10.6.8

EE25BTECH11052 - Shriyansh Kalpesh Chawda

Question

Construct a pair of tangents to a circle of radius 4cm from a point P lying outside the circle at a distance of 6cm from the centre. (10, 2023)

Solution

Let the center of the circle be at origin. The equation is $x^2 + y^2 = 16$ and Point P is at distance 6 from center along x-axis.

$$O = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1}$$

$$P = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \tag{2}$$

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{3}$$

where

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad f = -16 \tag{4}$$

Eigenvalue Decomposition of V:

The eigenvalue equation is:

$$\mathbf{V}\mathbf{p} = \lambda \mathbf{p} \tag{5}$$

The characteristic equation is:

$$\det(\mathbf{V} - \lambda \mathbf{I}) = 0 \tag{6}$$

$$\det \begin{pmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} = 0 \tag{7}$$

$$(1-\lambda)^2 = 0 \tag{8}$$

$$\lambda_1 = \lambda_2 = 1 \tag{9}$$

For $\lambda_1 = 1$, the eigenvector $\mathbf{p}_1$:

$$(\mathbf{V} - \lambda_1 \mathbf{I})\mathbf{p}_1 = 0 \tag{10}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p_{11} \\ p_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{11}$$

Choose $\mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (normalized) For $\lambda_2 = 1$, the eigenvector $\mathbf{p}_2$:

$$\mathbf{p}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ (orthogonal to } \mathbf{p}_1\text{)} \tag{12}$$

The eigenvector matrix (orthogonal) is:

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \tag{13}$$

Now using Spectral Decomposition:

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^\top \tag{14}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{15}$$
$$= \mathbf{I} \tag{16}$$

where $\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Step 3: Principal Axes Representation

The conic in principal axes coordinates: Since $\mathbf{P} = \mathbf{I}$ and $\mathbf{u} = \mathbf{0}$, the transformation is trivial.

$$\text{Let } \mathbf{y} = \mathbf{P}^\top(\mathbf{x} - \mathbf{c}), \text{ where } \mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{17}$$

In principal axes:

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 = -f \implies y_1^2 + y_2^2 = 16 \tag{18}$$

This confirms the circle has semi-axes along eigenvector directions with radii:

$$a = b = \sqrt{\frac{-f}{\lambda_1}} = \sqrt{\frac{16}{1}} = 4 \tag{19}$$

Step 4: Finding Contact Points using Eigenvector Framework

Transform point P to principal coordinates:

$$\mathbf{y}_P = \mathbf{P}^\top(P - \mathbf{c}) = \mathbf{I}\begin{pmatrix} 6 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \tag{20}$$

For tangent from external point, contact point $\mathbf{q}$ satisfies:

(a) $\mathbf{q}$ lies on circle: $\mathbf{q}^\top\mathbf{V}\mathbf{q} + f = 0$
(b) Tangent passes through P: $(\mathbf{V}\mathbf{q})^\top P + f = 0$

From condition (b) with $\mathbf{V} = \mathbf{I}$:

$$\mathbf{q}^\top P + f = 0 \tag{21}$$
$$\begin{pmatrix} q_1 & q_2 \end{pmatrix}\begin{pmatrix} 6 \\ 0 \end{pmatrix} = 16 \tag{22}$$
$$6q_1 = 16 \implies q_1 = \frac{8}{3} \tag{23}$$

From condition (a):

$$q_1^2 + q_2^2 = 16 \tag{24}$$

$$\left(\frac{8}{3}\right)^2 + q_2^2 = 16 \tag{25}$$

$$\frac{64}{9} + q_2^2 = 16 \tag{26}$$

$$q_2^2 = \frac{80}{9} \tag{27}$$

$$q_2 = \pm\frac{4\sqrt{5}}{3} \tag{28}$$

Expressing contact points using eigenvectors:

$$\mathbf{q}_1 = \frac{8}{3}\mathbf{p}_1 + \frac{4\sqrt{5}}{3}\mathbf{p}_2 = \frac{8}{3}\begin{pmatrix}1\\0\end{pmatrix} + \frac{4\sqrt{5}}{3}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\frac{8}{3}\\ \frac{4\sqrt{5}}{3}\end{pmatrix} \tag{29}$$

$$\mathbf{q}_2 = \frac{8}{3}\mathbf{p}_1 - \frac{4\sqrt{5}}{3}\mathbf{p}_2 = \frac{8}{3}\begin{pmatrix}1\\0\end{pmatrix} - \frac{4\sqrt{5}}{3}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\frac{8}{3}\\ -\frac{4\sqrt{5}}{3}\end{pmatrix} \tag{30}$$

