12.872

EE25BTECH11026-Harsha

Question:

Let $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 3 \\ -2 & -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$. For $\mathbf{Ax} = \mathbf{b}$ to be solvable, which one of the following options is the correct condition on $b_1, b_2, \textit{and}\, b_3$.

1) $b_1 + b_2 + b_3 = 1$
2) $3b_1 + b_2 + 2b_3 = 0$
3) $b_1 + 3b_2 + b_3 = 2$
4) $b_1 + b_2 + b_3 = 2$

Solution:

Let us solve the given question theoretically and then verify the solution computationally.

Given,

$$\mathbf{Ax} = \mathbf{b} \tag{4.1}$$

Multiplying $\mathbf{A}^\top$ on both sides,

$$\begin{pmatrix} 1 & 1 & -2 \\ 1 & 3 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 3 \\ -2 & -3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 1 & -2 \\ 1 & 3 & -3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{4.2}$$

$$\implies \begin{pmatrix} 6 & 10 \\ 10 & 19 \end{pmatrix} \mathbf{x} = \begin{pmatrix} b_1 + b_2 - 2b_3 \\ b_1 + 3b_2 - 3b_3 \end{pmatrix} \tag{4.3}$$

Forming the augmented matrix,

$$\left(\begin{array}{cc|c} 6 & 10 & b_1 + b_2 - 2b_3 \\ 10 & 19 & b_1 + 3b_2 - 3b_3 \end{array}\right) \xleftrightarrow{R_2 \leftarrow R_2 - \frac{5}{3}R_1} \left(\begin{array}{cc|c} 6 & 10 & b_1 + b_2 - 2b_3 \\ 0 & \frac{7}{3} & -\frac{2}{3}b_1 + \frac{4}{3}b_2 + \frac{1}{3}b_3 \end{array}\right) \xleftrightarrow[R_1 \leftarrow R_1 - 10R_2]{R_2 \leftarrow \frac{3}{7}R_2} \tag{4.4}$$

$$\left(\begin{array}{cc|c} 6 & 0 & \frac{27}{7}b_1 - \frac{33}{7}b_2 - \frac{24}{7}b_3 \\ 0 & 1 & -\frac{2}{7}b_1 + \frac{4}{7}b_2 + \frac{1}{7}b_3 \end{array}\right) \xleftrightarrow{R_1 \leftarrow \frac{1}{6}R_1} \left(\begin{array}{cc|c} 1 & 0 & \frac{9}{14}b_1 - \frac{11}{14}b_2 - \frac{4}{7}b_3 \\ 0 & 1 & -\frac{2}{7}b_1 + \frac{4}{7}b_2 + \frac{1}{7}b_3 \end{array}\right) \tag{4.5}$$

$$\therefore \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{9}{14}b_1 - \frac{11}{14}b_2 - \frac{4}{7}b_3 \\ -\frac{2}{7}b_1 + \frac{4}{7}b_2 + \frac{1}{7}b_3 \end{pmatrix} \tag{4.6}$$

From (4.1),

$$x_1 + x_2 = b_1 \Rightarrow \frac{9}{14}b_1 - \frac{11}{14}b_2 - \frac{4}{7}b_3 + \left(-\frac{2}{7}b_1 + \frac{4}{7}b_2 + \frac{1}{7}b_3\right) = b_1 \tag{4.7}$$

$$\therefore 3b_1 + b_2 + 2b_3 = 0 \tag{4.8}$$