# Matgeo-2.10.28

Harichandana Varanasi-ai25btech11039

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# Question

**Q 2.10.28.** For non-zero vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, the relation

$$\left|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}\right| = \|\mathbf{a}\| \, \|\mathbf{b}\| \, \|\mathbf{c}\| \tag{1}$$

holds if and only if

- $\mathbf{a} \cdot \mathbf{b} = 0, \quad \mathbf{b} \cdot \mathbf{c} = 0$
- $\mathbf{b} \cdot \mathbf{c} = 0, \quad \mathbf{c} \cdot \mathbf{a} = 0$
- $\mathbf{c} \cdot \mathbf{a} = 0, \quad \mathbf{a} \cdot \mathbf{b} = 0$
- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a} = 0$

## Solution

*Let*

$$A = \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{pmatrix}, \qquad G = A^\top A = \begin{pmatrix} \mathbf{a}^\top \mathbf{a} & \mathbf{a}^\top \mathbf{b} & \mathbf{a}^\top \mathbf{c} \\ \mathbf{b}^\top \mathbf{a} & \mathbf{b}^\top \mathbf{b} & \mathbf{b}^\top \mathbf{c} \\ \mathbf{c}^\top \mathbf{a} & \mathbf{c}^\top \mathbf{b} & \mathbf{c}^\top \mathbf{c} \end{pmatrix} \tag{2}$$

*be the column and Gram matrices of* $\mathbf{a}, \mathbf{b}, \mathbf{c}$*. The given magnitude equals* $|\det A|$*, so*

$$|\det A|^2 = (\det A)^2 = \det A^\top A = \det G. \tag{3}$$

*By Hadamard's inequality for the positive semidefinite matrix* $G$*,*

$$\det G \le (\mathbf{a}^\top \mathbf{a})(\mathbf{b}^\top \mathbf{b})(\mathbf{c}^\top \mathbf{c}) = \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 \|\mathbf{c}\|^2, \tag{4}$$

*with equality iff* $G$ *is diagonal, i.e., the columns of* $A$ *are pairwise orthogonal:*

$$\mathbf{a}^\top \mathbf{b} = 0, \qquad \mathbf{b}^\top \mathbf{c} = 0, \qquad \mathbf{c}^\top \mathbf{a} = 0. \tag{5}$$

# Solution

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*Taking square roots in 4.2 and 4.3 yields*

$$|\det A| = \|\mathbf{a}\| \, \|\mathbf{b}\| \, \|\mathbf{c}\| \iff 4.4 \mathit{holds}. \tag{6}$$

*Hence, the correct option is* $\boxed{\textit{(d)}}$.

# Plot

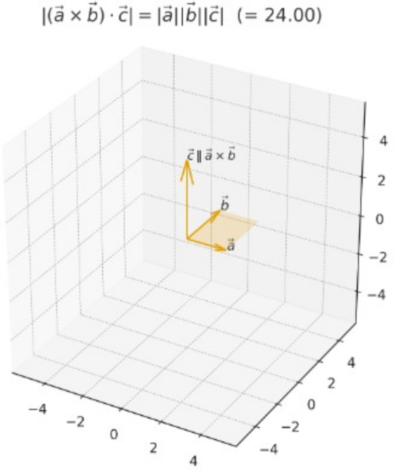


Figure: Illustration of $|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| = |\mathbf{a}|\,|\mathbf{b}|\,|\mathbf{c}|$ with $\mathbf{a} \perp \mathbf{b}$ and $\mathbf{c} \parallel (\mathbf{a} \times \mathbf{b})$.