7.4.34

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Question. Find the equation of the circle whose radius is 5 and which touches the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ at the point $(5, 5)$.
Solution:
Let us solve the given equation theoretically and then verify the solution computationally. The general circle equation can be given as:

$$\|\mathbf{x}\|^2 + 2\mathbf{u}^{\mathbf{T}}\mathbf{x} + f = 0 \tag{1}$$

Let the equation of the first circle be

$$\|\mathbf{x}\|^2 + 2\mathbf{u_1^T}\mathbf{x} + f_1 = 0 \tag{2}$$

The equation of the second circle be:

$$\|\mathbf{x}\|^2 + 2\mathbf{u_2^T}\mathbf{x} + f_2 = 0 \tag{3}$$

From the given information:

$$\mathbf{u_1} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} \ \ and \ \ f_1 = -20 \tag{4}$$

Let $\mathbf{c_1}$ and $\mathbf{c_2}$ be the centre of the circle 1 and circle 2:

$$\mathbf{c_1} = -\mathbf{u_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{5}$$

And also

$$f_1 = \|\mathbf{u_1}\|^2 - r_1 \tag{6}$$

$$r_1 = 5 \tag{7}$$

Let P be the point of contact

$$\mathbf{P} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} \tag{8}$$

If the two circle touch each other externally at $\mathbf{P}$. So the $\mathbf{P}, \mathbf{c_1}$ and $\mathbf{c_2}$ will be collinear and $\mathbf{P}$ will divide the points $\mathbf{c_1}$ and $\mathbf{c_2}$ in the ratio $\frac{r_1}{r_2} : 1$

$$\frac{r_1}{r_2} = 1 \tag{9}$$

$$\mathbf{P} = \frac{\mathbf{c_1} + \mathbf{c_2}}{2} \tag{10}$$

$$\begin{pmatrix}5\\5\end{pmatrix} = \frac{\begin{pmatrix}1\\2\end{pmatrix} + \mathbf{c_2}}{2} \tag{11}$$

$$\mathbf{c_2} = \begin{pmatrix}9\\8\end{pmatrix} \tag{12}$$

If two circles touch each other internally we get:

$$\mathbf{c_2} = \begin{pmatrix}1\\2\end{pmatrix} \tag{13}$$

This is the same as circle 1, so the two circles touch each other externally.
Now

$$r_2 = 5 \;\; and \;\; \mathbf{c_2} = \begin{pmatrix}9\\8\end{pmatrix} \tag{14}$$

$$\mathbf{u_2} = -\mathbf{c_2} = \begin{pmatrix}-9\\-8\end{pmatrix} \tag{15}$$

$$f_2 = ||\mathbf{u_2}||^2 - r_2 \tag{16}$$

$$f_2 = 120 \tag{17}$$

Now the required equation of circle is

$$||\mathbf{x}||^2 + 2\mathbf{u_2^T}\mathbf{x} + f_2 = 0 \tag{18}$$

$$||\mathbf{x}||^2 + 2\begin{pmatrix}-9\\-8\end{pmatrix}^T \mathbf{x} + 120 = 0 \tag{19}$$

From the figure it is clearly verified that the theoretical solution matches with the computational solution.

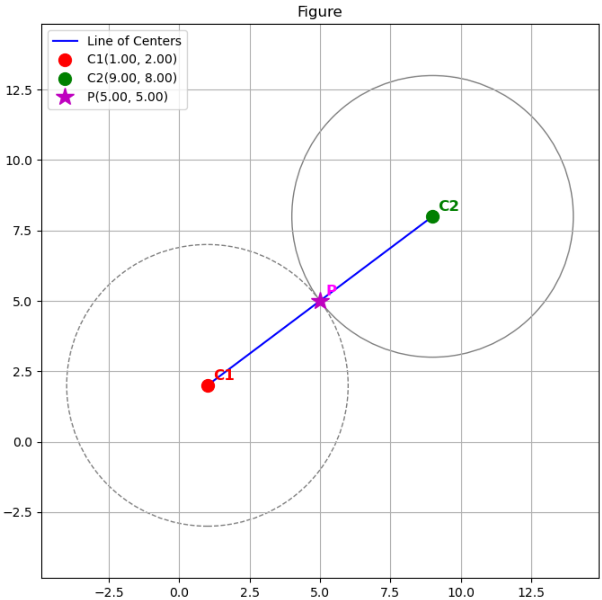
Figure
Line of Centers
C1(1.00, 2.00)
C2(9.00, 8.00)
P(5.00, 5.00)
C1
C2
P
12.5
10.0
7.5
5.0
2.5
0.0
−2.5
−2.5
0.0
2.5
5.0
7.5
10.0
12.5