Analysis of Algorithms

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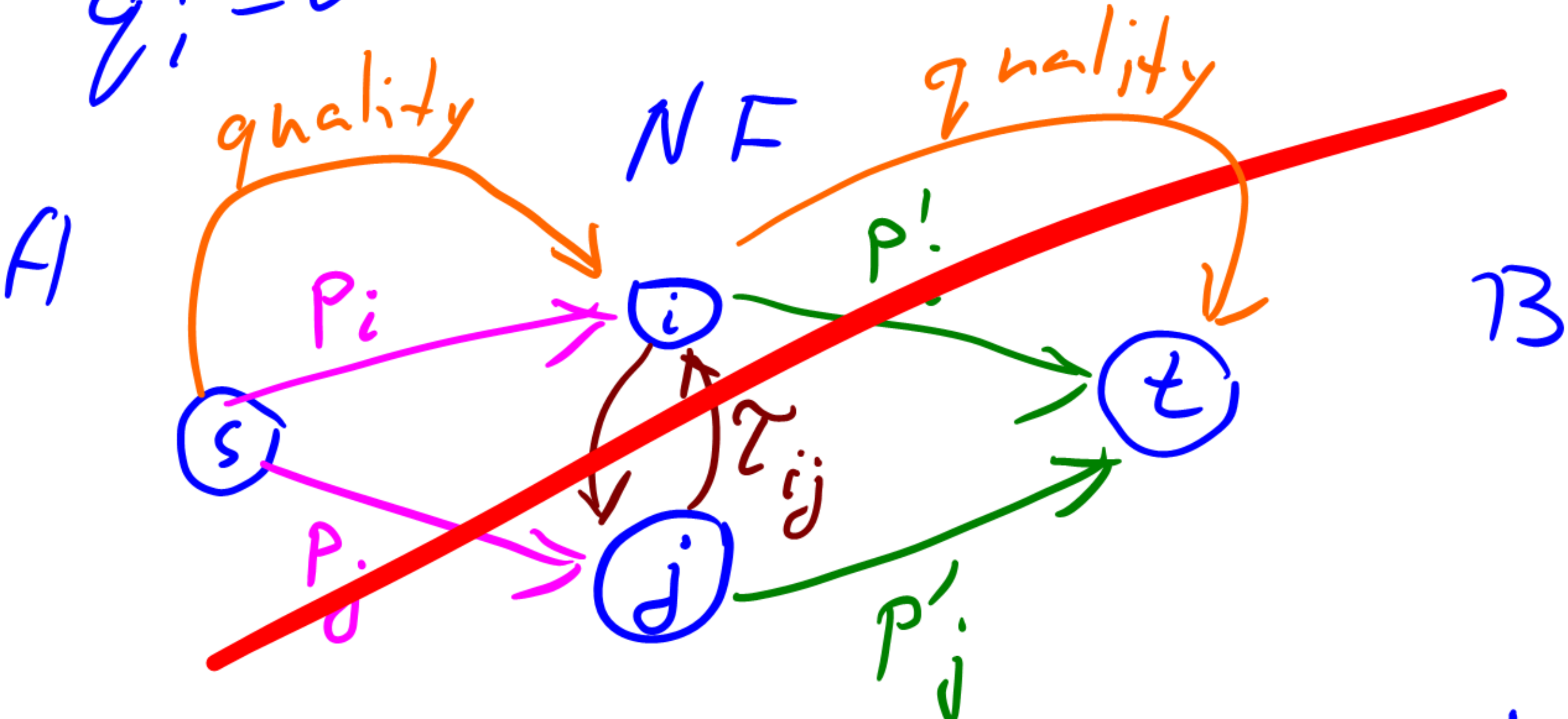
Lecture 10 University of Southern California

Linear Programming

Reading: chapter 8

HW4, #6

$q_i = 0$



$$cap(A,B) = \sum_{j \in B} p_j + \sum_{\substack{i \in A \\ j \in B}} T_{ij} + \sum_{i \in A} p'_i$$

DP

Linear Programming

In this lecture we describe linear programming that is used to express a wide variety of different kinds of problems. LP can solve the max-flow problem and the shortest distance, find optimal strategies in games, and many other things.

We will primarily discuss the setting and how to code up various problems as linear programs.

Solving by Reduction

Formally, to reduce a problem Y to a problem X (we write $Y \leq_p X$) we want a function f that maps Y to X such that:

- f is a <u>polynomial</u> time computable
- $\forall$ *instance* $y \in Y$ is solvable if and only if $f(y) \in X$ is solvable.

$Y \leq_p NF$

$Y \leq_p LP$

A Production Problem

A company wishes to produce two types of souvenirs: type-A will result in a profit of \$1.00, and type-B in a profit of \$1.20.

To manufacture a type-A souvenir requires 2 minutes on machine I and 1 minute on machine II.

A type-B souvenir requires 1 minute on machine I and 3 minutes on machine II.

There are 3 hours available on machine I and 5 hours available on machine II.

How many souvenirs of each type should the company make in order to maximize its profit?

