Trigonometry With Tears

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§1 Basics

We start by defining $\sin(\theta)$ and $\cos(\theta)$. On a unit circle, $x = \cos(\theta)$ and $y = \sin(\theta)$. From here, we can derive the periods of both functions as well. The x, y values repeat every rotation around the circle(2π radians).

We can then also define $\tan(\theta)$ as the ratio between $\sin(\theta)$ and $\cos(\theta)$. We can see that due to the signs of x, y, the period of tangent is π radians.

Sidenote: The period of a function $\text{trig}(\theta)$ is an angle α such that $\text{trig}(\theta + \alpha) = \text{trig}(\theta)$. This means that even though 2π is a period of sin, 4π also is. This applies for any integer multiple of 2π. From here on out, the period of a function refers to the primitive period, i.e. the lowest positive period of a function.

§1.1 Functions

Definition 1.1 (The basic trigonometric functions)**.**

$$\sin(\theta)$$

$$\cos(\theta)$$

$$\tan(\theta)$$

$$\csc(\theta) = \frac{1}{\sin(\theta)}$$

$$\sec(\theta) = \frac{1}{\sin(\theta)}$$

$$\csc(\theta) = \frac{1}{\cos(\theta)}$$

$$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\cos(\theta)}{\sin(\theta)}$$

§2 Identities and Formulas

These are the essential formulas. Almost all others can be derived from these:

Theorem 2.1

$\sin(\theta)^2 + \cos(\theta)^2 = 1$ due to the Pythagorean Theorem using legs of a right triangle in a unit circle.

$1 + \tan(\theta)^2 = \sec(\theta)$. This can be proved by writing $\tan(\theta)$ and $\sec(\theta)$ in terms of $\sin(\theta)$ and $\cos(\theta)$.

$1 + \cot(\theta)^2 = \csc(\theta)^2$. This can be proved by writing $\cot(\theta)$ and $\csc(\theta)$ in terms of $\sin(\theta)$ and $\cos(\theta)$.

§2.1 Angle Sum/Difference Formulas

Let's start off by going away from trigonometry for a second, and use complex numbers. Any complex number can be represented by $r\text{cis}(\theta) = r(\cos(\theta) + i\sin(\theta))$ We can notice that this looks very similar to a Cartesian coordinate in the form of (x,y), but instead, this is in the form of $(\cos(\theta), \sin(\theta))$ The r isn't needed because $\sin(\theta)$ and $\cos(\theta)$ have a maximum value of 1.

There is a geometric proof for the angle sum identities, but the complex proof is much simpler(pun intended). Using the properties of complex numbers, we know $\text{cis}(\theta + \alpha) = \text{cis}(\theta)\text{cis}(\alpha)$.

Rewriting the cis's in terms of sin and cos gives $\text{cis}(\theta + \alpha) = (\cos(\theta) + i\sin(\theta))(\cos(\alpha) + i\sin(\alpha))$. Expanding gives us $\cos(\theta)\cos(\alpha) - \sin(\theta)\sin(\alpha) + i(\sin(\theta)\cos(\alpha) + \cos(\theta)\sin(\alpha))$

Because $\text{cis}(\theta + \alpha) = \cos(\theta + \alpha) + i\sin(\theta + \alpha)$, we can equate the real and imaginary terms of the expansion with this to get

$$\cos(\theta + \alpha) = \cos(\theta)\cos(\alpha) - \sin(\theta)\sin(\alpha)$$

and

$$\sin(\theta + \alpha) = \sin(\theta)\cos(\alpha) + \cos(\theta)\sin(\alpha)$$

Finding the difference is just plugging in $-\alpha$ and using the even and odd properties of cos and sin, respectively.

This might seem like a lot of work, but it's really easy to remember and derive by hand compared to seeing it in this pdf.

