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Matrix 4.8.11

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Question

Find the vector equation of the plane determined by the points

$$A(3,-1,2), \quad B(5,2,4), \quad C(-1,-1,6),$$

and hence find the distance of this plane from the origin.

Solution

Step 1: Position vectors

$$\mathbf{A} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix}, \qquad \mathbf{C} = \begin{pmatrix} -1 \\ -1 \\ 6 \end{pmatrix}. \tag{1}$$

Step 2: Plane as $\mathbf{N}^\top x = 1$

Write the plane in the normalized form

$$\mathbf{N}^\top x = 1, \tag{2}$$

where $\mathbf{N} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$ is the (unknown) column vector of coefficients. Since A, B, C lie on the plane we must have

$$\mathbf{N}^\top \mathbf{A} = 1, \qquad \mathbf{N}^\top \mathbf{B} = 1, \qquad \mathbf{N}^\top \mathbf{C} = 1. \tag{3}$$

Step 3: Linear system (matrix form)

Equations (**??**) give the linear system

$$\begin{bmatrix} 3 & -1 & 2 \\ 5 & 2 & 4 \\ -1 & -1 & 6 \end{bmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{4}$$

We solve (**??**) by row reduction. Write the augmented matrix:

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & 1 \\ 5 & 2 & 4 & 1 \\ -1 & -1 & 6 & 1 \end{array}\right]. \tag{5}$$

Step 4: Row reduction (RREF steps)

Perform elementary row operations; each step is shown.

(i) Eliminate the first column below the pivot in row 1:

$$R_2 \leftarrow R_2 - \frac{5}{3}R_1, \qquad R_3 \leftarrow R_3 + \frac{1}{3}R_1,$$

giving

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & 1 \\ 0 & \frac{11}{3} & \frac{2}{3} & -\frac{2}{3} \\ 0 & -\frac{4}{3} & \frac{20}{3} & \frac{4}{3} \end{array}\right]. \tag{6}$$

(ii) Clear denominators in rows 2 and 3 by multiplying those rows by 3:

$$R_2 \leftarrow 3R_2, \qquad R_3 \leftarrow 3R_3,$$

so

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & 1 \\ 0 & 11 & 2 & -2 \\ 0 & -4 & 20 & 4 \end{array}\right]. \tag{7}$$

(iii) Eliminate the (3,2)-entry using row 2:

$$R_3 \leftarrow R_3 + \frac{4}{11}R_2,$$

which yields

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & 1 \\ 0 & 11 & 2 & -2 \\ 0 & 0 & \frac{228}{11} & \frac{36}{11} \end{array}\right]. \tag{8}$$

(iv) Scale Row3 to make the pivot 1:

$$R_3 \leftarrow \frac{11}{228}R_3,$$

so Row 3 becomes

$$\left[0\ 0\ 1 \mid \frac{36}{228}\right] = [0\ 0\ 1 \mid \tfrac{3}{19}]. \tag{9}$$

Thus

$$n_3 = \frac{3}{19}. \tag{10}$$

(v) Back substitution to clear above pivots:
Eliminate the 3rd column entry in Row2:

$$R_2 \leftarrow R_2 - 2R_3 \quad \Longrightarrow \quad \left[0\ 11\ 0 \mid -2 - 2\cdot\tfrac{3}{19}\right] = \left[0\ 11\ 0 \mid -\tfrac{44}{19}\right].$$

Hence

$$11\,n_2 = -\frac{44}{19} \quad\Longrightarrow\quad n_2 = -\frac{4}{19}. \tag{11}$$

Eliminate the 2nd and 3rd column entries in Row1:

$$R_1 \leftarrow R_1 + R_2 - 2R_3,$$

which produces the equation

$$3\,n_1 = 1 + n_2 - 2n_3 = 1 - \frac{4}{19} - \frac{6}{19} = \frac{9}{19}. \tag{12}$$

So

$$n_1 = \frac{3}{19}. \tag{13}$$

Step 5: Plane equation and distance

Thus the solution is

$$\mathbf{N} \;=\; \begin{pmatrix} \frac{3}{19} \\ -\frac{4}{19} \\ \frac{3}{19} \end{pmatrix}. \tag{14}$$

Equivalently, scaling by 19,

$$\mathbf{N} \;=\; \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix}, \qquad \mathbf{N}^{\top} x = 19, \tag{15}$$

so (**??**) holds with $\mathbf{N}$ as in (**??**).

The perpendicular distance D from the origin to the plane $\mathbf{N}^{\top} x = 1$ equals

$$D \;=\; \frac{|1|}{\|\mathbf{N}\|}. \tag{16}$$

Compute

$$\|\mathbf{N}\| = \sqrt{3^2 + (-4)^2 + 3^2} = \sqrt{34}, \tag{17}$$

so, since $\|\mathbf{N}\| = \|\mathbf{N}\|/19$,

$$D \;=\; \frac{1}{\|\mathbf{N}\|} \;=\; \frac{19}{\sqrt{34}} \;=\; \frac{19\sqrt{34}}{34} \approx 3.260. \tag{18}$$

Final Answer

$$\boxed{\mathbf{N}^{\top} x = 1, \quad \mathbf{N} = \begin{pmatrix} \frac{3}{19} \\ -\frac{4}{19} \\ \frac{3}{19} \end{pmatrix}} \tag{19}$$

$$\boxed{\text{Distance from origin to plane} = \frac{19}{\sqrt{34}} \approx 3.260} \tag{20}$$

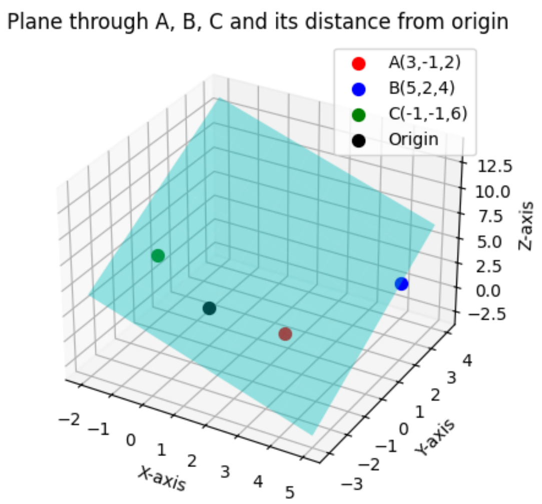

Figure 1: (Sketch: plane through A, B, C).