10.4.3

INDHIRESH S- EE25BTECH11027

Question.The point at which the normal to the curve $y = x + \frac{1}{x}$, $x > 0$ is perpendicular to the line $3x - 4y - 7 = 0$

Solution:

Let us solve the given equation theoretically and then verify the solution computationally. The given curve be rearranged as:

$$x^2 - xy + 1 = 0. \tag{1}$$

This can be expressed in the form:

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2}$$

Where:

$$\mathbf{v} = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad and \quad f = 1 \tag{3}$$

The required direction of normal which is perpendicular to the line $3x - 4y - 7 = 0$

$$\mathbf{m} = \begin{pmatrix} 1 \\ -\frac{4}{3} \end{pmatrix} \tag{4}$$

$$\mathbf{m} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \tag{5}$$

Now the equation of normal to the conic at the point of contact $\mathbf{q}$ is given by:

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^T\mathbf{R}(\mathbf{x} - \mathbf{q}) = 0 \tag{6}$$

In the normal equation $\mathbf{V}\mathbf{q} + \mathbf{u}$ is proportional to the direction vector of the normal.So,

$$\mathbf{V}\mathbf{q} + \mathbf{u} = k\mathbf{m} \tag{7}$$

$$\mathbf{q} = \mathbf{V}^{-1}(k\mathbf{m} - \mathbf{u}) \tag{8}$$

$\mathbf{q}$ lies on the curve.So substituting Eq.8 in Eq.2:

$$(\mathbf{V}^{-1}(k\mathbf{m} - \mathbf{u}))^T\mathbf{V}\mathbf{V}^{-1}(k\mathbf{m} - \mathbf{u}) + 2\mathbf{u}^T\mathbf{V}^{-1}(k\mathbf{m} - \mathbf{u}) + f = 0 \tag{9}$$

$$(\mathbf{V}^{-1}(k\mathbf{m} - \mathbf{u}))^T(k\mathbf{m} - \mathbf{u}) + f = 0 \tag{10}$$

$$\left(\begin{pmatrix} 0 & -2 \\ -2 & -4 \end{pmatrix}\left(k\begin{pmatrix} 3 \\ -4 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)\right)^T \left(k\begin{pmatrix} 3 \\ -4 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) + 1 = 0 \tag{11}$$

$$k^2 \begin{pmatrix} 3 & -4 \end{pmatrix} \begin{pmatrix} 0 & -2 \\ -2 & -4 \end{pmatrix} \begin{pmatrix} 3 \\ -4 \end{pmatrix} + 1 = 0 \tag{12}$$

$$k^2 = \frac{1}{16} \tag{13}$$

$$k = \frac{1}{4} \;\; and \;\; k = -\frac{1}{4} \tag{14}$$

Now substitute the corresponding values in the Eq.8 to get the point

$$\mathbf{q} = \begin{pmatrix} 0 & -2 \\ -2 & -4 \end{pmatrix} \left(k \begin{pmatrix} 3 \\ -4 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) \tag{15}$$

$$\mathbf{q} = k \begin{pmatrix} 0 & -2 \\ -2 & -4 \end{pmatrix} \begin{pmatrix} 3 \\ -4 \end{pmatrix} \tag{16}$$

$$\mathbf{q} = k \begin{pmatrix} 8 \\ 10 \end{pmatrix} \tag{17}$$

When $k = \frac{1}{4}$

$$\mathbf{q} = \begin{pmatrix} 2 \\ \frac{5}{2} \end{pmatrix} \tag{18}$$

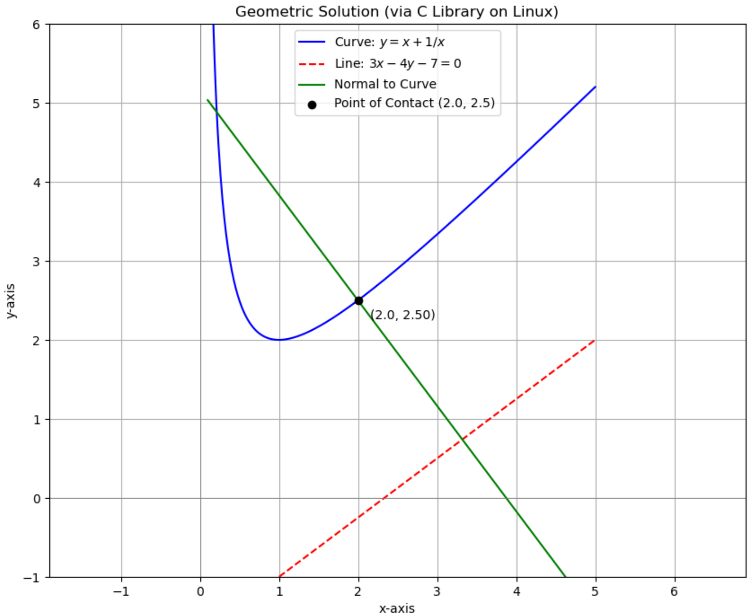
When $k = -\frac{1}{4}$

$$\mathbf{q} = \begin{pmatrix} -2 \\ -\frac{5}{2} \end{pmatrix} \tag{19}$$

Given that $x > 0$. So the point of contact is

$$\mathbf{q} = \begin{pmatrix} 2 \\ \frac{5}{2} \end{pmatrix} \tag{20}$$

From the figure it is clearly verified that the theoretical solution matches with the computational solution.


Geometric Solution (via C Library on Linux)
Curve: $y = x + 1/x$
Line: $3x - 4y - 7 = 0$
Normal to Curve
Point of Contact (2.0, 2.5)
(2.0, 2.50)
x-axis
y-axis