8.4.29

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Question

A hyperbola, having the transverse axis of length $2\sin\theta$, is confocal with the ellipse $3x^2 + 4y^2 = 12$. Then the equation is

1. $x^2 \csc^2\theta - y^2 \sec^2\theta = 1$
2. $x^2 \sec^2\theta - y^2 \csc^2\theta = 1$
3. $x^2 \sin^2\theta - y^2 \cos^2\theta = 1$
4. $x^2 \cos^2\theta - y^2 \sin^2\theta = 1$

Theoretical Solution

From the data given,

$$\text{Equation of ellipse is given by : } \mathbf{x}^\top \mathbf{M_e} \mathbf{x} = 1 \tag{1}$$

where,

$$\mathbf{M_e} = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \tag{2}$$

Focal length of the ellipse (f) is given by,

$$f_e^2 = \frac{\lambda_2 - \lambda_1}{\|M_e\|} \tag{3}$$

where, λ_1 and λ_2 are the eigen values of the matrix $\mathbf{M_e}$.For a diagonal matrix it's eigen values are given by their diagonal elements.

Theoretical Solution

$$\therefore f_e^2 = \frac{\frac{1}{3} - \frac{1}{4}}{\frac{1}{12}} = 1 \implies f_e = 1 \tag{4}$$

As ellipse and hyperbola are confocal,their focal lengths are same.Let the equation of hyperbola be

$$\mathbf{x}^\top \mathbf{M_H} \mathbf{x} = 1 \tag{5}$$

where,

$$\mathbf{M_H} = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} \tag{6}$$

where μ_1 and μ_2 are the eigen values of matrix $\mathbf{M_H}$.

Theoretical Solution

The focal length of hyperbola f_H is given by,

$$f_H^2 = -\frac{\mu_1 - \mu_2}{\|M_H\|} \tag{7}$$

As the value of transverse axis is $2\sin\theta$,

$$\mu_1 = \csc^2\theta \tag{8}$$

Also,

$$\mu_2 - \mu_1 = \mu_1\mu_2 \tag{9}$$

$$\implies \mu_2 = -sec^2\theta \tag{10}$$

Thus, the desired equation is

$$\mathbf{x}^\top \mathbf{M_H} \mathbf{x} = 1 \tag{11}$$

where, $\mathbf{M_H} = \begin{pmatrix} \csc^2\theta & 0 \\ 0 & -\sec^2\theta \end{pmatrix}$.

C Code -Finding the equation of hyperbola

```c
#include <stdio.h>
#include <math.h>

void hyperbola_params(double theta, double *arr) {
    double a = sin(theta); // transverse semi-axis
    double b = cos(theta); // conjugate semi-axis
    arr[0] = a * a; // a^2
    arr[1] = b * b; // b^2
}
```

Python+C code

```
import ctypes
import numpy as np
import matplotlib.pyplot as plt
from math import pi, sqrt

# Load the shared library
lib = ctypes.CDLL("./libconfocal.so")

# Define argument and return types for the C function
lib.hyperbola_params.argtypes = [ctypes.c_double, ctypes.POINTER(
    ctypes.c_double)]
lib.hyperbola_params.restype = None

def get_hyperbola_params(theta):
    arr = (ctypes.c_double * 2)()
    lib.hyperbola_params(theta, arr)
    return arr[0], arr[1] # returns (a^2, b^2)
```

Python+C code

```
def plot_confocal(theta=pi/6):
    # Ellipse parameters
    a_e = 2.0
    b_e = sqrt(3.0)
    c = sqrt(a_e*a_e - b_e*b_e) # foci distance = 1
    # Get hyperbola parameters from C
    a2, b2 = get_hyperbola_params(theta)
    a_h = sqrt(a2)
    b_h = sqrt(b2)
    # Print hyperbola equation
    print(f"Hyperbola: x^2/{a2:.3f} - y^2/{b2:.3f} = 1")
    # Ellipse parametric curve
    t = np.linspace(0, 2*np.pi, 800)
    x_ell = a_e * np.cos(t)
    y_ell = b_e * np.sin(t)
```

Python+C code

```
# Hyperbola parametric curve
u = np.linspace(-1.2, 1.2, 600)
sec_u = 1.0/np.cos(u)
tan_u = np.tan(u)
x_h_pos = a_h * sec_u
y_h_pos = b_h * tan_u
x_h_neg = -a_h * sec_u
y_h_neg = b_h * tan_u
mask_pos = np.isfinite(x_h_pos) & np.isfinite(y_h_pos) & (np.
    abs(x_h_pos)<50) & (np.abs(y_h_pos)<50)
mask_neg = np.isfinite(x_h_neg) & np.isfinite(y_h_neg) & (np.
    abs(x_h_neg)<50) & (np.abs(y_h_neg)<50)
```

