Question: Let $\mathbf{a} = 4\hat{\imath} + 5\hat{\jmath} - \hat{k}$, $\mathbf{b} = \hat{\imath} - 4\hat{\jmath} + 5\hat{k}$, $\mathbf{c} = 3\hat{\imath} + \hat{\jmath} - \hat{k}$. Find $\mathbf{d}$ perpendicular to both $\mathbf{b}$ and $\mathbf{c}$ and satisfying $\mathbf{d} \cdot \mathbf{a} = 21$.

Solution:
Write vectors as column matrices:

$$\mathbf{a} = \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}.$$

Since $\mathbf{d}$ is perpendicular to both $\mathbf{b}$ and $\mathbf{c}$,

$$\mathbf{d} = \lambda(\mathbf{b} \times \mathbf{c}).$$

Compute the cross product:

$$\mathbf{b} \times \mathbf{c} = \begin{pmatrix} (-4)(-1) - 5(1) \\ -(1(-1) - 5(3)) \\ 1(1) - (-4)(3) \end{pmatrix} = \begin{pmatrix} -1 \\ 16 \\ 13 \end{pmatrix}.$$

Thus

$$\mathbf{d} = \lambda \begin{pmatrix} -1 \\ 16 \\ 13 \end{pmatrix}.$$

Now apply the condition $\mathbf{d} \cdot \mathbf{a} = 21$:

$$\mathbf{d} \cdot \mathbf{a} = \lambda \begin{pmatrix} -1 & 16 & 13 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix}.$$

$$= \lambda(-4 + 80 - 13) = \lambda(63).$$

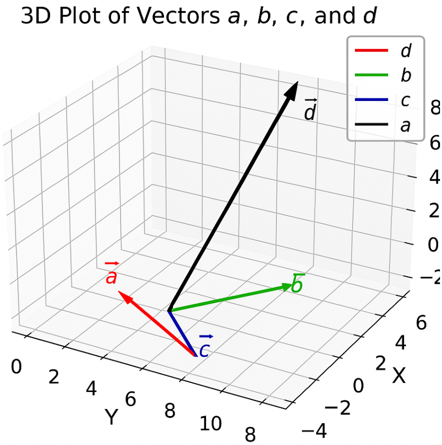
So

$$\lambda(63) = 21 \quad \Rightarrow \quad \lambda = \tfrac{1}{3}.$$

Hence

$$\mathbf{d} = \tfrac{1}{3} \begin{pmatrix} -1 \\ 16 \\ 13 \end{pmatrix} = -\tfrac{1}{3}\hat{\imath} + \tfrac{16}{3}\hat{\jmath} + \tfrac{13}{3}\hat{k}.$$

$$\boxed{\mathbf{d} = -\tfrac{1}{3}\hat{\imath} + \tfrac{16}{3}\hat{\jmath} + \tfrac{13}{3}\hat{k}}$$



3D Plot of Vectors a, b, c, and d
d
b
c
a
$\vec{d}$
$\vec{a}$
$\vec{b}$
$\vec{c}$
8
6
4
2
0
−2
6
4
2
0
−2
−4
X
0
2
4
6
8
10
Y

Fig. 1: plot