Problem 2.10.47

EE25BTECH11018-Darisy Sreetej

September 21, 2025

Problem

The value of a so that the volume of parallelopiped formed by $\hat{i} + a\hat{j} + \hat{k}, \hat{j} + a\hat{k}$ and $a\hat{i} + \hat{k}$ becomes minimum is

1. -3
2. 3
3. $\frac{1}{\sqrt{3}}$
4. $\sqrt{3}$

Formula

Volume of the parallelopiped

$$V = \mathbf{p} \cdot (\mathbf{q} \times \mathbf{r}) = \sqrt{\left|\mathbf{G}\right|}$$

where $\mathbf{G}$ is the Gram matrix of the vectors

Obtaining the Gram Matrix

Let us consider,

$$\mathbf{p} = \hat{i} + a\hat{j} + \hat{k}$$
$$\mathbf{q} = \hat{j} + a\hat{k}$$
$$\mathbf{r} = a\hat{i} + \hat{k}$$

The Gram matrix $\mathbf{G}$ for the vectors $\mathbf{p}, \mathbf{q}, \mathbf{r}$ is:

$$\mathbf{G} = \begin{pmatrix} \mathbf{p}^\top\mathbf{p} & \mathbf{p}^\top\mathbf{q} & \mathbf{p}^\top\mathbf{r} \\ \mathbf{q}^\top\mathbf{p} & \mathbf{q}^\top\mathbf{q} & \mathbf{q}^\top\mathbf{r} \\ \mathbf{r}^\top\mathbf{p} & \mathbf{r}^\top\mathbf{q} & \mathbf{r}^\top\mathbf{r} \end{pmatrix} \tag{1.1}$$

Now, calculate the dot products:

$$\mathbf{p}^\top\mathbf{p} = 1^2 + a^2 + 1^2 = 2 + a^2 \tag{1.2}$$

Obtaining the Gram Matrix

$$\mathbf{p}^\top \mathbf{q} = (1)(0) + (a)(1) + (1)(a) = 2a \tag{1.3}$$

$$\mathbf{p}^\top \mathbf{r} = (1)(a) + (a)(0) + (1)(1) = a + 1 \tag{1.4}$$

$$\mathbf{q}^\top \mathbf{p} = \mathbf{p}^\top \mathbf{q} = 2a \tag{1.5}$$

$$\mathbf{q}^\top \mathbf{q} = a^2 + 1^2 = 1 + a^2 \tag{1.6}$$

$$\mathbf{q}^\top \mathbf{r} = (0)(a) + (1)(0) + (a)(1) = a \tag{1.7}$$

$$\mathbf{r}^\top \mathbf{p} = \mathbf{p}^\top \mathbf{r} = a + 1 \tag{1.8}$$

Obtaining the Gram Matrix

$$\mathbf{r}^\top \mathbf{q} = \mathbf{q}^\top \mathbf{r} = a \tag{1.9}$$

$$\mathbf{r}^\top \mathbf{r} = a^2 + 1^2 = a^2 + 1 \tag{1.10}$$

Thus, the Gram matrix **G** is:

$$\mathbf{G} = \begin{pmatrix} 2 + a^2 & 2a & a + 1 \\ 2a & 1 + a^2 & a \\ a + 1 & a & 1 + a^2 \end{pmatrix} \tag{1.11}$$

Calculating Volume

The characteristic equation is obtained by solving the determinant equation $\left|\mathbf{G} - \lambda\mathbf{I}\right| = 0$. The characteristic polynomial for the matrix is:

$$\lambda^3 - (3a^2 + 4)\lambda^2 + (3a^4 + 2a^2 + 4)\lambda - (a^6 - 2a^4 + 2a^3 + a^2 - 2a + 1) = 0 \tag{1.12}$$

The determinant of $\mathbf{G}$ is the product of its eigenvalues:

$$\left|\mathbf{G}\right| = \lambda_1\lambda_2\lambda_3 = (a^6 - 2a^4 + 2a^3 + a^2 - 2a + 1). \tag{1.13}$$

The scalar triple product is the square root of the determinant of $\mathbf{G}$:

$$\mathbf{p} \cdot (\mathbf{q} \times \mathbf{r}) = \sqrt{\left|\mathbf{G}\right|} = \sqrt{(a^6 - 2a^4 + 2a^3 + a^2 - 2a + 1)} = a^3 - a + 1 \tag{1.14}$$

$$V = a^3 - a + 1 \tag{1.15}$$

Finding 'a' for minimum volume

Now , consider

$$f(a) = a^3 - a + 1 \tag{1.16}$$
$$f'(a) = 3a^2 + 1 \tag{1.17}$$

$$\text{Set } f'(a) = 0 \Rightarrow a^2 = \frac{1}{\sqrt{3}} \Rightarrow a = \frac{1}{\sqrt{3}} or - \frac{1}{\sqrt{3}}$$

$$\text{Second derivative } f''(a) = 6a \tag{1.18}$$
$$\text{At } a = \frac{1}{\sqrt{3}}, f'' > 0 \Rightarrow minimum \tag{1.19}$$
$$\text{At } a = -\frac{1}{\sqrt{3}}, f'' < 0 \Rightarrow maximum \tag{1.20}$$

Therefore , $a = \frac{1}{\sqrt{3}}$ for which the Volume of the parallelopiped becomes minimum.

