CS 292C Computer-Aided Reasoning for Software

Lecture 7: Satisfiability Modulo Theories

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Fall 2019

Summary of previous lecture

- 3rd paper review is out
- Applications of SAT (Max SAT, Partial Max SAT, etc.)

Outline of this lecture

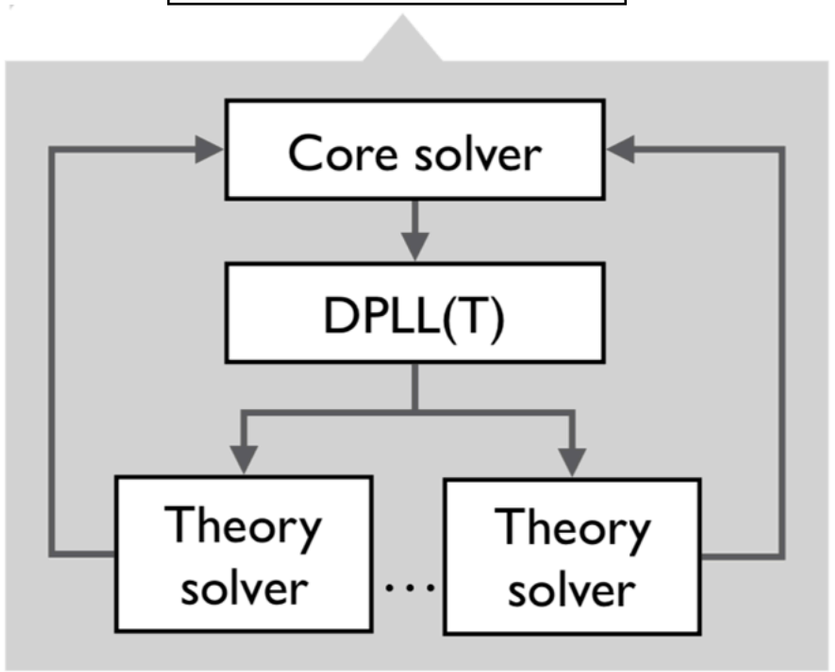
- Introduction to Satisfiability Modulo Theories (SMT)
- Syntax and semantics of first-order logic
- Overview of key theories

Satisfiability Modulo Theories

Theories: $x = g(y)$ $2x + y \leq 5$ $a[i] = x$ $(b >> 2) = c$

First order logic

SMT solver



Syntax of First-Order Logic (FOL)

Logical symbols

- Connectives: ¬,∧,∨,→,↔
- Parentheses: ()
- Quantifiers: ∀,∃

Non-logical symbols

- Constants: x,y,z
- N-ary functions: f, g
- N-ary predicates: p, q
- Variables: u,v,w

Syntax of First-Order Logic (FOL)

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quantifier-free fragment of FOL.

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quantifier-free fragment of FOL.

quantifier-free **ground** formulas.

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- A **term** is a constant or an n-ary function with n terms.
- An **atom** is $\top$, $\bot$, or an n-ary predicate applied to n terms.
- A **literal** is an atom or its negation.
- A (quantifier-free ground) **formula** is a literal or the application of logical connectives to formulas.

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$$\text{isPrime}(x) \rightarrow \neg\, \text{isInteger}(\text{sqrt}(x))$$

Semantics of FOL ⟨U, I⟩

Universe

- A non-empty set of values
- Finite or (un)countably infinite

Interpretation

- Maps a constant symbol c to an element of U: I[c] in U
- Maps an n-ary function symbol f to a function $f_I : U^n \rightarrow U$
- Maps an n-ary predicate symbol p to an n-ary relation $p \subseteq U^n$

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U = {☀, ☁}

I[x] = ☀

I[y] = ☁

I[f] = {☀ ↦ ☁, ☁ ↦ ☀}

I[p] = {⟨☀,☀⟩, ⟨☀,☁⟩}

⟨U, I⟩ ⊨ p(f(y), f(f(x))) ?

Emina's example doesn't apply to Santa Barbara :)

Satisfiability and validity of FOL

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F is **satisfiable** iff $M \models F$ for some structure $M = \langle U, I \rangle$.

F is **valid** iff $M \models F$ for all structures $M = \langle U, I \rangle$.

Satisfiability and validity of FOL

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F is **valid** iff $M \models F$ for all structures $M = \langle U, I \rangle$.

Duality of satisfiability and validity:

F is valid iff ¬*F* is unsatisfiable.

First-order theories

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.A formula F is **satisfiable modulo T** iff $M \models F$ for someT-model M.

A formula F is **valid modulo T** iff $M \models F$ for all T-models M.

