5.6.9

AI25BTECH11023 - Pratik R

Question

Let $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, show that $(a\mathbf{I} + b\mathbf{A})^n = a^n\mathbf{I} + na^{n-1}b\mathbf{A}$.

Solution:

Given

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{0.1}$$

calculating $\mathbf{A}^2$ we get

$$\mathbf{A}^2 = \mathbf{0} \tag{0.2}$$

Using binomial expansion

$$(a\mathbf{I} + b\mathbf{A})^n = \binom{n}{0}(a\mathbf{I})^n + \binom{n}{1}(a\mathbf{I})^{n-1}(b\mathbf{A})^1 + \binom{n}{2}(a\mathbf{I})^{n-2}(b\mathbf{A})^2 + ...\binom{n}{n}(b\mathbf{A})^n \tag{0.3}$$

Since $\mathbf{A}^2 = 0, \mathbf{A}^3 = 0, \mathbf{A}^4 = 0, ...\mathbf{A}^n = 0$

$$\therefore (a\mathbf{I} + b\mathbf{A})^n = \binom{n}{0}(a\mathbf{I})^n + \binom{n}{1}(a\mathbf{I})^{n-1}(b\mathbf{A})^1 \tag{0.4}$$

$$= a^n\mathbf{I} + na^{n-1}b\mathbf{A} \tag{0.5}$$

Hence proved.