3. Normal Probability Distribution

Introduction

In Chapters 1 and 2 we introduced some of the important concepts associated with statistical inference. A sampling experiment was conducted to help with visualising some important concepts of random variation while a Datadesk programme simulated the experiment 10,000 times to illustrate the distribution of possible outcomes and their relative frequency of occurrence. We referred to the distribution as a reference distribution or more formally as a **probability distribution**. We then compared the test statistic (nine satisfied users) with the probability distribution and made a decision to reject the status quo (defined as the null hypothesis) of no change because the test statistic fell into the critical region of the distribution. We also discussed the two risks associated with our decision making process.

The above procedure is, in essence, the basis behind all statistical tests of which there are many. Usability experiments can have different underlying probability distributions that describe the random variation for different experimental designs. However, if the reader understands the general concepts, as stated in the above paragraph, a significant component of the principles that underlie scientific research will have been grasped.

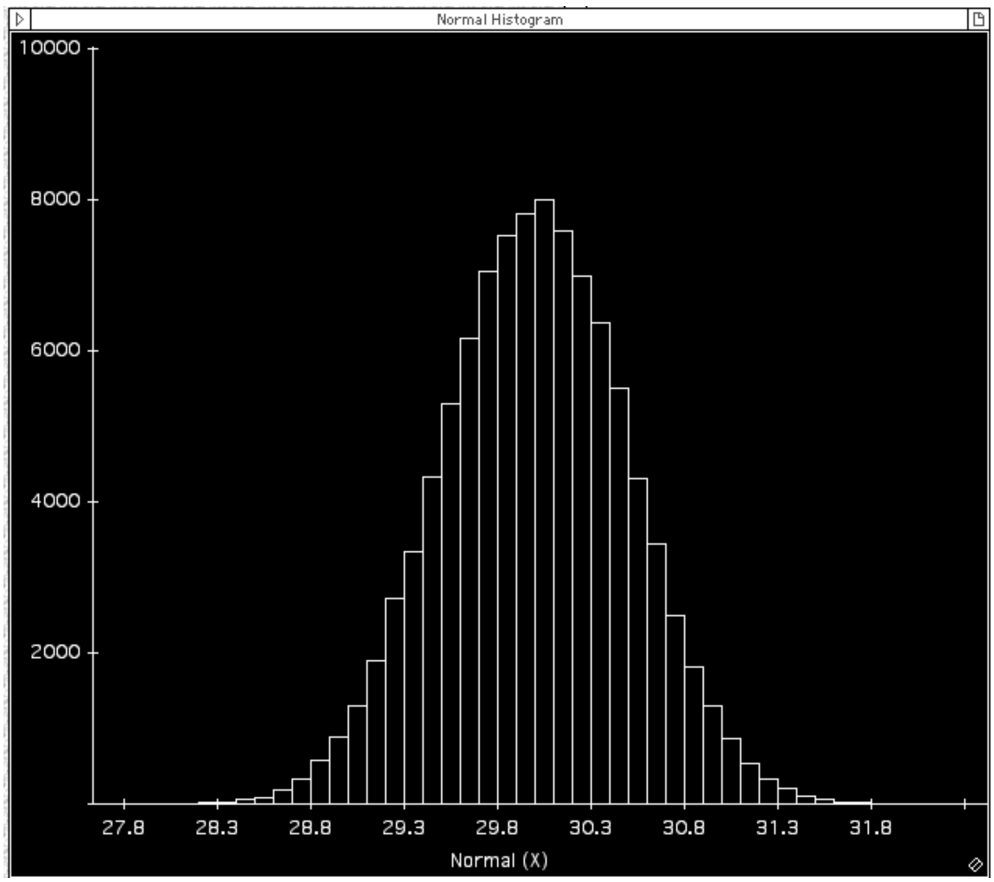
We will now turn ourselves to an important probability distribution known as the normal or bell shaped distribution which explains, in one form or another, the distribution of the vast majority of experimental data.

