

GATE 2007 MA

AI25BTECH11012 - UNNATHI GARIGE

Q.1-Q.20 carry one mark each.

- 1) Consider \mathbb{R}^2 with the usual topology. Let

$$S = \{(x, y) \in \mathbb{R}^2 : x \text{ is an integer}\}$$

Then S is

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- a) open but NOT closed
- b) both open and closed
- c) neither open nor closed
- d) closed but NOT open

- 2) Suppose $X = \{\alpha, \beta, \delta\}$. Let

$$\mathcal{T}_1 = \{\emptyset, X, \{\alpha\}, \{\alpha, \beta\}\} \quad \text{and} \quad \mathcal{T}_2 = \{\emptyset, X, \{\alpha\}, \{\beta, \delta\}\}.$$

Then

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- a) both $\mathcal{T}_1 \cap \mathcal{T}_2$ and $\mathcal{T}_1 \cup \mathcal{T}_2$ are topologies
- b) neither $\mathcal{T}_1 \cap \mathcal{T}_2$ nor $\mathcal{T}_1 \cup \mathcal{T}_2$ is a topology
- c) $\mathcal{T}_1 \cup \mathcal{T}_2$ is a topology but $\mathcal{T}_1 \cap \mathcal{T}_2$ is NOT a topology
- d) $\mathcal{T}_1 \cap \mathcal{T}_2$ is a topology but $\mathcal{T}_1 \cup \mathcal{T}_2$ is NOT a topology

- 3) For a positive integer n , let $f_n : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} \frac{1}{4n+5} & \text{if } 0 \leq x \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{f_n(x)\}$ converges to zero

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- a) uniformly but NOT in L^1 norm
- b) uniformly and also in L^1 norm
- c) pointwise but NOT uniformly
- d) in L^1 norm but NOT pointwise

- 4) Let P_1 and P_2 be two projection operators on a vector space.

Then

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- a) $P_1 + P_2$ is a projection if $P_1 P_2 = P_2 P_1 = 0$
- b) $P_1 - P_2$ is a projection if $P_1 P_2 = P_2 P_1 = 0$
- c) $P_1 + P_2$ is a projection
- d) $P_1 - P_2$ is a projection

5) Consider the system of linear equations

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$$x + y + z = 3,$$

$$x - y - z = 4,$$

$$-5y + kz = 6$$

Then the value of k for which this system has an infinite number of solutions is

- a) $k = -5$
- b) $k = 0$
- c) $k = 1$
- d) $k = 3$

6) Let

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$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ x & y & z \end{pmatrix}$$

and let $V = \{(x, y, z) \in \mathbb{R}^3 : \det(A) = 0\}$. Then the dimension of V equals:

- a) 0
- b) 1
- c) 2
- d) 3

7) Let

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$$S = \{0\} \cup \left\{ \frac{1}{4n+7} : n = 1, 2, \dots \right\}$$

Then the number of analytic functions which vanish only on S is:

- a) infinite
- b) 0
- c) 1
- d) 2

8) It is given that $\sum_{n=0}^{\infty} a_n z^n$ converges at $z = 3 + i4$. Then the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ is:

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- a) ≤ 5
- b) ≥ 5
- c) < 5
- d) > 5

9) The value of α for which $G = \langle \alpha, 1, 3, 9, 19, 27 \rangle$ is a cyclic group under multiplication modulo 56 is:

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- a) 5
- b) 15
- c) 25
- d) 35

10) Consider \mathbb{Z}_{24} as the additive group modulo 24. Then the number of elements of order 8 in the group \mathbb{Z}_{24} is:

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- a) 1
- b) 2
- c) 3
- d) 4

11) Define $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$f(x, y) = \begin{cases} 1, & \text{if } xy = 0, \\ 2, & \text{otherwise.} \end{cases}$$

If $S = \{(x, y) : f \text{ is continuous at the point } (x, y)\}$, then:

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- a) S is open
- b) S is connected
- c) $S = \emptyset$
- d) S is closed

12) Consider the linear programming problem

$$\begin{aligned} &\text{Maximize } z = c_1x_1 + c_2x_2, \quad c_1, c_2 > 0, \\ &\text{subject to} \\ &x_1 + x_2 \leq 3 \\ &2x_1 + 3x_2 \leq 4 \\ &x_i \geq 0 \end{aligned}$$

Then:

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- a) the primal has an optimal solution but the dual does NOT have an optimal solution
- b) both the primal and the dual have optimal solutions
- c) the dual has an optimal solution but the primal does NOT have an optimal solution
- d) neither the primal nor the dual have optimal solutions

