Number of Palindromes Revisited

- A palindrome is a composition for $n \in \mathbb{Z}^+$ that reads the same left to right as right to left (p. 109).
- Let a_n denote the number of palindromes for n.
- Clearly, $a_1 = 1$ and $a_2 = 2$.
- Given each palindrome for n, we can do two things to obtain a palindrome for $n + 2$.
  - Add 1 to the *first* and *last* summands.
    - So $1 + 3 + 1$ becomes $2 + 3 + 2$.
  - Insert summand 1 to the start and end.
    - So $1 + 3 + 1$ becomes $1 + 1 + 3 + 1 + 1$.

The Proof (continued)

- This mapping is a one-to-two correspondence (why?).
- Hence
$$a_{n+2} = 2a_n, \quad n \geq 1.$$
- The characteristic equation
$$r^2 - 2 = 0$$
has two roots $\pm\sqrt{2}$.

The Proof (continued)

- The general solution is hence

$$a_n = c_1 \left(\sqrt{2}\right)^n + c_2 \left(-\sqrt{2}\right)^n .$$

- Solve[a]

$$\begin{aligned} 1 &= a_1 = \sqrt{2}\,(c_1 - c_2), \\ 2 &= a_2 = 2(c_1 + c_2), \end{aligned}$$

for $c_1 = (1 + \frac{1}{\sqrt{2}})/2$ and $c_2 = (1 - \frac{1}{\sqrt{2}})/2$.

[a]This time, we do not retrofit.

The Proof (concluded)

- The number of palindromes for n therefore equals

$$
\begin{aligned}
a_n &= \frac{1+\frac{1}{\sqrt{2}}}{2}\left(\sqrt{2}\right)^n + \frac{1-\frac{1}{\sqrt{2}}}{2}\left(-\sqrt{2}\right)^n \\
&= \begin{cases} \frac{1+\frac{1}{\sqrt{2}}}{2}\, 2^{n/2} + \frac{1-\frac{1}{\sqrt{2}}}{2}\, 2^{n/2}, & \text{if } n \text{ is even,} \\ \frac{1+\frac{1}{\sqrt{2}}}{2}\sqrt{2}\, 2^{(n-1)/2} - \frac{1-\frac{1}{\sqrt{2}}}{2}\sqrt{2}\, 2^{(n-1)/2}, & \text{if } n \text{ is odd,} \end{cases} \\
&= \begin{cases} 2^{n/2}, & \text{if } n \text{ is even,} \\ 2^{(n-1)/2}, & \text{if } n \text{ is odd,} \end{cases} \\
&= 2^{\lfloor n/2 \rfloor}.
\end{aligned}
$$

- It matches Theorem 20 (p. 111).

An Example: A Third-Order Relation

- Consider
$$2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n$$
with $a_0 = 0$, $a_1 = 1$, and $a_2 = 2$.
- The characteristic equation
$$2r^3 - r^2 - 2r + 1 = 0$$
has three distinct real roots: 1, -1, and 0.5.
- The general solution is
$$\begin{aligned} a_n &= c_1 1^n + c_2(-1)^n + c_3(1/2)^n \\ &= c_1 + c_2(-1)^n + c_3(1/2)^n. \end{aligned}$$

An Example: A Third-Order Relation (concluded)

- Solve the three initial conditions via Eq. (86) on p. 555:

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0.5 \\ 1^2 & (-1)^2 & 0.5^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

- The solutions are

$$\begin{aligned} c_1 &= 2.5, \\ c_2 &= 1/6, \\ c_3 &= -8/3. \end{aligned}$$

The Case of Complex Roots

- Consider

$$a_n = 2(a_{n-1} - a_{n-2})$$

with $a_0 = 1$ and $a_1 = 2$.

- The characteristic equation

$$r^2 - 2r + 2 = 0$$

has two distinct complex roots $1 \pm i$.

- The general solution is

$$a_n = c_1(1+i)^n + c_2(1-i)^n.$$

The Case of Complex Roots (concluded)

- Solve the two initial conditions for $c_1 = (1 - i)/2$ and $c_2 = (1 + i)/2$.
- The particular solution becomes[a]

$$
\begin{aligned}
a_n &= (1+i)^{n-1} + (1-i)^{n-1} \\
&= (\sqrt{2}\,)^n \left[\, \cos(n\pi/4) + \sin(n\pi/4) \,\right].
\end{aligned}
$$

[a] An equivalent one is $a_n = (\sqrt{2}\,)^{n+1} \cos((n-1)\pi/4)$ by Mr. Tunglin Wu (B00902040) on May 17, 2012.

kth-Order Linear Homogeneous Recurrence Relations with Constant Coefficients: Repeated Real Roots

- Consider the recurrence relation

$$C_n a_n + C_{n-1} a_{n-1} + \cdots + C_{n-k} a_{n-k} = 0,$$

where $C_n, C_{n-1}, \ldots$ are real constants, $C_n \neq 0$, $C_{n-k} \neq 0$.

- Let r be a characteristic root of **multiplicity** m, where $2 \leq m \leq k$, of the characteristic equation

$$f(x) = C_n x^k + C_{n-1} x^{k-1} + \cdots + C_{n-k} = 0.$$

- The general solution that involves r has the form

$$(A_0 + A_1 n + A_2 n^2 + \cdots + A_{m-1} n^{m-1})\, r^n \tag{92}$$

with $A_0, A_1, \ldots, A_{m-1}$ are constants to be determined.

