MatGeo Assignment 3.2.23

AI25BTECH11007

Question:

Construct a triangle ABC in which

$$BC = 5\,\text{cm}, \quad \angle B = 60^\circ, \quad \text{and} \quad AC + AB = 7.5\,\text{cm}.$$

Solution:

Set up points and given data

Let

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \mathbf{C} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}.$$

$$\mathbf{AC} = \mathbf{c}, \qquad \mathbf{AC} = \mathbf{b}, \qquad \mathbf{b} + \mathbf{c} = 7.5.$$

Position vector of point A

Since $\angle B = 60^\circ$, the vector $\mathbf{BA}$ has length c and direction 60° above the x-axis. Thus

$$\mathbf{A} = \mathbf{c}\begin{pmatrix} \cos 60^\circ \\ \sin 60^\circ \end{pmatrix} = \begin{pmatrix} \frac{\mathbf{c}}{2} \\ \frac{\mathbf{c}\sqrt{3}}{2} \end{pmatrix}.$$

Expression for AC

$$\mathbf{AC} = \mathbf{C} - \mathbf{A} = \begin{pmatrix} 5 - \frac{\mathbf{c}}{2} \\ -\frac{\mathbf{c}\sqrt{3}}{2} \end{pmatrix},$$

and

$$b^2 = \left(5 - \tfrac{\mathbf{c}}{2}\right)^2 + \tfrac{3\mathbf{c}^2}{4}.$$

Apply $b + c = 7.5$

Since $b = 7.5 - c$, we have

$$(7.5 - c)^2 = \left(5 - \tfrac{c}{2}\right)^2 + \tfrac{3c^2}{4}.$$

Expanding and simplifying gives,

$$56.25 - 15c + c^2 = 25 - 5c + c^2,$$

$$c = 3.125.$$

Hence

$$b = 7.5 - 3.125 = 4.375.$$

Coordinates of vertices

$$\mathbf{A} = \begin{pmatrix} \frac{3.125}{2} \\ \frac{3.125\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} 1.5625 \\ 2.7050\ldots \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}.$$

Verification

$$\mathbf{BA} \cdot \mathbf{BC} = \tfrac{5c}{2}, \qquad |\mathbf{BA}| = c, \quad |\mathbf{BC}| = 5,$$

$$\cos \angle B = \frac{\frac{5c}{2}}{5c} = \tfrac{1}{2} = \cos 60^\circ.$$

Final Answer,

$$\boxed{\mathbf{A} = \begin{pmatrix} 1.5625 \\ 2.7050 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}}$$

with $AB = 3.125$ cm, $AC = 4.375$ cm, and $BC = 5$ cm.

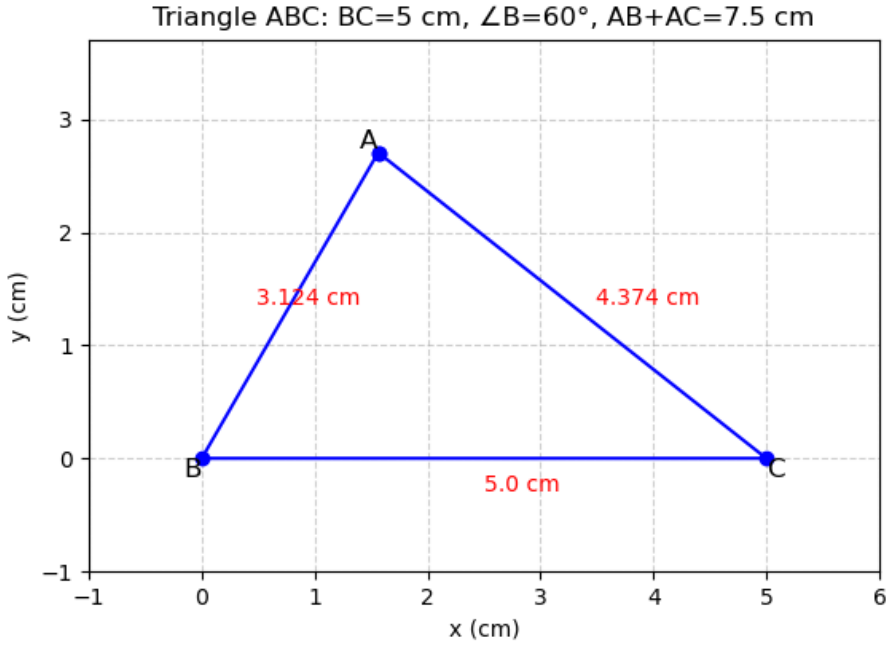

Fig. 0.1: Construction Plot