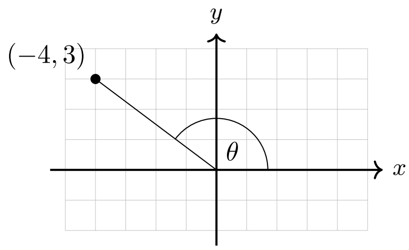
Group Practice Problems #1 - The Sine and Cosine Functions:

- Sketch the angle θ whose terminal side passes through the point $(-4, 3)$. Determine $\sin(\theta)$ and $\cos(\theta)$.



To calculate $\sin(\theta)$ and $\cos(\theta)$, we will need to calculate the distance from the origin to the point:

$$\begin{aligned} r &= \sqrt{x^2+y^2} && \text{Formula for the distance from the origin to the point } (x,y) \\ &= \sqrt{(-4)^2+3^2} && \text{Substitute} \\ &= \sqrt{16+9} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

We can now substitute the values directly into the definitions of the functions:

$$\sin(\theta) = \frac{y}{r} = \frac{3}{5}$$
$$\cos(\theta) = \frac{x}{r} = -\frac{4}{5}$$

- Suppose $\sin(\theta) > 0$ while $\cos(\theta) < 0$. What is the quadrant of the angle θ?

The sine function corresponds to the y-direction, so the condition $\sin(\theta) > 0$ tells us that the y-coordinate is positive. The cosine function corresponds to the x-direction, so the condition $\cos(\theta) < 0$ tells us that the x-coordinate is negative. Points that have a negative x-coordinate (left) and a positive y-coordinate (up) are in the second quadrant.

- Suppose $\cos(\theta) = 0$. What are four possible values of θ? (Hint: Think about coterminal angles.)

Sine $\cos(\theta) = 0$ and the cosine function corresponds to the x direction, we know that the terminal point must lie somewhere along the y-axis. If the point is on the upper part of the plane, then it will be an angle coterminal with $\frac{\pi}{2}$. Some examples of such angles are $\frac{\pi}{2}$, $\frac{5\pi}{2}$, and $-\frac{3\pi}{2}$. If the point is on the lower part of the plane, the it will be coterminal with $\frac{3\pi}{2}$. Some examples of such angles are $\frac{3\pi}{2}$, $\frac{7\pi}{2}$, and $-\frac{\pi}{2}$.

Comments and Observations:

- The primary outcome of these problems is to get comfortable with the geometric relationships of the sine and cosine functions. It is essential to understand that the cosine function corresponds to the x-coordinate and the sine function corresponds to the y-coordinate. From there, it is easier to understand how different points and angles correspond to each other, including thinking about the signs of the functions.

Group Practice Problems #2 - The Pythagorean Identity:

- Suppose that $\sin(\theta) = \frac{5}{8}$ and that $\frac{\pi}{2} < \theta < \pi$. Determine $\cos(\theta)$.

We start by plugging in the value to the Pythagorean Identity.

$$\begin{aligned}
\sin^2(\theta) + \cos^2(\theta) &= 1 && \text{The Pythagorean Identity} \\
\left(\frac{5}{8}\right)^2 + \cos^2(\theta) &= 1 && \text{Substitute} \\
\frac{25}{64} + \cos^2(\theta) &= 1 && \\
\cos^2(\theta) &= \frac{39}{64} && \text{Subtract } \frac{25}{64} \text{ from both sides} \\
\cos(\theta) &= \pm\sqrt{\frac{39}{64}} && \text{Take the square root of both sides} \\
\cos(\theta) &= \pm\frac{\sqrt{39}}{8} && \text{Simplify the radical}
\end{aligned}$$

From the given information, we know that the angle is in the second quadrant. In the second quadrant, the x-coordinate is negative, which means that cosine is negative. Therefore, we will choose the negative value of the cosine function to get $\cos(\theta) = -\frac{\sqrt{39}}{8}$.

- Suppose that $\cos(\theta) = -\frac{2}{3}$ and that $\pi < \theta < \frac{3\pi}{2}$. Determine $\sin(\theta)$.

We start by plugging in the value to the Pythagorean Identity.

$$\begin{aligned}
\sin^2(\theta) + \cos^2(\theta) &= 1 && \text{The Pythagorean Identity} \\
\sin^2(\theta) + \left(-\frac{2}{3}\right)^2 &= 1 && \text{Substitute} \\
\sin^2(\theta) + \frac{4}{9} &= 1 && \\
\sin^2(\theta) &= \frac{5}{9} && \text{Subtract } \frac{4}{9} \text{ from both sides} \\
\sin(\theta) &= \pm\sqrt{\frac{5}{9}} && \text{Take the square root of both sides} \\
\sin(\theta) &= \pm\frac{\sqrt{5}}{3} && \text{Simplify the radical}
\end{aligned}$$

From the given information, we know that the angle is in the third quadrant. In the third quadrant, the y-coordinate is negative, which means that sine is negative. Therefore, we will choose the negative value of the sine function to get $\sin(\theta) = -\frac{\sqrt{5}}{3}$.

