2.8.9

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Question:

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three vectors such that $|\mathbf{a}| = 3$, $|\mathbf{b}| = 4$, $|\mathbf{c}| = 5$, and each one of them is perpendicular to the sum of the other two. Find $|\mathbf{a} + \mathbf{b} + \mathbf{c}|$.

Solution:

Let

$$|\mathbf{a}| = 3, \quad |\mathbf{b}| = 4, \quad |\mathbf{c}| = 5 \tag{1}$$

Since each vector is perpendicular to the sum of the other two, we have:

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = 0, \quad \mathbf{b} \cdot (\mathbf{c} + \mathbf{a}) = 0, \quad \mathbf{c} \cdot (\mathbf{a} + \mathbf{b}) = 0 \tag{2}$$

Introduce notation:

$$s = \mathbf{a} \cdot \mathbf{b}, \quad t = \mathbf{b} \cdot \mathbf{c}, \quad u = \mathbf{c} \cdot \mathbf{a} \tag{3}$$

From (2), the equations become:

$$s + u = 0, \quad t + s = 0, \quad u + t = 0 \tag{4}$$

From the first equation,

$$u = -s \tag{5}$$

From the second equation,

$$t = -s \tag{6}$$

Substitute (5) and (6) into the third equation:

$$(-s) + (-s) = -2s = 0 \quad \Rightarrow \quad s = 0 \tag{7}$$

Hence,

$$s = t = u = 0 \tag{8}$$

This shows that $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are mutually perpendicular.
Now,

$$\|\mathbf{a} + \mathbf{b} + \mathbf{c}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2 + 2(s + t + u) \tag{9}$$

Substitute values from (1) and (8):

$$= 3^2 + 4^2 + 5^2 + 2(0 + 0 + 0) \tag{10}$$

$$= 9 + 16 + 25 = 50 \tag{11}$$

Therefore,

$$\|\mathbf{a} + \mathbf{b} + \mathbf{c}\| = \sqrt{50} = 5\sqrt{2} \tag{12}$$

Final Answer:

$$\boxed{5\sqrt{2}}$$

Graphical Representation:

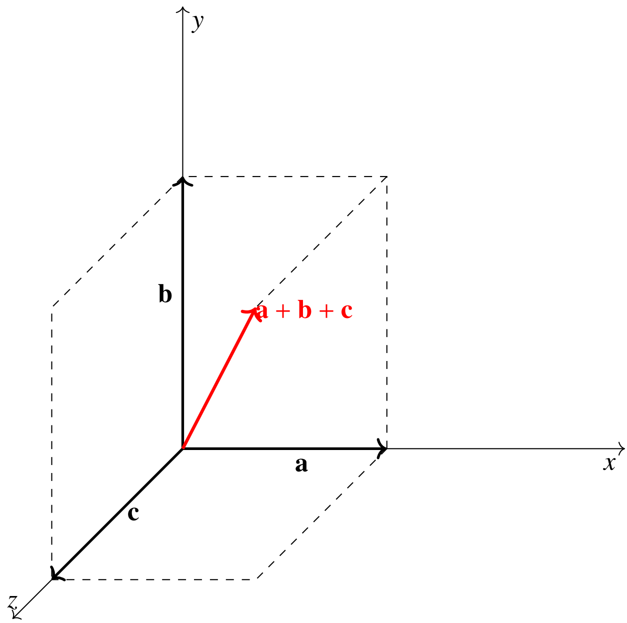

Fig. 4