5.8.19

EE25BTECH11021 - Dhanush sagar

Question:
Let a, b, c be real numbers. Consider the following system of equations in x, y, z:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$
$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

The system has:

1) no solution
2) unique solution
3) infinitely many solutions
4) finitely many solutions

Solution: Let

$$A = \frac{x^2}{a^2}, \tag{1}$$
$$B = \frac{y^2}{b^2}, \tag{2}$$
$$C = \frac{z^2}{c^2}. \tag{3}$$

Then the system becomes

$$A + B - C = 1, \tag{4}$$
$$A - B + C = 1, \tag{5}$$
$$-A + B + C = 1. \tag{6}$$

The augmented matrix is

$$\begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & 2 & 0 \\ -1 & 1 & 1 & 1 \end{pmatrix} \xrightarrow{R_3 \to R_3 + R_1} \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix}$$
$$\xrightarrow{R_3 \to R_3 + R_2} \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 2 & 2 \end{pmatrix} \xrightarrow{R_2 \to -\frac{1}{2} R_2} \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 2 \end{pmatrix}$$
$$\xrightarrow{R_3 \to \frac{1}{2} R_3} \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_2 \to R_2 + R_3} \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$
$$\xrightarrow{R_1 \to R_1 + R_3} \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_1 \to R_1 - R_2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \tag{7}$$

From the final matrix we read

$$A = 1, \qquad B = 1, \qquad C = 1. \tag{8}$$

Therefore,

$$\frac{x^2}{a^2} = 1, \qquad \frac{y^2}{b^2} = 1, \qquad \frac{z^2}{c^2} = 1, \tag{9}$$

which gives

$$x = \pm a, \qquad y = \pm b, \qquad z = \pm c. \tag{10}$$

Hence there are $2^3 = 8$ distinct solutions for (x, y, z), so the correct choice is

$$\boxed{\text{(d) finitely many solutions}} \tag{11}$$