2.10.50

EE25BTECH11021 - Dhanush Sagar

Question

A variable plane at a distance of one unit from the origin cuts the coordinate axes at A, B and C.

If the centroid $D(x, y, z)$ of triangle ABC satisfies the relation

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = k,$$

then the value of k is :

1) 3
2) 1
3) $\frac{1}{3}$
4) 9

Solution

Write the plane in vector form as

$$\mathbf{n}^\top \mathbf{x} = 1, \tag{4.1}$$

so the constant on the right is 1 (no scalar c appears).

The plane meets the coordinate axes at

$$\mathbf{A} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}, \qquad \mathbf{C} = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}. \tag{4.2}$$

Define

$$\mathbf{e} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \left(M\right) = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}. \tag{4.3}$$

Since $\mathbf{A}, \mathbf{B}, \mathbf{C}$ lie on the plane, we have

$$\mathbf{n}^\top \mathbf{A} = 1, \tag{4.4}$$

$$\mathbf{n}^\top \mathbf{B} = 1, \tag{4.5}$$

$$\mathbf{n}^\top \mathbf{C} = 1. \tag{4.6}$$

These combine to the single matrix equation

$$\mathbf{n}^\top \left(M\right) = \mathbf{e}^\top, \tag{4.7}$$

and transposing gives

$$\left(M\right)^\top \mathbf{n} = \mathbf{e}. \tag{4.8}$$

Because $\left(M\right)$ is diagonal, invertible, and equals its transpose, we obtain

$$\mathbf{n} = \left(M\right)^{-1}\mathbf{e}. \tag{4.9}$$

The perpendicular distance d of the plane $\mathbf{n}^\top\mathbf{x} = 1$ from the origin is

$$d = \frac{|1|}{||\mathbf{n}||} = \frac{1}{||\left(M\right)^{-1}\mathbf{e}||}. \tag{4.10}$$

Hence the quadratic-form relation

$$\mathbf{e}^\top\left(M\right)^{-2}\mathbf{e} = \frac{1}{d^2}. \tag{4.11}$$

The centroid of the triangle ABC is

$$\mathbf{D} = \frac{\mathbf{A}+\mathbf{B}+\mathbf{C}}{3} = \tfrac{1}{3}\left(M\right)\mathbf{e}, \tag{4.12}$$

so the coordinates of the centroid are

$$x = \tfrac{a}{3}, \quad y = \tfrac{b}{3}, \quad z = \tfrac{c}{3}, \quad \text{and} \quad \mathbf{D} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{4.13}$$

Compute the desired sum:

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{(a/3)^2} + \frac{1}{(b/3)^2} + \frac{1}{(c/3)^2} \tag{4.14}$$

$$= 9\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right). \tag{4.15}$$

To express this in matrix form note that

$$\left(M\right)^{-2} = \begin{pmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & \frac{1}{b^2} & 0 \\ 0 & 0 & \frac{1}{c^2} \end{pmatrix}, \tag{4.16}$$

$$\left(M\right)^{-2}\mathbf{e} = \begin{pmatrix} \frac{1}{a^2} \\ \frac{1}{b^2} \\ \frac{1}{c^2} \end{pmatrix}, \tag{4.17}$$

$$\mathbf{e}^\top\left(M\right)^{-2}\mathbf{e} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}. \tag{4.18}$$

Therefore

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = 9\,\mathbf{e}^\top\left(M\right)^{-2}\mathbf{e}. \tag{4.19}$$

Combining with the distance relation gives the compact formula

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{9}{d^2}. \tag{4.20}$$

For the given problem $d = 1$, so

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = 9, \tag{4.21}$$

and thus

$$\boxed{k = 9}.$$

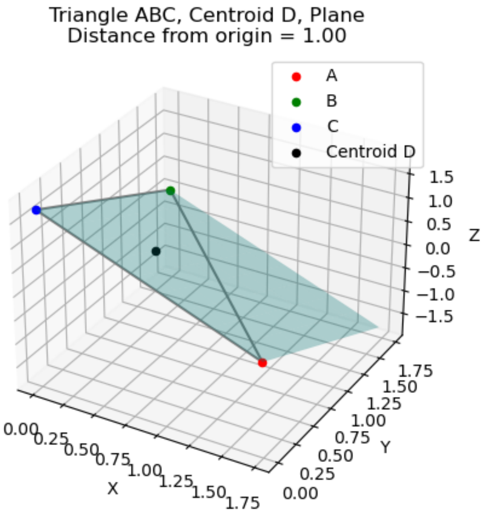

Fig. 4.1