8.4.22

EE25BTECH11019 - Darji Vivek M.

Question:

The radius of the circle passing through the foci of the ellipse

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

and having its centre at $(0, 3)$ is -

1) 4
2) 3
3) $\sqrt{\frac{1}{2}}$
4) $\frac{7}{2}$

Solution:

Use the matrix form (matrix method). Let $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$. The ellipse is

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} = 1, \qquad \mathbf{V} = \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{9} \end{pmatrix}.$$

Eigenvalues of $\mathbf{V}$ (diagonal entries) are

$$\lambda_1 = \frac{1}{16}, \qquad \lambda_2 = \frac{1}{9}.$$

For the principal-form ellipse $\mathbf{x}^\top \mathbf{V} \mathbf{x} = 1$ the semi-axes satisfy

$$a^2 = \frac{1}{\lambda_1} = 16, \qquad b^2 = \frac{1}{\lambda_2} = 9.$$

Hence the focal distance from origin is

$$c = \sqrt{a^2 - b^2} = \sqrt{16 - 9} = \sqrt{7}.$$

Thus the foci (in matrix/vector form) are

$$F_1 = \begin{pmatrix} \sqrt{7} \\ 0 \end{pmatrix}, \qquad F_2 = \begin{pmatrix} -\sqrt{7} \\ 0 \end{pmatrix}.$$

The required circle has centre $C = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ and passes through, say, F_1. Therefore its radius is

$$R = \|F_1 - C\| = \sqrt{(\sqrt{7} - 0)^2 + (0 - 3)^2} = \sqrt{7 + 9} = \sqrt{16} = \boxed{4}.$$

Pyhton plot

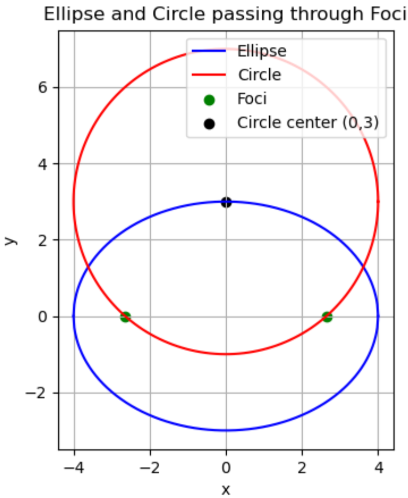

Fig. 4.1: plot if p=2,q=2