Matgeo Presentation - Problem 10.7.7

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October 2, 2025

Question

The slope of the line touching both the parabolas $y^2 = 4x$ and $x^2 = -32y$ is

Solution

The equation of a parabola in Matrix form is

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{0.1}$$

For $y^2=4x$

$$\mathbf{V_1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{0.2}$$

$$\mathbf{u_1} = -2\mathbf{e_1} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{0.3}$$

$$f_1 = 0 \tag{0.4}$$

$$\implies \mathbf{x}^\top \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -2 \\ 0 \end{pmatrix} \mathbf{x} = 0 \tag{0.5}$$

For $x^2=-32y$

$$\mathbf{V_2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{0.6}$$

$$\mathbf{u_2} = 16\mathbf{e_2} = \begin{pmatrix} 0 \\ 16 \end{pmatrix} \tag{0.7}$$

Solution

$$f_2 = 0 \tag{0.8}$$

$$\implies \mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 0 \\ 16 \end{pmatrix} \mathbf{x} = 0 \tag{0.9}$$

a line $\mathbf{x}$=$\mathbf{h}$ +k$\mathbf{m}$ touches (0.1) if

$$\mathbf{m}^\top (\mathbf{Vq} + \mathbf{u}) = 0 \,(\text{where } \mathbf{q} \text{ is the point of contact}) \tag{0.10}$$

$$\mathbf{m}^\top (\mathbf{V_1 q_1} + \mathbf{u_1}) = 0 \tag{0.11}$$

$$\mathbf{m}^\top (\mathbf{V_2 q_2} + \mathbf{u_2}) = 0 \tag{0.12}$$

let

$$\mathbf{q_2} - \mathbf{q_1} = c\mathbf{m} \,(\text{for some scalar } c) \tag{0.13}$$

Solution

substitute (0.13) in (0.12)

$$\implies \mathbf{m}^\top \left(\mathbf{v_2}\left(\mathbf{q_1} + c\mathbf{m}\right) + \mathbf{u_2}\right) = 0 \tag{0.14}$$

$$\implies \mathbf{m}^\top\mathbf{v_2}\mathbf{q_1} + \mathbf{m}^\top\mathbf{v_2}c\mathbf{m} + \mathbf{m}^\top\mathbf{u_2} = 0 \tag{0.15}$$

$$\implies \begin{pmatrix}1 & m\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\mathbf{q_1} + \begin{pmatrix}1 & m\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}c\\cm\end{pmatrix} + \begin{pmatrix}1 & m\end{pmatrix}\begin{pmatrix}0\\16\end{pmatrix} = 0 \tag{0.16}$$

$$\implies \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{q_1} + \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}c\\cm\end{pmatrix} + 16m = 0 \tag{0.17}$$

$$\implies \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{q_1} = -c - 16m \tag{0.18}$$

Solution

on expanding (0.11)

$$\implies \mathbf{m}^\top \mathbf{V_1} \mathbf{q_1} + \mathbf{m}^\top \mathbf{u_1} = 0 \tag{0.19}$$

$$\implies \begin{pmatrix} 1 & m \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{q_1} + \begin{pmatrix} 1 & m \end{pmatrix} \begin{pmatrix} -2 \\ 0 \end{pmatrix} = 0 \tag{0.20}$$

$$\implies \begin{pmatrix} 0 & m \end{pmatrix} \mathbf{q_1} = 2 \tag{0.21}$$

Equations (0.18) and (0.21) can be written as

$$\begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix} \mathbf{q_1} = \begin{pmatrix} -c - 16m \\ 2 \end{pmatrix} \tag{0.22}$$

The augmented matrix can be written as

$$\left(\begin{array}{cc|c} 1 & 0 & -c - 16m \\ 0 & m & 2 \end{array} \right) \tag{0.23}$$

$$\implies \mathbf{q_1} = \begin{pmatrix} -c - 16m \\ \frac{2}{m} \end{pmatrix} \tag{0.24}$$

