Matgeo Presentation - Problem 12.797

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Problem Statement

Let $\mathbf{A}$ be an $n \times n$ real matrix. Consider the following statements:

(I) If $\mathbf{A}$ is symmetric, then there exists $c \geq 0$ such that $\mathbf{A} + c\mathbf{I}_n$ is symmetric and positive definite, where $\mathbf{I}_n$ is the $n \times n$ identity matrix.

(II) If $\mathbf{A}$ is symmetric and positive definite, then there exists a symmetric and positive definite $\mathbf{B}$ such that $\mathbf{A} = \mathbf{B}^2$.

Which of the above statements is/are true?

(a) Only (I)
(b) Only (II)
(c) Both (I) and (II)
(d) Neither (I) nor (II)

Data

Name	**Description**
A	Matrix

Table : Matrix

Solution

Checking statement (**I**)
If **A** is symmetric, its eigenvalues are real. Let the minimum eigenvalue of **A** be λ_{min}.Then choose $c > -\lambda_{\text{min}}$.
The Eigen values of **A** are given as :

$$\left|\mathbf{A} - \lambda_i \mathbf{I}\right| = 0 \tag{0.1}$$

The Eigen values of $\mathbf{A} + c\mathbf{I}_n$ are given as :

$$\left|\mathbf{A} - (\lambda_k - c)\mathbf{I}\right| = 0 \tag{0.2}$$

$$\lambda_k = \lambda_i + c \tag{0.3}$$

$$\lambda_i + c > 0 \tag{0.4}$$

Since $\lambda_i + c > 0$ for all i, $\mathbf{A} + c\mathbf{I}_n$ is positive definite and symmetric.
Hence, statement (**I**) is **true**.

Solution

Checking statement (**II**)

If **A** is symmetric and positive definite, then it can be diagonalized as:

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^\top \tag{0.5}$$

where **P** is orthogonal and **D** is a diagonal matrix with positive entries (since **A** is positive definite). Define

$$\mathbf{B} = \mathbf{P}\mathbf{D}^{1/2}\mathbf{P}^\top \tag{0.6}$$

Then,

$$\mathbf{B}^2 = \mathbf{P}\mathbf{D}^{1/2}\mathbf{P}^\top\mathbf{P}\mathbf{D}^{1/2}\mathbf{P}^\top = \mathbf{P}\mathbf{D}\mathbf{P}^\top = \mathbf{A} \tag{0.7}$$

Hence, **B** is symmetric and positive definite. Therefore, statement (**II**) is also **true**.

Final Answer: (c) Both (I) and (II)