6.3.3

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Question

Find the shortest distance between the lines given by
$\mathbf{r} = (8+3\lambda)\hat{i} - (9+16\lambda)\hat{j} + (10+7\lambda)\hat{k}$ and
$\mathbf{r} = 15\hat{i} + 29\hat{j} + 5\hat{k} + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$

Solution:

The given lines can be written in vector form as

$$\mathbf{X} = \begin{pmatrix} 8 \\ -9 \\ 10 \end{pmatrix} + k \begin{pmatrix} 3 \\ -16 \\ 7 \end{pmatrix} \tag{1}$$

$$\mathbf{X} = \begin{pmatrix} 15 \\ 29 \\ 5 \end{pmatrix} + k \begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix} \tag{2}$$

$$\tag{3}$$

which are of the form

$$\mathbf{X_1} = \mathbf{A} + k_1\mathbf{m_1} \tag{4}$$

$$\mathbf{X_2} = \mathbf{B} + k_2\mathbf{m_2} \tag{5}$$

let $\mathbf{M} = \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix}$ and $\mathbf{K} = \begin{pmatrix} k_1 \\ -k_2 \end{pmatrix}$ be the values of k for which shortest distance between the two lines occurs

$$\implies \mathbf{M} = \begin{pmatrix} 3 & 3 \\ -16 & 8 \\ 7 & -5 \end{pmatrix} \text{ and } \mathbf{B} - \mathbf{A} = \begin{pmatrix} 7 \\ 38 \\ -5 \end{pmatrix} \tag{6}$$

$$\begin{pmatrix} \mathbf{M} & \mathbf{B} - \mathbf{A} \end{pmatrix} = \begin{pmatrix} 3 & 3 & 7 \\ -16 & 8 & 38 \\ 7 & -5 & -5 \end{pmatrix} \xleftrightarrow{R_2 \to R_2 + \frac{16}{3} \times R_1} \begin{pmatrix} 3 & 3 & 7 \\ 0 & 24 & \frac{226}{3} \\ 7 & -5 & -5 \end{pmatrix} \tag{7}$$

$$\xleftrightarrow{R_3 \to R_3 - \frac{7}{3} \times R_1} \begin{pmatrix} 3 & 3 & 7 \\ 0 & 24 & \frac{226}{3} \\ 0 & -12 & -\frac{64}{3} \end{pmatrix} \tag{8}$$

$$\xleftrightarrow{R_3 \to R_3 + \frac{1}{2} \times R_2} \begin{pmatrix} 3 & 3 & 7 \\ 0 & 24 & \frac{226}{3} \\ 0 & 0 & -\frac{49}{3} \end{pmatrix} \tag{9}$$

The above matrix now is in row echelon form.Rank of a matix in echelon form is number of non zero rows.so,The rank of the above matrix is 3
$\implies$ given lines are skew.

$$\implies \mathbf{M}^T\mathbf{M}\mathbf{K} = \mathbf{M}^T(\mathbf{B} - \mathbf{A}) \tag{10}$$

$$\begin{pmatrix} 3 & -16 & 7 \\ 3 & 8 & -5 \end{pmatrix} \begin{pmatrix} 3 & 3 \\ -16 & 8 \\ 7 & -5 \end{pmatrix} \mathbf{K} = \begin{pmatrix} 3 & -16 & 7 \\ 3 & 8 & -5 \end{pmatrix} \begin{pmatrix} 7 \\ 38 \\ -5 \end{pmatrix} \tag{11}$$

$$\implies \begin{pmatrix} 314 & -154 \\ -154 & 98 \end{pmatrix} \mathbf{K} = \begin{pmatrix} -622 \\ 350 \end{pmatrix} \tag{12}$$

The augmented matrix of above equation is given by

$$\left(\begin{array}{cc|c} 314 & -154 & -622 \\ -154 & 98 & 350 \end{array}\right) \xleftrightarrow{R_1 \to R_1 + 2R_2} \left(\begin{array}{cc|c} 6 & 42 & 78 \\ -154 & 98 & 350 \end{array}\right) \tag{13}$$

$$\xleftrightarrow{R_2 \to R_2 + \frac{77}{3} \times R_2} \left(\begin{array}{cc|c} 6 & 42 & 78 \\ 0 & 1176 & 2352 \end{array}\right) \tag{14}$$

$$\xleftrightarrow{R_1 \to \frac{1}{6} \times R_1} \left(\begin{array}{cc|c} 1 & 7 & 13 \\ 0 & 1176 & 2352 \end{array}\right) \tag{15}$$

$$\text{On back substitution we get,} \tag{16}$$

$$\mathbf{K} = \begin{pmatrix} k_1 \\ -k_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{17}$$

$$\implies \mathbf{X_1} = \begin{pmatrix} 5 \\ 7 \\ 3 \end{pmatrix} \text{ and } \mathbf{X_2} = \begin{pmatrix} 9 \\ 13 \\ 15 \end{pmatrix} \tag{18}$$

$$\tag{19}$$

The minimum distance between the lines is given by

$$\|\mathbf{X_2} - \mathbf{X_1}\| = \left\| \begin{pmatrix} 4 \\ 6 \\ 12 \end{pmatrix} \right\| = 14 \tag{20}$$

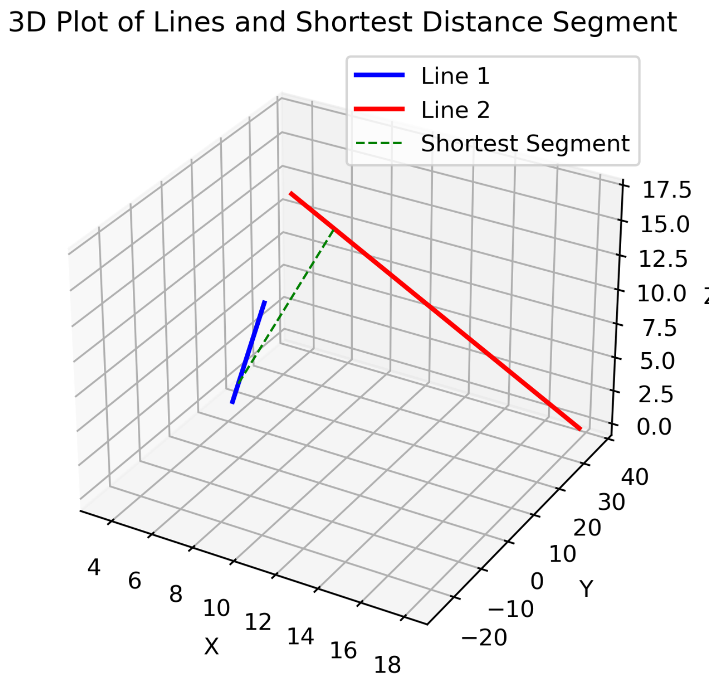

Fig. 0: Caption