Problem 4.13.101. Let p, q amd r be nonzero real numbers that are the $10^{th}, 100^{th}$ and 1000^{th} terms of a harmonic progression, respectively. Consider the following system of linear equations

$$x + y + z = 1 \tag{1}$$
$$10x + 100y + 1000z = 0 \tag{2}$$
$$qrx + pry + pqz = 0 \tag{3}$$

(I) If $\frac{q}{r} = 10$, then the system of linear equations has
(II) If $\frac{p}{r} \neq 100$, then the system of linear equations has
(III) If $\frac{p}{q} \neq 10$, then the system of linear equations has
(IV) If $\frac{p}{q} = 10$, then the system of linear equations has

(A) $x = 0,\ y = \frac{10}{9},\ z = -\frac{1}{9}$ as a solution
(B) $x = \frac{10}{9},\ y = -\frac{1}{9},\ z = 0$ as a solution
(C) infinitely many solutions
(D) no solution
(E) at least one solution

Input Data

Given scalars:	$p,\ q,\ r$
HP relation (reciprocals in AP):	$\frac{1}{p} = a + 9d,\ \frac{1}{q} = a + 99d,\ \frac{1}{r} = a + 999d$
Coefficient matrix (M) rows:	$\mathbf{R}_1$=$(1, 1, 1)$, $\mathbf{R}_2$=$(10, 100, 1000)$,$\mathbf{R}_3$=(qr, pr, pq)
RHS vector:	$\mathbf{b}$=$\begin{pmatrix}1\\0\\0\end{pmatrix}$

Table 1: Input data (scalars and vectors) derived from problem statement

Solution:

Given system of equations is:

$\mathbf{x}$ = $\begin{pmatrix}x\\y\\z\end{pmatrix}$.

$$\begin{pmatrix}1 & 1 & 1\\10 & 100 & 1000\\qr & pr & pq\end{pmatrix}\mathbf{x} = \begin{pmatrix}1\\0\\0\end{pmatrix}.$$

From the system of equations, the augmented matrix formed is:

$$[\,(M)\mid \mathbf{b}\,] = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 10 & 100 & 1000 & 0 \\ qr & pr & pq & 0 \end{pmatrix} \tag{4}$$

Eliminate first-column below row1: do $R_2 \leftarrow R_2 - 10R_1$ and $R_3 \leftarrow R_3 - (qr)R_1$:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 10 & 100 & 1000 & 0 \\ qr & pr & pq & 0 \end{pmatrix} \xrightarrow{R_2 - 10R_1,\ R_3 - qrR_1} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 90 & 990 & -10 \\ 0 & pr-qr & pq-qr & -qr \end{pmatrix} \tag{5}$$

Now eliminate the (3,2) entry using row2. Set

$$s = \frac{pr-qr}{90}, \tag{6}$$

and do $R_3 \leftarrow R_3 - sR_2$. Compute the new third-row entries explicitly:

$$\begin{aligned} (3,3)\text{:}\quad & (pq-qr) - s\cdot 990 && (7)\\ &= pq - qr - 990\cdot\frac{pr-qr}{90} && (8)\\ &= pq - qr - 11(pr-qr) && (9)\\ &= pq - 11\,pr + 10\,qr \ := \ D, && (10) \end{aligned}$$

$$\begin{aligned} (3,4)\text{:}\quad & -qr - s(-10) && (11)\\ &= -qr + 10\cdot\frac{pr-qr}{90} && (12)\\ &= -qr + \frac{pr-qr}{9} && (13)\\ &= \frac{pr-10\,qr}{9} \ := \ E. && (14) \end{aligned}$$

Thus the matrix in row-echelon form is

$$[\,(M)\mid \mathbf{b}\,] = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 90 & 990 & -10 \\ 0 & 0 & D & E \end{pmatrix}, \tag{15}$$

with

$$D = pq - 11\,pr + 10\,qr, \qquad E = \frac{pr-10\,qr}{9}. \tag{16}$$

Conclusions from the echelon form (standard linear algebra facts):

If $D \neq 0,$ the system has a unique solution.
If $D = 0$ but $E \neq 0,$ the system is inconsistent (no solution).
If $D = 0$ and $E = 0,$ the system has infinitely many solutions (rank 2).

