12.145

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Question. If a square matrix $\mathbf{A}$ is real and symmetric, then the eigenvalues

1) are always real
2) are always real and positive
3) are always real and non-negative
4) occur in complex conjupairs

Solution:
The correct statement is (1). This is a fundamental property of real symmetric matrices. Let $\mathbf{A}$ be a real and symmetric matrix, which means

$$\mathbf{A} = \mathbf{A}^T \ \ and \ \ \bar{\mathbf{A}} = \mathbf{A} \tag{1}$$

Let λ be an eigenvalue of $\mathbf{A}$ with a corresponding non-zero eigenvector $\mathbf{x}$. The eigenvalue equation is:

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \tag{2}$$

To prove that λ is real, we must show it is equal to its own complex conjugate, i.e., $\lambda = \bar{\lambda}$. We take the conjugate transpose (Hermitian conjugate) on both sides:

$$(\mathbf{A}\mathbf{x})^H = (\lambda\mathbf{x})^H \tag{3}$$

$$\mathbf{x}^H\mathbf{A}^H = \bar{\lambda}\mathbf{x}^H \tag{4}$$

For a real and symmetric matrix, its conjugate transpose is itself:

$$\mathbf{A}^H = \bar{\mathbf{A}}^T = \mathbf{A}^T = \mathbf{A} \tag{5}$$

Substituting this into the Eq.4 gives:

$$\mathbf{x}^H\mathbf{A} = \bar{\lambda}\mathbf{x}^H \tag{6}$$

Now, we pre-multiply the Eq.2 by $\mathbf{x}^H$:

$$\mathbf{x}^H\mathbf{A}\mathbf{x} = \lambda(\mathbf{x}^H\mathbf{x}) \tag{7}$$

And we post-multiply Eq.6 by $\mathbf{x}$:

$$\mathbf{x}^H\mathbf{A}\mathbf{x} = \bar{\lambda}(\mathbf{x}^H\mathbf{x}) \tag{8}$$

By comparing Eq.7 and Eq.8 , we see that:

$$\lambda(\mathbf{x}^H\mathbf{x}) = \bar{\lambda}(\mathbf{x}^H\mathbf{x}) \tag{9}$$

This can be rearranged to:

$$(\lambda - \bar{\lambda})(\mathbf{x}^H\mathbf{x}) = 0 \tag{10}$$

Since an eigenvector $\mathbf{x}$ is non-zero by definition, its magnitude $\|\mathbf{x}\|^2$ is a positive real number.So,

$$\lambda - \bar{\lambda} = 0 \tag{11}$$

$$\lambda = \bar{\lambda} \tag{12}$$

A number that is equal to its own complex conjugate must be a real number.
From above statement it is clear that eigenvalues are always real

Options (2) and (3) are incorrect because a real symmetric matrix can have negative eigenvalues.
Example:
Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{13}$$

This matrix is real and symmetric.Now finding the eigen value for the matrix:

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \tag{14}$$

The eigenvalues are

$$\lambda = 1 \;\; and \;\; \lambda = -1 \tag{15}$$

Therefore the eigenvalues can be negative
Option (1) is correct