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ee25btech11056 - Suraj.N

Question : The conic has vertices $(0, \pm 13)$ and foci $(0, \pm 5)$. Find the equation of the conic.
Solution :

Name	Description	vector form
$\mathbf{B_1}$	vertex 1 of conic	$\begin{pmatrix} 0 \\ 13 \end{pmatrix}$
$\mathbf{B_2}$	vertex 2 of conic	$\begin{pmatrix} 0 \\ -13 \end{pmatrix}$
$\mathbf{F_1}$	focus 1 of conic	$\begin{pmatrix} 0 \\ 5 \end{pmatrix}$
$\mathbf{F_2}$	focus 2 of conic	$\begin{pmatrix} 0 \\ -5 \end{pmatrix}$

Table : Ellipse

The conic has two foci , so it cannot be a parabola .
Equation for any conic with directrix $\mathbf{n}^\top \mathbf{x} = c$, eccentricity e and focus $\mathbf{F}$ is given by

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$
$$\tag{2}$$

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top \tag{3}$$
$$\mathbf{u} = ce^2\mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F} \tag{4}$$
$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \tag{5}$$

The normal vector of the directrix is along the direction vector of $\mathbf{F_1} - \mathbf{F_2}$

$$\mathbf{n} = \mathbf{F_1} - \mathbf{F_2} \equiv \mathbf{e_2} \tag{6}$$

From (3) we can form the matrix $\mathbf{V}$

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - e^2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{7}$$
$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 - e^2 \end{pmatrix} \tag{8}$$

As $\mathbf{V}$ is an upper triangular matrix , we get the eigen values as the diagonal entries

$$\lambda_1 = 1 - e^2 \qquad\qquad \lambda_2 = 1 \tag{9}$$

Clearly $\left|\mathbf{V}\right| \neq 0$, $\mathbf{V}^{-1}$ exists.

The center of the conic $\mathbf{c}$ can be found

$$\mathbf{c} = \frac{\mathbf{F_1} + \mathbf{F_2}}{2} = \mathbf{0} \tag{10}$$

The relation between the $\mathbf{c}$, $\mathbf{V}$ and $\mathbf{u}$ is given by

$$\mathbf{V}\mathbf{c} + \mathbf{u} = \mathbf{0} \qquad \left|\mathbf{V}\right| \neq 0 \tag{11}$$
$$\mathbf{c} = \mathbf{0} \tag{12}$$
$$\mathbf{u} = \mathbf{0} \tag{13}$$

From (4) we get

$$ce^2\mathbf{e_2} = \mathbf{F_1} \tag{14}$$
$$\begin{pmatrix} 0 \\ ce^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix} \tag{15}$$
$$ce^2 = 5 \tag{16}$$
$$c = \frac{5}{e^2} \tag{17}$$

$$f_0 = \mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f \tag{18}$$

as $\mathbf{u} = \mathbf{0}$ and from (5), we get

$$f_0 = c^2e^2 - 25 \tag{19}$$

The length of the major axis is the distance between the two vertices

$$\|\mathbf{B_1} - \mathbf{B_2}\| = 26 \tag{20}$$

The length of major axes is also given as

$$2\sqrt{\left|\frac{f_0}{\lambda_1}\right|} \tag{21}$$

So,

$$2\sqrt{\left|\frac{c^2e^2 - 25}{1 - e^2}\right|} = 26 \tag{22}$$

From (17) we get

$$\sqrt{\frac{25}{e^2}} = 13 \tag{23}$$
$$\frac{5}{e} = 13 \tag{24}$$
$$e = \frac{5}{13} \tag{25}$$

As e <1 , the conic is an **ellipse**
The value of c and directix equation are given as

$$c = \frac{169}{5} \qquad \mathbf{n}^\top \mathbf{x} = \pm\frac{169}{5} \tag{26}$$

Using the obtained values of c and e , we get

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{144}{169} \end{pmatrix} \tag{27}$$

$$\mathbf{u} = \mathbf{0} \tag{28}$$

$$f = -144 \tag{29}$$

Substituting these in (1) , we get the equation of **ellipse** as

$$\mathbf{x}^\top \begin{pmatrix} \frac{1}{144} & 0 \\ 0 & \frac{1}{169} \end{pmatrix} \mathbf{x} = 1 \tag{30}$$

$$\tag{31}$$

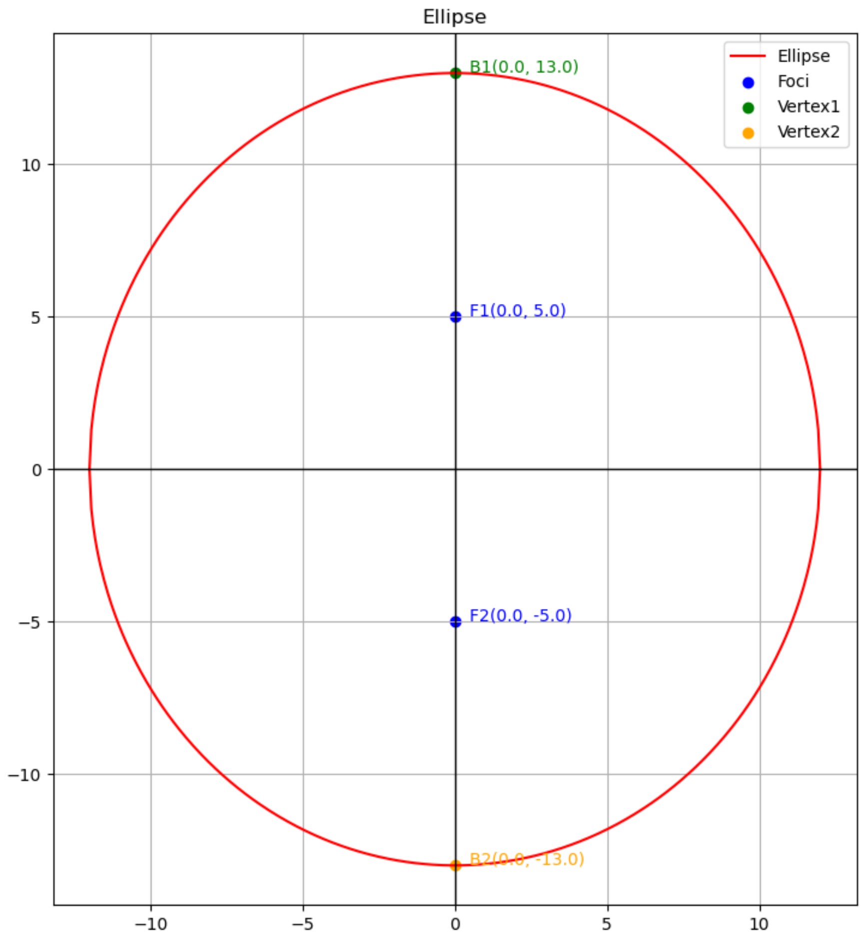
Ellipse
B1(0.0, 13.0)
F1(0.0, 5.0)
F2(0.0, -5.0)
B2(0.0, -13.0)
Ellipse
Foci
Vertex1
Vertex2
10
5
0
−5
−10
−10
−5
0
5
10

Fig : Ellipse