2.10.79

EE25BTECH11051 - Shreyas Goud Burra

Question

In a triangle PQR, let

$$\mathbf{a} = \mathbf{QR}, \mathbf{b} = \mathbf{RP}, \mathbf{c} = \mathbf{PQ}$$

det $\mathbf{a}$ = 3, det $\mathbf{b}$ = 4, and

$$\frac{\mathbf{a}.(\mathbf{c}-\mathbf{b})}{\mathbf{c}.(\mathbf{a}-\mathbf{b})} = \frac{|\mathbf{a}|}{|\mathbf{a}|+|\mathbf{b}|}$$

then the value of $|\mathbf{a}\times\mathbf{b}|$ is ______

Solution:

Let us find the solution theoretically first and then verify it computationally. It is given that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the sides of a triangle. This implies

$$\mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{QR}+\mathbf{RP}+\mathbf{PQ} = 0 \tag{0.1}$$

It is also given that,

$$|\mathbf{a}| = 3 \text{ and } |\mathbf{b}| = 4 \tag{0.2}$$

Let the given equation,

$$\frac{\mathbf{a}.(\mathbf{c}-\mathbf{b})}{\mathbf{c}.(\mathbf{a}-\mathbf{b})} = \frac{\|\mathbf{a}\|}{\|\mathbf{a}\|+\|\mathbf{b}\|} \tag{0.3}$$

This gives,

$$\frac{\mathbf{a}.(\mathbf{c}-\mathbf{b})}{\mathbf{c}.(\mathbf{a}-\mathbf{b})} = \frac{3}{7} \tag{0.4}$$

On further simplifying this gives us,

$$7(\mathbf{a}^{\mathrm{T}}\mathbf{c}-\mathbf{a}^{\mathrm{T}}\mathbf{b}) = 3(\mathbf{c}^{\mathrm{T}}\mathbf{a}-\mathbf{c}^{\mathrm{T}}\mathbf{b}) \tag{0.5}$$

$$4\mathbf{a}^{\mathrm{T}}\mathbf{c}-7\mathbf{a}^{\mathrm{T}}\mathbf{b}+3\mathbf{c}^{\mathrm{T}}\mathbf{b} = 0 \tag{0.6}$$

On multiplying $\mathbf{a}^{\mathbf{T}}$ on both sides of 0.1

$$\mathbf{a}^{\mathrm{T}}\mathbf{a}+\mathbf{a}^{\mathrm{T}}\mathbf{b}+\mathbf{a}^{\mathrm{T}}\mathbf{c} = 0 \implies \mathbf{a}^{\mathrm{T}}\mathbf{b}+\mathbf{a}^{\mathrm{T}}\mathbf{c} = -9 \tag{0.7}$$

On multiplying $\mathbf{b^T}$ on both sides of 0.1

$$\mathbf{b}^{\mathrm{T}}\mathbf{a} + \mathbf{b}^{\mathrm{T}}\mathbf{b} + \mathbf{b}^{\mathrm{T}}\mathbf{c} = 0 \implies \mathbf{b}^{\mathrm{T}}\mathbf{a} + \mathbf{b}^{\mathrm{T}}\mathbf{c} = -16 \tag{0.8}$$

On solving the equations 0.6, 0.7 and 0.8

$$\begin{pmatrix} -7 & 3 & 4 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{a}^{\mathrm{T}}\mathbf{b} \\ \mathbf{b}^{\mathrm{T}}\mathbf{c} \\ \mathbf{c}^{\mathrm{T}}\mathbf{a} \end{pmatrix} = \begin{pmatrix} 0 \\ -9 \\ -16 \end{pmatrix} \tag{0.9}$$

On using Gauss Jordan method to solve this

$$\begin{pmatrix} \mathbf{a}^{\mathrm{T}}\mathbf{b} \\ \mathbf{b}^{\mathrm{T}}\mathbf{c} \\ \mathbf{c}^{\mathrm{T}}\mathbf{a} \end{pmatrix} = \begin{pmatrix} -7 & 3 & 4 & | & 0 \\ 1 & 0 & 1 & | & -9 \\ 1 & 1 & 0 & | & -16 \end{pmatrix} \tag{0.10}$$

