CSC242: Homework 2.4
AIMA Chapter 80–8.3

1. Briefly define the following terms related to first-order logic:

   (a) Domain (of discourse)

   ANSWER: The objects that exist in the world being represented by a sentence or set of sentences of first-order logic.

   (b) Term

   ANSWER: A logical expression that refers to an object; constant symbols and functional expressions (function symbol applied to arguments that are terms).

   (c) Atomic sentence or atomic formula

   ANSWER: A predicate symbol applied to a list of arguments, each of which is a variable or term.

2. Describe the components of a first-order interpretation.

   ANSWER:

   - A mapping from constant symbols to objects
   - A mapping from function symbols to total functions (themselves total mappings from tuples of objects to objects)
   - A mapping from predicate symbols to relations (sets of tuples of objects)

3. Translate the following sentence of first-order logic into a *reasonable* sentence of English:

$$\forall x, y, l \;\; SpeaksLanguage(x, l) \wedge SpeaksLanguage(y, l) \Rightarrow \\ Understands(x, y) \wedge Understands(y, x). \tag{1}$$

   ANSWER: People who speak the same language understand each other. Any two people who speak a common language understand each other.

4. Explain why this sentence is entailed by the sentence

$$\forall x, y, l \; SpeaksLanguage(x, l) \wedge SpeaksLanguage(y, l) \Rightarrow Understands(x, y). \tag{2}$$

   ANSWER: It is easy to show that for any sentences Φ and Ψ:

$$\Phi \wedge \Psi \equiv \Psi \wedge \Phi.$$

   You should know how to prove this...

   Now consider any two individuals, A and B, that both understand some common language. Then by (2), any extended interpretation which maps x to A and y to B must also satisfy $Understands(A, B)$.

   But any interpretation that satisfies

$$SpeaksLanguage(A, L) \wedge SpeaksLanguage(B, L)$$

   also satisfies

$$SpeaksLanguage(B, L) \wedge SpeaksLanguage(A, L)$$

   thanks to the property stated above. And in that case, the interpretation must also satisfy $Understands(B, A)$, again thanks to (2).

   Therefore any interpretation which satisifies the antecedent of (1) (the left hand side of the implication) also satisfies consequent (the right-hand side), so the interpretation also satisfies (1). Hence, whenever (2) holds, so does (1).

5. Translate the following English sentences into first-order logic using the predicates $Understands$ and $FriendOf$:

   (a) Mutual understanding leads to mutual friendship.
   ANSWER:

$$\forall x, y \; Understands(x, y) \wedge Understands(y, x) \Rightarrow FriendOf(x, y) \wedge FriendOf(y, x)$$

   (b) My friend's friends are also my friends (that is, friendship is transitive).
   ANSWER:

$$\forall x, y, z \; FriendOf(x, y) \wedge FriendOf(y, z) \Rightarrow FriendOf(x, z)$$

6. Explain what is wrong with the following proposed definition of adjacent squares in the wumpus world:

$$\forall x, y Adjacent([x, y], [x + 1, y]) \wedge Adjacent([x, y], [x, y + 1]).$$

ANSWER: There are several problems:

- It allows one to prove, say, $Adjacent([1, 1], [1, 2])$ but not $Adjacent([1, 2], [1, 1])$, so we would need an additional symmetry axiom for that.
- It does not allow one to prove that $Adjacent([1, 1], [1, 3])$ is false, so it needs to be re-written as a biconditional.
- Finally, it does not work at the boundaries of the world, so some extra conditions must be added for the edge cases.

7. Write out the axioms for reasoning about the wumpus' location, using a constant symbol $Wumpus$, unary predicate $Smelly$, and binary predicates In, $Adjacent$, and equality if you need it. Remember that there is only one wumpus.

ANSWER: By analogy with AIMA Eq. 8.4 (p. 306), the main axiom is:

$$\forall r\ Smelly(r) \iff \exists s\ Adjacent(r, s) \wedge In(Wumpus, s)$$

Note that if the predicates could be applied to objects other than rooms, then we might need to use a unary type predicate like $Room$ to restrict the axiom to apply only to rooms (as opposed other things that might be $Adjacent$ and $Smelly$).

From this, you can derive useful rules, such as:

$$\forall r \forall s\ Adjacent(r, s) \wedge In(Wumpus, r) \Rightarrow Smelly(s)$$

Note however that the following does not follow from our axiom:

$$\forall r \forall s\ Adjacent(r, s) \wedge Smelly(r) \Rightarrow In(Wumpus, s)$$

Why not? Think about it... Answer: because there could be other things that make a room smelly. If wumpuses are the only thing that makes a room smelly, then we need to state that as an axiom.

For "there is only one wumpus," we need something like the following:

$$\exists r\ In(Wumpus, r) \wedge \forall s\ (r \neq s) \Rightarrow \neg In(Wumpus, s)$$

In other words, there is a room that the wumpus is in, and it can't be in two places at the same time. Yes, you have to say things like that. Note that $x \neq y$ is the same as $\neg(x = y)$ (AIMA p. 299).

From this you can get useful rules, such as

$$\forall r, s\ In(Wumpus, r) \wedge In(Wumpus, s) \Rightarrow r = s$$

You should try to derive that for yourself.

8. Assuming predicates $Manager(p, q)$ and $Friendly(p)$, and constants $Juanita$ and $Kyle$, express each of the following as sentences of first-order logic. You may use the abbreviation "$\exists!$" to mean "there exists exactly one."

   (a) Juanita manages a friendly person (perhaps more than one, and perhaps unfriendly people also).
   ANSWER:
   $$\exists x\ Manages(Juanita, x) \wedge Friendly(x)$$

   (b) Juanita manages exactly one friendly person, but may manage unfriendly peopel also).
   ANSWER:
   $$\exists! x\ Manages(Juanita, x) \wedge Friendly(x)$$

   (c) Juanita manages exactly one person and that person is friendly.
   ANSWER:
   $$\exists x\ Manages(Juanita, x) \wedge Friendly(x) \wedge [\forall y\ Manages(Juanita, y) \Rightarrow y = x]$$

   (d) Juanita and Kyle manage exactly one person together.
   ANSWER:
   $$\exists! x\ Manages(Juanita, x) \wedge Manages(Kyle, x)$$

   (e) Juanita manages at least one person with Kyle, and no people with anybody else.
   ANSWER:
   $$\begin{aligned} &\exists x\ Manages(Juanita, x) \wedge Manages(Kyle, x) \wedge \\ &\quad \forall d, p\ [Manages(Juanita, d) \wedge Manages(p, d)] \Rightarrow [p = Juanita \vee p = Kyle] \end{aligned}$$

9. Arithmetic statements can be written using first-logic with the binary predcate symbol $<$, the function symbols $+$ and $\times$, and the constant symbols 0 and 1 (and equality). Additional predicates can be defined using biconditionals (as you might recall from CSC173).

   (a) Represent the statement "x is an even number" by defining a unary predicate $Even$.
   ANSWER:
   $$\forall x\ Even(x) \iff \exists y\ x = y + y$$

   (b) Represent the statement "x is prime" by defining a unary predicate $Prime$.
   ANSWER:
   $$\forall x\ Prime(x) \iff \forall y, z\ x = y \times z \Rightarrow [y = 1 \vee z = 1]$$

(c) Goldbach's conjecture (Wikipedia) is the conjecture (unproven as yet) that every even number is the sum of two primes. Represent this conjecture as a logical sentence.

ANSWER:

$$\forall x \; Even(x) \Rightarrow \exists y, z \; Prime(y) \wedge Prime(z) \wedge x = y + z$$