AI25BTECH11034 - SUJAL CHAUHAN

4.4.30

Question:
Show that four points $\mathbf{A}(4,5,1)$, $\mathbf{B}(0,-1,-1)$, $\mathbf{C}(3,9,4)$, $\mathbf{D}(-4,4,4)$ are coplanar.

Theory:

If N points in $\mathbb{R}^3$ are given as

$$X_i = (x_i, y_i, z_i), \quad i = 1, 2, \ldots, N,$$

Let equation of the given plane be
all four point must satisfy the equation of plane

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{D} \end{pmatrix} = \mathbf{K} \tag{1}$$

Now equation will be

$$\mathbf{K}^T \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{2}$$

making augment matrix which is

$$\begin{pmatrix} x_1 & y_1 & z_1 & | & 1 \\ x_2 & y_2 & z_2 & | & 1 \\ x_3 & y_3 & z_3 & | & 1 \\ x_4 & y_4 & z_4 & | & 1 \end{pmatrix} \tag{3}$$

then the condition for coplanarity is that the augmented matrix

$$A = \begin{pmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{pmatrix} \tag{4}$$

satisfies

$$\operatorname{rank}(A) \leq 3. \tag{5}$$

Solution:
Equation of plane:

$$\mathbf{n}^\top \mathbf{x} = 1 \tag{6}$$

Now we have: Four point which satisfy the plane can be expressed as:

$$\begin{pmatrix} 4 & 5 & 1 \\ 0 & -1 & -1 \\ 3 & 9 & 4 \\ -4 & 4 & 4 \end{pmatrix} \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{7}$$

For the given four points to be coplanar, the rank of the following matrix must be less than 4:

$$\mathbf{A} = \begin{pmatrix} 4 & 5 & 1 & 1 \\ 0 & -1 & -1 & 1 \\ 3 & 9 & 4 & 1 \\ -4 & 4 & 4 & 1 \end{pmatrix} \tag{8}$$

$$\xrightarrow{R_3 \to R_3 - \frac{3}{4}R_1,\, R_4 \to R_4 + R_1} \begin{pmatrix} 4 & 5 & 1 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & \frac{21}{4} & \frac{13}{4} & \frac{1}{4} \\ 0 & 9 & 5 & 2 \end{pmatrix} \tag{9}$$

$$\xrightarrow{R_1 \to \frac{1}{4}R_1} \begin{pmatrix} 1 & \frac{5}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & -1 & -1 & 1 \\ 0 & \frac{21}{4} & \frac{13}{4} & \frac{1}{4} \\ 0 & 9 & 5 & 2 \end{pmatrix} \tag{10}$$

$$\xrightarrow{R_2 \to -R_2} \begin{pmatrix} 1 & \frac{5}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 1 & -1 \\ 0 & \frac{21}{4} & \frac{13}{4} & \frac{1}{4} \\ 0 & 9 & 5 & 2 \end{pmatrix} \tag{11}$$

$$\xrightarrow{R_3 \to R_3 - \frac{21}{4}R_2,\, R_4 \to R_4 - 9R_2} \begin{pmatrix} 1 & \frac{5}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -2 & \frac{22}{4} \\ 0 & 0 & -4 & 11 \end{pmatrix} \tag{12}$$

$$\xrightarrow{R_3 \to -\frac{1}{2}R_3} \begin{pmatrix} 1 & \frac{5}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -\frac{11}{4} \\ 0 & 0 & -4 & 11 \end{pmatrix} \tag{13}$$

$$\xrightarrow{R_4 \to R_4 + 4R_3} \begin{pmatrix} 1 & \frac{5}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -\frac{11}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{14}$$

Thus,

$$\operatorname{rank}(\mathbf{A}) = 3 \;<\; 4 \quad \Longrightarrow \quad \text{The given points are coplanar.} \tag{15}$$

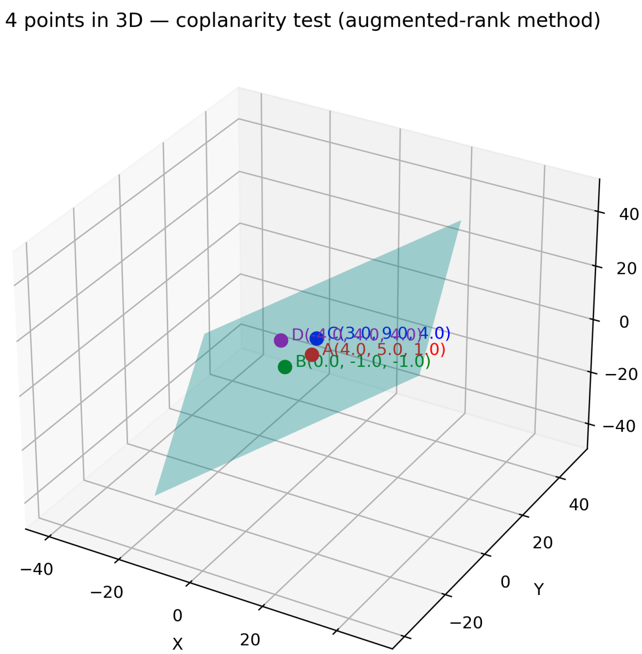

Figure 1: Geometric visualization of points A, B, C, D lying on the same plane.