Extra

EE25BTECH11023 - Venkata Sai

Question:
Find the equations of tangents drawn from origin to the circle $x^2+y^2-2rx-2hy+h^2 = 0$,are

1) $x = 0$
2) $y = 0$

3) $\left(h^2 - r^2\right)x - 2rhy = 0$
4) $\left(h^2 - r^2\right)x + 2rhy = 0$

Solution:
Equation of a circle is given by

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0, \tag{1}$$

The parametric equation of this line is

$$\mathbf{x} = \mathbf{h} + k\mathbf{m}. \tag{2}$$

$$(\mathbf{h} + k\mathbf{m})^T \mathbf{V} (\mathbf{h} + k\mathbf{m}) + 2\mathbf{u}^T (\mathbf{h} + k\mathbf{m}) + f \tag{3}$$

$$\mathbf{h}^\top \mathbf{V} \mathbf{h} + k\left(\mathbf{m}^\top \mathbf{V} \mathbf{h} + \mathbf{h}^\top \mathbf{V} \mathbf{m}\right) + k^2 \mathbf{m}^\top \mathbf{V} \mathbf{m} + 2\mathbf{u}^\top \mathbf{h} + 2k\,\mathbf{u}^\top \mathbf{m} + f \tag{4}$$

$$\left(\mathbf{m}^\top \mathbf{V} \mathbf{m}\right)k^2 + 2\left(\mathbf{m}^\top \mathbf{V} \mathbf{h} + \mathbf{u}^\top \mathbf{m}\right)k + \left(\mathbf{h}^\top \mathbf{V} \mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f\right) \tag{5}$$

Hence the quadratic in k is

$$Ak^2 + Bk + C = 0, \tag{6}$$

where

$$A = \mathbf{m}^\top \mathbf{V} \mathbf{m}, \tag{7}$$
$$B = 2\left(\mathbf{m}^\top \mathbf{V} \mathbf{h} + \mathbf{u}^\top \mathbf{m}\right), \tag{8}$$
$$C = \left(\mathbf{h}^\top \mathbf{V} \mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f\right) \tag{9}$$

Discriminant is 0 as tangent intersects the circle at only one point

$$B^2 - 4AC = 0 \tag{10}$$

$$A = \mathbf{m}^\top \mathbf{V} \mathbf{m}, \tag{11}$$
$$B = 2\left(\mathbf{m}^\top \mathbf{V} \mathbf{h} + \mathbf{u}^\top \mathbf{m}\right), \tag{12}$$
$$C = \left(\mathbf{h}^\top \mathbf{V} \mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f\right) \tag{13}$$

Discriminant is 0 as tangent intersects the circle at only one point

$$B^2 - 4AC = 0 \tag{14}$$

Substituting the expressions for A, B, and C and expanding:

$$\left(2\left(\mathbf{m}^\top\mathbf{V}\mathbf{h} + \mathbf{u}^\top\mathbf{m}\right)\right)^2 - 4\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)\left(\mathbf{h}^\top\mathbf{V}\mathbf{h} + 2\mathbf{u}^\top\mathbf{h} + f\right) = 0. \tag{15}$$

$$\left(\mathbf{m}^\top\mathbf{V}\mathbf{h} + \mathbf{u}^\top\mathbf{m}\right)\left(\mathbf{m}^\top\mathbf{V}\mathbf{h} + \mathbf{u}^\top\mathbf{m}\right)^\top - \left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)\left(\mathbf{h}^\top\mathbf{V}\mathbf{h} + 2\mathbf{u}^\top\mathbf{h} + f\right) = 0 \tag{16}$$

$$\left(\mathbf{m}^\top\mathbf{V}\mathbf{h} + \mathbf{u}^\top\mathbf{m}\right)\left(\mathbf{h}^\top\mathbf{V}\mathbf{m} + \mathbf{m}^\top\mathbf{u}\right) - \left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)\left(\mathbf{h}^\top\mathbf{V}\mathbf{h} + 2\mathbf{u}^\top\mathbf{h} + f\right) = 0 \tag{17}$$

$$\left(\mathbf{m}^\top\mathbf{V}\mathbf{h}\right)\left(\mathbf{h}^\top\mathbf{V}\mathbf{m}\right) + \left(\mathbf{m}^\top\mathbf{V}\mathbf{h}\right)\left(\mathbf{m}^\top\mathbf{u}\right) + \left(\mathbf{u}^\top\mathbf{m}\right)\left(\mathbf{h}^\top\mathbf{V}\mathbf{m}\right) + \left(\mathbf{u}^\top\mathbf{m}\right)\left(\mathbf{m}^\top\mathbf{u}\right)$$
$$- \left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)\left(\mathbf{h}^\top\mathbf{V}\mathbf{h}\right) + 2\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)\left(\mathbf{u}^\top\mathbf{h}\right) + f\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right) = 0 \tag{18}$$

$$\left(\mathbf{m}^\top\mathbf{V}\mathbf{h}\right)\left(\mathbf{h}^\top\mathbf{V}\mathbf{m}\right) + \left(\mathbf{m}^\top\mathbf{V}\mathbf{h}\right)\left(\mathbf{m}^\top\mathbf{u}\right) + \left(\mathbf{u}^\top\mathbf{m}\right)\left(\mathbf{h}^\top\mathbf{V}\mathbf{m}\right) + \left(\mathbf{u}^\top\mathbf{m}\right)\left(\mathbf{m}^\top\mathbf{u}\right)$$
$$- \left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)\left(\mathbf{h}^\top\mathbf{V}\mathbf{h}\right) - 2\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)\left(\mathbf{u}^\top\mathbf{h}\right) - f\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right) = 0 \tag{19}$$