13) Let

$$f(x) = x^{10} + x - 1, x \in \mathbb{R}$$

and let $x_k = k, k = 0, 1, 2, \dots, 10$. Then the value of the divided difference

$$f[x_0, x_1, x_2, \dots, x_{10}]$$

is:

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- a) -1
- b) 0
- c) 1
- d) 10

14) Let X, Y be jointly distributed random variables having the joint probability density function

$$f(x, y) = \begin{cases} 1, & \text{if } 0 < x + y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $P(Y \geq \max(X, 1 - X))$ is

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- a) $\frac{1}{2}$
- b) 1
- c) $\frac{1}{4}$
- d) $\frac{1}{6}$

15) Let X_1, X_2, \dots be a sequence of independent and identically distributed chi-square random variables, each having 4 degrees of freedom. Define

$$S_n = \sum_{i=1}^n X_i$$

If $\frac{S_n}{n} \rightarrow \mu$ as $n \rightarrow \infty$, then $\mu =$

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- a) 8 b) 16 c) 24 d) 32

- 16) Let $\{E_n : n = 1, 2, \dots\}$ be a decreasing sequence of Lebesgue measurable sets on \mathbb{R} and let F be a Lebesgue measurable set on \mathbb{R} such that $E_n \cap F = \emptyset$. Suppose that F has Lebesgue measure 2 and the Lebesgue measure of E_n equals

$$\frac{2n+2}{3n+1}, n = 1, 2, \dots$$

Then the Lebesgue measure of the set $(\bigcap_{n=1}^{\infty} E_n) \cup F$ equals GATE MA 2007

- a) $\frac{5}{3}$ b) 2 c) $\frac{7}{3}$ d) $\frac{8}{3}$

- 17) The extremum for the variational problem

$$\int_0^{\frac{\pi}{8}} ((y')^2 + 2yy' - 16y^2) dx, \quad y(0) = 0, \quad y\left(\frac{\pi}{8}\right) = 1,$$

occurs for the curve

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- a) $y = \sin(4x)$
 b) $y = \sqrt{2} \sin(2x)$
 c) $y = 1 - \cos(4x)$
 d) $y = \frac{1 - \cos(8x)}{2}$

- 18) Suppose $y_p(x) = x \cos(2x)$ is a particular solution of

$$y'' + \alpha y = \sin(2x).$$

Then the constant α equals

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- a) -4 b) -2 c) 2 d) 4

- 19) If $F(s) = \tan^{-1}(s) + k$ is the Laplace transform of some function $f(t)$, $t \geq 0$, then $k =$

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- a) $-\pi$ b) $\frac{\pi}{2}$ c) 0 d) $\frac{\pi}{2}$

- 20) Let

$$S = \{(0, 1, 1), (1, 0, 1), (-1, 2, 1)\} \subseteq \mathbb{R}^3.$$

Suppose \mathbb{R}^3 is endowed with the standard inner product. Define

$$M = \{x \in \mathbb{R}^3 : \langle x, y \rangle = 0 \text{ for all } y \in S\}$$

Then the dimension of M equals

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- a) 0 b) 1 c) 2 d) 3

Q.21-Q.75 carry one mark each.

21) Let X be an uncountable set and let

$$\tau = \{U \subseteq X : X \setminus U \text{ is countable or } X \setminus U \text{ is finite}\}.$$

Then the topological space (X, τ)

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- a) is separable
- b) is Hausdorff
- c) has a countable basis
- d) has a countable basis at each point

22) Suppose (X, τ) is a topological space. Let $\{S_\alpha\}_{\alpha \in A}$ be a sequence of subsets of X .

Then

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- a) $(S_1 \cup S_2)'' = S_1'' \cup S_2''$
- b) $(\bigcap S_\alpha)'' = \bigcap S_\alpha''$
- c) $\overline{\bigcup S_\alpha} = \bigcup_\alpha \overline{S_\alpha}$
- d) $\overline{S_1 \cup S_2} = \overline{S_1} \cup \overline{S_2}$

23) Let (X, d) be a metric space. Consider the metric ρ on X defined by

$$\rho(x, y) = \min(d(x, y), 1), \quad x, y \in X.$$

Suppose τ and τ_1 are topologies on X defined by d and ρ respectively. Then

- a) τ_1 is a proper subset of τ
- b) τ is a proper subset of τ_1
- c) neither τ_2 nor τ_1 is a subset of the other
- d) $\tau_1 = \tau$

24) A basis of the vector space

$$W = \{(x, y, z, w) \in \mathbb{R}^4 : x + y + z = 0, y + z + w = 0, 2x + y - z + w = 0\}$$

is

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- a) $\{(1, 1, 1, 1), (2, 1, 1, 1)\}$
- b) $\{(1, -1, 0, 1), (0, 1, -1, 0)\}$
- c) $\{(1, 0, -1, 0), (2, 1, 1, 1)\}$
- d) $\{(1, 0, -1, 0), (0, 1, -1, 0)\}$

