Problem 4.12.19

The point $\mathbf{P} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ undergoes the following two successive transformations:

1) Reflection about the line $y = x$
2) Translation through a distance of 2 units along the positive x-axis

Find the final coordinates of the point.

Options:

a) $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$

b) $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

c) $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$

d) $\begin{pmatrix} 3.5 \\ 3.5 \end{pmatrix}$

Solution

Let the original point be:

$$\mathbf{P} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

The reflection of $\mathbf{P}$ about the line $y = x$ is:

$$\mathbf{R} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

Let the final point after translation be:

$$\mathbf{Q} = \begin{pmatrix} x \\ 4 \end{pmatrix}$$

To find x, we use the fact that $\mathbf{P}, \mathbf{R}, \mathbf{Q}$ are collinear. So the rank of the matrix formed by their differences must be 1:

$$\mathbf{R} - \mathbf{P} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}, \quad \mathbf{Q} - \mathbf{R} = \begin{pmatrix} x - 1 \\ 0 \end{pmatrix}$$

Form the matrix:

$$\mathbf{M} = \begin{pmatrix} -3 & x - 1 \\ 3 & 0 \end{pmatrix}$$

Apply row operations to reduce to echelon form:

$$R_1 = \begin{pmatrix} -3 & x - 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 3 & 0 \end{pmatrix}$$

Add $R_1 + R_2$:

$$R_1 \leftarrow R_1 + R_2 = \begin{pmatrix} 0 & x - 1 \end{pmatrix}$$

Now the matrix becomes:

$$\begin{pmatrix} 3 & 0 \\ 0 & x-1 \end{pmatrix}$$

For rank to be 1 (collinearity), second row must be zero:

$$x - 1 = 0 \Rightarrow x = 1$$

Then:

$$\mathbf{Q} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

Now apply translation:

$$\mathbf{T} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad \mathbf{F} = \mathbf{Q} + \mathbf{T} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

Final Answer

$$\boxed{\text{Final coordinates: } \begin{pmatrix} 3 \\ 4 \end{pmatrix}} \quad \Rightarrow \quad \boxed{\text{Option (b)}}$$

Plot

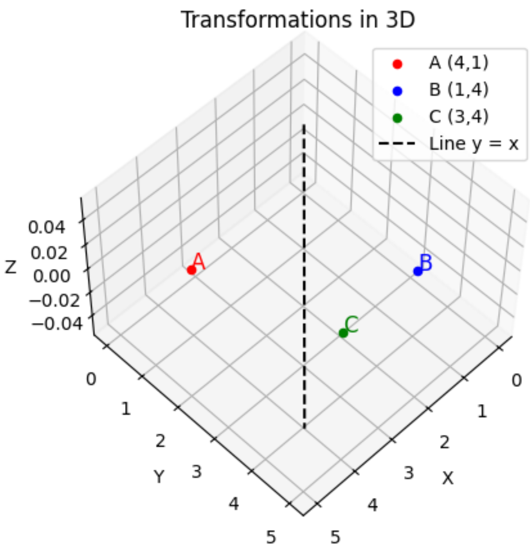

Fig. 1: Transformation of point **P** through reflection and translation