- Suppose that $\cos(\theta) = -1$. Determine $\sin(\theta)$ and explain why you can determine this value without knowing additional information about the angle θ.

We will first plug $\cos(\theta) = -1$ into the Pythagorean Identity. We start by plugging in the value to the Pythagorean Identity.

$$\begin{aligned}
\sin^2(\theta) + \cos^2(\theta) &= 1 && \text{The Pythagorean Identity} \\
\sin^2(\theta) + (-1)^2 &= 1 && \text{Substitute} \\
\sin^2(\theta) + 1 &= 1 && \\
\sin^2(\theta) &= 0 && \text{Subtract 1 from both sides} \\
\sin(\theta) &= 0 && \text{Take the square root of both sides}
\end{aligned}$$

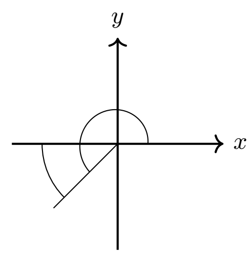
In this situation, there is no decision to make about the value of the sine function, and so there is no need for additional information about the quadrant to help us to determine which value is the correct one to use.

Comments and Observations:

- We will see a more geometric approach to solving these types of problems (a method using right triangles) in a couple sections.

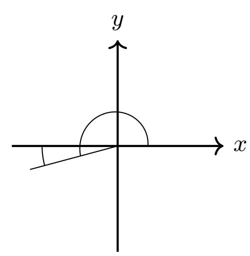
Group Practice Problems #3 - Reference Angles: Sketch the angle and the reference angle. Then determine the measure of the reference angle.

- $225°$



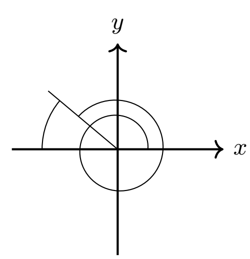
The reference angle is the amount of the angle that goes beyond $180°$. Therefore, the measure of the reference angle is $225° - 180° = 45°$.

- $\frac{13\pi}{12}$



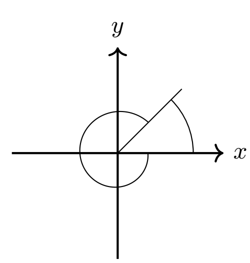
The reference angle is the amount of the angle that goes beyond π radians. Therefore, the measure of the reference angle is $\frac{13\pi}{12} - \pi = \frac{\pi}{12}$.

- $500°$



The reference angle is the amount needed to get back to the x-axis. After taking the full winding around the origin, we see that we are looking for the difference between the given angle and an angle of $360°+180° = 540°$. Therefore, the measure of the reference angle is $540° - 500° = 40°$.

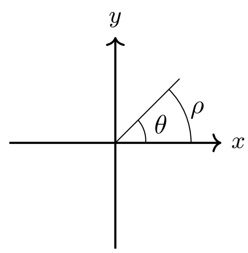
- $-\frac{7\pi}{4}$



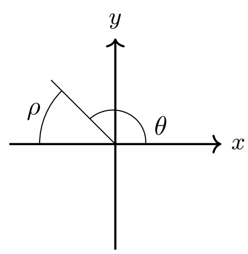
We can see that the reference angle is the amount needed to get to a full 2π radian rotation. Therefore, the reference angle is given by $2\pi - \frac{7\pi}{4} = \frac{\pi}{4}$.

Comments and Observations:

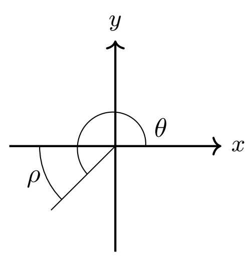
- It is helpful to visualize this in terms of the geometry rather than trying to memorize some formulas. But if you really wanted to do this in a formulaic manner, you would first write the angle θ in the interval $[0, 2\pi)$ that is coterminal to the given angle (assuming you're working in radians), and then the reference angle ρ can be obtained by you apply the following formulas based on the quadrant (compare the diagram and the formulas, and you should be able to use logic to see where the formulas come from):
  - Quadrant I: $\rho = \theta$.



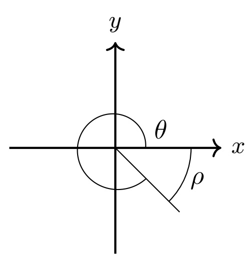
  - Quadrant II: $\rho = \pi - \theta$.



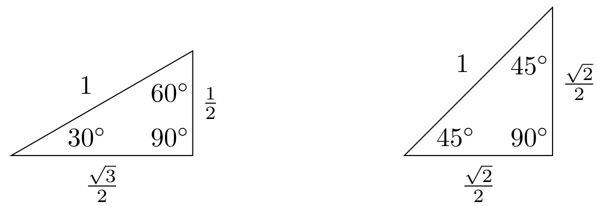
  - Quadrant III: $\rho = \theta - \pi$.



  - Quadrant IV: $\rho = 2\pi - \theta$.



