HADAMARD INEQUALITY(FOR DETERMINANT OF ORDER 3)

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$$\left|G\right| = \begin{vmatrix} ||\mathbf{a}||^2 & \mathbf{a}^\top\mathbf{b} & \mathbf{a}^\top\mathbf{c} \\ \mathbf{b}^\top\mathbf{a} & ||\mathbf{b}||^2 & \mathbf{b}^\top\mathbf{c} \\ \mathbf{c}^\top\mathbf{a} & \mathbf{c}^\top\mathbf{b} & ||\mathbf{c}||^2 \end{vmatrix} \tag{1}$$

$$= ||\mathbf{a}||^2\, ||\mathbf{b}||^2\, ||\mathbf{c}||^2 \begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos\gamma \\ \cos\beta & \cos\gamma & 1 \end{vmatrix} \tag{2}$$

$$= ||\mathbf{a}||^2\, ||\mathbf{b}||^2\, ||\mathbf{c}||^2 \left(1 - (\cos\alpha)^2 - (\cos\beta)^2 - (\cos\gamma)^2 + 2\cos\alpha\cos\beta\cos\gamma\right) \tag{3}$$

$$= ||\mathbf{a}||^2\, ||\mathbf{b}||^2\, ||\mathbf{c}||^2 \left(1 - (\cos\alpha)^2 - (\cos\beta)^2 + (\cos\alpha)^2(\cos\beta)^2 - (\cos\alpha)^2(\cos\beta)^2 + (\cos\gamma)^2 - 2\cos\alpha\cos\beta\cos\gamma\right) \tag{4}$$

$$= ||\mathbf{a}||^2\, ||\mathbf{b}||^2\, ||\mathbf{c}||^2 \left(1 - (\cos\alpha)^2 - (\cos\beta)^2\left(1 - (\cos\alpha)^2\right) - (\cos\alpha\cos\beta - \cos\gamma)^2\right) \le ||\mathbf{a}||^2\, ||\mathbf{b}||^2\, ||\mathbf{c}||^2 \tag{5}$$

$$\implies \left(1 - (\cos\alpha)^2 - (\cos\beta)^2\left(1 - (\cos\alpha)^2\right) - (\cos\alpha\cos\beta - \cos\gamma)^2\right) \le 1 \tag{6}$$

$$\implies \left((\cos\alpha)^2 + (\cos\beta)^2\left(1 - (\cos\alpha)^2\right) + (\cos\alpha\cos\beta - \cos\gamma)^2\right) \ge 0 \tag{7}$$

Equality holds when,

$$\cos\alpha = 0, \cos\beta = 0, \cos\alpha\cos\beta - \cos\gamma = 0 \tag{8}$$

Or,

$$\cos\alpha = \cos\beta = \cos\gamma = 0 \tag{9}$$

Hence $\mathbf{a}^\top\mathbf{b} = \mathbf{b}^\top\mathbf{c} = \mathbf{c}^\top\mathbf{a} = 0$ to hold equality.