Group Practice Problems #1 - Identifying Expressions: Use a double angle identity to rewrite the expression in terms of a single sinusoidal function. If possible, evaluate that expression.

- $\cos^2(75°) - \sin^2(75°)$

We can see that this matches one of the versions of the double angle formula for the cosine function:

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

In this case, we have $\theta = 75°$, so that

$$\begin{aligned}
\cos^2(75°) - \sin^2(75°) &= \cos(2 \cdot 75°) && \text{Double angle formula} \\
&= \cos(150°) \\
&= -\frac{\sqrt{3}}{2}
\end{aligned}$$

- $6\sin(5x)\cos(5x)$

This nearly matches the double angle formula for the sine function:

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

We need to manipulate the expression slightly to get isolate the appropriate form.

$$\begin{aligned}
6\sin(5x)\cos(5x) &= 3 \cdot 2\sin(5x)\cos(5x) && \text{Factor out a 3} \\
&= 3 \cdot \sin(2 \cdot 5x) && \text{Double angle formula} \\
&= 3\sin(10x)
\end{aligned}$$

Comments and Observations:

- An important skill that will become more useful as you move into calculus is recognizing when you can rewrite an expression into a useful form in order to perform a substitution or other algebraic manipulation. The second problem highlights an example of this.

Group Practice Problems #2 - Values of Trigonometric Functions: Determine $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$ from the given information.

- $\sin(\theta) = \frac{5}{13}$ and θ is in the second quadrant.

We will use the Pythagorean identity to determine the value of $\cos(\theta)$.

$$\begin{aligned}
\cos(\theta) &= \pm\sqrt{1-\sin^2(\theta)} && \text{Pythagorean identity} \\
&= \pm\sqrt{1-\left(\frac{5}{13}\right)^2} && \text{Substitute} \\
&= \pm\sqrt{1-\frac{25}{169}} \\
&= \pm\sqrt{\frac{144}{169}} \\
&= \pm\frac{12}{13}
\end{aligned}$$

Since we are given that θ is in the second quadrant, we know that $\cos(\theta)$ will be negative, so that $\cos(\theta) = -\frac{12}{13}$. This allows us to compute the indicated values.

$$\begin{aligned}
\sin(2\theta) &= 2\sin(\theta)\cos(\theta) = 2\cdot\frac{5}{13}\cdot\left(-\frac{12}{13}\right) = -\frac{120}{169} \\
\cos(2\theta) &= \cos^2(\theta)-\sin^2(\theta) = \left(-\frac{12}{13}\right)^2-\left(\frac{5}{13}\right)^2 = \frac{144}{169}-\frac{25}{169} = \frac{119}{169} \\
\tan(2\theta) &= \frac{\sin(2\theta)}{\cos(2\theta)} = \frac{-120/169}{119/169} = -\frac{120}{119}
\end{aligned}$$

- $\cos(\theta) = -\frac{3}{7}$ and θ is in the third quadrant.

We will use the Pythagorean identity to determine the value of $\sin(\theta)$.

$$\begin{aligned}
\sin(\theta) &= \pm\sqrt{1-\cos^2(\theta)} && \text{Pythagorean identity} \\
&= \pm\sqrt{1-\left(-\frac{3}{7}\right)^2} && \text{Substitute} \\
&= \pm\sqrt{1-\frac{9}{49}} \\
&= \pm\sqrt{\frac{40}{49}} \\
&= \pm\frac{2\sqrt{10}}{7}
\end{aligned}$$

Since we are given that θ is in the third quadrant, we know that $\sin(\theta)$ will be negative, so that $\sin(\theta) = -\frac{2\sqrt{10}}{7}$. This allows us to compute the indicated values.

$$\begin{aligned}
\sin(2\theta) &= 2\sin(\theta)\cos(\theta) = 2\cdot\left(-\frac{2\sqrt{10}}{7}\right)\cdot\left(-\frac{3}{7}\right) = -\frac{6\sqrt{10}}{49} \\
\cos(2\theta) &= \cos^2(\theta)-\sin^2(\theta) = \left(-\frac{3}{7}\right)^2-\left(-\frac{2\sqrt{10}}{7}\right)^2 = \frac{9}{49}-\frac{40}{49} = -\frac{31}{49} \\
\tan(2\theta) &= \frac{\sin(2\theta)}{\cos(2\theta)} = \frac{-6\sqrt{10}/49}{-31/49} = \frac{6\sqrt{10}}{49}
\end{aligned}$$

Comments and Observations:

- You can also use geometric methods to determine the value of the other trigonometric function. See Section 5.4 for examples of this.

Group Practice Problems #3 - Tangent Identity: Prove the following double angle identity for tangent twice. For the first time, use the double angle identities for the sine and cosine functions. For the second time, use the sum of angles identity for the tangent function. (See the previous in-class worksheet.)

