

10.7.94

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Question:

A circle touches the X axis and also touches the circle with centre at $(0, 3)$ and radius 2. The locus of the centre of the circle is

- 1) an ellipse
- 2) a circle
- 3) a hyperbola
- 4) a parabola

Solution:

Let the center of the moving circle be

$$\mathbf{c} = \begin{pmatrix} x \\ y \end{pmatrix}$$

and its radius be r . The circle touches the X-axis, so its radius is the y-coordinate of its center.

$$r = y = \mathbf{e}_2^\top \mathbf{c} \quad (y > 0) \quad (1)$$

The fixed circle has center

$$\mathbf{c}_f = 3\mathbf{e}_2 \quad (2)$$

and radius

$$r_f = 2 \quad (3)$$

The distance between the centers of two touching circles is the sum of their radii (for external tangency).

$$\|\mathbf{c} - \mathbf{c}_f\| = r + r_f \quad (4)$$

$$\|\mathbf{c} - 3\mathbf{e}_2\| = \mathbf{e}_2^\top \mathbf{c} + 2 \quad (5)$$

Squaring both sides,

$$(\mathbf{c} - 3\mathbf{e}_2)^\top (\mathbf{c} - 3\mathbf{e}_2) = (\mathbf{e}_2^\top \mathbf{c} + 2)^2 \quad (6)$$

$$\mathbf{c}^\top \mathbf{c} - 6\mathbf{e}_2^\top \mathbf{c} + 9 = (\mathbf{e}_2^\top \mathbf{c})^2 + 4\mathbf{e}_2^\top \mathbf{c} + 4 \quad (7)$$

$$x^2 + y^2 - 6y + 9 = y^2 + 4y + 4 \quad (8)$$

$$x^2 - 10y + 5 = 0 \quad (9)$$

The locus in the standard form of the conic is

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 2 \begin{pmatrix} 0 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 5 = 0 \quad (10)$$

The matrix of the quadratic part is

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (11)$$

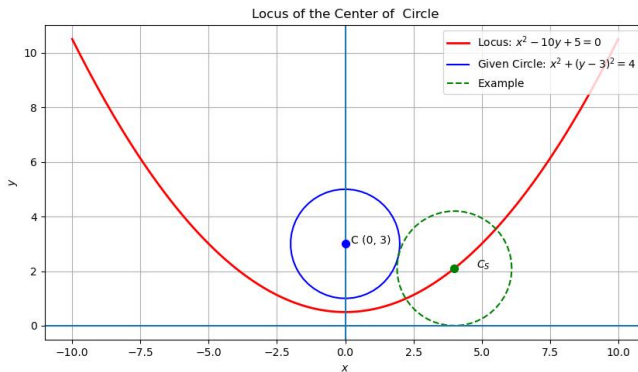
The type of conic section is determined by the eigenvalues of \mathbf{V} . For a diagonal matrix, the eigenvalues are the diagonal entries.

$$\lambda_1 = 1, \lambda_2 = 0 \quad (12)$$

$$|\mathbf{V}| = \lambda_1 \lambda_2 = 1 \cdot 0 = 0 \quad (13)$$

Since one of the eigenvalues is zero, the locus is a parabola.

The correct option is **4) a parabola.**



Plot