8.4.26

EE25BTECH11023 - Venkata Sai

Question:
The locus of the mid-point of the line segment joining the focus to a moving point on the parabola $y^2 = 4ax$ is another parabola with directrix

1) $x = 0$
2) $x = -a/2$
3) $x = a$
4) $x = a/2$

Solution:
The equation of a conic with directrix $\mathbf{n}^\top \mathbf{x} = c$, eccentricity e and focus $\mathbf{F}$ is given by

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$

On comparing with $y^2 - 4ax = 0$

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} = y^2 \tag{2}$$

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} = \left[\begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x}\right]^2 \tag{3}$$

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} = \left(\begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x}\right)^\top \left(\begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x}\right) \tag{4}$$

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} = \mathbf{x}^\top \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x} \tag{5}$$

$$\mathbf{x}^\top \left(\mathbf{V} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)\mathbf{x} = 0 \implies \mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{6}$$

$$2\mathbf{u}^\top \mathbf{x} = -4ax \tag{7}$$

$$2\mathbf{u}^\top \mathbf{x} = -4a\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{x} \implies 2\mathbf{u}^\top \mathbf{x} = \begin{pmatrix} -4a & 0 \end{pmatrix}\mathbf{x} \tag{8}$$

$$\left(2\mathbf{u}^\top - \begin{pmatrix} -4a & 0 \end{pmatrix}\right)\mathbf{x} = 0 \implies \mathbf{u}^\top = \begin{pmatrix} -2a & 0 \end{pmatrix} \tag{9}$$

$$\mathbf{u} = \begin{pmatrix} -2a \\ 0 \end{pmatrix} \tag{10}$$

$$f = 0 \tag{11}$$

$$\mathbf{F} = \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{12}$$

Let $\mathbf{X}$ be the point of locus of the midpoint

$$\mathbf{X} = \frac{\mathbf{x} + \mathbf{F}}{2} \implies \mathbf{x} = 2\mathbf{X} - \mathbf{F} \tag{13}$$

From (1) and (13)

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{14}$$

$$(\mathbf{2X} - \mathbf{F})^\top \mathbf{V}(\mathbf{2X} - \mathbf{F}) + 2\mathbf{u}^\top (\mathbf{2X} - \mathbf{F}) + f = 0 \tag{15}$$

$$\left(\mathbf{2X}^\top - \mathbf{F}^\top\right)\mathbf{V}(\mathbf{2X} - \mathbf{F}) + 2\mathbf{u}^\top (\mathbf{2X} - \mathbf{F}) + f = 0 \tag{16}$$

$$4\mathbf{X}^\top \mathbf{V}\mathbf{X} - 2\mathbf{X}^\top \mathbf{V}\mathbf{F} - 2\mathbf{F}^\top \mathbf{V}\mathbf{X} + \mathbf{F}^\top \mathbf{V}\mathbf{F} + 4\mathbf{u}^\top \mathbf{X} - 2\mathbf{u}^\top \mathbf{F} + f = 0 \tag{17}$$

As $\mathbf{V}$ is a symmetric matrix

$$4\mathbf{X}^\top \mathbf{V}\mathbf{X} - 2\mathbf{F}^\top \mathbf{V}\mathbf{X} - 2\mathbf{F}^\top \mathbf{V}\mathbf{X} + \mathbf{F}^\top \mathbf{V}\mathbf{F} + 4\mathbf{u}^\top \mathbf{X} - 2\mathbf{u}^\top \mathbf{F} + f = 0 \tag{18}$$

$$4\mathbf{X}^\top \mathbf{V}\mathbf{X} - 4\mathbf{F}^\top \mathbf{V}\mathbf{X} + 4\mathbf{u}^\top \mathbf{X} + \mathbf{F}^\top \mathbf{V}\mathbf{F} - 2\mathbf{u}^\top \mathbf{F} + f = 0 \tag{19}$$

$$\mathbf{X}^\top (\mathbf{4V})\mathbf{X} + 2\left(2\left(\mathbf{u}^\top - \mathbf{F}^\top \mathbf{V}\right)\right)\mathbf{X} + \mathbf{F}^\top \mathbf{V}\mathbf{F} - 2\mathbf{u}^\top \mathbf{F} + f = 0 \tag{20}$$

$$\mathbf{X}^\top (\mathbf{4V})\mathbf{X} + 2\left(2(\mathbf{u} - \mathbf{V}\mathbf{F})^\top\right)\mathbf{X} + \mathbf{F}^\top \mathbf{V}\mathbf{F} - 2\mathbf{u}^\top \mathbf{F} + f = 0 \tag{21}$$

$$\mathbf{V}' = 4\mathbf{V} = 4\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 4\end{pmatrix} \tag{22}$$

