5.2.44

EE25BTECH11042 - Nipun Dasari

Question:
Solve the following system of rational equations

$$\frac{10}{x+y} + \frac{2}{x-y} = 4 \tag{0.1}$$

$$\frac{15}{x+y} - \frac{5}{x-y} = -2 \tag{0.2}$$

Solution:
$\frac{A}{x+y} + \frac{B}{x-y} = C$ becomes:

$$c(x^2 - y^2) - (a+b)x + (a-b)y = 0 \tag{0.3}$$

Matrix form: $\mathbf{x}^\top V\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} = 0$, where:

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad V = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} -(a+b)/2 \\ (a-b)/2 \end{pmatrix} \tag{0.4}$$

The intersection points of the two hyperbolas lie on a Common chord, $c_1H_2 - c_2H_1 = 0$, where $H_1 = 0$ and $H_2 = 0$ are the equations of each of hyperbolas. This results in the linear equation $\mathbf{n}^\top\mathbf{x} = 0$,

$$d = c_1(a_2 + b_2) - c_2(a_1 + b_1) \tag{0.5}$$

$$e = c_2(a_1 - b_1) - c_1(a_2 - b_2) \tag{0.6}$$

$$\text{where d and e are obtained by eliminating the quadratic terms } \mathbf{n} = \begin{pmatrix} d \\ e \end{pmatrix} \tag{0.7}$$

The solution is the non-trivial intersection point of this common chord and either hyperbola

$$y = -\frac{d}{e}x \tag{0.8}$$

$$c_1\left(x^2 - \left(-\frac{d}{e}x\right)^2\right) - (a_1 + b_1)\,x + (a_1 - b_1)\left(-\frac{d}{e}x\right) = 0 \tag{0.9}$$

$$\implies x\left(x\left(c_1\left(\frac{e^2 - d^2}{e^2}\right)\right) - \left(\frac{e\,(a_1 + b_1) + d\,(a_1 - b_1)}{e}\right)\right) = 0 \tag{0.10}$$

$$\therefore x\left(c_1\left(\frac{e^2 - d^2}{e^2}\right)\right) = \left(\frac{e\,(a_1 + b_1) + d\,(a_1 - b_1)}{e}\right) \tag{0.11}$$

$$\implies x = \frac{e\,(e\,(a_1 + b_1) + d\,(a_1 - b_1))}{c_1\,(e^2 - d^2)} \tag{0.12}$$

For the given system, the coefficients are:

$$a_1 = 10, b_1 = 2, c_1 = 4 \quad \text{and} \quad a_2 = 15, b_2 = -5, c_2 = -2 \tag{0.13}$$

Using (0.5) and (0.6), we calculate the common chord coefficients:

$$d = 4\,(15 - 5) - (-2)\,(10 + 2) = 4\,(10) + 2\,(12) = 64 \tag{0.14}$$

$$e = (-2)\,(10 - 2) - 4\,(15 - -5) = -2(8) - 4(20) = -96 \tag{0.15}$$

Substituting these into the formula for x from (0.12):

$$x = \frac{(-96)\,((-96)\,(12) + (64)\,(8))}{4\left((-96)^2 - (64)^2\right)} \tag{0.16}$$

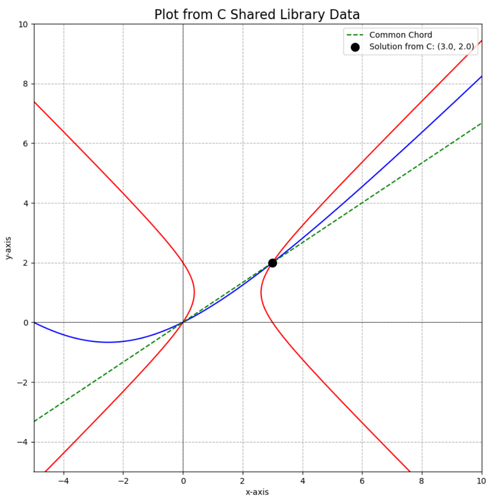
$$= \frac{-96\,(-1152 + 512)}{4\,(9216 - 4096)} \tag{0.17}$$

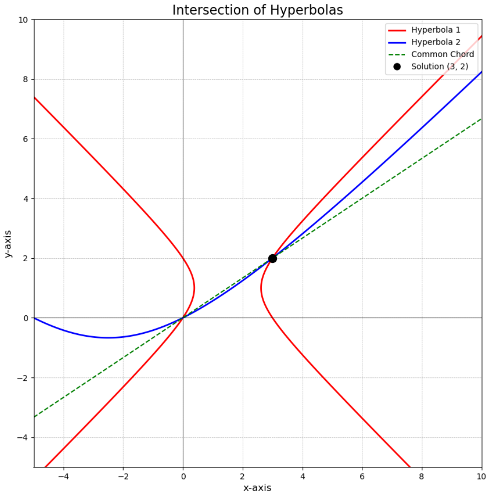
$$= \frac{-96\,(-640)}{4\,(5120)} = \frac{61440}{20480} = 3 \tag{0.18}$$

Using the value of x from (0.18) in the formula for y (0.8):

$$y = -\frac{64}{-96}(3) = \frac{2}{3}3 = 2 \tag{0.19}$$

Thus, from (0.18) and (0.19), the solution is $x = 3$ and $y = 2$.




Intersection of Hyperbolas
Hyperbola 1
Hyperbola 2
Common Chord
Solution (3, 2)
x-axis
y-axis
−4
−2
0
2
4
6
8
10