8.2.33

EE25BTECH11065 - Yoshita J

Question:
Find the equation of the conic with length of major axis 26, foci $(\pm 5, 0)$.

Solution:
The given foci are $\mathbf{F}_1 = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$ and $\mathbf{F}_2 = \begin{pmatrix} -5 \\ 0 \end{pmatrix}$.

The center of the conic is the midpoint of the foci:

$$\mathbf{u} = \frac{\mathbf{F}_1 + \mathbf{F}_2}{2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1}$$

The length of the major axis is given as $2a = 26$ So, The distance from the center to a focus is $c = 5$.
Eccentricity:

$$e = \frac{c}{a} = \frac{5}{13} \tag{2}$$

The general equation of a conic is given by :

$$g(\mathbf{x}) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \tag{3}$$

where, x is a vertex on the major axis,

Since the center $\mathbf{u} = \mathbf{0}$, the equation simplifies to

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + f = 0. \tag{4}$$

where,

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^T \tag{5}$$

This simplifies to:

$$\mathbf{V} = \begin{pmatrix} 1 - e^2 & 0 \\ 0 & 1 \end{pmatrix} \tag{6}$$

Substituting,

$$\mathbf{V} = \begin{pmatrix} 1-(5/13)^2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 144/169 & 0 \\ 0 & 1 \end{pmatrix} \tag{7}$$

Simplifying equation (5) and (4),

$$\begin{aligned} \begin{pmatrix} 13 & 0 \end{pmatrix} \begin{pmatrix} 144/169 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 13 \\ 0 \end{pmatrix} + f = 0 \\ 144 + f = 0 \Rightarrow f = -144 \end{aligned} \tag{8}$$

Final equation of the conic,

$$\mathbf{x}^T \begin{pmatrix} 144/169 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 144 = 0.$$