Supplemental notes for *Tape et al.* (2009):
"Multiscale estimation of GPS velocity fields"

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Contents

1 Text Excerpts

Representing functions on a sphere is standard problem in geophysics. Here we assume that an inverse problem has been performed to obtain a continuous vector field on the surface of a sphere. This set of notes pertains to how one can compute the spatial derivatives of the vector field.

Motivation (SCEC poster)

As the number of GPS stations in regional networks increases (Figure 1), we are afforded the opportunity to detect smaller-amplitude transients, where potential targets include small earthquakes, post-seismic deformation, aseismic events, and seasonal signals. Detection methods should not simply rely on large deviations in a single GPS time series, independent of its neighbors. Here we propose to represent GPS time series data using an approach that simultaneously considers multiple spatial and temporal scales. On this poster we illustrate the multiscale spatial representation of the velocity field derived from the SCEC Crustal Motion Map, v3.0 (CMM3) (*Shen et al.*, 2003).

Overview (SCEC poster)

In the spatial domain, we parameterize the continuous velocity field using a set of local basis functions, similar to the approach of *Beavan and Haines* (2001). Novel aspects of our approach include: (1) use of multiscale basis functions; (2) a minimum scale at which we estimate the velocity field at a particular location that is controlled by the local station coverage; (3) computation of strain rate tensors directly from the estimated surface velocity field, where the strain rates take into account the vertical component of the velocity (if it is available).

By focusing on quantities inferred from multiple sites (e.g., strain), our suggested approach does not rely on individual stations to detect transient signals, and instead focuses on locally correlated behavior. A transient detected at a given time and location will be characterized by a spatial scale, a time scale, and an empirically determined associated detection threshold.

Here we illustrate only the spatial aspects of our proposed multiscale approach, using both data from CCMM1 (Figure **??**) as well as synthetic data (Figure **??**). The spatial basis functions are spherical splines (Figure **??**). The multiscale aspects are achieved by using splines from progressively finer meshes, but only where justifiable based on the local site density. We estimate the coefficients by including all scales simultaneously, while forcing power to the longest scales using a minimum model length constraint (i.e., damped least squares).

Early draft

As the number of GPS stations in a regional network increases, and as the sampling rate of the position measurements increases, it becomes a formidable task to efficiently represent the enormous amount of data. We seek to represent the time-varying GPS time series at the stations in a network by using a multiresolution approach, that is, one that considers multiple scalelengths and multiple time intervals. The objective is to monitor the volume of data in a manner that allows for the detection of "transient" signals, in other words, a signal that deviates from a designated background pattern over a particular scalelength and time window. Thus, a transient could be an earthquake, an aseismic event (*Dragert et al.*, 2001; *Ozawa et al.*, 2002), or a seasonal signal (*Heki*, 2004), among others.

Several studies have used GPS observations to detect transient events of crustal deformation: for example, the subduction zones of Japan (*Miyazaki et al.*, 2003, 2004), Andes (*Melbourne et al.*, 2002), and Cascadia (*Dragert et al.*, 2001; *Rogers and Dragert*, 2003). These are the type of events we would like to be able to detect in an automated fashion, whereby we would have a characterization of the temporal scale and spatial scale of the event as well. In this paper, inspired by *Beavan and Haines* (2001), we estimate a spatially continuous GPS velocity field directly from the data using spherical spline basis functions. We use spherical spline basis functions (*Wang and Dahlen*, 1995; *Wang et al.*, 1998) because they provide a natural local basis that has multiscale possibilities.

Miscellaneous references

REFERENCES: *Bennett et al.* (2003); *McCaffrey et al.* (2007); *Flesch et al.* (2000); *Shen-Tu et al.* (1998, 1999); *Savage et al.* (2001); *Prescott et al.* (2001)

1.1 Introduction - OLD, May 2008

The increasing density and spatial extent of GPS networks provide an unprecedented spatial and temporal view of the deformation of Earth's crust. These networks are presently found in regions where there is (1) active crustal deformation, i.e., at plate boundary regions; (2) land; and (3) sufficient infrastructure to develop and operate the networks. Examples of such regions include Japan (*Sagiya*, 2004), Taiwan (*Yu et al.*, 1997; *Hsu et al.*, 2009), south Asia (*Holt et al.*, 2000), the Middle East (*Vernant et al.*, 2004), the Mediterranean (*Clarke et al.*, 1998; *Kahle et al.*, 2000; *McClusky et al.*, 2000), New Zealand (*Haines et al.*, 1998; *Beavan and Haines*, 2001), and southern California (*Shen and Jackson*, 1993; *Feigl et al.*, 1993), among others. Figure 1 shows the coverage of continuous GPS stations in Japan, Taiwan, and California.

Prior to the advent of GPS in crustal deformation studies, earthquake focal mechanisms and geologic measurements provided different views of the strain field (e.g., *Haines and Holt*, 1993). In two study areas, the United States and Europe, *Ward* (1998a,b) compared strain-rate estimates from these different data types with those from space-based measurements, in order to assess the level of agreement in different sub-regions. BETTER WORDING: Common uses of strain fields include comparison of geologic slip rates, seismic strain release, and geodetic strain accumulation in an effort to look for regions of high seismic hazard and to understand the role of localized deformation on faults versus distributed deformation (*Thatcher*, 1995).

In Table **??**, we categorize GPS-based studies of continental deformation based on two criteria:

time-independent vs time-dependent, and "physical" versus "non-physical" parameterizations. By "physical", we mean that the study imposes a physical description of the system, typically in the form of elastic block models with specific sets of faults. "Non-physical" parameterizations, such as the one in this paper, use convenient mathematical functions to estimate the velocity fields. Both approaches have their strengths. Our focus on the non-physical approach is driven by the desire to detect subtle signals in addition to the "steady-state" (or, "secular") plate motion.

We explicitly make the distinction between variable scale versus multi-scale approaches. A variable scale approach involves estimates of a particular quantity at a single scale at a given position, whereby the scale can vary from one location to another. A multi-scale approach involves a superposition of multiple scales at a given position. All studies of GPS data to date have employed a variable scale approach.

It is a formidable task to efficiently represent large quantities of GPS data, even once a steady-state velocity field has been estimated from the original time series. In this paper, we parameterize the spatial part of the velocity field using a set of local basis functions, similar to the approach of *Beavan and Haines* (2001). Novel aspects of our approach include:

1. an overt and consistent decomposition of the velocity field into multiple scales at all locations
2. a minimum scale at which we estimate the velocity field at a particular location that is controlled by the local station coverage
3. inclusion of the vertical velocity observations if they are available

By focusing on quantities inferred from multiple sites (e.g., strain rate), our suggested approach does not rely on individual stations, and instead focuses on locally correlated behavior. At a given time, this behavior may occur at different spatial extents, or scales, and an accurate representation of it requires a multiscale analysis.

A primary motivation for this study is to use our spatial multiscale estimation procedure within the framework of a time-varying "event detector". Several studies have used GPS observations to detect instances of crustal deformation: for example, the subduction zones of Japan (*Heki*, 1997; *Ozawa et al.*, 2002; *Miyazaki et al.*, 2003, 2004), Andes (*Melbourne et al.*, 2002), and Cascadia (*Dragert et al.*, 2001; *Rogers and Dragert*, 2003). These are the type of events we would like to be able to detect in an automated fashion, whereby we would have a characterization of both the temporal and spatial scale of the event.

In Section **??** we demonstrate the multiscale approach using spherical wavelets to decompose the velocity fields. The multiscale aspects are achieved by using wavelets from progressively finer meshes, but only where justifiable based on the local site density. From each estimated velocity field we can automatically compute a strain-rate map and other scalar quantities that are useful in interpreting crustal deformation, such as dilatation rate and rotation rate (Section 5). In Section **??** we present four synthetic examples to validate the approach, and we then illustrate the method using the three-component NASA REASoN velocity field for southern California (Section **??**).

2 Functions on a sphere

Our axis convention is such that $\hat{\mathbf{r}}$ points radially outward from the center of the sphere, $\hat{\boldsymbol{\theta}}$ points south, and $\hat{\boldsymbol{\phi}}$ points east.

The surface gradient, gradient, surface Laplacian, and Laplacian are defined in spherical polar coordinates as (*Dahlen and Tromp*, 1998, Section A.7.1)

$$
\begin{aligned}
\boldsymbol{\nabla}_1 &= \hat{\boldsymbol{\theta}} \frac{\partial}{\partial\theta} + \hat{\boldsymbol{\phi}} (\sin\theta)^{-1} \frac{\partial}{\partial\phi} && (1)\\
\boldsymbol{\nabla} &= \hat{\mathbf{r}}\, \partial_r + r^{-1} \boldsymbol{\nabla}_1 && (2)\\
\nabla_1^2 &= \boldsymbol{\nabla}_1 \cdot \boldsymbol{\nabla}_1 \;=\; \frac{\partial^2}{\partial\theta^2} + (\sin\theta)^{-2} \frac{\partial^2}{\partial\phi^2} + (\cot\theta) \frac{\partial}{\partial\theta} && (3)\\
\nabla^2 &= \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \;=\; \frac{\partial^2}{\partial r^2} + 2r^{-1} \frac{\partial}{\partial r} + r^{-2} \nabla_1^2 && (4)
\end{aligned}
$$

The surface gradient, the squared-magnitude of the surface gradient, and the surface Laplacian of a function $g(\theta, \phi)$ are given by

$$
\begin{aligned}
\boldsymbol{\nabla}_1 g &= \hat{\boldsymbol{\theta}}\, g_\theta + \hat{\boldsymbol{\phi}} (\sin\theta)^{-1} g_\phi && (5)\\
|\boldsymbol{\nabla}_1 g|^2 &= \boldsymbol{\nabla}_1 g \cdot \boldsymbol{\nabla}_1 g \;=\; (g_\theta)^2 + (\sin\theta)^{-2} (g_\phi)^2 && (6)\\
\nabla_1^2 g &= (\boldsymbol{\nabla}_1 \cdot \boldsymbol{\nabla}_1)\, g \;=\; g_{\theta\theta} + (\cot\theta) g_\theta + (\sin\theta)^{-2} g_{\phi\phi}\,. && (7)
\end{aligned}
$$

2.1 Curl of a velocity field in spherical coordinates (3D)

The curl of a velocity field is a vector field. In Cartesian coordinates $\hat{\mathbf{x}}$-$\hat{\mathbf{y}}$-$\hat{\mathbf{z}}$:

$$
\boldsymbol{\nabla} \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = \begin{bmatrix} \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \\ \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \\ \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \end{bmatrix}. \tag{8}
$$

In orthogonal curvilinear coordinates $\hat{\mathbf{u}}_1$-$\hat{\mathbf{u}}_2$-$\hat{\mathbf{u}}_3$ (see Mathworld)

$$
\boldsymbol{\nabla} \times \mathbf{v} = \frac{1}{h_1\, h_2\, h_3} \begin{vmatrix} h_1\, \hat{\mathbf{u}}_1 & h_2\, \hat{\mathbf{u}}_2 & h_3\, \hat{\mathbf{u}}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1\, v_1 & h_2\, v_2 & h_3\, v_3 \end{vmatrix} = \begin{bmatrix} \frac{1}{h_2\, h_3} \left[\frac{\partial}{\partial u_2} (h_3\, v_3) - \frac{\partial}{\partial u_3} (h_2\, v_2) \right] \\ \frac{1}{h_1\, h_3} \left[\frac{\partial}{\partial u_3} (h_1\, v_1) - \frac{\partial}{\partial u_1} (h_3\, v_3) \right] \\ \frac{1}{h_1\, h_2} \left[\frac{\partial}{\partial u_1} (h_2\, v_2) - \frac{\partial}{\partial u_2} (h_1\, v_1) \right] \end{bmatrix}. \tag{9}
$$

where

$$
h_i \equiv \left| \frac{\partial \mathbf{r}}{\partial u_i} \right|. \tag{10}
$$

In local spherical coordinates $\hat{\mathbf{r}}$-$\hat{\boldsymbol{\theta}}$-$\hat{\boldsymbol{\phi}}$, we have

$$
\boldsymbol{\nabla} \times \mathbf{v} = \frac{1}{h_r\, h_\theta\, h_\phi} \begin{vmatrix} h_r\, \hat{\mathbf{r}} & h_\theta\, \hat{\boldsymbol{\theta}} & h_\phi\, \hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ h_r\, v_r & h_\theta\, v_\theta & h_\phi\, v_\phi \end{vmatrix} = \begin{bmatrix} \frac{1}{h_\theta\, h_\phi} \left[\frac{\partial}{\partial \theta} (h_\phi\, v_\phi) - \frac{\partial}{\partial \phi} (h_\theta\, v_\theta) \right] \\ \frac{1}{h_r\, h_\phi} \left[\frac{\partial}{\partial \phi} (h_r\, v_r) - \frac{\partial}{\partial r} (h_\phi\, v_\phi) \right] \\ \frac{1}{h_r\, h_\theta} \left[\frac{\partial}{\partial r} (h_\theta\, v_\theta) - \frac{\partial}{\partial \theta} (h_r\, v_r) \right] \end{bmatrix}. \tag{11}
$$

The expression for h_i are given by

$$
\begin{aligned}
h_r &= \left|\frac{\partial \mathbf{r}}{\partial r}\right| = \left|\frac{\partial}{\partial r}\left(r\hat{\mathbf{r}}\right)\right| = \left|\frac{\partial r}{\partial r}\hat{\mathbf{r}} + r\frac{\partial \hat{\mathbf{r}}}{\partial r}\right| = |\hat{\mathbf{r}}| = 1 \\
h_\theta &= \left|\frac{\partial \mathbf{r}}{\partial \theta}\right| = \left|\frac{\partial}{\partial \theta}\left(r\hat{\mathbf{r}}\right)\right| = \left|\frac{\partial r}{\partial \theta}\hat{\mathbf{r}} + r\frac{\partial \hat{\mathbf{r}}}{\partial \theta}\right| = \left|\hat{\boldsymbol{\theta}}\, r\right| = r \\
h_\phi &= \left|\frac{\partial \mathbf{r}}{\partial \phi}\right| = \left|\frac{\partial}{\partial \phi}\left(r\hat{\mathbf{r}}\right)\right| = \left|\frac{\partial r}{\partial \phi}\hat{\mathbf{r}} + r\frac{\partial \hat{\mathbf{r}}}{\partial \phi}\right| = \left|\hat{\boldsymbol{\phi}}\, r\,\sin\theta\right| = r\,\sin\theta
\end{aligned}
$$

where we have used equations in *Dahlen and Tromp* (1998, Section A.7). Thus, Equation (11) becomes

$$
\begin{aligned}
\boldsymbol{\nabla}\times\mathbf{v} &= \begin{bmatrix} \frac{1}{r^2\,\sin\theta}\left[\frac{\partial}{\partial\theta}\left(r\,\sin\theta\, v_\phi\right) - \frac{\partial}{\partial\phi}\left(r\, v_\theta\right)\right] \\ \frac{1}{r\,\sin\theta}\left[\frac{\partial}{\partial\phi}\left(v_r\right) - \frac{\partial}{\partial r}\left(r\,\sin\theta\, v_\phi\right)\right] \\ \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\, v_\theta\right) - \frac{\partial}{\partial\theta}\left(v_r\right)\right] \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{r\,\sin\theta}\frac{\partial}{\partial\theta}\left(v_\phi\,\sin\theta\right) - \frac{1}{r\,\sin\theta}\frac{\partial v_\theta}{\partial\phi} \\ \frac{1}{r\,\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\, v_\phi\right) \\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\, v_\theta\right) - \frac{1}{r}\frac{\partial v_r}{\partial\theta} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{r\,\sin\theta}\left(v_\phi\cos\theta + \sin\theta\frac{\partial v_\phi}{\partial\theta}\right) - \frac{1}{r\,\sin\theta}\frac{\partial v_\theta}{\partial\phi} \\ \frac{1}{r\,\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{1}{r}\left(v_\phi + r\,\frac{\partial v_\phi}{\partial r}\right) \\ \frac{1}{r}\left(v_\theta + r\,\frac{\partial v_\theta}{\partial r}\right) - \frac{1}{r}\frac{\partial v_r}{\partial\theta} \end{bmatrix} .
\end{aligned}
$$

Thus, in explicit form, the curl of a vector field is

$$
\begin{aligned}
\boldsymbol{\nabla}\times\mathbf{v} &= \left[\frac{1}{r}\cot\theta\, v_\phi - \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial v_\theta}{\partial\phi} + \frac{1}{r}\frac{\partial v_\phi}{\partial\theta}\right]\hat{\mathbf{r}} \\
&\quad + \left[-\frac{1}{r}v_\phi + \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial v_\phi}{\partial r}\right]\hat{\boldsymbol{\theta}} \\
&\quad + \left[\frac{1}{r}v_\theta - \frac{1}{r}\frac{\partial v_r}{\partial\theta} + \frac{\partial v_\theta}{\partial r}\right]\boldsymbol{\phi}\ . \qquad (12)
\end{aligned}
$$

This is presented in *Malvern* (1969, Eq. II.4.S7) and *Dahlen and Tromp* (1998, A.142).

The magnitude-squared of the curl is then

$$
\begin{aligned}
|\boldsymbol{\nabla}\times\mathbf{v}|^2 &= \left[\frac{1}{r}\cot\theta\, v_\phi - \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial v_\theta}{\partial\phi} + \frac{1}{r}\frac{\partial v_\phi}{\partial\theta}\right]^2 \\
&\quad + \left[-\frac{1}{r}v_\phi + \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial v_\phi}{\partial r}\right]^2 \\
&\quad + \left[\frac{1}{r}v_\theta - \frac{1}{r}\frac{\partial v_r}{\partial\theta} + \frac{\partial v_\theta}{\partial r}\right]^2\ . \qquad (13)
\end{aligned}
$$

2.2 Velocity gradient tensor for spherical coordinates (3D)

This section follows *Dahlen and Tromp* (1998, Section A.7), which is based in part on *Malvern* (1969, Appendix 2).

