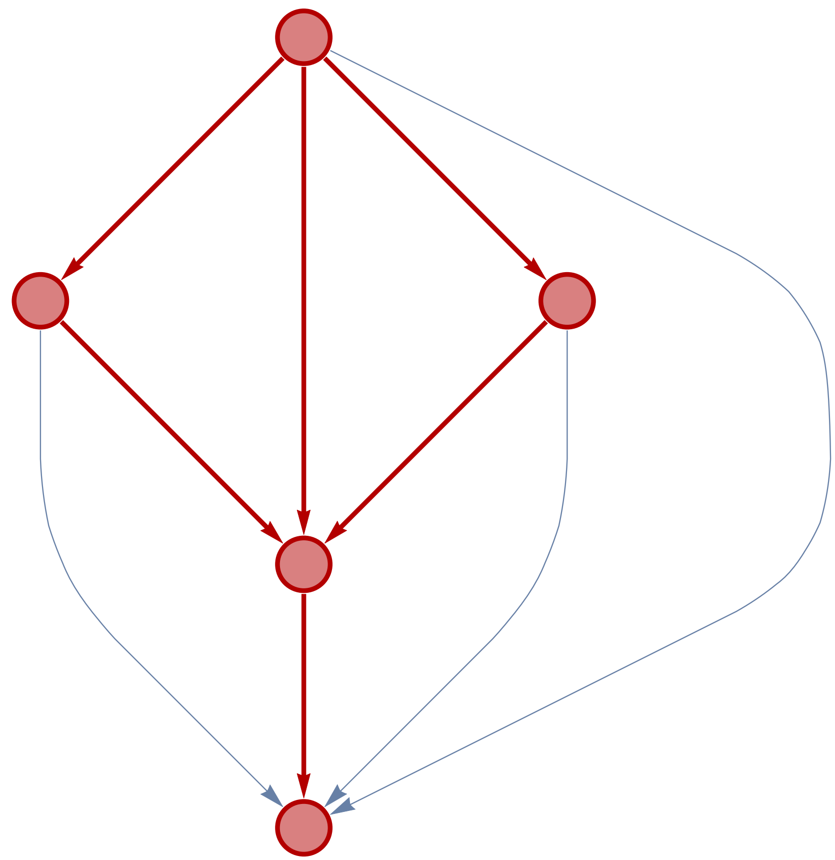
Partial Order and Its Digraph Representation

- The digraph representation of a partial order[a] must be acyclic.[b]
- Any acyclic digraph entails a partial order.
  - Take the transitive closure of the digraph.
  - The resulting digraph clearly remains acyclic.
  - Add a loop to every node.
  - It is not hard to check that the digraph's associated relation satisfies the definition of partial order.

---

[a] Recall p. 370.
[b] Recall p. 375.

Transitive Closure of a Digraph



Diameter

- Let $G(V, E)$ be an undirected graph.
- The **distance** between nodes $x, y \in V$ (or $d(x, y)$) is the minimum length of all the paths between x and y.
- The **diameter** $d(G)$ of G is the maximum distance over all pairs of nodes of G.
  - So the distance between any two nodes is at most $d(G)$.
- Diameter can be computed by an efficient all-pair-shortest-paths algorithm.[a]

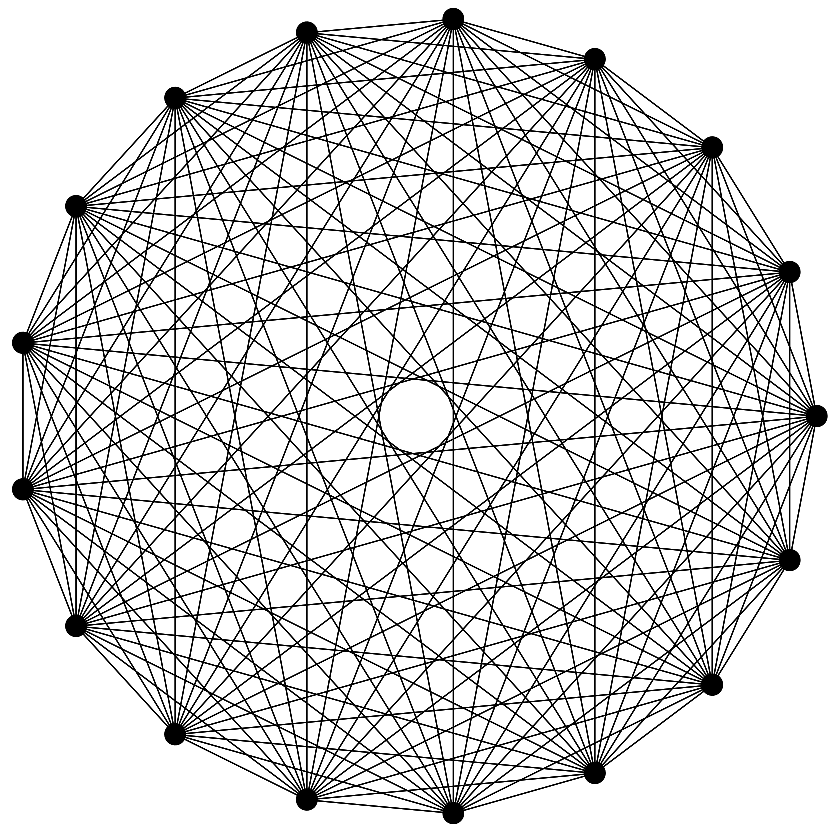
[a] Roy (1959); Floyd (1962); Warshall (1962).

Complete Graphs

- Let V be a set of n nodes.
- The **complete graph** on V, denoted K_n, is a loop-free[a] undirected graph.
  - There is an edge between any pair of distinct nodes.
  - K_n has $\binom{n}{2}$ edges.
- The diameter of K_n is clearly one.

[a]Depending on applications, sometimes (self-)loops are allowed.

K_{17}



Complete Graphs (concluded)

- There are $\binom{n}{i}$ ways to pick i nodes from K_n.[a]
- As there are $\binom{i}{2}$ pairs of nodes, there are $2^{\binom{i}{2}}$ ways to pick the edges.
- Hence K_n has
$$\sum_{i=1}^{n} \binom{n}{i} 2^{\binom{i}{2}}$$
subgraphs.
- Can you simplify it?

[a] K_n is a labeled graph.

An Inequality Relating $|V|$ and $|E|$

Lemma 70 *Let $G = (V, E)$ be a simple undirected graph. Then* $|V| \geq \frac{1+\sqrt{1+8\times|E|}}{2}$.

- G has at most $\binom{|V|}{2}$ edges (the complete graph).
- So V must be big enough such that $\binom{|V|}{2} \geq |E|$.
- This results in $|V|^2 - |V| \geq 2 \times |E|$, or

$$\left(|V| - \frac{1}{2}\right)^2 \geq \frac{1}{4} + 2 \times |E| \geq \frac{1+8\times|E|}{4}.$$

Complements

- The **complement** of graph G, denoted $\overline{G}$, is the subgraph of K_n consisting of the nodes in G and all edges that are *not* in G.
- $\overline{K_n}$, consisting of n nodes and no edges, is called a **null graph**.

