2.8.9

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Question:
Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three vectors such that $|\mathbf{a}| = 3,\ |\mathbf{b}| = 4,\ |\mathbf{c}| = 5$, and each one of them is perpendicular to the sum of the other two. Find $|\mathbf{a} + \mathbf{b} + \mathbf{c}|$.

Solution:
Let the Gram matrix G of the three vectors $(a), (b), (c)$ be

$$G = \begin{pmatrix} (a,a) & (a,b) & (a,c) \\ (b,a) & (b,b) & (b,c) \\ (c,a) & (c,b) & (c,c) \end{pmatrix} = \begin{pmatrix} 9 & x & z \\ x & 16 & y \\ z & y & 25 \end{pmatrix} \tag{1}$$

where

$$x = (a,b), \quad y = (b,c), \quad z = (c,a). \tag{2}$$

The conditions "each vector is perpendicular to the sum of the other two" give

$$(a, (b) + (c)) = 0, \tag{3}$$
$$(b, (c) + (a)) = 0, \tag{4}$$
$$(c, (a) + (b)) = 0. \tag{5}$$

In terms of x, y, z, equations (3)–(5) become

$$x + z = 0, \tag{6}$$
$$x + y = 0, \tag{7}$$
$$y + z = 0. \tag{8}$$

From (6) we get $z = -x$, and from (7) we get $y = -x$. Substituting into (8) gives

$$(-x) + (-x) = 0 \quad \Rightarrow \quad x = 0. \tag{9}$$

Hence

$$x = y = z = 0. \tag{10}$$

So $(a), (b), (c)$ are pairwise orthogonal.
Therefore

$$|(a) + (b) + (c)|^2 = (a + b + c) \cdot (a + b + c) \tag{11}$$
$$= (a,a) + (b,b) + (c,c) \tag{12}$$
$$= |a|^2 + |b|^2 + |c|^2 \tag{13}$$
$$= 9 + 16 + 25 \tag{14}$$
$$= 50. \tag{15}$$

Thus

$$|(a) + (b) + (c)| = \sqrt{50} = 5\sqrt{2}. \tag{16}$$

Graphical Representation:

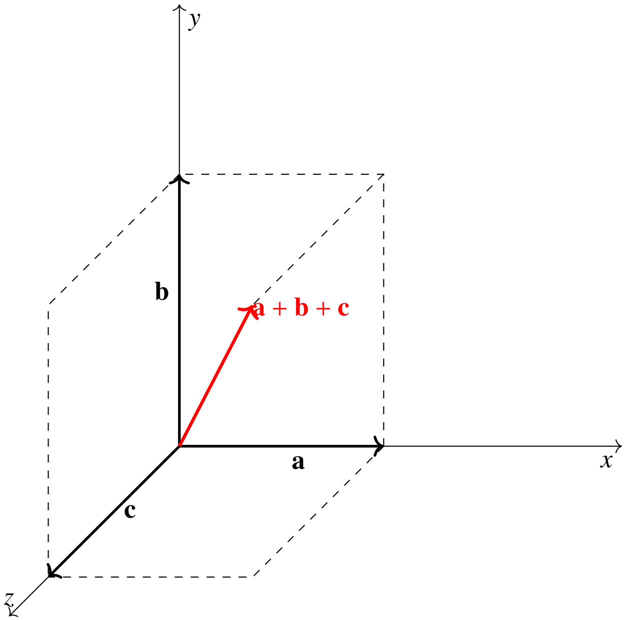

Fig. 4