1.6.19

AI25BTECH11004-B.JASWANTH

Question

The vectors $\lambda\hat{i}+\lambda\hat{j}+2\hat{k}$,$1\hat{i}+\lambda\hat{j}-1\hat{k}$ and $2\hat{i}-1\hat{j}+\lambda\hat{k}$ are coplanar if $\lambda =$

Solution:

Name	vector
vector**A**	$\begin{pmatrix}\lambda\\ \lambda\\ 2\end{pmatrix}$
vector**B**	$\begin{pmatrix}1\\ \lambda\\ -1\end{pmatrix}$
vector**C**	$\begin{pmatrix}2\\ -1\\ \lambda\end{pmatrix}$

TABLE 0: variables used

Form the 3×3 matrix whose columns are the given vectors:

$$A=\begin{pmatrix}\lambda & 1 & 2\\ \lambda & \lambda & -1\\ 2 & -1 & \lambda\end{pmatrix}.$$

The three vectors are coplanar if the columns are linearly dependent, i.e. if there exists a nonzero vector $u=\begin{pmatrix}x\\ y\\ z\end{pmatrix}$ with $Au=0$. Writing $Au=0$ gives the system

$$\begin{cases}\lambda x+y+2z=0,\\ \lambda x+\lambda y-z=0,\\ 2x-y+\lambda z=0.\end{cases}$$

Subtract the first equation from the second to eliminate x:

$$(\lambda x+\lambda y-z)-(\lambda x+y+2z)=0 \implies (\lambda-1)y-3z=0 \implies z=\frac{\lambda-1}{3}y.$$

Substitute this z into the first equation to express x in terms of y:

$$\lambda x + y + 2\left(\frac{\lambda - 1}{3} y\right) = 0 \implies \lambda x + \frac{2\lambda + 1}{3} y = 0 \implies x = -\frac{2\lambda + 1}{3\lambda} y,$$

(valid when $\lambda \neq 0$; the case $\lambda = 0$ is checked separately below).

Now substitute x and z (both expressed in terms of y) into the third equation:

$$2x - y + \lambda z = 0.$$

Using $x = -\dfrac{2\lambda + 1}{3\lambda} y$ and $z = \dfrac{\lambda - 1}{3} y$ we get

$$-\frac{4\lambda + 2}{3\lambda} y - y + \frac{\lambda(\lambda - 1)}{3} y = 0.$$

Multiply through by 3λ and factor y:

$$y(\lambda^3 - \lambda^2 - 7\lambda - 2) = 0.$$

A nontrivial solution requires $y \neq 0$, hence

$$\lambda^3 - \lambda^2 - 7\lambda - 2 = 0.$$

Factor the cubic. One checks $\lambda = -2$ is a root, and polynomial division yields

$$\lambda^3 - \lambda^2 - 7\lambda - 2 = (\lambda + 2)(\lambda^2 - 3\lambda - 1).$$

The quadratic factor has roots

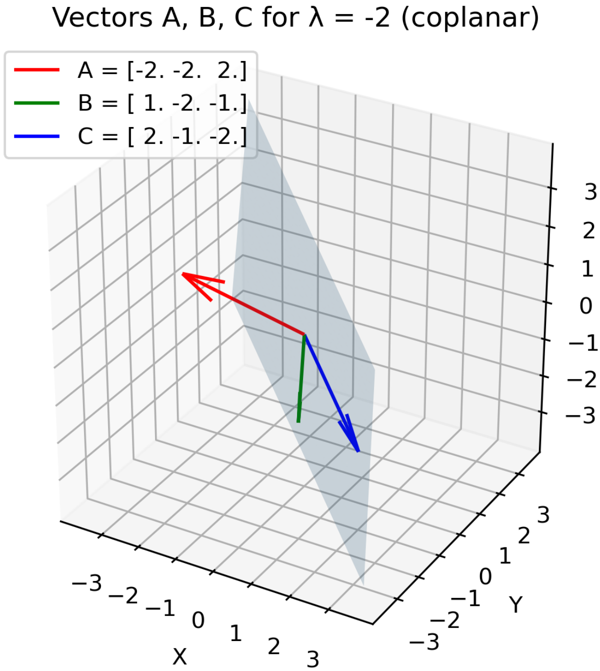
$$\lambda = \frac{3 \pm \sqrt{9 + 4}}{2} = \frac{3 \pm \sqrt{13}}{2}.$$

Finally check the special case $\lambda = 0$: the system becomes

$$\begin{cases} y + 2z = 0, \\ -z = 0, \\ 2x - y = 0, \end{cases}$$

which forces $x = y = z = 0$, so $\lambda = 0$ does *not* give a nontrivial solution.

$$\boxed{\text{Therefore the vectors are coplanar exactly for } \lambda = -2,\ \frac{3 + \sqrt{13}}{2},\ \frac{3 - \sqrt{13}}{2}.}$$


Vectors A, B, C for λ = -2 (coplanar)
A = [-2. -2. 2.]
B = [1. -2. -1.]
C = [2. -1. -2.]
X
Y
−3
−2
−1
0
1
2
3