Problem 8.4.26

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Problem

The locus of the mid-point of the line segment joining the focus to a moving point on the parabola $y^2 = 4ax$ is another parabola with directrix

1. $x = 0$
2. $x = -a/2$
3. $x = a$
4. $x = a/2$

Formula

The equation of a conic with directrix $\mathbf{n}^\top \mathbf{x} = c$, eccentricity e and focus $\mathbf{F}$ is given by

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1.1}$$

On comparing with $y^2 - 4ax = 0$

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} = y^2 \tag{1.2}$$

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} = \left[\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x}\right]^2 \tag{1.3}$$

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} = \left(\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x}\right)^\top \left(\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x}\right) \tag{1.4}$$

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} = \mathbf{x}^\top \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} \tag{1.5}$$

$$\mathbf{x}^\top \left(\mathbf{V} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) \mathbf{x} = 0 \implies \mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.6}$$

Locus

$$2\mathbf{u}^\top \mathbf{x} = -4ax \tag{1.7}$$

$$2\mathbf{u}^\top \mathbf{x} = -4a\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} \implies 2\mathbf{u}^\top \mathbf{x} = \begin{pmatrix}-4a & 0\end{pmatrix}\mathbf{x} \tag{1.8}$$

$$\left(2\mathbf{u}^\top - \begin{pmatrix}-4a & 0\end{pmatrix}\right)\mathbf{x} = 0 \implies \mathbf{u}^\top = \begin{pmatrix}-2a & 0\end{pmatrix} \tag{1.9}$$

$$\mathbf{u} = \begin{pmatrix}-2a \\ 0\end{pmatrix} \tag{1.10}$$

$$f = 0 \tag{1.11}$$

$$\mathbf{F} = \begin{pmatrix}a \\ 0\end{pmatrix} \tag{1.12}$$

Let $\mathbf{X}$ be the point of locus of the midpoint

$$\mathbf{X} = \frac{\mathbf{x} + \mathbf{F}}{2} \implies \mathbf{x} = 2\mathbf{X} - \mathbf{F} \tag{1.13}$$

Simplify

From (1) and (13)

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1.14}$$

$$(2\mathbf{X} - \mathbf{F})^\top \mathbf{V}(2\mathbf{X} - \mathbf{F}) + 2\mathbf{u}^\top (2\mathbf{X} - \mathbf{F}) + f = 0 \tag{1.15}$$

$$\left(2\mathbf{X}^\top - \mathbf{F}^\top\right) \mathbf{V}(2\mathbf{X} - \mathbf{F}) + 2\mathbf{u}^\top (2\mathbf{X} - \mathbf{F}) + f = 0 \tag{1.16}$$

$$4\mathbf{X}^\top \mathbf{V}\mathbf{X} - 2\mathbf{X}^\top \mathbf{V}\mathbf{F} - 2\mathbf{F}^\top \mathbf{V}\mathbf{X} + \mathbf{F}^\top \mathbf{V}\mathbf{F} + 4\mathbf{u}^\top \mathbf{X} - 2\mathbf{u}^\top \mathbf{F} + f = 0 \tag{1.17}$$

As $\mathbf{V}$ is a symmetric matrix

$$4\mathbf{X}^\top \mathbf{V}\mathbf{X} - 2\mathbf{F}^\top \mathbf{V}\mathbf{X} - 2\mathbf{F}^\top \mathbf{V}\mathbf{X} + \mathbf{F}^\top \mathbf{V}\mathbf{F} + 4\mathbf{u}^\top \mathbf{X} - 2\mathbf{u}^\top \mathbf{F} + f = 0 \tag{1.18}$$

$$4\mathbf{X}^\top \mathbf{V}\mathbf{X} - 4\mathbf{F}^\top \mathbf{V}\mathbf{X} + 4\mathbf{u}^\top \mathbf{X} + \mathbf{F}^\top \mathbf{V}\mathbf{F} - 2\mathbf{u}^\top \mathbf{F} + f = 0 \tag{1.19}$$

$$\mathbf{X}^\top (4\mathbf{V})\,\mathbf{X} + 2\left(2\left(\mathbf{u}^\top - \mathbf{F}^\top \mathbf{V}\right)\right)\mathbf{X} + \mathbf{F}^\top \mathbf{V}\mathbf{F} - 2\mathbf{u}^\top \mathbf{F} + f = 0 \tag{1.20}$$