Degrees

- Let $G = (V, E)$ be an undirected graph.
- For each node $v \in G$, the **degree** of v, or $\deg(v)$, is the number of edges in G that are incident with v.
- A self-loop contributes *two* incident edges.

A Useful Identity

Lemma 71 (The handshaking theorem)

$$\sum_{v \in V} \deg(v) = 2 \times |\, E \,|. \tag{102}$$

- An edge is counted twice, once at each end.

Corollary 72 *For finite graphs, the number of nodes of odd degree must be even.*

Existence of Nodes with Identical Degree

- Let $G = (V, E)$ be a simple undirected graph without isolated nodes.
- Let $n \stackrel{\Delta}{=} |\, V \,| \geq 2$.
- Observe that $1 \leq \deg(v) \leq n - 1$.
- But there are n nodes.
- By the pigeonhole principle (p. 307), there must be 2 nodes with the same degree.

Regular Graphs

- A d**-regular graph** is an undirected graph such that every node has degree d.
- An d-regular graph $G = (V, E)$ must have an even number of nodes if d is odd.
  - By Eq. (102) on p. 662,
  $$2 \times |\, E \,| = d \times |\, V \,|.$$
  - As d is odd, $|\, V \,|$ must be even.

The Hypercube

- The nodes of the n-dimensional **hypercube** Q_n are represented as n-bit numbers.[a]
  - There are 2^n nodes.
- Two nodes are connected if they differ in one dimension.
  - For example, there is an edge between 00100 and 00110.
- The diameter is n.
- It is n-regular.

---

[a] Recall p. 596.

The Hypercube (concluded)

- There are
$$\frac{n2^n}{2} = n2^{n-1}$$
undirected edges.

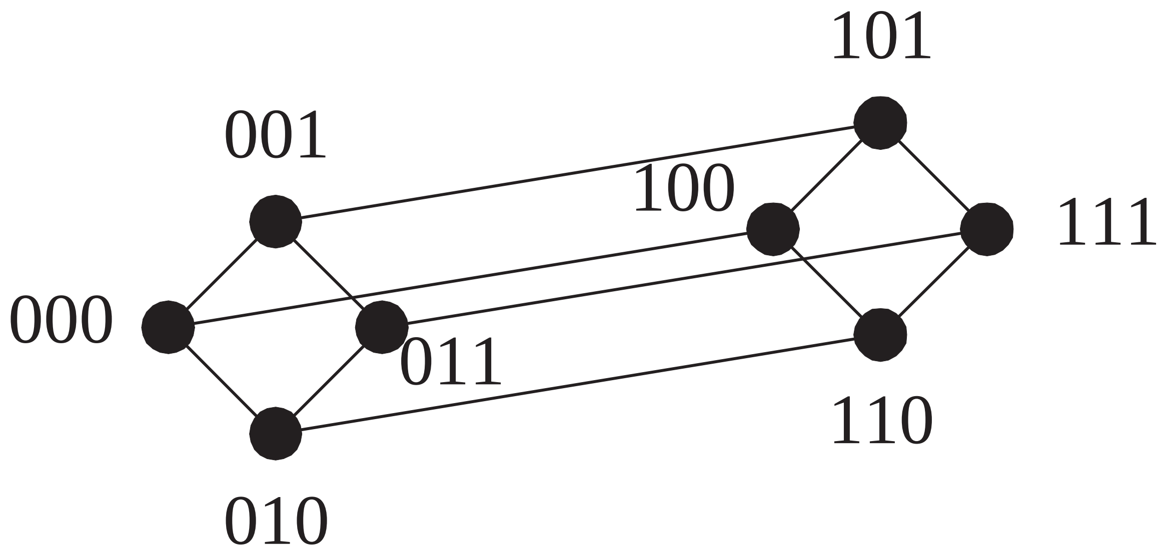
- The hypercube was once a popular topology for **massively parallel processors** (**MPPs**).

- The record is $n = 16$ set by Thinking Machine Corp.'s Connection Machine CM-2.[a]

---

[a] Hillis (1985).

Illustration with Q_3



Bipartite Graphs

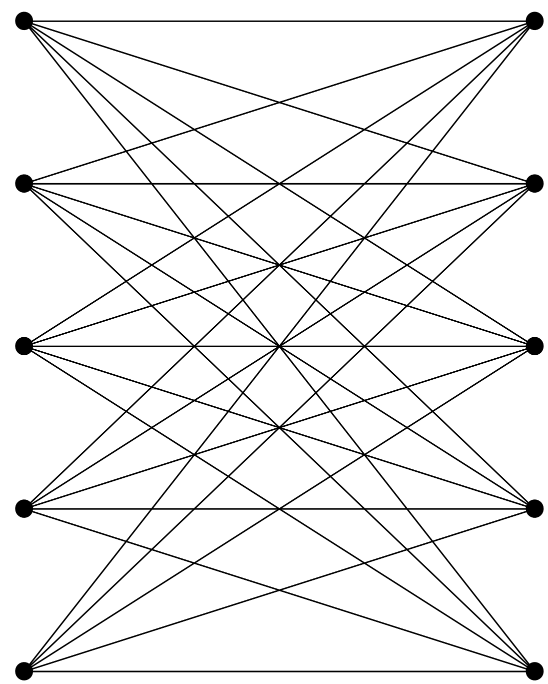
- A graph $G = (V, E)$ is called **bipartite** if:
  - $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$.
  - Every edge is of the form $\{ x, y \}$ with $x \in V_1$ and $y \in V_2$.
- Express the above bipartite graph as

$$G = (V_1, V_2, E).$$

Bipartite Graphs (continued)

- If each node in V_1 is joined with every node in V_2, we have a **complete bipartite graph**.
- If $|V_1| = m$ and $|V_2| = n$, the complete bipartite graph is denoted by $K_{m,n}$.

$K_{5,5}$



Bipartite Graphs (concluded)