Our velocity field, $\mathbf{v}$, can be expressed in terms of physical components as

$$\mathbf{v} \;=\; v_r\,\hat{\mathbf{r}} \;+\; v_\theta\,\hat{\boldsymbol{\theta}} \;+\; v_\phi\,\hat{\boldsymbol{\phi}}, \tag{14}$$

where $v_r = \hat{\mathbf{r}} \cdot \mathbf{v}$, $v_\theta = \hat{\boldsymbol{\theta}} \cdot \mathbf{v}$, and $v_\phi = \hat{\boldsymbol{\phi}} \cdot \mathbf{v}$. The surface gradients of $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ are given by (*Dahlen and Tromp*, 1998, p. 833)

$$\begin{aligned}
\boldsymbol{\nabla}_1\,\hat{\mathbf{r}} &= \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\phi}}\hat{\boldsymbol{\phi}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
\boldsymbol{\nabla}_1\,\hat{\boldsymbol{\theta}} &= -\hat{\boldsymbol{\theta}}\hat{\mathbf{r}} + \hat{\boldsymbol{\phi}}\hat{\boldsymbol{\phi}}\cot\theta = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \cot\theta \end{bmatrix} \\
\boldsymbol{\nabla}_1\,\hat{\boldsymbol{\phi}} &= -\hat{\boldsymbol{\phi}}\hat{\mathbf{r}} - \hat{\boldsymbol{\phi}}\hat{\boldsymbol{\theta}}\cot\theta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & -\cot\theta & 0 \end{bmatrix},
\end{aligned}$$

where the tensor indices are given by

$$\begin{bmatrix} \hat{\mathbf{r}}\hat{\mathbf{r}} & \hat{\mathbf{r}}\hat{\boldsymbol{\theta}} & \hat{\mathbf{r}}\hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{\theta}}\hat{\mathbf{r}} & \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{\phi}}\hat{\mathbf{r}} & \hat{\boldsymbol{\phi}}\hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}}\hat{\boldsymbol{\phi}} \end{bmatrix}.$$

Thus, the surface gradients of $\mathbf{v}$ in each direction are given by

$$\begin{aligned}
\boldsymbol{\nabla}_1(v_r\,\hat{\mathbf{r}}) &= (\boldsymbol{\nabla}_1\,v_r)\,\hat{\mathbf{r}} + v_r\,(\boldsymbol{\nabla}_1\,\hat{\mathbf{r}}) \\
&= \left[\left(\hat{\boldsymbol{\theta}}\,\frac{\partial}{\partial\theta} + \hat{\boldsymbol{\phi}}\,\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right)v_r\right]\hat{\mathbf{r}} \;+\; v_r\left[\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\phi}}\hat{\boldsymbol{\phi}}\right] \\
&= \frac{\partial v_r}{\partial\theta}\,\hat{\boldsymbol{\theta}}\hat{\mathbf{r}} \;+\; \frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi}\,\hat{\boldsymbol{\phi}}\hat{\mathbf{r}} \;+\; v_r\,\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} \;+\; v_r\,\hat{\boldsymbol{\phi}}\hat{\boldsymbol{\phi}}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\nabla}_1(v_\theta\,\hat{\boldsymbol{\theta}}) &= (\boldsymbol{\nabla}_1\,v_\theta)\,\hat{\boldsymbol{\theta}} + v_\theta\,(\boldsymbol{\nabla}_1\,\hat{\boldsymbol{\theta}}) \\
&= \left[\left(\hat{\boldsymbol{\theta}}\,\frac{\partial}{\partial\theta} + \hat{\boldsymbol{\phi}}\,\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right)v_\theta\right]\hat{\boldsymbol{\theta}} \;+\; v_\theta\left[-\hat{\boldsymbol{\theta}}\hat{\mathbf{r}} \;+\; \hat{\boldsymbol{\phi}}\hat{\boldsymbol{\phi}}\,\cot\theta\right] \\
&= \frac{\partial v_\theta}{\partial\theta}\,\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} \;+\; \frac{1}{\sin\theta}\frac{\partial v_\theta}{\partial\phi}\,\hat{\boldsymbol{\phi}}\hat{\boldsymbol{\theta}} \;-\; v_\theta\,\hat{\boldsymbol{\theta}}\hat{\mathbf{r}} \;+\; v_\theta\,\cot\theta\,\hat{\boldsymbol{\phi}}\hat{\boldsymbol{\phi}}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\nabla}_1(v_\phi\,\hat{\boldsymbol{\phi}}) &= (\boldsymbol{\nabla}_1\,v_\phi)\,\hat{\boldsymbol{\phi}} + v_\phi\,(\boldsymbol{\nabla}_1\,\hat{\boldsymbol{\phi}}) \\
&= \left[\left(\hat{\boldsymbol{\theta}}\,\frac{\partial}{\partial\theta} + \hat{\boldsymbol{\phi}}\,\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right)v_\phi\right]\hat{\boldsymbol{\phi}} \;+\; v_\phi\left[-\hat{\boldsymbol{\phi}}\hat{\mathbf{r}} \;-\; \hat{\boldsymbol{\phi}}\hat{\boldsymbol{\theta}}\,\cot\theta\right] \\
&= \frac{\partial v_\phi}{\partial\theta}\,\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\phi}} \;+\; \frac{1}{\sin\theta}\frac{\partial v_\phi}{\partial\phi}\,\hat{\boldsymbol{\phi}}\hat{\boldsymbol{\phi}} \;-\; v_\phi\,\hat{\boldsymbol{\phi}}\hat{\mathbf{r}} \;-\; v_\phi\,\cot\theta\,\hat{\boldsymbol{\phi}}\hat{\boldsymbol{\theta}}\,,
\end{aligned}$$

and therefore the surface gradient of $\mathbf{v}$ is given by

$$\begin{aligned}
\boldsymbol{\nabla}_1\,\mathbf{v} &= \boldsymbol{\nabla}_1(v_r\,\hat{\mathbf{r}}) \;+\; \boldsymbol{\nabla}_1(v_\theta\,\hat{\boldsymbol{\theta}}) \;+\; \boldsymbol{\nabla}_1(v_\phi\,\hat{\boldsymbol{\phi}}) \\
&= \frac{\partial v_r}{\partial\theta}\,\hat{\boldsymbol{\theta}}\hat{\mathbf{r}} \;+\; \frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi}\,\hat{\boldsymbol{\phi}}\hat{\mathbf{r}} \;+\; v_r\,\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} \;+\; v_r\,\hat{\boldsymbol{\phi}}\hat{\boldsymbol{\phi}} \\
&\quad + \frac{\partial v_\theta}{\partial\theta}\,\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} \;+\; \frac{1}{\sin\theta}\frac{\partial v_\theta}{\partial\phi}\,\hat{\boldsymbol{\phi}}\hat{\boldsymbol{\theta}} \;-\; v_\theta\,\hat{\boldsymbol{\theta}}\hat{\mathbf{r}} \;+\; v_\theta\,\cot\theta\,\hat{\boldsymbol{\phi}}\hat{\boldsymbol{\phi}} \\
&\quad + \frac{\partial v_\phi}{\partial\theta}\,\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\phi}} \;+\; \frac{1}{\sin\theta}\frac{\partial v_\phi}{\partial\phi}\,\hat{\boldsymbol{\phi}}\hat{\boldsymbol{\phi}} \;-\; v_\phi\,\hat{\boldsymbol{\phi}}\hat{\mathbf{r}} \;-\; v_\phi\,\cot\theta\,\hat{\boldsymbol{\phi}}\hat{\boldsymbol{\theta}} \\
&= \left(v_r + \frac{\partial v_\theta}{\partial\theta}\right)\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} \;+\; \left(v_r + v_\theta\,\cot\theta + \frac{1}{\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right)\hat{\boldsymbol{\phi}}\hat{\boldsymbol{\phi}} \;+\; \left(\frac{\partial v_r}{\partial\theta} - v_\theta\right)\hat{\boldsymbol{\theta}}\hat{\mathbf{r}} \\
&\quad + \left(\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - v_\phi\right)\hat{\boldsymbol{\phi}}\hat{\mathbf{r}} \;+\; \left(\frac{\partial v_\phi}{\partial\theta}\right)\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\phi}} \;+\; \left(\frac{1}{\sin\theta}\frac{\partial v_\theta}{\partial\phi} - v_\phi\,\cot\theta\right)\hat{\boldsymbol{\phi}}\hat{\boldsymbol{\theta}}\,.
\end{aligned}$$

The gradient of the vector field is given by

$$
\begin{aligned}
\nabla \mathbf{v} &= \left[\hat{\mathbf{r}}\,\frac{\partial}{\partial r} + r^{-1}\,\nabla_1\right]\left[v_r\,\hat{\mathbf{r}} \;+\; v_\theta\,\hat{\boldsymbol{\theta}} \;+\; v_\phi\,\hat{\boldsymbol{\phi}}\right] \\
&= \frac{\partial v_r}{\partial r}\,\hat{\mathbf{r}}\hat{\mathbf{r}} \;+\; \frac{\partial v_\theta}{\partial r}\,\hat{\mathbf{r}}\hat{\boldsymbol{\theta}} \;+\; \frac{\partial v_\phi}{\partial r}\,\hat{\mathbf{r}}\hat{\boldsymbol{\phi}} \;+\; r^{-1}\,\nabla_1\,\mathbf{v} \\
&= \frac{\partial v_r}{\partial r}\,\hat{\mathbf{r}}\hat{\mathbf{r}} \;+\; \frac{\partial v_\theta}{\partial r}\,\hat{\mathbf{r}}\hat{\boldsymbol{\theta}} \;+\; \frac{\partial v_\phi}{\partial r}\,\hat{\mathbf{r}}\hat{\boldsymbol{\phi}} \\
&\quad + r^{-1}\left(v_r + \frac{\partial v_\theta}{\partial \theta}\right)\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} \;+\; r^{-1}\left(v_r + v_\theta\,\cot\theta + \frac{1}{\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right)\hat{\boldsymbol{\phi}}\hat{\boldsymbol{\phi}} \;+\; r^{-1}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right)\hat{\boldsymbol{\theta}}\hat{\mathbf{r}} \\
&\quad + r^{-1}\left(\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - v_\phi\right)\hat{\boldsymbol{\phi}}\hat{\mathbf{r}} \;+\; r^{-1}\left(\frac{\partial v_\phi}{\partial \theta}\right)\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\phi}} \;+\; r^{-1}\left(\frac{1}{\sin\theta}\frac{\partial v_\theta}{\partial \phi} - v_\phi\,\cot\theta\right)\hat{\boldsymbol{\phi}}\hat{\boldsymbol{\theta}} \\
&= \left(\frac{\partial v_r}{\partial r}\right)\hat{\mathbf{r}}\hat{\mathbf{r}} \;+\; r^{-1}\left(v_r + \frac{\partial v_\theta}{\partial \theta}\right)\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} \;+\; r^{-1}\left(v_r + v_\theta\,\cot\theta + \frac{1}{\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right)\hat{\boldsymbol{\phi}}\hat{\boldsymbol{\phi}} \\
&\quad + \left(\frac{\partial v_\theta}{\partial r}\right)\hat{\mathbf{r}}\hat{\boldsymbol{\theta}} \;+\; r^{-1}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right)\hat{\boldsymbol{\theta}}\hat{\mathbf{r}} \;+\; \left(\frac{\partial v_\phi}{\partial r}\right)\hat{\mathbf{r}}\hat{\boldsymbol{\phi}} \;+\; r^{-1}\left(\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - v_\phi\right)\hat{\boldsymbol{\phi}}\hat{\mathbf{r}} \\
&\quad + r^{-1}\left(\frac{\partial v_\phi}{\partial \theta}\right)\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\phi}} \;+\; r^{-1}\left(\frac{1}{\sin\theta}\frac{\partial v_\theta}{\partial \phi} - v_\phi\,\cot\theta\right)\hat{\boldsymbol{\phi}}\hat{\boldsymbol{\theta}}\,.
\end{aligned}
$$

We define the velocity gradient tensor, $\mathbf{L}$, by

$$
\mathbf{L} \;\equiv\; (\nabla\mathbf{v})^T \;=\; \mathbf{v}\nabla \;=\; \begin{bmatrix}
\frac{\partial v_r}{\partial r} & \frac{1}{r}\left(-v_\theta + \frac{\partial v_r}{\partial \theta}\right) & \frac{1}{r}\left(-v_\phi + \frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi}\right) \\
\frac{\partial v_\theta}{\partial r} & \frac{1}{r}\left(v_r + \frac{\partial v_\theta}{\partial \theta}\right) & \frac{1}{r}\left(-v_\phi\,\cot\theta + \frac{1}{\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right) \\
\frac{\partial v_\phi}{\partial r} & \frac{1}{r}\left(\frac{\partial v_\phi}{\partial \theta}\right) & \frac{1}{r}\left(v_r + v_\theta\,\cot\theta + \frac{1}{\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right)
\end{bmatrix}. \tag{15}
$$

2.3 Strain rate tensor

The velocity gradient tensor may be decomposed as

$$
\mathbf{L} = \mathbf{D} + \mathbf{W}, \tag{16}
$$

where $\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T)$ is the symmetric strain rate tensor, and $\mathbf{W} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T)$ is the anti-symmetric rotation-rate tensor. Thus, the symmetric strain rate tensor is[1]

$$
\begin{aligned}
\mathbf{D}\,(r,\theta,\phi) &= \tfrac{1}{2}\left[(\nabla\mathbf{v})^T + \nabla\mathbf{v}\right] \\
&= \begin{bmatrix}
\frac{\partial v_r}{\partial r} & \frac{1}{2r}\left(-v_\theta + \frac{\partial v_r}{\partial \theta} + r\frac{\partial v_\theta}{\partial r}\right) & \frac{1}{2r}\left(-v_\phi + \frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial v_\phi}{\partial r}\right) \\
D_{12} & \frac{1}{r}\left(v_r + \frac{\partial v_\theta}{\partial \theta}\right) & \frac{1}{2r}\left(-v_\phi\,\cot\theta + \frac{1}{\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{\partial v_\phi}{\partial \theta}\right) \\
D_{13} & D_{23} & \frac{1}{r}\left(v_r + v_\theta\,\cot\theta + \frac{1}{\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right)
\end{bmatrix}.
\end{aligned} \tag{17}
$$

[1]Note that *Malvern* (1969) has a mistake in (II.4.S8), which carries into the $D_{\theta\phi}$ term in (II.4.S9): to the right of the $-v_\phi/r$ term it should be $\cot\theta$ instead of $\cot\phi$.

2.4 Rotation rate tensor and rotation rate vector

The anti-symmetric rotation-rate tensor is

$$\begin{aligned}
\mathbf{W}(r,\theta,\phi) &= \tfrac{1}{2}\left[(\boldsymbol{\nabla}\mathbf{v})^T - \boldsymbol{\nabla}\mathbf{v}\right] \\
&= \begin{bmatrix} 0 & \frac{1}{2r}\left(-v_\theta + \frac{\partial v_r}{\partial \theta} - r\frac{\partial v_\theta}{\partial r}\right) & \frac{1}{2r}\left(-v_\phi + \frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - r\frac{\partial v_\phi}{\partial r}\right) \\ -W_{12} & 0 & \frac{1}{2r}\left(-v_\phi\,\cot\theta + \frac{1}{\sin\theta}\frac{\partial v_\theta}{\partial \phi} - \frac{\partial v_\phi}{\partial \theta}\right) \\ -W_{13} & -W_{23} & 0 \end{bmatrix}.
\end{aligned} \tag{18}$$

The angular velocity rotation vector, $\mathbf{w}$, is determined from the components of the rotation-rate tensor as *Malvern* (1969, p. 147)

$$\mathbf{w} = -W_{\theta\phi}\,\hat{\mathbf{r}} + W_{r\phi}\,\hat{\boldsymbol{\theta}} - W_{r\theta}\,\hat{\boldsymbol{\phi}}\ . \tag{19}$$

Upon comparison of Equation (19) with Equation (12), we obtain (*Malvern*, 1969, p. 147)

$$\mathbf{w} = \tfrac{1}{2}\,\boldsymbol{\nabla}\times\mathbf{v}\ . \tag{20}$$

The magnitude-squared of rotation is therefore

$$w^2 = |\mathbf{w}|^2 = W_{\theta\phi}^2 + W_{\phi r}^2 + W_{r\theta}^2. \tag{21}$$

The units of w should be radians/year, assuming the velocity field is expressed in meters/year.

The *location* of the rotation pole on the sphere can be found by converting the vector in Equation (19) from a local basis to a global basis (e.g., *Cox and Hart*, 1986, p. 155).

Section 7 shows how we define the rotation matrix associated with the rotation vector.

2.5 Dilatation

The dilatation rate, or volumetric strain rate, is the divergence of the velocity field, or, equivalently, the first invariant of the velocity gradient tensor:

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{v} &= \mathrm{Tr}\,(\mathbf{L}) = \mathrm{Tr}\,(\mathbf{D}) \\
&= \frac{\partial v_r}{\partial r} + r^{-1}\left(v_r + \frac{\partial v_\theta}{\partial \theta}\right) + r^{-1}\left[v_r + v_\theta\,\cot\theta + \frac{1}{\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right] \\
&= r^{-1}\left[2v_r + v_\theta\,\cot\theta + r\frac{\partial v_r}{\partial r} + \frac{\partial v_\theta}{\partial \theta} + \frac{1}{\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right],
\end{aligned} \tag{22}$$

which is listed in *Malvern* (1969, Eq. II.4.S5) and *Dahlen and Tromp* (1998, p. 836).

We now consider a purely rotational field and show that the dilatation is zero. Consider a rotation vector at the North Pole, $\boldsymbol{\Omega} = \Omega\,\hat{\mathbf{z}}$. A point on the sphere is given by

$$\mathbf{P}(\theta,\phi) = R\sin\theta\cos\phi\,\hat{\mathbf{x}} + R\sin\theta\sin\phi\,\hat{\mathbf{y}} + R\cos\theta\,\hat{\mathbf{z}}\ . \tag{23}$$

The velocity at R is then

$$
\begin{aligned}
\mathbf{v}(\theta,\phi) &= \boldsymbol{\Omega} \times \mathbf{P}(\theta,\phi) \\
&= \Omega\, R\, \sin\theta\, (-\sin\phi\, \hat{\mathbf{x}} + \cos\phi\, \hat{\mathbf{y}}) \ , \qquad (24)
\end{aligned}
$$

or, in local coordinates, $\mathbf{v}(\theta,\phi) = \Omega\, R\, \sin\theta\, \hat{\boldsymbol{\phi}}$.

Substituting this into Equation (22), we obtain

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{v}(\theta,\phi) &= R^{-1} \left[2v_r + v_\theta \cot\theta + R\frac{\partial v_r}{\partial r} + \frac{\partial v_\theta}{\partial \theta} + \frac{1}{\sin\theta}\frac{\partial v_\phi}{\partial \phi} \right] \\
&= R^{-1} \left[\frac{1}{\sin\theta}\frac{\partial}{\partial \phi} (\Omega\, R\, \sin\theta) \right] \\
&= 0 \qquad (25)
\end{aligned}
$$

Thus the dilatation of a rotational field is zero.

