Problem 7.4.30

ee25btech11023-Venkata Sai

September 21, 2025

Problem

A circle is given by $x^2 + (y-1)^2 = 1$, another circle C touches it externally and also the X axis, then the locus of its centre is

1. $\{(x, y) : x^2 = 4y\} \bigcup \{(x, y) : y \leq 0\}$
2. $\left\{(x, y) : x^2 + (y-1)^2 = 4\right\} \bigcup \{(x, y) : y \leq 0\}$
3. $\{(x, y) : x^2 = 4y\} \bigcup \{(0, y) : y \leq 0\}$
4. $\{(x, y) : x^2 = 4y\} \bigcup \{(0, y) : y \leq 0\}$

Formula

As the circle touches $X-$axis , Distance of a point from x-axis is given by

$$r = \left|\mathbf{n}^\top \mathbf{c}\right| \tag{1.1}$$

where $\mathbf{n}$ is the unit vector normal to x-axis
For the given circle with radius r_1 and center c_1

$$x^2 + (y-1)^2 = 1 \tag{1.2}$$

$$\mathbf{p} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathbf{n} \text{ and } r_1 = 1 \tag{1.3}$$

Distance between their centers equal to sum of their radius

$$\|\mathbf{c}-\mathbf{p}\| = r + r_1 \tag{1.4}$$

$$\|\mathbf{c}-\mathbf{n}\| = \left|\mathbf{n}^\top \mathbf{c}\right| + 1 \tag{1.5}$$

$$\|\mathbf{c}-\mathbf{n}\|^2 = \left(\left|\mathbf{n}^\top \mathbf{c}\right| + 1\right)^2 \tag{1.6}$$

Squaring

$$(\mathbf{c}-\mathbf{n})\left(\mathbf{c}^\top-\mathbf{n}^\top\right)=\left(|\mathbf{n}^\top\mathbf{c}|+1\right)^2 \tag{1.7}$$

$$\mathbf{c}^\top\mathbf{c}+\mathbf{n}\mathbf{n}^\top-\mathbf{c}^\top\mathbf{n}-\mathbf{n}^\top\mathbf{c}=\left(|\mathbf{n}^\top\mathbf{c}|\right)^2+2|\mathbf{n}^\top\mathbf{c}|+1 \tag{1.8}$$

$$\mathbf{c}^\top\mathbf{c}+\mathbf{n}\mathbf{n}^\top-\mathbf{c}^\top\mathbf{n}-\mathbf{n}^\top\mathbf{c}=\left(\mathbf{n}^\top\mathbf{c}\right)^\top\left(\mathbf{n}^\top\mathbf{c}\right)+2|\mathbf{n}^\top\mathbf{c}|+1 \tag{1.9}$$

$$\mathbf{c}^\top\mathbf{c}+\|\mathbf{n}\|^2-2\mathbf{n}^\top\mathbf{c}=\left(\mathbf{c}^\top\mathbf{n}\mathbf{n}^\top\mathbf{c}\right)+2|\mathbf{n}^\top\mathbf{c}|+1 \tag{1.10}$$

$$\mathbf{c}^\top\mathbf{c}+1=\left(\mathbf{c}^\top\mathbf{n}\mathbf{n}^\top\mathbf{c}\right)+2\mathbf{n}^\top\mathbf{c}+2|\mathbf{n}^\top\mathbf{c}|+1 \tag{1.11}$$

$$\mathbf{c}^\top\mathbf{c}-\left(\mathbf{c}^\top\mathbf{n}\mathbf{n}^\top\mathbf{c}\right)=2\mathbf{n}^\top\mathbf{c}+2|\mathbf{n}^\top\mathbf{c}| \tag{1.12}$$

$$\mathbf{c}^\top\left(\mathbf{I}-\mathbf{n}\mathbf{n}^\top\right)\mathbf{c}=2\mathbf{n}^\top\mathbf{c}+2|\mathbf{n}^\top\mathbf{c}| \tag{1.13}$$

Outer Product

$$\mathbf{c}^\top \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \right) \mathbf{c} = 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}^\top \mathbf{c} + 2 \left| \begin{pmatrix} 0 \\ 1 \end{pmatrix}^\top \mathbf{c} \right| \tag{1.14}$$

$$\mathbf{c}^\top \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) \mathbf{c} = 2 \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{c} + 2 \left| \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{c} \right| \tag{1.15}$$

$$\begin{pmatrix} x & y \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \pm 2 \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{1.16}$$

$$\begin{pmatrix} x & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 4 \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \ (\textbf{or}) \ \begin{pmatrix} x & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} 0 \\ y \end{pmatrix} - 2 \begin{pmatrix} 0 \\ y \end{pmatrix} \tag{1.17}$$

$$\begin{pmatrix} x & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 4y \ (\textbf{or}) \ \begin{pmatrix} x & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \tag{1.18}$$

