4.12.45

AI25BTECH110031
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Question(4.12.45) Find the equation of the set of points P the sum of whose distances from $A(4,0,0)$ and $B(-4,0,0)$ is equal to 10.

Solution:

We want the locus of points $\mathbf{p} \in \mathbb{R}^3$ such that

$$\|\mathbf{p} - \mathbf{A}\| + \|\mathbf{p} - \mathbf{B}\| = 10, \tag{0.1}$$

where

$$\mathbf{A} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix}. \tag{0.2}$$

Step 1 - Decomposition of p:

Define the unit vector along the foci axis:

$$\mathbf{e} = \frac{\mathbf{A} - \mathbf{B}}{\|\mathbf{A} - \mathbf{B}\|} = \frac{\begin{pmatrix} 8 \\ 0 \\ 0 \end{pmatrix}}{8} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{0.3}$$

Decompose $\mathbf{p}$ into parallel and perpendicular components:

$$\mathbf{p} = (\mathbf{e}^\top \mathbf{p})\, \mathbf{e} + (I - \mathbf{e}\mathbf{e}^\top)\mathbf{p}. \tag{0.4}$$

Let:

$$\alpha := \mathbf{e}^\top \mathbf{p}, \quad R := I - \mathbf{e}\mathbf{e}^\top. \tag{0.5}$$

Then the perpendicular squared component is:

$$s := \|R\mathbf{p}\|^2. \tag{0.6}$$

Step 2 - Distances to the foci:

$$\|\mathbf{p} - \mathbf{A}\| = \sqrt{(\alpha - 4)^2 + s}, \quad \|\mathbf{p} - \mathbf{B}\| = \sqrt{(\alpha + 4)^2 + s}. \tag{0.7}$$

The condition becomes:

$$\sqrt{(\alpha - 4)^2 + s} + \sqrt{(\alpha + 4)^2 + s} = 10. \tag{0.8}$$

Step 3 - Eliminate square roots:

Square both sides and rearrange to eliminate the radicals. After algebraic manipulation, we obtain:

$$-36\,\alpha^2 - 100\,s + 900 = 0. \tag{0.9}$$

The equation becomes:

$$-36\,\mathbf{p}^\top(\mathbf{e}\mathbf{e}^\top)\mathbf{p} - 100\,\mathbf{p}^\top R\mathbf{p} + 900 = 0. \tag{0.10}$$

$$\mathbf{p}^\top\left(-36\,\mathbf{e}\mathbf{e}^\top - 100\,R\right)\mathbf{p} + 900 = 0. \tag{0.11}$$

Step 4 - Simplify using $R = I - \mathbf{e}\mathbf{e}^\top$:

$$-36\,\mathbf{e}\mathbf{e}^\top - 100(I - \mathbf{e}\mathbf{e}^\top) = -100I + 64\,\mathbf{e}\mathbf{e}^\top. \tag{0.12}$$

Thus:

$$\mathbf{p}^\top\left(I - \frac{64}{100}\mathbf{e}\mathbf{e}^\top\right)\mathbf{p} = 9. \tag{0.13}$$

$$\boxed{\mathbf{p}^\top\begin{pmatrix}\frac{1}{25} & 0 & 0\\ 0 & \frac{1}{9} & 0\\ 0 & 0 & \frac{1}{9}\end{pmatrix}\mathbf{p} = 1,} \tag{0.14}$$

which is the equation of a prolate spheroid with semi-axes $5, 3, 3$.

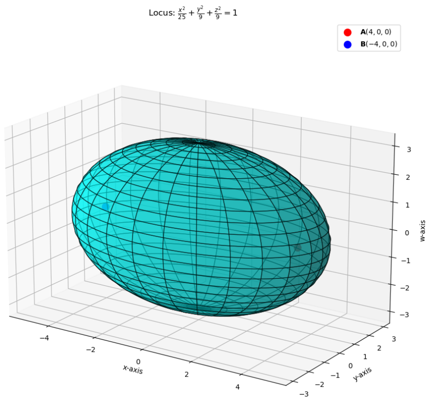

Fig. 0.1