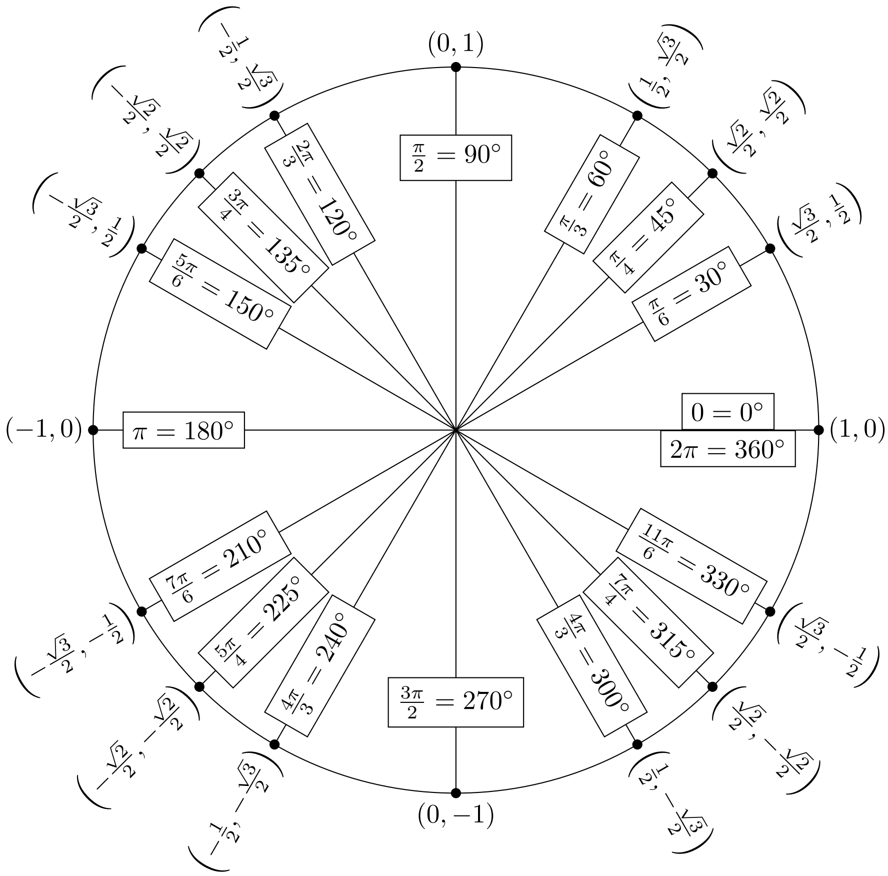
Group Practice Problems #4 - Special Triangles: Draw the 30-60-90 and 45-45-90 triangles with hypotenuses of length 1 and label the sides and the angles.



Comments and Observations:

- It is extremely useful to have these triangles memorized. There's not that much to memorize, and it will serve you all the way to the end of the course (and potentially beyond).

Group Practice Problems #5 - Unit Circle Diagram: Draw the unit circle diagram, including labeling the common angles in both degrees and radians, and specifying the coordinates of the corresponding points.



Comments and Observations:

- This is a lot of information to "memorize" but if you understand the construction of these values from the 30-60-90 and 45-45-90 triangles (as well as thinking through the reference angles), you can actually reconstruct all of these values relatively quickly and easily.
- From the first quadrant values and by knowing that the sine function is the y-coordinate and the cosine function is the x-coordinate, we can construct the following chart of values. This chart will be used many times in the coming sections.

θ (radians) θ (degrees)	0 $0°$	$\frac{\pi}{6}$ $30°$	$\frac{\pi}{4}$ $45°$	$\frac{\pi}{3}$ $60°$	$\frac{\pi}{2}$ $90°$
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Group Practice Problems #6 - Sine and Cosine Values: Determine the exact values of the following sine and cosine functions.

- $\sin(30°)$

The sine function is the y-coordinate of the corresponding point on the unit circle, so $\sin(30°) = \frac{1}{2}$.

- $\cos\left(\frac{\pi}{2}\right)$

The cosine function is the x-coordinate of the corresponding point on the unit circle, so $\cos(\frac{\pi}{2}) = 0$.

- $\cos(210°)$

The cosine function is the x-coordinate of the corresponding point on the unit circle, so $\cos(210°) = -\frac{\sqrt{3}}{2}$.

Alternatively, we can see that the reference angle for $210°$ is $30°$ and that $210°$ is in the third quadrant, which shows that the cosine function (which corresponds to the x-coordinate) will be negative. Since $\cos(30°) = \frac{\sqrt{3}}{2}$, we have that $\cos(210°) = -\cos(30°) = -\frac{\sqrt{3}}{2}$.

- $\sin(7\pi)$

We first note that 7π is coterminal with π. From here, we know that the sine function is the y-coordinate of the corresponding point on the unit circle, so $\sin(7\pi) = \sin(\pi) = 0$.

Comments and Observations:

- We used some words to explain the process of obtaining these values, but these are values that you ought to be comfortable calculating in your head since it's mostly just learning to recognize the corresponding angles.