

10.2.7

EE25BTECH11001 - Aarush Dilawri

Question:

At what points on the curve $x^2 + y^2 - 2x - 4y + 1 = 0$, the tangents are parallel to the y-axis?

Solution:

The general equation of a conic is $\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0$ (0.1)

Comparing with $x^2 + y^2 - 2x - 4y + 1 = 0$, we get

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \quad f = 1 \quad (0.2)$$

The centre and radius of the circle are given by

$$\mathbf{c} = -\mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad r = \sqrt{\|\mathbf{u}\|^2 - f} = \sqrt{(-1)^2 + (-2)^2 - 1} = \sqrt{4} = 2 \quad (0.3)$$

The points of contact of the tangent are given by

$$\mathbf{q}_{i,j} = \pm r \frac{\mathbf{n}_j}{\|\mathbf{n}_j\|} - \mathbf{u} \quad i, j = 1, 2 \quad (0.4)$$

Since the tangents are parallel to the y-axis, the normal vector is

$$\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (0.5)$$

Substituting in (0.4),

$$\mathbf{q}_{1,1} = \pm 2 \frac{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\left\| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|} - \begin{pmatrix} -1 \\ -2 \end{pmatrix} \quad (0.6)$$

$$= \pm 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (0.7)$$

Therefore, the two points of contact are

$$\mathbf{q}_1 = \begin{pmatrix} 1+2 \\ 2+0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad (0.8)$$

$$\mathbf{q}_2 = \begin{pmatrix} 1-2 \\ 2+0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (0.9)$$

Hence, the required points are $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

See Figure,