Plot

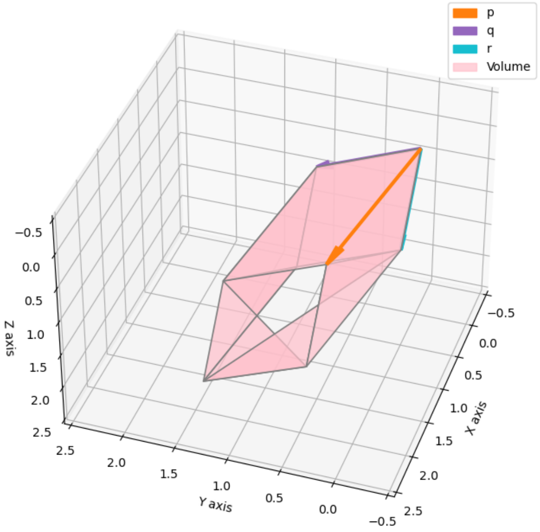
p
q
r
Volume
−0.5
0.0
0.5
1.0
1.5
2.0
2.5
Z axis
2.5
2.0
1.5
1.0
0.5
0.0
−0.5
Y axis
−0.5
0.0
0.5
1.0
1.5
2.0
2.5
X axis

Figure: *

C Code

```c
// Determinant of Gram matrix calculation with inline dot product
      calculations
double gram_determinant(double a) {
    double G11 = 2 + a * a;
    double G12 = 2 * a;
    double G13 = a + 1;
    double G21 = 2 * a;
    double G22 = a * a + 1;
    double G23 = a;
    double G31 = a + 1;
    double G32 = a;
    double G33 = 1 + a * a;
    double det = G11 * (G22 * G33 - G23 * G32) - G12 * (G21 * G33
         - G23 * G31)+ G13 * (G21 * G32 - G22 * G31);
               return det;
}
```

C code

```c
        // Volume calculation (sqrt of determinant)
double volume(double a) {
    double det = gram_determinant(a);
    if (det < 0) det = -det; // use absolute value for volume
    return sqrt(det);
}
int main() {
    double options[] = {-3, 3, 1.0 / sqrt(3), sqrt(3)};
    int num_options = 4;
    double min_vol = 1e9;
    double min_a = 0;
    printf("%-10s %-15s %-15s\n", "a", "Determinant", "Volume");
    for (int i = 0; i < num_options; i++) {
        double a = options[i];
        double det = gram_determinant(a);
        double vol = volume(a);
        printf("%-10.6f %-15.6f %-15.6f\n", a, det, vol);
```

C code

```
if (vol < min_vol) {
            min_vol = vol;
            min_a = a;
        }
    }
    printf("\nMinimum volume is at a = %.6f with volume = %.6f\n"
        , min_a, min_vol);
    return 0;
}
```

Python Code for Solving

```python
import ctypes
import numpy as np

lib = ctypes.CDLL('./volume.so')

lib.volume.argtypes = [ctypes.c_double]
lib.volume.restype = ctypes.c_double
a_values = np.array([-3, 3, 1.0 / np.sqrt(3), np.sqrt(3)])
min_volume = float('inf')
min_a = None
for a in a_values:
    vol = lib.volume(a)
    if vol < min_volume:
        min_volume = vol
        min_a = a

print(f"The value of a for which the volume is minimum: {min_a:.6
    f}")
```

Python Code for Plotting

```python
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D
from mpl_toolkits.mplot3d.art3d import Poly3DCollection
import matplotlib.patches as mpatches

a_val = 1 / np.sqrt(3)
p = np.array([1, a_val, 1])
q = np.array([0, 1, a_val])
r = np.array([a_val, 0, 1])
origin = np.array([0, 0, 0])
vertices = [
    origin, p, q, r, p + q, q + r, r + p, p + q + r ]
```

Python Code for Plotting

```
faces = [
    [vertices[0], vertices[1], vertices[4], vertices[2]],
    [vertices[0], vertices[1], vertices[6], vertices[3]],
    [vertices[0], vertices[2], vertices[5], vertices[3]],
    [vertices[7], vertices[5], vertices[4], vertices[6]],
    [vertices[7], vertices[6], vertices[1], vertices[3]],
    [vertices[7], vertices[5], vertices[2], vertices[4]]
]

fig = plt.figure(figsize=(10, 8))
ax = fig.add_subplot(111, projection='3d')
color_p = '#FF7F0E' # orange
color_q = '#9467BD' # purple
color_r = '#17BECF' # teal
shaded_color = '#ffc0cb' # pink
# Plot vectors
ax.quiver(*origin, *p, color=color_p, linewidth=3,
    arrow_length_ratio=0.15)
```

Python Code for Plotting

```
ax.quiver(*origin, *q, color=color_q, linewidth=3,
    arrow_length_ratio=0.15)
ax.quiver(*origin, *r, color=color_r, linewidth=3,
    arrow_length_ratio=0.15)
# Plot shaded volume - no automatic label, so use patch for
    legend
poly3d = Poly3DCollection(faces, facecolors=shaded_color,
    edgecolors='gray', linewidths=1.2, alpha=0.7)
ax.add_collection3d(poly3d)


p_patch = mpatches.Patch(color=color_p, label='p')
q_patch = mpatches.Patch(color=color_q, label='q')
r_patch = mpatches.Patch(color=color_r, label='r')
vol_patch = mpatches.Patch(color=shaded_color, label='Volume',
    alpha=0.7)
```

Python Code for Plotting

```
ax.set_xlim([-0.5, 2.5])
ax.set_ylim([-0.5, 2.5])
ax.set_zlim([-0.5, 2.5])
ax.set_xlabel('X axis')
ax.set_ylabel('Y axis')
ax.set_zlabel('Z axis')
ax.legend(handles=[p_patch, q_patch, r_patch, vol_patch])
plt.show()
plt.savefig("fig.png")
```