2.7.15

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Question

Find the volume of a parallelepiped whose edges are given by:

$$\mathbf{a} = -3\hat{i} + 7\hat{j} + 5\hat{k}, \quad \mathbf{b} = -5\hat{i} + 7\hat{j} - 3\hat{k}, \quad \mathbf{c} = 7\hat{i} - 5\hat{j} - 3\hat{k}.$$

Theoretical Solution

Let the three vectors representing the edges be:

$$\mathbf{a} = \begin{pmatrix} -3 \\ 7 \\ 5 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -5 \\ 7 \\ -3 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 7 \\ -5 \\ -3 \end{pmatrix}.$$

The volume is given by:

$$V = \sqrt{\det(G)},$$

where G is the Gram matrix formed by dot products of the vectors.
The Gram matrix is:

$$G = \begin{pmatrix} 83 & 49 & -71 \\ 49 & 83 & -61 \\ -71 & -61 & 83 \end{pmatrix}$$

Theoretical Solution

We use row reduction to make G upper triangular.
Perform the operations:

$$\begin{pmatrix} 83 & 49 & -71 \\ 49 & 83 & -61 \\ -71 & -61 & 83 \end{pmatrix} \xrightarrow{R_2 \to R_2 - \frac{49}{83}R_1} \begin{pmatrix} 83 & 49 & -71 \\ 0 & \frac{4488}{83} & \frac{-1584}{83} \\ -71 & -61 & 83 \end{pmatrix}$$

Theoretical Solution

$$\begin{pmatrix} 83 & 49 & -71 \\ 0 & \frac{4488}{83} & \frac{-1584}{83} \\ -71 & -61 & 83 \end{pmatrix} \xrightarrow{R_3 \to R_3 + \frac{71}{83} R_1} \begin{pmatrix} 83 & 49 & -71 \\ 0 & \frac{4488}{83} & \frac{-1584}{83} \\ 0 & \frac{-1584}{83} & \frac{1848}{83} \end{pmatrix}$$

Theoretical Solution

$$\begin{pmatrix} 83 & 49 & -71 \\ 0 & \frac{4488}{83} & \frac{-1584}{83} \\ 0 & \frac{-1584}{83} & \frac{1848}{83} \end{pmatrix} \xrightarrow{R_3 \to R_3 + \frac{6}{17} R_2} \begin{pmatrix} 83 & 49 & -71 \\ 0 & \frac{4488}{83} & \frac{-1584}{83} \\ 0 & 0 & \frac{264}{17} \end{pmatrix}$$

Theoretical Solution

The matrix is now upper triangular, so the determinant is the product of the diagonal entries:

$$\det(G) = 83 \times \frac{4488}{83} \times \frac{264}{17}.$$

Simplifying step-by-step:

$$\det(G) = 4488 \times \frac{264}{17} = (17 \times 264) \times \frac{264}{17} = 264 \times 264 = 69696$$

The volume is:

$$V = \sqrt{\det(G)} = \sqrt{69696} = \mathbf{264} \text{ cubic units.}$$

Plot

Parallelepiped defined by a, b, c