Problem 5.7.5

If

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \tag{1}$$

find A^2.

Input Variables

Variable	Value
A	$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$
$\mathbf{u}$	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Table 1: Input variables

Solution

Method 1: Direct Matrix Multiplication

$$\begin{aligned}
A^2 &= A \cdot A && (2) \\
&= \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} && (3) \\
&= \begin{pmatrix} 1 \cdot 1 + (-1)(-1) & 1 \cdot (-1) + (-1)(1) \\ (-1)(1) + 1(-1) & (-1)(-1) + 1 \cdot 1 \end{pmatrix} && (4) \\
&= \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} && (5)
\end{aligned}$$

Method 2: Vector–Matrix Representation

$$
\begin{aligned}
A &= \mathbf{u}\mathbf{u}^T && (6)\\
A^2 &= (\mathbf{u}\mathbf{u}^T)(\mathbf{u}\mathbf{u}^T) && (7)\\
&= \mathbf{u}(\mathbf{u}^T\mathbf{u})\mathbf{u}^T && (8)\\
\mathbf{u}^T\mathbf{u} &= 1^2 + (-1)^2 = 2 && (9)\\
\implies A^2 &= 2\mathbf{u}\mathbf{u}^T && (10)\\
&= 2A && (11)\\
&= \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} && (12)
\end{aligned}
$$

Final Answer

$$A^2 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \tag{13}$$