2.10.62

EE25BTECH11033 - Kavin

Question:
Find all values of λ such that $x, y, z \neq (0, 0, 0)$ and

$$\left(\hat{i} + \hat{j} + 3\hat{k}\right)x + \left(3\hat{i} - 3\hat{j} + \hat{k}\right)y + \left(-4\hat{i} + 5\hat{j}\right)z = \lambda\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

where $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along the coordinate axes.

Solution:
The given vector equation is:

$$(\hat{i} + \hat{j} + 3\hat{k})x + (3\hat{i} - 3\hat{j} + \hat{k})y + (-4\hat{i} + 5\hat{j})z = \lambda(x\hat{i} + y\hat{j} + z\hat{k}) \tag{1}$$

which can be expressed as,

$$x\begin{pmatrix}1\\1\\3\end{pmatrix} + y\begin{pmatrix}3\\-3\\1\end{pmatrix} + z\begin{pmatrix}-4\\5\\0\end{pmatrix} = \lambda\begin{pmatrix}x\\y\\z\end{pmatrix} \tag{2}$$

$$\implies \begin{pmatrix}1 & 3 & -4\\1 & -3 & 5\\3 & 1 & 0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \lambda\begin{pmatrix}x\\y\\z\end{pmatrix} \tag{3}$$

$$\implies A\mathbf{v} = \lambda\mathbf{v} \tag{4}$$

This is a homogeneous system of linear equations. It can be expressed in matrix form as $(A - \lambda I)\mathbf{v} = 0$, where:

$$A = \begin{pmatrix}1 & 3 & -4\\1 & -3 & 5\\3 & 1 & 0\end{pmatrix}, I = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}, \mathbf{v} = \begin{pmatrix}x\\y\\z\end{pmatrix} \tag{5}$$

The problem states that $(x, y, z) \neq (0, 0, 0)$, which means we are looking for a **non-trivial solution** for the vector $\mathbf{v}$. This is a **eigenvalue problem**. The values of λ for which non-trivial solutions exist are the eigenvalues of the matrix A.
A non-trivial solution exists if and only if the determinant of the coefficient matrix is zero. This gives us the characteristic equation:

$$\left|A - \lambda I\right| = 0 \tag{6}$$

$$\begin{vmatrix} 1-\lambda & 3 & -4 \\ 1 & -3-\lambda & 5 \\ 3 & 1 & -\lambda \end{vmatrix} = 0 \tag{7}$$

Now, we calculate the determinant by expanding along the first row:

$$(1-\lambda)\begin{vmatrix} -3-\lambda & 5 \\ 1 & -\lambda \end{vmatrix} - 3\begin{vmatrix} 1 & 5 \\ 3 & -\lambda \end{vmatrix} + (-4)\begin{vmatrix} 1 & -3-\lambda \\ 3 & 1 \end{vmatrix} = 0 \tag{8}$$

$$(1-\lambda)((-3-\lambda)(-\lambda)-5) - 3(-\lambda-15) - 4(1-3(-3-\lambda)) = 0 \tag{9}$$

$$(1-\lambda)(\lambda^2+3\lambda-5) + 3(\lambda+15) - 4(10+3\lambda) = 0 \tag{10}$$

$$(\lambda^2+3\lambda-5-\lambda^3-3\lambda^2+5\lambda) + (3\lambda+45) - (40+12\lambda) = 0 \tag{11}$$

$$-\lambda^3 - 2\lambda^2 + 8\lambda - 5 + 3\lambda + 45 - 40 - 12\lambda = 0 \tag{12}$$

Combine like terms to get the characteristic polynomial:

$$-\lambda^3 - 2\lambda^2 - \lambda = 0 \tag{13}$$

$$\lambda^3 + 2\lambda^2 + \lambda = 0 \tag{14}$$

Factoring out λ:

$$\lambda(\lambda^2 + 2\lambda + 1) = 0 \tag{15}$$

The quadratic term is a perfect square:

$$\lambda(\lambda+1)^2 = 0 \tag{16}$$

The solutions for λ are:

$$\lambda = 0 \quad \text{or} \quad \lambda = -1 \tag{17}$$

Thus, the required values of λ are 0 and -1.