

Problem 8.4.26

ee25btech11023-Venkata Sai

September 28, 2025

1 Problem

2 Solution

- Formula
- Locus
- Simplify
- Finding the variables
- Eigen vector
- Plot

3 C Code

4 Python Code

Problem

The locus of the mid-point of the line segment joining the focus to a moving point on the parabola $y^2 = 4ax$ is another parabola with directrix

- ① $x = 0$
- ② $x = -a/2$
- ③ $x = a$
- ④ $x = a/2$

Formula

The equation of a conic with directrix $\mathbf{n}^\top \mathbf{x} = c$, eccentricity e and focus \mathbf{F} is given by

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \quad (1.1)$$

On comparing with $y^2 - 4ax = 0$

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} = y^2 \quad (1.2)$$

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} = \left[\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} \right]^2 \quad (1.3)$$

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} = \left(\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} \right)^\top \left(\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} \right) \quad (1.4)$$

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} = \mathbf{x}^\top \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} \quad (1.5)$$

$$\mathbf{x}^\top \left(\mathbf{V} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) \mathbf{x} = 0 \implies \mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (1.6)$$

Locus

$$2\mathbf{u}^\top \mathbf{x} = -4ax \quad (1.7)$$

$$2\mathbf{u}^\top \mathbf{x} = -4a \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} \implies 2\mathbf{u}^\top \mathbf{x} = \begin{pmatrix} -4a & 0 \end{pmatrix} \mathbf{x} \quad (1.8)$$

$$\left(2\mathbf{u}^\top - \begin{pmatrix} -4a & 0 \end{pmatrix}\right) \mathbf{x} = 0 \implies \mathbf{u}^\top = \begin{pmatrix} -2a & 0 \end{pmatrix} \quad (1.9)$$

$$\mathbf{u} = \begin{pmatrix} -2a \\ 0 \end{pmatrix} \quad (1.10)$$

$$f = 0 \quad (1.11)$$

$$\mathbf{F} = \begin{pmatrix} a \\ 0 \end{pmatrix} \quad (1.12)$$

Let \mathbf{X} be the point of locus of the midpoint

$$\mathbf{X} = \frac{\mathbf{x} + \mathbf{F}}{2} \implies \mathbf{x} = 2\mathbf{X} - \mathbf{F} \quad (1.13)$$

Simplify

From (1) and (13)

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \quad (1.14)$$

$$(2\mathbf{X} - \mathbf{F})^\top \mathbf{V} (2\mathbf{X} - \mathbf{F}) + 2\mathbf{u}^\top (2\mathbf{X} - \mathbf{F}) + f = 0 \quad (1.15)$$

$$(2\mathbf{X}^\top - \mathbf{F}^\top) \mathbf{V} (2\mathbf{X} - \mathbf{F}) + 2\mathbf{u}^\top (2\mathbf{X} - \mathbf{F}) + f = 0 \quad (1.16)$$

$$4\mathbf{X}^\top \mathbf{V} \mathbf{X} - 2\mathbf{X}^\top \mathbf{V} \mathbf{F} - 2\mathbf{F}^\top \mathbf{V} \mathbf{X} + \mathbf{F}^\top \mathbf{V} \mathbf{F} + 4\mathbf{u}^\top \mathbf{X} - 2\mathbf{u}^\top \mathbf{F} + f = 0 \quad (1.17)$$

