## EE25BTECH11023 - Venkata Sai

## **Question:**

The locus of the mid-point of the line segment joining the focus to a moving point on the parabola  $y^2 = 4ax$  is another parabola with directrix

1) 
$$x = 0$$
 2)  $x = -a/2$  3)  $x = a$  4)  $x = a/2$ 

## **Solution:**

The equation of a conic with directrix  $\mathbf{n}^{\mathsf{T}}\mathbf{x} = c$ , eccentricity e and focus **F** is given by

$$\mathbf{x}^{\mathsf{T}}\mathbf{V}\mathbf{x} + 2\mathbf{u}^{\mathsf{T}}\mathbf{x} + f = 0 \tag{1}$$

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On comparing with  $y^2 - 4ax = 0$ 

$$\mathbf{x}^{\mathsf{T}}\mathbf{V}\mathbf{x} = y^2 \tag{2}$$

$$\mathbf{x}^{\mathsf{T}}\mathbf{V}\mathbf{x} = \begin{bmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} \end{bmatrix}^2 \tag{3}$$

$$\mathbf{x}^{\top}\mathbf{V}\mathbf{x} = \begin{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} \end{pmatrix}^{\top} \begin{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} \end{pmatrix}$$
(4)

$$\mathbf{x}^{\mathsf{T}}\mathbf{V}\mathbf{x} = \mathbf{x}^{\mathsf{T}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} \tag{5}$$

$$\mathbf{x}^{\mathsf{T}} \left( \mathbf{V} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) \mathbf{x} = 0 \implies \mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{6}$$

$$2\mathbf{u}^{\mathsf{T}}\mathbf{x} = -4ax\tag{7}$$

$$2\mathbf{u}^{\mathsf{T}}\mathbf{x} = -4a \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} \implies 2\mathbf{u}^{\mathsf{T}}\mathbf{x} = \begin{pmatrix} -4a & 0 \end{pmatrix} \mathbf{x} \tag{8}$$

$$(2\mathbf{u}^{\mathsf{T}} - (-4a \quad 0))\mathbf{x} = 0 \implies \mathbf{u}^{\mathsf{T}} = (-2a \quad 0)$$
(9)

$$\mathbf{u} = \begin{pmatrix} -2a \\ 0 \end{pmatrix} \tag{10}$$

$$f = 0 \tag{11}$$

$$\mathbf{F} = \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{12}$$

Let X be the point of locus of the midpoint

$$\mathbf{X} = \frac{\mathbf{x} + \mathbf{F}}{2} \implies \mathbf{x} = 2\mathbf{X} - \mathbf{F} \tag{13}$$

From (1) and (13)

$$\mathbf{x}^{\mathsf{T}}\mathbf{V}\mathbf{x} + 2\mathbf{u}^{\mathsf{T}}\mathbf{x} + f = 0 \tag{14}$$

$$(2\mathbf{X} - \mathbf{F})^{\mathsf{T}} \mathbf{V} (2\mathbf{X} - \mathbf{F}) + 2\mathbf{u}^{\mathsf{T}} (2\mathbf{X} - \mathbf{F}) + f = 0$$
(15)

$$\left(\mathbf{2X}^{\top} - \mathbf{F}^{\top}\right) \mathbf{V} \left(\mathbf{2X} - \mathbf{F}\right) + 2\mathbf{u}^{\top} \left(\mathbf{2X} - \mathbf{F}\right) + f = 0$$
(16)

$$4\mathbf{X}^{\mathsf{T}}\mathbf{V}\mathbf{X} - 2\mathbf{X}^{\mathsf{T}}\mathbf{V}\mathbf{F} - 2\mathbf{F}^{\mathsf{T}}\mathbf{V}\mathbf{X} + \mathbf{F}^{\mathsf{T}}\mathbf{V}\mathbf{F} + 4\mathbf{u}^{\mathsf{T}}\mathbf{X} - 2\mathbf{u}^{\mathsf{T}}\mathbf{F} + f = 0$$
 (17)

As V is a symmetric matrix

$$4\mathbf{X}^{\mathsf{T}}\mathbf{V}\mathbf{X} - 2\mathbf{F}^{\mathsf{T}}\mathbf{V}\mathbf{X} - 2\mathbf{F}^{\mathsf{T}}\mathbf{V}\mathbf{X} + \mathbf{F}^{\mathsf{T}}\mathbf{V}\mathbf{F} + 4\mathbf{u}^{\mathsf{T}}\mathbf{X} - 2\mathbf{u}^{\mathsf{T}}\mathbf{F} + f = 0$$
(18)