The Proof

- If $f(x)$ has a root r of multiplicity m, then

$$f(r) = f'(r) = \cdots = f^{(m-1)}(r) = 0.$$

- Because $r \neq 0$ is a root of multiplicity m, it is easy to check that

$$\begin{aligned}
0 &= r^{n-k} f(r), \\
0 &= r(r^{n-k} f(r))', \\
0 &= r(r(r^{n-k} f(r))')', \\
&\vdots \\
0 &= \overbrace{r(\cdots r(r(}^{m-1}\, r^{n-k} f(r) \,\overbrace{)')' \cdots)'}^{m-1}.
\end{aligned}$$

The Proof (continued)

- Note that we differentiate and then multiply by r before iterating.
- These give

$$
\begin{aligned}
0 &= C_n r^n + C_{n-1} r^{n-1} + \cdots + C_{n-k} r^{n-k}, \\
0 &= C_n n r^n + C_{n-1} (n-1)\, r^{n-1} + \cdots + C_{n-k} (n-k)\, r^{n-k}, \\
0 &= C_n n^2 r^n + C_{n-1} (n-1)^2 r^{n-1} + \cdots + C_{n-k} (n-k)^2 r^{n-k}, \\
&\ \ \vdots
\end{aligned}
$$

The Proof (continued)

- Now, $a_n = n^k r^n$, $0 \leq k \leq m-1$, is indeed a solution because the kth row on the previous page says

$$\begin{aligned} & 0 \\ = \quad & C_n n^k r^n + C_{n-1}(n-1)^k r^{n-1} + \cdots + C_{n-k}(n-k)^k r^{n-k} \\ = \quad & C_n a_n + C_{n-1} a_{n-1} + \cdots + C_{n-k} a_{n-k}. \end{aligned}$$

The Proof (continued)

- From Eq. (84) on p. 550, $r^n, nr^n, n^2 r^n, \ldots, n^{m-1} r^n$ form a fundamental set if[a]

$$\begin{vmatrix} 1 & 0 & \cdots & 0 \\ r & r & \cdots & r \\ r^2 & 2r^2 & \cdots & 2^{m-1} r^2 \\ \vdots & \vdots & \ddots & \vdots \\ r^{m-1} & (m-1)\, r^{m-1} & \cdots & (m-1)^{m-1} r^{m-1} \end{vmatrix} \neq 0.$$

[a] The ith row sets $n = i - 1$, $i = 1, 2, \ldots, m$.

The Proof (concluded)

- The above is a Vandermonde matrix in disguise.
- In fact, after deleting the first row and column, the determinant equals

$$
(m-1)!\, r^{1+2+\cdots+(m-1)}
\times \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 2^{m-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & (m-1) & \cdots & (m-1)^{m-2} \end{vmatrix} \neq 0.
$$

Nonhomogeneous Recurrence Relations

- Consider
$$C_n a_n + C_{n-1} a_{n-1} + \cdots + C_{n-k} a_{n-k} = f(n). \qquad (93)$$
- Suppose $a_n = a_{n-1} + f(n)$.
- Then the solution is
$$a_n = a_0 + \sum_{i=1}^{n} f(i).$$
- A closed-form formula exists if one for $\sum_{i=1}^{n} f(i)$ does.

Nonhomogeneous Recurrence Relations (concluded)

- In general, no failure-free methods exist except for special $f(n)$s.
  - See pp. 441–2 of the textbook (4th ed.).
  - See p. 532 of Rosen (2012) when $f(n)$ is the product of a polynomial in n and the nth power of a constant.

Examples ($c, c_1, c_2, \ldots$ Are Arbitrary Constants)

$a_{n+1} - a_n = 0$	$a_n = c$
$a_{n+1} - a_n = 1$	$a_n = n + c$
$a_{n+1} - a_n = n$	$a_n = n(n-1)/2 + c$
$a_{n+2} - 3a_{n+1} + 2a_n = 0$	$a_n = c_1 + c_2 2^n$
$a_{n+2} - 3a_{n+1} + 2a_n = 1$	$a_n = c_1 + c_2 2^n - n$
$a_{n+2} - a_n = 0$	$a_n = c_1 + c_2(-1)^n$
$a_{n+1} = a_n/(1 + a_n)$	$a_n = c/(1 + cn)$

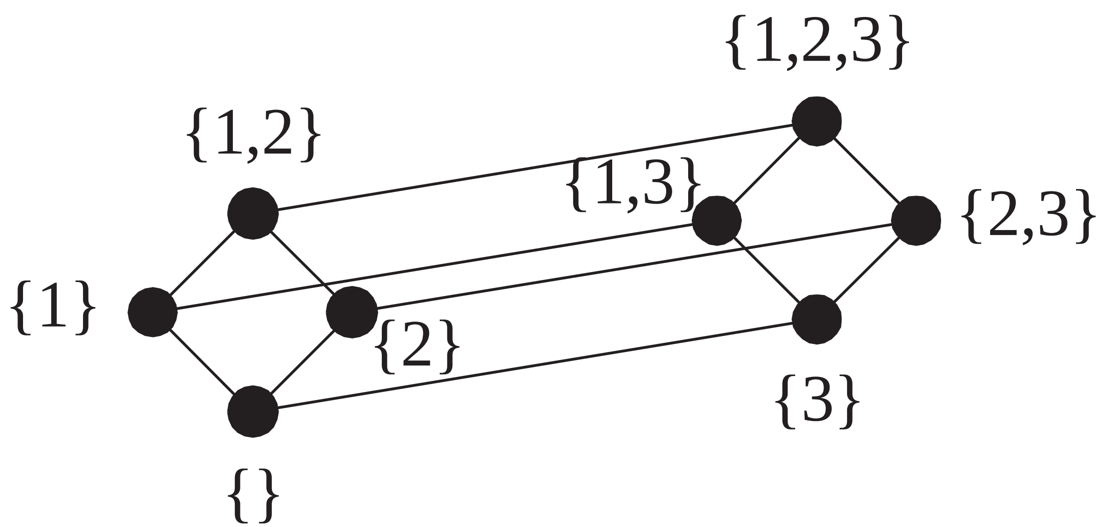
Trial and Error

- Consider $a_{n+1} = 2a_n + 2^n$ with $a_1 = 1$.
- Calculations show that $a_2 = 4$ and $a_3 = 12$.
- Conjecture:
$$a_n = n2^{n-1}. \tag{94}$$
- Verify that, indeed,
$$(n+1)\,2^n = 2(n2^{n-1}) + 2^n,$$
and $a_1 = 1$.