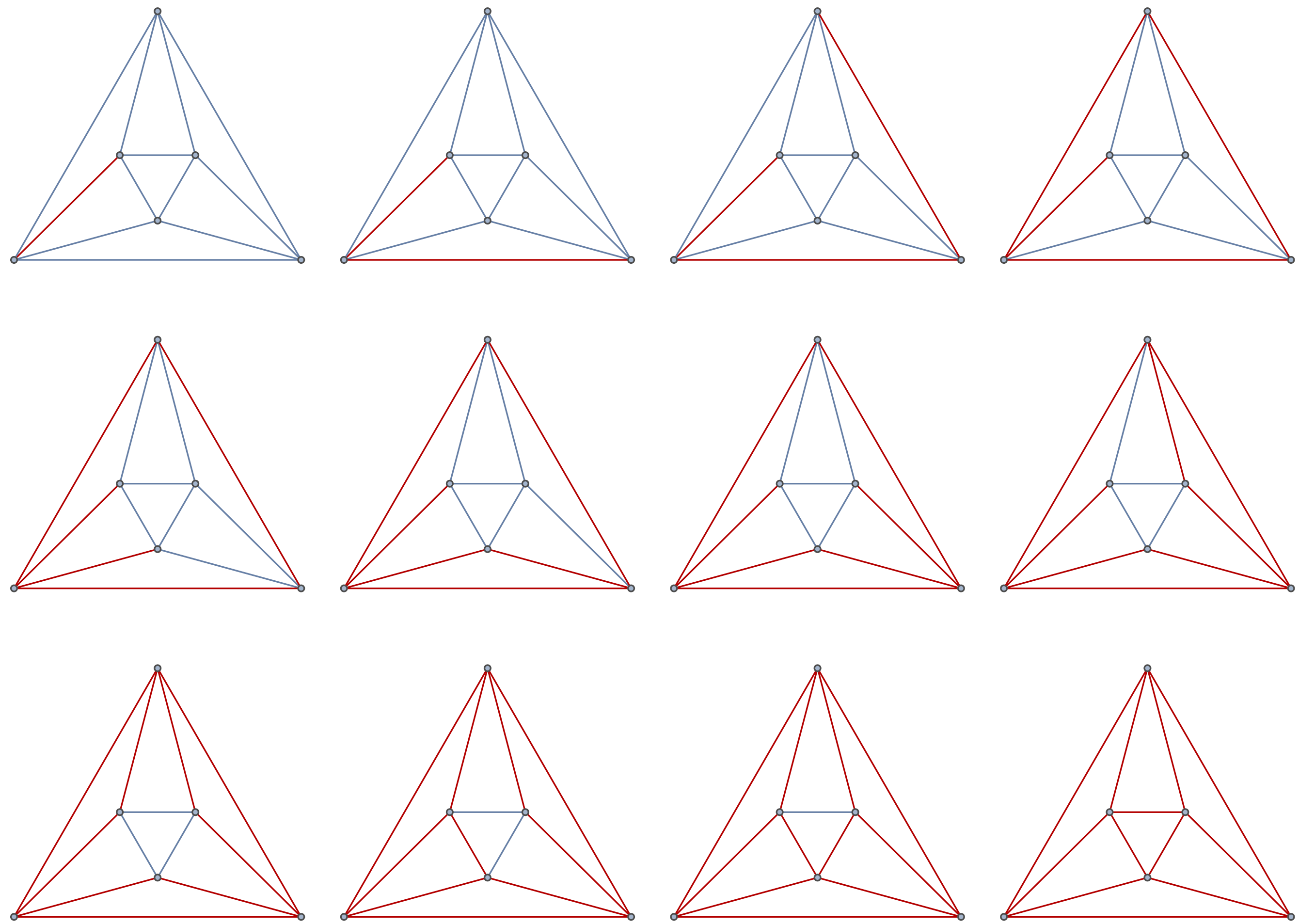
- Let graph $G = (V, E) = (V_1, V_2, E)$ be bipartite.
- Then G has at most $|\, V_1 \,| \times |\, V_2 \,|$ edges.
- Let $|\, V \,| = n$, $|\, E \,| = e$, and $|\, V_1 \,| = m$.
- Then $e \leq (n - m)\, m$, which is maximized at (1) $m = n/2$ when n is even and (2) $m = (n \pm 1)/2$ when n is odd.
- In either case,
$$e \leq (n/2)^2.$$
- Hence a graph with $e > (n/2)^2$ cannot be bipartite.

Euler Circuits and Trails[a]

- Let $G = (V, E)$ be an undirected graph or multigraph with no isolated nodes.
- G is said to have an **Euler circuit** if there is a circuit in G that traverses *every edge* of the graph exactly once.
  - You can draw the edges without lifting the pen.
- If there is an open trail from x to y in G and this trail traverses every edge of the graph exactly once, the trail is called an **Euler trail**.

[a]Euler in 1736, the year graph theory was born.

Euler Circuits



Characterization of Having Euler Circuits

Theorem 73 (Euler, 1736) *Let $G = (V, E)$ be an undirected graph or multigraph. Then G has an Euler circuit if and only if G is connected and every node in G has an even degree.*

- Testing if a graph is Eulerian hence is trivial.
- The proof will be constructive.
- Let $e \stackrel{\Delta}{=} |\, E \,|$.

The Proof ($\Rightarrow$)

- Clearly G is connected.
- Each time the Euler circuit enters a *non-starting* node v, it must exit it before coming back again, if ever.
- This contributes a count of 2 to $\deg(v)$.
- Because every edge is traversed, $\deg(v)$ must be even.
- The Euler circuit must start from the starting node s and end at the same starting node.
- Each exit is matched by one entry.
- So $\deg(s)$ is even, too.

The Proof ($\Leftarrow$)

- The $e = 1, 2$ cases are easy, by inspection.
- Assume the result is true when there are $< e$ edges.
- If G has e edges, select a node $s \in G$ as the starting and ending node.
- Construct a circuit C from s.
  - Start from s.
  - Traverse any hitherto untraversed edge, and repeat.
  - We must eventually return to s.
    * This is because every node has an even degree and hence the last visit to it must be an exit, except s.

The Proof ($\Leftarrow$) (continued)

- If C traverses every edge, we are done.
- Otherwise, remove the edges of C and isolated nodes to yield a new graph H.
- The degree of each node in H remains even.
  - This is key to induction.

The Proof ($\Leftarrow$) (continued)[a]

- Suppose H is connected and s is not isolated.
  - Construct an Euler circuit c of H by the induction hypothesis.
  - Node s is on this Euler circuit because $s \in H$ and H is connected.
  - The desired Euler circuit: Start from s and travel on C until we end at s and then traverse c until we end at s again.

[a]With input from Mr. Cheng-Yu Lee (`B91902103`) on December 1, 2003.

The Proof ($\Leftarrow$) (continued)

- Suppose H is disconnected or s is isolated.
  - Construct an Euler circuit c_i in each component of H by the induction hypothesis.
  - Each component must have at least one node in common with C because originally G is connected.
  - Let s_i be the *first* node with which C visits c_i when starting from s.[a]
  - Relabel $c_1, c_2, \ldots$s so that s_i's first visit occurs before s_{i+1}'s.

