5.5.5

EE25BTECH11052 - Shriyansh Kalpesh Chawda

Question

If $A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{pmatrix}$, then which of the following is true?

1) $A^{-1} = B$
2) $A^{-1} = 6B$
3) $B^{-1} = B$
4) $\mathbf{B^{-1}} = \frac{\mathbf{1}}{\mathbf{6}}\mathbf{A}$

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Solution

Given the matrices:

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{pmatrix} \tag{1}$$

We will find the inverse of A using Gauss-Jordan elimination method.
The augmented matrix $[A|I]$ is given by:

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \tag{2}$$

Performing elementary Row Operations
$R_2 \to R_2 - 2R_1$:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 5 & 4 & -2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \tag{3}$$

Swap rows $R_2 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 5 & 4 & -2 & 1 & 0 \end{array}\right] \tag{4}$$

$R_1 \to R_1 + R_2$ and $R_3 \to R_3 - 5R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & -6 & -2 & 1 & -5 \end{array}\right] \tag{5}$$

$R_3 \to -\frac{1}{6}R_3$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{6} & \frac{5}{6} \end{array}\right] \tag{6}$$

$R_1 \to R_1 - 2R_3$ and $R_2 \to R_2 - 2R_3$:

$$[I|A^{-1}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{6} & \frac{5}{6} \end{array}\right] \tag{7}$$

Hence,the Inverse Matrix A^{-1} is given by

$$A^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & -\frac{1}{6} & \frac{5}{6} \end{pmatrix} \tag{8}$$

By factoring out a scalar,The relation with B is given by:

$$A^{-1} = \frac{1}{6}\begin{pmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{pmatrix} \implies A^{-1} = \frac{1}{6}B \tag{9}$$

Now Pre-multiplying both side by A:

$$AA^{-1} = A\left(\frac{1}{6}B\right) \tag{10}$$

$$I = \frac{1}{6}AB \tag{11}$$

$$6I = AB \tag{12}$$

Now Post-multiplying both sides by B^{-1}:

$$6IB^{-1} = ABB^{-1} \tag{13}$$

$$6B^{-1} = A \tag{14}$$

$$\mathbf{B^{-1} = \frac{1}{6}A} \tag{15}$$