Application: Number of Edges of a Hasse Diagram

- Let a_n be the number of edges of the Hasse diagram for the partial order $(2^{\{1,2,\ldots,n\}}, \subseteq)$.
- Consider the Hasse diagrams H_1 for $(2^{\{1,2,\ldots,n\}}, \subseteq)$ and H_2 for $(\{T \cup \{n+1\} : T \subseteq \{1,2,\ldots,n\}\}, \subseteq)$.
  - H_1 and H_2 are "isomorphic."
- The Hasse diagram for $(2^{\{1,2,\ldots,n+1\}}, \subseteq)$ is constructed by adding an edge from each node T of H_1 to node $T \cup \{n+1\}$ of H_2.
- Hence $a_{n+1} = 2a_n + 2^n$ with $a_1 = 1$.
- The desired number was solved in Eq. (94) on p. 595.

Illustration with $(2^{\{1,2,3\}}, \subseteq)$



Trial and Error Again

- Consider $a_{n+1} - Aa_n = B$.
- Calculations show that

$$\begin{aligned} a_1 &= Aa_0 + B, \\ a_2 &= Aa_1 + B = A^2 a_0 + B(A+1), \\ a_3 &= Aa_2 + B = A^3 a_0 + B(A^2 + A + 1). \end{aligned}$$

- Conjecture that is easily verified by substitution:

$$a_n = \begin{cases} A^n a_0 + B\,\frac{A^n - 1}{A-1}, & \text{if } A \neq 1, \\ a_0 + Bn, & \text{if } A = 1. \end{cases} \tag{95}$$

Will the Number of Students Explode?

- A professor teaches a required course that N *new* students take every year.
- He flunks $0 < f < 1$ of the students taking his course.
- Failed students retake the course annually until passage.
- Does the class size a_n for year n explode or have a limit?
- It satisfies
$$a_{n+1} = f a_n + N.$$
- With $a_0 = 0$, $B = N$, and $A = f$ in Eq. (95) on p. 598,
$$a_n = N \, \frac{f^n - 1}{f - 1} \to \frac{N}{1 - f}.$$

Financial Application: Compound Interest[a]

- Consider $a_{n+1} = (1 + r)\, a_n$.
  - Deposit grows at a period interest rate of $r > 0$.
  - The initial deposit is a_0 dollars.
- The solution is obviously

$$a_n = (1 + r)^n a_0.$$

- The deposit thus grows exponentially with time.

---

[a] "In the fifteenth century mathematics was mainly concerned with questions of commercial arithmetic and the problems of the architect," wrote Joseph Alois Schumpeter (1883–1950) in *Capitalism, Socialism and Democracy* (1942).

Financial Application: Amortization

- The initial loan amount is a_0 dollars.
- The monthly payment is M dollars.
- Let the outstanding loan principal *after* the nth payment be a_n.
- Then
$$a_{n+1} = (1 + r)\, a_n - M.$$
- By Eq. (95) on p. 598, the solution is
$$a_n = (1 + r)^n a_0 - M\, \frac{(1 + r)^n - 1}{r}.$$

The Proof (concluded)

- What is the unique monthly payment M for the loan to be paid off after k monthly payments?
- Set $a_k = 0$ to obtain

$$a_k = (1+r)^k a_0 - M \, \frac{(1+r)^k - 1}{r} = 0.$$

- Hence

$$M = \frac{(1+r)^k a_0 r}{(1+r)^k - 1}.$$

- This is a standard formula for home mortgages and annuities.[a]

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[a] Lyuu (2002).

Trial and Error a Third Time

- Consider the more general $a_{n+1} - Aa_n = BC^n$.
- Calculations show that

$$\begin{aligned} a_1 &= Aa_0 + B, \\ a_2 &= Aa_1 + BC = A^2 a_0 + B(A + C), \\ a_3 &= Aa_2 + BC^2 = A^3 a_0 + B(A^2 + AC + C^2). \end{aligned}$$

- Conjecture that is easily verified by substitution:

$$a_n = \begin{cases} A^n a_0 + B\,\frac{A^n - C^n}{A - C}, & \text{if } A \neq C \\ A^n a_0 + BA^{n-1} n, & \text{if } A = C \end{cases}. \tag{96}$$

Application: Runs of Binary Strings

- A run is a maximal consecutive list of identical objects.[a]
  - Binary string "`0 0 1 1 1 0`" has 3 runs.
- Let r_n denote the *total* number of runs of the 2^n n-bit binary strings.
- First, $r_1 = 1 + 1 = 2$.
  - Each of "`0`" and "`1`" has 1 run.
- Next, $r_2 = 1 + 1 + 2 + 2 = 6$.
  - "`00`" and "`11`" each has 1 run, while "`01`" and "`10`" each has 2 runs.

[a] Recall p. 113.

The Proof (continued)

- In general, suppose we append a bit to *every* $(n-1)$-bit string $b_1 b_2 \cdots b_{n-1}$ to make $b_1 b_2 \cdots b_{n-1} b_n$.
- First, suppose $b_{n-1} = b_n$ (i.e., the last 2 bits are identical).
- Then the total number of runs does not change.
  - The total number of runs remains r_{n-1}.

The Proof (continued)

- Next, suppose $b_{n-1} \neq b_n$ (i.e., the last 2 bits are distinct).
- Then the total number of runs *increases* by 1 for *each* $(n-1)$-bit string.
  - There are 2^{n-1} of them.
- So the total number of runs becomes

$$r_{n-1} + 2^{n-1}.$$

The Proof (continued)

- Hence
$$r_n = 2r_{n-1} + 2^{n-1}, \quad n \geq 2. \tag{97}$$

- By Eq. (96) on p. 603,
$$r_n = 2^n r_0 + 2^{n-1} n.$$

- To make sure that $r_1 = 2$, it is easy to see that $r_0 = 1/2$.

