2.10.76

EE25BTECH11048 - Revanth Siva Kumar

Question: For any three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, prove or disprove

$$(\mathbf{a} - \mathbf{b}) \cdot ((\mathbf{b} - \mathbf{c}) \times (\mathbf{c} - \mathbf{a})) = 2\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

Solution: We write the scalar triple product in determinant form:

$$(\mathbf{a} - \mathbf{b}) \cdot ((\mathbf{b} - \mathbf{c}) \times (\mathbf{c} - \mathbf{a})) = \det \begin{pmatrix} (\mathbf{a} - \mathbf{b})^T \\ (\mathbf{b} - \mathbf{c})^T \\ (\mathbf{c} - \mathbf{a})^T \end{pmatrix}. \tag{1}$$

Now observe that

$$(\mathbf{a} - \mathbf{b}) + (\mathbf{b} - \mathbf{c}) + (\mathbf{c} - \mathbf{a}) = \mathbf{0}. \tag{2}$$

Thus the three rows of the determinant are linearly dependent. From matrix theory, the determinant of a matrix with linearly dependent rows is zero. Hence

$$(\mathbf{a} - \mathbf{b}) \cdot ((\mathbf{b} - \mathbf{c}) \times (\mathbf{c} - \mathbf{a})) = 0. \tag{3}$$

On the other hand, the right-hand side is

$$2\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 2 \det \begin{pmatrix} \mathbf{a}^T \\ \mathbf{b}^T \\ \mathbf{c}^T \end{pmatrix}, \tag{4}$$

which is not identically zero for arbitrary $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

Conclusion: The given statement is **false**. The left-hand side is always zero, while the right-hand side can be nonzero.