Matrices in Geometry - 5.13.61

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Problem Statement

Let $\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{pmatrix}$ and $\mathbf{I}$ be the identity matrix of order 3. If $\mathbf{Q} = q_{ij}$ is a matrix such that $\mathbf{P}^{50} - \mathbf{Q} = \mathbf{I}$, then $\frac{q_{31}+q_{32}}{q_{21}}$ equals (*JEEAdv*.2016)

1. 52
2. 103
3. 201
4. 205

Solution

Given,

The matrix $\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{pmatrix}$ and $\mathbf{Q} = \mathbf{P}^{50} - \mathbf{I}$

Let us express the matrix $\mathbf{P}$ as

$$\mathbf{P} = \mathbf{I} + \mathbf{N} \tag{1}$$

where

$$\mathbf{N} = \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 16 & 4 & 0 \end{pmatrix} \tag{2}$$

Now we see that

$$\mathbf{N}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 16 & 0 & 0 \end{pmatrix} \tag{3}$$

$$\mathbf{N}^3 = \mathbf{0} \tag{4}$$

Solution

So that now by binomial expansion we have,

$$\mathbf{P}^{50} = (\mathbf{I} + \mathbf{N})^{50} \tag{5}$$

from (4),

$$\implies \mathbf{P}^{50} = \mathbf{I} + 50\mathbf{N} + 1225\mathbf{N}^2 \tag{6}$$

$$\implies \mathbf{Q} = 50\mathbf{N} + 1225\mathbf{N}^2 \tag{7}$$

$$\mathbf{Q} = \begin{pmatrix} 0 & 0 & 0 \\ 200 & 0 & 0 \\ 20400 & 200 & 0 \end{pmatrix} \tag{8}$$

$$\implies \boxed{\frac{q_{31}+q_{32}}{q_{21}} = 103} \tag{9}$$

Conclusion

$\therefore$ The value of the given expression $\frac{q_{31}+q_{32}}{q_{21}} = 103$.
Hence, the correct answer is (2).