Rotation Matrix

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$$\mathbf{y} = \mathbf{P}\mathbf{x} \text{ for all } \mathbf{x} .$$

$$\|\mathbf{y}\| = \|\mathbf{x}\| \tag{1}$$
$$\|\mathbf{y}\|^2 = \|\mathbf{x}\|^2 \tag{2}$$
$$(\mathbf{P}\mathbf{x})^\top \mathbf{P}\mathbf{x} = \mathbf{x}^\top \mathbf{x} \tag{3}$$
$$\mathbf{x}^\top (\mathbf{P}^\top \mathbf{P} - \mathbf{I})\mathbf{x} = \mathbf{0} \tag{4}$$

Let $\mathbf{A} = \mathbf{P}^\top \mathbf{P} - \mathbf{I}$

$\mathbf{A}$ is symmetric because

$$(\mathbf{P}^\top \mathbf{P} - \mathbf{I})^\top = \mathbf{P}^\top \mathbf{P} - \mathbf{I}. \tag{5}$$

Using eigen-decomposition for symmetric A
By the spectral theorem, any real symmetric matrix $\mathbf{A}$ can be diagonalized by an orthogonal matrix $\mathbf{Q}$:

$$\mathbf{Q}^\top \mathbf{A} \mathbf{Q} = \mathbf{D}, \tag{6}$$

where $\mathbf{D}$ is a diagonal matrix with the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of $\mathbf{A}$.

Express $\mathbf{x}^\top \mathbf{A}\mathbf{x}$ in terms of eigenvalues
For any vector $\mathbf{x}$, define $\mathbf{y} = \mathbf{Q}^\top \mathbf{x}$. Then,

$$\mathbf{x}^\top \mathbf{A}\mathbf{x} = \mathbf{x}^\top \mathbf{Q}\mathbf{D}\mathbf{Q}^\top \mathbf{x} \tag{7}$$
$$= \mathbf{y}^\top \mathbf{D}\mathbf{y} \tag{8}$$
$$= \sum_{i=1}^{n} \lambda_i y_i^2 \tag{9}$$

Given $\mathbf{x}^\top \mathbf{A}\mathbf{x} = 0$ for all $\mathbf{x}$
Since $\mathbf{x}^\top \mathbf{A}\mathbf{x} = 0$ for every $\mathbf{x}$, it follows that

$$\sum_{i=1}^{n} \lambda_i y_i^2 = 0 \quad \text{for all } \mathbf{y}. \tag{10}$$

Show all eigenvalues must be zero
This quadratic form is zero for all $\mathbf{y}$, meaning the sum of $\lambda_i y_i^2$ is always zero regardless of $\mathbf{y}$. Select vectors $\mathbf{y}$ which are zero everywhere except in the i-th coordinate, say $\mathbf{y} = \mathbf{e}_i$ (the i-th standard basis vector):

$$\sum_{i=1}^{n} \lambda_i y_i^2 = \lambda_i \cdot 1^2 = \lambda_i = 0. \tag{11}$$

Since this holds for each i, we conclude

$$\lambda_i = 0 \quad \text{for } i = 1, 2, \dots, n. \tag{12}$$

Conclusion – A is the zero matrix
Because all eigenvalues of the symmetric matrix $\mathbf{A}$ are zero, it follows that

$$\mathbf{D} = \mathbf{0} \implies \mathbf{A} = \mathbf{Q}\mathbf{D}\mathbf{Q}^\top = \mathbf{0}. \tag{13}$$

Therefore,

$$\mathbf{P}^\top\mathbf{P} - \mathbf{I} = 0 \implies \mathbf{P}^\top\mathbf{P} = \mathbf{I}. \tag{14}$$

Therefore the rotation matrix is orthogonal

Deriving the Rotational matrix in 2D .
A rotation in $\mathbb{R}^n$ can be generated by exponentiating a skew-symmetric matrix $\mathbf{G}$:

$$\mathbf{P}(\theta) = e^{\theta\mathbf{G}}, \tag{15}$$

where

$$\mathbf{G}^\top = -\mathbf{G}, \tag{16}$$

and θ is the rotation angle.
In 2D, the fundamental skew-symmetric matrix is

$$\mathbf{G} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{17}$$

Note that $\mathbf{G}$ corresponds to a 90° rotation in the plane.
Using the Taylor series expansion of the matrix exponential,

$$e^{\theta\mathbf{G}} = \mathbf{I} + \theta\mathbf{G} + \frac{\theta^2}{2!}\mathbf{G}^2 + \frac{\theta^3}{3!}\mathbf{G}^3 + \cdots \tag{18}$$

