Matrices in Geometry 10.5.5

EE25BTECH11037 - Divyansh

Question: Construct a tangent to a circle of radius 4*cm* from a point on the concentric circle of radius 6*cm* and measure its length. Also verify the measurement by actual calculation.

Solution: Let center be the origin. Then the circle with radius 4 *cm* is

$$\mathbf{C} \ : \ \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \ ; \ \mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \ \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ , \ f = -16 \tag{1}$$

Let the external point from which the tangent is drawn be $\mathbf{h} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$.

Let us calculate the matrix Σ

$$\Sigma = (\mathbf{V}\mathbf{h} + \mathbf{u})(\mathbf{V}\mathbf{h} + \mathbf{u})^\top - g(\mathbf{h})\,\mathbf{V} \tag{2}$$

$$g(\mathbf{h}) = \mathbf{h}^\top \mathbf{V}\mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f = \|\mathbf{h}\|^2 + f = 36 - 16 = 20 \tag{3}$$

$$\Sigma = \mathbf{h}\mathbf{h}^\top - g(\mathbf{h})\,\mathbf{V} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \begin{pmatrix} 6 & 0 \end{pmatrix} - 20 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4}$$

$$\implies \Sigma = \begin{pmatrix} 16 & 0 \\ 0 & -20 \end{pmatrix} \tag{5}$$

The eigenvalues of the matrix Σ are clearly $\lambda_1 = 16$ and $\lambda_2 = -20$
The normalized eigenvectors form the matrix $\mathbf{P}$.

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{6}$$

The direction vectors of the two tangents are

$$\mathbf{m} = \mathbf{P} \begin{pmatrix} |\lambda_2| \\ \pm\sqrt{|\lambda_1|} \end{pmatrix} = \begin{pmatrix} 2\sqrt{5} \\ \pm 4 \end{pmatrix} \tag{7}$$

The length of the tangent is given by

$$\|\mathbf{T} - \mathbf{h}\| = |\mu|\,\|\mathbf{m}\| \ , \ (\mu \text{ is a parameter}) \tag{8}$$

$$\mu = -\frac{\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u})}{\|\mathbf{m}\|^2} = -\frac{\begin{pmatrix} 2\sqrt{5} & 4 \end{pmatrix} \begin{pmatrix} 6 \\ 0 \end{pmatrix}}{\left\| \begin{pmatrix} 2\sqrt{5} \\ 4 \end{pmatrix} \right\|^2} = -\frac{\sqrt{5}}{3} \tag{9}$$

$$\therefore \|\mathbf{T} - \mathbf{h}\| = \frac{\sqrt{5}}{3} \times 6 = 2\sqrt{5} \approx 4.47 \ cm \tag{10}$$

From the figure given below, we can verify that both lengths are equal and equal to 4.47 cm.

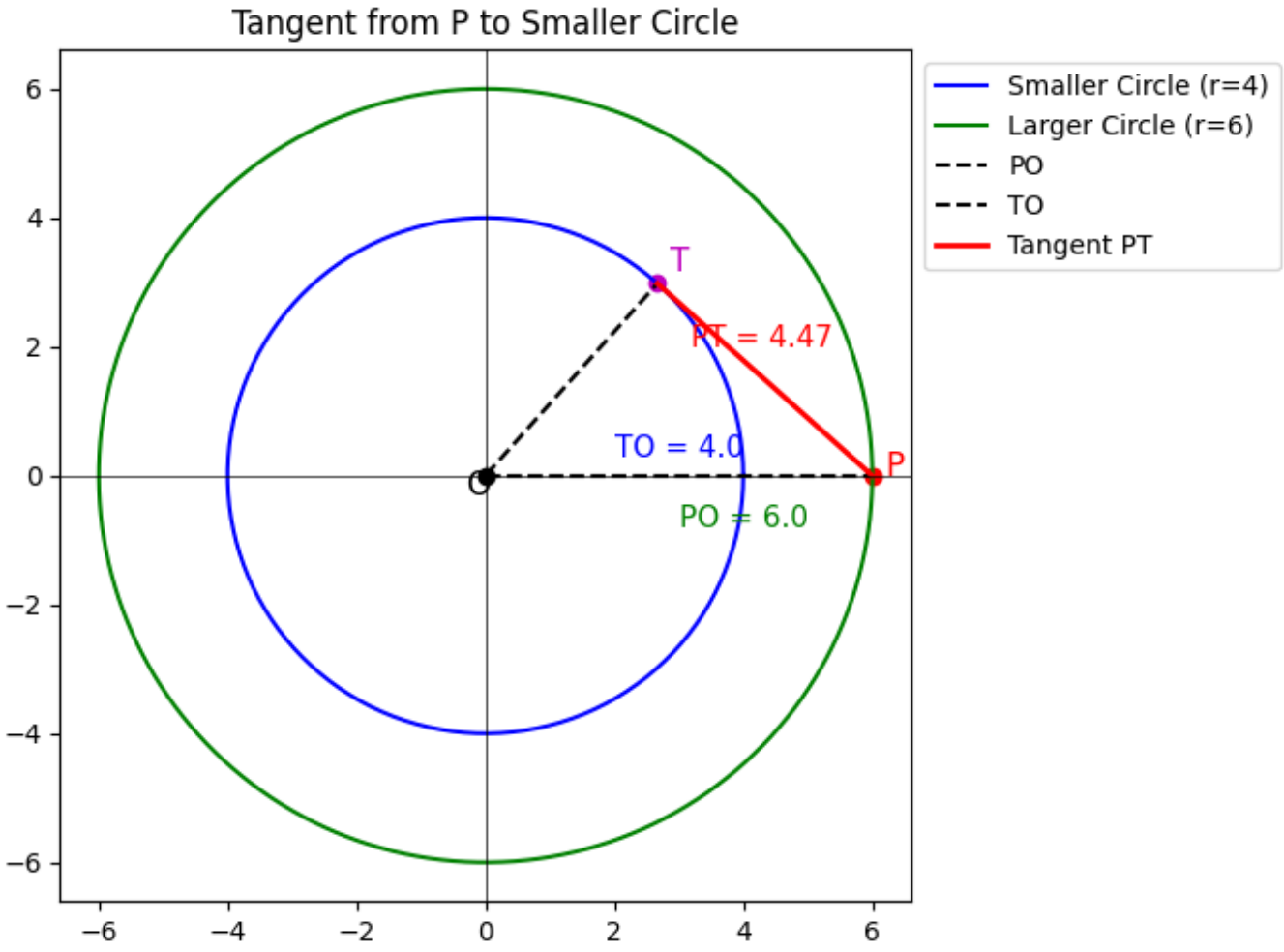

Fig. 1: Graph for 10.5.5