2.10.55

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Problem Statement
The edges of a parallelepiped are of unit length and are parallel to non-coplanar unit vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ such that $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a} = \frac{1}{2}$. Then, the volume of the parallelepiped is

(a) $\dfrac{1}{\sqrt{2}}$
(b) $\dfrac{1}{2\sqrt{2}}$
(c) $\dfrac{\sqrt{5}}{2}$
(d) $\dfrac{1}{\sqrt{3}}$

Solution:

Symbol	Value / Definition	Description
$\mathbf{a}, \mathbf{b}, \mathbf{c}$	$\lvert\mathbf{a}\rvert = \lvert\mathbf{b}\rvert = \lvert\mathbf{c}\rvert = 1$	Non-coplanar unit vectors for the parallelepiped edges.
$\mathbf{a} \cdot \mathbf{b}, \mathbf{b} \cdot \mathbf{c}, \mathbf{c} \cdot \mathbf{a}$	$\dfrac{1}{2}$	The dot product between any pair of the vectors.
A	$\begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{pmatrix}$	A 3×3 matrix with the edge vectors as its columns.
V	$\lvert\det(A)\rvert$	The volume of the parallelepiped (the value to be found).

Using Gram matrix, $G = A^\top A$.

$$G = A^\top A = \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{pmatrix}^\top \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{pmatrix} = \begin{pmatrix} \mathbf{a} \cdot \mathbf{a} & \mathbf{a} \cdot \mathbf{b} & \mathbf{a} \cdot \mathbf{c} \\ \mathbf{b} \cdot \mathbf{a} & \mathbf{b} \cdot \mathbf{b} & \mathbf{b} \cdot \mathbf{c} \\ \mathbf{c} \cdot \mathbf{a} & \mathbf{c} \cdot \mathbf{b} & \mathbf{c} \cdot \mathbf{c} \end{pmatrix} \tag{2.10.55.1}$$

Substituting the given values into the Gram matrix:

$$G = \begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix} \tag{2.10.55.2}$$

The determinant of the Gram matrix is related to the determinant of A by:

$$\det(G) = \det(A^\top A) = (\det(A))^2 = V^2 \tag{2.10.55.3}$$

Therefore, the volume is $V = \sqrt{\det(G)}$. Calculating determinant of G we get,

$$\det(G) \tag{2.10.55.4}$$

$$= \det\begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix} \xrightarrow[R_3 \to R_3 - \frac{1}{2}R_1]{R_2 \to R_2 - \frac{1}{2}R_1} \det\begin{pmatrix} 1 & 1/2 & 1/2 \\ 0 & 3/4 & 1/4 \\ 0 & 1/4 & 3/4 \end{pmatrix} \tag{2.10.55.5}$$

$$= \det\begin{pmatrix} 1 & 1/2 & 1/2 \\ 0 & 3/4 & 1/4 \\ 0 & 1/4 & 3/4 \end{pmatrix} \xrightarrow{R_3 \to R_3 - \frac{1}{3}R_2} \det\begin{pmatrix} 1 & 1/2 & 1/2 \\ 0 & 3/4 & 1/4 \\ 0 & 0 & 2/3 \end{pmatrix} \tag{2.10.55.6}$$

$$= \frac{1}{2} = \det(G) \tag{2.10.55.7}$$

Therefore, volume V is,

$$V = \sqrt{\det(G)} = \frac{1}{\sqrt{2}} \tag{2.10.55.8}$$

Thus, the volume of the parallelepiped is $\frac{1}{\sqrt{2}}$, which corresponds to option (a). See Figure 2.10.55.1.

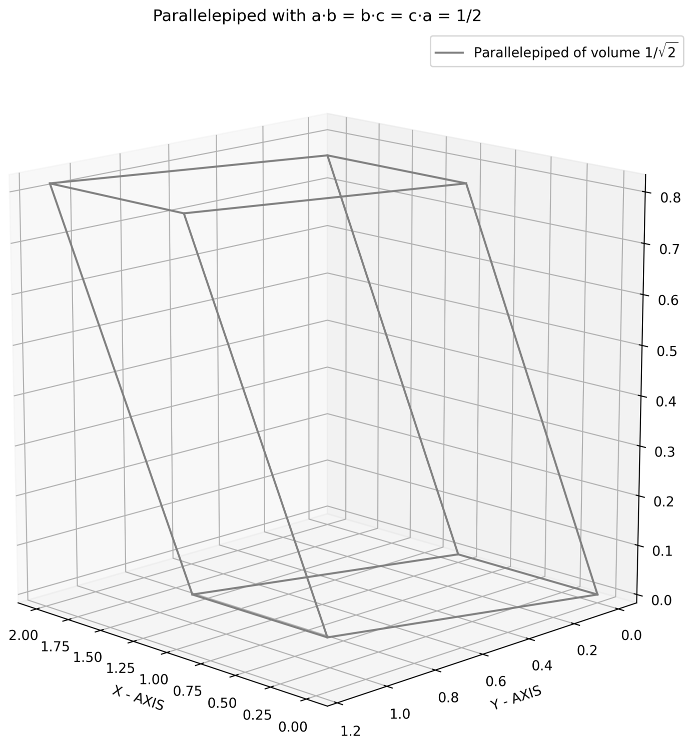
Parallelepiped with a·b = b·c = c·a = 1/2
Parallelepiped of volume $1/\sqrt{2}$
X - AXIS
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
Y - AXIS
0.0
0.2
0.4
0.6
0.8
1.0
1.2
Z - AXIS
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8

Fig. 2.10.55.1