5.13.62

EE25BTECH11036 - M Chanakya Srinivas

Question

How many 3×3 matrices M, with entries from the set $\{0, 1, 2\}$, satisfy

$$\operatorname{tr}(M^\top M) = 5? \tag{1}$$

Solution

Let the matrix $M \in \mathbb{R}^{3\times 3}$ be represented column-wise as:

$$M = \begin{pmatrix} \mathbf{c_1} & \mathbf{c_2} & \mathbf{c_3} \end{pmatrix} \tag{2}$$

where each column $\mathbf{c_j} \in \mathbb{R}^3$, and entries $c_{ij} \in \{0, 1, 2\}$.

Each column vector is of the form:

$$\mathbf{c_j} = \begin{pmatrix} c_{1j} \\ c_{2j} \\ c_{3j} \end{pmatrix}, \quad j = 1, 2, 3 \tag{3}$$

Since each entry in the vector is from $\{0, 1, 2\}$, the number of possible 3D vectors is:

$$3^3 = 27 \tag{4}$$

Step 1: Compute $M^\top M$

$$M^\top M = \begin{pmatrix} \mathbf{c_1}^\top \mathbf{c_1} & \mathbf{c_1}^\top \mathbf{c_2} & \mathbf{c_1}^\top \mathbf{c_3} \\ \mathbf{c_2}^\top \mathbf{c_1} & \mathbf{c_2}^\top \mathbf{c_2} & \mathbf{c_2}^\top \mathbf{c_3} \\ \mathbf{c_3}^\top \mathbf{c_1} & \mathbf{c_3}^\top \mathbf{c_2} & \mathbf{c_3}^\top \mathbf{c_3} \end{pmatrix} \tag{5}$$

The trace is the sum of diagonal entries:

$$\operatorname{tr}(M^\top M) = \mathbf{c_1}^\top \mathbf{c_1} + \mathbf{c_2}^\top \mathbf{c_2} + \mathbf{c_3}^\top \mathbf{c_3} = \|\mathbf{c_1}\|^2 + \|\mathbf{c_2}\|^2 + \|\mathbf{c_3}\|^2 \tag{6}$$

We are given:

$$\|\mathbf{c_1}\|^2 + \|\mathbf{c_2}\|^2 + \|\mathbf{c_3}\|^2 = 5 \tag{7}$$

Let:

$$n_1 = \|\mathbf{c_1}\|^2, \quad n_2 = \|\mathbf{c_2}\|^2, \quad n_3 = \|\mathbf{c_3}\|^2 \Rightarrow n_1 + n_2 + n_3 = 5 \tag{8}$$

Step 2: Count vectors with given norm

Let $N(n)$ be the number of vectors $\mathbf{v} \in \mathbb{R}^3$ with entries from $\{0, 1, 2\}$ and squared norm $||\mathbf{v}||^2 = n$.

By enumeration:

$$\begin{aligned}
N(0) &= 1 \quad (\begin{pmatrix}0\\0\\0\end{pmatrix}) \\
N(1) &= 3 \quad (\text{3 vectors with one 1}) \\
N(2) &= 3 \quad (\text{3 vectors with one 2}) \\
N(3) &= 6 \quad (\text{e.g., } \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}, \ldots) \\
N(4) &= 3 \quad (\text{e.g., } \begin{pmatrix}2\\0\\0\end{pmatrix}, \ldots) \\
N(5) &= 6
\end{aligned}$$

Step 3: Final Count Using Vector Norm Histogram

Each column $\mathbf{c_i} \in \mathbb{R}^3$ can take values from the set $\{0, 1, 2\}^3$, so total:

$$\text{27 vectors} \tag{9}$$

We want to count all ordered triples of vectors $(\mathbf{c_1}, \mathbf{c_2}, \mathbf{c_3})$ such that:

$$||\mathbf{c_1}||^2 + ||\mathbf{c_2}||^2 + ||\mathbf{c_3}||^2 = 5 \tag{10}$$

That is:

$$\text{Total} = \sum_{\substack{n_1+n_2+n_3=5 \\ 0 \leq n_i \leq 5}} N(n_1) \cdot N(n_2) \cdot N(n_3) \tag{11}$$

This is a bounded discrete sum over all integer solutions $(n_1, n_2, n_3) \in \{0, 1, \ldots, 5\}$ such that $n_1 + n_2 + n_3 = 5$.

Using enumeration or a small program, we evaluate this sum:

$$\boxed{198} \tag{12}$$

Final Answer

$$\boxed{198} \tag{13}$$