1.8.5

AI25BTECH110031
Shivam Sawarkar

Question(1.8.5) If $\mathbf{A}$ and $\mathbf{B}$ be the points $(3, 4, 5)$ and $(-1, 3, -7)$ respectively, find the equation of the set of a point $\mathbf{P}$ such that $\mathbf{PA}^2 + \mathbf{PB}^2 = k^2$

Solution: Given ponits

$$\mathbf{A} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} -1 \\ 3 \\ -7 \end{pmatrix} \tag{0.1}$$

According to the question,

$$\mathbf{PA}^2 + \mathbf{PB}^2 = k^2 \tag{0.2}$$

where, $\mathbf{PA} = ||P - A||$ and $\mathbf{PB} = ||P - B||$

The squared distances can be written as dot products:

$$\mathbf{PA}^2 = (\mathbf{P} - \mathbf{A})^T(\mathbf{P} - \mathbf{A}) \tag{0.3}$$

$$\mathbf{PB}^2 = (\mathbf{P} - \mathbf{B})^T(\mathbf{P} - \mathbf{B}) \tag{0.4}$$

Thus:

$$\mathbf{PA}^2 + \mathbf{PB}^2 = (\mathbf{P} - \mathbf{A})^T(\mathbf{P} - \mathbf{A}) + (\mathbf{P} - \mathbf{B})^T(\mathbf{P} - \mathbf{B}) \tag{0.5}$$

$$\mathbf{PA}^2 + \mathbf{PB}^2 = 2\mathbf{P}^T\mathbf{P} - 2\mathbf{A}^T\mathbf{P} + \mathbf{A}^T\mathbf{A} + \mathbf{P}^T\mathbf{P} - 2\mathbf{B}^T\mathbf{P} + \mathbf{B}^T\mathbf{B} \tag{0.6}$$

$$2\mathbf{P}^T\mathbf{P} - 2(\mathbf{A} + \mathbf{B})^T\mathbf{P} + \mathbf{A}^T\mathbf{A} + \mathbf{B}^T\mathbf{B} - k^2 = 0 \tag{0.7}$$

Complete the square,
Let,

$$\mathbf{M} := \frac{\mathbf{A} + \mathbf{B}}{2} \tag{0.8}$$

$$R^2 := ||M||^2 - \frac{\mathbf{A}^T\mathbf{A} + \mathbf{B}^T\mathbf{B} - k^2}{2} \tag{0.9}$$

$$\tag{0.10}$$

Then the equation becomes

$$||P - M||^2 = R^2 \tag{0.11}$$

$$\left\| P - \frac{\mathbf{A} + \mathbf{B}}{2} \right\|^2 = \left\| \frac{\mathbf{A} + \mathbf{B}}{2} \right\|^2 - \frac{||A||^2 + ||B||^2 - k^2}{2} \tag{0.12}$$

Substitute the known values

$$\|A\| = 3^2 + 4^2 + 5^2 = 50 \tag{0.13}$$

$$\|B\| = (-1)^2 + 3^2 + (-7)^2 = 59 \tag{0.14}$$

$$\frac{\mathbf{A} + \mathbf{B}}{2} = \begin{pmatrix} 1 \\ 3.5 \\ -1 \end{pmatrix} \tag{0.15}$$

The equation of the locus is:

$$\left\| P - \begin{pmatrix} 1 \\ 3.5 \\ -1 \end{pmatrix} \right\|^2 = \frac{2k^2 - 161}{4} \tag{0.16}$$

The plot show the locus for $k = 20$

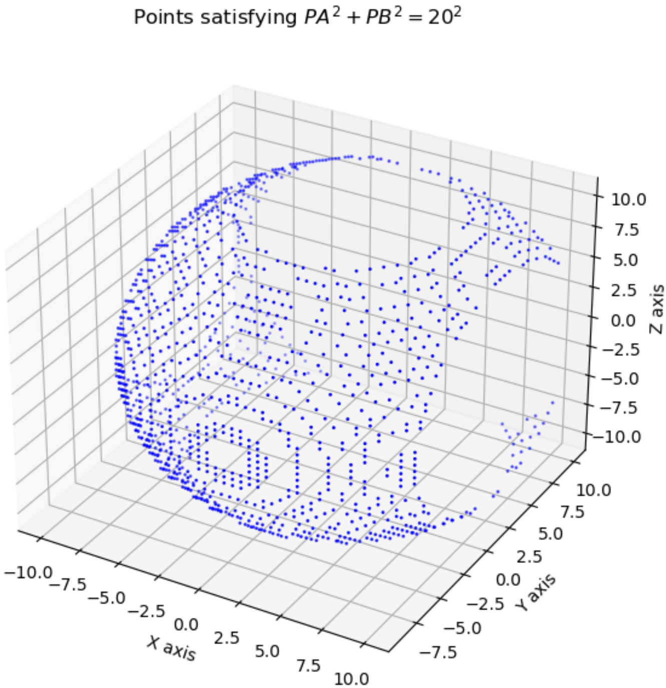

Fig. 0.1