2.10.2

AI25BTECH11013-Gautham

Question:
Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be vectors of lengths 3, 4, and 5 respectively such that $\mathbf{A} \perp \mathbf{B}+\mathbf{C}$, $\mathbf{B} \perp \mathbf{C}+\mathbf{A}$, and $\mathbf{C} \perp \mathbf{A}+\mathbf{B}$. Find the length of the vector $\mathbf{A}+\mathbf{B}+\mathbf{C}$.

Solution:

Given the orthogonality conditions:

$$\mathbf{A} \perp \mathbf{B}+\mathbf{C} \implies \mathbf{A}^T(\mathbf{B}+\mathbf{C}) = 0 \tag{0.1}$$
$$\mathbf{B} \perp \mathbf{C}+\mathbf{A} \implies \mathbf{B}^T(\mathbf{C}+\mathbf{A}) = 0 \tag{0.2}$$
$$\mathbf{C} \perp \mathbf{A}+\mathbf{B} \implies \mathbf{C}^T(\mathbf{A}+\mathbf{B}) = 0 \tag{0.3}$$

Expanding, we get:

$$\mathbf{A}^T\mathbf{B} + \mathbf{A}^T\mathbf{C} = 0 \tag{0.4}$$
$$\mathbf{B}^T\mathbf{C} + \mathbf{B}^T\mathbf{A} = 0 \tag{0.5}$$
$$\mathbf{C}^T\mathbf{A} + \mathbf{C}^T\mathbf{B} = 0 \tag{0.6}$$

Adding :

$$(\mathbf{A}^T\mathbf{B} + \mathbf{A}^T\mathbf{C}) + (\mathbf{B}^T\mathbf{C} + \mathbf{B}^T\mathbf{A}) + (\mathbf{C}^T\mathbf{A} + \mathbf{C}^T\mathbf{B}) = 0 \tag{0.7}$$

Grouping like terms and noting dot products are symmetric:

$$2(\mathbf{A}^T\mathbf{B} + \mathbf{B}^T\mathbf{C} + \mathbf{C}^T\mathbf{A}) = 0 \implies \mathbf{A}^T\mathbf{B} + \mathbf{B}^T\mathbf{C} + \mathbf{C}^T\mathbf{A} = 0 \tag{0.8}$$

Now compute the squared length of $\mathbf{A}+\mathbf{B}+\mathbf{C}$:

$$\begin{aligned}
||\mathbf{A}+\mathbf{B}+\mathbf{C}||^2 &= (\mathbf{A}+\mathbf{B}+\mathbf{C})^T(\mathbf{A}+\mathbf{B}+\mathbf{C}) && (0.9)\\
&= \mathbf{A}^T\mathbf{A} + \mathbf{B}^T\mathbf{B} + \mathbf{C}^T\mathbf{C} + 2(\mathbf{A}^T\mathbf{B} + \mathbf{B}^T\mathbf{C} + \mathbf{C}^T\mathbf{A}) && (0.10)\\
&= ||\mathbf{A}||^2 + ||\mathbf{B}||^2 + ||\mathbf{C}||^2 + 2(\mathbf{A}^T\mathbf{B} + \mathbf{B}^T\mathbf{C} + \mathbf{C}^T\mathbf{A}) && (0.11)\\
&= 3^2 + 4^2 + 5^2 + 2 \times 0 && (0.12)\\
&= 9 + 16 + 25 && (0.13)\\
&= 50 && (0.14)
\end{aligned}$$

Therefore,

$$||\mathbf{A}+\mathbf{B}+\mathbf{C}|| = \sqrt{50} = 5\sqrt{2} \tag{0.15}$$