4.13.57

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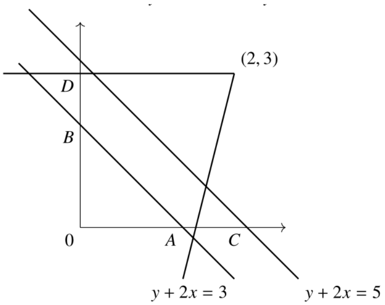
Question

Find the equation of the line passing through the point $(2, 3)$ and making intercept of length 2 units between the lines $y + 2x = 3$ and $y + 2x = 5$. (1991)



Solution

The point is given by:

$$\mathbf{P} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

The two lines are expressed in the vector form $\mathbf{n} \cdot \mathbf{x} = d$.

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 3 \tag{1}$$

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 5 \tag{2}$$

The common normal vector is $\mathbf{n}$ and the distance constants d_1 and d_2:

$$\mathbf{n} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad d_1 = 3, \quad d_2 = 5 \tag{3}$$

The line passes through $\mathbf{P}$ with an unknown direction vector $\mathbf{v}$ is:

Solution

$$\mathbf{x} = \mathbf{P} + t\mathbf{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} v_x \\ v_y \end{pmatrix} \tag{4}$$

The unknown line intersect L_1 at point $\mathbf{B}$ (parameter t_B) and L_2 at point $\mathbf{D}$ (parameter t_D).

$$\mathbf{n} \cdot (\mathbf{P} + t_B\mathbf{v}) = d_1 \implies t_B = \frac{d_1 - \mathbf{n} \cdot \mathbf{P}}{\mathbf{n} \cdot \mathbf{v}} \tag{5}$$

$$\mathbf{n} \cdot (\mathbf{P} + t_D\mathbf{v}) = d_2 \implies t_D = \frac{d_2 - \mathbf{n} \cdot \mathbf{P}}{\mathbf{n} \cdot \mathbf{v}} \tag{6}$$

Now ,Using the length of intercept.

$$\|\mathbf{D} - \mathbf{B}\| = \|(\mathbf{P} + t_D\mathbf{v}) - (\mathbf{P} + t_B\mathbf{v})\| = |t_D - t_B| \cdot \|\mathbf{v}\| = 2 \tag{7}$$

Solution

Using (0.3) and (0.4)

$$t_D - t_B = \frac{d_2 - \mathbf{n} \cdot \mathbf{P}}{\mathbf{n} \cdot \mathbf{v}} - \frac{d_1 - \mathbf{n} \cdot \mathbf{P}}{\mathbf{n} \cdot \mathbf{v}} = \frac{d_2 - d_1}{\mathbf{n} \cdot \mathbf{v}} \tag{8}$$

Substituting this into the distance equation:

$$\left| \frac{d_2 - d_1}{\mathbf{n} \cdot \mathbf{v}} \right| \cdot \|\mathbf{v}\| = 2 \tag{9}$$

$$\frac{2}{|\mathbf{n} \cdot \mathbf{v}|} \cdot \|\mathbf{v}\| = 2 \implies \|\mathbf{v}\| = |\mathbf{n} \cdot \mathbf{v}| \tag{10}$$

We express the condition $\|\mathbf{v}\| = |\mathbf{n} \cdot \mathbf{v}|$ as a matrix quadratic form. Squaring both sides gives:

Solution

$$\|\mathbf{v}\|^2 = (\mathbf{n} \cdot \mathbf{v})^2 \tag{11}$$

$$\mathbf{v}^\top \mathbf{I} \mathbf{v} = \mathbf{v}^\top (\mathbf{n}\mathbf{n}^\top) \mathbf{v} \tag{12}$$

$$\mathbf{v}^\top \mathbf{I} \mathbf{v} - \mathbf{v}^\top (\mathbf{n}\mathbf{n}^\top) \mathbf{v} = 0 \tag{13}$$

$$\mathbf{v}^\top (\mathbf{n}\mathbf{n}^\top - \mathbf{I}) \mathbf{v} = 0, \tag{14}$$

Let,

$$\mathbf{Q} = \mathbf{n}\mathbf{n}^\top - \mathbf{I} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix} \tag{15}$$

We now solve the quadratic form equation for $\mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$:

$$\mathbf{v}^\top \mathbf{Q} \mathbf{v} = \begin{pmatrix} v_x & v_y \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = 0 \tag{16}$$

Solution

$$3v_x^2 + 4v_x v_y = 0 \implies v_x(3v_x + 4v_y) = 0 \tag{17}$$

This yields two possible solutions for the components of the direction vector.

(a) $v_x = 0$: The direction vector is vertical.Hence $\mathbf{v_1}$ and vertical line passing through the point $(2, 3)$ is :

$$\mathbf{v_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{18}$$

$$\mathbf{x} = 2 \tag{19}$$

(b) $3v_x + 4v_y = 0$: This implies

$$v_y = -\frac{3}{4}v_x \tag{20}$$

$$\mathbf{v_2} = \begin{pmatrix} 4 \\ -3 \end{pmatrix} \tag{21}$$

Solution

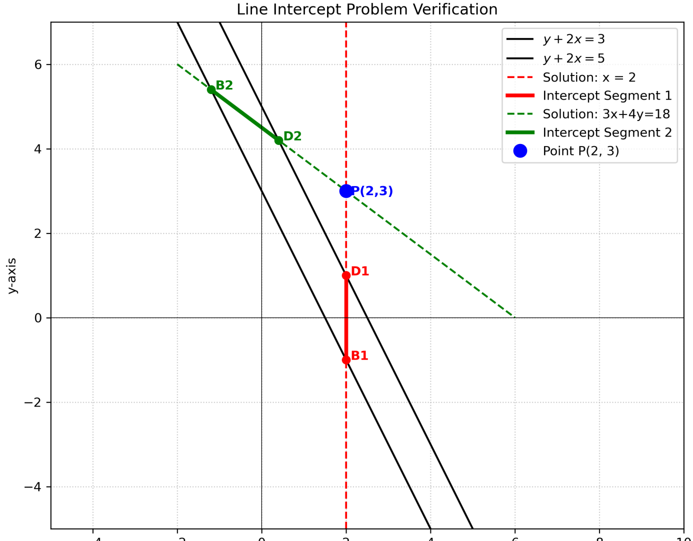
So, corresponding to a slope of $m = -3/4$. Using the point-slope form :

$$y - 3 = -\frac{3}{4}(x - 2) \tag{22}$$
$$4(y - 3) = -3(x - 2) \tag{23}$$
$$4y - 12 = -3x + 6 \tag{24}$$
$$3x + 4y = 18 \tag{25}$$

The two lines that satisfy the given conditions are :

$$x = 2 \quad and \quad 3x + 4y = 18$$

Plot



Line Intercept Problem Verification
y-axis
$y+2x=3$
$y+2x=5$
Solution: x = 2
Intercept Segment 1
Solution: 3x+4y=18
Intercept Segment 2
Point P(2, 3)
B2
D2
P(2,3)
D1
B1
6
4
2
0
−2
−4