$$4\mathbf{X}^{\mathsf{T}}\mathbf{V}\mathbf{X} - 4\mathbf{F}^{\mathsf{T}}\mathbf{V}\mathbf{X} + 4\mathbf{u}^{\mathsf{T}}\mathbf{X} + \mathbf{F}^{\mathsf{T}}\mathbf{V}\mathbf{F} - 2\mathbf{u}^{\mathsf{T}}\mathbf{F} + f = 0$$
 (19)

$$\mathbf{X}^{\mathsf{T}} \left( \mathbf{4} \mathbf{V} \right) \mathbf{X} + 2 \left( 2 \left( \mathbf{u}^{\mathsf{T}} - \mathbf{F}^{\mathsf{T}} \mathbf{V} \right) \right) \mathbf{X} + \mathbf{F}^{\mathsf{T}} \mathbf{V} \mathbf{F} - 2 \mathbf{u}^{\mathsf{T}} \mathbf{F} + f = 0$$
 (20)

$$\mathbf{X}^{\mathsf{T}} \left( \mathbf{4V} \right) \mathbf{X} + 2 \left( 2 \left( \mathbf{u} - \mathbf{VF} \right)^{\mathsf{T}} \right) \mathbf{X} + \mathbf{F}^{\mathsf{T}} \mathbf{VF} - 2 \mathbf{u}^{\mathsf{T}} \mathbf{F} + f = 0$$
 (21)

$$\mathbf{V}' = 4\mathbf{V} = 4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \tag{22}$$

$$\mathbf{u}' = 2\left(\mathbf{u} - \mathbf{V}\mathbf{F}\right) = 2\left(\begin{pmatrix} -2a \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix}\right) = 2\left(\begin{pmatrix} -2a \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} -4a \\ 0 \end{pmatrix} \tag{23}$$

$$f' = \mathbf{F}^{\mathsf{T}} \mathbf{V} \mathbf{F} - 2 \mathbf{u}^{\mathsf{T}} \mathbf{F} + f \tag{24}$$

$$f' = \begin{pmatrix} a \\ 0 \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} - 2 \begin{pmatrix} -2a \\ 0 \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} a \\ 0 \end{pmatrix} + 0 \tag{25}$$

$$f' = \begin{pmatrix} a & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} - 2 \begin{pmatrix} -2a & 0 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} + 0 \tag{26}$$

$$f' = \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} - 2 \begin{pmatrix} -2a & 0 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} + 0 \tag{27}$$

$$f' = -2(-2a^2) = 4a^2 \tag{28}$$

Finding eigen values of V'

$$|\mathbf{V}' - \lambda \mathbf{I}| = 0 \tag{29}$$

$$\begin{vmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} = 0 \implies \begin{vmatrix} -\lambda & 0 \\ 0 & 4 - \lambda \end{vmatrix} = 0 \tag{30}$$

$$(-\lambda)(4-\lambda) = 0 \implies \lambda_1 = 0 \text{ and } \lambda_2 = 4$$
 (31)

 $\mathbf{p_1}$  is an eigen vector of  $\mathbf{V}'$ 

$$(\mathbf{V}' - \lambda \mathbf{I}) \mathbf{p} = \mathbf{0} \tag{32}$$

From (30) and substituting  $\lambda$ =0

$$\begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{p_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{33}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{34}$$

$$0 = 0, y = 0 \implies \mathbf{p_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{35}$$

The Equation of a Directrix is given by

$$\mathbf{n}^{\mathsf{T}}\mathbf{x} = c \tag{36}$$

where

$$\mathbf{n} = \sqrt{\lambda_2} \mathbf{p_1} \text{ (and) } c = \frac{\left( \|\mathbf{u}'\|^2 - \lambda_2 f \right)}{2\mathbf{u}'^{\mathsf{T}} \mathbf{n}}$$
(37)

$$\mathbf{n} = \sqrt{4} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies \mathbf{n} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{38}$$

$$c = \frac{\left((-4a)^2 + (0)^2 - 4\left(4a^2\right)\right)}{2\binom{-4a}{0}^{\mathsf{T}}\binom{2}{0}} = \frac{\left(16a^2 - 16a^2\right)}{2\binom{-4a}{0}^{\mathsf{T}}\binom{2}{0}} = 0$$
(39)

From (36)

$$(2 0) \begin{pmatrix} x \\ y \end{pmatrix} = 0 \implies 2x = 0 \implies x = 0$$
 (40)

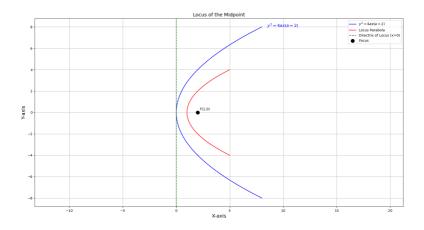


Fig. 4.1