3.1 The Normal Probability Distribution

The normal distribution is the one of the most important and well known probability distributions. It describes a wide range of human characteristics including weights, heights and IQ. It also describes the pattern of white noise in electronic circuits and countless other processes in medicine, science and engineering. The normal distribution is sometimes called the **bell shaped** distribution as it is symmetric about its mean. Its formula includes the mean (**u**) and standard deviation (**σ**) of the distribution and is usually represented by the compact notation **N**[u, σ^2].

Example 1

The histogram below is is based on the time in seconds required to boot up 100,000 portable computers - all of which have identical configurations. The bell shape of the plot suggests that boot-up times can be explained by a Normal distribution. The average time to boot up, calculated as the addition of all 100,000 times divided by 100,000, is 30 seconds while the standard deviation of the boot up times is 0.5 seconds. Therefore we can describe boot-up times as **N**[30,0.5^2]. As the sample size is substantial we can regard the mean and standard deviation as population parameters. That is **u** = 10 seconds and **σ** = 0.5 seconds.



N(30, 0.5^2) [i.e. u = 30, **σ** = 0.5]

Figure 3.1: Distribution of boot-up times

Note that in the above plot virtually all of the times lie between 28.5 and 31.5 seconds. This is explained by a feature of the normal distribution that over 99.9% of observations lie in the range mean ± 3 standard deviation i.e. 30 ± 3 (0.5) = 30 ± 1.5 seconds.

3.2 Discrete and Continuous Probability Distributions

In the last two chapters we examined the distribution of the number of white beads (i.e. satisfied customers) selected in a class sampling experiment. This type of distribution is known as a **discrete probability distribution** as the outcomes (number of white beads selected) are discrete. That is they take integer values 0, 1, 2, 3... etc and tend to have a smaller range of possible outcomes as we cannot have 1.5 satisfied customers or 3.58 satisfied customers and so on. The distribution was visualised using a bar chart where each bar represented an outcome of interest as shown in Figure 3.2 (left).

However, the normal distribution is known as a **continuous probability distribution** as there are many many possible outcomes. In example 1 we could have a boot-up time of 30.454 seconds or 30.675 seconds etc. Continuous distributions are normally visualised as a smooth curve rather than a bar chart due to the large number of possible outcomes as shown in Figure 3.2 (right).

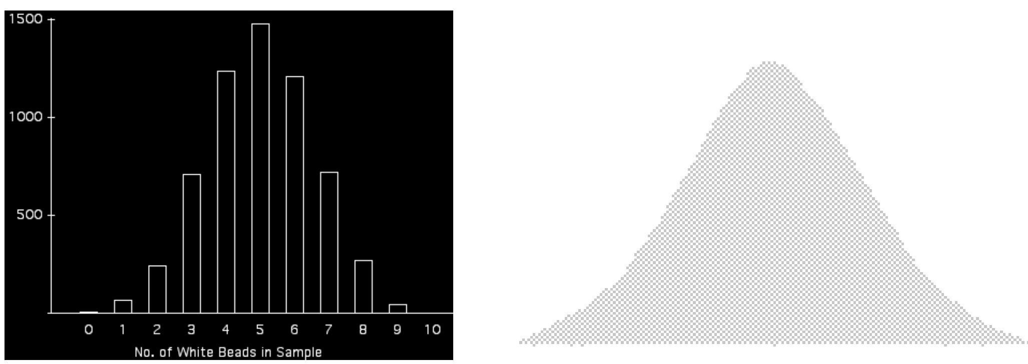


Figure 3.2: Visualisation of discrete (left) and continuous (right) probability distributions

In general, continuous distributions represent quantities that are measurable whereas discrete distributions are based on categories or what is known as categorical data. For example, the time taken for 1,000 users of a particular software application to complete an on-line survey could be described using a continuous probability distribution if we measure the time taken to the nearest millisecond. However, the distribution of returns classified by the gender of the respondent is a discrete distribution as the outcomes male and female are categorical rather than measurable quantities.

