2.10.25

AI25BTECH11036–SNEHAMRUDULA

2.10.25. In $\triangle PQR$

, let $\mathbf{a} = \overrightarrow{QR}$, $\mathbf{b} = \overrightarrow{RP}$, $\mathbf{c} = \overrightarrow{PQ}$.

If $|\mathbf{a}| = 12$, $|\mathbf{b}| = 4\sqrt{3}$, $\mathbf{b}^T\mathbf{c} = 24$, then which of the following is (are) true?

(a) $\frac{|\mathbf{c}|^2}{2} - |\mathbf{a}| = 12$

(b) $\frac{|\mathbf{c}|^2}{2} + |\mathbf{a}| = 30$

(c) $|\mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{a}| = 48\sqrt{3}$

(d) $\mathbf{a}^T\mathbf{b} = -72$

solution

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0} \qquad \Rightarrow \qquad \mathbf{c} = -\mathbf{a} - \mathbf{b}$$

a) (i):

$$\mathbf{b}^T\mathbf{c} = 24$$

$$\mathbf{b}^T(-\mathbf{a} - \mathbf{b}) = 24 \;\Rightarrow\; -\mathbf{b}^T\mathbf{a} - |\mathbf{b}|^2 = 24$$

$$|\mathbf{b}|^2 = (4\sqrt{3})^2 = 48$$

$$\mathbf{a}^T\mathbf{b} = -(48 + 24) = -72$$

b) (ii):

$$|\mathbf{c}|^2 = (-\mathbf{a} - \mathbf{b})^T(-\mathbf{a} - \mathbf{b}) = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a}^T\mathbf{b}$$

$$|\mathbf{a}|^2 = 12^2 = 144, \quad |\mathbf{b}|^2 = 48, \quad \mathbf{a}^T\mathbf{b} = -72$$

$$|\mathbf{c}|^2 = 144 + 48 + 2(-72) = 48 \qquad \frac{|\mathbf{c}|^2}{2} = 24$$

$$\frac{|\mathbf{c}|^2}{2} - |\mathbf{a}| = 24 - 12 = 12 \quad \Rightarrow (a) \text{ true}$$

$$\frac{|\mathbf{c}|^2}{2} + |\mathbf{a}| = 24 + 12 = 36 \quad \Rightarrow (b) \text{ false}$$

c) (iii):

$$\mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{a} = \mathbf{a} \times \mathbf{b} + (-\mathbf{a} - \mathbf{b}) \times \mathbf{a}$$

$$= \mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{a} - \mathbf{b} \times \mathbf{a} = \mathbf{a} \times \mathbf{b} - \mathbf{b} \times \mathbf{a}$$

$$= \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{b} = 2(\mathbf{a} \times \mathbf{b})$$

$$|\mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{a}| = 2|\mathbf{a} \times \mathbf{b}| = 2|\mathbf{a}||\mathbf{b}| \sin\theta$$

$$\cos\theta = \frac{\mathbf{a}^T\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{-72}{12\cdot 4\sqrt{3}} = -\frac{\sqrt{3}}{2}$$

$$\sin\theta = \tfrac{1}{2}$$

$$|\mathbf{a}\times\mathbf{b} + \mathbf{c}\times\mathbf{a}| = 48\sqrt{3} \quad \Rightarrow (c) \text{ true}$$

True = $\{(a), (c), (d)\}$,	False = $\{(b)\}$