Bonus question

AI25BTECH11014 - Gooty Suhas

Problem

Prove that if $s_1 s_2 > 0$, the points lie on the same side of the line, and if $s_1 s_2 < 0$, they lie on opposite sides.

Solution

Let the points be:

$$\mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Let the line be defined by:

$$\mathbf{n}^T \mathbf{x} + c = 0$$

where $\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$ is the normal vector.

Step 1: Signed Distance from Line

The signed distances of points A and B from the line are:

$$s_1 = \mathbf{n}^T \mathbf{A} + c, \quad s_2 = \mathbf{n}^T \mathbf{B} + c$$

Step 2: Point Dividing Line Segment

Let point P divide AB in the ratio $m : n$. Then:

$$\mathbf{P} = \frac{n\mathbf{A} + m\mathbf{B}}{m + n}$$

Substitute $\mathbf{P}$ into the line equation:

$$\mathbf{n}^T \left(\frac{n\mathbf{A} + m\mathbf{B}}{m + n} \right) + c = 0$$

Multiply both sides by $m + n$:

$$n(\mathbf{n}^T \mathbf{A}) + m(\mathbf{n}^T \mathbf{B}) + c(m + n) = 0$$

Group terms:

$$n(s_1) + m(s_2) = 0 \Rightarrow \frac{m}{n} = -\frac{s_1}{s_2}$$

Step 3: Ratio Interpretation

Hence, the ratio in which the line divides the segment AB is:

$$m : n = -s_1 : s_2$$

Step 4: Side Condition Proof

- If $s_1 \cdot s_2 > 0$, then both signed distances have the same sign. So, points A and B lie on the **same side** of the line.
- If $s_1 \cdot s_2 < 0$, then the signs are opposite. So, points A and B lie on **different sides**.

$$\boxed{\text{If } s_1 s_2 > 0 \Rightarrow \text{Same side}, \quad \text{If } s_1 s_2 < 0 \Rightarrow \text{Opposite sides}}$$