Problem 2.7.4

Problem. If

$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \qquad \mathbf{c} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \tag{1}$$

find $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.

Solution.

Input variable	Value
$\mathbf{a}$	$\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$
$\mathbf{b}$	$\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$
$\mathbf{c}$	$\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$

Table 1

We are asked to compute:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}). \tag{2}$$

Step 1 — Vectors as column matrices

$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \qquad \mathbf{c} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}. \tag{3}$$

Step 2 — Form the Gram matrix

The Gram matrix is

$$G = \begin{pmatrix} \mathbf{a}^T\mathbf{a} & \mathbf{a}^T\mathbf{b} & \mathbf{a}^T\mathbf{c} \\ \mathbf{b}^T\mathbf{a} & \mathbf{b}^T\mathbf{b} & \mathbf{b}^T\mathbf{c} \\ \mathbf{c}^T\mathbf{a} & \mathbf{c}^T\mathbf{b} & \mathbf{c}^T\mathbf{c} \end{pmatrix}. \tag{4}$$

Compute each entry:

$$\mathbf{a}^T\mathbf{a} = 14, \qquad \mathbf{b}^T\mathbf{b} = 6, \qquad \mathbf{c}^T\mathbf{c} = 14, \tag{5}$$
$$\mathbf{a}^T\mathbf{b} = 3, \qquad \mathbf{b}^T\mathbf{c} = 1, \qquad \mathbf{c}^T\mathbf{a} = 13. \tag{6}$$

Thus,

$$G = \begin{pmatrix} 14 & 3 & 13 \\ 3 & 6 & 1 \\ 13 & 1 & 14 \end{pmatrix}. \tag{7}$$

Step 3 — Gram determinant identity

We know

$$\det(G) = \big(\det([\mathbf{a}\ \mathbf{b}\ \mathbf{c}])\big)^2 \tag{8}$$
$$= \big(\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})\big)^2. \tag{9}$$

Direct computation gives

$$\det(G) = 100. \tag{10}$$

Hence

$$|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = \sqrt{100} = 10. \tag{11}$$

Step 4 — Find the sign

Form the matrix

$$A = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{pmatrix}. \tag{12}$$

Then

$$\det(A) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}). \tag{13}$$

Compute:

$$\det(A) = -10. \tag{14}$$

Final Answer

$$\boxed{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -10} \tag{15}$$

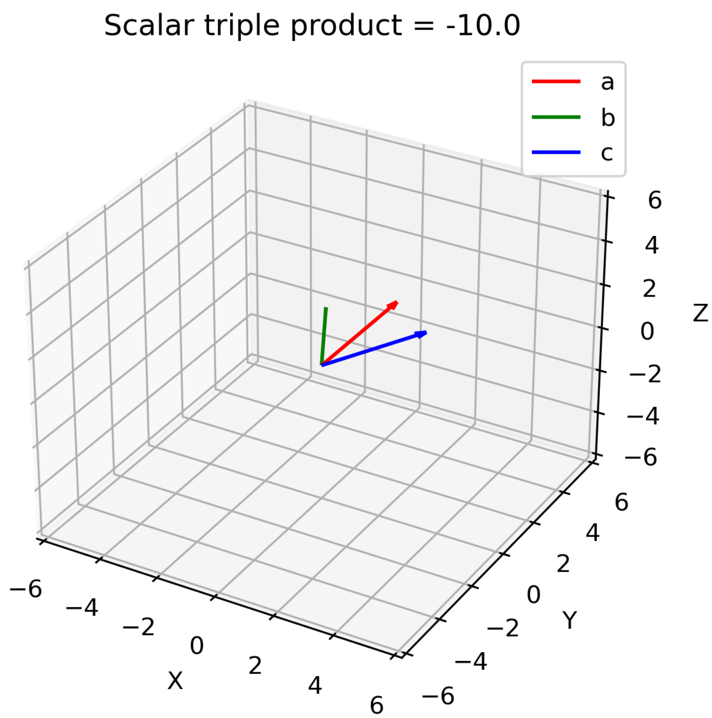

Figure 1