12.234

EE25BTECH11012-BEERAM MADHURI

Question:
Consider the set of vectors in three-dimensional real vector space $\mathbb{R}^3$,

$$S = \{(1, 1, 1), (1, -1, 1), (1, 1, -1)\}.$$

Which one of the following statements is true?

a) S is not a linearly independent set.
b) S is a basis for $\mathbb{R}^3$.
c) The vectors in S are orthogonal.
d) An orthogonal set of vectors cannot be generated from S.

Solution:
let the vectors in S be:

Point	**Vector**
$\mathbf{v_1}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
$\mathbf{v_2}$	$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$
$\mathbf{v_3}$	$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

TABLE 0: Variables used

Let A be the matrix with its columns as vectors of S

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \tag{0.1}$$

these vectors are linearly independent if and only if

$$\det(A) \neq 0 \tag{0.2}$$

$$det(A) = 1(0) - 1(-2) + 1(2) \tag{0.3}$$
$$= 4 \neq 0 \tag{0.4}$$

$\therefore$ Vectors are linearly independent
$\therefore$ Since there are 3 linearly independent vectors in $\mathbb{R}^3$
they form a basis for $\mathbb{R}^3$

Let the vector be $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$.

$$\mathbf{v_1^T v_2} \neq 0 \tag{0.5}$$
$$\mathbf{v_1^T v_3} \neq 0 \tag{0.6}$$
$$\mathbf{v_2^T v_3} \neq 0 \tag{0.7}$$

$\therefore$ These vectors are not orthogonal
Applying Gram-Schmidt process :
let the orthogonal vectors be $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}$ generated from $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$

$$\mathbf{u_1} = \mathbf{v_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{0.8}$$
$$\mathbf{u_2} = \mathbf{v_2} - (\mathbf{u_1^T v_2})\hat{\mathbf{u}}_1 \tag{0.9}$$
$$\mathbf{u_2} = \mathbf{v_2} - \left(\frac{\mathbf{v_2}^T \mathbf{u_1}}{\mathbf{u_1}^T \mathbf{u_1}}\right)\mathbf{u_1} \tag{0.10}$$
$$= \begin{pmatrix} 2/3 \\ -4/3 \\ 2/3 \end{pmatrix} \tag{0.11}$$

$$\mathbf{u_3} = \mathbf{v_3} - (\hat{\mathbf{u}}_2^T \mathbf{v_3})\hat{\mathbf{u}}_2 \tag{0.12}$$
$$= \mathbf{v_3} - \left(\frac{\mathbf{u_2}^T \mathbf{v_3}}{\mathbf{u_2}^T \mathbf{u_2}}\right)\mathbf{u_2} \tag{0.13}$$
$$= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{0.14}$$
$$\mathbf{u_1}^T \mathbf{u_2} = 0 \tag{0.15}$$
$$\mathbf{u_2}^T \mathbf{u_3} = 0 \tag{0.16}$$
$$\mathbf{u_1}^T \mathbf{u_3} = 0 \tag{0.17}$$

$\therefore$ an orthogonal set of vectors can be generated from S.
$\therefore$ Options b and d are correct.