2.10.56

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Question. Let two non-collinear unit vectors $\hat{a}$ and $\hat{b}$ form an acute angle. A point **P** moves so that at any time t the position vector $\overrightarrow{OP}$ (where **O** is the origin) is given by $\hat{\mathbf{a}}\cos t + \hat{\mathbf{b}}\sin t$. When **P** is farthest from origin **O**, let M be the length of $\overrightarrow{OP}$ and $\hat{\mathbf{u}}$ be the unit vector along $\overrightarrow{OP}$. Then,

1) $\hat{\mathbf{u}} = \frac{\hat{\mathbf{a}}+\hat{\mathbf{b}}}{|\hat{\mathbf{a}}+\hat{\mathbf{b}}|}$ and $M = (1 + \hat{\mathbf{a}} \cdot \hat{\mathbf{b}})^{\frac{1}{2}}$
2) $\hat{\mathbf{u}} = \frac{\hat{\mathbf{a}}-\hat{\mathbf{b}}}{|\hat{\mathbf{a}}-\hat{\mathbf{b}}|}$ and $M = (1 + \hat{\mathbf{a}} \cdot \hat{\mathbf{b}})^{\frac{1}{2}}$
3) $\hat{\mathbf{u}} = \frac{\hat{\mathbf{a}}+\hat{\mathbf{b}}}{|\hat{\mathbf{a}}+\hat{\mathbf{b}}|}$ and $M = (1 + 2\hat{\mathbf{a}} \cdot \hat{\mathbf{b}})^{\frac{1}{2}}$
4) $\hat{\mathbf{u}} = \frac{\hat{\mathbf{a}}-\hat{\mathbf{b}}}{|\hat{\mathbf{a}}-\hat{\mathbf{b}}|}$ and $M = (1 + 2\hat{\mathbf{a}} \cdot \hat{\mathbf{b}})^{\frac{1}{2}}$

Solution:
Let us solve the given equation theoretically and then verify the solution computationally.
Given equation:

$$\mathbf{P} = \mathbf{a}\cos t + \mathbf{b}\sin t \tag{1}$$

Which can be written as :

$$\mathbf{P} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \tag{2}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \mathbf{x} \tag{3}$$

Let

$$\mathbf{x} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \quad and \quad \mathbf{G} = \begin{pmatrix} 1 & (\mathbf{a})^T(\mathbf{b}) \\ (\mathbf{a})^T(\mathbf{b}) & 1 \end{pmatrix} \tag{4}$$

From given if **P** is farthest from origin , then length of **P** is given as M.From this we can say that

$$M = \max \|\mathbf{P}\| \tag{5}$$

Now,

$$\|\mathbf{P}\| = \sqrt{(\mathbf{P})^T(\mathbf{P})} \tag{6}$$

$$\|\mathbf{P}\| = \sqrt{\left(\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \mathbf{x}\right)^T \left(\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \mathbf{x}\right)} \tag{7}$$

$$\|\mathbf{P}\| = \sqrt{\mathbf{x}^T \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix}^T \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \mathbf{x}} \tag{8}$$

Let **G** be a gram matrix:

$$\mathbf{G} = \begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix}^T \begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix} = \begin{pmatrix} 1 & (\mathbf{a})^T(\mathbf{b}) \\ (\mathbf{a})^T(\mathbf{b}) & 1 \end{pmatrix} \tag{9}$$

$$||\mathbf{P}||^2 = \mathbf{x}^T \begin{pmatrix} 1 & (\mathbf{a})^T(\mathbf{b}) \\ (\mathbf{a})^T(\mathbf{b}) & 1 \end{pmatrix} \mathbf{x} \tag{10}$$

$$||\mathbf{P}||^2 = \mathbf{x}^T \mathbf{G} \mathbf{x} \tag{11}$$

Now we should find the maximum value of $x^T G x$ such that $||x|| = 1$

By **Rayleigh-Ritz theorem**, the maximum value of the quadratic form if $\mathbf{x}$ is a unit vector will be the largest eigenvalue (λ_{max}) of the matrix G.
So,

$$max\,||\mathbf{P}|| = \sqrt{\lambda_{max}} \tag{12}$$

Now we will calculate the Eigen value for the matrix G:

$$\left|\mathbf{G} - \lambda \mathbf{I}\right| = 0 \tag{13}$$

$$\left|\begin{pmatrix} 1-\lambda & (\mathbf{a})^T(\mathbf{b}) \\ (\mathbf{a})^T(\mathbf{b}) & 1-\lambda \end{pmatrix}\right| = 0 \tag{14}$$

$$(1-\lambda)^2 - ((\mathbf{a})^T(\mathbf{b}))^2 = 0 \tag{15}$$

$$1 - \lambda = (\mathbf{a})^T(\mathbf{b}) \;\; or \;\; 1 - \lambda = -(\mathbf{a})^T(\mathbf{b}) \tag{16}$$

$$\lambda = 1 + (\mathbf{a})^T(\mathbf{b}) \;\; or \;\; \lambda = 1 - (\mathbf{a})^T(\mathbf{b}) \tag{17}$$

