2.10.47

EE25BTECH11018 - DARISY SREETEJ

Question: The value of a so that the volume of parallelopiped formed by $\hat{i}+a\hat{j}+\hat{k}$, $\hat{j}+a\hat{k}$ and $a\hat{i}+\hat{k}$ becomes minimum is

1) -3
2) 3
3) $\frac{1}{\sqrt{3}}$
4) $\sqrt{3}$

Solution: Let us consider,

$$\mathbf{p} = \hat{i} + a\hat{j} + \hat{k}$$
$$\mathbf{q} = \hat{j} + a\hat{k}$$
$$\mathbf{r} = a\hat{i} + \hat{k}$$

then, the Volume of the parallelopiped formed by $\mathbf{p}, \mathbf{q}, \mathbf{r}$ is ,

$$V = \mathbf{p} \cdot (\mathbf{q} \times \mathbf{r}) \tag{4.1}$$

The Gram matrix $\mathbf{G}$ for the vectors $\mathbf{p}, \mathbf{q}, \mathbf{r}$ is:

$$\mathbf{G} = \begin{pmatrix} \mathbf{p}^\top\mathbf{p} & \mathbf{p}^\top\mathbf{q} & \mathbf{p}^\top\mathbf{r} \\ \mathbf{q}^\top\mathbf{p} & \mathbf{q}^\top\mathbf{q} & \mathbf{q}^\top\mathbf{r} \\ \mathbf{r}^\top\mathbf{p} & \mathbf{r}^\top\mathbf{q} & \mathbf{r}^\top\mathbf{r} \end{pmatrix} \tag{4.2}$$

Now, calculate the dot products:

$$\mathbf{p}^\top\mathbf{p} = 1^2 + a^2 + 1^2 = 2 + a^2 \tag{4.3}$$

$$\mathbf{p}^\top\mathbf{q} = (1)(0) + (a)(1) + (1)(a) = 2a \tag{4.4}$$

$$\mathbf{p}^\top\mathbf{r} = (1)(a) + (a)(0) + (1)(1) = a + 1 \tag{4.5}$$

$$\mathbf{q}^\top\mathbf{p} = \mathbf{p}^\top\mathbf{q} = 2a \tag{4.6}$$

$$\mathbf{q}^\top\mathbf{q} = a^2 + 1^2 = 1 + a^2 \tag{4.7}$$

$$\mathbf{q}^\top\mathbf{r} = (0)(a) + (1)(0) + (a)(1) = a \tag{4.8}$$

$$\mathbf{r}^\top \mathbf{p} = \mathbf{p}^\top \mathbf{r} = a + 1 \tag{4.9}$$

$$\mathbf{r}^\top \mathbf{q} = \mathbf{q}^\top \mathbf{r} = a \tag{4.10}$$

$$\mathbf{r}^\top \mathbf{r} = a^2 + 1^2 = a^2 + 1 \tag{4.11}$$

Thus, the Gram matrix $\mathbf{G}$ is:

$$\mathbf{G} = \begin{pmatrix} 2 + a^2 & 2a & a + 1 \\ 2a & 1 + a^2 & a \\ a + 1 & a & 1 + a^2 \end{pmatrix} \tag{4.12}$$

The characteristic equation is obtained by solving the determinant equation $\left|\mathbf{G} - \lambda \mathbf{I}\right| = 0$. The characteristic polynomial for the matrix is:

$$\lambda^3 - (3a^2 + 4)\lambda^2 + (3a^4 + 2a^2 + 4)\lambda - (a^6 - 2a^4 + 2a^3 + a^2 - 2a + 1) = 0 \tag{4.13}$$

To find the eigenvalues, we solve the cubic equation:

$$\lambda^3 - (3a^2 + 4)\lambda^2 + (3a^4 + 2a^2 + 4)\lambda - (a^6 - 2a^4 + 2a^3 + a^2 - 2a + 1) = 0$$

The determinant of $\mathbf{G}$ is the product of its eigenvalues:

$$\left|\mathbf{G}\right| = \lambda_1 \lambda_2 \lambda_3 = (a^6 - 2a^4 + 2a^3 + a^2 - 2a + 1). \tag{4.14}$$

The box product (scalar triple product) is the square root of the determinant of $\mathbf{G}$:

$$\mathbf{p} \cdot (\mathbf{q} \times \mathbf{r}) = \sqrt{\left|\mathbf{G}\right|} = \sqrt{(a^6 - 2a^4 + 2a^3 + a^2 - 2a + 1)} = a^3 - a + 1 \tag{4.15}$$

$$V = a^3 - a + 1 \tag{4.16}$$

Now , consider

$$f(a) = a^3 - a + 1 \tag{4.17}$$

$$f'(a) = 3a^2 + 1 \tag{4.18}$$

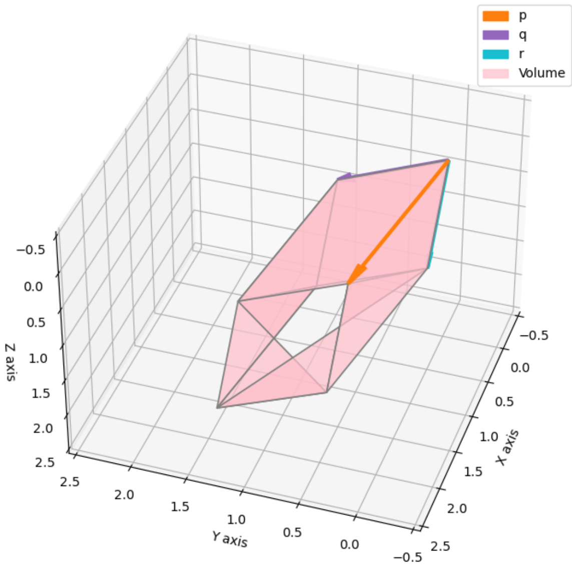
$$\text{Set } f'(a) = 0 \Rightarrow a^2 = \frac{1}{\sqrt{3}} \Rightarrow a = \frac{1}{\sqrt{3}} or - \frac{1}{\sqrt{3}}$$

$$\text{Second derivative } f''(a) = 6a \tag{4.19}$$

$$\text{At } a = \frac{1}{\sqrt{3}}, f'' > 0 \Rightarrow minimum \tag{4.20}$$

$$\text{At } a = -\frac{1}{\sqrt{3}}, f'' < 0 \Rightarrow maximum \tag{4.21}$$

Therefore , $a = \frac{1}{\sqrt{3}}$ for which the Volume of the parallelopiped becomes minimum.



Parallelopiped with Vectors $\mathbf{p}, \mathbf{q}, \mathbf{r}$ for which $a = \frac{1}{\sqrt{3}}$(Volume is minimum)