1.6.25

AI25BTECH11010 - Dhanush Kumar

If the sum of two unit vectors is a unit vector, prove that the magnitude of their difference is $\sqrt{3}$.

Solution:

Let

$$\mathbf{u}, \mathbf{v} \in \mathbb{R}^n, \quad \|\mathbf{u}\| = 1, \ \|\mathbf{v}\| = 1. \tag{1}$$

Form the matrix

$$M = \begin{pmatrix} \mathbf{u} & \mathbf{v} \end{pmatrix}, \tag{2}$$

whose Gram matrix is

$$G = M^T M \tag{3}$$

$$= \begin{pmatrix} \mathbf{u}^T\mathbf{u} & \mathbf{u}^T\mathbf{v} \\ \mathbf{v}^T\mathbf{u} & \mathbf{v}^T\mathbf{v} \end{pmatrix} \tag{4}$$

$$= \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \tag{5}$$

where $\rho = \mathbf{u}^T\mathbf{v}$.

Now,

$$\|\mathbf{u} + \mathbf{v}\|^2 = \begin{pmatrix} 1 & 1 \end{pmatrix} G \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{6}$$

$$= \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{7}$$

$$= 2 + 2\rho. \tag{8}$$

Since $\mathbf{u} + \mathbf{v}$ is a unit vector,

$$2 + 2\rho = 1 \quad \Rightarrow \quad \rho = -\tfrac{1}{2}. \tag{9}$$

Next,

$$\|\mathbf{u} - \mathbf{v}\|^2 = \begin{pmatrix} 1 & -1 \end{pmatrix} G \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{10}$$

$$= \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{11}$$

$$= 2 - 2\rho. \tag{12}$$

Substituting $\rho = -\frac{1}{2}$,

$$\|\mathbf{u} - \mathbf{v}\|^2 = 2 - 2\left(-\frac{1}{2}\right) \tag{13}$$

$$= 3. \tag{14}$$

Hence,

$$\|\mathbf{u} - \mathbf{v}\| = \sqrt{3}. \tag{15}$$

$\therefore$ The required result is proved.

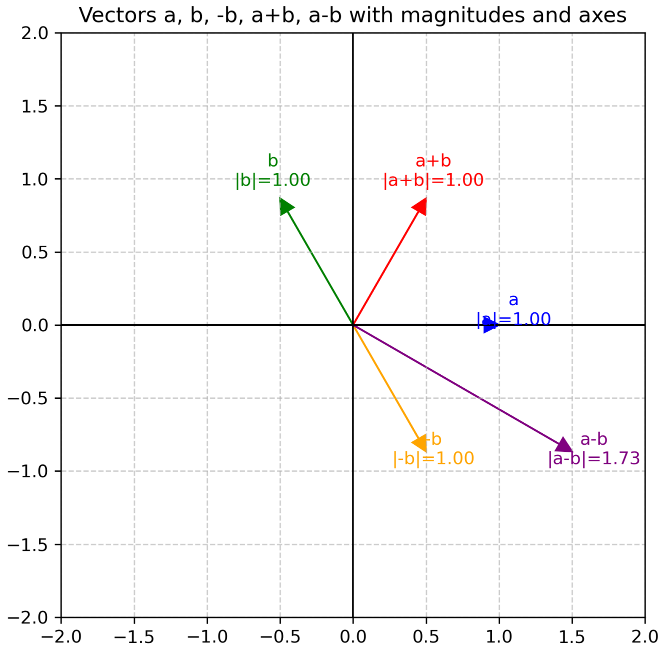

Fig. 0