10.6.1

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Question:

Draw a circle of radius 2.5cm. Take a point P outside the circle at a distance of 7cm from the center. Then construct a pair of tangents to the circle from point P.

Solution:

The tangent directions $\mathbf{m}$ from an external point $\mathbf{h}$ to the circle $g(\mathbf{x}) = \mathbf{x}^\top \mathbf{x} - r^2 = 0$ satisfy $\mathbf{m}^\top \mathbf{\Sigma} \mathbf{m} = 0$, where

$$\mathbf{\Sigma} = \mathbf{h}\mathbf{h}^\top - g(\mathbf{h})\,\mathbf{I} \tag{1}$$

With the point $\mathbf{h} = d\mathbf{e_1}$, we have $g(\mathbf{h}) = d^2 - r^2$. From (1),

$$\mathbf{\Sigma} = \begin{pmatrix} d^2 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} d^2 - r^2 & 0 \\ 0 & d^2 - r^2 \end{pmatrix}$$
$$= \begin{pmatrix} r^2 & 0 \\ 0 & -\left(d^2 - r^2\right) \end{pmatrix} \tag{2}$$

Since $\mathbf{\Sigma}$ is a diagonal matrix, its eigenvalues are the diagonal entries,

$$\lambda_1 = r^2 \text{ and } \lambda_2 = -\left(d^2 - r^2\right) \tag{3}$$

The matrix of orthonormal eigenvectors is

$$\mathbf{P} = \mathbf{I} \tag{4}$$

The unit direction vectors of the tangents are given by the formula:

$$\mathbf{m} = \frac{1}{\sqrt{\lambda_1 - \lambda_2}} \mathbf{P} \begin{pmatrix} \sqrt{-\lambda_2} \\ \pm\sqrt{\lambda_1} \end{pmatrix} \tag{5}$$

$$\lambda_1 - \lambda_2 = r^2 - \left(-\left(d^2 - r^2\right)\right) = d^2 \tag{6}$$
$$-\lambda_2 = d^2 - r^2 \tag{7}$$

Substituting these into (5):

$$\mathbf{m} = \frac{1}{\sqrt{d^2}}\mathbf{I}\begin{pmatrix} \sqrt{d^2 - r^2} \\ \pm\sqrt{r^2} \end{pmatrix} = \frac{1}{d}\begin{pmatrix} \sqrt{d^2 - r^2} \\ \pm r \end{pmatrix} \tag{8}$$

The points of contact are $\mathbf{q} = \mathbf{h} + \kappa\mathbf{m}$. The parameter κ is found by substituting the line equation into the circle equation:

$$g(\mathbf{h} + \kappa\mathbf{m}) = (\mathbf{h} + \kappa\mathbf{m})^\top (\mathbf{h} + \kappa\mathbf{m}) - r^2 = 0 \tag{9}$$
$$\kappa^2 \left(\mathbf{m}^\top\mathbf{m}\right) + 2\kappa\left(\mathbf{h}^\top\mathbf{m}\right) + g(\mathbf{h}) = 0 \tag{10}$$

For a tangent, this quadratic in κ has a single repeated root. Since $\mathbf{m}$ is a unit vector,

$$\mathbf{m}^\top\mathbf{m} = 1 \tag{11}$$

The value of κ for the point of contact is:

$$\kappa = \frac{-2\left(\mathbf{h}^\top\mathbf{m}\right)}{2(1)} = -\mathbf{h}^\top\mathbf{m} \tag{12}$$

We require $\kappa < 0$ for the tangent to point from $\mathbf{h}$ to the circle, which implies $\mathbf{h}^\top\mathbf{m} > 0$.

$$\mathbf{h}^\top\mathbf{m} = (d\mathbf{e_1})^\top \frac{1}{d}\begin{pmatrix} \sqrt{d^2 - r^2} \\ \pm r \end{pmatrix} = \sqrt{d^2 - r^2} > 0 \tag{13}$$

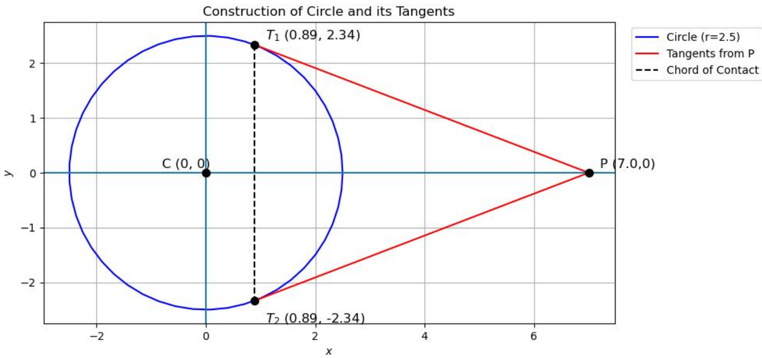
The condition is satisfied, and so $\kappa = -\sqrt{d^2 - r^2}$.

$$\mathbf{q} = d\mathbf{e_1} - \sqrt{d^2 - r^2}\left(\frac{1}{d}\begin{pmatrix} \sqrt{d^2 - r^2} \\ \pm r \end{pmatrix}\right) \tag{14}$$
$$= \begin{pmatrix} \frac{r^2}{d} \\ \mp\frac{r\sqrt{d^2 - r^2}}{d} \end{pmatrix} \tag{15}$$

Substituting $r = 2.5$ and $d = 7$:

$$\mathbf{q} = \begin{pmatrix} \frac{(2.5)^2}{7} \\ \pm\frac{2.5\sqrt{7^2-(2.5)^2}}{7} \end{pmatrix} \tag{16}$$

$$= \begin{pmatrix} \frac{6.25}{7} \\ \pm\frac{2.5\sqrt{42.75}}{7} \end{pmatrix} = \begin{pmatrix} \frac{25}{28} \\ \pm\frac{2.5\sqrt{42.75}}{7} \end{pmatrix} \tag{17}$$



Plot