A Production Problem

	Type-A	Type-B	Time Available
Profit/Unit	\$1.00	\$1.20	
Machine I	2 min $2x$	$+$ 1 min y	$\le$ 180 min
Machine II	1 min x	$+$ 3 min $3y$	$\le$ 300 min

let $x \ge 0$ be the number of type-A

$y \ge 0$ — " — " type-B

Objective function (profit)

$$\max_{x,y} \left(x \cdot 1 + y \cdot 1.2 \right)$$

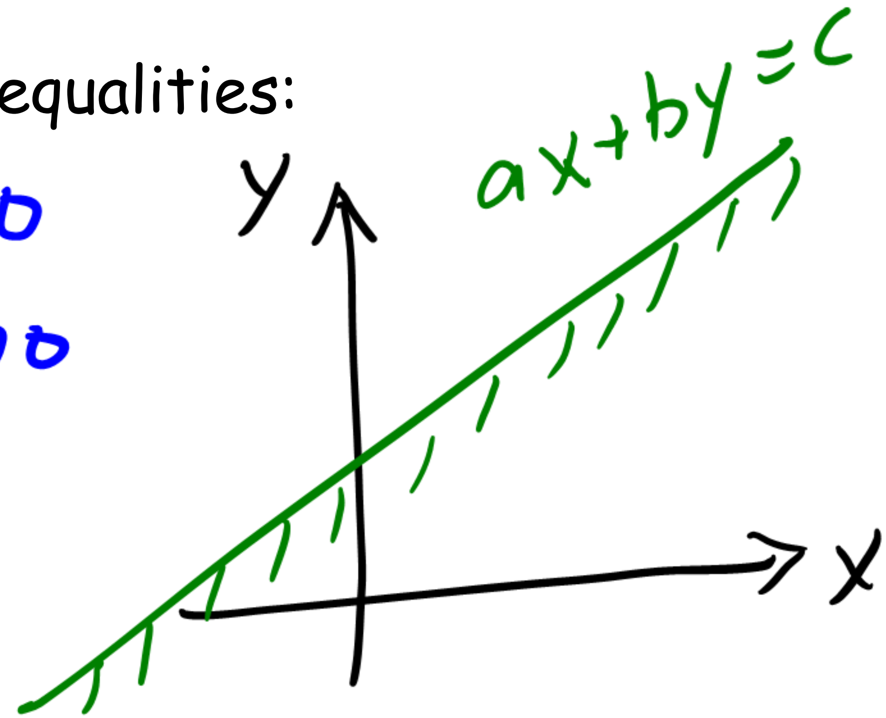
A Linear Program

We want to maximize the objective function

$$\max_{x,y} (x + 1.2y)$$

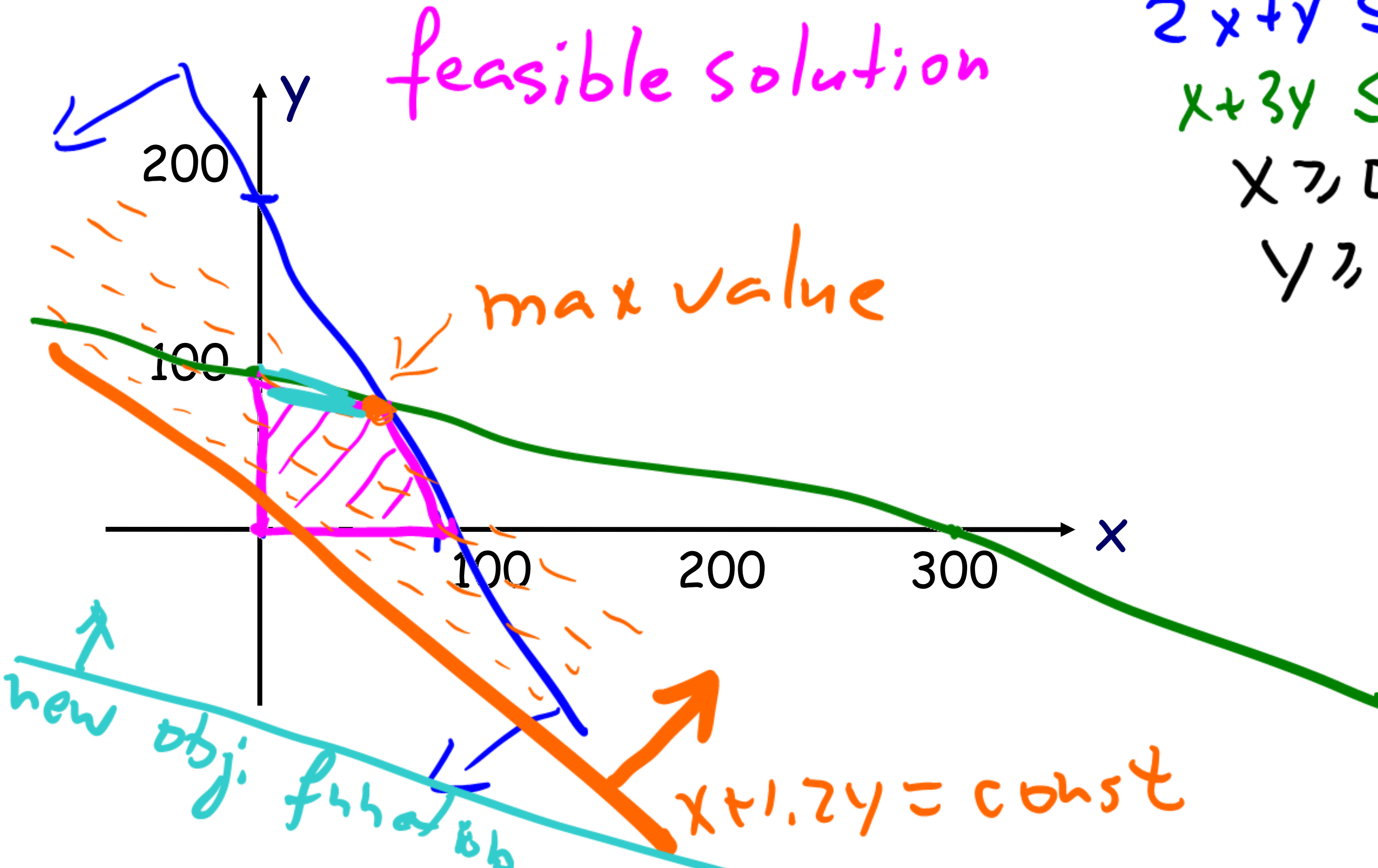
subject to the system of inequalities:

