10.6.8

EE25BTECH11052 - Shriyansh Kalpesh Chawda

Question

Construct a pair of tangents to a circle of radius 4cm from a point P lying outside the circle at a distance of 6cm from the centre. (10, 2023)

Solution

Let the center of the circle be at origin and point P be at distance 6 from center along x-axis.

$$O = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1}$$

$$P = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \tag{2}$$

The equation of the circle $x^2 + y^2 = 16$ can be written as:

$$\mathbf{x}^\top \mathbf{x} - 16 = 0 \tag{3}$$

The parameters of the circle are:

$$\mathbf{V} = \mathbf{I}, \quad \mathbf{u} = \mathbf{0}, \quad f = -16 \tag{4}$$

Let the tangent equation passing through P be:

$$\mathbf{x} = P + k\mathbf{m} \tag{5}$$

where $\mathbf{m}$ is the direction vector of the tangent.

Substituting into the circle equation:

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{x} - 16 \tag{6}$$

$$(P + k\mathbf{m})^\top (P + k\mathbf{m}) - 16 = 0 \tag{7}$$

$$k^2 \mathbf{m}^\top \mathbf{m} + 2kP^\top \mathbf{m} + P^\top P - 16 = 0 \tag{8}$$

$$k^2 \mathbf{m}^\top \mathbf{m} + 2kP^\top \mathbf{m} + g(P) = 0 \tag{9}$$

For tangency, the discriminant must be zero:

$$4(P^\top \mathbf{m})^2 - 4\mathbf{m}^\top \mathbf{m} \cdot g(P) = 0 \tag{10}$$

$$(P^\top \mathbf{m})^2 - g(P)\mathbf{m}^\top \mathbf{m} = 0 \tag{11}$$

Since $P = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$, we have $g(P) = 36 - 16 = 20$.

This can be written as:

$$\mathbf{m}^\top Q \mathbf{m} = 0 \tag{12}$$

where

$$Q = \begin{pmatrix} -g(P) & 0 \\ 0 & (P^\top P) \end{pmatrix} = \begin{pmatrix} -20 & 0 \\ 0 & 36 \end{pmatrix} \tag{13}$$

Eigenvalue Decomposition of Q:
Since Q is diagonal, the eigenvalues are:

$$\lambda_1 = -20, \quad \lambda_2 = 36 \tag{14}$$

The eigenvector matrix is:

$$X = \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{15}$$

Let $\mathbf{z} = X^\top \mathbf{m} = \mathbf{m}$. Then:

$$\mathbf{z}^\top Q \mathbf{z} = 0 \tag{16}$$
$$-20z_1^2 + 36z_2^2 = 0 \tag{17}$$
$$\frac{z_1^2}{z_2^2} = \frac{36}{20} = \frac{9}{5} \tag{18}$$
$$\frac{z_1}{z_2} = \pm\frac{3}{\sqrt{5}} \tag{19}$$

The direction vectors for the tangents are:

$$\mathbf{m}_1 = \begin{pmatrix} 3 \\ \sqrt{5} \end{pmatrix}, \quad \mathbf{m}_2 = \begin{pmatrix} 3 \\ -\sqrt{5} \end{pmatrix} \tag{20}$$

The normal vectors are:

$$\mathbf{n}_1 = \begin{pmatrix} -\sqrt{5} \\ 3 \end{pmatrix}, \quad \mathbf{n}_2 = \begin{pmatrix} \sqrt{5} \\ 3 \end{pmatrix} \tag{21}$$

The points of contact are given by:

$$\mathbf{q}_i = r\frac{\mathbf{n}_i}{\|\mathbf{n}_i\|} \tag{22}$$

Since $\|\mathbf{n}_i\| = \sqrt{5+9} = \sqrt{14}$ and $r = 4$:

$$\mathbf{q}_1 = 4\frac{1}{\sqrt{14}}\begin{pmatrix} -\sqrt{5} \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{-4\sqrt{5}}{\sqrt{14}} \\ \frac{12}{\sqrt{14}} \end{pmatrix} \tag{23}$$
$$\mathbf{q}_2 = 4\frac{1}{\sqrt{14}}\begin{pmatrix} \sqrt{5} \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{4\sqrt{5}}{\sqrt{14}} \\ \frac{12}{\sqrt{14}} \end{pmatrix} \tag{24}$$

However, we need $P^\top \mathbf{q} = 16$ (from tangency condition). Let's verify and correct:

