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Question : Let $\mathbf{A}$ be an $n \times n$ real matrix. Consider the following statements:

1) If $\mathbf{A}$ is symmetric, then there exists $c \geq 0$ such that $\mathbf{A} + c\mathbf{I}_n$ is symmetric and positive definite, where $\mathbf{I}_n$ is the $n \times n$ identity matrix.
2) If $\mathbf{A}$ is symmetric and positive definite, then there exists a symmetric and positive definite $\mathbf{B}$ such that $\mathbf{A} = \mathbf{B}^2$.
   Which of the above statements is/are true?

a) Only (I) b) Only (II) c) Both (I) and (II) d) Neither (I) nor (II)

Solution :

Name	Description
$\mathbf{A}$	Matrix

Table : Matrix

Checking statement (**I**)
If $\mathbf{A}$ is symmetric, its eigenvalues are real. Let the minimum eigenvalue of $\mathbf{A}$ be $\lambda_{\min}$.Then choose $c > -\lambda_{\min}$.
The Eigen values of $\mathbf{A}$ are given as :

$$\left|\mathbf{A} - \lambda_i \mathbf{I}\right| = 0 \tag{1}$$

The Eigen values of $\mathbf{A} + c\mathbf{I}_n$ are given as :

$$\left|\mathbf{A} - (\lambda_k - c)\mathbf{I}\right| = 0 \tag{2}$$
$$\lambda_k = \lambda_i + c \tag{3}$$
$$\lambda_i + c > 0 \tag{4}$$

Since $\lambda_i + c > 0$ for all i, $\mathbf{A} + c\mathbf{I}_n$ is positive definite and symmetric. Hence, statement (**I**) is **true**.
Checking statement (**II**)
If $\mathbf{A}$ is symmetric and positive definite, then it can be diagonalized as:

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^\top \tag{5}$$

where $\mathbf{P}$ is orthogonal and $\mathbf{D}$ is a diagonal matrix with positive entries (since $\mathbf{A}$ is positive definite). Define

$$\mathbf{B} = \mathbf{P}\mathbf{D}^{1/2}\mathbf{P}^\top \tag{6}$$

Then,

$$\mathbf{B}^2 = \mathbf{P}\mathbf{D}^{1/2}\mathbf{P}^\top\mathbf{P}\mathbf{D}^{1/2}\mathbf{P}^\top = \mathbf{P}\mathbf{D}\mathbf{P}^\top = \mathbf{A} \tag{7}$$

Hence, $\mathbf{B}$ is symmetric and positive definite. Therefore, statement (**II**) is also **true**.
Final Answer: (c) Both (I) and (II)