9.6.5

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Question : Solve

$$\frac{1}{3x+y}+\frac{1}{3x-y}=\frac{3}{4}, \quad \frac{1}{2(3x+y)}-\frac{1}{2(3x-y)}=-\frac{1}{8}.$$

Solution :

Conic	Value
Hyperbola 1	$\mathbf{x}^\top \begin{pmatrix} 9 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 0 \\ -4 \end{pmatrix}^\top \mathbf{x} = 0$
Hyperbola 2	$\mathbf{x}^\top \begin{pmatrix} 9 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -4 \\ 0 \end{pmatrix}^\top \mathbf{x} = 0$

Table : Hyperbola

By rearranging the two equations we get the equation of two hyperbolas as :

$$9x^2 - y^2 - 8y = 0 \tag{1}$$
$$9x^2 - y^2 - 8x = 0 \tag{2}$$

The conic parameters for the two hyperbolas can be expressed as :

$$\mathbf{V_1} = \begin{pmatrix} 9 & 0 \\ 0 & -1 \end{pmatrix} \qquad \mathbf{u_1} = \begin{pmatrix} 0 \\ -4 \end{pmatrix} \qquad f1 = 0 \tag{3}$$
$$\mathbf{V_2} = \begin{pmatrix} 9 & 0 \\ 0 & -1 \end{pmatrix} \qquad \mathbf{u_2} = \begin{pmatrix} -4 \\ 0 \end{pmatrix} \qquad f2 = 0 \tag{4}$$

The intersection of two conics is defined as :

$$\mathbf{x}^\top(\mathbf{V_1} + \mu\mathbf{V_2})\mathbf{x} + 2(\mathbf{u_1} + \mu\mathbf{u_2})^\top\mathbf{x} + (f1 + \mu f2) = 0 \tag{5}$$

The above equation represents a pair of straight lines if :

$$\begin{vmatrix} \mathbf{V_1} + \mu\mathbf{V_2} & \mathbf{u_1} + \mu\mathbf{u_2} \\ (\mathbf{u_1} + \mu\mathbf{u_2})^\top & f1 + \mu f2 \end{vmatrix} = 0 \tag{6}$$

Substituting the values in the above equation :

$$\begin{vmatrix} 9+9\mu & 0 & -4\mu \\ 0 & -1-\mu & -4 \\ -4\mu & -4 & 0 \end{vmatrix} = 0 \tag{7}$$

Applyint row reduction to find determinant:

$$\begin{vmatrix} 9+9\mu & 0 & -4\mu \\ 0 & -1-\mu & -4 \\ -4\mu & -4 & 0 \end{vmatrix} \xleftrightarrow{R_3 \to R_3 + \frac{4\mu}{9+9\mu}R_1} \begin{vmatrix} 9+9\mu & 0 & -4\mu \\ 0 & -1-\mu & -4 \\ 0 & -4 & -\frac{16\mu^2}{9+9\mu} \end{vmatrix} \xleftrightarrow{R_3 \to R_3 - \frac{4}{1+\mu}R_2} \begin{vmatrix} 9+9\mu & 0 & -4\mu \\ 0 & -1-\mu & -4 \\ 0 & 0 & \frac{-16\mu^2+144}{9+9\mu} \end{vmatrix} \tag{8}$$

By finding the determinant we get

$$(1+\mu)(-16\mu^2+144)=0 \tag{9}$$
$$\mu=-1, \mu=\pm 3 \tag{10}$$

Substituting $\mu=-1$ in (5) we get equation of line as :

$$2\begin{pmatrix}4\\-4\end{pmatrix}^{\top}\mathbf{x}=0 \tag{11}$$
$$\begin{pmatrix}-1 & 1\end{pmatrix}\mathbf{x}=0 \tag{12}$$

The parameters of the line are :

$$\mathbf{h}=\begin{pmatrix}0\\0\end{pmatrix} \qquad \mathbf{m}=\begin{pmatrix}1\\1\end{pmatrix} \tag{13}$$

Substituting the parameters of line and the first hyperbola in the below equation :

$$\kappa_i=\frac{1}{\mathbf{m}^{\top}\mathbf{V_1}\mathbf{m}}\left(-\mathbf{m}^{\top}(\mathbf{V_1}\mathbf{h}+\mathbf{u_1}) \pm \sqrt{\left[\mathbf{m}^{\top}(\mathbf{V_1}\mathbf{h}+\mathbf{u_1})\right]^2-g(\mathbf{h})\left(\mathbf{m}^{\top}\mathbf{V_1}\mathbf{m}\right)}\right) \tag{14}$$
$$\kappa_i=0,1 \tag{15}$$

Therefore the points of intersections of the line and first hyperbola are :

$$\mathbf{P_1}=\begin{pmatrix}1\\1\end{pmatrix} \qquad \mathbf{P_2}=\begin{pmatrix}0\\0\end{pmatrix} \tag{16}$$

But if we substiture $\mathbf{P_2}$ in the original equation we get 0 in the denominator , which is undefined. Therefore the solution for the given equations is :

$$\mathbf{P}=\begin{pmatrix}1\\1\end{pmatrix} \tag{17}$$

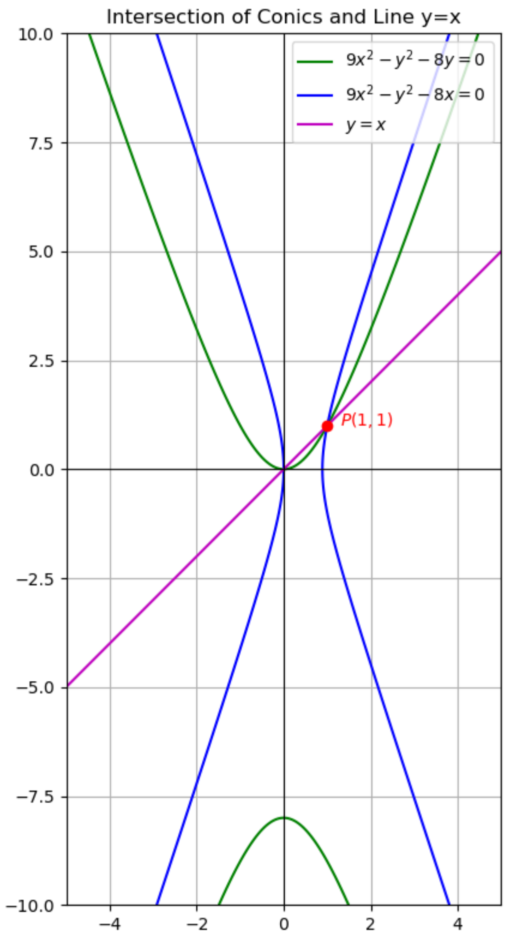

Fig : Hyperbola and Line