

7.4.39

EE25BTECH11032 - Kartik Lahoti

Question:

If $\left(m_i, \frac{1}{m_i}\right)$, $m_i > 0, i = 1, 2, 3, 4$ are four distinct points on a circle, then show that $m_1 m_2 m_3 m_4 = 1$

Solution:

Let the circle equation be

$$\|\mathbf{x}\|^2 + 2\mathbf{u}^\top \mathbf{x} + f = 0 \quad (0.1)$$

where, $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ with a and b as constants.

Let $\mathbf{P} = \begin{pmatrix} m \\ \frac{1}{m} \end{pmatrix}$ be a arbitrary vector in space.

Putting \mathbf{P} in the circle, we get

$$\|\mathbf{P}\|^2 + 2\mathbf{u}^\top \mathbf{P} + f = 0 \quad (0.2)$$

$$m^2 + \frac{1}{m^2} + 2am + \frac{2b}{m} + f = 0 \quad (0.3)$$

Let,

$$p(m) = m^4 + 2am^3 + fm^2 + 2bm + 1 = 0 \quad (0.4)$$

A general polynomial of degree n , has companion matrix as

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & 0 & \cdots & 0 & -c_2 \\ 0 & 0 & 1 & \cdots & 0 & -c_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -c_{n-1} \end{pmatrix} \quad (0.5)$$

The eigen values of the Companion Matrix \mathbf{C} are the roots of the polynomial.

For the question ,

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -2b \\ 0 & 1 & 0 & -f \\ 0 & 0 & 1 & -2a \end{pmatrix} \quad (0.6)$$

$$|m\mathbf{I} - \mathbf{C}| = p(m) \quad (0.7)$$

Here Eigen values of \mathbf{C} are m_i where $i \in \{1, 2, 3, 4\}$

Introducing Reversal Matrix

$$\mathbf{J} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad (0.8)$$

$$\mathbf{J}^2 = \mathbf{I} \quad (0.9)$$

The Matrix \mathbf{J} flips Rows when pre-multiplied and flips column when post-multiplied.

Then,

$$|\frac{1}{m}\mathbf{I} - \mathbf{JCJ}| = p\left(\frac{1}{m}\right) \quad (0.10)$$

Since $p\left(\frac{1}{m}\right)$ has eigen values $1/m_i$, we can say

$$\mathbf{JCJ} = \mathbf{C}^{-1} \quad (0.11)$$

$$(0.12)$$

Taking determinant, using 0.9

$$|\mathbf{C}| = |\mathbf{C}^{-1}| \quad (0.13)$$

$$|\mathbf{C}|^2 = 1 \quad (0.14)$$

Since \mathbf{C} is a real companion matrix of a monic quartic whose constant term is 1 ,

$$|C| = (-1)^4 1 \quad (0.15)$$

Also,

$$|C| = m_1 m_2 m_3 m_4 \quad (0.16)$$

$$\therefore m_1 m_2 m_3 m_4 = 1 \quad (0.17)$$

Hence Proved

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