9.2.33

EE25BTECH11065 - Yoshita J

Question

Find the area of the region enclosed by the parabola $x^2 = y$ and the line $y = x + 2$, using the matrix formulation of conics and the intersection-of-line-with-conic formula.

Solution:

The given ellipse can be expressed as conics with parameters,

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0, \qquad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

where,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{u} = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}, \qquad f = 0. \tag{1}$$

The line parameters are

$$\mathbf{x} = \mathbf{h} + \kappa\,\mathbf{m},\ \ \kappa \in \mathbb{R}.$$

where,

$$\mathbf{h} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \qquad \mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{2}$$

Substituting the given parameters to find the intersection point,

$$\kappa = \frac{1}{\mathbf{m}^\top \mathbf{V}\mathbf{m}}\Big(-\ \mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u}) \pm \sqrt{\left(\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u})\right)^2 -\ g(\mathbf{h})\ (\mathbf{m}^\top \mathbf{V}\mathbf{m})}\Big), \tag{3}$$

where

$$g(\mathbf{h}) = \mathbf{h}^\top \mathbf{V}\mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f. \tag{4}$$

Solving,

$$\mathbf{m}^\top \mathbf{V}\mathbf{m} = \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1. \tag{5}$$

$$\mathbf{V}\mathbf{h} + \mathbf{u} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}. \tag{6}$$

$$\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u}) = \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} = -\frac{1}{2}. \tag{7}$$

$$g(\mathbf{h}) = \mathbf{h}^\top \mathbf{V}\mathbf{h} + 2\mathbf{u}^\top \mathbf{h} = 0 + 2\begin{pmatrix} 0 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = -2. \tag{8}$$

Now the discriminant,

$$\left(\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u})\right)^2 - g(\mathbf{h})\left(\mathbf{m}^\top \mathbf{V}\mathbf{m}\right) = \left(-\tfrac{1}{2}\right)^2 - (-2)\cdot 1 = \tfrac{1}{4} + 2 = \tfrac{9}{4}, \tag{9}$$

so

$$\sqrt{\cdot} = \tfrac{3}{2}. \tag{10}$$

Hence

$$\kappa = -(-\tfrac{1}{2}) \pm \tfrac{3}{2} = \tfrac{1}{2} \pm \tfrac{3}{2} \implies \kappa_1 = 2,\ \kappa_2 = -1. \tag{11}$$

Points of intersection,

$$\mathbf{x}_i = \mathbf{h} + \kappa_i \mathbf{m} \tag{12}$$

$$\mathbf{x}_1 = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \qquad \mathbf{x}_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix} - 1\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{13}$$

Thus the intersection points are

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 \\ 4 \end{pmatrix}. \tag{14}$$

Area of the enclosed region,

$$A = \int_{-1}^{2} \left[(x+2) - x^2\right] dx = \frac{9}{2}. \tag{15}$$

Therefore the area of the region enclosed by $x^2 = y$ and $y = x + 2$ is

$$\boxed{\tfrac{9}{2}}$$

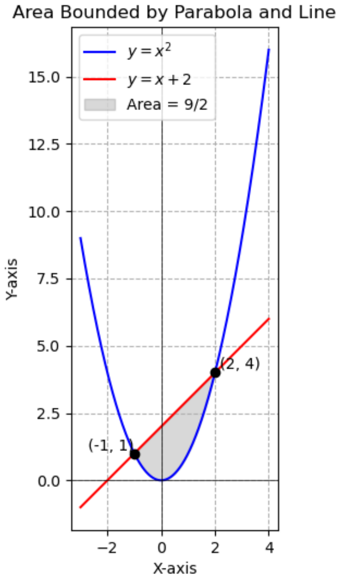

Fig. 0: A plane passing through point A with normal vector n.