2.10.19

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For three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ which of the following expression is not equal to any of the remaining three?

a. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$
b. $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w})$
c. $(\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u}$
d. $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

Solution

As we know that dot product is cumulative so, (1) and (2) are equal
That is,

$$\mathbf{u}^\top(\mathbf{v} \times \mathbf{w}) = (\mathbf{v} \times \mathbf{w})^\top \mathbf{u} \tag{1}$$

We prove

$$\mathbf{u}^\top(\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v})^\top \mathbf{w} \tag{2}$$

Solution

using the cross-product (skew-)matrix.
Define, for $\mathbf{a} = (a_1, a_2, a_3)^T$,

$$S(\mathbf{a}) = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \tag{3}$$

which satisfies $(\mathbf{a})\mathbf{b} = \mathbf{a} \times \mathbf{b}$ for all $\mathbf{b} \in \mathbb{R}^3$.

$$\mathbf{u}^\top(\mathbf{v}\times\mathbf{w}) = \mathbf{u}^T(S(\mathbf{v})\mathbf{w}) \quad (\text{since } S(\mathbf{v})\mathbf{w} = \mathbf{v}\times\mathbf{w}) \tag{4}$$

$$= (\mathbf{u}^T S(\mathbf{v}))\mathbf{w} \tag{5}$$

$$= (S(\mathbf{v})^T\mathbf{u})^T\mathbf{w} \quad (\text{transpose identity: } (A^T x)^T = x^T A) \tag{6}$$

$$= (-S(\mathbf{v})\mathbf{u})^T\mathbf{w} \quad (\text{since } S(\mathbf{v})^T = -S(\mathbf{v})) \tag{7}$$

$$= (\mathbf{u}\times\mathbf{v})^T\mathbf{w} \quad (\text{because } -S(\mathbf{v})\mathbf{u} = -(\mathbf{v}\times\mathbf{u}) = \mathbf{u}\times\mathbf{v}) \tag{8}$$

Thus

$$\mathbf{u}^\top(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\times\mathbf{v})^\top\mathbf{w} \tag{9}$$

This shows that (a), (c) and (d) are equal

Example

Let

$$\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}. \tag{10}$$

Case 1: $\mathbf{u}^\top(\mathbf{v} \times \mathbf{w})$

$$\mathbf{v} \times \mathbf{w} = \begin{pmatrix} v_{23} & w_{23} \\ v_{31} & w_{31} \\ v_{12} & w_{12} \end{pmatrix} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix} = \begin{pmatrix} 1 \times (-1) - 2 \times 0 \\ 2 \times 1 - 0 \times (-1) \\ 0 \times 0 - 1 \times 1 \end{pmatrix} \tag{11}$$

So

$$\mathbf{v} \times \mathbf{w} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \tag{12}$$

Examlpe

Now compute the dot product:

$$\mathbf{u}^\top(\mathbf{v} \times \mathbf{w}) = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \tag{13}$$

$$= (1)(-1) + (-1)(2) + (1)(-1) \tag{14}$$

$$= -4 \tag{15}$$

$$\boxed{\mathbf{u}^\top(\mathbf{v} \times \mathbf{w}) = -4} \tag{16}$$

Example

Case 2: $\mathbf{v}^\top(\mathbf{u}\times\mathbf{w})$
Compute $\mathbf{u}\times\mathbf{w}$:

$$\mathbf{u}\times\mathbf{w} = \begin{pmatrix} u_{23} & w_{23} \\ u_{31} & w_{31} \\ u_{12} & w_{12} \end{pmatrix} = \begin{pmatrix} u_2w_3 - u_3w_2 \\ u_3w_1 - u_1w_3 \\ u_1w_2 - u_2w_1 \end{pmatrix} = \begin{pmatrix} (-1)\times(-1) - 1\times 0 \\ 1\times 1 - 1\times(-1) \\ 1\times 0 - (-1)\times 1 \end{pmatrix} \tag{17}$$

So

$$\mathbf{u}\times\mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}. \tag{18}$$

Now compute dot product:

$$\mathbf{v}^\top(\mathbf{u}\times\mathbf{w}) = \begin{pmatrix} 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \tag{19}$$

$$= 4. \tag{20}$$

Example

Case 3: $(\mathbf{v} \times \mathbf{w})^\top \mathbf{u}$

We already have

$$\mathbf{v} \times \mathbf{w} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \tag{21}$$

Now compute:

$$(\mathbf{v} \times \mathbf{w})^\top \mathbf{u} = \begin{pmatrix} -1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \tag{22}$$

$$= (-1)(1) + (2)(-1) + (-1)(1) \tag{23}$$

$$= -4. \tag{24}$$

$$\boxed{(\mathbf{v} \times \mathbf{w})^\top \mathbf{u} = -4} \tag{25}$$

Example

Case 4: $(\mathbf{u} \times \mathbf{v})^\top \mathbf{w}$

Compute $\mathbf{u} \times \mathbf{v}$:

$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix} u_{23} & v_{23} \\ u_{31} & v_{31} \\ u_{12} & v_{12} \end{pmatrix} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} = \begin{pmatrix} (-1) \times 2 - 1 \times 1 \\ 1 \times 0 - (-1) \times 2 \\ 1 \times 1 - (-1) \times 0 \end{pmatrix} \tag{26}$$

