8.4.4

EE25BTECH11001 - Aarush Dilawri

Question:
Equation of the ellipse whose axes are the coordinates and which passes through the point $(-3, 1)$ and has eccentricity $\sqrt{\frac{2}{5}}$ is

Solution:
The general equation of a conic can be written as:

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{1}$$

Since the ellipse is centered at origin, we have

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad f = -1 \tag{2}$$

Let the major axis be along the X-axis:

$$\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e = \sqrt{\frac{2}{5}} \tag{3}$$

Then, using the formula:

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2\mathbf{n}\mathbf{n}^T \tag{4}$$

we get

$$\mathbf{V} = \begin{pmatrix} 1 - \frac{2}{5} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & 0 \\ 0 & 1 \end{pmatrix} \tag{5}$$

The ellipse passes through the point $(-3, 1)$, so scale $\mathbf{V}$ such that:

$$\mathbf{x}_0^T\mathbf{V}\mathbf{x}_0 = 1, \quad \mathbf{x}_0 = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \tag{6}$$

This gives

$$\mathbf{V} = \frac{5}{32}\begin{pmatrix} \frac{3}{5} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{32} & 0 \\ 0 & \frac{5}{32} \end{pmatrix} \tag{7}$$

Hence, the equation of the ellipse is:

$$\mathbf{x}^T \begin{pmatrix} \frac{3}{32} & 0 \\ 0 & \frac{5}{32} \end{pmatrix} \mathbf{x} = 1 \tag{8}$$

Or equivalently:

$$3x^2 + 5y^2 = 32 \tag{9}$$

See Fig. 0 ,

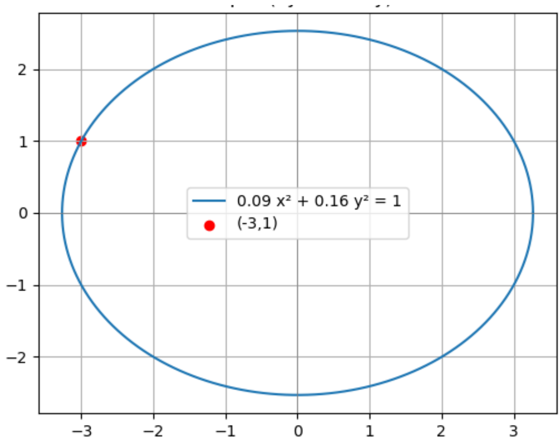

Fig. 0