5.13.39

EE25BTECH11013 - Bhargav

Question:

Let $\mathbf{P} = \begin{pmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{pmatrix}$ where $\alpha \in \mathbf{R}$. Suppose $\mathbf{Q} = \begin{pmatrix} q_{ij} \end{pmatrix}$ is a matrix such that $\mathbf{PQ} = \mathrm{k}\mathbf{I}$, where k $\neq 0$ and $\mathbf{I}$ is the identity of order 3. If q_{23} = -$\frac{k}{8}$ and $\det \mathbf{Q} = \frac{k^2}{2}$, then

1) a = 0, k = 8
2) 4a - k + 8 = 0
3) $\det(\mathbf{P}adj(\mathbf{Q})) = 2^9$
4) $\det(\mathbf{Q}adj(\mathbf{P})) = 2^{13}$

Solution:

It is given that

$$\mathbf{PQ} = k\mathbf{I}, \det \mathbf{Q} = \frac{k^2}{2} \tag{4.1}$$

Taking the determinant

$$(\det \mathbf{P}) \cdot \frac{k^2}{2} = k^3 \tag{4.2}$$

$$\left| \begin{pmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{pmatrix} \right| = 2k \tag{4.3}$$

$$\begin{pmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{pmatrix} \xleftrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - \frac{2}{3}R_1} \begin{pmatrix} 3 & -1 & -2 \\ 0 & \frac{2}{3} & \alpha + \frac{4}{3} \\ 0 & -4 & 2 \end{pmatrix} \tag{4.4}$$

From equation (4.3) we get

$$3 \times \left(\frac{2}{3} \times 2 - (-4) \times \left(\alpha + \frac{4}{3} \right) \right) = 2k \tag{4.5}$$

$$20 + 12\alpha = 2k \tag{4.6}$$

Using the relation $\mathbf{PQ} = k\mathbf{I}$, we get the following augmented matrix

$$\left(\begin{array}{ccc|ccc} 3 & -1 & -2 & 1 & 0 & 0 \\ 2 & 0 & \alpha & 0 & 1 & 0 \\ 3 & -5 & 0 & 0 & 0 & 1 \end{array}\right) \xleftrightarrow[R_2 \leftarrow R_2 - 2R_1]{R_1 \leftarrow \frac{1}{3}R_1} \left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & \alpha + \frac{4}{3} & -\frac{2}{3} & 1 & 0 \\ 3 & -5 & 0 & 0 & 0 & 1 \end{array}\right) \tag{4.7}$$

$$\xleftrightarrow[R_2\leftarrow\frac{3}{2}R_2]{R_3\leftarrow R_3-3R_1}\left(\begin{array}{ccc|ccc}1 & -\frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & 0 & 0\\ 0 & 1 & \frac{3}{2}\alpha+2 & -1 & \frac{3}{2} & 0\\ 0 & -4 & 2 & -1 & 0 & 1\end{array}\right)\xleftrightarrow[R_3\leftarrow R_3+4R_2]{R_1\leftarrow\frac{1}{3}R_2}\left(\begin{array}{ccc|ccc}1 & 0 & \frac{1}{2}\alpha & 0 & \frac{1}{2} & 0\\ 0 & 1 & \frac{3}{2}\alpha+2 & -1 & \frac{3}{2} & 0\\ 0 & 0 & 6\alpha+10 & -5 & 6 & 1\end{array}\right) \tag{4.8}$$

$$\xleftrightarrow{R_3\leftarrow\frac{1}{6\alpha+10}R_3}\left(\begin{array}{ccc|ccc}1 & 0 & \frac{1}{2}\alpha & 0 & \frac{1}{2} & 0\\ 0 & 1 & \frac{3}{2}\alpha+2 & -1 & \frac{3}{2} & 0\\ 0 & 0 & 1 & -\frac{5}{6\alpha+10} & \frac{6}{6\alpha+10} & \frac{1}{6\alpha+10}\end{array}\right) \tag{4.9}$$

$$\xleftrightarrow[R_2\leftarrow R_2-\left(\frac{3}{2}\alpha+2\right)R_3]{R_1\leftarrow R_1-\frac{1}{2}\alpha R_3}\left(\begin{array}{ccc|ccc}1 & 0 & 0 & \frac{5\alpha}{12\alpha+20} & \frac{3\alpha+10}{6\alpha+20} & -\frac{\alpha}{12\alpha+20}\\ 0 & 1 & 0 & -1+\frac{5(3\alpha+4)}{12\alpha+20} & \frac{3}{2}-\frac{6(3\alpha+4)}{12\alpha+20} & -\frac{3\alpha+4}{12\alpha+20}\\ 0 & 0 & 1 & -\frac{5}{6\alpha+10} & \frac{6}{6\alpha+10} & \frac{1}{6\alpha+10}\end{array}\right) \tag{4.10}$$

From this augmented matrix,

$$q_{23} = -k\frac{3\alpha+4}{12\alpha+20} = -\frac{k}{8}\,(Given) \tag{4.11}$$

$$\implies \alpha = -1 \tag{4.12}$$

Substituting the value of α in equation (4.6), we get

$$k = 4 \tag{4.13}$$

n is the order of matrix B

$$\left|\mathbf{A}\mathrm{adj}\left(\mathbf{B}\right)\right| = \left|\mathbf{A}\cdot\mathbf{B}^{n-1}\right| \tag{4.14}$$

$|\mathbf{P}| = 8$, $|\mathbf{Q}| = 8$

$$|(\mathbf{P}adj(\mathbf{Q}))| = |\mathbf{P}|\,|\mathbf{Q}|^2 = 8\times 64 = 2^9 \tag{4.15}$$

$$|(\mathbf{Q}adj(\mathbf{P}))| = |\mathbf{Q}|\,|\mathbf{P}|^2 = 8\times 64 = 2^9 \tag{4.16}$$

So options (2) and (3) are correct