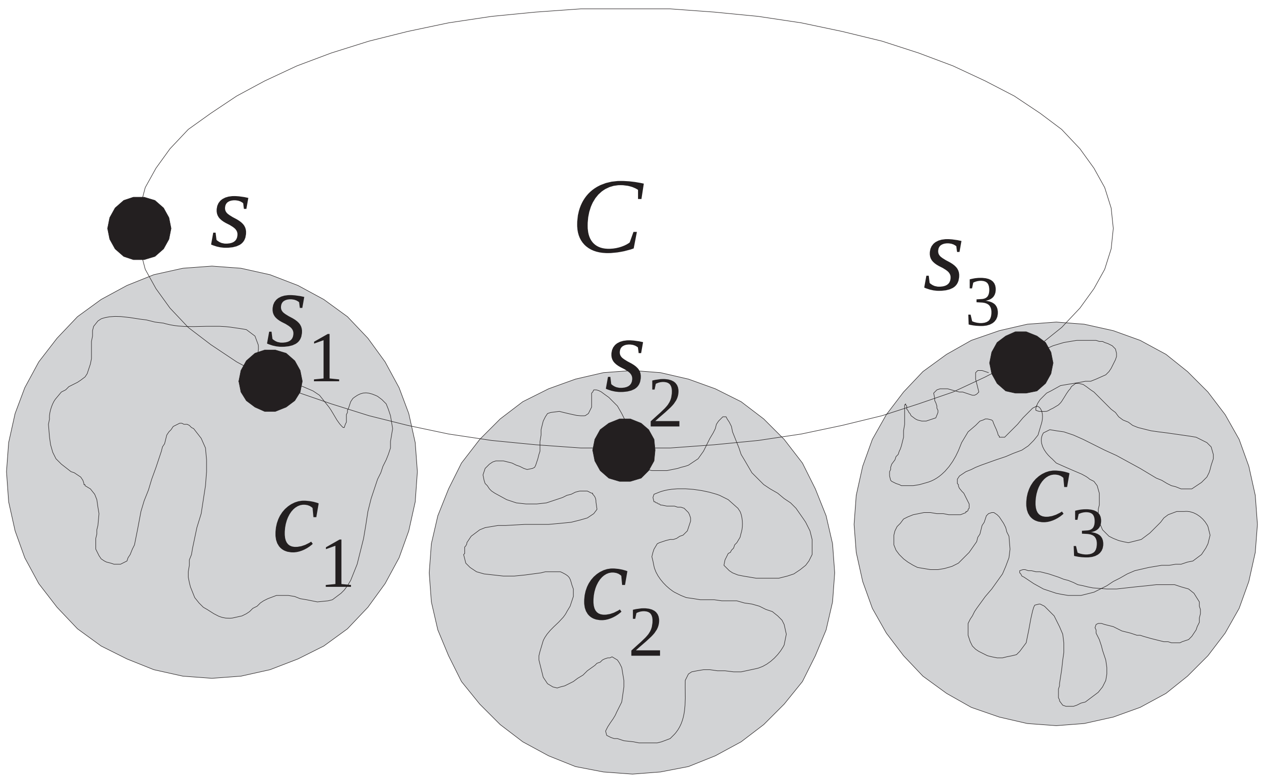
---

[a] C may visit many *nodes* of c_i (but not a single *edge* by definition). Thanks to a lively class discussion on May 31, 2012.

The Proof ($\Leftarrow$) (concluded)

- (continued)
  - The desired Euler circuit: Start from s and travel on C until we reach s_1, traverse c_1, return to s_1, continue on C until we reach s_2, and so on.

Constructing an Euler Circuit



Characterization of Having Euler Trails

Corollary 74 *Let $G = (V, E)$ be an undirected graph or multigraph. Then G has an Euler trail if and only if G is connected and has exactly two nodes of odd degree.*

- Let x, y be the two nodes of odd degree.
- Add edge $\{ x, y \}$ to G.
- Construct an Euler circuit for G, which exists by Theorem 73.
- Remove the edge $\{ x, y \}$ from the circuit to arrive at an Euler trail.

In and Out Degrees

- Let G be a directed graph.
- The **in degree** of $v \in V$ is the number of edges in G that are incident *into* v.
- The **out degree** of $v \in V$ is the number of edges in G that are incident *from* v.
  - The in and out degrees of a node may not equal.
- Similar to (undirected) regular graphs,[a] a directed d-regular graph has in-degree d *and* out-degree d for every node.

---

[a]Recall p. 664.

Characterization of Having Directed Euler Circuits

Theorem 75 *Let $G = (V, E)$ be a digraph. Then G has a directed Euler circuit if and only if G is connected and the in degree equals the out degree at every node.*

- Follow the same proof as Theorem 73 (p. 674).
- The only difference is that, whereas we maintained even node degrees, we now maintain the equality of in and out degrees.

Euler Circuits: Additional Remarks[a]

- Counting the number of Euler circuits for digraphs can be solved efficiently.[b]
- Counting the number of Euler circuits for undirected graphs is computationally hard—it is #P-complete.[c]
- Asymptotic formulas exist for the number of Euler circuits on K_n when n is odd.[d]
- Very useful in approximation algorithms.[e]

---

[a] Contributed by Mr. Eric Ruei-Min Lee (`B00902106`) on June 4, 2012.
[b] Harary & Palmer (1973).
[c] Brightwell & Winkler (2004).
[d] McKay & Robinson (1995).
[e] Vazirani (2003).

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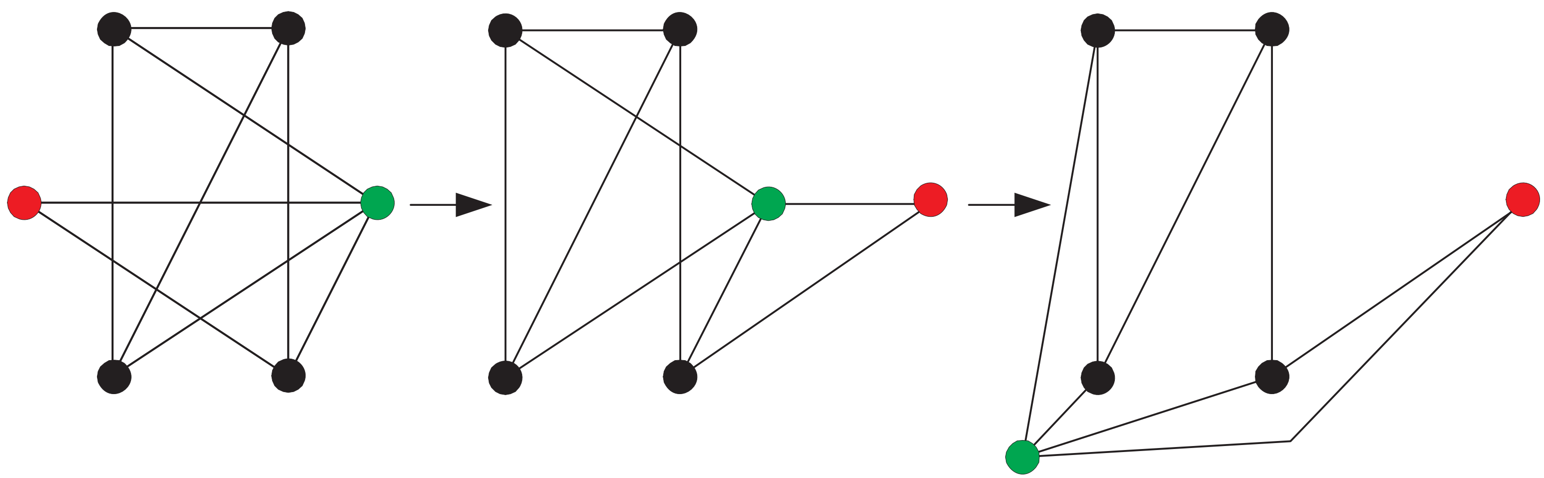
Planar Graphs

- A graph or multigraph G is called **planar** if it can be drawn in the plane with the edges intersecting only at nodes of G.
- Planarity can be tested efficiently.[a]

[a] Hopcroft & Tarjan (1974).

A Planar Graph



Such a drawing of G is called an **embedding** of G in the plane.

