

8.4.23

EE25BTECH11020 - Darsh Pankaj Gajare

Question:

The curve described parametrically by $x = t^2 + t + 1$ and $y = t^2 - t + 1$ represents:

- 1) a pair of straight lines 2) an ellipse 4) a hyperbola
3) a parabola

Solution:

TABLE I

x	$\begin{pmatrix} x \\ y \end{pmatrix}$
a	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
b	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$
c	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The parametric form can be written as

$$\mathbf{x} = \mathbf{a}t^2 + \mathbf{b}t + \mathbf{c}. \quad (1)$$

$$\mathbf{x} = \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix}^\top \begin{pmatrix} t^2 \\ t \end{pmatrix} + \mathbf{c} \quad (2)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} t^2 \\ t \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (3)$$

Solving for $\begin{pmatrix} t^2 \\ t \end{pmatrix}$ using the inverse matrix,

$$\begin{pmatrix} t^2 \\ t \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \quad (4)$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right). \quad (5)$$

Multiplying,

$$\begin{pmatrix} t^2 \\ t \end{pmatrix} = \frac{1}{2} \begin{pmatrix} x + y - 2 \\ x - y \end{pmatrix}. \quad (6)$$

Eliminating t :

$$\begin{pmatrix} t^2 \\ t \end{pmatrix} = \frac{1}{2} \begin{pmatrix} x + y - 2 \\ x - y \end{pmatrix} \implies \frac{1}{2}(x + y - 2) = \left(\frac{1}{2}(x - y) \right)^2 \quad (7)$$

$$\Rightarrow (x - y)^2 = 2(x + y - 2) \quad (8)$$

$$\text{Write as quadratic form: } \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (9)$$

$$\mathbf{V} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \quad f = 4 \quad (10)$$

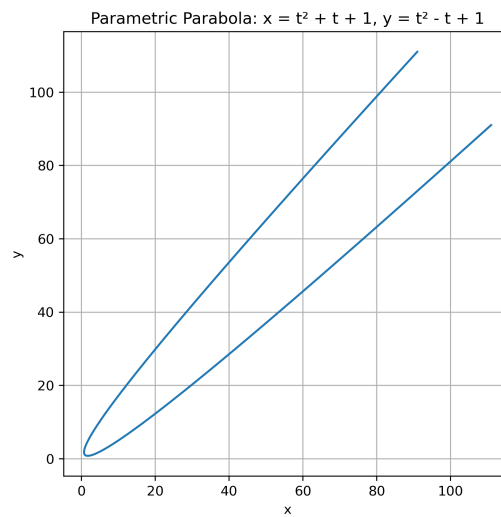
Extract quadratic coefficients:

$$A = V_{11} = 1, \quad B = 2V_{12} = -2, \quad C = V_{22} = 1 \quad (11)$$

Discriminant:

$$\Delta = B^2 - 4AC = (-2)^2 - 4(1)(1) = 0 \quad (12)$$

Since $\Delta = 0$ the conic is a parabola. Plot using C libraries:



Plot using Python:

