9.4.39

EE25BTECH11026-Harsha

Question:
The difference of squares of two numbers is 180. The square of the smaller number is 8 times the larger number. Find the two numbers.
Solution:
Let us solve the given question theoretically and then verify the solution computationally.

Let x and y be the 2 numbers such that $x > y$.
The given equations are,

$$x^2 - y^2 = 180 \tag{0.1}$$

$$y^2 = 8x \tag{0.2}$$

As the given equations are homogeneous, converting them into quadratic form,

$$\implies \mathbf{x}^\top \mathbf{V_1} \mathbf{x} + c = 0 \tag{0.3}$$

where $\mathbf{x}^\top = \begin{pmatrix} x & y \end{pmatrix}$ and $\mathbf{V_1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $c = -180$
And also,

$$\mathbf{x}^\top \mathbf{V_2} \mathbf{x} + 2\mathbf{b}^\top \mathbf{x} = 0 \tag{0.4}$$

where $\mathbf{x}^\top = \begin{pmatrix} x & y \end{pmatrix}^\top$, $\mathbf{V_2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -4 \\ 0 \end{pmatrix}$.
To identify the intersection of conics, we can employ the approach of degenerating the conics.

To work with degeneracy in matrix form we form the standard augmented 3×3 matrix for each conic:

$$\mathbf{M_i} = \begin{pmatrix} \mathbf{V_i} & \mathbf{b_i} \\ \mathbf{b_i}^\top & c_i \end{pmatrix} \tag{0.5}$$

From (0.5),

$$\implies \mathbf{M_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -180 \end{pmatrix} \quad \mathbf{M_2} = \begin{pmatrix} 0 & 0 & -4 \\ 0 & 1 & 0 \\ -4 & 0 & 0 \end{pmatrix} \tag{0.6}$$

$$\therefore \mathbf{x}^\top \left(\mathbf{M_1} + \lambda \mathbf{M_2}\right) \mathbf{x} = 0 \tag{0.7}$$

To degenerate the conic into a line, we can find the solutions of λ when $\|\mathbf{M_1}+\lambda\mathbf{M_2}\| = 0$

$$\therefore \|\mathbf{M_1} + \lambda\mathbf{M_2}\| = 0 \tag{0.8}$$

$$\implies (\lambda - 1)\left(4\lambda^2 + 45\right) = 0 \tag{0.9}$$

$$\therefore \lambda = 1 \tag{0.10}$$

Substituting λ in (0.8),

$$\implies \mathbf{x}^\top (\mathbf{M_1} + \mathbf{M_2})\,\mathbf{x} \tag{0.11}$$

$$\implies x^2 - 8x - 180 = 0 \tag{0.12}$$

$$\implies x = 18, -10 \tag{0.13}$$

for $x = -10$, there is no real solution of y in (0.2),

$$\implies y = \pm 12 \tag{0.14}$$

$$\therefore \text{The two numbers are } (18, 12) \textit{ and } (18, -12) \tag{0.15}$$

From the figure, it is clearly verified that the theoretical solution matches with the computational solution.