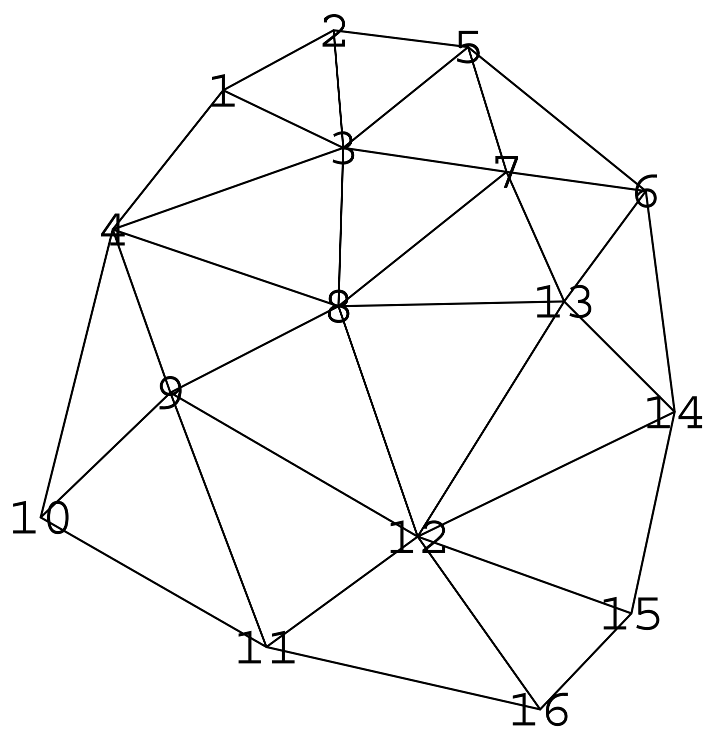
Euler's Theorem[a]

- Let $G = (V, E)$ be a connected planar graph or multigraph with $|V| = v$ and $|E| = e$.
- Let r be the number of regions in the plane determined by a planar embedding of G.
- One of these regions has infinite area.
  - It is called the **infinite region**.
- Then

$$v - e + r = 2. \tag{103}$$

[a]Euler (1752).

A Planar Graph with $v = 16$, $e = 35$, $r = 21$



The Proof[a]

- The theorem holds if $e = 0, 1$.[b]
- Assume the theorem holds for any connected planar graph with e edges, where $0 \leq e \leq k$.
- Let $G = (V, E)$ be a connected planar graph with v nodes, r regions, and $e = k + 1$ edges.
- Pick an arbitrary edge $\{ x, y \}$ from E.
- Delete $\{ x, y \}$ to obtain graph H:

$$G = H + \{ x, y \}.$$

---

[a] See Imre Lakatos's (1922–1974) *Proofs and Refutations: The Logic of Mathematical Discovery* (1989) for a most penetrating presentation.

[b] See p. 545 of the textbook (5th ed.).

The Proof When H Is Connected

- The dotted edge on p. 692 is $\{ x, y \}$.
- So H has v nodes, k edges, and $r - 1$ regions.
- H clearly remains planar.
- The induction hypothesis applied to H says

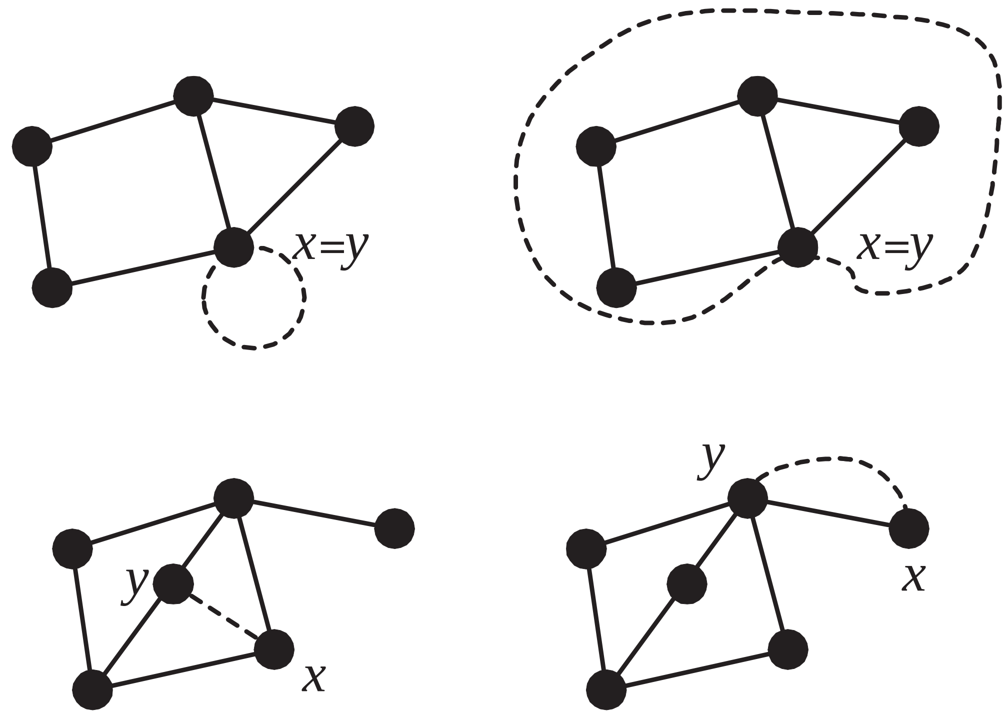
$$v - k + (r - 1) = 2.$$

- Or

$$v - (k + 1) + r = 2.$$

- The theorem is proved because G has v nodes, $e = k + 1$ edges, and r regions.

A Planar G from a Planar H



The Proof When H Is Not Connected

- The dotted edge on p. 694 is $\{ x, y \}$.
- So H has v nodes, k edges, and r regions.
- H has two components H_1 and H_2,[a] both planar.
- Let H_i have v_i nodes, e_i edges, and r_i regions, $i = 1, 2$.
- The induction hypothesis applied to H_i says

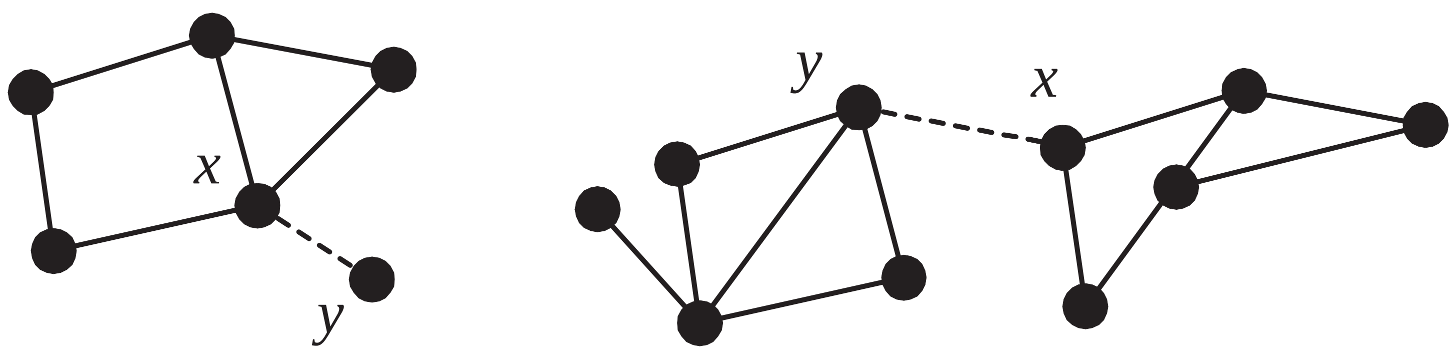
$$v_i - e_i + r_i = 2.$$

- Therefore,

$$(v_1 + v_2) - (e_1 + e_2) + (r_1 + r_2) = 4. \tag{104}$$

[a] Thanks to a lively class discussion on December 1, 2003.

