7.4.8

EE25BTECH11001 - Aarush Dilawri

Question:
For each natural number k, let C_k denote the circle with radius k centimetres and centre at the origin. On the circle C_k , a particle moves k centimetres in the counter- clockwise direction. After completing its motion on C_k , the particle moves to C_{k+1} in the radial direction. The motion of the particle continues in this manner. The particle starts at $(1, 0)$. If the particle crosses the positive direction of the X axis for the first time on the Circle C_n, then $n =$ _
Solution:

$$\text{Let } \mathbf{p}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{0.1}$$

We model a rotation by an angle θ using the rotation matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{0.2}$$

Note the group property of rotations:

$$R(\theta_1)\, R(\theta_2) = R(\theta_1 + \theta_2), \qquad R(\theta)^k = R(k\theta). \tag{0.3}$$

On the circle C_k the particle moves an arc of length k on a circle of radius k, so the angular increment on C_k is

$$\Delta\theta_k = \frac{\text{arc length}}{\text{radius}} = \frac{k}{k} = 1 \quad \text{(radian)}. \tag{0.4}$$

Thus each circular motion rotates the particle by 1 radian. We track the position of the particle at the instant it finishes its motion on C_k (that is, after the arc motion but before the radial jump to C_{k+1}). Starting at $\mathbf{p}_0$ on C_1, after finishing C_1 the position is

$$\mathbf{P}_1 = 1\, R(1)\, \mathbf{p}_0. \tag{0.5}$$

Then the particle moves radially to C_2, scaling the radius from 1 to 2, so just before moving on C_2 the vector is $2R(1)\mathbf{p}_0$. After moving on C_2 (an additional rotation by 1) the particle is at

$$\mathbf{P}_2 = 2\, R(1)R(1)\, \mathbf{p}_0 = 2\, R(2)\, \mathbf{p}_0. \tag{0.6}$$

By induction, after finishing its motion on C_k the particle is at

$$\mathbf{P}_k = k\, R(k)\, \mathbf{p}_0. \tag{0.7}$$

Therefore the angular coordinate of the particle after completing C_k is exactly k radians. The motion on C_n runs the angle from $(n-1)$ to n (radians). Hence the particle crosses the positive x-axis during the motion on C_n precisely when some integer multiple of 2π lies in the interval $(n-1, n]$, i.e. when there exists $m \in \mathbb{N}$ such that

$$n - 1 \ < \ 2\pi m \ \leq \ n. \tag{0.8}$$

We look for the smallest natural number n for which this happens. Take $m = 1$ (the first positive multiple of 2π). Compute

$$2\pi \approx 6.283185307\ldots \tag{0.9}$$

and observe

$$6 \ < \ 2\pi \ \leq \ 7. \tag{0.10}$$

Thus 2π lies in the interval $(6, 7]$, so the condition holds for $n = 7$ (with $m = 1$). For any $n \leq 6$ the interval $(n-1, n]$ is contained in $[0, 6]$ and cannot contain $2\pi \approx 6.283\ldots$.

Therefore the particle crosses the positive x-axis for the first time while moving on C_n with

$$\boxed{n \ = \ 7.} \tag{0.11}$$

See Figure,