On doing $R_1 \to R_1 + 8R_2$ and $R_3 \to R_3 - R_2$

$$\begin{pmatrix} \mathbf{a}^{\mathrm{T}}\mathbf{b} \\ \mathbf{b}^{\mathrm{T}}\mathbf{c} \\ \mathbf{c}^{\mathrm{T}}\mathbf{a} \end{pmatrix} = \begin{pmatrix} 1 & 3 & 12 & | & -72 \\ 1 & 0 & 1 & | & -9 \\ 0 & 1 & -1 & | & -7 \end{pmatrix} \tag{0.11}$$

On doing $R_2 \to R_2 - R_1$

$$\begin{pmatrix} \mathbf{a}^{\mathrm{T}}\mathbf{b} \\ \mathbf{b}^{\mathrm{T}}\mathbf{c} \\ \mathbf{c}^{\mathrm{T}}\mathbf{a} \end{pmatrix} = \begin{pmatrix} 1 & 3 & 12 & | & -72 \\ 0 & -3 & -11 & | & 63 \\ 0 & 1 & -1 & | & -7 \end{pmatrix} \tag{0.12}$$

On doing $R_1 \to R_1 + R_2$ and $R_3 \to R_3 + \frac{1}{3}R_2$

$$\begin{pmatrix} \mathbf{a}^{\mathrm{T}}\mathbf{b} \\ \mathbf{b}^{\mathrm{T}}\mathbf{c} \\ \mathbf{c}^{\mathrm{T}}\mathbf{a} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & | & -9 \\ 0 & -3 & -11 & | & 63 \\ 0 & 0 & -\frac{14}{3} & | & 14 \end{pmatrix} \tag{0.13}$$

On doing $R_1 \to R_1 + \frac{3}{14}R_3$ and $R_2 \to R_2 - \frac{33}{14}R_3$

$$\begin{pmatrix} \mathbf{a}^{\mathrm{T}}\mathbf{b} \\ \mathbf{b}^{\mathrm{T}}\mathbf{c} \\ \mathbf{c}^{\mathrm{T}}\mathbf{a} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & | & -6 \\ 0 & -3 & 0 & | & 30 \\ 0 & 0 & -\frac{14}{3} & | & 14 \end{pmatrix} \tag{0.14}$$

From this, we get,

$$\mathbf{a}^{\mathrm{T}}\mathbf{b} = -6 \tag{0.15}$$

From the definition of cross product, and from 0.15 we get,

$$\|\mathbf{a} \times \mathbf{b}\|^2 = \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 - (\mathbf{a}^\top \mathbf{b})^2 \implies \|\mathbf{a} \times \mathbf{b}\|^2 = 4^2.3^2 - (-6)^2 \tag{0.16}$$

The final answer,

$$\|\mathbf{a} \times \mathbf{b}\| = 6\sqrt{3} \tag{0.17}$$

The plot for the given question is given below,

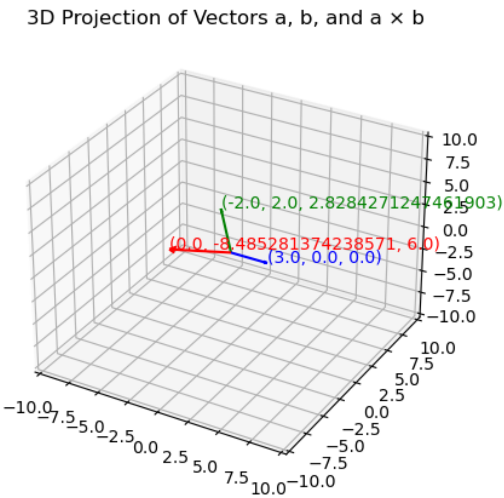

Fig. 0.1: 3D Plot