Assignment 9: 4.13.59

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Question:
Determine all values of α for which the point $\left(\alpha, \alpha^2\right)$ lies inside the triangle formed by the lines. (1992)

$$\begin{aligned} 2x + 3y - 1 &= 0 \\ x + 2y - 3 &= 0 \\ 5x - 6y - 1 &= 0 \end{aligned}$$

Solution:
Given:

$$\mathbf{n_1}^\top \mathbf{x} = c_1 \qquad \mathbf{n_1} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} c_1 = 1 \tag{1}$$

$$\mathbf{n_2}^\top \mathbf{x} = c_2 \qquad \mathbf{n_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} c_2 = 3 \tag{2}$$

$$\mathbf{n_3}^\top \mathbf{x} = c_3 \qquad \mathbf{n_3} = \begin{pmatrix} 5 \\ -6 \end{pmatrix} c_3 = 1 \tag{3}$$

$$\mathbf{P} = \begin{pmatrix} \alpha \\ \alpha^2 \end{pmatrix} \tag{4}$$

For finding vertices:

$$\begin{pmatrix} n_1 & n_2 \end{pmatrix}^\top \mathbf{V_3} = \begin{pmatrix} c1 \\ c2 \end{pmatrix} \tag{5}$$

$$\implies \mathbf{V_3} = \begin{pmatrix} n_1 & n_2 \end{pmatrix}^{-\top} \begin{pmatrix} c1 \\ c2 \end{pmatrix} \tag{6}$$

$$\mathbf{V_2} = \begin{pmatrix} n_3 & n_1 \end{pmatrix}^{-\top} \begin{pmatrix} c3 \\ c1 \end{pmatrix} \tag{7}$$

$$\mathbf{V_1} = \begin{pmatrix} n_2 & n_3 \end{pmatrix}^{-\top} \begin{pmatrix} c2 \\ c3 \end{pmatrix} \tag{8}$$

The normal matrices are invertible as the lines are not parallel. Let us define $d_i = \mathbf{n_i}^\top \mathbf{V_i} - c_i$ as the sign denoting which side of the line the vertex opposite to it lies on. Also define matrix $\mathbf{D} = \text{diag}\left(d_1, d_2, d_3\right)$ For point to lie inside triangle, we need $d_i \cdot \left(\mathbf{n_i}^\top \mathbf{P} - c_i\right) > 0$. In matrix form, this is written as:

$$\mathbf{D} = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \tag{9}$$

$$\mathbf{D} \begin{pmatrix} \mathbf{n_1}^\top \mathbf{P} - c_1 \\ \mathbf{n_2}^\top \mathbf{P} - c_2 \\ \mathbf{n_3}^\top \mathbf{P} - c_3 \end{pmatrix} > \mathbf{0} \tag{10}$$

Let

$$\mathbf{N} = \begin{pmatrix} n_1 & n_2 & n_3 \end{pmatrix}^\top \tag{11}$$

$$\mathbf{C} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \tag{12}$$

Thus representing everything in terms of matrices,

$$\mathbf{D}\left(\mathbf{NP} - \mathbf{C}\right) > \mathbf{0} \tag{13}$$

is the required inequality. On substituting values, we get

$$\alpha \tag{14}$$