$$\mathbf{X}^\top (4\mathbf{V})\,\mathbf{X} + 2\left(2(\mathbf{u} - \mathbf{V}\mathbf{F})^\top\right)\mathbf{X} + \mathbf{F}^\top \mathbf{V}\mathbf{F} - 2\mathbf{u}^\top \mathbf{F} + f = 0 \tag{1.21}$$

$$\mathbf{V}' = 4\mathbf{V} = 4\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \tag{1.22}$$

Finding the variables

$$\mathbf{u}' = 2\left(\mathbf{u} - \mathbf{V}\mathbf{F}\right) = 2\left(\begin{pmatrix} -2a \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ 0 \end{pmatrix}\right) = 2\left(\begin{pmatrix} -2a \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} -4a \\ 0 \end{pmatrix} \tag{1.23}$$

$$f' = \mathbf{F}^\top \mathbf{V}\mathbf{F} - 2\mathbf{u}^\top \mathbf{F} + f \tag{1.24}$$

$$f' = \begin{pmatrix} a \\ 0 \end{pmatrix}^\top \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} a \\ 0 \end{pmatrix} - 2\begin{pmatrix} -2a \\ 0 \end{pmatrix}^\top \begin{pmatrix} a \\ 0 \end{pmatrix} + 0 \tag{1.25}$$

$$f' = \begin{pmatrix} a & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} a \\ 0 \end{pmatrix} - 2\begin{pmatrix} -2a & 0 \end{pmatrix}\begin{pmatrix} a \\ 0 \end{pmatrix} + 0 \tag{1.26}$$

$$f' = \begin{pmatrix} 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ 0 \end{pmatrix} - 2\begin{pmatrix} -2a & 0 \end{pmatrix}\begin{pmatrix} a \\ 0 \end{pmatrix} + 0 \tag{1.27}$$

$$f' = -2\left(-2a^2\right) = 4a^2 \tag{1.28}$$

Eigen vector

Finding eigen values of $\mathbf{V}'$

$$|\mathbf{V}' - \lambda\mathbf{I}| = 0 \tag{1.29}$$

$$\left|\begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right| = 0 \implies \begin{vmatrix} -\lambda & 0 \\ 0 & 4-\lambda \end{vmatrix} = 0 \tag{1.30}$$

$$(-\lambda)(4-\lambda) = 0 \implies \lambda_1 = 0 \text{ and } \lambda_2 = 4 \tag{1.31}$$

$\mathbf{p_1}$ is an eigen vector of $\mathbf{V}'$

$$(\mathbf{V}' - \lambda\mathbf{I})\,\mathbf{p} = \mathbf{0} \tag{1.32}$$

From (30) and substituting λ=0

$$\begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{p_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.33}$$

$$0 = 0, y = 0 \implies \mathbf{p_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.34}$$

Conclusion

The Equation of a Directrix is given by

$$\mathbf{n}^{\top}\mathbf{x} = c \tag{1.35}$$

where

$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{p_1} \text{ (and) } c = \frac{\left(\|\mathbf{u}'\|^2 - \lambda_2 f\right)}{2\mathbf{u}'^{\top}\mathbf{n}} \tag{1.36}$$

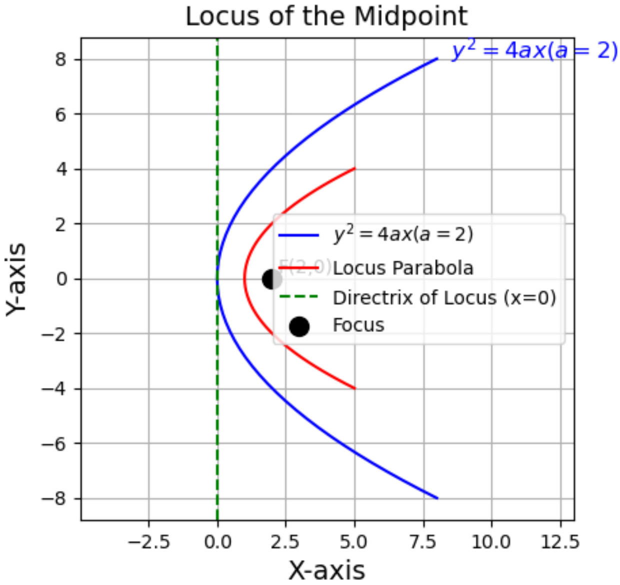
$$\mathbf{n} = \sqrt{4}\begin{pmatrix}1\\0\end{pmatrix} \implies \mathbf{n} = \begin{pmatrix}2\\0\end{pmatrix} \tag{1.37}$$

