12.485

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Question : Let

$$\mathbf{M} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

Which of the following is correct

1) Rank of $\mathbf{M}$ is 1 and $\mathbf{M}$ is diagonalizable
2) Rank of $\mathbf{M}$ is 2 and $\mathbf{M}$ is diagonalizable
3) 1 is the only eigenvalue and $\mathbf{M}$ is diagonalizable
4) 1 is the only eigenvalue and $\mathbf{M}$ is not diagonalizable

Solution :

Name	Value
$\mathbf{M}$	$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$

Table : Matrix

First convert $\mathbf{M}$ into echelon form by applying row reduction

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \xleftrightarrow{R_2 \to R_2 - R_1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{1}$$

From the echelon form we see that there is one nonzero row, hence

$$\text{rank}(\mathbf{M}) = 1 \tag{2}$$

Next find the eigenvalues. Because $\mathbf{M}$ is upper triangular, its eigenvalues are the diagonal entries:

$$\lambda_1 = 0 \qquad \lambda_2 = 1 \tag{3}$$

Now find eigenvectors by solving

$$(\mathbf{M} - \lambda \mathbf{I})\mathbf{v} = \mathbf{0} \tag{4}$$

For $\lambda = 0$ solve

$$\mathbf{M}\mathbf{v} = \mathbf{0} \tag{5}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{6}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{7}$$

This gives

$$y = 0 \tag{8}$$

And x can be anything

Thus an eigenvector for $\lambda = 0$ is

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{9}$$

For $\lambda = 1$ solve

$$(\mathbf{M} - \mathbf{I})\mathbf{v} = \mathbf{0} \tag{10}$$

$$\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{11}$$

$$\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{12}$$

This gives

$$-x + y = 0 \tag{13}$$

$$y = x \tag{14}$$

Thus an eigenvector for $\lambda = 1$ is

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{15}$$

Since the eigenvalues λ_1 and λ_2 are distinct, the matrix $\mathbf{M}$ is diagonalizable.
Form the matrix $\mathbf{P}$ with eigenvectors as columns and the diagonal matrix $\mathbf{D}$ of eigenvalues:

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{16}$$

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{17}$$

Compute $\mathbf{P}^{-1}$

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right) \xleftrightarrow{R_1 \to R_1 - R_2} \left(\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{array}\right) \tag{18}$$

The right block gives

$$\mathbf{P}^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \tag{19}$$

Finally, the diagonalization:

$$\mathbf{M} = \mathbf{PDP}^{-1} \tag{20}$$

$$\mathbf{M} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \tag{21}$$

Conclusion : The matrix $\mathbf{M}$ has $rank(\mathbf{M}) = 1$ and is **diagonalizable**.
Therefore the correct option is (1).