5.2.44

EE25BTECH11042 - Nipun Dasari

Question:
Solve the following system of rational equations

$$\frac{10}{x+y}+\frac{2}{x-y}=4 \tag{0.1}$$

$$\frac{15}{x+y}-\frac{5}{x-y}=-2 \tag{0.2}$$

Solution:
$\frac{a}{x+y}+\frac{b}{x-y}=c$ becomes:

$$c(x^2-y^2)-(a+b)x+(a-b)y=0 \tag{0.3}$$

Matrix form: $\mathbf{x}^\top V\mathbf{x}+2\mathbf{u}^\top\mathbf{x}=0$, where:

$$\mathbf{x}=\begin{pmatrix}x\\y\end{pmatrix},\quad V=\begin{pmatrix}c&0\\0&-c\end{pmatrix},\quad \mathbf{u}=\begin{pmatrix}-(a+b)/2\\(a-b)/2\end{pmatrix} \tag{0.4}$$

The intersection points of the two hyperbolas lie on a Common chord, $c_1H_2-c_2H_1=0$, where $H_1=0$ and $H_2=0$ are the equations of each of hyperbolas. This results in the linear equation $\mathbf{n}^\top\mathbf{x}=0$,

$$d=c_1(a_2+b_2)-c_2(a_1+b_1) \tag{0.5}$$

$$e=c_2(a_1-b_1)-c_1(a_2-b_2) \tag{0.6}$$

$$\text{where d and e are obtained by eliminating the quadratic terms } \mathbf{n}=\begin{pmatrix}d\\e\end{pmatrix} \tag{0.7}$$

The solution is the non-trivial intersection point of this common chord and either hyperbola

$$y=-\frac{d}{e}x \tag{0.8}$$

$$c_1\left(x^2-\left(-\frac{d}{e}x\right)^2\right)-(a_1+b_1)\,x+(a_1-b_1)\left(-\frac{d}{e}x\right)=0 \tag{0.9}$$

$$\implies x\left(x\left(c_1\left(\frac{e^2-d^2}{e^2}\right)\right)-\left(\frac{e\,(a_1+b_1)+d\,(a_1-b_1)}{e}\right)\right)=0 \tag{0.10}$$

$$\therefore x\left(c_1\left(\frac{e^2-d^2}{e^2}\right)\right)=\left(\frac{e\,(a_1+b_1)+d\,(a_1-b_1)}{e}\right) \tag{0.11}$$

$$\implies x=\frac{e\,(e\,(a_1+b_1)+d\,(a_1-b_1))}{c_1\,(e^2-d^2)} \tag{0.12}$$

For the given system, the coefficients are:

$$a_1 = 10, b_1 = 2, c_1 = 4 \quad \text{and} \quad a_2 = 15, b_2 = -5, c_2 = -2 \tag{0.13}$$

Using (0.5) and (0.6), we calculate the common chord coefficients:

$$d = 4(15-5) - (-2)(10+2) = 4(10) + 2(12) = 64 \tag{0.14}$$

$$e = (-2)(10-2) - 4(15--5) = -2(8) - 4(20) = -96 \tag{0.15}$$

Substituting these into the formula for x from (0.12):

$$x = \frac{(-96)((-96)(12) + (64)(8))}{4\left((-96)^2 - (64)^2\right)} \tag{0.16}$$

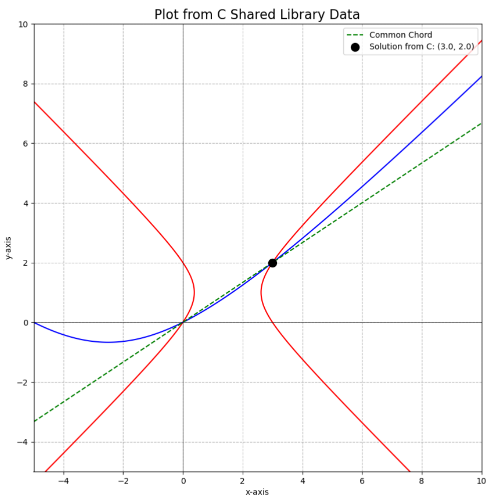
$$= \frac{-96(-1152+512)}{4(9216-4096)} \tag{0.17}$$

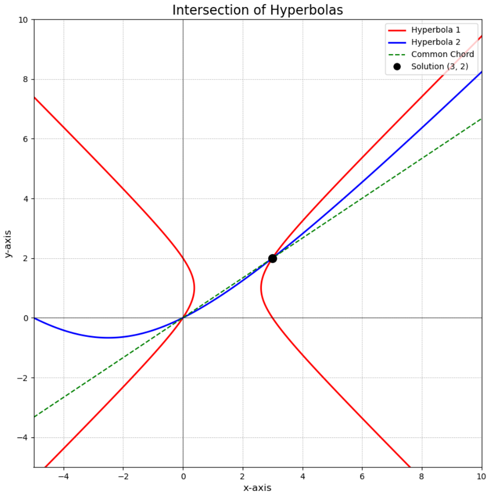
$$= \frac{-96(-640)}{4(5120)} = \frac{61440}{20480} = 3 \tag{0.18}$$

Using the value of x from (0.18) in the formula for y (0.8):

$$y = -\frac{64}{-96}(3) = \frac{2}{3}3 = 2 \tag{0.19}$$

Thus, from (0.18) and (0.19), the solution is $x = 3$ and $y = 2$.




Intersection of Hyperbolas
Hyperbola 1
Hyperbola 2
Common Chord
Solution (3, 2)
x-axis
y-axis
−4
−2
0
2
4
6
8
10