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Question

Given 3 vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are coplanar then show $\det(\mathbf{M}) = 0$ where $\mathbf{M} = (\mathbf{A} \ \mathbf{B} \ \mathbf{C})$

Solution:

Equation of plane through 3 coplanar points is

$$\mathbf{n}^T \mathbf{x} = 0 \quad (1)$$

$$\Rightarrow \mathbf{n}^T \mathbf{A} = \mathbf{n}^T \mathbf{B} = \mathbf{n}^T \mathbf{C} = 0 \quad (2)$$

$$\mathbf{M} = (\mathbf{A} \ \mathbf{B} \ \mathbf{C}) \quad (3)$$

$$\Rightarrow \mathbf{n}^T \mathbf{M} = (\mathbf{n}^T \mathbf{A} \ \mathbf{n}^T \mathbf{B} \ \mathbf{n}^T \mathbf{C}) \quad (4)$$

$$\Rightarrow \mathbf{n}^T \mathbf{M} = (0 \ 0 \ 0) \quad (5)$$

$$\Rightarrow \mathbf{n}^T \mathbf{M} = \mathbf{0} \quad (6)$$

From (6) it means \mathbf{M} has a non trivial vector in it's null space

means $\text{rank}(\mathbf{M}) < 3$.

For a 3×3 square matrix like \mathbf{M} if $\det(\mathbf{M}) \neq 0$ means \mathbf{M} is invertible which means \mathbf{M} is a full rank matrix

$\Rightarrow \text{rank}(\mathbf{M}) = 3$. (if $\det(\mathbf{M}) \neq 0$)

But from above discussion $\text{rank}(\mathbf{M}) < 3$

$\Rightarrow \mathbf{M}$ is not invertible

$\Rightarrow \det(\mathbf{M}) = 0$

proof 2:

3 vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are coplanar means they are linearly dependent.

let's assume

$$\mathbf{C} = \alpha \mathbf{A} + \beta \mathbf{B}. \quad (7)$$

$$\det(\mathbf{M}) = \det((\mathbf{A} \ \mathbf{B} \ \mathbf{C})) \quad (8)$$

$$= \det((\mathbf{A} \ \mathbf{B} \ \alpha \mathbf{A} + \beta \mathbf{B})) \quad (9)$$

$$= \alpha \det((\mathbf{A} \ \mathbf{B} \ \mathbf{A})) + \beta \det((\mathbf{A} \ \mathbf{B} \ \mathbf{B})) = 0 \quad (10)$$

$$\Rightarrow \det(\mathbf{M}) = 0 \quad (11)$$