2.7.15

EE25BTECH11003 - Adharvan Kshathriya Bommagani

Question:
Find the volume of a parallelepiped whose edges are given by $-3\hat{i}+7\hat{j}+5\hat{k}$, $-5\hat{i}+7\hat{j}-3\hat{k}$ and $7\hat{i}-5\hat{j}-3\hat{k}$.

Solution:

Let **a**, **b** and **c** be three vectors representing the edges of the given parallelepiped.

$$\mathbf{a} = \begin{pmatrix} -3 \\ 7 \\ 5 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} -5 \\ 7 \\ -3 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 7 \\ -5 \\ -3 \end{pmatrix} \tag{1}$$

The volume is given by $V = \sqrt{\det(G)}$, where G is the Gram matrix formed by the dot products of the vectors. Based on calculations, the Gram matrix is:

$$G = \begin{pmatrix} 83 & 49 & -71 \\ 49 & 83 & -61 \\ -71 & -61 & 83 \end{pmatrix}$$

We use **row reduction** to convert G into an upper triangular matrix U. The determinant is unchanged by adding a multiple of one row to another.

Step 1:

$$\begin{pmatrix} 83 & 49 & -71 \\ 49 & 83 & -61 \\ -71 & -61 & 83 \end{pmatrix} \xrightarrow{R_2 \to R_2 - \frac{49}{83} R_1} \begin{pmatrix} 83 & 49 & -71 \\ 0 & \frac{4488}{83} & \frac{-1584}{83} \\ -71 & -61 & 83 \end{pmatrix}$$

Step 2:

$$\begin{pmatrix} 83 & 49 & -71 \\ 0 & \frac{4488}{83} & \frac{-1584}{83} \\ -71 & -61 & 83 \end{pmatrix} \xrightarrow{R_3 \to R_3 + \frac{71}{83} R_1} \begin{pmatrix} 83 & 49 & -71 \\ 0 & \frac{4488}{83} & \frac{-1584}{83} \\ 0 & \frac{-1584}{83} & \frac{1848}{83} \end{pmatrix}$$

Step 3:

$$\begin{pmatrix} 83 & 49 & -71 \\ 0 & \frac{4488}{83} & \frac{-1584}{83} \\ 0 & \frac{-1584}{83} & \frac{1848}{83} \end{pmatrix} \xrightarrow{R_3 \to R_3 + \frac{6}{17} R_2} \begin{pmatrix} 83 & 49 & -71 \\ 0 & \frac{4488}{83} & \frac{-1584}{83} \\ 0 & 0 & \frac{264}{17} \end{pmatrix}$$

The matrix is now upper triangular. The determinant of G is the product of the diagonal entries of U.

$$\begin{aligned}\det(G) &= 83 \times \frac{4488}{83} \times \frac{264}{17} \\ &= 4488 \times \frac{264}{17} \\ &= (17 \times 264) \times \frac{264}{17} \\ &= 264 \times 264 = \mathbf{69696}\end{aligned}$$

The volume is the square root of the determinant.

$$\text{Volume} = \sqrt{\det(G)} = \sqrt{69696}$$
$$\text{Volume} = \mathbf{264} \text{ cubic units}$$

Therefore, the volume of the parallelepiped is 264 cubic units.

Parallelopiped Defined by Vectors a, b and c :

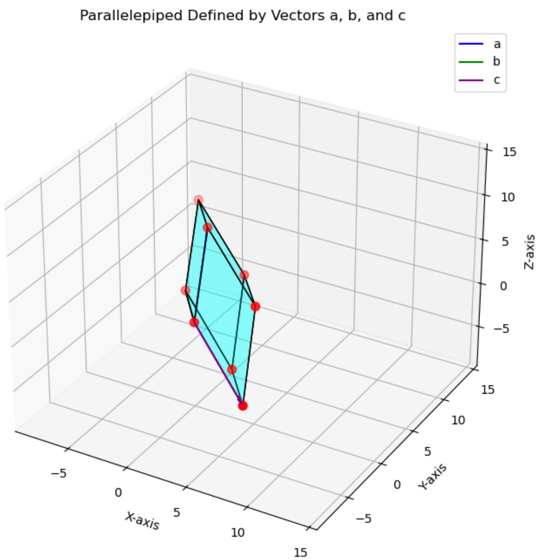

Fig. 0: Figure for 2.7.15