CS203

Probability - way of quantyfying uncertainity.

working Sat - 2, 9, 16 March, 11-1

Grading :- 2×10 Quizzes, 20 Prog Ass, 60 Endsem.

Quiz: 1) 11/13 March 2) 8^{th} April.

Sets: A collection of objects. (No ordering)

- Member of, subset, superset, empty set, universal set.
- Finite sets, Countably infinite (can create a 1-1 mapping to Natural numbers), uncountable set: Ex $[0,1)$

Diagnolization argument for $[0,1)$

List a set of numbers, and chose a number with different value at each diagnol element after decimal.

$0.\boxed{1}\ 3\ 4\ 5\ 7\ 9$

$0.\ 7\ \boxed{9}\ 8\ 3\ 2\ 1$

⋮

- Product set : $A \times B = \{(a_1, b_1), (a_2, b_2) \ldots \mid a_i \in A, b_i \in B\}$
- Disjoint set : No intersection but union is U.

If more than 2 sets we call the subsets partitions.

- Associavity : $\otimes$: $(A \otimes B) \otimes C = A \otimes (B \otimes C)$
- Commutativity : $\otimes$: $A \otimes B = B \otimes A$
- Distributivity : $\odot$ over $\otimes$: $A \odot (B \otimes C) = (A \odot B) \otimes (A \odot C)$

Probability Models :

Experiment - Process yields an outcome

→ Exactly one outcome (elementary events)

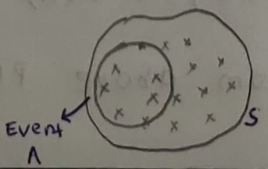
→ Known beforehand

Sample Space (S) : set of all possible outcomes

Ω

Event : set of outcomes

Designing a sample space is crutial.



Outcomes: They are mutually exclusive, collectively exhaustive

Sample space Conditions:

1. S must be an event

2. If A is an event, $\bar{A}$ must also be an event.

3. If $\{A_1, A_2 \dots\}$ is a countable set

$$S = \bigcup_{i=1}^{\infty} A_i$$

Probability Laws:

"Probability measure" $P(\cdot)$

$P(A) \longrightarrow$ Likelihood of event A.

Suppose we spread a unit mass, across the sample space

The mass of an event is the probability. (Hence also known as probability mass).

Probability Axioms:

1. (Non negativity): $\forall A_i \; P(A_i) \geq 0$

2. (Additivity): A, B are disjoint sets $P(A \cup B) = P(A) + P(B)$

3. (Normalization): $P(S) = 1$

Properties:

1) $P(\emptyset) = 0$

Pf: $P(S) = P(S \cup \emptyset) = P(S) + P(\emptyset)$

$\Rightarrow P(\emptyset) = 0$

2) If $A \subset B$, $P(A) \leq P(B)$

$$P(B) = P(A \vee (B \setminus A)) = P(A) + \underbrace{P(B \setminus A)}_{\geq 0}$$

$\Rightarrow P(B) \geq P(A)$

3) $P(A \vee B) = P(A) + P(B) - P(A \cap B)$

Pf: From above $P(B \setminus A) \Rightarrow P(B) = P(B \setminus A \cup B \cap A)$

$$P(B) = P(B/A) + P(A \cap B)$$

$$P(B/A) = P(B) - P(A \cap B)$$

$$\Rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

4) For disjoint sets : $P(A_1 \cup A_2 \ldots A_n) = P(A_1) + P(A_2) + \ldots$

By associavity $A_1 \cup A_2 \ldots A_n = A_1 \cup (A_2 \ldots A_n)$

Union bound

$$\underbrace{P\left(\bigcup_{i=1}^{n} A_i\right)} \leq \sum_{i=1}^{n} P(A_i)$$

Atleast one of the event happening

$$P\left(\bigcup_{i=1}^{n} A_i\right) = 1 - \underbrace{P\left(\bigcap_{i=1}^{n} \bar{A_i}\right)}$$

None of the events happening

Bonferroni Inequality

$$P\left(\bigcap_{i=1}^{n} A_i\right) \geq 1 - \sum_{i=1}^{n} P(\bar{A_i})$$

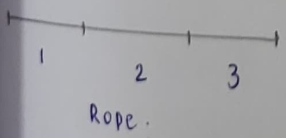
Discrete Prob. Law:

$$P(\{s_1, \ldots, s_n\}) = P(s_1) + P(s_2) + \ldots + P(s_n)$$

Uniform discrete prob. Law:

$$P(A) = \frac{|A|}{|S|}$$

Can't use for infinite events



Prob cutting in 1 : $\frac{L(1)}{L(\text{Rope})}$ ($L \to$ Length)

$\rightarrow C(n,k) = C(n-1,k) + C(n-1,k-1)$

$\hookrightarrow$ in / out

$\binom{n-1}{k-1} \quad \binom{n-1}{k}$

$\rightarrow k \cdot C(n,k) = n \cdot C(n-1,k-1)$

$\hookrightarrow$ in LHS choose k ppl and make 1 the leader. In LHS choose leader, then remaining k-1 ppl.

Ex: r types of cakes in a bakery, need to buy n cakes.

