

Given:

Lemma 1: $a * b \geq 0$ if $a, b \geq 0$

Lemma 2: Distributive Property of Multiplication over Addition

Lemma 3: Addition Property of Equality

Lemma 4: Every real number has a unique inverse

Lemma 5: Multiplicative Property of Equality

Lemma 6: Square Root is always increasing for increasing values of x

Proof: $\frac{d(\sqrt{x})}{dx} = \frac{1}{2}x^{-1/2}$ Derivative is always positive for positive x because the square root is always positive, so its multiplicative inverse is always positive.

Lemma 7: Reflexive Property of Equality

Lemma 8: AM-GM

Lemma 9: Multiplicative Property of Inequality

Proof:

Statement	Reason
1. $(a - b) * (a - b) = (a - b)^2 \geq 0$	1. Lemma 1
2. $a^2 - 2ab + b^2 \geq 0$	2. Lemma 2
3. $a^2 + 2ab + b^2 \geq 4ab$	3. Lemma 3
4. $(a + b)^2 \geq 4ab$	4. Lemma 2
5. $\frac{(a+b)^2}{4} \geq ab$	5. Lemmas 4, 5
6. $\frac{a+b}{2} \geq \sqrt{ab}$	6. Definition of square root and Lemma 6

Prove:

Given $ab = 1$, prove $a^6 + 4b^6 \geq 4$

Statement	Reason
1. $ab = 1$	1. Given
2. $(ab)^n = (ab)^n = (1)^n$	2. Lemma 7
3. $(ab)^6 = a^6 * b^6 = 1$	3. Statement 2
4. $a^6 = \frac{1}{b^6}$	4. Lemmas 4, 5
5. $a^6 + 4b^6 = a^6 + \frac{4}{a^6}$	5. Statement 4
6. $\frac{a^6 + \frac{4}{a^6}}{2} \geq \sqrt{a^6 * \frac{4}{a^6}} = \sqrt{4} = 2$	6. Lemma 8
7. $\boxed{a^6 + 4b^6 \geq 4}$	7. Lemma 9