2.10.79

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Question

In a triangle PQR, let

$$\mathbf{a} = \mathbf{QR},\ \mathbf{b} = \mathbf{RP},\ \mathbf{c} = \mathbf{PQ}$$

det $\mathbf{a} = 3$, det $\mathbf{b} = 4$, and

$$\frac{\mathbf{a}.(\mathbf{c} - \mathbf{b})}{\mathbf{c}.(\mathbf{a} - \mathbf{b})} = \frac{|\mathbf{a}|}{|\mathbf{a}| + |\mathbf{b}|}$$

then the value of $|\mathbf{a} \times \mathbf{b}|$ is ________

Given Information

Let us find the solution theoretically first and then verify it computationally.
It is given that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the sides of a triangle. This implies

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{QR} + \mathbf{RP} + \mathbf{PQ} = 0 \tag{1}$$

It is also given that,

$$|\mathbf{a}| = 3 \text{ and } |\mathbf{b}| = 4 \tag{2}$$

Solution

Let the given equation,

$$\frac{\mathbf{a}.(\mathbf{c}-\mathbf{b})}{\mathbf{c}.(\mathbf{a}-\mathbf{b})} = \frac{\|\mathbf{a}\|}{\|\mathbf{a}\| + \|\mathbf{b}\|} \tag{3}$$

This gives,

$$\frac{\mathbf{a}.(\mathbf{c}-\mathbf{b})}{\mathbf{c}.(\mathbf{a}-\mathbf{b})} = \frac{3}{7} \tag{4}$$

On further simplifying this gives us,

$$7(\mathbf{a}^\top \mathbf{c} - \mathbf{a}^\top \mathbf{b}) = 3(\mathbf{c}^\top \mathbf{a} - \mathbf{c}^\top \mathbf{b}) \tag{5}$$

$$4\mathbf{a}^\top \mathbf{c} - 7\mathbf{a}^\top \mathbf{b} + 3\mathbf{c}^\top \mathbf{b} = 0 \tag{6}$$

On multiplying $\mathbf{a^T}$ on both sides of 1

$$\mathbf{a}^\top \mathbf{a} + \mathbf{a}^\top \mathbf{b} + \mathbf{a}^\top \mathbf{c} = 0 \implies \mathbf{a}^\top \mathbf{b} + \mathbf{a}^\top \mathbf{c} = -9 \tag{7}$$

On multiplying $\mathbf{b^T}$ on both sides of 1

$$\mathbf{b}^\top \mathbf{a} + \mathbf{b}^\top \mathbf{b} + \mathbf{b}^\top \mathbf{c} = 0 \implies \mathbf{b}^\top \mathbf{a} + \mathbf{b}^\top \mathbf{c} = -16 \tag{8}$$

On solving the equations 6, 7 and 8

$$\begin{pmatrix} -7 & 3 & 4 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{a}^\top \mathbf{b} \\ \mathbf{b}^\top \mathbf{c} \\ \mathbf{c}^\top \mathbf{a} \end{pmatrix} = \begin{pmatrix} 0 \\ -9 \\ -16 \end{pmatrix} \tag{9}$$

On using Gauss Jordan method to solve this

$$\begin{pmatrix} \mathbf{a}^\top \mathbf{b} \\ \mathbf{b}^\top \mathbf{c} \\ \mathbf{c}^\top \mathbf{a} \end{pmatrix} = \left(\begin{array}{ccc|c} -7 & 3 & 4 & 0 \\ 1 & 0 & 1 & -9 \\ 1 & 1 & 0 & -16 \end{array}\right) \tag{10}$$

On doing $R_1 \rightarrow R_1 + 8R_2$ and $R_3 \rightarrow R_3 - R_2$

$$\begin{pmatrix} \mathbf{a}^\top \mathbf{b} \\ \mathbf{b}^\top \mathbf{c} \\ \mathbf{c}^\top \mathbf{a} \end{pmatrix} = \left(\begin{array}{ccc|c} 1 & 3 & 12 & -72 \\ 1 & 0 & 1 & -9 \\ 0 & 1 & -1 & -7 \end{array}\right) \tag{11}$$

On doing $R_2 \rightarrow R_2 - R_1$

$$\begin{pmatrix} \mathbf{a}^\top \mathbf{b} \\ \mathbf{b}^\top \mathbf{c} \\ \mathbf{c}^\top \mathbf{a} \end{pmatrix} = \left(\begin{array}{ccc|c} 1 & 3 & 12 & -72 \\ 0 & -3 & -11 & 63 \\ 0 & 1 & -1 & -7 \end{array}\right) \tag{12}$$

