Matrices in Geometry 10.7.86

EE25BTECH11037 - Divyansh

Question: Let $\mathbf{C_1}$ and $\mathbf{C_2}$ be two circles with $\mathbf{C_2}$ lying inside $\mathbf{C_1}$. A circle C lying inside $\mathbf{C_1}$ touches $\mathbf{C_1}$ internally and $\mathbf{C_2}$ externally. Identify the locus of center of $\mathbf{C}$.

Solution: Let the center of $\mathbf{C}$, $\mathbf{C_1}$ and $\mathbf{C_2}$ be $\mathbf{O}$, $\mathbf{O_1}$ and $\mathbf{O_2}$, respectively.
Let the radii of circles $\mathbf{C}$, $\mathbf{C_1}$ and $\mathbf{C_2}$ be r, r_1 and r_2
It is given that $\mathbf{C}$ touches the circle $\mathbf{C_1}$ internally and $\mathbf{C_2}$ externally. Therefore,

$$\|\mathbf{O} - \mathbf{O_1}\| = r_1 - r \tag{1}$$

$$\|\mathbf{O} - \mathbf{O_2}\| = r_2 + r \tag{2}$$

Adding these two equations, we get

$$\|\mathbf{O} - \mathbf{O_1}\| + \|\mathbf{O} - \mathbf{O_2}\| = r_1 + r_2 \tag{3}$$

Substitute $\mathbf{O}$ as $\mathbf{x}$

$$\|\mathbf{x} - \mathbf{O_1}\| + \|\mathbf{x} - \mathbf{O_2}\| = r_1 + r_2 \tag{4}$$

This is the equation of an ellipse because it is of form

$$\|\mathbf{x} - \mathbf{S_1}\| + \|\mathbf{x} - \mathbf{S_2}\| = 2a \tag{5}$$

with foci as $\mathbf{O_1}$, $\mathbf{O_2}$ and length of the major axis as $r_1 + r_2$.

$$\|\mathbf{x} - \mathbf{O_1}\| + \|\mathbf{x} - \mathbf{O_2}\| = K, \ \ K = r_1 + r_2 \tag{6}$$

To eliminate square roots from the norms, we rearrange and square the equation.

$$\|\mathbf{x} - \mathbf{O_1}\| = K - \|\mathbf{x} - \mathbf{O_2}\| \tag{7}$$

Squaring both sides and using the property $\|\mathbf{v}\|^2 = \mathbf{v}^\top\mathbf{v}$:

$$\|\mathbf{x} - \mathbf{O_1}\|^2 = (K - \|\mathbf{x} - \mathbf{O_2}\|)^2 \tag{8}$$

$$(\mathbf{x} - \mathbf{O_1})^\top (\mathbf{x} - \mathbf{O_1}) = K^2 - 2K\|\mathbf{x} - \mathbf{O_2}\| + \|\mathbf{x} - \mathbf{O_2}\|^2 \tag{9}$$

Expanding the terms and simplifying gives the following.

$$\|\mathbf{x}\|^2 - 2\mathbf{O_1}^\top\mathbf{x} + \|\mathbf{O_1}\|^2 = K^2 - 2K\|\mathbf{x} - \mathbf{O_2}\| + \|\mathbf{x}\|^2 - 2\mathbf{O_2}^\top\mathbf{x} + \|\mathbf{O_2}\|^2 \tag{10}$$

Rearrange the equation to isolate the remaining norm term:

$$2K\|\mathbf{x} - \mathbf{O_2}\| = (K^2 + \|\mathbf{O_2}\|^2 - \|\mathbf{O_1}\|^2) + 2(\mathbf{O_1} - \mathbf{O_2})^\top \mathbf{x} \tag{11}$$

Let $S = K^2 + \|\mathbf{O_2}\|^2 - \|\mathbf{O_1}\|^2$ and $\mathbf{v} = 2(\mathbf{O_1} - \mathbf{O_2})$. The equation becomes:

$$2K\|\mathbf{x} - \mathbf{O_2}\| = S + \mathbf{v}^\top\mathbf{x} \tag{12}$$

Squaring both sides again:

$$4K^2\|\mathbf{x} - \mathbf{O_2}\|^2 = (S + \mathbf{v}^\top\mathbf{x})^2 \tag{13}$$

$$4K^2\left(\mathbf{x}^\top\mathbf{x} - 2\mathbf{O_2}^\top\mathbf{x} + \|\mathbf{O_2}\|^2\right) = S^2 + 2S(\mathbf{v}^\top\mathbf{x}) + (\mathbf{v}^\top\mathbf{x})^2 \tag{14}$$

