3.3.5

AI25BTECH11021 - Abhiram Reddy N

Question

Construct a $\triangle ABC$ in which $CA = 6\,cm$, $AB = 5\,cm$, and $\angle BAC = 45°$.

Answer

***Step** 1: Define the coordinate system and vectors*

To solve the problem using vectors and matrices, place point A at the origin of the coordinate system:

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We want to find points B and C such that:

$$|\mathbf{C} - \mathbf{A}| = 6, \quad |\mathbf{B} - \mathbf{A}| = 5, \quad \text{and} \quad \angle BAC = 45°.$$

***Step** 2: Position vector of point B*

Since $AB = 5\,cm$, and $\angle BAC = 45°$, let us place $\mathbf{B}$ on the positive x-axis for convenience:

$$\mathbf{B} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

This choice sets the direction of $\mathbf{AB}$ along the x-axis.

***Step** 3: Position vector of point C*

Point C must be at a distance of 6 from A and must make a 45° angle with vector $\mathbf{AB}$. Using trigonometry, we can express $\mathbf{C}$ as:

$$\mathbf{C} = 6 \begin{pmatrix} \cos 45° \\ \sin 45° \end{pmatrix} = 6 \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} 3\sqrt{2} \\ 3\sqrt{2} \end{pmatrix}$$

***Step** 4: Verification using dot product*

The angle θ between vectors $\mathbf{AB}$ and $\mathbf{AC}$ is given by:

$$\cos\theta = \frac{(\mathbf{B} - \mathbf{A}) \cdot (\mathbf{C} - \mathbf{A})}{|\mathbf{B} - \mathbf{A}|\,|\mathbf{C} - \mathbf{A}|}$$

Calculate:

$$(\mathbf{B}-\mathbf{A})\cdot(\mathbf{C}-\mathbf{A}) = \begin{pmatrix}5\\0\end{pmatrix}\cdot\begin{pmatrix}3\sqrt{2}\\3\sqrt{2}\end{pmatrix} = 5\times 3\sqrt{2}+0 = 15\sqrt{2}$$

Magnitudes:

$$|\mathbf{B}-\mathbf{A}| = 5, \quad |\mathbf{C}-\mathbf{A}| = 6$$

Therefore,

$$\cos\theta = \frac{15\sqrt{2}}{5\times 6} = \frac{15\sqrt{2}}{30} = \frac{\sqrt{2}}{2} = \cos 45^\circ$$

This confirms the angle is 45°.

Step *5: Summary of points*

$$\boxed{\mathbf{A} = \begin{pmatrix}0\\0\end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix}5\\0\end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix}3\sqrt{2}\\3\sqrt{2}\end{pmatrix}}$$

These points construct the required triangle $\triangle ABC$ satisfying all given conditions.

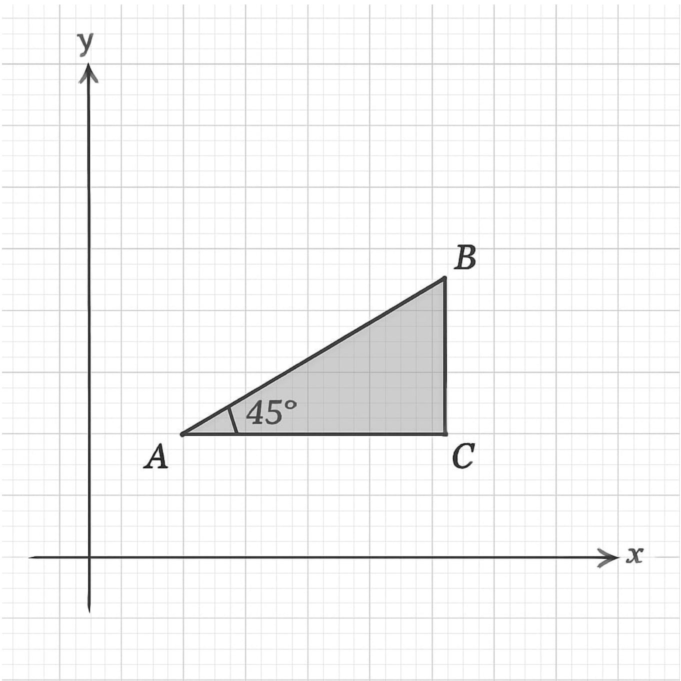

Fig. 0.1: plot