MA: Mathematics

Duration : Three Hours   **Maximum Marks : 150**

Read the following instructions carefully.

1. This question paper contains 85 objective type questions. Q.1 to Q.20 carry *one* mark each and Q.21 to Q.85 carry two marks each.

2. Attempt all the questions.

3. Questions must be answered on **Objective Response Sheet (ORS)** by darkening the appropriate bubble (marked A, B, C, D) using HB pencil against the question number on the left hand side of the **ORS**. *Each question has only one correct answer.*

4. Wrong answers will carry **NEGATIVE** marks. In Q.1 to Q.20, **0.25** mark will be deducted for each wrong answer. In Q.21 to Q.76, Q.78, Q.80, Q.82 and in Q.84, **0.5** mark will be deducted for each wrong answer. However, there is no negative marking in Q.77, Q.79, Q.81, Q.83 and in Q.85. More than one answer bubbled against a question will be taken as an incorrect response. Unattempted questions will not carry any marks.

5. Write your registration number, your name and name of the examination centre at the specified locations on the right half of the **ORS**.

6. Using HB pencil, darken the appropriate bubble under each digit of your registration number and the letters corresponding to your paper code.

7. Calculator is allowed in the examination hall.

8. Charts, graph sheets or tables are NOT allowed in the examination hall.

9. Rough work can be done on the question paper itself. Additionally blank pages are given at the end of the question paper for rough work.

10. This question paper contains **24** printed pages including pages for rough work. Please check all pages and report, if there is any discrepancy.

Notations and Definitions used in the paper

- $\mathbb{R}$: The set of real numbers.
- $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}, \ i = 1, 2, \ldots, n\}$
- $\mathbb{C}$: The set of complex numbers.
- ϕ : The empty set.
- For any subset E of X (or a topological space X):
    - $\bar{E}$: The closure of E in X.
    - E° : The interior of E in X.
    - E^c : The complement of E in X.
- $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$
- A^t : The transpose of a matrix A.

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Q.1   Consider $\mathbb{R}^2$ with the usual topology. Let $S = \{(x, y) \in \mathbb{R}^2 : x \text{ is an integer}\}$. Then S is

(A) open but NOT closed

(B) both open and closed

(C) neither open nor closed

(D) closed but NOT open

Q.2   Suppose $X = \{\alpha, \beta, \delta\}$. Let

$$\mathcal{T}_1 = \{\emptyset, X, \{\alpha\}, \{\alpha, \beta\}\} \quad \text{and} \quad \mathcal{T}_2 = \{\emptyset, X, \{\alpha\}, \{\beta, \delta\}\}.$$

Then

 (A) both $\mathcal{T}_1 \cap \mathcal{T}_2$ and $\mathcal{T}_1 \cup \mathcal{T}_2$ are topologies

 (B) neither $\mathcal{T}_1 \cap \mathcal{T}_2$ nor $\mathcal{T}_1 \cup \mathcal{T}_2$ is a topology

 (C) $\mathcal{T}_1 \cup \mathcal{T}_2$ is a topology but $\mathcal{T}_1 \cap \mathcal{T}_2$ is NOT a topology

 (D) $\mathcal{T}_1 \cap \mathcal{T}_2$ is a topology but $\mathcal{T}_1 \cup \mathcal{T}_2$ is NOT a topology

Q.3   For a positive integer n, let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} \frac{1}{4n+5} & \text{if } 0 \leq x \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{f_n(x)\}$ converges to zero

 (A) uniformly but NOT in L^1 norm

 (B) uniformly and also in L^1 norm

 (C) pointwise but NOT uniformly

 (D) in L^1 norm but NOT pointwise

Q.4   Let P_1 and P_2 be two projection operators on a vector space. Then

 (A) $P_1 + P_2$ is a projection if $P_1 P_2 = P_2 P_1 = 0$

 (B) $P_1 - P_2$ is a projection if $P_1 P_2 = P_2 P_1 = 0$

 (C) $P_1 + P_2$ is a projection

 (D) $P_1 - P_2$ is a projection

Q.5   Consider the system of linear equations

$$x + y + z = 3$$
$$x - y - z = 4$$
$$-5y + kz = 6$$

Then the value of k for which this system has an infinite number of solutions is

(A) $k = -5$ (B) $k = 0$ (C) $k = 1$ (D) $k = 3$

Q.6   Let
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ x & y & z \end{bmatrix}$$
and let $V = \{(x, y, z) \in \mathbb{R}^3 : \det(A) = 0\}$. Then the dimension of V equals:

(A)0   (B) 1   (C) 2   (D) 3

Q.7   Let $S = \{0\} \cup \left\{ \frac{1}{4n+7} : n = 1, 2, \dots \right\}$. Then the number of analytic functions which vanish only on S is:

(A) infinite   (B) 0   (C) 1   (D) 2

Q.8   It is given that $\sum_{n=0}^{\infty} a_n z^n$ converges at $z = 3 + i4$. Then the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ is:

(A) ≤ 5 (B) ≥ 5 (C) < 5 (D) > 5

Q.9   The value of α for which $G = \langle \alpha, 1, 3, 9, 19, 27 \rangle$ is a cyclic group under multiplication modulo 56 is:

