Bonus question

EE25BTECH11043 - Nishid Khandagre

Question: If vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are coplanar, then the matrix $(\mathbf{A}\ \mathbf{B}\ \mathbf{C})$ is singular.

Solution:
Given $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are coplanar.

$\therefore$ $\mathbf{A}, \mathbf{B}, \mathbf{C}$ lie in the same plane passing through the origin.
Equation of a plane passing through the origin:

$$lx + my + nz = 0 \tag{0.1}$$

$$\begin{pmatrix} l & m & n \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \tag{0.2}$$

There must be a non-zero normal vector for this plane.
Let $\mathbf{n} = \begin{pmatrix} l \\ m \\ n \end{pmatrix}$ be a non-zero normal vector to this plane.
Then the equation of the plane is

$$\mathbf{n}^T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \tag{0.3}$$

Since $\mathbf{A}, \mathbf{B}, \mathbf{C}$ lie in this plane:

$$\mathbf{n}^T \mathbf{A} = 0 \tag{0.4}$$
$$\mathbf{n}^T \mathbf{B} = 0 \tag{0.5}$$
$$\mathbf{n}^T \mathbf{C} = 0 \tag{0.6}$$

$$\mathbf{n}^T \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = \mathbf{0} \tag{0.7}$$

Let $\mathbf{M} = (\mathbf{A}\ \mathbf{B}\ \mathbf{C})$.

$$\mathbf{n}^T \mathbf{M} = \mathbf{0} \tag{0.8}$$

It means rows or columns of matrix $\mathbf{M}$ is linearly dependent. Hence,

$$\det(\mathbf{M}) = 0 \tag{0.9}$$

Therefore, matrix $\mathbf{M}$ is singular.
Therefore, if $\mathbf{A}$,$\mathbf{B}$,$\mathbf{C}$ are coplanar, then the matrix $\begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C}\end{pmatrix}$ is singular.