9.6.2

EE25BTECH11052 - Shriyansh Kalpesh Chawda

Question

Find the area of the circle $4x^2 + 4y^2 = 9$ which is interior to the parabola $x^2 = 4y$.

Solution

The conic parameters for the two curves can be expressed as follows:

For the circle $4x^2 + 4y^2 - 9 = 0$:

$$\mathbf{V_1} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \mathbf{u_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, f_1 = -9. \tag{1}$$

For the parabola $x^2 - 4y = 0$:

$$\mathbf{V_2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u_2} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, f_2 = 0. \tag{2}$$

The intersection of two conics with parameters $\mathbf{V_i}$, $\mathbf{u_i}$, f_i, i = 1, 2 is defined as:

$$\mathbf{x}^\top(\mathbf{V_1} + \mu\mathbf{V_2})\mathbf{x} + 2(\mathbf{u_1} + \mu\mathbf{u_2})^\top\mathbf{x} + (f_1 + \mu f_2) = 0 \tag{3}$$

For a degenerate conic, the determinant of the quadratic part's matrix must be zero.

$$\det(\mathbf{V_1} + \mu\mathbf{V_2}) = 0 \tag{4}$$

$$\det\left(\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} + \mu\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{vmatrix} 4+\mu & 0 \\ 0 & 4 \end{vmatrix} = 4(4+\mu) = 0 \tag{5}$$

$$\mu = -4 \tag{6}$$

Substituting $\mu = -4$ in (3) :

$$\mathbf{x}^\top(\mathbf{V_1} - 4\mathbf{V_2})\mathbf{x} + 2(\mathbf{u_1} - 4\mathbf{u_2})^\top\mathbf{x} + (f_1 - 4f_2) = 0 \tag{7}$$

$$\mathbf{x}^\top\left(\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)\mathbf{x} + 2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} - 4\begin{pmatrix} 0 \\ -2 \end{pmatrix}\right)^\top\mathbf{x} + (-9 - 4(0)) = 0 \tag{8}$$

$$\mathbf{x}^\top\begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}\mathbf{x} + 2\begin{pmatrix} 0 \\ 8 \end{pmatrix}^\top\mathbf{x} - 9 = 0 \tag{9}$$

Letting $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, the equation becomes:

$$\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + 2\begin{pmatrix} 0 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - 9 = 0 \tag{10}$$

$$\implies 4y^2 + 16y - 9 = 0 \tag{11}$$

Solving for y yields $y = 1/2$ and $y = -9/2$. For the area to be interior to the parabola $x^2 = 4y$, we must have $y \geq 0$. Therefore, the intersection occurs at the line $y = 1/2$.

Substituting $y = 1/2$ into the parabola's equation:

$$x^2 = 4(1/2) = 2 \implies x = \pm\sqrt{2}. \tag{12}$$

Hence, the points of intersection are:

$$\mathbf{a_1} = \begin{pmatrix} \sqrt{2} \\ 1/2 \end{pmatrix}, \mathbf{a_2} = \begin{pmatrix} -\sqrt{2} \\ 1/2 \end{pmatrix} \tag{13}$$

The desired area of the region is given by:

$$A = \int_{-\sqrt{2}}^{\sqrt{2}} \left(\sqrt{\frac{9}{4} - x^2} - \frac{x^2}{4} \right) dx \tag{14}$$

Due to symmetry,

$$= 2\left[\int_0^{\sqrt{2}} \sqrt{\frac{9}{4} - x^2}\, dx - \int_0^{\sqrt{2}} \frac{x^2}{4}\, dx \right] \tag{15}$$

The first integral uses the standard formula for $\sqrt{a^2 - x^2}$ (from trig substitution).

$$\int_0^{\sqrt{2}} \sqrt{\frac{9}{4} - x^2}\, dx = \frac{\sqrt{2}}{4} + \frac{9}{8}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) \tag{16}$$

The second integral uses the simple power rule.

$$\int_0^{\sqrt{2}} \frac{x^2}{4}\, dx = \frac{\sqrt{2}}{6} \tag{17}$$

Substituting these results back:

$$A = 2\left[\left(\frac{\sqrt{2}}{4} + \frac{9}{8}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) \right) - \frac{\sqrt{2}}{6} \right] \tag{18}$$

$$= \frac{\sqrt{2}}{6} + \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) \tag{19}$$