

9.4.39

EE25BTECH11026-Harsha

Question:

The difference of squares of two numbers is 180. The square of the smaller number is 8 times the larger number. Find the two numbers.

Solution:

Let us solve the given question theoretically and then verify the solution computationally.

Let x and y be the 2 numbers such that $x > y$.

The given equations are,

$$x^2 - y^2 = 180 \quad (0.1)$$

$$y^2 = 8x \quad (0.2)$$

As the given equations are homogeneous, converting them into quadratic form,

$$\Rightarrow \mathbf{x}^T \mathbf{F}_1 \mathbf{x} + c = 0 \quad (0.3)$$

where $\mathbf{x}^T = (x \quad y)$ and $\mathbf{F}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $c = -180$

And also,

$$\mathbf{x}^T \mathbf{F}_2 \mathbf{x} + 2\mathbf{b}^T \mathbf{x} = 0 \quad (0.4)$$

where $\mathbf{x}^T = (x \quad y)^T$, $\mathbf{F}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -4 \\ 0 \end{pmatrix}$.

To identify the intersection of conics, we can employ the approach of degenerating the conics.

To work with degeneracy in matrix form we form the standard augmented 3×3 matrix for each conic:

$$\mathbf{M}_i = \begin{pmatrix} \mathbf{F}_i & \mathbf{b}_i \\ \mathbf{b}_i^T & c_i \end{pmatrix} \quad (0.5)$$

$$\Rightarrow \mathbf{M}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -180 \end{pmatrix} \quad \mathbf{M}_2 = \begin{pmatrix} 0 & 0 & -4 \\ 0 & 1 & 0 \\ -4 & 0 & 0 \end{pmatrix} \quad (0.6)$$

$$\therefore \mathbf{x}^T (\mathbf{M}_1 + \lambda \mathbf{M}_2) \mathbf{x} = 0 \quad (0.7)$$

To degenerate the conic into a line, we can find the solutions of λ when $\|\mathbf{M}_1 + \lambda\mathbf{M}_2\| = 0$

$$\therefore \|\mathbf{M}_1 + \lambda\mathbf{M}_2\| = 0 \quad (0.8)$$

$$\Rightarrow (\lambda - 1)(4\lambda^2 + 45) = 0 \quad (0.9)$$

$$\therefore \lambda = 1 \quad (0.10)$$

Substituting λ in the equation,

$$\mathbf{x}^\top (\mathbf{M}_1 + \mathbf{M}_2) \mathbf{x} \quad (0.11)$$

$$\Rightarrow x^2 - 8x - 180 = 0 \quad (0.12)$$

$$\Rightarrow x = 18, -10 \quad (0.13)$$

for $x = -10$, there is no real solution of y ,

$$\Rightarrow y = \pm 12 \quad (0.14)$$

$$\therefore \text{The two numbers are } (18, 12) \text{ and } (18, -12) \quad (0.15)$$

From the figure, it is clearly verified that the theoretical solution matches with the computational solution.

