10.3.21

EE25BTECH11020 - Darsh Pankaj Gajare

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Question:
Find the point at which the line $y = x + 1$ is a tangent to the curve $y^2 = 4x$.
Solution: The given conic can be expressed as

$$\mathbf{x}^\top V\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f = 0 \tag{0.1}$$

where

$$V = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \quad f = 0 \tag{0.2}$$

and $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$.
The given line is

$$y = x + 1 \tag{0.3}$$

which can be parameterized as

$$\mathbf{x} = \mathbf{h} + t\mathbf{m} \tag{0.4}$$

where

$$\mathbf{h} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{0.5}$$

Substituting $\mathbf{x} = \mathbf{h} + t\mathbf{m}$ in the conic equation,

$$(\mathbf{h} + t\mathbf{m})^\top V(\mathbf{h} + t\mathbf{m}) + 2\mathbf{u}^\top(\mathbf{h} + t\mathbf{m}) + f = 0 \tag{0.6}$$

Expanding,

$$t^2 \left(\mathbf{m}^\top V\mathbf{m}\right) + 2t \left(\mathbf{m}^\top V\mathbf{h} + \mathbf{u}^\top\mathbf{m}\right) + \left(\mathbf{h}^\top V\mathbf{h} + 2\mathbf{u}^\top\mathbf{h} + f\right) = 0 \tag{0.7}$$

Compute each term:

$$\mathbf{m}^\top V\mathbf{m} = 1, \qquad \mathbf{m}^\top V\mathbf{h} = 1, \qquad \mathbf{u}^\top\mathbf{m} = -2, \tag{0.8}$$

$$\mathbf{h}^\top V\mathbf{h} = 1, \qquad \mathbf{u}^\top\mathbf{h} = 0 \tag{0.9}$$

Substituting,

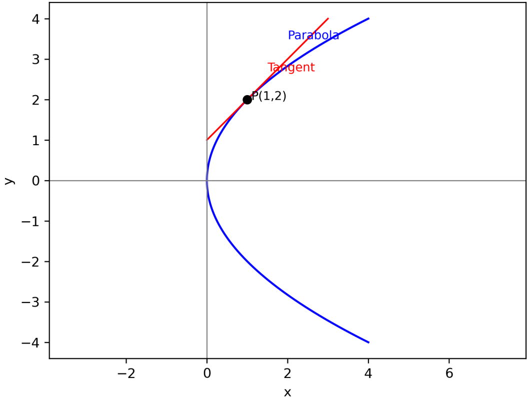
$$t^2 - 2t + 1 = 0 \tag{0.10}$$

$$\Rightarrow (t-1)^2 = 0 \implies t = 1 \tag{0.11}$$

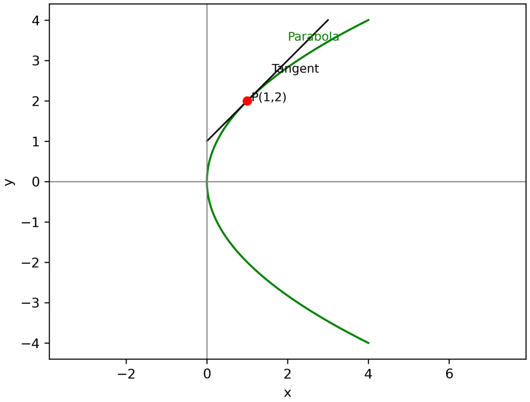
Hence, the point of contact is

$$\mathbf{q} = \mathbf{h} + t\mathbf{m} \tag{0.12}$$

$$\mathbf{q} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{0.13}$$



Plot using C libraries:



Plot using Python: