8.4.15

EE25BTECH11012-BEERAM MADHURI

Question:
In a triangle ABC with fixed base BC, the vertex A moves such that

$$\cos B + \cos C = 4\sin^2 \frac{A}{2}.$$

If a, b and c denote the lengths of the sides of the triangle opposite to the angles A, B and C, respectively, then

a) $b + c = 4a$
b) $b + c = 2a$
c) locus of the point A is an ellipse
d) locus of the point A is a pair of straight lines

Solution:
Given,

$$\cos B + \cos C = 4\sin^2 \frac{A}{2} \tag{0.1}$$

$$\cos B + \cos C = 4\frac{(1 - \cos A)}{2} \tag{0.2}$$

$$2\cos A + \cos B + \cos C = 2 \tag{0.3}$$

By Projection rule:

$$c\cos B + b\cos C = a \tag{0.4}$$

$$c\cos A + a\cos C = b \tag{0.5}$$

$$b\cos A + a\cos B = c \tag{0.6}$$

Combining all these into a Matrix:

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{bmatrix} \begin{bmatrix} \cos A \\ \cos B \\ \cos C \end{bmatrix} = \begin{bmatrix} 2 \\ a \\ b \\ c \end{bmatrix} \tag{0.7}$$

$$AX = b \tag{0.8}$$

for this system to be consistent

$$\text{rank}(A) = \text{rank}([A|b]) \leq 3 \tag{0.9}$$

if $\text{rank}([A|b]) \leq 3$
its columns are linearly dependent.

$\therefore \det([A|b]) = 0$

$$(2a - b - c)(-a^2 + (b - c)a + (b - c)^2) = 0 \tag{0.10}$$
$$\text{as, a, b, c are sides of triangle} \tag{0.11}$$
$$-a^2 + (b - c)a + (b - c)^2 \neq 0 \tag{0.12}$$
$$\therefore 2a - b - c = 0 \tag{0.13}$$
$$\therefore b + c = 2a \tag{0.14}$$

$$b + c = 2a \tag{0.15}$$
$$\|A - C\| + \|A - B\| = 2\|B - C\| \tag{0.16}$$

given B and C are fixed.

$\therefore$ Locus of 'A' is an ellipse,

as, the sum of its distances from 2 fixed points is constant (represents an ellipse).

$\therefore$ Options b and c are correct.

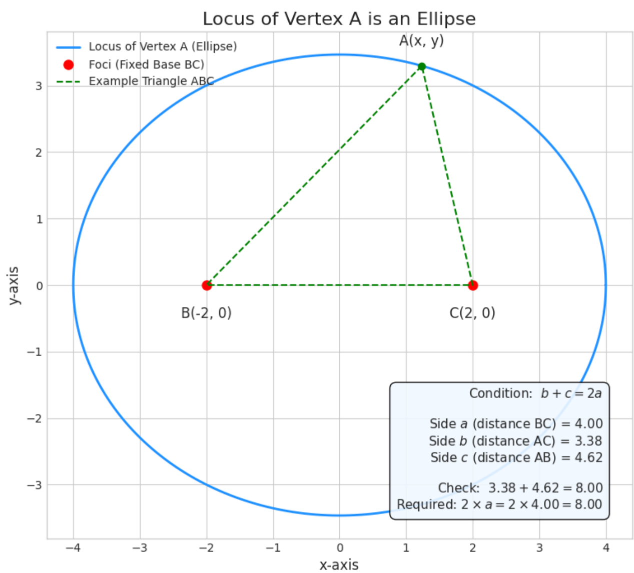

Fig. 0.1: 8.4.15