MATGEO Presentation: 4.13.59

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Problem Statement

Determine all values of α for which the point $\left(\alpha, \alpha^2\right)$ lies inside the triangle formed by the lines. (1992)

$$2x + 3y - 1 = 0 \tag{2.1}$$

$$x + 2y - 3 = 0 \tag{2.2}$$

$$5x - 6y - 1 = 0 \tag{2.3}$$

Given data

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$$\mathbf{n_1}^\top \mathbf{x} = c_1 \qquad \mathbf{n_1} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} c_1 = 1 \tag{3.1}$$

$$\mathbf{n_2}^\top \mathbf{x} = c_2 \qquad \mathbf{n_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} c_2 = 3 \tag{3.2}$$

$$\mathbf{n_3}^\top \mathbf{x} = c_3 \qquad \mathbf{n_3} = \begin{pmatrix} 5 \\ -6 \end{pmatrix} c_3 = 1 \tag{3.3}$$

$$\mathbf{P} = \begin{pmatrix} \alpha \\ \alpha^2 \end{pmatrix} \tag{3.4}$$

Formulae

For finding vertices:

$$\begin{pmatrix} n_1 & n_2 \end{pmatrix}^\top \mathbf{V_3} = \begin{pmatrix} c1 \\ c2 \end{pmatrix} \tag{3.5}$$

$$\begin{pmatrix} n_3 & n_1 \end{pmatrix}^\top \mathbf{V_2} = \begin{pmatrix} c3 \\ c1 \end{pmatrix} \tag{3.6}$$

$$\begin{pmatrix} n_2 & n_3 \end{pmatrix}^\top \mathbf{V_1} = \begin{pmatrix} c2 \\ c3 \end{pmatrix} \tag{3.7}$$

Let us define $d_i = \mathbf{n_i}^\top \mathbf{V_i} - c_i$ as the sign denoting which side of the line the vertex opposite to it lies on. Also define matrix $\mathbf{D} = \operatorname{diag}\left(d_1, d_2, d_3\right)$ For point to lie inside triangle, we need $d_i \cdot \left(\mathbf{n_i}^\top \mathbf{P} - c_i\right) > 0$. In matrix form, this is written as:

$$\mathbf{D} = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \tag{3.8}$$

$$\mathbf{D} \begin{pmatrix} \mathbf{n_1}^\top \mathbf{P} - c_1 \\ \mathbf{n_2}^\top \mathbf{P} - c_2 \\ \mathbf{n_3}^\top \mathbf{P} - c_3 \end{pmatrix} > \mathbf{0} \tag{3.9}$$

Let

$$\mathbf{N} = \begin{pmatrix} n_1 & n_2 & n_3 \end{pmatrix}^\top \tag{3.10}$$

$$\mathbf{C} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \tag{3.11}$$

Thus representing everything in terms of matrices,

$$\mathbf{D} \left(\mathbf{NP} - \mathbf{C} \right) > \mathbf{0} \tag{3.12}$$

is the required inequality.

Solving

First, we find the vertices of the triangle using Gaussian elimination:

$$\mathbf{V_1} : \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 5 & -6 & 1 \end{array}\right) \xrightarrow{R_2 \to R_2 - 5R_1} \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -16 & -14 \end{array}\right) \implies \mathbf{V_1} = \begin{pmatrix} 5/4 \\ 7/8 \end{pmatrix} \tag{3.13}$$

$$\mathbf{V_2} : \left(\begin{array}{cc|c} 2 & 3 & 1 \\ 5 & -6 & 1 \end{array}\right) \xrightarrow{R_2 \to R_2 - \frac{5}{2}R_1} \left(\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -27/2 & -3/2 \end{array}\right) \implies \mathbf{V_2} = \begin{pmatrix} 1/3 \\ 1/9 \end{pmatrix} \tag{3.14}$$

$$\mathbf{V_3} : \left(\begin{array}{cc|c} 2 & 3 & 1 \\ 1 & 2 & 3 \end{array}\right) \xrightarrow{R_2 \to R_2 - \frac{1}{2}R_1} \left(\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \end{array}\right) \implies \mathbf{V_3} = \begin{pmatrix} -7 \\ 5 \end{pmatrix} \tag{3.15}$$

