Matrices in Geometry 5.13.63

EE25BTECH11037 - Divyansh

Question: Let $\mathbf{M} = \begin{pmatrix} \sin^4(\theta) & -1-\sin^2(\theta) \\ 1+\cos^2(\theta) & \cos^4(\theta) \end{pmatrix} = \alpha\mathbf{I} + \beta\mathbf{M}^{-1}$
Where $\alpha = \alpha(\theta)$ and $\beta = \beta(\theta)$ are real numbers, and $\mathbf{I}$ is the 2×2 identity matrix. If α^* is the minimum of the set $(\alpha(\theta) : \theta \in [0, 2\pi))$ and β^* is the minimum of the set $(\beta(\theta) : \theta \in [0, 2\pi))$. Then the value of $\alpha^* + \beta^*$ is

1) $-\frac{31}{16}$
2) $-\frac{17}{16}$
3) $-\frac{37}{16}$
4) $-\frac{29}{16}$

Solution: Using the Cayley-Hamilton Theorem,

$$\mathbf{M}^2 - tr(\mathbf{M})\mathbf{M} + det(\mathbf{M})\mathbf{I} = 0 \tag{1}$$

$$\implies \mathbf{M} - tr(\mathbf{M})\mathbf{I} + det(\mathbf{M})\mathbf{M}^{-1} = 0 \tag{2}$$

The given expression is

$$\mathbf{M} - \alpha\mathbf{I} - \beta\mathbf{M}^{-1} = 0 \tag{3}$$

On comparing, we get

$$\alpha = tr(\mathbf{M}) \ , \ \beta = -det(\mathbf{M}) \tag{4}$$

$$\alpha(\theta) = \sin^4(\theta) + \cos^4(\theta) = 1 - 2\sin^2(\theta)\cos^2(\theta) \tag{5}$$

$$\implies \alpha = 1 - \sin^2(2\theta)/2 \tag{6}$$

$$\alpha^* = min(\alpha(\theta)) = 1 - 1/2 = \frac{1}{2}, \tag{7}$$

$$\left(\because \text{ for minimizing } \alpha, \ \sin^2(2\theta) \text{ should be maximum}\right)$$

$$\beta(\theta) = -det(\mathbf{M}) = -\left(\sin^4(\theta)\cos^4(\theta) + \sin^2(\theta)\cos^2(\theta) + 2\right) \tag{8}$$

$$\implies \beta = -\left((\sin(2\theta)/2)^4 + (\sin(2\theta)/2)^2 + 2\right) \tag{9}$$

$$\beta^* = -\left((1/2)^4 + (1/2)^2 + 2\right) = - -\frac{37}{16} \tag{10}$$

$$\left(\because \text{ for minimizing } \beta, \ \sin^2(2\theta) \text{ should be maximum}\right)$$

Now,

$$\alpha^* + \beta^* = \frac{1}{2} - \frac{37}{16} = -\frac{29}{16} \tag{11}$$

Thus, the correct option is option 4).
We get α^*,β^* at $\theta = \frac{\pi}{4}$, substituting this in $\mathbf{M}$, we get

$$\mathbf{M} = \begin{pmatrix} \left(\frac{1}{\sqrt{2}}\right)^4 & -1-\left(\frac{1}{\sqrt{2}}\right)^2 \\ 1+\left(\frac{1}{\sqrt{2}}\right)^2 & \left(\frac{1}{\sqrt{2}}\right)^4 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{-3}{2} \\ \frac{3}{2} & \frac{1}{4} \end{pmatrix} \tag{12}$$

Let us draw graphs to find α^*,β^*:

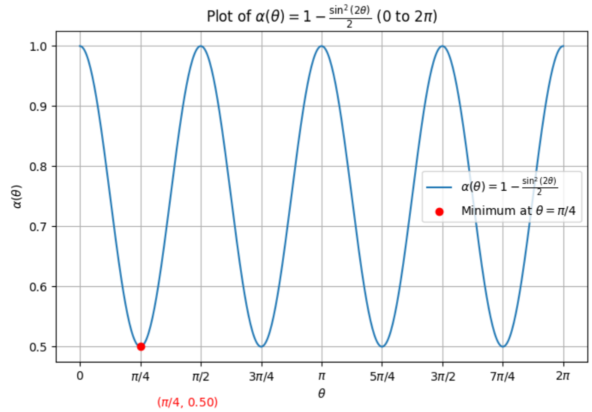

Fig. 1: Graph for $\alpha(\theta)$

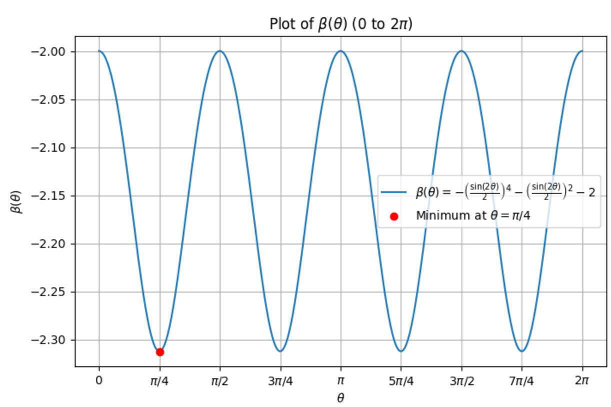

Fig. 2: Graph for $\beta(\theta)$