4.3.50

Namaswi-EE25BTECH11060

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Question

Let a, b, c be real numbers with $a^2 + b^2 + c^2 = 1$. Show that the equation

$$\begin{vmatrix} ax - by - c & bx + ay & cx + a \\ bx + ay & -ax + by - c & cy + b \\ cx + a & cy + b & -ax - by + c \end{vmatrix} = 0$$

represents a straight line.

Solution

Let us denote it as a 3×3 matrix:

$$M = \begin{pmatrix} ax - by - c & bx + ay & cx + a \\ bx + ay & -ax + by - c & cy + b \\ cx + a & cy + b & -ax - by + c \end{pmatrix}$$

Solution

A determinant represents a plane if it depends quadratically on x and y. Here, if we can reduce it to a determinant that is linear in x and y, it will represent a straight line.
so,Replace

$$R_3 \to R_3 - cR_1 - bR_2$$

First element of new R_3 :

$$(cx + a) - c(ax - by - c) - b(bx + ay) \tag{1}$$
$$= cx + a - cax + cby + c^2 - b^2x - aby \tag{2}$$
$$x(c - ca - b^2) + y(cb - ab) + (a + c^2) \tag{3}$$

Solution

Second element of new R_3 :

$$
\begin{aligned}
&(cy+b) - c(bx+ay) - b(-ax+by-c) && (4)\\
&= cy + b - cbx - cay + abx - b^2y + bc && (5)\\
&= x(-cb+ab) + y(c - ca - b^2) + (b+bc) && (6)
\end{aligned}
$$

Third element of new R_3 :

$$
\begin{aligned}
&(-ax - by + c) - c(cx+a) - b(cy+b) && (7)\\
&= -ax - by + c - c^2x - ac - bcy - b^2 && (8)\\
&= x(-a-c^2) + y(-b-bc) + (c - ac - b^2) && (9)
\end{aligned}
$$

Solution

But since

$$a^2 + b^2 + c^2 = 1 \implies 1 - a^2 = b^2 + c^2, \tag{10}$$

all quadratic terms cancel. Similarly, the 2nd and 3rd entries of the new R_3 become constants or linear in x, y.
The determinant now depends linearly on x and y, so we can write:

$$\det(M) = 0 \quad \implies \quad px + qy + r = 0,$$

for some real constants p, q, r.
Hence,the determinant represents a **straight line**.

C Code

```c
    #include <stdio.h>
int main() {
    // Define constants a, b, c (such that a^2 + b^2 + c^2 = 1)
    double a, b, c;
    printf("Enter values for a, b, c (a^2 + b^2 + c^2 = 1): ");
    scanf("%lf %lf %lf", &a, &b, &c);

    // Variables x and y
    double x, y;
    printf("Enter values for x and y: ");
    scanf("%lf %lf", &x, &y);
```

C Code

```c
 // Original matrix M
double M[3][3];
M[0][0] = a*x - b*y - c; M[0][1] = b*x + a*y; M[0][2] = c*x +
     a;
M[1][0] = b*x + a*y; M[1][1] = -a*x + b*y - c; M[1][2] = c*y
    + b;
M[2][0] = c*x + a; M[2][1] = c*y + b; M[2][2] = -a*x - b*y +
    c;

// Apply row operation: R3 -> R3 - c*R1 - b*R2
double newR3[3];
newR3[0] = M[2][0] - c*M[0][0] - b*M[1][0];
newR3[1] = M[2][1] - c*M[0][1] - b*M[1][1];
newR3[2] = M[2][2] - c*M[0][2] - b*M[1][2];
```

C Code

```
    // Determinant using expansion along R3 (simplified after
        operation)
    double det = M[0][0]*(M[1][1]*newR3[2] - M[1][2]*newR3[1])
               - M[0][1]*(M[1][0]*newR3[2] - M[1][2]*newR3[0])
               + M[0][2]*(M[1][0]*newR3[1] - M[1][1]*newR3[0]);

    printf("Determinant after row operation: %lf\n", det);

    // Since quadratic terms cancel, det is linear in x and y
    printf("This determinant represents a straight line in x and
        y.\n");

    return 0;
}
```