MatGeo Assignment 2.6.13

AI25BTECH11007

Question:
Using vectors, find the area of $\triangle ABC$ with vertices A(1, 2, 3),B(2, -1, 4) and C(4, 5, -1).

Solution:
Compute vectors $\mathbf{B} - \mathbf{A}$ and $\mathbf{C} - \mathbf{A}$:

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}, \tag{0.1}$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix}. \tag{0.2}$$

Recall the identity:

$$\|\mathbf{B} - \mathbf{A} \times \mathbf{C} - \mathbf{A}\|^2 = \|\mathbf{B} - \mathbf{A}\|^2 \|\mathbf{C} - \mathbf{A}\|^2 - ((\mathbf{B} - \mathbf{A})^T (\mathbf{C} - \mathbf{A}))^2. \tag{0.3}$$

Compute the inner products:

$$\|\mathbf{B} - \mathbf{A}\|^2 = (\mathbf{B} - \mathbf{A})^T (\mathbf{B} - \mathbf{A}) = 1^2 + (-3)^2 + 1^2 = 11, \tag{0.4}$$

$$\|\mathbf{C} - \mathbf{A}\|^2 = (\mathbf{C} - \mathbf{A})^T (\mathbf{C} - \mathbf{A}) = 3^2 + 3^2 + (-4)^2 = 34, \tag{0.5}$$

$$(\mathbf{B} - \mathbf{A})^T (\mathbf{C} - \mathbf{A}) = (1)(3) + (-3)(3) + (1)(-4) = -10. \tag{0.6}$$

Substitute into (**??**):

$$\begin{aligned} \|\mathbf{B} - \mathbf{A} \times \mathbf{C} - \mathbf{A}\|^2 &= (11)(34) - (-10)^2 \\ &= 374 - 100 \\ &= 274. \end{aligned} \tag{0.7}$$

Hence

$$\|\mathbf{B} - \mathbf{A} \times \mathbf{C} - \mathbf{A}\| = \sqrt{274}. \tag{0.8}$$

The area of $\triangle ABC$ is

$$\text{Area}(\triangle ABC) = \tfrac{1}{2} \|\mathbf{B} - \mathbf{A} \times \mathbf{C} - \mathbf{A}\| = \frac{\sqrt{274}}{2}. \tag{0.9}$$

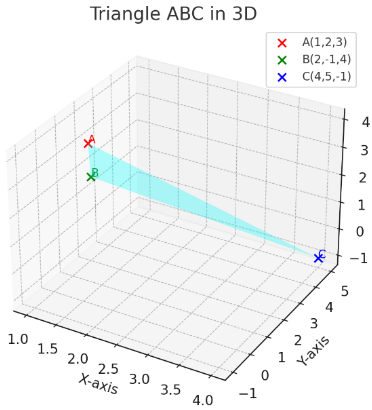

Fig. 0.1: Image Visual