2.10.19

AI25BTECH110031
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Question(2.10.19) For three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ which of the following expression is not equal to any of the remaining three?

a $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$
b $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w})$
c $(\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u}$
d $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

Solution As we know that dot product is cumulative (1) and (2) are equal
That is,

$$\mathbf{u}^\top (\mathbf{v} \times \mathbf{w}) = (\mathbf{v} \times \mathbf{w})^\top \mathbf{u} \tag{0.1}$$

We prove

$$\mathbf{u}^\top (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v})^\top \mathbf{w} \tag{0.2}$$

using the cross-product (skew-)matrix.

Define, for $\mathbf{a} = (a_1, a_2, a_3)^T$,

$$S(\mathbf{a}) = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \tag{0.3}$$

which satisfies $(\mathbf{a})\mathbf{b} = \mathbf{a} \times \mathbf{b}$ for all $\mathbf{b} \in \mathbb{R}^3$.

$$\begin{aligned}
\mathbf{u}^\top (\mathbf{v} \times \mathbf{w}) &= \mathbf{u}^T (S(\mathbf{v})\mathbf{w}) \quad (\text{since } S(\mathbf{v})\mathbf{w} = \mathbf{v} \times \mathbf{w}) && (0.4) \\
&= (\mathbf{u}^T S(\mathbf{v}))\mathbf{w} && (0.5) \\
&= (S(\mathbf{v})^T \mathbf{u})^T \mathbf{w} \quad (\text{transpose identity: } (A^T x)^T = x^T A) && (0.6) \\
&= (-S(\mathbf{v})\mathbf{u})^T \mathbf{w} \quad (\text{since } S(\mathbf{v})^T = -S(\mathbf{v})) && (0.7) \\
&= (\mathbf{u} \times \mathbf{v})^T \mathbf{w} \quad (\text{because } -S(\mathbf{v})\mathbf{u} = -(\mathbf{v} \times \mathbf{u}) = \mathbf{u} \times \mathbf{v}) && (0.8)
\end{aligned}$$

Thus

$$\mathbf{u}^\top (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v})^\top \mathbf{w} \tag{0.9}$$

This shows that (a), (c) and (d) are equal

For example, Let

$$\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}. \tag{0.10}$$

Case 1: $\mathbf{u}^\top(\mathbf{v} \times \mathbf{w})$

$$\mathbf{v} \times \mathbf{w} = \begin{pmatrix} v_{23} & w_{23} \\ v_{31} & w_{31} \\ v_{12} & w_{12} \end{pmatrix} = \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix} = \begin{pmatrix} 1 \times (-1) - 2 \times 0 \\ 2 \times 1 - 0 \times (-1) \\ 0 \times 0 - 1 \times 1 \end{pmatrix} \tag{0.11}$$

So

$$\mathbf{v} \times \mathbf{w} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \tag{0.12}$$

Now compute the dot product:

$$\mathbf{u}^\top(\mathbf{v} \times \mathbf{w}) = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \tag{0.13}$$

$$= (1)(-1) + (-1)(2) + (1)(-1) \tag{0.14}$$

$$= -4 \tag{0.15}$$

$$\boxed{\mathbf{u}^\top(\mathbf{v} \times \mathbf{w}) = -4} \tag{0.16}$$

Case 2: $\mathbf{v}^\top(\mathbf{u} \times \mathbf{w})$
Compute $\mathbf{u} \times \mathbf{w}$:

$$\mathbf{u} \times \mathbf{w} = \begin{pmatrix} u_{23} & w_{23} \\ u_{31} & w_{31} \\ u_{12} & w_{12} \end{pmatrix} = \begin{pmatrix} u_2w_3 - u_3w_2 \\ u_3w_1 - u_1w_3 \\ u_1w_2 - u_2w_1 \end{pmatrix} = \begin{pmatrix} (-1) \times (-1) - 1 \times 0 \\ 1 \times 1 - 1 \times (-1) \\ 1 \times 0 - (-1) \times 1 \end{pmatrix} \tag{0.17}$$

So

$$\mathbf{u} \times \mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}. \tag{0.18}$$

