2.3.10

EE25BTECH11007- Aniket

Question
If $\mathbf{a}$ and $\mathbf{b}$ are unit vectors, find the angle θ between $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a} - \sqrt{2}\,\mathbf{b}$ is a unit vector.

Solution
Since $\mathbf{a}$ and $\mathbf{b}$ are unit vectors,

$$\|\mathbf{a}\| = 1 \tag{1}$$
$$\|\mathbf{b}\| = 1 \tag{2}$$

The condition that $\mathbf{a} - \sqrt{2}\,\mathbf{b}$ is also a unit vector gives

$$\left\|\mathbf{a} - \sqrt{2}\,\mathbf{b}\right\| = 1. \tag{3}$$

Squaring both sides:

$$\left\|\mathbf{a} - \sqrt{2}\,\mathbf{b}\right\|^2 = (\mathbf{a} - \sqrt{2}\,\mathbf{b})^\top(\mathbf{a} - \sqrt{2}\,\mathbf{b}) = 1. \tag{4}$$

$$(\mathbf{a} - \sqrt{2}\,\mathbf{b})^\top(\mathbf{a} - \sqrt{2}\,\mathbf{b}) = \mathbf{a}^\top(\mathbf{a} - \sqrt{2}\,\mathbf{b}) \;-\; \sqrt{2}\,\mathbf{b}^\top(\mathbf{a} - \sqrt{2}\,\mathbf{b}). \tag{5}$$

Using (1) and (2) and $\mathbf{a}^\top\mathbf{b} = \mathbf{b}^\top\mathbf{a}$:

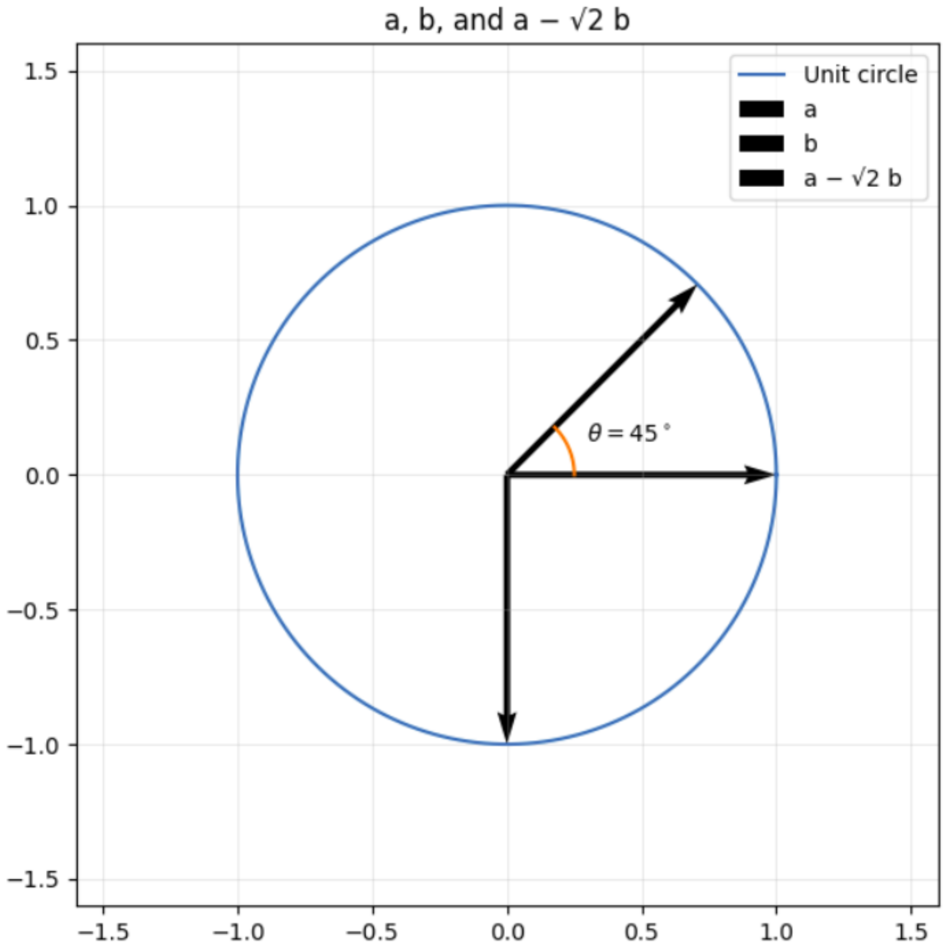
$$1 = (\mathbf{a} - \sqrt{2}\,\mathbf{b})^\top(\mathbf{a} - \sqrt{2}\,\mathbf{b}) = 1 - \sqrt{2}\,\mathbf{a}^\top\mathbf{b} \;-\; \sqrt{2}\,\mathbf{a}^\top\mathbf{b} \;+\; 2 = 3 - 2\sqrt{2}\,(\mathbf{a}^\top\mathbf{b}). \tag{6}$$

$$2\sqrt{2}\,(\mathbf{a}^\top\mathbf{b}) = 2 \implies \mathbf{a}^\top\mathbf{b} = \frac{1}{\sqrt{2}}. \tag{7}$$

Using the angle formula from dot product,

$$\cos\theta = \frac{\mathbf{a}^\top\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|} = \frac{1/\sqrt{2}}{1 \cdot 1} = \frac{1}{\sqrt{2}}, \tag{8}$$

$$\boxed{\theta = \frac{\pi}{4} = 45^\circ} \tag{9}$$


a, b, and a − √2 b
Unit circle
a
b
a − √2 b
θ = 45°
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5