

2.10.56

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Question. Let two non-collinear unit vectors \hat{a} and \hat{b} form an acute angle. A point \mathbf{P} moves so that at any time t the position vector \overrightarrow{OP} (where \mathbf{O} is the origin) is given by $\hat{a} \cos t + \hat{b} \sin t$. When \mathbf{P} is farthest from origin \mathbf{O} , let M be the length of \overrightarrow{OP} and \hat{u} be the unit vector along \overrightarrow{OP} . Then,

$$1) \hat{u} = \frac{\hat{a} + \hat{b}}{|\hat{a} + \hat{b}|} \text{ and } M = (1 + \hat{a} \cdot \hat{b})^{\frac{1}{2}}$$

$$2) \hat{u} = \frac{\hat{a} - \hat{b}}{|\hat{a} - \hat{b}|} \text{ and } M = (1 + \hat{a} \cdot \hat{b})^{\frac{1}{2}}$$

$$3) \hat{u} = \frac{\hat{a} + \hat{b}}{|\hat{a} + \hat{b}|} \text{ and } M = (1 + 2\hat{a} \cdot \hat{b})^{\frac{1}{2}}$$

$$4) \hat{u} = \frac{\hat{a} - \hat{b}}{|\hat{a} - \hat{b}|} \text{ and } M = (1 + 2\hat{a} \cdot \hat{b})^{\frac{1}{2}}$$

Solution:

Let us solve the given equation theoretically and then verify the solution computationally.
Given equation:

$$\mathbf{P} = \mathbf{a} \cos t + \mathbf{b} \sin t \quad (1)$$

Which can be written as :

$$\mathbf{P} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \quad (2)$$

From given if \mathbf{P} is farthest from origin , then length of \mathbf{P} is given as M. From this we can say that

$$M = \max \|\mathbf{P}\| \quad (3)$$

Now,

$$\|\mathbf{P}\| = \sqrt{(\mathbf{P})^T (\mathbf{P})} \quad (4)$$

$$\|\mathbf{P}\| = \sqrt{\left(\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \right)^T \left(\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \right)} \quad (5)$$

$$\|\mathbf{P}\| = \sqrt{\left(\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}^T \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix}^T \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \right)} \quad (6)$$

$$\|\mathbf{P}\| = \sqrt{\|\mathbf{a}\|^2 \cos^2 t + \|\mathbf{b}\|^2 \sin^2 t + 2(\mathbf{a})^T (\mathbf{b})(\cos t)(\sin t)} \quad (7)$$

$$\|\mathbf{P}\|^2 = \begin{pmatrix} \cos t & \sin t \end{pmatrix} \begin{pmatrix} 1 & (\mathbf{a})^T (\mathbf{b}) \\ (\mathbf{a})^T (\mathbf{b}) & 1 \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \quad (8)$$

Let

$$\mathbf{x} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \text{ and } \mathbf{G} = \begin{pmatrix} 1 & (\mathbf{a})^T(\mathbf{b}) \\ (\mathbf{a})^T(\mathbf{b}) & 1 \end{pmatrix} \quad (9)$$

Then,

$$\|\mathbf{P}\|^2 = \mathbf{x}^T \mathbf{G} \mathbf{x} \quad (10)$$

Now we should find the maximum value of $\mathbf{x}^T \mathbf{G} \mathbf{x}$ such that $\|\mathbf{x}\| = 1$

By **Rayleigh-Ritz theorem**, the maximum value of the quadratic form if \mathbf{x} is a unit vector will be the largest eigenvalue (λ_{max}) of the matrix \mathbf{G} .

So,

$$\max \|\mathbf{P}\| = \sqrt{\lambda_{max}} \quad (11)$$

Now we will calculate the Eigen value for the matrix \mathbf{G} :

$$|G - \lambda I| = 0 \quad (12)$$

$$\left| \begin{pmatrix} 1 - \lambda & (\mathbf{a})^T(\mathbf{b}) \\ (\mathbf{a})^T(\mathbf{b}) & 1 - \lambda \end{pmatrix} \right| = 0 \quad (13)$$

$$(1 - \lambda)^2 - ((\mathbf{a})^T(\mathbf{b}))^2 = 0 \quad (14)$$

$$1 - \lambda = (\mathbf{a})^T(\mathbf{b}) \text{ or } 1 - \lambda = -(\mathbf{a})^T(\mathbf{b}) \quad (15)$$

$$\lambda = 1 + (\mathbf{a})^T(\mathbf{b}) \text{ or } \lambda = 1 - (\mathbf{a})^T(\mathbf{b}) \quad (16)$$

It is already given that $(\mathbf{a})^T(\mathbf{b}) > 0$ (\mathbf{a} and \mathbf{b} form an acute angle) . so,

$$\lambda_{max} = 1 + (\mathbf{a})^T(\mathbf{b}) \quad (17)$$

From Eq.9

$$\max \|\mathbf{P}\| = \sqrt{1 + (\mathbf{a})^T(\mathbf{b})} \quad (18)$$

The above equation can be written as

$$\max \|\mathbf{P}\| = \sqrt{1 + \mathbf{a} \cdot \mathbf{b}} \quad (19)$$

From Eq.2:

$$M = \sqrt{1 + \mathbf{a} \cdot \mathbf{b}} \quad (20)$$

Now let us find the value of t for which $\|\mathbf{P}\|$ is max

With eigenvalue equation, We'll use matrix \mathbf{G} and largest eigenvalue λ_{max} such that,

$$(\mathbf{G} - \lambda I) \mathbf{x} = 0 \quad (21)$$

$$\begin{pmatrix} -(\mathbf{a})^T(\mathbf{b}) & (\mathbf{a})^T(\mathbf{b}) \\ (\mathbf{a})^T(\mathbf{b}) & -(\mathbf{a})^T(\mathbf{b}) \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (22)$$

By solving this we'll get

$$\cos t = \sin t \quad (23)$$

We already know that:

$$\sin^2 t + \cos^2 t = 1 \quad (24)$$

So,

$$\sin t = \frac{1}{\sqrt{2}} \text{ and } \cos t = \frac{1}{\sqrt{2}} \quad (25)$$

From above result

$$t = \frac{\pi}{4} \quad (26)$$

Now unit vector \mathbf{u} along \mathbf{P} is given by:

$$\mathbf{u} = \frac{\mathbf{P}}{\|\mathbf{P}\|} \quad (27)$$

$$\mathbf{u} = \frac{\mathbf{a} \cos t + \mathbf{b} \sin t}{\|\mathbf{a} \cos t + \mathbf{b} \sin t\|} \quad (28)$$

Now substituting the value of t in above equation:

$$\mathbf{u} = \frac{\mathbf{a} \frac{1}{\sqrt{2}} + \mathbf{b} \frac{1}{\sqrt{2}}}{\left\| \mathbf{a} \frac{1}{\sqrt{2}} + \mathbf{b} \frac{1}{\sqrt{2}} \right\|} \quad (29)$$

$$\mathbf{u} = \frac{\mathbf{a} + \mathbf{b}}{\|\mathbf{a} + \mathbf{b}\|} \quad (30)$$

From Eq.18 and Eq.28 (a) is correct

From the figure it is clearly verified that the theoretical solution matches with the computational solution.