So

$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} \tag{27}$$

Now compute:

$$(\mathbf{u} \times \mathbf{v})^\top \mathbf{w} = \begin{pmatrix} -3 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{28}$$

$$= (-3)(1) + (-2)(0) + (1)(-1) \tag{29}$$

$$= -4. \tag{30}$$

Example

Final Results

$$\mathbf{u}^\top(\mathbf{v}\times\mathbf{w}) = -4, \tag{31}$$

$$\mathbf{v}^\top(\mathbf{u}\times\mathbf{w}) = 4, \tag{32}$$

$$(\mathbf{v}\times\mathbf{w})^\top\mathbf{u} = -4, \tag{33}$$

$$(\mathbf{u}\times\mathbf{v})^\top\mathbf{w} = -4 \tag{34}$$

Thus (a), (c) and (d) are same

Plot

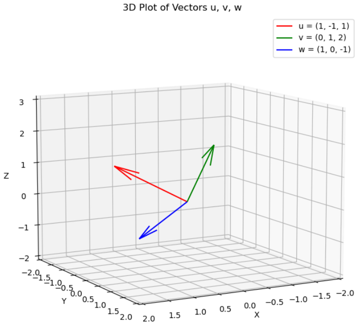

Figure:

C Code

```c
#ifndef VECTOROPS_H
#define VECTOROPS_H

// Function to compute cross product of two vectors
void cross(int a[3], int b[3], int result[3]) {
    result[0] = a[1]*b[2] - a[2]*b[1];
    result[1] = a[2]*b[0] - a[0]*b[2];
    result[2] = a[0]*b[1] - a[1]*b[0];
}

// Function to compute dot product of two vectors
int dot(int a[3], int b[3]) {
    return a[0]*b[0] + a[1]*b[1] + a[2]*b[2];
}

#endif
```

C Code

```
#include <stdio.h>
#include "vectorops.h"

int main() {
    int u[3], v[3], w[3];
    int temp[3];
    int A, B, C, D;

    // Input vectors
    printf("Enter vector u (x y z): ");
    scanf("%d %d %d", &u[0], &u[1], &u[2]);
    printf("Enter vector v (x y z): ");
    scanf("%d %d %d", &v[0], &v[1], &v[2]);
    printf("Enter vector w (x y z): ");
    scanf("%d %d %d", &w[0], &w[1], &w[2]);
```

C Code

```
    cross(v, w, temp);
    A = dot(u, temp);
    cross(u, w, temp);
    B = dot(v, temp);
    cross(v, w, temp);
    C = dot(temp, u);
    cross(u, v, temp);
    D = dot(temp, w);
    printf("\nResults:\n");
    printf("A = u · (v × w) = %d\n", A);
    printf("B = v · (u × w) = %d\n", B);
    printf("C = (v × w) · u = %d\n", C);
    printf("D = (u × v) · w = %d\n", D);
    printf("B is different");
    return 0;
}
```

Python + C Code

```
import ctypes
import numpy as np
import matplotlib.pyplot as plt

# Load shared library
lib = ctypes.CDLL("./libvectorops.so")

# Define function signatures
lib.dot.argtypes = [ctypes.POINTER(ctypes.c_int), ctypes.POINT
lib.dot.restype = ctypes.c_int

lib.cross.argtypes = [ctypes.POINTER(ctypes.c_int), ctypes.PO
lib.cross.restype = None
```

Python + C Code

```
def dot(a, b):
    a_arr = (ctypes.c_int * 3)(*a)
    b_arr = (ctypes.c_int * 3)(*b)
    return lib.dot(a_arr, b_arr)
def cross(a, b):
    a_arr = (ctypes.c_int * 3)(*a)
    b_arr = (ctypes.c_int * 3)(*b)
    result = (ctypes.c_int * 3)()
    lib.cross(a_arr, b_arr, result)
    return [result[0], result[1], result[2]]
def compute(u, v, w):
    A = dot(u, cross(v, w))      # u · (v × w)
    B = dot(v, cross(u, w))      # v · (u × w)
    C = dot(cross(v, w), u)      # (v × w) · u
    D = dot(cross(u, v), w)      # (u × v) · w
    return A, B, C, D
```

Python + C Code

```
u = list(map(int, input("Enter vector u (x y z): ").split(
v = list(map(int, input("Enter vector v (x y z): ").split(
w = list(map(int, input("Enter vector w (x y z): ").split(

A, B, C, D = compute(u, v, w)

print("\nResults:")
print(f"A = u · (v × w) = {A}")
print(f"B = v · (u × w) = {B}")
print(f"C = (v × w) · u = {C}")
print(f"D = (u × v) · w = {D}")
print("\n=> Expression B is different.")
```

Python Code

```
def dot(a, b):
    return a[0]*b[0] + a[1]*b[1] + a[2]*b[2]

def cross(a, b):
    return [
        a[1]*b[2] - a[2]*b[1],
        a[2]*b[0] - a[0]*b[2],
        a[0]*b[1] - a[1]*b[0]
    ]
```

Python Code

```
def main():
    u = list(map(int, input("Enter vector u (x y z): ").split(
    v = list(map(int, input("Enter vector v (x y z): ").split(
    w = list(map(int, input("Enter vector w (x y z): ").split(
    A = dot(u, cross(v, w))
    B = dot(v, cross(u, w))
    C = dot(cross(v, w), u)
    D = dot(cross(u, v), w)
    print("\nResults:")
    print(f"A = u · (v × w) = {A}")
    print(f"B = v · (u × w) = {B}")
    print(f"C = (v × w) · u = {C}")
    print(f"D = (u × v) · w = {D}")
    print("\n=> Expression B is different.")
```