2.10.31

EE25BTECH11002 - Achat Parth Kalpesh

Question:
Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be three non coplanar vectors and $\mathbf{p}, \mathbf{q}, \mathbf{r}$ are vectors defined by the relations

$$\mathbf{p} = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}, \mathbf{q} = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}, \mathbf{r} = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} \tag{0.1}$$

then the value of the expression $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{p} + (\mathbf{b} + \mathbf{c}) \cdot \mathbf{q} + (\mathbf{c} + \mathbf{a}) \cdot \mathbf{r}$ is equal to

1) 0 2) 1 3) 2 4) 3

Solution:
Let the given expression be:

$$E = (\mathbf{a} + \mathbf{b}) \cdot \mathbf{p} + (\mathbf{b} + \mathbf{c}) \cdot \mathbf{q} + (\mathbf{c} + \mathbf{a}) \cdot \mathbf{r} \tag{4.1}$$

$$= (\mathbf{a} \cdot \mathbf{p} + \mathbf{b} \cdot \mathbf{p}) + (\mathbf{b} \cdot \mathbf{q} + \mathbf{c} \cdot \mathbf{q}) + (\mathbf{c} \cdot \mathbf{r} + \mathbf{a} \cdot \mathbf{r}) \tag{4.2}$$

Let ,

$$\mathbf{V} = \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{pmatrix} \tag{4.3}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p} & \mathbf{q} & \mathbf{r} \end{pmatrix} \tag{4.4}$$

$$\mathbf{V}^\top \mathbf{P} = \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} \begin{pmatrix} \mathbf{p} & \mathbf{q} & \mathbf{r} \end{pmatrix} = \begin{pmatrix} \mathbf{a} \cdot \mathbf{p} & \mathbf{a} \cdot \mathbf{q} & \mathbf{a} \cdot \mathbf{r} \\ \mathbf{b} \cdot \mathbf{p} & \mathbf{b} \cdot \mathbf{q} & \mathbf{b} \cdot \mathbf{r} \\ \mathbf{c} \cdot \mathbf{p} & \mathbf{c} \cdot \mathbf{q} & \mathbf{c} \cdot \mathbf{r} \end{pmatrix} \tag{4.5}$$

$$\mathbf{a} \cdot \mathbf{p} = \mathbf{a} \cdot \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} = \frac{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} = 1 \tag{4.6}$$

$$\mathbf{a} \cdot \mathbf{q} = \mathbf{a} \cdot \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} = \frac{[\mathbf{a}\ \mathbf{c}\ \mathbf{a}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} = 0 \tag{4.7}$$

since the scalar triple product with a repeated vector is zero. Thus, the matrix product becomes the identity matrix:

$$\begin{pmatrix} \mathbf{a} \cdot \mathbf{p} & \mathbf{a} \cdot \mathbf{q} & \mathbf{a} \cdot \mathbf{r} \\ \mathbf{b} \cdot \mathbf{p} & \mathbf{b} \cdot \mathbf{q} & \mathbf{b} \cdot \mathbf{r} \\ \mathbf{c} \cdot \mathbf{p} & \mathbf{c} \cdot \mathbf{q} & \mathbf{c} \cdot \mathbf{r} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I} \tag{4.8}$$

Substituting these results back into the expanded expression for E:

$$E = (1 + 0) + (1 + 0) + (1 + 0) \tag{4.9}$$
$$= 1 + 1 + 1 \tag{4.10}$$
$$= 3 \tag{4.11}$$

The value of the expression is 3.