10.7.104

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Question : Let a, b and λ be positive real numbers. Suppose P is an end point of the latus rectum of the parabola $y^2 = 4\lambda x$, and suppose the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ passes through the point P. If the tangents to the parabola and the ellipse at the point P are perpendicular to each other, then the eccentricity of ellipse is

Solution :

Name	**Value**
Parabola	$\mathbf{x}^\top \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -2\lambda \\ 0 \end{pmatrix}^\top \mathbf{x} = 0$
Ellipse	$\mathbf{x}^\top \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix} \mathbf{x} - 1 = 0$

Table : Parabola and Ellipse

The parameters of the parabola are :

$$\mathbf{V_1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{u_1} = \begin{pmatrix} -2\lambda \\ 0 \end{pmatrix} \qquad f_1 = 0 \tag{1}$$

The parameters of the ellipse are :

$$\mathbf{V_2} = \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix} \qquad \mathbf{u_2} = \mathbf{0} \qquad f_2 = -1 \tag{2}$$

The end point of the latus rectum of parabola is

$$\mathbf{P} = \begin{pmatrix} \lambda \\ 2\lambda \end{pmatrix} \tag{3}$$

The equation of tangent to the parabola at $\mathbf{P}$ is given as :

$$(\mathbf{V_1 P} + \mathbf{u_1})^\top \mathbf{x} + \mathbf{u_1}^\top \mathbf{P} + f_1 = 0 \qquad \mathbf{n_1} = \mathbf{V_1 P} + \mathbf{u_1} \tag{4}$$

The equation of tangent to the ellipse at $\mathbf{P}$ is given as :

$$(\mathbf{V_2 P} + \mathbf{u_2})^\top \mathbf{x} + \mathbf{u_2}^\top \mathbf{P} + f_2 = 0 \qquad \mathbf{n_2} = \mathbf{V_2 P} + \mathbf{u_2} \tag{5}$$

As the tangents at $\mathbf{P}$ are perpendicular , the normal vectors of the tangents are also perpendicular

$$\mathbf{n_1}^\top \mathbf{n_2} = 0 \tag{6}$$

$$(\mathbf{V_1}\mathbf{P} + \mathbf{u_1})^\top \mathbf{V_2}\mathbf{P} = 0 \tag{7}$$

$$(\mathbf{P}^\top \mathbf{V_1}^\top + \mathbf{u_1}^\top)\mathbf{V_2}\mathbf{P} = 0 \tag{8}$$

$$\begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0 \tag{9}$$

$$\frac{a^2}{b^2} = \frac{1}{2} \tag{10}$$

From (2) , the eigen values of $\mathbf{V_2}$ are the diagonal entries as it is an upper triangular matrix and also $a < b$

$$\lambda_1 = \frac{1}{b^2} \qquad \lambda_2 = \frac{1}{a^2} \tag{11}$$

The eccentricity e of ellipse is given as

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \tag{12}$$

$$e = \sqrt{1 - \frac{a^2}{b^2}} \tag{13}$$

From (10) , we get

$$e = \frac{1}{\sqrt{2}} \tag{14}$$

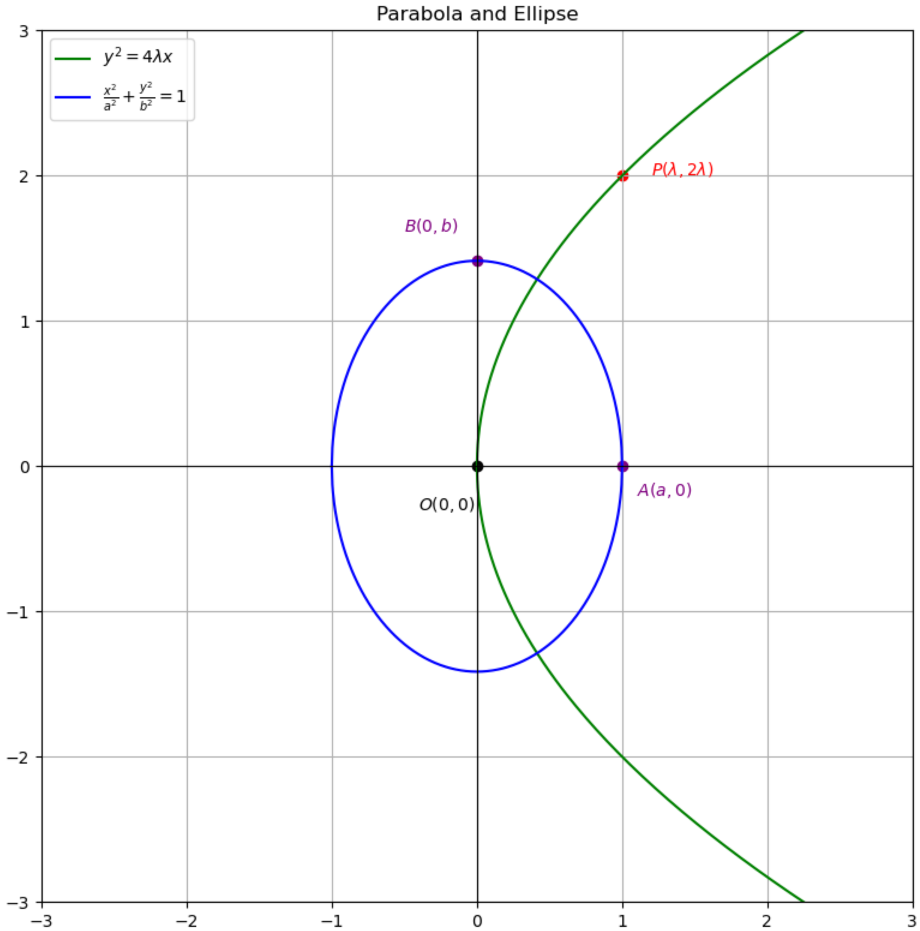

Fig : Parabola and Ellipse