$$c = \frac{\left((-4a)^2 + (0)^2 - 4\left(4a^2\right)\right)}{2\begin{pmatrix}-4a\\0\end{pmatrix}^{\top}\begin{pmatrix}2\\0\end{pmatrix}} = \frac{\left(16a^2 - 16a^2\right)}{2\begin{pmatrix}-4a\\0\end{pmatrix}^{\top}\begin{pmatrix}2\\0\end{pmatrix}} = 0 \tag{1.38}$$

From (36)

$$\begin{pmatrix}2 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = 0 \implies 2x = 0 \implies x = 0 \tag{1.39}$$

Plot



Locus of the Midpoint
$y^2 = 4ax(a = 2)$
$y^2 = 4ax(a = 2)$
Locus Parabola
Directrix of Locus (x=0)
Focus
F(2,0)
Y-axis
X-axis
8
6
4
2
0
−2
−4
−6
−8
−2.5
0.0
2.5
5.0
7.5
10.0
12.5

C Code

```c
void get_plot_data(double* out_data) {
    double a = 2.0;
    int num_points = 51;
    double t;
    out_data[0] = a;
    out_data[1] = 0.0;
    int index = 2;
    for (int i = 0; i < num_points; i++) {
        t = -2.0 + (4.0 * i) / (num_points - 1);
        out_data[index] = a * t * t;
        out_data[index + 1] = 2 * a * t;
        out_data[index + (num_points * 2)] = (a + a * t * t) /
            2.0;
        out_data[index + (num_points * 2) + 1] = a * t;
        index += 2;
    }
}
```

Python Code for Solving

```
import ctypes
import numpy as np

def get_data_from_c():
    lib = ctypes.CDLL('./coord.so')

    data_size = 2 + (2 * 51 * 2)
    double_array = ctypes.c_double * data_size
    lib.get_plot_data.argtypes = [ctypes.POINTER(ctypes.c_double)
        ]

    out_data_c = double_array()
    lib.get_plot_data(out_data_c)

    return np.array(out_data_c)
```

Python Code for Plotting

```
# Code by /sdcard/github/matgeo/codes/CoordGeoVV Sharma
# September 12, 2023
# Revised July 21, 2024
# Released under GNU GPL
# Section Formula
import sys
sys.path.insert(0, '/workspaces/urban-potato/matgeo/codes/
    CoordGeo/')
import numpy as np
import matplotlib.pyplot as plt

from call import get_data_from_c

all_data = get_data_from_c()
num_points = 51
focus = all_data[0:2]
parabola_orig = all_data[2 : 2 + num_points*2].reshape((
    num_points, 2))
```

Python Code for Plotting

```
parabola_locus = all_data[2 + num_points*2 :].reshape((num_points
    , 2))
fig, ax = plt.subplots(figsize=(10, 8))

ax.plot(parabola_orig[:, 0], parabola_orig[:, 1], 'b-', label='$y
    ^2=4ax$')
ax.plot(parabola_locus[:, 0], parabola_locus[:, 1], 'r-', label='
    Locus Parabola')
ax.text(8 + 0.5, 8, '$y^2=4ax$', fontsize=12, color='blue')
ax.axvline(x=0, color='g', linestyle='--', label='Directrix of
    Locus (x=0)')

ax.scatter(focus[0], focus[1], color='black', s=100, zorder=5,
    label='Focus')
ax.text(focus[0] + 0.2, focus[1] + 0.2, 'F')
ax.set_title('Locus of the Midpoint')
ax.set_xlabel('X-axis');ax.set_ylabel('Y-axis')
ax.grid(True);ax.axis('equal');ax.legend()
plt.show()
```