$$\mathbf{u}' = 2(\mathbf{u} - \mathbf{V}\mathbf{F}) = 2\left(\begin{pmatrix}-2a\\0\end{pmatrix} - \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}a\\0\end{pmatrix}\right) = 2\left(\begin{pmatrix}-2a\\0\end{pmatrix} - \begin{pmatrix}0\\0\end{pmatrix}\right) = \begin{pmatrix}-4a\\0\end{pmatrix} \tag{23}$$

$$f' = \mathbf{F}^\top \mathbf{V}\mathbf{F} - 2\mathbf{u}^\top \mathbf{F} + f \tag{24}$$

$$f' = \begin{pmatrix}a\\0\end{pmatrix}^\top \begin{pmatrix}0 & 0\\0 & 4\end{pmatrix}\begin{pmatrix}a\\0\end{pmatrix} - 2\begin{pmatrix}-2a\\0\end{pmatrix}^\top \begin{pmatrix}a\\0\end{pmatrix} + 0 \tag{25}$$

$$f' = \begin{pmatrix}a & 0\end{pmatrix}\begin{pmatrix}0 & 0\\0 & 4\end{pmatrix}\begin{pmatrix}a\\0\end{pmatrix} - 2\begin{pmatrix}-2a & 0\end{pmatrix}\begin{pmatrix}a\\0\end{pmatrix} + 0 \tag{26}$$

$$f' = \begin{pmatrix}0 & 0\end{pmatrix}\begin{pmatrix}a\\0\end{pmatrix} - 2\begin{pmatrix}-2a & 0\end{pmatrix}\begin{pmatrix}a\\0\end{pmatrix} + 0 \tag{27}$$

$$f' = -2\left(-2a^2\right) = 4a^2 \tag{28}$$

Finding eigen values of $\mathbf{V}'$

$$|\mathbf{V}' - \lambda\mathbf{I}| = 0 \tag{29}$$

$$\left|\begin{pmatrix}0 & 0\\0 & 4\end{pmatrix} - \begin{pmatrix}\lambda & 0\\0 & \lambda\end{pmatrix}\right| = 0 \implies \begin{vmatrix}-\lambda & 0\\0 & 4-\lambda\end{vmatrix} = 0 \tag{30}$$

$$(-\lambda)(4-\lambda) = 0 \implies \lambda_1 = 0 \text{ and } \lambda_2 = 4 \tag{31}$$

$\mathbf{p_1}$ is an eigen vector of $\mathbf{V}'$

$$(\mathbf{V}' - \lambda\mathbf{I})\mathbf{p} = \mathbf{0} \tag{32}$$

From (30) and substituting λ=0

$$\begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \mathbf{p_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{33}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{34}$$

$$0 = 0, y = 0 \implies \mathbf{p_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{35}$$

The Equation of a Directrix is given by

$$\mathbf{n}^\top \mathbf{x} = c \tag{36}$$

where

$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{p_1} \text{ (and) } c = \frac{\left(\|\mathbf{u}'\|^2 - \lambda_2 f\right)}{2\mathbf{u}'^\top \mathbf{n}} \tag{37}$$

$$\mathbf{n} = \sqrt{4}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies \mathbf{n} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{38}$$

$$c = \frac{\left((-4a)^2 + (0)^2 - 4\left(4a^2\right)\right)}{2\begin{pmatrix} -4a \\ 0 \end{pmatrix}^\top \begin{pmatrix} 2 \\ 0 \end{pmatrix}} = \frac{\left(16a^2 - 16a^2\right)}{2\begin{pmatrix} -4a \\ 0 \end{pmatrix}^\top \begin{pmatrix} 2 \\ 0 \end{pmatrix}} = 0 \tag{39}$$

From (36)

$$\begin{pmatrix} 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \implies 2x = 0 \implies x = 0 \tag{40}$$

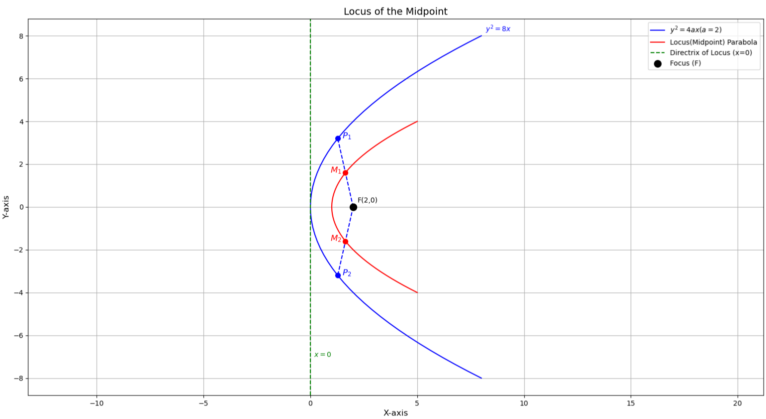

Fig. 4.1