Because

$$\mathbf{G}^2 = -\mathbf{I}, \tag{19}$$

the powers of $\mathbf{G}$ cycle as

$$\mathbf{G}^0 = \mathbf{I}, \qquad \mathbf{G}^1 = \mathbf{G}, \qquad \mathbf{G}^2 = -\mathbf{I}, \qquad \mathbf{G}^3 = -\mathbf{G}, \qquad \mathbf{G}^4 = \mathbf{I}, \text{ and so on.} \tag{20}$$

This lets us rewrite the series as

$$e^{\theta\mathbf{G}} = \mathbf{I}\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + \mathbf{G}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right). \tag{21}$$

Recognizing the Taylor series for $\cos\theta$ and $\sin\theta$, we obtain

$$e^{\theta\mathbf{G}} = \mathbf{I}\cos\theta + \mathbf{G}\sin\theta. \tag{22}$$

Substituting back $\mathbf{G}$ and $\mathbf{I}$,

$$\mathbf{P}(\theta) = \cos\theta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin\theta\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{23}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{24}$$

Doing the same thing for 3D ,
Define the axis vector and skew-symmetric generator
Let $\mathbf{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$ be a unit vector (axis of rotation) such that

$$\|\mathbf{u}\| = 1. \tag{25}$$

Define the skew-symmetric matrix $\mathbf{G}$:

$$\mathbf{G} = \begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix}. \tag{26}$$

Rotation matrix via exponential
A rotation by angle θ about $\mathbf{u}$ is given by

$$\mathbf{P}(\theta) = e^{\theta\mathbf{G}} = \sum_{n=0}^{\infty} \frac{(\theta\mathbf{G})^n}{n!}. \tag{27}$$

Powers of $\mathbf{G}$
It can be shown that

$$\mathbf{G}^2 = \mathbf{u}\mathbf{u}^\top - \mathbf{I}, \tag{28}$$

$$\mathbf{G}^3 = \mathbf{G}(\mathbf{u}\mathbf{u}^\top - \mathbf{I}) = -\mathbf{G}. \tag{29}$$

Series expansion
Expanding the exponential,

$$e^{\theta\mathbf{G}} = \mathbf{I} + \theta\mathbf{G} + \frac{\theta^2}{2!}\mathbf{G}^2 + \frac{\theta^3}{3!}\mathbf{G}^3 + \cdots \tag{30}$$

Grouping even and odd powers

$$e^{\theta\mathbf{G}} = \mathbf{I} + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)\mathbf{G} + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \cdots\right)\mathbf{G}^2. \tag{31}$$

Recognizing the Taylor series:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots , \tag{32}$$

$$1 - \cos\theta = \frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \cdots , \tag{33}$$

we obtain

$$e^{\theta\mathbf{G}} = \mathbf{I} + \sin\theta\,\mathbf{G} + (1 - \cos\theta)\mathbf{G}^2. \tag{34}$$

Substituting $\mathbf{G}^2$
Since $\mathbf{G}^2 = \mathbf{u}\mathbf{u}^\top - \mathbf{I}$,

$$\mathbf{P}(\theta) = \cos\theta\,\mathbf{I} + (1 - \cos\theta)\,\mathbf{u}\mathbf{u}^\top + \sin\theta\,\mathbf{G}. \tag{35}$$

Final Rotational Matrix
Thus, the rotation matrix about axis $\mathbf{u}$ is

$$\mathbf{P}(\theta) = \begin{pmatrix} \cos\theta + u_x^2(1 - \cos\theta) & u_x u_y(1 - \cos\theta) - u_z\sin\theta & u_x u_z(1 - \cos\theta) + u_y\sin\theta \\ u_y u_x(1 - \cos\theta) + u_z\sin\theta & \cos\theta + u_y^2(1 - \cos\theta) & u_y u_z(1 - \cos\theta) - u_x\sin\theta \\ u_z u_x(1 - \cos\theta) - u_y\sin\theta & u_z u_y(1 - \cos\theta) + u_x\sin\theta & \cos\theta + u_z^2(1 - \cos\theta) \end{pmatrix}. \tag{36}$$