$$\mathbf{m}^\top\left(\mathbf{V}\mathbf{h}\mathbf{h}^\top\mathbf{V} + \mathbf{V}\mathbf{h}\mathbf{u}^\top + \mathbf{u}\mathbf{h}^\top\mathbf{V} + \mathbf{u}\mathbf{u}^\top - \left(\mathbf{h}^\top\mathbf{V}\mathbf{h}\right)\mathbf{V} - 2\left(\mathbf{u}^\top\mathbf{h}\right)\mathbf{V} - f\mathbf{V}\right)\mathbf{m} = 0 \tag{20}$$

$$\mathbf{m}^\top\left(\mathbf{V}\mathbf{h}\left(\mathbf{V}\mathbf{h}\right)^\top + \mathbf{V}\mathbf{h}\mathbf{u}^\top + \mathbf{u}\left(\mathbf{V}\mathbf{h}\right)^\top + \mathbf{u}\mathbf{u}^\top - \left(\mathbf{h}^\top\mathbf{V}\mathbf{h}\right)\mathbf{V} - 2\left(\mathbf{u}^\top\mathbf{h}\right)\mathbf{V} - f\mathbf{V}\right)\mathbf{m} = 0 \tag{21}$$

$$\mathbf{m}^\top\left(\left(\mathbf{V}\mathbf{h} + \mathbf{u}\right)\left(\mathbf{V}\mathbf{h} + \mathbf{u}\right)^\top - \left(\mathbf{h}^\top\mathbf{V}\mathbf{h} + 2\mathbf{u}^\top\mathbf{h} + f\right)\mathbf{V}\right)\mathbf{m} = 0 \tag{22}$$

$$\mathbf{m}^\top\left(\left(\mathbf{V}\mathbf{h} + \mathbf{u}\right)\left(\mathbf{V}\mathbf{h} + \mathbf{u}\right)^\top - g\left(\mathbf{h}\right)\right)\mathbf{m} = 0 \tag{23}$$

where

$$g\left(\mathbf{h}\right) = \left(\mathbf{h}^\top\mathbf{V}\mathbf{h} + 2\mathbf{u}^\top\mathbf{h} + f\right)\mathbf{V} \tag{24}$$

On comparing the given circle with matrix equation

$$x^2 + y^2 = \mathbf{x}^\top\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\mathbf{x} \implies \mathbf{V} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{25}$$

$$-2rx - 2hy = 2\mathbf{u}^\top\mathbf{x} \implies \mathbf{u}^\top = \begin{pmatrix}-r & -h\end{pmatrix} \implies \mathbf{u} = \begin{pmatrix}-r\\ -h\end{pmatrix} \tag{26}$$

$$h^2 = f \tag{27}$$

Given tangents are drawn from the origin

$$\mathbf{h} = \begin{pmatrix}0\\ 0\end{pmatrix} \tag{28}$$

$$\mathbf{V}\mathbf{h} + \mathbf{u} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0\\ 0\end{pmatrix} + \begin{pmatrix}-r\\ -h\end{pmatrix} = \begin{pmatrix}-r\\ -h\end{pmatrix} \tag{29}$$

$$\mathbf{V}\mathbf{h} = 0 \implies \mathbf{h}^\top\mathbf{V}\mathbf{h} = 0 \tag{30}$$

$$g\left(\mathbf{h}\right) = \left(\mathbf{h}^\top \mathbf{V}\mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f\right)\mathbf{V} = \left(0 + 2\begin{pmatrix} -r \\ -h \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} + h^2\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{31}$$

$$g\left(\mathbf{h}\right) = h^2 \tag{32}$$

$$\mathbf{m}^\top \sum \mathbf{m} \implies \sum = \begin{pmatrix} -r \\ -h \end{pmatrix}\begin{pmatrix} -r & -h \end{pmatrix} - h^2 \mathbf{I} \tag{33}$$

$$\sum = \begin{pmatrix} r^2 & rh \\ rh & h^2 \end{pmatrix} - \begin{pmatrix} h^2 & 0 \\ 0 & h^2 \end{pmatrix} = \begin{pmatrix} r^2 - h^2 & rh \\ rh & 0 \end{pmatrix} \tag{34}$$

$$\left|\sum - \lambda \mathbf{I}\right| = 0 \tag{35}$$

$$\left|\begin{pmatrix} r^2 - h^2 - \lambda & rh \\ rh & -\lambda \end{pmatrix}\right| = 0 \tag{36}$$

$$\left(r^2 - h^2 - \lambda\right)\left(-\lambda\right) - r^2 h^2 = 0 \tag{37}$$

$$\lambda^2 - \left(r^2 - h^2\right)\lambda - r^2 h^2 = 0 \tag{38}$$

$$\left(\lambda - r^2\right)\left(\lambda + h^2\right) = 0 \tag{39}$$

the eigen values are

$$\lambda_1 = r^2 \text{ and } \lambda_2 = -h^2 \tag{40}$$

Because the tangent passes through origin

$$\mathbf{m} = \mathbf{x} \tag{41}$$

$$\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} r^2 - h^2 & rh \\ rh & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \tag{42}$$

$$\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} \left(r^2 - h^2\right)x + \left(rh\right)y \\ rhx \end{pmatrix} = 0 \tag{43}$$

$$x\left(\left(r^2 - h^2\right)x + rhy\right) + y\left(rhx\right) = 0 \tag{44}$$

$$x^2\left(r^2 - h^2\right) + 2rhxy = 0 \tag{45}$$

$$x\left(\left(r^2 - h^2\right)x + 2rhy\right) = 0 \tag{46}$$

$$x = 0 \text{ and } \left(r^2 - h^2\right)x + 2rhy = 0 \tag{47}$$