(A) 5   (B) 15   (C) 25   (D) 35

Q.10   Consider $\mathbb{Z}_{24}$ as the additive group modulo 24. Then the number of elements of order 8 in the group $\mathbb{Z}_{24}$ is:

(A)1   (B) 2   (C) 3   (D) 4

Q.11   Define $f : \mathbb{R}^2 \to \mathbb{R}$ by
$$f(x, y) = \begin{cases} 1, & \text{if } xy = 0, \\ 2, & \text{otherwise.} \end{cases}$$
If $S = \{(x, y) : f \text{ is continuous at the point } (x, y)\}$, then:

(A) S is open   (B) S is connected
(C) $S = \emptyset$ (D) S is closed

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Q.12   Consider the linear programming problem

$$\text{Maximize } z = c_1 x_1 + c_2 x_2, \quad c_1, c_2 > 0,$$
$$\text{subject to}$$
$$x_1 + x_2 \leq 3$$
$$2x_1 + 3x_2 \leq 4$$
$$x_i \geq 0$$

Then:

(A) the primal has an optimal solution but the dual does NOT have an optimal solution
(B) both the primal and the dual have optimal solutions
(C) the dual has an optimal solution but the primal does NOT have an optimal solution
(D) neither the primal nor the dual have optimal solutions

Q.13   Let $f(x) = x^{10} + x - 1$, $x \in \mathbb{R}$ and let $x_k = k$, $k = 0, 1, 2, \ldots, 10$. Then the value of the divided difference

$$f[x_0, x_1, x_2, \ldots, x_{10}]$$

is:

(A) -1 (B) 0   (C) 1   (D) 10

Q.14   Let X, Y be jointly distributed random variables having the joint probability density function

$$f(x, y) = \begin{cases} 1, & \text{if } 0 < x + y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $P(Y \geq \max(X, 1 - X))$ is

(A) $\frac{1}{2}$ (B) 1   (C) $\frac{1}{4}$ (D) $\frac{1}{6}$

Q.15   Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed chi-square random variables, each having 4 degrees of freedom. Define

$$S_n = \sum_{i=1}^{n} X_i$$

If $\frac{S_n}{n} \to \mu$ as $n \to \infty$, then $\mu =$

(A) 8   (B) 16   (C) 24   (D) 32

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Q.16   Let $\{E_n : n = 1, 2, \dots\}$ be a decreasing sequence of Lebesgue measurable sets on $\mathbb{R}$ and let F be a Lebesgue measurable set on $\mathbb{R}$ such that $E_n \cap F = \emptyset$. Suppose that F has Lebesgue measure 2 and the Lebesgue measure of E_n equals $\dfrac{2n + 2}{3n + 1}$, $n = 1, 2, \dots$

Then the Lebesgue measure of the set $\left(\bigcap_{n=1}^{\infty} E_n\right) \cup F$ equals

(A) $\frac{5}{3}$ (B) 2   (C) $\frac{7}{3}$ (D) $\frac{8}{3}$

Q.17   The extremum for the variational problem

$$\int_0^{\frac{\pi}{8}} \left((y')^2 + 2yy' - 16y^2\right) dx, \quad y(0) = 0, \ y\left(\frac{\pi}{8}\right) = 1,$$

occurs for the curve
(A) $y = \sin(4x)$ (B) $y = \sqrt{2}\sin(2x)$
(C) $y = 1 - \cos(4x)$ (D) $y = \dfrac{1 - \cos(8x)}{2}$

Q.18   Suppose $y_p(x) = x\cos(2x)$ is a particular solution of

$$y'' + \alpha y = \sin(2x).$$

Then the constant α equals

(A) -4 (B) -2 (C) 2   (D) 4

Q.19   If $F(s) = \tan^{-1}(s) + k$ is the Laplace transform of some function $f(t)$, $t \geq 0$, then $k =$

(A) $-\pi$ (B) $\dfrac{\pi}{2}$ (C) 0   (D) $\dfrac{\pi}{2}$

Q.20   Let $S = \{(0, 1, 1), (1, 0, 1), (-1, 2, 1)\} \subseteq \mathbb{R}^3$. Suppose $\mathbb{R}^3$ is endowed with the standard inner product $\langle \cdot, \cdot \rangle$. Define $M = \{x \in \mathbb{R}^3 : \langle x, y \rangle = 0 \text{ for all } y \in S\}$. Then the dimension of M equals

(A) 0   (B) 1   (C) 2   (D) 3

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Q.21-Q.75 carry one mark each.

Q.21   Let X be an uncountable set and let

$$\tau = \{U \subseteq X : X \setminus U \text{ is countable or } X \setminus U \text{ is finite}\}.$$

Then the topological space (X, τ)

(A) is separable

(B) is Hausdorff

(C) has a countable basis

(D) has a countable basis at each point

Q.22   Suppose (X, τ) is a topological space. Let $\{S_\alpha\}_{\alpha \in A}$ be a sequence of subsets of X.
Then

