2.10.55

EE25BTECH11026-Harsha

Question:
The edges of a parallelopiped are of unit length and are parallel to non-coplanar unit vectors $\hat{a}, \hat{b}, \hat{c}$ such that $\hat{a}.\hat{b} = \hat{b}.\hat{c} = \hat{c}.\hat{a} = \frac{1}{2}$. Then, the volume of the parallelopiped is

1) $\frac{1}{\sqrt{2}}$
2) $\frac{1}{2\sqrt{2}}$
3) $\frac{\sqrt{3}}{2}$
4) $\frac{1}{\sqrt{3}}$

Solution:
Let us solve the given equation theoretically and then verify the solution computationally.

According to the question,the edges of the parallelopiped are parallel to the unit vectors $\hat{a}, \hat{b}, \hat{c}$ and

$$\hat{a}^T \hat{b} = \hat{b}^T \hat{c} = \hat{c}^T \hat{a} = \frac{1}{2}$$

As we know that the volume of parallelopiped is given by

$$V = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$

and

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}][\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^T = \mathbf{G}$$

where $\mathbf{G}$ is the Gram Matrix.

$$\therefore \mathbf{G} = \begin{pmatrix} \hat{a}^T\hat{a} & \hat{a}^T\hat{b} & \hat{a}^T\hat{c} \\ \hat{b}^T\hat{a} & \hat{b}^T\hat{b} & \hat{b}^T\hat{c} \\ \hat{c}^T\hat{a} & \hat{c}^T\hat{b} & \hat{c}^T\hat{c} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}$$

For calculating the det($\mathbf{G}$), we can use the concept of eigen values.

Eigen values are those scalars which satisfies the following condition, For any non-zero eigen-vector $\mathbf{v}$ and coefficient matrix $\mathbf{M}$,

$$\mathbf{M}\mathbf{v} = \lambda\mathbf{v} \ , where\ \lambda\ is\ an\ eigen\ value.$$

$$\mathbf{G} = (1-\rho)\mathbf{I}\ + \rho\ \mathbf{1}\mathbf{1}^T,\ where\ \rho = \frac{1}{2}\ and\ \mathbf{1} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

Let $\mathbf{1}\mathbf{1}^T = \mathbf{J}$. As we could see that the eigen-vector of $\mathbf{J}$ is $\mathbf{1}$ and by the rule,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = 3\mathbf{1}$$

So, 3 is a eigen value of $\mathbf{J}$. Also, we know that the sum of eigen values is equal to trace of a matrix,we can say that the sum of the other eigen values would be 0. Also, we know that any orthogonal vector to $\mathbf{1}$,say $\begin{pmatrix} -1 & 1 & 0 \end{pmatrix}^T$,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \mathbf{0}$$

yields 0.Thus we can say that 0 is also one of the eigen value of J. As sum of the other eigen values other than 3 is zero, the other eigen value must be zero.

$$\therefore eigen\ values\ of\ \mathbf{J}\ are\ \{3, 0, 0\}$$

Modifying the above equation on $\mathbf{G}$,

$$\therefore \mathbf{Gv} = \frac{1}{2}\mathbf{Iv} + \frac{1}{2}\mathbf{Jv}$$

$$\implies \mathbf{Gv} = \frac{(1+\mu)}{2}\mathbf{v}$$

where μ is the eigen value of $\mathbf{J}$. Here the eigen value of $\mathbf{G}$ is $\frac{1+\mu}{2}$ and substituting the obtained eigen values of $\mathbf{J}$ in this equation, we get the eigen values of $\mathbf{G}$ to be $\{2, \frac{1}{2}, \frac{1}{2}\}$

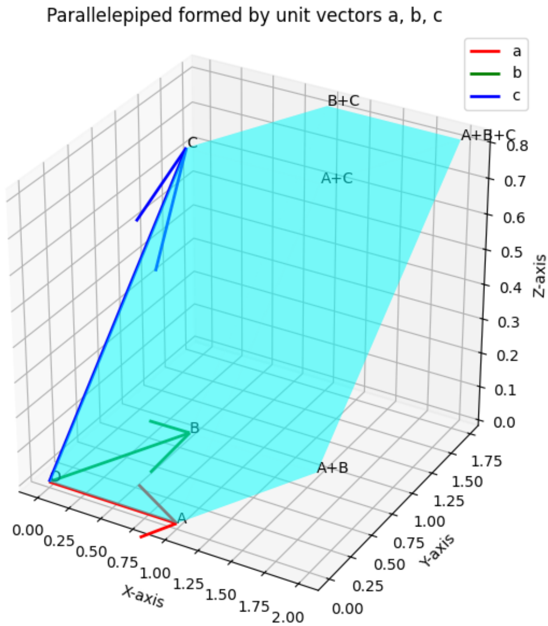
As we know that for eigen values of $\mathbf{G}$ being $\{\mu_1, \mu_2, \mu_3\}$,

$$det(\mathbf{G}) = \mu_1\mu_2\mu_3$$

$$\therefore det(\mathbf{G}) = 2 \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}$$

$$\implies V = \sqrt{det(\mathbf{G})} = \frac{1}{\sqrt{2}}\ units$$

From the figure, taking an example of vectors $\mathbf{a}$ and $\mathbf{b}$,it is clearly verified that the theoretical solution matches with the computational solution.



Parallelepiped formed by unit vectors a, b, c
a
b
c
B+C
A+B+C
C
A+C
B
A+B
O
A
X-axis
Y-axis
Z-axis
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8