Problem 2.7.4

Problem. If

$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \qquad \mathbf{c} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \tag{1}$$

find $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.

Solution.

Input variable	Value
$\mathbf{a}$	$\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$
$\mathbf{b}$	$\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$
$\mathbf{c}$	$\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$

Table 1

Write the vectors in component form:

$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \qquad \mathbf{c} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}. \tag{2}$$

Using the minor notation from the problem statement, where

$$\mathbf{B_{ij}} = \begin{pmatrix} b_i \\ b_j \end{pmatrix}, \qquad \mathbf{C_{ij}} = \begin{pmatrix} c_i \\ c_j \end{pmatrix}, \tag{3}$$

we write the cross product :

$$\mathbf{b} \times \mathbf{c} \ = \begin{pmatrix} |\mathbf{B_{23}}\ \mathbf{C_{23}}| \\ |\mathbf{B_{31}}\ \mathbf{C_{31}}| \\ |\mathbf{B_{12}}\ \mathbf{C_{12}}| \end{pmatrix}. \tag{4}$$

Substituting the components gives

$$\mathbf{b} \times \mathbf{c} = \begin{pmatrix} 3 \\ 5 \\ -7 \end{pmatrix}. \tag{5}$$

Now use the transpose (row-vector) method for the dot product:

$$\mathbf{a}^T(\mathbf{b}\times\mathbf{c}) = \begin{pmatrix}2\\1\\3\end{pmatrix}^T \begin{pmatrix}3\\5\\-7\end{pmatrix} = \begin{pmatrix}2 & 1 & 3\end{pmatrix} \begin{pmatrix}3\\5\\-7\end{pmatrix} = 2\cdot 3 + 1\cdot 5 + 3\cdot(-7) = -10. \tag{6}$$

Thus

$$\boxed{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = -10}. \tag{7}$$

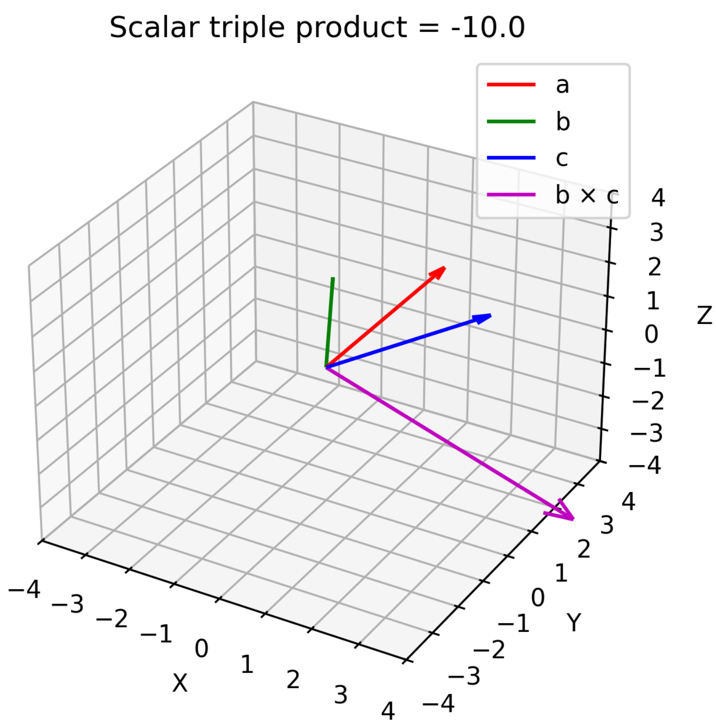

Figure 1