12.768

EE25BTECH11026-Harsha

Question:
In the figure, the vectors **u** and **v** are related as **Au** = **v** by a transformation matrix **A**. The correct choice of **A** is

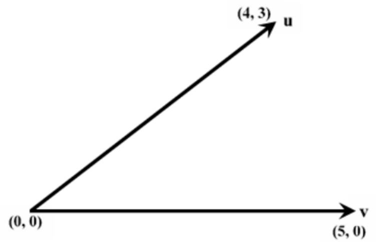

Fig. 0.1: Figure-1

1) $\begin{pmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & -\frac{4}{5} \end{pmatrix}$ 2) $\begin{pmatrix} \frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix}$ 3) $\begin{pmatrix} \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{pmatrix}$ 4) $\begin{pmatrix} \frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & -\frac{4}{5} \end{pmatrix}$

Solution:
Let us solve the given question theoretically and then verify the solution computationally.

Given,

$$\mathbf{u} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \tag{4.1}$$

From 0.1,

$$\|\mathbf{u}\| = \|\mathbf{v}\| = 5 \; units \tag{4.2}$$

This implies, **A** is a rotation matrix.
Rotation matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{4.3}$$

where θ is the angle between the vectors in counter-clockwise sense.

$$\therefore \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \tag{4.4}$$

$$\begin{pmatrix} 4\cos\theta - 3\sin\theta \\ 4\sin\theta + 3\cos\theta \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \tag{4.5}$$

The above equation can be re-arranged as,

$$\begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \tag{4.6}$$

We need to solve for $\cos\theta$ and $\sin\theta$ to get the transformation matrix $\mathbf{A}$.

We can see that in (4.6), the columns of the coefficient matrix are orthogonal to each other and also the column vectors have the same norm.

$$\therefore \frac{1}{5} \begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 5 \\ 0 \end{pmatrix} \tag{4.7}$$

$$\implies \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.8}$$

In equation (4.8), the coefficient matrix is an orthogonal matrix.

$$\implies \mathbf{A}\mathbf{x} = \mathbf{b} \Rightarrow \mathbf{A}^\top\mathbf{A}\mathbf{x} = \mathbf{A}^\top\mathbf{b} \Rightarrow \mathbf{x} = \mathbf{A}^\top\mathbf{b} \qquad \left(\because \mathbf{A}^\top\mathbf{A} = \mathbf{I}\right) \tag{4.9}$$

$$\therefore \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.10}$$

$$\therefore \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ -\frac{3}{5} \end{pmatrix} \tag{4.11}$$

$$\implies \mathbf{A} = \begin{pmatrix} \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{pmatrix} \tag{4.12}$$