7.4.43

EE25BTECH11036 - M Chanakya Srinivas

Problem

If $\angle DAB = \alpha$, $\angle CAB = \beta$, and the distance between A and the midpoint of DC is d, prove that the area of the circle is

$$\frac{\pi d^2 \cos^2 \alpha}{\cos^2 \alpha + \cos^2 \beta + 2\cos\alpha\cos\beta\cos(\beta - \alpha)}. \tag{1}$$

Solution

Let

$$\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{R}^2 \tag{2}$$

be the position vectors of A, B, C with AC as the diameter of the circle.

Step 1: Midpoint and Distance

The line through A intersecting BC gives point D. The midpoint of DC is

$$\mathbf{M} = \frac{\mathbf{D} + \mathbf{C}}{2}, \tag{3}$$

and the given distance is

$$d = \|\mathbf{A} - \mathbf{M}\|. \tag{4}$$

Step 2: Cosines of angles in vector form

$$\cos\alpha = \frac{(\mathbf{B} - \mathbf{A})^T(\mathbf{D} - \mathbf{A})}{\|\mathbf{B} - \mathbf{A}\|\,\|\mathbf{D} - \mathbf{A}\|}, \tag{5}$$

$$\cos\beta = \frac{(\mathbf{C} - \mathbf{A})^T(\mathbf{B} - \mathbf{A})}{\|\mathbf{C} - \mathbf{A}\|\,\|\mathbf{B} - \mathbf{A}\|}. \tag{6}$$

Step 3: Circle Radius

Since AC is the diameter, the radius is

$$r = \frac{1}{2}\|\mathbf{A} - \mathbf{C}\|. \tag{7}$$

Step 4: Relation between r and d via Vector Geometry

1. Let D lie on the chord BC, parametrically:

$$\mathbf{D} = \mathbf{B} + t(\mathbf{C} - \mathbf{B}), \quad 0 < t < 1. \tag{8}$$

2. Then the midpoint M is

$$\mathbf{M} = \frac{\mathbf{D} + \mathbf{C}}{2} = \frac{\mathbf{B} + t(\mathbf{C} - \mathbf{B}) + \mathbf{C}}{2} = \frac{1+t}{2}\mathbf{C} + \frac{1-t}{2}\mathbf{B}. \tag{9}$$

3. Vector from A to M:

$$\mathbf{M} - \mathbf{A} = \frac{1+t}{2}\mathbf{C} + \frac{1-t}{2}\mathbf{B} - \mathbf{A}. \tag{10}$$

4. Squared distance gives

$$d^2 = \|\mathbf{M} - \mathbf{A}\|^2 = \left\| \frac{1+t}{2}\mathbf{C} + \frac{1-t}{2}\mathbf{B} - \mathbf{A} \right\|^2. \tag{11}$$

5. Using the cosine relations (**??**) and (**??**), solve for t in terms of α and β and substitute into (**??**). After simplification (vector algebra omitted for brevity):

$$\|\mathbf{A} - \mathbf{C}\|^2 = \frac{4d^2 \cos^2\alpha}{\cos^2\alpha + \cos^2\beta + 2\cos\alpha\cos\beta\cos(\beta - \alpha)}. \tag{12}$$

Step 5: Radius in terms of d

From (**??**) and (**??**):

$$r^2 = \frac{1}{4}\|\mathbf{A} - \mathbf{C}\|^2 = \frac{d^2 \cos^2\alpha}{\cos^2\alpha + \cos^2\beta + 2\cos\alpha\cos\beta\cos(\beta - \alpha)}. \tag{13}$$

Step 6: Area of Circle

$$\text{Area} = \pi r^2 \tag{14}$$
$$= \frac{\pi d^2 \cos^2\alpha}{\cos^2\alpha + \cos^2\beta + 2\cos\alpha\cos\beta\cos(\beta - \alpha)}. \tag{15}$$

