

AI25BTECH11034 - SUJAL CHAUHAN

4.4.30

Question:

Show that four points $A(4, 5, 1)$, $B(0, -1, -1)$, $C(3, 9, 4)$, $D(-4, 4, 4)$ are coplanar.

Theory:

If N points in \mathbb{R}^3 are given as

$$X_i = (x_i, y_i, z_i), \quad i = 1, 2, \dots, N,$$

then the condition for coplanarity is that the augmented matrix

$$A = \begin{pmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ x_N & y_N & z_N & 1 \end{pmatrix} \quad (1)$$

satisfies

$$\text{rank}(A) \leq 3. \quad (2)$$

Solution:

For the given four points to be coplanar, the rank of the following matrix must be less than 4:

$$\mathbf{A} = \begin{pmatrix} 4 & 5 & 1 & 1 \\ 0 & -1 & -1 & 1 \\ 3 & 9 & 4 & 1 \\ -4 & 4 & 4 & 1 \end{pmatrix} \quad (3)$$

$$\xrightarrow{R_3 \rightarrow R_3 - \frac{3}{4}R_1, R_4 \rightarrow R_4 + R_1} \begin{pmatrix} 4 & 5 & 1 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & \frac{21}{4} & \frac{13}{4} & \frac{1}{4} \\ 0 & 9 & 5 & 2 \end{pmatrix} \quad (4)$$

$$\xrightarrow{R_1 \rightarrow \frac{1}{4}R_1} \begin{pmatrix} 1 & \frac{5}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & -1 & -1 & 1 \\ 0 & \frac{21}{4} & \frac{13}{4} & \frac{1}{4} \\ 0 & 9 & 5 & 2 \end{pmatrix} \quad (5)$$

$$\xrightarrow{R_2 \rightarrow -R_2} \begin{pmatrix} 1 & \frac{5}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 1 & -1 \\ 0 & \frac{21}{4} & \frac{13}{4} & \frac{1}{4} \\ 0 & 9 & 5 & 2 \end{pmatrix} \quad (6)$$

$$\xrightarrow{R_3 \rightarrow R_3 - \frac{21}{4}R_2, R_4 \rightarrow R_4 - 9R_2} \begin{pmatrix} 1 & \frac{5}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -2 & \frac{22}{4} \\ 0 & 0 & -4 & 11 \end{pmatrix} \quad (7)$$

$$\xrightarrow{R_3 \rightarrow -\frac{1}{2}R_3} \begin{pmatrix} 1 & \frac{5}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -\frac{11}{4} \\ 0 & 0 & -4 & 11 \end{pmatrix} \quad (8)$$

$$\xrightarrow{R_4 \rightarrow R_4 + 4R_3} \begin{pmatrix} 1 & \frac{5}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -\frac{11}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (9)$$

Thus,

$$\text{rank}(\mathbf{A}) = 3 < 4 \implies \text{The given points are coplanar.} \quad (10)$$

4 points in 3D — coplanarity test (augmented-rank method)

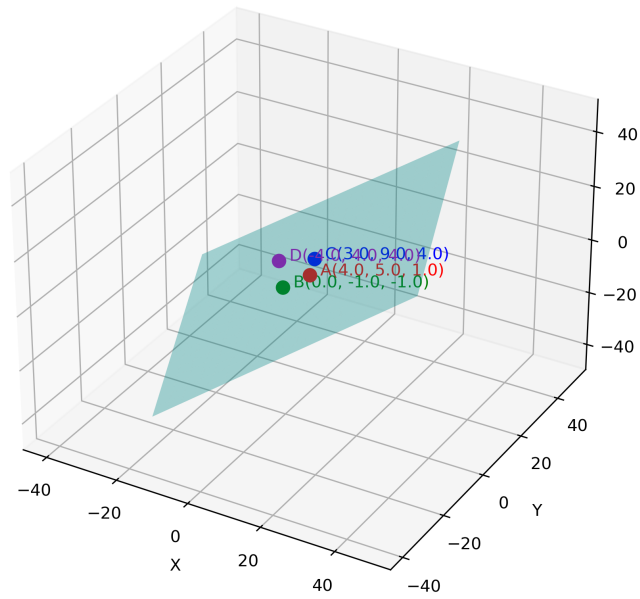


Figure 1: Geometric visualization of points A, B, C, D lying on the same plane.