8.4.35

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Question

The ellipse $E_1: \frac{x^2}{9} + \frac{y^2}{4} = 1$ is inscribed in a rectangle $\mathbf{R}$ whose sides are parallel to coordinate axes. another ellipse E_2 passing through the point $(0, 4)$ circumscribes the rectangle $\mathbf{R}$. The eccentricity of the ellipse E_2 is

1. $\frac{\sqrt{2}}{2}$
2. $\frac{\sqrt{3}}{2}$
3. $\frac{1}{2}$
4. $\frac{3}{4}$

Theoretical Solution

Given for E_1 :

	Input			Output		
Conic	**V**	**u**	**f**	**F**	**Directrix**	**Latus Rectu**
$\frac{x^2}{9}+\frac{y^2}{4}=1$	$\begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}$	$\mathbf{0}$	-36	$\pm\sqrt{5}\mathbf{e_1}$	$\mathbf{e_1}^\top\mathbf{x}=\pm\frac{9}{\sqrt{5}}$	$\frac{8}{3}$

Table: 8.4.35

Theoretical Solution

E_1 is inscribed in Rectangle
$\therefore$ The coordinates of Vertices of Rectangles are intersection points of,

$$\mathbf{e_1}^\top \mathbf{x} = 3 \tag{1}$$
$$\mathbf{e_2}^\top \mathbf{x} = 2 \tag{2}$$
$$\mathbf{e_1}^\top \mathbf{x} = -3 \tag{3}$$
$$\mathbf{e_2}^\top \mathbf{x} = -2 \tag{4}$$

Let $\mathbf{P}$ intersection of 1 and 2,

$$\mathbf{P} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \tag{5}$$

Let ,

$$\mathbf{Q} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \tag{6}$$

General Form of Conic equation is given as ,

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{7}$$

Since the rectangle in inscribed inside E_2 , E_2 is an ellipse centered at origin.
For a conic centered at Origin

$$\mathbf{u} = \mathbf{0} \tag{8}$$

$$\therefore \mathbf{x}^\top \mathbf{V}'\mathbf{x} = 1 \tag{9}$$

where, $\mathbf{V}' = -\frac{1}{f}\mathbf{V}$

Theoretical Solution

Since E_2 has axes parallel to coordinate axes , $\mathbf{V}'$ must be diagonal.

$$\mathbf{V}' = \begin{pmatrix} v_{11} & 0 \\ 0 & v_{22} \end{pmatrix} \tag{10}$$

Now, E_2 passes through $\mathbf{Q}$ and $\mathbf{P}$

$$\mathbf{Q}^\top \mathbf{V}' \mathbf{Q} = 1 \tag{11}$$
$$\mathbf{P}^\top \mathbf{V}' \mathbf{P} = 1 \tag{12}$$

Theoretical Solution

we get following system of eqn ,

$$\begin{pmatrix} 9 & 4 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{22} \end{pmatrix} = 1 \tag{13}$$

$$\begin{pmatrix} 0 & 16 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{22} \end{pmatrix} = 1 \tag{14}$$

Theoretical Solution

$$\left(\begin{array}{cc|c} 9 & 4 & 1 \\ 0 & 16 & 1 \end{array}\right) \xleftrightarrow{R_1 \rightarrow \frac{R_1}{9}} \left(\begin{array}{cc|c} 1 & \frac{4}{9} & \frac{1}{9} \\ 0 & 16 & 1 \end{array}\right) \tag{15}$$

$$\left(\begin{array}{cc|c} 1 & \frac{4}{9} & \frac{1}{9} \\ 0 & 16 & 1 \end{array}\right) \xleftrightarrow{R_2 \rightarrow \frac{R_2}{16}} \left(\begin{array}{cc|c} 1 & \frac{4}{9} & \frac{1}{9} \\ 0 & 1 & \frac{1}{16} \end{array}\right) \tag{16}$$

Theoretical Solution

$$\left(\begin{array}{cc|c} 1 & \frac{4}{9} & \frac{1}{9} \\ 0 & 1 & \frac{1}{16} \end{array}\right) \xleftrightarrow{R_1 \to R_1 - \frac{4}{9}R_2} \left(\begin{array}{cc|c} 1 & 0 & \frac{1}{12} \\ 0 & 1 & \frac{1}{16} \end{array}\right) \tag{17}$$

On solving , We get

$$\mathbf{V}' = \begin{pmatrix} \frac{1}{12} & 0 \\ 0 & \frac{1}{16} \end{pmatrix} \tag{18}$$

$$\mathbf{V} = \|n\|^2 \mathbf{I} - e^2 \mathbf{n}^\top \mathbf{n} \tag{19}$$

Here, $\mathbf{n} = \mathbf{e_2}$

Theoretical Solution

Substituting the values, we get

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 - e^2 \end{pmatrix} \tag{20}$$

Now,

$$\mathbf{V}' = k\mathbf{V} \tag{21}$$

$$\begin{pmatrix} \frac{1}{12} & 0 \\ 0 & \frac{1}{16} \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k\left(1 - e^2\right) \end{pmatrix} \tag{22}$$

$$\tag{23}$$

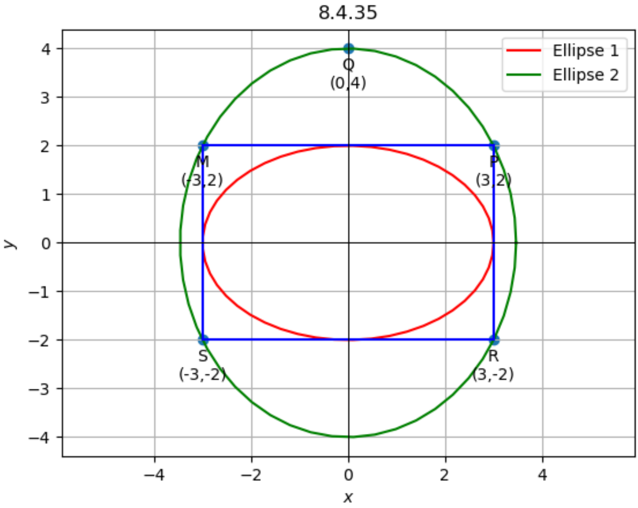
Theoretical Solution

$$\implies 1 - e^2 = \frac{12}{16} \tag{24}$$

$$\implies e = \frac{1}{2} \tag{25}$$

Hence Answer : Option (3) $\frac{1}{2}$

Plot


8.4.35
Ellipse 1
Ellipse 2
Q
(0,4)
M
(-3,2)
P
(3,2)
S
(-3,-2)
R
(3,-2)
4
3
2
1
0
−1
−2
−3
−4
y
−4
−2
0
2
4
x