4.13.41

EE25BTECH11036 - M Chanakya Srinivas

Problem

Find the area of the parallelogram formed by the lines

$$y = mx, \quad y = mx + 1, \quad y = nx, \quad y = nx + 1$$

.

Solution

Step 1: Represent lines in parametric vector form

$$\mathbf{r}_1 = \kappa_1 \begin{pmatrix} 1 \\ m \end{pmatrix}, \qquad \mathbf{r}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \kappa_2 \begin{pmatrix} 1 \\ m \end{pmatrix}, \tag{1}$$

$$\mathbf{r}_3 = \mu_1 \begin{pmatrix} 1 \\ n \end{pmatrix}, \qquad \mathbf{r}_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mu_2 \begin{pmatrix} 1 \\ n \end{pmatrix}. \tag{2}$$

Here, $\mathbf{r}_1, \mathbf{r}_2$ represent the pair of lines with slope m, and $\mathbf{r}_3, \mathbf{r}_4$ represent the pair of lines with slope n.

Step 2: Compute vertices of parallelogram using intersection

Intersection of $\mathbf{r}_1$ and $\mathbf{r}_3$:

$$\mathbf{P} : \kappa_1 \begin{pmatrix} 1 \\ m \end{pmatrix} = \mu_1 \begin{pmatrix} 1 \\ n \end{pmatrix} \quad \Rightarrow \quad \mathbf{P} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Intersection of $\mathbf{r}_2$ and $\mathbf{r}_3$:

$$\mathbf{Q} : \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \kappa_2 \begin{pmatrix} 1 \\ m \end{pmatrix} = \mu_1 \begin{pmatrix} 1 \\ n \end{pmatrix} \quad \Rightarrow \begin{pmatrix} \kappa_2 \\ 1 + m\kappa_2 \end{pmatrix} = \begin{pmatrix} \mu_1 \\ n\mu_1 \end{pmatrix}$$

Solve augmented matrix form :

$$\underbrace{\begin{pmatrix} 1 & -1 \\ m & -n \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} \kappa_2 \\ \mu_1 \end{pmatrix}}_{\mathbf{z}} = \underbrace{\begin{pmatrix} 0 \\ -1 \end{pmatrix}}_{\mathbf{b}}$$

Step 3: Row-reduction / consistency

$$R_2 \to R_2 - mR_1 : \qquad (m - n)\mu_1 = m - 1 \quad \Rightarrow \quad \mu_1 = \frac{1}{m - n}$$

$$R_1 \to R_1 : \qquad \kappa_2 - \mu_1 = 0 \quad \Rightarrow \quad \kappa_2 = \frac{1}{m - n}$$

Hence

$$\mathbf{Q} = \begin{pmatrix} -1/(m-n) \\ -m/(m-n) \end{pmatrix}$$

Similarly, intersections give the other vertices

$$\mathbf{R} = \begin{pmatrix} 1/(m-n) \\ m/(m-n) \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Step 4: Area using vector cross product (Chapter 4 vector style)

$$\mathbf{PQ} = \mathbf{Q} - \mathbf{P} = \begin{pmatrix} -1/(m-n) \\ -m/(m-n) \end{pmatrix}, \tag{3}$$

$$\mathbf{PR} = \mathbf{R} - \mathbf{P} = \begin{pmatrix} 1/(m-n) \\ m/(m-n) \end{pmatrix}, \tag{4}$$

$$\text{Area} = \left\| \mathbf{PQ} \times \mathbf{PR} \right\| \tag{5}$$

$$= \left| \left(-\frac{1}{m-n}\right)\left(\frac{m}{m-n}\right) - \left(-\frac{m}{m-n}\right)\left(\frac{1}{m-n}\right) \right| \tag{6}$$