- Hence
$$r_n = 2^{n-1} + 2^{n-1} n = 2^{n-1}(n+1).$$

The Proof (concluded)

- The recurrence (97) is identical to that for the number of edges of a Hasse diagram (p. 596).
- But the initial condition was different: $a_1 = 1$.
- That was reflected in the different solution (94) on p. 595: $a_n = n2^{n-1}$.

Method of Undetermined Coefficients

- Recall Eq. (93) on p. 592, repeated below:

$$C_n a_n + C_{n-1} a_{n-1} + \cdots + C_{n-k} a_{n-k} = f(n). \tag{98}$$

- Let $a_n^{(h)}$ denote the general solution of the associated *homogeneous* relation (with $f(n) = 0$).
- Let $a_n^{(p)}$ denote a particular solution of the *nonhomogeneous* relation.
- Then

$$a_n = a_n^{(h)} + a_n^{(p)}.$$

- All the entries in the table on p. 594 fit the claim.

Conditions for the General Solution

Similar to Theorem 68 (p. 550), we have the following.

Theorem 69 *Let $a_n^{(p)}$ be any particular solution of the nonhomogeneous recurrence relation Eq. (98) on p. 609. Let*

$$a_n^{(h)} = C_1 a_n^{(1)} + C_2 a_n^{(2)} + \cdots + C_k a_n^{(k)}$$

be the general solution of its homogeneous version as specified in Theorem 68. Then $a_n^{(h)} + a_n^{(p)}$ is the general solution of Eq. (98) on p. 609.

Solution Techniques

- Typically, one finds the general solution of its homogeneous version $a_n^{(h)}$ first.
- Then one finds a particular solution $a_n^{(p)}$ of the nonhomogeneous recurrence relation Eq. (98) on p. 609.
- Make sure $a_n^{(p)}$ is "independent" of $a_n^{(h)}$.
- Also $a_n^{(p)}$, after substitutions into the recurrence relation, should cancel all terms involving n.
- Finally, use the initial conditions to nail the coefficients of $a_n^{(h)}$.
- Output $a_n^{(h)} + a_n^{(p)}$.

$a_{n+1} - Aa_n = B$ Revisited

- Recall that the general solution is $a_n^{(h)} = cA^n$ by Eq. (82) on p. 544.
- A particular solution is (verify it)

$$a_n^{(p)} = \begin{cases} B/(1-A), & \text{if } A \neq 1, \\ Bn, & \text{if } A = 1. \end{cases} \tag{99}$$

- So $a_n = cA^n + a_n^{(p)}$.
- In particular,

$$c = a_0 - a_0^{(p)} = \begin{cases} a_0 - B/(1-A), & \text{if } A \neq 1, \\ a_0, & \text{if } A = 1. \end{cases}$$

$a_{n+1} - Aa_n = B$ Revisited (concluded)

- The solution matches Eq. (95) on p. 598.
- We can also write the solution as

$$a_n = \begin{cases} A^n[\,a_0 - a_0^{(p)}\,] + a_n^{(p)}, & \text{if } A \neq 1, \\ a_0 + a_n^{(p)}, & \text{if } A = 1. \end{cases} \tag{100}$$

Nonhomogeneous $a_n - 3a_{n-1} = 5 \times 7^n$ with $a_0 = 2$

- $a_n^{(h)} = c \times 3^n$, because the characteristic equation has the nonzero root 3.
- We propose $a_n^{(p)} = a \times 7^n$.
- Place $a \times 7^n$ into the relation to obtain $a \times 7^n - 3a \times 7^{n-1} = 5 \times 7^n$.
- Hence $a = 35/4$ and $a_n^{(p)} = (35/4) \times 7^n = (5/4) \times 7^{n+1}$.
- The general solution is $a_n = c \times 3^n + (5/4) \times 7^{n+1}$.
- Now, $c = -27/4$ because $a_0 = 2 = c + (5/4) \times 7$.
- So the solution is $a_n = -(27/4) \times 3^n + (5/4) \times 7^{n+1}$.

Nonhomogeneous $a_n - 3a_{n-1} = 5 \times 3^n$ with $a_0 = 2$

- As before, $a_n^{(h)} = c \times 3^n$.
- But $a \times 3^n$ and $f(n) = 5 \times 3^n$ are *not* "independent" this time.
- So propose $a_n^{(p)} = an \times 3^n$.
- Plug $an \times 3^n$ into the relation to obtain $an \times 3^n - 3a(n-1) \times 3^{n-1} = 5 \times 3^n$.
- Hence $a = 5$ and $a_n^{(p)} = 5n \times 3^n$.
- The general solution is $a_n = c \times 3^n + 5n \times 3^n$.
- Finally, $c = 2$ with use of $a_0 = 2$.

Nonhomogeneous $a_{n+1} - 2a_n = n + 1$ with $a_0 = 4$

- From Eq. (95) on p. 598, $a_n^{(h)} = c \times 2^n$.
- Guess $a_n^{(p)} = an + b$.
- Substitute this particular solution into the relation to yield
$$a(n+1) + b - 2(an + b) = n + 1.$$
- Rearrange the above to obtain
$$(-a - 1)\, n + (a - b - 1) = 0.$$
- This holds for all n if $a = -1$ and $b = -2$.

