10.3.12

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Question

Question:
If the line $y = \sqrt{3}x + K$ touches the parabola $x^2 = 16y$, then find the value of K.

Solution

The equation of the conic (*parabola*) can be written as

$$\mathbf{x}^{\mathbf{T}}\mathbf{V}\mathbf{x} + 2\mathbf{u}^{\mathbf{T}}\mathbf{x} + f = 0 \tag{1}$$

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -8 \end{pmatrix}, f = 0, \mathbf{m}^{\mathbf{T}} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \tag{2}$$

$$\mathbf{x} = \mathbf{h} + k_i\mathbf{m} \tag{3}$$

The value of k_i can be found out by solving the line and conic equation

$$\begin{aligned} (\mathbf{h} + k_i\mathbf{m})^\top \mathbf{V}(\mathbf{h} + k_i\mathbf{m}) + 2\mathbf{u}^\top(\mathbf{h} + k_i\mathbf{m}) + f &= 0 && (4)\\ \implies k_i^2\mathbf{m}^\top\mathbf{V}\mathbf{m} + 2k_i\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u}) + \mathbf{h}^\top\mathbf{V}\mathbf{h} + 2\mathbf{u}^\top\mathbf{h} + f &= 0 && (5)\\ \text{or, } k_i^2\mathbf{m}^\top\mathbf{V}\mathbf{m} + 2k_i\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u}) + g(\mathbf{h}) &= 0 && (6) \end{aligned}$$

Solution

Solving the above quadratic gives the equation

$$k_i = \frac{1}{\mathbf{m}^\top \mathbf{V}\mathbf{m}} \left(-\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) \pm \sqrt{[\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u})]^2 - g(\mathbf{h})\left(\mathbf{m}^\top \mathbf{V}\mathbf{m}\right)} \right) \tag{7}$$

Since the tangent passes through one point of the conic, and $g(\mathbf{q}) = 0$

$$\mathbf{m}^\mathbf{T} (\mathbf{V}\mathbf{q} + \mathbf{u}) = 0 \tag{8}$$

$$\mathbf{m}^\mathbf{T}\mathbf{V}\mathbf{q} = -\mathbf{m}^\mathbf{T}\mathbf{u} \tag{9}$$

$$\mathbf{q} = -\frac{\left(\mathbf{m}^\mathbf{T}\mathbf{V}\right)^T \mathbf{m}^\mathbf{T}\mathbf{u}}{\left\|\mathbf{m}^\mathbf{T}\mathbf{V}\right\|^2} \tag{10}$$

On solving, we get

$$\mathbf{q} = \begin{pmatrix} 8\sqrt{3} \\ t \end{pmatrix}, t \in \mathbf{R} \tag{11}$$

Solution

Since $\mathbf{q}$ lies on the conic,

$$g(\mathbf{q}) = 0 \tag{12}$$

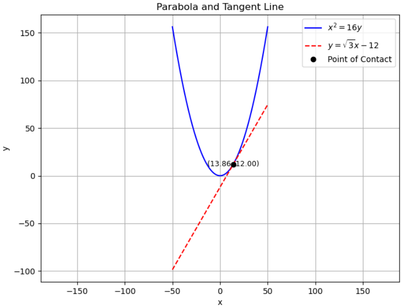
$$\implies \mathbf{q}^{\mathsf{T}}\mathbf{V}\mathbf{q} + 2\mathbf{u}^{\mathsf{T}}\mathbf{q} + f = 0 \tag{13}$$

Substituting and solving gives $t = 12$

$$\therefore \mathbf{q} = \begin{pmatrix} 8\sqrt{3} \\ 12 \end{pmatrix} \tag{14}$$

Therefore $k = -12$

Plot



Parabola and Tangent Line
$x^2 = 16y$
$y = \sqrt{3}x - 12$
Point of Contact
(13.86, 12.00)
150
100
50
0
−50
−100
−150
−100
−50
0
50
100
150
x
y

C Code

```c
#include <math.h>

double compute_x() {
    return 8 * sqrt(3);
}

double compute_y(double x) {
    return (x * x) / 16.0;
}

double compute_k(double x, double y) {
    return y - sqrt(3) * x;
}
```

Python + C Code

```
import ctypes
import numpy as np
import matplotlib.pyplot as plt
lib = ctypes.CDLL('./libcode.so')
lib.compute_x.argtypes = []
lib.compute_x.restype = ctypes.c_double
lib.compute_y.argtypes = [ctypes.c_double]
lib.compute_y.restype = ctypes.c_double
lib.compute_k.argtypes = [ctypes.c_double, ctypes.c_double]
lib.compute_k.restype = ctypes.c_double
x = lib.compute_x()
y = lib.compute_y(x)
K = lib.compute_k(x, y)
q = np.array([x, y])
x_vals = np.linspace(-50, 50, 400)
y_parabola = x_vals**2 / 16
```

```
y_line = np.sqrt(3) * x_vals + K
plt.figure(figsize=(8, 6))
plt.plot(x_vals, y_parabola, label=r'$x^2 = 16y$', color='blue')
plt.plot(x_vals, y_line, label=rf'$y = \sqrt{{3}}x {K:.0f}$',
    color='red', linestyle='--')
plt.plot(q[0], q[1], 'ko', label='Point of Contact')
plt.text(q[0], q[1], f'({q[0]:.2f}, {q[1]:.2f})', fontsize=9, ha=
    'center', va='center')
plt.title("Parabola and Tangent Line")
plt.xlabel("x")
plt.ylabel("y")
plt.legend()
plt.grid(True)
plt.axis('equal')
plt.savefig("/mnt/c/Users/bharg/Documents/backupmatrix/
    ee25btech11013/matgeo/10.3.12/figs/Figure_1.png")
plt.show()
```

Python Code

```
import numpy as np
import matplotlib.pyplot as plt

V = np.array([[1, 0], [0, 0]])
u = np.array([[0], [-8]])
f = 0
m = np.array([[1], [np.sqrt(3)]])
x = 8 * np.sqrt(3)
y = x**2 / 16
q = np.array([x, y])
K = y - np.sqrt(3) * x
x_vals = np.linspace(-50, 50, 400)
y_parabola = x_vals**2/16
y_line = np.sqrt(3)*x_vals + K
plt.figure(figsize=(8, 6))
plt.plot(x_vals, y_parabola, label=r'$x^2 = 16y$', color='blue')
plt.plot(x_vals, y_line, label=rf'$y = \sqrt{{3}}x {K:.0f}$',
    color='red', linestyle='--')
```

Python Code

```
plt.plot(q[0], q[1], 'ko', label='Point of Contact')
plt.text(q[0], q[1], f'({q[0]:.2f}, {q[1]:.2f})')
plt.title("Parabola and Tangent Line")
plt.xlabel("x")
plt.ylabel("y")
plt.legend()
plt.grid(True)
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plt.savefig("/mnt/c/Users/bharg/Documents/backupmatrix/
    ee25btech11013/matgeo/10.3.12/figs/Figure_1.png")
plt.show()
```