$$2x + y \le 180$$
$$x + 3y \le 300$$
$$x \ge 0$$
$$y \ge 0$$



A Production Problem

$$2x + y \le 180$$
$$x + 3y \le 300$$
$$x \ge 0$$
$$y \ge 0$$



We need to find the feasible point that is farthest in the "objective" direction $x + 1.2y$

Fundamental Theorem

If a linear programming problem has a solution, then it must occur at a vertex, or corner point, of the feasible set *S* associated with the problem.

If the objective function *P* is optimized at *two* adjacent vertices of *S*, then it is optimized at *every point* on the line segment joining these vertices, in which case there are infinitely many solutions to the problem.

Existence of Solution

Suppose we are given a LP problem with a feasible set S and an objective function P. There are 3 cases to consider

① S is empty, LP has no solution

$\max(x)$
$x \leq -1$
$x \geq 0$

② S is unbounded

LP may or may not have a solution

$\max(x)$
$x \geq 0$
sol $= \infty$

$\max(x)$
$x \leq 0$
sol $= 0$

③ S is bounded

LP has a solution(s)

??

Standard LP form

We say that a maximization linear program with n variables is in standard form if for every variable x_k we have the inequality $x_k \geq 0$ and all other m linear inequalities. A LP in standard form is written as

min

$c^T x = x^T c$

$$\max\ (c_1 x_1 + \dots + c_n x_n)$$

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

subject to

$\geq$ or $==$

$$Ax \leq b \begin{cases} a_{11}x_1 + \dots + a_{1n}x_n \leq b_1 \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n \leq b_m \end{cases}$$

$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ - & - & - \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$

≥ 5

$$x \geq 0 \left\{ x_1 \geq 0, \dots, x_n \geq 0 \right.$$

≤ 2

Standard LP in Matrix Form

The vector c is the column vector $(c_1, \ldots, c_n)$.
The vector x is the column vector $(x_1, \ldots, x_n)$.
The matrix A is the $n \times m$ matrix of coefficients of the left-hand sides of the inequalities, and
$b = (b_1, \ldots, b_m)$ is the vector of right-hand sides of the inequalities.

$$\max\ (c^T x)$$

subject to

$$A x \leq b$$
$$\boxed{x \geq 0}$$

Exercise: Convert to Matrix Form

$\max(x_1 + 1.2\, x_2)$
$2x_1 + x_2 \leq 180$
$x_1 + 3x_2 \leq 300$
$x_1 \geq 0$
$x_2 \geq 0$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 1{,}2 \end{pmatrix}$$

$$b = \begin{pmatrix} 180 \\ 300 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

Algorithms for LP

The standard algorithm for solving LPs is the Simplex Algorithm, due to Dantzig, 1947.

This algorithm starts by finding a vertex of the polytope, and then moving to a neighbor with increased cost as long as this is possible. By linearity and convexity, once it gets stuck it has found the optimal solution.

Unfortunately simplex does not run in polynomial time it does well in practice, but poorly in theory.

Algorithms for LP

In 1974 Khachian has shown that LP could be done in polynomial time by something called the Ellipsoid Algorithm (but it tends to be fairly slow in practice).

In 1984 Karmarkal discovered a faster polynomial-time algorithm called "interior-point". While simplex only moves along the outer faces of the polytope, "interior-point" algorithm moves inside the polytope.

MATLAB

https://www.mathworks.com/help/optim/ug/linprog.html

linprog

Linear programming solver

Finds the minimum of a problem specified by

$- \max$

$$\min_x f^T x \text{ such that } \begin{cases} A \cdot x \leq b, \\ Aeq \cdot x = beq, \\ lb \leq x \leq ub. \end{cases}$$

f, *x*, *b*, *beq*, *lb*, and *ub* are vectors, and *A* and *Aeq* are matrices.

Description

`x = linprog(f,A,b)` solves min `f'*x` such that `A*x ≤ b`.

`x = linprog(f,A,b,Aeq,beq)` includes equality constraints `Aeq*x = beq`. Set `A = []` and `b = []` if no inequalities exist.

