2.8.16

EE25BTECH11033 - Kavin

Question:
Prove that the lines $x = py + q, z = ry + s$ and $x = p'y + q', z = r'y + s'$ are perpendicular if $pp' + rr' + 1 = 0$.

Solution:
Let line L_1 be the intersection of the planes,

$$x - py - q = 0 \ , \ z - ry - s = 0$$

Let line L_2 be the intersection of the planes,

$$x - p'y - q' = 0 \ , \ z - r'y - s' = 0$$

The direction vectors of the lines L_1 and L_2 are given by the cross product of the direction vectors of the normals of the intersecting planes.
Let $\mathbf{n_1}$, $\mathbf{n_2}$ be the normals for the planes $x - py - q = 0$ and $z - ry - s = 0$ respectively.

$$direction\ vector\ of\ \mathbf{n_1} = \begin{pmatrix} 1 \\ -p \\ 0 \end{pmatrix} \tag{1}$$

$$direction\ vector\ of\ \mathbf{n_2} = \begin{pmatrix} 0 \\ -r \\ 1 \end{pmatrix} \tag{2}$$

Let $\mathbf{n_3}$, $\mathbf{n_4}$ be the normals for the planes $x - p'y - q' = 0$ and $z - r'y - s' = 0$ respectively.

$$direction\ vector\ of\ \mathbf{n_3} = \begin{pmatrix} 1 \\ -p' \\ 0 \end{pmatrix} \tag{3}$$

$$direction\ vector\ of\ \mathbf{n_4} = \begin{pmatrix} 0 \\ -r' \\ 1 \end{pmatrix} \tag{4}$$

$$\therefore direction\ vector\ of\ L_1 \ = \mathbf{n_1} \times \mathbf{n_2} \tag{5}$$

$$direction\ vector\ of\ L_2 \ = \mathbf{n_3} \times \mathbf{n_4} \tag{6}$$

The *cross product* or *vector product* of $\mathbf{n_1}, \mathbf{n_2}$ is defined as

$$\mathbf{n_1} \times \mathbf{n_2} = \begin{pmatrix} \begin{vmatrix} (\mathbf{n_1})_{23} & (\mathbf{n_2})_{23} \end{vmatrix} \\ \begin{vmatrix} (\mathbf{n_1})_{31} & (\mathbf{n_2})_{31} \end{vmatrix} \\ \begin{vmatrix} (\mathbf{n_1})_{12} & (\mathbf{n_2})_{12} \end{vmatrix} \end{pmatrix} \tag{7}$$

$$\begin{vmatrix} (\mathbf{n_1})_{23} & (\mathbf{n_2})_{23} \end{vmatrix} = \begin{vmatrix} -p & -r \\ 0 & 1 \end{vmatrix} = -p \tag{8}$$

$$\begin{vmatrix} (\mathbf{n_1})_{31} & (\mathbf{n_2})_{31} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \tag{9}$$

$$\begin{vmatrix} (\mathbf{n_1})_{12} & (\mathbf{n_2})_{12} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -p & -r \end{vmatrix} = -r, \tag{10}$$

$$\mathbf{n_1} \times \mathbf{n_2} = \begin{pmatrix} \begin{vmatrix} (\mathbf{n_1})_{23} & (\mathbf{n_2})_{23} \end{vmatrix} \\ \begin{vmatrix} (\mathbf{n_1})_{31} & (\mathbf{n_2})_{31} \end{vmatrix} \\ \begin{vmatrix} (\mathbf{n_1})_{12} & (\mathbf{n_2})_{12} \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -p \\ -1 \\ -r \end{pmatrix} \tag{11}$$

$$\implies direction\ vector\ of\ L_1 \ = \begin{pmatrix} -p \\ -1 \\ -r \end{pmatrix} \tag{12}$$

The *cross product* or *vector product* of $\mathbf{n_3}, \mathbf{n_4}$ is defined as

$$\mathbf{n_3} \times \mathbf{n_4} = \begin{pmatrix} \begin{vmatrix} (\mathbf{n_3})_{23} & (\mathbf{n_4})_{23} \end{vmatrix} \\ \begin{vmatrix} (\mathbf{n_3})_{31} & (\mathbf{n_4})_{31} \end{vmatrix} \\ \begin{vmatrix} (\mathbf{n_3})_{12} & (\mathbf{n_4})_{12} \end{vmatrix} \end{pmatrix} \tag{13}$$

$$\begin{vmatrix}(\mathbf{n_3})_{23} & (\mathbf{n_4})_{23}\end{vmatrix} = \begin{vmatrix}-p' & -r' \\ 0 & 1\end{vmatrix} = -p' \tag{14}$$

$$\begin{vmatrix}(\mathbf{n_3})_{31} & (\mathbf{n_4})_{31}\end{vmatrix} = \begin{vmatrix}0 & 1 \\ 1 & 0\end{vmatrix} = -1 \tag{15}$$

$$\begin{vmatrix}(\mathbf{n_3})_{12} & (\mathbf{n_4})_{12}\end{vmatrix} = \begin{vmatrix}1 & 0 \\ -p' & -r'\end{vmatrix} = -r', \tag{16}$$

$$\mathbf{n_3} \times \mathbf{n_4} = \begin{pmatrix}\begin{vmatrix}(\mathbf{n_3})_{23} & (\mathbf{n_4})_{23}\end{vmatrix} \\ \begin{vmatrix}(\mathbf{n_3})_{31} & (\mathbf{n_4})_{31}\end{vmatrix} \\ \begin{vmatrix}(\mathbf{n_3})_{12} & (\mathbf{n_4})_{12}\end{vmatrix}\end{pmatrix} = \begin{pmatrix}-p' \\ -1 \\ -r'\end{pmatrix} \tag{17}$$

$$\implies direction\ vector\ of\ L_2 = \begin{pmatrix}-p' \\ -1 \\ -r'\end{pmatrix} \tag{18}$$

If the lines are perpendicular, then their dot product of direction vectors must be zero.

$$\implies (direction\ vector\ of\ L_1)^\top (direction\ vector\ of\ L_2) = 0 \tag{19}$$

$$\implies \begin{pmatrix}-p & -1 & -r\end{pmatrix}\begin{pmatrix}-p' \\ -1 \\ -r'\end{pmatrix} = 0 \tag{20}$$

$$\implies pp' + rr' + 1 = 0 \tag{21}$$

$\therefore$the lines $x = py + q, z = ry + s$ and $x = p'y + q', z = r'y + s'$ are perpendicular if $pp' + rr' + 1 = 0$