

2.10.37

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October 5, 2025

Question

If \mathbf{a} , \mathbf{b} and \mathbf{c} are three non-coplanar vectors, then find the value of

$$(\mathbf{a} + \mathbf{b} + \mathbf{c}) \cdot [(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} + \mathbf{c})].$$

Solution

We know that the scalar triple product is defined as

$$[\mathbf{p} \ \mathbf{q} \ \mathbf{r}] = \mathbf{p} \cdot (\mathbf{q} \times \mathbf{r}).$$

Expanding the cross product,

$$(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} + \mathbf{c}) = \mathbf{a} \times \mathbf{a} + \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{a} + \mathbf{b} \times \mathbf{c} \quad (1)$$

$$= 0 + \mathbf{a} \times \mathbf{c} - \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c}. \quad (2)$$

Hence,

$$(\mathbf{a} + \mathbf{b} + \mathbf{c}) \cdot [(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} + \mathbf{c})] = (\mathbf{a} + \mathbf{b} + \mathbf{c}) \cdot (-\mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}). \quad (3)$$

Expanding using linearity of the scalar triple product,

$$= -[\mathbf{a} \ \mathbf{a} \ \mathbf{b}] - [\mathbf{b} \ \mathbf{a} \ \mathbf{b}] - [\mathbf{c} \ \mathbf{a} \ \mathbf{b}] + [\mathbf{a} \ \mathbf{a} \ \mathbf{c}] + [\mathbf{b} \ \mathbf{a} \ \mathbf{c}] + [\mathbf{c} \ \mathbf{a} \ \mathbf{c}] + [\mathbf{a} \ \mathbf{b} \ \mathbf{c}] + [\mathbf{b} \ \mathbf{b} \ \mathbf{c}] + [\mathbf{c} \ \mathbf{b} \ \mathbf{c}]. \quad (4)$$

All terms containing repeated vectors vanish, so we have

$$= -[\mathbf{c} \ \mathbf{a} \ \mathbf{b}] + [\mathbf{b} \ \mathbf{a} \ \mathbf{c}] + [\mathbf{a} \ \mathbf{b} \ \mathbf{c}]. \quad (5)$$

Now, using the properties of the scalar triple product:

$$[\mathbf{c} \ \mathbf{a} \ \mathbf{b}] = [\mathbf{a} \ \mathbf{b} \ \mathbf{c}], \quad [\mathbf{b} \ \mathbf{a} \ \mathbf{c}] = -[\mathbf{a} \ \mathbf{b} \ \mathbf{c}].$$

Hence,

$$-[\mathbf{c} \ \mathbf{a} \ \mathbf{b}] + [\mathbf{b} \ \mathbf{a} \ \mathbf{c}] + [\mathbf{a} \ \mathbf{b} \ \mathbf{c}] = -[\mathbf{a} \ \mathbf{b} \ \mathbf{c}] - [\mathbf{a} \ \mathbf{b} \ \mathbf{c}] + [\mathbf{a} \ \mathbf{b} \ \mathbf{c}] \quad (6)$$

$$= -[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]. \quad (7)$$

$$\boxed{(\mathbf{a} + \mathbf{b} + \mathbf{c}) \cdot [(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} + \mathbf{c})] = -[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]}$$