Using the HP condition,

Because p, q, r are the $10^{\text{th}}, 100^{\text{th}}, 1000^{\text{th}}$ terms of an HP,

$$\frac{1}{p} = a + 9d, \qquad \frac{1}{q} = a + 99d, \qquad \frac{1}{r} = a + 999d \tag{17}$$

for some real a, d. Evaluate $D/(pqr)$ to simplify algebra:

$$\frac{D}{pqr} = \frac{1}{r} - \frac{11}{q} + \frac{10}{p} \tag{18}$$

$$= (a + 999d) - 11(a + 99d) + 10(a + 9d) \tag{19}$$

$$= a + 999d - 11a - 1089d + 10a + 90d = 0. \tag{20}$$

Hence

$$\boxed{D \equiv 0 \text{ for every valid HP triple } (p, q, r).} \tag{21}$$

Therefore the coefficient matrix is singular and a unique solution is impossible.

Next compute $E/(pqr)$:

$$\frac{E}{pqr} = \frac{1}{9}\Big(\frac{1}{q} - \frac{10}{p}\Big) \tag{22}$$

$$= \frac{1}{9}\big((a + 99d) - 10(a + 9d)\big) \tag{23}$$

$$= \frac{1}{9}(-9a + 9d) = d - a. \tag{24}$$

Thus

$$\boxed{E = pqr\,(d - a).} \tag{25}$$

So the system is consistent (infinitely many solutions) exactly when $E = 0$, i.e. when $d = a$. Equivalently,

$$d = a \quad \Longrightarrow \quad \frac{1}{p} = 10a,\ \frac{1}{q} = 100a,\ \frac{1}{r} = 1000a \quad \Longrightarrow \quad p : q : r = 100 : 10 : 1. \tag{26}$$

Parametric solution when consistent. If $d = a$ (equivalently $p : q : r = 100 : 10 : 1$) then the third equation is redundant and we can solve the first two:

$$x + y + z = 1, \tag{27}$$

$$10x + 100y + 1000z = 0. \tag{28}$$

Set $z = t$. Then $y = 1 - t - x$. Substitute into the second:

$$10x + 100(1 - t - x) = -1000t \tag{29}$$
$$-90x + 100 - 100t = -1000t \tag{30}$$
$$-90x = -900t - 100 \tag{31}$$
$$x = 10t + \frac{10}{9}. \tag{32}$$

Thus the solution family is

$$\mathbf{x} = \begin{pmatrix} 10t + \frac{10}{9} \\ -11t - \frac{1}{9} \\ t \end{pmatrix}, \qquad t \in \mathbb{R}. \tag{33}$$

Two convenient particular choices:

$$t = -\frac{1}{9} \implies \mathbf{x} = \begin{pmatrix} 0 \\ \frac{10}{9} \\ -\frac{1}{9} \end{pmatrix} \quad \text{(matches option A)}, \tag{34}$$
$$t = 0 \implies \mathbf{x} = \begin{pmatrix} \frac{10}{9} \\ -\frac{1}{9} \\ 0 \end{pmatrix} \quad \text{(matches option B)}. \tag{35}$$

So when consistent both A and B are valid particular solutions, and there are infinitely many of them (C).

Now check cases (I)–(IV)

(I) If $\frac{q}{r} = 10$.

From reciprocals,

$$\frac{1/q}{1/r} = \frac{r}{q} = \frac{1}{10} \quad \implies \quad \frac{a + 99d}{a + 999d} = \frac{1}{10}. \tag{36}$$

Multiply out:

$$10(a + 99d) = a + 999d \quad \implies \quad 10a + 990d = a + 999d \quad \implies \quad 9a = 9d, \tag{37}$$

so $a = d$. Therefore $E = 0$ and we are in the consistent case. Conclusion:

$$\text{(I)} \quad \boxed{\text{infinitely many solutions (option C). Also A and B are solutions.}} \tag{38}$$

(II) If $\frac{p}{r} \neq 100$.

Now $p/r \neq 100$ means $p \neq 100r$. Under the HP parametrisation, $p = 100r$ is equivalent to $a = d$ (see derivation above). Hence $p \neq 100r$ is equivalent to $a \neq d$. Then $E =$

$pqr(d-a) \neq 0$. Since we already have $D \equiv 0$, $D = 0$ and $E \neq 0$ implies inconsistency. Conclusion:

$$\text{(II)} \quad \boxed{\text{no solution (option D).}} \tag{39}$$

(III) If $\frac{p}{q} \neq 10$.

