

4.11.23

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Question:

Find the co-ordinates of the point where the line $\frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6}$ crosses the plane passing through the points $(\frac{7}{2}, 0, 0)$, $(0, 7, 0)$, and $(0, 0, 7)$.

Solution:

For the intersection of a line

$$\mathbf{x} = \mathbf{p} + \lambda \mathbf{m}$$

with the plane

$$\mathbf{n}^\top \mathbf{x} = c$$

$$\mathbf{n}^\top (\mathbf{p} + \lambda \mathbf{m}) = c \quad (1)$$

$$\mathbf{n}^\top \mathbf{p} + \lambda \mathbf{n}^\top \mathbf{m} = c \quad (2)$$

$$\lambda = \frac{c - \mathbf{n}^\top \mathbf{p}}{\mathbf{n}^\top \mathbf{m}} \quad (3)$$

$$\mathbf{x} = \mathbf{p} + \left(\frac{c - \mathbf{n}^\top \mathbf{p}}{\mathbf{n}^\top \mathbf{m}} \right) \mathbf{m} \quad (4)$$

Let the equation of the plane be

$$\begin{pmatrix} n_1 & n_2 & n_3 \end{pmatrix} \mathbf{x} = c \quad (5)$$

The three points

$$\mathbf{P}_1 = \begin{pmatrix} \frac{7}{2} \\ 0 \\ 0 \end{pmatrix}, \mathbf{P}_2 = \begin{pmatrix} 0 \\ 7 \\ 0 \end{pmatrix}, \mathbf{P}_3 = \begin{pmatrix} 0 \\ 0 \\ 7 \end{pmatrix}$$

satisfy this equation, giving the system:

$$\frac{7}{2}n_1 + 0n_2 + 0n_3 = c \quad (6)$$

$$0n_1 + 7n_2 + 0n_3 = c \quad (7)$$

$$0n_1 + 0n_2 + 7n_3 = c \quad (8)$$

This system of equations gives the augmented matrix

$$\left(\begin{array}{ccc|c} \frac{7}{2} & 0 & 0 & c \\ 0 & 7 & 0 & c \\ 0 & 0 & 7 & c \end{array} \right) \xrightarrow[\mathbf{R}_3 \rightarrow \frac{1}{7}\mathbf{R}_3]{\mathbf{R}_1 \rightarrow \frac{2}{7}\mathbf{R}_1, \mathbf{R}_2 \rightarrow \frac{1}{7}\mathbf{R}_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{2c}{7} \\ 0 & 1 & 0 & \frac{c}{7} \\ 0 & 0 & 1 & \frac{c}{7} \end{array} \right) \quad (9)$$

From the row-reduced echelon form,

$$n_1 = \frac{2c}{7}, n_2 = \frac{c}{7}, n_3 = \frac{c}{7} \quad (10)$$

Substituting (10) in (5),

$$\left(\frac{2c}{7} \quad \frac{c}{7} \quad \frac{c}{7}\right)\mathbf{x} = c \quad (11)$$

Assuming $c \neq 0$, the equation simplifies to the normal form of the plane, $\mathbf{n}^\top \mathbf{x} = c$, which is

$$(2 \quad 1 \quad 1)\mathbf{x} = 7 \quad (12)$$

The vector equation of the line passing through

$$\mathbf{p} = \begin{pmatrix} 3 \\ -4 \\ -5 \end{pmatrix}$$

with direction vector

$$\mathbf{m} = \begin{pmatrix} -1 \\ 1 \\ 6 \end{pmatrix}$$

is

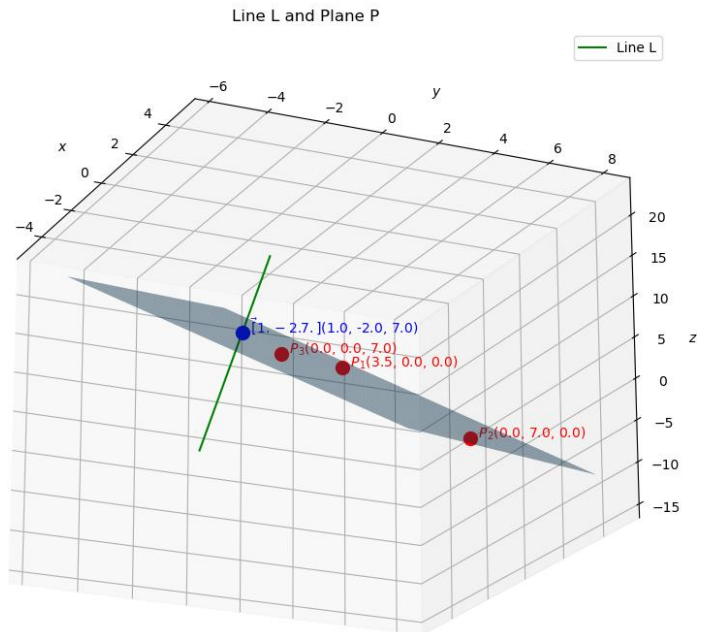
$$\mathbf{x} = \begin{pmatrix} 3 \\ -4 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 6 \end{pmatrix} \quad (13)$$

Using (4),

$$\mathbf{x} = \begin{pmatrix} 3 \\ -4 \\ -5 \end{pmatrix} + \frac{7 - (2 \quad 1 \quad 1)\begin{pmatrix} 3 \\ -4 \\ -5 \end{pmatrix}}{(2 \quad 1 \quad 1)\begin{pmatrix} -1 \\ 1 \\ 6 \end{pmatrix}} \begin{pmatrix} -1 \\ 1 \\ 6 \end{pmatrix} \quad (14)$$

$$\mathbf{x} = \begin{pmatrix} 3 \\ -4 \\ -5 \end{pmatrix} + 2 \begin{pmatrix} -1 \\ 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 7 \end{pmatrix} \quad (15)$$

The co-ordinates of the point of intersection are $\begin{pmatrix} 1 \\ -2 \\ 7 \end{pmatrix}$.



Plot