7.4.8

EE25BTECH11001 - Aarush Dilawri

October 11, 2025

Question:
For each natural number k, let C_k denote the circle with radius k centimetres and centre at the origin. On the circle C_k , a particle moves k centimetres in the counter- clockwise direction. After completing its motion on C_k , the particle moves to C_{k+1} in the radial direction. The motion of the particle continues in this manner. The particle starts at $(1, 0)$. If the particle crosses the positive direction of the X axis for the first time on the Circle C_n, then $n =$ _

Solution:

$$\text{Let} \quad \mathbf{p}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1}$$

We model a rotation by an angle θ using the rotation matrix

$$\mathbf{R}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{2}$$

Note the group property of rotations:

$$\mathbf{R}(\theta_1)\, \mathbf{R}(\theta_2) = \mathbf{R}(\theta_1 + \theta_2), \qquad \mathbf{R}(\theta)^k = \mathbf{R}(k\theta). \tag{3}$$

Solution

On the circle C_k the particle moves an arc of length k on a circle of radius k, so the angular increment on C_k is

$$\Delta\theta_k = \frac{\text{arc length}}{\text{radius}} = \frac{k}{k} = 1 \quad \text{(radian)}. \tag{4}$$

Thus each circular motion rotates the particle by 1 radian. We track the position of the particle at the instant it finishes its motion on C_k (that is, after the arc motion but before the radial jump to C_{k+1}).

Solution

Starting at $\mathbf{p}_0$ on C_1, after finishing C_1 the position is

$$\mathbf{P}_1 = 1\,\mathbf{R}\,(1)\;\mathbf{p}_0. \tag{5}$$

Then the particle moves radially to C_2, scaling the radius from 1 to 2, so just before moving on C_2 the vector is $2\mathbf{R}\,(1)\,\mathbf{p}_0$. After moving on C_2 (an additional rotation by 1) the particle is at

$$\mathbf{P}_2 = 2\,\mathbf{R}\,(1)\,\mathbf{R}\,(1)\;\mathbf{p}_0 \;=\; 2\,\mathbf{R}\,(2)\;\mathbf{p}_0. \tag{6}$$

By induction, after finishing its motion on C_k the particle is at

$$\mathbf{P}_k = k\,\mathbf{R}\,(k)\;\mathbf{p}_0. \tag{7}$$

Solution

Therefore the angular coordinate of the particle after completing C_k is exactly k radians. The motion on C_n runs the angle from $(n-1)$ to n (radians). Hence the particle crosses the positive x-axis during the motion on C_n precisely when some integer multiple of 2π lies in the interval $(n-1, n]$, i.e. when there exists $m \in \mathbb{N}$ such that

$$n - 1 \; < \; 2\pi m \; \leq \; n. \tag{8}$$

Solution

We look for the smallest natural number n for which this happens. Take $m = 1$ (the first positive multiple of 2π). Compute

$$2\pi \approx 6.283185307\ldots \tag{9}$$

and observe

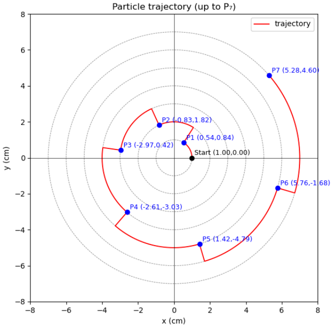
$$6 \; < \; 2\pi \; \leq \; 7. \tag{10}$$

Thus 2π lies in the interval $(6, 7]$, so the condition holds for $n = 7$ (with $m = 1$). For any $n \leq 6$ the interval $(n - 1, n]$ is contained in $[0, 6]$ and cannot contain $2\pi \approx 6.283\ldots$.

Therefore the particle crosses the positive x-axis for the first time while moving on C_n with

$$\boxed{n \; = \; 7.} \tag{11}$$

Graphical Representation



C Code (code.c)

```c
#include <stdio.h>
#include <math.h>

void particle_endpoints(int n, double *px, double *py, double *theta_out
    ) {
    double theta = 0.0;
    for (int k = 1; k <= n; ++k) {
        double r = (double)k;
        // Arc length on C_k = k    = 1 rad
        double delta = 1.0;
        theta += delta;
        px[k-1] = r * cos(theta);
        py[k-1] = r * sin(theta);
        if (theta_out != NULL) theta_out[k-1] = theta;
    }
}
```