Problem 5.3.26

For what value of k, does the system of linear equations

$$2x + 3y = 7, \qquad (k-1)x + (k+2)y = 3k \tag{1}$$

have an infinite number of solutions?

Input Variables and Vectors

Symbol	Description	Value/Expression
x, y	Unknown variables	Real numbers
k	Parameter in system	To be determined
$\mathbf{x}$	Unknown vector	$\begin{pmatrix} x \\ y \end{pmatrix}$
A	Coefficient matrix	$\begin{pmatrix} 2 & 3 \\ k-1 & k+2 \end{pmatrix}$
$\mathbf{b}$	RHS vector	$\begin{pmatrix} 7 \\ 3k \end{pmatrix}$
$[A\|b]$	Augmented matrix	$\begin{pmatrix} 2 & 3 & 7 \\ k-1 & k+2 & 3k \end{pmatrix}$

Table 1

Solution

$$\begin{pmatrix} 2 & 3 \\ k-1 & k+2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 7 \\ 3k \end{pmatrix}, \quad \text{where } \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}. \tag{2}$$

$$\text{The augmented matrix is } \begin{pmatrix} 2 & 3 & 7 \\ k-1 & k+2 & 3k \end{pmatrix}. \tag{3}$$

$$R_2 \to R_2 - \tfrac{k-1}{2} R_1 \tag{4}$$

$$\begin{pmatrix} 2 & 3 & 7 \\ k-1 & k+2 & 3k \end{pmatrix} \to \begin{pmatrix} 2 & 3 & 7 \\ 0 & (k+2) - \tfrac{3}{2}(k-1) & 3k - \tfrac{7}{2}(k-1) \end{pmatrix}. \tag{5}$$

$$= \begin{pmatrix} 2 & 3 & 7 \\ 0 & \frac{-k+7}{2} & \frac{-k+7}{2} \end{pmatrix}. \tag{6}$$

$$\text{For infinite solutions: } \operatorname{rank}(A) = \operatorname{rank}([A|b]) < 2. \tag{7}$$

$$\tfrac{-k+7}{2} = 0 \quad \Longrightarrow \quad k = 7. \tag{8}$$

$$\text{When } k = 7, \quad \begin{pmatrix} 2 & 3 & 7 \\ 0 & 0 & 0 \end{pmatrix}, \tag{9}$$

$$\operatorname{rank}(A) = \operatorname{rank}([A|b]) = 1 < 2. \tag{10}$$

$$\therefore \quad \text{The system has infinitely many solutions when } \boxed{k = 7}. \tag{11}$$

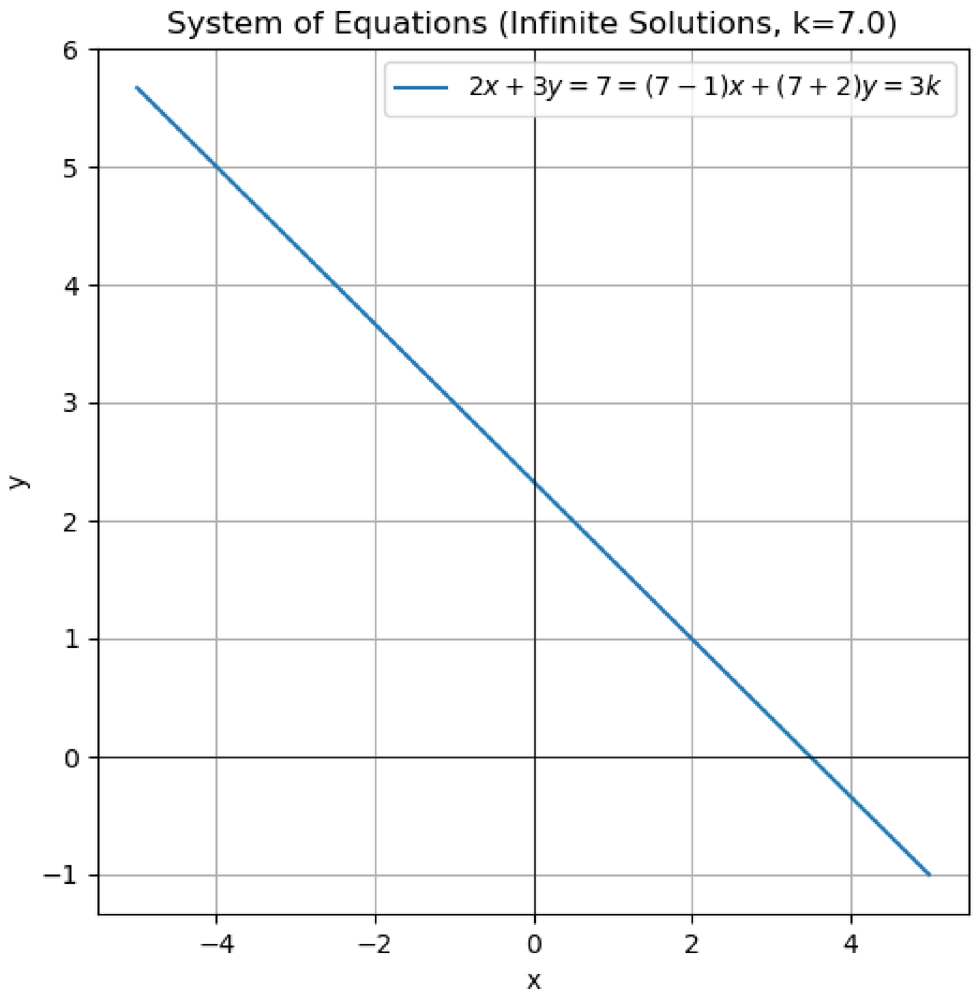
System of Equations (Infinite Solutions, k=7.0)
$2x + 3y = 7 = (7 - 1)x + (7 + 2)y = 3k$
6
5
4
3
2
1
0
−1
y
−4
−2
0
2
4
x

Figure 1