Solutions to Assignment 4

October 25, 2000

Exercise 1 *(20 pts) Let* $G = (V, \Sigma, R, S)$ *be a context-free grammar such that* $V = \{E, T, F\}$, $\Sigma = \{a, +, *, (,)\}$, $S = E$ *and* R *is*

$$\begin{array}{lcl} E & \rightarrow & E + T \mid T \\ T & \rightarrow & T \times F \mid F \\ F & \rightarrow & (E) \mid a \end{array}$$

Give parse trees and leftmost derivations for the following strings.

1. *a*
2. *a*(a+a)*
3. *a+a+a*
4. *((a+(a)))*

Solution We present the leftmost derivations below.

1. $S \Rightarrow E \Rightarrow T \Rightarrow F \Rightarrow a$.
2. $S \Rightarrow E \Rightarrow T \Rightarrow T*F \Rightarrow F*F \Rightarrow a*F \Rightarrow a*(E) \Rightarrow a*(E+T) \Rightarrow a*(T+T) \Rightarrow a*(F+T) \Rightarrow a*(a+T) \Rightarrow a*(a+F) \Rightarrow a*(a+a)$
3. $S \Rightarrow E+T \Rightarrow E+T+T \Rightarrow T+T+T \Rightarrow T+T+T \Rightarrow F+T+T \Rightarrow a+T+T \Rightarrow a+F+T \Rightarrow a+a+T \Rightarrow a+a+F \Rightarrow a+a+T \Rightarrow a+a+a$
4. $S \Rightarrow T \Rightarrow F \Rightarrow (E) \Rightarrow (T) \Rightarrow (F) \Rightarrow ((E)) \Rightarrow ((E+T)) \Rightarrow ((T+T)) \Rightarrow ((F+T)) \Rightarrow ((a+T)) \Rightarrow ((a+F)) \Rightarrow ((a+(E))) \Rightarrow ((a+(T))) \Rightarrow ((a+(F))) \Rightarrow ((a+(a)))$

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Exercise 2 *Answer each part for the following context-free grammar.*

$$\begin{array}{lcl} R & \rightarrow & XRX \mid S \\ S & \rightarrow & aTb \mid bTa \\ T & \rightarrow & XTX \mid X \mid \epsilon \\ X & \rightarrow & a \mid b \end{array}$$

1. *What are the variables and terminals of G? Which is the start symbol? (The set of variables is* $\{R, S, X, T\}$ *and the set of terminals is* $\{a, b\}$*. R is the start symbol.)*
2. *Give three examples of strings in* $L(G)$*. (ab, ba, aab)*
3. *Give three examples of strings not in* $L(G)$*. (a, b, aa)*

4. *True or False:* $T \Rightarrow aba$. *(False)*

5. *True or False:* $T \Rightarrow^* aba$. *(True)*

6. *True or False:* $T \Rightarrow T$. *(False)*

7. *True or False:* $T \Rightarrow^* T$. *(True)*

8. *True or False:* $XXX \Rightarrow^* aba$. *(True)*

9. *True or False:* $X \Rightarrow^* aba$. *(False)*

10. *True or False:* $T \Rightarrow^* XX$. *(True)*

11. *True or False:* $T \Rightarrow^* XXX$. *(True)*

12. *True or False:* $S \Rightarrow^* \epsilon$. *(False)*

13. *Give a description of* $L(G)$ *in English. Every word in* $L(G)$ *is of form* w_1awbw_2 *or* w_1bwaw_2 *for some* $w, w_1, w_2 \in \{a, b\}^*$ *such that* w_1 *and* w_2 *have the same length.*

Exercise 3 *Give context-free grammars that generate the following languages.*

1. $\{w \mid w$ *contains at least three 1's*$\}$

2. $\{w \mid w$ *starts and ends with the same symbol*$\}$

3. $\{w \mid$ *the length of* w *is odd*$\}$

4. $\{w \mid$ *the length of* w *is odd and its middle is* $0\}$

5. $\{w \mid w$ *contains more 1's than 0's*$\}$

6. $\{w \mid w$ *is a palindrome, i.e.,* $w = w^R\}$

7. *The empty set*

Solution

1. Here is a context-free grammar for $L = \{w \mid w \text{ contains at least three 1's}\}$:

$$\begin{array}{rcl} S & \rightarrow & T1T1T1T \\ T & \rightarrow & 0T \mid 1T \mid \epsilon \end{array}$$

