

5.2.44

EE25BTECH11042 - Nipun Dasari

Question:

Solve the following system of rational equations

$$\frac{10}{x+y} + \frac{2}{x-y} = 4 \quad (0.1)$$

$$\frac{15}{x+y} - \frac{5}{x-y} = -2 \quad (0.2)$$

Solution:

$\frac{a}{x+y} + \frac{a}{x-y} = C$ becomes:

$$c(x^2 - y^2) - (a+b)x + (a-b)y = 0 \quad (0.3)$$

Matrix form: $\mathbf{x}^T V \mathbf{x} + 2\mathbf{u}^T \mathbf{x} = 0$, where:

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad V = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} -(a+b)/2 \\ (a-b)/2 \end{pmatrix} \quad (0.4)$$

The intersection points of the two hyperbolas lie on a Common chord, $c_1 H_2 - c_2 H_1 = 0$, where $H_1 = 0$ and $H_2 = 0$ are the equations of each of hyperbolas. This results in the linear equation $\mathbf{n}^T \mathbf{x} = 0$,

$$d = c_1(a_2 + b_2) - c_2(a_1 + b_1) \quad (0.5)$$

$$e = c_2(a_1 - b_1) - c_1(a_2 - b_2) \quad (0.6)$$

$$\text{where } d \text{ and } e \text{ are obtained by eliminating the quadratic terms } \mathbf{n} = \begin{pmatrix} d \\ e \end{pmatrix} \quad (0.7)$$

The solution is the non-trivial intersection point of this common chord and either hyperbola

$$y = -\frac{d}{e}x \quad (0.8)$$

$$c_1 \left(x^2 - \left(-\frac{d}{e}x \right)^2 \right) - (a_1 + b_1)x + (a_1 - b_1) \left(-\frac{d}{e}x \right) = 0 \quad (0.9)$$

$$\Rightarrow x \left(x \left(c_1 \left(\frac{e^2 - d^2}{e^2} \right) \right) - \left(\frac{e(a_1 + b_1) + d(a_1 - b_1)}{e} \right) \right) = 0 \quad (0.10)$$

$$\therefore x \left(c_1 \left(\frac{e^2 - d^2}{e^2} \right) \right) = \left(\frac{e(a_1 + b_1) + d(a_1 - b_1)}{e} \right) \quad (0.11)$$

$$\Rightarrow x = \frac{e(e(a_1 + b_1) + d(a_1 - b_1))}{c_1(e^2 - d^2)} \quad (0.12)$$

For the given system, the coefficients are:

$$a_1 = 10, b_1 = 2, c_1 = 4 \quad \text{and} \quad a_2 = 15, b_2 = -5, c_2 = -2 \quad (0.13)$$

Using (0.5) and (0.6), we calculate the common chord coefficients:

$$d = 4(15 - 5) - (-2)(10 + 2) = 4(10) + 2(12) = 64 \quad (0.14)$$

$$e = (-2)(10 - 2) - 4(15 - 5) = -2(8) - 4(20) = -96 \quad (0.15)$$

Substituting these into the formula for x from (0.12):

$$x = \frac{(-96)((-96)(12) + (64)(8))}{4((-96)^2 - (64)^2)} \quad (0.16)$$

$$= \frac{-96(-1152 + 512)}{4(9216 - 4096)} \quad (0.17)$$

$$= \frac{-96(-640)}{4(5120)} = \frac{61440}{20480} = 3 \quad (0.18)$$

Using the value of x from (0.18) in the formula for y (0.8):

$$y = -\frac{64}{-96}(3) = \frac{2}{3}3 = 2 \quad (0.19)$$

Thus, from (0.18) and (0.19), the solution is $x = 3$ and $y = 2$.



