Matrices in Geometry 7.4.44

EE25BTECH11037 - Divyansh

Question: Let C be any circle with centre $\left(0, \sqrt{2}\right)$. Prove that at most two rational points can be there on C. (A rational point is a point both of whose coordinates are rational numbers).

Solution:
The equation of the given circle C can be written as

$$C : \|\mathbf{x} - \mathbf{O}\| = r \tag{1}$$

where r is the radius of circle C and $\mathbf{O} = \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix}$ is the center of the circle.
Let $\mathbf{P}$ be a rational point on the circle, then

$$\|\mathbf{P} - \mathbf{O}\| = r \tag{2}$$

Upon squaring both sides,

$$\|\mathbf{P} - \mathbf{O}\|^2 = r^2 \implies \mathbf{P}^\top\mathbf{P} - 2\mathbf{P}^\top\mathbf{O} + \mathbf{O}^\top\mathbf{O} = r^2 \tag{3}$$

Substituting $\mathbf{P} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{O} = \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix}$

$$\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - 2\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix} + \begin{pmatrix} 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix} = r^2 \tag{4}$$

$$\implies x^2 + y^2 - 2\sqrt{2}y + 2 = r^2 \tag{5}$$

Rearranging the terms,

$$x^2 + y^2 - r^2 + 2 = 2\sqrt{2}y \tag{6}$$

$$\mathbf{P} \in R^2 \implies x, y \in R \implies \text{LHS is rational} \implies \text{RHS should be rational} \tag{7}$$

$$\implies y = 0 \tag{8}$$

$$\therefore x^2 = r^2 - 2 \implies x = \pm\sqrt{r^2 - 2} : r > \sqrt{2} \tag{9}$$

We get the points

$$\mathbf{P} = \begin{pmatrix} \sqrt{r^2 - 2} \\ 0 \end{pmatrix} \text{ OR } \mathbf{P} = \begin{pmatrix} -\sqrt{r^2 - 2} \\ 0 \end{pmatrix} : r^2 - 2 \text{ is a perfect square} \tag{10}$$

This proves that at most two rational points can be present in C.
Let us try to show this using a graph with $r = \sqrt{6}$.

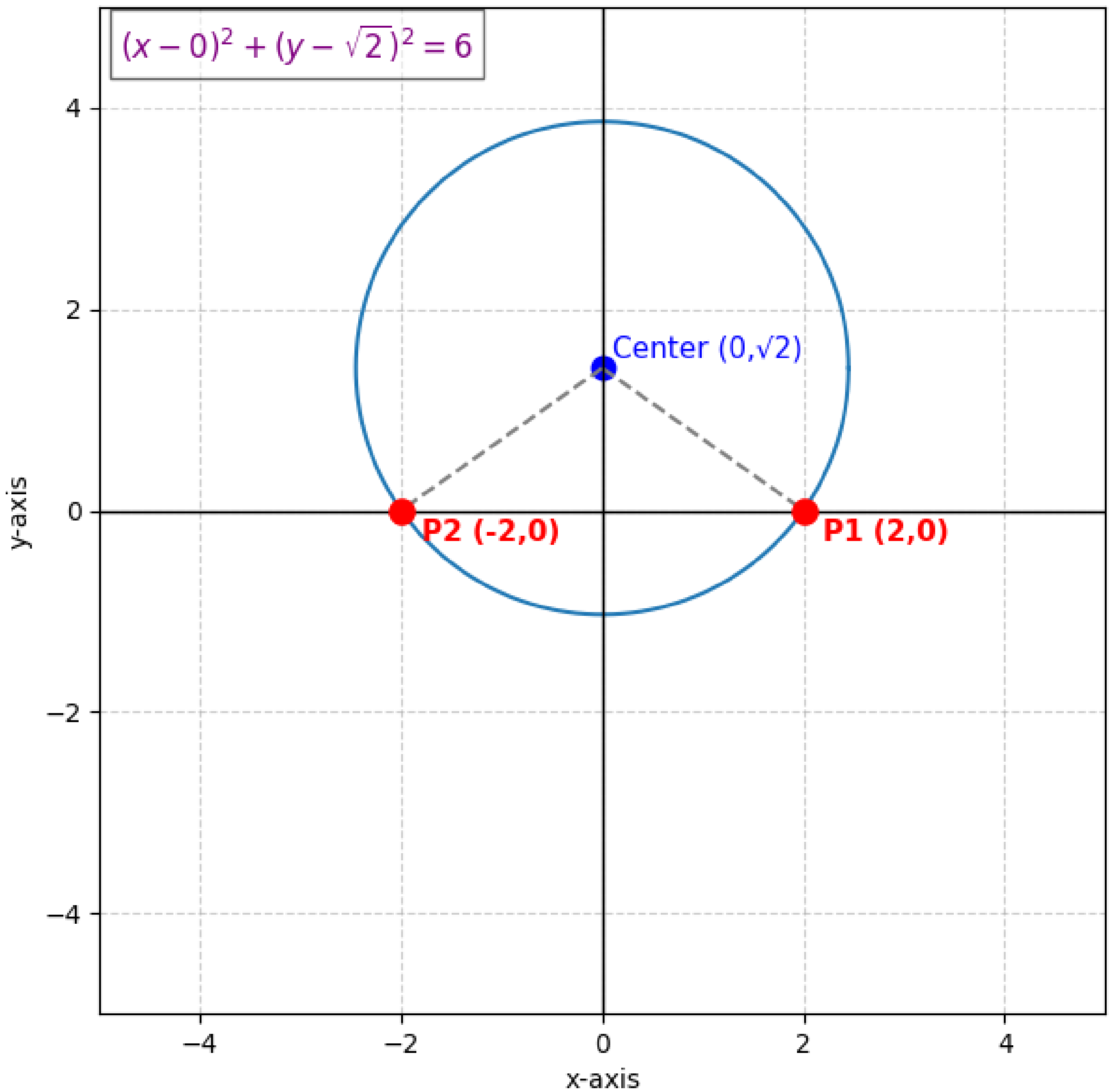

Fig. 1: Graph for 7.4.44 with $r = \sqrt{6}$