Therefore calculating the relative frequency or probability of a particular outcome occurring (e.g. obtaining nine satisfied users) as we did for the class experiment is not possible for continuous distributions. There are just so many possible values that the probability of any exact outcome occurring can be close to zero. Instead we must find the likelihood of an outcome falling in a particular region of the distribution. This involves finding the **area** under the curve of the required region which in turn can be regarded as a probability as the total area under a probability distribution curve is 1. For example, to find the probability of a result falling in the interval [$\mathbf{x}_1$, $\mathbf{x}_2$] as shown below in Figure 3.3 we calculate the area between $\mathbf{x}_1$ and $\mathbf{x}_2$.

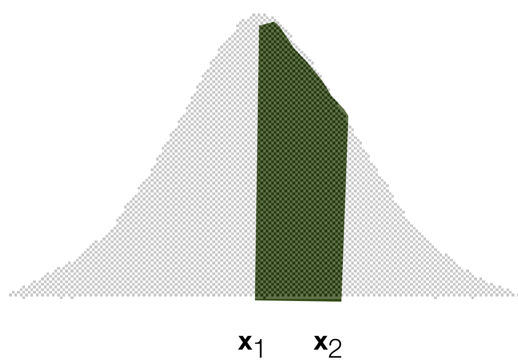


Figure 3.3: Finding the area between $\mathbf{x}_1$ and $\mathbf{x}_2$

That is, calculating the probabilities of outcomes of interest based on continuous distributions implies calculating the area under the distribution curve that encloses the interval as shown in Figure 3.3.

3.3 The Standard Normal (or Z) Distribution

Calculating probabilities by finding the area under the normal distribution curve is difficult as the distribution has a complicated function which is difficult to integrate (integration is a branch of calculus which is used to calculate areas). To overcome this problem the data can be transformed from a normal distribution with mean u and standard deviation $\boldsymbol{\sigma}$ to a normal distribution which has a mean of 0 and a variance of 1. This new transformed distribution is called the **standard normal** or **Z distribution** and is usually represented by the notation $N(0,1^2)$.

Transforming a normal distribution to a standard normal distribution allows us to use special tables known as **standard normal tables** to calculate probabilities. Transforming from a normal distribution with mean u and variance σ to a standard normal distribution with mean 0 and variance 1 involves calculating what is known as a **z score** using the following formula

$$\boxed{z = (x-\mathbf{u})/\sigma}$$

Where **x** is a sample result while **u and** σ are the 'true' mean and standard deviation, respectively.

The plot below visualises a standard normal distribution with u = 0 and σ = 1. Note that as with the general normal distribution virtually all of the data points in the standard normal fall within the limits 0 ± 3σ. Since σ = 1 this means virtually all the observations fall within the interval between z = -3 and +3.

