2.10.44

EE25BTECH11015 - Bhoomika V

Question :-
If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are unit vectors, then

$$||\mathbf{a} - \mathbf{b}||^2 + ||\mathbf{b} - \mathbf{c}||^2 + ||\mathbf{a} - \mathbf{b}||^2$$

does not exceed

a) 4
b) 9
c) 8
d) 6

Solution:
Let

$$x = \mathbf{a} \cdot \mathbf{b}, \quad y = \mathbf{a} \cdot \mathbf{c}, \quad z = \mathbf{b} \cdot \mathbf{c}.$$

Since $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are unit vectors, their Gram matrix is

$$= \begin{pmatrix} \mathbf{a} \cdot \mathbf{a} & \mathbf{a} \cdot \mathbf{b} & \mathbf{a} \cdot \mathbf{c} \\ \mathbf{a} \cdot \mathbf{b} & \mathbf{b} \cdot \mathbf{b} & \mathbf{b} \cdot \mathbf{c} \\ \mathbf{a} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{c} & \mathbf{c} \cdot \mathbf{c} \end{pmatrix}.$$

$$G = \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix}.$$

Now consider

$$(1, 1, 1)\, G\, (1, 1, 1)^T = (\mathbf{a} + \mathbf{b} + \mathbf{c}) \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c}) \geq 0.$$

Expanding,

$$|\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2(x + y + z) = 3 + 2(x + y + z) \geq 0,$$

$$\implies \quad x + y + z \geq -\tfrac{3}{2}. \tag{4.1}$$

Now ,

$$|\mathbf{a} - \mathbf{b}|^2 + |\mathbf{b} - \mathbf{c}|^2 + |\mathbf{c} - \mathbf{a}|^2 = (2 - 2x) + (2 - 2z) + (2 - 2y).$$

So,

$$= 6 - 2(x + y + z).$$

From Equation (4.1)

$$6 - 2(x + y + z) \leq 6 - 2\left(-\tfrac{3}{2}\right) = 9.$$

Thus, $||\mathbf{a} - \mathbf{b}||^2 + ||\mathbf{b} - \mathbf{c}||^2 + ||\mathbf{a} - \mathbf{b}||^2$ does not exceed 9.