Group Practice Problems #1 - Values of Trigonometric Functions: Determine the exact values of the following trigonometric functions.

- $\sin(75°)$

$$\begin{aligned}
\sin(75°) &= \sin(45° + 30°) && \text{Rewrite the angle as the sum of two angles} \\
&= \sin(45°) \cdot \cos(30°) + \cos(45°) \cdot \sin(30°) && \text{Sum of angles formula} \\
&= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} && \text{Substitute} \\
&= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} \\
&= \frac{\sqrt{6} + \sqrt{2}}{4}
\end{aligned}$$

- $\cos(105°)$

$$\begin{aligned}
\cos(105°) &= \cos(45° + 60°) && \text{Rewrite the angle as the sum of two angles} \\
&= \cos(45°) \cdot \cos(60°) - \sin(45°) \cdot \sin(60°) && \text{Sum of angles formula} \\
&= \frac{\sqrt{2}}{2} \cdot \frac{1}{2} - \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} && \text{Evaluate the functions} \\
&= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} \\
&= \frac{\sqrt{2} - \sqrt{6}}{4}
\end{aligned}$$

- $\sin\left(\frac{11\pi}{12}\right)$

$$\begin{aligned}
\sin\left(\frac{11\pi}{12}\right) &= \sin\left(\frac{3\pi}{12} + \frac{8\pi}{12}\right) && \text{Rewrite the angle as the sum of two angles} \\
&= \sin\left(\frac{\pi}{4} + \frac{2\pi}{3}\right) && \text{Simplify} \\
&= \sin\left(\frac{\pi}{4}\right) \cdot \cos\left(\frac{2\pi}{3}\right) + \cos\left(\frac{\pi}{4}\right) \cdot \sin\left(\frac{2\pi}{3}\right) && \text{Sum of angles formula} \\
&= \frac{\sqrt{2}}{2} \cdot \left(-\frac{1}{2}\right) + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} && \text{Evaluate the functions} \\
&= -\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} \\
&= \frac{-\sqrt{2} + \sqrt{6}}{4}
\end{aligned}$$

Comments and Observations:

- There are many ways to break the angle into a sum of two angles. The order and values may vary, but the final answers should still match.

Group Practice Problems #2 - Identifying Expressions: Use either a sum or difference identity to rewrite the given expression in terms of a single sinusoidal function.

- $\sin(4t)\cos(2t) + \sin(2t)\cos(4t)$

This expression matches the pattern of the sum of angles formula for the sine function.

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

We can see that this matches the expression with $\alpha = 4t$ and $\beta = 2t$.

$$\sin(4t)\cos(2t) + \sin(2t)\cos(4t) = \sin(4t + 2t) = \sin(6t)$$

- $\cos(11x)\cos(3x) - \sin(11x)\sin(3x)$

This expression matches the pattern of the sum of angles formula for the sine function.

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

We can see that this matches the expression with $\alpha = 11x$ and $\beta = 3x$.

$$\cos(11x)\cos(3x) - \sin(11x)\sin(3x) = \cos(11x + 3x) = \cos(14x)$$

Comments and Observations:

- These problems are about increasing your familiarity with these formulas.

Group Practice Problems #3 - Tangent Identity: Prove the following sum of angles identity for tangent using the sum and difference identities for the sine and cosine functions.

- $\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$

$$\begin{aligned}
\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} && \text{Definition of tangent}\\
&= \frac{\sin(\alpha)\cos(\beta)+\sin(\beta)\cos(\alpha)}{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)} && \text{Sum of angles formulas}\\
&= \frac{\sin(\alpha)\cos(\beta)+\sin(\beta)\cos(\alpha)}{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)} \cdot \frac{\frac{1}{\cos(\alpha)\cos(\beta)}}{\frac{1}{\cos(\alpha)\cos(\beta)}} && \text{Multiply by 1}\\
&= \frac{\frac{\sin(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)}+\frac{\sin(\beta)\cos(\alpha)}{\cos(\alpha)\cos(\beta)}}{\frac{\cos(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)}-\frac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}} && \text{Distributive property}\\
&= \frac{\frac{\sin(\alpha)}{\cos(\alpha)}+\frac{\sin(\beta)}{\cos(\beta)}}{1-\frac{\sin(\alpha)}{\cos(\alpha)}\cdot\frac{\sin(\beta)}{\cos(\beta)}} && \text{Simplify}\\
&= \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)} && \text{Definition of tangent}
\end{aligned}$$

Comments and Observations:

- This proof can also be done in the opposite direction.