3 Helmholtz decomposition

References: *Dahlen and Tromp* (1998, Section B.12.1), *Bayer et al.* (2001)

A *Helmholtz decomposition* partitions a vector field into divergence-free toroidal field and a curl-free spheroidal field.

$$\begin{aligned}
\mathbf{v} &= v_r\,\hat{\mathbf{r}} + \mathbf{v}^{\Omega} && (26)\\
&= U\,\hat{\mathbf{r}} + \boldsymbol{\nabla}_1 V - \hat{\mathbf{r}} \times \boldsymbol{\nabla}_1 W && (27)\\
&= \mathbf{v}^{\mathrm{S}} + \mathbf{v}^{\mathrm{T}}\,, && (28)
\end{aligned}$$

where $\mathbf{v}^{\Omega}$ is the horizontal velocity, and the spheroidal and toroidal vector fields are given by

$$\begin{aligned}
\mathbf{v}^{\mathrm{S}} &= U\,\hat{\mathbf{r}} + \boldsymbol{\nabla}_1 V && (29)\\
\mathbf{v}^{\mathrm{T}} &= -\hat{\mathbf{r}} \times \boldsymbol{\nabla}_1 W\;. && (30)
\end{aligned}$$

and

$$U = v_r\;. \tag{31}$$

Using the $\hat{\mathbf{r}}$-$\hat{\boldsymbol{\theta}}$-$\hat{\boldsymbol{\phi}}$ basis and the formulas in Section 2, we have

$$\begin{aligned}
\mathbf{v}^{\mathrm{S}} &= U\,\hat{\mathbf{r}} + \left[\hat{\boldsymbol{\theta}}\,\frac{\partial}{\partial\theta} + \hat{\boldsymbol{\phi}}\,(\sin\theta)^{-1}\,\frac{\partial}{\partial\phi}\right] V\\
&= U\,\hat{\mathbf{r}} + \frac{\partial V}{\partial\theta}\,\hat{\boldsymbol{\theta}} + (\sin\theta)^{-1}\,\frac{\partial V}{\partial\phi}\,\hat{\boldsymbol{\phi}} && (32)\\
\mathbf{v}^{\mathrm{T}} &= -\hat{\mathbf{r}} \times \boldsymbol{\nabla}_1 W\\
&= -\hat{\mathbf{r}} \times \left[\hat{\boldsymbol{\theta}}\,\frac{\partial}{\partial\theta} + \hat{\boldsymbol{\phi}}\,(\sin\theta)^{-1}\,\frac{\partial}{\partial\phi}\right] W\\
&= -\hat{\mathbf{r}} \times \left[\hat{\boldsymbol{\theta}}\,\frac{\partial W}{\partial\theta} + \hat{\boldsymbol{\phi}}\,(\sin\theta)^{-1}\,\frac{\partial W}{\partial\phi}\right]\\
&= (\sin\theta)^{-1}\,\frac{\partial W}{\partial\phi}\,\hat{\boldsymbol{\theta}} - \frac{\partial W}{\partial\theta}\,\hat{\boldsymbol{\phi}}\;. && (33)
\end{aligned}$$

In terms of U, V, and W, the components of $\mathbf{v}$ are then

$$\begin{aligned}
v_r(\theta,\phi) &= U(\theta,\phi) && (34)\\
v_\theta(\theta,\phi) &= \frac{\partial V}{\partial\theta} + (\sin\theta)^{-1}\,\frac{\partial W}{\partial\phi} && (35)\\
v_\phi(\theta,\phi) &= -\frac{\partial W}{\partial\theta} + (\sin\theta)^{-1}\,\frac{\partial V}{\partial\phi}\;. && (36)
\end{aligned}$$

The rotation vector $\mathbf{w}$ is given by

$$
\begin{aligned}
2\mathbf{w} &= \boldsymbol{\nabla} \times \mathbf{v} = \boldsymbol{\nabla} \times \mathbf{v}^{\mathrm{S}} + \boldsymbol{\nabla} \times \mathbf{v}^{\mathrm{T}} \\
&= \boldsymbol{\nabla} \times (U\,\hat{\mathbf{r}} + \boldsymbol{\nabla}_1 V) + \boldsymbol{\nabla} \times (-\hat{\mathbf{r}} \times \boldsymbol{\nabla}_1 W) \\
&= \boldsymbol{\nabla} \times U\,\hat{\mathbf{r}} + \boldsymbol{\nabla} \times \boldsymbol{\nabla}_1 V - \boldsymbol{\nabla} \times (\hat{\mathbf{r}} \times \boldsymbol{\nabla}_1 W) \\
&= -\hat{\mathbf{r}}\,\boldsymbol{\nabla}_1 U - \boldsymbol{\nabla} \times (\hat{\mathbf{r}} \times \boldsymbol{\nabla}_1 W)\ .
\end{aligned}
$$

We can also write the rotation vector as the sum of a poloidal term and a toroidal term:

$$
\begin{aligned}
\mathbf{w} &= \tfrac{1}{2}\left(\mathbf{w}^{\mathrm{P}} + \mathbf{w}^{\mathrm{T}}\right) && (37) \\
\mathbf{w}^{\mathrm{P}} &= -\boldsymbol{\nabla} \times (\hat{\mathbf{r}} \times \boldsymbol{\nabla}_1 W) && (38) \\
\mathbf{w}^{\mathrm{T}} &= -\hat{\mathbf{r}}\,\boldsymbol{\nabla}_1 U\ . && (39)
\end{aligned}
$$

Note that the toroidal part, $\mathbf{w}^{\mathrm{T}}$ only contains the vertical component of the velocity field (Eq. 31).

3.1 Expansion of v in terms of U, V, and W

Let $g_k(\theta,\phi)$, $k = 1,\ldots,m$ be a set of scalar-valued basis functions defined on the sphere. We expand the potential fields in terms of $g_k(\theta,\phi)$:

$$
\begin{aligned}
U(\theta,\phi) &= \sum_{k=1}^{m} p_k\, g_k(\theta,\phi) && (40) \\
V(\theta,\phi) &= \sum_{k=1}^{m} b_k\, g_k(\theta,\phi) && (41) \\
W(\theta,\phi) &= \sum_{k=1}^{m} c_k\, g_k(\theta,\phi)\ . && (42)
\end{aligned}
$$

For the spheroidal and toroidal vector fields, we substitute Equations (40) and (42) into Equations (32) and (33). For the spheroidal vector field, we have

$$
\begin{aligned}
\mathbf{v}^{\mathrm{S}}(\theta,\phi) &= U(\theta,\phi)\,\hat{\mathbf{r}} + \frac{\partial V}{\partial \theta}\,\hat{\boldsymbol{\theta}} + (\sin\theta)^{-1}\,\frac{\partial V}{\partial \phi}\,\hat{\boldsymbol{\phi}} \\
&= \hat{\mathbf{r}} \sum_{k=1}^{m} p_k\, g_k(\theta,\phi) + \hat{\boldsymbol{\theta}}\,\frac{\partial}{\partial\theta}\left[\sum_{k=1}^{m} b_k\, g_k(\theta,\phi)\right] + \hat{\boldsymbol{\phi}}\,(\sin\theta)^{-1}\,\frac{\partial}{\partial\phi}\left[\sum_{k=1}^{m} b_k\, g_k(\theta,\phi)\right] \\
&= \hat{\mathbf{r}} \sum_{k=1}^{m} p_k\, g_k(\theta,\phi) + \hat{\boldsymbol{\theta}} \sum_{k=1}^{m} b_k \frac{\partial g_k}{\partial\theta} + \hat{\boldsymbol{\phi}}\,(\sin\theta)^{-1} \sum_{k=1}^{m} b_k \frac{\partial g_k}{\partial\phi}\ , && (43)
\end{aligned}
$$

and for the toroidal vector field, we have

$$\begin{aligned}
\mathbf{v}^{\mathrm{T}}(\theta,\phi) &= (\sin\theta)^{-1}\,\frac{\partial W}{\partial\phi}\,\hat{\boldsymbol{\theta}} - \frac{\partial W}{\partial\theta}\,\hat{\boldsymbol{\phi}} \\
&= \hat{\boldsymbol{\theta}}\,(\sin\theta)^{-1}\,\frac{\partial}{\partial\phi}\left[\sum_{k=1}^{m} c_k\,g_k(\theta,\phi)\right] - \hat{\boldsymbol{\phi}}\,\frac{\partial}{\partial\theta}\left[\sum_{k=1}^{m} c_k\,g_k(\theta,\phi)\right] \\
&= \hat{\boldsymbol{\theta}}\,(\sin\theta)^{-1}\sum_{k=1}^{m} c_k\,\frac{\partial g_k}{\partial\phi} - \hat{\boldsymbol{\phi}}\sum_{k=1}^{m} c_k\,\frac{\partial g_k}{\partial\theta}\;. \qquad (44)
\end{aligned}$$

Let n be the number of GPS stations and m the number of basis functions, and let $f|_i$ denote $f(\theta_i,\,\phi_i)$. In matrix notation, we can write the spheroidal field (Eq. 43) as

$$\begin{bmatrix} v_r^{\mathrm{S}}|_1 \\ \vdots \\ v_r^{\mathrm{S}}|_n \\ v_\theta^{\mathrm{S}}|_1 \\ \vdots \\ v_\theta^{\mathrm{S}}|_n \\ v_\phi^{\mathrm{S}}\Big|_1 \\ \vdots \\ v_\phi^{\mathrm{S}}\Big|_n \end{bmatrix} = \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{G}^{\theta} \\ \mathbf{0} & (\sin\theta)^{-1}\,\mathbf{G}^{\phi} \end{bmatrix} \begin{bmatrix} p_1 \\ \vdots \\ p_m \\ b_1 \\ \vdots \\ b_m \end{bmatrix}, \qquad (45)$$

where the matrix is $3n \times 2m$ and contains six $n \times m$ blocks, three of which are $\mathbf{0}$. We define the elements of the $n \times m$ $\mathbf{G}$ matrices as

$$\begin{aligned}
G_{ik} &= g_k(\theta_i,\,\phi_i) && (46)\\
G_{ik}^{\theta} &= \frac{\partial g_k}{\partial\theta}(\theta_i,\,\phi_i) && (47)\\
G_{ik}^{\phi} &= \frac{\partial g_k}{\partial\phi}(\theta_i,\,\phi_i)\;. && (48)
\end{aligned}$$

Similarly, the toroidal field (Eq. 44) is

$$\begin{bmatrix} v_\theta^{\mathrm{T}}|_1 \\ \vdots \\ v_\theta^{\mathrm{T}}|_n \\ v_\phi^{\mathrm{T}}\Big|_1 \\ \vdots \\ v_\phi^{\mathrm{T}}\Big|_n \end{bmatrix} = \begin{bmatrix} (\sin\theta_1)^{-1}\,\mathbf{G}^{\phi} \\ -\mathbf{G}^{\theta} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}, \qquad (49)$$

where the matrix is $2n \times m$ and contains two $n \times m$ blocks.

Substituting Equations (45) and (49) into Equation (28), we obtain a single matrix expression for $\mathbf{v}$:

$$
\begin{aligned}
\mathbf{v} &= \mathbf{v}^{\mathrm{S}} + \mathbf{v}^{\mathrm{T}} \\
\begin{bmatrix} v_r\big|_1 \\ \vdots \\ v_r\big|_n \\ v_\theta\big|_1 \\ \vdots \\ v_\theta\big|_n \\ v_\phi\big|_1 \\ \vdots \\ v_\phi\big|_n \end{bmatrix} &= \begin{bmatrix} v_r^{\mathrm{S}}\big|_1 \\ \vdots \\ v_r^{\mathrm{S}}\big|_n \\ v_\theta^{\mathrm{S}}\big|_1 \\ \vdots \\ v_\theta^{\mathrm{S}}\big|_n \\ v_\phi^{\mathrm{S}}\Big|_1 \\ \vdots \\ v_\phi^{\mathrm{S}}\Big|_n \end{bmatrix} + \begin{bmatrix} v_r^{\mathrm{T}}\big|_1 \\ \vdots \\ v_r^{\mathrm{T}}\big|_n \\ v_\theta^{\mathrm{T}}\big|_1 \\ \vdots \\ v_\theta^{\mathrm{T}}\big|_n \\ v_\phi^{\mathrm{T}}\Big|_1 \\ \vdots \\ v_\phi^{\mathrm{T}}\Big|_n \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{G} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{G}^\theta & \mathbf{0} \\ \mathbf{0} & (\sin\theta)^{-1}\,\mathbf{G}^\phi & \mathbf{0} \end{bmatrix} \begin{bmatrix} p_1 \\ \vdots \\ p_m \\ b_1 \\ \vdots \\ b_m \\ c_1 \\ \vdots \\ c_m \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (\sin\theta_1)^{-1}\,\mathbf{G}^\phi \\ \mathbf{0} & \mathbf{0} & -\mathbf{G}^\theta \end{bmatrix} \begin{bmatrix} p_1 \\ \vdots \\ p_m \\ b_1 \\ \vdots \\ b_m \\ c_1 \\ \vdots \\ c_m \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{G} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{G}^\theta & (\sin\theta_1)^{-1}\,\mathbf{G}^\phi \\ \mathbf{0} & (\sin\theta)^{-1}\,\mathbf{G}^\phi & -\mathbf{G}^\theta \end{bmatrix} \begin{bmatrix} p_1 \\ \vdots \\ p_m \\ b_1 \\ \vdots \\ b_m \\ c_1 \\ \vdots \\ c_m \end{bmatrix}
\end{aligned} \tag{50}
$$

In continuous form, this is (see Eqs. 43 and 44)

$$
\mathbf{v}(\theta,\phi) = \sum_{k=1}^{m} \left\{ \left[p_k\, g_k(\theta,\phi)\right]\hat{\mathbf{r}} + \left[b_k \frac{\partial g_k}{\partial\theta} + (\sin\theta)^{-1}\, c_k\, \frac{\partial g_k}{\partial\phi}\right]\hat{\boldsymbol{\theta}} + \left[(\sin\theta)^{-1} b_k \frac{\partial g_k}{\partial\phi} - c_k\, \frac{\partial g_k}{\partial\theta}\right]\hat{\boldsymbol{\phi}} \right\} \tag{51}
$$

Thus, the horizontal components, v_θ and v_ϕ, can be expressed as

$$\begin{bmatrix} v_\theta|_1 \\ \vdots \\ v_\theta|_i \\ \vdots \\ v_\theta|_n \\ \\ v_\phi|_1 \\ \vdots \\ v_\phi|_i \\ \vdots \\ v_\phi|_n \end{bmatrix} = \begin{bmatrix} \mathbf{G}^\theta & (\sin\theta)^{-1}\,\mathbf{G}^\phi \\ (\sin\theta)^{-1}\,\mathbf{G}^\phi & -\mathbf{G}^\theta \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_k \\ \vdots \\ b_m \\ \\ c_1 \\ \vdots \\ c_k \\ \vdots \\ c_m \end{bmatrix},$$

where the $2n \times 2m$ matrix is

$$\begin{bmatrix}
\left.\frac{\partial g_1}{\partial\theta}\right|_1 & \vdots & \left.\frac{\partial g_k}{\partial\theta}\right|_1 & \vdots & \left.\frac{\partial g_m}{\partial\theta}\right|_1 & (\sin\theta_1)^{-1}\left.\frac{\partial g_1}{\partial\phi}\right|_1 & \vdots & (\sin\theta_1)^{-1}\left.\frac{\partial g_k}{\partial\phi}\right|_1 & \vdots & (\sin\theta_1)^{-1}\left.\frac{\partial g_m}{\partial\phi}\right|_1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\left.\frac{\partial g_1}{\partial\theta}\right|_i & \vdots & \left.\frac{\partial g_k}{\partial\theta}\right|_i & \vdots & \left.\frac{\partial g_m}{\partial\theta}\right|_i & (\sin\theta_i)^{-1}\left.\frac{\partial g_1}{\partial\phi}\right|_i & \vdots & (\sin\theta_i)^{-1}\left.\frac{\partial g_k}{\partial\phi}\right|_i & \vdots & (\sin\theta_i)^{-1}\left.\frac{\partial g_m}{\partial\phi}\right|_i \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\left.\frac{\partial g_1}{\partial\theta}\right|_n & \vdots & \left.\frac{\partial g_k}{\partial\theta}\right|_n & \vdots & \left.\frac{\partial g_m}{\partial\theta}\right|_n & (\sin\theta_n)^{-1}\left.\frac{\partial g_1}{\partial\phi}\right|_n & \vdots & (\sin\theta_n)^{-1}\left.\frac{\partial g_k}{\partial\phi}\right|_n & \vdots & (\sin\theta_n)^{-1}\left.\frac{\partial g_m}{\partial\phi}\right|_n \\
\\
(\sin\theta_1)^{-1}\left.\frac{\partial g_1}{\partial\phi}\right|_1 & \vdots & (\sin\theta_1)^{-1}\left.\frac{\partial g_k}{\partial\phi}\right|_1 & \vdots & (\sin\theta_1)^{-1}\left.\frac{\partial g_m}{\partial\phi}\right|_1 & -\left.\frac{\partial g_1}{\partial\theta}\right|_1 & \vdots & -\left.\frac{\partial g_k}{\partial\theta}\right|_1 & \vdots & -\left.\frac{\partial g_m}{\partial\theta}\right|_1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(\sin\theta_i)^{-1}\left.\frac{\partial g_1}{\partial\phi}\right|_i & \vdots & (\sin\theta_i)^{-1}\left.\frac{\partial g_k}{\partial\phi}\right|_i & \vdots & (\sin\theta_i)^{-1}\left.\frac{\partial g_m}{\partial\phi}\right|_i & -\left.\frac{\partial g_1}{\partial\theta}\right|_i & \vdots & -\left.\frac{\partial g_k}{\partial\theta}\right|_i & \vdots & -\left.\frac{\partial g_m}{\partial\theta}\right|_i \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(\sin\theta_n)^{-1}\left.\frac{\partial g_1}{\partial\phi}\right|_n & \vdots & (\sin\theta_n)^{-1}\left.\frac{\partial g_k}{\partial\phi}\right|_n & \vdots & (\sin\theta_n)^{-1}\left.\frac{\partial g_m}{\partial\phi}\right|_n & -\left.\frac{\partial g_1}{\partial\theta}\right|_n & \vdots & -\left.\frac{\partial g_k}{\partial\theta}\right|_n & \vdots & -\left.\frac{\partial g_m}{\partial\theta}\right|_n
\end{bmatrix}$$

To compute U, V, and W at the n observation points, we have (see Eq. 46)

$$\begin{bmatrix} U|_1 & V|_1 & W|_1 \\ \vdots & \vdots & \vdots \\ U|_n & V|_n & W|_n \end{bmatrix} = \mathbf{G} \begin{bmatrix} p_1 & b_1 & c_1 \\ \vdots & \vdots & \vdots \\ p_m & b_m & c_m \end{bmatrix} \tag{52}$$

3.2 Expansion of v in terms of P, B, and C

We now start with Equation (51) and rearrange terms to define a new vector basis **P**, **B**, and **C**. To avoid clutter, we omit the (θ, ϕ) labels.

$$
\begin{aligned}
\mathbf{v} &= \sum_{k=1}^{m} \left\{ [p_k\, g_k]\, \hat{\mathbf{r}} + \left[b_k \frac{\partial g_k}{\partial \theta} + (\sin\theta)^{-1}\, c_k\, \frac{\partial g_k}{\partial \phi} \right] \hat{\boldsymbol{\theta}} + \left[(\sin\theta)^{-1} b_k \frac{\partial g_k}{\partial \phi} - c_k\, \frac{\partial g_k}{\partial \theta} \right] \hat{\boldsymbol{\phi}} \right\} \\
&= \sum_{k=1}^{m} \left\{ p_k\, g_k\, \hat{\mathbf{r}} + b_k\, \frac{\partial g_k}{\partial \theta}\, \hat{\boldsymbol{\theta}} + b_k\, (\sin\theta)^{-1} \frac{\partial g_k}{\partial \phi}\, \hat{\boldsymbol{\phi}} + c_k\, (\sin\theta)^{-1} \frac{\partial g_k}{\partial \phi}\, \hat{\boldsymbol{\theta}} - c_k\, \frac{\partial g_k}{\partial \theta}\, \hat{\boldsymbol{\phi}} \right\} \\
&= \sum_{k=1}^{m} \left\{ p_k\, [\hat{\mathbf{r}}\, g_k] + b_k \left[\hat{\boldsymbol{\theta}}\, \frac{\partial g_k}{\partial \theta} + \hat{\boldsymbol{\phi}}\, (\sin\theta)^{-1} \frac{\partial g_k}{\partial \phi} \right] + c_k \left[\hat{\boldsymbol{\theta}}\, (\sin\theta)^{-1} \frac{\partial g_k}{\partial \phi} - \hat{\boldsymbol{\phi}}\, \frac{\partial g_k}{\partial \theta} \right] \right\} \\
&= \sum_{k=1}^{m} \left[p_k\, \mathbf{P}_k + b_k\, \mathbf{B}_k + c_k\, \mathbf{C}_k \right]
\end{aligned}
$$

Thus,

$$
\mathbf{v}(\theta,\phi) = \sum_{k=1}^{m} \left[p_k\, \mathbf{P}_k(\theta,\phi) + b_k\, \mathbf{B}_k(\theta,\phi) + c_k\, \mathbf{C}_k(\theta,\phi) \right], \tag{53}
$$

where

$$
\mathbf{P}_k(\theta,\phi) = \hat{\mathbf{r}}\, g_k(\theta,\phi) \tag{54}
$$

$$
\mathbf{B}_k(\theta,\phi) = \boldsymbol{\nabla}_1\, g_k(\theta,\phi) = \hat{\boldsymbol{\theta}}\, \frac{\partial g_k}{\partial \theta} + \hat{\boldsymbol{\phi}}\, (\sin\theta)^{-1} \frac{\partial g_k}{\partial \phi} \tag{55}
$$

$$
\mathbf{C}_k(\theta,\phi) = -\hat{\mathbf{r}} \times \boldsymbol{\nabla}_1\, g_k(\theta,\phi) = -\hat{\mathbf{r}} \times \left[\hat{\boldsymbol{\theta}}\, \frac{\partial g_k}{\partial \theta} + \hat{\boldsymbol{\phi}}\, (\sin\theta)^{-1} \frac{\partial g_k}{\partial \phi} \right] = \hat{\boldsymbol{\theta}}\, (\sin\theta)^{-1} \frac{\partial g_k}{\partial \phi} - \hat{\boldsymbol{\phi}}\, \frac{\partial g_k}{\partial \theta} \tag{56}
$$

3.3 Computing $\boldsymbol{\nabla}\mathbf{v}$ with the new model vectors (UNFINISHED)

We have introduced the vectors **p**, **b**, and **c**, which obtained from solving a least-squares problem. In Equations (50) and (51) we showed the formula for the $\hat{\mathbf{r}}$-$\hat{\boldsymbol{\theta}}$-$\hat{\boldsymbol{\phi}}$ components of **v** in terms of the elements of **p**, **b**, and **c**.

