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EE25BTECH11042 - Nipun Dasari

Question:
Find the coordinates of the point where the line through the points P(4, 3, 2) and Q(5, 1, 6) crosses the XZ plane. Also find the angle which this line makes with the XZ plane. Solve using matrices and vectors only.
Solution:
First, we represent the points P and Q as position vectors.

$$\mathbf{p} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} \text{and} \mathbf{q} = \begin{pmatrix} 5 \\ 1 \\ 6 \end{pmatrix} \tag{0.1}$$

The direction vector, $\mathbf{d}$

$$\mathbf{d} = \mathbf{q} - \mathbf{p} = \begin{pmatrix} 5 \\ 1 \\ 6 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} \tag{0.2}$$

Line can be written as $\mathbf{r} = \mathbf{p} + \lambda \mathbf{d}$.

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 + \lambda \\ 3 - 2\lambda \\ 2 + 4\lambda \end{pmatrix} \tag{0.3}$$

Equation for x-z plane is

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^\top \mathbf{x} = 0 \tag{0.4}$$

On solving for point of intersection

$$3 - 2\lambda = 0 \implies 2\lambda = 3 \implies \lambda = \frac{3}{2} \tag{0.5}$$

By (0.5)

$$\mathbf{r}_{\text{intersection}} = \begin{pmatrix} 4 + \frac{3}{2} \\ 3 - 2(\frac{3}{2}) \\ 2 + 4(\frac{3}{2}) \end{pmatrix} = \begin{pmatrix} \frac{11}{2} \\ 0 \\ 8 \end{pmatrix} \tag{0.6}$$

The intersection point is $(\frac{11}{2}, 0, 8)$.
The angle θ between a line with direction vector $\mathbf{d}$ and a plane with normal vector $\mathbf{n}$ is given by:

$$\sin(\theta) = \frac{|\mathbf{d}^T \mathbf{n}|}{||\mathbf{d}||\, ||\mathbf{n}||} \tag{0.7}$$

The direction vector of the line is $\mathbf{d} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$. The normal vector to the XZ plane ($y = 0$) is $\mathbf{n} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

$$\mathbf{d}^T\mathbf{n} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}^T \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = (1)(0) + (-2)(1) + (4)(0) = -2 \tag{0.8}$$

$$\|\mathbf{d}\| = \sqrt{1^2 + (-2)^2 + 4^2} = \sqrt{1 + 4 + 16} = \sqrt{21} \tag{0.9}$$

$$\|\mathbf{n}\| = \sqrt{0^2 + 1^2 + 0^2} = 1 \tag{0.10}$$

By (0.7),(0.8),(0.9),(0.10)

$$\sin\theta = \frac{|-2|}{\sqrt{21}1} = \frac{2}{\sqrt{21}} \tag{0.11}$$

Therefore, the angle the line makes with the XZ plane is:

$$\theta = \arcsin\left(\frac{2}{\sqrt{21}}\right) \tag{0.12}$$