4.11.3

Vishwambhar - EE25BTECH11025

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Question

Find the equation of the line passing through (2,-1,2) and (5,3,4) and the equation of the plane passing through (2,0,3), (1,1,5), and (3,2,4). Also, find their point of intersection.

Given

Let:

$$\mathbf{P}_1 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}; \mathbf{P}_2 = \begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix} \tag{1}$$

$$\mathbf{A} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}; \mathbf{B} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}; \mathbf{C} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} \tag{2}$$

Vector forms

Direction vector of the line:

$$\mathbf{m} = \mathbf{P}_2 - \mathbf{P}_1 = \begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} \tag{3}$$

Vector form of the line can be written as:

$$\mathbf{x} = \mathbf{P}_1 + \kappa \mathbf{m} \tag{4}$$

Vector form of the line can be written as:

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}^\top \mathbf{n} = \mathbf{1} \tag{5}$$

$$\begin{pmatrix} 2 & 0 & 3 \\ 1 & 1 & 5 \\ 3 & 2 & 4 \end{pmatrix} \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{6}$$

Augmented matrix

Augmented matrix can be written as:

$$\left(\begin{array}{ccc|c} 2 & 0 & 3 & 1 \\ 1 & 1 & 5 & 1 \\ 3 & 2 & 4 & 1 \end{array}\right) R_2 \leftrightarrow R_1 \left(\begin{array}{ccc|c} 1 & 1 & 5 & 1 \\ 2 & 0 & 3 & 1 \\ 3 & 2 & 4 & 1 \end{array}\right) \frac{R_2 \to R_2 - 2R_1}{R_3 \to R_3 - 3R_1} \tag{7}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 5 & 1 \\ 0 & -2 & -7 & -1 \\ 0 & -1 & -11 & -2 \end{array}\right) \frac{R_2 \leftrightarrow R_3}{R_2 \to -R_2} \left(\begin{array}{ccc|c} 1 & 1 & 5 & 1 \\ 0 & 1 & 11 & 2 \\ 0 & -2 & -7 & -1 \end{array}\right) \tag{8}$$

Augmented matrix

$$\xrightarrow[R_3 \to R_3 + 2R_2]{R_1 \to R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 0 & -6 & -1 \\ 0 & 1 & 11 & 2 \\ 0 & 0 & 15 & 3 \end{array}\right) R_3 \to \frac{1}{15} R_3 \tag{9}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -6 & -1 \\ 0 & 1 & 11 & 2 \\ 0 & 0 & 1 & \frac{1}{5} \end{array}\right) \xrightarrow[R_2 \to R_2 - 11R_3]{R_1 \to R_1 + 6R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{5} \\ 0 & 1 & 0 & \frac{-1}{5} \\ 0 & 0 & 1 & \frac{1}{5} \end{array}\right) \tag{10}$$

Therefore, the plane equation is:

$$\begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \mathbf{x} = 5 \mathbf{n}^\top \mathbf{x} = c \tag{11}$$

Substituting (4) in (11):

$$\mathbf{n}^\top \left(\mathbf{P}_1 + \kappa \mathbf{m} \right) = c \tag{12}$$

$$(\mathbf{n}^\top \mathbf{P}_1) + (\kappa \mathbf{n}^\top \mathbf{m}) = c \tag{13}$$

$$\kappa = \frac{c - (\mathbf{n}^\top \mathbf{P}_1)}{\mathbf{n}^\top \mathbf{m}} \tag{14}$$

Point of intersection

The point of intersection is (from(4)):

$$\mathbf{x} = \mathbf{P}_1 + \left(\frac{c - (\mathbf{n}^\top \mathbf{P}_1)}{\mathbf{n}^\top \mathbf{m}} \right) \mathbf{m} \tag{15}$$

Substituting the values from (11), (1) and (3):

$$\mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \left(\frac{0}{-3} \right) \begin{pmatrix} 3 & 4 & 2 \end{pmatrix} \tag{16}$$

$$\mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \tag{17}$$

C Code

```c
#include <stdio.h>
void get_data(double *out_data){
    double P0[3] = {2.0, -1.0, 2.0};
    double P1[3] = {5.0, 3.0, 4.0};
    double Q1[3] = {2.0, 0.0, 3.0};
    double Q2[3] = {1.0, 1.0, 5.0};
    double Q3[3] = {3.0, 2.0, 4.0};
    double u[3], v[3], n[3];
    for(int i=0;i<3;i++){
        u[i] = Q2[i] - Q1[i];
        v[i] = Q3[i] - Q1[i];
    }
    n[0] = u[1]*v[2] - u[2]*v[1];
    n[1] = u[2]*v[0] - u[0]*v[2];
    n[2] = u[0]*v[1] - u[1]*v[0];
```