`x = linprog(f,A,b,Aeq,beq,lb,ub)` defines a set of lower and upper bounds on the design variables, `x`, so that the solution is always in the range `lb ≤ x ≤ ub`. Set `Aeq = []` and `beq = []` if no equalities exist.

Discussion Problem 1

A cargo plane can carry a maximum weight of 100 tons and a maximum volume of 60 cubic meters. There are three materials to be transported, and the cargo company may choose to carry any amount of each, up to the maximum available limits given below.

	Density	Volume	Price
Material 1	2 tons/m^3	40 m^3	\$1,000 per m^3
Material 2	1 tons/m^3	30 m^3	\$2,000 per m^3
Material 3	3 tons/m^3	20 m^3	\$12,000 per m^3

Write a linear program that optimizes revenue within the constraints.

let x_1, x_2, x_3 be the volumes ...

objective function: $\max(1000x_1 + 2000x_2 + 12000x_3)$

subject to:

$$2 \cdot x_1 + 1 \cdot x_2 + 3 \cdot x_3 \le 100$$

$$x_1 + x_2 + x_3 \le 60$$

$$0 \le x_1 \le 40, \quad 0 \le x_2 \le 30, \quad 0 \le x_3 \le 20$$

matrix form

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad c = \begin{pmatrix} 1000 \\ 2000 \\ 12000 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 100 \\ 60 \\ 40 \\ 30 \\ 20 \end{pmatrix}$$

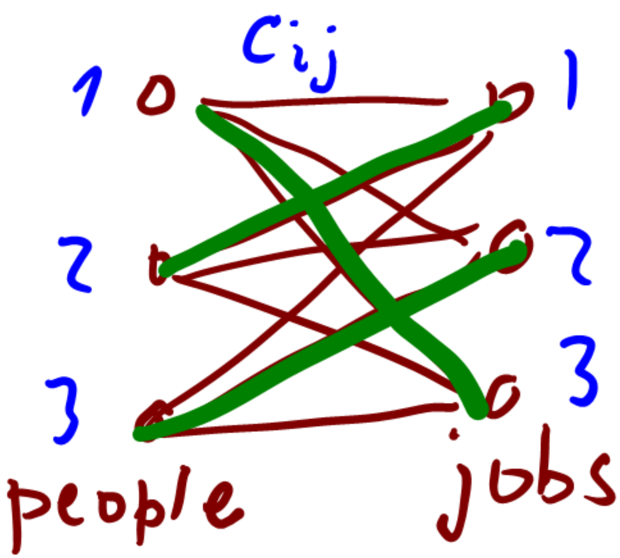
$$0 \le x_1 \le 40 \Rightarrow \begin{cases} x_1 \ge 0 \\ x_1 \le 40 \end{cases}$$

$$x_1 \le 40 \Rightarrow x_1 + 0 \cdot x_2 + 0 \cdot x_3 \le 40$$

$$x_2 \le 30 \Rightarrow 0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 \le 30$$

Discussion Problem 2

There are n people and n jobs. You are given a cost matrix, C, where c_{ij} represents the cost of assigning person i to do job j. You need to assign all the jobs to people and also only one job to a person. You also need to minimize the total cost of your assignment. Write a linear program that minimizes the total cost of your assignment.



$c_{13} + c_{21} + c_{32} \Rightarrow \min$

① Define variables.

Let x_{ij} be an assignment between i-th person and j-th job.

② Objective function.

$$\min \sum_{i,j=1}^{n} x_{ij} \cdot c_{ij}$$

③ Constraints.

pick a person $i = 1, 2, \ldots, n$

$$x_{i1} + x_{i2} + \ldots + x_{in} = 1$$

pick a job $j = 1, \ldots, n$

$$x_{1j} + x_{2j} + \ldots + x_{nj} = 1$$

4 Constraints on x_{ij}

$\cancel{x_{ij} \geq 0}$

$$x_{ij} = \begin{cases} 1 \\ 0 \end{cases}$$

$$x_{ij} \in \{1, 0\}$$

Integer LP, vars $\in \mathbb{N}$
we do not know how to solve
ILP in polynomial time.

BREAK!

Discussion Problem 3

Convert the following LP to standard form

$\max\ (5x_1 - 2x_2 + 9x_3)$

$3x_1 + x_2 + 4x_3 = 8$

$2x_1 + 7x_2 - 6x_3 \le 4$

$x_1 \le 0,\ x_3 \ge 1$ x_2 - ?

$3x_1 + x_2 + 4x_3 \ge 8$ ← ??

$3x_1 + x_2 + 4x_3 \le 8$

$z_1 = -x_1,\ z_1 \geq 0$

$z_3 = x_3 - 1,\ z_3 \geq 0$

$-\infty < x_2 < +\infty$

$x_2 = z_2 - z_4$

$z_2 \geq 0,\ z_4 \geq 0$

x_2 - free variable

replace x_i by z_i

$$\max\left(5(-z_1) - 2(z_2 - z_4) + 9(z_3 + 1)\right)$$

$$\boxed{\max\left(-5z_1 - 2z_2 + 9z_3 + 2z_4\right)}$$ new

$$-3x_1 - x_2 - 4x_3 \le -8$$

$$-3(-z_1) - (z_2 - z_4) - 4(z_3 + 1) \le -8$$

$$\boxed{3z_1 - z_2 + z_4 - 4z_3 \ge -4}$$ new

$$\boxed{-3z_1 + z_2 - z_4 + 4z_3 \le 4}$$ new

$$2x_1 + 7x_2 - 6x_3 \le 4$$

$$\boxed{-2z_1 + 7z_2 - 7z_4 - 6z_3 \le 10}$$ new

$$z_1, z_2, z_3, z_4 \ge 0$$

Discussion Problem 4

Explain why LP <u>cannot</u> contain constrains in the form of *strong* inequalities.