A Planar G from Planar H_1 and H_2



The Proof When H Is Not Connected (concluded)

- Now,

$$\begin{aligned} v_1 + v_2 &= v, \\ e_1 + e_2 &= k = e - 1, \\ r_1 + r_2 &= r + 1. \end{aligned}$$

  – The infinite region is counted twice.

- Hence Eq. (104) on p. 693 becomes

$$v - (e - 1) + (r + 1) = 4.$$

- Again, $v - e + r = 2$.

A Useful Corollary

Corollary 76 *Let* $G = (V, E)$ *be a connected planar simple*[a] *graph with* $|\,V\,| = v$ *and* $|\,E\,| = e > 2$. *Then*

$$(3/2)\,r \leq e \leq 3v - 6.$$

- Let there be r regions.
- Each edge is shared by ≤ 2 regions.
  - The edge $\{\,x, y\,\}$ on p. 694 is shared by one region.
  - One can replace the above with "$= 2$" if that edge is considered to be shared by 2 regions.[b]
- The boundary of each region (including the infinite region) contains at least 3 edges.

[a] It is loop-free without parallel edges (recall p. 645).
[b] See p. 546 of the textbook.

The Proof (concluded)

- Hence
$$2e \geq \sum_{\text{region } R} |\, R\text{'s boundary} \,| \geq 3r.$$
  - This proves the first inequality of the corollary.
- Euler's theorem implies
$$2 = v - e + r \leq v - e + (2/3)\, e = v - (1/3)\, e,$$
proving the second inequality.

K_5 Is Not Planar

- K_5 has $v = 5$ nodes and $e = 10$ edges.
- Suppose it is planar.
- By Corollary 76,

$$10 = e \leq 3v - 6 = 9,$$

a contradiction.

$K_{3,3}$ Is Not Planar

- $K_{3,3}$ has $v = 6$ nodes and $e = 9$ edges.
- Suppose it is planar.
- By Euler's formula (103) on p. 688, the number of regions is
$$r = 2 + e - v = 5.$$

The Proof (concluded)

- But $K_{3,3}$ has no 3 nodes forming a complete subgraph.
- So the border of a region must contain at least 4 edges.
- The sum of those edges is at least $4r = 20$.
- Hence
$$2e \geq \sum_{\text{region } R} |\, R\text{'s boundary} \,| \geq 20,$$
contradicting $e = 9$.

Kuratowski's[a] Theorem

Theorem 77 (Kuratowski, 1930) *A graph is nonplanar if and only if it contains a subgraph that is "homeomorphic" to either K_5 or $K_{3,3}$.*

Corollary 78 *(1) Shrinking any edge of a planar graph to a single node preserves planarity. (2) Shrinking any connected component of a planar graph to a single node preserves planarity.*

---

[a]Kasimir Kuratowski (1896–1980).

Kasimir Kuratowski (1896–1980)

Hamiltonian[a] Paths and Cycles

- Let $G = (V, E)$ be a graph with $|V| \geq 3$.
- A **Hamiltonian cycle** is a *cycle* in G that contains every node (exactly once) in V.
- A **Hamiltonian path** is a *path* in G that contains every node (exactly once) in V.
- Testing if G has a Hamiltonian path or cycle is computationally hard—it is NP-complete.[b]

---

[a] William Rowan Hamilton (1805–1865).
[b] Karp (1972).

William Rowan Hamilton (1805–1865)

Richard Karp[a] (1935–)

[a]Turing Award (1985).

Application: Tournaments

- Let K_n^* be a *directed* graph with n nodes.
- If for each distinct pair x, y of nodes, either $(x, y) \in K_n^*$ or $(y, x) \in K_n^*$ but not both, then K_n^* is called a tournament.[a]
- A tournament is not necessarily transitive.
  - A digraph (V, E) is transitive if
$$(a, b) \in E \wedge (b, c) \in E \Rightarrow (a, c) \in E.$$
- But the next theorem says that players can be ranked in at least one way.

---

[a] Recall p. 343.

Tournaments Are Hamiltonian[a]

Theorem 79 (Redei, 1934) *A tournament always contains a directed Hamiltonian path.*

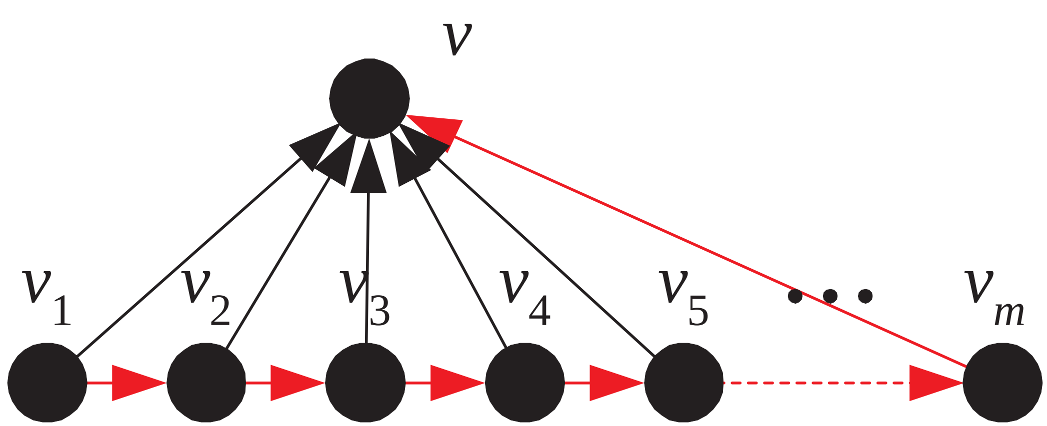
- Let $p_m = (v_1, v_2, \ldots, v_m)$ be a path of maximum length.
- Assume $m < n = |\, V\, |$ and proceed to derive a contradiction.
- Let v be a node *not* on p_m.
- If $(v, v_1) \in K_n^*$, then p_m can be lengthened to
$$(v, v_1, v_2, \ldots, v_m).$$
- Hence $(v, v_1) \notin K_n^*$ and $(v_1, v) \in K_n^*$.

---

[a]Similar results appear on p. 345 and p. 382.

The Proof (continued)

- If there exists a $2 \leq j \leq m$ such that $(v_{j-1}, v) \in K_n^*$ and $(v, v_j) \in K_n^*$, then the path $(v_1, \ldots, v_{j-1}, v, v_j, \ldots, v_m)$ is longer than p_m, a contradiction.[a]
- As $(v_1, v) \in K_n^*$, we conclude that for each $2 \leq j \leq m$, $(v_{j-1}, v) \in K_n^*$ but $(v, v_j) \notin K_n^*$ by induction.



[a]Improved by a lively discussion on June 5, 2014.

The Proof (concluded)

- Hence $(v_j, v) \in K_n^*$ for $2 \leq j \leq m$.
- In particular, $(v_m, v) \in K_n^*$.
- We now add (v_m, v) to lengthen p_m, a contradiction.
- Remark: Now that K_n^* is Hamiltonian, how to find a Hamiltonian path efficiently?