(A) $(S_1 \bigcup S_2)^{"} = S_1^{"} \bigcup S_2^{"}$

(B) $(\bigcap S_{"})^{"} = \bigcap S_{"}$

(C) $\overline{\bigcup S_{"}} = \bigcup_\alpha \overline{S_{"}}$

(D) $\overline{S_1 \bigcup S_2} = \overline{S_1} \cup \overline{S_2}$

Q.23   Let (X, d) be a metric space. Consider the metric ρ on X defined by

$$\rho(x, y) = \min(d(x, y), 1), \quad x, y \in X.$$

Suppose τ and τ_1 are topologies on X defined by d and ρ respectively. Then

(A) τ_1 is a proper subset of τ_2 (B) τ_2 is a proper subset of τ_1
(C) neither τ_2 nor τ_1 is a subset of the other   (D) $\tau_1 = \tau_2$

Q.24   A basis of the vector space $W = \{(x, y, z, w) \in \mathbb{R}^4 : x+y+z = 0, y+z+w = 0, 2x+y-z+w = 0\}$
is
(A) $\{(1, 1, 1, 1), (2, 1, 1, 1)\}$ (B) $\{(1, -1, 0, 1), (0, 1, -1, 0)\}$
(C) $\{(1, 0, -1, 0), (2, 1, 1, 1)\}$ (D) $\{(1, 0, -1, 0), (0, 1, -1, 0)\}$

Q.25   Consider $\mathbb{R}^3$ with the standard inner product. Let

$$S = \{(1, 1, 1), (2, -1, 2), (-1, 2, 1)\}.$$

For a subset W of $\mathbb{R}^3$, let $L(W)$ denote the linear span of W in $\mathbb{R}^3$. Then an orthonormal set T with
$L(S) = L(T)$ is
(A) $\left\{\frac{1}{\sqrt{3}}(1, 1, 1), \frac{1}{\sqrt{6}}(1, 0, -2), \frac{1}{\sqrt{2}}(1, -1, 0)\right\}$ (B)$\{(0, 0, 0), (0, 1, 0), (0, 0, 1)\}$
(C) $\left\{\frac{1}{\sqrt{3}}(1, 1, 1), \frac{1}{\sqrt{2}}(1, 0, -1)\right\}$ (D) $\left\{\frac{1}{\sqrt{3}}(1, 1, 1), \frac{1}{\sqrt{2}}(1, -1, 0)\right\}$

Q.26 Let A be a 3×3 matrix. Suppose that the eigenvalues of A are $-1, 0, 1$ with respective eigenvectors $(1, -1, 0)^T$, $(1, 1, -2)^T$ and $(1, 1, 1)^T$. Then $6A$ equals

(A) $\begin{bmatrix} -1 & 5 & 2 \\ 5 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$
(B) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 5 & 3 \\ 5 & 1 & 3 \\ 3 & 3 & 3 \end{bmatrix}$
(D) $\begin{bmatrix} -3 & 9 & 0 \\ 9 & -3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

Q.27 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by

$$T((x, y, z)) = (x + y - z, x + y + z, y - z).$$

Then the matrix of the linear transformation T with respect to the ordered basis $B = \{(0, 1, 0), (0, 0, 1), (1, 0, 0)\}$ of $\mathbb{R}^3$ is

(A) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$
(B) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$
(D) $\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

Q.28 Let $Y(x) = (y_1(x), y_2(x))^T$ and let

$$A = \begin{bmatrix} -3 & 1 \\ k & -1 \end{bmatrix}.$$

Further, let S be the set of values of k for which all the solutions of the system of equations $Y'(x) = AY(x)$ tend to zero as $x \to \infty$. Then S is given by

(A) $\{k : k \leq -1\}$
(B) $\{k : k \leq 3\}$
(C) $\{k : k < -1\}$
(D) $\{k : k < 3\}$

Q.29 Let
$$u(x, y) = f(xe^y) + g(y^2 \cos y),$$

where f and g are infinitely differentiable functions. Then the partial differential equation of minimum order satisfied by u is

(A) $u_x + xu_{xx} = u_y$
(B) $u_y + xu_{xx} = xu_y$
(C) $u_y - xu_{xx} = u_x$
(D) $u_y - xu_{xx} = xu_y$

Q.30 Let C be the boundary of the triangle formed by the points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. Then the value of the line integral

$$\oint_C -2y\,dx + (3x - 4y^2)\,dy + (z^2 + 3y)\,dz$$

is

(A)0   (B) 1   (C) 2   (D) 4

Q.31 Let X be a complete metric space and let $E \subset X$. Consider the following statements:

(S_1) E is compact,

(S_2) E is closed and bounded,

(S_3) E is closed and totally bounded,

(S_4) Every sequence in E has a subsequence converging in E.

Which one of the above statements does **NOT** imply all the other statements?

(A)S_1 (B) S_2 (C) S_3 (D) S_4

Q.32 Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \sin(nx).$$

Then the series

(A) converges uniformly on $\mathbb{R}$

(B) converges pointwise but NOT uniformly on $\mathbb{R}$

(C) converges in L^1 norm to an integrable function on $[0, 2\pi]$ but does NOT converge uniformly on $\mathbb{R}$

(D) does NOT converge pointwise

Q.33 Let $f(z)$ be an analytic function. Then the value of

$$\int_0^{2\pi} f(e^{it}) \cos(t)\,dt$$

equals

(A)0   (B) $2\pi f(0)$ (C) $2\pi f'(0)$ (D) $\pi f'(0)$

Q.34 Let G_1 and G_2 be the images of the disc $\{z \in \mathbb{C} : |z + 1| < 1\}$ under the transformations

$$w = \frac{(1 - i)z + 2}{(1 + i)z + 2} \quad \text{and} \quad w = \frac{(1 + i)z + 2}{(1 - i)z + 2}$$

respectively. Then

(A) $G_1 = \{w \in \mathbb{C} : \mathrm{Im}(w) < 0\}$ and $G_2 = \{w \in \mathbb{C} : \mathrm{Im}(w) > 0\}$

