Matgeo Presentation - Problem 4.7.45

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QUESTION

For any three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, prove or disprove

$$(\mathbf{a}-\mathbf{b}) \cdot \big((\mathbf{b}-\mathbf{c}) \times (\mathbf{c}-\mathbf{a})\big) = 2\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

Solution:

We write the scalar triple product in determinant form:

$$(\mathbf{a}-\mathbf{b})\cdot\big((\mathbf{b}-\mathbf{c})\times(\mathbf{c}-\mathbf{a})\big)=\det\begin{pmatrix}(\mathbf{a}-\mathbf{b})^T\\(\mathbf{b}-\mathbf{c})^T\\(\mathbf{c}-\mathbf{a})^T\end{pmatrix}. \tag{0.1}$$

Now observe that

$$(\mathbf{a}-\mathbf{b})+(\mathbf{b}-\mathbf{c})+(\mathbf{c}-\mathbf{a})=\mathbf{0}. \tag{0.2}$$

Thus the three rows of the determinant are linearly dependent. From matrix theory, the determinant of a matrix with linearly dependent rows is zero. Hence

$$(\mathbf{a}-\mathbf{b})\cdot\big((\mathbf{b}-\mathbf{c})\times(\mathbf{c}-\mathbf{a})\big)=0. \tag{0.3}$$

Solution:

On the other hand, the right-hand side is

$$2\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 2 \det \begin{pmatrix} \mathbf{a}^T \\ \mathbf{b}^T \\ \mathbf{c}^T \end{pmatrix}, \tag{0.4}$$

which is not identically zero for arbitrary $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

Conclusion

The given statement is **false**.
The left-hand side $(\mathbf{a}-\mathbf{b})\cdot\big((\mathbf{b}-\mathbf{c})\times(\mathbf{c}-\mathbf{a})\big)$is always zero, while the right-hand side $2\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ can be nonzero.