2.10.71

EE25BTECH11043 - Nishid Khandagre

Question: If the vectors $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are not coplanar, then prove that the vector $\left(\mathbf{a} \times \mathbf{b}\right) \times \left(\mathbf{c} \times \mathbf{d}\right) + \left(\mathbf{a} \times \mathbf{c}\right) \times \left(\mathbf{d} \times \mathbf{b}\right) + \left(\mathbf{a} \times \mathbf{d}\right) \times \left(\mathbf{b} \times \mathbf{c}\right)$ is parallel to $\mathbf{a}$.

Solution:

The vector triple product $\mathbf{A} \times \left(\mathbf{B} \times \mathbf{C}\right)$ can be written as:

$$\mathbf{A} \times \left(\mathbf{B} \times \mathbf{C}\right) = \mathbf{B}\left(\mathbf{A}^\top \mathbf{C}\right) - \mathbf{C}\left(\mathbf{A}^\top \mathbf{B}\right) \tag{0.1}$$

Also, we know

$$\left(\mathbf{A} \times \mathbf{B}\right) \times \mathbf{C} = -\mathbf{C} \times \left(\mathbf{A} \times \mathbf{B}\right) \tag{0.2}$$

$$\left(\mathbf{A} \times \mathbf{B}\right) \times \left(\mathbf{C} \times \mathbf{D}\right) = \left(\mathbf{A}^\top \left(\mathbf{C} \times \mathbf{D}\right)\right)\mathbf{B} - \left(\mathbf{B}^\top \left(\mathbf{C} \times \mathbf{D}\right)\right)\mathbf{A} \tag{0.3}$$

by using (0.3)

$$\left(\mathbf{a} \times \mathbf{b}\right) \times \left(\mathbf{c} \times \mathbf{d}\right) = \left(\mathbf{a}^\top \left(\mathbf{c} \times \mathbf{d}\right)\right)\mathbf{b} - \left(\mathbf{b}^\top \left(\mathbf{c} \times \mathbf{d}\right)\right)\mathbf{a} \tag{0.4}$$

$$= [\mathbf{a}\ \mathbf{c}\ \mathbf{d}]\,\mathbf{b} - [\mathbf{b}\ \mathbf{c}\ \mathbf{d}]\,\mathbf{a} \tag{0.5}$$

$$\left(\mathbf{a} \times \mathbf{c}\right) \times \left(\mathbf{d} \times \mathbf{b}\right) = \left(\mathbf{a}^\top \left(\mathbf{d} \times \mathbf{b}\right)\right)\mathbf{c} - \left(\mathbf{c}^\top \left(\mathbf{d} \times \mathbf{b}\right)\right)\mathbf{a} \tag{0.6}$$

$$= [\mathbf{a}\ \mathbf{d}\ \mathbf{b}]\,\mathbf{c} - [\mathbf{c}\ \mathbf{d}\ \mathbf{b}]\,\mathbf{a} \tag{0.7}$$

$$= -[\mathbf{a}\ \mathbf{b}\ \mathbf{d}]\,\mathbf{c} + [\mathbf{b}\ \mathbf{c}\ \mathbf{d}]\,\mathbf{a} \tag{0.8}$$

$$\left(\mathbf{a} \times \mathbf{d}\right) \times \left(\mathbf{b} \times \mathbf{c}\right) = \left(\mathbf{a}^\top \left(\mathbf{b} \times \mathbf{c}\right)\right)\mathbf{d} - \left(\mathbf{d}^\top \left(\mathbf{b} \times \mathbf{c}\right)\right)\mathbf{a} \tag{0.9}$$

$$= [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,\mathbf{d} - [\mathbf{d}\ \mathbf{b}\ \mathbf{c}]\,\mathbf{a} \tag{0.10}$$

$$= [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,\mathbf{d} - [\mathbf{b}\ \mathbf{c}\ \mathbf{d}]\,\mathbf{a} \tag{0.11}$$

