12.859

EE25BTECH11013 - Bhargav

Question:
Let $\mathbf{O} = \{\mathbf{P} : \mathbf{P}$ is a 3×3 real matrix with $\mathbf{P}^T\mathbf{P} = \mathbf{I}_3, \det(\mathbf{P}) = 1\}$. Which of the following options is/are correct?

a) There exists $\mathbf{P} \in \mathbf{O}$ with $\lambda = \frac{1}{2}$ as an eigenvalue.
b) There exists $\mathbf{P} \in \mathbf{O}$ with $\lambda = 2$ as an eigenvalue.
c) If λ is the only real eigenvalue of $\mathbf{P} \in \mathbf{O}$, then $\lambda = 1$.
d) There exists $\mathbf{P} \in O$ with $\lambda = -1$ as an eigenvalue.

Solution:
Let $\mathbf{v}$ be the eigenvector corresponding to the eigenvalue λ.

$$\mathbf{P}\mathbf{v} = \lambda\mathbf{v} \tag{0.1}$$

Since orthogonal transformations preserve the length of vectors ($|\mathbf{P}| = 1$)

$$||\mathbf{P}\mathbf{v}|| = ||\mathbf{v}|| \tag{0.2}$$

$$||\mathbf{P}\mathbf{v}|| = |\lambda|\,||\mathbf{v}|| \tag{0.3}$$

Using the equations (0.2) and (0.3),

$$||\mathbf{P}\mathbf{v}|| = ||\mathbf{v}|| = |\lambda|\,||\mathbf{v}|| \tag{0.4}$$

$$\implies ||\mathbf{v}|| = |\lambda|\,||\mathbf{v}|| \tag{0.5}$$

Thus, $|\lambda| = 1$
Eigenvalues can be either -1 or 1 or both.

Thus, options (c) and (d) are correct.

This can be verified by examples.

1. For $\lambda_1 = 1$

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$\mathbf{P}^T\mathbf{P} = \mathbf{I}$

Eigenvalue of **P** is 1.

2. For $\lambda_2 = -1$

$$\mathbf{P} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$\mathbf{P}^T\mathbf{P} = \mathbf{I}$

Eigenvalue of **P** is -1.