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Question : Let $S = \{\mathbf{A} = \begin{pmatrix} 0 & 1 & c \\ 1 & a & d \\ 1 & b & e \end{pmatrix} : a, b, c, d, e \in \{0, 1\}$ and $|\mathbf{A}| \in \{-1, 1\}\}$. Find the number of elements in S.

Solution :

Name	Matrix
A	$\begin{pmatrix} 0 & 1 & c \\ 1 & a & d \\ 1 & b & e \end{pmatrix}$ with $a, b, c, d, e \in \{0, 1\}$

Table : Matrix

Rearranging the rows of **A**

$$\begin{pmatrix} 0 & 1 & c \\ 1 & a & d \\ 1 & b & e \end{pmatrix} \xleftrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 0 & 1 & c \\ 1 & b & e \\ 1 & a & d \end{pmatrix} \xleftrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & b & e \\ 0 & 1 & c \\ 1 & a & d \end{pmatrix} \tag{1}$$

Applyting row operation to **A** to reduce it into Echelon form

$$\begin{pmatrix} 1 & b & e \\ 0 & 1 & c \\ 1 & a & d \end{pmatrix} \xleftrightarrow{R_3 \to R_3 - R_1} \begin{pmatrix} 1 & b & e \\ 0 & 1 & c \\ 0 & a-b & d-e \end{pmatrix} \xleftrightarrow{R_3 \to R_3 - (a-b)R_2} \begin{pmatrix} 1 & \text{b} & \text{e} \\ 0 & 1 & \text{c} \\ 0 & 0 & d-e-c(a-b) \end{pmatrix} \tag{2}$$

Finding the determinant by the first column

$$\left|\mathbf{A}\right| = d - e - c(a - b) \tag{3}$$

Taking cases to find the possibilities of matrix **A**
Case 1 : $\left|\mathbf{A}\right| = 1$

if c = 0
the value of b and a can be 0 or 1.

$$d - e = 1 \tag{4}$$

So,

$$d = 1 \tag{5}$$
$$e = 0 \tag{6}$$

By permutation we get ,

$$2 \times 2 \times 1 \times 1 = 4 \tag{7}$$

if c = 1, we get 4 possibilities

$$d - e - (a - b) = 1 \tag{8}$$

So,

$$\begin{aligned} d &= 1 & e &= 0 \qquad (9) \\ b = a &= 1 & b = a &= 0 \qquad (10) \end{aligned}$$

$$\begin{aligned} a &= 0 & b &= 1 \qquad (11) \\ d = e &= 1 & d = e &= 0 \qquad (12) \end{aligned}$$

Case 2 : $\left|\mathbf{A}\right| = -1$

if c = 0
the value of b and a can be 0 or 1.

$$d - e = -1 \tag{13}$$

So,

$$d = 0 \tag{14}$$
$$e = 1 \tag{15}$$

By permutation we get ,

$$2 \times 2 \times 1 \times 1 = 4 \tag{16}$$

if c = 1, we get 4 possibilities

$$d - e - (a - b) = -1 \tag{17}$$

So,

$$\begin{aligned} d &= 0 & e &= 1 \qquad (18) \\ b = a &= 1 & b = a &= 0 \qquad (19) \end{aligned}$$

$$\begin{aligned} a &= 1 & b &= 0 \qquad (20) \\ d = e &= 1 & d = e &= 0 \qquad (21) \end{aligned}$$

By adding all the possibilities , we get

$$4 + 4 + 4 + 4 = 16 \tag{22}$$

Therefore, the number of elements in $S = 16$.