

Question

Problem

Given points $A(1, -2)$, $B(2, 3)$, $C(a, 2)$ and $D(-4, -3)$ which form a parallelogram.

- Find the value of a .
- Find the height of the parallelogram taking AB as base.

Step 1: Represent the Vectors

Represent the points as vectors:

$$\mathbf{A} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} a \\ 2 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}. \quad (1)$$

Condition for parallelogram (diagonals bisect):

$$\mathbf{A} + \mathbf{C} = \mathbf{B} + \mathbf{D}. \quad (2)$$

Step 2: Solve for a

Substituting:

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} a \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -4 \\ -3 \end{pmatrix}. \quad (3)$$

Simplify RHS:

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -4 \\ -3 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}. \quad (4)$$

Thus:

$$\begin{pmatrix} 1 + a \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}. \quad (5)$$

So,

$$1 + a = -2 \implies a = -3. \quad (6)$$

Hence,

$$\mathbf{c} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}. \quad (7)$$

Step 3: Base and Side Vectors

Base vector:

$$\mathbf{AB} = \mathbf{B} - \mathbf{A} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}. \quad (8)$$

Side vector:

$$\mathbf{AD} = \mathbf{D} - \mathbf{A} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}. \quad (9)$$

Step 4: Projection of **AD** on **AB**

Projection of **AD** on **AB**:

$$\text{proj}_{\mathbf{AB}}(\mathbf{AD}) = \frac{\mathbf{AB}^T \mathbf{AD}}{\mathbf{AB}^T \mathbf{AB}} \mathbf{AB}. \quad (10)$$

Compute inner products:

$$\mathbf{AB}^T \mathbf{AD} = 1(-5) + 5(-1) = -10, \quad \mathbf{AB}^T \mathbf{AB} = 1^2 + 5^2 = 26. \quad (11)$$

So,

$$\text{proj}_{\mathbf{AB}}(\mathbf{AD}) = \frac{-10}{26} \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} -\frac{5}{13} \\ -\frac{25}{13} \end{pmatrix}. \quad (12)$$

Step 5: Perpendicular Component

Perpendicular component:

$$\mathbf{r} = \mathbf{AD} - \text{proj}_{\mathbf{AB}}(\mathbf{AD}) = \begin{pmatrix} -5 \\ -1 \end{pmatrix} - \begin{pmatrix} -\frac{5}{13} \\ -\frac{25}{13} \end{pmatrix} = \begin{pmatrix} -\frac{60}{13} \\ \frac{12}{13} \end{pmatrix}. \quad (13)$$

Height:

$$h = \|\mathbf{r}\| = \sqrt{\left(-\frac{60}{13}\right)^2 + \left(\frac{12}{13}\right)^2}. \quad (14)$$

Final Answer

$$h = \frac{\sqrt{3744}}{13} = \frac{12\sqrt{26}}{13} = \frac{24}{\sqrt{26}}$$

$$a = -3,$$

$$h = \frac{12\sqrt{26}}{13}$$

C Code (1/2)

```
#include <stdio.h>
#include <math.h>

double norm(double v[2]) {
    return sqrt(v[0]*v[0] + v[1]*v[1]);
}

int main() {
    double A[2] = {1, -2};
    double B[2] = {2, 3};
    double D[2] = {-4, -3};
    double C[2];

    // Find 'a' using parallelogram condition:  $A + C = B + D$ 
    C[0] = (B[0] + D[0]) - A[0];
    C[1] = 2; // given y-coordinate
    printf("a = %lf\n", C[0]);
}
```


C Code (2/2)

```
// Vectors
double u[2] = {B[0]-A[0], B[1]-A[1]};
double v[2] = {C[0]-A[0], C[1]-A[1]};

// Projection of v on u
double dot_uv = u[0]*v[0] + u[1]*v[1];
double dot_uu = u[0]*u[0] + u[1]*u[1];
double proj[2] = { (dot_uv/dot_uu)*u[0], (dot_uv/dot_uu)*u[1]
};

// Perpendicular component
double w[2] = { v[0]-proj[0], v[1]-proj[1] };
double h = norm(w);

printf("Height = %lf\n", h);
return 0;
}
```

Python Code (1/2)

```
import numpy as np
import matplotlib.pyplot as plt

def norm(v):
    return np.sqrt(np.dot(v, v))

# Points
A = np.array([1, -2], dtype=float)
B = np.array([2, 3], dtype=float)
C = np.array([None, 2], dtype=float)
D = np.array([-4, -3], dtype=float)

# Find a using parallelogram condition
a_val = (B[0] + D[0]) - A[0]
C[0] = a_val
print("a =", a_val)
```

Python Code (2/2)

```
# Base and AC vectors
```

```
u = B - A
```

```
v = C - A
```

```
# Projection
```

```
proj = (np.dot(v,u)/np.dot(u,u)) * u
```

```
w = v - proj
```

```
h = norm(w)
```

```
print("Height =", h)
```

```
# Plot
```

```
fig, ax = plt.subplots()
```

```
x_vals = [A[0],B[0],C[0],D[0],A[0]]
```

```
y_vals = [A[1],B[1],C[1],D[1],A[1]]
```

```
ax.plot(x_vals, y_vals, 'k-')
```

```
ax.plot([C[0], (A+proj)[0]], [C[1], (A+proj)[1]], 'r-')
```

```
plt.show()
```

Parallelogram Plot