25) Consider \mathbb{R}^3 with the standard inner product. Let

$$S = \{(1, 1, 1), (2, -1, 2), (-1, 2, 1)\}$$

For a subset W of \mathbb{R}^3 , let $L(W)$ denote the linear span of W in \mathbb{R}^3 . Then an orthonormal set T with $L(S) = L(T)$ is

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- a) $\left\{ \frac{1}{\sqrt{3}}(1, 1, 1), \frac{1}{\sqrt{6}}(1, 0, -2), \frac{1}{\sqrt{2}}(1, -1, 0) \right\}$
- b) $\{(0, 0, 0), (0, 1, 0), (0, 0, 1)\}$
- c) $\left\{ \frac{1}{\sqrt{3}}(1, 1, 1), \frac{1}{\sqrt{2}}(1, 0, -1) \right\}$
- d) $\left\{ \frac{1}{\sqrt{3}}(1, 1, 1), \frac{1}{\sqrt{2}}(1, -1, 0) \right\}$

- 26) Let A be a 3×3 matrix. Suppose that the eigenvalues of A are $-1, 0, 1$ with respective eigenvectors $(1, -1, 0)^T$, $(1, 1, -2)^T$ and $(1, 1, 1)^T$.

Then $6A$ equals

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a) $\begin{pmatrix} -1 & 5 & 2 \\ 5 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 5 & 3 \\ 5 & 1 & 3 \\ 3 & 3 & 3 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

d) $\begin{pmatrix} -3 & 9 & 0 \\ 9 & -3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

- 27) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear transformation defined by

$$T((x, y, z)) = (x + y - z, x + y + z, y - z).$$

Then the matrix of the linear transformation T with respect to the ordered basis

$$B = \{(0, 1, 0), (0, 0, 1), (1, 0, 0)\} \text{ of } \mathbb{R}^3$$

is

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a) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$

- 28) Let $Y(x) = (y_1(x), y_2(x))^T$ and let

$$A = \begin{pmatrix} -3 & 1 \\ k & -1 \end{pmatrix}.$$

Further, let S be the set of values of k for which all the solutions of the system of equations $Y'(x) = AY(x)$ tend to zero as $x \rightarrow \infty$.

Then S is given by

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a) $\{k : k \leq -1\}$

c) $\{k : k < -1\}$

b) $\{k : k \leq 3\}$

d) $\{k : k < 3\}$

- 29) Let

$$u(x, y) = f(xe^y) + g(y^2 \cos y),$$

where f and g are infinitely differentiable functions. Then the partial differential equation of minimum order satisfied by u is

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- a) $u_x + xu_{xx} = u_y$
 b) $u_y + xu_{xx} = xu_y$

- c) $u_y - xu_{xx} = u_x$
 d) $u_y - xu_{xx} = xu_y$

- 30) Let C be the boundary of the triangle formed by the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$.
 Then the value of the line integral GATE MA 2007

$$\oint_C -2y \, dx + (3x - 4y^2) \, dy + (z^2 + 3y) \, dz$$

is

- a) 0 b) 1 c) 2 d) 4

- 31) Let X be a complete metric space and let $E \subset X$.

Consider the following statements:

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- a) E is compact,
 b) E is closed and bounded,
 c) E is closed and totally bounded,
 d) Every sequence in E has a subsequence converging in E .

Which one of the above statements does **NOT** imply all the other statements?

- a) a b) b c) c d) d

- 32) Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \sin(nx).$$

Then the series

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- a) converges uniformly on \mathbb{R}
 b) converges pointwise but NOT uniformly on \mathbb{R}
 c) converges in L^1 norm to an integrable function on $[0, 2\pi]$ but does NOT converge uniformly on \mathbb{R}
 d) does NOT converge pointwise

- 33) Let $f(z)$ be an analytic function. Then the value of

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$$\int_0^{2\pi} f(e^{it}) \cos(t) \, dt$$

equals

- a) 0 b) $2\pi f(0)$ c) $2\pi f'(0)$ d) $\pi f'(0)$

- 34) Let G_1 and G_2 be the images of the disc $\{z \in \mathbb{C} : |z+1| < 1\}$ under the transformations

$$w = \frac{(1-i)z+2}{(1+i)z+2} \quad \text{and} \quad w = \frac{(1+i)z+2}{(1-i)z+2}$$

respectively. Then

- a) $G_1 = \{w \in \mathbb{C} : \text{Im}(w) < 0\}$ and $G_2 = \{w \in \mathbb{C} : \text{Im}(w) > 0\}$
 b) $G_1 = \{w \in \mathbb{C} : \text{Im}(w) > 0\}$ and $G_2 = \{w \in \mathbb{C} : \text{Im}(w) < 0\}$
 c) $G_1 = \{w \in \mathbb{C} : |w| > 2\}$ and $G_2 = \{w \in \mathbb{C} : |w| < 2\}$
 d) $G_1 = \{w \in \mathbb{C} : |w| < 2\}$ and $G_2 = \{w \in \mathbb{C} : |w| > 2\}$

