5.13.1

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Problem

If the system of linear equations

$$x + 2ay + az = 0 \tag{0.1}$$

$$x + 3by + bz = 0 \tag{0.2}$$

$$x + 4cy + cz = 0 \tag{0.3}$$

has a non-zero solution, then a, b, c

a) satisfy $a + 2b + 3c = 0$
b) are in A.P.
c) are in G.P.
d) are in H.P.

Solution

We write the system in matrix form:

$$\mathbf{M}_0 = \begin{bmatrix} 1 & 2a & a \\ 1 & 3b & b \\ 1 & 4c & c \end{bmatrix}$$

Apply row operations. First, subtract Row 1 from Rows 2 and 3:

$$\mathbf{M}_1 = \begin{bmatrix} 1 & 2a & a \\ 0 & 3b - 2a & b - a \\ 0 & 4c - 2a & c - a \end{bmatrix}$$

Now subtract Row 2 from Row 3:

$$\mathbf{M}_2 = \begin{bmatrix} 1 & 2a & a \\ 0 & 3b - 2a & b - a \\ 0 & 4c - 3b & c - b \end{bmatrix}$$

For a non-zero solution to exist, the rows must be linearly dependent. So Row 3 must be a scalar multiple of Row 2:

$$\frac{4c - 3b}{3b - 2a} = \frac{c - b}{b - a}$$

Cross-multiplying:

$$(4c - 3b)(b - a) = (c - b)(3b - 2a)$$

Expanding both sides:

$$4bc - 4ac - 3b^2 + 3ab = 3bc - 2ac - 3b^2 + 2ab \tag{4.1}$$

Cancel $-3b^2$ and rearrange:

$$4bc - 4ac + 3ab = 3bc - 2ac + 2ab$$

Bring all terms to one side:

$$(4bc - 3bc) + (3ab - 2ab) + (-4ac + 2ac) = 0 \Rightarrow bc + ab - 2ac = 0$$

Correct Condition

$$\boxed{ab + bc = 2ac}$$

Verification of Option d

If a, b, c are in H.P., then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.:

$$\frac{2}{b} = \frac{1}{a} + \frac{1}{c}$$

Multiply both sides by abc:

$$2ac = bc + ab \Rightarrow ab + bc = 2ac$$

Which matches our derived condition:

$$\boxed{ab + bc = 2ac}$$

Thus, option d) is correct.

Final Answer

$$\boxed{\text{Option d) } a, b, c \text{ are in H.P.}}$$