It is already given that $(\mathbf{a})^T(\mathbf{b}) > 0$(**a** and **b** form an acute angle) . so,

$$\lambda_{max} = 1 + (\mathbf{a})^T(\mathbf{b}) \tag{18}$$

From Eq.12

$$max\,||\mathbf{P}|| = \sqrt{1 + (\mathbf{a})^T(\mathbf{b})} \tag{19}$$

The above equation can be written as

$$max\,||\mathbf{P}|| = \sqrt{1 + \mathbf{a}.\mathbf{b}} \tag{20}$$

From Eq.5:

$$M = \sqrt{1 + \mathbf{a}.\mathbf{b}} \tag{21}$$

Now let us find the value of t for which $||\mathbf{P}||$ is max
With eigenvalue equation,We'll use matrix G and largest eigenvalue λ_{max} such that,

$$\left(\mathbf{G} - \lambda \mathbf{I}\right)\mathbf{x} = 0 \tag{22}$$

$$\left(\begin{pmatrix} 1 & (\mathbf{a})^T(\mathbf{b}) \\ (\mathbf{a})^T(\mathbf{b}) & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right)\mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{23}$$

$$\begin{pmatrix} 1-\lambda & (a)^T(b) \\ (a)^T(b) & 1-\lambda \end{pmatrix}\mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{24}$$

By substituting $\lambda = 1 + (\mathbf{a})^T(\mathbf{b})$. We get:

$$\begin{pmatrix} -(\mathbf{a})^T(\mathbf{b}) & (\mathbf{a})^T(\mathbf{b}) \\ (\mathbf{a})^T(\mathbf{b}) & -(\mathbf{a})^T(\mathbf{b}) \end{pmatrix}\mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{25}$$

$$\begin{pmatrix} -(\mathbf{a})^T(\mathbf{b}) & (\mathbf{a})^T(\mathbf{b}) \\ (\mathbf{a})^T(\mathbf{b}) & -(\mathbf{a})^T(\mathbf{b}) \end{pmatrix}\begin{pmatrix} \cos t \\ \sin t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{26}$$

$$\begin{pmatrix} -(\mathbf{a})^T(\mathbf{b})\cos t + (\mathbf{a})^T(\mathbf{b})\sin t \\ (\mathbf{a})^T(\mathbf{b})\cos t - (\mathbf{a})^T(\mathbf{b})\sin t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{27}$$

$$-(\mathbf{a})^T(\mathbf{b})\cos t + (\mathbf{a})^T(\mathbf{b})\sin t = 0 \tag{28}$$

$$(\mathbf{a})^T(\mathbf{b})\cos t = (\mathbf{a})^T(\mathbf{b})\sin t \tag{29}$$

$$\cos t = \sin t \tag{30}$$

We already know that:

$$\sin^2 t + \cos^2 t = 1 \tag{31}$$

So,

$$\sin t = \frac{1}{\sqrt{2}} \quad and \quad \cos t = \frac{1}{\sqrt{2}} \tag{32}$$

From above result

$$t = \frac{\pi}{4} \tag{33}$$

Now unit vector $\mathbf{u}$ along $\mathbf{P}$ is given by:

$$\mathbf{u} = \frac{\mathbf{P}}{||\mathbf{P}||} \tag{34}$$

$$\mathbf{u} = \frac{\mathbf{a}\cos t + \mathbf{b}\sin t}{||\mathbf{a}\cos t + \mathbf{b}\sin t||} \tag{35}$$

Now subtituiting the value of t in above equation:

$$\mathbf{u} = \frac{\mathbf{a}\frac{1}{\sqrt{2}} + \mathbf{b}\frac{1}{\sqrt{2}}}{\left\|\mathbf{a}\frac{1}{\sqrt{2}} + \mathbf{b}\frac{1}{\sqrt{2}}\right\|} \tag{36}$$

$$\mathbf{u} = \frac{\mathbf{a} + \mathbf{b}}{\|\mathbf{a} + \mathbf{b}\|} \tag{37}$$

From Eq.21 and Eq.37 (a) is correct
From the figure it is clearly verified that the theoretical solution matches with the computational solution.