

# 4.3.37

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## Question:

Find the equations of the two lines passing through the origin which intersect the line  $\frac{x-3}{2} = \frac{y-3}{1} = \frac{z}{1}$  at angles of  $\frac{\pi}{3}$  each.

## Solution:

The given line can be expressed as

$$\mathbf{x} = \mathbf{h} + \kappa \mathbf{m} \quad (0.1)$$

$$\text{where } \mathbf{h} = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{m} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \quad (0.2)$$

Any point  $\mathbf{P}$  on this line can be given as

$$\mathbf{P} = \mathbf{h} + \kappa \mathbf{m} \quad (0.3)$$

The line through the origin and  $\mathbf{P}$  will have direction vector  $\mathbf{P}$ .

Since the angle between  $\mathbf{m}$  and  $\mathbf{P}$  is  $\frac{\pi}{3}$ ,

$$\cos \theta = \frac{\mathbf{m}^\top \mathbf{P}}{\|\mathbf{m}\| \|\mathbf{P}\|} \quad (0.4)$$

$$\implies (\mathbf{m}^\top \mathbf{P})^2 = \cos^2 \theta (\mathbf{m}^\top \mathbf{m})(\mathbf{P}^\top \mathbf{P}). \quad (0.5)$$

Substituting  $\mathbf{P} = \mathbf{h} + \kappa \mathbf{m}$  and solving, we get a quadratic equation in  $\kappa$

$$\kappa^2 (\mathbf{m}^\top \mathbf{m})^2 \sin^2 \theta + 2\kappa (\mathbf{m}^\top \mathbf{m})(\mathbf{m}^\top \mathbf{h}) \sin^2 \theta + (\mathbf{m}^\top \mathbf{h})^2 - \mathbf{m}^\top \mathbf{m} \cos^2 \theta \mathbf{h}^\top \mathbf{h} = 0 \quad (0.6)$$

Plugging in the values,

$$27\kappa^2 + 81\kappa + 54 = 0 \quad (0.7)$$

$$\kappa^2 + 3\kappa + 2 = 0 \quad (0.8)$$

$$\implies \kappa = -1, -2 \quad (0.9)$$

Therefore, the direction vectors of the lines are

$$\mathbf{x} = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} - 2 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \quad (0.10)$$

Therefore, the equations of the lines are

$$\mathbf{x} = \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \mu \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \quad (0.11)$$

See Figure,