The Proof (concluded)

- Hence $a_n^{(p)} = -n - 2$.
- The general solution is

$$a_n = c \times 2^n - n - 2.$$

- Use the initial condition

$$4 = a_0 = c - 2$$

  to obtain $c = 6$.
- The solution to the complete relation is

$$a_n = 6 \times 2^n - n - 2.$$

Nonhomogeneous $a_{n+1} - a_n = 2n + 3$ with $a_0 = 1$

- This equation is very similar to the previous one:

$$a_{n+1} - 2a_n = n + 1.$$

- First, $a_n^{(h)} = d \times 1^n = d$.
- If one guesses $a_n^{(p)} = an + b$ as before, then

$$a_{n+1} - a_n = a(n+1) + b - an - b = a,$$

  which cannot be right.[a]
- So we guess $a_n^{(p)} = an^2 + bn + c$.

[a] Contributed by Mr. Yen-Chieh Sung (`B01902011`) on June 17, 2013.

The Proof (continued)

- Substitute this particular solution into the relation to yield

$$a(n+1)^2 + b(n+1) + c - (an^2 + bn + c) = 2n + 3.$$

- Simplify the above to obtain

$$2an + (a + b) = 2n + 3.$$

- The solutions are $a = 1$ and $b = 2$.
- Hence $a_n^{(p)} = n^2 + 2n + c$.
- The general solution is $a_n = n^2 + 2n + c$.[a]

[a] We merge d into c.

The Proof (concluded)

- Use the initial condition

$$1 = a_0 = c$$

to obtain $c = 1$.

- The solution to the complete relation is

$$a_n = n^2 + 2n + 1 = (n+1)^2.$$

- It is very different from the solution to the previous example:

$$a_n = 6 \times 2^n - n - 2.$$

Nonhomogeneous $a_{n+2} - 3a_{n+1} + 2a_n = 2$ with $a_0 = 0$ and $a_1 = 2$

- The characteristic equation $r^2 - 3r + 2 = 0$ has roots 2 and 1.
- So $a_n^{(h)} = c_1 1^n + c_2 2^n = c_1 + c_2 2^n$.
- Guess $a_n^{(p)} = an + b$.
- Substitute $a_n^{(p)}$ into the relation to yield

$$a(n+2) + b - 3[\, a(n+1) + b\,] + 2(an + b) = 2.$$

- Rearrange the above to obtain $a = -2$.
- Hence $a_n^{(p)} = -2n + b$.

The Proof (concluded)

- The general solution is now $a_n = c_1 + c_2 2^n - 2n$.[a]
- Use the initial conditions

$$\begin{aligned} 0 = a_0 &= c_1 + c_2, \\ 2 = a_1 &= c_1 + 2c_2 - 2. \end{aligned}$$

  to obtain $c_1 = -4$ and $c_2 = 4$.
- The solution to the complete relation is

$$a_n = -4 + 2^{n+2} - 2n.$$

[a] We merge b into c_1.

The Method of Generating Functions[a]

- Consider the relation $a_n - 3a_{n-1} = n$ with $a_0 = 1$.
- Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function for $a_0, a_1, \ldots$.
- From the recurrence equation,

$$\sum_{n=1}^{\infty} a_n x^n - \sum_{n=1}^{\infty} 3a_{n-1} x^n = \sum_{n=1}^{\infty} n x^n.$$

- $f(x) - a_0 - 3x f(x) = \sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}$ from p. 478.
- Hence

$$f(x) = \frac{\frac{x}{(1-x)^2} + 1}{1 - 3x}.$$

[a]Recall p. 566.

The Method of Generating Functions (continued)

- Now,

$$\begin{aligned} f(x) &= \frac{1}{1-3x} + \frac{x}{(1-x)^2(1-3x)} \\ &= \frac{7/4}{1-3x} + \frac{-1/4}{1-x} + \frac{-1/2}{(1-x)^2} \end{aligned}$$

  by a **partial fraction decomposition**.

  – The following equivalent form is *not* a partial fraction decomposition:

$$\frac{7/4}{-3x+1} + \frac{x-3}{(1-x)^2}.$$

The Method of Generating Functions (continued)

- Now,

$$
\begin{aligned}
\frac{7/4}{1-3x} &= (7/4)\,\frac{1}{1-3x} \\
&= (7/4)\sum_{n=0}^{\infty}(3x)^n, \\
\frac{-1/4}{1-x} &= -(1/4)\,\frac{1}{1-x} \\
&= -(1/4)\sum_{n=0}^{\infty}x^n, \\
\frac{-1/2}{(1-x)^2} &= -(1/2)\,\frac{1}{(1-x)^2} \\
&= -(1/2)\sum_{n=0}^{\infty}(n+1)\,x^n, \quad \text{from p. 477.}
\end{aligned}
$$

The Method of Generating Functions (concluded)

- Now,

$$
\begin{aligned}
& f(x) \\
= \quad & (7/4) \sum_{n=0}^{\infty} 3^n x^n - (1/4) \sum_{n=0}^{\infty} x^n - (1/2) \sum_{n=0}^{\infty} (n+1)\, x^n.
\end{aligned}
$$

- So

$$
a_n = (7/4)\, 3^n - (1/4) - (1/2)(n+1).
$$

- The methodology should be clear.

The Method of Generating Functions for $a_{n+1} - a_n = 3^n$ with $a_0 = 1$

- Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function for $a_0, a_1, \ldots$.
- From the recurrence equation, $\sum_{n=0}^{\infty} a_{n+1} x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=0}^{\infty} 3^n x^{n+1}$.
- $f(x) - a_0 - x f(x) = x \sum_{n=0}^{\infty} (3x)^n = \frac{x}{1-3x}$.
- This implies that

$$f(x) = \frac{\frac{x}{1-3x} + 1}{1-x} = \frac{1/2}{1-3x} + \frac{1/2}{1-x} = (1/2) \sum_{n=0}^{\infty} (3^n + 1)\, x^n.$$

- Hence $a_n = (3^n + 1)/2$.

