12.81

AI25BTECH11003 - Bhavesh Gaikwad

Question: Let $\mathbf{M}$ be a 3×3 real symmetric matrix with eigenvalues -1, 1, 2 and the corresponding unit eigenvectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$, respectively. Let $\mathbf{x}$ and $\mathbf{y}$ be two vectors in $\mathbb{R}^3$ such that

$$\mathbf{Mx} = \mathbf{u} + 2(\mathbf{v} + \mathbf{w}) \text{ and } \mathbf{M}^2\mathbf{y} = \mathbf{u} - (\mathbf{v} + 2\mathbf{w})$$

Considering the usual inner product in $\mathbb{R}^3$, the value of $|\mathbf{x}+\mathbf{y}|^2$, where $|\mathbf{x}+\mathbf{y}|$ is the length of the vector $\mathbf{x}+\mathbf{y}$, is

(ST 2022)

a) 1.25 b) 0.25 c) 0.75 d) 1

Solution:

Given:

$$\mathbf{Mu} = -\mathbf{u},\ \mathbf{Mv} = \mathbf{v},\ \mathbf{Mw} = 2\mathbf{w} \tag{0.1}$$

Multiplying with $\mathbf{M}$ from the left side to all equations in Equation 0.1

$$\mathbf{M}^2\mathbf{u} = -\mathbf{Mu} = \mathbf{u} \tag{0.2}$$

$$\mathbf{M}^2\mathbf{v} = \mathbf{Mv} = \mathbf{v} \tag{0.3}$$

$$\mathbf{M}^2\mathbf{w} = 2\mathbf{Mw} = 4\mathbf{w} \tag{0.4}$$

$$\mathbf{M}^2\mathbf{u} = \mathbf{u},\ \mathbf{M}^2\mathbf{v} = \mathbf{v},\ \mathbf{M}^2\mathbf{w} = 4\mathbf{w} \tag{0.5}$$

We know,

$$\mathbf{Mx} = \mathbf{u} + 2(\mathbf{v} + \mathbf{w}) \tag{0.6}$$

From Equation 0.1,

$$\mathbf{Mx} = -\mathbf{Mu} + 2\mathbf{Mv} + \mathbf{Mw} \tag{0.7}$$

$$\mathbf{M}(\mathbf{x} + \mathbf{u} - 2\mathbf{v} - \mathbf{w}) = 0 \tag{0.8}$$

Since, Eigen values of $\mathbf{M}$ exists and are non-zero, Thus $\mathbf{M} \neq \mathbf{O}$.

$$\therefore\ \mathbf{x} = 2\mathbf{v} + \mathbf{w} - \mathbf{u} \tag{0.9}$$

We know,

$$\mathbf{M}^2\mathbf{y} = \mathbf{u} - (\mathbf{v} + 2\mathbf{w}) \tag{0.10}$$

From Equation 0.5

$$\mathbf{M}^2\mathbf{y} = \mathbf{M}^2\mathbf{u} - \mathbf{M}^2\mathbf{v} - \frac{1}{2}\mathbf{M}^2\mathbf{w} \tag{0.11}$$

$$\mathbf{M}^2(\mathbf{y} - \mathbf{u} + \mathbf{v} + \frac{1}{2}\mathbf{w}) = 0 \tag{0.12}$$

Since, Eigen values of $\mathbf{M}$ exists and are non-zero, Thus $\mathbf{M}^2 \neq \mathbf{O}$.

$$\mathbf{y} = \mathbf{u} - \mathbf{v} - \frac{1}{2}\mathbf{w} \tag{0.13}$$

$$\mathbf{x} + \mathbf{y} = \mathbf{v} + \frac{1}{2}\mathbf{w} \tag{0.14}$$

$$\|\mathbf{x} + \mathbf{y}\|^2 = \left(\mathbf{v} + \frac{1}{2}\mathbf{w}\right)^\top \left(\mathbf{v} + \frac{1}{2}\mathbf{w}\right) \tag{0.15}$$

$$\|\mathbf{x} + \mathbf{y}\|^2 = \mathbf{v}^\top\mathbf{v} + \frac{\mathbf{w}^\top\mathbf{v}}{2} + \frac{\mathbf{v}^\top\mathbf{w}}{2} + \frac{\mathbf{w}^\top\mathbf{w}}{4} \tag{0.16}$$

$$\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{v}\|^2 + \mathbf{w}^\top\mathbf{v} + \frac{\|\mathbf{w}\|^2}{4} \tag{0.17}$$

Since eigen vectors are orthonormal and $\mathbf{v}\,\&\,\mathbf{w}$ are unit vectors.

$$\|\mathbf{x} + \mathbf{y}\|^2 = 1 + 0 + \frac{1}{4} \tag{0.18}$$

$$\|\mathbf{x} + \mathbf{y}\|^2 = 1.25 \tag{0.19}$$

$$\boxed{\text{Option-A is correct.}}$$