# 2.5.20

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**Question** : Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three vectors with $\|\mathbf{a}\| = 1,\ \|\mathbf{b}\| = 2,\ \|\mathbf{c}\| = 3$. If the projection of $\mathbf{b}$ on $\mathbf{a}$ equals the projection of $\mathbf{c}$ on $\mathbf{a}$, and $\mathbf{b} \perp \mathbf{c}$, find

$$\left\|3\mathbf{a} - 2\mathbf{b} + 2\mathbf{c}\right\|. \tag{0.1}$$

**Solution (matrix / rank style).** : Let us denote the scalar products

$$x = \mathbf{a}\cdot\mathbf{b}, \qquad y = \mathbf{a}\cdot\mathbf{c}, \qquad z = \mathbf{b}\cdot\mathbf{c}. \tag{0.2}$$

Given: projection of $\mathbf{b}$ on $\mathbf{a}$ equals projection of $\mathbf{c}$ on $\mathbf{a}$. Since $\|\mathbf{a}\| = 1$, this implies

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{c} \quad \Rightarrow \quad x = y. \tag{0.3}$$

Also $\mathbf{b} \perp \mathbf{c} \Rightarrow z = 0$. Using the magnitudes,

$$\mathbf{a}\cdot\mathbf{a} = 1, \quad \mathbf{b}\cdot\mathbf{b} = 4, \quad \mathbf{c}\cdot\mathbf{c} = 9. \tag{0.4}$$

Form the Gram (inner-product) matrix of $(\mathbf{a}, \mathbf{b}, \mathbf{c})$:

$$G = \begin{pmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{pmatrix} = \begin{pmatrix} 1 & x & x \\ x & 4 & 0 \\ x & 0 & 9 \end{pmatrix}, \tag{0.5}$$

where we used $x = y$ and $z = 0$.

Now denote the coefficient vector of $3\mathbf{a} - 2\mathbf{b} + 2\mathbf{c}$ relative to $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ by

$$\mathbf{u} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}. \tag{0.6}$$

Then the squared norm is the quadratic form

$$\left\|3\mathbf{a} - 2\mathbf{b} + 2\mathbf{c}\right\|^2 = \mathbf{u}^\top G\,\mathbf{u}. \tag{0.7}$$

Compute $G\mathbf{u}$ (this step may be viewed as simple row-combination / row-operation arithmetic):

$$G\mathbf{u} = \begin{pmatrix} 1 & x & x \\ x & 4 & 0 \\ x & 0 & 9 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1\cdot 3 + x\cdot(-2) + x\cdot 2 \\ x\cdot 3 + 4\cdot(-2) + 0\cdot 2 \\ x\cdot 3 + 0\cdot(-2) + 9\cdot 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3x - 8 \\ 3x + 18 \end{pmatrix}. \tag{0.8}$$

Now take the inner product with

$$\mathbf{u}^\top = (3, -2, 2) \tag{0.9}$$

:

$$\mathbf{u}^\top(G\mathbf{u}) = (3, -2, 2) \cdot (3,\ 3x - 8,\ 3x + 18) \tag{0.10}$$

$$= 3 \cdot 3 + (-2)(3x - 8) + 2(3x + 18). \tag{0.11}$$

$$= 9 - 6x + 16 + 6x + 36 \tag{0.12}$$

$$= 9 + 16 + 36 + (-6x + 6x) = 61. \tag{0.13}$$

Crucially the terms in $x$ cancel out, so the value does not depend on $x$.
Therefore

$$\left\|3\mathbf{a} - 2\mathbf{b} + 2\mathbf{c}\right\|^2 = 61 \quad \implies \quad \boxed{\left\|3\mathbf{a} - 2\mathbf{b} + 2\mathbf{c}\right\| = \sqrt{61}}. \tag{0.14}$$

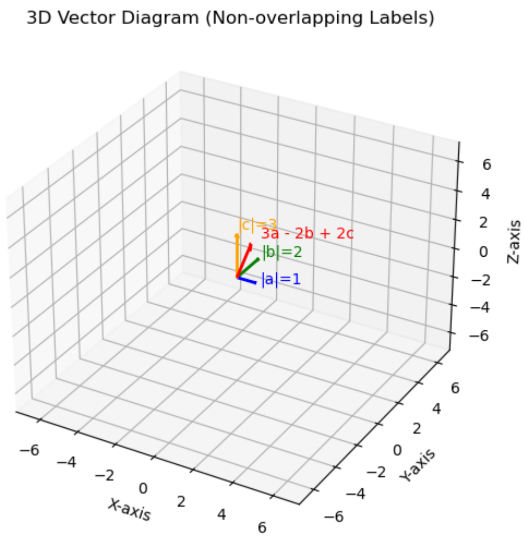


Fig: Representation of vectors