MatGeo Assignment - Problem 2.10.55

EE25BTECH11024

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Problem Statement

The edges of a parallelepiped are of unit length and are parallel to non-coplanar unit vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ such that $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a} = \frac{1}{2}$. Then, the volume of the parallelepiped is

1. $\dfrac{1}{\sqrt{2}}$
2. $\dfrac{1}{2\sqrt{2}}$
3. $\dfrac{\sqrt{5}}{2}$
4. $\dfrac{1}{\sqrt{3}}$

Solution:

Symbol	Value / Definition	Description
$\mathbf{a}, \mathbf{b}, \mathbf{c}$	$\|\mathbf{a}\| = \|\mathbf{b}\| = \|\mathbf{c}\| = 1$	Non-coplanar unit vectors for the parallelepiped edges.
$\mathbf{a} \cdot \mathbf{b}, \mathbf{b} \cdot \mathbf{c}, \mathbf{c} \cdot \mathbf{a}$	$\frac{1}{2}$	The dot product between any pair of the vectors.
A	$\begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{pmatrix}$	A 3×3 matrix with the edge vectors as its columns.
V	$\left\| \det(A) \right\|$	The volume of the parallelepiped (the value to be found).

Solution:

Using Gram matrix,

$$G = A^\top A = \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{pmatrix}^\top \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{pmatrix} = \begin{pmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{pmatrix} \tag{1}$$

Substituting the given values into the Gram matrix:

$$G = \begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix} \tag{2}$$

The determinant of the Gram matrix is related to the determinant of A by:

$$\det(G) = \det(A^\top A) = (\det(A))^2 = V^2 \tag{3}$$

Therefore, the volume is $V = \sqrt{\det(G)}$.

Solution:

Calculating determinant of G we get,

$$\det(G) \tag{4}$$

$$= \det\begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix} \xrightarrow[R_3 \to R_3 - \frac{1}{2}R_1]{R_2 \to R_2 - \frac{1}{2}R_1} \det\begin{pmatrix} 1 & 1/2 & 1/2 \\ 0 & 3/4 & 1/4 \\ 0 & 1/4 & 3/4 \end{pmatrix} \tag{5}$$

$$= \det\begin{pmatrix} 1 & 1/2 & 1/2 \\ 0 & 3/4 & 1/4 \\ 0 & 1/4 & 3/4 \end{pmatrix} \xrightarrow{R_3 \to R_3 - \frac{1}{3}R_2} \det\begin{pmatrix} 1 & 1/2 & 1/2 \\ 0 & 3/4 & 1/4 \\ 0 & 0 & 2/3 \end{pmatrix} \tag{6}$$

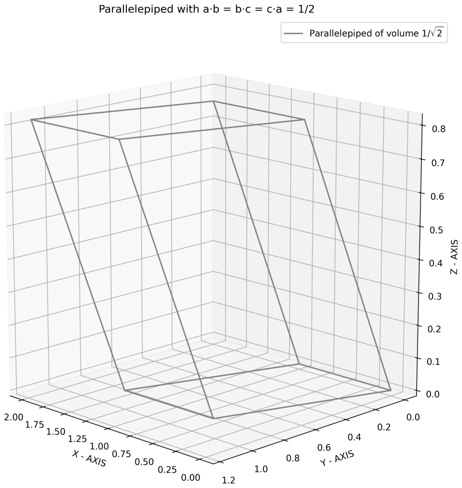
$$= \frac{1}{2} = \det(G) \tag{7}$$

Therefore, volume V is,

$$V = \sqrt{\det(G)} = \frac{1}{\sqrt{2}} \tag{8}$$

Final Answer

Thus, the volume of the parallelepiped is $\frac{1}{\sqrt{2}}$, which corresponds to option (a).
See Figure 1.


Parallelepiped with a·b = b·c = c·a = 1/2
Parallelepiped of volume 1/√2
X - AXIS
Y - AXIS
Z - AXIS
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
0.0
0.2
0.4
0.6
0.8
1.0
1.2
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8

Python Code: plot.py (Native)

```
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

a = np.array([1, 0, 0])
b = np.array([0.5, np.sqrt(3)/2, 0])
c = np.array([0.5, 1/(2*np.sqrt(3)), np.sqrt(2/3)])

# Vertices of parallelepiped
p1 = np.array([0,0,0])
p2 = a
p3 = b
p4 = c
p5 = a+b
p6 = b+c
p7 = c+a
p8 = a+b+c
```