35) Let $f(z) = 2^z - 2^{-z}$. Then the maximum value of $|f(z)|$ on the unit disc

$$D = \{z \in \mathbb{C} : |z| \leq 1\}$$

equals

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- a) 1 b) 2 c) 3 d) 4

36) Let

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$$f(z) = \frac{1}{z^2 - 3z + 2}$$

Then the coefficient of $\frac{1}{z}$ in the Laurent series expansion of $f(z)$ for $|z| > 2$ is

- a) 0 b) 1 c) 3 d) 5

37) Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an arbitrary analytic function satisfying $f(0) = 0$ and $f(1) = 2$.

Then

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- a) there exists a sequence $\{z_n\}$ such that $|z_n| > n$ and $|f(z_n)| > n$
 b) there exists a sequence $\{z_n\}$ such that $|z_n| > n$ and $|f(z_n)| < n$
 c) there exists a bounded sequence $\{z_n\}$ such that $|f(z_n)| > n$
 d) there exists a sequence $\{z_n\}$ such that $z_n \rightarrow 0$ and $f(z_n) \rightarrow 2$

38) Define $f : \mathbb{C} \rightarrow \mathbb{C}$ by

$$f(z) = \begin{cases} 0, & \text{if } \text{Re}(z) = 0 \text{ or } \text{Im}(z) = 0, \\ \frac{1}{z}, & \text{otherwise.} \end{cases}$$

Then the set of points where f is analytic is

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- a) $\{z : \text{Re}(z) \neq 0 \text{ and } \text{Im}(z) \neq 0\}$ c) $\{z : \text{Re}(z) \neq 0 \text{ or } \text{Im}(z) \neq 0\}$
 b) $\{z : \text{Re}(z) \neq 0\}$ d) $\{z : \text{Im}(z) \neq 0\}$

39) Let $U(n)$ be the set of all positive integers less than n and relatively prime to n . Then $U(n)$ is a group under multiplication modulo n . For $n = 248$, the number of elements in $U(n)$ is

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- a) 60 b) 120 c) 180 d) 240

40) Let $\mathbb{R}[x]$ be the polynomial ring in x with real coefficients and let $I = \langle x^2 + 1 \rangle$ be the ideal generated by the polynomial $x^2 + 1$ in $\mathbb{R}[x]$. Then

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- a) I is a maximal ideal
 b) I is a prime ideal but NOT a maximal ideal

- c) I is NOT a prime ideal
 d) $\mathbb{R}[x]/I$ has zero divisors

41) Consider \mathbb{Z}_5 and \mathbb{Z}_{20} as rings modulo 5 and 20, respectively. Then the number of homomorphisms $\varphi : \mathbb{Z}_5 \rightarrow \mathbb{Z}_{20}$ is GATE MA 2007

- a) 1 b) 2 c) 4 d) 5

42) Let \mathbb{Q} be the field of rational numbers and consider \mathbb{Z}_2 as a field modulo 2. Let

$$f(x) = x^3 - 9x^2 + 9x + 3.$$

Then $f(x)$ is GATE MA 2007

- a) irreducible over \mathbb{Q} but reducible over \mathbb{Z}_2
 b) irreducible over both \mathbb{Q} and \mathbb{Z}_2
 c) reducible over \mathbb{Q} but irreducible over \mathbb{Z}_2
 d) reducible over both \mathbb{Q} and \mathbb{Z}_2

43) Let \mathbb{Q} be the field of rational numbers and consider \mathbb{Z}_2 as a field modulo 2. Let

$$f(x) = x^3 - 9x^2 + 9x + 3.$$

Then $f(x)$ is GATE MA 2007

- a) irreducible over \mathbb{Q} but reducible over \mathbb{Z}_2
 b) irreducible over both \mathbb{Q} and \mathbb{Z}_2
 c) reducible over \mathbb{Q} but irreducible over \mathbb{Z}_2
 d) reducible over both \mathbb{Q} and \mathbb{Z}_2

