12.144

EE25BTECH11026-Harsha

Question:

Let $\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 2 \\ 0 & 2 & 6 \end{pmatrix}$ and let $\lambda_1 \geq \lambda_2 \geq \lambda_3$ be the eigen values of $\mathbf{A}$.

(a) The triple $(\lambda_1, \lambda_2, \lambda_3)$ equals

1) $(9, 4, 2)$
2) $(8, 4, 3)$
3) $(9, 3, 3)$
4) $(7, 5, 3)$

(b) The Matrix $\mathbf{P}$ such that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

is

1) $\begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$

2) $\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$

3) $\begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{pmatrix}$

4) $\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{pmatrix}$

Solution:

Let us solve the given question theoretically and then verify the solution computationally.

(a) The eigen values of $\mathbf{A}$ is obtained using characteristic polynomial, which is given by,

$$det|\mathbf{A} - \lambda\mathbf{I}| = 0 \tag{4.1}$$

$$\therefore \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 6-\lambda & 2 \\ 0 & 2 & 6-\lambda \end{vmatrix} = 0 \tag{4.2}$$

$$\therefore (3-\lambda)\left((6-\lambda)^2 - 4\right) = 0 \tag{4.3}$$

$$\implies (\lambda - 3)(\lambda - 4)(\lambda - 8) = 0 \tag{4.4}$$

$$\therefore (\lambda_1, \lambda_2, \lambda_3) = (8, 4, 3) \tag{4.5}$$

(b) The given relation can be computed as ,

$$\mathbf{A} = \mathbf{PDP}^{-1} \tag{4.6}$$

where $\mathbf{D}$ is the diagonal matrix of eigenvalues of $\mathbf{A}$.

From (4.6), we can infer that it is the Eigen-value decomposition of matrix $\mathbf{A}$.

Therefore, $\mathbf{P}$ is the ortho-normalized matrix of collection of eigen vectors of $\mathbf{A}$.

$$\mathbf{P} = \begin{pmatrix} \mathbf{v_1} & \mathbf{v_2} & \mathbf{v_3} \end{pmatrix} \tag{4.7}$$

where $\mathbf{v_1}, \mathbf{v_2}$ *and* $\mathbf{v_3}$ are the normalized eigen vectors of $\mathbf{A}$.

Eigenvectors $\mathbf{v}$ for any square matrix $\mathbf{A}$ is defined as

$$\mathbf{Av} = \lambda \mathbf{v} \tag{4.8}$$

where λ is a scalar and is called the eigen value of $\mathbf{A}$.

As we could observe that matrix $\mathbf{A}$ has zeroes along the first row and first column except the the first pivot,

$$\implies \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 2 \\ 0 & 2 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{4.9}$$

$$\therefore \mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{4.10}$$

To obtain the other eigen vectors of $\mathbf{A}$, we can use the fact that the $\mathbf{A}$ is symmetric.

Let us consider two eigen vectors of symmetric matrix $\mathbf{A}$ to be $\mathbf{u}$ and $\mathbf{w}$ such that,

$$\mathbf{Au} = \lambda \mathbf{u} \quad \textit{and} \quad \mathbf{Aw} = \mu \mathbf{w} \tag{4.11}$$

Consider the scalar $\mathbf{u}^\top \mathbf{A}\mathbf{w}$.Because $\mathbf{A}$ is symmetric,

$$\mathbf{u}^\top \mathbf{A}\mathbf{w} = (\mathbf{A}\mathbf{u})^\top \mathbf{w} = (\lambda\mathbf{u})^\top \mathbf{w} = \lambda\mathbf{u}^\top\mathbf{w} \tag{4.12}$$

Similarly,

$$\mathbf{u}^\top (\mathbf{A}\mathbf{w}) = \mu\mathbf{u}^\top\mathbf{w} \tag{4.13}$$

From (4.12) and (4.13),

$$\lambda\mathbf{u}^\top\mathbf{w} = \mu\mathbf{u}^\top\mathbf{w} \implies (\lambda - \mu)\,\mathbf{u}^\top\mathbf{w} = 0 \tag{4.14}$$

As λ and μ are distinct,

$$\mathbf{u}^\top\mathbf{w} = 0 \tag{4.15}$$

$\implies$ $\mathbf{u}$ and $\mathbf{w}$ are orthogonal.

Therefore, the other eigenvectors of $\mathbf{A}$ would be orthogonal to the eigen vector $\begin{pmatrix}1\\0\\0\end{pmatrix}$.

Any vector of form $\begin{pmatrix}0\\a\\b\end{pmatrix}$ will be orthogonal to $\begin{pmatrix}1\\0\\0\end{pmatrix}$.

As we could observe that the 2×2 block from $\mathbf{A}$, i.e,

$$\mathbf{B} = \begin{pmatrix}6 & 2\\2 & 6\end{pmatrix} \tag{4.16}$$

is also symmetric,

$$\therefore \begin{pmatrix}1\\1\end{pmatrix} \text{ is an eigen-vector of } \mathbf{B} \tag{4.17}$$

$$\implies \text{Eigen vector of } \mathbf{A}\ (\mathbf{e_2}) = \begin{pmatrix}0\\1\\1\end{pmatrix} \tag{4.18}$$

As we know that from (4.15), we could say that the other eigen-vector is orthogonal to both $\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\\1\end{pmatrix}$.

The third eigen-vector $\mathbf{e_3}$ is the vector-product of $\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\\1\end{pmatrix}$.

$$\implies \mathbf{e_3} = \mathbf{e_2} \times \mathbf{e_1} \tag{4.19}$$

$$\therefore \mathbf{e_3} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \tag{4.20}$$

$$\therefore \text{The eigen-vectors of A:} \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \tag{4.21}$$

As we require unit eigen-vectors,

$$\implies \mathbf{v_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{v_2} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \qquad \mathbf{v_3} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \tag{4.22}$$

$$\therefore \mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \tag{4.23}$$