Samyak Gondane-AI25BTECH11029

Question

If $\mathbf{A} = \begin{pmatrix} 5a & -b \\ 3 & 2 \end{pmatrix}$ and $\mathbf{A}\text{adj}(\mathbf{A}) = \mathbf{A}\mathbf{A}^T$, then $5a + b$ is equal to

Solution

Given:

$$A = \begin{pmatrix} 5a & -b \\ 3 & 2 \end{pmatrix} \tag{1}$$

$$\text{Adj}(A) = \begin{pmatrix} 2 & b \\ -3 & 5a \end{pmatrix} \tag{2}$$

Compute $AA^\top$

First, compute the transpose:

$$A^\top = \begin{pmatrix} 5a & 3 \\ -b & 2 \end{pmatrix} \tag{3}$$

Now multiply:

$$AA^\top = \begin{pmatrix} 5a & -b \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 5a & 3 \\ -b & 2 \end{pmatrix} \tag{4}$$

Compute each entry:

$$AA^\top = \begin{pmatrix} (5a)^2 + (-b)^2 & 5a \cdot 3 + (-b) \cdot 2 \\ 3 \cdot 5a + 2 \cdot (-b) & 3^2 + 2^2 \end{pmatrix} = \begin{pmatrix} 25a^2 + b^2 & 15a - 2b \\ 15a - 2b & 13 \end{pmatrix} \tag{5}$$

Equate $AA^T = \mathbf{A}Adj\mathbf{A}$

$$\begin{pmatrix} 25a^2 + b^2 & 15a - 2b \\ 15a - 2b & 13 \end{pmatrix} = \begin{pmatrix} 2 & b \\ -3 & 5a \end{pmatrix} \tag{6}$$

Compare bottom-right entries:

$$13 = 5a \Rightarrow a = \frac{13}{5} \tag{7}$$

Compare top-right entries:

$$15a - 2b = b \Rightarrow 15a = 3b \Rightarrow b = 13 \tag{8}$$

Final Step: Compute $5a + b$

$$5a + b = 5 \times \frac{13}{5} + 13 = 26 = \boxed{26} \tag{9}$$