4.13.64

EE25BTECH11060 - V.Namaswi

Question
Let a, b, c be real numbers with $a^2 + b^2 + c^2 = 1$. Show that the equation

$$\begin{vmatrix} ax - by - c & bx + ay & cx + a \\ bx + ay & -ax + by - c & cy + b \\ cx + a & cy + b & -ax - by + c \end{vmatrix} = 0$$

represents a straight line.
Solution
Let us denote it as a 3×3 matrix:

$$M = \begin{pmatrix} ax - by - c & bx + ay & cx + a \\ bx + ay & -ax + by - c & cy + b \\ cx + a & cy + b & -ax - by + c \end{pmatrix}$$

A determinant represents a plane if it depends quadratically on x and y. Here, if we can reduce it to a determinant that is linear in x and y, it will represent a straight line.
so,Replace

$$R_3 \to R_3 - cR_1 - bR_2$$

First element of new R_3 :

$$(cx + a) - c(ax - by - c) - b(bx + ay) \tag{1}$$
$$= cx + a - cax + cby + c^2 - b^2x - aby \tag{2}$$
$$x(c - ca - b^2) + y(cb - ab) + (a + c^2) \tag{3}$$

Second element of new R_3 :

$$(cy + b) - c(bx + ay) - b(-ax + by - c) \tag{4}$$
$$= cy + b - cbx - cay + abx - b^2y + bc \tag{5}$$
$$= x(-cb + ab) + y(c - ca - b^2) + (b + bc) \tag{6}$$

Third element of new R_3 :

$$(-ax - by + c) - c(cx + a) - b(cy + b) \tag{7}$$
$$= -ax - by + c - c^2x - ac - bcy - b^2 \tag{8}$$
$$= x(-a - c^2) + y(-b - bc) + (c - ac - b^2) \tag{9}$$

But since

$$a^2 + b^2 + c^2 = 1 \implies 1 - a^2 = b^2 + c^2, \tag{10}$$

all quadratic terms cancel. Similarly, the 2nd and 3rd entries of the new R_3 become constants or linear in x, y.
The determinant now depends linearly on x and y, so we can write:

$$\det(M) = 0 \quad \implies \quad px + qy + r = 0,$$

for some real constants p, q, r.
Hence,the determinant represents a **straight line**.