Matgeo Presentation - Problem 12.693

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October 3, 2025

Problem Statement

Suppose the circles

$$x^2 + y^2 + ax + 6 = 0$$
$$x^2 + y^2 + bx - 4 = 0$$

intersect each other orthogonally at the point $(1, 2)$. Then $a + b = \_$

Data

Name	Value
Circle 1	$\mathbf{x}^\top \mathbf{x} + 2 \begin{pmatrix} \frac{a}{2} \\ 0 \end{pmatrix}^\top \mathbf{x} + 6 = 0$
Circle 2	$\mathbf{x}^\top \mathbf{x} + 2 \begin{pmatrix} \frac{b}{2} \\ 0 \end{pmatrix}^\top \mathbf{x} - 4 = 0$
$\mathbf{P}$	$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Table : Circles and Point

Solution

The conic parameters for the two circles can be expressed as :

$$\mathbf{V_1} = \mathbf{I} \qquad \mathbf{u_1} = \begin{pmatrix} \frac{a}{2} \\ 0 \end{pmatrix} \qquad f1 = 6 \tag{0.1}$$

$$\mathbf{V_2} = \mathbf{I} \qquad \mathbf{u_2} = \begin{pmatrix} \frac{b}{2} \\ 0 \end{pmatrix} \qquad f2 = -4 \tag{0.2}$$

The point of intersection of the two circles is **P**
The equation of tangent to Circle 1 at **P** is given as :

$$(\mathbf{V_1}\mathbf{P} + \mathbf{u_1})^\top \mathbf{x} + \mathbf{u_1}^\top \mathbf{P} + f_1 = 0 \qquad \mathbf{n_1} = \mathbf{V_1}\mathbf{P} + \mathbf{u_1} \tag{0.3}$$

The equation of tangent to Circle 2 at **P** is given as :

$$(\mathbf{V_2}\mathbf{P} + \mathbf{u_2})^\top \mathbf{x} + \mathbf{u_2}^\top \mathbf{P} + f_2 = 0 \qquad \mathbf{n_2} = \mathbf{V_2}\mathbf{P} + \mathbf{u_2} \tag{0.4}$$

Solution

As the tangents at $\mathbf{P}$ are perpendicular , the normal vectors of the tangents are also perpendicular

$$\mathbf{n_1}^\top \mathbf{n_2} = 0 \tag{0.5}$$

$$(\mathbf{V_1}\mathbf{P} + \mathbf{u_1})^\top (\mathbf{V_2}\mathbf{P} + \mathbf{u_2}) = 0 \tag{0.6}$$

$$(\mathbf{P} + \mathbf{u_1})^\top (\mathbf{P} + \mathbf{u_2}) = 0 \tag{0.7}$$

$$(\mathbf{P}^\top + \mathbf{u_1}^\top)(\mathbf{P} + \mathbf{u_2}) = 0 \tag{0.8}$$

$$\begin{pmatrix} \frac{a+2}{2} & 2 \end{pmatrix} \begin{pmatrix} \frac{b+2}{2} \\ 2 \end{pmatrix} = 0 \tag{0.9}$$

$$2(a+b) + ab + 20 = 0 \tag{0.10}$$

As $\mathbf{P}$ lies on both the circles we get :

$$\mathbf{P}^\top \mathbf{P} + 2\mathbf{u_1}^\top \mathbf{P} + f_1 = 0 \tag{0.11}$$

$$\mathbf{P}^\top \mathbf{P} + 2\mathbf{u_2}^\top \mathbf{P} + f_2 = 0 \tag{0.12}$$

Solution

By subtracting the above two equations we get :

$$(\mathbf{u_1}^\top - \mathbf{u_2}^\top)\mathbf{P} = \frac{f_2 - f_1}{2} \tag{0.13}$$

If we substitue (0.11) in (0.8) we get :

$$(\mathbf{u_2}^\top - \mathbf{u_1}^\top)\mathbf{P} + \mathbf{u_1}^\top\mathbf{u_2} - f_1 = 0 \tag{0.14}$$

$$(\mathbf{u_1}^\top - \mathbf{u_2}^\top)\mathbf{P} = \mathbf{u_1}^\top\mathbf{u_2} - f_1 \tag{0.15}$$

From (0.13) and (0.15) , we get :

$$\mathbf{u_1}^\top\mathbf{u_2} = \frac{f_1 + f_2}{2} \tag{0.16}$$

$$ab = 4 \tag{0.17}$$

Solution

By substituting (0.17) in (0.10) , we get :

$$a + b = -12 \tag{0.18}$$

Plot

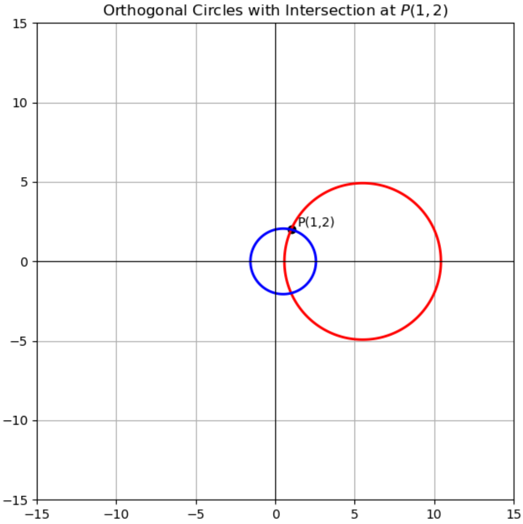

Fig : Circles