2.10.31

EE25BTECH11002 - Achat Parth Kalpesh

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Question

Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be three non coplanar vectors and $\mathbf{p}, \mathbf{q}, \mathbf{r}$ are vectors defined by the relations

$$\mathbf{p} = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}, \mathbf{q} = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}, \mathbf{r} = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} \tag{1}$$

then the value of the expression $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{p} + (\mathbf{b} + \mathbf{c}) \cdot \mathbf{q} + (\mathbf{c} + \mathbf{a}) \cdot \mathbf{r}$ is equal to

1. 0
2. 1
3. 2
4. 3

Theoretical Solution

Let the given expression be:

$$E = (\mathbf{a}+\mathbf{b})\cdot\mathbf{p} + (\mathbf{b}+\mathbf{c})\cdot\mathbf{q} + (\mathbf{c}+\mathbf{a})\cdot\mathbf{r} \tag{2}$$

$$= (\mathbf{a}\cdot\mathbf{p}+\mathbf{b}\cdot\mathbf{p}) + (\mathbf{b}\cdot\mathbf{q}+\mathbf{c}\cdot\mathbf{q}) + (\mathbf{c}\cdot\mathbf{r}+\mathbf{a}\cdot\mathbf{r}) \tag{3}$$

Let ,

$$\mathbf{V} = \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{pmatrix} \tag{4}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p} & \mathbf{q} & \mathbf{r} \end{pmatrix} \tag{5}$$

$$\mathbf{V}^\top\mathbf{P} = \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} \begin{pmatrix} \mathbf{p} & \mathbf{q} & \mathbf{r} \end{pmatrix} = \begin{pmatrix} \mathbf{a}\cdot\mathbf{p} & \mathbf{a}\cdot\mathbf{q} & \mathbf{a}\cdot\mathbf{r} \\ \mathbf{b}\cdot\mathbf{p} & \mathbf{b}\cdot\mathbf{q} & \mathbf{b}\cdot\mathbf{r} \\ \mathbf{c}\cdot\mathbf{p} & \mathbf{c}\cdot\mathbf{q} & \mathbf{c}\cdot\mathbf{r} \end{pmatrix} \tag{6}$$

Theoretical Solution

$$\mathbf{a} \cdot \mathbf{p} = \mathbf{a} \cdot \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} = \frac{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} = 1 \tag{7}$$

$$\mathbf{a} \cdot \mathbf{q} = \mathbf{a} \cdot \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} = \frac{[\mathbf{a}\ \mathbf{c}\ \mathbf{a}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} = 0 \tag{8}$$

Theoretical Solution

since the scalar triple product with a repeated vector is zero. Thus, the matrix product becomes the identity matrix:

$$\begin{pmatrix} \mathbf{a}\cdot\mathbf{p} & \mathbf{a}\cdot\mathbf{q} & \mathbf{a}\cdot\mathbf{r} \\ \mathbf{b}\cdot\mathbf{p} & \mathbf{b}\cdot\mathbf{q} & \mathbf{b}\cdot\mathbf{r} \\ \mathbf{c}\cdot\mathbf{p} & \mathbf{c}\cdot\mathbf{q} & \mathbf{c}\cdot\mathbf{r} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I} \tag{9}$$

Substituting these results back into the expanded expression for E:

$$\begin{aligned} E &= (1+0)+(1+0)+(1+0) && (10) \\ &= 1+1+1 && (11) \\ &= 3 && (12) \end{aligned}$$

The value of the expression is 3.