

7.4.32

EE25BTECH11025 - Ganachari Vishwambhar

Question:

$ABCD$ is a square of side length 2 units. C_1 is the circle touching all the sides of the square $ABCD$ and C_2 is the circumcircle of square $ABCD$. L is a fixed line in same plane and \mathbf{R} is a fixed point.

1) If \mathbf{P} is any point of C_1 and \mathbf{Q} is another point on C_2 , then $\frac{PA^2+PB^2+PC^2+PD^2}{QA^2+QB^2+QC^2+QD^2}$

- a) 0.75 b) 1.25 c) 1 d) 0.5

2) If a circle is such that it touches the line L and the circle C_1 externally, such that both the circles are on the same side of the line, then locus of centre of the circle

- a) ellipse b) hyperbola c) parabola d) circle

3) A L' through \mathbf{A} is drawn parallel to BD . Point S moves such that its distances from the line BD and the vertex \mathbf{A} are equal. If locus of S cuts L' at T_2 and T_3 and AC at T_1 , then area of $\triangle T_1 T_2 T_3$ is

- a) 1/2 sq.units b) 2/3 sq.units c) 1 sq.units d) 2 sq.units

Solution:

Let:

The centre of incircle and circumcircle be \mathbf{O} .

The radius of incircle be r_1 and that of circumcircle be r_2 .

Given:

$$r_1 = 1 \quad (1)$$

$$r_2 = \sqrt{2} \quad (2)$$

1) Let \mathbf{P} be any point on incircle and \mathbf{Q} be any point on circumcircle.
 $\mathbf{X} \in \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$

$$\|\mathbf{X} - \mathbf{P}\|^2 = \|\mathbf{X}\|^2 + \|\mathbf{P}\|^2 - 2\mathbf{P} \cdot \mathbf{X} \quad (3)$$

$$(4)$$

Summation over all $\mathbf{X} | \mathbf{X} \in \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$:

$$\sum \|\mathbf{X} - \mathbf{P}\|^2 = \sum \|\mathbf{X}\|^2 + 4 \cdot \|\mathbf{P}\|^2 - 2\mathbf{P} \cdot \sum \mathbf{X} \quad (5)$$

$$(6)$$

For, $\mathbf{P} = \mathbf{P}$

$$\sum \|\mathbf{X} - \mathbf{P}\|^2 = \sum \|\mathbf{X}\|^2 + 4 \cdot \|\mathbf{P}\|^2 - 2\mathbf{P} \sum \mathbf{X} \quad (7)$$

$$4(1^2 + 1^2) + 4(1) - 2\mathbf{P} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right) \quad (8)$$

$$\therefore \|\mathbf{A} - \mathbf{P}\|^2 + \|\mathbf{B} - \mathbf{P}\|^2 + \|\mathbf{C} - \mathbf{P}\|^2 + \|\mathbf{D} - \mathbf{P}\|^2 = 12 \quad (9)$$

For, $\mathbf{P} = \mathbf{Q}$

$$\sum \|\mathbf{X} - \mathbf{Q}\|^2 = \sum \|\mathbf{X}\|^2 + 4 \cdot \|\mathbf{Q}\|^2 - 2\mathbf{Q} \sum \mathbf{X} \quad (10)$$

$$4(1^2 + 1^2) + 4(2) - 2\mathbf{Q} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right) \quad (11)$$

$$\therefore \|\mathbf{A} - \mathbf{Q}\|^2 + \|\mathbf{B} - \mathbf{Q}\|^2 + \|\mathbf{C} - \mathbf{Q}\|^2 + \|\mathbf{D} - \mathbf{Q}\|^2 = 12 \quad (12)$$

conclusion:

$$\frac{12}{16} = 0.75 \quad (13)$$

Hence, option(a) is correct.

- 2) Let the radius of the moving circle be r , the centre of the circle be \mathbf{X} and the line equation be $\hat{\mathbf{n}}^\top \mathbf{X} = c$,

$$|\hat{\mathbf{n}}^\top \mathbf{X} - c| = r \quad (14)$$

$$\|\mathbf{X}\| = r + 1 \quad (15)$$

$$\|\mathbf{X}\| = |\hat{\mathbf{n}}^\top \mathbf{X} - (c - 1)| \quad (16)$$

$$\|\mathbf{X}\|^2 = |\hat{\mathbf{n}}^\top \mathbf{X} - (c - 1)|^2 \quad (17)$$

$$\mathbf{X}^\top \mathbf{X} = (\hat{\mathbf{n}}^\top \mathbf{X})^2 + (c - 1)^2 - 2\hat{\mathbf{n}}^\top \mathbf{X}(c - 1) \quad (18)$$

$$\mathbf{X}^\top \mathbf{X} - (\hat{\mathbf{n}}^\top \mathbf{X})^2 + 2\hat{\mathbf{n}}^\top \mathbf{X}(c - 1) - (c - 1)^2 \quad (19)$$

$$\mathbf{X}^\top (I - \hat{\mathbf{n}}\hat{\mathbf{n}}^\top) \mathbf{X} + 2(c - 1)\hat{\mathbf{n}}^\top \mathbf{X} - (c - 1)^2 \quad (20)$$

Equation (20) is the equation of parabola.

