# Assignment1

All final answers will be given to 5 significant figures (5 s.f.)

## Question 1

### 1a.

Sensitivity:

$$0.98 \cdot 0.92 \cdot 0.93 = 0.83845$$

Specificity:

$$1 - [(1 - 0.91) \cdot (1 - 0.88) \cdot (1 - 0.87)] = 0.99860$$

### 1b.

Sensitivity:

$$1 - [(1 - 0.98) \cdot (1 - 0.92) \cdot (1 - 0.93)] = 0.99989$$

Specificity:

$$0.91 \cdot 0.88 \cdot 0.87 = 0.69670$$

### 1c.

Let:

- sensitivity be $Se$,
- specificity be $Sp$, and
- prevalence, $Pre = 50/1000$

Then, PPV is defined as:

$$PPV = \frac{Se \cdot Pre}{(Se \cdot Pre) + (1 - Sp) \cdot (1 - Pre)}$$

If the tests are combined in a serial manner, $PPV = 0.96917$. Conversely, if the tests are combined in a parallel manner, $PPV = 0.14785$.

## Question 2

Let $N_i$ represent the event that the $i$-th neuron fires. Let $H_i$ represent the event that the $i$-th neuron received a stimulus.

Clearly, if the $i-1$-th neuron fires, the $i$-th neuron will receive a stimulus. Therefore, $N_{i-1} = H_i$.

It is given that neuron 1 is given a stimulus (i.e., $H_1$ has occurred).

## 2a.

Calculating the probabilities of each neuron firing given neuron 1 is stimulated:

$$P(N_1) = 0.9$$

$$\begin{aligned} P(N_2) = P(N_3) &= P(N_2|H_1) \cdot P(H_1) + P(N_2|\neg H_1) \cdot P(\neg H_1) \\ &= 0.9 \cdot 0.9 + 0.05 \cdot 0.1 \\ &= 0.815 \end{aligned}$$

$$\begin{aligned} P(N_4) = P(N_5) &= P(N_4|H_3) \cdot P(H_3) + P(N_4|\neg H_3) \cdot P(\neg H_3) \\ &= 0.9 \cdot 0.815 + 0.05 \cdot 0.185 \\ &= 0.74275 \end{aligned}$$

$$\begin{aligned} P(H_6) = P(N_5 \cup N_6) &= 1 - [(1 - 0.74275) \cdot (1 - 0.74275)] \\ &= 0.93382 \end{aligned}$$

$$\begin{aligned} \therefore P(N_6) &= (0.9) \cdot (0.93382) + (0.05) \cdot (1 - 0.93382) \\ &= 0.84375 \end{aligned}$$

## 2b.

Given neuron 4 did not fire, $P(H_6) = P(N_5) = 0.74275$ as calculated above.

Therefore, $P(N_6|\neg N_4)$ can be calculated as:

$$\begin{aligned} P(N_6|\neg N_4) &= (0.9 \cdot 0.74275) + (0.05 \cdot (1 - 0.74275)) \\ &= 0.68134 \end{aligned}$$

## 2c.

By Bayes Theorem,

$$P(H_5|\neg N_6) = \frac{P(\neg N_6|H_5) \cdot P(H_5)}{P(\neg N_6)}$$

We know the following from (2a.):

- $P(\neg N_6) = 1 - 0.84375 = 0.15325$
- $P(H_5) = P(N_2) = 0.815$

Solving for $P(\neg N_6|H_5)$,

$$P(N_5|H_5) = 0.9$$

$$\begin{aligned} P(H_6|H_5) &= 1 - [(1 - P(N_5|H_5)) \cdot (1 - P(N_4|H_5)] \\ &= 1 - [(1 - 0.9) \cdot (1 - 0.74275)] \\ &= 0.97428 \end{aligned}$$

$$\begin{aligned} P(\neg N_6|H_5) &= 1 - [(0.9 \cdot 0.94728) + (0.05 \cdot (1 - 0.94728))] \\ &= 0.074439 \end{aligned}$$

