2.10.44

EE25BTECH11015 - Bhoomika V

Question :-
If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are unit vectors, then

$$\|\mathbf{a}-\mathbf{b}\|^2 + \|\mathbf{b}-\mathbf{c}\|^2 + \|\mathbf{a}-\mathbf{b}\|^2$$

does not exceed

a) 4
b) 9
c) 8
d) 6

Solution:
Let

$$x = \mathbf{a}\cdot\mathbf{b}, \quad y = \mathbf{a}\cdot\mathbf{c}, \quad z = \mathbf{b}\cdot\mathbf{c}.$$

Since $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are unit vectors, their Gram matrix is

$$G = \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix}.$$

Now consider

$$(1,1,1)\,G\,(1,1,1)^T = (\mathbf{a}+\mathbf{b}+\mathbf{c})\cdot(\mathbf{a}+\mathbf{b}+\mathbf{c}) \geq 0.$$

Expanding,

$$|\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2(x+y+z) = 3 + 2(x+y+z) \geq 0,$$

$$\implies \quad x+y+z \geq -\tfrac{3}{2}. \tag{4.1}$$

Now ,

$$|\mathbf{a}-\mathbf{b}|^2 + |\mathbf{b}-\mathbf{c}|^2 + |\mathbf{c}-\mathbf{a}|^2 = (2-2x) + (2-2z) + (2-2y).$$

So,

$$= 6 - 2(x+y+z).$$

From Equation (4.1)

$$6 - 2(x+y+z) \leq 6 - 2\left(-\tfrac{3}{2}\right) = 9.$$

Thus, $\|\mathbf{a}-\mathbf{b}\|^2 + \|\mathbf{b}-\mathbf{c}\|^2 + \|\mathbf{a}-\mathbf{b}\|^2$ does not exceed 9.