Hence, correct option is (c).

- 3) Let the point moving point be \mathbf{S} and the line equation be $\mathbf{n}^\top \mathbf{S} = 0$.

$$\frac{|\mathbf{n}^\top \mathbf{S}|}{\|\mathbf{n}\|} = \|\mathbf{S} - \mathbf{A}\| \quad (21)$$

$$\frac{|\mathbf{n}^\top \mathbf{S}|^2}{\|\mathbf{n}\|^2} = \|\mathbf{S} - \mathbf{A}\|^2 \quad (22)$$

$$\frac{|\mathbf{S}^\top \mathbf{n} \mathbf{n}^\top \mathbf{S}|}{\|\mathbf{n}\|} = (\mathbf{S} - \mathbf{A})^\top (\mathbf{S} - \mathbf{A}) \quad (23)$$

$$\mathbf{S}^\top (I - \hat{\mathbf{n}}\hat{\mathbf{n}}^\top) \mathbf{S} - 2\mathbf{A}^\top \mathbf{S} + \mathbf{A}^\top \mathbf{A} = 0 \quad (24)$$

Equation 24 is the locus of the moving point.

Let:

$$\mathbf{A} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \mathbf{B} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (25)$$

$$\mathbf{C} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}; \mathbf{D} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (26)$$

\mathbf{m}_1 be the direction vector of line AC, \mathbf{m}_2 be the direction vector of line L' .

$$\mathbf{m}_1 = \begin{pmatrix} -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (27)$$

$$\mathbf{m}_2 = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \quad (28)$$

The equation of line L'

$$\mathbf{S} = \mathbf{A} + t\mathbf{m}_2 \quad (29)$$

The equation of the line AC

$$\mathbf{S} = \lambda\mathbf{m}_1 \quad (30)$$

Substituting equation (30) in (24) we get $\lambda = \frac{1}{2}$:

$$\mathbf{T}_1 = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (31)$$

Substituting (29) in (24) we get $t = \frac{-1}{2}, \frac{1}{2}$

$$\mathbf{T}_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad (32)$$

$$\mathbf{T}_3 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad (33)$$

Now, finding area of the triangle:

$$\Delta T_1 T_2 T_3 = \frac{1}{2} \left\| (\mathbf{T}_2 - \mathbf{T}_1 \quad \mathbf{T}_3 - \mathbf{T}_1) \right\| \quad (34)$$

$$\Delta T_1 T_2 T_3 = 1 \quad (35)$$

Option (c) is correct.

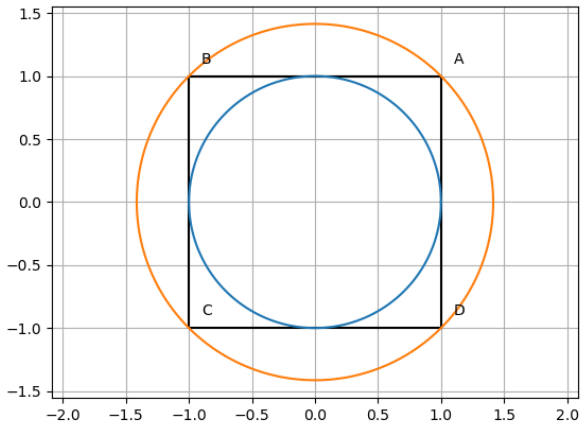


Fig. 1: Plot of the given square and circles

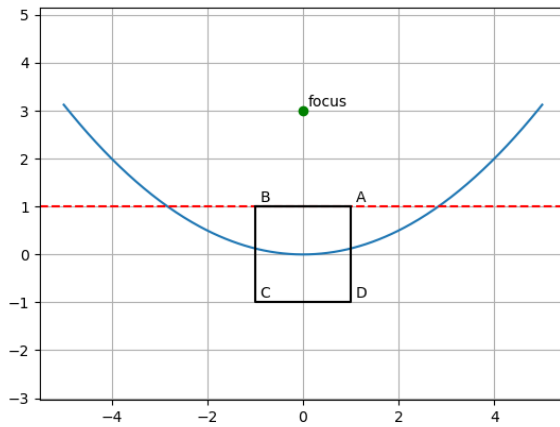


Fig. 2: Plot of the given circles, square and locus of the point

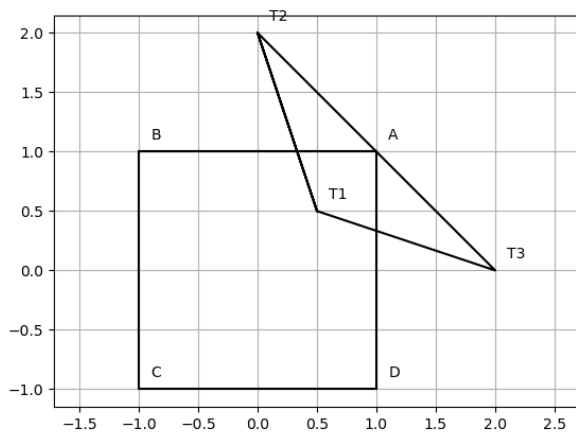


Fig. 3: Plot of the given circles, square and locus of the point