4.13.66

EE25BTECH11062 - Vivek K Kumar

Question:
A straight line L with negative slope passes through the point $(8, 2)$ and cuts the positive coordinate axes at points P and Q. Find the absolute minimum value of $OP + OQ$, as L varies, where O is the origin.

Solution:

Point	**Value**
$\mathbf{m}$	$\begin{pmatrix} 1 \\ -m \end{pmatrix}$ $\quad m > 0$
$\mathbf{h}$	$\begin{pmatrix} 8 \\ 2 \end{pmatrix}$

TABLE 0: Variables used

It is known that

$$\mathbf{e_1}^\top \mathbf{P} = 0 \tag{0.1}$$
$$\mathbf{e_2}^\top \mathbf{Q} = 0 \tag{0.2}$$

Given line L can be represented as

$$\mathbf{x} = \mathbf{h} + k\mathbf{m} \tag{0.3}$$
$$\mathbf{e_1}^\top \mathbf{P} = \mathbf{e_1}^\top \mathbf{h} + k_1 \mathbf{e_1}^\top \mathbf{m} \tag{0.4}$$
$$k_1 = -\frac{\mathbf{e_1}^\top \mathbf{h}}{\mathbf{e_1}^\top \mathbf{m}} \tag{0.5}$$
$$\mathbf{P} = \mathbf{h} - \frac{\mathbf{e_1}^\top \mathbf{h}}{\mathbf{e_1}^\top \mathbf{m}} \mathbf{m} \tag{0.6}$$
$$\mathbf{e_2}^\top \mathbf{Q} = \mathbf{e_2}^\top \mathbf{h} + k_2 \mathbf{e_2}^\top \mathbf{m} \tag{0.7}$$
$$k_2 = -\frac{\mathbf{e_2}^\top \mathbf{h}}{\mathbf{e_2}^\top \mathbf{m}} \tag{0.8}$$
$$\mathbf{Q} = \mathbf{h} - \frac{\mathbf{e_2}^\top \mathbf{h}}{\mathbf{e_2}^\top \mathbf{m}} \mathbf{m} \tag{0.9}$$

Substituting values

$$\mathbf{P} = \begin{pmatrix} 8 \\ 2 \end{pmatrix} - 8 \begin{pmatrix} 1 \\ -m \end{pmatrix} \tag{0.10}$$
$$= \begin{pmatrix} 0 \\ 2 + 8m \end{pmatrix} \tag{0.11}$$
$$\mathbf{Q} = \begin{pmatrix} 8 \\ 2 \end{pmatrix} - \frac{2}{m} \begin{pmatrix} 1 \\ -m \end{pmatrix} \tag{0.12}$$
$$= \begin{pmatrix} 8 + \frac{2}{m} \\ 0 \end{pmatrix} \tag{0.13}$$

We have to find the minimum of $||\mathbf{P}|| + ||\mathbf{Q}||$

$$||\mathbf{P}|| + ||\mathbf{Q}|| = 2 + 8m + 8 + \frac{2}{m} \tag{0.14}$$
$$= 10 + 8m + \frac{2}{m} \tag{0.15}$$

Applying AM-GM inequality

$$\frac{8m + \frac{2}{m}}{2} \geq \sqrt{8m.\frac{2}{m}} \tag{0.16}$$
$$\geq 4 \tag{0.17}$$
$$\implies 10 + 8m + \frac{2}{m} \geq 18 \tag{0.18}$$

Hence we can write

$$||\mathbf{P}|| + ||\mathbf{Q}|| \geq 18 \tag{0.19}$$

Hence, $min\left(||\mathbf{P}|| + ||\mathbf{Q}||\right) = 18$

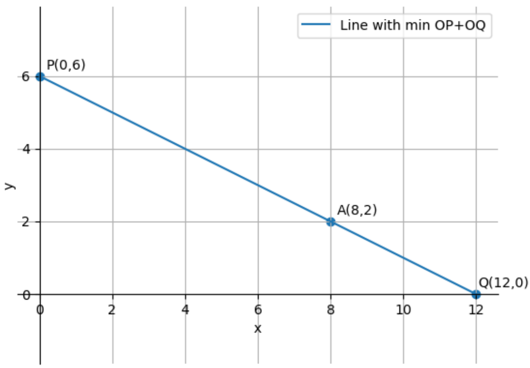

Fig. 0.1: Given points on a line