4.12.6

AI25BTECH11001 - ABHISEK MOHAPATRA

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$$\left| G \right| = \begin{vmatrix} \|\mathbf{a}\|^2 & \mathbf{a}^\top \mathbf{b} & \mathbf{a}^\top \mathbf{c} \\ \mathbf{b}^\top \mathbf{a} & \|\mathbf{b}\|^2 & \mathbf{b}^\top \mathbf{c} \\ \mathbf{c}^\top \mathbf{a} & \mathbf{c}^\top \mathbf{b} & \|\mathbf{c}\|^2 \end{vmatrix} \tag{1}$$

$$= \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 \|\mathbf{c}\|^2 \left(1 - (\cos \alpha)^2 - (\cos \beta)^2 - (\cos \gamma)^2 + 2 \cos \alpha \cos \beta \cos \gamma \right) \tag{2}$$

$$= \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 \|\mathbf{c}\|^2 \begin{pmatrix} 1 - (\cos \alpha)^2 - (\cos \beta)^2 + (\cos \alpha)^2 (\cos \beta)^2 \\ - (\cos \alpha)^2 (\cos \beta)^2 + (\cos \gamma)^2 - 2 \cos \alpha \cos \beta \cos \gamma \end{pmatrix} \tag{3}$$

$$= \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 \|\mathbf{c}\|^2 \begin{pmatrix} 1 - (\cos \alpha)^2 - (\cos \beta)^2 \left(1 - (\cos \alpha)^2 \right) \\ - (\cos \alpha \cos \beta - \cos \gamma)^2 \end{pmatrix} \tag{4}$$

$$\leq \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 \|\mathbf{c}\|^2 \tag{5}$$

$$\implies \left(1 - (\cos\alpha)^2 - (\cos\beta)^2 \left(1 - (\cos\alpha)^2\right) - (\cos\alpha\cos\beta - \cos\gamma)^2\right) \leq 1 \tag{6}$$

$$\implies \left((\cos\alpha)^2 + (\cos\beta)^2 \left(1 - (\cos\alpha)^2\right) + (\cos\alpha\cos\beta - \cos\gamma)^2\right) \geq 0 \tag{7}$$

Equality holds when,

$$\cos\alpha = 0, \cos\beta = 0, \cos\alpha\cos\beta - \cos\gamma = 0 \tag{8}$$

Or,

$$\cos\alpha = \cos\beta = \cos\gamma = 0 \tag{9}$$

Hence $\mathbf{a}^\top\mathbf{b} = \mathbf{b}^\top\mathbf{c} = \mathbf{c}^\top\mathbf{a} = 0$ to hold equality.