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Question

The lines $ax + 2y + 1 = 0, bx - 3y + 1 = 0$ and $cx + 4y + 1 = 0$ are concurrent if a, b, c are in G.P.

Solution

Given three lines:

$$L_1 : ax + 2y = -1 \tag{1}$$
$$L_2 : bx - 3y = -1 \tag{2}$$
$$L_3 : cx + 4y = -1 \tag{3}$$

representing as an augmented matrix:

$$\begin{pmatrix} a & 2 & -1 \\ b & -3 & -1 \\ c & 4 & -1 \end{pmatrix} \tag{4}$$

Assume a, b, c are in geometric progression:

$$b^2 = ac \quad \Rightarrow \quad c = \frac{b^2}{a} \tag{5}$$

Substitute into the third row:

$$\begin{pmatrix} a & 2 & -1 \\ b & -3 & -1 \\ \frac{b^2}{a} & 4 & -1 \end{pmatrix} \tag{6}$$

Normalize the First Row

$$R_1 \to \frac{1}{a} R_1 = \begin{pmatrix} 1 & \frac{2}{a} & \frac{-1}{a} \\ b & -3 & -1 \\ \frac{b^2}{a} & 4 & -1 \end{pmatrix} \tag{7}$$

Eliminate First Column in R_2 and R_3

$$R_2 \to R_2 - b \cdot R_1 = \begin{pmatrix} 1 & \frac{2}{a} & \frac{-1}{a} \\ 0 & -3 - \frac{2b}{a} & -1 + \frac{b}{a} \\ \frac{b^2}{a} & 4 & -1 \end{pmatrix} \tag{8}$$

$$R_3 \to R_3 - \frac{b^2}{a} \cdot R_1 = \begin{pmatrix} 1 & \frac{2}{a} & \frac{-1}{a} \\ 0 & -3 - \frac{2b}{a} & -1 + \frac{b}{a} \\ 0 & 4 - \frac{2b^2}{a^2} & -1 + \frac{b^2}{a^2} \end{pmatrix} \tag{9}$$

For the system to be consistent (i.e., lines concurrent), rows 2 and 3 must be linearly dependent. This means the second and third rows must be scalar multiples of each other.

Let us compare the second and third rows:

$$Row\,2 : \begin{pmatrix} 0 & A & B \end{pmatrix} \quad Row\,3 : \begin{pmatrix} 0 & C & D \end{pmatrix} \tag{10}$$

Where:

$$A = -3 - \frac{2b}{a}, \quad B = -1 + \frac{b}{a} \tag{11}$$

$$C = 4 - \frac{2b^2}{a^2}, \quad D = -1 + \frac{b^2}{a^2} \tag{12}$$

For linear dependence:

$$\frac{C}{A} = \frac{D}{B} \tag{13}$$

$$A \cdot D = B \cdot C \tag{14}$$

Substitute the expressions and simplify:

$$(-3 - \frac{2b}{a})(-1 + \frac{b^2}{a^2}) = (-1 + \frac{b}{a})(4 - \frac{2b^2}{a^2}) \tag{15}$$

Expanding both sides and simplifying leads to:

$$-7a^2 + 2ab + 5b^2 = 0 \tag{16}$$

Final Condition for Concurrency

$$\boxed{-7a^2 + 2ab + 5b^2 = 0} \tag{17}$$

If the condition is satisfied, the matrix reduces to:

$$\begin{pmatrix} 1 & \frac{2}{a} & \frac{-1}{a} \\ 0 & 1 & \frac{B}{A} \\ 0 & 0 & 0 \end{pmatrix} \tag{18}$$

This confirms the system is consistent and the lines are concurrent.