Assignment 9: 4.13.59

EE25BTECH11055 - Subhodeep Chakraborty

Question:
Determine all values of α for which the point $\left(\alpha, \alpha^2\right)$ lies inside the triangle formed by the lines. (1992)

$$2x + 3y - 1 = 0 \tag{1}$$
$$x + 2y - 3 = 0 \tag{2}$$
$$5x - 6y - 1 = 0 \tag{3}$$

Solution:
Given:

$$\mathbf{n_1}^\top \mathbf{x} = c_1 \qquad \mathbf{n_1} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} c_1 = 1 \tag{4}$$
$$\mathbf{n_2}^\top \mathbf{x} = c_2 \qquad \mathbf{n_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} c_2 = 3 \tag{5}$$
$$\mathbf{n_3}^\top \mathbf{x} = c_3 \qquad \mathbf{n_3} = \begin{pmatrix} 5 \\ -6 \end{pmatrix} c_3 = 1 \tag{6}$$
$$\mathbf{P} = \begin{pmatrix} \alpha \\ \alpha^2 \end{pmatrix} \tag{7}$$

For finding vertices:

$$\begin{pmatrix} n_1 & n_2 \end{pmatrix}^\top \mathbf{V_3} = \begin{pmatrix} c1 \\ c2 \end{pmatrix} \tag{8}$$
$$\begin{pmatrix} n_3 & n_1 \end{pmatrix}^\top \mathbf{V_2} = \begin{pmatrix} c3 \\ c1 \end{pmatrix} \tag{9}$$
$$\begin{pmatrix} n_2 & n_3 \end{pmatrix}^\top \mathbf{V_1} = \begin{pmatrix} c2 \\ c3 \end{pmatrix} \tag{10}$$

Let us define $d_i = \mathbf{n_i}^\top \mathbf{V_i} - c_i$ as the sign denoting which side of the line the vertex opposite to it lies on. Also define matrix $\mathbf{D} = \text{diag}\left(d_1, d_2, d_3\right)$
For point to lie inside triangle, we need $d_i \cdot (\mathbf{n_i}^\top \mathbf{P} - c_i) > 0$. In matrix form, this is written as:

$$\mathbf{D} = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \tag{11}$$
$$\mathbf{D} \begin{pmatrix} \mathbf{n_1}^\top \mathbf{P} - c_1 \\ \mathbf{n_2}^\top \mathbf{P} - c_2 \\ \mathbf{n_3}^\top \mathbf{P} - c_3 \end{pmatrix} > \mathbf{0} \tag{12}$$

Let

$$\mathbf{N} = \begin{pmatrix} n_1 & n_2 & n_3 \end{pmatrix}^\top \tag{13}$$
$$\mathbf{C} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \tag{14}$$

Thus representing everything in terms of matrices,

$$\mathbf{D}\left(\mathbf{N}\mathbf{P}-\mathbf{C}\right) > \mathbf{0} \tag{15}$$

is the required inequality. On substituting values,
First, we find the vertices of the triangle using Gaussian elimination:

$$\mathbf{V_1} : \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 5 & -6 & 1 \end{array}\right) \xrightarrow{R_2 \to R_2 - 5R_1} \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -16 & -14 \end{array}\right) \implies \mathbf{V_1} = \begin{pmatrix} 5/4 \\ 7/8 \end{pmatrix} \tag{16}$$

$$\mathbf{V_2} : \left(\begin{array}{cc|c} 2 & 3 & 1 \\ 5 & -6 & 1 \end{array}\right) \xrightarrow{R_2 \to R_2 - \frac{5}{2}R_1} \left(\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -27/2 & -3/2 \end{array}\right) \implies \mathbf{V_2} = \begin{pmatrix} 1/3 \\ 1/9 \end{pmatrix} \tag{17}$$

$$\mathbf{V_3} : \left(\begin{array}{cc|c} 2 & 3 & 1 \\ 1 & 2 & 3 \end{array}\right) \xrightarrow{R_2 \to R_2 - \frac{1}{2}R_1} \left(\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \end{array}\right) \implies \mathbf{V_3} = \begin{pmatrix} -7 \\ 5 \end{pmatrix} \tag{18}$$

Next, we determine the signs $d_i = \mathbf{n_i}^\top \mathbf{V_i} - c_i$ for each line evaluated at its opposite vertex:

$$d_1 = \mathbf{n_1}^\top \mathbf{V_1} - c_1 = 2(5/4) + 3(7/8) - 1 = 33/8 \tag{19}$$

$$d_2 = \mathbf{n_2}^\top \mathbf{V_2} - c_2 = (1/3) + 2(1/9) - 3 = -22/9 \tag{20}$$

$$d_3 = \mathbf{n_3}^\top \mathbf{V_3} - c_3 = 5(-7) - 6(5) - 1 = -66 \tag{21}$$

For the point $\mathbf{P} = \begin{pmatrix} \alpha \\ \alpha^2 \end{pmatrix}$ to be inside, the condition $\mathbf{D}(\mathbf{N}\mathbf{P}-\mathbf{C}) > \mathbf{0}$ must hold.

