

Matrices in Geometry 4.11.13

EE25BTECH11035 - Kushal B N

Question:

Find the equation of the plane passing through the points (2,5,-3), (-2,-3,5), and (5,3,-3). Also find the point of intersection of this plane with the line passing through points (3,1,5) and (-1,-3,-1).

Given:

The points $\mathbf{A} \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}$, $\mathbf{B} \begin{pmatrix} -2 \\ -3 \\ 5 \end{pmatrix}$ and $\mathbf{C} \begin{pmatrix} 5 \\ 3 \\ -3 \end{pmatrix}$ which pass through a plane.

The points $\mathbf{D} \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$ and $\mathbf{E} \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix}$ which pass through a line.

Solution:

If \mathbf{n} is the normal vector to the plane, then by the plane equation we can write

$$\mathbf{n}^T \mathbf{A} = c \quad (1)$$

$$\mathbf{n}^T \mathbf{B} = c \quad (2)$$

$$\mathbf{n}^T \mathbf{C} = c \quad (3)$$

which can be written as

$$\begin{pmatrix} A & B & C \end{pmatrix}^T \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (4)$$

Forming the augmented matrix for this

$$\left(\begin{array}{ccc|c} 2 & 5 & -3 & 1 \\ -2 & -3 & 5 & 1 \\ 5 & 3 & -3 & 1 \end{array} \right) \quad (5)$$

$$\left(\begin{array}{ccc|c} 2 & 5 & -3 & 1 \\ -2 & -3 & 5 & 1 \\ 5 & 3 & -3 & 1 \end{array} \right) \xrightarrow{R_1 \leftarrow \frac{R_1}{2}} \left(\begin{array}{ccc|c} 1 & \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \\ -2 & -3 & 5 & 1 \\ 5 & 3 & -3 & 1 \end{array} \right) \quad (6)$$

$$\xrightarrow{R_2 \leftarrow \frac{R_2}{2}} \left(\begin{array}{ccc|c} 1 & \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \\ 0 & 1 & 1 & \frac{1}{2} \\ 0 & \frac{-19}{2} & \frac{9}{2} & \frac{-3}{2} \end{array} \right) \xrightarrow{\begin{array}{l} R_3 \leftarrow R_3 + \frac{19}{2} R_2 \\ R_1 \leftarrow R_1 - \frac{5}{2} R_2 \end{array}} \left(\begin{array}{ccc|c} 1 & 0 & -4 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 14 & 8 \end{array} \right) \quad (7)$$

$$\xrightarrow{R_3 \leftarrow \frac{1}{14} R_3} \left(\begin{array}{ccc|c} 1 & 0 & -4 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & \frac{4}{7} \end{array} \right) \xrightarrow{\begin{array}{l} R_1 \leftarrow R_1 + 4R_3 \\ R_2 \leftarrow R_2 - R_3 \end{array}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{2}{7} \\ 0 & 1 & 0 & \frac{3}{7} \\ 0 & 0 & 1 & \frac{4}{7} \end{array} \right) \quad (8)$$

Thus, multiplying by 7, the plane equation can be expressed as

$$(2 \ 3 \ 4) \mathbf{x} = 7 \quad (9)$$

Now, the line passing through the two given points in parametric form

$$\mathbf{x} = \mathbf{P} + \lambda \mathbf{m} \quad (10)$$

where $\mathbf{P} = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$ and $\mathbf{m} = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} - \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix} \equiv \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$,

Now substituting these in (11) to get \mathbf{x} and then that in (10), we get

$$\begin{pmatrix} 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 3 + 2\lambda \\ 1 + 2\lambda \\ 5 + 3\lambda \end{pmatrix} = 7 \quad (11)$$

$$\Rightarrow \boxed{\lambda = -1} \quad (12)$$

which gives the intersection point as $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

Final Answer:

\therefore The equation of the plane is $\begin{pmatrix} 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 7$ and the point of intersection of this plane with the line

through the given two points is $\mathbf{P} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

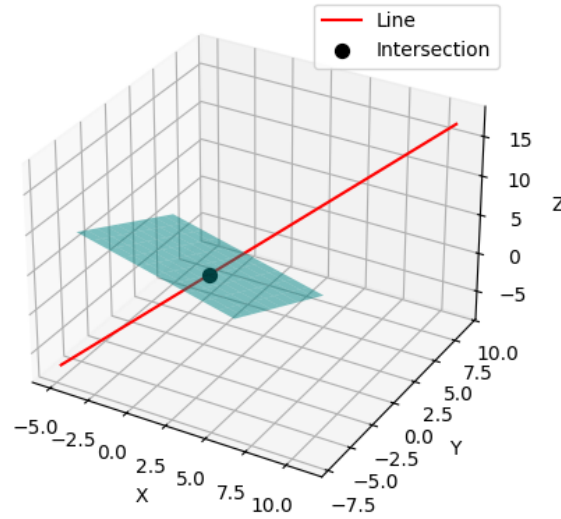


Fig. 1