TRANSFORMING ASYMMETRIC INTO SYMMETRIC TRAVELING SALESMAN PROBLEMS

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We describe how to transform an asymmetric traveling salesman problem into a symmetric one at the cost of almost doubled problem size. Use and consequences are discussed shortly.

Traveling salesman problem

1. Introduction

A transformation of an asymmetric traveling salesman problem (TSP) into a symmetric one can be useful in two ways. First it can be used to solve the occasional asymmetric TSP with only an algorithm for the symmetric TSP available, say some version of Held and Karp's method [3]. Second, and more important, one can solve partly (or nearly) symmetric TSP's without having to rely on algorithms specially developed for the asymmetric TSP, such as Balas and Christofides' algorithm [1].

Karp [4] gave a transformation, pointed out by Tarjan, that transforms a directed hamiltonian circuit problem with n nodes into an undirected hamiltonian circuit problem with $3n$ nodes. We will show that an asymmetric TSP with n nodes is equivalent to a symmetric TSP with at most $2n$ nodes.

In Section 2 we discuss the transformation of an asymmetric TSP and in Section 3 of a partially symmetric TSP. Some remarks on use and consequences constitute the final section.

2. Transforming the asymmetric TSP

TSP(C) denotes a TSP on weight matrix $C = ((c_{ij}))$, $i, j \in N$ with N the node set $\{1, 2, \ldots, n\}$. Let the weight matrix $\bar{C}$ be equal to C except for $\bar{c}_{ii} = -M$ $(i \in N)$ where M is a very large number, and let $U = ((u_{ij}))$ with $u_{ij} = \infty$ for $i, j \in N$.

We propose to transform the asymmetric TSP(C) into a symmetric TSP on the weight matrix

$$\tilde{C} = \begin{bmatrix} U & \bar{C}' \\ \bar{C} & U \end{bmatrix}$$

which we call TSP($\tilde{C}$). The node set of TSP($\tilde{C}$) is $\{1, 2, \ldots, n, n+1, \ldots, 2n\}$.

The optimal solutions of TSP($\tilde{C}$) belong to the class of solutions that have finite value and contain n edges of weight $-M$. With $\tilde{C}$ symmetric the solutions occur in pairs. It is easily seen that one solution of each pair has the form

$$i_1 \to (i_1 + n) \to i_2 \to (i_2 + n) \to \cdots \to i_n \to (i_n + n) \to i_1,$$

with $i_k \in N$ for $k = 1, 2, \ldots, n$. Of each pair we will consider only this solution.

Clearly there is a one-to-one correspondence between the set of these TSP($\tilde{C}$) solutions and the solutions of TSP(C). One must delete the nodes with index greater than n from the TSP($\tilde{C}$) solution and add $n \cdot M$ to its value in order to obtain the corresponding solution for TSP(C) and its value.

This completes the proof of the following theorem:

Theorem. *Any optimal solution of* TSP($\tilde{C}$) *corresponds to an optimal solution of* TSP(C).

The theorem implies that we can solve TSP($\tilde{C}$) instead of TSP(C).

3. Transforming the partially asymmetric TSP

In the transformation from C to $\tilde{C}$ every node is replaced by two nodes, one of which serves as ingoing part and the other as outgoing part. We can use the same approach to transform partially asymmetric TSP's into symmetric ones by doubling only nodes with rows and columns that cover the asymmetric elements of C.

We illustrate the method on a TSP where the asymmetric weights in C occur in the rows and columns of nodes 1 and 2, i.e. with $c_{ij} = c_{ji}$ for i, $j = 3,\ldots,n$. The method is easily generalized for other partially asymmetric TSP's.

Let the weight matrix C be

$$\begin{bmatrix} \infty & c_{12} & r_1 \\ c_{21} & \infty & r_2 \\ k_1 & k_2 & S \end{bmatrix}$$

where

$$r_1 = (c_{13},\ldots,c_{1n}), \quad r_2 = (c_{23},\ldots,c_{2n}),$$
$$k_1 = (c_{31},\ldots,c_{n1})', \quad k_2 = (c_{32},\ldots,c_{n2})'$$

and S the symmetric matrix of internode weights on the nodes $3,\ldots,n$.