44) Consider \mathbb{Z}_5 as a field modulo 5 and let GATE MA 2007

$$f(x) = x^4 + 4x^3 + 4x^2 + 4x + 1.$$

Then the zeros of $f(x)$ over \mathbb{Z}_5 are 1 and 3 with respective multiplicity

- a) 1 and 4 b) 2 and 3 c) 2 and 2 d) 1 and 2

45) Consider the Hilbert space GATE MA 2007

$$\ell^2 = \left\{ x = \{x_n\}; x_n \in \mathbb{R}, \sum x_n^2 < \infty \right\}.$$

Let

$$E = \left\{ x = \{x_n\} \mid |x_n| < \frac{1}{n} \text{ for all } n \right\}$$

be a subset of ℓ^2 . Then

- a) $E^\circ = \left\{ x \mid |x_n| < \frac{1}{n} \text{ for all } n \right\}$
 b) $E^\circ = E$
 c) $E^\circ = \left\{ x \mid |x_n| < \frac{1}{n} \text{ for all but finitely many } n \right\}$
 d) $E^\circ = \emptyset$

46) Let X be a normed linear space and let $E_1, E_2 \subseteq X$. Define

$$E_1 + E_2 = \{x + y : x \in E_1, y \in E_2\}.$$

Then $E_1 + E_2$ is:

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- a) open if E_1 or E_2 is open
- b) NOT open unless both E_1 and E_2 are open
- c) closed if E_1 or E_2 is closed
- d) closed if both E_1 and E_2 are closed

47) For each $a \in \mathbb{R}$, consider the linear programming problem:

$$\text{Max. } z = x_1 + 2x_2 + 3x_3 + 4x_4$$

subject to

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$$ax_1 + 2x_2 \leq 1$$

$$x_1 + 2x_2 + 3x_3 \leq 2$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Let $S = \{a \in \mathbb{R} : \text{the given LP problem has a basic feasible solution}\}$. Then:

- a) $S = \emptyset$
- b) $S = \mathbb{R}$
- c) $S = (0, \infty)$
- d) $S = (-\infty, 0)$

48) Consider the linear programming problem:

$$\text{Max. } z = x_1 + 5x_2 + 3x_3$$

subject to

$$2x_1 - 3x_2 + 5x_3 \leq 3$$

$$x_1 - x_2 \leq 5$$

$$x_1, x_2, x_3 \geq 0$$

Then the dual of this LP problem:

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- a) has a feasible solution but does NOT have a basic feasible solution
- b) has a basic feasible solution
- c) has infinite number of feasible solutions
- d) has no feasible solution

49) Consider a transportation problem with two warehouses and two markets. The warehouse capacities are $a_1 = 2$ and $a_2 = 4$, and the market demands are $b_1 = 3$ and $b_2 = 3$. Let x_{ij} be the quantity shipped from warehouse i to market j , and c_{ij} be the corresponding unit cost. Suppose that $c_{11} = 1$, $c_{21} = 1$, and $c_{22} = 2$. Then $(x_{11}, x_{12}, x_{21}, x_{22}) = (2, 0, 1, 3)$ is optimal for every:

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- a) $c_{12} \in [1, 2]$
- b) $c_{12} \in [0, 3]$
- c) $c_{12} \in [1, 3]$
- d) $c_{12} \in [2, 4]$

50) The smallest degree of the polynomial that interpolates the data is:

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x	-2	-1	0	1	2	3
$f(x)$	-58	-21	-12	-13	-6	27

TABLE 50

- a) 3 b) 4 c) 5 d) 6

51) Suppose that x_n is sufficiently close to 3. Which of the following iterations $x_{n+1} = g(x_n)$ will converge to the fixed point $x = 3$? GATE MA 2007

- a) $x_{n+1} = -16 + 6x_n + \frac{3}{x_n}$ c) $x_{n+1} = \frac{3}{x_n} - \frac{x_n}{2}$
b) $x_{n+1} = \sqrt{3 + 2x_n}$ d) $x_{n+1} = \frac{x_n^2 - 3}{2}$

52) Consider the quadrature formula:

$$\int_{x_1}^{x_2} f(x) dx \approx \frac{1}{2} [f(x_1) + f(x_2)],$$

where x_1 and x_2 are quadrature points. Then the highest degree of the polynomial for which the above formula is exact equals: GATE MA 2007

- a) 1 b) 2 c) 3 d) 4

53) Let A , B and C be three events such that:

$$P(A) = 0.4, \quad P(B) = 0.5, \quad P(A \cup B) = 0.6, \quad P(C) = 0.6, \quad \text{and } P(A \cap B \cap C^c) = 0.1.$$

Then $P(A \cap B \cap C) =$

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- a) $\frac{1}{2}$ b) $\frac{1}{3}$ c) $\frac{1}{4}$ d) $\frac{1}{5}$

