2.10.52

EE25BTECH11023 - Venkata Sai

Question:
Let $\mathbf{a} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$. A vector in the plane of $\mathbf{a}$ and $\mathbf{b}$ whose projection on $\mathbf{c}$ is $\frac{1}{\sqrt{3}}$, is

1) $4\hat{\mathbf{i}} - \hat{\mathbf{j}} + 4\hat{\mathbf{k}}$
2) $3\hat{\mathbf{i}} + \hat{\mathbf{j}} - 3\hat{\mathbf{k}}$
3) $2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$
4) $4\hat{\mathbf{i}} + \hat{\mathbf{j}} - 4\hat{\mathbf{k}}$

Solution:
Given

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \tag{1}$$

Let $\mathbf{r}$ be coplanar to $\mathbf{a}$ and $\mathbf{b}$

$$\mathbf{r} = \mathbf{a} + t\mathbf{b} \tag{2}$$

Given the projection of $\mathbf{r}$ on $\mathbf{c}$ is $\frac{1}{\sqrt{3}}$

$$\frac{|\mathbf{r}^\top \mathbf{c}|}{||\mathbf{c}||} = \frac{1}{\sqrt{3}} \tag{3}$$

$$\implies \mathbf{r}^\top \mathbf{c} = \pm \frac{||\mathbf{c}||}{\sqrt{3}} \tag{4}$$

$$\mathbf{r}^\top \mathbf{c} = (\mathbf{a} + t\mathbf{b})^\top \mathbf{c} \tag{5}$$

$$= \left(\mathbf{a}^\top + t\mathbf{b}^\top\right) c \tag{6}$$

$$= \mathbf{a}^\top \mathbf{c} + t\left(\mathbf{b}^\top \mathbf{c}\right) \tag{7}$$

$$\mathbf{r}^\top \mathbf{c} - \mathbf{a}^\top \mathbf{c} = t\left(\mathbf{b}^\top \mathbf{c}\right) \tag{8}$$

$$\implies t = \frac{\mathbf{r}^\top \mathbf{c} - \mathbf{a}^\top \mathbf{c}}{\mathbf{b}^\top \mathbf{c}} \tag{9}$$

From (2)

$$\mathbf{r} = \mathbf{a} + \left(\frac{\mathbf{r}^\top \mathbf{c} - \mathbf{a}^\top \mathbf{c}}{\mathbf{b}^\top \mathbf{c}}\right)\mathbf{b} \tag{10}$$

$$\mathbf{r} = \mathbf{a} + \left(\frac{\pm\frac{||\mathbf{c}||}{\sqrt{3}} - \mathbf{a}^\top \mathbf{c}}{\mathbf{b}^\top \mathbf{c}}\right)\mathbf{b} \tag{11}$$

$$\|\mathbf{c}\|^2 = \mathbf{c}^\top \mathbf{c} = \begin{pmatrix} 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \tag{12}$$

$$= 1 + 1 + 1 = 3 \implies \|\mathbf{c}\| = \sqrt{3} \tag{13}$$

$$\mathbf{a}^\top \mathbf{c} = \begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 1 + 2 - 1 = 2 \tag{14}$$

$$\mathbf{b}^\top \mathbf{c} = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 1 - 1 - 1 = -1 \tag{15}$$

$$\tag{16}$$

Substitute (13),(14),(15) in (11)

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \left(\frac{\pm 1 - 2}{-1} \right) \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \tag{17}$$

$$\implies \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \implies \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \text{ or } \begin{pmatrix} 4 \\ -1 \\ 4 \end{pmatrix} \tag{18}$$

Hence Option(1) is the correct answer

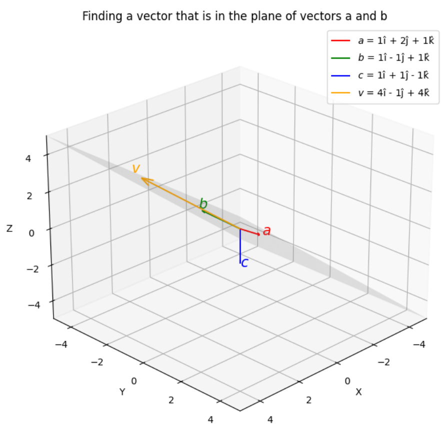

Fig. 4.1