5.13.45

EE25BTECH11019 – Darji Vivek M.

Question:

Let $x \in \mathbb{R}$

$$\mathbf{P} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}, \qquad \mathbf{Q} = \begin{pmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 5 \end{pmatrix}, \qquad \mathbf{R} = \mathbf{P}\mathbf{Q}\mathbf{P}^{-1}.$$

Then which of the following options is/are correct?

Question:

1. $\det \mathbf{R} = \det \begin{pmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 5 \end{pmatrix} + 8$ for all $x \in \mathbb{R}$.
2. For $x = 1$, there exists a unit vector $\alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}$ such that $\mathbf{R}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.
3. There exists a real number x such that $\mathbf{PQ} = \mathbf{QP}$.
4. For $x = 0$, if $\mathbf{R}\begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = 6\begin{pmatrix} 1 \\ a \\ b \end{pmatrix}$, then $a + b = 5$.

Solution

Given

$$\mathbf{R} = \mathbf{P}\mathbf{Q}\mathbf{P}^{-1}$$

$$\mathbf{P} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 5 \end{pmatrix}$$

(a)
Since $\left|\mathbf{R}\right| = \left|\mathbf{Q}\right|$,

$$\left|\mathbf{Q}\right| = 2\begin{vmatrix} 4 & 0 \\ x & 5 \end{vmatrix} - x\begin{vmatrix} 0 & 0 \\ x & 5 \end{vmatrix} + x\begin{vmatrix} 0 & 4 \\ x & x \end{vmatrix}$$

$$= 2(20) - x(0) + x(0 - 4x) \Rightarrow \left|\mathbf{Q}\right| = 40 - 4x^2$$

$$\Rightarrow \left|\mathbf{R}\right| = 40 - 4x^2$$

Hence, option (a) is **false**.

(b)
For $x = 1$,

$$\left|\mathbf{R}\right| = 40 - 4(1)^2 = 36 \neq 0$$

Since $\left|\mathbf{R}\right| \neq 0$, $\mathbf{R}$ is invertible, so $\mathbf{R}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ has only the trivial solution. Hence, no unit vector exists $\Rightarrow$ (b) **false**.

(c)

$$\mathbf{PQ} = \begin{pmatrix}1 & 1 & 1\\ 0 & 2 & 2\\ 0 & 0 & 3\end{pmatrix}\begin{pmatrix}2 & x & x\\ 0 & 4 & 0\\ x & x & 5\end{pmatrix} = \begin{pmatrix}2+x & 2x+4 & x+5\\ 2x & 8+2x & 10\\ 3x & 3x & 15\end{pmatrix}$$

$$\mathbf{QP} = \begin{pmatrix}2 & x & x\\ 0 & 4 & 0\\ x & x & 5\end{pmatrix}\begin{pmatrix}1 & 1 & 1\\ 0 & 2 & 2\\ 0 & 0 & 3\end{pmatrix} = \begin{pmatrix}2 & 2+2x & 2+5x\\ 0 & 8 & 8\\ x & 3x & 3x+15\end{pmatrix}$$

Comparing entries, no x satisfies $\mathbf{PQ} = \mathbf{QP}$. Hence, (c) **false**.

(d)
For $x = 0$,

$$\mathbf{Q} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix} \Rightarrow \text{Eigenvalues of } \mathbf{R} = \{2, 4, 5\}$$

If $\mathbf{R} \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = 6 \begin{pmatrix} 1 \\ a \\ b \end{pmatrix}$, then 6 must be an eigenvalue, which is not possible.
Hence, (d) **false**.

$$\boxed{\text{All options are incorrect.}}$$

C code

```
#include <stdio.h>

// This function fills matrix Q for given x
// Q = [[2, x, x],
//      [0, 4, 0],
//      [x, x, 5]]
void fill_Q(float x, float Q[3][3]) {
    Q[0][0] = 2;  Q[0][1] = x;  Q[0][2] = x;
    Q[1][0] = 0;  Q[1][1] = 4;  Q[1][2] = 0;
    Q[2][0] = x;  Q[2][1] = x;  Q[2][2] = 5;
}
```

C code

```
https://github.com/vivekd03/ee1030-2025/blob/main/
    ee25btech11019/matgeo/5.13.45/codes/13.c
```

Python

```
https://github.com/vivekd03/ee1030-2025/blob/main/
    ee25btech11019/matgeo/5.13.45/codes/13.py
```