10.6.11

ee25btech11056 - Suraj.N

Question : Draw a circle of radius 4 cm. Draw two tangents to the circle inclined at an angle of 60° to each other.

Solution :

Name	Value
Circle	$\mathbf{x}^\top\mathbf{x} - 16 = 0$
$\mathbf{P}$	$\begin{pmatrix} 8 \\ 0 \end{pmatrix}$

Table : Circle

The parameters of the circle with center $\mathbf{0}$ are :

$$\mathbf{V} = \mathbf{I} \qquad \mathbf{u} = \mathbf{0} \qquad f = -16 \tag{1}$$

Let the point from which tangent is being drawn be $\mathbf{p}$.

Let the point of contact be $\mathbf{q}$ and

$$\mathbf{q}^\top\mathbf{q} = 16 \tag{2}$$

From the condition of tangency we get

$$\mathbf{q}^\top(\mathbf{q} - \mathbf{p}) = 0 \tag{3}$$
$$\mathbf{p}^\top\mathbf{q} = \mathbf{q}^\top\mathbf{q} \tag{4}$$
$$\mathbf{p}^\top\mathbf{q} = 16 \tag{5}$$

If the angle between the tangents is 60° then the angle betweent the normals at the points of contact is 120°.

Therefore,

$$\cos\left(\tfrac{120°}{2}\right) = \frac{\mathbf{p}^\top\mathbf{q}}{\|\mathbf{p}\|\,\|\mathbf{q}\|} \tag{6}$$
$$\|\mathbf{p}\| = 8 \tag{7}$$
$$\mathbf{p}^\top\mathbf{p} - 64 = 0 \tag{8}$$

Therefore the locus of point $\mathbf{p}$ is a circle with center $\mathbf{0}$ and radius 8 cm.

Consider point $\mathbf{P} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$ (lies on the locus) from which tangents are drawn.

Let the tangent equation passing through $\mathbf{P}$ be

$$\mathbf{x} = \mathbf{P} + k\mathbf{m} \tag{9}$$

Finding the point of contact :

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{x} - 16 \tag{10}$$
$$(\mathbf{P} + k\mathbf{m})^\top (\mathbf{P} + k\mathbf{m}) - 16 = 0 \tag{11}$$
$$k^2 \mathbf{m}^\top \mathbf{m} + 2k\mathbf{P}^\top \mathbf{m} + \mathbf{P}^\top \mathbf{P} - 16 = 0 \tag{12}$$
$$k^2 \mathbf{m}^\top \mathbf{m} + 2k\mathbf{P}^\top \mathbf{m} + g(\mathbf{P}) = 0 \tag{13}$$
$$\tag{14}$$

As the tangent intersects the conic at only one point(the point of contact),the discriminant for the quadratic in k is equal to 0

$$g(\mathbf{P}) = 48 \tag{15}$$
$$\mathbf{m}^\top \begin{pmatrix} -16 & 0 \\ 0 & 48 \end{pmatrix} \mathbf{m} = 0 \tag{16}$$
$$\mathbf{Q} = \begin{pmatrix} -16 & 0 \\ 0 & 48 \end{pmatrix} \tag{17}$$
$$\tag{18}$$

As $\mathbf{Q}$ is an upper triangular matrix , the eigen values are the diagonal entries :

$$\lambda_1 = -16 \qquad \lambda_2 = 48 \tag{19}$$

Applying eigen value decomposition for $\mathbf{Q}$

$$\mathbf{Q} = \mathbf{X}\mathbf{D}\mathbf{X}^\top \tag{20}$$
$$\mathbf{D} = \begin{pmatrix} -16 & 0 \\ 0 & 48 \end{pmatrix} \tag{21}$$

$\mathbf{X}$ is an orthogonal matrix whose columns are the corresponding normalized eigenvectors of $\mathbf{Q}$
As $\mathbf{Q}$ is a diagonal matrix ,

$$\mathbf{X} = \mathbf{I} \tag{22}$$

From (16) ,

$$\mathbf{m}^\top \mathbf{X}\mathbf{D}\mathbf{X}^\top \mathbf{m} = 0 \tag{23}$$
$$\mathbf{z} = \mathbf{X}^\top \mathbf{m} \tag{24}$$
$$\mathbf{z}^\top \mathbf{D}\mathbf{z} = 0 \tag{25}$$
$$\begin{pmatrix} z_1 & z_2 \end{pmatrix} \begin{pmatrix} -16 & 0 \\ 0 & 48 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0 \tag{26}$$
$$\frac{z_1}{z_2} = \pm\sqrt{3} \tag{27}$$

