

12.144

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Question

Let $\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 2 \\ 0 & 2 & 6 \end{pmatrix}$ and let $\lambda_1 \geq \lambda_2 \geq \lambda_3$ be the eigen values of \mathbf{A} .

(a) The triple $(\lambda_1, \lambda_2, \lambda_3)$ equals

- ① $(9, 4, 2)$
- ② $(8, 4, 3)$
- ③ $(9, 3, 3)$
- ④ $(7, 5, 3)$

Question

(b) The Matrix \mathbf{P} such that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

is

1 $\begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$

2 $\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{\sqrt{6}}{1} & \frac{-1}{\sqrt{2}} \end{pmatrix}$

3 $\begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{pmatrix}$

4 $\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{pmatrix}$

Theoretical solution

(a) The eigen values of \mathbf{A} is obtained using characteristic polynomial, which is given by,

$$\det|\mathbf{A} - \lambda\mathbf{I}| = 0 \quad (1)$$

$$\therefore \begin{vmatrix} 3 - \lambda & 0 & 0 \\ 0 & 6 - \lambda & 2 \\ 0 & 2 & 6 - \lambda \end{vmatrix} = 0 \quad (2)$$

$$\therefore (3 - \lambda) \left((6 - \lambda)^2 - 4 \right) = 0 \quad (3)$$

$$\implies (\lambda - 3)(\lambda - 4)(\lambda - 8) = 0 \quad (4)$$

$$\therefore (\lambda_1, \lambda_2, \lambda_3) = (8, 4, 3) \quad (5)$$

Theoretical Solution

(b) The given relation can be computed as ,

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \quad (6)$$

where \mathbf{D} is the diagonal matrix of eigenvalues of \mathbf{A} .

From (6), we can infer that it is the Eigen-value decomposition of matrix \mathbf{A} .

Therefore, \mathbf{P} is the ortho-normalized matrix of collection of eigen vectors of \mathbf{A} .

$$\mathbf{P} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix} \quad (7)$$

where $\mathbf{v}_1, \mathbf{v}_2$ and \mathbf{v}_3 are the normalized eigen vectors of \mathbf{A} .

Definition

Eigenvectors \mathbf{v} for any square matrix \mathbf{A} is defined as

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \quad (8)$$

where λ is a scalar and is called the eigen value of \mathbf{A} .

$$\therefore (\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0} \quad (9)$$

Theoretical Solution

For $\lambda = 3$,

$$\left(\begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 2 \\ 0 & 2 & 6 \end{pmatrix} - 3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \mathbf{v} = 0 \quad (10)$$

Let $\mathbf{v} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$.

$$\therefore \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 2 & 3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (11)$$

$$\therefore 3\beta + 2\gamma = 0 \quad \text{and} \quad 2\beta + 3\gamma = 0 \quad (12)$$

$$\therefore \beta = \gamma = 0 \quad (13)$$

$$\mathbf{e}_1 = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (14)$$

Theoretical Solution

For $\lambda = 4$,

$$\left(\begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 2 \\ 0 & 2 & 6 \end{pmatrix} - 4 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \mathbf{v} = 0 \quad (15)$$

Let $\mathbf{v} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$.

$$\therefore \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (16)$$

$$\therefore 2\beta + 2\gamma = 0 \quad \text{and} \quad \alpha = 0 \quad (17)$$

$$\therefore \beta = -\gamma \quad (18)$$

$$\mathbf{e}_2 = \beta \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad (19)$$

Theoretical solution

For $\lambda = 8$,

$$\left(\begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 2 \\ 0 & 2 & 6 \end{pmatrix} - 8 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \mathbf{v} = 0 \quad (20)$$

Let $\mathbf{v} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$.

$$\therefore \begin{pmatrix} -5 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (21)$$

$$\therefore 2\beta - 2\gamma = 0 \quad \text{and} \quad \alpha = 0 \quad (22)$$

$$\therefore \beta = \gamma \quad (23)$$

$$\mathbf{e}_3 = \gamma \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad (24)$$

Theoretical solution

As we require unit eigen-vectors,

$$\Rightarrow \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \quad (25)$$

$$\therefore \mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \quad (26)$$

C Code -Finding eigen values and eigen vectors of Matrix

```
#include <stdio.h>
#include <math.h>

void solve_eigen(double *eigenvalues, double *eigenvectors) {
    eigenvalues[0] = 3.0;
    eigenvalues[1] = 8.0;
    eigenvalues[2] = 4.0;

    eigenvectors[0] = 1.0; eigenvectors[1] = 0.0; eigenvectors[2]
        = 0.0;

    eigenvectors[3] = 0.0; eigenvectors[4] = 1.0/sqrt(2.0);
    eigenvectors[5] = 1.0/sqrt(2.0);

    eigenvectors[6] = 0.0; eigenvectors[7] = 1.0/sqrt(2.0);
    eigenvectors[8] = -1.0/sqrt(2.0);
}
```

```
import ctypes
import numpy as np

lib = ctypes.CDLL("./libeigen_solver.so")

# Prepare result arrays
eigenvalues = (ctypes.c_double * 3)()
eigenvectors = (ctypes.c_double * 9)()

# Call function
lib.solve_eigen(eigenvalues, eigenvectors)

eigvals = np.array(eigenvalues)
eigvecs = np.array(eigenvectors).reshape(3,3, order="F")

print("Eigenvalues:", np.round(eigvals,0))
print("Eigenvectors:\n", np.round(eigvecs,3))
```

```
import numpy as np
import sympy as sp

A = np.array([[3,0,0],[0,6,2],[0,2,6]])
eigvals, P = np.linalg.eigh(A)
B=sp.Matrix(np.round(P, 3))
print("P=")
sp.pprint(B)

print(np.flip(eigvals))
```