5.13.39

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Question

Let $\mathbf{P} = \begin{pmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{pmatrix}$ where $\alpha \in \mathbf{R}$. Suppose $\mathbf{Q} = \begin{pmatrix} q_{ij} \end{pmatrix}$ is a matrix such that $\mathbf{PQ} = \mathrm{k}\mathbf{I}$, where k $\neq$ 0 and $\mathbf{I}$ is the identity of order 3. If $q_{23} =$ -$\frac{k}{8}$ and $\det \mathbf{Q} = \frac{k^2}{2}$, then

1. a $=$ 0, k $=$ 8
2. 4a - k $+$ 8 $=$ 0
3. $\det\left(\mathbf{P}adj(\mathbf{Q})\right) = 2^9$
4. $\det\left(\mathbf{Q}adj(\mathbf{P})\right) = 2^{13}$

Solution

It is given that

$$\mathbf{PQ} = k\mathbf{I}, \det \mathbf{Q} = \frac{k^2}{2} \tag{1}$$

Taking the determinant

$$(\det \mathbf{P}) \cdot \frac{k^2}{2} = k^3 \tag{2}$$

$$\left| \begin{pmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{pmatrix} \right| = 2k \tag{3}$$

Solution

$$\begin{pmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{pmatrix} \xleftrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - \frac{2}{3} R_1} \begin{pmatrix} 3 & -1 & -2 \\ 0 & \frac{2}{3} & \alpha + \frac{4}{3} \\ 0 & -4 & 2 \end{pmatrix} \tag{4}$$

From equation (3) we get

$$3 \times \left(\frac{2}{3} \times 2 - (-4) \times \left(\alpha + \frac{4}{3} \right) \right) = 2k \tag{5}$$

$$20 + 12\alpha = 2k \tag{6}$$

Solution

Using the relation $\mathbf{PQ} = k\mathbf{I}$, we get the following augmented matrix

$$\left(\begin{array}{ccc|ccc} 3 & -1 & -2 & 1 & 0 & 0 \\ 2 & 0 & \alpha & 0 & 1 & 0 \\ 3 & -5 & 0 & 0 & 0 & 1 \end{array}\right) \xleftrightarrow[R_2 \leftarrow R_2 - 2R_1]{R_1 \leftarrow \frac{1}{3}R_1} \left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & \alpha + \frac{4}{3} & -\frac{2}{3} & 1 & 0 \\ 3 & -5 & 0 & 0 & 0 & 1 \end{array}\right) \tag{7}$$

$$\xleftrightarrow[R_2 \leftarrow \frac{3}{2}R_2]{R_3 \leftarrow R_3 - 3R_1} \left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & \frac{3}{2}\alpha + 2 & -1 & \frac{3}{2} & 0 \\ 0 & -4 & 2 & -1 & 0 & 1 \end{array}\right) \xleftrightarrow[R_3 \leftarrow R_3 + 4R_2]{R_1 \leftarrow \frac{1}{3}R_2} \tag{8}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2}\alpha & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{2}\alpha + 2 & -1 & \frac{3}{2} & 0 \\ 0 & 0 & 6\alpha + 10 & -5 & 6 & 1 \end{array}\right) \xleftrightarrow{R_3 \leftarrow \frac{1}{6\alpha + 10}R_3} \tag{9}$$

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2}\alpha & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{2}\alpha+2 & -1 & \frac{3}{2} & 0 \\ 0 & 0 & 1 & -\frac{5}{6\alpha+10} & \frac{6}{6\alpha+10} & \frac{1}{6\alpha+10} \end{array}\right) \tag{10}$$

$$\xleftrightarrow[R_2 \leftarrow R_2 - \left(\frac{3}{2}\alpha+2\right)R_3]{R_1 \leftarrow R_1 - \frac{1}{2}\alpha R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5\alpha}{12\alpha+20} & \frac{3\alpha+10}{6\alpha+20} & -\frac{\alpha}{12\alpha+20} \\ 0 & 1 & 0 & -1+\frac{5(3\alpha+4)}{12\alpha+20} & \frac{3}{2}-\frac{6(3\alpha+4)}{12\alpha+20} & -\frac{3\alpha+4}{12\alpha+20} \\ 0 & 0 & 1 & -\frac{5}{6\alpha+10} & \frac{6}{6\alpha+10} & \frac{1}{6\alpha+10} \end{array}\right) \tag{11}$$