Next, we determine the signs $d_i = \mathbf{n_i}^\top \mathbf{V_i} - c_i$ for each line evaluated at its opposite vertex:

$$d_1 = \mathbf{n_1}^\top \mathbf{V_1} - c_1 = 2(5/4) + 3(7/8) - 1 = 33/8 \tag{3.16}$$

$$d_2 = \mathbf{n_2}^\top \mathbf{V_2} - c_2 = (1/3) + 2(1/9) - 3 = -22/9 \tag{3.17}$$

$$d_3 = \mathbf{n_3}^\top \mathbf{V_3} - c_3 = 5(-7) - 6(5) - 1 = -66 \tag{3.18}$$

For the point $\mathbf{P} = \begin{pmatrix} \alpha \\ \alpha^2 \end{pmatrix}$ to be inside, the condition $\mathbf{D}(\mathbf{NP} - \mathbf{C}) > \mathbf{0}$ must hold.

$$\mathbf{NP} - \mathbf{C} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \\ 5 & -6 \end{pmatrix} \begin{pmatrix} \alpha \\ \alpha^2 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3\alpha^2 + 2\alpha - 1 \\ 2\alpha^2 + \alpha - 3 \\ -6\alpha^2 + 5\alpha - 1 \end{pmatrix} \tag{3.19}$$

Multiplying by the diagonal matrix **D**:

$$\mathbf{D}(\mathbf{NP}-\mathbf{C}) = \begin{pmatrix} 33/8 & 0 & 0 \\ 0 & -22/9 & 0 \\ 0 & 0 & -66 \end{pmatrix} \begin{pmatrix} 3\alpha^2+2\alpha-1 \\ 2\alpha^2+\alpha-3 \\ -6\alpha^2+5\alpha-1 \end{pmatrix} \tag{3.20}$$

$$= \begin{pmatrix} (33/8)(3\alpha^2+2\alpha-1) \\ (-22/9)(2\alpha^2+\alpha-3) \\ (-66)(-6\alpha^2+5\alpha-1) \end{pmatrix} > \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.21}$$

This yields the system of inequalities:

$$3\alpha^2+2\alpha-1>0 \implies \alpha \in (-\infty,-1) \cup (1/3,\infty) \tag{3.22}$$

$$2\alpha^2+\alpha-3<0 \implies \alpha \in (-3/2,1) \tag{3.23}$$

$$6\alpha^2-5\alpha+1>0 \implies \alpha \in (-\infty,1/3) \cup (1/2,\infty) \tag{3.24}$$

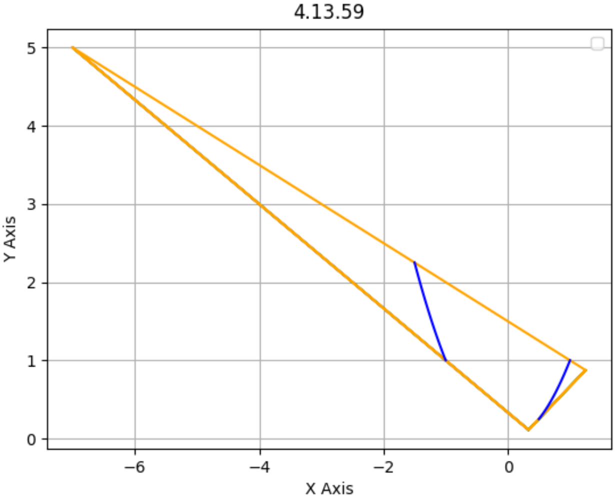
Result

The value of α

must satisfy all three conditions. Taking the intersection of the solution sets

$$\alpha \in (-3/2, -1) \cup (1/2, 1) \tag{3.25}$$

Plot



C code for generating points on line

```
void point_gen(const double* P1, const double* P2, double t, double
     * result_point) {
    result_point[0] = P1[0] + t * (P2[0] - P1[0]);
    result_point[1] = P1[1] + t * (P2[1] - P1[1]);
    result_point[2] = P1[2] + t * (P2[2] - P1[2]);
}
```

