8.4.24

EE25BTECH11021 - Dhanush sagar

Question:
If the line $x - 1 = 0$ is the directrix of the parabola

$$y^2 - kx + 8 = 0,$$

then one of the values of k is:

1) 18
2) 8
3) 4
4) 14

Solution:
We are given the parabola

$$y^2 - kx + 8 = 0 \tag{1}$$

with directrix $x - 1 = 0$. Represent the parabola in matrix form:

$$\mathbf{x}^\top V\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f = 0 \tag{2}$$

For a conic with directrix $\mathbf{n}^\top\mathbf{x} = c$, eccentricity e and focus $\mathbf{F}$, the matrix formulas are:

$$\mathbf{V} = \|\mathbf{n}\|^2 I - e^2\mathbf{n}\mathbf{n}^\top \tag{3}$$

$$\mathbf{u} = ce^2\mathbf{n} - \|\mathbf{n}\|^2\mathbf{F} \tag{4}$$

$$f = \|\mathbf{n}\|^2\|\mathbf{F}\|^2 - c^2e^2 \tag{5}$$

For the parabola $y^2 - kx + 8 = 0$, we write the matrices as

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{u} = \begin{pmatrix} -k/2 \\ 0 \end{pmatrix}, \qquad f = 8 \tag{6}$$

The directrix is $\mathbf{n}^\top\mathbf{x} = c \implies \mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, c = 1$, and for a parabola $e = 1$. Then

$$\mathbf{V} = \|\mathbf{n}\|^2 I - e^2\mathbf{n}\mathbf{n}^\top = 1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{7}$$

The vector $\mathbf{u}$ gives the focus:

$$\mathbf{u} = ce^2\mathbf{n} - \|\mathbf{n}\|^2\mathbf{F} \implies \mathbf{F} = c\mathbf{n} - \mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} -k/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 + k/2 \\ 0 \end{pmatrix} \tag{8}$$

The constant term is

$$f = \|\mathbf{n}\|^2\|\mathbf{F}\|^2 - c^2e^2 = 1 \cdot \left(\begin{pmatrix} 1 + k/2 \\ 0 \end{pmatrix}^\top \begin{pmatrix} 1 + k/2 \\ 0 \end{pmatrix}\right) - 1 = (1 + k/2)^2 - 1 \tag{9}$$

Equating with the given $f = 8$:

$$(1 + k/2)^2 - 1 = 8 \implies (1 + k/2)^2 = 9 \tag{10}$$

Solving the matrix equation:

$$1 + k/2 = 3 \implies k = 4 \tag{11}$$

$$1 + k/2 = -3 \implies k = -8 \tag{12}$$

Hence, one of the values of k is

$$\boxed{4} \tag{13}$$

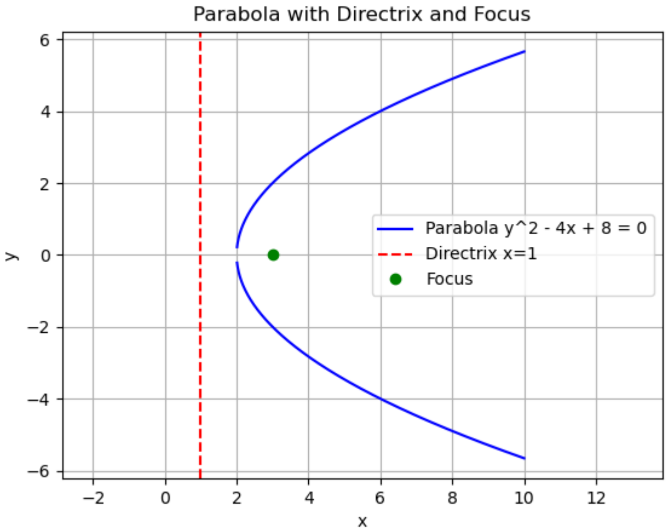

Fig. 4.1