

2.8.36

Puni Aditya - EE25BTECH11046

Question:

The value of the expression $\|\mathbf{a} \times \mathbf{b}\| + (\mathbf{a}^\top \mathbf{b})$ is _____

Solution:

Let \mathbf{a} and \mathbf{b} be two vectors, and let θ be the angle between them.

The magnitude of the cross product is

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin(\theta) \quad (1)$$

The dot product or inner product is

$$\mathbf{a}^\top \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos(\theta) \quad (2)$$

Now, we substitute these definitions into the given expression:

$$\|\mathbf{a} \times \mathbf{b}\| + (\mathbf{a}^\top \mathbf{b}) = \|\mathbf{a}\| \|\mathbf{b}\| \sin(\theta) + \|\mathbf{a}\| \|\mathbf{b}\| \cos(\theta) \quad (3)$$

$$\implies \|\mathbf{a} \times \mathbf{b}\| + (\mathbf{a}^\top \mathbf{b}) = \|\mathbf{a}\| \|\mathbf{b}\| (\sin(\theta) + \cos(\theta)) \quad (4)$$

$$\sin(\theta) + \cos(\theta) = \sqrt{2} \sin\left(\theta + \frac{\pi}{4}\right) \quad (5)$$

$$\therefore \|\mathbf{a} \times \mathbf{b}\| + (\mathbf{a}^\top \mathbf{b}) = \sqrt{2} \|\mathbf{a}\| \|\mathbf{b}\| \sin\left(\theta + \frac{\pi}{4}\right)$$

Example: Let

$$\mathbf{a} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\|\mathbf{a}\| = 1 \text{ and } \|\mathbf{b}\| = \sqrt{2}$$

From (3),

$$\|\mathbf{a} \times \mathbf{b}\| + (\mathbf{a}^\top \mathbf{b}) = \left\| \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\| + \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad (6)$$

$$\|\mathbf{a} \times \mathbf{b}\| + (\mathbf{a}^\top \mathbf{b}) = 1 + 1 = 2 \quad (7)$$

$$\cos(\theta) = \frac{(0 \ 1 \ 0) \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}{\left\| \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\| \left\| \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\|} \quad (8)$$

$$\cos(\theta) = \frac{1}{\sqrt{2}} \quad (9)$$

$$\theta = \frac{\pi}{4} \quad (10)$$

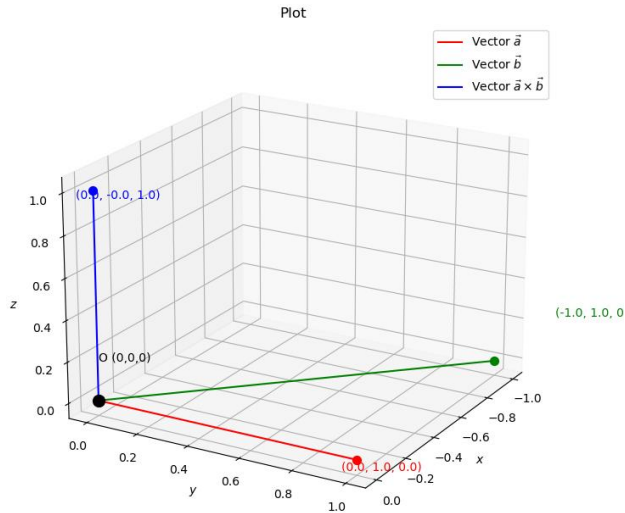
$$\text{From (10)} \quad (11)$$

$$\sqrt{2} \|\mathbf{a}\| \|\mathbf{b}\| \sin\left(\theta + \frac{\pi}{4}\right) = \sqrt{2} \times 1 \times \sqrt{2} \times \sin\left(\frac{\pi}{2}\right) \quad (12)$$

$$\sqrt{2} \|\mathbf{a}\| \|\mathbf{b}\| \sin\left(\theta + \frac{\pi}{4}\right) = 2 \quad (13)$$

From (7) and (13),

$$\|\mathbf{a} \times \mathbf{b}\| + (\mathbf{a}^\top \mathbf{b}) = \sqrt{2} \|\mathbf{a}\| \|\mathbf{b}\| \sin\left(\theta + \frac{\pi}{4}\right)$$



Example