

9.6.2

EE25BTECH11052 - Shriyansh Kalpesh Chawda

Question

Find the area of the circle $4x^2 + 4y^2 = 9$ which is interior to the parabola $x^2 = 4y$.

Solution

The conic parameters for the two curves can be expressed as follows:

For the circle $4x^2 + 4y^2 - 9 = 0$:

$$\mathbf{V}_1 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \mathbf{u}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, f_1 = -9. \quad (1)$$

For the parabola $x^2 - 4y = 0$:

$$\mathbf{V}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, f_2 = 0. \quad (2)$$

The intersection of two conics with parameters $\mathbf{V}_i, \mathbf{u}_i, f_i, i = 1, 2$ is defined as:

$$\mathbf{x}^T (\mathbf{V}_1 + \mu \mathbf{V}_2) \mathbf{x} + 2(\mathbf{u}_1 + \mu \mathbf{u}_2)^T \mathbf{x} + (f_1 + \mu f_2) = 0 \quad (3)$$

For a degenerate conic, the determinant of the quadratic part's matrix must be zero.

$$\det(\mathbf{V}_1 + \mu \mathbf{V}_2) = 0 \quad (4)$$

$$\det \left(\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} + \mu \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) = \begin{vmatrix} 4 + \mu & 0 \\ 0 & 4 \end{vmatrix} = 4(4 + \mu) = 0 \quad (5)$$

$$\mu = -4 \quad (6)$$

Substituting $\mu = -4$ in (3) :

$$\mathbf{x}^T (\mathbf{V}_1 - 4\mathbf{V}_2) \mathbf{x} + 2(\mathbf{u}_1 - 4\mathbf{u}_2)^T \mathbf{x} + (f_1 - 4f_2) = 0 \quad (7)$$

$$\mathbf{x}^T \left(\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} - 4 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) \mathbf{x} + 2 \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} - 4 \begin{pmatrix} 0 \\ -2 \end{pmatrix} \right)^T \mathbf{x} + (-9 - 4(0)) = 0 \quad (8)$$

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 0 \\ 8 \end{pmatrix}^T \mathbf{x} - 9 = 0 \quad (9)$$

Letting $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, the equation becomes:

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 2 \begin{pmatrix} 0 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - 9 = 0 \quad (10)$$

$$\implies 4y^2 + 16y - 9 = 0 \quad (11)$$

Solving for y yields $y = 1/2$ and $y = -9/2$. For the area to be interior to the parabola $x^2 = 4y$, we must have $y \geq 0$. Therefore, the intersection occurs at the line $y = 1/2$.

Substituting $y = 1/2$ into the parabola's equation:

$$x^2 = 4(1/2) = 2 \implies x = \pm \sqrt{2}. \quad (12)$$

Hence, the points of intersection are:

$$\mathbf{a}_1 = \left(\frac{\sqrt{2}}{1/2} \right), \mathbf{a}_2 = \left(-\frac{\sqrt{2}}{1/2} \right) \quad (13)$$

The desired area of the region is given by:

$$A = \int_{-\sqrt{2}}^{\sqrt{2}} \left(\sqrt{\frac{9}{4} - x^2} - \frac{x^2}{4} \right) dx \quad (14)$$

Due to symmetry,

$$= 2 \left[\int_0^{\sqrt{2}} \sqrt{\frac{9}{4} - x^2} dx - \int_0^{\sqrt{2}} \frac{x^2}{4} dx \right] \quad (15)$$

The first integral uses the standard formula for $\sqrt{a^2 - x^2}$ (from trig substitution).

$$\int_0^{\sqrt{2}} \sqrt{\frac{9}{4} - x^2} dx = \frac{\sqrt{2}}{4} + \frac{9}{8} \sin^{-1} \left(\frac{2\sqrt{2}}{3} \right) \quad (16)$$

The second integral uses the simple power rule.

$$\int_0^{\sqrt{2}} \frac{x^2}{4} dx = \frac{\sqrt{2}}{6} \quad (17)$$

Substituting these results back:

$$A = 2 \left[\left(\frac{\sqrt{2}}{4} + \frac{9}{8} \sin^{-1} \left(\frac{2\sqrt{2}}{3} \right) \right) - \frac{\sqrt{2}}{6} \right] \quad (18)$$

$$= \frac{\sqrt{2}}{6} + \frac{9}{4} \sin^{-1} \left(\frac{2\sqrt{2}}{3} \right) \quad (19)$$

Area of Circle Interior to Parabola
 $4x^2 + 4y^2 = 9$ interior to $x^2 = 4y$

