2.8.24

EE25BTECH11042 - Nipun Dasari

Question:
The $\mathbf{a}+\mathbf{b}$ bisects the angle between $\mathbf{a}$ and $\mathbf{b}$ if ___________
Solution:
Theorem: The $\mathbf{a}+\mathbf{b}$ bisects the angle between $\mathbf{a}$ and $\mathbf{b}$ if and only if $\|\mathbf{a}\| = \|\mathbf{b}\|$
We prove the above in two parts:
Assume a $\mathbf{c}$ such that

$$\mathbf{c} = \mathbf{a} + \mathbf{b} \tag{0.1}$$

Let α and β be angles made by $\mathbf{c}$ with $\mathbf{a}$ and $\mathbf{b}$ respectively.
Part 1
Given :

$$\|\mathbf{a}\| = \|\mathbf{b}\| \tag{0.2}$$

To prove : $\mathbf{a}+\mathbf{b}$ bisects the angle between $\mathbf{a}$ and $\mathbf{b}$
Proof:
The angle θ between $\mathbf{p}$ and $\mathbf{q}$ is given by:

$$\cos\theta = \frac{\mathbf{p}^\top \mathbf{q}}{\|\mathbf{p}\| \|\mathbf{q}\|} \tag{0.3}$$

By (0.3) and (0.1)

$$\implies \cos\alpha = \frac{\mathbf{a}^\top (\mathbf{a}+\mathbf{b})}{\|\mathbf{a}\| \|\mathbf{a}+\mathbf{b}\|} \tag{0.4}$$

$$\implies \cos\beta = \frac{\mathbf{b}^\top (\mathbf{a}+\mathbf{b})}{\|\mathbf{b}\| \|\mathbf{a}+\mathbf{b}\|} \tag{0.5}$$

By (0.2)

$$\mathbf{a}^\top\mathbf{a} = \mathbf{b}^\top\mathbf{b} \tag{0.6}$$

$$\mathbf{a}^\top\mathbf{a} + \mathbf{a}^\top\mathbf{b} = \mathbf{b}^\top\mathbf{b} + \mathbf{b}^\top\mathbf{a} \tag{0.7}$$

$$\frac{\mathbf{a}^\top\mathbf{a} + \mathbf{a}^\top\mathbf{b}}{\|\mathbf{a}\| \|\mathbf{a}+\mathbf{b}\|} = \frac{\mathbf{b}^\top\mathbf{b} + \mathbf{b}^\top\mathbf{a}}{\|\mathbf{b}\| \|\mathbf{a}+\mathbf{b}\|} \tag{0.8}$$

$$\therefore \cos\alpha = \cos\beta \tag{0.9}$$

$$\therefore \alpha = \beta \tag{0.10}$$

Part 2
Given:

$$\alpha = \beta \tag{0.11}$$

To prove:

$$\|\mathbf{a}\| = \|\mathbf{b}\| \tag{0.12}$$

Proof:
By (0.9)

$$\cos\alpha = \cos\beta \tag{0.13}$$

$$\frac{\mathbf{a}^\top(\mathbf{a}+\mathbf{b})}{\|\mathbf{a}\|\,\|\mathbf{a}+\mathbf{b}\|} = \frac{\mathbf{b}^\top(\mathbf{a}+\mathbf{b})}{\|\mathbf{b}\|\,\|\mathbf{a}+\mathbf{b}\|} \tag{0.14}$$

$$\|\mathbf{b}\|\left(\|\mathbf{a}\|^2 + \mathbf{a}^\top\mathbf{b}\right) = \|\mathbf{a}\|\left(\|\mathbf{b}\|^2 + \mathbf{a}^\top\mathbf{b}\right) \tag{0.15}$$

$$\|\mathbf{b}\|\,\|\mathbf{a}\|^2 + \|\mathbf{b}\|\left(\mathbf{a}^\top\mathbf{b}\right) = \|\mathbf{a}\|\,\|\mathbf{b}\|^2 + \|\mathbf{a}\|\left(\mathbf{a}^\top\mathbf{b}\right) \tag{0.16}$$

Rearrange the terms to group common factors:

$$\|\mathbf{b}\|\,\|\mathbf{a}\|^2 - \|\mathbf{a}\|\,\|\mathbf{b}\|^2 = \|\mathbf{a}\|\left(\mathbf{a}^\top\mathbf{b}\right) - \|\mathbf{b}\|\left(\mathbf{a}^\top\mathbf{b}\right) \tag{0.17}$$

$$\|\mathbf{a}\|\,\|\mathbf{b}\|\left(\|\mathbf{a}\| - \|\mathbf{b}\|\right) = (\mathbf{a}^\top\mathbf{b})(\|\mathbf{a}\| - \|\mathbf{b}\|) \tag{0.18}$$

$$(\|\mathbf{a}\| - \|\mathbf{b}\|)(\|\mathbf{a}\|\,\|\mathbf{b}\| - \mathbf{a}^\top\mathbf{b}) = 0 \tag{0.19}$$

This equation gives two possibilities:

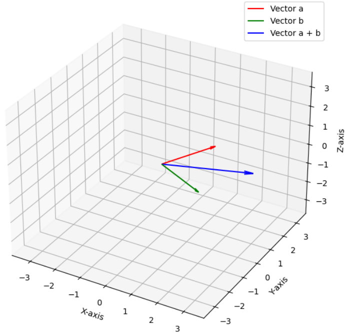
$$\|\mathbf{a}\| - \|\mathbf{b}\| = 0 \implies \|\mathbf{a}\| = \|\mathbf{b}\| \tag{0.20}$$

$$\|\mathbf{a}\|\,\|\mathbf{b}\| - \mathbf{a}^\top\mathbf{b} = 0 \implies \mathbf{a}^\top\mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\| \tag{0.21}$$

(0.21) is incorrect as parallel vectors are not being assumed.
Thus proved

Angle Bisected (Rhombus Case) (a=[1,2,0], b=[2,-1,0])
Magnitudes (from C):
||a|| = 2.236068
||b|| = 2.236068
Angles (deg, from Python):
Angle(a, a+b) = 45.00
Angle(b, a+b) = 45.00
Angle(a, b) = 90.00

Result (from C): Magnitudes are equal (within EPSILON),
so a+b bisects the angle (alpha ~ beta).



Angle Not Bisected (Parallelogram Case) (a=[3,0,0], b=[1,1,0])
Magnitudes (from C):
||a|| = 3.000000
||b|| = 1.414214
Angles (deg, from Python):
Angle(a, a+b) = 14.04
Angle(b, a+b) = 30.96
Angle(a, b) = 45.00

Result (from C): Magnitudes are NOT equal,
so a+b does NOT bisect the angle.