4 Reductions useful for GPS velocity fields

In the previous section, we computed the velocity gradient tensor ($\mathbf{L}$), strain rate tensor ($\mathbf{D}$), and rotation-rate tensor ($\mathbf{W}$) for (local) spherical coordinates, assuming a 3D (local) velocity field ($\mathbf{v}$) and a 3D gradient operator ($\boldsymbol{\nabla}$).

Because GPS stations are distributed laterally and not on top of each other, there are no observations of $\partial v_r/\partial r$, $\partial v_\theta/\partial r$, and $\partial v_\phi/\partial r$. Furthermore, we may want to neglect the vertical component of the velocity field, either because it is poorly known or small in magnitude compared with the horizontal components. In this section we examine various reductions from the general case.

4.1 The earth as a sphere

From here on out, we assume that the earth is a sphere. In other words, all points are situated at $r = R$, and the unit normal at each point is

$$\hat{\mathbf{n}} = \hat{\mathbf{r}} \ .$$

In the general case, $\hat{\mathbf{n}}$ may point in odd directions (e.g., on the side of Mt. Everest). At the wavelengths we are looking at, $\lambda > 10$ km, we are saying that effects associated with topographic gradients can be ignored.

4.2 Free surface condition (3D)

Because the observations are made at the *surface*, we are able to reduce Equation (17). At the surface, $r = R$, we have the free-surface condition

$$\boldsymbol{\sigma}\hat{\mathbf{n}} = \mathbf{0}, \tag{57}$$

where $\boldsymbol{\sigma}$ is the second-order stress tensor, and $\hat{\mathbf{n}} = \hat{\mathbf{r}} = (1, 0, 0)$ is the vertical-pointing outward unit normal in the r-θ-ϕ local-coordinate reference frame. This provides the constraints

$$\sigma_{11} = 0, \qquad \sigma_{21} = 0, \qquad \sigma_{31} = 0. \tag{58}$$

4.2.1 Assumption of elastic rheology

We now assume that Hooke's law is valid, i.e., that stress is linearly related to strain,

$$\boldsymbol{\sigma} \ = \ \mathbf{c} : \boldsymbol{\varepsilon}, \tag{59}$$

where $\mathbf{c}$ is the fourth-order elastic tensor, and $\boldsymbol{\varepsilon}$ is the second-order strain tensor. For an isotropic solid, the stress tensor reduces to (e.g., *Shearer*, 1999, Eq. 2.24):

$$\boldsymbol{\sigma} = \begin{bmatrix} \lambda \operatorname{Tr}(\boldsymbol{\varepsilon}) + 2\mu\varepsilon_{11} & 2\mu\varepsilon_{12} & 2\mu\varepsilon_{13} \\ \sigma_{12} & \lambda \operatorname{Tr}(\boldsymbol{\varepsilon}) + 2\mu\varepsilon_{22} & 2\mu\varepsilon_{23} \\ \sigma_{13} & \sigma_{23} & \lambda \operatorname{Tr}(\boldsymbol{\varepsilon}) + 2\mu\varepsilon_{33} \end{bmatrix}. \tag{60}$$

Combining the constraints in Equation (58) with Equation (60), and solving for the strain components, we obtain

$$\begin{aligned} \varepsilon_{11} &= F(\varepsilon_{22} + \varepsilon_{33}) \\ \varepsilon_{12} &= 0 \\ \varepsilon_{13} &= 0. \end{aligned}$$

where

$$F = \frac{-\lambda}{\lambda + 2\mu} \tag{61}$$

is a constant; for a Poisson solid ($\lambda = \mu$),

$$F = -1/3. \tag{62}$$

Hence, the strain tensor for an isotropic solid is given by

$$\boldsymbol{\varepsilon}_0 = \begin{bmatrix} F(\varepsilon_{22} + \varepsilon_{33}) & 0 & 0 \\ 0 & \varepsilon_{22} & \varepsilon_{23} \\ 0 & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix}, \tag{63}$$

where the 0-subscript denotes that it is evaluated at the surface.

Considering a (infinitesimal?) time increment, Δt, we have $\boldsymbol{\varepsilon} \rightarrow \mathbf{D}$. Our strain rate tensor is now

$$\mathbf{D}_0^{\text{elastic}} = \begin{bmatrix} F(D_{22} + D_{33}) & 0 & 0 \\ 0 & D_{22} & D_{23} \\ 0 & D_{23} & D_{33} \end{bmatrix}. \tag{64}$$

The dilatation rate is thus

$$tr\left(\mathbf{D}_0^{\text{elastic}}\right) = (D_{22} + D_{33})(1 + F). \tag{65}$$

With these assumptions, for an observation point at the surface ($r = R$), Equation (17) reduces to

$$\mathbf{D}_0^{\text{elastic}} = \begin{bmatrix} F\left(D_{22}+D_{33}\right) & 0 & 0 \\ 0 & \frac{1}{R}\left(v_r + \frac{\partial v_\theta}{\partial \theta}\right) & \frac{1}{2R}\left(-v_\phi \cot\theta + \frac{1}{\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{\partial v_\phi}{\partial \theta}\right) \\ 0 & D_{23} & \frac{1}{R}\left(v_r + v_\theta \cot\theta + \frac{1}{\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right) \end{bmatrix} . \tag{66}$$

where we have used the constraints

$$\begin{aligned} \frac{\partial v_r}{\partial r} &= F\left[\frac{1}{R}\left(v_r + \frac{\partial v_\theta}{\partial \theta}\right) + \frac{1}{R}\left(v_r + v_\theta \cot\theta + \frac{1}{\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right)\right] \\ \frac{1}{2R}\left(-v_\theta + \frac{\partial v_r}{\partial \theta} + R\frac{\partial v_\theta}{\partial r}\right) &= 0 \\ \frac{1}{2R}\left(-v_\phi + \frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} + R\frac{\partial v_\phi}{\partial r}\right) &= 0 \, , \end{aligned}$$

which can be solved for the radial gradients (at $r = R$) to obtain

$$\frac{\partial v_r}{\partial r} = \frac{F}{R}\left(2v_r + v_\theta \cot\theta + \frac{\partial v_\theta}{\partial \theta} + \frac{1}{\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right) \tag{67}$$

$$\frac{\partial v_\theta}{\partial r} = \frac{1}{R}\left(v_\theta - \frac{\partial v_r}{\partial \theta}\right) \tag{68}$$

$$\frac{\partial v_\phi}{\partial r} = \frac{1}{R}\left(v_\phi - \frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi}\right) \, , \tag{69}$$

Substituting these equations into Equation (18), the rotation-rate tensor (at $r = R$) reduces to

$$\mathbf{W}_0^{\text{elastic}} = \begin{bmatrix} 0 & \frac{1}{R}\left(-v_\theta + \frac{\partial v_r}{\partial \theta}\right) & \frac{1}{R}\left(-v_\phi + \frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi}\right) \\ -W_{12} & 0 & \frac{1}{2R}\left(-v_\phi \cot\theta + \frac{1}{\sin\theta}\frac{\partial v_\theta}{\partial \phi} - \frac{\partial v_\phi}{\partial \theta}\right) \\ -W_{13} & -W_{23} & 0 \end{bmatrix} . \tag{70}$$

The velocity gradient tensor (Eq. 15) reduces to

$$\mathbf{L}_0^{\text{elastic}} = \begin{bmatrix} F\left(L_{22}+L_{33}\right) & \frac{1}{R}\left(-v_\theta + \frac{\partial v_r}{\partial \theta}\right) & \frac{1}{R}\left(-v_\phi + \frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi}\right) \\ -L_{12} & \frac{1}{R}\left(v_r + \frac{\partial v_\theta}{\partial \theta}\right) & \frac{1}{R}\left(-v_\phi \cot\theta + \frac{1}{\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right) \\ -L_{13} & \frac{1}{R}\left(\frac{\partial v_\phi}{\partial \theta}\right) & \frac{1}{R}\left(v_r + v_\theta \cot\theta + \frac{1}{\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right) \end{bmatrix} . \tag{71}$$

4.2.2 Assumption of viscous rheology

In the previous section, we assumed an elastic rheology (Eq. 59). If instead, we now assume a linear viscous rheology of the form (*Malvern*, 1969, p. 298):

$$\boldsymbol{\sigma} = \operatorname{Tr}\left(\boldsymbol{\sigma}\right)\mathbf{I} + \Lambda \operatorname{Tr}\left(\mathbf{D}\right)\mathbf{I} + 2\eta\mathbf{D} \, , \tag{72}$$

where η and Λ characterize the viscosity.

With the assumption of incompressibility,

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \mathrm{Tr}\left(\mathbf{L}\right) = \mathrm{Tr}\left(\mathbf{D}\right) = 0 \ , \tag{73}$$

and therefore

$$D_{11} = -(D_{22} + D_{33}) \ . \tag{74}$$

In comparison with Equation (66), this corresponds to $F = -1$. With the incompressibility constraint, Equation (72) is written as

$$\begin{bmatrix} \frac{2}{3}\sigma_{11} - \frac{1}{3}\sigma_{22} - \frac{1}{3}\sigma_{33} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \frac{2}{3}\sigma_{22} - \frac{1}{3}\sigma_{11} - \frac{1}{3}\sigma_{33} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \frac{2}{3}\sigma_{33} - \frac{1}{3}\sigma_{11} - \frac{1}{3}\sigma_{22} \end{bmatrix} = 2\eta \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{12} & D_{22} & D_{23} \\ D_{13} & D_{23} & D_{33} \end{bmatrix} . \tag{75}$$

Applying the free-surface constraints in Equation (58), we obtain

$$\begin{bmatrix} -\frac{1}{3}\sigma_{22} - \frac{1}{3}\sigma_{33} & 0 & 0 \\ 0 & \frac{2}{3}\sigma_{22} - \frac{1}{3}\sigma_{33} & \sigma_{23} \\ 0 & \sigma_{23} & \frac{2}{3}\sigma_{33} - \frac{1}{3}\sigma_{22} \end{bmatrix} = 2\eta \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{12} & D_{22} & D_{23} \\ D_{13} & D_{23} & D_{33} \end{bmatrix} . \tag{76}$$

Thus, we obtain the additional constraints, $D_{12} = D_{13} = 0$, and our strain rate tensor is now

$$\mathbf{D}_0^{\mathrm{viscous}} = \begin{bmatrix} -(D_{22} + D_{33}) & 0 & 0 \\ 0 & D_{22} & D_{23} \\ 0 & D_{23} & D_{33} \end{bmatrix} , \tag{77}$$

and the algebra follows the elastic case, but with $F = -1$.

5 Scalar quantities derived from tensors

For plotting purposes, we are interested in computing scalar quantities from the estimated tensor fields. In general, a tensor will not be in its eigenbasis, and thus we are interested in computing quantities that are invariant under coordinate transformations.

5.1 Trace

We denote the square tensor $\mathbf{T}$ in its eigenbasis as $\mathbf{D}$:

$$\mathbf{T} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}, \tag{78}$$

where $\mathbf{V}$ is a $N \times N$ matrix of the eigenvectors of $\mathbf{T}$, and $\mathbf{D}$ is a diagonal matrix of corresponding eigenvalues λ_k. Using the property

$$\mathrm{Tr}\left(\mathbf{AB}\right) = \mathrm{Tr}\left(\mathbf{BA}\right), \tag{79}$$

we can write

$$\mathrm{Tr}\left(\mathbf{T}\right) = \mathrm{Tr}\left(\mathbf{V}\mathbf{D}\mathbf{V}^{-1}\right) = \mathrm{Tr}\left(\mathbf{V}^{-1}\mathbf{V}\mathbf{D}\right) = \mathrm{Tr}\left(\mathbf{D}\right) = \sum_{k=1}^{N} \lambda_k, \tag{80}$$

which is generalized to

$$\mathrm{Tr}\left(\mathbf{T}^p\right) = \sum_{k=1}^{N} \lambda_k^p \,. \tag{81}$$

Because the eigenvalues of tensor $\mathbf{T}$ are invariant to coordinate changes, the operation $\mathrm{Tr}\left(\mathbf{T}^p\right)$ is also invariant for any p.

5.2 Matrix norms

A common matrix norm is the Frobenius norm, defined as (*Golub and Van Loan*, 1989, p. 56)

$$\|\mathbf{T}\|_F = \left(\sum_{i=1}^{N} \sum_{j=1}^{N} |T_{ij}|^2 \right)^{1/2} . \tag{82}$$

If $\mathbf{T}$ is real-valued, then

$$\|\mathbf{T}\|_F^2 = \mathbf{T} : \mathbf{T}, \tag{83}$$

where ":" denotes the scalar product of two tensors (*Malvern*, 1969, p. 35). If $\mathbf{T}$ is symmetric, then

$$\|\mathbf{T}\|_F^2 = \mathbf{T} : \mathbf{T} = tr\left(\mathbf{T}\mathbf{T}\right) = \sum_{k=1}^{N} \lambda_k^2. \tag{84}$$

5.3 Invariants obtained from the Cayley–Hamilton theorem

The Cayley–Hamilton theorem states that any square matrix will satisfy its own characteristic equation. In three dimensions it is given by

$$\mathbf{T}^3 - I_1\,\mathbf{T}^2 + I_2\,\mathbf{T} - I_3\,\mathbf{I} = \mathbf{0}, \tag{85}$$

where the three scalar invariants for tensor $\mathbf{T}$ are given by

$$\begin{aligned}
I_1(\mathbf{T}) &= \mathrm{Tr}\left(\mathbf{T}\right) && (86)\\
I_2(\mathbf{T}) &= \tfrac{1}{2}\left\{[\mathrm{Tr}\left(\mathbf{T}\right)]^2 - \mathrm{Tr}\left(\mathbf{T}\mathbf{T}\right)\right\} && (87)\\
I_3(\mathbf{T}) &= \det(\mathbf{T}). && (88)
\end{aligned}$$

These are discussed in *Malvern* (1969, p. 89); however, his expressions assume that $\mathbf{T}$ is symmetric, in which case, $\mathrm{Tr}\left(\mathbf{T}\mathbf{T}\right) = \mathbf{T} : \mathbf{T}$. Furthermore, he uses a different sign convention for I_2.

We now consider the deviatoric part of $\mathbf{T}$,

$$\begin{aligned}
\mathbf{T} &= \mathbf{T}^S + \mathbf{T}^D\\
\mathbf{T}^S &= \tfrac{1}{3}\mathrm{Tr}\left(\mathbf{T}\right)\mathbf{I}\\
\mathbf{T}^D &= \mathbf{T} - \tfrac{1}{3}\mathrm{Tr}\left(\mathbf{T}\right)\mathbf{I}.
\end{aligned}$$

In a general basis, the expressions for the first and second invariants of tensors $\mathbf{T}$ and $\mathbf{T}^D$ are

$$\begin{aligned}
I_1(\mathbf{T}) &= T_{11} + T_{22} + T_{33} && (89)\\
I_2(\mathbf{T}) &= T_{11}T_{22} + T_{22}T_{33} + T_{11}T_{33} - (T_{12}T_{21} + T_{13}T_{31} + T_{23}T_{32}) && (90)\\
I_1(\mathbf{T}^D) &= 0 && (91)\\
I_2(\mathbf{T}^D) &= -\tfrac{1}{2}\,\mathrm{Tr}\left(\mathbf{T}^D\mathbf{T}^D\right)\\
&= \tfrac{1}{3}\left(T_{11}T_{22} + T_{11}T_{33} + T_{22}T_{33}\right) - \tfrac{1}{3}\left(T_{11}^2 + T_{22}^2 + T_{33}^2\right) - \left(T_{12}T_{21} + T_{13}T_{31} + T_{23}T_{32}\right). && (92)
\end{aligned}$$

In the eigenbasis of tensor $\mathbf{T}$, with eigenvalues λ_k, the first and second invariants are

$$
\begin{aligned}
I_1(\mathbf{T}) &= \lambda_1 + \lambda_2 + \lambda_3 && (93)\\
I_2(\mathbf{T}) &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 && (94)\\
I_1(\mathbf{T}^D) &= 0 && (95)\\
I_2(\mathbf{T}^D) &= \tfrac{1}{3}\left(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3\right) - \tfrac{1}{3}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right), && (96)
\end{aligned}
$$

and the third invariants are

$$
\begin{aligned}
I_3(\mathbf{T}) &= \lambda_1\,\lambda_2\,\lambda_3 && (97)\\
I_3(\mathbf{T}^D) &= \frac{1}{27}\left(\lambda_1 + \lambda_2 - 2\lambda_3\right)\left(2\lambda_1 - \lambda_2 - \lambda_3\right)\left(\lambda_1 - 2\lambda_2 + \lambda_3\right). && (98)
\end{aligned}
$$

We can also write the invariants of $\mathbf{T}^D$ (I_1^D, I_2^D, and I_3^D) in terms of the invariants of $\mathbf{T}$ (I_1, I_2, and I_3):

$$
\begin{aligned}
I_1^D &\equiv I_1(\mathbf{T}^D) = 0 && (99)\\
I_2^D &\equiv I_2(\mathbf{T}^D) = I_2 - \tfrac{1}{3}(I_1)^2 && (100)\\
I_3^D &\equiv I_3(\mathbf{T}^D) = I_3 - \tfrac{1}{3}I_1 I_2 + \tfrac{2}{27}(I_1)^3\,. && (101)
\end{aligned}
$$

5.4 Comparison between matrix norms and the Cayley–Hamilton second invariant

We consider four scalar quantities derived from a symmetric 3×3 tensor $\mathbf{T}$: $\mathbf{T} : \mathbf{T}$, $I_2(\mathbf{T})$, $\mathbf{T}^D : \mathbf{T}^D$, $I_2(\mathbf{T}^D)$. Mathematically, a simple way to show the relationships among these quantities is to define the vector

$$
\mathbf{a} = \left[\begin{array}{c} T_{12}^2 + T_{13}^2 + T_{23}^2 \\ T_{11}^2 + T_{22}^2 + T_{33}^2 \\ T_{11}T_{22} + T_{11}T_{33} + T_{22}T_{33}. \end{array}\right]. \qquad (102)
$$

Then, the four scalar quantities can be written as (see Mathematica `tensor_invariants.nb`)

$$
\begin{aligned}
\mathbf{T} : \mathbf{T} &= (2, 1, 0) \cdot \mathbf{a} && (103)\\
I_2(\mathbf{T}) &= (-1, 0, 1) \cdot \mathbf{a} && (104)\\
\mathbf{T}^D : \mathbf{T}^D &= (2, \tfrac{2}{3}, -\tfrac{2}{3}) \cdot \mathbf{a} && (105)\\
I_2(\mathbf{T}^D) &= (-1, -\tfrac{1}{3}, \tfrac{1}{3}) \cdot \mathbf{a} && (106)
\end{aligned}
$$

Miscellaneous

Dilatation is defined in Equation (22). Mathematically, from Equations (**??**) and (65), the ratio of the dilatation rates for the viscous and elastic cases is given by

$$\frac{tr\left(\mathbf{D}_0^{\text{viscous}}\right)}{tr\left(\mathbf{D}_0^{\text{elastic}}\right)} = 1 + F = 1 + \frac{-\lambda}{\lambda + 2\mu} = \frac{2\mu}{\lambda + 2\mu} = \frac{2\beta^2}{\alpha^2} \ . \tag{107}$$

where F was defined in Equation (61), α is the P-wave speed, and β is the S-wave speed.

