4.11.39

EE25BTECH11062 - Vivek K Kumar

Question:
Find the area of the region bounded by the lines $3x - 2y + 1 = 0$, $2x + 3y - 21 = 0$ and $x - 5y + 9 = 0$

Solution:

Point	Value
$\mathbf{n_1}$	$\begin{pmatrix} 3 \\ -2 \end{pmatrix}$
$\mathbf{n_2}$	$\begin{pmatrix} 2 \\ 3 \end{pmatrix}$
$\mathbf{n_3}$	$\begin{pmatrix} 1 \\ -5 \end{pmatrix}$
$\mathbf{c_1}$	-1
$\mathbf{c_2}$	21
$\mathbf{c_3}$	-9

TABLE 0: Variables used

The given lines can be represented as

$$\mathbf{n_1}^\top \mathbf{x} = c_1 \tag{0.1}$$
$$\mathbf{n_2}^\top \mathbf{x} = c_2 \tag{0.2}$$
$$\mathbf{n_3}^\top \mathbf{x} = c_3 \tag{0.3}$$

Let the points of intersections of the given lines be represented as $\mathbf{A}, \mathbf{B}, \mathbf{C}$

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix}^\top \mathbf{A} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{0.4}$$

$$\begin{pmatrix} \mathbf{n_2} & \mathbf{n_3} \end{pmatrix}^\top \mathbf{B} = \begin{pmatrix} c_2 \\ c_3 \end{pmatrix} \tag{0.5}$$

$$\begin{pmatrix} \mathbf{n_3} & \mathbf{n_1} \end{pmatrix}^\top \mathbf{C} = \begin{pmatrix} c_3 \\ c_1 \end{pmatrix} \tag{0.6}$$

The area of the triangle can be then represented as

$$\frac{1}{2} \|\mathbf{A} - \mathbf{B}\| \, \|\mathbf{C} - \mathbf{B}\| \sqrt{1 - \left(\frac{\mathbf{n_2}^\top \mathbf{n_3}}{\|\mathbf{n_2}\| \, \|\mathbf{n_3}\|} \right)^2} \tag{0.7}$$

Solving for $\mathbf{A}, \mathbf{B}, \mathbf{C}$

$$\begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix} \mathbf{A} = \begin{pmatrix} -1 \\ 21 \end{pmatrix} \tag{0.8}$$

$$\implies \left(\begin{array}{cc|c} 3 & -2 & -1 \\ 2 & 3 & 21 \end{array}\right) \xleftrightarrow{R_2 \leftarrow R_2 - 2/3 R_1} \left(\begin{array}{cc|c} 3 & -2 & -1 \\ 0 & 13/3 & 65/3 \end{array}\right) \tag{0.9}$$

$$\mathbf{A} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \tag{0.10}$$

$$\begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix} \mathbf{B} = \begin{pmatrix} 21 \\ -9 \end{pmatrix} \tag{0.11}$$

$$\implies \left(\begin{array}{cc|c} 2 & 3 & 21 \\ 1 & -5 & -9 \end{array}\right) \xleftrightarrow{R_2 \leftarrow R_2 - 1/2 R_1} \left(\begin{array}{cc|c} 2 & 3 & 21 \\ 0 & -13/2 & -39/2 \end{array}\right) \tag{0.12}$$

$$\mathbf{B} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} \tag{0.13}$$

$$\begin{pmatrix} 1 & -5 \\ 3 & -2 \end{pmatrix} \mathbf{C} = \begin{pmatrix} -9 \\ -1 \end{pmatrix} \tag{0.14}$$

$$\implies \left(\begin{array}{cc|c} 1 & -5 & -9 \\ 3 & -2 & -1 \end{array}\right) \xleftrightarrow{R_2 \leftarrow R_2 - 3 R_1} \left(\begin{array}{cc|c} 1 & -5 & -9 \\ 0 & 13 & 26 \end{array}\right) \tag{0.15}$$

$$\mathbf{C} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{0.16}$$

Area of the triangle from the above equations is

$$\frac{1}{2} \left\| \begin{pmatrix} -3 & 2 \end{pmatrix}^\top \right\| \left\| \begin{pmatrix} -5 & -1 \end{pmatrix}^\top \right\| \sqrt{1 - \left(\frac{-13}{13\sqrt{2}} \right)^2} = \frac{13}{2} \tag{0.17}$$

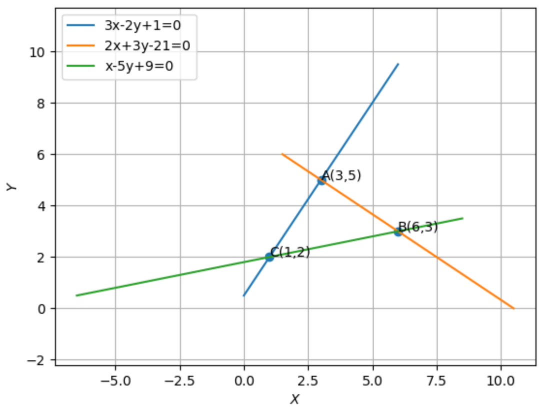

Fig. 0.1: Triangle enclosed by given lines