Conclusion

$$x^2 = 4y \;\; \text{(or)} \;\; x^2 = 0 \implies x = 0 \tag{1.19}$$

Case (1)

$$x^2 = 4y \implies y \geq 0 \tag{1.20}$$

Case (2)

$$x = 0 \tag{1.21}$$

$$\mathbf{n}^\top \mathbf{c} \leq 0 \implies \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq 0 \tag{1.22}$$

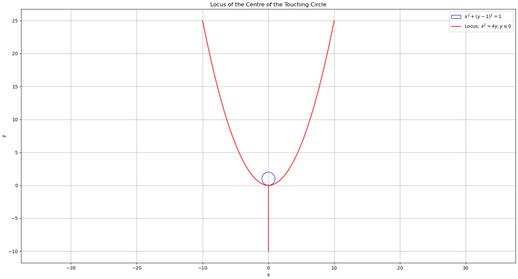
$$y \leq 0 \tag{1.23}$$

Hence from Case (1) and Case (2)

$$\left\{(x, y) : x^2 = 4y\right\} \bigcup \{(x, y) : x = 0 \text{ AND } y \leq 0\} \tag{1.24}$$

$$\left\{(x, y) : x^2 = 4y\right\} \bigcup \{(0, y) : y \leq 0\} \tag{1.25}$$

Plot



Locus of the Centre of the Touching Circle
$x^2 + (y-1)^2 = 1$
Locus: $x^2 = 4y, y \leq 0$
y
x
25
20
15
10
5
0
−5
−10
−30
−20
−10
0
10
20
30

Figure:

C Code

```c
void get_circle_params(double* out_data) {
    out_data[0] = 0.0;
    out_data[1] = 1.0;
    out_data[2] = 1.0;
}
```

Python Code for Solving

```python
import ctypes
import sympy

def find_locus_equation():

    lib = ctypes.CDLL('./code.so')

    double_array_3 = ctypes.c_double * 3
    lib.get_circle_params.argtypes = [ctypes.POINTER(ctypes.c_double)]
    out_data_c = double_array_3()
    lib.get_circle_params(out_data_c)

    c1_x, c1_y, r1 = list(out_data_c)
    c1_center = (c1_x, c1_y)

    h, k = sympy.symbols('h k', real=True)
    r = sympy.Abs(k)
    lhs = (h - c1_center[0])**2 + (k - c1_center[1])**2
```

Python Code for Solving

```
   rhs = (r + r1)**2
    equation = sympy.Eq(lhs, rhs)
    locus_eq = sympy.simplify(equation.lhs - equation.rhs)
    x, y = sympy.symbols('x y')
    final_locus = locus_eq.subs([(h, x), (k, y)])

    return sympy.Eq(final_locus, 0)
```

Python Code for Plotting

```python
# Code by /sdcard/github/matgeo/codes/CoordGeoVV Sharma
# September 12, 2023
# Revised July 21, 2024
# Released under GNU GPL
# Section Formula
import sys
sys.path.insert(0, '/workspaces/urban-potato/matgeo/codes/CoordGeo/')
import numpy as np
import matplotlib.pyplot as plt
from call import find_locus_equation

locus_equation = find_locus_equation()

print(fLocus equation: {locus_equation})
fig, ax = plt.subplots(figsize=(8, 8))

c1 = plt.Circle((0, 1), 1, color='blue', fill=False, label='$x^2 + (y-1)^2 = 1$')
```

Python Code for Plotting

```
ax.add_patch(c1)

x_vals = np.linspace(-10, 10, 400)
y_vals = x_vals**2 / 4
ax.plot(x_vals, y_vals, 'r-', label=f'Locus: $x^2=4y$')
ax.plot([0, 0], [-10, 0], 'r-')
ax.set_title(Locus of the Centre of the Touching Circle)
ax.set_xlabel(x); ax.set_ylabel(y)
ax.grid(True); ax.axis('equal'); ax.legend()
plt.show()
plt.savefig('fig1.png')
```