Solution

From this augmented matrix,

$$q_{23} = -k\frac{3\alpha + 4}{12\alpha + 20} = -\frac{k}{8}\,(Given) \tag{12}$$

$$\implies \alpha = -1 \tag{13}$$

Substituting the value of α in equation (6), we get

$$k = 4 \tag{14}$$

n is the order of matrix B

$$\left|\mathbf{A}\mathrm{adj}\left(\mathbf{B}\right)\right| = \left|\mathbf{A} \cdot \mathbf{B}^{n-1}\right| \tag{15}$$

$|\mathbf{P}| = 8$, $|\mathbf{Q}| = 8$

Answer

$$\left|(\mathbf{P}adj(\mathbf{Q}))\right| = |\mathbf{P}|\,|\mathbf{Q}|^2 = 8 \times 64 = 2^9 \tag{16}$$

$$\left|(\mathbf{Q}adj(\mathbf{P}))\right| = |\mathbf{Q}|\,|\mathbf{P}|^2 = 8 \times 64 = 2^9 \tag{17}$$

So options (2) and (3) are correct

C Code

```c
#include <stdio.h>


int determinant(int n, int mat[n][n]){
    int det = mat[0][0]*(mat[1][1]*mat[2][2] - mat[1][2]*mat
        [2][1])
            - mat[0][1]*(mat[1][0]*mat[2][2] - mat[1][2]*mat
                [2][0])
            + mat[0][2]*(mat[1][0]*mat[2][1] - mat[1][1]*mat
                [2][0]);
    return det;
}

int solution(int a1, int b1, int c1, int alpha){
    return ((c1-a1*alpha)/b1);
}
```

Python + C Code

```
import ctypes
import numpy as np
lib = ctypes.CDLL(./libcode.so)
array = ctypes.c_int * 3
matrix = array * 3
lib.determinant.argtypes = [matrix]
lib.determinant.restype = ctypes.c_int
lib.solution.argtypes = [ctypes.c_int, ctypes.c_int, ctypes.c_int
    , ctypes.c_int]
lib.solution.restype = ctypes.c_int
k = lib.solution(12 , -2, -20, -1)
P = np.array([[3, -1, -2], [2, 0, -1], [3, -5, 0]])
mat = matrix(*[ (ctypes.c_int * 3)(*row) for row in P ])
det_P = lib.determinant(mat)
det_Q = 2*k
print(Determinant of P and Q = , det_P)
print(det_Q*(det_P**2))
print(det_P*(det_Q**2))
```

```
import numpy as np
def determinant(mat):
    det = (mat[0][0]*(mat[1][1]*mat[2][2] - mat[1][2]*mat[2][1])
         - mat[0][1]*(mat[1][0]*mat[2][2] - mat[1][2]*mat[2][0])
         + mat[0][2]*(mat[1][0]*mat[2][1] - mat[1][1]*mat[2][0]))
    return det
def solution(a1, b1, c1, alpha):
    return (c1 - a1*alpha) / b1
k = solution(12, -2, -20, -1)
P = np.array([[3, -1, -2],
              [2, 0, -1],
              [3, -5, 0]])
det_P = determinant(P)
det_Q = 2 * k
print(Determinant of P =, det_P)
print(Determinant of Q =, det_Q)
print(det(Q) * det(P)^2 =, det_Q * (det_P**2))
print(det(P) * det(Q)^2 =, det_P * (det_Q**2))
```