$C(n+r-1, r-1) \longrightarrow$ Bar method

$\underbrace{x \; x}_{\text{type 1}} \} \underbrace{x \; x \; x}_{\text{type 2}} \} \underbrace{x \; x \; x}_{\text{type 3}} \} \underbrace{x}_{\text{type 4}}$

Ex: n teams playing knockouts, what is the probability that there is an India, Pakistan match in the competition. If odd teams, 1 team randomly qualifies

$$\frac{2}{n} <= \frac{n-1}{{}^nC_2}$$

Total no. of matches

Principal of Inclusion Exclusion (PIE)

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) \cdots$$

Matching Problem:

A_i: Event that i^{th} name has the current photo

$$P(A_i) = \frac{(n-1)!}{n!} = \frac{1}{n}$$

$$P(A_1 \cap A_2) = \frac{(n-2)!}{n!}$$

$$P(A_1 \cup A_2 \cdots) = n \times \frac{1}{n} - {}^nC_2 \times \frac{1}{n(n-1)} + {}^nC_3 \times \frac{1}{n(n-1)(n-2)}$$

$$= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!}$$

$$e^x = 1 + x + \frac{x^2}{2!} \cdots$$

$\Rightarrow$ Converges to $\left(1 - \frac{1}{e}\right)$

Coupon collector problem

We are throwing numbered balls from 1 to n, into r identical boxes. Probability that no box is left empty

A_i : No ball goes into i^{th} box

$$P(A_i) = \frac{(r-1)^n}{r^n}$$

H.W

Conditional Probability:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$\Rightarrow$ Axioms: 1) Non negativity

2) $P(\underbrace{A_1 \cup A_2}_{\text{Disjoint}} \mid B) = P(A_1 \mid B) + P(A_2 \mid B)$

3) $P(S \mid B) = 1$

f of 2: $P(A_1 \cup A_2 \mid B) = \dfrac{P((A_1 \cup A_2) \cap B)}{P(B)}$

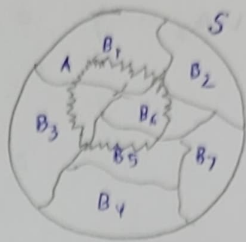
Disjoint as well

$$= \frac{P((A_1 \cap B) \cup (A_2 \cap B))}{P(B)}$$

$$= \frac{P(A_1 \cap B) + P(A_2 \cap B)}{P(B)}$$

$$= P(A_1 \mid B) + P(A_2 \mid B)$$

Total probability:



$$P(A) = \sum P(A \mid B_i) \times P(B_i)$$

$B_i \longrightarrow$ Partitions of S.

↳ Mutually exclusive & exhaustive

$$P(A \mid C) = \sum P(A \mid B_i \cap C) \cdot P(B_i \mid C)$$

Bayes theorem:

$$P(B_i \mid A) = \frac{P(A \mid B_i) \times P(B_i)}{\sum_{i=1}^{n} P(A \mid B_i)\, P(B_i)}$$

$$P(A \cap B \cap C) = P(A \mid B \cap C) \cdot P(B \mid C) \cdot P(C)$$

Ex: A Radar correctly detects an aircraft in it's territory with probability of 0.9, and probability for a false alarm is 0.1. Prob. of aircraft is present in that region is 0.05.

Prob. that there is a plane but no alarm being raised is.

$P(A)$ ↳ Plane present

$P(B)$ ↳ Radar raising alarm. cor

$$P(\neg B / A) = \frac{P(\neg B \cap A)}{P(A)} = \frac{P(A) - P(A \cap B)}{P(A)}$$

$$= 1 - \underbrace{P(B/A)}_{0.9}$$

$P(A \mid B)$ vs $P(A)$

$P(A \mid B) > P(A)$: A is attracted to B.

$P(A \mid B) < P(A)$: A is repelled by B.

$P(A \mid B) = P(A)$: A is indifferent

Independence:

A_1, A_2 are independent

$$P(A_2 \mid A_1) = P(A_2)$$

$$P(A_1 \cap A_2) = P(A_1) \cdot P(A_2)$$

Ex: A disease occurs with $\frac{1}{100,000}$ probability. The accuracy of the test is 99% (i.e. has disease correct result with 99%, if doesnt have disease correct result with 99%).

If the test is positive, what is the probability that the person has the disease.

$$\frac{1}{100,000} = P(D)$$

$$0.99 = P(T \mid D) = P(\neg T \mid \neg D)$$

$$P(D \mid T) = \frac{P(T \mid D) \times P(D)}{P(T)}$$

$$P(T \cap D) = 0.99 \times \frac{1}{100,000}$$

$$\frac{P(\neg(T \cup D))}{1 - P(D)} = 0.99 = \frac{1 + P(T \cap D) - P(T) - P(D)}{1 - P(D)}$$

$$0.99 \times \frac{99,999}{100\,000} = 1 + \frac{0.99 \times 1}{100000} - P(T) - \frac{1}{100000}$$

$$P(T) = 1 - \frac{0.01}{100000} - \frac{0.99 \times 99,999}{100000}$$

$$P(D \mid T) = \frac{0.99}{100000} \Big/ P(T) = \frac{0.99}{100000 - 0.01 - 0.99 \times 999\ldots} \quad \frac{0.99}{0.01 \times 99\ldots + 0.99}$$

There are 3 doors, host invities audience member to choose a door. Host opens one of the 2 unchosen doors to reveal a goat. Should the audience member need to change the chosen door to increase prob. of winning a car (2 goats, 1 car)

Let A be the initial door chosen

A	B	C	
C	G	G	win
G	G	C	lose
G	C	G	lose

Now if he switches door, probability of winning is $\frac{2}{3}$ from $\frac{1}{3}$ before!

Q) Probability of never getting a head in a coin toss

p : Prob of head

q : Prob of tail $= (1-p)$

$$= q + q \cdot q + q \cdot q \cdot q \dots = \frac{q}{1-q} = \frac{q}{p}$$

Prob of keep tossing till head (Eventually I'll always get a head)

$$= p + qp + q^2 p \dots = \frac{p}{1-q} = \frac{p}{p} = 1$$