$$\max(7x_1 - x_2 + 5x_3)$$
$$x_1 + x_2 + 4x_3 < 8$$
$$3x_1 - x_2 + 2x_3 > 3$$
$$2x_1 + 5x_2 - x_3 \leq -7$$
$$x_1, x_2, x_3 \geq 0$$

wrong - !

Example:

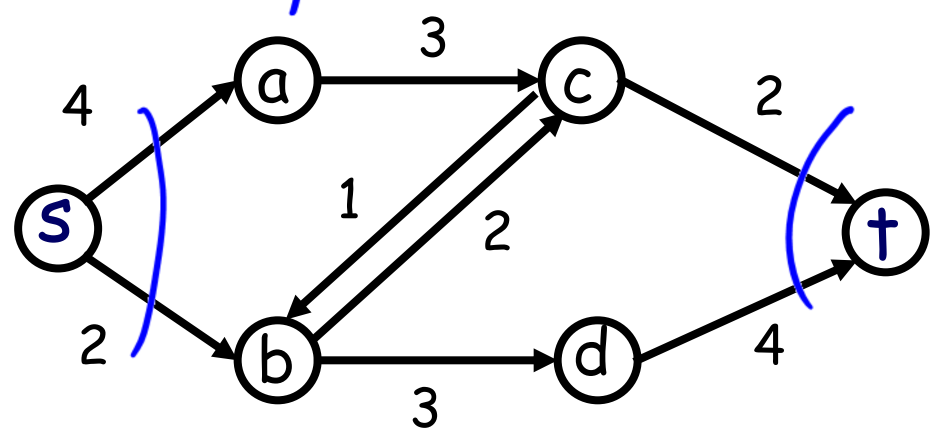
$$\boxed{\begin{array}{c}\max(x)\\ x < 1\\ x \geq 0\end{array}}$$

$x = 1$ is NOT a solution

Exercise: Max-Flow as LP

$NF \leq_p LP$

Write a max-flow problem as a linear program.



f_{uv} – flow on edge (u,v) for $\forall e \in E$

Objective function:

$\max(f_{sa} + f_{sb})$ or $\max(f_{ct} + f_{dt})$

Constraints

$0 \leq f_{sa} \leq 4$

$0 \leq f_{ac} \leq 3$

for $\forall e \in E$

$f_{sa} = f_{ac}$

$f_{sb} + f_{cb} = f_{bc} + f_{bd}$

for $\forall v \in V \setminus \{s,t\}$

Exercise: Shortest Path as LP

$SP \leq_p LP$

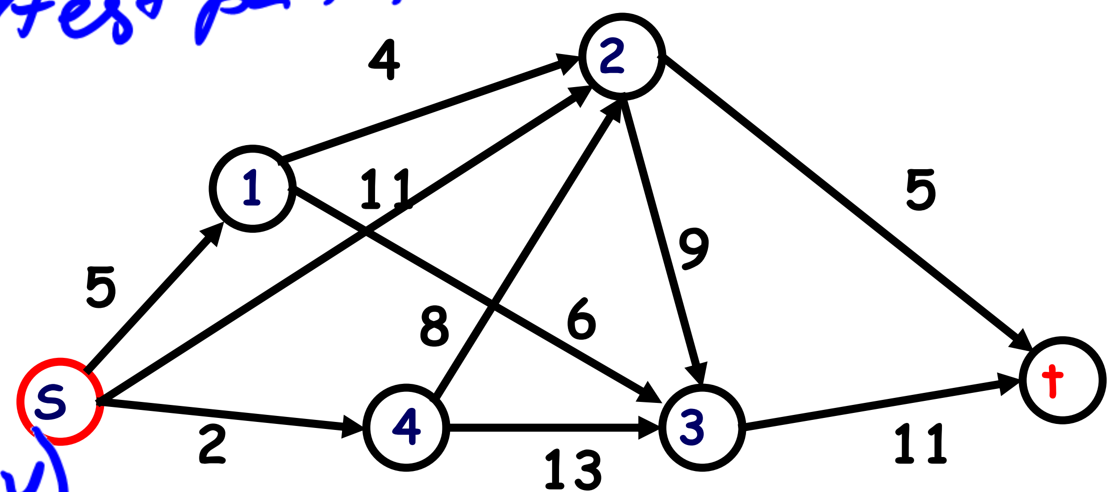
Write a shortest st-path problem as a linear program.