Python+C code

```
    fig, ax = plt.subplots(figsize=(8,6))
    ax.plot(x_ell, y_ell, label="Ellipse")
    ax.plot(x_h_pos[mask_pos], y_h_pos[mask_pos], linestyle='--',
        label="Hyperbola (right branch)")
    ax.plot(x_h_neg[mask_neg], y_h_neg[mask_neg], linestyle='--',
        label="Hyperbola (left branch)")
    ax.scatter([c, -c], [0,0], marker='x', s=80, label="Foci")
    ax.set_aspect('equal', 'box')
    ax.grid(True)
    ax.set_xlabel('x')
    ax.set_ylabel('y')
    ax.set_title('Ellipse and Confocal Hyperbola (via C + Python)
        ')
    ax.legend()
    plt.savefig("/home/user/Matrix Theory: workspace/
        Matgeo_assignments/8.4.29/figs/figure_1.png")
    plt.show()
plot_confocal(theta=pi/6)
```

Python code

```
import numpy as np
import matplotlib.pyplot as plt
from math import sin, cos, pi, sqrt

def plot_confocal_hyperbola(theta=pi/6):
    # Ellipse parameters
    a_e = 2.0
    b_e = sqrt(3.0)
    c = sqrt(a_e*a_e - b_e*b_e) # foci distance = 1

    # Hyperbola parameters
    a_h = abs(sin(theta)) # transverse semi-axis
    b_h = abs(cos(theta)) # conjugate semi-axis
    if a_h == 0:
        raise ValueError("theta leads to a_h = 0 (sin(theta)=0).
            Choose a different theta.")
```

Python code

```
eq_hyperbola = f"Hyperbola: x^2/{a_h**2:.3f} - y^2/{b_h**2:.3f}
     = 1"
    print(eq_hyperbola)

    # Parametric ellipse
    t = np.linspace(0, 2*np.pi, 800)
    x_ell = a_e * np.cos(t)
    y_ell = b_e * np.sin(t)

    # Parametric hyperbola branches
    u = np.linspace(-1.2, 1.2, 600)
    sec_u = 1.0/np.cos(u)
    tan_u = np.tan(u)
    x_h_pos = a_h * sec_u
    y_h_pos = b_h * tan_u
    x_h_neg = -a_h * sec_u
    y_h_neg = b_h * tan_u
```

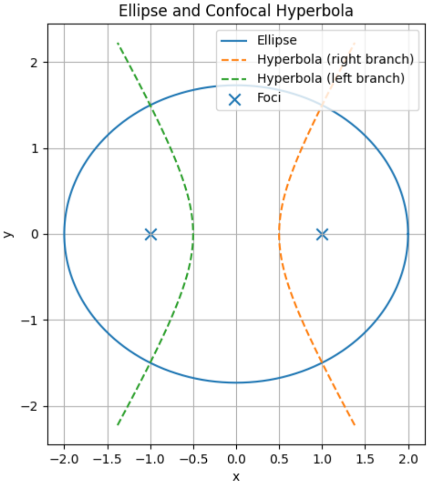
Python code

```
mask_pos = np.isfinite(x_h_pos) & np.isfinite(y_h_pos) & (np.abs(x_h_pos)<50) & (np.abs(y_h_pos)<50)
    mask_neg = np.isfinite(x_h_neg) & np.isfinite(y_h_neg) & (np.abs(x_h_neg)<50) & (np.abs(y_h_neg)<50)

    # Plot
    fig, ax = plt.subplots(figsize=(8,6))
    ax.plot(x_ell, y_ell, label="Ellipse")
    ax.plot(x_h_pos[mask_pos], y_h_pos[mask_pos], linestyle='--', label="Hyperbola (right branch)")
    ax.plot(x_h_neg[mask_neg], y_h_neg[mask_neg], linestyle='--', label="Hyperbola (left branch)")
    ax.scatter([c, -c], [0,0], marker='x', s=80, label="Foci")
```

Python code

```
ax.set_aspect('equal', 'box')
    ax.grid(True)
    ax.set_xlabel('x')
    ax.set_ylabel('y')
    ax.set_title('Ellipse and Confocal Hyperbola')
    ax.legend()
    plt.savefig("/home/user/Matrix Theory: workspace/
        Matgeo_assignments/8.4.29/figs/Figure_1.png")
    plt.show()

# Example run with theta = pi/6
plot_confocal_hyperbola(theta=pi/6)
```

Plot



Ellipse and Confocal Hyperbola
Ellipse
Hyperbola (right branch)
Hyperbola (left branch)
Foci
y
x
2
1
0
−1
−2
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0