4.12.26

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The line $\frac{x}{a} + \frac{y}{b} = 1$ moves in such a way that the condition

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$$

is satisfied, where c is a constant.
Prove that the locus of the foot of the perpendicular from the origin on the given line is $x^2 + y^2 = c^2$, using vector algebra with matrix notation and transpose.

Step 1: Vector Representation

Vector and Perpendicularity Conditions

1. **Line Equation:** The line L is $\frac{x}{a} + \frac{y}{b} = 1$.

$$L : \mathbf{r}^T \mathbf{n} = 1, \quad \text{where } \mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 1/a \\ 1/b \end{pmatrix} \quad \text{(Equation 1)}$$

2. **Foot of Perpendicular P:** Let $P(x_0, y_0)$ have position vector $\mathbf{p} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$.

3. **Perpendicularity $\mathbf{p} \parallel \mathbf{n}$:** The vector $\mathbf{p}$ is parallel to the normal $\mathbf{n}$.

$$\mathbf{p} = \lambda \mathbf{n} \quad \text{for some scalar } \lambda \quad \text{(Equation 2)}$$

This gives the component relations:

$$\frac{1}{a} = \frac{x_0}{\lambda} \quad \text{and} \quad \frac{1}{b} = \frac{y_0}{\lambda} \quad \text{(Equation 3)}$$

Substituting into the Line and Constraint Equations

4. **P lies on L:** Substitute $\mathbf{p}$ into the line equation (Eq. 1):

$$\mathbf{p}^T\mathbf{n} = 1 \implies \frac{x_0}{a} + \frac{y_0}{b} = 1$$

Substitute (Eq. 3) into the above:

$$\frac{x_0^2}{\lambda} + \frac{y_0^2}{\lambda} = 1 \implies x_0^2 + y_0^2 = \lambda \quad \text{(Equation 4)}$$

In matrix form: $\mathbf{p}^T\mathbf{p} = \lambda$.

5. **Apply the Constraint:** Substitute (Eq. 3) into the given condition $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$:

$$\left(\frac{x_0}{\lambda}\right)^2 + \left(\frac{y_0}{\lambda}\right)^2 = \frac{1}{c^2}$$

$$\frac{x_0^2 + y_0^2}{\lambda^2} = \frac{1}{c^2} \quad \text{(Equation 5)}$$

6. **Solve for Locus:** Substitute λ from (Eq. 4) into (Eq. 5):

C Code: Locus Verification Function

Main Formula: $x^2 + y^2 = c^2$

The main formula is the equation of the locus. The C function verifies if a given point (x, y) lies on this locus for a constant c.

```c
#include <stdio.h>
#include <math.h>

int verify_locus(double x, double y, double c) {
    double lhs = (x * x) + (y * y);
    double rhs = c * c;
    double tolerance = 1e-9;

    if (fabs(lhs - rhs) < tolerance) {
        return 1;
    } else {
        return 0;
    }
}
```

```c
int main() {
    double constant_c = 5.0; // Locus is x^2 + y^2 = 25

    double x1 = 3.0; // Point (3, 4) is on the locus
    double y1 = 4.0;

    if (verify_locus(x1, y1, constant_c)) {
        printf(Point (%.2f, %.2f) IS on the locus x^2 + y^2 = %.2f\n,
               x1, y1, constant_c * constant_c);
    } else {
        printf(Point (%.2f, %.2f) is NOT on the locus.\n, x1, y1);
    }

    double x2 = 1.0; // Point (1, 1) is not on the locus
    double y2 = 1.0;
```

Python Code: Setup and Locus Data

Generating Data for the Plot

We use $c = 5$ and an example line with intercepts $a = 25/3$ and $b = 25/4$.

```
import numpy as np
import matplotlib.pyplot as plt

# 1. Define the constant c and the locus (x^2 + y^2 = c^2)
c = 5.0
c_squared = c * c

# Generate points for the circle (Locus)
theta = np.linspace(0, 2 * np.pi, 100)
x_locus = c * np.cos(theta)
y_locus = c * np.sin(theta)

# 2. Define a specific variable line and its foot of the
    perpendicular P(x0, y0)
x0 = 3.0
y0 = 4.0
```

Python Code: Plotting the Graphs

Visualization of the Locus and an Example Line

```python
# 3. Create the plot
plt.figure(figsize=(8, 8))

# Plot the Locus (Circle)
plt.plot(x_locus, y_locus, 'r--', label=f'Locus: $x^2 + y^2 = {c_squared:.0f}$')

# Plot the Variable Line L
plt.plot(x_line, y_line, 'b-', label=f'Variable Line L: $\\frac{{x}}{{{a:.2f}}} + \\frac{{y}}{{{b:.2f}}} = 1$')

# Plot the Foot of the Perpendicular P and Origin O
plt.plot(x0, y0, 'go', markersize=8, label=f'Foot P({x0:.0f}, {y0:.0f})')
plt.plot([0, x0], [0, y0], 'g-', linestyle=':', linewidth=2, label='Perpendicular Segment $\\mathbf{OP}$')
plt.plot(0, 0, 'kx', markersize=8, label='Origin O(0, 0)')
```

Plot

figs/python_plot.png