2.8.31

EE25BTECH11044 - Sai Hasini Pappula

Question Given $A(1,-2)$, $B(2,3)$, $C(a,2)$ and $D(-4,-3)$ which form a parallelogram. Using only matrices and norms (no coordinate geometry formulas), find a and the height when AB is taken as base.

Solution

Represent the points as column vectors:

$$A = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad C = \begin{pmatrix} a \\ 2 \end{pmatrix}, \quad D = \begin{pmatrix} -4 \\ -3 \end{pmatrix}. \tag{0.1}$$

Parallelogram condition (diagonals bisect):

$$A + C = B + D. \tag{0.2}$$

Hence

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} a \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -4 \\ -3 \end{pmatrix}. \tag{0.3}$$

Compute the right-hand side and equate:

$$\begin{pmatrix} 1+a \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \quad \implies \quad a = -3. \tag{0.4}$$

Thus

$$C = \begin{pmatrix} -3 \\ 2 \end{pmatrix}. \tag{0.5}$$

Now let the base vector and the AC vector be

$$\mathbf{u} = AB = B - A = \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \qquad \mathbf{v} = AC = C - A = \begin{pmatrix} -4 \\ 4 \end{pmatrix}. \tag{0.6}$$

Form the orthogonal projector onto $\mathbf{u}$ (matrix form):

$$P_{\mathbf{u}} = \frac{\mathbf{u}\mathbf{u}^\top}{\mathbf{u}^\top\mathbf{u}}. \tag{0.7}$$

The component of $\mathbf{v}$ orthogonal to $\mathbf{u}$ is

$$\mathbf{w} = (I - P_{\mathbf{u}})\,\mathbf{v}. \tag{0.8}$$

The height h (distance from C to line through AB) is the norm of $\mathbf{w}$:

$$h = \|\mathbf{w}\| = \left\|(I - P_{\mathbf{u}})\mathbf{v}\right\|. \tag{0.9}$$

Compute the scalar products needed:

$$\mathbf{u}^\top \mathbf{u} = 1^2 + 5^2 = 26, \tag{0.10}$$

$$\mathbf{u}^\top \mathbf{v} = (-4)\cdot 1 + 4\cdot 5 = -4 + 20 = 16. \tag{0.11}$$

Thus the projector acting on $\mathbf{v}$ is

$$P_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{u}(\mathbf{u}^\top \mathbf{v})}{\mathbf{u}^\top \mathbf{u}} = \frac{16}{26}\mathbf{u} = \frac{8}{13}\begin{pmatrix}1\\5\end{pmatrix} = \begin{pmatrix}\frac{8}{13}\\ \frac{40}{13}\end{pmatrix}. \tag{0.12}$$

So

$$\mathbf{w} = \mathbf{v} - P_{\mathbf{u}}\mathbf{v} = \begin{pmatrix}-4\\4\end{pmatrix} - \begin{pmatrix}\frac{8}{13}\\ \frac{40}{13}\end{pmatrix} = \begin{pmatrix}-\frac{60}{13}\\ \frac{12}{13}\end{pmatrix}. \tag{0.13}$$

Therefore the height is

$$\begin{aligned} h = \|\mathbf{w}\| = \sqrt{\left(-\tfrac{60}{13}\right)^2 + \left(\tfrac{12}{13}\right)^2} = \frac{\sqrt{60^2+12^2}}{13} = \frac{\sqrt{3744}}{13} \\ = \frac{12\sqrt{26}}{13} = \frac{24}{\sqrt{26}}. \end{aligned} \tag{0.14}$$

Final results:

$$\boxed{a = -3}, \qquad \boxed{h = \frac{24}{\sqrt{26}} = \frac{12\sqrt{26}}{13}}. \tag{0.15}$$