4.11.25

EE25BTECH11048 - Revanth Siva Kumar.D

Question

Find the distance of the point $\left(1,-2,9\right)$ from the point of intersection of the line

$$\mathbf{r} = 4\hat{i} + 2\hat{j} + 7\hat{k} + \lambda\left(3\hat{i} + 4\hat{j} + 2\hat{k}\right)$$

and the plane

$$\mathbf{r} \cdot \left(\hat{i} - \hat{j} + \hat{k}\right) = 10.$$

Solution:

Step 1: General setup

Let the line be

$$\mathbf{r} = \mathbf{r}_0 + \lambda\mathbf{d}, \tag{1}$$

and the plane be

$$\mathbf{n}^T\mathbf{r} = c, \tag{2}$$

where $\mathbf{r}_0$ is a point on the line, $\mathbf{d}$ is its direction vector, $\mathbf{n}$ is the plane normal, and c is a constant. The given external point is $\mathbf{A}$.

Step 2: Intersection point of line and plane

Substitute $\mathbf{r} = \mathbf{r}_0 + \lambda\mathbf{d}$ into the plane equation:

$$\mathbf{n}^T(\mathbf{r}_0 + \lambda\mathbf{d}) = c. \tag{3}$$

This gives

$$\lambda = \frac{c - \mathbf{n}^T\mathbf{r}_0}{\mathbf{n}^T\mathbf{d}}. \tag{4}$$

Hence, the intersection point is

$$\mathbf{P} = \mathbf{r}_0 + \frac{c - \mathbf{n}^T\mathbf{r}_0}{\mathbf{n}^T\mathbf{d}}\mathbf{d}. \tag{5}$$

Step 3: Distance formula

The displacement vector is

$$\mathbf{v} = \mathbf{P} - \mathbf{A}, \tag{6}$$

$$\mathbf{v} = \mathbf{r}_0 + \frac{c - \mathbf{n}^T\mathbf{r}_0}{\mathbf{n}^T\mathbf{d}}\mathbf{d} - \mathbf{A} \tag{7}$$

and therefore the required distance is

$$d = \|\mathbf{v}\| = \sqrt{\mathbf{v}^T\mathbf{v}}. \tag{8}$$

Step 4: Substitution from the question

From the problem statement,

$$\mathbf{r}_0 = \begin{pmatrix} 4 \\ 2 \\ 7 \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}, \quad \mathbf{n} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad c = 10, \quad \mathbf{A} = \begin{pmatrix} 1 \\ -2 \\ 9 \end{pmatrix}. \tag{9}$$

Now compute:

$$\mathbf{n}^T \mathbf{d} = 1, \tag{10}$$

$$\mathbf{n}^T \mathbf{r}_0 = 9, \tag{11}$$

$$\lambda = \frac{10 - 9}{1} = 1. \tag{12}$$

So

$$\mathbf{P} = \begin{pmatrix} 4 \\ 2 \\ 7 \end{pmatrix} + 1 \cdot \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \\ 9 \end{pmatrix}, \tag{13}$$

$$\mathbf{v} = \mathbf{P} - \mathbf{A} = \begin{pmatrix} 6 \\ 8 \\ 0 \end{pmatrix}, \tag{14}$$

$$d = \sqrt{6^2 + 8^2 + 0^2} = 10. \tag{15}$$

Final Answer:

$$\boxed{10}$$

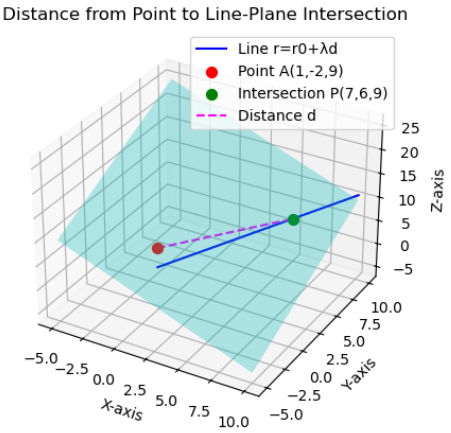

Fig. 1: Intersection point P, given point A, and distance AP.