On doing $R_1 \to R_1 + R_2$ and $R_3 \to R_3 + \frac{1}{3}R_2$

$$\begin{pmatrix} \mathbf{a}^\top\mathbf{b} \\ \mathbf{b}^\top\mathbf{c} \\ \mathbf{c}^\top\mathbf{a} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & | & -9 \\ 0 & -3 & -11 & | & 63 \\ 0 & 0 & -\frac{14}{3} & | & 14 \end{pmatrix} \tag{13}$$

On doing $R_1 \to R_1 + \frac{3}{14}R_3$ and $R_2 \to R_2 - \frac{33}{14}R_3$

$$\begin{pmatrix} \mathbf{a}^\top\mathbf{b} \\ \mathbf{b}^\top\mathbf{c} \\ \mathbf{c}^\top\mathbf{a} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & | & -6 \\ 0 & -3 & 0 & | & 30 \\ 0 & 0 & -\frac{14}{3} & | & 14 \end{pmatrix} \tag{14}$$

From this, we get,

$$\mathbf{a}^{\top}\mathbf{b} = -6 \tag{15}$$

From the definition of cross product, and from 15 we get,

$$\left\|\mathbf{a} \times \mathbf{b}\right\|^2 = \left\|\mathbf{a}\right\|^2 \left\|\mathbf{b}\right\|^2 - (\mathbf{a}^{\top}\mathbf{b})^2 \implies \left\|\mathbf{a} \times \mathbf{b}\right\|^2 = 4^2.3^2 - (-6)^2 \tag{16}$$

Final Answer

The final answer,

$$\|\mathbf{a} \times \mathbf{b}\| = 6\sqrt{3} \tag{17}$$

C code

```c
#include <stdio.h>
#include<math.h>

void cross(const double* a, const double* b, double* result) {
    result[0] = a[1]*b[2] - a[2]*b[1];
    result[1] = a[2]*b[0] - a[0]*b[2];
    result[2] = a[0]*b[1] - a[1]*b[0];
}
```

Python code

```
import numpy as np
import matplotlib.pyplot as plt
import ctypes
import os
import sys

a = np.array([3, 0, 0], dtype=np.float64)

y = 2
z = np.sqrt(12 - y**2)
b = np.array([-2, y, z], dtype=np.float64)

axb = np.array([0, 0, 0], dtype=np.float64)
```

Python code

```
cross_lib = ctypes.CDLL('./cross.so')
cross_lib.cross.argtypes = [
        ctypes.POINTER(ctypes.c_double),
        ctypes.POINTER(ctypes.c_double),
        ctypes.POINTER(ctypes.c_double)
    ]

cross_lib.cross.restype = ctypes.c_double

cross=cross_lib.cross(
        a.ctypes.data_as(ctypes.POINTER(ctypes.c_double)),
        b.ctypes.data_as(ctypes.POINTER(ctypes.c_double)),
        axb.ctypes.data_as(ctypes.POINTER(ctypes.c_double))
    )
```

Python code

```
fig=plt.figure()
ax=fig.add_subplot(111, projection='3d')

ax.quiver(0, 0, 0, a[0], a[1], a[2], color='b',
    arrow_length_ratio=0.1)
ax.quiver(0, 0, 0, b[0], b[1], b[2], color='g',
    arrow_length_ratio=0.1)
ax.quiver(0, 0, 0, axb[0], axb[1], axb[2], color='r',
    arrow_length_ratio=0.1)
```

Python code

```python

label = f'({a[0]}, {a[1]}, {a[2]})'
ax.text(a[0], a[1], a[2], s=label, color='b', fontsize=10)

label = f'({b[0]}, {b[1]}, {b[2]})'
ax.text(b[0], b[1], b[2], s=label, color='g', fontsize=10)

label = f'({axb[0]}, {axb[1]}, {axb[2]})'
ax.text(axb[0], axb[1], axb[2], s=label, color='r', fontsize=10)
```

Python code

```
ax.set_xlim([-10, 10])
ax.set_ylim([-10, 10])
ax.set_zlim([-10, 10])
plt.title('3D Projection of Vectors a, b, and a x b')
plt.savefig('/home/shreyas/GVV_Assignments/matgeo/2.10.79/figs/
    fig1.png')

plt.grid(True)
plt.show()
```

3D Plot

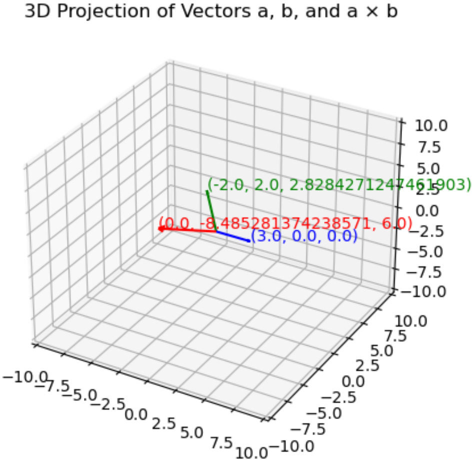

Figure: 3D Plot