2.10.56

INDHIRESH S- EE25BTECH11027

Question. Let two non-collinear unit vectors $\hat{a}$ and $\hat{b}$ form an acute angle. A point **P** moves so that at any time t the position vector $\overrightarrow{OP}$ (where **O** is the origin) is given by $\hat{\mathbf{a}}\cos t + \hat{\mathbf{b}}\sin t$. When **P** is farthest from origin **O**, let M be the length of $\overrightarrow{OP}$ and $\hat{\mathbf{u}}$ be the unit vector along $\overrightarrow{OP}$. Then,

1) $\hat{\mathbf{u}} = \frac{\hat{\mathbf{a}}+\hat{\mathbf{b}}}{|\hat{\mathbf{a}}+\hat{\mathbf{b}}|}$ and $M = (1 + \hat{\mathbf{a}} \cdot \hat{\mathbf{b}})^{\frac{1}{2}}$
2) $\hat{\mathbf{u}} = \frac{\hat{\mathbf{a}}-\hat{\mathbf{b}}}{|\hat{\mathbf{a}}-\hat{\mathbf{b}}|}$ and $M = (1 + \hat{\mathbf{a}} \cdot \hat{\mathbf{b}})^{\frac{1}{2}}$
3) $\hat{\mathbf{u}} = \frac{\hat{\mathbf{a}}+\hat{\mathbf{b}}}{|\hat{\mathbf{a}}+\hat{\mathbf{b}}|}$ and $M = (1 + 2\hat{\mathbf{a}} \cdot \hat{\mathbf{b}})^{\frac{1}{2}}$
4) $\hat{\mathbf{u}} = \frac{\hat{\mathbf{a}}-\hat{\mathbf{b}}}{|\hat{\mathbf{a}}-\hat{\mathbf{b}}|}$ and $M = (1 + 2\hat{\mathbf{a}} \cdot \hat{\mathbf{b}})^{\frac{1}{2}}$

Solution:
Let us solve the given equation theoretically and then verify the solution computationally.
Given equation:

$$\mathbf{P} = \mathbf{a}\cos t + \mathbf{b}\sin t \tag{1}$$

Which can be written as :

$$\mathbf{P} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \begin{pmatrix} cost \\ sint \end{pmatrix} \tag{2}$$

Let

$$\mathbf{x} = \begin{pmatrix} cost \\ sint \end{pmatrix} \quad and \quad \mathbf{G} = \begin{pmatrix} 1 & (\mathbf{a})^T(\mathbf{b}) \\ (\mathbf{a})^T(\mathbf{b}) & 1 \end{pmatrix} \tag{3}$$

From given if **P** is farthest from origin , then length of **P** is given as M.From this we can say that

$$M = \max ||\mathbf{P}|| \tag{4}$$

Now,

$$||\mathbf{P}|| = \sqrt{(\mathbf{P})^T(\mathbf{P})} \tag{5}$$

$$||\mathbf{P}|| = \sqrt{\left(\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \begin{pmatrix} cost \\ sint \end{pmatrix}\right)^T \left(\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \begin{pmatrix} cost \\ sint \end{pmatrix}\right)} \tag{6}$$

$$||\mathbf{P}|| = \sqrt{\begin{pmatrix} cost \\ sint \end{pmatrix}^T \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix}^T \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \begin{pmatrix} cost \\ sint \end{pmatrix}} \tag{7}$$

$$||\mathbf{P}|| = \sqrt{||\mathbf{a}||^2\ cos^2t + ||\mathbf{b}||^2\ sin^2t + 2(\mathbf{a})^T(\mathbf{b})(cost)(sint)} \tag{8}$$

$$||\mathbf{P}||^2 = \begin{pmatrix} cost & sint \end{pmatrix} \begin{pmatrix} 1 & (\mathbf{a})^T(\mathbf{b}) \\ (\mathbf{a})^T(\mathbf{b}) & 1 \end{pmatrix} \begin{pmatrix} cost \\ sint \end{pmatrix} \tag{9}$$

From Eq.3:

$$||\mathbf{P}||^2 = \mathbf{x}^T\mathbf{G}\mathbf{x} \tag{10}$$

Now we should find the maximum value of $x^T Gx$ such that $||x|| = 1$

By **Rayleigh-Ritz theorem**, the maximum value of the quadratic form if **x** is a unit vector will be the largest eigenvalue (λ_{max}) of the matrix G.
So,

$$max\,||\mathbf{P}|| = \sqrt{\lambda_{max}} \tag{11}$$

Now we will calculate the Eigen value for the matrix G:

$$\left|G - \lambda I\right| = 0 \tag{12}$$

$$\left|\begin{pmatrix} 1-\lambda & (\mathbf{a})^T(\mathbf{b}) \\ (\mathbf{a})^T(\mathbf{b}) & 1-\lambda \end{pmatrix}\right| = 0 \tag{13}$$

$$(1-\lambda)^2 - ((\mathbf{a})^T(\mathbf{b}))^2 = 0 \tag{14}$$

$$1-\lambda = (\mathbf{a})^T(\mathbf{b}) \;\; or \;\; 1-\lambda = -(\mathbf{a})^T(\mathbf{b}) \tag{15}$$

$$\lambda = 1 + (\mathbf{a})^T(\mathbf{b}) \;\; or \;\; \lambda = 1 - (\mathbf{a})^T(\mathbf{b}) \tag{16}$$

