Problem 9.8.34

Find the equation of the line passing through the points of intersection of the circles

$$3x^2 + 3y^2 - 2x + 12y - 9 = 0 \quad \text{and} \quad x^2 + y^2 + 6x + 2y - 15 = 0. \tag{1}$$

Input Variables (Conic form)

The general conic is

$$\mathbf{x}^\top V\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f = 0, \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}. \tag{2}$$

	C_1	C_2
V	$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$\mathbf{u}$	$\begin{pmatrix} -1 \\ 6 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 1 \end{pmatrix}$
f	-9	-15

Solution

All conics through the intersection points are given by the locus

$$\mathbf{x}^\top(V_1 + \mu V_2)\mathbf{x} + 2(\mathbf{u}_1 + \mu\mathbf{u}_2)^\top\mathbf{x} + (f_1 + \mu f_2) = 0. \tag{3}$$

Since we want a line, we eliminate the quadratic part by choosing μ such that

$$V_1 + \mu V_2 = (3 + \mu)I = 0, \tag{4}$$
$$\mu = -3. \tag{5}$$

Substituting $\mu = -3$, the equation of the required line becomes

$$2(\mathbf{u}_1 - 3\mathbf{u}_2)^\top\mathbf{x} + (f_1 - 3f_2) = 0. \tag{6}$$

Now we compute the coefficients:

$$\mathbf{u}_1 - 3\mathbf{u}_2 = \begin{pmatrix} -1 \\ 6 \end{pmatrix} - 3\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -10 \\ 3 \end{pmatrix}, \tag{7}$$
$$f_1 - 3f_2 = -9 - 3(-15) = 36. \tag{8}$$

Thus the line is

$$2\begin{pmatrix}-10 & 3\end{pmatrix}\mathbf{x} + 36 = 0. \tag{9}$$

Multiplying throughout by $-\frac{1}{2}$, we obtain

$$\begin{pmatrix}10 \\ -3\end{pmatrix}^{\top}\mathbf{x} - 18 = 0. \tag{10}$$

$$\boxed{\begin{pmatrix}10 \\ -3\end{pmatrix}^{\top}\mathbf{x} - 18 = 0} \tag{11}$$

This is the required line through the intersection points.

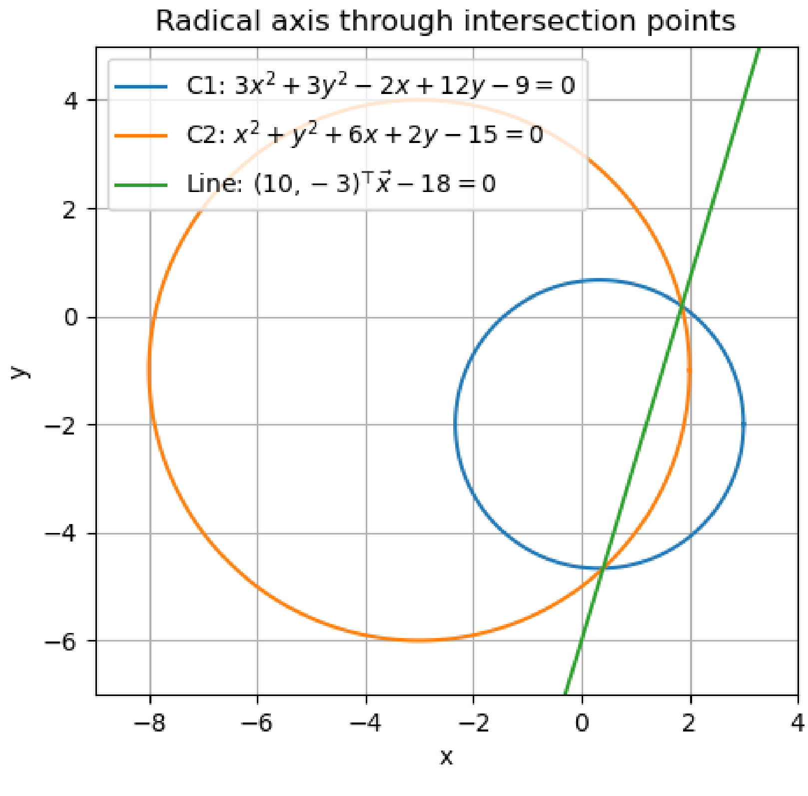

Figure 1