- $\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)}$

For the first proof, we will use the double angle identities for the sine and cosine functions.

$$\begin{aligned}
\tan(2\theta) &= \frac{\sin(2\theta)}{\cos(2\theta)} && \text{Definition of tangent} \\
&= \frac{2\sin(\theta)\cos(\theta)}{\cos^2(\theta)-\sin^2(\theta)} && \text{Double angle formulas} \\
&= \frac{2\sin(\theta)\cos(\theta)}{\cos^2(\theta)-\sin^2(\theta)} \cdot \frac{\frac{1}{\cos^2(\theta)}}{\frac{1}{\cos^2(\theta)}} && \text{Multiply by 1} \\
&= \frac{2\frac{\sin(\theta)}{\cos(\theta)}}{1-\frac{\sin^2(\theta)}{\cos^2(\theta)}} \\
&= \frac{2\tan(\theta)}{1-\tan^2(\theta)} && \text{Definition of tangent}
\end{aligned}$$

For the second proof, we will use this formula:

$$\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$$

To apply the formula, we will write 2θ as $\theta+\theta$.

$$\begin{aligned}
\tan(2\theta) &= \tan(\theta+\theta) && \text{Rewrite the expression} \\
&= \frac{\tan(\theta)+\tan(\theta)}{1-\tan(\theta)\tan(\theta)} && \text{Double angle formula} \\
&= \frac{2\tan(\theta)}{1-\tan^2(\theta)}
\end{aligned}$$

Comments and Observations:

- This exercise is a reminder that there may be multiple ways to prove a result.

Group Practice Problems #4 - Solving Equations: Solve the equation. Then list the solutions on the interval $[0, 2\pi)$. Describe your steps and state any identities that you use.

- $\cos(2t) = \cos(t)$

$$\begin{aligned} \cos(2t) &= \cos(t) && \text{Original Equation} \\ 2\cos^2(t) - 1 &= \cos(t) && \text{Double angle formula} \\ 2\cos^2(t) - \cos(t) - 1 &= 0 && \text{Subtract } \cos(t) \text{ from both sides} \end{aligned}$$

We will factor this using the ac method. We need two numbers that multiply to $ac = 2 \cdot (-1) = -2$ and add to $b = -1$. We can see that -2 and 1 accomplishes this.

$$\begin{aligned} 2\cos^2(t) - \cos(t) - 1 &= 0 \\ 2\cos^2(t) - 2\cos(t) + \cos(t) - 1 &= 0 && \text{Use the } ac \text{ method to rewrite the linear term} \\ 2\cos(t)(\cos(t) - 1) + (\cos(t) - 1) &= 0 && \text{Factor by grouping} \\ (2\cos(t) + 1)(\cos(t) - 1) &= 0 \end{aligned}$$

By the zero product property, we must have that $2\cos(t) + 1 = 0$ or $\cos(t) - 1 = 0$.

If $2\cos(t) + 1 = 0$, then we have that $\cos(t) = -\frac{1}{2}$. This gives solutions in the second and third quadrant, and we can see that the reference angle is $\frac{\pi}{3}$. Therefore,

$$t = \begin{cases} \pi - \frac{\pi}{3} + 2\pi k \\ \pi + \frac{\pi}{3} + 2\pi k \end{cases} = \begin{cases} \frac{2\pi}{3} + 2\pi k \\ \frac{4\pi}{3} + 2\pi k \end{cases}$$

where k is any integer.

If $\cos(t) - 1 = 0$, then $\cos(t) = 1$, which leads to the solution on the positive x-axis. Therefore, this gives solutions$t = 2\pi k$, where k is any integer.

The solutions on the interval $[0, 2\pi)$ are 0, $\frac{2\pi}{3}$, and $\frac{4\pi}{3}$.

- $2\sin(2t) + 3\cos(t) = 0$

$$\begin{aligned} 2\sin(2t) + 3\cos(t) &= 0 && \text{Original equation} \\ 2 \cdot 2\sin(t)\cos(t) + 3\cos(t) &= 0 && \text{Double angle formula} \\ 4\sin(t)\cos(t) + 3\cos(t) &= 0 \\ \cos(t)(4\sin(t) + 3) &= 0 && \text{Factor out } \cos(t) \end{aligned}$$

By the zero product property, we must either have $\cos(t) = 0$ or $4\sin(t) + 3 = 0$.

If $\cos(t) = 0$, then t must be a solution on the y-axis. This implies that

$$t = \begin{cases} \frac{\pi}{2} + 2\pi k \\ \frac{3\pi}{2} + 2\pi k \end{cases}$$

where k is any integer.

If $4\sin(t) + 3 = 0$, then $\sin(t) = -\frac{3}{4}$. This has solutions in the third and fourth quadrants. Notice that $\sin^{-1}(-\frac{3}{4}) \approx -0.84$, which implies that the reference angle is 0.84. This means that

$$t \approx \begin{cases} \pi + 0.84 + 2\pi k \\ 2\pi - 0.84 + 2\pi k \end{cases} \approx \begin{cases} 3.98 + 2\pi k \\ 5.44 + 2\pi k \end{cases}$$

where k is any integer.

