4.4.12

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Question:

Find the equation of the plane passing through the points $(2, 5, -3)$, $(-2, -3, 5)$ and $(5, 3, -3)$. Also find the point of intersection of this plane with the line passing through points $(3, 1, 5)$ and $(-1, -3, -1)$.

Solution:
The points are

$$\mathbf{A} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -2 \\ -3 \\ 5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 5 \\ 3 \\ -3 \end{pmatrix} \tag{1}$$

$$\begin{pmatrix} 2 & 5 & -3 \\ -2 & -3 & 5 \\ 5 & 3 & -3 \end{pmatrix} \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{2}$$

$$\left(\begin{array}{ccc|c} 2 & 5 & -3 & 1 \\ -2 & -3 & 5 & 1 \\ 5 & 3 & -3 & 1 \end{array}\right) \xrightarrow{R_2 \leftarrow R_2 + R_1,\ R_3 \leftarrow 2R_3 - 5R_1} \left(\begin{array}{ccc|c} 2 & 5 & -3 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & -19 & 9 & -3 \end{array}\right)$$

$$\xrightarrow{R_3 \leftarrow 2R_3 + 19R_2} \left(\begin{array}{ccc|c} 2 & 5 & -3 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 56 & 22 \end{array}\right)$$

$$\xrightarrow{R_1 \leftarrow \frac{1}{2}R_1,\ R_2 \leftarrow \frac{1}{2}R_2,\ R_3 \leftarrow \frac{1}{56}R_3} \left(\begin{array}{ccc|c} 1 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & \frac{11}{28} \end{array}\right)$$

$$\xrightarrow{R_2 \leftarrow R_2 - R_3} \left(\begin{array}{ccc|c} 1 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{17}{28} \\ 0 & 0 & 1 & \frac{11}{28} \end{array}\right)$$

$$\xrightarrow{R_1 \leftarrow R_1 + \frac{3}{2}R_3,\ R_1 \leftarrow R_1 - \frac{5}{2}R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{2}{7} \\ 0 & 1 & 0 & \frac{3}{7} \\ 0 & 0 & 1 & \frac{4}{7} \end{array}\right)$$

Hence the equation of the plane is

$$\begin{pmatrix} \frac{2}{7} & \frac{3}{7} & \frac{4}{7} \end{pmatrix} \mathbf{x} = 1 \tag{3}$$

The equation of the line passing through:

$$\mathbf{A} = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix} \tag{4}$$

$$\tag{5}$$

The direction vector of of the line

$$\mathbf{m} = \mathbf{A} - \mathbf{B} \tag{6}$$

$$= \begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix} \tag{7}$$

Vector equation of the line is

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{8}$$

Solving the equation of the plane ($\mathbf{n}^T \mathbf{x} = 1$) and the line ($\mathbf{x} = \mathbf{A} + \lambda \mathbf{m}$),

$$\mathbf{n}^T (\mathbf{A} + \lambda \mathbf{m}) = 1 \tag{9}$$

$$\mathbf{n}^T \mathbf{A} + \lambda \mathbf{n}^T \mathbf{m} = 1 \tag{10}$$

$$\lambda = \frac{1 - \mathbf{n}^T \mathbf{A}}{\mathbf{n}^T \mathbf{m}} \tag{11}$$

$$\textbf{Substituting the } \mathbf{n}, \mathbf{A}, \mathbf{m} \tag{12}$$

$$\lambda = \frac{1 - \begin{pmatrix} \frac{2}{7} \\ \frac{3}{7} \\ \frac{4}{7} \end{pmatrix}^T \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}}{\begin{pmatrix} \frac{2}{7} \\ \frac{3}{7} \\ \frac{4}{7} \end{pmatrix}^T \begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix}} \tag{13}$$

$$\lambda = \frac{1 - (\frac{2}{7}.3 + \frac{3}{7}.1 + \frac{4}{7}.5)}{(\frac{2}{7}.4 + \frac{3}{7}.4 + \frac{4}{7}.6)} \tag{14}$$

$$\lambda = \frac{1 - \frac{29}{7}}{\frac{44}{7}} \tag{15}$$

$$\lambda = \frac{-1}{2} \tag{16}$$

$$\text{From equation 8,} \tag{17}$$

$$\mathbf{x} = \begin{pmatrix} 3 + (\frac{-1}{2})4 \\ 1 + (\frac{-1}{2})4 \\ 5 + (\frac{-1}{2})6 \end{pmatrix} \tag{18}$$

$$= \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \tag{19}$$

the point of intersection is

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \tag{20}$$

Therefore,
the equation of the plane passing through the points (2, 5, -3) , (-2, -3, 5) and (5, 3, -3) is $\begin{pmatrix} \frac{2}{7} & \frac{3}{7} & \frac{4}{7} \end{pmatrix} \mathbf{x} = 1$

and the point of intersection of the line with the plane is $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

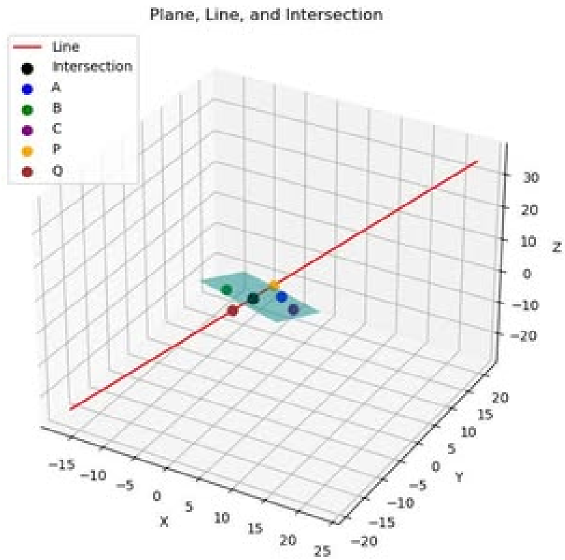

Fig. 0.1