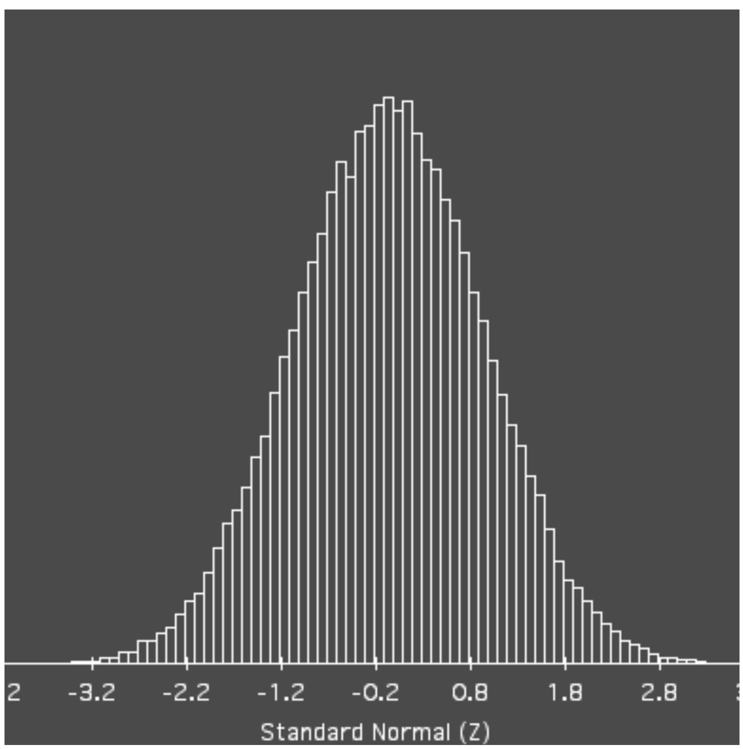


Figure 3.4: Standard normal or Z distribution.

3.4 Special Properties of the Normal Distribution

The standard normal distribution has the following three important properties

i) Slightly more than two observations in 3 (68 per cent) will lie in the interval between $z = 0 \pm 1$

ii) About 95 in 100 (95 per cent) will lie in the interval $z = 0 \pm 2$

iii) 99.97 in 100 (99.97 per cent) of observations will lie in the interval $z = 0 \pm 3$ as shown in Figure 3.5.

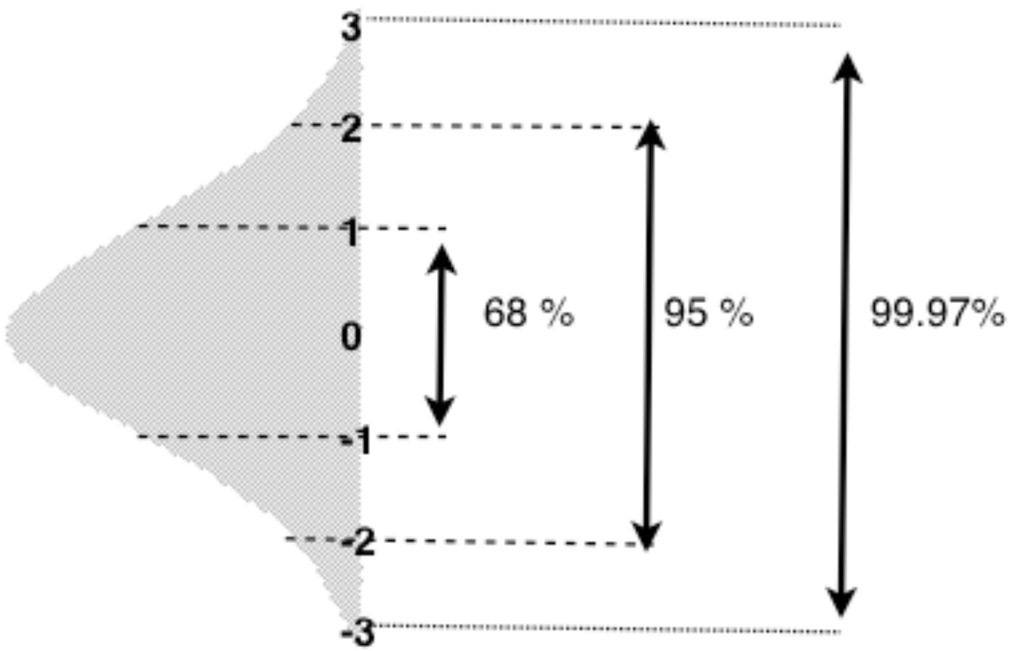


Figure 3.5: Special properties of the standard normal or Z distribution

3.5 Using the Standard Normal Probability Tables (Z tables) to calculate probabilities

An extract of the standard normal tables are shown in Table 3.2. These tables provide the probabilities of outcomes occurring that are **less than or equal** to some value z expressed in statistical notation as $P(Z \leq z)$.

