MatGeo Assignment 2.6.13

AI25BTECH11007

Question:
Given that vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ form a triangle such that

$$\mathbf{a} = \mathbf{b} + \mathbf{c},$$

find p, q, r, s given that

$$\mathbf{a} = p\hat{i} + q\hat{j} + r\hat{k}, \qquad \mathbf{b} = s\hat{i} + 3\hat{j} + 4\hat{k}, \qquad \mathbf{c} = 3\hat{i} + 1\hat{j} - 2\hat{k},$$

and the area of the triangle is $5\sqrt{6}$.

Solution:
From the condition given,

$$\mathbf{a} = \mathbf{b} + \mathbf{c},$$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} s \\ 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}. \tag{0.1}$$

Rearrange to write a linear system in the unknowns (p, s, q, r).

$$p - s = 3, \tag{0.2}$$
$$q = 4, \tag{0.3}$$
$$r = 2. \tag{0.4}$$

Thus $q = 4,\ r = 2$ and $p = s + 3$. The variable s is a free parameter. We express the family of solutions as

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} s+3 \\ 4 \\ 2 \end{pmatrix}, \qquad s \in \mathbb{R}. \tag{0.5}$$

Choose $\mathbf{b}$ and $\mathbf{c}$ as the two adjacent side-vectors of the triangle.
The area A of the triangle,

$$A = \tfrac{1}{2}\|\mathbf{b} \times \mathbf{c}\|. \tag{0.6}$$

$$\|\mathbf{b} \times \mathbf{c}\| = 2A = 10\sqrt{6}. \tag{0.7}$$

Substitute $\mathbf{b} = \begin{pmatrix} s \\ 3 \\ 4 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$. Compute the cross product.

$$\mathbf{b} \times \mathbf{c} = \begin{pmatrix} -10 \\ 2s + 12 \\ s - 9 \end{pmatrix}. \tag{0.8}$$

Therefore the squared norm is

$$\|\mathbf{b} \times \mathbf{c}\|^2 = (-10)^2 + (2s + 12)^2 + (s - 9)^2 = (10\sqrt{6})^2 \tag{0.9}$$

$$100 + (2s + 12)^2 + (s - 9)^2 = 600. \tag{0.10}$$

Hence

$$s = 5 \quad \text{or} \quad s = -11. \tag{0.11}$$

Back-substitute to obtain p, q, r

Case 1: $s = 5$**.** Then from

$$p = s + 3 = 8, \qquad q = 4, \qquad r = 2. \tag{0.12}$$

Case 2: $s = -11$**.** Then

$$p = s + 3 = -8, \qquad q = 4, \qquad r = 2. \tag{0.13}$$

$$(p, q, r, s) = (8, 4, 2, 5) \quad \text{or} \quad (p, q, r, s) = (-8, 4, 2, -11). \tag{0.14}$$