Now compute dot product:

$$\mathbf{v}^\top(\mathbf{u} \times \mathbf{w}) = \begin{pmatrix} 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \tag{0.19}$$

$$= (0)(1) + (1)(2) + (2)(1) \tag{0.20}$$

$$= 4. \tag{0.21}$$

$$\boxed{\mathbf{u}^\top(\mathbf{u}\times\mathbf{w}) = 4} \tag{0.22}$$

Case 3: $(\mathbf{v}\times\mathbf{w})^\top\mathbf{u}$
We already have

$$\mathbf{v}\times\mathbf{w} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \tag{0.23}$$

Now compute:

$$(\mathbf{v}\times\mathbf{w})^\top\mathbf{u} = \begin{pmatrix} -1 & 2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \tag{0.24}$$

$$= (-1)(1) + (2)(-1) + (-1)(1) \tag{0.25}$$

$$= -4. \tag{0.26}$$

$$\boxed{(\mathbf{v}\times\mathbf{w})^\top\mathbf{u} = -4} \tag{0.27}$$

Case 4: $(\mathbf{u}\times\mathbf{v})^\top\mathbf{w}$
Compute $\mathbf{u}\times\mathbf{v}$:

$$\mathbf{u}\times\mathbf{v} = \begin{pmatrix} u_{23} & v_{23} \\ u_{31} & v_{31} \\ u_{12} & v_{12} \end{pmatrix} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix} = \begin{pmatrix} (-1)\times 2 - 1\times 1 \\ 1\times 0 - (-1)\times 2 \\ 1\times 1 - (-1)\times 0 \end{pmatrix} \tag{0.28}$$

So

$$\mathbf{u}\times\mathbf{v} = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} \tag{0.29}$$

Now compute:

$$(\mathbf{u}\times\mathbf{v})^\top\mathbf{w} = \begin{pmatrix} -3 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{0.30}$$

$$= (-3)(1) + (-2)(0) + (1)(-1) \tag{0.31}$$

$$= -3 + 0 - 1 = -4. \tag{0.32}$$

$$\boxed{(\mathbf{u}\times\mathbf{v})^\top\mathbf{w} = -4} \tag{0.33}$$

Final Results

$$\mathbf{u}^\top(\mathbf{v}\times\mathbf{w}) = -4, \quad \mathbf{v}^\top(\mathbf{u}\times\mathbf{w}) = 4, \quad (\mathbf{v}\times\mathbf{w})^\top\mathbf{u} = -4, \quad (\mathbf{u}\times\mathbf{v})^\top\mathbf{w} = -4 \tag{0.34}$$

Thus (a), (c) and (d) are same

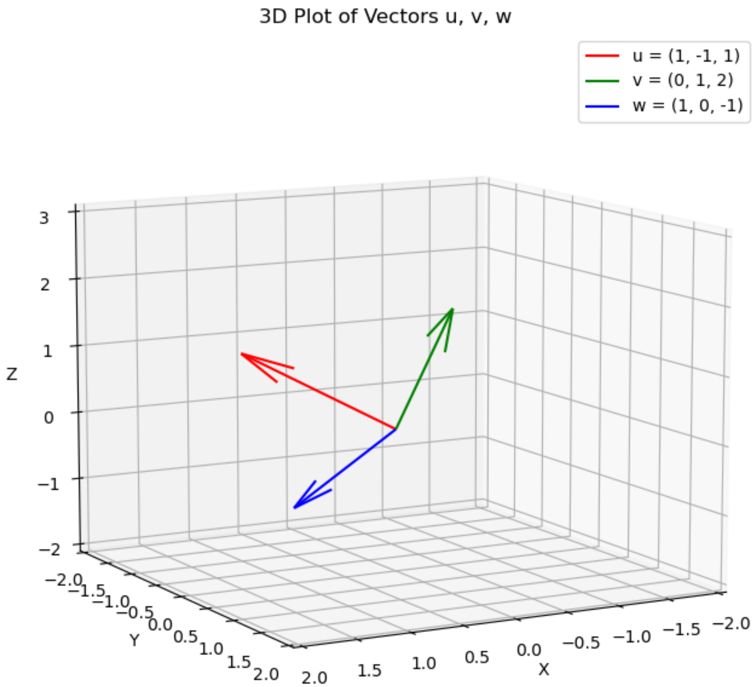

Fig. 4.1