(B) $G_1 = \{w \in \mathbb{C} : \mathrm{Im}(w) > 0\}$ and $G_2 = \{w \in \mathbb{C} : \mathrm{Im}(w) < 0\}$

(C) $G_1 = \{w \in \mathbb{C} : |w| > 2\}$ and $G_2 = \{w \in \mathbb{C} : |w| < 2\}$

(D) $G_1 = \{w \in \mathbb{C} : |w| < 2\}$ and $G_2 = \{w \in \mathbb{C} : |w| > 2\}$

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Q.35 Let $f(z) = 2^z - 2^{-z}$. Then the maximum value of $|f(z)|$ on the unit disc $D = \{z \in \mathbb{C} : |z| \le 1\}$ equals

(A) 1   (B) 2   (C) 3   (D) 4

Q.36 Let

$$f(z) = \frac{1}{z^2 - 3z + 2}$$

Then the coefficient of $\frac{1}{z}$ in the Laurent series expansion of $f(z)$ for $|z| > 2$ is

(A) 0   (B) 1   (C) 3   (D) 5

Q.37 Let $f : \mathbb{C} \to \mathbb{C}$ be an arbitrary analytic function satisfying $f(0) = 0$ and $f(1) = 2$. Then

(A) there exists a sequence $\{z_n\}$ such that $|z_n| > n$ and $|f(z_n)| > n$
(B) there exists a sequence $\{z_n\}$ such that $|z_n| > n$ and $|f(z_n)| < n$
(C) there exists a bounded sequence $\{z_n\}$ such that $|f(z_n)| > n$
(D) there exists a sequence $\{z_n\}$ such that $z_n \to 0$ and $f(z_n) \to 2$

Q.38 Define $f : \mathbb{C} \to \mathbb{C}$ by
$$f(z) = \begin{cases} 0, & \text{if } \mathrm{Re}(z) = 0 \text{ or } \mathrm{Im}(z) = 0, \\ \frac{1}{z}, & \text{otherwise.} \end{cases}$$
Then the set of points where f is analytic is

(A) $\{z : \mathrm{Re}(z) \ne 0 \text{ and } \mathrm{Im}(z) \ne 0\}$ (B) $\{z : \mathrm{Re}(z) \ne 0\}$
(C) $\{z : \mathrm{Re}(z) \ne 0 \text{ or } \mathrm{Im}(z) \ne 0\}$ (D) $\{z : \mathrm{Im}(z) \ne 0\}$

Q.39 Let $U(n)$ be the set of all positive integers less than n and relatively prime to n. Then $U(n)$ is a group under multiplication modulo n. For $n = 248$, the number of elements in $U(n)$ is

(A) 60   (B) 120   (C) 180   (D) 240

Q.40 Let $\mathbb{R}[x]$ be the polynomial ring in x with real coefficients and let $I = \langle x^2 + 1 \rangle$ be the ideal generated by the polynomial $x^2 + 1$ in $\mathbb{R}[x]$. Then

(A) I is a maximal ideal
(B) I is a prime ideal but NOT a maximal ideal
(C) I is NOT a prime ideal
(D) $\mathbb{R}[x]/I$ has zero divisors

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Q.41   Consider $\mathbb{Z}5$ and $\mathbb{Z}20$ as rings modulo 5 and 20, respectively. Then the number of homomorphisms $\varphi : \mathbb{Z}5 \to \mathbb{Z}20$ is

(A) 1   (B) 2   (C) 4   (D) 5

Let $\mathbb{Q}$ be the field of rational numbers and consider $\mathbb{Z}_2$ as a field modulo 2. Let

$$f(x) = x^3 - 9x^2 + 9x + 3.$$

Then $f(x)$ is

(A) irreducible over $\mathbb{Q}$ but reducible over $\mathbb{Z}_2$

(B) irreducible over both $\mathbb{Q}$ and $\mathbb{Z}_2$

(C) reducible over $\mathbb{Q}$ but irreducible over $\mathbb{Z}_2$

(D) educible over both $\mathbb{Q}$ and $\mathbb{Z}_2$

Q.42   Let $\mathbb{Q}$ be the field of rational numbers and consider $\mathbb{Z}_2$ as a field modulo 2. Let

$$f(x) = x^3 - 9x^2 + 9x + 3.$$

Then $f(x)$ is

(A) irreducible over $\mathbb{Q}$ but reducible over $\mathbb{Z}_2$

(B) irreducible over both $\mathbb{Q}$ and $\mathbb{Z}_2$

(C) reducible over $\mathbb{Q}$ but irreducible over $\mathbb{Z}_2$

(D) reducible over both $\mathbb{Q}$ and $\mathbb{Z}_2$

Q.43   Consider $\mathbb{Z}_5$ as a field modulo 5 and let

$$f(x) = x^4 + 4x^3 + 4x^2 + 4x + 1.$$

Then the zeros of $f(x)$ over $\mathbb{Z}_5$ are 1 and 3 with respective multiplicity