The Method of Generating Functions for $a_{n+1} - Aa_n = B$ Again

- Assume $A \neq 1$.
- We next obtain Eq. (100) on p. 613,

$$a_n = A^n [\, a_0 - a_0^{(p)} \,] + a_n^{(p)},$$

  by the method of generating functions.
- Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function for $a_0, a_1, \ldots$.

The Proof (continued)

- Then

$$\sum_{n=0}^{\infty} a_{n+1} x^{n+1} - \sum_{n=0}^{\infty} A a_n x^{n+1} = \sum_{n=0}^{\infty} B x^{n+1}.$$

- So

$$f(x) - a_0 - Axf(x) = Bx \frac{1}{1-x}$$

  from p. 474.

The Proof (continued)

- Simplify the identity to yield

$$
\begin{aligned}
f(x) &= \frac{a_0}{1-Ax} + \frac{Bx}{(1-x)(1-Ax)} \\
&= \frac{a_0}{1-Ax} + \frac{B}{1-A}\left(\frac{1}{1-x} - \frac{1}{1-Ax}\right) \\
&= \frac{a_0}{1-Ax} + a_n^{(p)}\left(\frac{1}{1-x} - \frac{1}{1-Ax}\right) \\
&= \left[a_0 - a_n^{(p)}\right]\frac{1}{1-Ax} + a_n^{(p)}\frac{1}{1-x},
\end{aligned}
$$

where $a_n^{(p)} = B/(1-A)$, matching Eq. (99) on p. 612.

The Proof (concluded)

- From p. 474,

$$f(x) = \left[a_0 - a_n^{(p)} \right] \sum_{n=0}^{\infty} A^n x^n + a_n^{(p)} \sum_{n=0}^{\infty} x^n.$$

  – Note that $a_n^{(p)}$ is independent of n.

- So

$$a_n = A^n \left[a_0 - a_n^{(p)} \right] + a_n^{(p)},$$

  matching the earlier solution (100) on p. 613 as desired.

Convolution

- Consider the following recurrence equation,

$$b_{n+1} = b_0 b_n + b_1 b_{n-1} + \cdots + b_{n-1} b_1 + b_n b_0.$$

- Let $f(x) = \sum_{n=0}^{\infty} b_n x^n$.
- Then

$$\sum_{n=0}^{\infty} b_{n+1} x^{n+1} = \sum_{n=0}^{\infty} (b_0 b_n + b_1 b_{n-1} + \cdots + b_n b_0)\, x^{n+1}.$$

- So $f(x) - b_0 = x f^2(x)$ from p. 484.

The Proof (continued)

- When $b_0 = 1$,

$$f(x) = \left(1 \pm \sqrt{1-4x}\,\right)/(2x).$$

- Pick

$$f(x) = \left(1 - \sqrt{1-4x}\,\right)/(2x)$$

to match b_0.[a]

- By Eq. (70) on p. 495,

$$\sqrt{1-4x} = \sum_{n=0}^{\infty} \binom{1/2}{n} (-4x)^n = \sum_{n=0}^{\infty} \binom{1/2}{n} (-4)^n x^n.$$

---

[a] $f(0) = \infty$ if one picked $f(x) = (1+\sqrt{1-4x}\,)/(2x)$ instead (Graham, Knuth, & Patashnik, 1989).

The Proof (continued)

- Now, by Eq. (66) on p. 493,

$$\begin{aligned}
\binom{1/2}{n}(-4)^n &= \frac{\frac{1}{2}\,(\frac{1}{2}-1)\cdots(\frac{1}{2}-n+1)}{n!}\,(-4)^n \\
&= \frac{-1\cdot 1\cdot 3\cdots(2n-3)}{n!}\,2^n \\
&= -\frac{1\cdot 3\cdots(2n-3)\cdot n!}{n!\,n!}\,2^n \\
&= -\frac{1\cdot 3\cdots(2n-1)\cdot 2\cdot 4\cdots 2n}{(2n-1)\,n!\,n!} \\
&= -\frac{1}{2n-1}\binom{2n}{n}.
\end{aligned}$$

The Proof (concluded)

- So

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{2(2n-1)}\, x^{n-1} \\ &= \sum_{n=1}^{\infty} \frac{\binom{2n-2}{n-1}}{n}\, x^{n-1} \\ &= \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{n+1}\, x^{n}, \qquad (101) \end{aligned}$$

  the Catalan numbers![a]

[a] Recall Eq. (19) on p. 120.

An Example

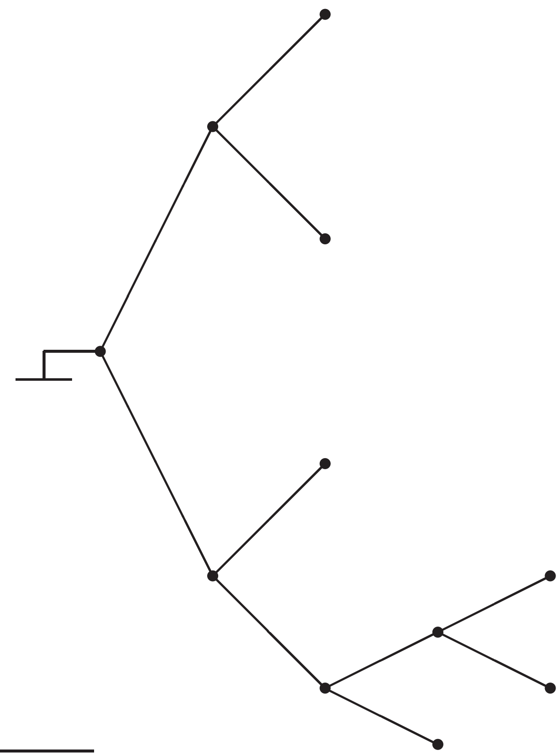
- It is easy to verify that

$$\begin{aligned} b_1 &= 1, \\ b_2 &= 2, \\ b_3 &= 5, \\ b_4 &= 14, \\ b_5 &= 42. \end{aligned}$$

