7.4.26

EE25BTECH11019 - Darji Vivek M.

Question:

If two distinct chords, drawn from the point (p, q) on the circle

$$x^2 + y^2 = px + qy$$

(where $pq \neq 0$) are bisected by the X-axis, then which of the following is true?

1) $p^2 = q^2$
2) $p^2 = 8q^2$
3) $p^2 < 8q^2$
4) $p^2 > 8q^2$

Solution:

Let

$$\mathbf{P} = \begin{pmatrix} p \\ q \end{pmatrix}, \qquad \mathbf{c} = \frac{1}{2}\mathbf{P}, \qquad r = \frac{1}{2}\sqrt{\mathbf{P}^\top \mathbf{P}}. \tag{1}$$

(So the circle in translated form is $\|\mathbf{x} - \mathbf{c}\| = r$.)

Let the midpoint of a chord through $\mathbf{P}$ lying on the x-axis be

$$\mathbf{M} = \begin{pmatrix} h \\ 0 \end{pmatrix}, \tag{2}$$

and the other end of the chord be

$$\mathbf{B} = 2\mathbf{M} - \mathbf{P}. \tag{3}$$

Since $\mathbf{B}$ lies on the circle,

$$(\mathbf{B} - \mathbf{c})^\top (\mathbf{B} - \mathbf{c}) = r^2. \tag{4}$$

Substitute $\mathbf{B} = 2\mathbf{M} - \mathbf{P}$ and $\mathbf{c} = \frac{1}{2}\mathbf{P}$:

$$\left(2\mathbf{M} - \frac{3}{2}\mathbf{P}\right)^\top \left(2\mathbf{M} - \frac{3}{2}\mathbf{P}\right) = \frac{1}{4}\mathbf{P}^\top \mathbf{P}. \tag{5}$$

Expand and simplify:

$$4\mathbf{M}^\top \mathbf{M} - 6\mathbf{M}^\top \mathbf{P} + \frac{9}{4}\mathbf{P}^\top \mathbf{P} = \frac{1}{4}\mathbf{P}^\top \mathbf{P} \tag{6}$$

$$4\mathbf{M}^\top \mathbf{M} - 6\mathbf{M}^\top \mathbf{P} + 2\mathbf{P}^\top \mathbf{P} = 0. \tag{7}$$

With $\mathbf{M} = \begin{pmatrix} h \\ 0 \end{pmatrix}$ we have $\mathbf{M}^\top \mathbf{M} = h^2$ and $\mathbf{M}^\top \mathbf{P} = ph$. Thus

$$4h^2 - 6ph + 2\mathbf{P}^\top \mathbf{P} = 0.$$

Divide by 2:

$$2h^2 - 3ph + \mathbf{P}^\top \mathbf{P} = 0.$$

This quadratic in h must have two distinct real roots (two distinct chords), so its discriminant is positive:

$$\Delta = 9p^2 - 8\mathbf{P}^\top\mathbf{P} > 0 \tag{8}$$

$$= 9p^2 - 8(p^2 + q^2) > 0 \tag{9}$$

$$\implies p^2 - 8q^2 > 0. \tag{10}$$

Hence

$$\boxed{p^2 > 8q^2}$$

(option (d)).

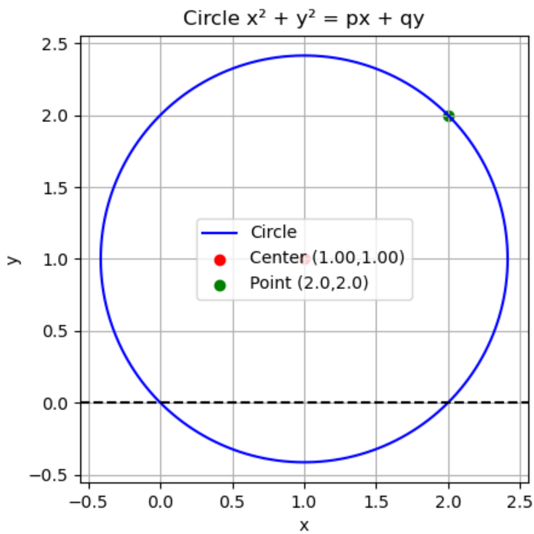

Fig. 4.1: plot if p=2,q=2