Problem 8.3.12

Find the equation of the set of all points the sum of whose distances from the points $(3, 0)$ and $(9, 0)$ is 12.

Input Variables

Variable	Value
$\mathbf{F_1}$	$\begin{pmatrix} 3 \\ 0 \end{pmatrix}$
$\mathbf{F_2}$	$\begin{pmatrix} 9 \\ 0 \end{pmatrix}$
$2a$	12

Table 1

Solution

Step 1: Center and directions

$$\mathbf{c} = \frac{\mathbf{F_1} + \mathbf{F_2}}{2} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \tag{1}$$

$$\mathbf{p_1} = \frac{\mathbf{F_2} - \mathbf{F_1}}{\|\mathbf{F_2} - \mathbf{F_1}\|} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{p_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad P = I \tag{2}$$

Step 2: Semi-minor axis

$$c_f = \frac{\|\mathbf{F_2} - \mathbf{F_1}\|}{2} = 3 \tag{3}$$

$$a = 6 \tag{4}$$

$$b^2 = a^2 - c_f^2 = 36 - 9 = 27 \tag{5}$$

Step 3: Standard ellipse form

$$(\mathbf{x}-\mathbf{c})^\top D(\mathbf{x}-\mathbf{c}) = 1 \tag{6}$$

$$D = \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix} = \begin{pmatrix} 1/36 & 0 \\ 0 & 1/27 \end{pmatrix} \tag{7}$$

$$V = PDP^\top = D \tag{8}$$

Step 4: Convert to general quadratic form

$$(\mathbf{x}-\mathbf{c})^\top V(\mathbf{x}-\mathbf{c}) = 1 \tag{9}$$

$$\mathbf{x}^\top V\mathbf{x} - 2\mathbf{c}^\top V\mathbf{x} + \mathbf{c}^\top V\mathbf{c} - 1 = 0 \tag{10}$$

Comparing with $\mathbf{x}^\top V\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f = 0$:

$$\mathbf{u} = -V\mathbf{c} \tag{11}$$

$$f = \mathbf{c}^\top V\mathbf{c} - 1 \tag{12}$$

Compute:

$$\mathbf{u} = -\begin{pmatrix} 1/36 & 0 \\ 0 & 1/27 \end{pmatrix}\begin{pmatrix} 6 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/6 \\ 0 \end{pmatrix} \tag{13}$$

$$f = \begin{pmatrix} 6 & 0 \end{pmatrix}\begin{pmatrix} 1/36 & 0 \\ 0 & 1/27 \end{pmatrix}\begin{pmatrix} 6 \\ 0 \end{pmatrix} - 1 = 0 \tag{14}$$

Step 5: Clear denominators

$$V = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \tag{15}$$

$$\mathbf{u} = \begin{pmatrix} -18 \\ 0 \end{pmatrix} \tag{16}$$

$$f = 0 \tag{17}$$

Final Matrix Equation

$$\boxed{\mathbf{x}^\top V\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f = 0, \quad V = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix},\ \mathbf{u} = \begin{pmatrix} -18 \\ 0 \end{pmatrix},\ f = 0} \tag{18}$$

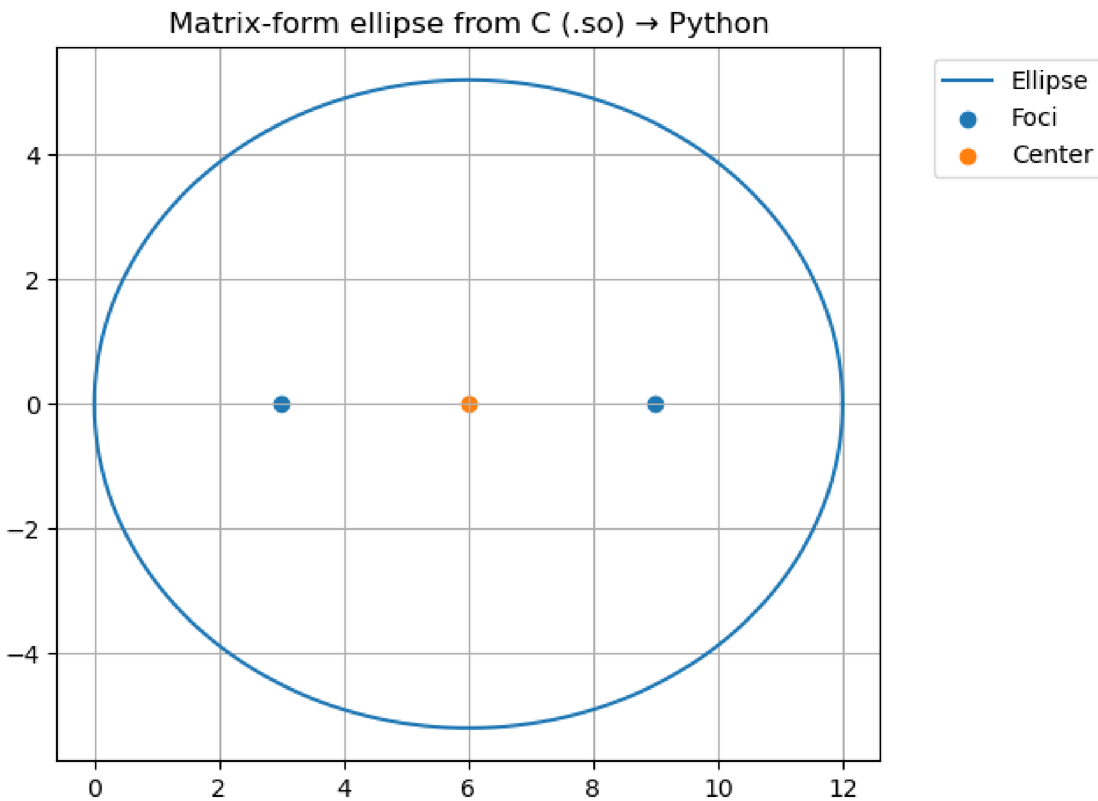
Matrix-form ellipse from C (.so) → Python
Ellipse
Foci
Center
4
2
0
−2
−4
0
2
4
6
8
10
12

Figure 1