54) Consider two identical boxes B_1 and B_2 , where the box B_i ($i = 1, 2$) contains $i + 1$ red and $5 - i + 1$ white balls. A fair die is cast. Let the number of dots shown on the top face of the die be N . If N is even or 5, then two balls are drawn with replacement from the box B_1 ; otherwise, two balls are drawn with replacement from the box B_2 . The probability that the two drawn balls are of different colours is: GATE MA 2007

- a) $\frac{7}{25}$ b) $\frac{9}{25}$ c) $\frac{12}{25}$ d) $\frac{16}{25}$

55) Let X_1, X_2, \dots be a sequence of independent and identically distributed random variables with

$$P(X_i = 1) = P(X_i = -1) = \frac{1}{2}.$$

Suppose for the standard normal random variable Z , $P(-0.1 < Z \leq 0.1) = 0.08$. If $S_n = \sum_{i=1}^n X_i$, then

$$\lim P\left(\frac{S_n}{\sqrt{n}} > \frac{n}{10}\right) =$$

- a) 0.42 b) 0.46 c) 0.5 d) 0.54

56) Let X_1, X_2, \dots, X_5 be a random sample of size 5 from a population having standard normal distribution. Let

$$\bar{X} = \frac{1}{5} \sum_{i=1}^5 X_i \quad \text{and} \quad T = \sum_{i=1}^5 (X_i - \bar{X})^2.$$

Then $E(T^2 \bar{X}^2) =$

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- a) 3 b) 3.6 c) 4.8 d) 5.2

57) Let $x_1 = 3.5$, $x_2 = 7.5$ and $x_3 = 5.2$ be observed values of a random sample of size three from a population having uniform distribution over the interval $(\theta, \theta + 5)$, where $\theta \in (0, \infty)$ is unknown and is to be estimated. Then which of the following is NOT a maximum likelihood estimate of θ ? GATE MA 2007

- a) 2.4 b) 2.7 c) 3 d) 3.3

58) The value of

$$\int_0^1 \int_y^1 x^2 e^{x^2} dx dy$$

equals

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- a) $\frac{1}{4}$ b) $\frac{1}{3}$ c) $\frac{1}{2}$ d) 1

59)

$$\lim_{n \rightarrow \infty} \left[(n+1) \int_0^1 x^n \ln(1+x) dx \right] =$$

GATE MA 2007

- a) 0 b) $\ln 2$ c) $\ln 3$ d) ∞

60) Consider the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^4, & \text{if } x \text{ is rational,} \\ 2x^4 - 1, & \text{if } x \text{ is irrational.} \end{cases}$$

Let S be the set of points where f is continuous. Then

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- a) $S = \{1\}$ b) $S = \{-1\}$ c) $S = \{-1, 1\}$ d) $S = \emptyset$

61) For a positive real number p , let $\{f_n : n = 1, 2, \dots\}$ be a sequence of functions defined on $[0, 1]$ by

$$f_n(x) = \begin{cases} n^{p+1}x, & 0 \leq x \leq \frac{1}{n} \\ \frac{1}{n^p}, & \frac{1}{n} < x \leq 1. \end{cases}$$

Let $f(x) = \lim_{n \rightarrow \infty} f_n(x)$, $x \in [0, 1]$. Then, on $[0, 1]$,

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- a) f is Riemann integrable
 b) the improper integral $\int_0^1 f(x)dx$ converges for $p \geq 1$
 c) the improper integral $\int_0^1 f(x)dx$ converges for $p < 1$
 d) f_n converges uniformly

62) Which of the following inequality is NOT true for $x \in \left[\frac{1}{4}, \frac{3}{4}\right]$

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- a) $e^{-x} > \sum_{j=0}^{\infty} \frac{(-x)^j}{j!}$ c) $e^{-x} = \sum_{j=0}^{\infty} \frac{(-x)^j}{j!}$
 b) $e^{-x} < \sum_{j=0}^{\infty} \frac{(-x)^j}{j!}$ d) $e^{-x} > \sum_{j=0}^{10} \frac{(-x)^j}{j!}$

63) Let $u(x, y)$ be the solution to the Cauchy problem

$$xu_x + u_y = 1, \quad u(x, 0) = 2 \ln(x), \quad x > 1.$$

Then $u(e, 1) =$

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- a) -1 b) 0 c) 1 d) e

64) Suppose

$$y(x) = \lambda \int_0^{2\pi} y(t) \sin(x+t) dt, \quad x \in [0, 2\pi]$$

has eigenvalues $\lambda = \frac{1}{\pi}$ and $\lambda = -\frac{1}{\pi}$ with corresponding eigenfunctions

