

7.4.33

EE25BTECH11026-Harsha

Question:

A circle C of radius 1 unit is inscribed in an equilateral triangle PQR . The points of contact of C with sides PQ , QR , RP are D , E , F respectively. The line PQ is given by the equation $\sqrt{3}x + y - 6 = 0$ and the point D is $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$. Further, it is given that the origin and the centre of C are on same side of line PQ . The equation of circle C is

$$1) (x - 2\sqrt{3})^2 + (y - 1)^2 = 1$$

$$3) (x - \sqrt{3})^2 + (y - 1)^2 = 1$$

$$2) (x - 2\sqrt{3})^2 + \left(y + \frac{1}{2}\right)^2 = 1$$

$$4) (x - \sqrt{3})^2 + (y + 1)^2 = 1$$

Solution:

Let us solve the given question theoretically and then verify the solution computationally.

According to the question,

$$\text{Equation of tangent } PQ : \mathbf{n}^T \mathbf{x} = c \quad (4.1)$$

where $\mathbf{n} = \begin{pmatrix} \sqrt{3} & 1 \end{pmatrix}^T$ and $c = 6$

$$\text{Point of tangency (D)} : \begin{pmatrix} \frac{3\sqrt{3}}{2} \\ \frac{3}{2} \end{pmatrix} \quad (4.2)$$

$$\text{radius}(r) = 1 \quad (4.3)$$

From the given data,

$$\mathbf{D} - \mathbf{u} = \lambda \mathbf{n}, \text{ for some scalar } \lambda \quad (4.4)$$

$$\implies \mathbf{u} = \mathbf{D} - \lambda \mathbf{n} \quad (4.5)$$

Also,

$$\frac{|\mathbf{n}^T \mathbf{O} - c|}{\|\mathbf{n}\|} = r \quad (4.6)$$

$$|\mathbf{n}^T \mathbf{u} - c| = r \|\mathbf{n}\| \quad (4.7)$$

$$\mathbf{n}^T \mathbf{u} = c \pm r \|\mathbf{n}\| \quad (4.8)$$

To decide the sign , we need to use the fact that the origin and centre of circle are on the same side of the line PQ.

$$\therefore (\mathbf{n}^\top \mathbf{u} - c) \left(\mathbf{n}^\top \begin{pmatrix} 0 \\ 0 \end{pmatrix} - c \right) > 0 \quad (4.9)$$

$$\implies \mathbf{n}^\top \mathbf{u} < c \quad (4.10)$$

$$\therefore \mathbf{n}^\top \mathbf{u} = c - r\|\mathbf{n}\| \quad (4.11)$$

Substituting value of \mathbf{u} ,

$$\mathbf{n}^\top (\mathbf{D} - \lambda \mathbf{n}) = c - r\|\mathbf{n}\| \quad (4.12)$$

$$\implies \lambda = \frac{\mathbf{n}^\top \mathbf{D} + r\|\mathbf{n}\| - c}{\mathbf{n}^\top \mathbf{n}} \quad (4.13)$$

$$\mathbf{u} = \mathbf{D} - \frac{\mathbf{n}^\top \mathbf{D} + r\|\mathbf{n}\| - c}{\mathbf{n}^\top \mathbf{n}} \mathbf{n} \quad (4.14)$$

Substituting the values,

$$\therefore \mathbf{u} = \begin{pmatrix} \frac{3\sqrt{3}}{2} \\ \frac{3}{2} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \quad (4.15)$$

$$\therefore \text{Required equation of circle : } \|\mathbf{x}\|^2 - 2\left(\sqrt{3} \quad 1\right)\mathbf{x} + 3 = 0 \quad (4.16)$$

From the figure, it is clearly verified that the theoretical solution matches with the computational solution.