(A) 1 and 4

(B) 2 and 3

(C) 2 and 2

(D) 1 and 2

Q.44   Consider the Hilbert space

$$\ell^2 = \left\{ x = \{x_n\}; \ x_n \in \mathbb{R}, \ \sum x_n^2 < \infty \right\}.$$

Let

$$E = \left\{ x = \{x_n\} \mid |x_n| < \frac{1}{n} \text{ for all } n \right\}$$

be a subset of ℓ^2. Then

(A) $E^\circ = \left\{ x \mid |x_n| < \frac{1}{n} \text{ for all } n \right\}$

(B) $E^\circ = E$

(C) $E^\circ = \left\{ x \mid |x_n| < \frac{1}{n} \text{ for all but finitely many } n \right\}$

(D) $E^\circ = \emptyset$

Q.45   Let X and Y be normed linear spaces and let $T : X \to Y$ be a linear map. Then T is continuous if

(A) Y is finite dimensional

(B) X is finite dimensional

(C) T is one to one

(D) T is onto

Q.46   Let X be a normed linear space and let $E_1, E_2 \subseteq X$. Define

$$E_1 + E_2 = \{x + y : x \in E_1, y \in E_2\}.$$

Then $E_1 + E_2$ is:

 (A) open if E_1 or E_2 is open

 (B) NOT open unless both E_1 and E_2 are open

 (C) closed if E_1 or E_2 is closed

 (D) closed if both E_1 and E_2 are closed

Q.47   For each $a \in \mathbb{R}$, consider the linear programming problem:

Max. $z = x_1 + 2x_2 + 3x_3 + 4x_4$

subject to

$$ax_1 + 2x_2 \leq 1$$
$$x_1 + 2x_2 + 3x_3 \leq 2$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Let $S = \{a \in \mathbb{R} : \text{the given LP problem has a basic feasible solution}\}$. Then:
(A) $S = \emptyset$ (B) $S = \mathbb{R}$
(C) $S = (0, \infty)$ (D) $S = (-\infty, 0)$

Q.48   Consider the linear programming problem:

Max. $z = x_1 + 5x_2 + 3x_3$

subject to

$$2x_1 - 3x_2 + 5x_3 \leq 3$$
$$x_1 - x_2 \leq 5$$
$$x_1, x_2, x_3 \geq 0$$

Then the dual of this LP problem:

 (A) has a feasible solution but does NOT have a basic feasible solution

 (B) has a basic feasible solution

 (C) has infinite number of feasible solutions

 (D) has no feasible solution

Q.49   Consider a transportation problem with two warehouses and two markets. The warehouse capacities are $a_1 = 2$ and $a_2 = 4$, and the market demands are $b_1 = 3$ and $b_2 = 3$. Let x_{ij} be the quantity shipped from warehouse i to market j, and c_{ij} be the corresponding unit cost. Suppose that $c_{11} = 1$, $c_{21} = 1$, and $c_{22} = 2$. Then $(x_{11}, x_{12}, x_{21}, x_{22}) = (2, 0, 1, 3)$ is optimal for every:

(A) $c_{12} \in [1, 2]$ (B) $c_{12} \in [0, 3]$
(C) $c_{12} \in [1, 3]$ (D) $c_{12} \in [2, 4]$

Q.50 The smallest degree of the polynomial that interpolates the data

x	-2	-1	0	1	2	3
$f(x)$	-58	-21	-12	-13	-6	27

is:

(A) 3 (B) 4 (C) 5 (D) 6

Q.51 Suppose that x_n is sufficiently close to 3. Which of the following iterations $x_{n+1} = g(x_n)$ will converge to the fixed point $x = 3$?

(A) $x_{n+1} = -16 + 6x_n + \dfrac{3}{x_n}$

(B) $x_{n+1} = \sqrt{3 + 2x_n}$

(C) $x_{n+1} = \dfrac{3}{x_n} - \dfrac{x_n}{2}$

(D) $x_{n+1} = \dfrac{x_n^2 - 3}{2}$

Q.52 Consider the quadrature formula:

$$\int_{x_1}^{x_2} f(x)\,dx \approx \frac{1}{2}\left[f(x_1) + f(x_2)\right],$$

where x_1 and x_2 are quadrature points. Then the highest degree of the polynomial for which the above formula is exact equals:

(A) 1 (B) 2 (C) 3 (D) 4

Q.53 Let A, B and C be three events such that:

$$P(A) = 0.4, \quad P(B) = 0.5, \quad P(A \cup B) = 0.6, \quad P(C) = 0.6, \quad \text{and } P(A \cap B \cap C^c) = 0.1.$$

Then $P(A \cap B \cap C) =$

(A) $\dfrac{1}{2}$ (B) $\dfrac{1}{3}$ (C) $\dfrac{1}{4}$ (D) $\dfrac{1}{5}$

Q.54 Consider two identical boxes B_1 and B_2, where the box B_i $(i = 1, 2)$ contains $i + 1$ red and $5 - i + 1$ white balls. A fair die is cast. Let the number of dots shown on the top face of the die be N. If N is even or 5, then two balls are drawn with replacement from the box B_1; otherwise, two balls are drawn with replacement from the box B_2. The probability that the two drawn balls are of different colours is:

