8.4.29

EE25BTECH11026-Harsha

Question:
A hyperbola, having the transverse axis of length $2\sin\theta$, is confocal with the ellipse $3x^2 + 4y^2 = 12$. Then the equation is

1) $x^2\csc^2\theta - y^2\sec^2\theta = 1$
2) $x^2\sec^2\theta - y^2\csc^2\theta = 1$
3) $x^2\sin^2\theta - y^2\cos^2\theta = 1$
4) $x^2\cos^2\theta - y^2\sin^2\theta = 1$

Solution:
Let us solve the given question theoretically and then verify the solution computationally.

From the data given,

$$\text{Equation of ellipse is given by} : \mathbf{x}^\top \mathbf{M_e}\mathbf{x} = 1 \tag{4.1}$$

where,

$$\mathbf{M_e} = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \tag{4.2}$$

Focal length of the ellipse (f) is given by,

$$f_e^2 = \frac{\lambda_2 - \lambda_1}{\|M_e\|} \tag{4.3}$$

where, λ_1 and λ_2 are the eigen values of the matrix $\mathbf{M_e}$. For a diagonal matrix it's eigen values are given by their diagonal elements.

$$\therefore f_e^2 = \frac{\frac{1}{3} - \frac{1}{4}}{\frac{1}{12}} = 1 \implies f_e = 1 \tag{4.4}$$

As ellipse and hyperbola are confocal, their focal lengths are same. Let the equation of hyperbola be

$$\mathbf{x}^\top \mathbf{M_H}\mathbf{x} = 1 \tag{4.5}$$

where,

$$\mathbf{M_H} = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} \tag{4.6}$$

where μ_1 and μ_2 are the eigen values of matrix $\mathbf{M_H}$.

The focal length of hyperbola f_H is given by,

$$f_H^2 = -\frac{\mu_1 - \mu_2}{\|M_H\|} \tag{4.7}$$

As the value of transverse axis is $2\sin\theta$,

$$\mu_1 = \csc^2\theta \tag{4.8}$$

Also,

$$\mu_2 - \mu_1 = \mu_1\mu_2 \tag{4.9}$$

$$\implies \mu_2 = -sec^2\theta \tag{4.10}$$

Thus, the desired equation is

$$\mathbf{x}^\top \mathbf{M_H}\mathbf{x} = 1 \tag{4.11}$$

where, $\mathbf{M_H} = \begin{pmatrix} \csc^2\theta & 0 \\ 0 & -\sec^2\theta \end{pmatrix}$.

From the figure, it is clearly verified that the theoretical solution matches with the computational solution.

Ellipse and Confocal Hyperbola for theta=π/6