2.9.8

EE25BTECH11057 - Rushil Shanmukha Srinivas

Question : $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are four non-zero vectors such that $\mathbf{a}\times\mathbf{b} = \mathbf{c}\times\mathbf{d}$ and $\mathbf{a}\times\mathbf{c} = 4\,\mathbf{b}\times\mathbf{d}$ then show that $(\mathbf{a} - 2\mathbf{d})$ is parallel to $(2\mathbf{b} - \mathbf{c})$ where $\mathbf{a} \neq 2\mathbf{d}$, $\mathbf{c} \neq 2\mathbf{b}$.

Solution : We are given nonzero vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ such that

$$\mathbf{a} \times \mathbf{b} = \mathbf{c} \times \mathbf{d}, \mathbf{a} \times \mathbf{c} = 4\,\mathbf{b} \times \mathbf{d}, \tag{0.1}$$

with $\mathbf{a} \neq 2\mathbf{d}$ and $\mathbf{c} \neq 2\mathbf{b}$.

We need to show $(\mathbf{a} - 2\mathbf{d})$ is parallel to $(2\mathbf{b} - \mathbf{c})$, i.e.

$$(\mathbf{a} - 2\mathbf{d}) \times (2\mathbf{b} - \mathbf{c}) = \mathbf{0}. \tag{0.2}$$

By bilinearity:

$$(\mathbf{a} - 2\mathbf{d}) \times (2\mathbf{b} - \mathbf{c}) = 2(\mathbf{a} \times \mathbf{b}) - (\mathbf{a} \times \mathbf{c}) - 4(\mathbf{d} \times \mathbf{b}) + 2(\mathbf{d} \times \mathbf{c}). \tag{0.3}$$

Also, $\mathbf{d} \times \mathbf{b} = -\mathbf{b} \times \mathbf{d}$ and $\mathbf{d} \times \mathbf{c} = -\mathbf{c} \times \mathbf{d}$.

Substitute these:

$$2(\mathbf{c} \times \mathbf{d}) - 4(\mathbf{b} \times \mathbf{d}) - 4(-\mathbf{b} \times \mathbf{d}) + 2(-\mathbf{c} \times \mathbf{d}) \tag{0.4}$$

$$= 2(\mathbf{c} \times \mathbf{d}) - 4(\mathbf{b} \times \mathbf{d}) + 4(\mathbf{b} \times \mathbf{d}) - 2(\mathbf{c} \times \mathbf{d}) = \mathbf{0}. \tag{0.5}$$

Let $\mathbf{u} = \mathbf{a} - 2\mathbf{d}$ and $\mathbf{v} = 2\mathbf{b} - \mathbf{c}$. Since $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, they are linearly dependent.

Equivalently, the matrix

$$M = \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} \tag{0.6}$$

has $\text{rank}(M) = 1$. This is exactly the criterion for $\mathbf{u}$ and $\mathbf{v}$ to be parallel.

Therefore,

$$\boxed{\mathbf{a} - 2\mathbf{d} \parallel 2\mathbf{b} - \mathbf{c}}. \tag{0.7}$$

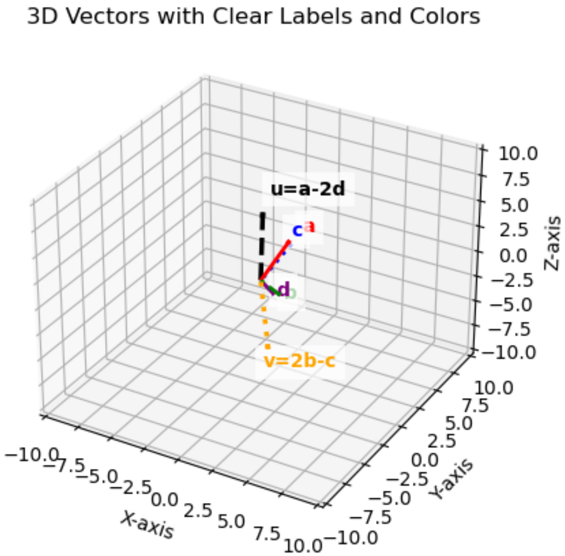

Fig: Representation of vectors