2.10.5

AI25BTECH11016-Varun

Question:

A, B, C and D, are four points in a plane respectively such that $(A - D) \cdot (B - C) = (B - D) \cdot (C - A) = 0$. The point D, then, is the ______ of $\triangle ABC$.

Solution:

Consider the equation,

$$(\mathbf{A} - \mathbf{D})^T (\mathbf{B} - \mathbf{C}) = 0 \tag{1}$$

This implies line joining A and D is perpendicular to line joining B and C

Consider the equation,

$$(\mathbf{B} - \mathbf{D})^T (\mathbf{C} - \mathbf{A}) = 0 \tag{2}$$

This implies line joining B and D is perpendicular to line joining A and C

In $\triangle ABC$,
side BC is perpendicular to AD
side AC is perpendicular to BD

We know that,

The altitudes(The perpendiculars drawn from a vertex to opposite sides) are concurrent at Orthocentre.

Therefore,

D must be Orthocentre of $\triangle ABC$

Verification by example:

Let us take the points
$\mathbf{A} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 3 \\ \frac{7}{3} \end{pmatrix}$.

Checking the First condition:

$$(\mathbf{A} - \mathbf{D})^T (\mathbf{B} - \mathbf{C}) = 0 \tag{3}$$

$$L.H.S = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ \frac{7}{3} \end{pmatrix} \right)^T \left(\begin{pmatrix} 5 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right) \tag{4}$$

$$= \begin{pmatrix} -2 \\ \frac{-4}{3} \end{pmatrix}^T \begin{pmatrix} 2 \\ -3 \end{pmatrix} \tag{5}$$

$$= 0 \tag{6}$$

$$= R.H.S \tag{7}$$

$$L.H.S = R.H.S \tag{8}$$

Checking the Second condition:

$$(\mathbf{B}-\mathbf{D})^T(\mathbf{C}-\mathbf{A}) = 0 \tag{9}$$

$$L.H.S = \left(\begin{pmatrix}5\\1\end{pmatrix} - \begin{pmatrix}3\\\frac{7}{3}\end{pmatrix}\right)^T \left(\begin{pmatrix}3\\4\end{pmatrix} - \begin{pmatrix}1\\1\end{pmatrix}\right) \tag{10}$$

$$= \begin{pmatrix}2\\\frac{-4}{3}\end{pmatrix}^T \begin{pmatrix}2\\3\end{pmatrix}$$

$$= 0$$

$$= R.H.S$$

$$L.H.S = R.H.S \tag{11}$$

Let's take two points F and E which are foot of perpendiculars of altitudes drawn from vertices A and B respectively.

1.The normal vector of $\mathbf{F}-\mathbf{A}$ is

$$\mathbf{n} = \begin{pmatrix}2\\-3\end{pmatrix} \tag{12}$$

The equation of the altitude from A (i.e AF) is

$$\mathbf{n}^T(\mathbf{x}-\mathbf{A}) = 0 \tag{13}$$

$$\begin{pmatrix}2\\-3\end{pmatrix}^T (\mathbf{x} - \begin{pmatrix}1\\1\end{pmatrix}) = 0 \tag{14}$$

$$\begin{pmatrix}2 & -3\end{pmatrix}(\mathbf{x} - \begin{pmatrix}1\\1\end{pmatrix}) = 0 \tag{15}$$

$$\begin{pmatrix}2 & -3\end{pmatrix}(\mathbf{x}) = -1 \tag{16}$$

$$\tag{17}$$

2.The normal vector of $\mathbf{E}-\mathbf{B}$ is

$$\mathbf{n} = \begin{pmatrix}2\\3\end{pmatrix} \tag{18}$$

The equation of the altitude from B (i.e BE) is

$$\mathbf{n}^T(\mathbf{x}-\mathbf{B}) = 0 \tag{19}$$

$$\begin{pmatrix}2\\3\end{pmatrix}^T (\mathbf{x} - \begin{pmatrix}5\\1\end{pmatrix}) = 0 \tag{20}$$

$$\begin{pmatrix}2 & 3\end{pmatrix}(\mathbf{x} - \begin{pmatrix}5\\1\end{pmatrix}) = 0 \tag{21}$$

$$\begin{pmatrix}2 & 3\end{pmatrix}(\mathbf{x}) = 13 \tag{22}$$

$$\tag{23}$$

The intersection point of altitudes **orthocenter:H** can be obtained by solving the above two equations

$$\begin{pmatrix} 2 & -3 \\ 2 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -1 \\ 13 \end{pmatrix} \tag{24}$$

$$\begin{pmatrix} 2 & -3 & -1 \\ 2 & 3 & 13 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 2 & -3 & -1 \\ 0 & 6 & 14 \end{pmatrix} \tag{25}$$

$$\xrightarrow{R_2 \leftarrow \frac{1}{6} R_2} \begin{pmatrix} 2 & -3 & -1 \\ 0 & 1 & \frac{7}{3} \end{pmatrix} \tag{26}$$

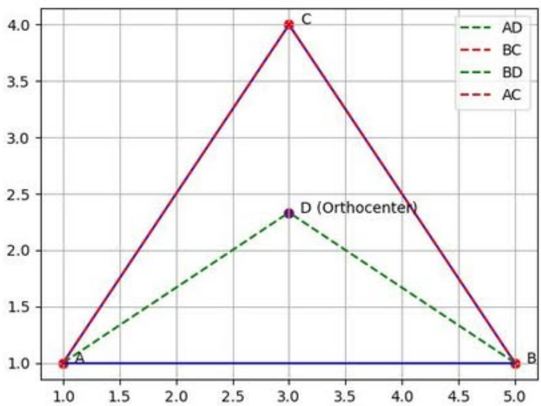
$$\xrightarrow{R_1 \leftarrow R_1 + 3R_2} \begin{pmatrix} 2 & 0 & 6 \\ 0 & 1 & \frac{7}{3} \end{pmatrix} \tag{27}$$

$$\xrightarrow{R_1 \leftarrow \frac{1}{2} R_1} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & \frac{7}{3} \end{pmatrix} \tag{28}$$

which gives,

$$H = \begin{pmatrix} 3 \\ \frac{7}{3} \end{pmatrix} \tag{29}$$

Therefore,
The D we have taken matches with the orthocenter H of the given triangle



4.0
3.5
3.0
2.5
2.0
1.5
1.0
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0
C
D (Orthocenter)
A
B
AD
BC
BD
AC

Fig. 0.1