10.7.111

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Question:
Let the straight line $y = 2x$ touch a circle with center $(0, a)$, $a > 0$, and radius r at a point $\mathbf{A_1}$. Let $\mathbf{B_1}$ be the point on the circle such that the line segment $\mathbf{A_1B_1}$ is a diameter of the circle. Let $a + r = 5 + \sqrt{5}$. Match the following:

(A) a equals	(1) $(-2, 4)$
(B) r equals	2. $\sqrt{5}$
(C)$\mathbf{A_1}$ equals	(3)$(-2, 6)$
(D) $\mathbf{B_1}$ equals	(4) 5
	(5) $(2, 4)$

The correct option is (2024)

a) A−4, B−2, C−1, D−3
b) A−2, B−4, C−1, D−3
c) A−4, B−2, C−5, D−3
d) A−2, B−4, C−3, D−5

Solution:
The equation of a Conic in Matrix form is

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$

For the given circle. let r be the radius of given circle

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -a \end{pmatrix}, f = a^2 - r^2 \tag{2}$$

Equation of Tangent is given by

$$\mathbf{n}^\top \mathbf{x} = c \tag{3}$$

$$\implies \mathbf{n} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, c = 0 \tag{4}$$

$$\implies \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \tag{5}$$

For a circle, the points of contact are

$$\mathbf{q_j} = \left(\pm r \frac{\mathbf{n_j}}{\left\| \mathbf{n_j} \right\|} - \mathbf{u} \right), j = 1, 2 \tag{6}$$

let $\mathbf{A_1}$ be the point of contact

$$\mathbf{A_1} = \left(-r \frac{\mathbf{n}}{\|\mathbf{n}\|} - \mathbf{u} \right) \tag{7}$$

$$= \left(\frac{r}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ -a \end{pmatrix} \right) \quad \text{(given } a \text{ is positive)} \tag{8}$$

$$\implies \mathbf{A_1} = \begin{pmatrix} \frac{2r}{\sqrt{5}} \\ \frac{-r}{\sqrt{5}} + a \end{pmatrix} \tag{9}$$

$\mathbf{A_1}$ lies on (5)

$$\frac{-r}{\sqrt{5}} + a = \frac{4r}{\sqrt{5}} \tag{10}$$

$$\implies a = \sqrt{5} r \tag{11}$$

given

$$a + r = 5 + \sqrt{5} \tag{12}$$

substitute (11) in (12)

$$\sqrt{5} r + r = 5 + \sqrt{5} \tag{13}$$

$$\implies r = \sqrt{5} \tag{14}$$

From (11)

$$\implies a = 5 \tag{15}$$

$$\implies \mathbf{u} = \begin{pmatrix} 0 \\ 5 \end{pmatrix} \tag{16}$$

From (9)

$$\mathbf{A_1} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \tag{17}$$

Given $\mathbf{A_1}$ and $\mathbf{B_1}$ is the diameter of the circle

$$\frac{\mathbf{A_1} + \mathbf{B_1}}{2} = \mathbf{u} \tag{18}$$

$$\mathbf{B_1} = (\mathbf{u} \quad \mathbf{A_1}) \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{19}$$

$$\mathbf{B_1} = \begin{pmatrix} -2 \\ 6 \end{pmatrix} \tag{20}$$

Answer is c) A–4, B–2, C–5, D–3

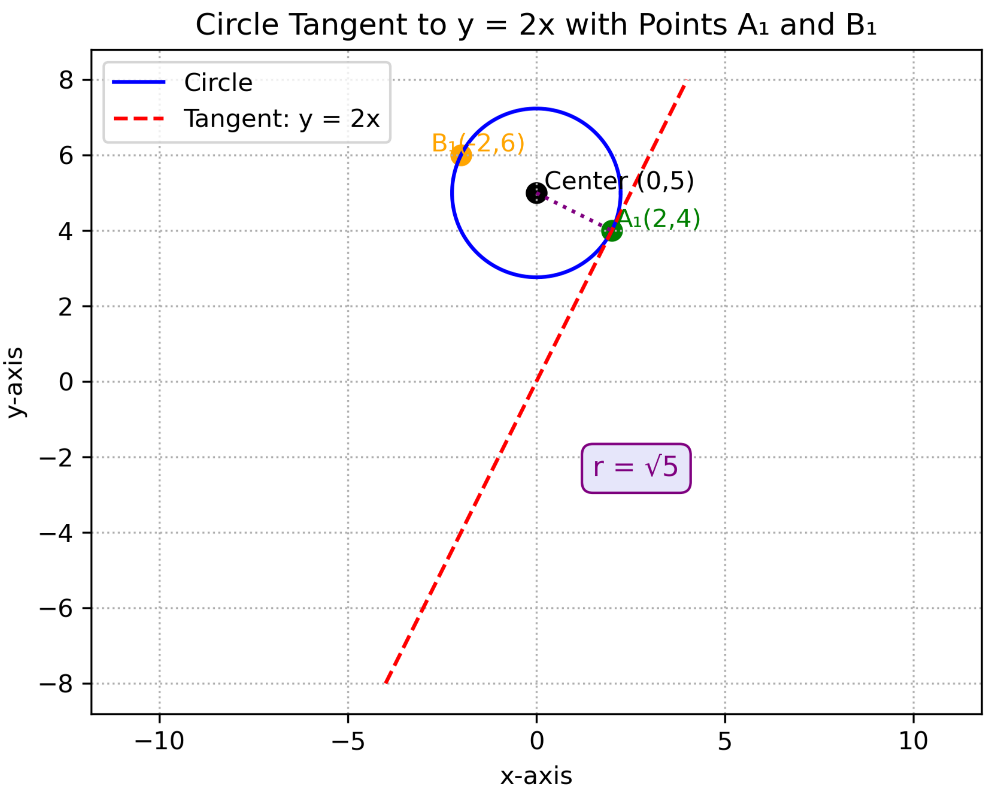

Fig. 4