7.4.33

EE25BTECH11026-Harsha

Question:
A circle C of radius 1 unit is inscribed in an equilateral triangle PQR. The points of contact of C with sides PQ, QR, RP are D, E, F respectively.The line PQ is given by the equation $\sqrt{3}x + y - 6 = 0$ and the point $\mathbf{D}$ is $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$. Further, it is given that the origin and the centre of C are on same side of line PQ. The equation of circle C is

1) $\left(x - 2\sqrt{3}\right)^2 + (y - 1)^2 = 1$
2) $\left(x - 2\sqrt{3}\right)^2 + \left(y + \frac{1}{2}\right)^2 = 1$
3) $\left(x - \sqrt{3}\right)^2 + (y - 1)^2 = 1$
4) $\left(x - \sqrt{3}\right)^2 + (y + 1)^2 = 1$

Solution:
Let us solve the given question theoretically and then verify the solution computationally.

According to the question,

$$\text{Equation of tangent PQ} : \mathbf{n}^\top \mathbf{x} = c \tag{4.1}$$

where $\mathbf{n} = \begin{pmatrix} \sqrt{3} & 1 \end{pmatrix}^\top$ and $c = 6$

$$\text{Point of tangency}\,(\mathbf{D}) : \begin{pmatrix} \frac{3\sqrt{3}}{2} \\ \frac{3}{2} \end{pmatrix} \tag{4.2}$$

$$radius(r) = 1 \tag{4.3}$$

As the point of tangency $\mathbf{D}$ and centre of circle $\mathbf{u}$ are along the direction of the vector $\mathbf{n}$,

$$\therefore \mathbf{D} - \mathbf{u} = \lambda\mathbf{n} \text{ , for some scalar } \lambda \tag{4.4}$$

$$\implies \mathbf{u} = \mathbf{D} - \lambda\mathbf{n} \tag{4.5}$$

Also,

$$\frac{|\mathbf{n}^\top \mathbf{O} - c|}{||\mathbf{n}||} = r \tag{4.6}$$

$$|\mathbf{n}^\top \mathbf{u} - c| = r||\mathbf{n}|| \tag{4.7}$$

$$\mathbf{n}^\top \mathbf{u} = c \pm r||\mathbf{n}|| \tag{4.8}$$

To decide the sign , we need to use the fact that the origin and centre of circle are on the same side of the line PQ.

$$\therefore \left(\mathbf{n}^\top \mathbf{u} - c\right)\left(\mathbf{n}^\top \begin{pmatrix} 0 \\ 0 \end{pmatrix} - c\right) > 0 \tag{4.9}$$

$$\implies \mathbf{n}^\top \mathbf{u} < c \tag{4.10}$$

$$\therefore \mathbf{n}^\top \mathbf{u} = c - r\|\mathbf{n}\| \tag{4.11}$$

Substituting value of $\mathbf{u}$,

$$\mathbf{n}^\top \left(\mathbf{D} - \lambda \mathbf{n}\right) = c - r\|\mathbf{n}\| \tag{4.12}$$

$$\implies \lambda = \frac{\mathbf{n}^\top \mathbf{D} + r\|n\| - c}{\mathbf{n}^\top \mathbf{n}} \tag{4.13}$$

$$\mathbf{u} = \mathbf{D} - \frac{\mathbf{n}^\top \mathbf{D} + r\|n\| - c}{\mathbf{n}^\top \mathbf{n}} \mathbf{n} \tag{4.14}$$

Substituting the values,

$$\therefore \mathbf{u} = \begin{pmatrix} \frac{3\sqrt{3}}{2} \\ \frac{3}{2} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \tag{4.15}$$

$$\therefore \text{Required equation of circle} : \|\mathbf{x}\|^2 - 2\begin{pmatrix} \sqrt{3} & 1 \end{pmatrix}\mathbf{x} + 3 = 0 \tag{4.16}$$

From the figure, it is clearly verified that the theoretical solution matches with the computational solution.