You might recall that tangent is the ratio of sine to cosine. So,

$$\begin{aligned}
\tan(\alpha+\theta) &= \frac{\sin(\theta\pm\alpha)}{\cos(\theta\pm\alpha)} \\
&= \frac{\sin(\theta)\cos(\alpha)\pm\cos(\theta)\sin(\alpha)}{\cos(\theta)\cos(\alpha)\mp\sin(\theta)\sin(\alpha)} \\
&= \frac{\frac{1}{\cos(\theta)\cos(\alpha)}}{\frac{1}{\cos(\theta)\cos(\alpha)}} * \frac{\sin(\theta)\cos(\alpha)\pm\cos(\theta)\sin(\alpha)}{\cos(\theta)\cos(\alpha)\mp\sin(\theta)\sin(\alpha)} \\
&= \frac{\tan(\alpha)\pm\tan(\theta)}{1\mp\tan(\alpha)\tan(\theta)}
\end{aligned}$$

Theorem 2.2 (Sum/Difference Formulas)

$$\cos(\theta\pm\alpha) = \cos(\theta)\cos(\alpha)\mp\sin(\theta)\sin(\alpha)$$

$$\sin(\theta\pm\alpha) = \sin(\theta)\cos(\alpha)\pm\cos(\theta)\sin(\alpha)$$

$$\tan(\alpha\pm\theta) = \frac{\tan(\alpha)\pm\tan(\theta)}{1\mp\tan(\alpha)\tan(\theta)}$$

§2.2 Double Angle, Triple Angle.....

To continue the theme of complex numbers in trigonometry, let me show you a *much* quicker way of deriving multiples of angles.

We know cis(A) represents a complex number. cis$(A)^n$ =cis(nA). In other words $(\cos(A) + i\sin(A))^n = \cos(nA) + i\sin(nA)$. Well, $(\cos(A) + i\sin(A))$ can be represented as $x + iy$. Raising this to the $n'th$ power is the same thing as $(x + y)^n$ but with a slight twist.

First, we know $(x + y)^n$ can be represented through Pascal's Triangle. The coefficients of $(x + y)^n$ are the coefficients of the $n'th$ row of Pascal's Triangle. Keep in mind that the first row is the 0th row.

Now, we're faced with a problem. We need $(x + iy)^n$, but Pascal's Triangle only covers $(x + y)^n$. This can be solved. By the Binomial Theorem, for the first element of the nth row of Pascal's Triangle, the exponent of iy is 0. For the second element, it's 1, for the third, it's 2, and for the fourth it's 3... This means the coefficient of the first element in the nth row is real while the second element is not, the third element is, and the fourth element is not... We also know that the nth row of Pascal's Triangle is equal to $\cos(nA) + i\sin(nA)$. This means we can equate the real and imaginary parts of the nth row of Pascal's Triangle with $\cos(nA) + i\sin(nA)$. From this we get $\cos(nA) =$ the odd indexed elements of the nth row of Pascal's Triangle while $\sin(nA) =$ the even indexed elements of the nth row of Pascal's Triangle.

If you managed to make it through that wall of text, but you are still a bit confused, here is an example problem: find $\cos(4x)$ in terms of $\cos(x)$. Well, we know that the 4th row of Pascal's Triangle is 1 4 6 4 1 which means we want 1,6, and 1 as they are the odd indexed elements. By the Binomial Theorem, this is $\cos(x)^4$, $\binom{4}{2}\cos(x)^2\sin(x)^2$, and $\sin(x)^4$. By our Pythagorean Identities, we know $\sin(x)^2 = 1 - \cos(x)^2$, and we also know $\sin(x)^4 = (\sin(x)^2)^2 = (1 - \cos(x)^2)^2$, so the final answer should become clear after some substitution.

§2.3 Product to Sum

Sometimes, a problem can be solved by converting a product of trigonometric functions to a sum. If you're taking a test that is based on trigonometry then you should memorize these for speed; otherwise, I have some good intuition for them.

Example 2.3

Consider $\cos(a+b)$ and $\cos(a-b)$. Expanding gives $\cos(a)\cos(b) - \sin(a)\sin(b)$ and $\cos(a)\cos(b) + \sin(a)\sin(b)$. Well, you can cancel out terms by adding to get

$$\cos(a)\cos(b) = \frac{1}{2}(\cos(a+b) + \cos(a-b))$$

Try proving the rest using the same method.

Theorem 2.4 (Product to Sum)

$$\cos a \cos b = \frac{1}{2}(\cos(a+b) + \cos(a-b))$$
$$\sin a \sin b = \frac{1}{2}(\cos(a+b) - \cos(a-b))$$
$$\sin a \cos b = \frac{1}{2}(\sin(a+b) + \sin(a-b))$$
$$\cos a \sin b = \frac{1}{2}(\sin(a+b) - \sin(a-b))$$

§2.4 Sum to Product

It's not even worth memorizing both sum to product and product to sum. Just learn one of them. As you can see below, they're both just variations of each other.