- They indeed match

$$\frac{\binom{0}{0}}{1}, \frac{\binom{2}{1}}{2}, \frac{\binom{4}{2}}{3}, \frac{\binom{6}{3}}{4}, \frac{\binom{8}{4}}{5}, \frac{\binom{10}{5}}{6}, \ldots.$$

A Binary Tree[a]

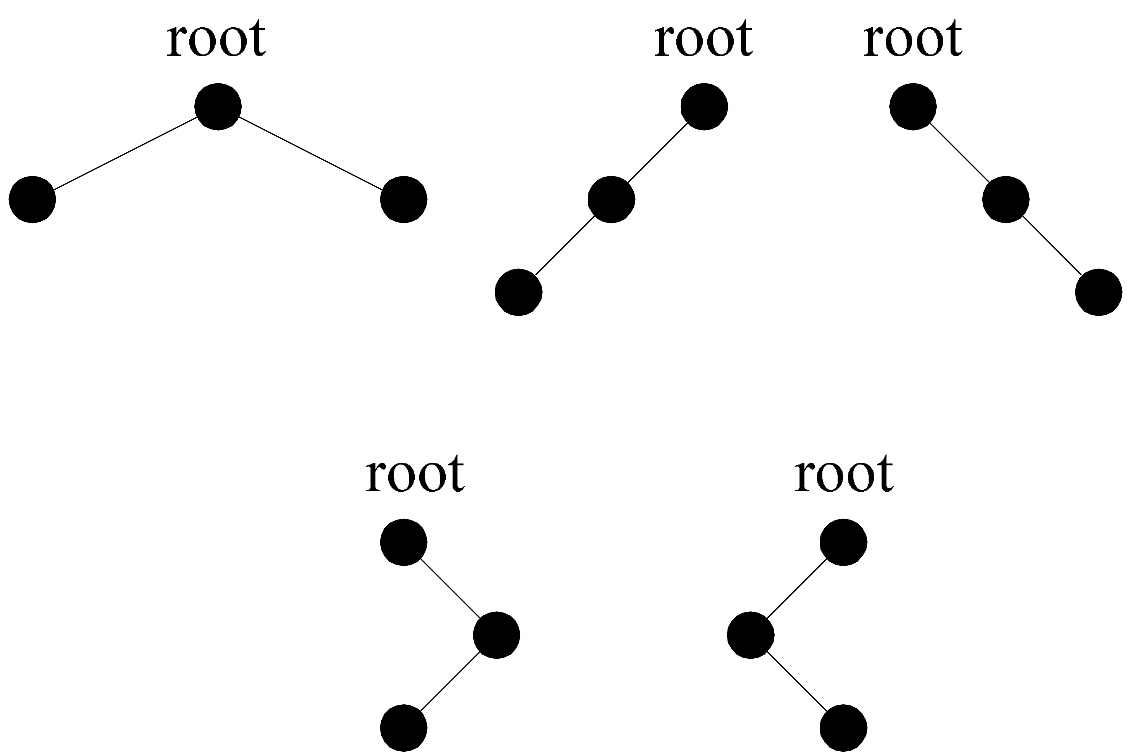


[a]Gustav Kirchhoff (1824–1887).

Number of Rooted Binary Trees

- There is a distinct node called the **root**.
- Every node has at most two descendants.
- A rooted binary tree is **ordered** if the left and right branches are considered distinct.
- What is the number b_n of rooted ordered binary trees on n nodes?

Illustration: $b_3 = 5$



Number of Rooted Binary Trees: The Formula

- $b_0 = 1$, as it is the empty tree.
- Recursively,

$$b_{n+1} = b_0 b_n + b_1 b_{n-1} + \cdots + b_{n-1} b_1 + b_n b_0.$$

  - $b_i b_{n-i}$: i nodes on the left and $n - i$ nodes on the right, $0 \leq i \leq n$.

- So b_n is the nth Catalan number by Eq. (101) on p. 635:

$$b_n = \frac{\binom{2n}{n}}{n+1}.$$

An Introduction to Graph Theory

If 50 million people believe a foolish thing,
it's still a foolish thing.
— George Bernard Shaw (1856–1950)

Graphs[a]

Vertex (= node)

- Let V be a finite nonempty set of nodes.
- Let $E \subseteq V \times V$ be a set of edges.
- $G = (V, E)$ is the directed graph (or digraph) made up of the node set V and the edge set E.
- When E is considered to consist of *unordered* pairs, (V, E) is called an **undirected graph**.[b]

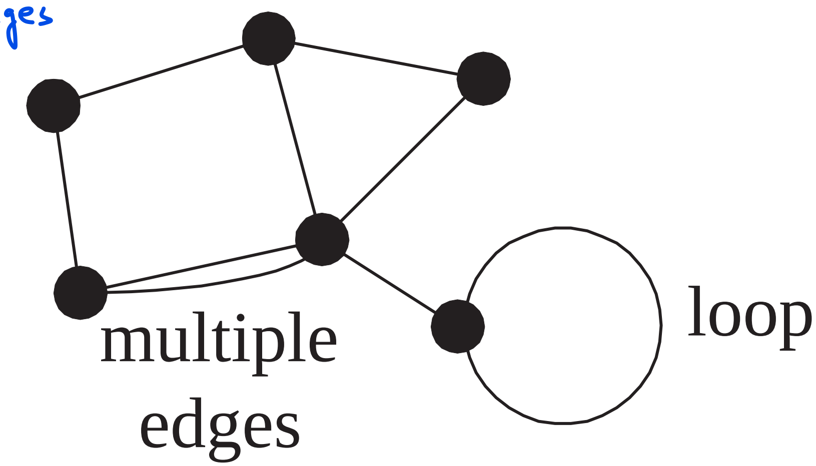
[a] Founded by Leonhard Euler in 1736.

