7.4.32

EE25BTECH11025 - Ganachari Vishwambhar

Question:
$ABCD$ is a square of side length 2 units. C_1 is the circle touching all the sides of the square $ABCD$ and C_2 is the circumcircle of square $ABCD$. L is a fixed line in same plane and $\mathbf{R}$ is a fixed point.

1) If $\mathbf{P}$ is any point of C_1 and $\mathbf{Q}$ is another point on C_2, then $\frac{PA^2+PB^2+PC^2+PD^2}{QA^2+QB^2+QC^2+QD^2}$

   a) 0.75 b) 1.25 c) 1 d) 0.5

2) If a circle is such that it touches the line L and the circle C_1 externally, such that both the circles are on the same side of the line, then locus of centre of the circle

   a) ellipse b) hyperbola c) parabola d) circle

3) A L' through $\mathbf{A}$ is drawn parallel to BD. Point S moves such that its distances from the line BD and the vertex $\mathbf{A}$ are equal. If locus of S cuts L' at T_2 and T_3 and AC at T_1, then area of $\triangle T_1T_2T_3$ is

   a) 1/2 sq.units b) 2/3 sq.units c) 1 sq.units d) 2 sq.units

Solution:
Let:
The centre of incircle and circumcircle be $\mathbf{O}$.
The radius of incircle be r_1 and that of circumcircle be r_2.
Given:

$$r_1 = 1 \tag{1}$$
$$r_2 = \sqrt{2} \tag{2}$$

1) Let $\mathbf{P}$ be any point on incircle and $\mathbf{Q}$ be any point on circumcircle.
   $X \epsilon \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$

$$|\mathbf{X} - \mathbf{P}|^2 = |\mathbf{X}|^2 + |\mathbf{P}|^2 - 2\mathbf{P}.\mathbf{X} \tag{3}$$
$$\tag{4}$$

   Summation over all $\mathbf{X}|\mathbf{X}\epsilon \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$:

$$\sum |\mathbf{X} - \mathbf{P}|^2 = \sum |\mathbf{X}|^2 + 4.|\mathbf{P}|^2 - 2\mathbf{P} \sum \mathbf{X} \tag{5}$$
$$\tag{6}$$

For, $\mathbf{P} = \mathbf{P}$

$$\sum |\mathbf{X} - \mathbf{P}|^2 = \sum |\mathbf{X}|^2 + 4.|\mathbf{P}|^2 - 2\mathbf{P}\sum \mathbf{X} \tag{7}$$

$$4\left(1^2 + 1^2\right) + 4\,(1) - 2\mathbf{P}\left(\begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}1\\-1\end{pmatrix} + \begin{pmatrix}-1\\1\end{pmatrix} + \begin{pmatrix}-1\\-1\end{pmatrix}\right) \tag{8}$$

$$\therefore |\mathbf{A} - \mathbf{P}|^2 + |\mathbf{B} - \mathbf{P}|^2 + |\mathbf{C} - \mathbf{P}|^2 + |\mathbf{D} - \mathbf{P}|^2 = 12 \tag{9}$$

For, $\mathbf{P} = \mathbf{Q}$

$$\sum |\mathbf{X} - \mathbf{Q}|^2 = \sum |\mathbf{X}|^2 + 4.|\mathbf{Q}|^2 - 2\mathbf{Q}\sum \mathbf{X} \tag{10}$$

$$4\left(1^2 + 1^2\right) + 4\,(2) - 2\mathbf{Q}\left(\begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}1\\-1\end{pmatrix} + \begin{pmatrix}-1\\1\end{pmatrix} + \begin{pmatrix}-1\\-1\end{pmatrix}\right) \tag{11}$$

$$\therefore |\mathbf{A} - \mathbf{Q}|^2 + |\mathbf{B} - \mathbf{Q}|^2 + |\mathbf{C} - \mathbf{Q}|^2 + |\mathbf{D} - \mathbf{Q}|^2 = 12 \tag{12}$$

conclusion:

$$\frac{12}{16} = 0.75 \tag{13}$$

Hence, option(a) is correct.

2) Let the radius of the moving circle be r, the centre of the circle be $\mathbf{X}$ and the line equation be $\hat{\mathbf{n}}^\top\mathbf{X} = c$,

$$|\hat{\mathbf{n}}^\top\mathbf{X} - c| = r \tag{14}$$

$$\|\mathbf{X}\| = r + 1 \tag{15}$$

$$\|\mathbf{X}\| = |\hat{\mathbf{n}}\mathbf{X} - (c - 1)\,| \tag{16}$$

$$\|\mathbf{X}\|^2 = |\hat{\mathbf{n}}\mathbf{X} - (c - 1)\,|^2 \tag{17}$$

$$\mathbf{X}^\top\mathbf{X} = \left(\hat{\mathbf{n}}^\top\mathbf{X}\right)^2 + (c - 1)^2 - 2\hat{\mathbf{n}}^\top\mathbf{X}\,(c - 1) \tag{18}$$

$$\mathbf{X}^\top\mathbf{X} - (\hat{\mathbf{n}}\mathbf{X})^2 + 2\hat{\mathbf{n}}^\top\mathbf{X}\,(c - 1) - (c - 1)^2 \tag{19}$$

$$\mathbf{X}^\top\left(I - \hat{\mathbf{n}}\hat{\mathbf{n}}^\top\right)\mathbf{X} + 2\,(c - 1)\,\hat{\mathbf{n}}^\top\mathbf{X} - (c - 1)^2 \tag{20}$$

Equation (20) is the equation of parabola.
Hence, correct option is (c).

