

4.13.57

EE25BTECH11052 - Shriyansh Kalpesh Chawda

Question:

Find the equation of the line passing through the point $(2, 3)$ and making intercept of length 2 units between the lines $y + 2x = 3$ and $y + 2x = 5$. (1991)

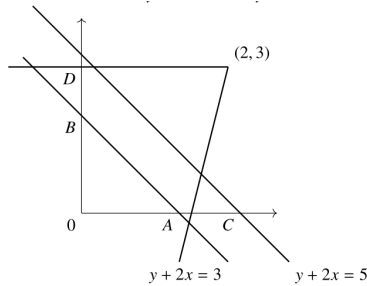


Fig. 0.1

Solution

The point is given by:

$$\mathbf{P} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

The two lines are expressed in the vector form $\mathbf{n} \cdot \mathbf{x} = d$.

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 3 \quad (0.1)$$

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 5 \quad (0.2)$$

The common normal vector is \mathbf{n} and the distance constants d_1 and d_2 :

$$\mathbf{n} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad d_1 = 3, \quad d_2 = 5 \quad (0.3)$$

The line passes through \mathbf{P} with an unknown direction vector \mathbf{v} is:

$$\mathbf{x} = \mathbf{P} + t\mathbf{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} v_x \\ v_y \end{pmatrix} \quad (0.4)$$

The unknown line intersect L_1 at point \mathbf{B} (parameter t_B) and L_2 at point \mathbf{D} (parameter t_D).

$$\mathbf{n} \cdot (\mathbf{P} + t_B \mathbf{v}) = d_1 \implies t_B = \frac{d_1 - \mathbf{n} \cdot \mathbf{P}}{\mathbf{n} \cdot \mathbf{v}} \quad (0.5)$$

$$\mathbf{n} \cdot (\mathbf{P} + t_D \mathbf{v}) = d_2 \implies t_D = \frac{d_2 - \mathbf{n} \cdot \mathbf{P}}{\mathbf{n} \cdot \mathbf{v}} \quad (0.6)$$

Now ,Using the length of intercept.

$$\|\mathbf{D} - \mathbf{B}\| = \|(\mathbf{P} + t_D \mathbf{v}) - (\mathbf{P} + t_B \mathbf{v})\| = |t_D - t_B| \cdot \|\mathbf{v}\| = 2 \quad (0.7)$$

Using (0.3) and (0.4)

$$t_D - t_B = \frac{d_2 - \mathbf{n} \cdot \mathbf{P}}{\mathbf{n} \cdot \mathbf{v}} - \frac{d_1 - \mathbf{n} \cdot \mathbf{P}}{\mathbf{n} \cdot \mathbf{v}} = \frac{d_2 - d_1}{\mathbf{n} \cdot \mathbf{v}} \quad (0.8)$$

Substituting this into the distance equation:

$$\left| \frac{d_2 - d_1}{\mathbf{n} \cdot \mathbf{v}} \right| \cdot \|\mathbf{v}\| = 2 \quad (0.9)$$

$$\frac{2}{|\mathbf{n} \cdot \mathbf{v}|} \cdot \|\mathbf{v}\| = 2 \implies \|\mathbf{v}\| = |\mathbf{n} \cdot \mathbf{v}| \quad (0.10)$$

We express the condition $\|\mathbf{v}\| = |\mathbf{n} \cdot \mathbf{v}|$ as a matrix quadratic form. Squaring both sides gives:

$$\|\mathbf{v}\|^2 = (\mathbf{n} \cdot \mathbf{v})^2 \quad (0.11)$$

$$\mathbf{v}^\top \mathbf{I} \mathbf{v} = \mathbf{v}^\top (\mathbf{n} \mathbf{n}^\top) \mathbf{v} \quad (0.12)$$

$$\mathbf{v}^\top \mathbf{I} \mathbf{v} - \mathbf{v}^\top (\mathbf{n} \mathbf{n}^\top) \mathbf{v} = 0 \quad (0.13)$$

$$\mathbf{v}^\top (\mathbf{n} \mathbf{n}^\top - \mathbf{I}) \mathbf{v} = 0, \quad (0.14)$$

Let,

$$\mathbf{Q} = \mathbf{n} \mathbf{n}^\top - \mathbf{I} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix} \quad (0.15)$$

We now solve the quadratic form equation for $\mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$:

$$\mathbf{v}^\top \mathbf{Q} \mathbf{v} = \begin{pmatrix} v_x & v_y \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = 0 \quad (0.16)$$

$$3v_x^2 + 4v_x v_y = 0 \implies v_x(3v_x + 4v_y) = 0 \quad (0.17)$$

This yields two possible solutions for the components of the direction vector.

(a) $v_x = 0$:

The direction vector is vertical.Hence \mathbf{v}_1 and vertical line passing through the point (2, 3)

is :

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (0.18)$$

$$\mathbf{x} = 2 \quad (0.19)$$

(b) $3v_x + 4v_y = 0$:

This implies

$$v_y = -\frac{3}{4}v_x \quad (0.20)$$

$$\mathbf{v}_2 = \begin{pmatrix} 4 \\ -3 \end{pmatrix} \quad (0.21)$$

So, corresponding to a slope of $m = -3/4$. Using the point-slope form :

$$y - 3 = -\frac{3}{4}(x - 2) \quad (0.22)$$

$$4(y - 3) = -3(x - 2) \quad (0.23)$$

$$4y - 12 = -3x + 6 \quad (0.24)$$

$$3x + 4y = 18 \quad (0.25)$$

The two lines that satisfy the given conditions are :

$$x = 2 \quad \text{and} \quad 3x + 4y = 18$$