The tangent at $\mathbf{q}$ is: $(\mathbf{V}\mathbf{q})^\top\mathbf{x} + f = 0$

Tangent 1 at $\mathbf{q}_1$:

$$\mathbf{V}\mathbf{q}_1 = \begin{pmatrix}\frac{8}{3}\\ \frac{4\sqrt{5}}{3}\end{pmatrix} \tag{31}$$

$$\begin{pmatrix}\frac{8}{3} & \frac{4\sqrt{5}}{3}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} - 16 = 0 \tag{32}$$

$$\frac{8}{3}x + \frac{4\sqrt{5}}{3}y = 16 \tag{33}$$

$$2x + \sqrt{5}y = 12 \tag{34}$$

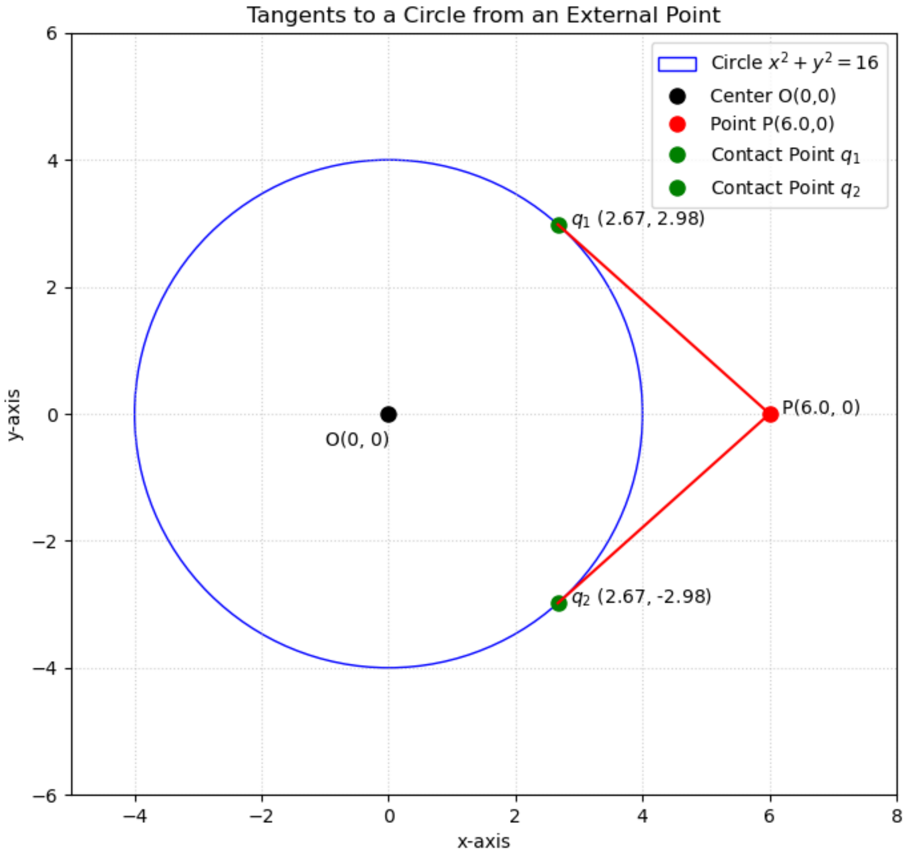
Tangent 2 at $\mathbf{q}_2$:

$$\begin{pmatrix}\frac{8}{3} & -\frac{4\sqrt{5}}{3}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} - 16 = 0 \tag{35}$$

$$2x - \sqrt{5}y = 12 \tag{36}$$

The equations of tangents are:

$$\boxed{2x + \sqrt{5}y = 12 \quad \text{and} \quad 2x - \sqrt{5}y = 12} \tag{37}$$


Tangents to a Circle from an External Point
Circle $x^2 + y^2 = 16$
Center O(0,0)
Point P(6.0,0)
Contact Point q_1
Contact Point q_2
q_1 (2.67, 2.98)
q_2 (2.67, -2.98)
P(6.0, 0)
O(0, 0)
y-axis
x-axis
6
4
2
0
−2
−4
−6
−4
−2
0
2
4
6
8