(A) $\dfrac{7}{25}$ (B) $\dfrac{9}{25}$ (C) $\dfrac{12}{25}$ (D) $\dfrac{16}{25}$

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Q.55 Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with

$$P(X_i = 1) = P(X_i = -1) = \frac{1}{2}.$$

Suppose for the standard normal random variable Z, $P(-0.1 < Z \leq 0.1) = 0.08$. If $S_n = \sum_{i=1}^{n} X_i$, then

$$\lim P \left(\frac{S_n}{\sqrt{n}} > \frac{n}{10} \right) =$$

(A) 0.42   (B) 0.46   (C) 0.50   (D) 0.54

Q.56 Let $X_1, X_2, \ldots, X_5$ be a random sample of size 5 from a population having standard normal distribution. Let

$$\bar{X} = \frac{1}{5} \sum_{i=1}^{5} X_i \quad \text{and} \quad T = \sum_{i=1}^{5} (X_i - \bar{X})^2.$$

Then $E(T^2 \bar{X}^2) =$

(A) 3   (B) 3.6   (C) 4.8   (D) 5.2

Q.57 Let $x_1 = 3.5$, $x_2 = 7.5$ and $x_3 = 5.2$ be observed values of a random sample of size three from a population having uniform distribution over the interval $(\theta, \theta + 5)$, where $\theta \in (0, \infty)$ is unknown and is to be estimated. Then which of the following is NOT a maximum likelihood estimate of θ?

(A) 2.4   (B) 2.7   (C) 3.0   (D) 3.3

Q.58 The value of

$$\int_0^1 \int_y^1 x^2 e^{x^2} \, dx \, dy$$

equals

(A) $\frac{1}{4}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) 1

Q.59

$$\lim_{n \to \infty} \left[(n+1) \int_0^1 x^n \ln(1+x) \, dx \right] =$$

(A) 0   (B) $\ln 2$ (C) $\ln 3$ (D) ∞

Q.60 Consider the function $f \colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^4, & \text{if } x \text{ is rational,} \\ 2x^4 - 1, & \text{if } x \text{ is irrational.} \end{cases}$$

Let S be the set of points where f is continuous. Then

(A) $S = \{1\}$ (B) $S = \{-1\}$ (C) $S = \{-1, 1\}$ (D) $S = \emptyset$

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Q.61   For a positive real number p, let $\{f_n : n = 1, 2, \dots\}$ be a sequence of functions defined on $[0, 1]$ by

$$f_n(x) = \begin{cases} n^{p+1}x, & 0 \le x \le \frac{1}{n} \\ \frac{1}{n^p}, & \frac{1}{n} < x \le 1. \end{cases}$$

Let $f(x) = \lim_{n \to \infty} f_n(x), \ x \in [0, 1]$. Then, on $[0, 1]$,

(A) f is Riemann integrable
(B) the improper integral $\int_0^1 f(x)dx$ converges for $p \ge 1$
(C) the improper integral $\int_0^1 f(x)dx$ converges for $p < 1$
(D) f_n converges uniformly

Q.62   Which of the following inequality is NOT true for $x \in \left[\frac{1}{4}, \frac{3}{4}\right]$

(A) $e^{-x} > \sum_{j=0}^{\infty} \frac{(-x)^j}{j!}$
(B) $e^{-x} < \sum_{j=0}^{\infty} \frac{(-x)^j}{j!}$
(C) $e^{-x} = \sum_{j=0}^{\infty} \frac{(-x)^j}{j!}$
(D) $e^{-x} > \sum_{j=0}^{10} \frac{(-x)^j}{j!}$

Q.63   Let $u(x, y)$ be the solution to the Cauchy problem

$$xu_x + u_y = 1, \quad u(x, 0) = 2\ln(x), \ x > 1.$$

Then $u(e, 1) =$

(A) -1 (B) 0 (C) 1 (D) e

Q.64   Suppose

$$y(x) = \lambda \int_0^{2\pi} y(t) \sin(x + t) \, dt, \ x \in [0, 2\pi]$$

has eigenvalues $\lambda = \frac{1}{\pi}$ and $\lambda = -\frac{1}{\pi}$ with corresponding eigenfunctions
$y_1(x) = \sin(x) + \cos(x)$ and $y_2(x) = \sin(x) - \cos(x)$, respectively. Then the integral equation

$$y(x) = f(x) + \frac{1}{\pi} \int_0^{2\pi} y(t) \sin(x + t) \, dt, \ x \in [0, 2\pi]$$

has a solution when $f(x) =$

(A) 1 (B) $\cos(x)$ (C) $\sin(x)$ (D) $1 + \sin(x) + \cos(x)$

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Q.65   Consider the Neumann problem

