

8.2.11

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29th September, 2025

Question

Find the coordinates of the focus, vertex, eccentricity, axis of the conic section, the equation of the directrix and the length of the latus rectum.

$$\frac{x^2}{100} + \frac{y^2}{400} = 1$$

is _____.

Theoretical Solution

We use an affine transformation to convert the conic equation to its standard form.

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0$$

The symmetric matrix \mathbf{V} is spectrally decomposed to align axes with eigenvectors.

$$\mathbf{V} = \mathbf{P} \mathbf{D} \mathbf{P}^\top, \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \mathbf{P}^\top \mathbf{P} = \mathbf{I} \quad (1)$$

Theoretical Solution

Substituting the decomposition into the conic equation.

$$\mathbf{x}^\top \mathbf{P} \mathbf{D} \mathbf{P}^\top \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \quad (2)$$

A rotation

$$\mathbf{x}_r = \mathbf{P}^\top \mathbf{x} \quad (3)$$

aligns the conic with the coordinate axes.

$$\mathbf{x} = \mathbf{P} \mathbf{x}_r \quad (4)$$

Applying the rotation to the conic equation.

$$(\mathbf{P}\mathbf{x}_r)^\top \mathbf{P}\mathbf{D}\mathbf{P}^\top (\mathbf{P}\mathbf{x}_r) + 2\mathbf{u}^\top (\mathbf{P}\mathbf{x}_r) + f = 0 \quad (5)$$

$$\mathbf{x}_r^\top \mathbf{P}^\top \mathbf{P}\mathbf{D}\mathbf{P}^\top \mathbf{P}\mathbf{x}_r + 2\left(\mathbf{P}^\top \mathbf{u}\right)^\top \mathbf{x}_r + f = 0 \quad (6)$$

$$\mathbf{x}_r^\top \mathbf{D}\mathbf{x}_r + 2\mathbf{u}_r^\top \mathbf{x}_r + f = 0 \quad (7)$$

Theoretical Solution

A translation

$$\mathbf{x}_c = \mathbf{x}_r + \mathbf{D}^{-1}\mathbf{u}_r \quad (8)$$

moves the conic's center to the origin.

$$f_c = f - \mathbf{u}_r^\top \mathbf{D}^{-1}\mathbf{u}_r \quad (9)$$

The center of the conic in the original coordinates is

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \quad (10)$$

Theoretical Solution

$$\mathbf{c} = -\left(\mathbf{P}\mathbf{D}\mathbf{P}^\top\right)^{-1}\mathbf{u} = -\mathbf{P}\mathbf{D}^{-1}\mathbf{P}^\top\mathbf{u} = -\mathbf{P}\mathbf{D}^{-1}\mathbf{u}_r \quad (11)$$

The complete transformation from original to centered coordinates is

$$\mathbf{x}_c = \mathbf{P}^\top (\mathbf{x} - \mathbf{c}) \quad (12)$$

$$\mathbf{x}_c = \mathbf{P}^\top \mathbf{x} + \mathbf{D}^{-1}\mathbf{u}_r = \mathbf{P}^\top \mathbf{x} - \mathbf{P}^\top \mathbf{c} = \mathbf{P}^\top (\mathbf{x} - \mathbf{c}) \quad (13)$$

$$\implies \mathbf{x} = \mathbf{P}\mathbf{x}_c + \mathbf{c} \quad (14)$$

Theoretical Solution

The given conic equation

$$\frac{x^2}{100} + \frac{y^2}{400} - 1 = 0 \quad (15)$$

$$\mathbf{v} = \begin{pmatrix} \frac{1}{100} & 0 \\ 0 & \frac{1}{400} \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, f = -1 \quad (16)$$

The major axis corresponds to smaller eigenvalue.

$$\lambda_1 = \frac{1}{400}, \lambda_2 = \frac{1}{100}, \mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (17)$$

Theoretical Solution

Applying the rotation to find the canonical coordinates.

$$\mathbf{x}_c = \mathbf{P}^\top \mathbf{x} \implies \begin{pmatrix} x_c \\ y_c \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix} \quad (18)$$

The standard form of the ellipse in canonical coordinates.

$$\frac{x_c^2}{-f/\lambda_1} + \frac{y_c^2}{-f/\lambda_2} = 1 \quad (19)$$

Theoretical Solution

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} = \sqrt{1 - \frac{1/400}{1/100}} = \frac{\sqrt{3}}{2} \quad (20)$$

$$\mathbf{f}_c = \pm \sqrt{\frac{f(\lambda_1 - \lambda_2)}{\lambda_1 \lambda_2}} \mathbf{e}_1 = \pm 10\sqrt{3} \mathbf{e}_1 \quad (21)$$

$$\mathbf{v}_c = \pm \sqrt{\frac{-f}{\lambda_1}} \mathbf{e}_1 = \pm 20 \mathbf{e}_1 \quad (22)$$

$$\mathbf{d}_c : \mathbf{e}_1^\top \mathbf{x}_c = \pm \sqrt{\frac{-f\lambda_2}{\lambda_1(\lambda_2 - \lambda_1)}} = \pm \frac{40}{\sqrt{3}} \quad (23)$$

$$L = \frac{-2f}{\lambda_2} \sqrt{\frac{\lambda_1}{-f}} = 10 \quad (24)$$

Transforming properties back to the original coordinate system using (14)

$$\mathbf{f} = \mathbf{P} \left(\pm 10 \sqrt{3} \mathbf{e}_1 \right) = \pm 10 \sqrt{3} \mathbf{e}_2 \quad (25)$$

$$\mathbf{v} = \mathbf{P} (\pm 20 \mathbf{e}_1) = \pm 20 \mathbf{e}_2 \quad (26)$$

$$\mathbf{d} : \mathbf{e}_2^\top \mathbf{x} = \pm \frac{40}{\sqrt{3}} \implies \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = \pm \frac{40}{\sqrt{3}} \quad (27)$$

Theoretical Solution

Property	Value
Eccentricity	$\frac{\sqrt{3}}{2}$
Axis	$x = 0$
Vertices	$(0, \pm 20)$
Foci	$(0, \pm 10\sqrt{3})$
Directrices	$y = \pm \frac{40}{\sqrt{3}}$
Latus Rectum	10

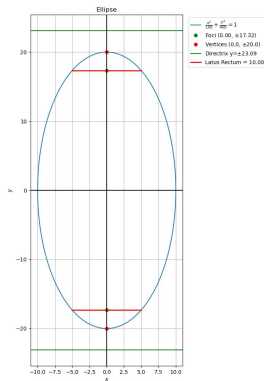


Figure: Plot