Matrices in Geometry 8.4.40

EE25BTECH11037 - Divyansh

Question: Let **P** be a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, 0 < b < a$. Let the line parallel to the X axis passing through **P** meet the circle $x^2 + y^2 = a^2$ at the point **Q** such that **P** and **Q** are on the same side of the X axis. For two positive real numbers r and s, find the locus of the point **R** on **PQ** such that $PR = r$ as **P** varies over the ellipse.

Solution:
The given ellipse is

$$\mathbf{E} : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, 0 < b < a \tag{1}$$

This can be written as

$$\mathbf{E} \ : \ \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \ : \ \mathbf{V} = \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ , \ f = -a^2b^2 \tag{2}$$

The line parallel to the X-axis and passing through a point $\mathbf{P} = \begin{pmatrix} x_P \\ y_P \end{pmatrix}$ on the ellipse is

$$\mathbf{L} \ : \ \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = c \tag{3}$$

P satisfies this line; therefore, $c = y_P$

Let $Q = \begin{pmatrix} x_Q \\ y_Q \end{pmatrix}$ be a point on **L**; therefore, $y_R = y_P$

$$\|\mathbf{P} - \mathbf{R}\| = r \implies x_R - x_P = r \implies x_P = x_R - r \implies \mathbf{P} = \mathbf{R} - \mathbf{c} \ , \ \mathbf{c} = \begin{pmatrix} r \\ 0 \end{pmatrix} \tag{4}$$

Since, **P** is a point on **E**

$$\mathbf{P}^\top \mathbf{V}\mathbf{P} + f = 0 \tag{5}$$

Substituting $\mathbf{P} = \mathbf{Q} - \mathbf{c}$

$$(\mathbf{R} - \mathbf{c})^\top \mathbf{V} (\mathbf{R} - \mathbf{c}) + f = 0 \implies \mathbf{R}^\top \mathbf{V}\mathbf{R} - 2\mathbf{R}^\top \mathbf{V}\mathbf{c} + \mathbf{c}^\top \mathbf{V}\mathbf{c} + f = 0 \tag{6}$$

$$\mathbf{R} = \begin{pmatrix} x \\ y \end{pmatrix} \ , \ \mathbf{V} = \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \ , \ \mathbf{c} = \begin{pmatrix} r \\ 0 \end{pmatrix} \ , \ f = -a^2b^2 \tag{7}$$

Simplifying this equation, we get

$$b^2x^2 + a^2y^2 - 2b^2xr + b^2r^2 - a^2b^2 = 0 \tag{8}$$

This can also be written as

$$\frac{(x-r)^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{9}$$

This is the equation of locus of the point **R**. Let us try to draw the locus for $a = 4, b = 2, r = 1$

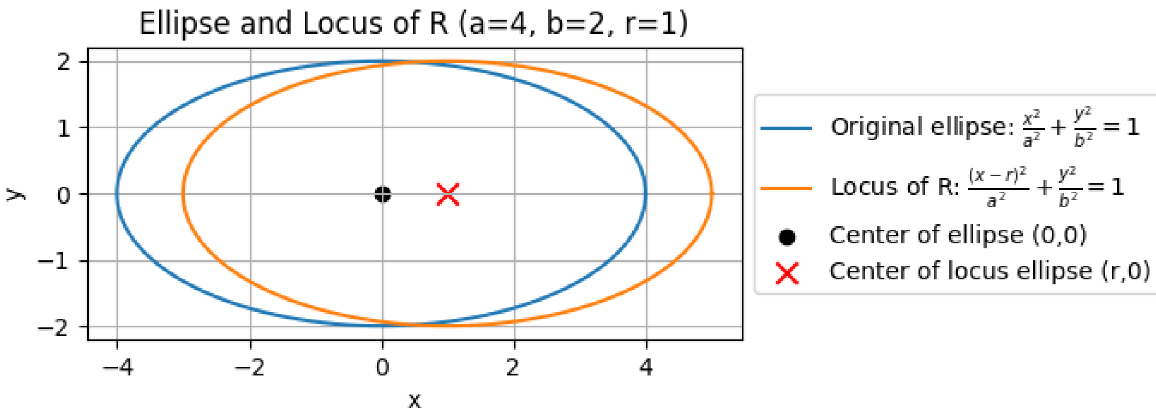

Fig. 1: Figure for 8.4.40 for $a = 4, b = 2, r = 1$