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EE25BTECH11052 - Shriyansh Kalpesh Chawda

Question:
The matrix

$$A = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1+i \\ 1-i & -1 \end{pmatrix}$$

is:

1) orthogonal 2) symmetric 3) anti-symmetric 4) unitary

(PH 2014)

Solution: Given matrix:

$$A = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1+i \\ 1-i & -1 \end{pmatrix} \tag{4.1}$$

Check 1: Symmetric ($A = A^\top$)

The transpose of A is:

$$A^\top = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix} \tag{4.2}$$

Since $A^\top \neq A$ (the off-diagonal elements are different: $1+i \neq 1-i$),
A **is NOT symmetric.**

Check 2: Anti-symmetric ($A = -A^\top$)

For anti-symmetric:

$$-A^\top = -\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix} \tag{4.3}$$

$$= \frac{1}{\sqrt{3}}\begin{pmatrix} -1 & -1+i \\ -1-i & 1 \end{pmatrix} \tag{4.4}$$

Since $A \neq -A^\top$,
A **is NOT anti-symmetric.**

Check 3: Orthogonal ($AA^\top = I$)

Note: For real matrices, orthogonal means $AA^\top = I$. However, A contains complex entries, so we need to check if it satisfies the real orthogonal property.

$$AA^\top = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1+i \\ 1-i & -1 \end{pmatrix} \cdot \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix} \tag{4.5}$$

$$= \frac{1}{3}\begin{pmatrix} 1 & 1+i \\ 1-i & -1 \end{pmatrix}\begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix} \tag{4.6}$$

Computing the $(1,1)$ entry:

$$(AA^\top)_{11} = \frac{1}{3}\left[1 \cdot 1 + (1+i)(1+i)\right] = \frac{1}{3}\left[1 + 1 + 2i + i^2\right] = \frac{1}{3}[1 + 2i] \neq 1 \tag{4.7}$$

Since this is complex (not real), $AA^\top \neq I$,
A **is NOT orthogonal.**

Check 4: Unitary ($A\overline{A^\top} = I$)

For a unitary matrix, we need $A\overline{A^\top} = I$, where $\overline{A^\top} = \overline{A^\top}$ is the conjugate transpose. The conjugate transpose is:

$$\overline{A^\top} = \overline{A^\top} = \overline{\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix}} \tag{4.8}$$

$$= \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1+i \\ 1-i & -1 \end{pmatrix} \tag{4.9}$$

Notice that $\overline{A^\top} = A$ (the matrix is Hermitian!), but let's verify unitarity:

$$A\overline{A^\top} = \frac{1}{3}\begin{pmatrix} 1 & 1+i \\ 1-i & -1 \end{pmatrix}\begin{pmatrix} 1 & 1+i \\ 1-i & -1 \end{pmatrix} \tag{4.10}$$

$$A\overline{A^\top} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \tag{4.11}$$

A **is UNITARY.**

$\boxed{\text{Option 4: The matrix } A \text{ is unitary}}$