

8.4.26

EE25BTECH11023 - Venkata Sai

Question:

The locus of the mid-point of the line segment joining the focus to a moving point on the parabola $y^2 = 4ax$ is another parabola with directrix

1) $x = 0$

2) $x = -a/2$

3) $x = a$

4) $x = a/2$

Solution:

The equation of a conic with directrix $\mathbf{n}^\top \mathbf{x} = c$, eccentricity e and focus \mathbf{F} is given by

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \quad (1)$$

On comparing with $y^2 - 4ax = 0$

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} = y^2 \quad (2)$$

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} = \left[\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} \right]^2 \quad (3)$$

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} = \left(\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} \right)^\top \left(\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} \right) \quad (4)$$

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} = \mathbf{x}^\top \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} \quad (5)$$

$$\mathbf{x}^\top \left(\mathbf{V} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) \mathbf{x} = 0 \implies \mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (6)$$

$$2\mathbf{u}^\top \mathbf{x} = -4ax \quad (7)$$

$$2\mathbf{u}^\top \mathbf{x} = -4a \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} \implies 2\mathbf{u}^\top \mathbf{x} = \begin{pmatrix} -4a & 0 \end{pmatrix} \mathbf{x} \quad (8)$$

$$\left(2\mathbf{u}^\top - \begin{pmatrix} -4a & 0 \end{pmatrix} \right) \mathbf{x} = 0 \implies \mathbf{u}^\top = \begin{pmatrix} -2a & 0 \end{pmatrix} \quad (9)$$

$$\mathbf{u} = \begin{pmatrix} -2a \\ 0 \end{pmatrix} \quad (10)$$

$$f = 0 \quad (11)$$

$$\mathbf{F} = \begin{pmatrix} a \\ 0 \end{pmatrix} \quad (12)$$

Let \mathbf{X} be the point of locus of the midpoint

$$\mathbf{X} = \frac{\mathbf{x} + \mathbf{F}}{2} \implies \mathbf{x} = 2\mathbf{X} - \mathbf{F} \quad (13)$$

From (1) and (13)

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \quad (14)$$

$$(\mathbf{2X} - \mathbf{F})^\top \mathbf{V} (\mathbf{2X} - \mathbf{F}) + 2\mathbf{u}^\top (\mathbf{2X} - \mathbf{F}) + f = 0 \quad (15)$$

$$(\mathbf{2X}^\top - \mathbf{F}^\top) \mathbf{V} (\mathbf{2X} - \mathbf{F}) + 2\mathbf{u}^\top (\mathbf{2X} - \mathbf{F}) + f = 0 \quad (16)$$

$$4\mathbf{X}^\top \mathbf{V} \mathbf{X} - 2\mathbf{X}^\top \mathbf{V} \mathbf{F} - 2\mathbf{F}^\top \mathbf{V} \mathbf{X} + \mathbf{F}^\top \mathbf{V} \mathbf{F} + 4\mathbf{u}^\top \mathbf{X} - 2\mathbf{u}^\top \mathbf{F} + f = 0 \quad (17)$$

As \mathbf{V} is a symmetric matrix

$$4\mathbf{X}^\top \mathbf{V} \mathbf{X} - 2\mathbf{F}^\top \mathbf{V} \mathbf{X} - 2\mathbf{F}^\top \mathbf{V} \mathbf{X} + \mathbf{F}^\top \mathbf{V} \mathbf{F} + 4\mathbf{u}^\top \mathbf{X} - 2\mathbf{u}^\top \mathbf{F} + f = 0 \quad (18)$$

$$4\mathbf{X}^\top \mathbf{V} \mathbf{X} - 4\mathbf{F}^\top \mathbf{V} \mathbf{X} + 4\mathbf{u}^\top \mathbf{X} + \mathbf{F}^\top \mathbf{V} \mathbf{F} - 2\mathbf{u}^\top \mathbf{F} + f = 0 \quad (19)$$

$$\mathbf{X}^\top (\mathbf{4V}) \mathbf{X} + 2 \left(2(\mathbf{u}^\top - \mathbf{F}^\top \mathbf{V}) \right) \mathbf{X} + \mathbf{F}^\top \mathbf{V} \mathbf{F} - 2\mathbf{u}^\top \mathbf{F} + f = 0 \quad (20)$$