The symmetric TSP($\tilde{C}$) that can be solved instead of TSP(C) has weight matrix

$$\tilde{C} = \begin{bmatrix} \infty & \infty & k_1' & -M & c_{21} \\ \infty & \infty & k_2' & c_{12} & -M \\ k_1 & k_2 & S & r_1' & r_2' \\ -M & c_{12} & r_1 & \infty & \infty \\ c_{21} & -M & r_2 & \infty & \infty \end{bmatrix}$$

An optimal solution of TSP($\tilde{C}$) will be of type

$$i_1 \to i_2 \to \cdots \to 1 \to (1+n)$$
$$\to \cdots \to 2 \to (2+n) \to \cdots \to i_n \to i_1.$$

By omitting the nodes with indices $(1+n)$ and $(2+n)$, and adding $2M$ to the value of the solution, one obtains an optimal solution for TSP(C).

For an arbitrary, partially asymmetric C we may consider the problem of minimizing the size of TSP($\tilde{C}$): determine the node set of minimum cardinality so that the corresponding rows and columns cover all asymmetries in C. It is a nice application of the (difficult) node cover problem.

4. Remarks

The symmetric TSP has at most the computational complexity of the asymmetric TSP, as symmetry is a special case of asymmetry. The transformation of Section 2 shows again that the opposite is also true: the asymmetric TSP is as complex as the symmetric TSP.

We consider the transformation not only interesting but also useful, especially for TSP's with only few asymmetries in the weight matrix. Most algorithms for the asymmetric TSP are based on the assignment relaxation, e.g. the method of Balas and Christofides [1]. This relaxation behaves rather poorly on symmetric problems and probably also on almost symmetric problems. It is clearly very impractical to solve the transformed problem with an assignment based algorithm, but one that is based on e.g. the 1-tree relaxation should perform fine.

The transformation can also be used on the minimal spanning tree problem. Here it is equivalent to solving the problem on a symmetric weight matrix D with $d_{ij} = d_{ji} = \min\{c_{ij}, c_{ji}\}$. Bazaraa and Goode [2] have given computational results for lower bounds of this type for the (asymmetric) TSP.

An asymmetric TSP can always be made symmetric for one of its nodes by appropriately reducing its rows or columns. This reduces the size of its symmetric transformation to $2n$-1.

Symmetric TSP-algorithms based on the 1-tree relaxation (see [3]) can handle problems with asymmetry for one node by choosing this node as the special node when constructing the 1-trees. This observation reduces the size of the symmetric transformation of an asymmetric TSP to $2n-2$ if a 1-tree based algorithm is used.

The size can also be reduced in other cases. For instance the multiple TSP (with or without fixed charges) can be solved as a TSP of size $n+m-1$, with m the number of salesman. The symmetric form of an asymmetric multiple TSP has size $2(n+m-2)-1$ by combining the two transformations.

Clearly the transformation disturbs the Euclidean property of the original weight matrix, if present. Furthermore the underlying problem remains asymmetric in spite of the symmetry of the weight matrix. As an illustration: each TSP($\tilde{C}$) solution of finite value of 2-optimal in the sense of Lin [5], so in this context the 2-optimality heuristic is useless.

In general the transformation is only of small practical interest as it almost doubles the problem size. Lacking a good algorithm for the asymmetric TSP we solved some small asymmetric problems with our 1-tree based algorithm for the symmetric TSP (Volgenant and Jonker [6]). On the average it took 2.8 seconds (on a CDC Cyber 750) to solve an asymmetric 30-cities problem, whereas a symmetric 30-cities problem is solved in 0.5 seconds and a symmetric 60-cities problem in 4.0 seconds.

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Our student Kees Jongens posed a question, this paper is an answer.

References

[1] E. Balas and N. Christofides, "A restricted Lagrangean approach to the traveling salesman problem", *Math. Programming* **21**, 19–46 (1981).

[2] M.S. Bazaraa and J.J. Goode, "The traveling salesman problem: A duality approach", *Math. Programming* **13**, 221–237 (1977).

[3] M. Held and R.M. Karp, "The traveling-salesman problem and minimum spanning trees", *Math. Programming* **1**, 6–26 (1971).

[4] R.M. Karp, "Reducibility among combinatorial problems", in: R.E. Miller and J.W. Thatcher, eds., *Complexity of Computer Computations*, New York, 1972.

[5] Lin, S., "Computer solutions of the traveling salesman problem", *Bell Syst. Techn. J.* **44**, 2245–2269 (1965).

[6] A. Volgenant and R. Jonker, "The symmetric traveling salesman problem and edge exchanges in minimal 1-trees", *European J. Oper. Res.* **12**, 394–403 (1983).