12.144

EE25BTECH11026-Harsha

Question:

Let $\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 2 \\ 0 & 2 & 6 \end{pmatrix}$ and let $\lambda_1 \geq \lambda_2 \geq \lambda_3$ be the eigen values of $\mathbf{A}$.

(a) The triple $(\lambda_1, \lambda_2, \lambda_3)$ equals

1) $(9, 4, 2)$
2) $(8, 4, 3)$
3) $(9, 3, 3)$
4) $(7, 5, 3)$

(b) The Matrix $\mathbf{P}$ such that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

is

1) $\begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$

2) $\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$

3) $\begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{pmatrix}$

4) $\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{pmatrix}$

Solution:

Let us solve the given question theoretically and then verify the solution computationally.

(a) The eigen values of $\mathbf{A}$ is obtained using characteristic polynomial, which is given by,

$$det|\mathbf{A} - \lambda\mathbf{I}| = 0 \tag{4.1}$$

$$\therefore \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 6-\lambda & 2 \\ 0 & 2 & 6-\lambda \end{vmatrix} = 0 \tag{4.2}$$

$$\therefore (3-\lambda)\left((6-\lambda)^2 - 4\right) = 0 \tag{4.3}$$

$$\implies (\lambda - 3)(\lambda - 4)(\lambda - 8) = 0 \tag{4.4}$$

$$\therefore (\lambda_1, \lambda_2, \lambda_3) = (8, 4, 3) \tag{4.5}$$

(b) The given relation can be computed as ,

$$\mathbf{A} = \mathbf{PDP}^{-1} \tag{4.6}$$

where $\mathbf{D}$ is the diagonal matrix of eigenvalues of $\mathbf{A}$.

From (4.6), we can infer that it is the Eigen-value decomposition of matrix $\mathbf{A}$.

Therefore, $\mathbf{P}$ is the ortho-normalized matrix of collection of eigen vectors of $\mathbf{A}$.

$$\mathbf{P} = \begin{pmatrix} \mathbf{v_1} & \mathbf{v_2} & \mathbf{v_3} \end{pmatrix} \tag{4.7}$$

where $\mathbf{v_1}, \mathbf{v_2}\, and\, \mathbf{v_3}$ are the normalized eigen vectors of $\mathbf{A}$.

Eigenvectors $\mathbf{v}$ for any square matrix $\mathbf{A}$ is defined as

$$\mathbf{Av} = \lambda \mathbf{v} \tag{4.8}$$

where λ is a scalar and is called the eigen value of $\mathbf{A}$.

$$\therefore (\mathbf{A} - \lambda \mathbf{I})\, \mathbf{v} = \mathbf{0} \tag{4.9}$$

For $\lambda = 3$,

$$\left(\begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 2 \\ 0 & 2 & 6 \end{pmatrix} - 3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \mathbf{v} = 0 \tag{4.10}$$

Let $\mathbf{v} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$.

$$\therefore \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 2 & 3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{4.11}$$

$$\therefore 3\beta + 2\gamma = 0 \quad and \quad 2\beta + 3\gamma = 0 \tag{4.12}$$

$$\therefore \beta = \gamma = 0 \tag{4.13}$$

$$\mathbf{e_1} = \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} \tag{4.14}$$

For $\lambda = 4$,

$$\left(\begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 2 \\ 0 & 2 & 6 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)\mathbf{v} = 0 \tag{4.15}$$

Let $\mathbf{v} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$.

$$\therefore \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{4.16}$$

$$\therefore 2\beta + 2\gamma = 0 \quad and \quad \alpha = 0 \tag{4.17}$$

$$\therefore \beta = -\gamma \tag{4.18}$$

$$\mathbf{e_2} = \beta\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \tag{4.19}$$

For $\lambda = 8$,

$$\left(\begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 2 \\ 0 & 2 & 6 \end{pmatrix} - 8\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)\mathbf{v} = 0 \tag{4.20}$$

Let $\mathbf{v} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$.

$$\therefore \begin{pmatrix} -5 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{4.21}$$

$$\therefore 2\beta - 2\gamma = 0 \quad and \quad \alpha = 0 \tag{4.22}$$

$$\therefore \beta = \gamma \tag{4.23}$$

$$\mathbf{e_3} = \gamma\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \tag{4.24}$$

As we require unit eigen-vectors,

$$\implies \mathbf{v_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{v_2} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \qquad \mathbf{v_3} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \tag{4.25}$$

$$\therefore \mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \tag{4.26}$$