Problem 12.557

ee25btech11023-Venkata Sai

October 10, 2025

Problem

Let $\mathbf{A} = \begin{pmatrix} 5 & -3 \\ 6 & -4 \end{pmatrix}$. Then the trace of $\mathbf{A}^{1000}$ equals

Given

Given

$$\mathbf{A} = \begin{pmatrix} 5 & -3 \\ 6 & -4 \end{pmatrix} \tag{3.1}$$

$$\tag{3.2}$$

To find eigen values

$$|\mathbf{A} - \lambda \mathbf{I}| = 0 \tag{3.3}$$

$$\left| \begin{pmatrix} 5 & -3 \\ 6 & -4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0 \tag{3.4}$$

$$\left| \begin{pmatrix} 5 & -3 \\ 6 & -4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0 \tag{3.5}$$

$$\left| \begin{pmatrix} 5-\lambda & -3 \\ 6 & -4-\lambda \end{pmatrix} \right| = 0 \tag{3.6}$$

$$(5-\lambda)(-4-\lambda) + 3(6) = 0 \tag{3.7}$$

Finding eigen values

$$\lambda^2 + 4\lambda - 5\lambda - 20 + 18 = 0 \tag{3.8}$$

$$\lambda^2 - \lambda - 2 = 0 \tag{3.9}$$

$$(\lambda - 2)(\lambda + 1) = 0 \tag{3.10}$$

$$\lambda_1 = 2 \ \text{(and)} \ \lambda_2 = -1 \tag{3.11}$$

For a given matrix **A**

$$\mathbf{A} = \mathbf{PDP}^{-1} \tag{3.12}$$

$$\mathbf{A}^2 = \left(\mathbf{PDP}^{-1}\right)^2 \tag{3.13}$$

$$= \mathbf{PDP}^{-1}\mathbf{PDP}^{-1} \tag{3.14}$$

$$= \mathbf{PDIDP}^{-1} \tag{3.15}$$

$$= \mathbf{PD}^2\mathbf{P}^{-1} \tag{3.16}$$

where

Formula

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.17}$$

$$\mathbf{A}^k = \mathbf{P}\mathbf{D}^k\mathbf{P}^{-1} \tag{3.18}$$

$$\text{trace}\left(\mathbf{A}^k\right) = \text{trace}\left(\mathbf{P}\mathbf{D}^k\mathbf{P}^{-1}\right) \tag{3.19}$$

$$= \text{trace}\left(\left(\mathbf{P}\mathbf{D}^k\right)\mathbf{P}^{-1}\right) \tag{3.20}$$

Since trace($\mathbf{AB}$)=trace($\mathbf{BA}$)

$$\text{trace}\left(\mathbf{A}^k\right) = \text{trace}\left(\left(\mathbf{P}\mathbf{D}^k\right)\mathbf{P}^{-1}\right) \tag{3.21}$$

$$= \text{trace}\left(\mathbf{P}^{-1}\left(\mathbf{P}\mathbf{D}^k\right)\right) \tag{3.22}$$

$$\text{trace}\left(\mathbf{A}^k\right) = \text{trace}\left(\mathbf{I}\mathbf{D}^k\right) = \text{trace}\left(\mathbf{D}^k\right) \tag{3.23}$$

Conclusion

$$\operatorname{trace}\left(\mathbf{A}^{1000}\right) = \operatorname{trace}\left(\mathbf{D}^{1000}\right) \tag{3.24}$$

$$= \operatorname{trace}\begin{pmatrix} 2^{1000} & 0 \\ 0 & (-1)^{1000} \end{pmatrix} \tag{3.25}$$

$$= 2^{1000} + 1 \tag{3.26}$$

Python Code for Solving

```
  import numpy as np

A = np.array([
    [ 5, -3],
    [ 6, -4]
])

eigenvalues = np.linalg.eigvals(A)
lambda_1, lambda_2 = eigenvalues
print(fTrace of A^1000={lambda_1}^1000+{lambda_2**1000})
```