Similarly $p/q = 10$ is equivalent to $a = d$ (check by $(a+9d)/(a+99d) = 1/10$ as in (I)). Therefore $p/q \neq 10$ implies $a \neq d$ and hence $E \neq 0$. With $D \equiv 0$ this gives inconsistency. Conclusion:

$$\text{(III)} \quad \boxed{\text{no solution (option D).}} \tag{40}$$

(IV) If $\frac{p}{q} = 10$.

As noted, $p/q = 10$ implies $a = d$. Thus $E = 0$ and the system is consistent with infinitely many solutions. Conclusion:

$$\text{(IV)} \quad \boxed{\text{infinitely many solutions (option C). Also A and B are solutions.}} \tag{41}$$

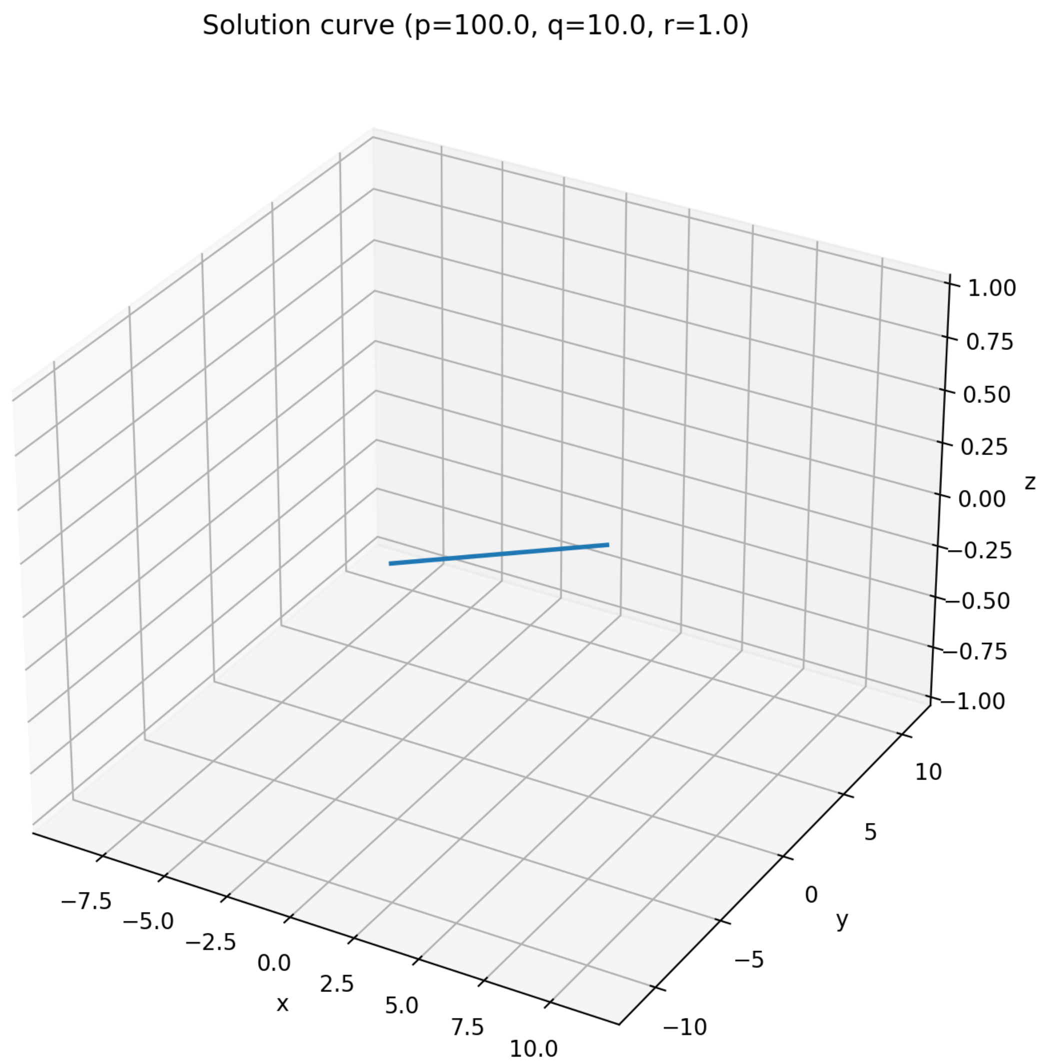
Solution curve (p=100.0, q=10.0, r=1.0)
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
z
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0
x
10
5
0
−5
−10
y

Figure 1