Finally,

$$P(H_5|N_6) = \frac{0.074439 \cdot 0.815}{0.15325}$$

$$= 0.38827$$

### 2d.

By Bayes Theorem,

$$P(\neg N_2 \neg N_3|N_6) = \frac{P(N_6|\neg N_2 \neg N_3) \cdot P(\neg N_2 \neg N_3)}{P(N_6)}$$

We know the following from (2a.):

- $P(N_6) = 0.84375$
- $P(\neg N_2 \neg N_3) = (1 - 0.815)^2 = 0.034225$

Given $N_2$ and $N_3$, $P(N_4) = P(N_5) = 0.05$.

Solving for $P(N_6|\neg N_2 \neg N_3)$ ,

$$\begin{aligned} P(H_6|\neg N_2 \neg N_3) &= 1 - [(1 - 0.05)^2] \\ &= 0.0975 \end{aligned}$$

$$\begin{aligned} \therefore P(N_6|\neg N_2 \neg N_3) &= (0.9 \cdot 0.0975) + (0.05 \cdot (1 - 0.0975)) \\ &= 0.13288 \end{aligned}$$

Finally,

$$\begin{aligned} P(\neg N_2 \neg N_3|N_6) &= \frac{0.13288 \cdot 0.034225}{0.84375} \\ &= 5.3898 \times 10^{-3} \end{aligned}$$

## Question 3

### 3a.

Let $S$ denote the overall success of the machine.

Let $A$ denote the event that the "component with failure probability of 0.5" works as intended. In other words, $P(A) = P(\neg A) = 0.5$.

Let $p$ be the probability that each of the other three components work.

By Total Probability,

$$P(S) = P(S|A)P(A) + P(S|\neg A)P(\neg A)$$

Given $\neg A$, all other three components must work for $S$ to occur. Therefore,

$$P(S|\neg A) = p^3$$

Given $A$, at least two other components need to work for $S$ to occur. Calculating the inverse, i.e.

$$\begin{aligned} P(\neg S|A) &= 3 \cdot ((1-p)^2 \cdot p) + (1-p)^3 \\ &= 2p^3 - 3p^2 + 1 \end{aligned}$$

Therefore,

$$P(S|A) = 1 - [2p^3 - 3p^2 + 1] = 3p^2 - 2p^3$$

Finally, the probability that the overall machine fails is:

$$\begin{aligned} P(\neg S) &= 1 - P(S) \\ &= 1 - \frac{3p^2 - p^3}{2} \end{aligned}$$

It is given that $p = 0.75$.

$$P(\neg S) = 0.36719$$

### 3b.

By Bayes Theorem,

$$P(\neg A|\neg S) = \frac{P(\neg S|\neg A) \cdot P(\neg A)}{P(\neg S)}$$

From (3a.):

$$P(\neg S) = 1 - \frac{3p^2 - p^3}{2}$$

It is also trivial to show that:

$$P(\neg S|\neg A) = 1 - p^3$$

Therefore,

$$P(\neg A|\neg S) = \frac{1 - p^3}{2 - 3p^2 + p^3}$$

Assuming $p = 0.75$, we get $P(\neg A|\neg S) = 0.78725$.

### 3c.

Let the probability that the test returns a positive result be $P(T)$, and the probability that the test returns a negative result be $P(\neg T)$.

Given the following:

- Specificity, $P(\neg T|\neg A) = 0.9$
- Sensitivity, $P(T|A) = 0.8$

By Bayes Formula and Total Probability,

$$
\begin{aligned}
P(\neg A|\neg T) &= \frac{P(\neg T|\neg A) \cdot P(\neg A)}{P(\neg T)} \\
&= \frac{P(\neg T|\neg A) \cdot P(\neg A)}{P(\neg T|\neg A) \cdot P(\neg A) + P(\neg T|A) \cdot P(A)} \\
&= \frac{0.9 \cdot 0.5}{(0.9 \cdot 0.5) + (0.2 \cdot 0.5)} \\
&= 0.81818
\end{aligned}
$$