2.9.4

EE25BTECH11052 - Shriyansh Kalpesh Chawda

Question:

If $\mathbf{a} = \hat{\imath} + \hat{\jmath} + \hat{k}, \quad \mathbf{a} \cdot \mathbf{b} = 1, \quad \text{and} \quad \mathbf{a} \times \mathbf{b} = \hat{\jmath} - \hat{k}$, then find $|\mathbf{b}|$. (12, 2022)

Solution:

Given in the question :

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{0.1}$$

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \tag{0.2}$$

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{0.3}$$

From the dot product:

$$\mathbf{a}^\top \mathbf{b} = 1 \implies \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = 1 \tag{0.4}$$

$$b_1 + b_2 + b_3 = 1 \tag{0.5}$$

Applying the Cross Product Formula given in the book (2.1.9):

The provided formula for the cross product is:

$$A \times B = \begin{pmatrix} |A_{23} B_{23}| \\ |A_{31} B_{31}| \\ |A_{12} B_{12}| \end{pmatrix}$$

where $|A_{ij} B_{ij}|$ represents the determinant of the 2x2 matrix formed by the column vectors A_{ij} and B_{ij}.

Step 3.1: Define the sub-matrices A_{ij} and B_{ij}: Based on the definition $A_{ij} = \begin{pmatrix} a_i \\ a_j \end{pmatrix}$, we define the following matrices for A:

$$A_{23} = \begin{pmatrix} a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{0.6}$$

$$A_{31} = \begin{pmatrix} a_3 \\ a_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{0.7}$$

$$A_{12} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{0.8}$$

Similarly, for the unknown vector B:

$$B_{23} = \begin{pmatrix} b_2 \\ b_3 \end{pmatrix} \tag{0.9}$$

$$B_{31} = \begin{pmatrix} b_3 \\ b_1 \end{pmatrix} \tag{0.10}$$

$$B_{12} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{0.11}$$

Step 3.2: Calculate the determinants: Now, we compute the determinants for each component of the cross product:

$$|\mathbf{A}_{23}\mathbf{B}_{23}| = \mathbf{A}_{23}^{\top}\mathbf{B}_{23} = \begin{pmatrix} 1 & b_2 \\ 1 & b_3 \end{pmatrix} = (1)(b_3) - (1)(b_2) = b_3 - b_2 \tag{0.12}$$

$$|\mathbf{A}_{31}\mathbf{B}_{31}| = \mathbf{A}_{31}^{\top}\mathbf{B}_{31} = \begin{pmatrix} 1 & b_3 \\ 1 & b_1 \end{pmatrix} = (1)(b_1) - (1)(b_3) = b_1 - b_3 \tag{0.13}$$

$$|\mathbf{A}_{12}\mathbf{B}_{12}| = \mathbf{A}_{12}^{\top}\mathbf{B}_{12} = \begin{pmatrix} 1 & b_1 \\ 1 & b_2 \end{pmatrix} = (1)(b_2) - (1)(b_1) = b_2 - b_1 \tag{0.14}$$

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} b_3 - b_2 \\ b_1 - b_3 \\ b_2 - b_1 \end{pmatrix}. \tag{0.15}$$

4. Formulating and Solving the System of Equations

By equating our calculated cross product with the given one, $\mathbf{a} \times \mathbf{b} = (0, 1, -1)$, we get a system of linear equations:

1) $b_3 - b_2 = 0$
2) $b_1 - b_3 = 1$
3) $b_2 - b_1 = -1$

We also use the given dot product information: $\mathbf{a} \cdot \mathbf{b} = 1$.

$$\begin{aligned} a_1 b_1 + a_2 b_2 + a_3 b_3 &= 1 \\ (1)\, b_1 + (1)\, b_2 + (1)\, b_3 &= 1 \end{aligned}$$

4) $b_1 + b_2 + b_3 = 1$

Now we solve this system of four equations:

- From equation (1), we find:

$$b_3 = b_2$$

- Substitute $b_3 = b_2$ into equation (2):

$$b_1 - b_2 = 1$$

  (Note: This is consistent with equation (3), as multiplying by -1 gives $b_2 - b_1 = -1$)
- Now we have a simplified system:

(i) $b_3 = b_2$
(ii) $b_1 = 1 + b_2$
(iii) $b_1 + b_2 + b_3 = 1$

- Substitute (i) and (ii) into (iii):

$$\begin{aligned}(1 + b_2) + b_2 + (b_2) &= 1\\ 1 + 3b_2 &= 1\\ 3b_2 &= 0\\ b_2 &= 0\end{aligned}$$

- Now find b_1 and b_3:

$$\begin{aligned}b_3 &= b_2 = 0\\ b_1 &= 1 + b_2 = 1 + 0 = 1\end{aligned}$$

So, the components of vector $\mathbf{b}$ are $(b_1, b_2, b_3) = (1, 0, 0)$. This means the vector is $\mathbf{b} = 1\hat{i} + 0\hat{j} + 0\hat{k} = \hat{i}$.

To find magnitude,

$$\mathbf{b}^\top \mathbf{b} = 1 \tag{4.1}$$

$$\begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = 1 \tag{4.2}$$

The magnitude of vector $\mathbf{b}$ is $\mathbf{1}$.