From the tangency condition $\mathbf{q}^\top(\mathbf{q} - P) = 0$ and $\mathbf{q}^\top\mathbf{q} = 16$:

$$P^\top \mathbf{q} = 16 \tag{25}$$

$$6q_1 = 16 \tag{26}$$

$$q_1 = \frac{8}{3} \tag{27}$$

From $q_1^2 + q_2^2 = 16$:

$$q_2^2 = 16 - \frac{64}{9} = \frac{80}{9} \tag{28}$$

$$q_2 = \pm\frac{4\sqrt{5}}{3} \tag{29}$$

Therefore, the contact points are:

$$\mathbf{q}_1 = \begin{pmatrix} \frac{8}{3} \\ \frac{4\sqrt{5}}{3} \end{pmatrix}, \quad \mathbf{q}_2 = \begin{pmatrix} \frac{8}{3} \\ -\frac{4\sqrt{5}}{3} \end{pmatrix} \tag{30}$$

Equations of Tangents:

The tangent at $\mathbf{q}$ is given by: $\mathbf{q}^\top\mathbf{x} = 16$.

Tangent 1 at $\mathbf{q}_1$:

$$\begin{pmatrix} \frac{8}{3} & \frac{4\sqrt{5}}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 16 \tag{31}$$

$$\frac{8}{3}x + \frac{4\sqrt{5}}{3}y = 16 \tag{32}$$

$$2x + \sqrt{5}y = 12 \tag{33}$$

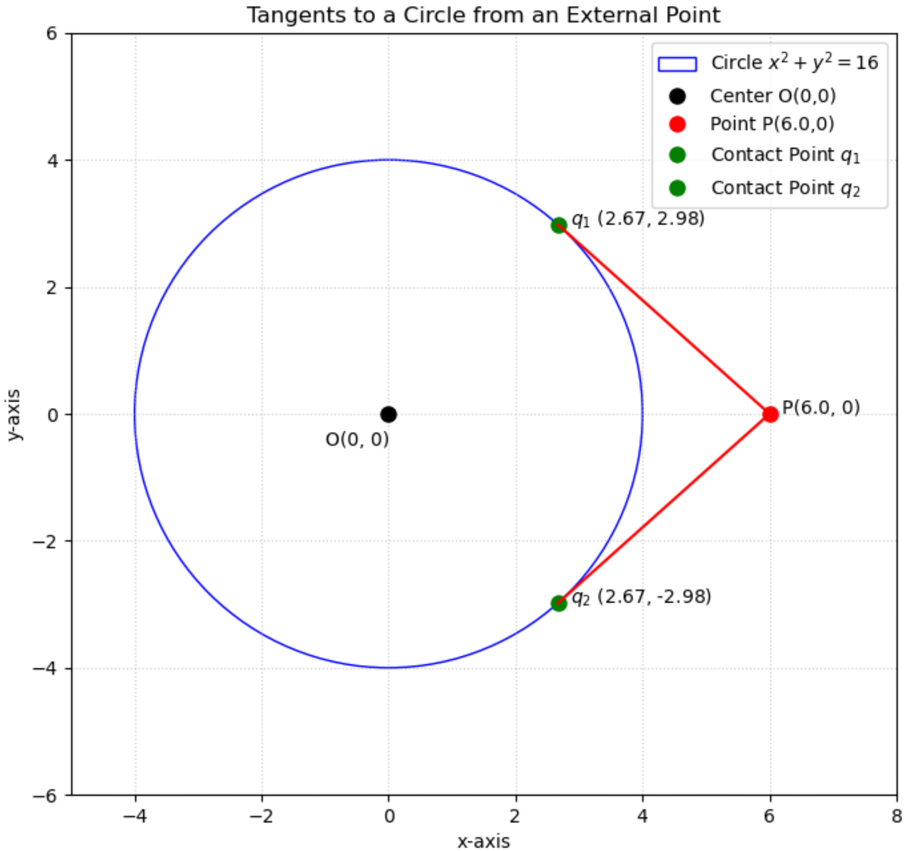
Tangent 2 at $\mathbf{q}_2$:

$$\begin{pmatrix} \frac{8}{3} & -\frac{4\sqrt{5}}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 16 \tag{34}$$

$$2x - \sqrt{5}y = 12 \tag{35}$$

The equations of the pair of tangents are:

$$\boxed{2x + \sqrt{5}y = 12 \quad \text{and} \quad 2x - \sqrt{5}y = 12} \tag{36}$$



Tangents to a Circle from an External Point
Circle $x^2 + y^2 = 16$
Center O(0,0)
Point P(6.0,0)
Contact Point q_1
Contact Point q_2
q_1 (2.67, 2.98)
q_2 (2.67, -2.98)
P(6.0, 0)
O(0, 0)
y-axis
x-axis