5.13.68

EE25BTECH11043 - Nishid Khandagre

Question: For what value of k do the following system of equations possess a non trivial solution over the set of rationals Q?

$$x + ky + 3z = 0$$
$$3x + ky - 2z = 0$$
$$2x + 3y - 4z = 0$$

For that value of k, find all the solutions of the system.

Solution:

$$\left(\begin{array}{ccc|c} 1 & k & 3 & 0 \\ 3 & k & -2 & 0 \\ 2 & 3 & -4 & 0 \end{array}\right) \tag{0.1}$$

Apply Gaussian elimination to find the row echelon form.
$R_2 \to R_2 - 3R_1$:
$R_3 \to R_3 - 2R_1$:

$$\left(\begin{array}{ccc|c} 1 & k & 3 & 0 \\ 0 & k - 3k & -2 - 9 & 0 \\ 0 & 3 - 2k & -4 - 6 & 0 \end{array}\right) \tag{0.2}$$

$$\left(\begin{array}{ccc|c} 1 & k & 3 & 0 \\ 0 & -2k & -11 & 0 \\ 0 & 3 - 2k & -10 & 0 \end{array}\right) \tag{0.3}$$

For a non-trivial solution, the rank of this coefficient matrix A must be less than the 3.
If $-2k = 0$, then $k = 0$. In this case, the matrix becomes:

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 0 & -11 & 0 \\ 0 & 3 & -10 & 0 \end{array}\right) \tag{0.4}$$

This matrix has rank 3, which would lead to only the trivial solution. So $k \neq 0$.

If $k \neq 0$, we can proceed:
$R_2 \to -R_2$:

$$\left(\begin{array}{ccc|c} 1 & k & 3 & 0 \\ 0 & 2k & 11 & 0 \\ 0 & 3 - 2k & -10 & 0 \end{array}\right) \tag{0.5}$$

$R_3 \to 2kR_3 - (3-2k)R_2$:

$$\left(\begin{array}{ccc|c} 1 & k & 3 & 0 \\ 0 & 2k & 11 & 0 \\ 0 & 0 & 2k-33 & 0 \end{array}\right) \tag{0.6}$$

For a non-trivial solution, the rank of A must be less than 3, meaning the (3,3) element in the row echelon form must be zero.

$$2k - 33 = 0 \tag{0.7}$$
$$2k = 33 \tag{0.8}$$
$$k = \frac{33}{2} \tag{0.9}$$

The augmented matrix for $k = \frac{33}{2}$:

$$\left(\begin{array}{ccc|c} 1 & 33/2 & 3 & 0 \\ 3 & 33/2 & -2 & 0 \\ 2 & 3 & -4 & 0 \end{array}\right) \tag{0.10}$$

$R_2 \to R_2 - 3R_1$:

$$\left(\begin{array}{ccc|c} 1 & 33/2 & 3 & 0 \\ 0 & -33 & -11 & 0 \\ 2 & 3 & -4 & 0 \end{array}\right) \tag{0.11}$$

$R_3 \to R_3 - 2R_1$:

$$\left(\begin{array}{ccc|c} 1 & 33/2 & 3 & 0 \\ 0 & -33 & -11 & 0 \\ 0 & -30 & -10 & 0 \end{array}\right) \tag{0.12}$$

$R_2 \to R_2/(-11)$:
$R_3 \to R_3/(-10)$:

$$\left(\begin{array}{ccc|c} 1 & 33/2 & 3 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right) \tag{0.13}$$

$R_3 \to R_3 - R_2$:

$$\left(\begin{array}{ccc|c} 1 & 33/2 & 3 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \tag{0.14}$$

The rank of the coefficient matrix is 2, which is less than 3, so there are non-trivial solutions.
From the second row: $3y + z = 0 \Rightarrow z = -3y$.
From the first row: $x + \frac{33}{2}y + 3z = 0$
Substitute $z = -3y$: $x = -\frac{15}{2}y$.

Let $y = 2t$ Then $x = -\frac{15}{2}(2t) = -15t$. And $z = -3(2t) = -6t$.

The solutions are of the form: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} -15 \\ 2 \\ -6 \end{pmatrix}$ for any $t \in \mathbb{Q}$.