

2.8.9

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Question:

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three vectors such that $|\mathbf{a}| = 3$, $|\mathbf{b}| = 4$, $|\mathbf{c}| = 5$, and each one of them is perpendicular to the sum of the other two. Find $|\mathbf{a} + \mathbf{b} + \mathbf{c}|$.

Solution:

From the identity:

$$\mathbf{a}^\top (\mathbf{b} + \mathbf{c}) = 0, \quad (1)$$

we expand:

$$\mathbf{a}^\top \mathbf{b} + \mathbf{a}^\top \mathbf{c} = 0. \quad (2)$$

Similarly, from the symmetry of dot products:

$$\mathbf{b}^\top \mathbf{c} + \mathbf{b}^\top \mathbf{a} = 0, \quad (3)$$

$$\mathbf{c}^\top \mathbf{a} + \mathbf{c}^\top \mathbf{b} = 0. \quad (4)$$

Let

$$\mathbf{x} = \mathbf{a}^\top \mathbf{b}, \quad \mathbf{y} = \mathbf{b}^\top \mathbf{c}, \quad \mathbf{z} = \mathbf{c}^\top \mathbf{a}. \quad (5)$$

Then equations (2), (3), and (4) become:

$$\mathbf{x} + \mathbf{z} = 0, \quad (6)$$

$$\mathbf{x} + \mathbf{y} = 0, \quad (7)$$

$$\mathbf{y} + \mathbf{z} = 0. \quad (8)$$

From equation (7):

$$\mathbf{y} = -\mathbf{x},$$

and from equation (6):

$$\mathbf{z} = -\mathbf{x}.$$

Substitute into equation (8):

$$\mathbf{y} + \mathbf{z} = -\mathbf{x} + (-\mathbf{x}) = -2\mathbf{x} = 0 \Rightarrow \mathbf{x} = 0.$$

Therefore:

$$\mathbf{x} = \mathbf{y} = \mathbf{z} = 0, \quad (9)$$

so $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are **pairwise orthogonal**.

The **Gram matrix** of $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is:

$$G = \begin{bmatrix} \mathbf{a}^\top \mathbf{a} & \mathbf{a}^\top \mathbf{b} & \mathbf{a}^\top \mathbf{c} \\ \mathbf{b}^\top \mathbf{a} & \mathbf{b}^\top \mathbf{b} & \mathbf{b}^\top \mathbf{c} \\ \mathbf{c}^\top \mathbf{a} & \mathbf{c}^\top \mathbf{b} & \mathbf{c}^\top \mathbf{c} \end{bmatrix} = \begin{bmatrix} \|\mathbf{a}\|^2 & 0 & 0 \\ 0 & \|\mathbf{b}\|^2 & 0 \\ 0 & 0 & \|\mathbf{c}\|^2 \end{bmatrix} = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 25 \end{pmatrix}. \quad (10)$$

Let

$$u = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Then

$$\|a + b + c\|^2 = (a + b + c)^T (a + b + c) = u^T G u. \quad (11)$$

Now compute:

$$\begin{aligned} u^T G u &= \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 9 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 25 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= 9 + 16 + 25 = 50. \end{aligned} \quad (12)$$

Therefore:

$$\|a + b + c\| = \sqrt{50} = 5\sqrt{2}. \quad (13)$$

Final Answer:

$$\boxed{5\sqrt{2}}$$

Graph presentation:

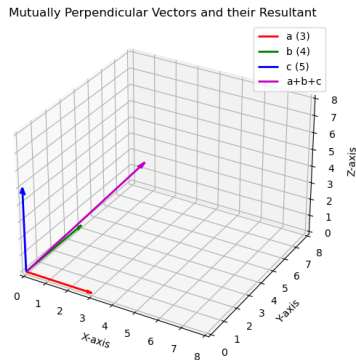


Fig. 1