Matrices in Geometry - 5.8.35

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Problem Statement

Let C be any circle with center $\left(0, \sqrt{2}\right)$. Prove that at most two rational points can be there on C.
(A rational point is a point both of whose coordinates are rational numbers).

Solution

The equation of the given circle C can be written as

$$C : \|\mathbf{x} - \mathbf{O}\| = r \tag{1}$$

where r is the radius of circle C and $\mathbf{O} = \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix}$ is the center of the circle. Let $\mathbf{P}$ be a rational point on the circle, then

$$\|\mathbf{P} - \mathbf{O}\| = r \tag{2}$$

Upon squaring both sides,

$$\|\mathbf{P} - \mathbf{O}\|^2 = r^2 \implies \mathbf{P}^\top \mathbf{P} - 2\mathbf{P}^\top \mathbf{O} + \mathbf{O}^\top \mathbf{O} = r^2 \tag{3}$$

Solution

Substituting $\mathbf{P} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{O} = \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix}$

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - 2 \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix} + \begin{pmatrix} 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix} = r^2 \tag{4}$$

$$\implies x^2 + y^2 - 2\sqrt{2}y + 2 = r^2 \tag{5}$$

Rearranging the terms,

$$x^2 + y^2 - r^2 + 2 = 2\sqrt{2}y \tag{6}$$

$$\mathbf{P} \in R^2 \implies x, y \in R \implies \text{LHS is rational} \tag{7}$$

$$\implies \text{RHS should be rational} \implies y = 0 \tag{8}$$

$$\therefore x^2 = r^2 - 2 \implies x = \pm\sqrt{r^2 - 2} : r > \sqrt{2} \tag{9}$$

Solution

We get the points

$$\mathbf{P} = \begin{pmatrix} \sqrt{r^2 - 2} \\ 0 \end{pmatrix} \text{ OR } \mathbf{P} = \begin{pmatrix} -\sqrt{r^2 - 2} \\ 0 \end{pmatrix} : r^2 - 2 \text{ is a perfect square} \quad (10)$$

This proves that at most two rational points can be present in C.
Let us try to show this using a graph with $r = \sqrt{6}$.

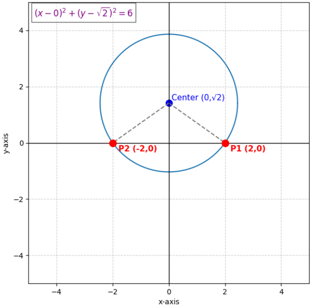

Figure: Graph for 7.4.44 with $r = \sqrt{6}$