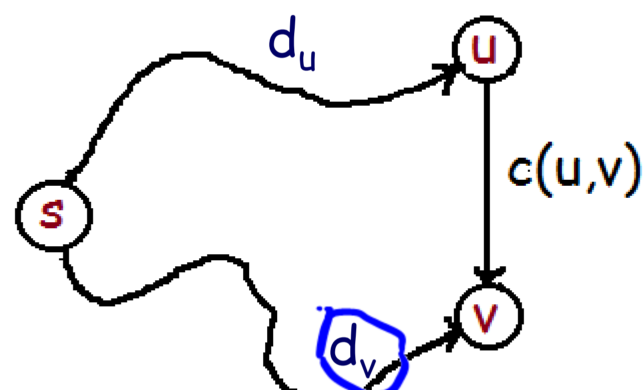
Let $d(v)$ be the shortest path from s to v.

$d(s) = 0$

$d(v) \leq d(u) + c(u,v)$

we use it for writing constraints





$d(1) \le d(s)+5$

$d(4) \le d(s)+2$

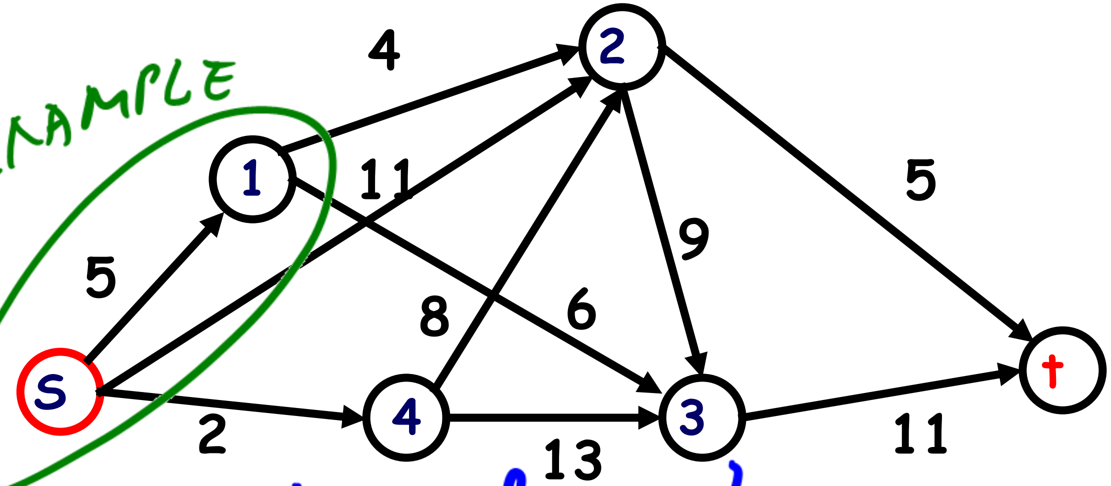
$d(2) \le d(1)+4$

$d(2) \le d(4)+8$

$d(2) \le d(s)+11$

for $\forall e \in E$

EXAMPLE



Objective function:

$\cancel{\min d(t)}$

$\max d(1)$

Example: $t=1$

LP: $\cancel{\min}$ max $d(1)$

$d(1) \le d(s)+5=5$

$d(1) \ge 0$

$\min d(1) = 0$

$\max d(1) = 5$

Discussion Problem 5

Write a 0-1 Knapsack Problem as a linear program.

Given n items with weights $w_1, w_2, \ldots, w_n$ and values $v_1, v_2, \ldots, v_n$. Put these items in a knapsack of capacity W to get the maximum total value in the knapsack.

$$\text{Given } \sum_{k=1}^{m} w_k \le W$$

$$\text{optimize } \sum_{k=1}^{m} v_k \to \max$$

Knapsack $\le_p$ LP

Knapsack as LP

Variables $x_i = \begin{cases} 1, & \text{item } i \text{ is selected} \\ 0 \end{cases}$

Obj. function: $\max\left(\sum_{i=1}^{n} x_i \cdot v_i\right)$

Constraints: $\sum_{i=1}^{n} x_i \cdot w_i \leq W$

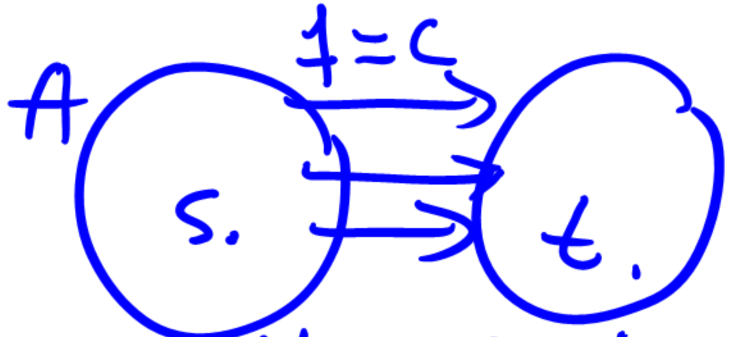
$x_i \in \{1, 0\}$

Knapsack $\leq_p$ ILP

Dual LP

To every linear program there is a dual linear program

$|f| \leq cap(A,B)$



1. max-flow $\leq$ min-cut

2. max-flow $=$ min-cut

Duality

<u>Definition.</u> The dual of the standard (primal) maximum problem

LP

$$\boxed{\begin{array}{l} \max c^T x \\ Ax \le b \text{ and } x \ge 0 \end{array}}$$

primal

is defined to be the standard minimum problem

LP

$$\boxed{\begin{array}{l} \min b^T y \\ A^T y \ge c \text{ and } y \ge 0 \end{array}}$$

dual

Exercise: duality

Consider the LP:

$$\max(7x_1 - x_2 + 5x_3)$$
$$x_1 + x_2 + 4x_3 \le 8$$
$$3x_1 - x_2 + 2x_3 \le 3$$
$$2x_1 + 5x_2 - x_3 \le -7$$
$$x_1, x_2, x_3 \ge 0$$

