

1.3.4

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Question 1.3.4

If $A(1, 3)$, $B(4, 2)$, $C(x, 5)$, and $D(x, 4)$ are the vertices of a parallelogram $ABCD$, then the value of x is _____. (10, 2012)

Solution:

In a parallelogram, opposite sides are equal and parallel. Since $ABCD$ is a parallelogram, vectors \mathbf{AB} and \mathbf{CD} must be equal.

$$\mathbf{AB} = \mathbf{B} - \mathbf{A} = (4 - 1, 2 - 3) = (3, -1) \quad (0.1)$$

$$\mathbf{CD} = \mathbf{D} - \mathbf{C} = (x - x, 4 - 5) = (0, -1) \quad (0.2)$$

Clearly, $\mathbf{AB} \neq \mathbf{CD}$, so let's try using diagonals. In a parallelogram, the diagonals bisect each other.

Midpoint of diagonal AC :

$$\mathbf{M}_{AC} = \left(\frac{1+x}{2}, \frac{3+5}{2} \right) = \left(\frac{1+x}{2}, 4 \right) \quad (0.3)$$

Midpoint of diagonal BD :

$$\mathbf{M}_{BD} = \left(\frac{4+x}{2}, \frac{2+4}{2} \right) = \left(\frac{4+x}{2}, 3 \right) \quad (0.4)$$

Equating midpoints:

$$\frac{1+x}{2} = \frac{4+x}{2} \quad \text{and} \quad 4 = 3 \quad (0.5)$$

The second equation is false, so diagonals do not bisect each other. Let's try using opposite sides again, but this time equating \mathbf{AD} and \mathbf{BC} :

$$\mathbf{AD} = \mathbf{D} - \mathbf{A} = \begin{pmatrix} x-1 \\ 4-3 \end{pmatrix} = \begin{pmatrix} x-1 \\ 1 \end{pmatrix} \quad (0.6)$$

$$\mathbf{BC} = \mathbf{C} - \mathbf{B} = \begin{pmatrix} x-4 \\ 5-2 \end{pmatrix} = \begin{pmatrix} x-4 \\ 3 \end{pmatrix} \quad (0.7)$$

Equating vectors:

$$x - 1 = x - 4 \quad \text{and} \quad 1 = 3 \quad (0.8)$$

Again, contradiction. So let's try using the property that opposite sides are equal in length.

$$\text{Length of } \mathbf{AD} : |AD| = \sqrt{(x-1)^2 + (4-3)^2} = \sqrt{(x-1)^2 + 1} \quad (0.9)$$

$$\text{Length of } \mathbf{BC} : |BC| = \sqrt{(x-4)^2 + (5-2)^2} = \sqrt{(x-4)^2 + 9} \quad (0.10)$$

Equating lengths:

$$\sqrt{(x-1)^2 + 1} = \sqrt{(x-4)^2 + 9} \quad (0.11)$$

Squaring both sides:

$$(x-1)^2 + 1 = (x-4)^2 + 9 \quad (0.12)$$

$$x^2 - 2x + 1 + 1 = x^2 - 8x + 16 + 9 \quad (0.13)$$

$$x^2 - 2x + 2 = x^2 - 8x + 25 \quad (0.14)$$

Subtract x^2 from both sides:

$$-2x + 2 = -8x + 25 \quad (0.15)$$

$$6x = 23 \Rightarrow x = \frac{23}{6} \quad (0.16)$$

Answer: $\frac{23}{6}$