6 Connections to previous studies

6.1 Connection to *Haines and Holt* (1993)

Haines and Holt (1993) consider the vector field

$$\mathbf{v} = v_r\,\hat{\mathbf{r}} \;+\; v_\lambda\,\hat{\boldsymbol{\lambda}} \;+\; v_\phi\,\hat{\boldsymbol{\phi}}\;, \tag{108}$$

where λ is latitude and ϕ is longitude, and

$$\begin{aligned}
\hat{\mathbf{r}} &= \cos\lambda\cos\phi\,\hat{\mathbf{x}} + \cos\lambda\sin\phi\,\hat{\mathbf{y}} + \sin\lambda\,\hat{\mathbf{z}} && (109)\\
\hat{\boldsymbol{\lambda}} &= -\sin\lambda\cos\phi\,\hat{\mathbf{x}} - \sin\lambda\sin\phi\,\hat{\mathbf{y}} + \cos\lambda\,\hat{\mathbf{z}} && (110)\\
\hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}\;. && (111)
\end{aligned}$$

Haines and Holt (1993) list the following equations in their Appendix A:[2]

$$\begin{aligned}
D_{\lambda\lambda} &= \hat{\boldsymbol{\lambda}}^T\mathbf{D}\,\hat{\boldsymbol{\lambda}} && (112)\\
D_{\phi\phi} &= \hat{\boldsymbol{\phi}}^T\mathbf{D}\,\hat{\boldsymbol{\phi}} && (113)\\
D_{\lambda\phi} &= \hat{\boldsymbol{\lambda}}^T\mathbf{D}\,\hat{\boldsymbol{\phi}} && (114)\\
D_{\lambda\lambda} &= \frac{1}{r}\,\hat{\boldsymbol{\lambda}}\cdot\frac{\partial\mathbf{v}}{\partial\lambda} && (115)\\
D_{\phi\phi} &= \frac{1}{r\cos\lambda}\,\hat{\boldsymbol{\phi}}\cdot\frac{\partial\mathbf{v}}{\partial\phi} && (116)\\
D_{\lambda\phi} &= \frac{1}{2r\cos\lambda}\,\hat{\boldsymbol{\lambda}}\cdot\frac{\partial\mathbf{v}}{\partial\phi} + \frac{1}{2r}\hat{\boldsymbol{\Phi}}\cdot\frac{\partial\mathbf{v}}{\partial\lambda}, && (117)
\end{aligned}$$

We now transition to our notation. In our $\hat{\mathbf{r}}$-$\hat{\boldsymbol{\theta}}$-$\hat{\boldsymbol{\phi}}$ basis, the second basis vector points *south*, that is, [3]

$$\begin{aligned}
\hat{\boldsymbol{\lambda}} &= -\hat{\boldsymbol{\theta}}\\
v_\lambda &= -v_\theta.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
D_{\lambda\lambda} &= \hat{\boldsymbol{\lambda}}^T\mathbf{D}\,\hat{\boldsymbol{\lambda}} = \left(-\hat{\boldsymbol{\theta}}\right)^T\mathbf{D}\,\left(-\hat{\boldsymbol{\theta}}\right) = \hat{\boldsymbol{\theta}}^T\mathbf{D}\,\hat{\boldsymbol{\theta}} = D_{\theta\theta}\\
D_{\phi\phi} &= \hat{\boldsymbol{\phi}}^T\mathbf{D}\,\hat{\boldsymbol{\phi}}\\
D_{\lambda\phi} &= \hat{\boldsymbol{\lambda}}^T\mathbf{D}\,\hat{\boldsymbol{\phi}} = \left(-\hat{\boldsymbol{\theta}}\right)^T\mathbf{D}\,\left(\hat{\boldsymbol{\phi}}\right) = -\hat{\boldsymbol{\theta}}^T\mathbf{D}\,\hat{\boldsymbol{\phi}} = -D_{\theta\phi}
\end{aligned}$$

[2]We have avoided their use of tensor notation in Equations (112)–(114), which makes their exposition very confusing, in my opinion. Also, we have replaced $\dot{\varepsilon}$ by D.

[3]Substituting these into Equation (163), we obtain Equation (14), which is $\mathbf{v}$ in a different basis:

$$\mathbf{v} = v_r\,\hat{\mathbf{r}} \;+\; v_\theta\,\hat{\boldsymbol{\theta}} \;+\; v_\phi\,\hat{\boldsymbol{\phi}}\;. \tag{118}$$

Equations (115)–(117) can now be written as

$$
\begin{aligned}
D_{\theta\theta} &= \frac{1}{r}\left(-\hat{\boldsymbol{\theta}}\right)\cdot\frac{\partial\mathbf{v}}{\partial\left(\frac{\pi}{2}-\theta\right)} \\
&= \frac{1}{r}\hat{\boldsymbol{\theta}}\cdot\frac{\partial\mathbf{v}}{\partial\theta} \\
&= \frac{1}{r}\,\hat{\boldsymbol{\theta}}\cdot\left[\left(-v_\theta+\frac{\partial v_r}{\partial\theta}\right)\hat{\mathbf{r}}+\left(v_r+\frac{\partial v_\theta}{\partial\theta}\right)\hat{\boldsymbol{\theta}}+\left(\frac{\partial v_\phi}{\partial\theta}\right)\hat{\boldsymbol{\phi}}\right] \\
&= \frac{1}{r}\left(v_r+\frac{\partial v_\theta}{\partial\theta}\right) \qquad (119)\\
D_{\phi\phi} &= \frac{1}{r\sin\theta}\,\hat{\boldsymbol{\phi}}\cdot\frac{\partial\mathbf{v}}{\partial\phi} \\
&= \frac{1}{r\sin\theta}\,\hat{\boldsymbol{\phi}}\cdot\left[\left(\frac{\partial v_r}{\partial\phi}-v_\phi\,\sin\theta\right)\hat{\mathbf{r}}+\left(\frac{\partial v_\theta}{\partial\phi}-v_\phi\,\cos\theta\right)\hat{\boldsymbol{\theta}}+\left(v_r\,\sin\theta+v_\theta\,\cos\theta+\frac{\partial v_\phi}{\partial\phi}\right)\hat{\boldsymbol{\phi}}\right] \\
&= \frac{1}{r\sin\theta}\left(v_r\,\sin\theta+v_\theta\,\cos\theta+\frac{\partial v_\phi}{\partial\phi}\right) \\
&= \frac{1}{r}\left(v_r+v_\theta\,\cot\theta+\frac{1}{\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right) \qquad (120)\\
-D_{\theta\phi} &= \frac{1}{2r\sin\theta}\left(-\hat{\boldsymbol{\theta}}\right)\cdot\frac{\partial\mathbf{v}}{\partial\phi}+\frac{1}{2r}\,\hat{\boldsymbol{\phi}}\cdot\frac{\partial\mathbf{v}}{\partial\left(\frac{\pi}{2}-\theta\right)} \\
&= \frac{-1}{2r\sin\theta}\,\hat{\boldsymbol{\theta}}\cdot\frac{\partial\mathbf{v}}{\partial\phi}-\frac{1}{2r}\,\hat{\boldsymbol{\phi}}\cdot\frac{\partial\mathbf{v}}{\partial\theta} \\
&= \frac{-1}{2r\sin\theta}\,\hat{\boldsymbol{\theta}}\cdot\left[\left(\frac{\partial v_r}{\partial\phi}-v_\phi\,\sin\theta\right)\hat{\mathbf{r}}+\left(\frac{\partial v_\theta}{\partial\phi}-v_\phi\,\cos\theta\right)\hat{\boldsymbol{\theta}}+\left(v_r\,\sin\theta+v_\theta\,\cos\theta+\frac{\partial v_\phi}{\partial\phi}\right)\hat{\boldsymbol{\phi}}\right] \\
&\quad -\frac{1}{2r}\,\hat{\boldsymbol{\phi}}\cdot\left[\left(-v_\theta+\frac{\partial v_r}{\partial\theta}\right)\hat{\mathbf{r}}+\left(v_r+\frac{\partial v_\theta}{\partial\theta}\right)\hat{\boldsymbol{\theta}}+\left(\frac{\partial v_\phi}{\partial\theta}\right)\hat{\boldsymbol{\phi}}\right] \\
&= \frac{-1}{2r\sin\theta}\left(\frac{\partial v_\theta}{\partial\phi}-v_\phi\,\cos\theta\right)-\frac{1}{2r}\frac{\partial v_\phi}{\partial\theta} \\
&= \frac{1}{2r}\left(v_\phi\,\cot\theta-\frac{1}{\sin\theta}\frac{\partial v_\theta}{\partial\phi}-\frac{\partial v_\phi}{\partial\theta}\right)\;, \qquad (121)
\end{aligned}
$$

where we have used the expressions

$$
\begin{aligned}
\frac{\partial \mathbf{v}}{\partial \theta} &= \frac{\partial}{\partial \theta}\left(v_r\,\hat{\mathbf{r}} \;+\; v_\theta\,\hat{\boldsymbol{\theta}} \;+\; v_\phi\,\hat{\boldsymbol{\phi}}\right) \\
&= \frac{\partial v_r}{\partial \theta}\,\hat{\mathbf{r}} + v_r\frac{\partial \hat{\mathbf{r}}}{\partial \theta} + \frac{\partial v_\theta}{\partial \theta}\,\hat{\boldsymbol{\theta}} + v_\theta\frac{\partial \hat{\boldsymbol{\theta}}}{\partial \theta} + \frac{\partial v_\phi}{\partial \theta}\,\hat{\boldsymbol{\phi}} + v_\phi\frac{\partial \hat{\boldsymbol{\phi}}}{\partial \theta} \\
&= \frac{\partial v_r}{\partial \theta}\,\hat{\mathbf{r}} + v_r\,\hat{\boldsymbol{\theta}} + \frac{\partial v_\theta}{\partial \theta}\,\hat{\boldsymbol{\theta}} - v_\theta\,\hat{\mathbf{r}} + \frac{\partial v_\phi}{\partial \theta}\,\hat{\boldsymbol{\phi}} \\
&= \left(-v_\theta + \frac{\partial v_r}{\partial \theta}\right)\hat{\mathbf{r}} + \left(v_r + \frac{\partial v_\theta}{\partial \theta}\right)\hat{\boldsymbol{\theta}} + \left(\frac{\partial v_\phi}{\partial \theta}\right)\hat{\boldsymbol{\phi}} \\
\frac{\partial \mathbf{v}}{\partial \phi} &= \frac{\partial}{\partial \phi}\left(v_r\,\hat{\mathbf{r}} \;+\; v_\theta\,\hat{\boldsymbol{\theta}} \;+\; v_\phi\,\hat{\boldsymbol{\phi}}\right) \\
&= \frac{\partial v_r}{\partial \phi}\,\hat{\mathbf{r}} + v_r\frac{\partial \hat{\mathbf{r}}}{\partial \phi} + \frac{\partial v_\theta}{\partial \phi}\,\hat{\boldsymbol{\theta}} + v_\theta\frac{\partial \hat{\boldsymbol{\theta}}}{\partial \phi} + \frac{\partial v_\phi}{\partial \phi}\,\hat{\boldsymbol{\phi}} + v_\phi\frac{\partial \hat{\boldsymbol{\phi}}}{\partial \phi} \\
&= \frac{\partial v_r}{\partial \phi}\,\hat{\mathbf{r}} + v_r\sin\theta\,\hat{\boldsymbol{\phi}} + \frac{\partial v_\theta}{\partial \phi}\,\hat{\boldsymbol{\theta}} + v_\theta\cos\theta\,\hat{\boldsymbol{\phi}} + \frac{\partial v_\phi}{\partial \phi}\,\hat{\boldsymbol{\phi}} + v_\phi\left(-\sin\theta\,\hat{\mathbf{r}} - \cos\theta\,\hat{\boldsymbol{\theta}}\right) \\
&= \left(\frac{\partial v_r}{\partial \phi} - v_\phi\,\sin\theta\right)\hat{\mathbf{r}} + \left(\frac{\partial v_\theta}{\partial \phi} - v_\phi\,\cos\theta\right)\hat{\boldsymbol{\theta}} + \left(v_r\,\sin\theta + v_\theta\,\cos\theta + \frac{\partial v_\phi}{\partial \phi}\right)\hat{\boldsymbol{\phi}}.
\end{aligned}
$$

Equations (121)–(120) are identical to those found in Equation (17). *Haines and Holt* (1993) do not list expressions for D_{rr}, $D_{r\theta}$, or $D_{r\phi}$, because they neglect the vertical velocity component and also derivatives of the horizontal velocity components with respect to the vertical direction. They justify these choices.

6.2 Connection to *Ward* (1998a)

6.2.1 Excerpt from *Ward* (1998a) Section 3.1

This article examines large areas of the world, so velocities, rotations and strains are modelled in spherical coordinates with r, θ and ϕ being the distance from the Earth's centre, and conventional latitude and longitude respectively. Consider nearby surface points $\mathbf{r} = (R, \theta, \phi)$ and $\mathbf{r}_0 = (R, \theta_0, \phi_0)$, with local north, east and vertical directions $(\hat{\mathbf{n}}, \hat{\mathbf{e}}, \hat{\mathbf{r}})$ and $(\hat{\mathbf{n}}_0, \hat{\mathbf{e}}_0, \hat{\mathbf{r}}_0)$. To first order, the geodetic velocity at $\mathbf{r}$ can be expressed by the deformation rate tensor $\boldsymbol{\nabla}\mathbf{v}(\mathbf{r})$ as

$$\mathbf{v}(\mathbf{r}) \sim \mathbf{v}(\mathbf{r}_0) + (\mathbf{r} - \mathbf{r}_0)\cdot\boldsymbol{\nabla}\mathbf{v}(\mathbf{r}_0). \tag{122}$$

If motions are everywhere tangent to the surface in the form

$$\mathbf{v}(\mathbf{r}) = \boldsymbol{\Omega}(\mathbf{r}) \times \mathbf{r} \tag{123}$$

then (122) becomes

$$\mathbf{v}(\mathbf{r}) \sim \boldsymbol{\Omega}(\mathbf{r}_0) \times \mathbf{r} + (\mathbf{r} - \mathbf{r}_0)\cdot\left[\boldsymbol{\nabla}\boldsymbol{\Omega}(\mathbf{r}_0) \times \mathbf{r}_0\right]. \tag{124}$$

The first term describes a rigid-body rotation about pole $\boldsymbol{\Omega}(\mathbf{r}_0)$. Decomposition of the tensor $\boldsymbol{\nabla}\boldsymbol{\Omega}(\mathbf{r}_0) \times \mathbf{r}_0$ into symmetric and anti-symmetric elements yields a strain rate tensor $\dot{\boldsymbol{\varepsilon}}(\mathbf{r}_0)$ and an additional rotation tensor $\mathbf{R}(\mathbf{r}_0)$. Clearly, if $\mathbf{\Omega}(\mathbf{r})$ is constant, then $\nabla\mathbf{\Omega}(\mathbf{r}_0) = \mathbf{0}$ and both $\dot{\boldsymbol{\varepsilon}}(\mathbf{r}_0)$ and $\mathbf{R}(\mathbf{r}_0)$ vanish. Restricting $\mathbf{\Omega}(\mathbf{r})$ to be a function of surface coordinates only,

$$\mathbf{\Omega}(\mathbf{r}) = \mathbf{\Omega}(\theta, \phi), \tag{125}$$

causes the nine components of $\dot{\boldsymbol{\varepsilon}}(\mathbf{r})$ and $\mathbf{R}(\mathbf{r})$ to be reduced to four:

$$\begin{aligned} \dot{\boldsymbol{\varepsilon}}(\mathbf{r}) &= [\dot{\varepsilon}_{nn}(\mathbf{r})\hat{\mathbf{n}}\hat{\mathbf{n}} + \dot{\varepsilon}_{ne}(\mathbf{r})\,(\hat{\mathbf{n}}\hat{\mathbf{e}} - \hat{\mathbf{e}}\hat{\mathbf{n}}) + \dot{\varepsilon}_{ee}(\mathbf{r})\hat{\mathbf{e}}\hat{\mathbf{e}}] & (126) \\ \mathbf{R}(\mathbf{r}) &= R(\mathbf{r})\,(\hat{\mathbf{n}}\hat{\mathbf{e}} - \hat{\mathbf{e}}\hat{\mathbf{n}}) & (127) \end{aligned}$$

6.2.2 *Ward* (1998a) Section 3.1 using our notation and convention

We will use the notation shown in Equations (153) and (155) for the rotation matrix and rotation vector. For consistency of notation, we first make the following direct variable changes:

$$\begin{aligned} \mathbf{r},\ \mathbf{r}_0 &\rightarrow \mathbf{x},\ \mathbf{x}_0 \\ \mathbf{\Omega} &\rightarrow \mathbf{q} \\ \mathbf{R} &\rightarrow \mathbf{W} \\ \dot{\boldsymbol{\varepsilon}} &\rightarrow \mathbf{D} \\ e &\rightarrow \phi \\ n &\rightarrow \lambda \\ \hat{\mathbf{e}},\ \hat{\mathbf{e}}_0 &\rightarrow \hat{\boldsymbol{\phi}},\ \hat{\boldsymbol{\phi}}_0 \\ \hat{\mathbf{n}},\ \hat{\mathbf{n}}_0 &\rightarrow \hat{\boldsymbol{\lambda}},\ \hat{\boldsymbol{\lambda}}_0 \\ R(\mathbf{r}) &\rightarrow W_{\lambda\phi}(\mathbf{x}) \end{aligned}$$

Let $\mathbf{x}$ denote the position of a point in a 3-D space, and $\mathbf{v}(\mathbf{x})$ its velocity. The first-order Taylor approximation of $\mathbf{v}(\mathbf{x})$ about position $\mathbf{x}_0$ is given by

$$\mathbf{v}(\mathbf{x}) \approx \mathbf{v}(\mathbf{x}_0) + \nabla\mathbf{v}(\mathbf{x}_0)\,(\mathbf{x} - \mathbf{x}_0)\,. \tag{128}$$

