1.8.5

Shivam Sawarkar
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Question (1.8.5)

If **A** and **B** be the points $(3, 4, 5)$ and $(-1, 3, -7)$ respectively, find the equation of the set of a point **P** such that $PA^2 + PB^2 = k^2$

$$\mathbf{A} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} -1 \\ 3 \\ -7 \end{pmatrix} \tag{1}$$

According to the question,

$$PA^2 + PB^2 = k^2 \tag{2}$$

where, $PA^2 = \|P - A\|$ and $PB^2 = \|P - B\|$

Solution

The squared distances can be written as dot products:

$$PA^2 = (\mathbf{P} - \mathbf{A})^T(\mathbf{P} - \mathbf{A}) \tag{3}$$
$$PB^2 = (\mathbf{P} - \mathbf{B})^T(\mathbf{P} - \mathbf{B}) \tag{4}$$

Thus:

$$PA^2 + PB^2 = (\mathbf{P} - \mathbf{A})^T(\mathbf{P} - \mathbf{A}) + (\mathbf{P} - \mathbf{B})^T(\mathbf{P} - \mathbf{B}) \tag{5}$$
$$PA^2 + PB^2 = \mathbf{P}^T\mathbf{P} - 2\mathbf{A}^T\mathbf{P} + \mathbf{A}^T\mathbf{A} + \mathbf{P}^T\mathbf{P} - 2\mathbf{B}^T\mathbf{P} + \mathbf{B}^T\mathbf{B} \tag{6}$$

Compliting Square

Let,

$$\mathbf{M} := \frac{\mathbf{A}+\mathbf{B}}{2} \tag{7}$$

$$R^2 := \|M\|^2 - \frac{\mathbf{A}^T\mathbf{A}+\mathbf{B}^T\mathbf{B}-k^2}{2} \tag{8}$$

Then the equation becomes

$$\|P-M\|^2 = R^2 \tag{9}$$

$$\left\|P-\frac{\mathbf{A}+\mathbf{B}}{2}\right\|^2 = \left\|\frac{\mathbf{A}+\mathbf{B}}{2}\right\|^2 - \frac{\|A\|^2+\|B\|^2-k^2}{2} \tag{10}$$

Substitute the known values

$$\|A\| = 3^2 + 4^2 + 5^2 = 50 \quad (11)$$

$$\|B\| = (-1)^2 + 3^2 + (-7)^2 = 59 \quad (12)$$

$$\frac{\mathbf{A}+\mathbf{B}}{2} = \begin{pmatrix} 1 \\ 3.5 \\ -1 \end{pmatrix} \quad (13)$$

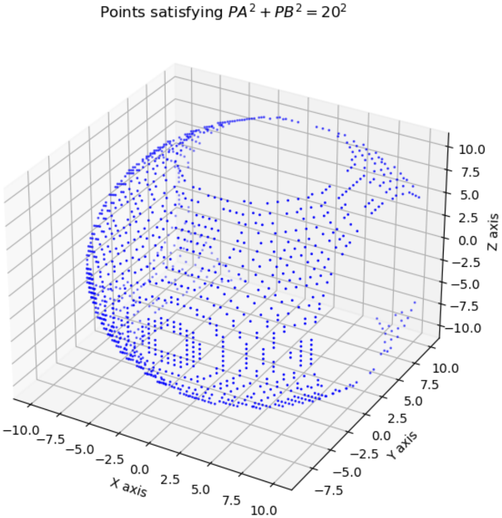
Result

The equation of the locus is:

$$\left\| P - \begin{pmatrix} 1 \\ 3.5 \\ -1 \end{pmatrix} \right\|^2 = \frac{2k^2 - 161}{4} \tag{14}$$

Plot

The plot show the locus for $k = 20$



C Code

```
#include <stdio.h>
#include <math.h>

int main() {
    double Ax, Ay, Az, Bx, By, Bz, K;
    double Mx, My, Mz, diffx, diffy, diffz;
    double diff_sq, r2, radius;

    // Input
    printf("Enter coordinates of A (x y z): ");
    scanf("%lf %lf %lf", &Ax, &Ay, &Az);

    printf("Enter coordinates of B (x y z): ");
    scanf("%lf %lf %lf", &Bx, &By, &Bz);
```

C Code

```
    printf("Enter constant K: ");
scanf("%lf", &K);

// Midpoint (center of sphere)
Mx = (Ax + Bx) / 2.0;
My = (Ay + By) / 2.0;
Mz = (Az + Bz) / 2.0;

// ||A-B||^2
diffx = Ax - Bx;
diffy = Ay - By;
diffz = Az - Bz;
diff_sq = diffx*diffx + diffy*diffy + diffz*diffz;

// r^2 formula
r2 = (2.0 * K * K - diff_sq) / 4.0;
```

C Code

```
    printf("Center M = (%.2f, %.2f, %.2f)\n", Mx, My, Mz);
    printf("r^2 = %.4f\n", r2);

    if (r2 < 0) {
        printf("No real sphere exists (r^2 < 0).\n");
    } else {
        radius = sqrt(r2);
        printf("Radius = %.4f\n", radius);

        printf("\nEquation of locus (vector form):\n");
        printf("|| P - (%.2f, %.2f, %.2f) ||^2 = %.4f\n",
        Mx, My, Mz, r2);
    }

    return 0;
}
```

Python Code

```
    import numpy as np

def sphere_from_sum_of_squares(A, B, K):
    A = np.asarray(A, dtype=float)
    B = np.asarray(B, dtype=float)
    M = (A + B) / 2.0
    diff = A - B
    diff_sq = np.dot(diff, diff)  # ||A-B||^2
    r2 = (2.0 * K**2 - diff_sq) / 4.0
    if r2 < 0:
        return {'center': M, 'r2': r2, 'radius': None, 'real':
    else:
        return {'center': M, 'r2': r2, 'radius': np.sqrt(r2),
```

Python Code

```
def main():
# Input values
A = tuple(map(float, input("Enter coordinates of A (x y z)
B = tuple(map(float, input("Enter coordinates of B (x y z)
K = float(input("Enter constant K: "))

res = sphere_from_sum_of_squares(A, B, K)

print("\n--- Results ---")
print("Point A =", A)
print("Point B =", B)
print("K =", K)
print("Center M =", res['center'])
print("r^2 =", res['r2'])
```

Python Code

```
if res['real']:
    print("Radius =", res['radius'])
    # Vector form of locus
    print("\nEquation of locus (vector form):")
    print(f"|| P - {res['center']} ||^2 = {res['r2']}")
else:
    print("No real sphere exists (r^2 < 0).")
```