Question 4.4.22:

Find the equation of a plane which passes through the point $(3, 2, 0)$ and contains the line

$$\frac{x-3}{1} = \frac{y-6}{5} = \frac{z-4}{4}. \tag{1}$$

solution:

Finding the plane using column vectors.

Let the normal be $\mathbf{n} = (a, b, c)^T$. We use the form

$$\mathbf{n}^T\mathbf{x} = 1. \tag{2}$$

The plane passes through the point $P = (3, 2, 0)$ and contains the line

$$\frac{x-3}{1} = \frac{y-6}{5} = \frac{z-4}{4}, \tag{3}$$

so take a point on the line $A = (3, 6, 4)$ and the direction vector $\mathbf{v} = (1, 5, 4)$.

The conditions are

$$\mathbf{n}^T P = 1 \tag{4}$$

$$\mathbf{n}^T A = 1 \tag{5}$$

$$\mathbf{n}^T \mathbf{v} = 0. \tag{6}$$

Put these three column vectors together into a matrix (columns are the given points/vectors):

$$\mathbf{M} = \begin{pmatrix} 3 & 3 & 1 \\ 2 & 6 & 5 \\ 0 & 4 & 4 \end{pmatrix} \qquad \text{(columns are } P,\ A,\ \mathbf{v}\text{)}. \tag{7}$$

Then the three scalar conditions above read compactly as

$$\mathbf{n}^T\mathbf{M} = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}. \tag{8}$$

Transposing both sides gives a standard linear system for $\mathbf{n}$:

$$\mathbf{M}^T\mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}. \tag{9}$$

Write this out:

$$\begin{pmatrix} 3 & 2 & 0 \\ 3 & 6 & 4 \\ 1 & 5 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}. \tag{10}$$

$$\mathbf{n} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}. \tag{11}$$

Thus a convenient normal vector is $\mathbf{n} = (1, -1, 1)^T$, and the plane equation in the requested form is

$$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}^T \mathbf{x} = 1 \tag{12}$$

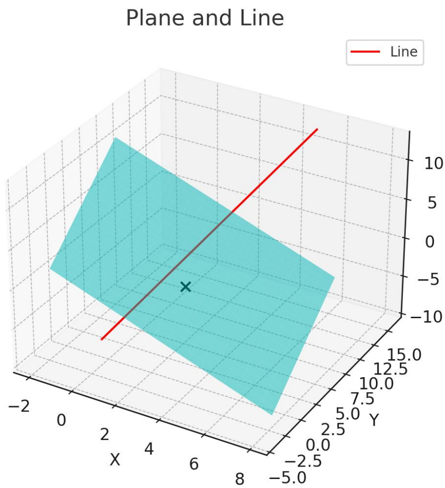

Fig. 1