Assume that motions are purely tangential, i.e., everywhere perpendicular to the position $\mathbf{x}$. Then, there exists a field $\mathbf{q}(\mathbf{x})$ such that

$$\mathbf{v}(\mathbf{x}) = \mathbf{q}(\mathbf{x}) \times \mathbf{x} = [\mathbf{q}(\mathbf{x})]_\times\,\mathbf{x}\,, \tag{129}$$

where the matrix $[\mathbf{q}(\mathbf{x})]_\times$ is defined in Equation (155). Using the identity $\nabla(\mathbf{a} \times \mathbf{b}) = \nabla\mathbf{a} \times \mathbf{b} + \mathbf{a} \times \nabla\mathbf{b}$ (Section 7), it follows that

$$\nabla\mathbf{v}(\mathbf{x}) = \nabla\mathbf{q}(\mathbf{x}) \times \mathbf{x} + \mathbf{q}(\mathbf{x}) \times \nabla\mathbf{x}. \tag{130}$$

This simplifies as

$$\begin{aligned}
\boldsymbol{\nabla}\mathbf{v}(\mathbf{x}) &= \boldsymbol{\nabla}\mathbf{q}(\mathbf{x}) \times \mathbf{x} + \mathbf{q}(\mathbf{x}) \times \boldsymbol{\nabla}\mathbf{x} \\
&= \boldsymbol{\nabla}\mathbf{q}(\mathbf{x}) \times \mathbf{x} + \mathbf{q}(\mathbf{x}) \times \mathbf{I}_3 \\
&= \boldsymbol{\nabla}\mathbf{q}(\mathbf{x}) \times \mathbf{x} + [\mathbf{q}(\mathbf{x})]_\times \mathbf{I}_3 \\
&= \boldsymbol{\nabla}\mathbf{q}(\mathbf{x}) \times \mathbf{x} + [\mathbf{q}(\mathbf{x})]_\times,
\end{aligned} \tag{131}$$

where $\mathbf{I}_3$ is the 3×3 identity matrix. Evaluating Equations (129) and (131) at $\mathbf{x} = \mathbf{x}_0$, and then substituting into Equation (128), we obtain

$$\begin{aligned}
\mathbf{v}(\mathbf{x}) &\approx \mathbf{v}(\mathbf{x}_0) + \boldsymbol{\nabla}\mathbf{v}(\mathbf{x}_0)\,(\mathbf{x} - \mathbf{x}_0) \\
&= \mathbf{q}(\mathbf{x}_0) \times \mathbf{x}_0 + \left\{\mathbf{L}_0^{\text{Ward}} + [\mathbf{q}(\mathbf{x}_0)]_\times\right\}(\mathbf{x} - \mathbf{x}_0) \\
&= \mathbf{q}(\mathbf{x}_0) \times \mathbf{x}_0 + \mathbf{L}_0^{\text{Ward}}\,(\mathbf{x} - \mathbf{x}_0) + [\mathbf{q}(\mathbf{x}_0)]_\times\,\mathbf{x} - [\mathbf{q}(\mathbf{x}_0)]_\times\,\mathbf{x}_0 \\
&= \mathbf{q}(\mathbf{x}_0) \times \mathbf{x}_0 + \mathbf{L}_0^{\text{Ward}}\,(\mathbf{x} - \mathbf{x}_0) + \mathbf{q}(\mathbf{x}_0) \times \mathbf{x} - \mathbf{q}(\mathbf{x}_0) \times \mathbf{x}_0 \\
&= \mathbf{q}(\mathbf{x}_0) \times \mathbf{x} + \mathbf{L}_0^{\text{Ward}}\,(\mathbf{x} - \mathbf{x}_0)\,.
\end{aligned} \tag{132}$$

where we have defined

$$\mathbf{L}_0^{\text{Ward}} \equiv \boldsymbol{\nabla}\mathbf{q}(\mathbf{x}_0) \times \mathbf{x}_0\ . \tag{133}$$

Our local coordinate system is $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}})$, denoting up, south, and east.

From now on, points are constrained to lie on the surface of a sphere or radius R, and are described in a spherical coordinates (local east, north, and vertical directions). The local bases at $\mathbf{x}$ and $\mathbf{x}_0$ are noted $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}})$ and $(\hat{\mathbf{r}}_0, \hat{\boldsymbol{\theta}}_0, \hat{\boldsymbol{\phi}}_0)$, respectively.

The first term in Equation (132) corresponds to a rigid-body rotation about the pole $\mathbf{q}(\mathbf{x}_0)$. Since for all $\mathbf{x}$, velocity $\mathbf{v}(\mathbf{x})$ has no component in the vertical direction, the tensor $\mathbf{q}(\mathbf{x}_0) \times \mathbf{x}_0$ in the second term must have zeroes in its last row. By assuming that the field $\mathbf{q}(\mathbf{x})$ depends on the surface coordinates only ($\mathbf{q}(\mathbf{x}) = \mathbf{q}(\theta, \phi)$), the last column of the tensor also vanishes, reducing its d.o.f. to four. Decomposition of the tensor $\boldsymbol{\nabla}\mathbf{q}(\mathbf{x}_0) \times \mathbf{x}_0$ into its symmetric and anti-symmetric parts yields a strain rate tensor $\mathbf{D}(\mathbf{x}_0)$ with three d.o.f., and a rotation tensor $\mathbf{W}(\mathbf{x}_0)$ with a single d.o.f.:

$$\mathbf{D}(\mathbf{x}_0) = D_{\theta\theta}(\mathbf{x}_0)\hat{\boldsymbol{\theta}}_0\hat{\boldsymbol{\theta}}_0 + D_{\theta\phi}(\mathbf{x}_0)\left(\hat{\boldsymbol{\theta}}_0\hat{\boldsymbol{\phi}}_0 + \hat{\boldsymbol{\phi}}_0\hat{\boldsymbol{\theta}}_0\right) + D_{\phi\phi}(\mathbf{x}_0)\hat{\boldsymbol{\phi}}_0\hat{\boldsymbol{\phi}}_0, \tag{134}$$

$$\mathbf{W}(\mathbf{x}_0) = W_{\phi\theta}(\mathbf{x}_0)\left(\hat{\boldsymbol{\phi}}_0\hat{\boldsymbol{\theta}}_0 - \hat{\boldsymbol{\theta}}_0\hat{\boldsymbol{\phi}}_0\right)\ , \tag{135}$$

which, in matrix notation, are[4]

$$\mathbf{D}(\mathbf{x}_0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & D_{\theta\theta} & D_{\theta\phi} \\ 0 & D_{\theta\phi} & D_{\phi\phi} \end{bmatrix} \tag{136}$$

$$\mathbf{W}(\mathbf{x}_0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -W_{\phi\theta} \\ 0 & W_{\phi\theta} & 0 \end{bmatrix}. \tag{137}$$

The convention for the rotation is such that $W_{\phi\theta} > 0$ denotes positive rotation about the local $\hat{\mathbf{r}}$. For example, if $W_{\phi\theta} = 1$, then application of $\mathbf{W}$ to $\hat{\boldsymbol{\phi}} = (0, 0, 1)$ is $\mathbf{W}\hat{\boldsymbol{\phi}} = (0,\, -1,\, 0) = -\hat{\boldsymbol{\theta}}$, which is a positive rotation of an east-pointing vector to a north-pointing vector.

To directly compare these equations with those of *Ward* (1998a), we must change from our basis, $(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}})$, to $(\hat{\mathbf{r}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\lambda}})$. (Note that the basis of *Ward* (1998a), $(\hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{r}})$, is not a right-handed ordering.) The new basis requires making the substitutions

$$\begin{aligned} \hat{\boldsymbol{\lambda}} &= -\hat{\boldsymbol{\theta}} \\ v_\lambda &= -v_\theta \\ D_{\lambda\lambda} &= D_{\theta\theta} \\ D_{\lambda\phi} &= -D_{\theta\phi} \\ W_{r\lambda} &= -W_{r\theta} \\ W_{\lambda\phi} &= -W_{\theta\phi} \\ W_{\phi\lambda} &= -W_{\phi\theta} \\ q_\lambda &= -q_\theta \end{aligned} \tag{138}$$

Omitting the 0-subscripts and substituting Equation (138) into Equations (134) and (135), we obtain

$$\mathbf{D}(\mathbf{x}) = D_{\theta\theta}(\mathbf{x})\hat{\boldsymbol{\lambda}}\hat{\boldsymbol{\lambda}} + D_{\lambda\phi}(\mathbf{x})\left(\hat{\boldsymbol{\lambda}}\hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\phi}}\hat{\boldsymbol{\lambda}}\right) + D_{\phi\phi}(\mathbf{x})\hat{\boldsymbol{\phi}}\hat{\boldsymbol{\phi}} \tag{139}$$

$$\mathbf{W}(\mathbf{x}) = W_{\phi\lambda}(\mathbf{x})\left(\hat{\boldsymbol{\phi}}\hat{\boldsymbol{\lambda}} - \hat{\boldsymbol{\lambda}}\hat{\boldsymbol{\phi}}\right) = W_{\lambda\phi}(\mathbf{x})\left(\hat{\boldsymbol{\lambda}}\hat{\boldsymbol{\phi}} - \hat{\boldsymbol{\phi}}\hat{\boldsymbol{\lambda}}\right). \tag{140}$$

With the direct variable changes (Eq. 128), Equations (139) and (140) are identical to *Ward* (1998a, Equation 6):

$$\dot{\boldsymbol{\varepsilon}}(\mathbf{x}) = \dot{\varepsilon}_{nn}(\mathbf{x})\hat{\mathbf{n}}\hat{\mathbf{n}} + \dot{\varepsilon}_{ne}(\mathbf{x})\left(\hat{\mathbf{n}}\hat{\mathbf{e}} + \hat{\mathbf{e}}\hat{\mathbf{n}}\right) + \dot{\varepsilon}_{ee}(\mathbf{x})\hat{\mathbf{e}}\hat{\mathbf{e}} \tag{141}$$

$$\mathbf{R}(\mathbf{x}) = R(\mathbf{x})\left(\hat{\mathbf{n}}\hat{\mathbf{e}} - \hat{\mathbf{e}}\hat{\mathbf{n}}\right). \tag{142}$$

[4] Comparing Equation (137) with the general expression in Equation (15) and the elastic expression in Equation (71), we see that the constraint limiting motion to the surface of the sphere, used by *Ward* (1998a), is stronger than the elastic (or viscous) rheology constraint.

Inversion for strains and rotations

The objective is to infer the value of $\mathbf{q}(\mathbf{x}_0)$, $W_{\theta\phi}(\mathbf{x}_0)$, $D_{\theta\theta}(\mathbf{x}_0)$, $D_{\theta\phi}(\mathbf{x}_0)$, $D_{\phi\phi}(\mathbf{x}_0)$, at a set of gridpoints $\mathbf{x}_0$, given a set of velocities $\mathbf{v}(\mathbf{x})$ measured at a set of unevenly distributed stations with location $\mathbf{x}$.

We first characterize the forward problem by deriving the 2×7 matrix $\mathbf{G}(\mathbf{x}, \mathbf{x}_0)$ such that

$$[v_\theta(\mathbf{x}), v_\phi(\mathbf{x})]^T = \mathbf{G}(\mathbf{x}, \mathbf{x}_0)\,[q_\theta(\mathbf{x}_0), q_\phi(\mathbf{x}_0), q_r(\mathbf{x}_0), W_{\phi\theta}(\mathbf{x}_0), D_{\theta\theta}(\mathbf{x}_0), D_{\theta\phi}(\mathbf{x}_0), D_{\phi\phi}(\mathbf{x}_0)]^T\,. \quad (143)$$

In what follows we show how to compute the coefficients of $\mathbf{G}(\mathbf{x}, \mathbf{x}_0)$ corresponding to the south component (θ) of the observed velocity $\mathbf{v}(\mathbf{x})$. Those for the east component (ϕ) can be derived in the same way. From Equation (132), the south component is given by

$$\begin{aligned}
v_\theta(\mathbf{x}) &= \hat{\boldsymbol{\theta}}^T \mathbf{v}(\mathbf{x}) \\
&\approx \hat{\boldsymbol{\theta}}^T \left[\mathbf{q}(\mathbf{x}_0) \times \mathbf{x} + [\boldsymbol{\nabla}\mathbf{q}(\mathbf{x}_0) \times \mathbf{x}_0]\,(\mathbf{x} - \mathbf{x}_0)\right] \\
&= R\,\hat{\boldsymbol{\theta}}^T \left\{\mathbf{q}(\mathbf{x}_0) \times \hat{\mathbf{x}} + [\mathbf{W}(\mathbf{x}_0) + \mathbf{D}(\mathbf{x}_0)]\,(\hat{\mathbf{r}} - \hat{\mathbf{r}}_0)\right\} \\
&= R\,\hat{\boldsymbol{\theta}}^T \left(\mathbf{q}(\mathbf{x}_0) \times \hat{\mathbf{r}}\right) + R\,\hat{\boldsymbol{\theta}}^T \mathbf{W}(\mathbf{x}_0)\,(\hat{\mathbf{r}} - \hat{\mathbf{r}}_0) + R\,\hat{\boldsymbol{\theta}}^T \mathbf{D}(\mathbf{x}_0)\,(\hat{\mathbf{r}} - \hat{\mathbf{r}}_0)\,. \qquad (144)
\end{aligned}$$

The first term in Equation (144) can be written as

$$\begin{aligned}
R\,\hat{\boldsymbol{\theta}}^T \left(\mathbf{q}(\mathbf{x}_0) \times \hat{\mathbf{r}}\right) &= -R\hat{\boldsymbol{\theta}}^T \left(\hat{\mathbf{r}} \times \mathbf{q}(\mathbf{x}_0)\right) \\
&= -R\,\hat{\boldsymbol{\theta}}^T [\hat{\mathbf{r}}]_\times\, \mathbf{q}(\mathbf{x}_0) \\
&= -R\,\mathbf{q}(\mathbf{x}_0)^T [\hat{\mathbf{r}}]_\times^T\, \hat{\boldsymbol{\theta}} \\
&= R\,\mathbf{q}(\mathbf{x}_0)^T \left(\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}}\right) \\
&= R\,\mathbf{q}(\mathbf{x}_0)^T \hat{\boldsymbol{\phi}}. \\
&= R\,q_r(\mathbf{x}_0) \left(\hat{\mathbf{r}}_0^T \hat{\boldsymbol{\phi}}\right) R\,q_\theta(\mathbf{x}_0) \left(\hat{\boldsymbol{\theta}}_0^T \hat{\boldsymbol{\phi}}\right) R\,q_\phi(\mathbf{x}_0) \left(\hat{\boldsymbol{\phi}}_0^T \hat{\boldsymbol{\phi}}\right).
\end{aligned}$$

Replacing $\mathbf{W}(\mathbf{x}_0)$ by its expression (Eq. 135), the second term in Equation (144) can be simplified as follows:

$$\begin{aligned}
R\,\hat{\boldsymbol{\theta}}^T \mathbf{W}(\mathbf{x}_0)\,(\hat{\mathbf{x}} - \hat{\mathbf{x}}_0) &= R\,W_{\phi\theta}(\mathbf{x}_0)\hat{\boldsymbol{\theta}}^T \left(\hat{\boldsymbol{\phi}}_0\hat{\boldsymbol{\theta}}_0 - \hat{\boldsymbol{\theta}}_0\hat{\boldsymbol{\phi}}_0\right)(\hat{\mathbf{r}} - \hat{\mathbf{r}}_0) \\
&= R\,W_{\phi\theta}(\mathbf{x}_0)\hat{\boldsymbol{\theta}}^T [\hat{\mathbf{r}}_0]_\times\,(\hat{\mathbf{r}} - \hat{\mathbf{r}}_0) \\
&= R\,W_{\phi\theta}(\mathbf{x}_0)\hat{\boldsymbol{\theta}}^T [\hat{\mathbf{r}}_0 \times (\hat{\mathbf{r}} - \hat{\mathbf{r}}_0)] \\
&= R\,W_{\phi\theta}(\mathbf{x}_0)\hat{\boldsymbol{\theta}}^T [(\hat{\mathbf{r}}_0 \times \hat{\mathbf{r}}) - (\hat{\mathbf{r}}_0 \times \hat{\mathbf{r}}_0)] \\
&= R\,W_{\phi\theta}(\mathbf{x}_0)\hat{\boldsymbol{\theta}}^T \left[\hat{\mathbf{r}}_0 \times \left(\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}}\right)\right] \\
&= R\,W_{\phi\theta}(\mathbf{x}_0)\hat{\boldsymbol{\theta}}^T \left[\hat{\boldsymbol{\theta}}\left(\hat{\mathbf{r}}_0^T \hat{\boldsymbol{\phi}}\right) - \hat{\boldsymbol{\phi}}\left(\hat{\mathbf{r}}_0^T \hat{\boldsymbol{\theta}}\right)\right] \\
&= R\,W_{\phi\theta}(\mathbf{x}_0)\left[\hat{\boldsymbol{\theta}}^T \hat{\boldsymbol{\theta}}\left(\hat{\mathbf{r}}_0^T \hat{\boldsymbol{\phi}}\right) - \hat{\boldsymbol{\theta}}^T \hat{\boldsymbol{\phi}}\left(\hat{\mathbf{r}}_0^T \hat{\boldsymbol{\theta}}\right)\right] \\
&= R\,W_{\phi\theta}(\mathbf{x}_0)\left(\hat{\mathbf{r}}_0^T \hat{\boldsymbol{\phi}}\right),
\end{aligned}$$

where we have used Equation (156) as well as the property

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}). \tag{145}$$

Finally, by simply replacing the expression for the strain rate tensor $\mathbf{D}$ (Eq. 134) in the third term in Equation (144) yields

$$\begin{aligned} R\hat{\boldsymbol{\theta}}^T \mathbf{D}(\mathbf{x}_0)\,(\hat{\mathbf{r}} - \hat{\mathbf{r}}_0) \;&=\; R\,D_{\theta\theta}(\mathbf{x}_0)\left(\hat{\boldsymbol{\theta}}^T \hat{\boldsymbol{\theta}}_0\right)\left(\hat{\boldsymbol{\theta}}_0^T \hat{\mathbf{r}}\right) \\ &\quad + R\,D_{\theta\phi}(\mathbf{x}_0)\left[\left(\hat{\boldsymbol{\theta}}^T \hat{\boldsymbol{\theta}}_0\right)\left(\hat{\boldsymbol{\phi}}_0^T \hat{\mathbf{r}}\right) + \left(\hat{\boldsymbol{\theta}}^T \hat{\boldsymbol{\phi}}_0\right)\left(\hat{\boldsymbol{\theta}}_0^T \hat{\mathbf{r}}\right)\right] \\ &\quad + R\,D_{\phi\phi}(\mathbf{x}_0)\left(\hat{\boldsymbol{\theta}}^T \hat{\boldsymbol{\phi}}_0\right)\left(\hat{\boldsymbol{\phi}}_0^T \hat{\mathbf{r}}\right). \end{aligned}$$