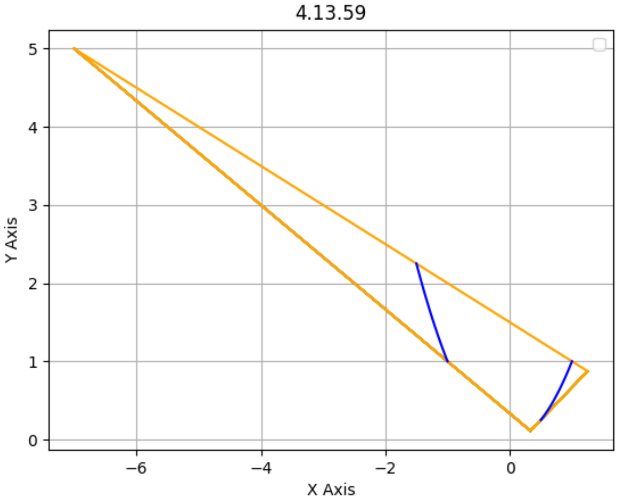
Python code for plotting using C

```
import ctypes
import numpy as np
import numpy.linalg as LA
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

libline = ctypes.CDLL("./line.so")

get_point = libline.point_gen
get_point.argtypes = [
    ctypes.POINTER(ctypes.c_double), # P1
    ctypes.POINTER(ctypes.c_double), # P2
    ctypes.c_double, # t
    ctypes.POINTER(ctypes.c_double), # result_point
]
get_point.restype = None
```

```
DoubleArray3 = ctypes.c_double * 3
a = DoubleArray3(5, 1, 6)
b = DoubleArray3(3, 4, 1)
c = DoubleArray3(13 / 5, 23 / 5, 0)

fig = plt.figure(figsize=(8, 6))
ax = fig.add_subplot(111, projection="3d")

t_values = np.linspace(0, 1, 100)
line_points_x, line_points_y, line_points_z = [], [], []

for t in t_values:
    result_arr = DoubleArray3()

    get_point(a, c, t, result_arr)

    line_points_x.append(result_arr[0])
    line_points_y.append(result_arr[1])
    line_points_z.append(result_arr[2])
```

```
ax.plot(
    line_points_x,
    line_points_y,
    line_points_z,
    color="gray",
)
ax.scatter(b[0], b[1], b[2], color="blue", label="b")
ax.scatter(a[0], a[1], a[2], color="red", label="a")
ax.scatter(c[0], c[1], c[2], color="green", label="Point")

ax.set_xlabel("X Axis")
ax.set_ylabel("Y Axis")
ax.set_zlabel("Z Axis")
ax.set_title("2.9.6")
ax.legend()
ax.grid(True)
plt.savefig("../figs/plot.png")
plt.show()
```

Plot



Pure Python code

```python
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

a = np.array([5, 1, 6]).T
b = np.array([3, 4, 1]).T
c = np.array([13 / 5, 23 / 5, 0])

fig = plt.figure(figsize=(8, 8))
ax = fig.add_subplot(111, projection="3d")

ax.plot([a[0], c[0]], [a[1], c[1]], [a[2], c[2]], color="blue", label="b")

ax.scatter(b[0], b[1], b[2], color="blue", label="b")
ax.scatter(a[0], a[1], a[2], color="red", label="a")
ax.scatter(c[0], c[1], c[2], color="green", label="Point")
```

Pure Python code

```
ax.text(a[0], a[1], a[2], "A")
ax.text(b[0], b[1], b[2], "B")
ax.text(c[0], c[1], c[2], "Point")

ax.set_xlabel("X-axis")
ax.set_ylabel("Y-axis")
ax.set_zlabel("Z-axis")
ax.set_title("2.9.6")
ax.set_xlim([-5, 5])
ax.set_ylim([-5, 5])
ax.set_zlim([-5, 5])
ax.legend()
ax.grid(True)

plt.savefig("../figs/python.png")
plt.show()
```

Plot