12.495

AI25BTECH11001 - ABHISEK MOHAPATRA

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Question: The directional derivative of the function

$$f(x,y) = \frac{x^2 + xy^2}{\sqrt{5}} \tag{0.1}$$

in the direction

$$\mathbf{d} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} \tag{0.2}$$

at $X = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is

a) $-\frac{1}{\sqrt{5}}$
b) $-\frac{2}{\sqrt{5}}$
c) 0
d) $-\frac{1}{3}$

Solution:

Let $\mathbf{R} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ be a rotaion matrix such that $\mathbf{Rd} = \mathbf{e}_1$.

$$\Rightarrow \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{0.3}$$

$$2\cos\theta - 4\sin\theta = 1 \tag{0.4}$$

$$-2\sin\theta - 4\cos\theta = 0 \tag{0.5}$$

Combing the two equations,

$$\begin{pmatrix} 2 & -4 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{0.6}$$

$$\Rightarrow \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = -\frac{1}{10} \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{0.7}$$

So,

$$\mathbf{R} = \frac{1}{10} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \tag{0.8}$$

$$\mathbf{X}^{`} = \mathbf{R}\mathbf{X} \tag{0.9}$$

$$\Rightarrow \mathbf{X} = \mathbf{R}^{-1}\mathbf{X}^{`} = 2\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 2\begin{pmatrix} x^{`} + 2y^{`} \\ -2x^{`} + y^{`} \end{pmatrix} \tag{0.10}$$

$$so, f(x,y) = \frac{1}{\sqrt{5}}\left(4\left(x^{`} + 2y^{`}\right)^2 + 8\left(x^{`} + 2y^{`}\right)\left(2x^{`} - y^{`}\right)^2\right) \tag{0.11}$$

$$so, \frac{\partial f(x,y)}{\partial x^{`}} = \frac{1}{\sqrt{5}}\left(8\left(x^{`} + 2y^{`}\right) + 8\left(2x^{`} - y^{`}\right)^2 + 32\left(x^{`} + 2y^{`}\right)\left(2x^{`} - y^{`}\right)\right) \tag{0.12}$$

for $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\mathbf{X_o} = \mathbf{R}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} -1 \\ 3 \end{pmatrix} \tag{0.13}$$

$$so, \frac{\partial f(1,1)}{\partial x`} = \frac{1}{\sqrt{5}}(4+2-8) = -\frac{2}{\sqrt{5}}(b) \tag{0.14}$$