1.8.5

AI25BTECH110031
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Question(1.8.5) If **A** and **B** be the points $(3, 4, 5)$ and $(-1, 3, -7)$ respectively, find the equation of the set of a point **P** such that $\mathbf{PA}^2 + \mathbf{PB}^2 = k^2$

Solution: Given ponits

$$\mathbf{A} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} -1 \\ 3 \\ -7 \end{pmatrix} \tag{0.1}$$

According to the question,

$$\mathbf{PA}^2 + \mathbf{PB}^2 = k^2 \tag{0.2}$$

where, $\mathbf{PA} = \|P - A\|$ and $\mathbf{PB} = \|P - B\|$

The squared distances can be written as dot products:

$$\mathbf{PA}^2 = (\mathbf{P} - \mathbf{A}).(\mathbf{P} - \mathbf{A}) \tag{0.3}$$

$$\mathbf{PB}^2 = (\mathbf{P} - \mathbf{B}).(\mathbf{P} - \mathbf{B}) \tag{0.4}$$

Thus:

$$\mathbf{PA}^2 + \mathbf{PB}^2 = (\mathbf{P} - \mathbf{A}).(\mathbf{P} - \mathbf{A}) + (\mathbf{P} - \mathbf{B}).(\mathbf{P} - \mathbf{B}) \tag{0.5}$$

$$\mathbf{PA}^2 + \mathbf{PB}^2 = \mathbf{P}.\mathbf{P} - 2\mathbf{A}.\mathbf{P} + \mathbf{A}.\mathbf{A} + \mathbf{P}.\mathbf{P} - 2\mathbf{B}.\mathbf{P} + \mathbf{B}.\mathbf{B} \tag{0.6}$$

$$\tag{0.7}$$

Substitute the known values

$$\mathbf{A}.\mathbf{A} = 3^2 + 4^2 + 5^2 = 50 \tag{0.8}$$

$$\mathbf{B}.\mathbf{B} = (-1)^2 + 3^2 + (-7)^2 = 59 \tag{0.9}$$

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 3 - 1 \\ 4 - 3 \\ 5 - 7 \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \\ -2 \end{pmatrix} \tag{0.10}$$

The equation of the locus is:

$$2\mathbf{P}.\mathbf{P} - 2\begin{pmatrix} 2 \\ 7 \\ -2 \end{pmatrix}.\mathbf{P} + 109 = K^2 \tag{0.11}$$

or equivalently,

$$2\mathbf{P}^T\mathbf{P} - 2\begin{pmatrix} 2 & 7 & -2 \end{pmatrix}.\mathbf{P} + 109 = K^2 \tag{0.12}$$

The plot show the locus for $k = 20$

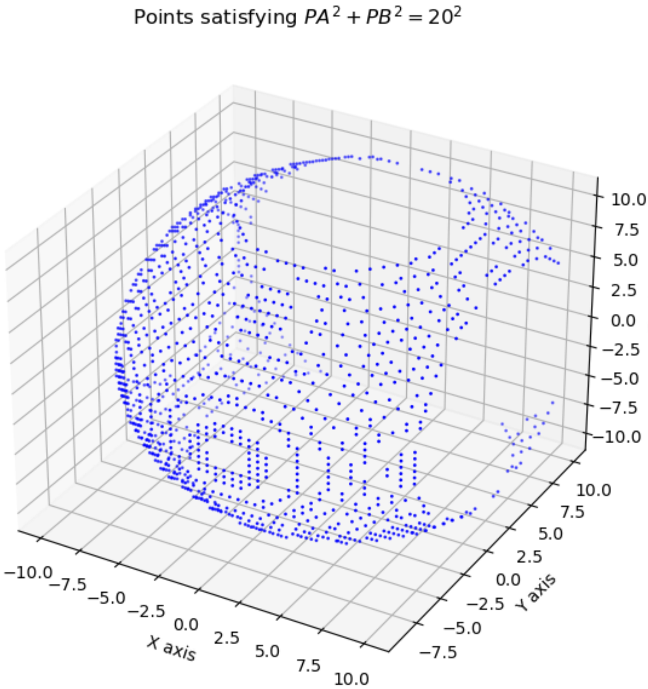

Fig. 0.1