2.9.4

EE25BTECH11052 - Shriyansh Kalpesh Chawda

Question:

If $\mathbf{a} = \hat{i} + \hat{j} + \hat{k}, \quad \mathbf{a} \cdot \mathbf{b} = 1, \quad \text{and} \quad \mathbf{a} \times \mathbf{b} = \hat{j} - \hat{k}$, then find $|\mathbf{b}|$. (12, 2022)

Solution:

We are given the vectors in component form:

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{0.1}$$

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \tag{0.2}$$

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{0.3}$$

From the dot product:

$$\mathbf{a}^\top \mathbf{b} = 1 \implies \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = 1 \tag{0.4}$$

$$b_1 + b_2 + b_3 = 1 \tag{0.5}$$

From the cross product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ b_1 & b_2 & b_3 \end{vmatrix} = (b_3 - b_2)\,\hat{i} + (b_1 - b_3)\,\hat{j} + (b_2 - b_1)\,\hat{k} \tag{0.6}$$

Comparing Equation (0.2) and (0.6)

$$b_3 - b_2 = 0 \tag{0.7}$$

$$b_1 - b_3 = 1 \tag{0.8}$$

Substituting values in (0.5):

$$(1 + b_3) + (b_3) + b_3 = 1 \tag{0.9}$$

$$1 + 3b_3 = 1 \tag{0.10}$$

$$3b_3 = 0 \implies b_3 = 0 \tag{0.11}$$

So, now for b_2 and b_1

$$b_2 = b_3 = 0 \tag{0.12}$$

$$b_1 = 1 + b_3 = 1 + 0 = 1 \tag{0.13}$$

So, $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$,

To find magnitude,

$$\mathbf{b}^\top \mathbf{b} = 1 \tag{0.14}$$

$$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1 \tag{0.15}$$

The magnitude of vector **b** is **1**.