EE25BTECH11001 - Aarush Dilawri

Question:

Given the points $\mathbf{A}(0,4)$ and $\mathbf{B}(0,-4)$, the equation of the locus of the point $\mathbf{p}(x,y)$, such that $(AP - BP)^2 = 6^2$

Solution:

$$\mathbf{A}, \mathbf{B}, \mathbf{P} \in \mathbb{R}^n \tag{0.1}$$

Let the given scalar be $\delta \ge 0$

$$(r_1 - r_2)^2 = \delta^2 \tag{0.2}$$

where $r_1 = ||\mathbf{P} - \mathbf{A}||$ and $r_2 = ||\mathbf{P} - \mathbf{B}||$.

Taking square root,

$$|r_1 - r_2| = \pm \delta \tag{0.3}$$

Let's define

$$D = s \delta \tag{0.4}$$

where $s \in \{+1, -1\}$ such that

$$r_1 - r_2 = s\delta = D \tag{0.5}$$

Let's find $r_1^2 - r_2^2$

$$\|\mathbf{P} - \mathbf{A}\|^2 - \|\mathbf{P} - \mathbf{B}\|^2 = (\mathbf{P} - \mathbf{A})^{\mathsf{T}} (\mathbf{P} - \mathbf{A}) - (\mathbf{P} - \mathbf{B})^{\mathsf{T}} (\mathbf{P} - \mathbf{B})$$
(0.6)

$$= -2 \mathbf{P}^{\mathsf{T}} \mathbf{u} + \mathbf{A}^{\mathsf{T}} \mathbf{A} - \mathbf{B}^{\mathsf{T}} \mathbf{B} = -2 \mathbf{P}^{\mathsf{T}} \mathbf{u} + \alpha. \tag{0.7}$$

where

$$\mathbf{u} = \mathbf{A} - \mathbf{B}$$
 and $\alpha = \mathbf{A}^{\mathsf{T}} \mathbf{A} - \mathbf{B}^{\mathsf{T}} \mathbf{B}$. (0.8)

Use $(r_1 - r_2)(r_1 + r_2) = r_1^2 - r_2^2$ and $r_1 - r_2 = D$ to get

$$D(r_1 + r_2) = -2\mathbf{P}^{\mathsf{T}}\mathbf{u} + \alpha \quad \Longrightarrow \quad r_1 + r_2 = \frac{-2\mathbf{P}^{\mathsf{T}}\mathbf{u} + \alpha}{D}. \tag{0.9}$$

Hence

$$r_1 = \frac{(r_1 - r_2) + (r_1 + r_2)}{2} = \frac{D}{2} + \frac{\alpha}{2D} - \frac{\mathbf{P}^\mathsf{T}\mathbf{u}}{D}.$$
 (0.10)

Square this expression and equate to the explicit quadratic form for r_1^2 :

$$\left(\frac{D}{2} + \frac{\alpha}{2D} - \frac{\mathbf{P}^{\mathsf{T}}\mathbf{u}}{D}\right)^{2} = (\mathbf{P} - \mathbf{A})^{\mathsf{T}}(\mathbf{P} - \mathbf{A}) = \mathbf{P}^{\mathsf{T}}\mathbf{P} - 2\mathbf{P}^{\mathsf{T}}\mathbf{A} + \mathbf{A}^{\mathsf{T}}\mathbf{A}. \tag{0.11}$$

Multiply both sides by D^2 and simplify. After collecting terms one obtains the general quadratic (conic) equation in the vector **P**:

$$\mathbf{P}^{\mathsf{T}}(\mathbf{u}\mathbf{u}^{\mathsf{T}} - D^{2}I)\mathbf{P} + (-(D^{2} + \alpha)\mathbf{u} + 2D^{2}\mathbf{A})^{\mathsf{T}}\mathbf{P} + \frac{(D^{2} + \alpha)^{2}}{4} - D^{2}\mathbf{A}^{\mathsf{T}}\mathbf{A} = 0. \quad (0.12)$$

Now substitute $\mathbf{A} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$ and $\delta = 6$.

$$\mathbf{u} = \mathbf{A} - \mathbf{B} = \begin{pmatrix} 0 \\ 8 \end{pmatrix} \tag{0.13}$$

$$\alpha = \mathbf{A}^{\mathsf{T}} \mathbf{A} - \mathbf{B}^{\mathsf{T}} \mathbf{B} = 16 - 16 = 0 \tag{0.14}$$

$$D = s \delta = \pm 6 \quad \Rightarrow \quad D^2 = 36 \tag{0.15}$$

The quadratic matrix equation becomes

$$\mathbf{P}^{\mathsf{T}}(\mathbf{u}\mathbf{u}^{\mathsf{T}} - 36I)\mathbf{P} + (-D^{2}\mathbf{u} + 2D^{2}\mathbf{A})^{\mathsf{T}}\mathbf{P} + \frac{D^{4}}{4} - 36\mathbf{A}^{\mathsf{T}}\mathbf{A} = 0$$
 (0.16)

Now compute each term.

$$\mathbf{u}\mathbf{u}^{\top} = \begin{pmatrix} 0 & 0 \\ 0 & 64 \end{pmatrix}, \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{0.17}$$

So

$$\mathbf{u}\mathbf{u}^{\mathsf{T}} - 36I = \begin{pmatrix} -36 & 0\\ 0 & 28 \end{pmatrix} \tag{0.18}$$

Next, the linear coefficient:

$$-D^{2}\mathbf{u} + 2D^{2}\mathbf{A} = -36 \begin{pmatrix} 0 \\ 8 \end{pmatrix} + 72 \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$
 (0.19)

$$= \begin{pmatrix} 0 \\ -288 \end{pmatrix} + \begin{pmatrix} 0 \\ 288 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{0.20}$$

So there is no linear term.

Finally, the constant term:

$$\frac{D^4}{4} - 36\mathbf{A}^{\mathsf{T}}\mathbf{A} = \frac{1296}{4} - 36(16) \tag{0.21}$$

$$= 324 - 576 \tag{0.22}$$

$$=-252$$
 (0.23)

Therefore, the locus is given by

$$\mathbf{P}^{\mathsf{T}} \begin{pmatrix} -36 & 0\\ 0 & 28 \end{pmatrix} \mathbf{P} - 252 = 0 \tag{0.24}$$

or equivalently

$$\mathbf{P}^{\mathsf{T}} \begin{pmatrix} -36 & 0 \\ 0 & 28 \end{pmatrix} \mathbf{P} = 252 \tag{0.25}$$

Expanding with $\mathbf{P} = \begin{pmatrix} x \\ y \end{pmatrix}$,

$$-36x^2 + 28y^2 = 252\tag{0.26}$$

Dividing through,

$$\frac{y^2}{9} - \frac{x^2}{7} = 1\tag{0.27}$$

Thus the locus is a hyperbola centered at the origin with equation

$$\frac{y^2}{9} - \frac{x^2}{7} = 1\tag{0.28}$$

See Figure,