2. Here is a context-free grammar for $L = \{w \mid w \text{ starts and ends with the same symbol}\}$:

$$\begin{array}{rcl} S & \rightarrow & 0T0 \mid 1T1 \\ T & \rightarrow & 0T \mid 1T \mid \epsilon \end{array}$$

3. Here is a context-free grammar for $L = \{w \mid \text{the length of } w \text{ is odd}\}$:

$$\begin{array}{rcl} S & \rightarrow & 0T \mid 1T \\ T & \rightarrow & 0S \mid 1S \mid \epsilon \end{array}$$

Note that S and T generate words with even and odd lengths, respectively.

4. Here is a context-free grammar for $L = \{w \mid \text{the length of } w \text{ is odd and its middle is } 0\}$:

$$S \rightarrow 0S0 \mid 0S1 \mid 1S0 \mid 1S1 \mid 0$$

5. Here is a context-free grammar for $L = \{w \mid w \text{ contains more 1's than 0's}\}$:

$$\begin{array}{rcl} S & \rightarrow & TS \mid 1T \mid 1S \\ T & \rightarrow & TT \mid 0T1 \mid 1T0 \mid \epsilon \end{array}$$

   Note that T generates all words in which there are equal number of 1's and 0's. If a word w contains more 1's that 0's, then w must be of one of the following forms.

   - $w = 1w_1$ such that w_1 contains more 1's than 0's.
   - $w = 1w_1$ such that w_1 contains equal number of 1's and 0's.
   - $w = w_1w_2$ such that w_1 contains equal number of 1's and 0's and w_2 contains more 1's than 0's.

6. Here is a context-free grammar for $L = \{w \mid w \text{ is a palindrome, i.e., } w = w^R\}$:

$$S \rightarrow 0S0 \mid 1S1 \mid 0 \mid 1 \mid \epsilon$$

7. Here is a context-free grammar for the empty set:

$$S \rightarrow S$$

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Exercise 4 *Please convert the following CFG into an equivalent CFG in Chomsky normal form, using the procedure given in Theorem 2.6.*

$$\begin{array}{rcl} A & \rightarrow & BAB \mid ABA \mid B \mid \epsilon \\ B & \rightarrow & 00 \mid \epsilon \end{array}$$

Solution We introduce a new start symbol S and obtain the following CFG.

$$\begin{array}{rcl} S & \rightarrow & A \\ A & \rightarrow & BAB \mid ABA \mid B \mid \epsilon \\ B & \rightarrow & 00 \mid \epsilon \end{array}$$

After ϵ-rule elimination, we generate the following CFG.

$$\begin{array}{rcl} S & \rightarrow & A \mid \epsilon \\ A & \rightarrow & BAB \mid ABA \mid B \mid BA \mid AB \mid AA \mid BB \\ B & \rightarrow & 00 \end{array}$$

After unit rule elimination, we generate the following CFG.

$$\begin{array}{rcl} S & \rightarrow & BAB \mid ABA \mid BA \mid AB \mid AA \mid BB \mid 00 \mid \epsilon \\ A & \rightarrow & BAB \mid ABA \mid BA \mid AB \mid AA \mid BB \mid 00 \\ B & \rightarrow & 00 \end{array}$$

We now convert the above rules into the proper form.

$$\begin{aligned}
S &\rightarrow BU \mid AV \mid BA \mid AB \mid AA \mid BB \mid WW \mid \epsilon \\
A &\rightarrow BU \mid AV \mid BA \mid AB \mid AA \mid BB \mid WW \\
B &\rightarrow WW \\
U &\rightarrow AB \\
V &\rightarrow BA \\
W &\rightarrow 0
\end{aligned}$$

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