right) && \text{Negative angle identity for tangent}\\
&\quad= \frac{\sin(\theta)}{\cos^2(\theta)}\cdot\tan(\theta)\left(\csc(\theta)+1\right)\left(\frac{1}{\sin(\theta)}-1\right)\\
&\quad= \frac{\sin(\theta)}{\cos^2(\theta)}\cdot\tan(\theta)\left(\csc(\theta)+1\right)\left(\csc(\theta)-1\right) && \text{Reciprocal identity for sine}\\
&\quad= \frac{\sin(\theta)}{\cos^2(\theta)}\cdot\tan(\theta)\left(\csc^2(\theta)-1\right)\\
&\quad= \frac{\sin(\theta)}{\cos^2(\theta)}\cdot\tan(\theta)\cdot\cot^2(\theta) && \text{Pythagorean Identity}\\
&\quad= \frac{\sin(\theta)}{\cos^2(\theta)}\cdot\frac{\sin(\theta)}{\cos(\theta)}\cdot\frac{\cos^2(\theta)}{\sin^2(\theta)} && \text{Definition of tangent and cotangent}
\end{aligned}$$

Comments and Observations:

- These are review exercises. All of these identities have been seen in the past and the point is to practice some algebra.

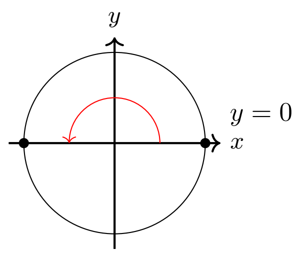
Group Practice Problems #2 - Solving Equations: Find all solutions to the equations, then list the solutions in the interval $[0, 2\pi)$. Describe your steps and state any identities that you use.

- $2\sin^2(t) + \sin(t) = 0$

$$\begin{aligned} 2\sin^2(t) + \sin(t) &= 0 && \text{Original equation} \\ \sin(t)(2\sin(t) + 1) &= 0 && \text{Factor out } \sin(t) \end{aligned}$$

By the zero product property, we must have that $\sin(t) = 0$ or $2\sin(t) + 1 = 0$.

To solve $\sin(t) = 0$, we consider the unit circle diagram.

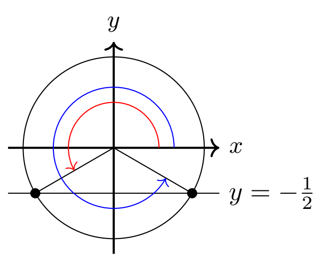


The intersections are on the x-axis, so that the solutions to this part of the equation are $t = 0$ and $t = \pi$.

To solve $2\sin(t) + 1 = 0$, we first isolate the trigonometric function.

$$\begin{aligned} 2\sin(t) + 1 &= 0 && \\ 2\sin(t) &= -1 && \text{Subtract 1 from both sides} \\ \sin(t) &= -\frac{1}{2} && \text{Divide both sides by 2} \end{aligned}$$

And now we consider the unit circle diagram.



We can see that the solutions are in the third and fourth quadrants, and we can identify that the reference angle is $\frac{\pi}{6}$. Therefore, the solutions are $t = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$ and $t = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$.

So the collection of all solutions is $t = 0, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}$.

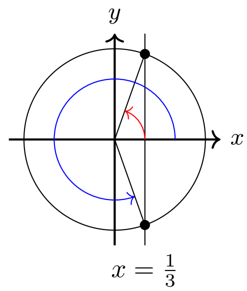
- $3\sec^2(t) - 5\sec(t) - 2 = 0$

We begin by factoring the quadratic equation using the ac method. We are looking for two numbers that multiply to $ac = 3 \cdot (-2) = -6$ and add to $b = -5$. We can see that the numbers -6 and 1 meet this requirement.

$$
\begin{aligned}
3\sec^2(t) - 5\sec(t) - 2 &= 0 \\
3\sec^2(t) + \sec(t) - 6\sec(t) - 2 &= 0 && \text{Use the } ac \text{ method to rewrite the linear term} \\
\sec(t)(3\sec(t)+1) - 2(3\sec(t)+1) &= 0 && \text{Factor by grouping} \\
(\sec(t)-2)(3\sec(t)+1) &= 0
\end{aligned}
$$

By the zero product property, we must have that $\sec(t) - 3 = 0$ or $3\sec(t) + 1 = 0$.

If $\sec(t) - 3 = 0$, then we must have $\sec(t) = 3$, or $\cos(t) = \frac{1}{3}$. We will draw the unit circle diagram.



Using a calculator, we get that $\cos^{-1}(\frac{1}{3}) \approx 1.23$. This is the first quadrant angle and the reference angle. The fourth quadrant angle is $2\pi - 1.23 \approx 5.05$. These are the two solutions from this part of the equation.

For $3\sec(t) + 1 = 0$, we find that $\sec(t) = -\frac{1}{3}$, which implies that $\cos(t) = -3$. This generates no additional solutions.

Comments and Observations:

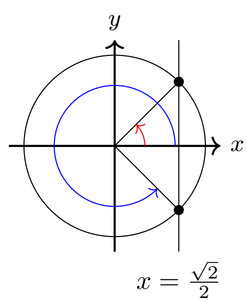
- It may not be necessary to draw the unit circles at this point, but they are included here for emphasis.

Group Practice Problems #3 - Solving Equations: Find all solutions to the equations, then list the solutions in the interval $[0, 2\pi)$. Describe your steps and state any identities that you use.

