2.10.2

AI25BTECH11013-Gautham

Question:
Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be vectors of lengths 3, 4, and 5 respectively such that $\mathbf{A} \perp \mathbf{B}+\mathbf{C}$, $\mathbf{B} \perp \mathbf{C}+\mathbf{A}$, and $\mathbf{C} \perp \mathbf{A}+\mathbf{B}$. Find the length of the vector $\mathbf{A}+\mathbf{B}+\mathbf{C}$.
Solution:

Let the Gram matrix G for the vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be:

$$G = \begin{pmatrix} \mathbf{A}^T\mathbf{A} & \mathbf{A}^T\mathbf{B} & \mathbf{A}^T\mathbf{C} \\ \mathbf{B}^T\mathbf{A} & \mathbf{B}^T\mathbf{B} & \mathbf{B}^T\mathbf{C} \\ \mathbf{C}^T\mathbf{A} & \mathbf{C}^T\mathbf{B} & \mathbf{C}^T\mathbf{C} \end{pmatrix} = \begin{pmatrix} 9 & a & b \\ a & 16 & c \\ b & c & 25 \end{pmatrix} \tag{0.1}$$

where $a = \mathbf{A}^T\mathbf{B}$, $b = \mathbf{A}^T\mathbf{C}$, and $c = \mathbf{B}^T\mathbf{C}$.
Given the orthogonality conditions:

$$\mathbf{A} \perp \mathbf{B}+\mathbf{C} \implies \mathbf{A}^T(\mathbf{B}+\mathbf{C}) = 0 \implies a+b = 0, \tag{0.2}$$
$$\mathbf{B} \perp \mathbf{C}+\mathbf{A} \implies \mathbf{B}^T(\mathbf{C}+\mathbf{A}) = 0 \implies c+a = 0, \tag{0.3}$$
$$\mathbf{C} \perp \mathbf{A}+\mathbf{B} \implies \mathbf{C}^T(\mathbf{A}+\mathbf{B}) = 0 \implies b+c = 0. \tag{0.4}$$

This system can be written as:

$$a+b = 0 \tag{0.5}$$
$$c+a = 0 \tag{0.6}$$
$$b+c = 0. \tag{0.7}$$

In matrix form:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{0.8}$$

Convert the coefficient matrix to upper triangular form by row operations:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \tag{0.9}$$

From the last row:

$$2c = 0 \implies c = 0 \tag{0.10}$$

From the second row:

$$-b+c = 0 \implies b = 0 \tag{0.11}$$

From the first row:

$$a+b = 0 \implies a = 0 \tag{0.12}$$

Thus, the Gram matrix is:

$$G = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 25 \end{pmatrix} \tag{0.13}$$

Let $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
Now, the squared length of $\mathbf{A}+\mathbf{B}+\mathbf{C}$ is:

$$||\mathbf{A}+\mathbf{B}+\mathbf{C}||^2 = \mathbf{u}^T\mathbf{G}\mathbf{u} \tag{0.14}$$
$$||\mathbf{A}+\mathbf{B}+\mathbf{C}||^2 = 50 \tag{0.15}$$

Therefore,

$$||\mathbf{A}+\mathbf{B}+\mathbf{C}|| = \sqrt{50} = 5\sqrt{2} \tag{0.16}$$