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Question:

Two matrices $\mathbf{A}$ and $\mathbf{B}$ are said to be similar if

$$\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

for some invertible matrix $\mathbf{P}$. Which of the following statements is NOT TRUE?

1) det $\mathbf{A}$ = det $\mathbf{B}$
2) Trace of $\mathbf{A}$ = Trace of $\mathbf{B}$
3) $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvectors
4) $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvalues

Solution:

Let $\mathbf{A}$ and $\mathbf{B}$ be similar matrices, such that

$$\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} \tag{1}$$

for an invertible matrix $\mathbf{P}$.
For the determinant in 1),

$$\left|\mathbf{B}\right| = \left|\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\right| \tag{2}$$
$$= \left|\mathbf{P}^{-1}\right|\left|\mathbf{A}\right|\left|\mathbf{P}\right| \tag{3}$$
$$= \frac{1}{\left|\mathbf{P}\right|}\left|\mathbf{A}\right|\left|\mathbf{P}\right| = \left|\mathbf{A}\right| \tag{4}$$

The statement is true.
For the trace in 2),
The cyclic property of trace of matrices states

$$\text{Tr}\left(\mathbf{XYZ}\right) = \text{Tr}\left(\mathbf{ZXY}\right) \tag{5}$$

Using (5)

$$\mathrm{Tr}\,(\mathbf{B}) = \mathrm{Tr}\left(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\right) \tag{6}$$

$$= \mathrm{Tr}\left(\mathbf{A}\mathbf{P}\mathbf{P}^{-1}\right) = \mathrm{Tr}\,(\mathbf{A}) \tag{7}$$

The statement is true.
For the eigenvalues in 4), we examine the characteristic polynomial.

$$\left|\mathbf{B} - \lambda\mathbf{I}\right| = \left|\mathbf{P}^{-1}\mathbf{A}\mathbf{P} - \lambda\mathbf{I}\right| \tag{8}$$

$$= \left|\mathbf{P}^{-1}\mathbf{A}\mathbf{P} - \lambda\mathbf{P}^{-1}\mathbf{I}\mathbf{P}\right| \tag{9}$$

$$= \left|\mathbf{P}^{-1}\left(\mathbf{A} - \lambda\mathbf{I}\right)\mathbf{P}\right| \tag{10}$$

$$= \left|\mathbf{P}^{-1}\right|\left|\mathbf{A} - \lambda\mathbf{I}\right|\left|\mathbf{P}\right| \tag{11}$$

$$= \left|\mathbf{A} - \lambda\mathbf{I}\right| \tag{12}$$

Since the characteristic polynomials are identical, the eigenvalues are the same. The statement is true.
For the eigenvectors in 3), let $\mathbf{v}$ be an eigenvector of $\mathbf{A}$ with eigenvalue λ, so that

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \tag{13}$$

$$\mathbf{B}\left(\mathbf{P}^{-1}\mathbf{v}\right) = \left(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\right)\left(\mathbf{P}^{-1}\mathbf{v}\right) \tag{14}$$

$$= \mathbf{P}^{-1}\mathbf{A}\left(\mathbf{P}\mathbf{P}^{-1}\right)\mathbf{v} \tag{15}$$

$$= \mathbf{P}^{-1}\mathbf{A}\mathbf{v} = \mathbf{P}^{-1}\left(\lambda\mathbf{v}\right) \tag{16}$$

$$= \lambda\left(\mathbf{P}^{-1}\mathbf{v}\right) \tag{17}$$

This shows that if $\mathbf{v}$ is an eigenvector of $\mathbf{A}$, then $\mathbf{P}^{-1}\mathbf{v}$ is the eigenvector of $\mathbf{B}$ for the same eigenvalue. Since $\mathbf{v} \neq \mathbf{P}^{-1}\mathbf{v}$ in general, the eigenvectors are not the same. The statement is not true.
Example: Let

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \implies \mathbf{P}^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \tag{18}$$

The matrix $\mathbf{A}$ has an eigenvalue $\lambda = 2$ with corresponding eigenvector

$$\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{19}$$

The similar matrix $\mathbf{B}$ is

$$\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \tag{20}$$

The corresponding eigenvector of $\mathbf{B}$ for the eigenvalue $\lambda = 2$ is

$$\mathbf{w} = \mathbf{P}^{-1}\mathbf{v} \tag{21}$$

$$\mathbf{w} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{22}$$

Clearly, $\mathbf{v} \neq \mathbf{w}$.
The statement that is NOT TRUE is **3) A and B have the same eigenvectors**.