10.6.8

EE25BTECH11052 - Shriyansh Kalpesh Chawda

Question

Construct a pair of tangents to a circle of radius 4cm from a point P lying outside the circle at a distance of 6cm from the centre. (10, 2023)

Solution

Let the center of the circle be at origin and point P be at distance 6 from center along x-axis.

$$O = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1}$$

$$P = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \tag{2}$$

The equation of the circle $x^2 + y^2 = 16$ can be written as:

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{3}$$

where

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad f = -16 \tag{4}$$

Eigenvalue Decomposition:

The eigenvalues of $\mathbf{V}$ satisfy:

$$\det(\mathbf{V} - \lambda \mathbf{I}) = 0 \tag{5}$$

$$(1 - \lambda)^2 = 0 \tag{6}$$

$$\lambda_1 = \lambda_2 = 1 \tag{7}$$

The corresponding orthonormal eigenvectors are:

$$\mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{p}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8}$$

These form the eigenvector matrix:

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{9}$$

The semi-axes of the circle are:

$$a = b = \sqrt{\frac{-f}{\lambda_1}} = \sqrt{\frac{16}{1}} = 4 \tag{10}$$

For tangent from external point P, the contact point $\mathbf{q}$ satisfies:

(a) $\mathbf{q}$ lies on circle: $\mathbf{q}^\top \mathbf{V} \mathbf{q} + f = 0$

(b) Tangent passes through P: $(\mathbf{Vq})^\top P + f = 0$

From condition (b) with $\mathbf{V} = \mathbf{I}$:

$$\mathbf{q}^\top P + f = 0 \tag{11}$$
$$\begin{pmatrix} q_1 & q_2 \end{pmatrix}\begin{pmatrix} 6 \\ 0 \end{pmatrix} = 16 \tag{12}$$
$$6q_1 = 16 \tag{13}$$
$$q_1 = \frac{8}{3} \tag{14}$$

From condition (a):

$$q_1^2 + q_2^2 = 16 \tag{15}$$
$$\left(\frac{8}{3}\right)^2 + q_2^2 = 16 \tag{16}$$
$$q_2^2 = \frac{80}{9} \tag{17}$$
$$q_2 = \pm\frac{4\sqrt{5}}{3} \tag{18}$$

The contact points can be expressed as linear combinations of eigenvectors:

$$\mathbf{q}_1 = \frac{8}{3}\mathbf{p}_1 + \frac{4\sqrt{5}}{3}\mathbf{p}_2 \tag{19}$$
$$= \frac{8}{3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{4\sqrt{5}}{3}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{20}$$
$$= \begin{pmatrix} \frac{8}{3} \\ \frac{4\sqrt{5}}{3} \end{pmatrix} \tag{21}$$

$$\mathbf{q}_2 = \frac{8}{3}\mathbf{p}_1 - \frac{4\sqrt{5}}{3}\mathbf{p}_2 \tag{22}$$
$$= \frac{8}{3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{4\sqrt{5}}{3}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{23}$$
$$= \begin{pmatrix} \frac{8}{3} \\ -\frac{4\sqrt{5}}{3} \end{pmatrix} \tag{24}$$

This shows that the contact points lie along the principal axes (eigenvector directions) of the circle, with coefficients $\frac{8}{3}$ and $\pm\frac{4\sqrt{5}}{3}$ along $\mathbf{p}_1$ and $\mathbf{p}_2$ respectively.

Equations of Tangents:
The tangent at $\mathbf{q}$ is given by: $(\mathbf{Vq})^\top \mathbf{x} + f = 0$.

Tangent 1 at $\mathbf{q}_1$:

$$\mathbf{V}\mathbf{q}_1 = \begin{pmatrix} \frac{8}{3} \\ \frac{4\sqrt{5}}{3} \end{pmatrix} \tag{25}$$

$$\begin{pmatrix} \frac{8}{3} & \frac{4\sqrt{5}}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - 16 = 0 \tag{26}$$

$$\frac{8}{3}x + \frac{4\sqrt{5}}{3}y = 16 \tag{27}$$

$$2x + \sqrt{5}y = 12 \tag{28}$$

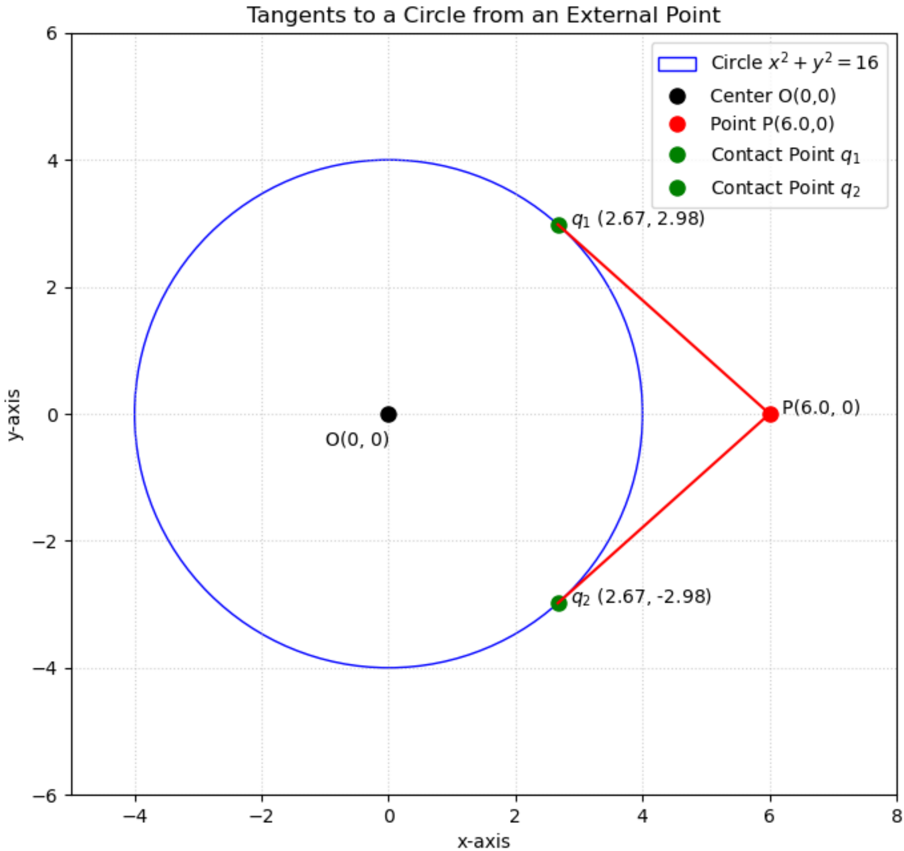
Tangent 2 at $\mathbf{q}_2$:

$$\begin{pmatrix} \frac{8}{3} & -\frac{4\sqrt{5}}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - 16 = 0 \tag{29}$$

$$2x - \sqrt{5}y = 12 \tag{30}$$

The equations of the pair of tangents are:

$$\boxed{2x + \sqrt{5}y = 12 \quad \text{and} \quad 2x - \sqrt{5}y = 12} \tag{31}$$



Tangents to a Circle from an External Point
Circle $x^2 + y^2 = 16$
Center O(0,0)
Point P(6.0,0)
Contact Point q_1
Contact Point q_2
q_1 (2.67, 2.98)
q_2 (2.67, -2.98)
P(6.0, 0)
O(0, 0)
y-axis
x-axis