5.13.58

EE25BTECH11032 - Kartik Lahoti

Question:

Let $\omega \neq 1$ be a cube root of unity and $\mathbb{S}$ be the set of all non-singular matrices of the form

$$\begin{pmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{pmatrix} \tag{0.1}$$

where each of a, b and c is either ω or ω^2. Then the number of distinct matrices in the set $\mathbb{S}$ is

Solution:

Let,

$$\mathbf{A} = \begin{pmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{pmatrix} \tag{0.2}$$

where $\mathbf{A} \in \mathbb{S}$

For $\mathbf{A}$ to be Non-singular , $\mathbf{A}$ should be a full rank matrix.

Thus,

$$rank\left(\mathbf{A}\right) = 3 \tag{0.3}$$

$$\begin{pmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{pmatrix} \xleftrightarrow{R_2 \to R_2 - \omega R_1} \begin{pmatrix} 1 & a & b \\ 0 & 1 - a\omega & c - b\omega \\ \omega^2 & \omega & 1 \end{pmatrix} \tag{0.4}$$

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 - a\omega & c - b\omega \\ \omega^2 & \omega & 1 \end{pmatrix} \xleftrightarrow{R_2 \to R_2 - \omega^2 R_1} \begin{pmatrix} 1 & a & b \\ 0 & 1 - a\omega & c - b\omega \\ 0 & \omega - a\omega^2 & 1 - b\omega^2 \end{pmatrix} \tag{0.5}$$

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 - a\omega & c - b\omega \\ 0 & \omega - a\omega^2 & 1 - b\omega^2 \end{pmatrix} \xleftrightarrow{R_3 \to R_3 - \omega R_2} \begin{pmatrix} 1 & a & b \\ 0 & 1 - a\omega & c - b\omega \\ 0 & 0 & 1 - c\omega \end{pmatrix} \tag{0.6}$$

For this Row Reduced Echelon Form Matrix to be a full rank matrix, the diagonal pivots should be non-zero.

$$1 - c\omega \neq 0 \implies c = \omega \tag{0.7}$$

$$1 - a\omega \neq 0 \implies a = \omega \tag{0.8}$$

Non-singularity does not depends on b thus, $b \in \left\{\omega, \omega^2\right\}$

$$\therefore n(\mathbb{S}) = 1 \times 2 \times 1 \tag{0.9}$$
$$n(\mathbb{S}) = 2 \tag{0.10}$$

Hence , number of Matrices in Set $\mathbb{S}$ is 2.
In the graph , Let 1 be equivalent to ω and -1 be equivalent to ω^2.