2.7.17

EE25BTECH11005 - Aditya Mishra

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Question

Problem: Show that the points $\mathbf{A} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$, $\mathbf{B} = \hat{\imath} - 3\hat{\jmath} - 5\hat{k}$, and $\mathbf{C} = 3\hat{\imath} - 4\hat{\jmath} - 4\hat{k}$ are vertices of a right-angled triangle. Find the area of the triangle.

$$\mathbf{A} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix}.$$

Calculate the side vectors:

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} 1-2 \\ -3+1 \\ -5-1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -6 \end{pmatrix}, \quad \mathbf{C} - \mathbf{A} = \begin{pmatrix} 3-2 \\ -4+1 \\ -4-1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}.$$

Check right angle at $\mathbf{A}$ by verifying:

$$(\mathbf{B} - \mathbf{A})^\top (\mathbf{C} - \mathbf{A}) = (-1)(1) + (-2)(-3) + (-6)(-5) = -1 + 6 + 30 = 35 \neq 0.$$

Similarly,

$$(\mathbf{A} - \mathbf{B}) = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}, \quad (\mathbf{C} - \mathbf{B}) = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix},$$

$$(\mathbf{A} - \mathbf{B})^\top (\mathbf{C} - \mathbf{B}) = 2 - 2 + 6 = 6 \neq 0.$$

At $\mathbf{C}$:

$$(\mathbf{A} - \mathbf{C}) = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix}, \quad (\mathbf{B} - \mathbf{C}) = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix},$$

$$(\mathbf{A}-\mathbf{C})^\top(\mathbf{B}-\mathbf{C}) = (-1)(-2) + 3(1) + 5(-1) = 2 + 3 - 5 = 0,$$

confirming the right angle at **C**.—

The area of $\triangle ABC$ is given by:

$$\text{Area} = \frac{1}{2}\left\|(\mathbf{A}-\mathbf{C}) \times (\mathbf{B}-\mathbf{C})\right\|.$$

Area computation using the identity:

$$\|\mathbf{a} \times \mathbf{b}\|^2 + (\mathbf{a}^\top\mathbf{b})^2 = \|\mathbf{a}\|^2\|\mathbf{b}\|^2,$$

for vectors $\mathbf{a} = \mathbf{A} - \mathbf{C}$, $\mathbf{b} = \mathbf{B} - \mathbf{C}$.

Calculate

$$\|\mathbf{a}\|^2 = (-1)^2 + 3^2 + 5^2 = 1 + 9 + 25 = 35,$$
$$\|\mathbf{b}\|^2 = (-2)^2 + 1^2 + (-1)^2 = 4 + 1 + 1 = 6,$$
$$(\mathbf{a}^\top\mathbf{b})^2 = 0^2 = 0.$$

Thus,

$$\|\mathbf{a} \times \mathbf{b}\|^2 = \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a}^\top\mathbf{b})^2 = 35 \times 6 - 0 = 210.$$

The area is:

$$\text{Area} = \frac{1}{2}\|\mathbf{a} \times \mathbf{b}\| = \frac{1}{2}\sqrt{210}.$$

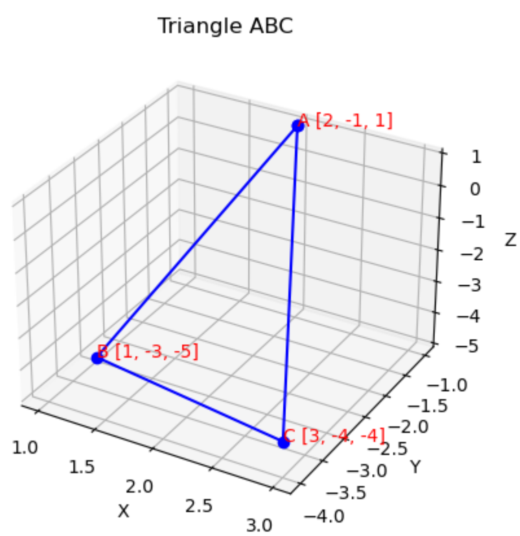

Figure 1: Plot