Solution

From (0.11)

$$\mathbf{q_2} = \mathbf{q_1} + c\mathbf{m} \tag{0.25}$$

$$\implies \mathbf{q_2} = \begin{pmatrix} -16m \\ \frac{2}{m} + cm \end{pmatrix} \tag{0.26}$$

substitute $\mathbf{q_1}$ in (0.5)

$$\implies \frac{1}{m^2} + 16m = -c \tag{0.27}$$

substitute $\mathbf{q_2}$ in (0.9)

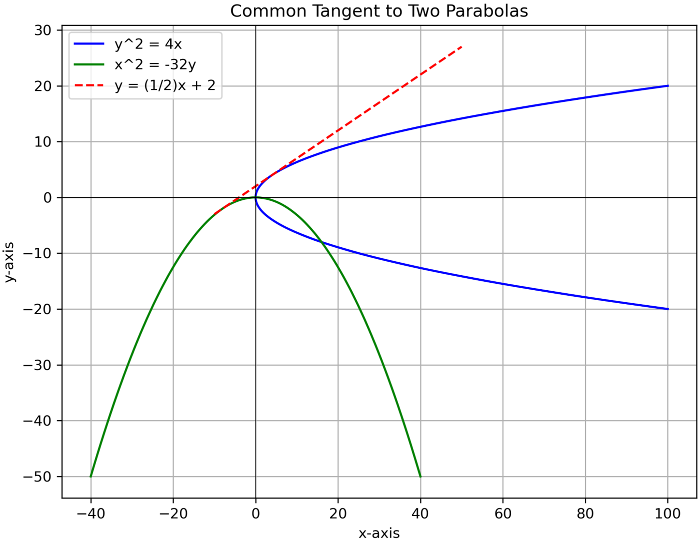
$$\implies 8m^2 + \frac{2}{m} = -cm \tag{0.28}$$

on solving (0.27) and (0.28) we get

$$m = \frac{1}{2} \tag{0.29}$$

$$\implies \mathbf{m} = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} \tag{0.30}$$

Plot



Common Tangent to Two Parabolas
y^2 = 4x
x^2 = -32y
y = (1/2)x + 2
30
20
10
0
−10
−20
−30
−40
−50
y-axis
−40
−20
0
20
40
60
80
100
x-axis

C Code: Slope.c

```
#include <stdio.h>
#include <math.h>

int main() {
    FILE *fp;
    double m;

    // Calculation based on derived formula: m^3 = 1/8
    m = cbrt(1.0/8.0); // cube root

    // Open file slope.dat for writing
    fp = fopen("slope.dat", "w");
    if(fp == NULL) {
        printf("Error␣opening␣file!\n");
        return 1;
    }

    // Write slope value into file
    fprintf(fp, "The␣slope␣of␣the␣common␣tangent␣is:␣%lf\n", m);

    fclose(fp);

    printf("Result␣written␣to␣slope.dat␣successfully.\n");
    return 0;
}
```

Python: plot.py

```
import numpy as np
import matplotlib.pyplot as plt

# Define parabola 1: y^2 = 4x -> x = y^2 / 4
y1 = np.linspace(-20, 20, 400)
x1 = (y1**2) / 4

# Define parabola 2: x^2 = -32y -> y = -x^2 / 32
x2 = np.linspace(-40, 40, 400)
y2 = -(x2**2) / 32

# Define common tangent: y = (1/2)x + 2
x_tan = np.linspace(-10, 50, 400)
y_tan = 0.5 * x_tan + 2

# Plot the parabolas
plt.figure(figsize=(8, 6))
plt.plot(x1, y1, 'b', label='y^2␣=␣4x')
plt.plot(x2, y2, 'g', label='x^2␣=␣-32y')

# Plot the tangent line
plt.plot(x_tan, y_tan, 'r--', label='y␣=␣(1/2)x␣+␣2')
plt.xlabel("x-axis")
plt.ylabel("y-axis")
plt.title("Common␣Tangent␣to␣Two␣Parabolas")
plt.axhline(0, color='black', linewidth=0.5)
plt.axvline(0, color='black', linewidth=0.5)
plt.legend()
plt.grid(True)
plt.savefig("slope_plot.png", dpi=300)
# Show the plot
plt.show()
```