Problem 12.110

Consider the four systems of algebraic equations (listed in Group I). The systems (Q), (R), and (S) are obtained from (P) by restricting the accuracy of data or coefficients or both, respectively, to two decimal places. (GG 2014)

Group I	Group II
(P) $x + 1.0000y = 2.0000$ $x + 1.0001y = 2.0001$ (Q) $x + 1.0000y = 2.00$ $x + 1.0001y = 2.00$ (R) $x + 1.00y = 2.0000$ $x + 1.00y = 2.0001$ (S) $x + 1.00y = 2.00$ $x + 1.00y = 2.00$	(1) Instability (2) Inconsistency (3) Non-uniqueness (4) Exact

Input Variables

System	Coefficient Matrix A	Data Vector $\mathbf{b}$
P	$\begin{pmatrix} 1 & 1.0000 \\ 1 & 1.0001 \end{pmatrix}$	$\begin{pmatrix} 2.0000 \\ 2.0001 \end{pmatrix}$
Q	$\begin{pmatrix} 1 & 1.0000 \\ 1 & 1.0001 \end{pmatrix}$	$\begin{pmatrix} 2.00 \\ 2.00 \end{pmatrix}$
R	$\begin{pmatrix} 1 & 1.00 \\ 1 & 1.00 \end{pmatrix}$	$\begin{pmatrix} 2.0000 \\ 2.0001 \end{pmatrix}$
S	$\begin{pmatrix} 1 & 1.00 \\ 1 & 1.00 \end{pmatrix}$	$\begin{pmatrix} 2.00 \\ 2.00 \end{pmatrix}$

Rank Criteria

Let n be the number of unknowns in the system ($n = 2$ here, since we have x, y). For a system $A\mathbf{x} = \mathbf{b}$, the solution type is decided by ranks:

$$\begin{aligned}
\operatorname{rank}(A) = \operatorname{rank}([A|\mathbf{b}]) = n \quad &\Rightarrow \text{Unique solution,} \\
\operatorname{rank}(A) = \operatorname{rank}([A|\mathbf{b}]) < n \quad &\Rightarrow \text{Infinitely many solutions,} \\
\operatorname{rank}(A) < \operatorname{rank}([A|\mathbf{b}]) \quad &\Rightarrow \text{No solution (Inconsistent).}
\end{aligned}$$

Solution

We analyze each case using rank and row-reduction of the augmented matrix $\begin{pmatrix} A & \mathbf{b} \end{pmatrix}$.

Case P

$$
\begin{aligned}
\begin{pmatrix} 1 & 1.0000 & 2.0000 \\ 1 & 1.0001 & 2.0001 \end{pmatrix} &\xrightarrow{R_2-R_1} \begin{pmatrix} 1 & 1.0000 & 2.0000 \\ 0 & 0.0001 & 0.0001 \end{pmatrix} && (1) \\
&\xrightarrow{\frac{1}{0.0001}R_2} \begin{pmatrix} 1 & 1.0000 & 2.0000 \\ 0 & 1 & 1 \end{pmatrix} && (2) \\
&\xrightarrow{R_1-R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} && (3)
\end{aligned}
$$

$\Rightarrow$ rank$(A) =$ rank$([A|\mathbf{b}]) = 2 \ = \ n$. Hence, Exact solution.

Case Q

$$
\begin{aligned}
\begin{pmatrix} 1 & 1.0000 & 2.00 \\ 1 & 1.0001 & 2.00 \end{pmatrix} &\xrightarrow{R_2-R_1} \begin{pmatrix} 1 & 1.0000 & 2.00 \\ 0 & 0.0001 & 0 \end{pmatrix} && (4) \\
&\xrightarrow{\frac{1}{0.0001}R_2} \begin{pmatrix} 1 & 1.0000 & 2.00 \\ 0 & 1 & 0 \end{pmatrix} && (5) \\
&\xrightarrow{R_1-R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix} && (6)
\end{aligned}
$$

$\Rightarrow \ \mathbf{x} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$. Compared with Case P solution $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, a small change in $\mathbf{b}$ gave a very different solution. Hence, Instability.

Case R

$$
\begin{pmatrix} 1 & 1.00 & 2.0000 \\ 1 & 1.00 & 2.0001 \end{pmatrix} \xrightarrow{R_2-R_1} \begin{pmatrix} 1 & 1.00 & 2.0000 \\ 0 & 0 & 0.0001 \end{pmatrix} \tag{7}
$$

$\Rightarrow$ rank$(A) = 1,$ rank$([A|\mathbf{b}]) = 2$. Thus, Inconsistency.

Case S

$$
\begin{pmatrix} 1 & 1.00 & 2.00 \\ 1 & 1.00 & 2.00 \end{pmatrix} \xrightarrow{R_2-R_1} \begin{pmatrix} 1 & 1.00 & 2.00 \\ 0 & 0 & 0 \end{pmatrix} \tag{8}
$$

$\Rightarrow$ rank$(A) = 1,$ rank$([A|\mathbf{b}]) = 1 < n = 2$. Thus, Non-uniqueness.

Final Answer

$$\boxed{P-4,\ Q-1,\ R-2,\ S-3} \tag{9}$$