The solutions on the interval $[0, 2\pi)$ are $\frac{\pi}{2}$, $\frac{3\pi}{2}$, and approximately 3.98 and 5.44.

Comments and Observations:

- In the last problem, the decision to use $2\pi - 0.84$ instead of simply -0.84 was because we were going to need solutions in the interval $[0, 2\pi)$, and recognizing this in advance saved us a little extra work at the end.

Group Practice Problems #5 - Power Reduction Identities: Use the formulas $\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$ and $\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$ to write the expression without exponents.

- $\sin^4(3x)$

$$\begin{aligned}
\sin^4(3x) &= \sin^2(3x)\cdot\sin^2(3x) && \text{Rewrite as squared sinusoidals}\\
&= \frac{1-\cos(2\cdot 3x)}{2}\cdot\frac{1-\cos(2\cdot 3x)}{2} && \text{Power reduction identity}\\
&= \frac{1-\cos(6x)}{2}\cdot\frac{1-\cos(6x)}{2}\\
&= \frac{1-2\cos(6x)+\cos^2(6x)}{4}\\
&= \frac{1-2\cos(6x)}{4}+\frac{1}{4}\cos^2(6x) && \text{Isolate the squared sinusoidal}\\
&= \frac{1-2\cos(6x)}{4}+\frac{1}{4}\cdot\frac{1+\cos(2\cdot 6x)}{2}\\
&= \frac{1-2\cos(6x)}{4}+\frac{1+\cos(12x)}{8}\\
&= \frac{2-4\cos(6x)+1+\cos(12x)}{8} && \text{Common denominator}\\
&= \frac{3-4\cos(6x)+\cos(12x)}{8}
\end{aligned}$$

- $\cos^4(8x)$

$$\begin{aligned}
\cos^4(8x) &= \cos^2(8x)\cdot\cos^2(8x) && \text{Rewrite as squared sinusoidals}\\
&= \frac{1+\cos(2\cdot 8x)}{2}\cdot\frac{1+\cos(2\cdot 8x)}{2} && \text{Power reduction identity}\\
&= \frac{1+\cos(16x)}{2}\cdot\frac{1+\cos(16x)}{2}\\
&= \frac{1+2\cos(16x)+\cos^2(16x)}{4}\\
&= \frac{1+2\cos(16x)}{4}+\frac{1}{4}\cos^2(16x) && \text{Isolate the squared sinusoidal}\\
&= \frac{1+2\cos(16x)}{4}+\frac{1}{4}\cdot\frac{1+\cos(2\cdot 16x)}{2}\\
&= \frac{1+2\cos(16x)}{4}+\frac{1+\cos(32x)}{8}\\
&= \frac{2+4\cos(16x)+1+\cos(32x)}{8} && \text{Common denominator}\\
&= \frac{3+4\cos(16x)+\cos(32x)}{8}
\end{aligned}$$

Comments and Observations:

- The most common mistakes happen near the squared term after making the first substitution. One mistake is that students fail to apply the power reduction identity to that term. Another error arises from making arithmetic errors with the fraction manipulations. And other times, the error happens from failing to correctly double the angle.

Group Practice Problems #6 - Half Angle Identities: Use the formulas $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}$ and $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$ to calculate the given expressions.

- $\sin(15°)$

$$\begin{aligned}
\sin(15°) &= \sin\left(\frac{30°}{2}\right) && \text{Rewrite as a half angle} \\
&= \pm\sqrt{\frac{1-\cos(30°)}{2}} \\
&= \pm\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} \\
&= \pm\sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}} \\
&= \pm\sqrt{\frac{2-\sqrt{3}}{4}} \\
&= \pm\frac{\sqrt{2-\sqrt{3}}}{2}
\end{aligned}$$

Since the original angle of $15°$ is in the first quadrant and sine is positive in that quadrant, we will pick the positive value. Therefore, $\sin(15°) = \frac{\sqrt{2-\sqrt{3}}}{2}$.

- $\cos(105°)$

$$\begin{aligned}
\cos(105°) &= \cos\left(\frac{210°}{2}\right) && \text{Rewrite as a half angle} \\
&= \pm\sqrt{\frac{1+\cos(210°)}{2}} \\
&= \pm\sqrt{\frac{1+\left(-\frac{\sqrt{3}}{2}\right)}{2}} \\
&= \pm\sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}} \\
&= \pm\sqrt{\frac{2-\sqrt{3}}{4}} \\
&= \pm\frac{\sqrt{2-\sqrt{3}}}{2}
\end{aligned}$$

Since the original angle of $105°$ is in the second quadrant and cosine is negative in that quadrant, we will pick the negative value. Therefore, $\cos(105°) = -\frac{\sqrt{2-\sqrt{3}}}{2}$.

Comments and Observations:

- The key to these problems is to correctly identify the θ that allows you to apply the identity. It is common for students to use the given angle as θ in the formula, and then become lost when trying to evaluate the cosine of that angle.