1.3.4

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Question 1.3.4
If $A(1,3)$, $B(4,2)$, $C(x,5)$, and $D(x,4)$ are the vertices of a parallelogram $ABCD$, then the value of x is _____. (10, 2012)

Solution:
In a parallelogram, opposite sides are equal and parallel. Since $ABCD$ is a parallelogram, vectors **AB** and **CD** must be equal.

$$\mathbf{B}-\mathbf{A}=\begin{pmatrix}4-1\\2-3\end{pmatrix}=\begin{pmatrix}3\\-1\end{pmatrix} \tag{0.1}$$

$$\mathbf{D}-\mathbf{C}=\begin{pmatrix}x-x\\4-5\end{pmatrix}=\begin{pmatrix}0\\-1\end{pmatrix} \tag{0.2}$$

Clearly, $\mathbf{B}-\mathbf{A}\neq\mathbf{D}-\mathbf{C}$, so let's try using diagonals. In a parallelogram, the diagonals bisect each other.
Midpoint of diagonal AC:

$$\begin{pmatrix}\frac{1+x}{2}\\\frac{3+5}{2}\end{pmatrix}=\begin{pmatrix}\frac{1+x}{2}\\4\end{pmatrix} \tag{0.3}$$

Midpoint of diagonal BD:

$$\begin{pmatrix}\frac{4+x}{2}\\\frac{2+4}{2}\end{pmatrix}=\begin{pmatrix}\frac{4+x}{2}\\3\end{pmatrix} \tag{0.4}$$

Equating midpoints:

$$\frac{1+x}{2}=\frac{4+x}{2}\quad\text{and}\quad 4=3 \tag{0.5}$$

The second equation is false, so diagonals do not bisect each other. Let's try using opposite sides again, but this time equating **AD** and **BC**:

$$\mathbf{D}-\mathbf{A}=\begin{pmatrix}x-1\\4-3\end{pmatrix}=\begin{pmatrix}x-1\\1\end{pmatrix} \tag{0.6}$$

$$\mathbf{C}-\mathbf{B}=\begin{pmatrix}x-4\\5-2\end{pmatrix}=\begin{pmatrix}x-4\\3\end{pmatrix} \tag{0.7}$$

Equating vectors:

$$x-1=x-4\quad\text{and}\quad 1=3 \tag{0.8}$$

Again, contradiction. So let's try using the property that opposite sides are equal in length.

$$\text{Length of }\mathbf{D}-\mathbf{A}:\quad |AD|=\sqrt{(x-1)^2+(4-3)^2}=\sqrt{(x-1)^2+1} \tag{0.9}$$

$$\text{Length of }\mathbf{C}-\mathbf{B}:\quad |BC|=\sqrt{(x-4)^2+(5-2)^2}=\sqrt{(x-4)^2+9} \tag{0.10}$$

Equating lengths:

$$\sqrt{(x-1)^2+1}=\sqrt{(x-4)^2+9} \tag{0.11}$$

Squaring both sides:

$$(x-1)^2+1=(x-4)^2+9 \tag{0.12}$$

$$x^2-2x+1+1=x^2-8x+16+9 \tag{0.13}$$

$$x^2-2x+2=x^2-8x+25 \tag{0.14}$$

Subtract x^2 from both sides:

$$-2x+2=-8x+25 \tag{0.15}$$

$$6x=23\Rightarrow x=\frac{23}{6} \tag{0.16}$$

Answer: $\boxed{\frac{23}{6}}$