12.859

EE25BTECH11013 - Bhargav

Question:
Let $\mathbf{O} = \{\mathbf{P} : \mathbf{P}$ is a 3×3 real matrix with $\mathbf{P}^T\mathbf{P} = \mathbf{I}_3, \det(\mathbf{P}) = 1\}$. Which of the following options is/are correct?

1) There exists $\mathbf{P} \in \mathbf{O}$ with $\lambda = \frac{1}{2}$ as an eigenvalue.
2) There exists $\mathbf{P} \in \mathbf{O}$ with $\lambda = 2$ as an eigenvalue.
3) If λ is the only real eigenvalue of $\mathbf{P} \in \mathbf{O}$, then $\lambda = 1$.
4) There exists $\mathbf{P} \in O$ with $\lambda = -1$ as an eigenvalue.

Solution:
Let $\mathbf{v}$ be the eigenvector corresponding to the eigenvalue λ.

$$\mathbf{P}\mathbf{v} = \lambda\mathbf{v} \tag{4.1}$$

Orthogonal transformations preserve the length of vectors ($|\mathbf{P}| = 1$)

$$||\mathbf{P}\mathbf{v}|| = ||\mathbf{v}|| \tag{4.2}$$

This can be proved in this way:

$$||\mathbf{P}\mathbf{v}||^2 = (\mathbf{P}\mathbf{v})^\top (\mathbf{P}\mathbf{v}) = \mathbf{v}^\top\mathbf{P}^\top\mathbf{P}\mathbf{v} \tag{4.3}$$

Since $\mathbf{P}^\top\mathbf{P} = \mathbf{I}$

$$||\mathbf{P}\mathbf{v}||^2 = \mathbf{v}^\top\mathbf{v} = ||\mathbf{v}||^2 \tag{4.4}$$

$$\implies ||\mathbf{P}\mathbf{v}|| = ||\mathbf{v}|| \tag{4.5}$$

From (4.1),

$$||\mathbf{P}\mathbf{v}|| = |\lambda|\,||\mathbf{v}|| \tag{4.6}$$

Using the equations (4.2) and (4.6),

$$||\mathbf{P}\mathbf{v}|| = ||\mathbf{v}|| = |\lambda|\,||\mathbf{v}|| \tag{4.7}$$

$$\implies ||\mathbf{v}|| = |\lambda|\,||\mathbf{v}|| \tag{4.8}$$

Thus, $|\lambda| = 1$
Eigenvalues can be either -1 or 1 or both.

Thus, options (3) and (4) are correct.

This can be verified by examples.

1. For $\lambda_1 = 1$

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\mathbf{P}^T\mathbf{P} = \mathbf{I}$

Eigenvalue of $\mathbf{P}$ is 1.

2. For $\lambda_2 = -1$

$$\mathbf{P} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$\mathbf{P}^T\mathbf{P} = \mathbf{I}$

Eigenvalue of $\mathbf{P}$ is -1.