2.7.14

EE25BTECH11002 - Achat Parth Kalpesh

Question:
If θ is the angle between the two vectors $\mathbf{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\mathbf{b} = 3\hat{i} - 2\hat{j} + \hat{k}$, find $\sin\theta$.
Solution:
Let the given vectors be represented by column matrices $\mathbf{a}$ and $\mathbf{b}$.

$$\mathbf{a} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \tag{0.1}$$

For calculating cross product, $\mathbf{A} \times \mathbf{B}$

$$\begin{aligned} \mathbf{A}_{ij} &= \begin{pmatrix} a_i \\ a_j \end{pmatrix}, \\ \mathbf{B}_{ij} &= \begin{pmatrix} b_i \\ b_j \end{pmatrix} \end{aligned} \tag{0.2}$$

$$\mathbf{A} \times \mathbf{B} = \begin{pmatrix} \begin{vmatrix} \mathbf{A}_{23} & \mathbf{B}_{23} \end{vmatrix} \\ \begin{vmatrix} \mathbf{A}_{31} & \mathbf{B}_{31} \end{vmatrix} \\ \begin{vmatrix} \mathbf{A}_{12} & \mathbf{B}_{12} \end{vmatrix} \end{pmatrix} \tag{0.3}$$

From (0.2) and (0.3), we calculate the components of the cross product $\mathbf{a} \times \mathbf{b}$:

$$\begin{vmatrix} A_{23} & B_{23} \end{vmatrix} = \begin{vmatrix} -2 & -2 \\ 3 & 1 \end{vmatrix} = (-2)(1) - (3)(-2) = 4 \tag{0.4}$$

$$\begin{vmatrix} A_{31} & B_{31} \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = (3)(3) - (1)(1) = 8 \tag{0.5}$$

$$\begin{vmatrix} A_{12} & B_{12} \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ -2 & -2 \end{vmatrix} = (1)(-2) - (-2)(3) = 4 \tag{0.6}$$

Therfore,

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 4 \\ 8 \\ 4 \end{pmatrix}. \tag{0.7}$$

$$\|\mathbf{a} \times \mathbf{b}\| = \sqrt{4^2 + 8^2 + 4^2} = \sqrt{16 + 64 + 16} = \sqrt{96} \tag{0.8}$$

$$\|\mathbf{a}\| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14} \tag{0.9}$$

$$\|\mathbf{b}\| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{9 + 4 + 1} = \sqrt{14} \tag{0.10}$$

$$\sin\theta = \frac{\|\mathbf{a}\times\mathbf{b}\|}{\|\mathbf{a}\|\cdot\|\mathbf{b}\|} \tag{0.11}$$

$$= \frac{\sqrt{96}}{\sqrt{14}\cdot\sqrt{14}} = \frac{\sqrt{16\times 6}}{14} \tag{0.12}$$

$$= \frac{4\sqrt{6}}{14} = \frac{2\sqrt{6}}{7} \tag{0.13}$$

Therefore, the value of $\sin\theta$ is $\frac{2\sqrt{6}}{7}$.

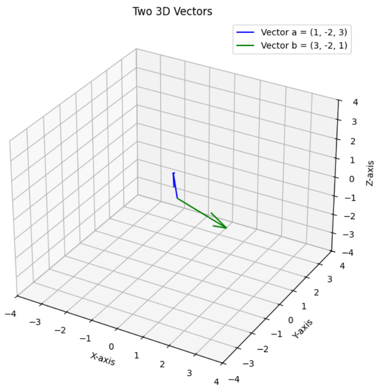

Fig. 0.1: Graph