3) Let the point moving point be S and the line equation be $\mathbf{n}^\top\mathbf{S} = 0$.

$$\frac{|\mathbf{n}^\top\mathbf{S}|}{\|\mathbf{n}\|} = \|\mathbf{S} - \mathbf{A}\| \tag{21}$$

$$\frac{|\mathbf{n}^\top\mathbf{S}|^2}{\|\mathbf{n}\|^2} = \|\mathbf{S} - \mathbf{A}\|^2 \tag{22}$$

$$\frac{|\mathbf{S}^\top\mathbf{n}\mathbf{n}^\top\mathbf{S}|}{\|\mathbf{n}\|} = (\mathbf{S} - \mathbf{A})^\top(\mathbf{S} - \mathbf{A}) \tag{23}$$

$$\mathbf{S}^\top\left(I - \hat{\mathbf{n}}\hat{\mathbf{n}}^\top\right)\mathbf{S} - 2\mathbf{A}^\top\mathbf{S} + \mathbf{A}^\top\mathbf{A} = 0 \tag{24}$$

Equation 24 is the locus of the moving point.

Let:

$$\mathbf{A} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \mathbf{B} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{25}$$

$$\mathbf{C} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}; \mathbf{D} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{26}$$

$\mathbf{m_1}$ be the direction vector of line AC, $\mathbf{m_2}$ be the direction vector of line L'.

$$\mathbf{m_1} = \begin{pmatrix} -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{27}$$

$$\mathbf{m_2} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \tag{28}$$

The equation of line L'

$$\mathbf{S} = \mathbf{A} + t\mathbf{m_2} \tag{29}$$

The equation of the line AC

$$\mathbf{S} = \lambda\mathbf{m_1} \tag{30}$$

Substituting equation (30) in (24) we get $\lambda = \frac{1}{2}$:

$$\mathbf{T_1} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{31}$$

Substituting (29) in (24) we get $t = \frac{-1}{2}, \frac{1}{2}$

$$\mathbf{T_2} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{32}$$

$$\mathbf{T_3} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{33}$$

Now, finding area of the triangle:

$$\triangle T_1T_2T_3 = \frac{1}{2}|\begin{pmatrix} \mathbf{T_2} - \mathbf{T_1} & \mathbf{T_3} - \mathbf{T_2} \end{pmatrix}| \tag{34}$$

$$\triangle T_1T_2T_3 = 1 \tag{35}$$

Option (c) is correct.

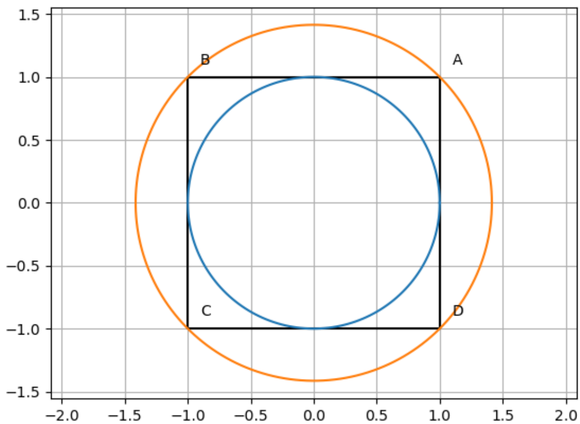

Fig. 1: Plot of the given square and circles

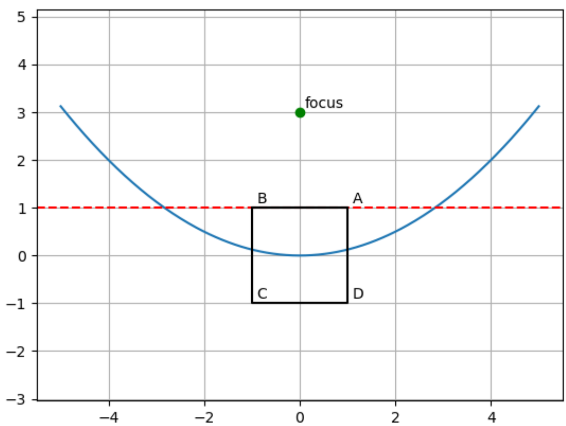

Fig. 2: Plot of the given circles, square and locus of the point

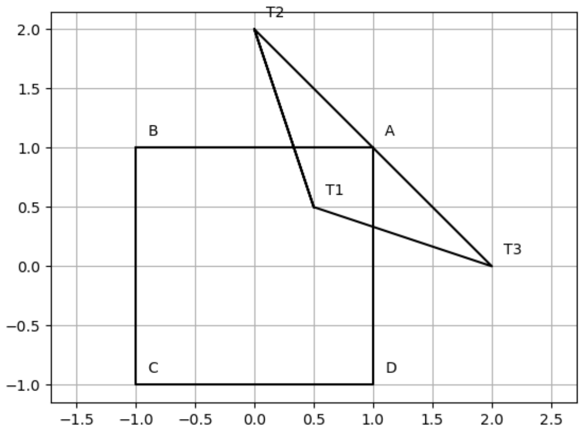

Fig. 3: Plot of the given circles, square and locus of the point