[b] Assumed unless stated otherwise.

Graphs (continued)

- A graph is **loop-free** if it contains no (self-)loops.
  ("acyclic" is digraph)
- A **multigraph** allows parallel edges between nodes.

multiple edges



Graphs (concluded)

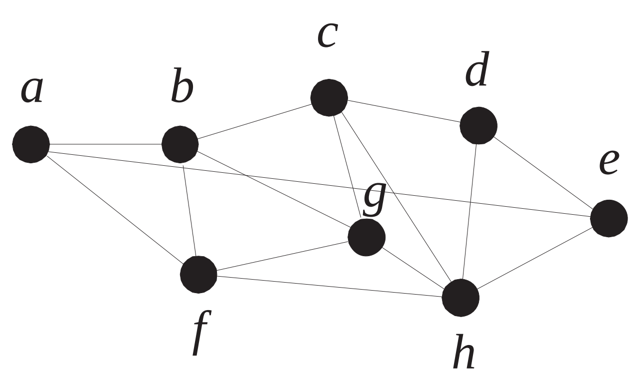
- A loop-free undirected graph without parallel edges between nodes is said to be **simple**.
- A node is **isolated** if it has no incident edges.
- For an undirected graph, we typically use $\{ x, y \}$ to represent an edge.
- For a digraph, we always use (x, y) to represent an edge.

Illustration of Graphs

- In the following graph G,

$$
\begin{aligned}
V &= \{ a, b, c, d, e, f, g, h \} \\
E &= \{ \{ a, b \}, \{ a, e \}, \{ a, f \}, \{ b, c \}, \{ b, g \}, \{ b, f \}, \\
&\quad \{ f, g \}, \{ f, h \}, \{ c, d \}, \{ c, h \}, \{ c, g \}, \\
&\quad \{ d, e \}, \{ d, h \}, \{ g, h \}, \{ h, e \} \}.
\end{aligned}
$$

→ undirected graph



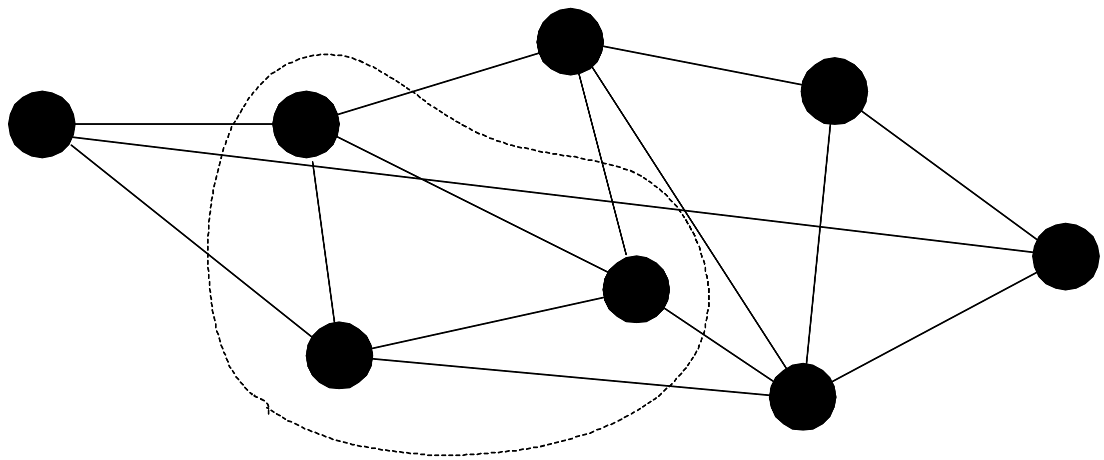
Applications of Graph Theory

- Representation of networks, both structured ones like interconnection networks and unstructured ones like telephone networks or social networks.
- Natural representation of relations (p. 363).
- A computation can be described as a digraph.
- Optimization problems such as circuit layout.
- Physical systems such as ferromagnetism.
- ...

Additional Notions

- Let $G = (V, E)$ be a graph (directed or otherwise).
- $G_1 = (V_1, E_1)$ is called a **subgraph** of G if
  - $\emptyset \neq V_1 \subseteq V$.
  - $E_1 \subseteq V_1 \times V_1$.
  - $E_1 \subseteq E$.
- G_1 is an **induced subgraph** of G if it is a subgraph of G and $E_1 = \underbrace{E}_{\text{in } E} \cap (\underbrace{V_1 \times V_1}_{\text{every possible edges}})$.
- An undirected graph G is **connected** if there is a path between any two distinct nodes of G.
- A **component** is a maximal subgraph that is connected.

Illustration of Subgraphs



All Kinds of Walks on Undirected Graphs

- A **walk** from x to y is a finite sequence of non-loop edges connecting x and y.
- The **length** of a walk is the number of *edges* in it.
- A walk from x to y where $x \neq y$ is called an **open walk**.
- A walk from x to itself is called a **closed walk**.
- A walk without repeated *edges* is called a **trail**.
- A closed trail is called a **circuit**.

All Kinds of Walks on Undirected Graphs (concluded)

- A walk without repeated *nodes* is a (**simple**) **path**.
- A closed path is called a **cycle**.
  - A cycle must be a circuit, but not vice versa.
- By convention, a cycle has at least 3 distinct edges.
- A cycle of even length is called an **even cycle**.
- A cycle of odd length is called an **odd cycle**.
- These definitions apply to digraphs with minimum changes.
- A digraph that has no cycles is **acyclic**.

Illustration of Walks

- (b, c, g, b, f) is a trail of length 4. "b" repeated, not path.
- (a, b, c) is a path of length 2.
- (a, b, c, d, e, a) is a cycle of length 5.
- $(g, b, c, g, h, e, a, f, g)$ is a circuit but not a cycle (as g is repeated).