4.13.50

Puni Aditya - EE25BTECH11046

Question:

Two equal sides of an isosceles triangle are given by the equations $7x - y + 3 = 0$ and $x + y - 3 = 0$ and its third side passes through the point $(1, -10)$. Determine the equation of the third side.

Solution: Let the two equal sides of the isosceles triangle be represented by

$$\mathbf{n_1}^\top \mathbf{x} = c_1$$
$$\mathbf{n_2}^\top \mathbf{x} = c_2$$

and the third side by the line

$$\mathbf{n}^\top \mathbf{x} = c$$

The third side of the isosceles, the base, is perpendicular to the angle bisector of the two equal sides.

$$\frac{\left|\mathbf{n}^\top \mathbf{n_1}\right|}{\|\mathbf{n}\|\,\|\mathbf{n_1}\|} = \frac{\left|\mathbf{n}^\top \mathbf{n_2}\right|}{\|\mathbf{n}\|\,\|\mathbf{n_2}\|} \tag{1}$$
$$\frac{\left|\mathbf{n}^\top \mathbf{u_1}\right|}{\|\mathbf{n}\|} = \frac{\left|\mathbf{n}^\top \mathbf{u_2}\right|}{\|\mathbf{n}\|} \tag{2}$$
$$\left|\mathbf{n}^\top \mathbf{u_1}\right| = \left|\mathbf{n}^\top \mathbf{u_2}\right| \tag{3}$$
$$\mathbf{n}^\top \mathbf{u_1} = \pm \mathbf{n}^\top \mathbf{u_2} \tag{4}$$
$$\mathbf{n}^\top (\mathbf{u_1} \mp \mathbf{u_2}) = 0 \tag{5}$$

Here, $\mathbf{u_1}$ and $\mathbf{u_2}$ represent the unit vectors of $\mathbf{n_1}$ and $\mathbf{n_2}$ respectively. A vector perpendicular to given vector $\begin{pmatrix} 1 \\ m \end{pmatrix}$ is

$$\mathbf{n} = \begin{pmatrix} -m \\ 1 \end{pmatrix} \tag{6}$$

For the given question,

$$\mathbf{n_1} = \begin{pmatrix} 7 \\ -1 \end{pmatrix} \text{ and } \mathbf{n_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{7}$$
$$\|\mathbf{n_1}\| = \sqrt{50} = 5\sqrt{2} \tag{8}$$
$$\|\mathbf{n_2}\| = \sqrt{2} \tag{9}$$
$$\mathbf{u_1} = \frac{1}{5\sqrt{2}} \begin{pmatrix} 7 \\ -1 \end{pmatrix}, \ \mathbf{u_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{5}{5\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{10}$$

$$\mathbf{u_1} - \mathbf{u_2} = \frac{1}{5\sqrt{2}}\left(\begin{pmatrix}7\\-1\end{pmatrix} - \begin{pmatrix}5\\5\end{pmatrix}\right) = \frac{1}{5\sqrt{2}}\begin{pmatrix}2\\-6\end{pmatrix} \tag{11}$$

$$\mathbf{n_a} = \begin{pmatrix}1\\-3\end{pmatrix} \tag{12}$$

$$\mathbf{u_1} + \mathbf{u_2} = \frac{1}{5\sqrt{2}}\left(\begin{pmatrix}7\\-1\end{pmatrix} + \begin{pmatrix}5\\5\end{pmatrix}\right) = \frac{1}{5\sqrt{2}}\begin{pmatrix}12\\4\end{pmatrix} \tag{13}$$

$$\mathbf{n_b} = \begin{pmatrix}1\\\frac{1}{3}\end{pmatrix} \tag{14}$$

For the bisector parallel to $\mathbf{n_a}$, using (6),

$$\mathbf{n_p} = \begin{pmatrix}3\\1\end{pmatrix} \tag{15}$$

For the bisector parallel to $\mathbf{n_b}$, using (6),

$$\mathbf{n_q} = \begin{pmatrix}-\frac{1}{3}\\1\end{pmatrix} \tag{16}$$

For a line passing through a given point $\mathbf{p}$,

$$\mathbf{p} = \begin{pmatrix}1\\-10\end{pmatrix} \tag{17}$$

$$\mathbf{n}^\top\mathbf{x} = \mathbf{n}^\top\mathbf{p} \tag{18}$$

For $\mathbf{n_p}$,

$$\mathbf{n} = \begin{pmatrix}3\\1\end{pmatrix} \tag{19}$$

$$\begin{pmatrix}3 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}3 & 1\end{pmatrix}\begin{pmatrix}1\\-10\end{pmatrix} \tag{20}$$

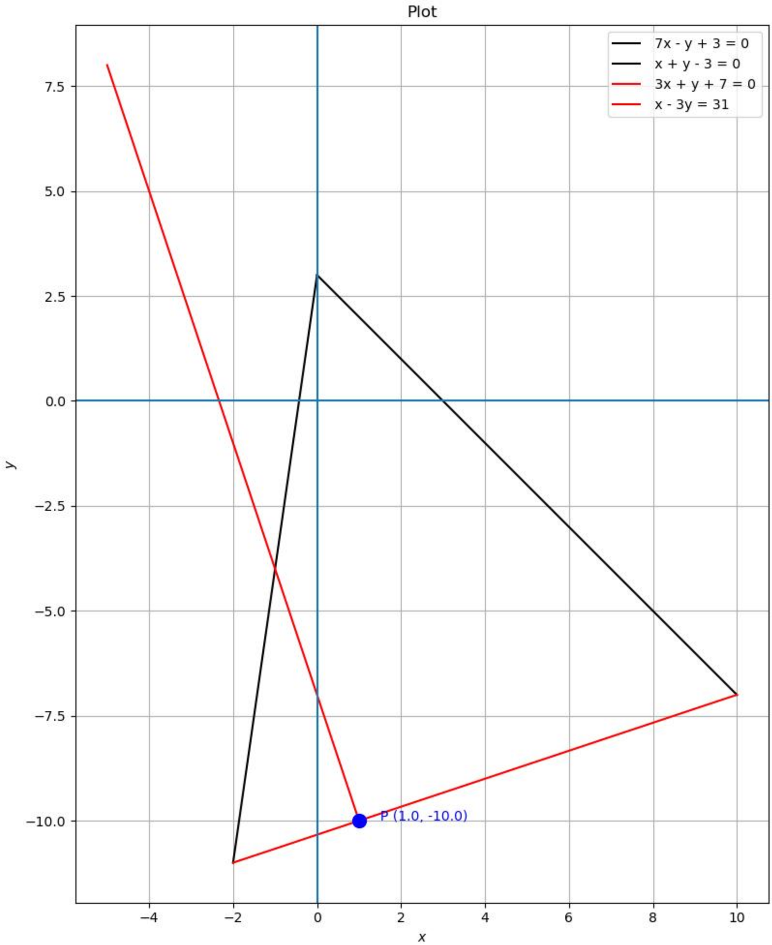
$$\begin{pmatrix}3 & 1\end{pmatrix}\mathbf{x} = -7 \tag{21}$$

For $\mathbf{n_q}$,

$$\mathbf{n} = \begin{pmatrix}1\\-3\end{pmatrix} \tag{22}$$

$$\begin{pmatrix}1 & -3\end{pmatrix}\mathbf{x} = \begin{pmatrix}1 & -3\end{pmatrix}\begin{pmatrix}1\\-10\end{pmatrix} \tag{23}$$

$$\begin{pmatrix}1 & -3\end{pmatrix}\mathbf{x} = 31 \tag{24}$$



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Isosceles Triangle) fig:fig)