Adding equations (1), (2), and (3):

$$\left(\mathbf{a} \times \mathbf{b}\right) \times \left(\mathbf{c} \times \mathbf{d}\right) + \left(\mathbf{a} \times \mathbf{c}\right) \times \left(\mathbf{d} \times \mathbf{b}\right) + \left(\mathbf{a} \times \mathbf{d}\right) \times \left(\mathbf{b} \times \mathbf{c}\right) \tag{0.12}$$

$$\tag{0.13}$$

$$= [\mathbf{a}\ \mathbf{c}\ \mathbf{d}]\,\mathbf{b} - [\mathbf{b}\ \mathbf{c}\ \mathbf{d}]\,\mathbf{a} - [\mathbf{a}\ \mathbf{b}\ \mathbf{d}]\,\mathbf{c} + [\mathbf{b}\ \mathbf{c}\ \mathbf{d}]\,\mathbf{a} + [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,\mathbf{d} - [\mathbf{b}\ \mathbf{c}\ \mathbf{d}]\,\mathbf{a} \tag{0.14}$$

$$= [\mathbf{a}\ \mathbf{c}\ \mathbf{d}]\,\mathbf{b} - [\mathbf{a}\ \mathbf{b}\ \mathbf{d}]\,\mathbf{c} + [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,\mathbf{d} - [\mathbf{b}\ \mathbf{c}\ \mathbf{d}]\,\mathbf{a} \tag{0.15}$$

Using the expansion of a vector $\mathbf{a}$ in terms of non-coplanar vectors $\mathbf{b}, \mathbf{c}, \mathbf{d}$:

$$\mathbf{a}\left[\mathbf{b}\ \mathbf{c}\ \mathbf{d}\right] = \left[\mathbf{a}\ \mathbf{c}\ \mathbf{d}\right]\mathbf{b} + \left[\mathbf{a}\ \mathbf{d}\ \mathbf{b}\right]\mathbf{c} + \left[\mathbf{a}\ \mathbf{b}\ \mathbf{c}\right]\mathbf{d} \tag{0.16}$$

Rearranging the scalar triple products on the right:

$$\mathbf{a}\left[\mathbf{b}\ \mathbf{c}\ \mathbf{d}\right] = \left[\mathbf{a}\ \mathbf{c}\ \mathbf{d}\right]\mathbf{b} - \left[\mathbf{a}\ \mathbf{b}\ \mathbf{d}\right]\mathbf{c} + \left[\mathbf{a}\ \mathbf{b}\ \mathbf{c}\right]\mathbf{d} \tag{0.17}$$

Substitute this into the sum:

$$= \mathbf{a}\left[\mathbf{b}\ \mathbf{c}\ \mathbf{d}\right] - \left[\mathbf{b}\ \mathbf{c}\ \mathbf{d}\right]\mathbf{a} \tag{0.18}$$

$$= \left[\mathbf{b}\ \mathbf{c}\ \mathbf{d}\right]\mathbf{a} - \left[\mathbf{b}\ \mathbf{c}\ \mathbf{d}\right]\mathbf{a} \tag{0.19}$$

$$= \mathbf{0} \tag{0.20}$$

The resultant vector is $\mathbf{0}$.
A zero vector is considered parallel to any vector. Thus, the given vector expression is parallel to $\mathbf{a}$.

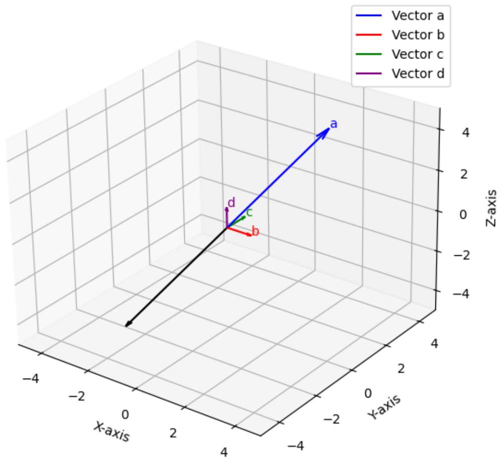

Fig. 0.1