

10.7.50

EE25BTECH11001 - Aarush Dilawri

Question:

Consider the family of circles $x^2 + y^2 = r^2$, $2 < r < 5$. If in the first quadrant, the common tangent to a circle of this family and the ellipse $4x^2 + 25y^2 = 100$ meets the coordinate axes at **A** and **B**, then find the equation of the locus of the midpoint of AB .

Solution:

$$\text{The family of circles is } \mathbf{X}^\top \mathbf{X} = r^2, \quad 2 < r < 5, \quad (1)$$

$$\text{and the ellipse is } \mathbf{X}^\top \mathbf{V} \mathbf{X} = 100, \quad \mathbf{V} = \begin{pmatrix} 4 & 0 \\ 0 & 25 \end{pmatrix}. \quad (2)$$

Let the common tangent meet the coordinate axes at

$$\mathbf{A} = a\mathbf{e}_1, \quad \mathbf{B} = b\mathbf{e}_2, \quad a, b > 0. \quad (3)$$

The equation of the line passing through **A** and **B** can be written as

$$\frac{\mathbf{e}_1^\top \mathbf{X}}{a} + \frac{\mathbf{e}_2^\top \mathbf{X}}{b} = 1. \quad (4)$$

This is of the form $\mathbf{n}^\top \mathbf{X} = c$, with

$$\mathbf{n} = \begin{pmatrix} \frac{1}{a} \\ \frac{1}{b} \end{pmatrix}, \quad c = 1. \quad (5)$$

Let the midpoint of **A** and **B** be

$$\mathbf{m} = \frac{\mathbf{A} + \mathbf{B}}{2}. \quad (6)$$

From this,

$$a = 2\mathbf{e}_1^\top \mathbf{m}, \quad b = 2\mathbf{e}_2^\top \mathbf{m}. \quad (7)$$

For the line $\mathbf{n}^\top \mathbf{X} = c$ to be tangent to $\mathbf{X}^\top \mathbf{V} \mathbf{X} = 100$, the condition is $c^2 = 100 \mathbf{n}^\top \mathbf{V}^{-1} \mathbf{n}$.

$$\text{Here } c = 1, \quad \mathbf{V}^{-1} = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{25} \end{pmatrix}. \text{ Substituting gives } 1 = 100 \left(\frac{1}{4a^2} + \frac{1}{25b^2} \right). \quad (8)$$

$$\Rightarrow \frac{25}{a^2} + \frac{4}{b^2} = 1. \quad (9)$$

Also, for the line $\mathbf{n}^\top \mathbf{X} = c$ to be tangent to the circle $\mathbf{X}^\top \mathbf{X} = r^2$, the distance from the origin must equal r .

$$\frac{|c|}{\|\mathbf{n}\|} = r. \quad (10)$$

$$\text{With } c = 1 \text{ this gives } \|\mathbf{n}\|^2 = \frac{1}{r^2}. \text{ So } \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{r^2}. \quad (11)$$

We now express the ellipse tangency condition in terms of \mathbf{m} .
Substitute $a = 2\mathbf{e}_1^\top \mathbf{m}$, $b = 2\mathbf{e}_2^\top \mathbf{m}$:

$$\frac{25}{4(\mathbf{e}_1^\top \mathbf{m})^2} + \frac{4}{4(\mathbf{e}_2^\top \mathbf{m})^2} = 1. \quad (12)$$

$$\Rightarrow \left(4(\mathbf{e}_1^\top \mathbf{m})^2 - 25\right)(\mathbf{e}_2^\top \mathbf{m})^2 = 4(\mathbf{e}_1^\top \mathbf{m})^2. \quad (13)$$

$$\text{Or equivalently, } 4(\mathbf{e}_1^\top \mathbf{m})^2 (\mathbf{e}_2^\top \mathbf{m})^2 - 4(\mathbf{e}_1^\top \mathbf{m})^2 - 25(\mathbf{e}_2^\top \mathbf{m})^2 = 0. \quad (14)$$

$$\text{Finally, let } \mathbf{m} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{e}_1^\top \mathbf{m} = x, \quad \mathbf{e}_2^\top \mathbf{m} = y. \quad (15)$$

$$\text{Substituting gives the locus equation } 4x^2y^2 - 4x^2 - 25y^2 = 0. \quad (16)$$

$$\text{Required locus: } 4x^2y^2 - 4x^2 - 25y^2 = 0. \quad (17)$$

See Fig. 0 ,

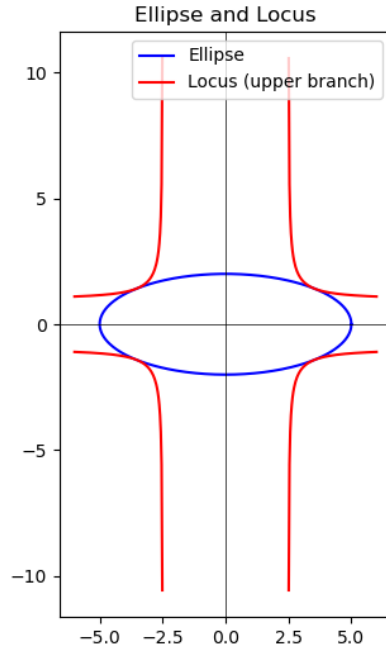


Fig. 0