Matgeo-q.2.10.25

AI25BTECH11036-SNEHAMRUDULA

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Question

Q 2.10.25. In $\triangle PQR$, let $\mathbf{a} = \overrightarrow{QR}$, $\mathbf{b} = \overrightarrow{RP}$ and $\mathbf{c} = \overrightarrow{PQ}$. If $\|\mathbf{a}\| = 12$, $\|\mathbf{b}\| = 4\sqrt{3}$, and $\mathbf{b} \cdot \mathbf{c} = 24$, then which of the following is/are true?

1. $\frac{\|\mathbf{c}\|^2}{2} - \|\mathbf{a}\| = 12$
2. $\frac{\|\mathbf{c}\|^2}{2} + \|\mathbf{a}\| = 30$
3. $\left\| \mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{a} \right\| = 48\sqrt{3}$
4. $\mathbf{a} \cdot \mathbf{b} = -72$

solutioon

$$\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0} \quad \Rightarrow \quad \mathbf{c}=-\mathbf{a}-\mathbf{b}$$

(i)

$$\mathbf{b}^T\mathbf{c}=24$$

$$\mathbf{b}^T(-\mathbf{a}-\mathbf{b})=24 \;\Rightarrow\; -\mathbf{b}^T\mathbf{a}-|\mathbf{b}|^2=24$$

$$|\mathbf{b}|^2=(4\sqrt{3})^2=48$$

$$\mathbf{a}^T\mathbf{b}=-(48+24)=-72$$

(ii)

$$|\mathbf{c}|^2=(-\mathbf{a}-\mathbf{b})^T(-\mathbf{a}-\mathbf{b})=|\mathbf{a}|^2+|\mathbf{b}|^2+2\mathbf{a}^T\mathbf{b}$$

$$|\mathbf{a}|^2=12^2=144, \quad |\mathbf{b}|^2=48, \quad \mathbf{a}^T\mathbf{b}=-72$$

$$|\mathbf{c}|^2=144+48+2(-72)=48 \qquad \frac{|\mathbf{c}|^2}{2}=24$$

$$\frac{|\mathbf{c}|^2}{2}-|\mathbf{a}|=24-12=12 \quad \Rightarrow (a) \text{ true}$$

$$\frac{|\mathbf{c}|^2}{2} + |\mathbf{a}| = 24 + 12 = 36 \quad \Rightarrow (b) \text{ false}$$

(iii)

$$\mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{a} = \mathbf{a} \times \mathbf{b} + (-\mathbf{a} - \mathbf{b}) \times \mathbf{a}$$

$$= \mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{a} - \mathbf{b} \times \mathbf{a} = \mathbf{a} \times \mathbf{b} - \mathbf{b} \times \mathbf{a}$$

$$= \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{b} = 2(\mathbf{a} \times \mathbf{b})$$

$$|\mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{a}| = 2|\mathbf{a} \times \mathbf{b}| = 2|\mathbf{a}||\mathbf{b}| \sin\theta$$

$$\cos\theta = \frac{\mathbf{a}^T\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{-72}{12 \cdot 4\sqrt{3}} = -\frac{\sqrt{3}}{2}$$

$$\sin\theta = \tfrac{1}{2}$$

$$|\mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{a}| = 48\sqrt{3} \quad \Rightarrow (c) \text{ true}$$

$$\boxed{\text{True} = \{(a), (c), (d)\}, \qquad \text{False} = \{(b)\}}$$

Graphical Representation