2.8.9

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Question:
Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three vectors such that $|\mathbf{a}| = 3$, $|\mathbf{b}| = 4$, $|\mathbf{c}| = 5$, and each one of them is perpendicular to the sum of the other two. Find $|\mathbf{a} + \mathbf{b} + \mathbf{c}|$.

Solution:
From the identity:

$$\mathbf{a}^\top(\mathbf{b} + \mathbf{c}) = 0, \tag{1}$$

we expand:

$$\mathbf{a}^\top\mathbf{b} + \mathbf{a}^\top\mathbf{c} = 0. \tag{2}$$

Similarly, from the symmetry of dot products:

$$\mathbf{b}^\top\mathbf{c} + \mathbf{b}^\top\mathbf{a} = 0, \tag{3}$$

$$\mathbf{c}^\top\mathbf{a} + \mathbf{c}^\top\mathbf{b} = 0. \tag{4}$$

Let

$$x = \mathbf{a}^\top\mathbf{b}, \quad y = \mathbf{b}^\top\mathbf{c}, \quad z = \mathbf{c}^\top\mathbf{a}. \tag{5}$$

Then equations (2), (3), and (4) become:

$$x + z = 0, \tag{6}$$

$$x + y = 0, \tag{7}$$

$$y + z = 0. \tag{8}$$

From equation (7):

$$y = -x,$$

and from equation (6):

$$z = -x.$$

Substitute into equation (8):

$$y + z = -x + (-x) = -2x = 0 \Rightarrow x = 0.$$

Therefore:

$$x = y = z = 0, \tag{9}$$

so $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are **pairwise orthogonal**.

The **Gram matrix** of $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is:

$$G = \begin{bmatrix} \mathbf{a}^\top\mathbf{a} & \mathbf{a}^\top\mathbf{b} & \mathbf{a}^\top\mathbf{c} \\ \mathbf{b}^\top\mathbf{a} & \mathbf{b}^\top\mathbf{b} & \mathbf{b}^\top\mathbf{c} \\ \mathbf{c}^\top\mathbf{a} & \mathbf{c}^\top\mathbf{b} & \mathbf{c}^\top\mathbf{c} \end{bmatrix} = \begin{bmatrix} ||\mathbf{a}||^2 & 0 & 0 \\ 0 & ||\mathbf{b}||^2 & 0 \\ 0 & 0 & ||\mathbf{c}||^2 \end{bmatrix} = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 25 \end{pmatrix}. \tag{10}$$

Let

$$\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Then

$$\|\mathbf{a} + \mathbf{b} + \mathbf{c}\|^2 = (\mathbf{a} + \mathbf{b} + \mathbf{c})^\top (\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{u}^\top G \mathbf{u}. \tag{11}$$

Now compute:

$$\begin{aligned} \mathbf{u}^\top G \mathbf{u} \hat{A} &= \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 9 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 25 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= 9 + 16 + 25 = 50. \end{aligned} \tag{12}$$

Therefore:

$$\|\mathbf{a} + \mathbf{b} + \mathbf{c}\| = \sqrt{50} = 5\sqrt{2}. \tag{13}$$

Final Answer:

$$\boxed{5\sqrt{2}}$$

Graph presentation:

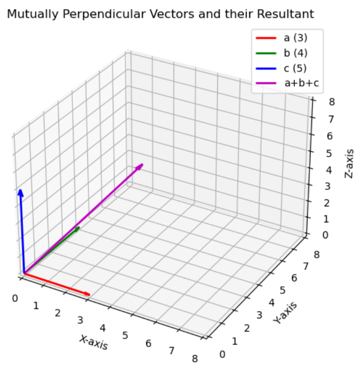

Fig. 1