

2.10.50

EE25BTECH11021 - Dhanush Sagar

Question

A variable plane at a distance of one unit from the origin cuts the coordinate axes at A, B and C .

If the centroid $D(x, y, z)$ of triangle ABC satisfies the relation

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = k,$$

then the value of k is :

- | | |
|------|------------------|
| 1) 3 | 3) $\frac{1}{3}$ |
| 2) 1 | 4) 9 |

Solution

Write the plane in vector form as

$$\mathbf{n}^T \mathbf{x} = 1, \quad (4.1)$$

so the constant on the right is 1 (no scalar c appears).

The plane meets the coordinate axes at

$$\mathbf{A} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}. \quad (4.2)$$

Define

$$\mathbf{e} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}. \quad (4.3)$$

Since $\mathbf{A}, \mathbf{B}, \mathbf{C}$ lie on the plane, we have

$$\mathbf{n}^T \mathbf{A} = 1, \quad (4.4)$$

$$\mathbf{n}^T \mathbf{B} = 1, \quad (4.5)$$

$$\mathbf{n}^T \mathbf{C} = 1. \quad (4.6)$$

These combine to the single matrix equation

$$\mathbf{n}^T \mathbf{M} = \mathbf{e}^T, \quad (4.7)$$

and transposing gives

$$\mathbf{M}^T \mathbf{n} = \mathbf{e}. \quad (4.8)$$

Because \mathbf{M} is diagonal, invertible, and equals its transpose, we obtain

$$\mathbf{n} = \mathbf{M}^{-1} \mathbf{e}. \quad (4.9)$$

The perpendicular distance d of the plane $\mathbf{n}^\top \mathbf{x} = 1$ from the origin is

$$d = \frac{|1|}{\|\mathbf{n}\|} = \frac{1}{\|\mathbf{M}^{-1} \mathbf{e}\|}. \quad (4.10)$$

Hence the quadratic-form relation

$$\mathbf{e}^\top \mathbf{M}^{-2} \mathbf{e} = \frac{1}{d^2}. \quad (4.11)$$

The centroid of the triangle ABC is

$$\mathbf{D} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} = \frac{1}{3} \mathbf{M} \mathbf{e}, \quad (4.12)$$

so the coordinates of the centroid are

$$x = \frac{a}{3}, \quad y = \frac{b}{3}, \quad z = \frac{c}{3}, \quad \text{and} \quad \mathbf{D} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \quad (4.13)$$

Compute the desired sum:

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{(a/3)^2} + \frac{1}{(b/3)^2} + \frac{1}{(c/3)^2} \quad (4.14)$$

$$= 9 \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right). \quad (4.15)$$

To express this in matrix form note that

$$\mathbf{M}^{-2} = \begin{pmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & \frac{1}{b^2} & 0 \\ 0 & 0 & \frac{1}{c^2} \end{pmatrix}, \quad (4.16)$$

$$\mathbf{M}^{-2} \mathbf{e} = \begin{pmatrix} \frac{1}{a^2} \\ \frac{1}{b^2} \\ \frac{1}{c^2} \end{pmatrix}, \quad (4.17)$$

$$\mathbf{e}^\top \mathbf{M}^{-2} \mathbf{e} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}. \quad (4.18)$$

Therefore

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = 9 \mathbf{e}^\top \mathbf{M}^{-2} \mathbf{e}. \quad (4.19)$$

Combining with the distance relation gives the compact formula

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{9}{d^2}. \quad (4.20)$$

For the given problem $d = 1$, so

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = 9, \quad (4.21)$$

and thus

$$k = 9.$$

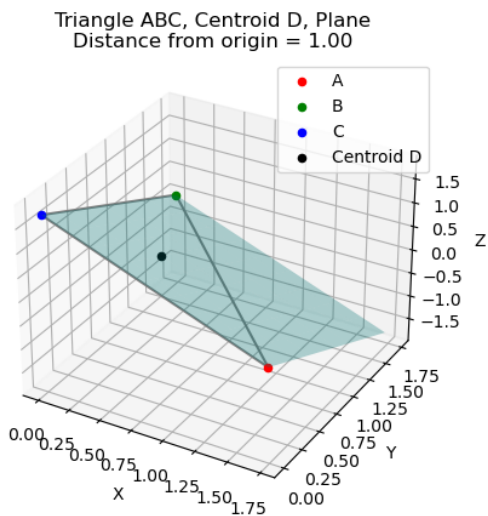


Fig. 4.1