Python Code (Native Implementation – plot.py)

```
fig = plt.figure(figsize=(10,10))
ax = fig.add_subplot(111, projection='3d')

# Bottom face
ax.plot([p1[0], p2[0]], [p1[1], p2[1]], [p1[2], p2[2]], 'grey')
ax.plot([p1[0], p3[0]], [p1[1], p3[1]], [p1[2], p3[2]], 'grey')
ax.plot([p2[0], p5[0]], [p2[1], p5[1]], [p2[2], p5[2]], 'grey')
ax.plot([p3[0], p5[0]], [p3[1], p5[1]], [p3[2], p5[2]], 'grey')

# Top face
ax.plot([p4[0], p7[0]], [p4[1], p7[1]], [p4[2], p7[2]], 'grey')
ax.plot([p4[0], p6[0]], [p4[1], p6[1]], [p4[2], p6[2]], 'grey')
ax.plot([p7[0], p8[0]], [p7[1], p8[1]], [p7[2], p8[2]], 'grey')
ax.plot([p6[0], p8[0]], [p6[1], p8[1]], [p6[2], p8[2]], 'grey')

# Vertical edges
ax.plot([p1[0], p4[0]], [p1[1], p4[1]], [p1[2], p4[2]], 'grey')
ax.plot([p2[0], p7[0]], [p2[1], p7[1]], [p2[2], p7[2]], 'grey')
ax.plot([p3[0], p6[0]], [p3[1], p6[1]], [p3[2], p6[2]], 'grey')
ax.plot([p5[0], p8[0]], [p5[1], p8[1]], [p5[2], p8[2]], 'grey')
```

Python Code (Native Implementation – plot.py)

```
ax.plot([], [], [], 'grey', label=r"Parallelepiped of volume $1/\sqrt{2}
    $")

ax.set_xlabel("X - AXIS")
ax.set_ylabel("Y - AXIS")
ax.set_zlabel("Z - AXIS")
ax.set_title("Parallelepiped with ab = bc = ca = 1/2")
ax.legend()
ax.set_box_aspect([1,1,1])
ax.view_init(elev=15, azim=135)
plt.savefig("parallelepiped.png", dpi=300)
plt.show()
```

C Code (Shared Library – findparalellepipedvol.c)

```c
#include <stdio.h>
#include <math.h>

double determinant3x3(double m[3][3]) {
    return m[0][0]*(m[1][1]*m[2][2] - m[1][2]*m[2][1])
         - m[0][1]*(m[1][0]*m[2][2] - m[1][2]*m[2][0])
         + m[0][2]*(m[1][0]*m[2][1] - m[1][1]*m[2][0]);
}

double parallelepiped_volume(double *a, double *b, double *c) {
    double m[3][3] = {
        {a[0], b[0], c[0]},
        {a[1], b[1], c[1]},
        {a[2], b[2], c[2]}
    };
    double det = determinant3x3(m);
    return fabs(det); // absolute value = volume
}
```

Python Code: call.py (C + Python)

```
import ctypes
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

lib = ctypes.CDLL("./find_parallelepiped_vol.so")
lib.parallelepiped_volume.argtypes = [
    ctypes.POINTER(ctypes.c_double),
    ctypes.POINTER(ctypes.c_double),
    ctypes.POINTER(ctypes.c_double)
]
lib.parallelepiped_volume.restype = ctypes.c_double

a = np.array([1,0,0], dtype=np.float64)
b = np.array([0.5, np.sqrt(3)/2, 0], dtype=np.double)
c = np.array([0.5, 1/np.sqrt(12), np.sqrt(2/3)], dtype=np.double)

a_ptr = a.ctypes.data_as(ctypes.POINTER(ctypes.c_double))
b_ptr = b.ctypes.data_as(ctypes.POINTER(ctypes.c_double))
c_ptr = c.ctypes.data_as(ctypes.POINTER(ctypes.c_double))
```

Python Code (C Integrated – call.py)

```
volume = lib.parallelepiped_volume(a_ptr, b_ptr, c_ptr)

O = np.array([0,0,0])
points = np.array([
    O, a, b, c, a+b, b+c, c+a, a+b+c
])

edges = [(0,1),(0,2),(0,3),
         (1,4),(1,6),
         (2,4),(2,5),
         (3,5),(3,6),
         (4,7),(5,7),(6,7)]

fig = plt.figure(figsize=(8,8))
ax = fig.add_subplot(111, projection='3d')

for i,j in edges:
    ax.plot([points[i,0],points[j,0]],
            [points[i,1],points[j,1]],
            [points[i,2],points[j,2]], 'grey')
```

Python Code (C Integrated – call.py)

```
ax.plot([], [], [], color='grey', linestyle='--',
       label=fr"Parallelepiped of volume {volume:.3f}")
ax.set_xlabel("X - AXIS")
ax.set_ylabel("Y - AXIS")
ax.set_zlabel("Z - AXIS")
ax.set_title("Parallelepiped with ab = bc = ca = 1/2")
ax.legend() ax.set_box_aspect([1,1,1])
ax.view_init(elev=15, azim=135)
plt.savefig("parallelepiped_volume.png", dpi=300)
plt.show()
```