Write the $\boxed{\text{dual}}$ problem.

primal LP		dual LP
$\max(c^T x)$ $A x \le b$ $x \ge 0$	$\Longleftrightarrow$	$\min(b^T y)$ $A^T y \ge c$ $y \ge 0$

$$x=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},\ c=\begin{pmatrix} 7 \\ -1 \\ 5 \end{pmatrix},\ b=\begin{pmatrix} 8 \\ 3 \\ -7 \end{pmatrix},\ y=\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$A=\begin{pmatrix} 1 & 1 & 4 \\ 3 & -1 & 2 \\ 2 & 5 & -1 \end{pmatrix},\quad A^T=\begin{pmatrix} 1 & 3 & 2 \\ 1 & -1 & 5 \\ 4 & 2 & -1 \end{pmatrix}$$

dual:

$$\min\left(8y_1+3y_2-7y_3\right)$$

$$y_1+3y_2+2y_3 \geq 7$$

$$y_1 - y_2+5y_3 \geq -1$$

$$4y_1+2y_2-y_3 \geq 5$$

From Primal to Dual

$Ax \le b$

$x \in \mathbb{R}^n$

$y \in \mathbb{R}^m$

Consider the max LP constrains

$$
\begin{aligned}
& y_1\,(a_{11}x_1+\ldots+a_{1n}x_n) \le b_1 \cdot y_1 \\
+ & \qquad\vdots \\
& y_m\,(a_{m1}x_1+\ldots+a_{mn}x_n) \le b_m \cdot y_m
\end{aligned}
$$

$$y_1(a_{11}x_1+\ldots+a_{1n}x_n)+\ldots+y_m(a_{m1}x_1+a_{mn}x_n) \le b_1y_1+\ldots+b_my_m$$

1) Multiply each equation by a new variable $y_k \ge 0$.
2) Add up those m equations.
3) Collect terms wrt to x_k.
4) Choose y_k in a way such that $A^T y \ge c$.

$$x_1\underbrace{(y_1 a_{11} + \dots + y_m a_{m1})}_{\geq c_1} + \dots + x_n\underbrace{(y_1 a_{1n} + \dots + y_m a_{mn})}_{\geq c_n} \leq$$

$$\leq b_1 y_1 + \dots + b_m y_m$$

New constraints

$$\left.\begin{array}{l} y_1 a_{11} + \dots + y_m a_{m1} \geq c_1 \\ \dots \\ y_1 a_{1n} + \dots + y_m a_{mn} \geq c_n \end{array}\right\} A^T y \geq c$$

Objective function:

$$x_1 \cdot c_1 + \dots + x_n \cdot c_n \leq b_1 y_1 + \dots + b_m y_m$$

$$\boxed{\underbrace{c^T x}_{\text{primal}} \leq \underbrace{b^T y}_{\text{dual}}}$$

max primal dual min

Weak Duality

$$\max(c^T x)$$
$$Ax \le b$$
$$x \ge 0$$

primal linear program

$$\min(b^T y)$$
$$A^T y \ge c$$
$$y \ge 0$$

dual linear program

Weak Duality. The optimum of the dual is an upper bound to the optimum of the primal.

$$\text{opt(primal)} \le \text{opt(dual)}$$

Weak Duality

$(Ax)^T = x^T \cdot A^T$

$\max(c^T x)$ $Ax \le b$ $x \ge 0$	$\Longleftrightarrow$	$\min(b^T y)$ $A^T y \ge c$ $y \ge 0$

Theorem (The weak duality).
Let P and D be primal and dual LP correspondingly.
If x is a feasible solution for P and y is a feasible solution for D, then $\boxed{c^T x \le b^T y}$.

Proof (in $\boxed{\text{matrix}}$ form).

$$c^T x = x^T c \le x^T (A^T y) = ((Ax))^T y \le b^T y$$

Weak Duality: opt(primal) ≤ opt(dual)

Corollary 1. If a standard problem and its dual are both feasible, then both are feasible bounded.

x is feasible, $\underbrace{c^T x}_{\text{number}} \leq b^T y \rightarrow \min \neq -\infty$

y is feasible, $\overset{+\infty}{\cancel{x}}$ $\underset{\max}{\leftarrow} c^T x \leq \underbrace{b^T y}_{\text{number}}$

Corollary 2. If one problem has an unbounded solution, then the dual of that problem is infeasible.

$\underbrace{c^T x}_{\text{unbounded}} \leq b^T y \Rightarrow b^T y \geq +\infty$

$x = +\infty$

y has no solution

Strong Duality

$\max(c^T x)$ $Ax \le b$ $x \ge 0$	$\Longleftrightarrow$	$\min(b^T y)$ $A^T y \ge c$ $y \ge 0$

Theorem (The strong duality).
Let P and D be primal and dual LP correspondingly.
If P and D are feasible, then $\boxed{c^T x = b^T y.}$

The proof of this theorem is beyond the scope of this course.

