Problem 9.4.4

Find the roots of the following quadratic equation graphically:

$$x^2 - 3x - 10 = 0 \tag{1}$$

Input Variables

The given quadratic can be written in the conic form

$$\mathbf{x}^T V \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \tag{2}$$

where

$$V = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} -\frac{3}{2} \\ 0 \end{pmatrix}, \quad f = -10 \tag{3}$$

Since the roots of the quadratic correspond to intersections with the x-axis, we represent the line

$$L : \mathbf{x} = \mathbf{h} + \kappa \mathbf{m} \tag{4}$$

with

$$\mathbf{h} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{5}$$

Symbol	**Value**
V	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
$\mathbf{u}$	$\begin{pmatrix} -\frac{3}{2} \\ 0 \end{pmatrix}$
f	-10
$\mathbf{h}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$\mathbf{m}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Solution

The points of intersection of a line with a conic are given by

$$\kappa = \frac{1}{\mathbf{m}^T V \mathbf{m}} \left(-\mathbf{m}^T (V\mathbf{h} + \mathbf{u}) \pm \sqrt{(\mathbf{m}^T (V\mathbf{h} + \mathbf{u}))^2 - g(\mathbf{h})(\mathbf{m}^T V \mathbf{m})} \right), \tag{6}$$

where

$$g(\mathbf{h}) = \mathbf{h}^T V \mathbf{h} + 2\mathbf{u}^T \mathbf{h} + f. \tag{7}$$

Step 1: Compute $m^T V m$

$$\mathbf{m}^T V \mathbf{m} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \tag{8}$$

Step 2: Compute $Vh + u$

$$V\mathbf{h} + \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{3}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} \\ 0 \end{pmatrix} \tag{9}$$

Step 3: Compute $m^T(Vh + u)$

$$\mathbf{m}^T (V\mathbf{h} + \mathbf{u}) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{3}{2} \\ 0 \end{pmatrix} = -\frac{3}{2} \tag{10}$$

Step 4: Compute $g(h)$

$$g(\mathbf{h}) = \mathbf{h}^T V \mathbf{h} + 2\mathbf{u}^T \mathbf{h} + f = -10 \tag{11}$$

Step 5: Substitute into formula for κ

$$\kappa = -(-\tfrac{3}{2}) \pm \sqrt{\left(-\tfrac{3}{2}\right)^2 - (-10)(1)} \tag{12}$$

$$= \tfrac{3}{2} \pm \sqrt{\tfrac{9}{4} + 10} \tag{13}$$

$$= \tfrac{3}{2} \pm \sqrt{\tfrac{49}{4}} \tag{14}$$

$$= \tfrac{3}{2} \pm \tfrac{7}{2} \tag{15}$$

Step 6: Evaluate roots

$$\kappa_1 = \tfrac{3}{2} + \tfrac{7}{2} = 5 \tag{16}$$

$$\kappa_2 = \tfrac{3}{2} - \tfrac{7}{2} = -2 \tag{17}$$

Step 7: Find intersection points The intersection points are obtained as

$$\mathbf{x}_1 = \mathbf{h} + \kappa_1 \mathbf{m} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 5 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \tag{18}$$

$$\mathbf{x}_2 = \mathbf{h} + \kappa_2 \mathbf{m} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{19}$$

Final Answer

Thus, the quadratic $x^2 - 3x - 10 = 0$ intersects the x-axis at

$$\boxed{x = -2 \quad \text{and} \quad x = 5} \tag{20}$$

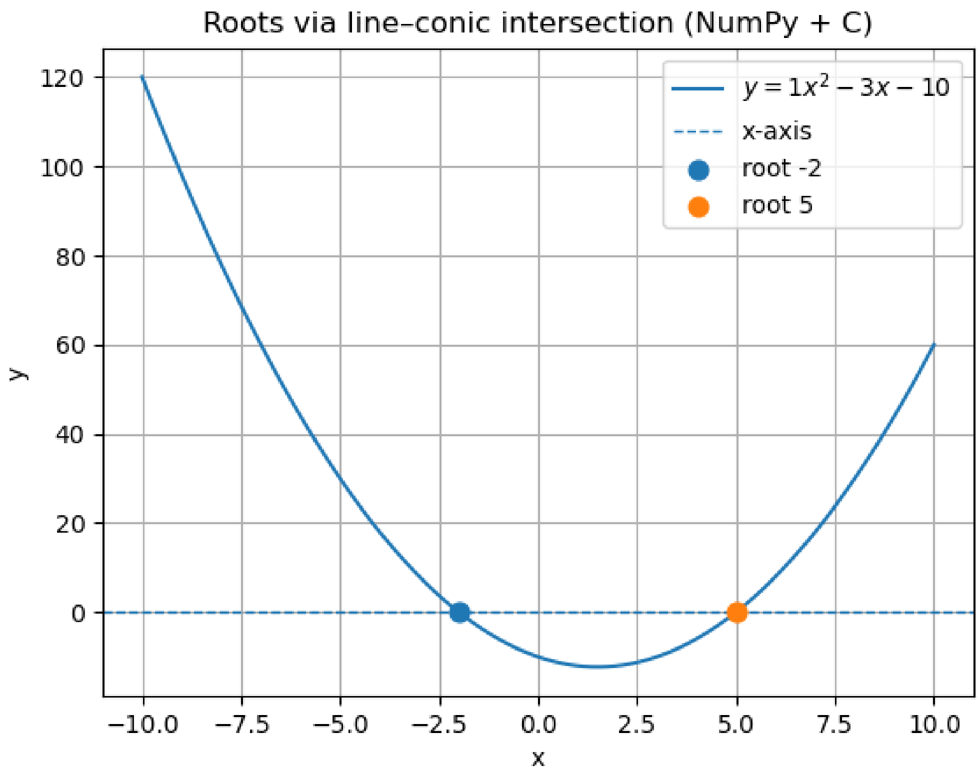

Figure 1