

12.391

AI25BTECH11001 - ABHISEK MOHAPATRA

Question: The directional derivative of the function

$$f(x, y) = \frac{x^2 + xy^2}{\sqrt{5}} \quad (1)$$

in the direction

$$\mathbf{d} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} \quad (2)$$

at $\mathbf{X} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is

- a) $-\frac{1}{\sqrt{5}}$
- b) $-\frac{2}{\sqrt{5}}$
- c) 0
- d) $-\frac{1}{3}$

Solution:

Let $\mathbf{R} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$ be a rotation matrix such that $\mathbf{R}\mathbf{d} = \mathbf{e}_1$.

$$\Rightarrow \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3)$$

$$2 \cos \theta - 4 \sin \theta = 1 \quad (4)$$

$$-2 \sin \theta - 4 \cos \theta = 0 \quad (5)$$

Combining the two equations,

$$\begin{pmatrix} 2 & -4 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (6)$$

$$\Rightarrow \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} = -\frac{1}{10} \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad (7)$$

So,

$$\mathbf{R} = \frac{1}{10} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \quad (8)$$

$$\mathbf{X}' = \mathbf{R}\mathbf{X} \quad (9)$$

$$\Rightarrow \mathbf{X} = \mathbf{R}^{-1}\mathbf{X}' = 2 \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 2 \begin{pmatrix} x' + 2y' \\ -2x' + y' \end{pmatrix} \quad (10)$$

$$so, f(x, y) = \frac{1}{\sqrt{5}} \left(4 (x' + 2y')^2 + 8 (x' + 2y') (2x' - y')^2 \right) \quad (11)$$

$$so, \frac{\partial f(x, y)}{\partial x'} = \frac{1}{\sqrt{5}} \left(8 (x' + 2y') + 8 (2x' - y')^2 + 32 (x' + 2y') (2x' - y') \right) \quad (12)$$

for $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\mathbf{X}_o = \mathbf{R} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} -1 \\ 3 \end{pmatrix} \quad (13)$$

$$so, \frac{\partial f(1, 1)}{\partial x'} = \frac{1}{\sqrt{5}} (4 + 2 - 8) = -\frac{2}{\sqrt{5}}(b) \quad (14)$$