Matrices in Geometry - 10.7.86

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Problem Statement

Let $\mathbf{C_1}$ and $\mathbf{C_2}$ be two circles with $\mathbf{C_2}$ lying inside $\mathbf{C_1}$. A circle C lying inside $\mathbf{C_1}$ touches $\mathbf{C_1}$ internally and $\mathbf{C_2}$ externally. Identify the locus of center of $\mathbf{C}$.

Solution

Let the center of $\mathbf{C}$, $\mathbf{C_1}$ and $\mathbf{C_2}$ be $\mathbf{O}$, $\mathbf{O_1}$ and $\mathbf{O_2}$, respectively.
Let the radii of circles $\mathbf{C}$, $\mathbf{C_1}$ and $\mathbf{C_2}$ be r, r_1 and r_2
It is given that $\mathbf{C}$ touches the circle $\mathbf{C_1}$ internally and $\mathbf{C_2}$ externally.
Therefore,

$$\|\mathbf{O} - \mathbf{O_1}\| = r_1 - r \tag{1}$$
$$\|\mathbf{O} - \mathbf{O_2}\| = r_2 + r \tag{2}$$

Adding these two equations, we get

$$\|\mathbf{O} - \mathbf{O_1}\| + \|\mathbf{O} - \mathbf{O_2}\| = r_1 + r_2 \tag{3}$$

Solution

Substitute $\mathbf{O}$ as $\mathbf{x}$

$$\|\mathbf{x} - \mathbf{O_1}\| + \|\mathbf{x} - \mathbf{O_2}\| = r_1 + r_2 \tag{4}$$

This is equation of an ellipse because it is of form

$$\|\mathbf{x} - \mathbf{S_1}\| + \|\mathbf{x} - \mathbf{S_2}\| = 2a \tag{5}$$

with focii as $\mathbf{O_1}$, $\mathbf{O_2}$ and length of the major axis as $r_1 + r_2$

Solution

$$\|\mathbf{x} - \mathbf{O_1}\| + \|\mathbf{x} - \mathbf{O}_2\| = K, \; K = r_1 + r_2 \tag{6}$$

To eliminate square roots from the norms, we rearrange and square the equation.

$$\|\mathbf{x} - \mathbf{O}_1\| = K - \|\mathbf{x} - \mathbf{O}_2\| \tag{7}$$

Squaring both sides and using the property $\|\mathbf{v}\|^2 = \mathbf{v}^\top \mathbf{v}$:

$$\|\mathbf{x} - \mathbf{O}_1\|^2 = (K - \|\mathbf{x} - \mathbf{O}_2\|)^2 \tag{8}$$

Solution

$$\|\mathbf{x} - \mathbf{O_1}\| + \|\mathbf{x} - \mathbf{O_2}\| = K, \;\; K = r_1 + r_2 \tag{9}$$

To eliminate square roots from the norms, we rearrange and square the equation.

$$\|\mathbf{x} - \mathbf{O_1}\| = K - \|\mathbf{x} - \mathbf{O_2}\| \tag{10}$$

Squaring both sides and using the property $\|\mathbf{v}\|^2 = \mathbf{v}^\top \mathbf{v}$:

$$\|\mathbf{x} - \mathbf{O_1}\|^2 = (K - \|\mathbf{x} - \mathbf{O_2}\|)^2 \tag{11}$$

Solution

Let $S = K^2 + \|\mathbf{O_2}\|^2 - \|\mathbf{O_1}\|^2$ and $\mathbf{v} = 2(\mathbf{O_1} - \mathbf{O_2})$. The equation becomes:

$$2K\|\mathbf{x} - \mathbf{O_2}\| = S + \mathbf{v}^\top \mathbf{x} \tag{12}$$

Squaring both sides again:

$$4K^2\|\mathbf{x} - \mathbf{O_2}\|^2 = (S + \mathbf{v}^\top \mathbf{x})^2 \tag{13}$$

$$4K^2\left(\mathbf{x}^\top \mathbf{x} - 2\mathbf{O_2}^\top \mathbf{x} + \|\mathbf{O_2}\|^2\right) = S^2 + 2S\left(\mathbf{v}^\top \mathbf{x}\right) + \left(\mathbf{v}^\top \mathbf{x}\right)^2 \tag{14}$$

Solution

Using the identity $\left(\mathbf{v}^\top \mathbf{x}\right)^2 = \mathbf{x}^\top \left(\mathbf{v}\mathbf{v}^\top\right)\mathbf{x}$, we group all terms to one side to match the form $\mathbf{x}^\top V\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0$.

$$\mathbf{x}^\top \left(4K^2 I - \mathbf{v}\mathbf{v}^\top\right)\mathbf{x} + 2\left(-4K^2\mathbf{O_2} - S\mathbf{v}\right)^\top \mathbf{x} + \left(4K^2\left\|\mathbf{O_2}\right\|^2 - S^2\right) = 0 \tag{15}$$

Solution

Compared with the general conic equation, we identify the matrix V, the vector $\mathbf{u}$, and the scalar f:

$$\mathbf{V} = 4K^2 I - \mathbf{v}\mathbf{v}^\top = 4\left(r_1 + r_2\right)^2 I - 4\left(\mathbf{O_1} - \mathbf{O_2}\right)\left(\mathbf{O_1} - \mathbf{O_2}\right)^\top \tag{16}$$

$$\begin{aligned} \mathbf{u} = -4K^2\mathbf{O_2} - S\mathbf{v} = -4\left(r_1 + r_2\right)^2 \mathbf{O_2} - \\ \left(\left(r_1 + r_2\right)^2 + \left\|\mathbf{O_2}\right\|^2 - \left\|\mathbf{O_1}\right\|^2\right) \cdot 2\left(\mathbf{O_1} - \mathbf{O_2}\right) \end{aligned} \tag{17}$$

$$\begin{aligned} f = 4K^2\left\|\mathbf{O_2}\right\|^2 - S^2 = 4\left(r_1 + r_2\right)^2 \left\|\mathbf{O_2}\right\|^2 - \\ \left(\left(r_1 + r_2\right)^2 + \left\|\mathbf{O_2}\right\|^2 - \left\|\mathbf{O_1}\right\|^2\right)^2 \end{aligned} \tag{18}$$

Solution

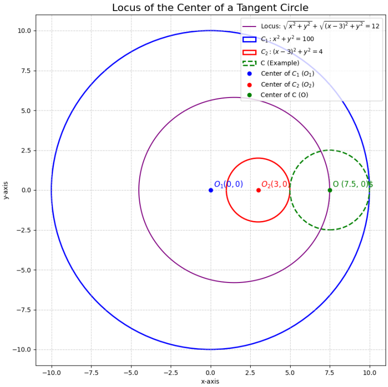

Figure: Graph for 10.7.86