Graph Coloring

- Let $G = (V, E)$ be an undirected graph.
- A **proper coloring** of G occurs when its nodes are colored so that adjacent nodes have different colors.
- The minimum number of colors needed to color G is the **chromatic number** of G and is written as $\chi(G)$.
- Four colors suffice to color any planar graph.[a]

---

[a] Appel & Haken (1976). Although the original proof uses a computer, a computer-generated *formal* proof has been given by Gonthier (2004)! This theorem was examined in 1850 by Francis Guthrie (1831–1899) and made its official birth in a letter from DeMorgan to Hamilton in 1852. Kenneth Appel (1932–2013), "Without computers, we would be stuck only proving theorems that have short proofs."

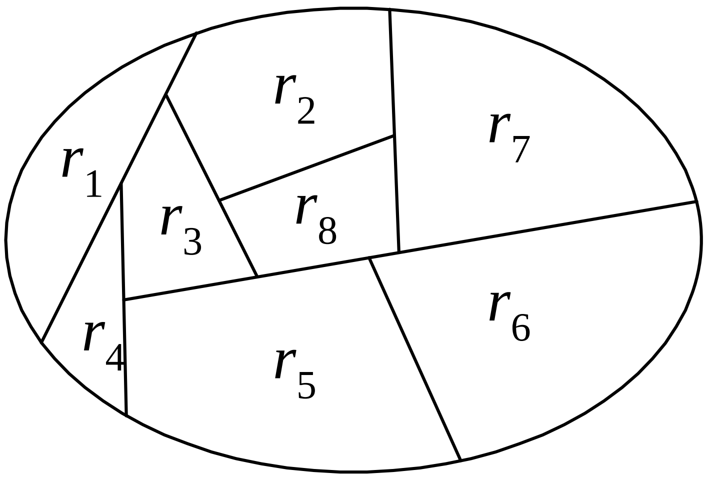
Graph Coloring (concluded)

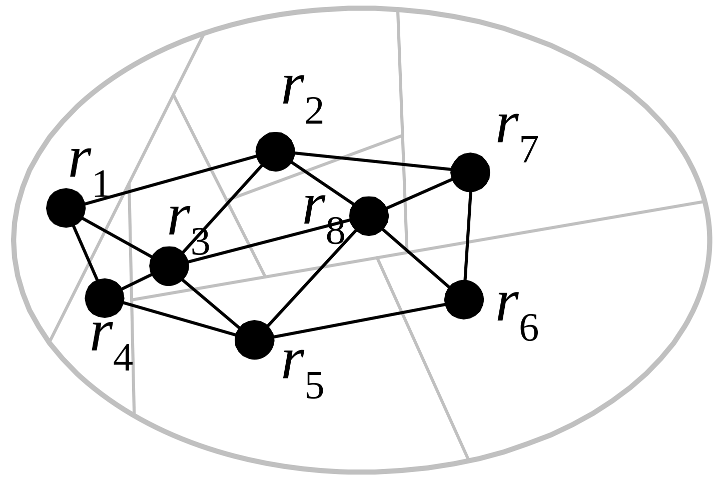
- The graph colorability problem for 3 colors and up is computationally hard—it is NP-complete.[a]
- The number of ways to color a graph on n nodes using k colors can be calculated in time $O(2^n n^{O(1)})$.[b]
- $\chi(G)$ can be calculated in time $O(2.2461^n)$.[c]

---

[a] Karp (1972).
[b] Bjöklund & Husfeldt (2006).
[c] Bjöklund & Husfeldt (2006).

Colors and Maps





The four-color theorem says that any map can be colored with just 4 colors.

Elementary Facts

- For all $n \geq 1$, $\chi(K_n) = n$.
  - Each node is adjacent to $n-1$ other nodes.
- If H is a subgraph of G, then $\chi(H) \leq \chi(G)$.
  - A proper coloring of G is also one of H.
- An undirected graph G is bipartite if and only if $\chi(G) \leq 2$.
  - Given a bipartite partition $V = V_1 \cup V_2$, color V_1 and V_2 with two different colors.

An Upper Bound on the Chromatic Number

Theorem 80 (Vizing, 1964; Gupta, 1966) *Every graph is $(\kappa + 1)$-colorable, where κ is the maximum degree of the nodes.*

```
while G(V, E) has uncolored nodes do
    Pick an arbitrary uncolored v ∈ V;
    Choose color c that is not used by v's ≤ κ neighbors;
    Color v with c;
end while
```

Comments on Vizing's Theorem

- This bound is tight because $\chi(K_n) = n$ (p. 713).
- Some neighbors may be colored with the same colors if they are not adjacent to each other.
- So $\kappa + 1$ may not be a lower bound for the chromatic number.[a]
- A **clique** is a subgraph that is also a complete graph.
- Is the size of the largest clique in a graph the chromatic number?[b]

---

[a] Contributed by Mr. Asger K. Pedersen (`T02202107`) on June 5, 2014.
[b] Contributed by Ms. Zhijing Jin (`T05902125`) on May 18, 2017.

Coloring 3-Colorable Graphs Efficiently

Theorem 81 (Wigderson, 1983) *Any 3-colorable graph can be colored in polynomial time with $O(\sqrt{n})$ colors.*

- Surprisingly, no one knows how to do better![a]

---

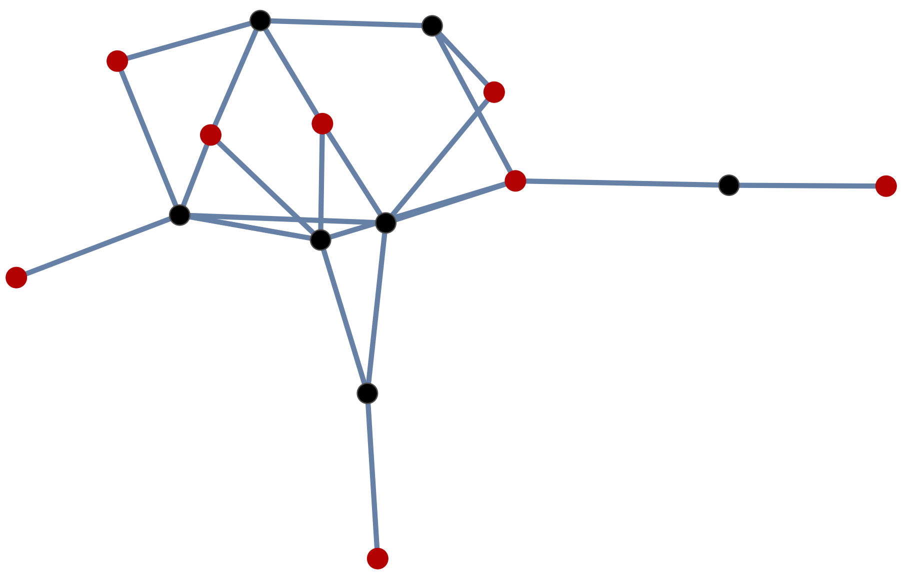
[a]Williamson & Shmoys (2011).

Independent Set

- Let $G = (V, E)$ be an undirected graph.
- An **independent set** for G is a set of nodes no two of which are adjacent.[a]
- The size of a largest independent set is called the **independence number** or $\alpha(G)$.

[a] An independent set is sometimes called a **stable set**.