$$\mathbf{X}^\top (\mathbf{4V}) \mathbf{X} + 2 \left(2(\mathbf{u} - \mathbf{V} \mathbf{F})^\top \right) \mathbf{X} + \mathbf{F}^\top \mathbf{V} \mathbf{F} - 2\mathbf{u}^\top \mathbf{F} + f = 0 \quad (21)$$

$$\mathbf{V}' = 4\mathbf{V} = 4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \quad (22)$$

$$\mathbf{u}' = 2(\mathbf{u} - \mathbf{V} \mathbf{F}) = 2 \left(\begin{pmatrix} -2a \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} \right) = 2 \left(\begin{pmatrix} -2a \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} -4a \\ 0 \end{pmatrix} \quad (23)$$

$$f' = \mathbf{F}^\top \mathbf{V} \mathbf{F} - 2\mathbf{u}^\top \mathbf{F} + f \quad (24)$$

$$f' = \begin{pmatrix} a \\ 0 \end{pmatrix}^\top \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} - 2 \begin{pmatrix} -2a \\ 0 \end{pmatrix}^\top \begin{pmatrix} a \\ 0 \end{pmatrix} + 0 \quad (25)$$

$$f' = \begin{pmatrix} a & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} - 2 \begin{pmatrix} -2a & 0 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} + 0 \quad (26)$$

$$f' = \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} - 2 \begin{pmatrix} -2a & 0 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} + 0 \quad (27)$$

$$f' = -2 \begin{pmatrix} -2a^2 \end{pmatrix} = 4a^2 \quad (28)$$

Finding eigen values of \mathbf{V}'

$$|\mathbf{V}' - \lambda \mathbf{I}| = 0 \quad (29)$$

$$\left| \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0 \implies \left| \begin{pmatrix} -\lambda & 0 \\ 0 & 4 - \lambda \end{pmatrix} \right| = 0 \quad (30)$$

$$(-\lambda)(4 - \lambda) = 0 \implies \lambda_1 = 0 \text{ and } \lambda_2 = 4 \quad (31)$$

\mathbf{p}_1 is an eigen vector of \mathbf{V}'

$$(\mathbf{V}' - \lambda \mathbf{I}) \mathbf{p} = \mathbf{0} \quad (32)$$

From (30) and substituting $\lambda=0$

$$\begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{p}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (33)$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (34)$$

$$0 = 0, y = 0 \implies \mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (35)$$

The Equation of a Directrix is given by

$$\mathbf{n}^\top \mathbf{x} = c \quad (36)$$

where

$$\mathbf{n} = \sqrt{\lambda_2} \mathbf{p}_1 \text{ (and) } c = \frac{(\|\mathbf{u}'\|^2 - \lambda_2 f)}{2\mathbf{u}'^\top \mathbf{n}} \quad (37)$$

$$\mathbf{n} = \sqrt{4} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies \mathbf{n} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad (38)$$

$$c = \frac{((-4a)^2 + (0)^2 - 4(4a^2))}{2 \begin{pmatrix} -4a \\ 0 \end{pmatrix}^\top \begin{pmatrix} 2 \\ 0 \end{pmatrix}} = \frac{(16a^2 - 16a^2)}{2 \begin{pmatrix} -4a \\ 0 \end{pmatrix}^\top \begin{pmatrix} 2 \\ 0 \end{pmatrix}} = 0 \quad (39)$$

From (36)

$$\begin{pmatrix} 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \implies 2x = 0 \implies x = 0 \quad (40)$$

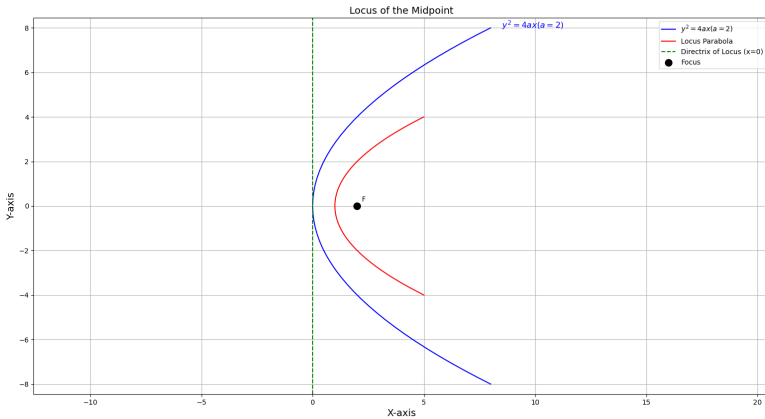


Fig. 4.1