$$\begin{aligned}
\frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)} &= \frac{\frac{\sin(\alpha)}{\cos(\alpha)}+\frac{\sin(\beta)}{\cos(\beta)}}{1-\frac{\sin(\alpha)}{\cos(\alpha)}\cdot\frac{\sin(\beta)}{\cos(\beta)}} && \text{Definition of tangent}\\
&= \frac{\frac{\sin(\alpha)}{\cos(\alpha)}+\frac{\sin(\beta)}{\cos(\beta)}}{1-\frac{\sin(\alpha)}{\cos(\alpha)}\cdot\frac{\sin(\beta)}{\cos(\beta)}} \cdot \frac{\cos(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)} && \text{Clear the denominators}\\
&= \frac{\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)}\\
&= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} && \text{Sum of angles formula}\\
&= \tan(\alpha+\beta) && \text{Definition of tangent}
\end{aligned}$$

Group Practice Problems #4 - Solving Equations: Solve the equation. Describe your steps and state any identities that you use.

- $\sin(3x)\sin(2x) - \cos(3x)\cos(2x) = \frac{\sqrt{2}}{2}$

$$\begin{aligned}
\sin(3x)\sin(2x) - \cos(3x)\cos(2x) &= \frac{\sqrt{2}}{2} && \text{Original equation}\\
\cos(3x)\cos(2x) - \sin(3x)\sin(2x) &= -\frac{\sqrt{2}}{2} && \text{Multiply both sides by } -1\\
\cos(3x+2x) &= -\frac{\sqrt{2}}{2} && \text{Sum of angles formula}\\
\cos(5x) &= -\frac{\sqrt{2}}{2}
\end{aligned}$$

We can see that this will have solutions in the second and third quadrants with a reference angle of $\frac{\pi}{4}$. Therefore, for any integer k we have

$$\begin{aligned}
5x &= \begin{cases} \pi - \frac{\pi}{4} + 2\pi k \\ \pi + \frac{\pi}{4} + 2\pi k \end{cases}\\
5x &= \begin{cases} \frac{3\pi}{4} + 2\pi k \\ \frac{5\pi}{4} + 2\pi k \end{cases}\\
x &= \begin{cases} \frac{3\pi}{20} + \frac{2\pi k}{5} \\ \frac{\pi}{4} + \frac{2\pi k}{5} \end{cases} && \text{Divide both sides by 5}
\end{aligned}$$

Comments and Observations:

- Students often get stuck on the very first step of this problem. They can recognize that it's almost a sum or difference formula, but they either ignore the signs (and get the problem wrong) or they don't know what to do. Multiplying both sides of the equation by a constant or factoring out a constant on one side is a step that takes experience to learn to recognize.
- After recognizing the formula, this problem is no different from problems in previous sections.

Group Practice Problems #5 - Rewriting a Sum of Sinusoidals: Rewrite the expression in the form $A\sin(Bx+C)$.

- $-8\sin(3x)+8\sqrt{3}\cos(3x)$

We first note that if this of the form $m\sin(Bx)+n\cos(Bx)$, then we have $m=-8$, $n=8\sqrt{3}$, and $B=3$. Then we have that

$$A=\sqrt{m^2+n^2}=\sqrt{(-8)^2+(8\sqrt{3})^2}=\sqrt{64+192}=\sqrt{256}=16.$$

Next, we have the following equations for C:

$$\begin{aligned}\cos(C)&=\frac{-8}{16}=-\frac{1}{2}\\ \sin(C)&=\frac{8\sqrt{3}}{16}=\frac{\sqrt{3}}{2}\end{aligned}$$

Since the cosine is negative and the sine is positive, the angle C is in the second quadrant. We can also see that the reference angle is $\frac{\pi}{3}$, so that we must have $C=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$. Therefore,

$$-8\sin(3x)+8\sqrt{3}\cos(3x)=16\sin\left(3x+\frac{2\pi}{3}\right).$$

- $2\sin(4x)-2\cos(4x)$

We first note that if this of the form $m\sin(Bx)+n\cos(Bx)$, then we have $m=2$, $n=-2$, and $B=4$. Then we have that

$$A=\sqrt{m^2+n^2}=\sqrt{2^2+(-2)^2}=\sqrt{4+4}=\sqrt{8}=2\sqrt{2}.$$

Next, we have the following equations for C:

$$\begin{aligned}\cos(C)&=\frac{2}{2\sqrt{2}}=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}\\ \sin(C)&=\frac{-2}{2\sqrt{2}}=-\frac{1}{\sqrt{2}}=-\frac{\sqrt{2}}{2}\end{aligned}$$

Since the cosine is positive and the sine is negative, the angle C is in the fourth quadrant. We can also see that the reference angle is $\frac{\pi}{4}$, so that we must have $C=-\frac{\pi}{4}$. Therefore,

$$2\sin(4x)-2\cos(4x)=2\sqrt{2}\sin\left(4x-\frac{\pi}{4}\right).$$

Comments and Observations:

- These calculations are very much about following a basic pattern and just applying the formulas. But it's a good synthesis of the sum of angles formula and thinking about the signs of the trigonometric functions. You can always check your answer by applying the sum of angles formula to the result and comparing it with the original problem. For example, here is that calculation with the first problem.

$$\begin{aligned}16\sin\left(3x+\frac{2\pi}{3}\right)&=16\left(\sin(3x)\cos\left(\frac{2\pi}{3}\right)+\cos(3x)\sin\left(\frac{2\pi}{3}\right)\right)\\ &=16\left(-\frac{1}{2}\sin(3x)+\frac{\sqrt{3}}{2}\cos(3x)\right)\\ &=-8\sin(3x)+8\sqrt{3}\cos(3x)\end{aligned}$$

Group Practice Problems #6 - Proving Identities: Prove the following identities. Describe the steps and state any identities that you use.

- $\sin(\alpha)\cos(\beta) = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right)$

$$\begin{aligned}
&\frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right) \\
&\qquad = \frac{1}{2}\left((\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)) + (\sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta))\right) && \text{Sum of angles formula} \\
&\qquad = \frac{1}{2}\left(2\sin(\alpha)\cos(\beta)\right) \\
&\qquad = \sin(\alpha)\cos(\beta)
\end{aligned}$$

- $\cos(\alpha)\cos(\beta) = \frac{1}{2}\left(\cos(\alpha+\beta) + \cos(\alpha-\beta)\right)$

$$\begin{aligned}
&\frac{1}{2}\left(\cos(\alpha+\beta) + \cos(\alpha-\beta)\right) \\
&\qquad = \frac{1}{2}\left((\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)) + (\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta))\right) && \text{Sum of angles formula} \\
&\qquad = \frac{1}{2}\left(2\cos(\alpha)\cos(\beta)\right) \\
&\qquad = \cos(\alpha)\cos(\beta)
\end{aligned}$$

Comments and Observations:

- These identities show that the product formulas arise from the sum of angles formulas. And while it proves that the formulas are valid, it doesn't give a real good hint as to how they are initially derived. The key fact is that the sum and difference of angles formulas only differ by a changed sign, and you can leverage that changed sign to cancel out some of the terms by adding the equations together. For example:

$$\begin{array}{rcl}
\sin(\alpha+\beta) & = & \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \\
\sin(\alpha-\beta) & = & \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) \\
\hline
\sin(\alpha+\beta) + \sin(\alpha-\beta) & = & 2\sin(\alpha)\cos(\beta)
\end{array}$$

  From here, we can divide both sides by 2 to get the product formula.

Group Practice Problems #7 - Combinations of Sinusoidals: Evaluate the expression using a sum-to-product identity.

- $\cos(15^\circ) - \cos(75^\circ)$

We will use the following formula:

$$\cos(u) - \cos(v) = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

Notice that we have $u = 15^\circ$ and $v = 75^\circ$. Therefore,

$$\begin{aligned}
\cos(15^\circ) - \cos(75^\circ) &= -2\sin\left(\frac{15^\circ + 75^\circ}{2}\right)\sin\left(\frac{15^\circ - 75^\circ}{2}\right) && \text{Substitute} \\
&= -2\sin(45^\circ)\sin(-30^\circ) \\
&= -2 \cdot \frac{\sqrt{2}}{2} \cdot \left(-\frac{1}{2}\right) && \text{Evaluate the functions} \\
&= \frac{\sqrt{2}}{2}
\end{aligned}$$

- $\sin(15^\circ) - \sin(105^\circ)$

We will use the following formula:

$$\sin(u) - \sin(v) = 2\sin\left(\frac{u-v}{2}\right)\cos\left(\frac{u+v}{2}\right)$$

Notice that we have $u = 15^\circ$ and $v = 105^\circ$. Therefore,

$$\begin{aligned}
\sin(15^\circ) - \sin(105^\circ) &= 2\sin\left(\frac{15^\circ - 105^\circ}{2}\right)\cos\left(\frac{15^\circ + 105^\circ}{2}\right) && \text{Substitute} \\
&= 2\sin(-45^\circ)\cos(60^\circ) \\
&= 2 \cdot \left(-\frac{\sqrt{2}}{2}\right) \cdot \frac{1}{2} && \text{Evaluate the functions} \\
&= -\frac{\sqrt{2}}{2}
\end{aligned}$$

Comments and Observations:

- These problems are primarily about practicing basic algebra and remembering how to calculate sine and cosine values. There's no expectation that these will become memorized formulas.