Matrices in Geometry - 5.13.63

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Problem Statement

Let $\mathbf{M} = \begin{pmatrix} \sin^4(\theta) & -1-\sin^2(\theta) \\ 1+\cos^2(\theta) & \cos^4(\theta) \end{pmatrix} = \alpha\mathbf{I} + \beta\mathbf{M}^{-1}$

Where $\alpha = \alpha(\theta)$ and $\beta = \beta(\theta)$ are real numbers, and $\mathbf{I}$ is the 2×2 identity matrix. If α^* is the minimum of the set $(\alpha(\theta) : \theta \in [0, 2\pi))$ and β^* is the minimum of the set $(\beta(\theta) : \theta \in [0, 2\pi))$. Then the value of $\alpha^* + \beta^*$ is

(a) $-\frac{31}{16}$

(b) $-\frac{17}{16}$

(c) $-\frac{37}{16}$

(d) $-\frac{29}{16}$

Solution

Using the Cayley-Hamilton Theorem,

$$\mathbf{M}^2 - tr\left(\mathbf{M}\right)\mathbf{M} + det\left(\mathbf{M}\right)\mathbf{I} = 0 \tag{1}$$

$$\implies \mathbf{M} - tr\left(\mathbf{M}\right)\mathbf{I} + det\left(\mathbf{M}\right)\mathbf{M}^{-1} = 0 \tag{2}$$

The given expression is

$$\mathbf{M} - \alpha\mathbf{I} - \beta\mathbf{M}^{-1} = 0 \tag{3}$$

Solution

On comparing, we get

$$\alpha = tr\left(\mathbf{M}\right)\ ,\ \beta = -det\left(\mathbf{M}\right) \tag{4}$$

$$\alpha\left(\theta\right) = \sin^4\left(\theta\right) + \cos^4\left(\theta\right) = 1 - 2\sin^2\left(\theta\right)\cos^2\left(\theta\right) \tag{5}$$

$$\implies \alpha = 1 - \sin^2\left(2\theta\right)/2 \tag{6}$$

$$\alpha^* = min\left(\alpha\left(\theta\right)\right) = 1 - 1/2 = \frac{1}{2}, \tag{7}$$

$$\left(\because \text{for minimizing } \alpha,\ \sin^2\left(2\theta\right) \text{ should be maximum}\right)$$

$$\beta\left(\theta\right) = -det\left(\mathbf{M}\right) = -\left(\sin^4\left(\theta\right)\cos^4\left(\theta\right) + \sin^2\left(\theta\right)\cos^2\left(\theta\right) + 2\right) \tag{8}$$

$$\implies \beta = -\left(\left(\sin\left(2\theta\right)/2\right)^4 + \left(\sin\left(2\theta\right)/2\right)^2 + 2\right) \tag{9}$$

$$\beta^* = -\left(\left(1/2\right)^4 + \left(1/2\right)^2 + 2\right) = - - \frac{37}{16} \tag{10}$$

$$\left(\because \text{for minimizing } \beta,\ \sin^2\left(2\theta\right) \text{ should be maximum}\right)$$

Solution

Now,

$$\alpha^* + \beta^* = \frac{1}{2} - \frac{37}{16} = -\frac{29}{16} \tag{11}$$

Thus, the correct option is option (d)