2.10.74

Puni Aditya - EE25BTECH11046

Question:
If $\mathbf{X} \cdot \mathbf{A} = 0$, $\mathbf{X} \cdot \mathbf{B} = 0$, and $\mathbf{X} \cdot \mathbf{C} = 0$ for some non-zero vector $\mathbf{X}$, then $[\mathbf{A}\ \mathbf{B}\ \mathbf{C}] = 0$.

Solution:
Given that for a for a non-zero vector $\mathbf{X}$:

$$\mathbf{X} \cdot \mathbf{A} = 0 \tag{1}$$

$$\mathbf{X} \cdot \mathbf{B} = 0 \tag{2}$$

$$\mathbf{X} \cdot \mathbf{C} = 0 \tag{3}$$

From (1), (2) and (3),

$$\mathbf{A}^\top \mathbf{X} = \mathbf{B}^\top \mathbf{X} = \mathbf{C}^\top \mathbf{X} = 0 \tag{4}$$

This forms the set of equations

$$\begin{pmatrix} \mathbf{A}^\top \\ \mathbf{B}^\top \\ \mathbf{C}^\top \end{pmatrix} \mathbf{X} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{5}$$

For a homogeneous set of equations to have non-trivial solution $\mathbf{X}$, the coefficient matrix is singular.

$$\implies \begin{bmatrix} \mathbf{A}^\top & \mathbf{B}^\top & \mathbf{C}^\top \end{bmatrix} = 0 \tag{6}$$

From (6)

$$[\mathbf{A}\ \mathbf{B}\ \mathbf{C}] = 0 \tag{7}$$

Hence, the given statement is true.

Example: Let

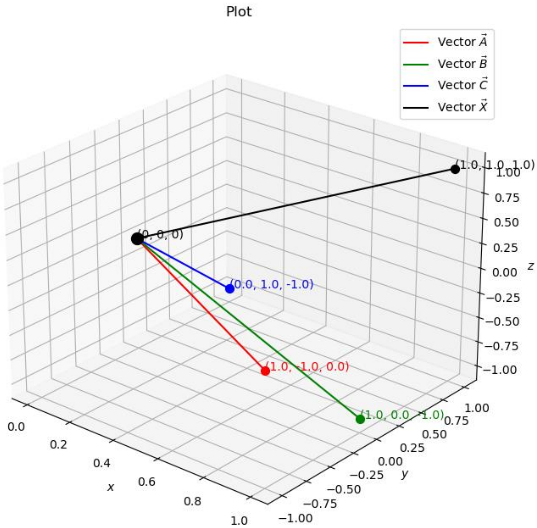
$$\mathbf{X} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\mathbf{A}^\top \mathbf{X} = \mathbf{B}^\top \mathbf{X} = \mathbf{C}^\top \mathbf{X} = 0 \text{ for this example.} \tag{8}$$

$$[\mathbf{A}\ \mathbf{B}\ \mathbf{C}] = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} \left(\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right) \tag{9}$$

$$[\mathbf{A}\ \mathbf{B}\ \mathbf{C}] = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{10}$$

$$[\mathbf{A}\ \mathbf{B}\ \mathbf{C}] = 0 \tag{11}$$



Example