As \mathbf{V} is a symmetric matrix

$$4\mathbf{X}^\top \mathbf{V} \mathbf{X} - 2\mathbf{F}^\top \mathbf{V} \mathbf{X} - 2\mathbf{F}^\top \mathbf{V} \mathbf{X} + \mathbf{F}^\top \mathbf{V} \mathbf{F} + 4\mathbf{u}^\top \mathbf{X} - 2\mathbf{u}^\top \mathbf{F} + f = 0 \quad (1.18)$$

$$4\mathbf{X}^\top \mathbf{V} \mathbf{X} - 4\mathbf{F}^\top \mathbf{V} \mathbf{X} + 4\mathbf{u}^\top \mathbf{X} + \mathbf{F}^\top \mathbf{V} \mathbf{F} - 2\mathbf{u}^\top \mathbf{F} + f = 0 \quad (1.19)$$

$$\mathbf{X}^\top (4\mathbf{V}) \mathbf{X} + 2 \left(2 (\mathbf{u}^\top - \mathbf{F}^\top \mathbf{V}) \right) \mathbf{X} + \mathbf{F}^\top \mathbf{V} \mathbf{F} - 2\mathbf{u}^\top \mathbf{F} + f = 0 \quad (1.20)$$

$$\mathbf{X}^\top (4\mathbf{V}) \mathbf{X} + 2 \left(2 (\mathbf{u} - \mathbf{V} \mathbf{F})^\top \right) \mathbf{X} + \mathbf{F}^\top \mathbf{V} \mathbf{F} - 2\mathbf{u}^\top \mathbf{F} + f = 0 \quad (1.21)$$

$$\mathbf{V}' = 4\mathbf{V} = 4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \quad (1.22)$$

Finding the variables

$$\begin{aligned}\mathbf{u}' = 2(\mathbf{u} - \mathbf{V}\mathbf{F}) &= 2\left(\begin{pmatrix} -2a \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix}\right) = 2\left(\begin{pmatrix} -2a \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) \\ &= \begin{pmatrix} -4a \\ 0 \end{pmatrix}\end{aligned}\quad (1.23)$$

$$f' = \mathbf{F}^\top \mathbf{V}\mathbf{F} - 2\mathbf{u}^\top \mathbf{F} + f \quad (1.24)$$

$$f' = \begin{pmatrix} a \\ 0 \end{pmatrix}^\top \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} - 2 \begin{pmatrix} -2a \\ 0 \end{pmatrix}^\top \begin{pmatrix} a \\ 0 \end{pmatrix} + 0 \quad (1.25)$$

$$f' = (a \ 0) \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} - 2(-2a \ 0) \begin{pmatrix} a \\ 0 \end{pmatrix} + 0 \quad (1.26)$$

$$f' = (0 \ 0) \begin{pmatrix} a \\ 0 \end{pmatrix} - 2(-2a \ 0) \begin{pmatrix} a \\ 0 \end{pmatrix} + 0 \quad (1.27)$$

$$f' = -2(-2a^2) = 4a^2 \quad (1.28)$$

Eigen vector

Finding eigen values of \mathbf{V}'

$$|\mathbf{V}' - \lambda \mathbf{I}| = 0 \quad (1.29)$$

$$\left| \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0 \implies \begin{vmatrix} -\lambda & 0 \\ 0 & 4 - \lambda \end{vmatrix} = 0 \quad (1.30)$$

$$(-\lambda)(4 - \lambda) = 0 \implies \lambda_1 = 0 \text{ and } \lambda_2 = 4 \quad (1.31)$$

\mathbf{p}_1 is an eigen vector of \mathbf{V}'

$$(\mathbf{V}' - \lambda \mathbf{I}) \mathbf{p} = \mathbf{0} \quad (1.32)$$

From (30) and substituting $\lambda=0$

$$\begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{p}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (1.33)$$

$$0 = 0, y = 0 \implies \mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1.34)$$

