10.6.1

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Question:

Draw a circle of radius 2.5cm. Take a point P outside the circle at a distance of 7cm from the center. Then construct a pair of tangents to the circle from point P.

Solution:

We first derive the formula for the chord of contact from the general tangent equation.

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0$$

The equation of the tangent at a point of contact $\mathbf{q}$ is:

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^\top \mathbf{x} + \mathbf{u}^\top \mathbf{q} + f = 0 \tag{1}$$

Since the tangent passes through the external point $\mathbf{h}$:

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^\top \mathbf{h} + \mathbf{u}^\top \mathbf{q} + f = 0 \tag{2}$$

$$\mathbf{h}^\top \mathbf{V}\mathbf{q} + \mathbf{u}^\top \mathbf{q} + \mathbf{u}^\top \mathbf{h} + f = 0 \tag{3}$$

The previous equation shows that any point of contact $\mathbf{q}$ lies on the following line, the chord of contact:

$$(\mathbf{V}\mathbf{h} + \mathbf{u})^\top \mathbf{x} + \mathbf{u}^\top \mathbf{h} + f = 0 \tag{4}$$

For the given circle, the circle is centered at the origin, so its conic parameters are:

$$\mathbf{V} = \mathbf{I}, \ \mathbf{u} = \mathbf{0}, \ f = -r^2, \ \mathbf{h} = d\mathbf{e_1} \tag{5}$$

Substituting these into (4):

$$\left(\mathbf{I}\left(d\mathbf{e_1}\right)+\mathbf{0}\right)^\top \mathbf{x}+\mathbf{0}^\top\left(d\mathbf{e_1}\right)-r^2=0 \tag{6}$$

$$\left(d\mathbf{e_1}\right)^\top \mathbf{x}-r^2=0 \tag{7}$$

This line, L, contains the points of contact. Its parametric form is $\mathbf{x}=\mathbf{h_L}+\kappa\mathbf{m_L}$.

$$dx-r^2=0 \implies \mathbf{h_L}=\frac{r^2}{d}\mathbf{e_1},\ \mathbf{m_L}=\mathbf{e_2} \tag{8}$$

The points of contact are the intersection of line L with the circle $g\left(\mathbf{x}\right)=\mathbf{x}^\top\mathbf{x}-r^2=0$. We use the intersection formula for the parameter κ:

$$\kappa_{1,2}=\frac{-\mathbf{m_L}^\top\left(\mathbf{V}\mathbf{h_L}+\mathbf{u}\right)\pm\sqrt{\left(\mathbf{m_L}^\top\left(\mathbf{V}\mathbf{h_L}+\mathbf{u}\right)\right)^2-\left(\mathbf{m_L}^\top\mathbf{V}\mathbf{m_L}\right)g\left(\mathbf{h_L}\right)}}{\mathbf{m_L}^\top\mathbf{V}\mathbf{m_L}} \tag{9}$$

Calculating the terms with $\mathbf{V}=\mathbf{I},\mathbf{u}=\mathbf{0}$:

$$\mathbf{m_L}^\top\mathbf{V}\mathbf{m_L}=\mathbf{e_2}^\top\mathbf{I}\mathbf{e_2}=1 \tag{10}$$

$$\mathbf{m_L}^\top\left(\mathbf{V}\mathbf{h_L}+\mathbf{u}\right)=\mathbf{e_2}^\top\left(\mathbf{I}\frac{r^2}{d}\mathbf{e_1}+\mathbf{0}\right)=0 \tag{11}$$

$$g\left(\mathbf{h_L}\right)=\left(\frac{r^2}{d}\mathbf{e_1}\right)^\top\left(\frac{r^2}{d}\mathbf{e_1}\right)-r^2=\frac{r^4}{d^2}-r^2 \tag{12}$$

Substituting these into (9),

$$\kappa=\frac{0\pm\sqrt{0-1\left(\frac{r^4}{d^2}-r^2\right)}}{1}=\pm\sqrt{r^2-\frac{r^4}{d^2}}=\pm\frac{r}{d}\sqrt{d^2-r^2} \tag{13}$$

The points of contact are $\mathbf{q}=\mathbf{h_L}+\kappa\mathbf{m_L}$.

$$\mathbf{q}=\frac{r^2}{d}\mathbf{e_1}\pm\frac{r}{d}\sqrt{d^2-r^2}\mathbf{e_2} \tag{14}$$

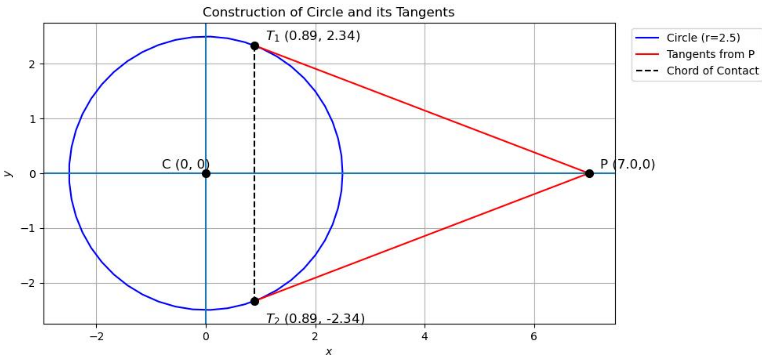
Substituting the given values $r = 2.5$ and $d = 7$:

$$\mathbf{q} = \frac{(2.5)^2}{7}\mathbf{e_1} \pm \frac{2.5}{7}\sqrt{7^2 - (2.5)^2}\mathbf{e_2} \tag{15}$$

$$= \frac{6.25}{7}\mathbf{e_1} \pm \frac{2.5}{7}\sqrt{42.75}\mathbf{e_2} \tag{16}$$

The coordinates of the two points of contact are:

$$\mathbf{q_{1,2}} = \begin{pmatrix} \frac{25}{28} \\ \pm\frac{2.5\sqrt{42.75}}{7} \end{pmatrix} \tag{17}$$



Plot