Thus, substituting these three terms into Equation (144), we obtain

$$\begin{aligned} v_\theta(\mathbf{x}) \;&\approx\; R\,\hat{\boldsymbol{\theta}}^T\left(\mathbf{q}(\mathbf{x}_0) \times \hat{\mathbf{r}}\right) + R\,\hat{\boldsymbol{\theta}}^T \mathbf{W}(\mathbf{x}_0)\,(\hat{\mathbf{r}} - \hat{\mathbf{r}}_0) + R\,\hat{\boldsymbol{\theta}}^T \mathbf{D}(\mathbf{x}_0)\,(\hat{\mathbf{r}} - \hat{\mathbf{r}}_0) \\ &=\; R\,q_\theta(\mathbf{x}_0)\left(\hat{\boldsymbol{\theta}}_0^T \hat{\boldsymbol{\phi}}\right) + R\,q_\phi(\mathbf{x}_0)\left(\hat{\boldsymbol{\phi}}_0^T \hat{\boldsymbol{\phi}}\right) + R\,q_r(\mathbf{x}_0)\left(\hat{\mathbf{r}}_0^T \hat{\boldsymbol{\phi}}\right) + R\,W_{\phi\theta}(\mathbf{x}_0)\left(\hat{\mathbf{r}}_0^T \hat{\boldsymbol{\phi}}\right) \\ &\quad + R\,D_{\theta\theta}(\mathbf{x}_0)\left(\hat{\boldsymbol{\theta}}^T \hat{\boldsymbol{\theta}}_0\right)\left(\hat{\boldsymbol{\theta}}_0^T \hat{\mathbf{r}}\right) + R\,D_{\phi\phi}(\mathbf{x}_0)\left(\hat{\boldsymbol{\theta}}^T \hat{\boldsymbol{\phi}}_0\right)\left(\hat{\boldsymbol{\phi}}_0^T \hat{\mathbf{r}}\right) \\ &\quad + R\,D_{\theta\phi}(\mathbf{x}_0)\left[\left(\hat{\boldsymbol{\theta}}^T \hat{\boldsymbol{\theta}}_0\right)\left(\hat{\boldsymbol{\phi}}_0^T \hat{\mathbf{r}}\right) + \left(\hat{\boldsymbol{\theta}}^T \hat{\boldsymbol{\phi}}_0\right)\left(\hat{\boldsymbol{\theta}}_0^T \hat{\mathbf{r}}\right)\right] \end{aligned} \tag{146}$$

which, in comparison with Equation (143), leads to the following set of equations:

$$\begin{aligned} G_{11} &= R\left(\hat{\boldsymbol{\theta}}_0^T \hat{\boldsymbol{\phi}}\right) \\ G_{12} &= R\left(\hat{\boldsymbol{\phi}}_0^T \hat{\boldsymbol{\phi}}\right) \\ G_{13} &= R\left(\hat{\mathbf{r}}_0^T \hat{\boldsymbol{\phi}}\right) \\ G_{14} &= R\left(\hat{\mathbf{r}}_0^T \hat{\boldsymbol{\phi}}\right) \\ G_{15} &= R\left(\hat{\boldsymbol{\theta}}^T \hat{\boldsymbol{\theta}}_0\right)\left(\hat{\boldsymbol{\theta}}_0^T \hat{\mathbf{r}}\right) \\ G_{16} &= R\left(\hat{\boldsymbol{\theta}}^T \hat{\boldsymbol{\theta}}_0\right)\left(\hat{\boldsymbol{\phi}}_0^T \hat{\mathbf{r}}\right) + R\left(\hat{\boldsymbol{\theta}}^T \hat{\boldsymbol{\phi}}_0\right)\left(\hat{\boldsymbol{\theta}}_0^T \hat{\mathbf{r}}\right) \\ G_{17} &= R\left(\hat{\boldsymbol{\theta}}^T \hat{\boldsymbol{\phi}}_0\right)\left(\hat{\boldsymbol{\phi}}_0^T \hat{\mathbf{r}}\right) \end{aligned}$$

with the model and data vectors (Eq. 143)

$$\mathbf{m} = \left[q_\theta(\mathbf{x}_0), q_\phi(\mathbf{x}_0), q_r(\mathbf{x}_0), W_{\phi\theta}(\mathbf{x}_0), D_{\theta\theta}(\mathbf{x}_0), D_{\theta\phi}(\mathbf{x}_0), D_{\phi\phi}(\mathbf{x}_0)\right]^T \tag{147}$$

$$\mathbf{d} = \left[v_\theta(\mathbf{x}), v_\phi(\mathbf{x})\right]^T\ . \tag{148}$$

Making the substitutions in Equation (138), we obtain

$$
\begin{aligned}
G_{11} &= -R\left(\hat{\boldsymbol{\lambda}}_0^T\hat{\boldsymbol{\phi}}\right)\\
G_{12} &= R\left(\hat{\boldsymbol{\phi}}_0^T\hat{\boldsymbol{\phi}}\right)\\
G_{13} &= R\left(\hat{\mathbf{r}}_0^T\hat{\boldsymbol{\phi}}\right)\\
G_{14} &= R\left(\hat{\mathbf{r}}_0^T\hat{\boldsymbol{\phi}}\right)\\
G_{15} &= -R\left(\hat{\boldsymbol{\lambda}}^T\hat{\boldsymbol{\lambda}}_0\right)\left(\hat{\boldsymbol{\lambda}}_0^T\hat{\mathbf{r}}\right)\\
G_{16} &= R\left(\hat{\boldsymbol{\lambda}}^T\hat{\boldsymbol{\lambda}}_0\right)\left(\hat{\boldsymbol{\phi}}_0^T\hat{\mathbf{r}}\right)+R\left(\hat{\boldsymbol{\lambda}}^T\hat{\boldsymbol{\phi}}_0\right)\left(\hat{\boldsymbol{\lambda}}_0^T\hat{\mathbf{r}}\right)\\
G_{17} &= -R\left(\hat{\boldsymbol{\lambda}}^T\hat{\boldsymbol{\phi}}_0\right)\left(\hat{\boldsymbol{\phi}}_0^T\hat{\mathbf{r}}\right)\\
\mathbf{m} &= [-q_\lambda(\mathbf{x}_0), q_\phi(\mathbf{x}_0), q_r(\mathbf{x}_0), W_{\lambda\phi}(\mathbf{x}_0), D_{\lambda\lambda}(\mathbf{x}_0), -D_{\lambda\phi}(\mathbf{x}_0), D_{\phi\phi}(\mathbf{x}_0)]^T\\
\mathbf{d} &= [-v_\lambda(\mathbf{x}), v_\phi(\mathbf{x})]^T\ .
\end{aligned}
$$

Making the direct substitutions in Equation (128), we obtain

$$
\begin{aligned}
G_{11} &= -R\left(\hat{\mathbf{n}}_0^T\hat{\mathbf{e}}\right)\\
G_{12} &= R\left(\hat{\mathbf{e}}_0^T\hat{\mathbf{e}}\right)\\
G_{13} &= R\left(\hat{\mathbf{r}}_0^T\hat{\mathbf{e}}\right)\\
G_{14} &= R\left(\hat{\mathbf{r}}_0^T\hat{\mathbf{e}}\right)\\
G_{15} &= -R\left(\hat{\mathbf{n}}^T\hat{\mathbf{n}}_0\right)\left(\hat{\mathbf{n}}_0^T\hat{\mathbf{r}}\right)\\
G_{16} &= R\left(\hat{\mathbf{n}}^T\hat{\mathbf{n}}_0\right)\left(\hat{\mathbf{e}}_0^T\hat{\mathbf{r}}\right)+R\left(\hat{\mathbf{n}}^T\hat{\mathbf{e}}_0\right)\left(\hat{\mathbf{n}}_0^T\hat{\mathbf{r}}\right)\\
G_{17} &= -R\left(\hat{\mathbf{n}}^T\hat{\mathbf{e}}_0\right)\left(\hat{\mathbf{e}}_0^T\hat{\mathbf{r}}\right)\\
\mathbf{m} &= [-\Omega_n(\mathbf{x}_0), \Omega_e(\mathbf{x}_0), \Omega_r(\mathbf{x}_0), R(\mathbf{x}_0), \dot{\varepsilon}_{nn}(\mathbf{x}_0), -\dot{\varepsilon}_{ne}(\mathbf{x}_0), \dot{\varepsilon}_{ee}(\mathbf{x}_0)]^T\\
\mathbf{d} &= [-v_n(\mathbf{x}), v_e(\mathbf{x})]^T\ .
\end{aligned}
$$

The equations listed in *Ward* (1998a) are

$$
\begin{aligned}
G'_{11} &= -R\left(\hat{\mathbf{n}}_0^T\hat{\mathbf{e}}\right)\\
G'_{12} &= -R\left(\hat{\mathbf{e}}_0^T\hat{\mathbf{e}}\right)\\
G'_{13} &= -R\left(\hat{\mathbf{r}}_0^T\hat{\mathbf{e}}\right)\\
G'_{14} &= -R\left(\hat{\mathbf{r}}_0^T\hat{\mathbf{e}}\right)\\
G'_{15} &= R\left(\hat{\mathbf{n}}^T\hat{\mathbf{n}}_0\right)\left(\hat{\mathbf{n}}_0^T\hat{\mathbf{r}}\right)\\
G'_{16} &= R\left(\hat{\mathbf{n}}^T\hat{\mathbf{n}}_0\right)\left(\hat{\mathbf{e}}_0^T\hat{\mathbf{r}}\right)+\left(\hat{\mathbf{n}}^T\hat{\mathbf{e}}_0\right)\left(\hat{\mathbf{n}}_0^T\hat{\mathbf{r}}\right)\\
G'_{17} &= R\left(\hat{\mathbf{n}}^T\hat{\mathbf{e}}_0\right)\left(\hat{\mathbf{e}}_0^T\hat{\mathbf{r}}\right)\\
\mathbf{m}' &= [\Omega_n(\mathbf{x}_0), \Omega_e(\mathbf{x}_0), \Omega_r(\mathbf{x}_0), R(\mathbf{x}_0), \dot{\varepsilon}_{nn}(\mathbf{x}_0), \dot{\varepsilon}_{ne}(\mathbf{x}_0), \dot{\varepsilon}_{ee}(\mathbf{x}_0)]^T\\
\mathbf{d}' &= [v_n(\mathbf{x}), v_e(\mathbf{x})]^T\ .
\end{aligned}
$$

Finally, by comparison, we see that, for the first component (θ or n),

$$\mathbf{G}\mathbf{m} = \mathbf{d} \tag{149}$$

is the same set of equations as

$$\mathbf{G}'\mathbf{m}' = \mathbf{d}'. \tag{150}$$

Note that the following is true:

$$W_{\phi\theta} = -W_{\phi\theta} = W_{\lambda\phi} = R\ , \tag{151}$$

where here the R is the rotation magnitude variable in *Ward* (1998a).

7 Equations and definitions

7.1 Cross product operations

7.1.1 $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$

7.1.2 Cross product between tensor and vector

We first write tensor $\mathbf{T}$ as three column vectors,

$$\mathbf{T} = \begin{bmatrix} | & | & | \\ \mathbf{T}_x & \mathbf{T}_y & \mathbf{T}_z \\ | & | & | \end{bmatrix}.$$

We now define the operation $\mathbf{a} \times \mathbf{T}$ as

$$\mathbf{a} \times \mathbf{T} = \mathbf{a} \times \begin{bmatrix} | & | & | \\ \mathbf{T}_x & \mathbf{T}_y & \mathbf{T}_z \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \mathbf{a} \times \mathbf{T}_x & \mathbf{a} \times \mathbf{T}_y & \mathbf{a} \times \mathbf{T}_z \\ | & | & | \end{bmatrix}$$

Note that the cross-product is non-commutative (as in the vector case):

$$\begin{aligned} \mathbf{T} \times \mathbf{a} &= \begin{bmatrix} | & | & | \\ \mathbf{T}_x & \mathbf{T}_y & \mathbf{T}_z \\ | & | & | \end{bmatrix} \times \mathbf{a} \\ &= \begin{bmatrix} | & | & | \\ \mathbf{T}_x \times \mathbf{a} & \mathbf{T}_y \times \mathbf{a} & \mathbf{T}_z \times \mathbf{a} \\ | & | & | \end{bmatrix} \\ &= \begin{bmatrix} | & | & | \\ -\mathbf{a} \times \mathbf{T}_x & -\mathbf{a} \times \mathbf{T}_y & -\mathbf{a} \times \mathbf{T}_z \\ | & | & | \end{bmatrix} \\ &= -\mathbf{a} \times \mathbf{T}. \end{aligned}$$

7.2 Gradient operator, $\boldsymbol{\nabla}$

7.2.1 Gradient of a scalar, $\boldsymbol{\nabla} f$

The gradient of a scalar field is a vector:

$$\begin{aligned} \boldsymbol{\nabla} f &= \left[\hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z}\right] f(x, y, z) \\ &= \hat{\mathbf{x}} \frac{\partial f}{\partial x} + \hat{\mathbf{y}} \frac{\partial f}{\partial y} + \hat{\mathbf{z}} \frac{\partial f}{\partial z} \\ &= \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} \end{aligned}$$

7.2.2 Gradient of a vector, $\boldsymbol{\nabla}\mathbf{a}$

We define the vector field $\mathbf{a}$ as

$$\mathbf{a} = a_x(x,y,z)\,\hat{\mathbf{x}} + a_y(x,y,z)\,\hat{\mathbf{y}} + a_z(x,y,z)\,\hat{\mathbf{z}}.$$

The gradient of the vector field is a tensor field (*Malvern*, 1969, Eq. 2.5.31b):

$$\begin{aligned}
\boldsymbol{\nabla}\mathbf{a} &= \left[\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right]\left[a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}}\right] \\
&= \frac{\partial a_x}{\partial x}\hat{\mathbf{x}}\hat{\mathbf{x}} + \frac{\partial a_y}{\partial x}\hat{\mathbf{x}}\hat{\mathbf{y}} + \frac{\partial a_z}{\partial x}\hat{\mathbf{x}}\hat{\mathbf{z}} \\
&\quad + \frac{\partial a_x}{\partial y}\hat{\mathbf{y}}\hat{\mathbf{x}} + \frac{\partial a_y}{\partial y}\hat{\mathbf{y}}\hat{\mathbf{y}} + \frac{\partial a_z}{\partial y}\hat{\mathbf{y}}\hat{\mathbf{z}} \\
&\quad + \frac{\partial a_x}{\partial z}\hat{\mathbf{z}}\hat{\mathbf{x}} + \frac{\partial a_y}{\partial z}\hat{\mathbf{z}}\hat{\mathbf{y}} + \frac{\partial a_z}{\partial z}\hat{\mathbf{z}}\hat{\mathbf{z}} \\
&= \begin{bmatrix} \frac{\partial a_x}{\partial x} & \frac{\partial a_y}{\partial x} & \frac{\partial a_z}{\partial x} \\ \frac{\partial a_x}{\partial y} & \frac{\partial a_y}{\partial y} & \frac{\partial a_z}{\partial y} \\ \frac{\partial a_x}{\partial z} & \frac{\partial a_y}{\partial z} & \frac{\partial a_z}{\partial z} \end{bmatrix},
\end{aligned}$$

where the tensor indices are given by

$$\begin{bmatrix} \hat{\mathbf{x}}\hat{\mathbf{x}} & \hat{\mathbf{x}}\hat{\mathbf{y}} & \hat{\mathbf{x}}\hat{\mathbf{z}} \\ \hat{\mathbf{y}}\hat{\mathbf{x}} & \hat{\mathbf{y}}\hat{\mathbf{y}} & \hat{\mathbf{y}}\hat{\mathbf{z}} \\ \hat{\mathbf{z}}\hat{\mathbf{x}} & \hat{\mathbf{z}}\hat{\mathbf{y}} & \hat{\mathbf{z}}\hat{\mathbf{z}} \end{bmatrix}.$$

We can also write

$$\boldsymbol{\nabla}\mathbf{a} = \begin{bmatrix} | & | & | \\ \boldsymbol{\nabla}a_x & \boldsymbol{\nabla}a_y & \boldsymbol{\nabla}a_z \\ | & | & | \end{bmatrix}.$$

7.2.3 Gradient of a tensor, $\boldsymbol{\nabla}\mathbf{T}$

Text here.

7.3 $\nabla(\mathbf{a}\times\mathbf{b}) = \nabla\mathbf{a}\times\mathbf{b} + \mathbf{a}\times\nabla\mathbf{b}$

Here we use the previous definitions for $\nabla\mathbf{a}$ and $\mathbf{a}\times\mathbf{T}$.