Theorem 2.5 (Sum to Product)

$$\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a+b}{2}\right)$$
$$\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a+b}{2}\right)$$
$$\sin(a) + \sin(b) = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a+b}{2}\right)$$
$$\sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a+b}{2}\right)$$

§3 Techniques

It would be practically impossible to include all possible techniques in this section, but I'll do my best to cover the most important ones.

§3.1 Solutions over $[0, 2\pi)$

Problem 3.2. Find the sum of angles $\theta \in [0, 2\pi)$ that satisfy $\cos(7\theta) = \frac{4}{7}$

Please don't expand $\cos(7\theta)$. Please. Instead, use the pattern of solutions in $\cos(n\theta) = 0$. For $n = 1$, the sum is 2π. For $n = 2$, the sum is 4π. For $n = 3$, the sum is 6π, so the sum is $(2n)\pi$ So, the answer is $\boxed{14}$. Note, that the reason why we could make cos equal to 0 is because y=4/7 and y=0 intersect cos the same number of times over $[0, 2\pi)$. This doesn't apply to y=1 because the line hits *cos* once for every 2 times the other lines hit cos.

Sidenote: You can extend this to sin, and also to $[0, 2\pi]$ easily using the same pattern technique. You can also do this for the number of solutions. Just notice a pattern for $\cos(n\theta) = 1$ For $n = 1$, there is one solution. For $n = 2$ there are 2 solutions. You can continue for higher values of n, and you will see this pattern continues. Since cos will equal 1 the same number of times it will equal $\frac{4}{7}$, this pattern can be extended. So, the answer is that $\cos(7\theta) = 4/7$ $\boxed{7}$ times.

§3.2 Roots of Unity

This is by far one of the most powerful techniques used to solve trigonometry problems such as trigonometric sums. I'll explain through an example.

Problem 3.4 (2019 FAMAT Alpha Individual)**.** Find the sum of all angles $\theta \in [0, 2\pi)$ such that

$$\cos(\theta) + \cos(2\theta) + \cos(3\theta) + \cdots + \cos(2019\theta) = \frac{-1}{2}$$

For the solution, remember $\text{cis}(\theta) = e^{i\theta} = \cos(\theta) + i\sin(\theta)$. So, we can rewrite cos in terms of cis because $\frac{\text{cis}(\theta)+\text{cis}(-\theta)}{2} = \cos(\theta)$ The summation then becomes

$$\sum_{n=1}^{2019} \frac{\text{cis}(n\theta) + \text{cis}(-n\theta)}{2} = \frac{-1}{2}$$

Multiply both sides by 2 and simplify $\text{cis}(-n\theta)$ to $\frac{1}{\text{cis}(n\theta)}$ to get

$$\sum_{n=1}^{2019} \text{cis}(n\theta) + \frac{1}{\text{cis}(n\theta)} = -1$$

By now, you might be able to see where this solution is going. Since $\text{cis}(n\theta) = e^{ni\theta}$, adding a bunch of powers of e is a finite geometric series. The formula for the sum of a finite geometric series is

$$\frac{a(1-r^n)}{1-r}$$

where r is the common ratio and a is the first term, but you should be able to prove this.

In our case, $a, r = \operatorname{cis}(\pm\theta)$, so our summation becomes

$$\frac{(\operatorname{cis}(\theta))(1-\operatorname{cis}(2019\theta))}{1-\operatorname{cis}(\theta)} + \frac{1-\frac{1}{\operatorname{cis}(2019\theta)}}{\operatorname{cis}(\theta)(1-\frac{1}{\operatorname{cis}(\theta)})} = -1$$

This simplifies to

$$\frac{(\operatorname{cis}(\theta))(1-\operatorname{cis}(2019\theta))}{1-\operatorname{cis}(\theta)} + \frac{\operatorname{cis}(2019\theta)-1}{\operatorname{cis}(\theta)-1} = -1$$

Factoring out $1-\operatorname{cis}2019\theta$ gets

$$1-\operatorname{cis}2019\theta = 1$$

or

$$\operatorname{cis}(2019\theta) = \cos(2019\theta) + i\sin(2019\theta) = 0$$

Now, we were asked for the sum of the solutions from $[0, 2\pi)$. You can then pair real terms and imaginary terms, so the question is either $\cos(2019\theta) = 0$ or $\sin(2019\theta) = 0$. If you derive the pattern mentioned in 3.1 for sin, you'll notice it's not as easy to implement as the one for cos, so let's use the pattern $2n\pi$ which gives us a final answer of $\boxed{4038\pi}$