2.10.28

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Question

Q 2.10.28. For non-zero vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, the relation

$$\left|(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}\right| = \|\mathbf{a}\|\,\|\mathbf{b}\|\,\|\mathbf{c}\|$$

holds if and only if

1) $\mathbf{a}\cdot\mathbf{b}=0,\ \mathbf{b}\cdot\mathbf{c}=0$
2) $\mathbf{b}\cdot\mathbf{c}=0,\ \mathbf{c}\cdot\mathbf{a}=0$
3) $\mathbf{c}\cdot\mathbf{a}=0,\ \mathbf{a}\cdot\mathbf{b}=0$
4) $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{c}=\mathbf{c}\cdot\mathbf{a}=0$

Solution

We need the condition for

$$|(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}| = \|\mathbf{a}\|\,\|\mathbf{b}\|\,\|\mathbf{c}\|.$$

Now,

$$\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\,\|\mathbf{b}\|\sin\theta,$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

So,

$$|(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}| \le \|\mathbf{a}\|\,\|\mathbf{b}\|\,\|\mathbf{c}\|.$$

Equality holds iff 1. $\sin\theta = 1 \quad\Rightarrow\quad \mathbf{a}\perp\mathbf{b}$, and 2. $\mathbf{c}\parallel(\mathbf{a}\times\mathbf{b}) \quad\Rightarrow\quad \mathbf{c}\perp\mathbf{a},\ \mathbf{c}\perp\mathbf{b}$.
Thus the conditions are

$$\mathbf{a}\cdot\mathbf{b}=0,\quad \mathbf{b}\cdot\mathbf{c}=0,\quad \mathbf{c}\cdot\mathbf{a}=0.$$

Hence, the correct option is

$$\boxed{\text{(d)}}$$

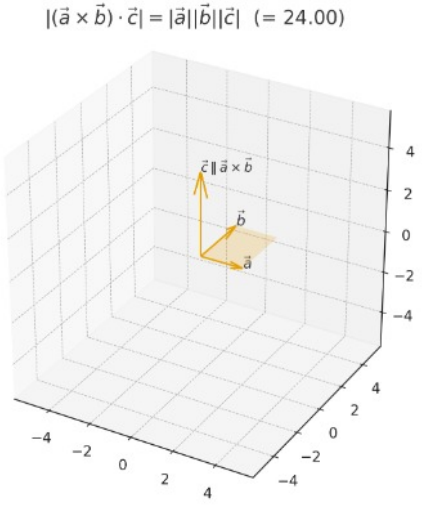

Fig. 4.1: Illustration of $|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| = |\mathbf{a}|\,|\mathbf{b}|\,|\mathbf{c}|$ with $\mathbf{a} \perp \mathbf{b}$ and $\mathbf{c} \parallel (\mathbf{a} \times \mathbf{b})$.