Solving for $\mathbf{m}$,

$$\mathbf{I}\mathbf{m} = \mathbf{z} \tag{28}$$

$$\mathbf{m} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{29}$$

$$\mathbf{m} = \begin{pmatrix} 1 \\ \frac{z_2}{z_1} \end{pmatrix} \tag{30}$$

From (27) , the direction vectors for the tangents are given as :

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix} \qquad \mathbf{m_2} = \begin{pmatrix} 1 \\ -\frac{1}{\sqrt{3}} \end{pmatrix} \tag{31}$$

The normal vectors for the tangents are given as :

$$\mathbf{n_1} = \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ 1 \end{pmatrix} \qquad \mathbf{n_2} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 1 \end{pmatrix} \tag{32}$$

The points of contacts are given as :

$$\mathbf{q_i} = \pm r \frac{\mathbf{n_i}}{||\mathbf{n_i}||} \tag{33}$$

From (5) , $\mathbf{P}^\top \mathbf{q} = 16$, so the points of contact are :

$$\mathbf{q_1} = \begin{pmatrix} 2 \\ 2\sqrt{3} \end{pmatrix} \qquad \mathbf{q_2} = \begin{pmatrix} 2 \\ -2\sqrt{3} \end{pmatrix} \tag{34}$$

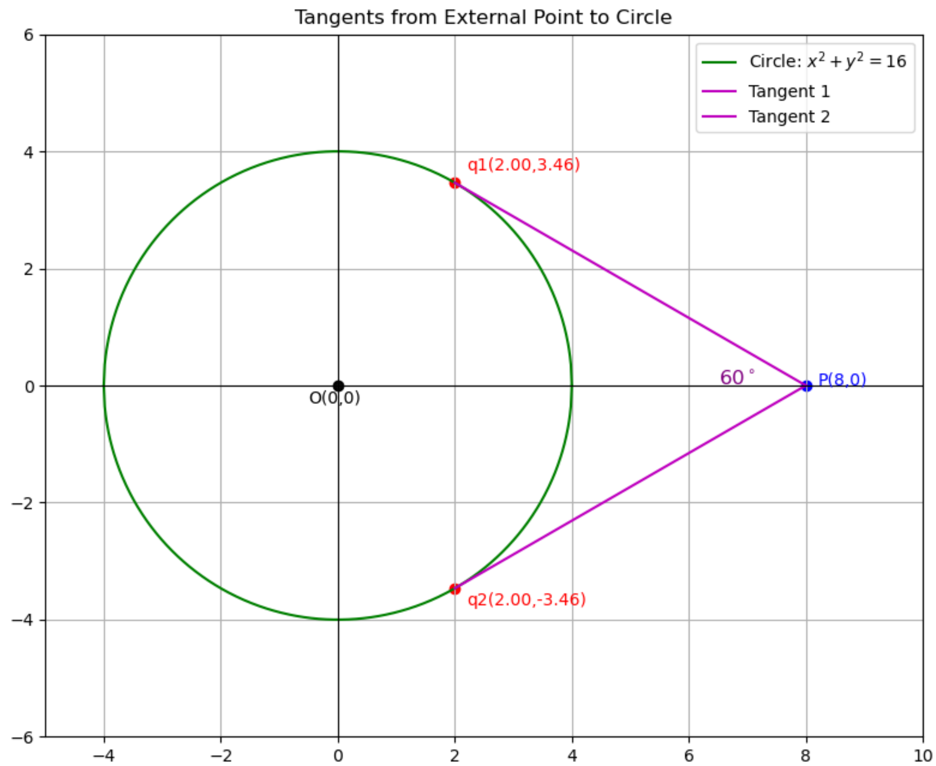

Fig : Circle and Tangents