$$u_{xx} + u_{yy} = 0, \quad 0 < x < \pi, \ -1 < y < 1,$$

$$u_y(0, y) = u_y(\pi, y) = 0,$$

$$u_y(x, -1) = 0, \quad u_y(x, 1) = \alpha + \beta \sin(x).$$

The problem admits solution for

(A) $\alpha = 0, \ \beta = 1$ (B) $\alpha = -1, \ \beta = \dfrac{\pi}{2}$

(C) $\alpha = 1, \ \beta = \dfrac{\pi}{2}$ (D) $\alpha = 1, \ \beta = -\pi$

Q.66   The functional

$$\int_0^1 (1+x)(y')^2 \, dx, \quad y(0) = 0, \ y(1) = 1,$$

possesses

(A) strong maxima

(B) strong minima

(C) weak maxima but NOT a strong maxima

(D) weak minima but NOT a strong minima

Q.67   The value of α for which the integral equation

$$u(x) = \alpha \int_0^1 e^{xt} u(t) \, dt,$$

has a non-trivial solution is

(A) -2 (B) -1 (C) 1   (D) 2

Q.68   Let $P_n(x)$ be the Legendre polynomial of degree n and let

$$P_{n+1}(0) = -\frac{m}{m+1} P_{n-1}(0), \quad m = 1, 2, \ldots$$

If $P_2(0) = -\dfrac{5}{16}$ then $\int_{-1}^1 \left[P_2^2(x) \right] dx =$

(A) $\dfrac{2}{13}$ (B) $\dfrac{2}{9}$ (C) $\dfrac{5}{16}$ (D) $\dfrac{2}{5}$

Q.69   For which of the following pair of functions $y_1(x)$ and $y_2(x)$, continuous functions $p(x)$ and $q(x)$ can be determined on $[-1, 1]$ such that $y_1(x)$ and $y_2(x)$ give two linearly independent solutions of

$$y'' + p(x)y' + q(x)y = 0, \quad x \in [-1, 1].$$

(A) $y_1(x) = x \sin(x), \ y_2(x) = \cos(x)$ (B) $y_1(x) = xe^x, \ y_2(x) = \sin(x)$

(C) $y_1(x) = e^{-x}, \ y_2(x) = e^{-1}$ (D) $y_1(x) = x^2, \ y_2(x) = \cos(x)$

Q.70 Let $J_0(s)$ and $J_1(s)$ be the Bessel functions of the first kind of orders zero and one, respectively. If

$$\mathcal{L}(J_0)(s) = \frac{1}{\sqrt{s^2 + 1}},$$

then $\mathcal{L}(J_1)(s) =$

(A) $\dfrac{s}{\sqrt{s^2 + 1}}$

(B) $\dfrac{1}{\sqrt{s^2 + 1}}$

(C) $1 - \dfrac{1}{\sqrt{s^2 + 1}}$

(D) $\dfrac{1}{\sqrt{s^2 + 1}} - 1$

Common Data Questions

Common Data for Questions 71, 72, 73:

Let $P[0, 1] = \{p : p$ is a polynomial function on $[0, 1]\}$. For $p \in P[0, 1]$, define

$$\|p\| = \sup\{|p(x)| : 0 \le x \le 1\}.$$

Consider the map $T : P[0, 1] \to P[0, 1]$ defined by

$$(Tp)(x) = \frac{d}{dx}\left(p(x)\right).$$

Then $P[0, 1]$ is a normed linear space and T is a linear map. The map T is said to be closed if the set $G = \{(p, Tp) : p \in P[0, 1]\}$ is a closed subset of $P[0, 1] \times P[0, 1]$.

Q.71 The linear map T is

(A) one to one and onto

(B) one to one but NOT onto

(C) onto but NOT one to one

(D) neither one to one nor onto

Q.72 The normed linear space $P[0, 1]$ is

(A) a finite dimensional normed linear space which is NOT a Banach space

(B) a finite dimensional Banach space

(C) an infinite dimensional normed linear space which is NOT a Banach space

(D) an infinite dimensional Banach space

Q.73 The map T is

(A) closed and continuous

(B) neither continuous nor closed

(C) continuous but NOT closed

(D) closed but NOT continuous

Common Data for Questions 74, 75:

Let X and Y be jointly distributed random variables such that the conditional distribution of Y, given $X = x$, is uniform on the interval $(x - 1, x + 1)$. Suppose $\mathbb{E}(X) = 1$ and $\text{Var}(X) = \dfrac{5}{3}$.

Q.74 The mean of the random variable Y is

(A) $\dfrac{1}{2}$

(B) 1

(C) $\dfrac{3}{2}$

(D) 2

Q.75 The variance of the random variable Y is

(A) $\dfrac{1}{2}$ (B) $\dfrac{2}{3}$ (C) 1   (D) 2

Linked Answer Questions: Q.76 to Q.85 carry two marks each.

Statement for Linked Answer Questions 76 & 77:

Suppose the equation
$$x^2 y'' - xy' + (1 + x^2)y = 0$$
has a solution of the form
$$y = x^r \sum_{n=0}^{\infty} c_n x^n, \quad c_0 \neq 0.$$

Q.76 The indicial equation for r is

(A) $r^2 - 1 = 0$ (B) $(r-1)^2 = 0$
(C) $(r+1)^2 = 0$ (D) $r^2 + 1 = 0$

Q.77 For $n \geq 2$, the coefficients c_n will satisfy the relation

(A) $n^2 c_n - c_{n-2} = 0$ (B) $n^2 c_n + c_{n-2} = 0$
(C) $c_n - n^2 c_{n-2} = 0$ (D) $c_n + n^2 c_{n-2} = 0$

Statement for Linked Answer Questions 78 & 79:

A particle of mass m slides down without friction along a curve $z = 1 + \dfrac{x^2}{2}$ in the xz-plane under the action of constant gravity. Suppose the z-axis points vertically upwards. Let $\dot{x}$ and $\ddot{x}$ denote $\dfrac{dx}{dt}$ and $\dfrac{d^2 x}{dt^2}$ respectively.