$$
\begin{aligned}
\mathbf{a} &= a_x(x,y,z)\,\hat{\mathbf{x}} + a_y(x,y,z)\,\hat{\mathbf{y}} + a_z(x,y,z)\,\hat{\mathbf{z}} \\
\mathbf{b} &= b_x(x,y,z)\,\hat{\mathbf{x}} + b_y(x,y,z)\,\hat{\mathbf{y}} + b_z(x,y,z)\,\hat{\mathbf{z}} \\
\mathbf{a}\times\mathbf{b} &= (a_yb_z - a_zb_y)\,\hat{\mathbf{x}} + (a_zb_x - a_xb_z)\,\hat{\mathbf{y}} + (a_xb_y - a_yb_x)\,\hat{\mathbf{z}} \\
\nabla(\mathbf{a}\times\mathbf{b}) &= \left[\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right]\left[(a_yb_z - a_zb_y)\,\hat{\mathbf{x}} + (a_zb_x - a_xb_z)\,\hat{\mathbf{y}} + (a_xb_y - a_yb_x)\,\hat{\mathbf{z}}\right] \\
&= \frac{\partial}{\partial x}(a_yb_z - a_zb_y)\,\hat{\mathbf{x}}\hat{\mathbf{x}} + \frac{\partial}{\partial x}(a_zb_x - a_xb_z)\,\hat{\mathbf{x}}\hat{\mathbf{y}} + \frac{\partial}{\partial x}(a_xb_y - a_yb_x)\,\hat{\mathbf{x}}\hat{\mathbf{z}} \\
&+ \frac{\partial}{\partial y}(a_yb_z - a_zb_y)\,\hat{\mathbf{y}}\hat{\mathbf{x}} + \frac{\partial}{\partial y}(a_zb_x - a_xb_z)\,\hat{\mathbf{y}}\hat{\mathbf{y}} + \frac{\partial}{\partial y}(a_xb_y - a_yb_x)\,\hat{\mathbf{y}}\hat{\mathbf{z}} \\
&+ \frac{\partial}{\partial z}(a_yb_z - a_zb_y)\,\hat{\mathbf{z}}\hat{\mathbf{x}} + \frac{\partial}{\partial z}(a_zb_x - a_xb_z)\,\hat{\mathbf{z}}\hat{\mathbf{y}} + \frac{\partial}{\partial z}(a_xb_y - a_yb_x)\,\hat{\mathbf{z}}\hat{\mathbf{z}} \\
&= \begin{bmatrix}
\frac{\partial}{\partial x}(a_yb_z - a_zb_y) & \frac{\partial}{\partial x}(a_zb_x - a_xb_z) & \frac{\partial}{\partial x}(a_xb_y - a_yb_x) \\
\frac{\partial}{\partial y}(a_yb_z - a_zb_y) & \frac{\partial}{\partial y}(a_zb_x - a_xb_z) & \frac{\partial}{\partial y}(a_xb_y - a_yb_x) \\
\frac{\partial}{\partial z}(a_yb_z - a_zb_y) & \frac{\partial}{\partial z}(a_zb_x - a_xb_z) & \frac{\partial}{\partial z}(a_xb_y - a_yb_x)
\end{bmatrix} = \\
&\begin{bmatrix}
\frac{\partial a_y}{\partial x}b_z + a_y\frac{\partial b_z}{\partial x} - \frac{\partial a_z}{\partial x}b_y - a_z\frac{\partial b_y}{\partial x} & \frac{\partial a_z}{\partial x}b_x + a_z\frac{\partial b_x}{\partial x} - \frac{\partial a_x}{\partial x}b_z - a_x\frac{\partial b_z}{\partial x} & \frac{\partial a_x}{\partial x}b_y + a_x\frac{\partial b_y}{\partial x} - \frac{\partial a_y}{\partial x}b_x - a_y\frac{\partial b_x}{\partial x} \\
\frac{\partial a_y}{\partial y}b_z + a_y\frac{\partial b_z}{\partial y} - \frac{\partial a_z}{\partial y}b_y - a_z\frac{\partial b_y}{\partial y} & \frac{\partial a_z}{\partial y}b_x + a_z\frac{\partial b_x}{\partial y} - \frac{\partial a_x}{\partial y}b_z - a_x\frac{\partial b_z}{\partial y} & \frac{\partial a_x}{\partial y}b_y + a_x\frac{\partial b_y}{\partial y} - \frac{\partial a_y}{\partial y}b_x - a_y\frac{\partial b_x}{\partial y} \\
\frac{\partial a_y}{\partial z}b_z + a_y\frac{\partial b_z}{\partial z} - \frac{\partial a_z}{\partial z}b_y - a_z\frac{\partial b_y}{\partial z} & \frac{\partial a_z}{\partial z}b_x + a_z\frac{\partial b_x}{\partial z} - \frac{\partial a_x}{\partial z}b_z - a_x\frac{\partial b_z}{\partial z} & \frac{\partial a_x}{\partial z}b_y + a_x\frac{\partial b_y}{\partial z} - \frac{\partial a_y}{\partial z}b_x - a_y\frac{\partial b_x}{\partial z}
\end{bmatrix} \\
&= \begin{bmatrix}
\frac{\partial a_y}{\partial x}b_z - \frac{\partial a_z}{\partial x}b_y & \frac{\partial a_z}{\partial x}b_x - \frac{\partial a_x}{\partial x}b_z & \frac{\partial a_x}{\partial x}b_y - \frac{\partial a_y}{\partial x}b_x \\
\frac{\partial a_y}{\partial y}b_z - \frac{\partial a_z}{\partial y}b_y & \frac{\partial a_z}{\partial y}b_x - \frac{\partial a_x}{\partial y}b_z & \frac{\partial a_x}{\partial y}b_y - \frac{\partial a_y}{\partial y}b_x \\
\frac{\partial a_y}{\partial z}b_z - \frac{\partial a_z}{\partial z}b_y & \frac{\partial a_z}{\partial z}b_x - \frac{\partial a_x}{\partial z}b_z & \frac{\partial a_x}{\partial z}b_y - \frac{\partial a_y}{\partial z}b_x
\end{bmatrix} \\
&+ \begin{bmatrix}
a_y\frac{\partial b_z}{\partial x} - a_z\frac{\partial b_y}{\partial x} & a_z\frac{\partial b_x}{\partial x} - a_x\frac{\partial b_z}{\partial x} & a_x\frac{\partial b_y}{\partial x} - a_y\frac{\partial b_x}{\partial x} \\
a_y\frac{\partial b_z}{\partial y} - a_z\frac{\partial b_y}{\partial y} & a_z\frac{\partial b_x}{\partial y} - a_x\frac{\partial b_z}{\partial y} & a_x\frac{\partial b_y}{\partial y} - a_y\frac{\partial b_x}{\partial y} \\
a_y\frac{\partial b_z}{\partial z} - a_z\frac{\partial b_y}{\partial z} & a_z\frac{\partial b_x}{\partial z} - a_x\frac{\partial b_z}{\partial z} & a_x\frac{\partial b_y}{\partial z} - a_y\frac{\partial b_x}{\partial z}
\end{bmatrix} \\
&= \begin{bmatrix}
| & | & | \\
(\nabla\mathbf{a})_x\times\mathbf{b} & (\nabla\mathbf{a})_y\times\mathbf{b} & (\nabla\mathbf{a})_z\times\mathbf{b} \\
| & | & |
\end{bmatrix} + \begin{bmatrix}
| & | & | \\
\mathbf{a}\times(\nabla\mathbf{b})_x & \mathbf{a}\times(\nabla\mathbf{b})_y & \mathbf{a}\times(\nabla\mathbf{b})_z \\
| & | & |
\end{bmatrix} \\
&= \nabla\mathbf{a}\times\mathbf{b} + \mathbf{a}\times\nabla\mathbf{b},
\end{aligned}
$$

where, in the second-to-last step, we use the notation $(\nabla\mathbf{a})_k$ to denote the kth column vector of the tensor $\nabla\mathbf{a}$ (Section 7.2.3).

Using the expressions in Section 7.1.2, we could have begun the derivation as

$$
\begin{aligned}
\boldsymbol{\nabla}(\mathbf{a}\times\mathbf{b}) &= \left[\begin{array}{ccc} | & | & | \\ \boldsymbol{\nabla}(\mathbf{a}\times\mathbf{b})_x & \boldsymbol{\nabla}(\mathbf{a}\times\mathbf{b})_y & \boldsymbol{\nabla}(\mathbf{a}\times\mathbf{b})_z \\ | & | & | \end{array}\right] \\
&= \left[\begin{array}{ccc} | & | & | \\ \boldsymbol{\nabla}(a_y b_z - a_z b_y) & \boldsymbol{\nabla}(a_z b_x - a_x b_z) & \boldsymbol{\nabla}(a_x b_y - a_y b_x) \\ | & | & | \end{array}\right] \\
&= \left[\begin{array}{ccc} \frac{\partial}{\partial x}(a_y b_z - a_z b_y) & \frac{\partial}{\partial x}(a_z b_x - a_x b_z) & \frac{\partial}{\partial x}(a_x b_y - a_y b_x) \\ \frac{\partial}{\partial y}(a_y b_z - a_z b_y) & \frac{\partial}{\partial y}(a_z b_x - a_x b_z) & \frac{\partial}{\partial y}(a_x b_y - a_y b_x) \\ \frac{\partial}{\partial z}(a_y b_z - a_z b_y) & \frac{\partial}{\partial z}(a_z b_x - a_x b_z) & \frac{\partial}{\partial z}(a_x b_y - a_y b_x) \end{array}\right]
\end{aligned}
$$

7.4 Rotation matrix and rotation vector

We denote the skew-symmetric matrix associated with the cross-product operation by vector $\mathbf{q}$,

$$\mathbf{q} = q_1\,\hat{\mathbf{x}} + q_2\,\hat{\mathbf{x}} + q_3\,\hat{\mathbf{x}}, \tag{152}$$

as

$$[\mathbf{q}]_\times = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix}. \tag{153}$$

Rotation in the negative sense is given by

$$[-\mathbf{q}]_\times = [\mathbf{q}]_\times^T\ . \tag{154}$$

The operation $\mathbf{q}\times\mathbf{b}$ is therefore

$$[\mathbf{q}]_\times\,\mathbf{b} = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} q_2 b_3 - q_3 b_2 \\ q_3 b_1 - q_1 b_3 \\ q_1 b_2 - q_2 b_1 \end{bmatrix} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ q_1 & q_2 & q_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{q}\times\mathbf{b}\ . \tag{155}$$

Here are some examples:

$$[\hat{\mathbf{r}}]_\times = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \hat{\boldsymbol{\phi}}\hat{\boldsymbol{\theta}} - \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\phi}} \tag{156}$$

$$[\hat{\mathbf{r}}]_\times^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\phi}} - \hat{\boldsymbol{\phi}}\hat{\boldsymbol{\theta}} = -\,[\hat{\mathbf{r}}]_\times \tag{157}$$

We also have the expected property

$$[\mathbf{q}]_\times \mathbf{I}_3 = \mathbf{q} \times \mathbf{I}_3 = [\mathbf{q}]_\times \ , \tag{158}$$

where $\mathbf{I}_3$ is the 3×3 identity matrix.

8 Bases for vector field

The local basis vectors that we use to express the velocity field (Eq. 14) are

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \tag{159}$$

$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} \ - \sin\theta\,\hat{\mathbf{z}} \tag{160}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{y}}, \tag{161}$$

where $\hat{\mathbf{r}}$ points up, $\hat{\boldsymbol{\theta}}$ points south, and $\hat{\boldsymbol{\phi}}$ points east.

We let $\mathbf{x}$ denote a vector in global coordinates $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$. We let $\mathbf{x}'$ denote a vector in local coordinates $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$. The transformation from global to local coordinates is given by $\mathbf{x}' = \mathbf{R}\,\mathbf{x}$, where

$$\mathbf{R} = \begin{bmatrix} \hat{\mathbf{r}}\cdot\hat{\mathbf{x}} & \hat{\mathbf{r}}\cdot\hat{\mathbf{y}} & \hat{\mathbf{r}}\cdot\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}}\cdot\hat{\mathbf{x}} & \hat{\boldsymbol{\theta}}\cdot\hat{\mathbf{y}} & \hat{\boldsymbol{\theta}}\cdot\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{x}} & \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{y}} & \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix}. \tag{162}$$

The transformation from local to global coordinates is obtained via the operation $\mathbf{x} = \mathbf{R}^T\,\mathbf{x}'$.

Other authors (*Haines and Holt*, 1993; *Ward*, 1998a) have instead used the basis $\hat{\mathbf{r}}$-$\hat{\boldsymbol{\phi}}$-$\hat{\boldsymbol{\lambda}}$, where $\hat{\boldsymbol{\lambda}}$ points north. In that case, the vector field (compare Eq. 14) is expressed as

$$\mathbf{v} = v_r\,\hat{\mathbf{r}} \ + \ v_\phi\,\hat{\boldsymbol{\phi}} \ + \ v_\lambda\,\hat{\boldsymbol{\lambda}}\,, \tag{163}$$

where λ is latitude and ϕ is longitude, and

$$\hat{\mathbf{r}} = \cos\lambda\cos\phi\,\hat{\mathbf{x}} + \cos\lambda\sin\phi\,\hat{\mathbf{y}} + \sin\lambda\,\hat{\mathbf{z}} \tag{164}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \tag{165}$$

$$\hat{\boldsymbol{\lambda}} = -\sin\lambda\cos\phi\,\hat{\mathbf{x}} - \sin\lambda\sin\phi\,\hat{\mathbf{y}} + \cos\lambda\,\hat{\mathbf{z}}\ . \tag{166}$$

This difference in notation leads to the following adjustments:

$$\begin{aligned}
\hat{\boldsymbol{\lambda}} &= -\hat{\boldsymbol{\theta}} && (167)\\
v_\lambda &= -v_\theta && (168)\\
D_{\lambda\lambda} &= \hat{\boldsymbol{\lambda}}^T \mathbf{D}\, \hat{\boldsymbol{\lambda}} = \hat{\boldsymbol{\theta}}^T \mathbf{D}\, \hat{\boldsymbol{\theta}} = D_{\theta\theta} && (169)\\
D_{\lambda\phi} &= \hat{\boldsymbol{\lambda}}^T \mathbf{D}\, \hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{\theta}}^T \mathbf{D}\, \hat{\boldsymbol{\phi}} = -D_{\theta\phi} && (170)\\
W_{r\lambda} &= \hat{\mathbf{r}}^T \mathbf{W}\, \hat{\boldsymbol{\lambda}} = -\hat{\mathbf{r}}^T \mathbf{W}\, \hat{\boldsymbol{\theta}} = -W_{r\theta} && (171)\\
W_{\phi\lambda} &= \hat{\boldsymbol{\phi}}^T \mathbf{W}\, \hat{\boldsymbol{\lambda}} = -\hat{\boldsymbol{\phi}}^T \mathbf{W}\, \hat{\boldsymbol{\theta}} = -W_{\phi\theta}. && (172)
\end{aligned}$$

We use these expressions in Section **??**.

9 Details on the estimation inverse problem

9.1 Connection to Bayesian least-squares

The Bayesian least-squares misfit function is given by (*Tarantola*, 2005, Eq. 3.35)

$$
\begin{aligned}
S(\mathbf{m}) &= \tfrac{1}{2}\|\mathbf{Gm}-\mathbf{d}\|_D^2 + \tfrac{1}{2}\|\mathbf{m}\|_M^2 && (173)\\
&= \tfrac{1}{2}\left(\mathbf{Gm}-\mathbf{d}\right)^T \mathbf{C}_D^{-1}\left(\mathbf{Gm}-\mathbf{d}\right) + \tfrac{1}{2}\left(\mathbf{m}-\mathbf{m}_{\text{prior}}\right)^T \mathbf{C}_M^{-1}\left(\mathbf{m}-\mathbf{m}_{\text{prior}}\right), && (174)
\end{aligned}
$$

where d is a vector of observations, $\mathbf{Gm}$ is a vector of predictions, $\mathbf{C}_M^{-1}$ is the inverse data covariance matrix, $\mathbf{C}_M^{-1}$ is the inverse model covariance matrix, and $\mathbf{m}_{\text{prior}}$ is the prior model, whose uncertainties are given by $\mathbf{C}_M$. We see that the misfit function in *Tape et al.* (2009) is equivalent to Equation (174) with

$$
\begin{aligned}
\mathbf{m}_{\text{prior}} &= \mathbf{0} && (175)\\
\mathbf{C}_M^{-1} &= \lambda^2\mathbf{D}. && (176)
\end{aligned}
$$

In the Bayesian context, the regularization term with $\mathbf{S}$ corresponds to the inverse of the prior model covariance. Adopting the notation of *Tarantola* (2005), we use the tilde notation to denote quantities associated with the posterior distributions: for example, $\widetilde{\mathbf{m}}$ for the mean model of the posterior distribution $\widetilde{\mathbf{C}}_{\text{M}}$. Pertinent equations are:

$$
\begin{aligned}
\widetilde{\mathbf{C}}_{\text{M}} &= \left(\mathbf{G}^T\mathbf{C}_{\text{D}}^{-1}\mathbf{G} + \mathbf{C}_{\text{M}}^{-1}\right)^{-1} && (177)\\
\widetilde{\mathbf{m}} &= \widetilde{\mathbf{C}}_{\text{M}}\mathbf{G}^T\mathbf{C}_{\text{D}}^{-1}\mathbf{d} && (178)\\
\widetilde{\mathbf{C}}_{\text{D}} &= \mathbf{G}\widetilde{\mathbf{C}}_{\text{M}}\mathbf{G}^T && (179)\\
\widetilde{\mathbf{d}} &= \mathbf{G}\widetilde{\mathbf{m}} && (180)
\end{aligned}
$$

The algebraic connection with the regularization notation above comes with

$$
\mathbf{C}_{\text{M}} = \left(\lambda^2\,\mathbf{S}\right)^{-1}\ , \qquad (181)
$$

which leads to

$$
\widetilde{\mathbf{C}}_{\text{D}} = \mathbf{G}\left(\mathbf{G}^T\mathbf{C}_{\text{D}}^{-1}\mathbf{G} + \lambda^2\,\mathbf{S}\right)^{-1}\mathbf{G}^T\ . \qquad (182)
$$

This quantity is used in computing the mask for the estimated velocity field (Section **??**).

9.2 Choice of regularization parameter: generalized cross-validation

We experimented with three different approaches of selecting the regularization parameter, λ, for the least-squares solution in Equation (**??**). The first is the L-curve approach (e.g., *Hansen*, 1998, p. 189), which is a plot of model norm ($\|\mathbf{m}_\lambda\|_2$) versus misfit norm ($\|\mathbf{Gm}_\lambda - \mathbf{d}\|_2$). (We use the λ subscript to indicate that the model depends on the choice of λ.) The maximum of the curvature of the function provides one estimate for λ.

We also used generalized cross-validation (GCV) to select the regularization parameter (*Golub et al.*, 1979; *Wahba*, 1990; *Hansen*, 1998; *Schneider*, 2001). Figure **??** shows a comparison of the regularization parameter selections from the L-curve approach, OCV, and GCV. For our examples, the GCV and OCV performed similarly, and tended to select a lower regularization parameter than that of the L-curve approach.

References for generalized cross-validation: *Golub et al.* (1979); *Wahba* (1990); *Hansen* (1998); *Schneider* (2001). The generalized cross-validation function is defined as

$$F(\lambda) = \left[\frac{\|\mathbf{G}\mathbf{m}_\lambda - \mathbf{d}\|_2}{\mathrm{tr}\left(\mathbf{I} - \mathbf{N}(\lambda)\right)}\right]^2 \tag{183}$$

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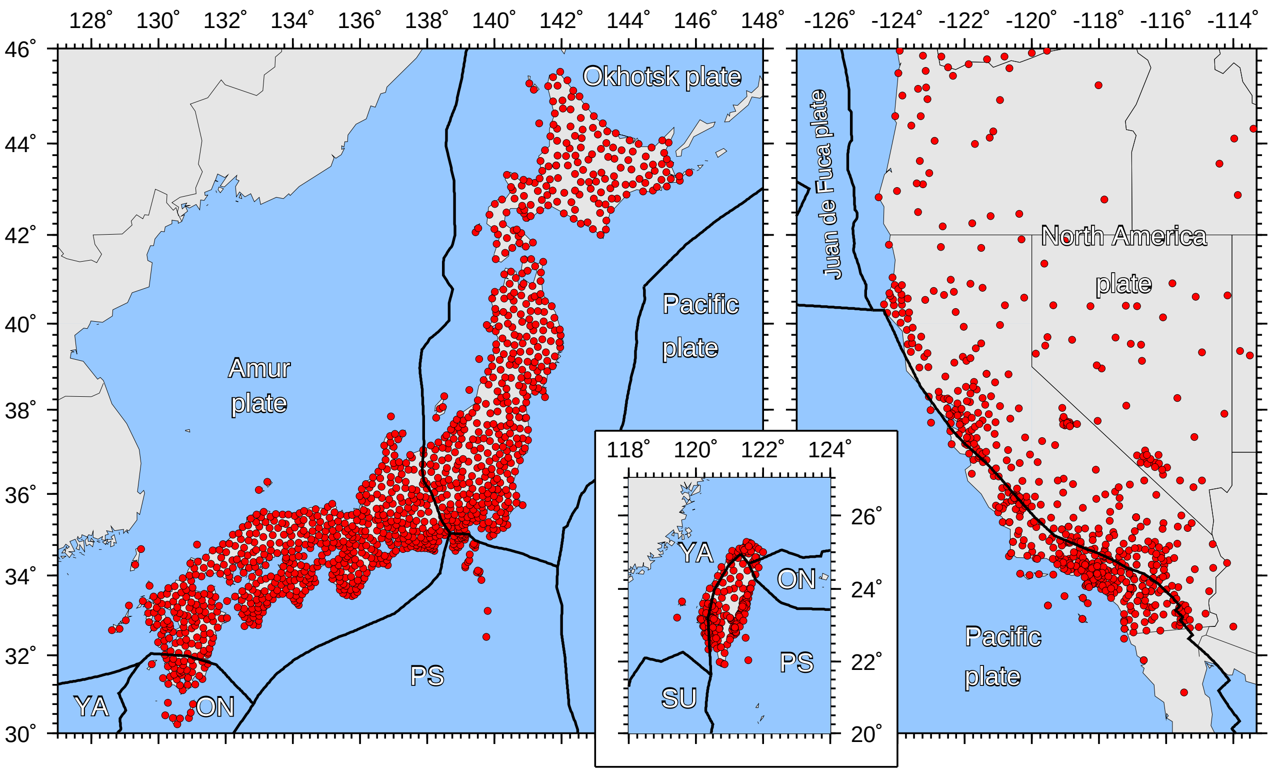

Figure 1: Continuously recording GPS station coverage in Japan (*Sagiya et al.*, 2000; *Sagiya*, 2004), Taiwan (e.g., *Hsu et al.*, 2009), and California (*Dong et al.*, 2009). Maps are plotted at the same scale. Plate boundaries are from *Bird* (2003).