

# 1.8.5

AI25BTECH110031

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**Question(1.8.5)** If  $\mathbf{A}$  and  $\mathbf{B}$  be the points  $(3, 4, 5)$  and  $(-1, 3, -7)$  respectively, find the equation of the set of a point  $\mathbf{P}$  such that  $PA^2 + PB^2 = k^2$

**Solution:** Given ponits

$$\mathbf{A} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} -1 \\ 3 \\ -7 \end{pmatrix} \quad (0.1)$$

According to the question,

$$PA^2 + PB^2 = k^2 \quad (0.2)$$

where,  $PA = \|P - A\|$  and  $PB = \|P - B\|$

The squared distances can be written as dot products:

$$PA^2 = (\mathbf{P} - \mathbf{A})^T (\mathbf{P} - \mathbf{A}) \quad (0.3)$$

$$PB^2 = (\mathbf{P} - \mathbf{B})^T (\mathbf{P} - \mathbf{B}) \quad (0.4)$$

Thus:

$$PA^2 + PB^2 = (\mathbf{P} - \mathbf{A})^T (\mathbf{P} - \mathbf{A}) + (\mathbf{P} - \mathbf{B})^T (\mathbf{P} - \mathbf{B}) \quad (0.5)$$

$$PA^2 + PB^2 = 2\mathbf{P}^T \mathbf{P} - 2\mathbf{A}^T \mathbf{P} + \mathbf{A}^T \mathbf{A} + \mathbf{P}^T \mathbf{P} - 2\mathbf{B}^T \mathbf{P} + \mathbf{B}^T \mathbf{B} \quad (0.6)$$

$$2\mathbf{P}^T \mathbf{P} - 2(\mathbf{A} + \mathbf{B})^T \mathbf{P} + \mathbf{A}^T \mathbf{A} + \mathbf{B}^T \mathbf{B} - k^2 = 0 \quad (0.7)$$

Complete the square,

Let,

$$\mathbf{M} := \frac{\mathbf{A} + \mathbf{B}}{2} \quad (0.8)$$

$$R^2 := \|\mathbf{M}\|^2 - \frac{\mathbf{A}^T \mathbf{A} + \mathbf{B}^T \mathbf{B} - k^2}{2} \quad (0.9)$$

$$(0.10)$$

Then the equation becomes

$$\|P - M\|^2 = R^2 \quad (0.11)$$

$$\left\| P - \frac{\mathbf{A} + \mathbf{B}}{2} \right\|^2 = \left\| \frac{\mathbf{A} + \mathbf{B}}{2} \right\|^2 - \frac{\|A\|^2 + \|B\|^2 - k^2}{2} \quad (0.12)$$

Substitute the known values

$$\|A\| = 3^2 + 4^2 + 5^2 = 50 \quad (0.13)$$

$$\|B\| = (-1)^2 + 3^2 + (-7)^2 = 59 \quad (0.14)$$

$$\frac{\mathbf{A} + \mathbf{B}}{2} = \begin{pmatrix} 1 \\ 3.5 \\ -1 \end{pmatrix} \quad (0.15)$$

The equation of the locus is:

$$\left\| P - \begin{pmatrix} 1 \\ 3.5 \\ -1 \end{pmatrix} \right\|^2 = \frac{2k^2 - 161}{4} \quad (0.16)$$

The plot show the locus for  $k = 20$

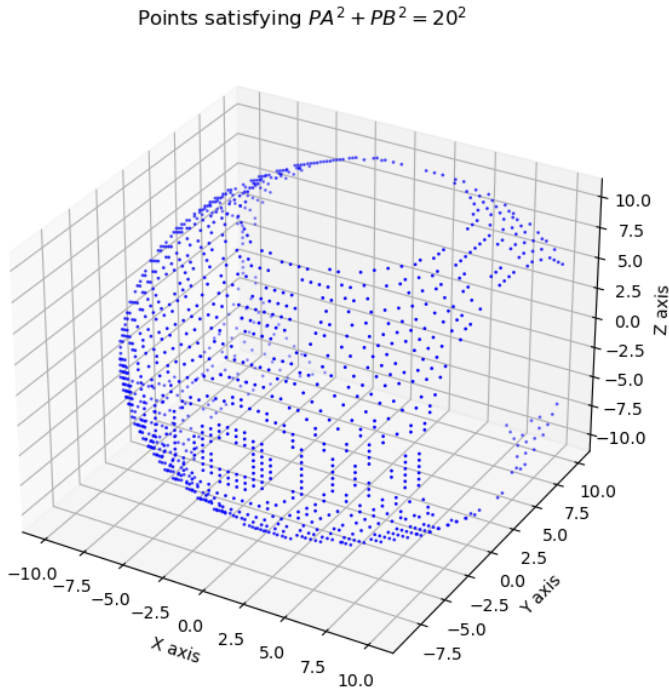


Fig. 0.1