Using the identity $(\mathbf{v}^\top\mathbf{x})^2 = \mathbf{x}^\top\left(\mathbf{v}\mathbf{v}^\top\right)\mathbf{x}$, we group all terms to one side to match the form $\mathbf{x}^\top V\mathbf{x}+2\mathbf{u}^\top\mathbf{x}+f=0$.

$$\mathbf{x}^\top\left(4K^2I-\mathbf{v}\mathbf{v}^\top\right)\mathbf{x}+2\left(-4K^2\mathbf{O_2}-S\,\mathbf{v}\right)^\top\mathbf{x}+\left(4K^2\left\|\mathbf{O_2}\right\|^2-S^2\right)=0 \tag{15}$$

Compared with the general conic equation, we identify the matrix V, the vector $\mathbf{u}$, and the scalar f:

$$\mathbf{V} = 4K^2I-\mathbf{v}\mathbf{v}^\top = 4\left(r_1+r_2\right)^2I-4\left(\mathbf{O_1}-\mathbf{O_2}\right)\left(\mathbf{O_1}-\mathbf{O_2}\right)^\top \tag{16}$$

$$\mathbf{u} = -4K^2\mathbf{O_2}-S\,\mathbf{v} = -4\left(r_1+r_2\right)^2\mathbf{O_2}-\left(\left(r_1+r_2\right)^2+\left\|\mathbf{O_2}\right\|^2-\left\|\mathbf{O_1}\right\|^2\right)\cdot 2\left(\mathbf{O_1}-\mathbf{O_2}\right) \tag{17}$$

$$f = 4K^2\left\|\mathbf{O_2}\right\|^2-S^2 = 4\left(r_1+r_2\right)^2\left\|\mathbf{O_2}\right\|^2-\left(\left(r_1+r_2\right)^2+\left\|\mathbf{O_2}\right\|^2-\left\|\mathbf{O_1}\right\|^2\right)^2 \tag{18}$$

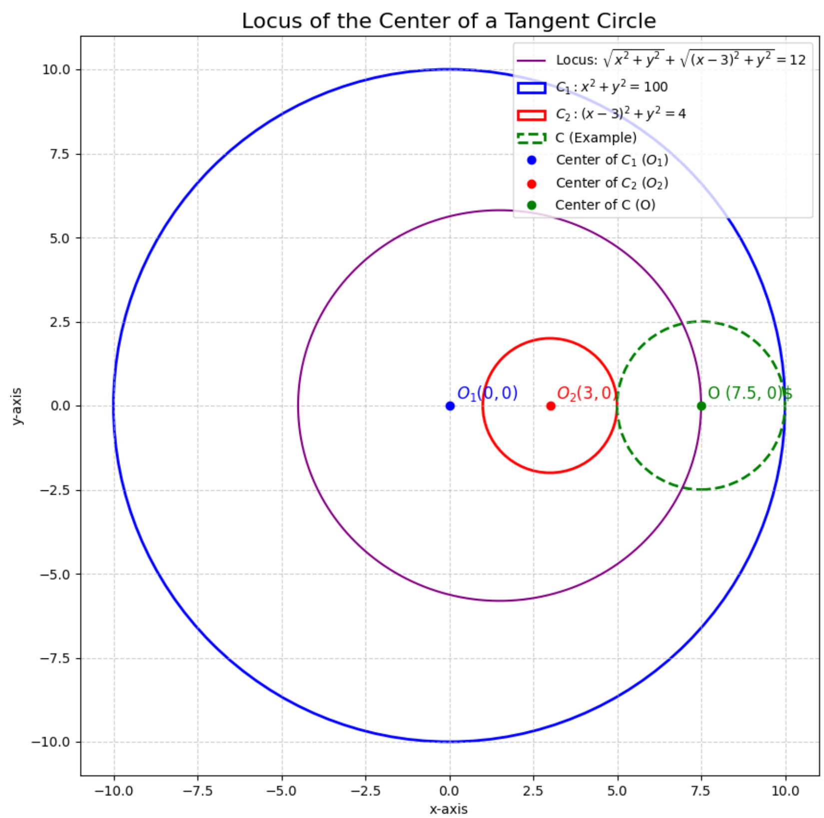

Fig. 1: Caption