ML 567 : SVM

Possibilities for the Feasibility

$$\max(c^T x)$$
$$Ax \le b$$
$$x \ge 0$$

feasible

$$\min(b^T y)$$
$$A^T y \ge c$$
$$y \ge 0$$

P\D	F.B.	F.U.	I.
F.B.	YES cor. 1	NO cor. 1	NO
F.U.	NO cor. 1	NO	YES cor. 2
I.	NO	YES cor. 2	? Yes

example below

feasible bounded - F.B.
feasible unbounded - F.U.
infeasible - I.

$\max(2x_1 + x_2)$

$x_1 - x_2 \geq 4$

$x_1 - x_2 \leq 2$

infeasible

$-x_1 + x_2 \leq -4$

$\min(-4y_1 + 2y_2)$

$-y_1 + y_2 \geq 2$

$y_1 - y_2 \geq 1$

$y_1 - y_2 \leq -2$

infeasible

Discussion Problem 6

Consider the LP:

$$\min(3x_1 + 8x_2 + x_3)$$

$$x_1 + 4x_2 - 2x_3 \leq 20$$

$$x_1 + x_2 + x_3 \geq 7$$

! ! $\boxed{x_2 + x_3 = 3}$

$$x_1, x_2, x_3 \geq 0$$

DIY

Write the dual problem.

Finding the Dual in Equality Form

$$\max\ (c^T x)$$
$$A x = b$$
$$x \geq 0$$

Very often linear programs are encountered in equality form $A x = b$. A problem can be transformed into inequality form by replacing each equation by two inequalities.

$$\begin{array}{ll} y^+ & A x \leq b \\ y^- & -A x \leq -b \end{array}$$

$$a = b \Rightarrow \left\{ \begin{array}{l} a \geq b \\ a \leq b \end{array} \right. \Rightarrow \left\{ \begin{array}{l} -b \leq -a \\ a \leq b \end{array} \right.$$

The dual can then be found by applying the definition of the dual to this problem. Let y^+ and y^- be the dual variables associated with each of the above inequality.

Finding the Dual in Equality Form

$$\max(c^T x)$$
$$A x = b$$
$$x \geq 0$$

$$\max(c^T x)$$
$$A x \leq b$$
$$-A x \leq -b$$
$$x \geq 0$$

$\Longleftrightarrow$

$$\min(b^T y^+ - b^T y^-)$$
$$A^T y^+ - A^T y^- \geq c$$
$$y^+ \geq 0,\ y^- \geq 0$$

$$\min(b^T (y^+ - y^-))$$
$$A^T (y^+ - y^-) \geq c$$
$$y^+ \geq 0,\ y^- \geq 0$$

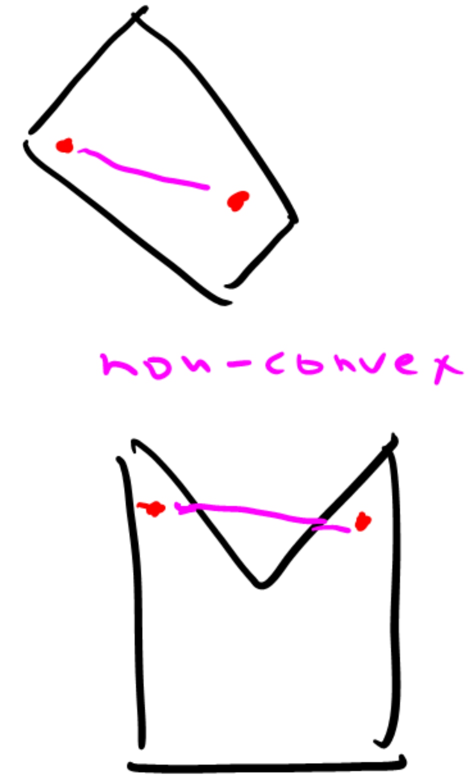
$z = y^+ - y^-$

$$\boxed{\begin{array}{c}\min(b^T z) \\ A^T z \geq c\end{array}}$$

$-\infty < z < +\infty$

CONVEX

Nonlinear Optimization



together

$$\min f(x_1, x_2, \ldots, x_n)$$

$$h_i(x_1, x_2, \ldots, x_n) \leq 0,\ i = 1,, \ldots, m.$$

$$x_k \geq 0,\ k = 1, , \ldots, n.$$

Here f and/or h are nonlinear functions.

The problem is solved using Lagrange multipliers λ_k.

Lagrange Duality (KKT-1951)

Primal in x:

$$\min f(x) \quad \text{subject to} \quad h_k(x) \le 0$$

Dual in λ: $=(\lambda_1, \lambda_2, \ldots, \lambda_n)$

$$\max g(\lambda) \quad \text{subject to} \quad \lambda_k \ge 0$$

The Lagrangian: $L(x,\lambda) = f(x) + \sum_k \lambda_k h_k(x)$

The dual: $g(\lambda) = \min_x L(x,\lambda)$

Weak Duality:
Let P and D be the optimum of primal and dual problems respectively. Then opt(P) $\le$ opt(D).
Equality (strong duality) holds for $\boxed{\text{convex}}$ functions under some conditions.