

12.393

AI25BTECH11003 - Bhavesh Gaikwad

Question: Values of a, b, c which render the matrix

$$\mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & a \\ \frac{1}{\sqrt{3}} & 0 & b \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & c \end{pmatrix}$$

orthonormal are, respectively,

(AE 2013)

- a) $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0$
- b) $\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}$
- c) $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$
- d) $\frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}$

Solution:

Given: \mathbf{Q} is an orthogonal matrix.

$$\mathbf{Q} = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & \sqrt{3} & \sqrt{6}a \\ \sqrt{2} & 0 & \sqrt{6}b \\ \sqrt{2} & -\sqrt{3} & \sqrt{6}c \end{pmatrix}$$

Condition for orthogonality,

$$\mathbf{A}^T \mathbf{A} = \mathbf{I} \quad (0.1)$$

Substituting \mathbf{Q} in Equation 0.1,

$$\mathbf{Q}^T \mathbf{Q} = \frac{1}{6} \begin{pmatrix} 6 & 0 & 2\sqrt{3}(a+b+c) \\ 0 & 6 & 3\sqrt{2}(a-c) \\ 2\sqrt{3}(a+b+c) & 3\sqrt{2}(a-c) & 6(a^2+b^2+c^2) \end{pmatrix} \quad (0.2)$$

$$\frac{1}{6} \begin{pmatrix} 6 & 0 & 2\sqrt{3}(a+b+c) \\ 0 & 6 & 3\sqrt{2}(a-c) \\ 2\sqrt{3}(a+b+c) & 3\sqrt{2}(a-c) & 6(a^2+b^2+c^2) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (0.3)$$

On Comparing we get,

$$\text{Let } \mathbf{P} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\mathbf{P}^\top \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0 \quad (0.4)$$

$$\mathbf{P}^\top \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 0 \quad (0.5)$$

$$\mathbf{P}^\top \mathbf{P} = \|\mathbf{P}\|^2 = 1 \quad (0.6)$$

$$\therefore \mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} \quad OR \quad \mathbf{P} = \begin{pmatrix} \frac{-1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \end{pmatrix} \quad (0.7)$$

Thus, Option B and D are correct.