1.6.25

AI25BTECH11010 - Dhanush Kumar

If the sum of two unit vectors is a unit vector, prove that the magnitude of their difference is $\sqrt{3}$.

Solution:

Let **u** and **v** be unit vectors.
Hence,

$$\mathbf{u}^T\mathbf{u} = 1, \qquad\qquad \mathbf{v}^T\mathbf{v} = 1. \qquad\qquad (1)$$

$$\|\mathbf{u} + \mathbf{v}\|^2 = (\mathbf{u} + \mathbf{v})^T(\mathbf{u} + \mathbf{v}) \qquad\qquad (2)$$

$$= \mathbf{u}^T\mathbf{u} + \mathbf{v}^T\mathbf{v} + 2\mathbf{u}^T\mathbf{v} \qquad\qquad (3)$$

$$1^2 = 1 + 1 + 2\mathbf{u}^T\mathbf{v} \qquad\qquad (4)$$

$$1 = 2 + 2\mathbf{u}^T\mathbf{v} \qquad\qquad (5)$$

$$\Rightarrow \quad \mathbf{u}^T\mathbf{v} = -\tfrac{1}{2}. \qquad\qquad (6)$$

Now,

$$\|\mathbf{u} - \mathbf{v}\|^2 = (\mathbf{u} - \mathbf{v})^T(\mathbf{u} - \mathbf{v}) \qquad\qquad (7)$$

$$= \mathbf{u}^T\mathbf{u} + \mathbf{v}^T\mathbf{v} - 2\mathbf{u}^T\mathbf{v} \qquad\qquad (8)$$

$$= 1 + 1 - 2\left(-\tfrac{1}{2}\right) \qquad\qquad (9)$$

$$= 2 + 1 = 3. \qquad\qquad (10)$$

Therefore,

$$\|\mathbf{u} - \mathbf{v}\| = \sqrt{3}. \qquad\qquad (11)$$
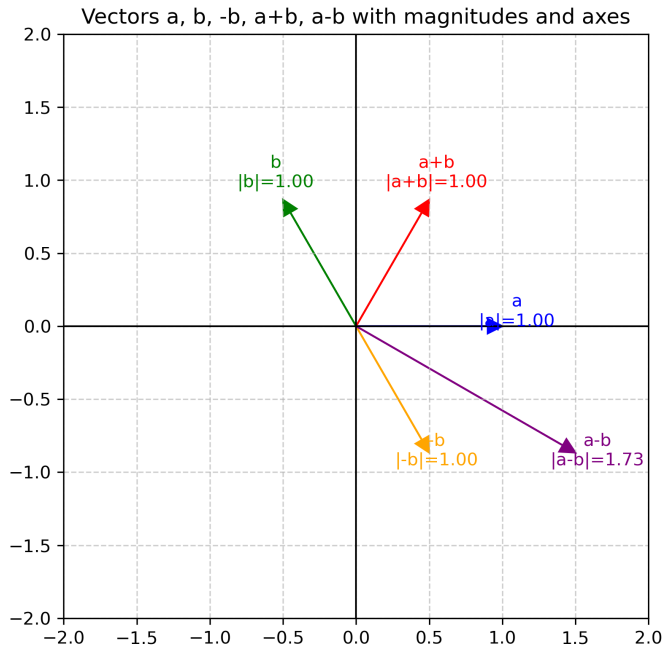
$\therefore$ The required result is proved.

Fig. 0