Presentation - Matgeo

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EE1030 - Matrix Theory

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Problem Statement

Problem 12.318 Let V be the vector space of all real polynomials of degree at most 20. Define the subspaces

$$W_1 = \{p \in V : p(1) = p(\tfrac{1}{2}) = p(5) = p(7) = 0\}, \tag{1.1}$$

$$W_2 = \{p \in V : p(\tfrac{1}{2}) = p(3) = p(4) = p(7) = 0\}. \tag{1.2}$$

Find $\dim(W_1 \cap W_2)$.

Description of Variables used

Symbol	Description
$p(x)$	Polynomial of degree ≤ 20
c_i	Coefficients of $p(x)$
a	Point of evaluation (root condition)
A	Constraint matrix from evaluations

Table

Theoretical Solution

Definitions

Vector Space: A set V together with two operations (vector addition and scalar multiplication) is called a vector space over the field $\mathbb{R}$ if for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and scalars $a, b \in \mathbb{R}$, the following conditions hold:

Closure under addition: $\mathbf{u} + \mathbf{v} \in V$.

Commutativity: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

Associativity: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

Existence of zero vector: $\exists\, \mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.

Existence of additive inverse: $\forall \mathbf{u} \in V$, $\exists (-\mathbf{u}) \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

Closure under scalar multiplication: $a\mathbf{u} \in V$.

Distributivity: $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ and $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$.

Compatibility: $a(b\mathbf{u}) = (ab)\mathbf{u}$.

Identity: $1 \cdot \mathbf{u} = \mathbf{u}$.

Theoretical Solution

A subset $W \subseteq V$ is called a subspace of V if:

- $\mathbf{0} \in W$ (contains the zero vector),
- If $\mathbf{u}, \mathbf{v} \in W$, then $\mathbf{u} + \mathbf{v} \in W$ (closed under addition),
- If $\mathbf{u} \in W$ and $\alpha \in \mathbb{R}$, then $\alpha\mathbf{u} \in W$ (closed under scalar multiplication).

Dimension of a Subspace: The dimension of a subspace W of V is the number of vectors in a basis of W, i.e.,

$$\dim(W) = \text{number of linearly independent vectors that span } W.$$

Theoretical Solution

Step 1: Represent the polynomial

$$p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{20} x^{20}, \tag{2.1}$$

$$\mathbf{c} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{20} \end{pmatrix} \in \mathbb{R}^{21}. \tag{2.2}$$

Step 2: Each condition $p(a) = 0$ gives

$$p(a) = \begin{pmatrix} 1 & a & a^2 & \cdots & a^{20} \end{pmatrix} \mathbf{c} = 0. \tag{2.3}$$

Step 3: For the intersection $W_1 \cap W_2$, the polynomial must vanish at

$$\{1, \tfrac{1}{2}, 5, 7, 3, 4\}.$$

Theoretical Solution

Thus we obtain the matrix equation

$$A\mathbf{c} = \mathbf{0}, \quad \text{where} \tag{2.4}$$

$$A = \begin{pmatrix} 1 & 1 & 1^2 & \cdots & 1^{20} \\ 1 & \frac{1}{2} & (\frac{1}{2})^2 & \cdots & (\frac{1}{2})^{20} \\ 1 & 5 & 5^2 & \cdots & 5^{20} \\ 1 & 7 & 7^2 & \cdots & 7^{20} \\ 1 & 3 & 3^2 & \cdots & 3^{20} \\ 1 & 4 & 4^2 & \cdots & 4^{20} \end{pmatrix}. \tag{2.5}$$

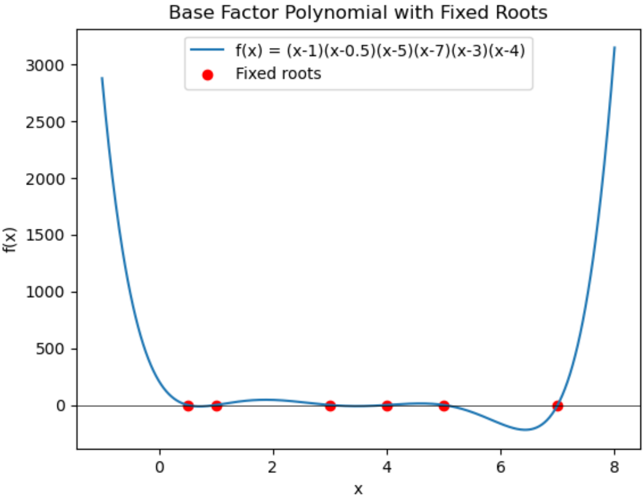
Step 4: The system $A\mathbf{c} = \mathbf{0}$ is a homogeneous system with 21 unknowns and 6 independent equations. Hence the number of free variables is

$$21 - 6 = 15. \tag{2.6}$$

Final Answer:

$$\dim(W_1 \cap W_2) = 15 \tag{2.7}$$

Plot


Base Factor Polynomial with Fixed Roots
f(x) = (x-1)(x-0.5)(x-5)(x-7)(x-3)(x-4)
Fixed roots
3000
2500
2000
1500
1000
500
0
f(x)
0
2
4
6
8
x

Figure

Code - C

```
#include <stdio.h>

// Base factor polynomial f(x) = (x-1)(x-0.5)(x-5)(x-7)(x-3)(x-4)
double base_factor(double x) {
    return (x-1.0)*(x-0.5)*(x-5.0)*(x-7.0)*(x-3.0)*(x-4.0);
}
```

Code - Python(with shared C code)

The code to obtain the required plot is

```
import ctypes
import numpy as np
import matplotlib.pyplot as plt

# Load the compiled C library
lib = ctypes.CDLL("./poly.so")
lib.base_factor.restype = ctypes.c_double

# Define a Python wrapper around the C function
def f(x):
    return lib.base_factor(ctypes.c_double(x))

# Generate values
xs = np.linspace(-1, 8, 600)
ys = [f(x) for x in xs]
```

Code - Python(with shared C code)

```
# Known fixed roots
roots = np.array([1.0, 0.5, 5.0, 7.0, 3.0, 4.0])

# Plot
plt.plot(xs, ys, label="f(x) = (x-1)(x-0.5)(x-5)(x-7)(x-3)(x-4)")
plt.scatter(roots, np.zeros_like(roots), color="red", label="Fixed roots")
plt.axhline(0, color="black", linewidth=0.5)
plt.xlabel("x")
plt.ylabel("f(x)")
plt.title("Base Factor Polynomial with Fixed Roots")
plt.legend()
plt.savefig("poly_dim.png")
plt.show()
```

Code - Python only

```
import numpy as np
import matplotlib.pyplot as plt

# Define the base factor polynomial
def f(x):
    return (x-1.0)*(x-0.5)*(x-5.0)*(x-7.0)*(x-3.0)*(x-4.0)

# Range of x values
xs = np.linspace(-1, 8, 600)
ys = f(xs)

# The six fixed roots
roots = np.array([1.0, 0.5, 5.0, 7.0, 3.0, 4.0])

# Plot the curve
plt.plot(xs, ys, label="f(x) = (x-1)(x-0.5)(x-5)(x-7)(x-3)(x-4)")
```

Code - Python only

```
# Mark the roots on the x−axis
plt.scatter(roots, np.zeros_like(roots), color="red", zorder=5, label="
    Fixed roots")

# Draw x−axis
plt.axhline(0, color="black", linewidth=0.8)

# Labels and title
plt.xlabel("x")
plt.ylabel("f(x)")
plt.title("Base Factor Polynomial with Fixed Roots")
plt.legend()
plt.savefig("new_poly_dim.png")
plt.show()
```