4.2.22

EE25BTECH11019 - Darji Vivek M.

Question:
Show that the two lines

$$a_1x + b_1y + c_1 = 0, \qquad a_2x + b_2y + c_2 = 0$$

with $b_1b_2 \neq 0$ are parallel iff $\frac{a_1}{b_1} = \frac{a_2}{b_2}$.
Solution:

$$\text{Form the } 2 \times 2 \text{ coefficient matrix of normals: } \mathbf{M} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}. \tag{1}$$

Assume $\frac{a_1}{b_1} = \frac{a_2}{b_2}$. Then there exists $k \in \mathbb{R}$ such that

$$a_2 = k\,a_1, \qquad b_2 = k\,b_1. \tag{2}$$

Write the rows of $\mathbf{M}$ as row vectors:

$$\text{Row}_1 = (a_1,\ a_2) = (a_1,\ ka_1) = a_1(1,\ k), \tag{3}$$

$$\text{Row}_2 = (b_1,\ b_2) = (b_1,\ kb_1) = b_1(1,\ k). \tag{4}$$

Perform the row operation $\text{Row}_2 \leftarrow \text{Row}_2 - \frac{b_1}{a_1}\text{Row}_1$ (assuming $a_1 \neq 0$; if $a_1 = 0$ use a symmetric argument swapping roles). Because Row_2 is $\frac{b_1}{a_1}$ times Row_1, this operation yields the zero row:

$$\text{Row}_2 \to \text{Row}_2 - \frac{b_1}{a_1}\text{Row}_1 = (0, 0). \tag{5}$$

Thus the row-echelon form of $\mathbf{M}$ has exactly one nonzero row, so

$$\text{rank}(\mathbf{M}) = 1. \tag{6}$$

Rank 1 means the two column vectors (or equivalently the two normal vectors) are linearly dependent - i.e. collinear - hence the associated lines have the same slope and are parallel.

Conversely, if $\text{rank}(\mathbf{M}) = 1$ then the two rows (or columns) are proportional, which gives $a_2 = ka_1$ and $b_2 = kb_1$ for some k, and therefore $\frac{a_1}{b_1} = \frac{a_2}{b_2}$.

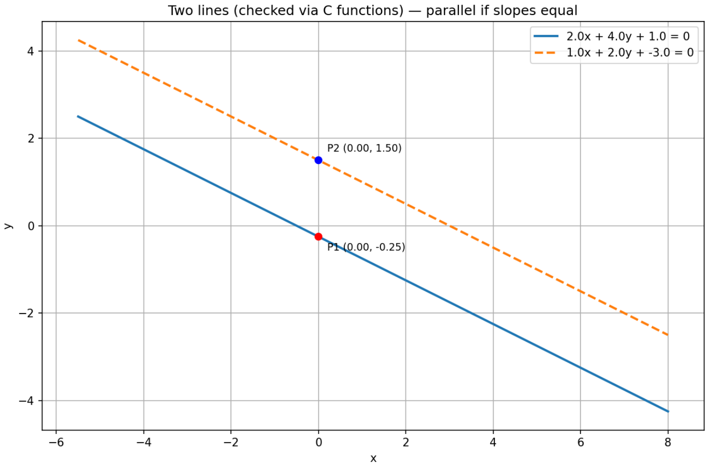
Two lines (checked via C functions) — parallel if slopes equal
2.0x + 4.0y + 1.0 = 0
1.0x + 2.0y + -3.0 = 0
P2 (0.00, 1.50)
P1 (0.00, -0.25)
y
x
4
2
0
−2
−4
−6
−4
−2
0
2
4
6
8

Fig. 0.1: plot