$$\mathbf{N}\mathbf{P}-\mathbf{C} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \\ 5 & -6 \end{pmatrix} \begin{pmatrix} \alpha \\ \alpha^2 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3\alpha^2 + 2\alpha - 1 \\ 2\alpha^2 + \alpha - 3 \\ -6\alpha^2 + 5\alpha - 1 \end{pmatrix} \tag{22}$$

Multiplying by the diagonal matrix $\mathbf{D}$:

$$\mathbf{D}(\mathbf{N}\mathbf{P}-\mathbf{C}) = \begin{pmatrix} 33/8 & 0 & 0 \\ 0 & -22/9 & 0 \\ 0 & 0 & -66 \end{pmatrix} \begin{pmatrix} 3\alpha^2 + 2\alpha - 1 \\ 2\alpha^2 + \alpha - 3 \\ -6\alpha^2 + 5\alpha - 1 \end{pmatrix} \tag{23}$$

$$= \begin{pmatrix} (33/8)(3\alpha^2 + 2\alpha - 1) \\ (-22/9)(2\alpha^2 + \alpha - 3) \\ (-66)(-6\alpha^2 + 5\alpha - 1) \end{pmatrix} > \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{24}$$

This yields the system of inequalities:

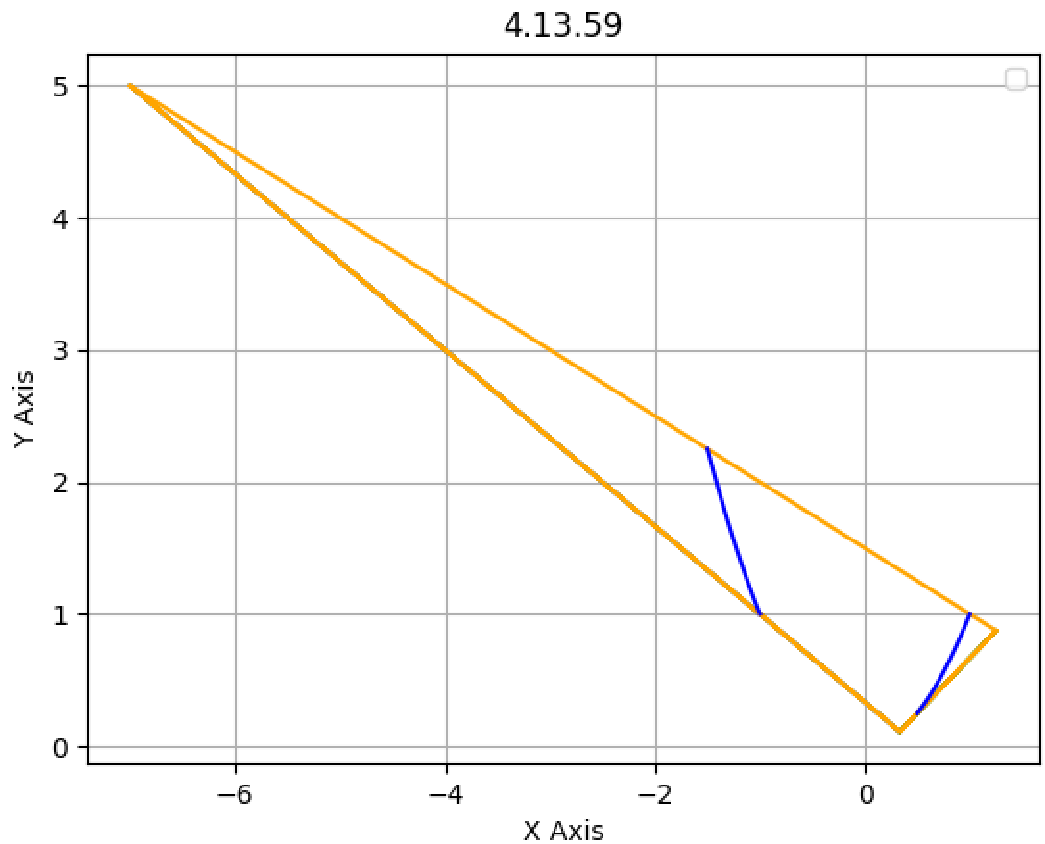
$$3\alpha^2 + 2\alpha - 1 > 0 \implies \alpha \in (-\infty, -1) \cup (1/3, \infty) \tag{25}$$

$$2\alpha^2 + \alpha - 3 < 0 \implies \alpha \in (-3/2, 1) \tag{26}$$

$$6\alpha^2 - 5\alpha + 1 > 0 \implies \alpha \in (-\infty, 1/3) \cup (1/2, \infty) \tag{27}$$

The value of α must satisfy all three conditions. Taking the intersection of the solution sets:

$$\alpha \in (-3/2, -1) \cup (1/2, 1) \tag{28}$$


4.13.59
5
4
3
2
1
0
Y Axis
−6
−4
−2
0
X Axis