C Code

```c
double dir[3];
for(int i=0;i<3;i++) dir[i] = P1[i] - P0[i];
double numerator = 0.0, denominator = 0.0, rhs = 0.0;
for(int i=0;i<3;i++){
    rhs += n[i] * Q1[i];
    numerator += n[i] * P0[i];
    denominator += n[i] * dir[i];
}
double t;
if (denominator == 0.0) {
    if (numerator == rhs) t = 0.0;
    else {
        t = 0.0/0.0;
    }
} else {
    t = (rhs - numerator) / denominator;
}
```

C Code

```c
    for(int i=0;i<3;i++) X[i] = P0[i] + t * dir[i];
    out_data[0] = P0[0]; out_data[1] = P0[1]; out_data[2] = P0[2];
    out_data[3] = P1[0]; out_data[4] = P1[1]; out_data[5] = P1[2];
    out_data[6] = Q1[0]; out_data[7] = Q1[1]; out_data[8] = Q1[2];
    out_data[9] = Q2[0]; out_data[10]= Q2[1]; out_data[11]= Q2[2];
    out_data[12]= Q3[0]; out_data[13]= Q3[1]; out_data[14]= Q3[2];
    out_data[15]= X[0]; out_data[16]= X[1]; out_data[17]= X[2];
}
```

Python Code 1

```
import ctypes
import numpy as np
lib = ctypes.CDLL("./problem.so")
pointP = [0.00,0.00]
pointQ = [0.00,0.00]
pointR = [0.00,0.00]

for i in range(0,2):
    pointP[i] = lib.get_pointP(i)
for i in range(0,2):
    pointQ[i] = lib.get_pointQ(i)
for i in range(0,2):
    pointR[i] = lib.get_pointR(i)
```

Python Code 1

```python
normal = [0,0]
print(pointP)
print(pointQ)
print(pointR)
for i in range(0,2):
    normal[i] = pointQ[i] + pointR[i] - (2*pointP[i])
z = np.array(['x','y'])
z_t = z.T
k = 0.00
for i in range(0,2):
    k += ((pointQ[i]**2)+(pointR[i]**2)-(2*(pointP[i]**2)))/2
print(normal,z_t,'=',k,"\nHence the locus of S is a line.")
```

Python Code 2

```
import numpy as np
import matplotlib.pyplot as plt
from call import get_data

P0, P1, Q1, Q2, Q3, X = get_data()
P0 = np.asarray(P0); P1 = np.asarray(P1)
Q1 = np.asarray(Q1); Q2 = np.asarray(Q2); Q3 = np.asarray(Q3)
X = np.asarray(X)
fig = plt.figure(figsize=(8,6))
ax = fig.add_subplot(111, projection='3d')
t = np.linspace(-1, 2, 50)
dirv = P1 - P0
line_pts = P0[None,:] + t[:,None] * dirv[None,:]
ax.plot(line_pts[:,0], line_pts[:,1], line_pts[:,2], label='Line'
    , linewidth=2)
```

Python Code 2

```
u = Q2 - Q1
v = Q3 - Q1
s = np.linspace( -0.5, 1.2, 10 )
r = np.linspace( -0.5, 1.2, 10 )
S,R = np.meshgrid(s, r)
plane_pts = Q1[None,None,:] + S[:,:,None]*u[None,None,:] + R[:,:,
    None]*v[None,None,:]
ax.plot_surface(plane_pts[:,:,0], plane_pts[:,:,1], plane_pts
    [:,:,2], alpha=0.5)
ax.scatter(*P0, color='red', s=40, label='P0 (line)')
ax.scatter(*P1, color='red', s=40, label='P1 (line)')
ax.scatter(*Q1, color='green', s=40, label='Q1 (plane)')
```

Python Code 2

```
ax.scatter(*Q2, color='green', s=40, label='Q2 (plane)')
ax.scatter(*Q3, color='green', s=40, label='Q3 (plane)')
ax.scatter(*X, color='black', s=70, label='Intersection')

ax.set_xlabel('X'); ax.set_ylabel('Y'); ax.set_zlabel('Z')
ax.legend()
plt.savefig("../figs/plot.png")
plt.show()
```

Plot

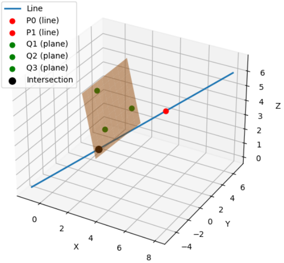

Figure: Plot of given plane and line