4.8.14

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Question:

Find the area of the region
$\{(x, y) : 0 \leq y \leq x^2, 0 \leq y \leq x + 2, -1 \leq x \leq 3\}$.
Solution:

The parabola $y = x^2$ can be written as

$$y - x^2 = 0$$

or, in conic matrix form:

$$\mathbf{x}^T V\mathbf{x} + 2\,\mathbf{u}^T\mathbf{x} + f = 0, \tag{0.1}$$

$$\mathbf{V} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, \quad f = 0. \tag{0.2}$$

The line $y = x + 2$ can be written as

$$\text{x - y + 2 = 0,}$$

The General Equation of a Line:

$$\mathbf{x} = k\mathbf{m} + \mathbf{h} \tag{0.3}$$

On comparing, we get:

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{h} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{0.4}$$

The Intersection of the given conic with the given line can be written as:

$$\mathbf{x}_i = \mathbf{h} + k_i\mathbf{m} \tag{0.5}$$

$$where,\ k_i = \left(\frac{1}{\mathbf{m}^\top\mathbf{V}\mathbf{m}}\right)\left(-\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u}) \pm \sqrt{[\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u})]^2 - g(h)(\mathbf{m}^\top\mathbf{V}\mathbf{m})}\right) \tag{0.6}$$

Let $\mathbf{K} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$

The Solution Matrix can be expressed as:

$$\mathbf{X} = \begin{pmatrix} \mathbf{h} & \mathbf{m} \end{pmatrix}\begin{pmatrix} \mathbf{1} & \mathbf{k} \end{pmatrix}^\top \tag{0.7}$$

Therefore, The points of intersection are:

$$\mathbf{x}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \& \quad \mathbf{x}_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \tag{0.8}$$

From Fig.0.1, the required area is given by:

$$\int_{-1}^{2} [(x+2) - (x^2)]\, dx = \int_{-1}^{2} [2 + x - x^2]\, dx \tag{0.9}$$

$$\int_{-1}^{2} [x^2]\, dx + \int_{2}^{3} [x+2]\, dx = \frac{15}{2} = 7.5\, sq.units \tag{0.10}$$

$$\boxed{\text{Therefore, the required area is 7.5 sq.units.}}$$

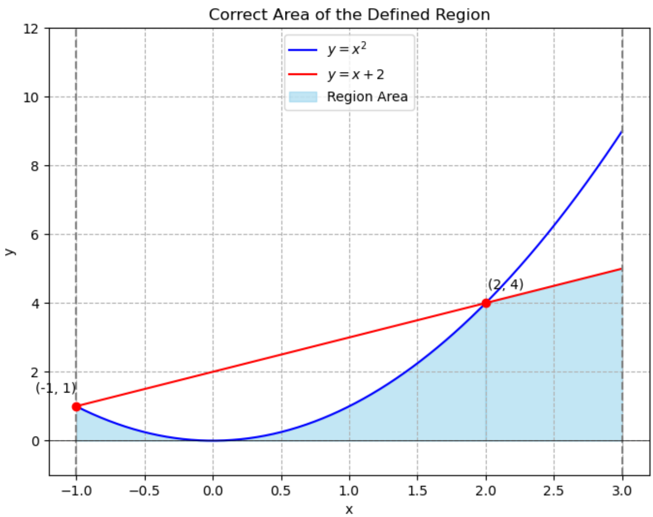

Fig. 0.1