2.10.28

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Question

Q 2.10.28. For non-zero vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, the relation

$$\left|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}\right| = \|\mathbf{a}\| \, \|\mathbf{b}\| \, \|\mathbf{c}\|$$

holds if and only if

1) $\mathbf{a} \cdot \mathbf{b} = 0, \ \mathbf{b} \cdot \mathbf{c} = 0$
2) $\mathbf{b} \cdot \mathbf{c} = 0, \ \mathbf{c} \cdot \mathbf{a} = 0$
3) $\mathbf{c} \cdot \mathbf{a} = 0, \ \mathbf{a} \cdot \mathbf{b} = 0$
4) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a} = 0$

Solution: Let

$$A = \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{pmatrix}$$

and consider the Gram matrix of $\mathbf{a}, \mathbf{b}, \mathbf{c}$:

$$G = A^\top A = \begin{pmatrix} \mathbf{a}^\top \mathbf{a} & \mathbf{a}^\top \mathbf{b} & \mathbf{a}^\top \mathbf{c} \\ \mathbf{b}^\top \mathbf{a} & \mathbf{b}^\top \mathbf{b} & \mathbf{b}^\top \mathbf{c} \\ \mathbf{c}^\top \mathbf{a} & \mathbf{c}^\top \mathbf{b} & \mathbf{c}^\top \mathbf{c} \end{pmatrix}.$$

Since the scalar triple product equals the determinant of the column matrix,

$$\det A = \det \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{pmatrix} = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c},$$

we have

$$|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|^2 = (\det A)^2 = \det(A^\top A) = \det G.$$

$$\det G \le (\mathbf{a}^\top \mathbf{a})(\mathbf{b}^\top \mathbf{b})(\mathbf{c}^\top \mathbf{c}) = \|\mathbf{a}\|^2 \, \|\mathbf{b}\|^2 \, \|\mathbf{c}\|^2,$$

with equality iff G is diagonal, i.e.,

$$\mathbf{a} \cdot \mathbf{b} = 0, \qquad \mathbf{b} \cdot \mathbf{c} = 0, \qquad \mathbf{c} \cdot \mathbf{a} = 0.$$

Taking square roots yields

$$|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| = \|\mathbf{a}\| \, \|\mathbf{b}\| \, \|\mathbf{c}\| \iff \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a} = 0.$$

Hence, the correct option is $\boxed{\text{(d)}}$.

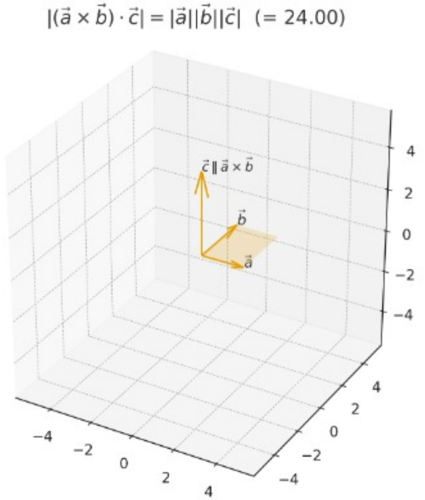

Fig. 4.1: Illustration of $|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| = |\mathbf{a}|\,|\mathbf{b}|\,|\mathbf{c}|$ with $\mathbf{a} \perp \mathbf{b}$ and $\mathbf{c} \parallel (\mathbf{a} \times \mathbf{b})$.