5.6.7

AI25BTECH11021 - Abhiram Reddy N

Question

Let

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}.$$

Verify that

$$A^3 - 6A^2 + 9A - 4I = 0, \tag{0.1}$$

by using the Cayley–Hamilton theorem, and hence find A^{-1}. When computing A^{-1} show the relation to the adjugate (transpose of cofactors).

Step 1: Characteristic polynomial

Compute the characteristic polynomial of A:

$$\begin{aligned} \chi_A(\lambda) &= \det(\lambda I - A) \\ &= \det \begin{pmatrix} \lambda - 2 & 1 & -1 \\ 1 & \lambda - 2 & 1 \\ -1 & 1 & \lambda - 2 \end{pmatrix}. \end{aligned} \tag{0.2}$$

Expanding (or by direct computation) one obtains the factorisation

$$\chi_A(\lambda) = (\lambda - 4)(\lambda - 1)^2 = \lambda^3 - 6\lambda^2 + 9\lambda - 4. \tag{0.3}$$

Step 2: Cayley–Hamilton theorem

By the Cayley–Hamilton theorem the matrix A satisfies its own characteristic polynomial, i.e.

$$A^3 - 6A^2 + 9A - 4I = 0, \tag{0.4}$$

which is the desired identity (**??**).

Step 3: Express A^{-1} from the polynomial

Assuming A is invertible (we will check the determinant shortly), multiply (**??**) on the right by A^{-1} to obtain

$$A^2 - 6A + 9I - 4A^{-1} = 0. \tag{0.5}$$

Rearrange (**??**) to solve for A^{-1}:

$$A^{-1} = \tfrac{1}{4}(A^2 - 6A + 9I). \tag{0.6}$$

Step 4: Determinant and adjugate (transpose of cofactors)

From the characteristic polynomial (**??**) we read off the eigenvalues $4, 1, 1$. Thus

$$\det(A) = 4, \tag{0.7}$$

so A is invertible. The adjugate matrix $\operatorname{adj}(A)$ satisfies

$$\operatorname{adj}(A) = \det(A)\, A^{-1} = 4A^{-1}.$$

From (**??**) we get

$$\operatorname{adj}(A) = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}. \tag{0.8}$$

The adjugate is the transpose of the cofactor matrix; in this case $\operatorname{adj}(A)$ is symmetric, so taking the transpose of the cofactor matrix yields the same matrix (**??**). Finally,

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{4} \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}, \tag{0.9}$$

which matches (**??**).

Answer. The Cayley–Hamilton identity (**??**) holds, and

$$\boxed{A^{-1} = \frac{1}{4}(A^2 - 6A + 9I) = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{3}{4} \end{pmatrix}}$$

which may equivalently be obtained from $A^{-1} = \dfrac{1}{\det(A)} \operatorname{adj}(A)$ with $\det(A) = 4$ and $\operatorname{adj}(A)$ given in (**??**) (the adjugate is the transpose of the cofactor matrix).