Matrices in Geometry 2.4.34

EE25BTECH11037 - Divyansh

Question: What type of quadrilateral do the points $\mathbf{A}\,(2, -2)$, $\mathbf{B}\,(7, 3)$, $\mathbf{C}\,(11, -1)$ and $\mathbf{D}\,(6, -6)$ taken in that order, form?

Given: $\mathbf{A}\begin{pmatrix} 2 \\ -2 \end{pmatrix}$, $\mathbf{B}\begin{pmatrix} 7 \\ 3 \end{pmatrix}$, $\mathbf{C}\begin{pmatrix} 11 \\ -1 \end{pmatrix}$ and $\mathbf{D}\begin{pmatrix} 6 \\ -6 \end{pmatrix}$.

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} \tag{1}$$

$$\mathbf{C} - \mathbf{B} = \begin{pmatrix} 4 \\ -4 \end{pmatrix} \tag{2}$$

$$\mathbf{D} - \mathbf{C} = \begin{pmatrix} -5 \\ -5 \end{pmatrix} \tag{3}$$

$$\mathbf{A} - \mathbf{D} = \begin{pmatrix} -4 \\ 4 \end{pmatrix} \tag{4}$$

$$\text{Checking opposite sides,} \quad \mathbf{D} - \mathbf{C} = -(\mathbf{A} - \mathbf{B}) \text{ and } \mathbf{A} - \mathbf{D} = -(\mathbf{B} - \mathbf{C}) \tag{5}$$

Each pair of opposite sides are parallel and equal in length; this implies that the quadrilateral is a parallelogram.

Now, checking for right angle, we check for inner product.

$$(\mathbf{B} - \mathbf{A})^\top (\mathbf{C} - \mathbf{B}) = \begin{pmatrix} 5 & 5 \end{pmatrix}\begin{pmatrix} 4 \\ -4 \end{pmatrix} = 0 \tag{6}$$

This implies that the angle at $\mathbf{B}$ is one right angle. A parallelogram with a right angle is a rectangle.

Checking for a square:

$$\|\mathbf{B} - \mathbf{A}\| = 5\sqrt{2} \text{ and } \|\mathbf{C} - \mathbf{B}\| = 4\sqrt{2} \tag{7}$$

We can see that $\|\mathbf{B} - \mathbf{A}\| \neq \|\mathbf{C} - \mathbf{B}\|$, and therefore it is not a square.

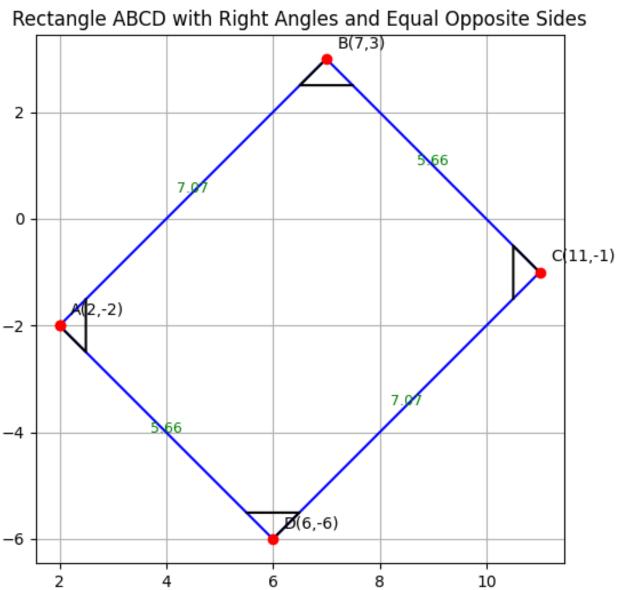Therefore, the quadrilateral $\mathbf{ABCD}$ is a rectangle.

Fig. 1: Plot for 2.4.34