5.13.4

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Question:

Let $\mathbf{A}$ be a 2×2 matrix with non-zero entries and let $\mathbf{A}^2 = \mathbf{I}$, where $\mathbf{I}$ is 2×2 identity matrix. Define
$Tr(\mathbf{A})$- sum of diagonal elements of $\mathbf{A}$ and
$|\mathbf{A}|$- determinant of matrix $\mathbf{A}$.
Statement - 1: $Tr(\mathbf{A}) = 0$.
Statement - 2: $|\mathbf{A}| = 1$

1) Statement - 1 is true, Statement - 2 is true; Statement - 2 is not a correct explanation for Statement-1.
2) Statement - 1 is true, Statement - 2 is false.
3) Statement - 1 is false, Statement - 2 is true.
4) Statement - 1 is true, Statement - 2 is true; Statement - 2 is a correct explanation for Statement-1.

Solution:
Given,
$\mathbf{A}$ is a 2×2 matrix with non-zero entries and $\mathbf{A}^2 = \mathbf{I}$
The Cayley-Hamilton Theorem states that every square matrix satisfies its own characteristic equation.

For a 2×2 matrix $\mathbf{A}$, the characteristic equation is given by λ^2-Tr(A) λ+det(A)=0.

$$\text{By the theorem,}\mathbf{A}^2 - Tr(A)\mathbf{A} + \det(\mathbf{A})\mathbf{I} = 0 \tag{4.1}$$

Substituting $\mathbf{A}^2 = \mathbf{I}$ into the equation:

$$\mathbf{I} - Tr(A)\mathbf{A} + |\mathbf{A}|\mathbf{I} = 0 \tag{4.2}$$

$$Tr(A)\mathbf{A} = det(A)\mathbf{I} + \mathbf{I} \tag{4.3}$$

Rearranging the terms:

$$\mathbf{A} = \mathbf{I}\left(\frac{1 + det(A)}{Tr(A)}\right) \tag{4.4}$$

If the trace, Tr($\mathbf{A}$), is not zero, we would have $\mathbf{A}$=$\mathbf{I}(\frac{1+det(A)}{Tr(A)})$. This would mean $\mathbf{A}$ is a scalar multiple of the identity matrix, which contradicts the problem statement that $\mathbf{A}$

has non-zero entries.

The only way for the equation to hold true for a general matrix **A** with non-zero entries is if the coefficient of **A** on the left side is zero(see eq. 4.3) , which means Tr(**A**)=0. In this case, the right side must also be zero, so 1+det(**A**)=0

$$det(\mathbf{A}) = -1. \tag{4.5}$$

Statement - 1 is true, Statement - 2 is false.