4.13.6

EE25BTECH11001 - Aarush Dilawri

Question:
Given the points $\mathbf{A}(0,4)$ and $\mathbf{B}(0,-4)$, the equation of the locus of the point $\mathbf{p}(x,y)$, such that $|AP - BP| = 6$
Solution:

$$\mathbf{A},\ \mathbf{B},\ \mathbf{P} \in \mathbb{R}^n \tag{0.1}$$

Let the given scalar be $\delta \geq 0$ and choose the sign $s \in \{+1, -1\}$ so that

$$r_1 - r_2 = s\,\delta, \tag{0.2}$$

where $r_1 = \|\mathbf{P} - \mathbf{A}\|$ and $r_2 = \|\mathbf{P} - \mathbf{B}\|$.
Define the difference vector

$$\mathbf{u} = \mathbf{A} - \mathbf{B}, \tag{0.3}$$

and the shorthand

$$D = s\,\delta, \qquad \alpha = \mathbf{A}^\top\mathbf{A} - \mathbf{B}^\top\mathbf{B}. \tag{0.4}$$

Compute the difference of squares:

$$\|\mathbf{P} - \mathbf{A}\|^2 - \|\mathbf{P} - \mathbf{B}\|^2 = (\mathbf{P} - \mathbf{A})^\top(\mathbf{P} - \mathbf{A}) - (\mathbf{P} - \mathbf{B})^\top(\mathbf{P} - \mathbf{B}) \tag{0.5}$$

$$= -2\,\mathbf{P}^\top\mathbf{u} + \mathbf{A}^\top\mathbf{A} - \mathbf{B}^\top\mathbf{B} = -2\,\mathbf{P}^\top\mathbf{u} + \alpha. \tag{0.6}$$

Use $(r_1 - r_2)(r_1 + r_2) = r_1^2 - r_2^2$ and $r_1 - r_2 = D$ to get

$$D\,(r_1 + r_2) = -2\,\mathbf{P}^\top\mathbf{u} + \alpha \quad \Longrightarrow \quad r_1 + r_2 = \frac{-2\,\mathbf{P}^\top\mathbf{u} + \alpha}{D}. \tag{0.7}$$

Hence

$$r_1 = \frac{(r_1 - r_2) + (r_1 + r_2)}{2} = \frac{D}{2} + \frac{\alpha}{2D} - \frac{\mathbf{P}^\top\mathbf{u}}{D}. \tag{0.8}$$

Square this expression and equate to the explicit quadratic form for r_1^2:

$$\left(\frac{D}{2} + \frac{\alpha}{2D} - \frac{\mathbf{P}^\top\mathbf{u}}{D}\right)^2 = (\mathbf{P} - \mathbf{A})^\top(\mathbf{P} - \mathbf{A}) = \mathbf{P}^\top\mathbf{P} - 2\,\mathbf{P}^\top\mathbf{A} + \mathbf{A}^\top\mathbf{A}. \tag{0.9}$$

Multiply both sides by D^2 and simplify. After collecting terms one obtains the general quadratic (conic) equation in the vector $\mathbf{P}$:

$$\mathbf{P}^\top(\mathbf{u}\mathbf{u}^\top - D^2 I)\mathbf{P} + \left(-(D^2 + \alpha)\,\mathbf{u} + 2D^2\,\mathbf{A}\right)^\top\mathbf{P} + \frac{(D^2 + \alpha)^2}{4} - D^2\,\mathbf{A}^\top\mathbf{A} = 0. \tag{0.10}$$

Now substitute $\mathbf{A} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$ and $\delta = 6$.

$$\mathbf{u} = \mathbf{A} - \mathbf{B} = \begin{pmatrix} 0 \\ 8 \end{pmatrix} \tag{0.11}$$

$$\alpha = \mathbf{A}^\top \mathbf{A} - \mathbf{B}^\top \mathbf{B} = 16 - 16 = 0 \tag{0.12}$$

$$D = s\,\delta = \pm 6 \quad \Rightarrow \quad D^2 = 36 \tag{0.13}$$

The quadratic matrix equation becomes

$$\mathbf{P}^\top (\mathbf{u}\mathbf{u}^\top - 36I)\mathbf{P} + (-D^2\mathbf{u} + 2D^2\mathbf{A})^\top \mathbf{P} + \frac{D^4}{4} - 36\mathbf{A}^\top \mathbf{A} = 0 \tag{0.14}$$

Now compute each term.

$$\mathbf{u}\mathbf{u}^\top = \begin{pmatrix} 0 & 0 \\ 0 & 64 \end{pmatrix}, \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{0.15}$$

So

$$\mathbf{u}\mathbf{u}^\top - 36I = \begin{pmatrix} -36 & 0 \\ 0 & 28 \end{pmatrix} \tag{0.16}$$

Next, the linear coefficient:

$$-D^2\mathbf{u} + 2D^2\mathbf{A} = -36\begin{pmatrix} 0 \\ 8 \end{pmatrix} + 72\begin{pmatrix} 0 \\ 4 \end{pmatrix} \tag{0.17}$$

$$= \begin{pmatrix} 0 \\ -288 \end{pmatrix} + \begin{pmatrix} 0 \\ 288 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{0.18}$$

So there is no linear term.
Finally, the constant term:

$$\frac{D^4}{4} - 36\mathbf{A}^\top \mathbf{A} = \frac{1296}{4} - 36(16) \tag{0.19}$$

$$= 324 - 576 \tag{0.20}$$

$$= -252 \tag{0.21}$$

Therefore, the locus is given by

$$\mathbf{P}^\top \begin{pmatrix} -36 & 0 \\ 0 & 28 \end{pmatrix} \mathbf{P} - 252 = 0 \tag{0.22}$$

or equivalently

$$\mathbf{P}^\top \begin{pmatrix} -36 & 0 \\ 0 & 28 \end{pmatrix} \mathbf{P} = 252 \tag{0.23}$$

Expanding with $\mathbf{P} = \begin{pmatrix} x \\ y \end{pmatrix}$,

$$-36x^2 + 28y^2 = 252 \tag{0.24}$$

Dividing through,

$$\frac{y^2}{9} - \frac{x^2}{7} = 1 \tag{0.25}$$

Thus the locus is a hyperbola centered at the origin with equation

$$\frac{y^2}{9} - \frac{x^2}{7} = 1 \tag{0.26}$$