4.6.9

EE25BTECH11021 - Dhanush Sagar

Question

Find the equation of the plane containing the two parallel lines $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z}{3}$ and $\frac{x}{4} = \frac{y-2}{-2} = \frac{z+1}{6}$. Also, determine whether the plane thus obtained contains the line $\frac{x-2}{3} = \frac{y-1}{1} = \frac{z-2}{5}$.

Solution

A plane can be written in matrix form as $\mathbf{n}^T(\mathbf{r} - \mathbf{P}) = 0$, where $\mathbf{n}$ is the normal vector, $\mathbf{r}$ is a general point on the plane, and $\mathbf{P}$ is a point on the plane:

$$\mathbf{n}^T(\mathbf{r} - \mathbf{P}) = 0 \tag{0.1}$$

The given lines are in symmetric form:

$$\text{L1} = \frac{x-1}{2} = \frac{y+1}{-1} = \frac{z}{3} \tag{0.2}$$

$$\text{L2} = \frac{x}{4} = \frac{y-2}{-2} = \frac{z+1}{6} \tag{0.3}$$

$$\text{L3} = \frac{x-2}{3} = \frac{y-1}{1} = \frac{z-2}{5} \tag{0.4}$$

Extract points and directions from L1 and L2. The vector joining points from the two lines is:

$$\mathbf{P}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{d}_1 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \quad \mathbf{P}_2 = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \quad \mathbf{v} = \mathbf{P}_2 - \mathbf{P}_1 = \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} \tag{0.5}$$

The plane's normal vector is orthogonal to both $\mathbf{d}_1$ and $\mathbf{v}$, so it lies in the nullspace of the constraint matrix:

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 3 \\ -1 & 3 & -1 \end{pmatrix} \tag{0.6}$$

Row-reduction to find the nullspace:

$$\mathbf{A} \xrightarrow{R_1 \to \frac{1}{2}R_1} \begin{pmatrix} 1 & -\frac{1}{2} & \frac{3}{2} \\ -1 & 3 & -1 \end{pmatrix} \tag{0.7}$$

$$\mathbf{A} \xrightarrow{R_2 \to R_2 + R_1} \begin{pmatrix} 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & \frac{5}{2} & \frac{1}{2} \end{pmatrix} \tag{0.8}$$

$$\mathbf{A} \xrightarrow{R_2 \to \frac{2}{5}R_2} \begin{pmatrix} 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 1 & \frac{1}{5} \end{pmatrix} \tag{0.9}$$

$$\mathbf{A} \xrightarrow{R_1 \to R_1 + \frac{1}{2}R_2} \begin{pmatrix} 1 & 0 & \frac{8}{5} \\ 0 & 1 & \frac{1}{5} \end{pmatrix} \tag{0.10}$$

Now, we need to form a vector $\mathbf{n}_0$ whose product with $\mathbf{A}$ gives a null vector.
Express leading variables in terms of the free variable n_3 to get a vector in the nullspace, which is the plane's normal vector:

$$n_1 + \frac{8}{5}n_3 = 0 \quad \Rightarrow \quad n_1 = -\frac{8}{5}n_3 \tag{0.11}$$

$$n_2 + \frac{1}{5}n_3 = 0 \quad \Rightarrow \quad n_2 = -\frac{1}{5}n_3 \tag{0.12}$$

Let $n_3 = 1$ for simplicity:

$$\mathbf{n}_0 = \begin{pmatrix} -\frac{8}{5} \\ -\frac{1}{5} \\ 1 \end{pmatrix} \tag{0.13}$$

Clearing denominators and adjusting the sign gives the normal vector:

$$\mathbf{n} = \begin{pmatrix} 8 \\ 1 \\ -5 \end{pmatrix} \tag{0.14}$$

Let $\mathbf{r}$ be a general point on the plane:

$$\mathbf{r} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} \tag{0.15}$$

The plane equation using point $\mathbf{P}_1$ and normal vector $\mathbf{n}$:

$$\mathbf{n}^T(\mathbf{r} - \mathbf{P}_1) = 0 \tag{0.16}$$

$$\mathbf{n}^T\mathbf{P}_1 = \begin{pmatrix} 8 & 1 & -5 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 7 \tag{0.17}$$

$$\boxed{\begin{pmatrix} 8 & 1 & -5 \end{pmatrix} \mathbf{r} = 7} \tag{0.18}$$

Check if the third line L3 lies in the plane by verifying the point and direction:

$$\mathbf{P}_3 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{d}_3 = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} \tag{0.19}$$

$$\mathbf{n}^T \mathbf{P}_3 = \begin{pmatrix} 8 & 1 & -5 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 7 \tag{0.20}$$

$$\mathbf{n}^T \mathbf{d}_3 = \begin{pmatrix} 8 & 1 & -5 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} = 0 \tag{0.21}$$

Therefore, the plane containing the first two lines has the matrix form:

$$\begin{pmatrix} 8 & 1 & -5 \end{pmatrix} \mathbf{r} = 7$$

and it also contains the third line.

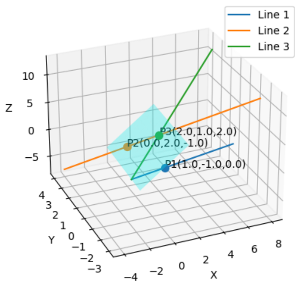

Fig. 0.1