$$= \frac{1}{|m-n|} \tag{7}$$

Answer

$$\boxed{\text{Area of the parallelogram} = \frac{1}{|m-n|}}$$

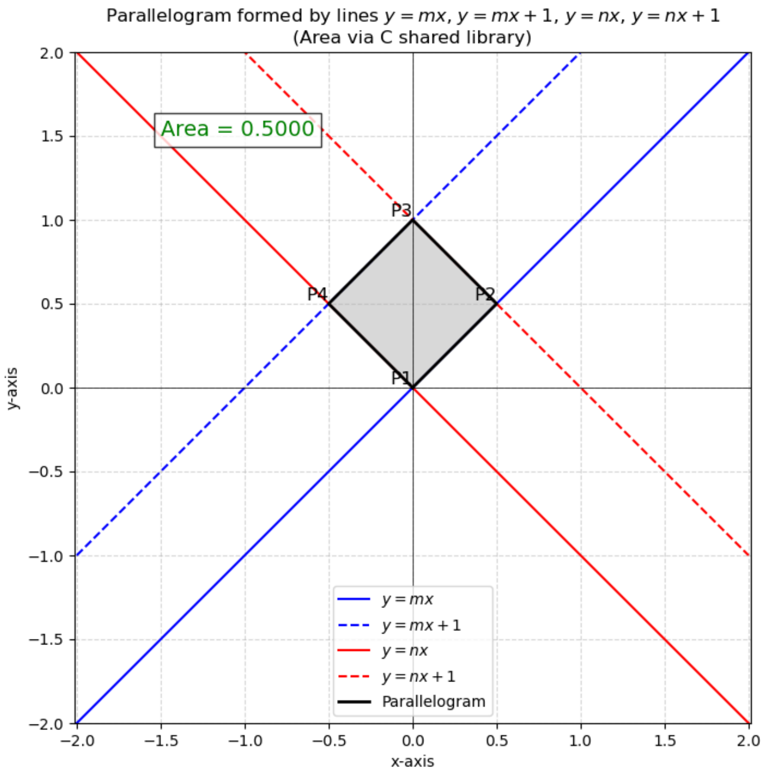
Parallelogram formed by lines $y = mx$, $y = mx + 1$, $y = nx$, $y = nx + 1$
(Area via C shared library)
Area = 0.5000
P1
P2
P3
P4
$y = mx$
$y = mx + 1$
$y = nx$
$y = nx + 1$
Parallelogram
x-axis
y-axis
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0

Fig. 1

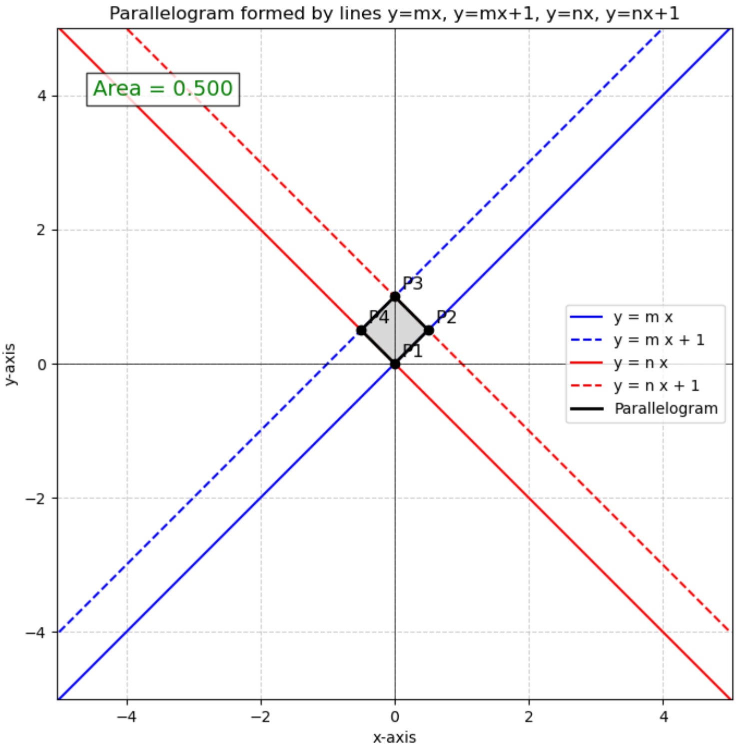
Parallelogram formed by lines y=mx, y=mx+1, y=nx, y=nx+1
Area = 0.500
P1
P2
P3
P4
y = m x
y = m x + 1
y = n x
y = n x + 1
Parallelogram
x-axis
y-axis
−4
−2
0
2
4

Fig. 2