8.4.28

EE25BTECH11025 - Ganachari Vishwambhar

Question:
The axis of the parabola is along the line $y = x$ and trhe distance of its vertex and focus from origin are $\sqrt{2}$ and $2\sqrt{2}$ respectively. If the vertex and focus both lie in the first quadrant, then the equation of the parabola is

1) $(x+y)^2 = (x-y-2)$
2) $(x-y)^2 = (x+y-2)$
3) $(x-y)^2 = 4(x+y-2)$
4) $(x-y)^2 = 8(x+y-2)$

Solution:
Let:
Focus of the parabola be $\mathbf{F}$
Vertex of the parabola be $\mathbf{V}$
Normal vector to the directrix be $\mathbf{n}$
The point of intersection of directrix and axis be $\mathbf{P}$
Direction vector and slope of axis be $\mathbf{m}_1$ and m_1
Direction vector and slope of directrix be $\mathbf{m}_2$ and m_2
Equation of axis be $\mathbf{x} = \lambda\mathbf{m}_1$
Given:

$$\|\mathbf{F}\| = 2\sqrt{2} \tag{1}$$

$$\|\mathbf{V}\| = \sqrt{2} \tag{2}$$

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ m_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3}$$

Finding focus($\mathbf{F}$):

$$\lambda\mathbf{m}_1 = \mathbf{F} \tag{4}$$

$$\lambda = \pm\frac{\|\mathbf{F}\|}{\|\mathbf{m}_1\|} \tag{5}$$

$$\mathbf{F} = \pm\frac{\|\mathbf{F}\|}{\|\mathbf{m}_1\|}\mathbf{m}_1 \tag{6}$$

Finding vertex($\mathbf{V}$)

$$\lambda\mathbf{m}_1 = \mathbf{V} \tag{7}$$

$$\lambda = \pm\frac{\|\mathbf{V}\|}{\|\mathbf{m}_1\|} \tag{8}$$

$$\mathbf{V} = \pm\frac{\|\mathbf{V}\|}{\|\mathbf{m}_1\|}\mathbf{m}_1 \tag{9}$$

Since directrix will be perpendicular to axis $m_1 m_2 = -1$

$$\mathbf{m_2} = \begin{pmatrix} 1 \\ m_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{-1}{m_1} \end{pmatrix} \tag{10}$$

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m_2} \tag{11}$$

Since $\mathbf{V}$ will be midpoint of $\mathbf{P}$ and $\mathbf{F}$ and also $\mathbf{P}$:

$$\frac{\mathbf{P} + \mathbf{F}}{2} = \mathbf{V} \tag{12}$$

$$\mathbf{P} = 2\mathbf{V} - \mathbf{F} \tag{13}$$

Now, finding the directrix equation in normal form;

$$\mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{P} \tag{14}$$

From equation of conic $\mathbf{x}^\top A \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0$
For parabola:

$$A = \|\mathbf{n}\|^2 I - \mathbf{n}\mathbf{n}^\top \tag{15}$$

$$\mathbf{u} = c\mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F} = \mathbf{n}^2 \mathbf{P}\mathbf{n} - \|\mathbf{n}\|^2 \frac{\|\mathbf{F}\|}{\|\mathbf{m}_1\|} \mathbf{m}_1 \tag{16}$$

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - \left(\mathbf{n}^\top \mathbf{P}\right)^2 \tag{17}$$

Substituting values given in question we get:

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{18}$$

$$\mathbf{u} = -4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{19}$$

$$f = 16 \tag{20}$$

Substituting (18), (19) and (20) in conic equation we get:

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + (-4) \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 16 = 0 \tag{21}$$

$$(x + y)^2 = 8(x + y - 2) \tag{22}$$

Hnece, option(4) is correct.

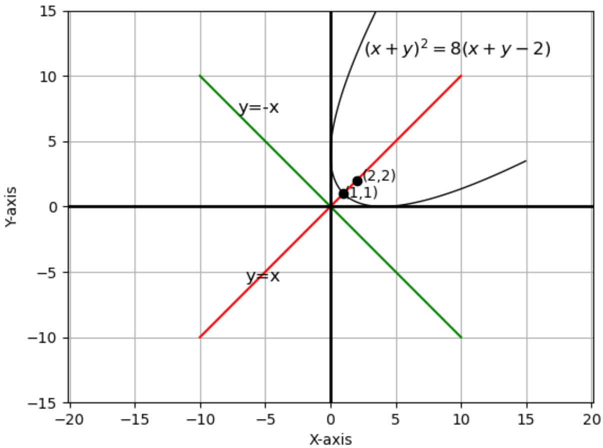

Fig. 1: Plot of the parabola