12.693

ee25btech11056 - Suraj.N

Question : Suppose the circles

$$x^2 + y^2 + ax + 6 = 0$$
$$x^2 + y^2 + bx - 4 = 0$$

intersect each other orthogonally at the point $(1, 2)$. Then $a + b =$ _

Solution :

Name	Value
Circle 1	$\mathbf{x}^\top\mathbf{x} + 2\begin{pmatrix}\frac{a}{2}\\0\end{pmatrix}^\top \mathbf{x} + 6 = 0$
Circle 2	$\mathbf{x}^\top\mathbf{x} + 2\begin{pmatrix}\frac{b}{2}\\0\end{pmatrix}^\top \mathbf{x} - 4 = 0$
$\mathbf{P}$	$\begin{pmatrix}1\\2\end{pmatrix}$

Table : Circles and Point

The conic parameters for the two circles can be expressed as :

$$\mathbf{V_1} = \mathbf{I} \qquad \mathbf{u_1} = \begin{pmatrix}\frac{a}{2}\\0\end{pmatrix} \qquad f1 = 6 \tag{1}$$

$$\mathbf{V_2} = \mathbf{I} \qquad \mathbf{u_2} = \begin{pmatrix}\frac{b}{2}\\0\end{pmatrix} \qquad f2 = -4 \tag{2}$$

The point of intersection of the two circles is $\mathbf{P}$

The equation of tangent to Circle 1 at $\mathbf{P}$ is given as :

$$(\mathbf{V_1}\mathbf{P} + \mathbf{u_1})^\top\mathbf{x} + \mathbf{u_1}^\top\mathbf{P} + f_1 = 0 \qquad \mathbf{n_1} = \mathbf{V_1}\mathbf{P} + \mathbf{u_1} \tag{3}$$

The equation of tangent to Circle 2 at $\mathbf{P}$ is given as :

$$(\mathbf{V_2}\mathbf{P} + \mathbf{u_2})^\top\mathbf{x} + \mathbf{u_2}^\top\mathbf{P} + f_2 = 0 \qquad \mathbf{n_2} = \mathbf{V_2}\mathbf{P} + \mathbf{u_2} \tag{4}$$

As the tangents at $\mathbf{P}$ are perpendicular , the normal vectors of the tangents are also perpendicular

$$\mathbf{n_1}^\top\mathbf{n_2} = 0 \tag{5}$$
$$(\mathbf{V_1}\mathbf{P} + \mathbf{u_1})^\top(\mathbf{V_2}\mathbf{P} + \mathbf{u_2}) = 0 \tag{6}$$
$$(\mathbf{P} + \mathbf{u_1})^\top(\mathbf{P} + \mathbf{u_2}) = 0 \tag{7}$$
$$(\mathbf{P}^\top + \mathbf{u_1}^\top)(\mathbf{P} + \mathbf{u_2}) = 0 \tag{8}$$
$$\begin{pmatrix}\frac{a+2}{2} & 2\end{pmatrix}\begin{pmatrix}\frac{b+2}{2}\\2\end{pmatrix} = 0 \tag{9}$$
$$2(a + b) + ab + 20 = 0 \tag{10}$$

As $\mathbf{P}$ lies on both the circles we get :

$$\mathbf{P}^\top\mathbf{P} + 2\mathbf{u_1}^\top\mathbf{P} + f_1 = 0 \tag{11}$$
$$\mathbf{P}^\top\mathbf{P} + 2\mathbf{u_2}^\top\mathbf{P} + f_2 = 0 \tag{12}$$

By subtracting the above two equations we get :

$$(\mathbf{u_1}^\top - \mathbf{u_2}^\top)\mathbf{P} = \frac{f_2 - f_1}{2} \tag{13}$$

If we substitue (11) in (8) we get :

$$(\mathbf{u_2}^\top - \mathbf{u_1}^\top)\mathbf{P} + \mathbf{u_1}^\top\mathbf{u_2} - f_1 = 0 \tag{14}$$
$$(\mathbf{u_1}^\top - \mathbf{u_2}^\top)\mathbf{P} = \mathbf{u_1}^\top\mathbf{u_2} - f_1 \tag{15}$$

From (13) and (15) , we get :

$$\mathbf{u_1}^\top\mathbf{u_2} = \frac{f_1 + f_2}{2} \tag{16}$$
$$ab = 4 \tag{17}$$

By substituting (17) in (10) , we get :

$$a + b = -12 \tag{18}$$

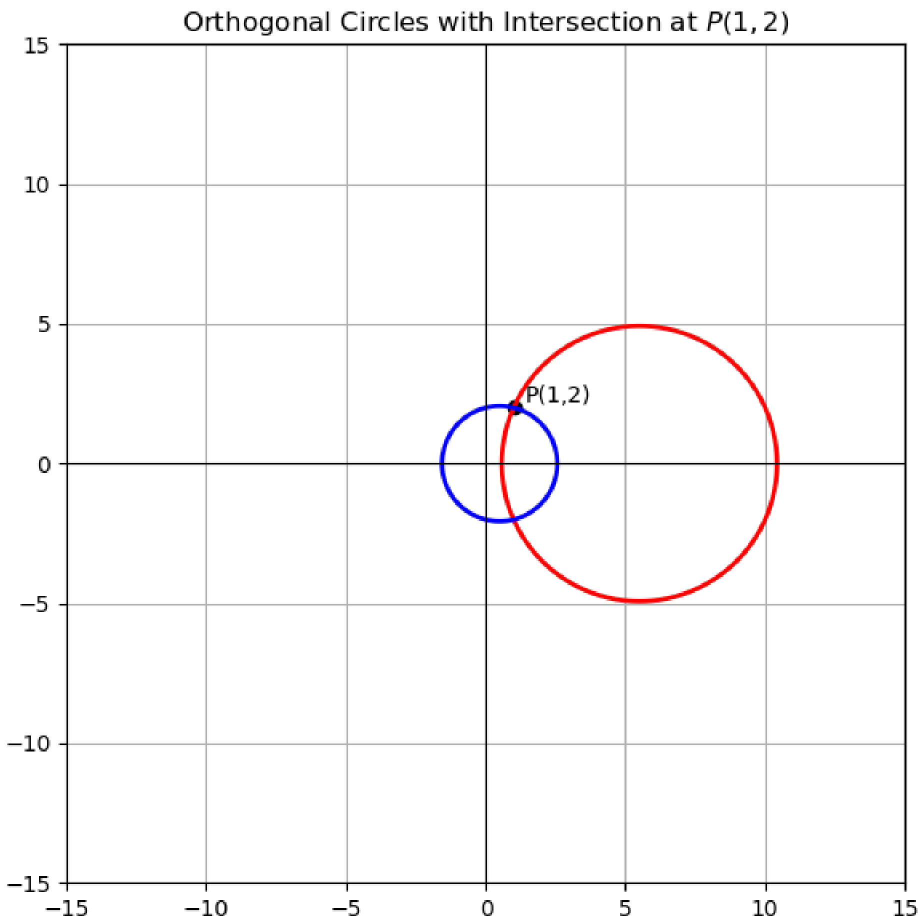
Orthogonal Circles with Intersection at $P(1,2)$
P(1,2)
15
10
5
0
−5
−10
−15
−15
−10
−5
0
5
10
15

Fig : Circles