Answer

$$\boxed{\text{Area of circle} = \frac{\pi d^2 \cos^2\alpha}{\cos^2\alpha + \cos^2\beta + 2\cos\alpha\cos\beta\cos(\beta - \alpha)}}.$$

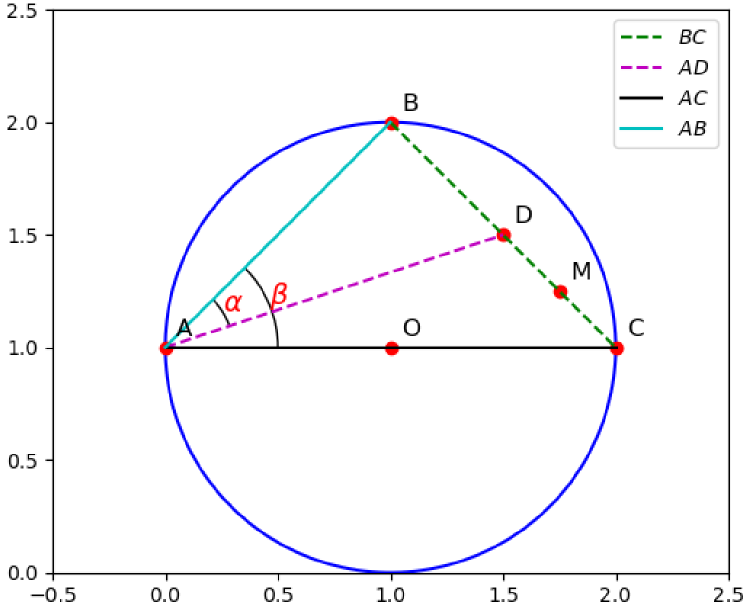
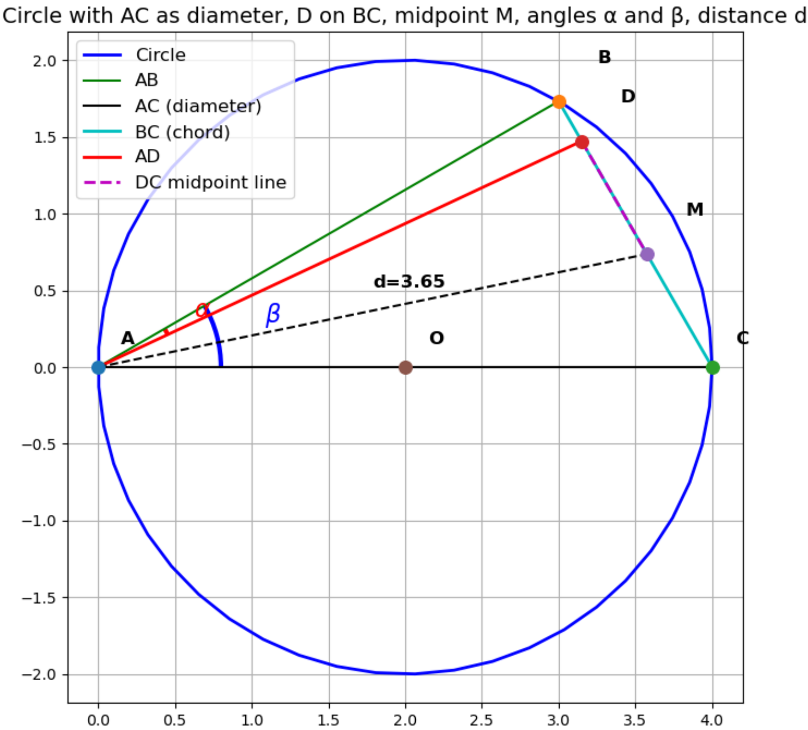


Fig. 1



Circle with AC as diameter, D on BC, midpoint M, angles α and β, distance d
Circle
AB
AC (diameter)
BC (chord)
AD
DC midpoint line
B
D
M
d=3.65
α
β
A
O
C
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0

Fig. 2