$y_1(x) = \sin(x) + \cos(x)$ and $y_2(x) = \sin(x) - \cos(x)$, respectively. Then the integral equation

$$y(x) = f(x) + \frac{1}{\pi} \int_0^{2\pi} y(t) \sin(x+t) dt, \quad x \in [0, 2\pi]$$

has a solution when $f(x) =$

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- a) 1 b) $\cos(x)$ c) $\sin(x)$ d) $1 + \sin(x) + \cos(x)$

65) Consider the Neumann problem

$$u_{xx} + u_{yy} = 0, \quad 0 < x < \pi, \quad -1 < y < 1,$$

$$u_y(0, y) = u_y(\pi, y) = 0,$$

$$\begin{aligned} \text{a)} \quad & \frac{s}{\sqrt{s^2 + 1}} \\ \text{b)} \quad & \frac{1}{\sqrt{s^2 + 1}} \end{aligned}$$

$$\begin{aligned} \text{c)} \quad & 1 - \frac{1}{\sqrt{s^2 + 1}} \\ \text{d)} \quad & \frac{1}{\sqrt{s^2 + 1}} - 1 \end{aligned}$$

Common Data Questions

Common Data for Questions 71, 72, 73:

Let $P[0, 1] = \{p : p \text{ is a polynomial function on } [0, 1]\}$. For $p \in P[0, 1]$, define

$$\|p\| = \sup\{|p(x)| : 0 \leq x \leq 1\}.$$

Consider the map $T : P[0, 1] \rightarrow P[0, 1]$ defined by

$$(Tp)(x) = \frac{d}{dx}(p(x)).$$

Then $P[0, 1]$ is a normed linear space and T is a linear map. The map T is said to be closed if the set $G = \{(p, Tp) : p \in P[0, 1]\}$ is a closed subset of $P[0, 1] \times P[0, 1]$.

71) The linear map T is GATE MA 2007

- | | |
|----------------------------|--------------------------------|
| a) one to one and onto | c) onto but NOT one to one |
| b) one to one but NOT onto | d) neither one to one nor onto |

72) The normed linear space $P[0, 1]$ is GATE MA 2007

- a) a finite dimensional normed linear space which is NOT a Banach space
- b) a finite dimensional Banach space
- c) an infinite dimensional normed linear space which is NOT a Banach space
- d) an infinite dimensional Banach space

73) The map T is GATE MA 2007

- | | |
|----------------------------------|------------------------------|
| a) closed and continuous | c) continuous but NOT closed |
| b) neither continuous nor closed | d) closed but NOT continuous |

Common Data for Questions 74, 75:

Let X and Y be jointly distributed random variables such that the conditional distribution of Y , given $X = x$, is uniform on the interval $(x - 1, x + 1)$. Suppose

$$\mathbb{E}(X) = 1 \text{ and } \text{Var}(X) = \frac{5}{3}.$$

74) The mean of the random variable Y is GATE MA 2007

- | | | | |
|------------------|------|------------------|------|
| a) $\frac{1}{2}$ | b) 1 | c) $\frac{3}{2}$ | d) 2 |
|------------------|------|------------------|------|

75) The variance of the random variable Y is

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- a) $\frac{1}{2}$ b) $\frac{2}{3}$ c) 1 d) 2

Linked Answer Questions: Q.76 to Q.85 carry two marks each.

Statement for Linked Answer Questions 76 & 77:

Suppose the equation

$$x^2 y'' - xy' + (1 + x^2)y = 0$$

has a solution of the form

$$y = x^r \sum_{n=0}^{\infty} c_n x^n, \quad c_0 \neq 0.$$

76) The indicial equation for r is

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- a) $r^2 - 1 = 0$ c) $(r + 1)^2 = 0$
b) $(r - 1)^2 = 0$ d) $r^2 + 1 = 0$

77) For $n \geq 2$, the coefficients c_n will satisfy the relation

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- a) $n^2 c_n - c_{n-2} = 0$ c) $c_n - n^2 c_{n-2} = 0$
b) $c_n - n^2 c_{n-2} = 0$ d) $c_n + n^2 c_{n-2} = 0$

Statement for Linked Answer Questions 78 & 79:

A particle of mass m slides down without friction along a curve $z = 1 + \frac{x^2}{2}$ in the

xz -plane under the action of constant gravity. Suppose the z -axis points vertically upwards. Let \dot{x} and \ddot{x} denote $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ respectively.