For example, to evaluate the probability of outcomes less than or equal to 1.65 i.e. $P(Z \leq 1.65)$ we consult the z column in Table 3.1 (column 1) and go to 1.6. To get the second decimal place for z i.e. 5 we then go to column 0.05 read the value **0.9505**. This means that 95% of the Z distribution is less than 1.65

z	**0.0**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	**0.9505**	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9700	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	**0.9750**	0.9756	0.9761	0.9767

P(Z ≤ 1.96 = 0.9750)

Table 3.2 Extract from Standard Normal Tables

P(Z ≤ 1.65 = 0.9505)

3.6 Determining Critical Values of the Standard Normal Distribution

In the previous paragraph we saw that 95% of the values of the standard normal distribution are less than 1.65. As the total probability for all outcomes is 1.0 this means that 5% of the distribution is greater than 1.65.

Consider the one-tailed hypothesis (right tail) using the notation outlined in Chapter 2;

H_o: $u = u_o$

H_1: $u > u_o$

where u_o is the pre-existing mean and assume a maximum false positive rate (α) of 5%. The hypothesis will therefore be rejected if a z value greater than 1.65 is obtained. Therefore 1.65 is the critical value and values greater than 1.65 lie in the critical region and lead to rejection of H_o.

Because of the symmetry of the normal distribution this means that for the one-tailed test (left tail) with the same false positive rate (α) of 5% i.e.

H_o: $u = u_o$

H_1: $u < u_o$

H_o will be rejected if a z value less than -1.65 is obtained.

Similarly, if the maximum false positive rate is 10% then the critical values for a one-tailed test are ±1.28. If the false positive rate is 1% then the critical values are ±1.28. The reader should confirm these figures using the standard normal tables. The critical values for 10%, 5% and 1% are shown in Table 1 for one and two-tailed tests. Please note that for two-tailed tests the false positive error α is divided equally between the two tails of the distribution.

α	One-Tailed	Two-Tailed
10%	1.28	1.65
5%	1.65	1.96
1%	2.33	2.57

Table 3.1 Critical z values for one and two-tailed tests

Exercises

If Z is a standard normal random variable, calculate using the standard normal tables;

i) $P(Z \leq 0.01)$

ii) $P(Z > -2.34)$

iii) $P(Z \leq -3.03)$

iv) $P(Z > 1.96)$

v) $P(-2.5 \leq Z \leq 2.0)$

vi) $P(-2.0 \leq Z \leq -1.0)$

vii) $P(1.54 \leq Z \leq 2.23)$

3.7 Worked Example

In example 1 boot-up times for a particular brand of portable computer was described as normal with a mean of 30 and a standard deviation of 0.5 i.e. $\mathbf{N}[30, 0.5^2]$. After installing the new software on **one** computer and powering up the time taken to boot-up was recorded as 28.7 seconds. The design team want to establish if the modification has reduced the boot up time significantly or is the result just reflective of random variation. The maximum risk (type 1 error or α) they are willing to accept is 5% and because the engineers believe that the new code should reduce the time to boot-up it is a one-tailed test.

Step 1: Specify the Hypothesis

The first step is to formally specify the null hypothesis which is that the mean time ($\mathbf{u_0}$) is 30 seconds. This is expressed as:

H_o: $\mathbf{u_0}$ = 30 seconds

The software team feel that the new modification, which requires less code, should reduce the boot up time. The alternative hypothesis is then

H_1: $\mathbf{u_0}$ < 30 seconds

The test is therefore one-tailed as it is focussing on one tail (i.e. the left) of the normal distribution.

Step 2: Calculate the test statistic

The test statistic $\mathbf{z}$ is calculated as:

$$\mathbf{z} = \frac{\text{sample data - } \mathbf{u}}{\text{standard deviation } (\sigma)} = \frac{28.7 - 30}{0.5} = -2.6$$

Step 3: Making the decision

The team have decided to set the maximum type 1 error (α) to 5%. The test statistic z is calculated as -2.6 which is well in excess of the critical value of -1.65 so we can reject the hypothesis of no change and conclude that the modification has significantly improved boot-up times. The exact area to the left of -2.6 using the standard normal tables is 0.0047 or 0.4% which in terms of significance testing can be regarded as the p-value associated with this test.

