8.4.35

EE25BTECH11032 - Kartik Lahoti

Question:

The ellipse $E_1: \frac{x^2}{9} + \frac{y^2}{4} = 1$ is inscribed in a rectangle $\mathbf{R}$ whose sides are parallel to coordinate axes. another ellipse E_2 passing through the point $(0, 4)$ circumscribes the rectangle $\mathbf{R}$. The eccentricity of the ellipse E_2 is

1) $\frac{\sqrt{2}}{2}$
2) $\frac{\sqrt{3}}{2}$
3) $\frac{1}{2}$
4) $\frac{3}{4}$

Solution:

Given for E_1:

	Input			Output		
Conic	**V**	**u**	**f**	**F**	**Directrix**	**Latus Rectum**
$\frac{x^2}{9} + \frac{y^2}{4} = 1$	$\begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}$	$\mathbf{0}$	-36	$\pm\sqrt{5}\mathbf{e_1}$	$\mathbf{e_1}^\top\mathbf{x} = \pm\frac{9}{\sqrt{5}}$	$\frac{8}{3}$

E_1 is inscribed in Rectangle
$\therefore$ The coordinates of Vertices of Rectangles are intersection points of,

$$\mathbf{e_1}^\top\mathbf{x} = 3 \tag{1}$$
$$\mathbf{e_2}^\top\mathbf{x} = 2 \tag{2}$$
$$\mathbf{e_1}^\top\mathbf{x} = -3 \tag{3}$$
$$\mathbf{e_2}^\top\mathbf{x} = -2 \tag{4}$$

Let $\mathbf{P}$ intersection of 1 and 2,

$$\mathbf{P} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \tag{5}$$

Let ,

$$\mathbf{Q} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \tag{6}$$

General Form of Conic equation is given as ,

$$g(\mathbf{x}) = \mathbf{x}^\top\mathbf{V}\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f = 0 \tag{7}$$

Since the rectangle in inscribed inside E_2 , E_2 is an ellipse centered at origin. For a conic centered at Origin

$$\mathbf{u} = \mathbf{0} \tag{8}$$

$$\therefore \mathbf{x}^\top \mathbf{V}' \mathbf{x} = 1 \tag{9}$$

where, $\mathbf{V}' = -\frac{1}{f}\mathbf{V}$
Since E_2 has axes parallel to coordinate axes , $\mathbf{V}'$ must be diagonal.

$$\mathbf{V}' = \begin{pmatrix} v_{11} & 0 \\ 0 & v_{22} \end{pmatrix} \tag{10}$$

Now, E_2 passes through $\mathbf{Q}$ and $\mathbf{P}$

$$\mathbf{Q}^\top \mathbf{V}' \mathbf{Q} = 1 \tag{11}$$
$$\mathbf{P}^\top \mathbf{V}' \mathbf{P} = 1 \tag{12}$$

we get following system of eqn ,

$$\begin{pmatrix} 9 & 4 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{22} \end{pmatrix} = 1, \begin{pmatrix} 0 & 16 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{22} \end{pmatrix} = 1 \tag{13}$$

$$\left(\begin{array}{cc|c} 9 & 4 & 1 \\ 0 & 16 & 1 \end{array}\right) \xleftrightarrow{R_1 \to \frac{R_1}{9}} \left(\begin{array}{cc|c} 1 & \frac{4}{9} & \frac{1}{9} \\ 0 & 16 & 1 \end{array}\right) \tag{14}$$

$$\left(\begin{array}{cc|c} 1 & \frac{4}{9} & \frac{1}{9} \\ 0 & 16 & 1 \end{array}\right) \xleftrightarrow{R_2 \to \frac{R_2}{16}} \left(\begin{array}{cc|c} 1 & \frac{4}{9} & \frac{1}{9} \\ 0 & 1 & \frac{1}{16} \end{array}\right) \tag{15}$$

$$\left(\begin{array}{cc|c} 1 & \frac{4}{9} & \frac{1}{9} \\ 0 & 1 & \frac{1}{16} \end{array}\right) \xleftrightarrow{R_1 \to R_1 - \frac{4}{9} R_2} \left(\begin{array}{cc|c} 1 & 0 & \frac{1}{12} \\ 0 & 1 & \frac{1}{16} \end{array}\right) \tag{16}$$

On solving , We get

$$\mathbf{V}' = \begin{pmatrix} \frac{1}{12} & 0 \\ 0 & \frac{1}{16} \end{pmatrix} \tag{17}$$

$$\mathbf{V} = \|n\|^2 \mathbf{I} - e^2 \mathbf{n}^\top \mathbf{n} \tag{18}$$

Here, $\mathbf{n} = \mathbf{e_2}$
Substituting the values, we get

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 - e^2 \end{pmatrix} \tag{19}$$

Now,

$$\mathbf{V}' = k\mathbf{V} \tag{20}$$

$$\begin{pmatrix} \frac{1}{12} & 0 \\ 0 & \frac{1}{16} \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k\left(1 - e^2\right) \end{pmatrix} \tag{21}$$

$$\tag{22}$$

$$\implies 1 - e^2 = \frac{12}{16} \tag{23}$$

$$\implies e = \frac{1}{2} \tag{24}$$

Hence Answer : Option (3) $\frac{1}{2}$