- $\sec(\theta) = 2\cos(\theta)$

$$\begin{aligned}
\sec(\theta) &= 2\cos(\theta) && \text{Original equation} \\
\frac{1}{\cos(\theta)} &= 2\cos(\theta) && \text{Definition of secant} \\
1 &= 2\cos^2(\theta) && \text{Multiply both sides by } \cos(\theta) \\
\frac{1}{2} &= \cos^2(\theta) && \text{Divide both sides by 2} \\
\pm\frac{1}{\sqrt{2}} &= \cos(\theta) && \text{Take the square root of both sides} \\
\pm\frac{\sqrt{2}}{2} &= \cos(\theta) && \text{Rationalize the denominator}
\end{aligned}$$

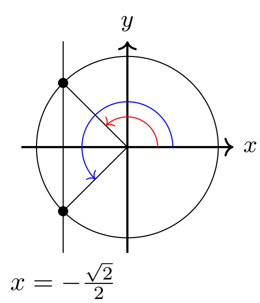
For $\cos(\theta) = \frac{\sqrt{2}}{2}$, we have the following unit circle diagram:



We can see that the reference angle is $\frac{\pi}{4}$, so that the solutions are

$$\theta = \begin{cases} \frac{\pi}{4} \\ 2\pi - \frac{\pi}{4} \end{cases} = \begin{cases} \frac{\pi}{4} \\ \frac{7\pi}{4} \end{cases}.$$

For the equation $\cos(\theta) = -\frac{\sqrt{2}}{2}$, we have this unit circle diagram:



The reference angle is still $\frac{\pi}{4}$, so that the solutions are

$$\theta = \begin{cases} \pi - \frac{\pi}{4} \\ \pi + \frac{\pi}{4} \end{cases} = \begin{cases} \frac{3\pi}{4} \\ \frac{5\pi}{4} \end{cases}.$$

So the collection of all solutions is $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

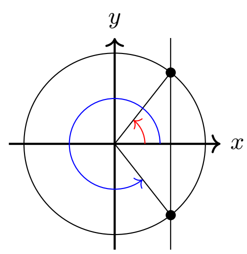
- $\sin^2(t) = \cos(t)$

$$\begin{aligned} \sin^2(t) &= \cos(t) && \text{Original equation} \\ 1-\cos^2(t) &= \cos(t) && \text{Pythagorean identity} \\ 0 &= \cos^2(t)+\cos(t)-1 && \text{Add } \cos^2(t) \text{ and subtract 1 from both sides} \end{aligned}$$

This does not factor, so we will use the quadratic formula.

$$\begin{aligned} \cos(t) &= \frac{-1 \pm \sqrt{1^2 - 4\cdot 1\cdot(-1)}}{2\cdot 1} && \text{Quadratic formula} \\ \cos(t) &= \frac{-1\pm\sqrt{5}}{2} \end{aligned}$$

We will first work with the "plus" case. Notice that $\frac{-1+\sqrt{5}}{2} \approx 0.6180$. This gives us the following unit circle diagram:



$x = \frac{-1+\sqrt{5}}{2}$

This shows that the solutions are in the first and fourth quadrants. Using a calculator, we get $\cos^{-1}(0.6180) \approx 0.90$, which is the reference angle. Therefore, the two solutions are $t \approx 0.90$ and $t \approx 2\pi - 0.9 \approx 5.38$.
For the "minus" case, notice that $\frac{-1-\sqrt{5}}{2} \approx -1.62$, so that there are no additional solutions arising from this part of the equation.

- $9\sin(w) - 2 = 4\sin^2(w)$

$$\begin{aligned} 9\sin(w)-2 &= 4\sin^2(w) && \text{Original equation} \\ 0 &= 4\sin^2(w) - 9\sin(w) + 2 && \text{Subtract } 9\sin(w) \text{ and add 2 to both sides} \end{aligned}$$

We will factor this using the *ac* method. We are looking for two numbers that multiply to $ac = 4\cdot 2 = 8$ and add to -9. We can see that -1 and -8 satisfy these properties.

$$\begin{aligned} 0 &= 4\sin^2(w) - 9\sin(w) + 2 \\ 0 &= 4\sin^2(w) - 8\sin(w) - \sin(w) + 2 && \text{Use the } ac \text{ method to rewrite the linear term} \\ 0 &= 4\sin(w)(\sin(w)-2) - (\sin(w)-2) && \text{Factor by grouping} \\ 0 &= (4\sin(w)-1)(\sin(w)-2) \end{aligned}$$

By the zero product property, we must have that $4\sin(w) - 1 = 0$ or $\sin(w) - 2 = 0$.

If $4\sin(w) - 1 = 0$, then we must have $\sin(w) = \frac{1}{4}$. This will have solutions in the first and second quadrants. Using a calculator, we find that $\sin^{-1}(\frac{1}{4}) = 0.25$ which is the reference angle. So the solutions from this part of the equation are

$$w = \begin{cases} 0.25 \\ \pi - 0.25 \end{cases} = \begin{cases} 0.25 \\ 2.89 \end{cases}.$$

The second equation becomes $\sin(w) = 2$, which has no solutions.

Comments and Observations:

- The presentations of the three problems are done with fewer steps shown on each. The unit circle diagram is important to keep in mind, but after enough practice it's not something that needs to be drawn out every time.