2.3.3

AI25BTECH11039-Harichandana Varanasi

Question

Q. Find the angle between unit vectors $\mathbf{a}$ and $\mathbf{b}$ such that $\sqrt{3}\,\mathbf{a}-\mathbf{b}$ is also a unit vector.

Solution

Let

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \qquad \mathbf{a}^\top \mathbf{a} = \mathbf{b}^\top \mathbf{b} = 1.$$

If $\sqrt{3}\,\mathbf{a}-\mathbf{b}$ is a unit vector, then

$$1 = \|\sqrt{3}\,\mathbf{a}-\mathbf{b}\|^2 = (\sqrt{3}\,\mathbf{a}-\mathbf{b})^\top(\sqrt{3}\,\mathbf{a}-\mathbf{b}) = 3\,\mathbf{a}^\top\mathbf{a} + \mathbf{b}^\top\mathbf{b} - 2\sqrt{3}\,\mathbf{a}^\top\mathbf{b}.$$

Hence

$$4 - 2\sqrt{3}\,(\mathbf{a}^\top\mathbf{b}) = 1 \implies \mathbf{a}^\top\mathbf{b} = \frac{\sqrt{3}}{2}.$$

Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$; $\cos\theta = \mathbf{a}^\top\mathbf{b}$, so

$$\cos\theta = \frac{\sqrt{3}}{2} \quad \Rightarrow \quad \boxed{\theta = 30^\circ}.$$

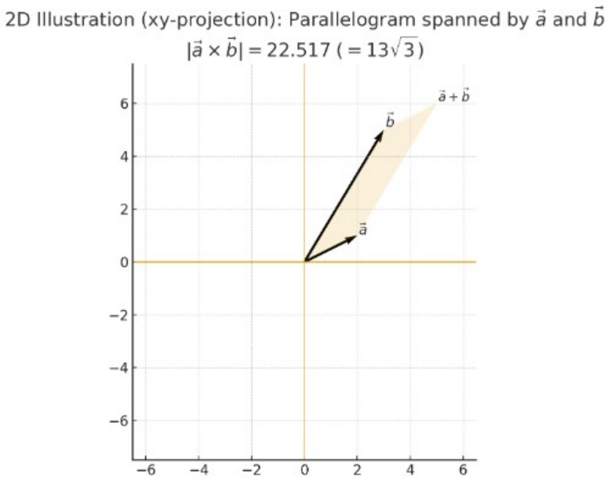

Fig. 0.1: xy-projection of $\mathbf{a}$ and $\mathbf{b}$; $|\mathbf{a}\times\mathbf{b}| = 13\sqrt{3}$.