4.13.45

EE25BTECH11041 - Naman Kumar

Question a):
Two vertices of a triangle are $(5, -1)$ and $(2, -3)$. If the orthocentre of the triangle is the origin, find the coordinates of the third point.
Solution:
Given,

$$\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \tag{1}$$

Where,

$\mathbf{O}$	Point vector of orthocenter
$\mathbf{A}$ and $\mathbf{B}$	known vector of Points of triangle
$\mathbf{m}_1$	Direction vector of line from $\mathbf{C}$ to $\mathbf{B}$
$\mathbf{m}_2$	Direction vector of line from $\mathbf{C}$ to $\mathbf{A}$
$\mathbf{A}_1$	Altitude from $\mathbf{A}$ to $\mathbf{O}$
$\mathbf{A}_2$	Altitude from $\mathbf{B}$ to $\mathbf{O}$
$\mathbf{L}$	Line from $\mathbf{B}$ to $\mathbf{A}$
$\mathbf{C}$	Required point

TABLE I

From General Triangle Properties.

$$m_1^T A_1 = 0, m_2^T A_2 = 0 \tag{2}$$

$$(\mathbf{A} - \mathbf{O})^T (\mathbf{C} - \mathbf{B}) = 0; \tag{3}$$

$$\mathbf{A}^T (\mathbf{C} - \mathbf{B}) = 0, \text{Since } \mathbf{O} \text{ is origin} \tag{4}$$

and

$$(\mathbf{B} - \mathbf{O})^T (\mathbf{C} - \mathbf{A}) = 0; \tag{5}$$

$$(\mathbf{B})^T (\mathbf{C} - \mathbf{A}) = 0, \text{Since } \mathbf{O} \text{ is origin} \tag{6}$$

Modifying (4) and (6)

$$\mathbf{A}^T \mathbf{C} - \mathbf{A}^T \mathbf{B} = 0, \mathbf{B}^T \mathbf{C} - \mathbf{B}^T \mathbf{A} = 0 \tag{7}$$

$$\mathbf{A}^T \mathbf{C} = \mathbf{A}^T \mathbf{B} \tag{8}$$

$$\mathbf{B}^T \mathbf{C} = \mathbf{B}^T \mathbf{A} \tag{9}$$

This can be written as

$$\begin{pmatrix} \mathbf{A}^T \\ \mathbf{B}^T \end{pmatrix} \mathbf{C} = \begin{pmatrix} \mathbf{A}^T \mathbf{B} \\ \mathbf{B}^T \mathbf{A} \end{pmatrix} \tag{10}$$

$$\begin{pmatrix} 5 & -1 \\ 2 & -3 \end{pmatrix} \mathbf{C} = \begin{pmatrix} \mathbf{A}^T \mathbf{B} \\ \mathbf{B}^T \mathbf{A} \end{pmatrix} \tag{11}$$

Let

$$\left(\begin{array}{cc|c} 5 & -1 & \mathbf{A}^T\mathbf{B} \\ 2 & -3 & \mathbf{B}^T\mathbf{A} \end{array}\right) = \left(\begin{array}{cc|c} 5 & -1 & 13 \\ 2 & -3 & 13 \end{array}\right) \tag{12}$$

By Gaussian Elimination

$$\left(\begin{array}{cc|c} 5 & -1 & 13 \\ 2 & -3 & 13 \end{array}\right) \xrightarrow{R_2-\frac{2}{5}R_1} \left(\begin{array}{cc|c} 5 & -1 & 13 \\ 0 & \frac{-13}{5} & \frac{39}{5} \end{array}\right) \tag{13}$$

$$\xrightarrow{-\frac{5}{13}R_2} \left(\begin{array}{cc|c} 5 & -1 & 13 \\ 0 & 1 & -3 \end{array}\right) \tag{14}$$

In equation (11)

$$\begin{pmatrix} 5 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 13 \\ -3 \end{pmatrix} \tag{15}$$

Therefore, $\mathbf{C}$ is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \tag{16}$$

But,Now

$$\mathbf{B} = \mathbf{C} \tag{17}$$

Which is not possible for a triangle,
Slope of line $\mathbf{A}$ to $\mathbf{B}$ or $\mathbf{L}$ be $\mathbf{m}$

$$\mathbf{m} = \mathbf{A} - \mathbf{B} \tag{18}$$

$$\mathbf{m} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \tag{19}$$

We can see, that

$$\mathbf{m}^T\mathbf{A_2} = 0, \mathbf{m_2}^T\mathbf{A_2} = 0 \tag{20}$$

Therefore, for this to be possible when

$$\mathbf{L} \parallel \mathbf{m_2}, \tag{21}$$

$$\mathbf{A} - \mathbf{C} \parallel \mathbf{A} - \mathbf{B} \tag{22}$$

Since both lines have a point in common $\mathbf{A}$, therefore they must be collinear.
So, $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ is just a straight line

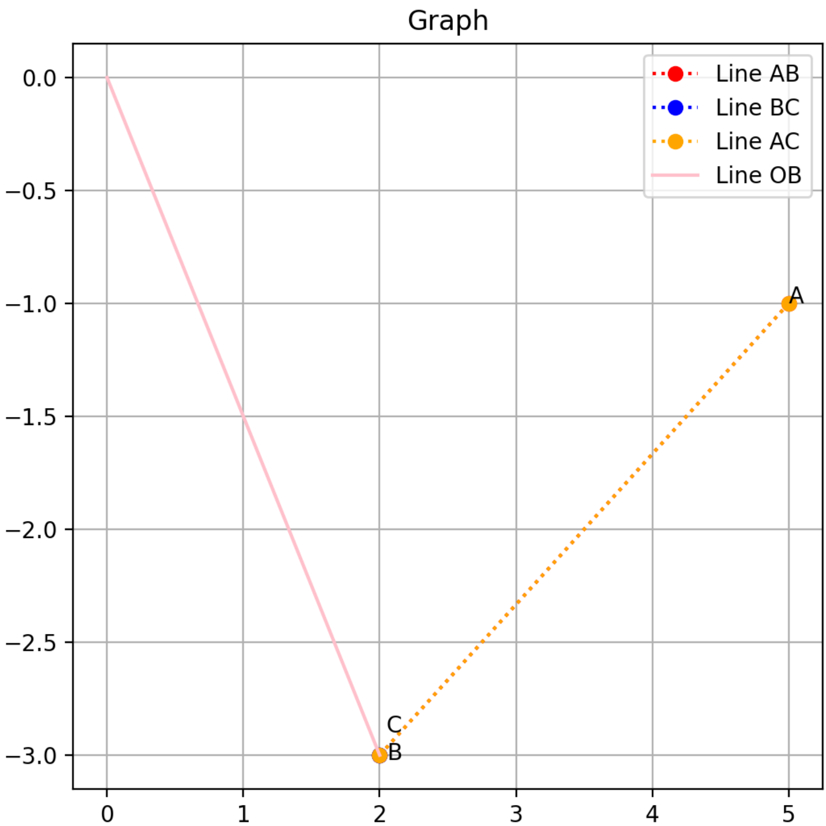
Graph
Line AB
Line BC
Line AC
Line OB
A
B
C
0.0
−0.5
−1.0
−1.5
−2.0
−2.5
−3.0
0
1
2
3
4
5

Fig. 1