1.3.6

AI25BTECH11027 - NAGA BHUVANA

Question:
Show that the points $\mathbf{A}\,(6, 2)$,$\mathbf{B}\,(2, 1)$,$\mathbf{C}\,(1, 5)$ and $\mathbf{D}\,(5, 6)$ are vertices of a square.
Solution:
Given that

$$\mathbf{A} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 5 \\ 6 \end{pmatrix} \tag{1}$$

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} 2 - 6 \\ 1 - 2 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \end{pmatrix} \tag{2}$$

$$\mathbf{C} - \mathbf{D} = \begin{pmatrix} 1 - 5 \\ 5 - 6 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \end{pmatrix} \tag{3}$$

$$\mathbf{B} - \mathbf{A} = \mathbf{C} - \mathbf{D} \tag{4}$$

By the above property we can say that $\mathbf{ABCD}$ is a parallelogram.
Consider the sides

$$\mathbf{A} - \mathbf{D} = \begin{pmatrix} 6 - 5 \\ 2 - 6 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix} \tag{5}$$

$$(\mathbf{B} - \mathbf{A})^T = \begin{pmatrix} -4 & -1 \end{pmatrix} \tag{6}$$

$$\|\mathbf{B} - \mathbf{A}\| = \sqrt{17} \tag{7}$$
$$\|\mathbf{A} - \mathbf{D}\| = \sqrt{17} \tag{8}$$
$$\tag{9}$$

Consider the angle θ between the sides $\mathbf{B} - \mathbf{A}$ and $\mathbf{A} - \mathbf{D}$ of the parallelogram

$$\cos\theta = \frac{\left(B - A\right)^T \left(A - D\right)}{\|\mathbf{B} - \mathbf{A}\|\|\mathbf{A} - \mathbf{D}\|} \tag{10}$$

$$\cos\theta = \frac{\begin{pmatrix} -4 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -4 \end{pmatrix}}{\sqrt{17}\,\sqrt{17}} \tag{11}$$

$$\cos\theta = \frac{(-4)(1) + (-1)(-4)}{17} \tag{12}$$

$$\tag{13}$$

$$\cos\theta = 0 \tag{14}$$

$$\tag{15}$$

$$\theta = 90° $$

Property:
A parallelogram with one angle 90° is a rectangle
Hence the parallelogram is a rectangle

$$\mathbf{A} - \mathbf{C} = \begin{pmatrix} 5 \\ -3 \end{pmatrix} \tag{16}$$

$$\implies (\mathbf{A} - \mathbf{C})^T = \begin{pmatrix} 5 & -3 \end{pmatrix} \tag{17}$$

$$\mathbf{B} - \mathbf{D} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} \tag{18}$$

Let the angle between the diagonals of the rectangle be α
Now Consider the inner product of the diagonals of rectangle $\mathbf{A} - \mathbf{C}$ and $\mathbf{B} - \mathbf{D}$

$$\cos\alpha = \frac{(A-C)^T (B-D)}{\|\mathbf{A}-\mathbf{C}\|\|\mathbf{B}-\mathbf{D}\|} = \frac{\begin{pmatrix} 5 & -3 \end{pmatrix}\begin{pmatrix} -3 \\ -5 \end{pmatrix}}{\sqrt{34}\,\sqrt{34}} \tag{19}$$
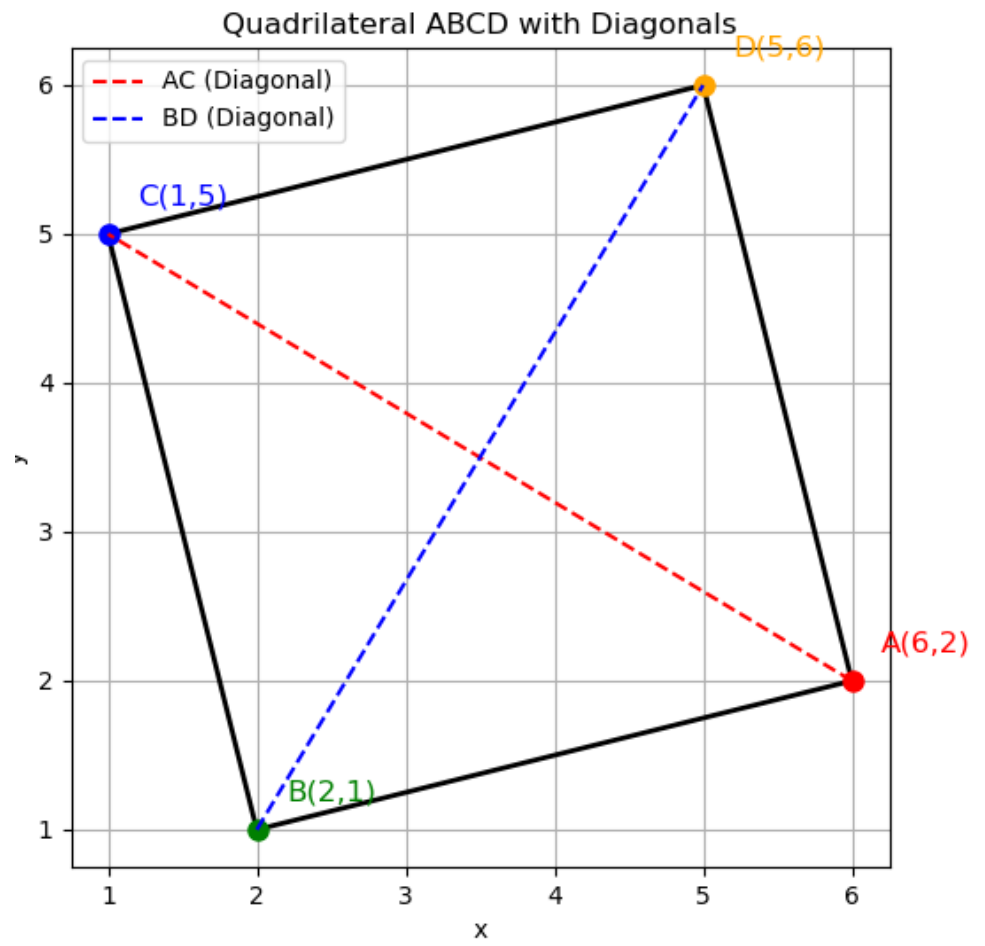
$$\cos\alpha = 0 \tag{20}$$

$$\cos\alpha = 90° \tag{21}$$

Property:
Rectangle with diagonals at right angle is a square
Hence given points forms a square

[H] Graphical Representation