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EE25BTECH11032 - Kartik Lahoti

Question:

If $\left(m_i, \frac{1}{m_i}\right)$, $m_i > 0, i = 1, 2, 3, 4$ are four distinct points on a circle, then show that $m_1 m_2 m_3 m_4 = 1$

Solution:

Let the circle equation be

$$\|\mathbf{x}\|^2 + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{0.1}$$

where, $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ with a and b as constants.

Let $\mathbf{P} = \begin{pmatrix} m \\ \frac{1}{m} \end{pmatrix}$ be a arbitrary vector in space.

Putting $\mathbf{P}$ in the circle , we get

$$\|\mathbf{P}\|^2 + 2\mathbf{u}^\top \mathbf{P} + f = 0 \tag{0.2}$$

$$m^2 + \frac{1}{m^2} + 2am + \frac{2b}{m} + f = 0 \tag{0.3}$$

Let,

$$p(m) = m^4 + 2am^3 + fm^2 + 2bm + 1 = 0 \tag{0.4}$$

A general polynomial of degree n, has companion matrix as

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & 0 & \cdots & 0 & -c_2 \\ 0 & 0 & 1 & \cdots & 0 & -c_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -c_{n-1} \end{pmatrix} \tag{0.5}$$

The eigen values of the Companion Matrix $\mathbf{C}$ are the roots of the polynomial.

For the question ,

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -2b \\ 0 & 1 & 0 & -f \\ 0 & 0 & 1 & -2a \end{pmatrix} \tag{0.6}$$

$$\left| m\mathbf{I} - \mathbf{C} \right| = p(m) \tag{0.7}$$

Here Eigen values of $\mathbf{C}$ are m_i where $i \in \{1, 2, 3, 4\}$
Introducing Reversal Matrix

$$\mathbf{J} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{0.8}$$

$$\mathbf{J}^2 = \mathbf{I} \tag{0.9}$$

The Matrix $\mathbf{J}$ flips Rows when pre-multiplied and flips column when post-multiplied. Then,

$$\left| \tfrac{1}{m}\mathbf{I} - \mathbf{JCJ} \right| = p\left(\frac{1}{m}\right) \tag{0.10}$$

Since $p\left(\frac{1}{m}\right)$ has eigen values $1/m_i$, we can say

$$\mathbf{JCJ} = \mathbf{C}^{-1} \tag{0.11}$$

$$\tag{0.12}$$

Taking determinant, using 0.9

$$\left|\mathbf{C}\right| = \left|\mathbf{C}^{-1}\right| \tag{0.13}$$

$$\left|\mathbf{C}\right|^2 = 1 \tag{0.14}$$

Since $\mathbf{C}$ is a real companion matrix of a monic quartic whose constant term is 1 ,

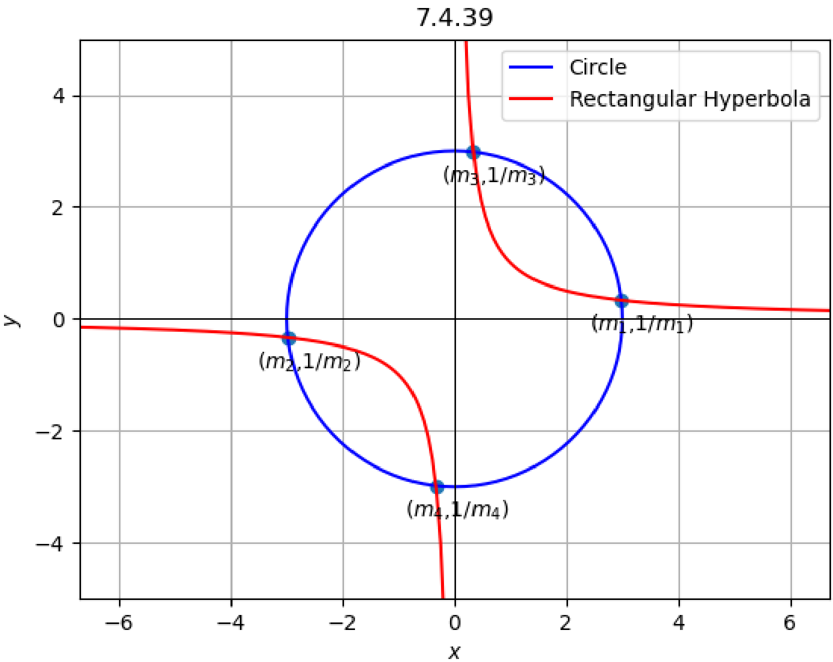
$$\left|\mathbf{C}\right| = (-1)^4\, 1 \tag{0.15}$$

Also,

$$\left|\mathbf{C}\right| = m_1 m_2 m_3 m_4 \tag{0.16}$$

$$\therefore m_1 m_2 m_3 m_4 = 1 \tag{0.17}$$

Hence Proved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Circle
Rectangular Hyperbola
$(m_3, 1/m_3)$
$(m_1, 1/m_1)$
$(m_2, 1/m_2)$
$(m_4, 1/m_4)$
4
2
0
−2
−4
−6
−4
−2
0
2
4
6
x
y