It is already given that $(\mathbf{a})^T(\mathbf{b}) > 0$(**a** and **b** form an acute angle) . so,

$$\lambda_{max} = 1 + (\mathbf{a})^T(\mathbf{b}) \tag{17}$$

From Eq.9

$$max\,||\mathbf{P}|| = \sqrt{1 + (\mathbf{a})^T(\mathbf{b})} \tag{18}$$

The above equation can be written as

$$max\,||\mathbf{P}|| = \sqrt{1 + \mathbf{a}.\mathbf{b}} \tag{19}$$

From Eq.4:

$$M = \sqrt{1 + \mathbf{a}.\mathbf{b}} \tag{20}$$

Now let us find the value of t for which $||\mathbf{P}||$ is max
With eigenvalue equation,We'll use matrix G and largest eigenvalue λ_{max} such that,

$$\left(G - \lambda I\right)x = 0 \tag{21}$$

$$\begin{pmatrix} -(\mathbf{a})^T(\mathbf{b}) & (\mathbf{a})^T(\mathbf{b}) \\ (\mathbf{a})^T(\mathbf{b}) & -(\mathbf{a})^T(\mathbf{b}) \end{pmatrix} \begin{pmatrix} cost \\ sint \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{22}$$

By solving this we'll get

$$cost = sint \tag{23}$$

We already know that:

$$sin^2t + cos^2t = 1 \tag{24}$$

So,

$$sint = \frac{1}{\sqrt{2}} \;\; and \;\; cost = \frac{1}{\sqrt{2}} \tag{25}$$

From above result

$$t = \frac{\pi}{4} \tag{26}$$

Now unit vector **u** along **P** is given by:

$$\mathbf{u} = \frac{\mathbf{P}}{||\mathbf{P}||} \tag{27}$$

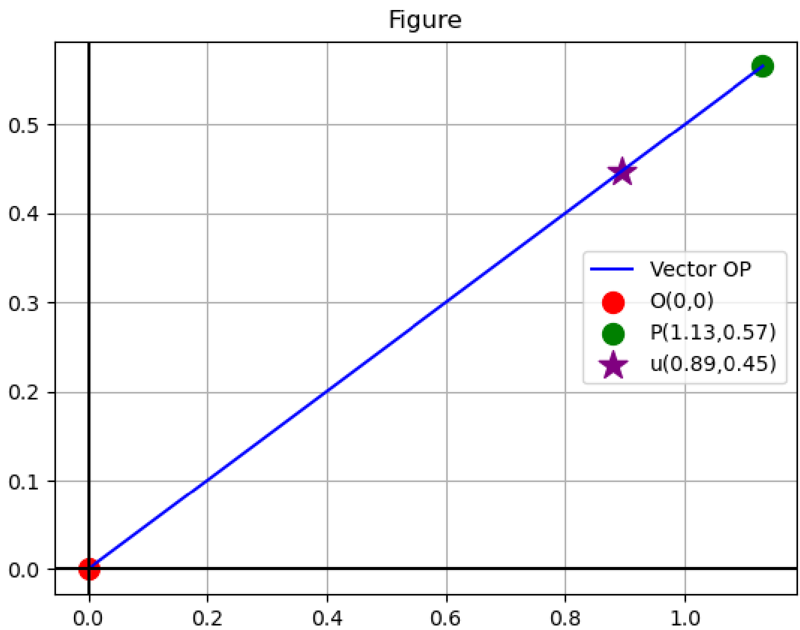
$$\mathbf{u} = \frac{\mathbf{a}\cos t + \mathbf{b}\sin t}{||\mathbf{a}\cos t + \mathbf{b}\sin t||} \tag{28}$$

Now subtituiting the value of t in above equation:

$$\mathbf{u} = \frac{\mathbf{a}\frac{1}{\sqrt{2}} + \mathbf{b}\frac{1}{\sqrt{2}}}{\left\|\mathbf{a}\frac{1}{\sqrt{2}} + \mathbf{b}\frac{1}{\sqrt{2}}\right\|} \tag{29}$$

$$\mathbf{u} = \frac{\mathbf{a} + \mathbf{b}}{||\mathbf{a} + \mathbf{b}||} \tag{30}$$

From Eq.18 and Eq.28 (a) is correct
From the figure it is clearly verified that the theoretical solution matches with the computational solution.



Figure
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Vector OP
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u(0.89,0.45)