Exercises

1. Repeat worked example 1 assuming a boot-up time of 29.1 seconds and a maximum false positive rates of a) 10% and b) 1%.

2. Repeat exercise 1 assuming that the software development team have no prior information on the impact of the modification i.e the test is two-tailed.

3.8 Worked Example using Normal Distribution Simulation Software

In this section we will use the application *Normal Distribution* to illustrate the workings of example 2. A screen shot of the application is shown in Figure 3.7 where the normal and standard normal plots are shown at the top of the plot. The software allows the user to input the following quantities:

i) Mean (**u**) and standard deviation (σ) of the reference distribution using the sliders called *Mean* and *Standard Deviation*.

ii) Number of simulations in the field *Number of Simulations*

iii) The sample result in the field X

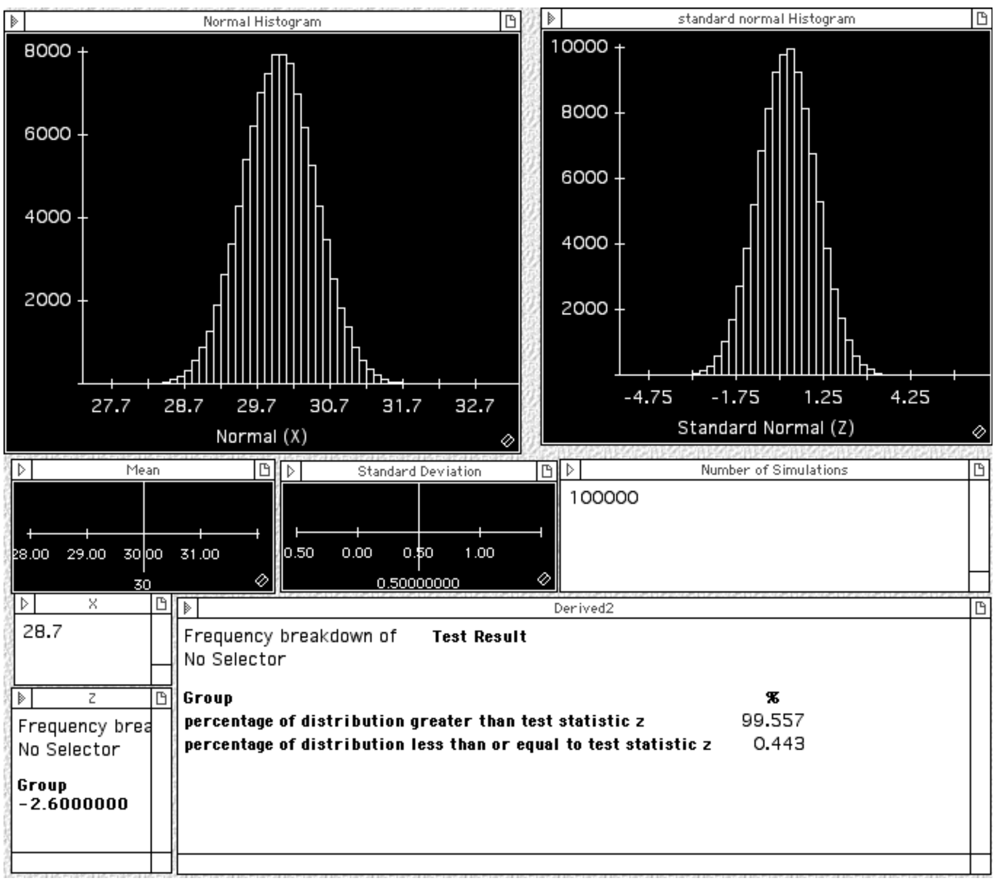


Figure 3.7: Screen shot of application *Normal Distribution*

The application converts the sample data (x) to a **z** value using the formula $\mathbf{z} = (x-\mathbf{u})/\sigma$ and puts the result in the field entitled *z*. The percentage of the **z** (i.e. standard normal distribution) greater than or less than or equal to the calculated z value is placed in the field *Test Result*.

