4.3.36

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Question

Show that the points $\left(\hat{i} - \hat{j} + 3\hat{k}\right)$ and $3\left(\hat{i} + \hat{j} + \hat{k}\right)$ are equidistant from the plane $\mathbf{r} \cdot \left(5\hat{i} + 2\hat{j} - 7\hat{k}\right) + 9 = 0$ and lie on opposite sides of it.

Theoretical Solution

Let the given points be $\mathbf{P_1} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{P_2} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$. The equation of the given plane is

$$\begin{pmatrix} 5 & 2 & -7 \end{pmatrix} \mathbf{x} + 9 = 0 \tag{1}$$

This can be written in the standard form $\mathbf{n}^\top \mathbf{x} = k$. Here, $\mathbf{n} = \begin{pmatrix} 5 \\ 2 \\ -7 \end{pmatrix}$ and $k = -9$.

$$\begin{pmatrix} 5 & 2 & -7 \end{pmatrix} \mathbf{x} = -9 \tag{2}$$

Theoretical Solution

The perpendicular distance of a point with position vector $\mathbf{P}$ from the plane $\mathbf{n}^\top\mathbf{x} = k$ is given by the formula

$$D = \frac{\left|\mathbf{n}^\top\mathbf{P} - k\right|}{\|\mathbf{n}\|} \tag{3}$$

$$\|\mathbf{n}\| = \sqrt{5^2 + 2^2 + (-7)^2} \tag{4}$$

$$= \sqrt{25 + 4 + 49} = \sqrt{78} \tag{5}$$

Distance D_1 of the point $\mathbf{P_1}$ from the plane is

$$D_1 = \frac{\left|\begin{pmatrix} 5 & 2 & -7 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} - (-9)\right|}{\sqrt{78}} \tag{6}$$

$$= \frac{|5 - 2 - 21 + 9|}{\sqrt{78}} \tag{7}$$

$$\frac{|-9|}{} \quad 9$$

Theoretical Solution

Distance D_2 of the point $\mathbf{P_2}$ from the plane is

$$D_2 = \frac{\left| \begin{pmatrix} 5 & 2 & -7 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} - (-9) \right|}{\sqrt{78}} \tag{9}$$

$$= \frac{|15 + 6 - 21 + 9|}{\sqrt{78}} \tag{10}$$

$$= \frac{|9|}{\sqrt{78}} = \frac{9}{\sqrt{78}} \tag{11}$$

From (8) and (11), $D_1 = D_2$. Thus, the points are equidistant from the plane.

$$\mathbf{n}^\top \mathbf{P_1} - k = \begin{pmatrix} 5 & 2 & -7 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} - (-9) = -9 \tag{12}$$

Theoretical Solution

$$\mathbf{n}^\top \mathbf{P_2} - k = \begin{pmatrix} 5 & 2 & -7 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} - (-9) = 9 \tag{13}$$

From (12) and (13),

$$\left(\mathbf{n}^\top \mathbf{P_1} - k\right)\left(\mathbf{n}^\top \mathbf{P_2} - k\right) = -81 < 0 \tag{14}$$

$\because \left(\mathbf{n}^\top \mathbf{P_1} - k\right)\left(\mathbf{n}^\top \mathbf{P_2} - k\right) < 0$, the points $\mathbf{P_1}$ and $\mathbf{P_2}$ lie on opposite sides of the plane.

Plot

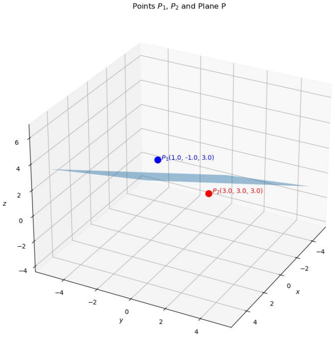

Figure: Plot