8.2.33

EE25BTECH11065 - Yoshita J

Question

Find the equation of the conic with length of major axis 26, foci $(\pm 5, 0)$.

Solution

The equation of a conic is

$$x^T V x + 2u^T x + f = 0 \tag{1}$$

where

$$V = \|n\|^2 I - e^2 nn^T \tag{2}$$

The foci are

$$F_1 = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \quad F_2 = \begin{pmatrix} -5 \\ 0 \end{pmatrix} \tag{3}$$

The centre is

$$u = \frac{F_1 + F_2}{2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4}$$

The axis vector is

$$n = F_1 - F_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{5}$$

Therefore, substituting $n = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in (2), we get

$$V = \begin{pmatrix} 1 - e^2 & 0 \\ 0 & 1 \end{pmatrix} \tag{6}$$

From the formula for the length of the major axis,

$$2\sqrt{\frac{|f|}{\lambda_1}} \tag{7}$$

where $\lambda_1 = 1 - e^2$. Hence

$$26 = 2\sqrt{\frac{|f|}{1 - e^2}} \tag{8}$$

The relation between focus and eccentricity is

$$\pm ce^2 = 5 \tag{9}$$

The distance c is

$$c = \pm \frac{1}{e}\sqrt{\frac{|f|}{|e^2 - 1|}} \tag{10}$$

Thus from (8)–(10), solving the unknowns (c, e, f) we get

$$e = \tfrac{5}{13}, \quad c = \pm 5, \quad |f| = 144. \tag{11}$$

Let $x = \begin{pmatrix} 0 \\ \alpha \end{pmatrix}$ be a vertex on the minor axis. Substituting in (1):

$$\frac{12^2}{1} + f = 0 \implies f = -144. \tag{12}$$

Hence the equation of the conic is

$$x^T \begin{pmatrix} \frac{144}{169} & 0 \\ 0 & 1 \end{pmatrix} x - 144 = 0. \tag{13}$$

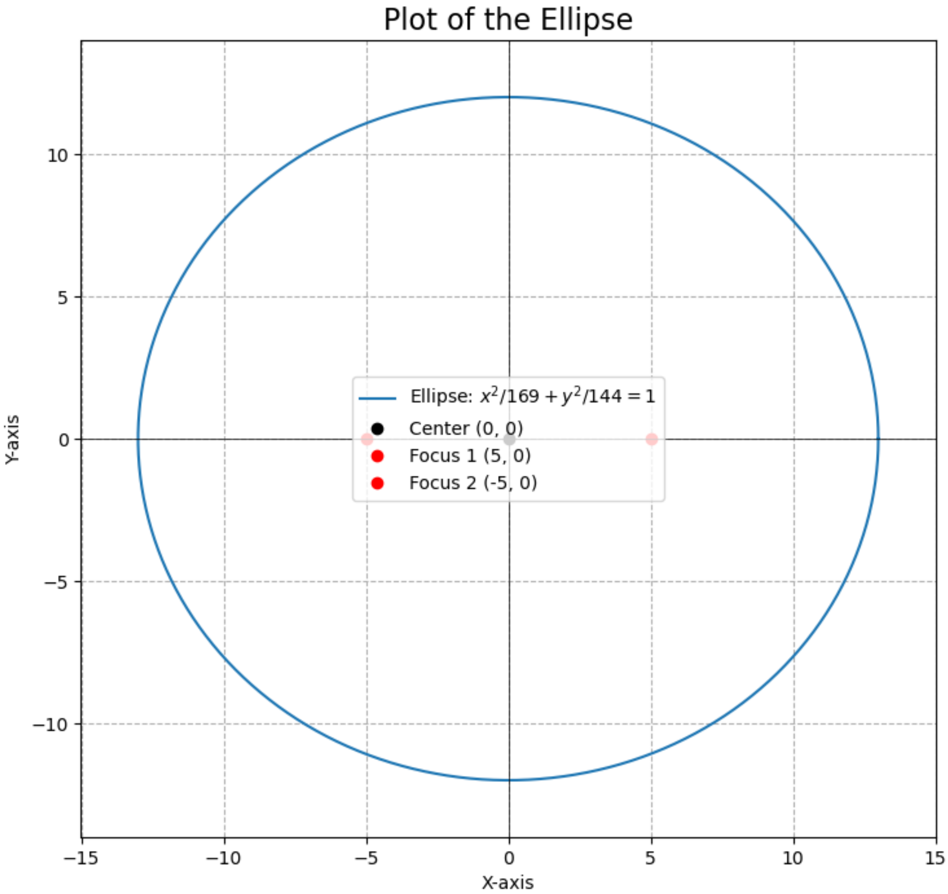

Fig. 0: Ellipse with major axis 26 and foci at $(\pm 5, 0)$