Question 4.2.3

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September 30, 2025

Question:

Given $\mathbf{A} = \begin{pmatrix} 2 & -3 \\ -4 & 7 \end{pmatrix}$, compute $\mathbf{A}^{-1}$ and show that $2\mathbf{A}^{-1} = 9\mathbf{I} - \mathbf{A}$.

Solution:

We start by finding the characteristic equation of $\mathbf{A}$.

$$
\begin{aligned}
f(\lambda) &= \det(\mathbf{A} - \lambda \mathbf{I}) \\
&= \det \begin{pmatrix} \lambda - 2 & 3 \\ 4 & \lambda - 7 \end{pmatrix} \\
&= (\lambda - 2)(\lambda - 7) - 12 \\
&= \lambda^2 - 9\lambda + 2 \\
\implies f(\lambda) &= (\lambda - 1)(\lambda - 8) = \lambda^2 - 9\lambda + 2 \qquad (1)
\end{aligned}
$$

If we were to set $\lambda = 0$, we get $\det(\mathbf{A}) = 2 \neq 0$. Thus, $\mathbf{A}$ is invertible and $\mathbf{A}^{-1}$ exists. To find it, we use the Cayley-Hamilton theorem that states that the characteristic equation $f(\lambda) = 0$ is also satisfied by the matrix itself, i.e, $f(\mathbf{A}) = 0$. Thus,

$$
\begin{aligned}
f(\mathbf{A}) &= \mathbf{A}^2 - 9\mathbf{A} + 2\mathbf{I} = 0 \\
\implies 2\mathbf{I} &= 9\mathbf{A} - \mathbf{A}^2 \\
\implies 2\mathbf{I}\mathbf{A}^{-1} &= 9\mathbf{A}\mathbf{A}^{-1} - \mathbf{A}^2\mathbf{A}^{-1} \\
\implies 2\mathbf{A}^{-1} &= 9\mathbf{I} - \mathbf{A} \qquad (2) \\
\implies \mathbf{A}^{-1} &= \frac{1}{2}\begin{pmatrix} 9-2 & 3 \\ 4 & 9-7 \end{pmatrix} \\
\implies \mathbf{A}^{-1} &= \begin{pmatrix} \frac{7}{2} & \frac{3}{2} \\ 2 & 1 \end{pmatrix} \qquad (3)
\end{aligned}
$$

Thus, we have computed $\mathbf{A}^{-1}$ and shown that $2\mathbf{A}^{-1} = 9\mathbf{I} - \mathbf{A}$.