78) The Lagrangian of the motion is

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- a) $\frac{1}{2} m \dot{x}^2 (1 + x^2) - mg \left(1 + \frac{x^2}{2} \right)$ c) $\frac{1}{2} m x^2 \dot{x}^2 - mg \left(1 + \frac{x^2}{2} \right)$
b) $\frac{1}{2} m \dot{x}^2 (1 + x^2) + mg \left(1 + \frac{x^2}{2} \right)$ d) $\frac{1}{2} m \dot{x}^2 (1 - x^2) - mg \left(1 + \frac{x^2}{2} \right)$

79) The Lagrangian equation of motion is

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- a) $\ddot{x}(1 + x^2) = -x(g + \dot{x}^2)$
b) $\ddot{x}(1 + x^2) = x(g - \dot{x}^2)$
c) $\ddot{x} = -gx$
d) $\ddot{x}(1 - x^2) = -x(g - \dot{x}^2)$

Statement for Linked Answer Questions 80 & 81:

Let $u(x, t)$ be the solution of the one dimensional wave equation

$$u_{tt} = 4u_{xx}, \quad -\infty < x < \infty, \quad t > 0,$$

$$u(x, 0) = \begin{cases} 16 - x^2, & |x| \leq 4, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad u_t(x, 0) = \begin{cases} 1, & |x| \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

80) For $1 < t < 3$, $u(2, t) =$

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- a) $\left[16 - (2 - 2t)^2\right]^+ + \frac{1}{2} [1 - \min\{1, t - 1\}]$
- b) $\left[32 - (2 - 2t)^2 - (2 + 2t)^2\right]^+ + t$
- c) $\left[32 - (2 - 2t)^2 - (2 + 2t)^2\right]^+ + 1$
- d) $\left[16 - (2 - 2t)^2\right]^+ + \frac{1}{2} [1 - \max\{1, t - 1\}]$

81) The value of $u(2, 2)$

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- a) equals 15
- b) equals 16
- c) equals 0
- d) does NOT exist

Statement for Linked Answer Questions 82 & 83: Suppose $E = \{(x, y) : xy \neq 0\}$.

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} 0, & \text{if } xy = 0, \\ y \sin\left(\frac{1}{x}\right) + x \sin\left(\frac{1}{y}\right), & \text{otherwise.} \end{cases}$$

Let S_1 be the set of points in \mathbb{R}^2 where f_x exists and S_2 be the set of points in \mathbb{R}^2 where f_y exists. Also, let E_1 be the set of points where f_x is continuous and E_2 be the set of points where f_y is continuous.

82) S_1 and S_2 are given by

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- a) $S_1 = E \cup \{(x, y) : y = 0\}$, $S_2 = E \cup \{(x, y) : x = 0\}$
- b) $S_1 = E \cup \{(x, y) : x = 0\}$, $S_2 = E \cup \{(x, y) : y = 0\}$
- c) $S_1 = S_2 = \mathbb{R}^2$
- d) $S_1 = S_2 = E \cup \{(0, 0)\}$

83) E_1 and E_2 are given by

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- a) $E_1 = S_1$, $E_2 = S_1 \cap S_2$
- b) $E_1 = S_1 \cap S_2 \setminus \{(0, 0)\}$, $E_2 = S_1$
- c) $E_1 = S_2$, $E_2 = S_1$
- d) $E_1 = S_2$, $E_2 = S_2$

Statement for Linked Answer Questions 84 & 85:

Let

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 2 \\ 0 & 2 & 6 \end{pmatrix}$$

and let $\lambda_1 \geq \lambda_2 \geq \lambda_3$ be the eigenvalues of A .84) The triple $(\lambda_1, \lambda_2, \lambda_3)$ equals

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a) (9, 4, 2)

c) (9, 3, 3)

b) (8, 4, 3)

d) (7, 5, 3)

85) The matrix P such that

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$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

is

a) $\begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 1 & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{-1} & \frac{\sqrt{6}}{1} \end{pmatrix}$

b) $\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{-2} & \frac{\sqrt{6}}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{\sqrt{6}}{1} & 0 \\ \frac{1}{\sqrt{3}} & \frac{\sqrt{6}}{1} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{\sqrt{6}}{1} & \frac{\sqrt{2}}{-1} \\ \frac{1}{\sqrt{3}} & \frac{\sqrt{6}}{\sqrt{6}} & \frac{\sqrt{2}}{\sqrt{2}} \end{pmatrix}$

c) $\begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{0}{\sqrt{2}} & \frac{\sqrt{2}}{\sqrt{2}} & 0 \end{pmatrix}$

d) $\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{\sqrt{2}}{-1} \\ \frac{1}{\sqrt{2}} & 0 & \frac{\sqrt{2}}{\sqrt{2}} \end{pmatrix}$

END OF THE QUESTION PAPER