A Maximum Independent Set



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Trees

I love a tree more than a man.
— Ludwig van Beethoven (1770–1827)

Most mathematicians work with calculus-type
"smooth" problems, not discrete things like
cleverly arranged arrays of zeros and ones.
— Diaconis and Graham,
Magical Mathematics (2012)

Trees[a]

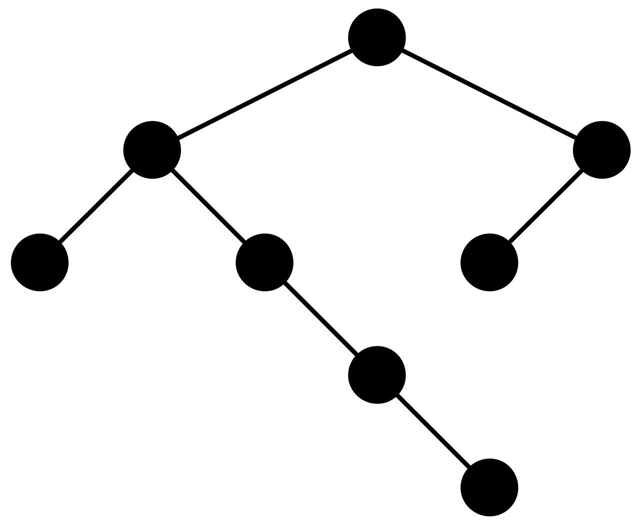
- A **tree** is a loop-free undirected graph that is connected and contains no cycles.
- A **forest** is a loop-free undirected graph whose components are trees.

Lemma 82 *A loop-free connected undirected graph has cycles if and only if it is not a tree.*

- By definition of tree.

[a]Kirchhoff (1847).

A Tree



Spanning Trees

- A **spanning tree** for a connected graph $G = (V, E)$ is a subgraph of G with the same node set V that is also a tree.
  - A (minimum) spanning tree is computationally easy to construct.[a]
  - The number of spanning trees of a connected labeled graph is easy to compute.[b]
- An undirected graph has a spanning tree if and only if it is connected.

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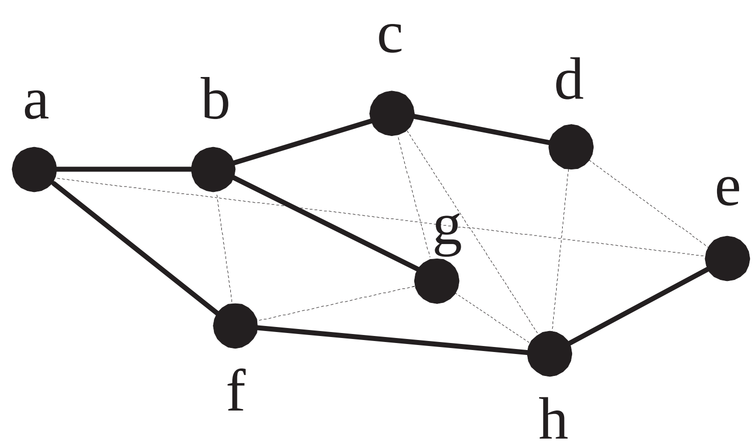
[a] Borůvka (1926).
[b] Kirchhoff (1847).

A Spanning Tree

The solid lines constitute the edges of a spanning tree.



Properties of Trees

- If x and y are distinct nodes in a tree, then there is a unique path that connects them.
  - There is at least one such path because a tree is connected.
  - But more than one such path implies the existence of a cycle, a contradiction.

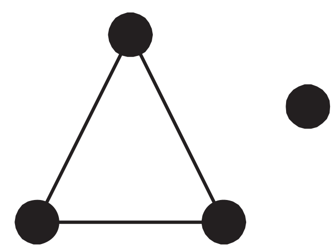
Properties of Trees (continued)

Theorem 83 *For a tree* (V, E), $|V| = |E| + 1$.

- Obviously true when $|E| = 0$ as it is a single node.
- In general, a tree with $|E| = k + 1$ edges breaks into two trees (V_1, E_1) and (V_2, E_2) by the deletion of an edge.
- So
$$|E| = |E_1| + |E_2| + 1.$$
- By the induction hypothesis, $|V_1| = |E_1| + 1$ and $|V_2| = |E_2| + 1$.
- Hence,
$$|V| = |V_1| + |V_2| = |E_1| + |E_2| + 2 = |E| + 1.$$

Properties of Trees (continued)

- Theorem 83 (p. 726) may hold for nontrees.
- Consider the following graph:



- It satisfies Theorem 83.
- But the graph is not connected.
- Additional conditions are needed.

Properties of Trees (concluded)

The following statements are equivalent for a loop-free undirected graph $G = (V, E)$.

1. G is a tree.
2. G is connected, but the removal of any edge disconnects G into two subgraphs that are trees.
3. G contains no cycles, and $|\,V\,| = |\,E\,| + 1$.
4. G is connected, and $|\,V\,| = |\,E\,| + 1$.
5. G contains no cycles, and if $x, y \in V$ with $\{\,x, y\,\} \notin E$, then the graph obtained by adding edge $\{\,x, y\,\}$ to G has precisely one cycle.

Trees and Forests

Corollary 84 *For a forest* (V, E), $|V| = |E| + \kappa$, *where* κ *is the number of trees in the forest.*

- From Theorem 83 (p. 726), $|V_i| = |E_i| + 1$ for each tree in the forest.
- Hence

$$
\begin{aligned}
|V| &= \sum_{i=1}^{\kappa} |V_i| \\
&= \sum_{i=1}^{\kappa} (|E_i| + 1) \\
&= |E| + \kappa.
\end{aligned}
$$

Trees and Cycles

Corollary 85 *If a loop-free connected undirected graph is not a tree, then* $|V| \leq |E|$.

- Suppose $|V| \geq |E| + 1$ instead.
- Because the graph is not a tree, $|V| > |E| + 1$ by Property 4 on p. 728.
- But then the graph cannot be connected (why?), a contradiction.

Trees Have the *Most* Nodes among Connected Graphs

Corollary 86 *Among loop-free connected undirected graphs with the same number of edges, trees have the most nodes.*

- Consider a graph with e edges.
- From Corollary 85, a nontree must have $\leq e$ nodes.
- In comparison, a tree with e edges has $e + 1$ nodes by Theorem 83 (p. 726).

Coloring of Trees

Theorem 87 *Every tree is 2-colorable.*

- Pick any node v.
- Color any node reachable from v via an odd number of edges red.
- Color any node reachable from v via an even number of edges blue.
- Because there is a unique path between tree nodes,[a] this coloring is well-defined.

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[a] Recall p.725.

Planarity of Trees

Lemma 88 *Trees are planar.*

- A tree contains no cycles.
- So it cannot contain a subgraph homeomorphic to either $K_{3,3}$ or K_5.
- The lemma follows by Kuratowski's theorem (p. 701).

Theorem 83 (p. 726) Reproved

- Theorem 83 says $|V| = |E| + 1$ for a tree.
- A tree (V, E) is planar by Lemma 88 (p. 733).
- Euler's theorem (p. 688) says $|V| - |E| + 1 = 2$.
- But this is exactly what Theorem 83 says,

$$|V| = |E| + 1.$$