The worked example had values of **u** = 30 and σ = 0.5. The normal and standard normal distributions for these two parameters are simulated 100,000 times and shown in the screenshot in Figure 3.7.

The sample data (i.e. x) of 28.7 seconds is equivalent to a **z** value of -2.6 The percentage of the simulated standard normal distribution less than -2.6 is 0.443%. This is virtually the same as calculated from the z tables and can be regarded as the p-value associated with the test. Therefore we can conclude that the modification has significantly reduced boot up time as the test statistic falls well into the critical region of -1.65 as defined by our acceptable false positive rate of 5%.

Exercise

In the previous example assume the false positive rate is set to 1% rather than 5%. Test the hypothesis assuming i) one-tailed and ii) two-tailed test and describe briefly your conclusions

3.9 Type 2 error

The type 2 error discussed in the previous two chapters is the risk of concluding no change in the mean of the distribution when there has been a change. The error depends on the true but generally unknown alternative value of the true mean, **u** (or what we have called the **parameter**) and has to be computed for each possible alternative value of **u**. The lower the type 2 error the greater the sensitivity of the test in detecting a true change in **u**. A plot of the type 2 error for a range of values for **u** is called the **operating characteristic** (OC) curve while a plot of (1-**β**) is known as the **power curve** associated with the test. The power curve illustrates the probability of correctly rejecting Ho based on our decision rule.

The current version of the application *Normal Distribution* does not currently compute the power or OC curves but the procedure is the same as outlined in the previous chapter i.e.

- Set the false positive rate upfront to calculate the critical value.
- Calculate the probabilities of outcomes occurring less than this critical value for a range of alternative means.

Exercise

1. The boot-up times for a particular brand of portable computer was described as normal with a mean of 20 and a standard deviation of 0.25 i.e. **N**[20,0.25^2]. Using the application *Normal Distribution* calculate the percentage of the z distribution less than

i) 19.5. seconds

ii) 19.1 seconds

iii) If your false positive error rate is set to 1% state whether or not you would reject the hypothesis of no change from the results recorded in i) and ii).

2. If Z is a standard normal random variable, calculate using the standard normal tables;

i) $P(Z > 0.01)$

ii) $P(Z \leq -2.34)$

iii) $P(Z > -3.03)$

iv) $P(-3 \leq Z \leq 3)$

3. The boot-up times of Mac OS X on MacBook Pro 2.66 GHz machines are described by Apple Inc. as Normally distributed with a mean of 30 seconds and a standard deviation of 2 seconds i.e. it is described as N[30,2^2]

In an attempt to reduce the time taken to boot-up the systems software development team have developed a new routine for which it is hoped will decrease boot-up times. After installing the new software on **one** prototype computer and powering up the time taken to boot-up was recorded as 25 seconds. The design team want to establish if the modification has reduce the boot up time significantly or is the result just reflective of random variation.

Using the standard normal tables test the hypothesis that the new code has no impact on the boot up times using maximum risk (type 1 errors) of:

i) 10% ii) 5% iii) 1%

State clearly your conclusions for each of the three tests.

4. Using the experimental set-up in Question 3 above use the application *Normal Distribution* to test the hypothesis of no change for each of the three error rates.

Standard Normal Tables: $P(Z \leq z)$

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.006	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.008	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0227	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0300	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.063	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.166	0.1635	0.1611
-0.8	0.2119	0.209	0.2061	0.2033	0.2005	0.1977	0.1949	0.1921	0.1894	0.1867
-0.7	0.242	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.305	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.281	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.352	0.3483
-0.2	0.4207	0.4168	0.4129	0.409	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
-0.0	0.5000	0.4960	04920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8079	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9700	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9773	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990