Q.78 The Lagrangian of the motion is

(A) $\dfrac{1}{2} m \dot{x}^2 (1 + x^2) - mg\left(1 + \dfrac{x^2}{2}\right)$ (B) $\dfrac{1}{2} m \dot{x}^2 (1 + x^2) + mg\left(1 + \dfrac{x^2}{2}\right)$

(C) $\dfrac{1}{2} m x^2 \dot{x}^2 - mg\left(1 + \dfrac{x^2}{2}\right)$ (D) $\dfrac{1}{2} m \dot{x}^2 (1 - x^2) - mg\left(1 + \dfrac{x^2}{2}\right)$

Q.79 The Lagrangian equation of motion is

(A) $\ddot{x}(1 + x^2) = -x(g + \dot{x}^2)$ (B) $\ddot{x}(1 + x^2) = x(g - \dot{x}^2)$

(C) $\ddot{x} = -gx$ (D) $\ddot{x}(1 - x^2) = -x(g - \dot{x}^2)$

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Statement for Linked Answer Questions 80 & 81:

Let $u(x,t)$ be the solution of the one dimensional wave equation

$$u_{tt} = 4u_{xx}, \quad -\infty < x < \infty, \ t > 0,$$

$$u(x,0) = \begin{cases} 16 - x^2, & |x| \le 4, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad u_t(x,0) = \begin{cases} 1, & |x| \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

Q.80   For $1 < t < 3$, $u(2,t) =$

(A) $\left[16 - (2-2t)^2\right]^+ + \dfrac{1}{2}\left[1 - \min\{1, t-1\}\right]$

(B) $\left[32 - (2-2t)^2 - (2+2t)^2\right]^+ + t$

(C) $\left[32 - (2-2t)^2 - (2+2t)^2\right]^+ + 1$

(D) $\left[16 - (2-2t)^2\right]^+ + \dfrac{1}{2}\left[1 - \max\{1, t-1\}\right]$

Q.81   The value of $u(2,2)$

(A) equals 15

(B) equals 16

(C) equals 0

(D) does NOT exist

Statement for Linked Answer Questions 82 & 83:

Suppose $E = \{(x,y) : xy \neq 0\}$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 0, & \text{if } xy = 0, \\ y\sin\left(\frac{1}{x}\right) + x\sin\left(\frac{1}{y}\right), & \text{otherwise.} \end{cases}$$

Let S_1 be the set of points in $\mathbb{R}^2$ where f_x exists and S_2 be the set of points in $\mathbb{R}^2$ where f_y exists. Also, let E_1 be the set of points where f_x is continuous and E_2 be the set of points where f_y is continuous.

Q.82 S_1 and S_2 are given by

(A) $S_1 = E \cup \{(x,y) : y = 0\}, \quad S_2 = E \cup \{(x,y) : x = 0\}$

(B) $S_1 = E \cup \{(x,y) : x = 0\}, \quad S_2 = E \cup \{(x,y) : y = 0\}$

(C) $S_1 = S_2 = \mathbb{R}^2$

(D) $S_1 = S_2 = E \cup \{(0,0)\}$

Q.83 E_1 and E_2 are given by

(A) $E_1 = S_1, \quad E_2 = S_1 \cap S_2$

(B) $E_1 = S_1 \cap S_2 \setminus \{(0,0)\}, \quad E_2 = S_1$

(C) $E_1 = S_2, \quad E_2 = S_1$

(D) $E_1 = S_2, \quad E_2 = S_2$

Statement for Linked Answer Questions 84 & 85:

Let

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 2 \\ 0 & 2 & 6 \end{bmatrix}$$

and let $\lambda_1 \geq \lambda_2 \geq \lambda_3$ be the eigenvalues of A.

Q.84   The triple $(\lambda_1, \lambda_2, \lambda_3)$ equals

(A) $(9, 4, 2)$ (B) $(8, 4, 3)$

(C) $(9, 3, 3)$ (D) $(7, 5, 3)$

Q.85   The matrix P such that

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

is

(A) $\begin{bmatrix} \dfrac{1}{\sqrt{3}} & 0 & \dfrac{-2}{\sqrt{6}} \\ \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{6}} \\ \dfrac{1}{\sqrt{3}} & \dfrac{-1}{\sqrt{2}} & \dfrac{1}{\sqrt{6}} \end{bmatrix}$

(B) $\begin{bmatrix} \dfrac{1}{\sqrt{3}} & \dfrac{-2}{\sqrt{6}} & 0 \\ \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{6}} & \dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{6}} & \dfrac{-1}{\sqrt{2}} \end{bmatrix}$

(C) $\begin{bmatrix} 0 & 0 & 1 \\ \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} & 0 \\ \dfrac{1}{\sqrt{2}} & \dfrac{-1}{\sqrt{2}} & 0 \end{bmatrix}$

(D) $\begin{bmatrix} 0 & 1 & 0 \\ \dfrac{1}{\sqrt{2}} & 0 & \dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} & 0 & \dfrac{-1}{\sqrt{2}} \end{bmatrix}$

END OF THE QUESTION PAPER

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