2.10.50

EE25BTECH11021 - Dhanush Sagar

Question

A variable plane at a distance of one unit from the origin cuts the coordinate axes at A, B and C.

If the centroid $D(x, y, z)$ of triangle ABC satisfies the relation

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = k,$$

then the value of k is :

1) 3
2) 1
3) $\frac{1}{3}$
4) 9

Solution

Let the plane meet the coordinate axes at

$$\mathbf{A} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}.$$

Define

$$\mathbf{M} = \text{diag}(a, b, c), \quad \mathbf{e} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

The normal vector of the plane is given by

$$\mathbf{n} = \mathbf{M}^{-1}\mathbf{e} = \begin{pmatrix} \frac{1}{a} \\ \frac{1}{b} \\ \frac{1}{c} \end{pmatrix} \tag{4.1}$$

The squared norm of the normal vector is

$$\|\mathbf{n}\|^2 = \mathbf{e}^T\mathbf{M}^{-2}\mathbf{e} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \tag{4.2}$$

The perpendicular distance of the plane from the origin is

$$d = \frac{|1|}{\|\mathbf{n}\|} = \frac{1}{\sqrt{\mathbf{e}^T\mathbf{M}^{-2}\mathbf{e}}} \tag{4.3}$$

Thus, the relation between a, b, c and d is

$$\mathbf{e}^T \mathbf{M}^{-2} \mathbf{e} = \frac{1}{d^2} \tag{4.4}$$

The centroid of the triangle ABC is

$$D = \frac{1}{3} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{3} \mathbf{M} \mathbf{e} \tag{4.5}$$

Hence, the coordinates of the centroid are

$$x = \frac{a}{3}, \quad y = \frac{b}{3}, \quad z = \frac{c}{3} \tag{4.6}$$

Now, the required expression is

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{(a/3)^2} + \frac{1}{(b/3)^2} + \frac{1}{(c/3)^2} \tag{4.7}$$

Simplifying, we get

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = 9 \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right) \tag{4.8}$$

In matrix form, this becomes

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = 9 \, \mathbf{e}^T \mathbf{M}^{-2} \mathbf{e} \tag{4.9}$$

Using the relation obtained earlier,

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{9}{d^2} \tag{4.10}$$

For $d = 1$, we finally obtain

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = 9 \tag{4.11}$$

$$\boxed{k = 9}$$

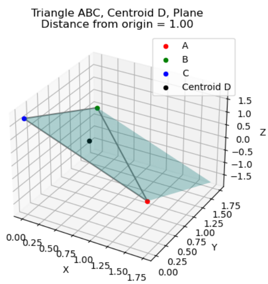

Fig. 4.1