4.11.5

INDHIRESH S- EE25BTECH11027

Question. Find the equation of the plane passing through the points $(2, 5, -3)$, $(-2, -3, 5)$, and $(5, 3, -3)$. Also, find the point of intersection of this plane with the line passing through points $(3, 1, 5)$ and $(-1, -3, -1)$.
Solution:
Let us solve the given equation theoretically and then verify the solution computationally.
Let the given points be:

$$\mathbf{A} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -2 \\ -3 \\ 5 \end{pmatrix} \; and \; \mathbf{C} = \begin{pmatrix} 5 \\ 3 \\ -3 \end{pmatrix} \tag{1}$$

For equation of plane:

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}^T \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{2}$$

$$\begin{pmatrix} 2 & -2 & 5 \\ 5 & -3 & 3 \\ -3 & 5 & -3 \end{pmatrix}^T \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{3}$$

$$\begin{pmatrix} 2 & 5 & -3 \\ -2 & -3 & 5 \\ 5 & 3 & -3 \end{pmatrix} \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \left(\begin{array}{ccc|c} 2 & 5 & -3 & 1 \\ -2 & -3 & 5 & 1 \\ 5 & 3 & -3 & 1 \end{array}\right) \tag{4}$$

$$\left(\begin{array}{ccc|c} 2 & 5 & -3 & 1 \\ -2 & -3 & 5 & 1 \\ 5 & 3 & -3 & 1 \end{array}\right) \xleftrightarrow[R_2 \leftarrow R_2 + R_1]{R_3 \leftarrow R_3 - \frac{5}{2}R_1} \left(\begin{array}{ccc|c} 2 & 5 & -3 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & \frac{-19}{2} & \frac{9}{2} & \frac{-3}{2} \end{array}\right) \tag{5}$$

$$\left(\begin{array}{ccc|c} 2 & 5 & -3 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & \frac{-19}{2} & \frac{9}{2} & \frac{-3}{2} \end{array}\right) \xleftrightarrow{R_2 \leftarrow \frac{1}{2}R_2} \left(\begin{array}{ccc|c} 2 & 5 & -3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & \frac{-19}{2} & \frac{9}{2} & \frac{-3}{2} \end{array}\right) \tag{6}$$

$$\left(\begin{array}{ccc|c} 2 & 5 & -3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & \frac{-19}{2} & \frac{9}{2} & \frac{-3}{2} \end{array}\right) \xleftrightarrow{R_3 \leftarrow R_3 + \frac{19}{2}R_2} \left(\begin{array}{ccc|c} 2 & 5 & -3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 14 & 8 \end{array}\right) \tag{7}$$

$$\left(\begin{array}{ccc|c}2 & 5 & -3 & 1\\0 & 1 & 1 & 1\\0 & 0 & 14 & 8\end{array}\right) \xleftrightarrow{R_3\leftarrow\frac{R_3}{14}} \left(\begin{array}{ccc|c}2 & 5 & -3 & 1\\0 & 1 & 1 & 1\\0 & 0 & 1 & \frac{4}{7}\end{array}\right) \tag{8}$$

$$\left(\begin{array}{ccc|c}2 & 5 & -3 & 1\\0 & 1 & 1 & 1\\0 & 0 & 1 & \frac{4}{7}\end{array}\right) \xleftrightarrow[R_2\leftarrow R_2-R_3]{R_1\leftarrow R_1+3R_3} \left(\begin{array}{ccc|c}2 & 5 & 0 & \frac{19}{7}\\0 & 1 & 0 & \frac{3}{7}\\0 & 0 & 1 & \frac{4}{7}\end{array}\right) \tag{9}$$

$$\left(\begin{array}{ccc|c}2 & 5 & 0 & \frac{19}{7}\\0 & 1 & 0 & \frac{3}{7}\\0 & 0 & 1 & \frac{4}{7}\end{array}\right) \xleftrightarrow{R_1\leftarrow R_1-5R_2} \left(\begin{array}{ccc|c}2 & 0 & 0 & \frac{4}{7}\\0 & 1 & 0 & \frac{3}{7}\\0 & 0 & 1 & \frac{4}{7}\end{array}\right) \tag{10}$$

$$\left(\begin{array}{ccc|c}2 & 0 & 0 & \frac{4}{7}\\0 & 1 & 0 & \frac{3}{7}\\0 & 0 & 1 & \frac{4}{7}\end{array}\right) \xleftrightarrow{R_1\leftarrow\frac{R_1}{2}} \left(\begin{array}{ccc|c}1 & 0 & 0 & \frac{2}{7}\\0 & 1 & 0 & \frac{3}{7}\\0 & 0 & 1 & \frac{4}{7}\end{array}\right) \tag{11}$$

The equation of plane can be given as:

$$\mathbf{n}^T\mathbf{x} = 1 \tag{12}$$

Therefore the equation of plane can be given as:

$$\begin{pmatrix}2\\3\\4\end{pmatrix}^T \mathbf{x} = 7 \tag{13}$$

Now let us find the equation of line passing through the points $(3, 1, 5)$ and $(-1, -3, -1)$

$$\mathbf{x} = \mathbf{h} + k\mathbf{m} \tag{14}$$

$$\mathbf{m} = \begin{pmatrix}3-(-1)\\1-(-3)\\5-(-1)\end{pmatrix} \tag{15}$$

$$\mathbf{m} = \begin{pmatrix}4\\4\\6\end{pmatrix} \tag{16}$$

$$\mathbf{x} = \begin{pmatrix}3\\1\\5\end{pmatrix} + k\begin{pmatrix}4\\4\\6\end{pmatrix} \tag{17}$$

Now substitute Eq.22 in Eq.18:

$$\begin{pmatrix}2\\3\\4\end{pmatrix}^T \left(\begin{pmatrix}3\\1\\5\end{pmatrix} + k\begin{pmatrix}4\\4\\6\end{pmatrix}\right) = 7 \tag{18}$$

$$\begin{pmatrix}2\\3\\4\end{pmatrix}^T \begin{pmatrix}3+4k\\1+4k\\5+6k\end{pmatrix} = 7 \tag{19}$$

Solving this we get

$$k = \frac{-1}{2} \tag{20}$$

Now substitute the value of k in Eq.22

$$\mathbf{x} = \begin{pmatrix}3-2\\1-2\\5-3\end{pmatrix} \tag{21}$$

Therefore the point of intersection is:

$$\mathbf{x} = \begin{pmatrix}1\\-1\\2\end{pmatrix} \tag{22}$$

From the figure it is clearly verified that the theoretical solution matches with the computational solution.