Common theories

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Equality (and uninterpreted functions)

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- x = g(y)

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Linear arithmetic (over R and Z)

Common theories

Equality (and uninterpreted functions)

- $x = g(y)$

Fixed-width bitvectors

- $(b >> 1) = c$

Linear arithmetic (over R and Z)

- $2x + y \leq 5$

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Arrays

Common theories

Equality (and uninterpreted functions)

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Linear arithmetic (over R and Z)

- $2x + y \leq 5$

Arrays

- a[i] = x

Theory of equality with uninterpreted functions

- **Signature: {=, x, y, z, ..., f, g, ..., p, q, ...}**
  - The binary predicate = is *interpreted*.
  - All constant, function, and predicate symbols are *uninterpreted*.
- **Axioms**
  - $\forall x.\ x = x$
  - $\forall x,y.\ x=y \rightarrow y=x$
  - $\forall x,y,z.\ x=y \wedge y=z \rightarrow x=z$
  - $\forall x_1,...,x_n,y_1,...,y_n.(x_1 = y_1 \wedge ... \wedge x_n = y_n) \rightarrow (f(x_1,...,x_n) = f(y_1,\ldots,y_n))$
  - $\forall x_1,...,x_n,y_1,...,y_n.(x_1 = y_1 \wedge ... \wedge x_n = y_n) \rightarrow (p(x_1,...,x_n) \leftrightarrow p(y_1,...,y_n))$
- **Deciding T=**
  - Conjunctions of literals modulo T= is decidable in polynomial time.

$T_=$ example: checking program equivalence

```
int fun1(int y) {
  int x, z;
  z = y;
  y = x;
  x = z;
  return x * x;
}

int fun2(int y) {
  return y * y
}
```

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A formula that is unsatisfiable iff programs are equivalent:

$(z_1 = y_0 \wedge y_1 = x_0 \wedge x_1 = z_1 \wedge r_1 = x_1 * x_1) \wedge$
$(r_2 = y_0 * y_0) \wedge$
$\neg(r_2 = r_1)$

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Using 32-bit integers, a SAT solver fails to return an answer in 5 min.

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A formula that is unsatisfiable iff programs are equivalent:

$(z_1 = y_0 \wedge y_1 = x_0 \wedge x_1 = z_1 \wedge r_1 = \mathrm{mul}(x_1, x_1)) \wedge$
$(r_2 = \mathrm{mul}(y_0, y_0)) \wedge$
$\neg(r_2 = r_1)$

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A formula that is unsatisfiable iff programs are equivalent:

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Using T=, an SMT solver proves unsatisfiability in a fraction of a second.

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int fun1(int y) {
  int x, z;
  z = y;
  y = x;
  x = z;
  return x * x;
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int fun2(int y) {
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}
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Using T=, an SMT solver proves unsatisfiability in a fraction of a second.

$T_=$ example: checking program equivalence

```
int fun1(int y) {
 int x, y;
 x = x ^ y;
 y = x ^ y;
 x = x ^ y;
 return x * x;
}
```

```
int fun2(int y) {
 return y * y
}
```

$T_=$ example: checking program equivalence

```
int fun1(int y) {
 int x, y;
 x = x ^ y;
 y = x ^ y;
 x = x ^ y;
 return x * x;
}
```

```
int fun2(int y) {
 return y * y
}
```

Is the uninterpreted function abstraction going to work?

- No, we need the theory of fixed-width bitvectors to reason about ^ (xor).

Theory of fixed-width bitvector

Signature

- Fixed-width words modeling machine ints, longs, ...
- Arithmetic operations: bvadd, bvsub, bvmul, ...
- Bitwise operations: bvand, bvor, bvnot, ...
- Comparison predicates: bvlt, bvgt, …
- Equality: =
- Expanded with all constant symbols: x, y, z, ...

Deciding T_{BV}

- NP-complete.

Theory of linear integer and real

Signature

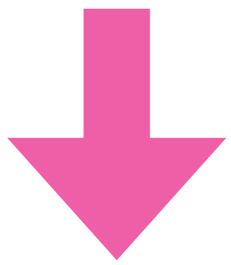
- Integers (or reals)
- Arithmetic operations: multiplication by an integer (or real) number, +, -.
- Predicates: =, ≤.
- Expanded with all constant symbols: x, y, z, ...

Deciding T_{LIA} and T_{LRA}

- NP-complete for linear integer arithmetic (LIA). Polynomial time for linear real arithmetic (LRA).
- Polynomial time for difference logic (conjunctions of the form $x - y \leq c$, where c is an integer or real number).

LIA example: compiler optimization

```
for (i=1; i<=10; i++) {
  a[j+i] = a[j];
}
```



```
int v = a[j];
for (i=1; i<=10; i++) {
  a[j+i] = v;
}
```

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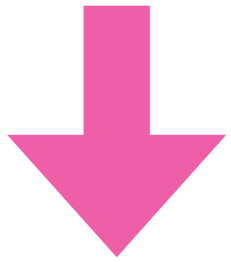

```
int v = a[j];
for (i=1; i<=10; i++) {
  a[j+i] = v;
}
```

A LIA formula that is unsatisfiable iff this transformation is valid:

$(i \geq 1) \wedge (i \leq 10) \wedge$
$(j + i = j)$

Polyhedral model

https://en.wikipedia.org/wiki/Polytope_model

Theory of arrays

Signature

- Array operations: read, write
- Equality: =
- Expanded with all constant symbols: x, y, z, ...

Axioms

- ∀a, i, v. read(write(a, i, v), i) = v
- ∀a, i, j, v. ¬(i = j) → (read(write(a, i, v), j) = read(a, j))
- ∀a, b. (∀i. read(a, i) = read(b, i)) → a = b

Deciding TA

- Satisfiability problem: NP-complete.
- Used in many software verification tools to model memory.

TODOs by next lecture

- The 3rd reading assignment will be due
- The 2nd homework will be out
- Start to work on the proposal for your final project