Matrices in Geometry - 4.11.13

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Problem Statement

Find the equation of the plane passing through the points (2,5,-3), (-2,-3,5), and (5,3,-3). Also find the point of intersection of this plane with the line passing through points (3,1,5) and (-1,-3,-1).

Solution

Given,

The points $\mathbf{A}\begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}$, $\mathbf{B}\begin{pmatrix} -2 \\ -3 \\ 5 \end{pmatrix}$ and $\mathbf{C}\begin{pmatrix} 5 \\ 3 \\ -3 \end{pmatrix}$ which pass through a plane.

The points $\mathbf{D}\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$ and $\mathbf{E}\begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix}$ which pass through a line.

Solution

If $\mathbf{n}$ is the normal vector to the plane, then by the plane equation we can write

$$\mathbf{n}^\top \mathbf{A} = c \tag{1}$$

$$\mathbf{n}^\top \mathbf{B} = c \tag{2}$$

$$\mathbf{n}^\top \mathbf{C} = c \tag{3}$$

which can be writeen as

$$\begin{pmatrix} A & B & C \end{pmatrix}^\top \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{4}$$

Solution

Forming the augmented matrix for this

$$\left(\begin{array}{ccc|c} 2 & 5 & -3 & 1 \\ -2 & -3 & 5 & 1 \\ 5 & 3 & -3 & 1 \end{array}\right) \tag{5}$$

$$\left(\begin{array}{ccc|c} 2 & 5 & -3 & 1 \\ -2 & -3 & 5 & 1 \\ 5 & 3 & -3 & 1 \end{array}\right) \xleftrightarrow{R_1 \leftarrow \frac{R_1}{2}} \left(\begin{array}{ccc|c} 1 & \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \\ -2 & -3 & 5 & 1 \\ 5 & 3 & -3 & 1 \end{array}\right) \tag{6}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{R_2}{2}} \left(\begin{array}{ccc|c} 1 & \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \\ 0 & 1 & 1 & 1 \\ 0 & \frac{-19}{2} & \frac{9}{2} & \frac{-3}{2} \end{array}\right) \xleftrightarrow[R_1 \leftarrow R_1 - \frac{5}{2} R_2]{R_3 \leftarrow R_3 + \frac{19}{2} R_2} \left(\begin{array}{ccc|c} 1 & 0 & -4 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 14 & 8 \end{array}\right) \tag{7}$$

Solution

$$\xleftrightarrow{R_3 \leftarrow \frac{1}{14} R_3} \left(\begin{array}{ccc|c} 1 & 0 & -4 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & \frac{4}{7} \end{array}\right) \xleftrightarrow[R_2 \leftarrow R_2 - R_3]{R_1 \leftarrow R_1 + 4R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{2}{7} \\ 0 & 1 & 0 & \frac{3}{7} \\ 0 & 0 & 1 & \frac{4}{7} \end{array}\right) \tag{8}$$

Thus, multiplying by 7, the plane equation can be expressed as

$$\begin{pmatrix} 2 & 3 & 4 \end{pmatrix} \mathbf{x} = 7 \tag{9}$$

Now, the line passing through the two given points in parametric form

$$\mathbf{x} = \mathbf{P} + \lambda \mathbf{m} \tag{10}$$

where $\mathbf{P} = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$ and $\mathbf{m} = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} - \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix} \equiv \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$,

Solution

Now substituting the parametric form in the plane equation $\mathbf{n}^\top \mathbf{x} = c$ where here $\mathbf{n} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ and $c = 7$,

$$\mathbf{n}^\top (\mathbf{P} + \lambda \mathbf{m}) = c \tag{11}$$

$$\mathbf{n}^\top \mathbf{P} + \lambda \mathbf{n}^\top \mathbf{m} = c \tag{12}$$

$$\implies \lambda = \frac{c - \mathbf{n}^\top \mathbf{P}}{\mathbf{n}^\top \mathbf{m}} \tag{13}$$

Solution

So that,

$$\mathbf{x} = \mathbf{P} + \left(\frac{c - \mathbf{n}^\top \mathbf{P}}{\mathbf{n}^\top \mathbf{m}}\right)\mathbf{m} \tag{14}$$

Substituting the values in equation (14) we get the intersection point as,

$$\implies \boxed{\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}} \tag{15}$$

Conclusion

$\therefore$ The equation of the plane is $\begin{pmatrix} 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 7$ and the point of intersection of this plane with the line through the given two points is $\mathbf{P} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.