2.10.50

EE25BTECH11021 - Dhanush Sagar

Question

A variable plane at a distance of one unit from the origin cuts the coordinate axes at A, B and C.

If the centroid $D(x, y, z)$ of triangle ABC satisfies the relation

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = k,$$

then the value of k is :

1) 3
2) 1
3) $\frac{1}{3}$
4) 9

Solution

The plane is written in vector form as

$$\mathbf{n}^\top \mathbf{x} = c, \tag{4.1}$$

where $\mathbf{n} \in \mathbb{R}^3$ is the normal vector. The plane cuts the coordinate axes at

$$\mathbf{A} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}, \qquad \mathbf{C} = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}. \tag{4.2}$$

Define

$$\mathbf{e} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{M} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}. \tag{4.3}$$

Since $\mathbf{A}, \mathbf{B}, \mathbf{C}$ lie on the plane,

$$\mathbf{n}^\top \mathbf{M} = c\, \mathbf{e}^\top. \tag{4.4}$$

Taking transpose,

$$\mathbf{M}^\top \mathbf{n} = c\, \mathbf{e}. \tag{4.5}$$

Since $\mathbf{M}$ is diagonal,

$$\mathbf{n} = c\, \mathbf{M}^{-1} \mathbf{e}. \tag{4.6}$$

The perpendicular distance of the plane from the origin is

$$d = \frac{|c|}{||\mathbf{n}||} = \frac{|c|}{|c|\, ||\mathbf{M}^{-1}\mathbf{e}||} = \frac{1}{||\mathbf{M}^{-1}\mathbf{e}||}, \tag{4.7}$$

hence

$$\mathbf{e}^{\top}\mathbf{M}^{-2}\mathbf{e} = \frac{1}{d^2}. \tag{4.8}$$

The centroid of $\triangle ABC$ is

$$\mathbf{D} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} = \tfrac{1}{3}\mathbf{M}\mathbf{e}. \tag{4.9}$$

Thus the centroid coordinates are

$$x = \tfrac{a}{3}, \quad y = \tfrac{b}{3}, \quad z = \tfrac{c}{3}, \quad \text{so} \quad \mathbf{D} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{4.10}$$

Now we compute

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = 9\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right). \tag{4.11}$$

To connect with the matrix form, observe that

$$\mathbf{M}^{-2} = \begin{pmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & \frac{1}{b^2} & 0 \\ 0 & 0 & \frac{1}{c^2} \end{pmatrix}, \tag{4.12}$$

$$\mathbf{M}^{-2}\mathbf{e} = \begin{pmatrix} \frac{1}{a^2} \\ \frac{1}{b^2} \\ \frac{1}{c^2} \end{pmatrix}, \tag{4.13}$$

$$\mathbf{e}^{\top}\mathbf{M}^{-2}\mathbf{e} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}. \tag{4.14}$$

Therefore

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = 9\,\mathbf{e}^{\top}\mathbf{M}^{-2}\mathbf{e}. \tag{4.15}$$

Using the distance relation,

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{9}{d^2}. \tag{4.16}$$

For $d = 1$,

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = 9, \tag{4.17}$$

so

$$\boxed{k = 9}.$$

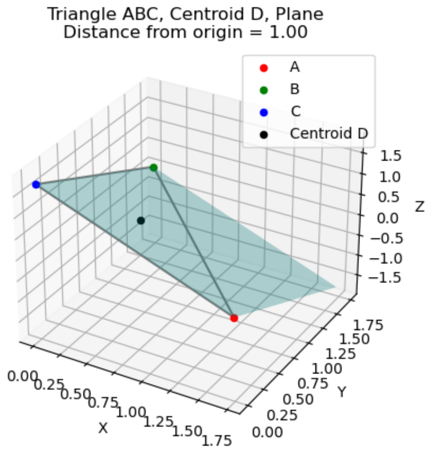
Triangle ABC, Centroid D, Plane
Distance from origin = 1.00
A
B
C
Centroid D
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
Z
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
Y
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
X

Fig. 4.1