Conclusion

The Equation of a Directrix is given by

$$\mathbf{n}^\top \mathbf{x} = c \quad (1.35)$$

where

$$\mathbf{n} = \sqrt{\lambda_2} \mathbf{p}_1 \quad (\text{and}) \quad c = \frac{(\|\mathbf{u}'\|^2 - \lambda_2 f)}{2\mathbf{u}'^\top \mathbf{n}} \quad (1.36)$$

$$\mathbf{n} = \sqrt{4} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies \mathbf{n} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad (1.37)$$

$$c = \frac{((-4a)^2 + (0)^2 - 4(4a^2))}{2 \begin{pmatrix} -4a \\ 0 \end{pmatrix}^\top \begin{pmatrix} 2 \\ 0 \end{pmatrix}} = \frac{(16a^2 - 16a^2)}{2 \begin{pmatrix} -4a \\ 0 \end{pmatrix}^\top \begin{pmatrix} 2 \\ 0 \end{pmatrix}} = 0 \quad (1.38)$$

From (36)

$$\begin{pmatrix} 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \implies 2x = 0 \implies x = 0 \quad (1.39)$$

Plot

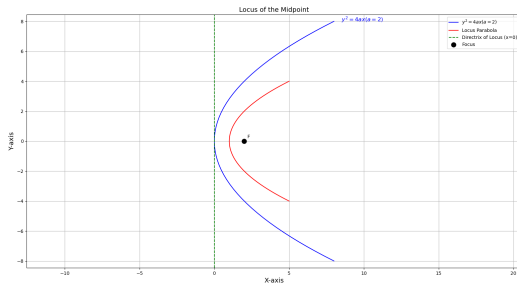


Figure:

C Code

```
void get_plot_data(double* out_data) {
    double a = 2.0;
    int num_points = 51;
    double t;
    out_data[0] = a;
    out_data[1] = 0.0;
    int index = 2;
    for (int i = 0; i < num_points; i++) {
        t = -2.0 + (4.0 * i) / (num_points - 1);
        out_data[index] = a * t * t;
        out_data[index + 1] = 2 * a * t;
        out_data[index + (num_points * 2)] = (a + a * t * t) /
            2.0;
        out_data[index + (num_points * 2) + 1] = a * t;
        index += 2;
    }
}
```

Python Code for Solving

```
import ctypes
import numpy as np

def get_data_from_c():
    lib = ctypes.CDLL('./coord.so')

    data_size = 2 + (2 * 51 * 2)
    double_array = ctypes.c_double * data_size
    lib.get_plot_data.argtypes = [ctypes.POINTER(ctypes.c_double)
    ]

    out_data_c = double_array()
    lib.get_plot_data(out_data_c)

    return np.array(out_data_c)
```

Python Code for Plotting

```
# Code by /sdcard/github/matgeo/codes/CoordGeoVV Sharma
# September 12, 2023
# Revised July 21, 2024
# Released under GNU GPL
# Section Formula
import sys
sys.path.insert(0, '/workspaces/urban-potato/matgeo/codes/
    CoordGeo/')
import numpy as np
import matplotlib.pyplot as plt

from call import get_data_from_c

all_data = get_data_from_c()
num_points = 51
focus = all_data[0:2]
parabola_orig = all_data[2 : 2 + num_points*2].reshape((
    num_points, 2))
```

Python Code for Plotting

```
parabola_locus = all_data[2 + num_points*2 :].reshape((num_points
    , 2))
fig, ax = plt.subplots(figsize=(10, 8))

ax.plot(parabola_orig[:, 0], parabola_orig[:, 1], 'b-', label='$y^2=4ax$')
ax.plot(parabola_locus[:, 0], parabola_locus[:, 1], 'r-', label='Locus Parabola')
ax.text(8 + 0.5, 8, '$y^2=4ax$', fontsize=12, color='blue')
ax.axvline(x=0, color='g', linestyle='--', label='Directrix of Locus (x=0)')

ax.scatter(focus[0], focus[1], color='black', s=100, zorder=5, label='Focus')
ax.text(focus[0] + 0.2, focus[1] + 0.2, 'F')
ax.set_title('Locus of the Midpoint')
ax.set_xlabel('X-axis'); ax.set_ylabel('Y-axis')
ax.grid(True); ax.axis('equal'); ax.legend()
plt.show()
```