

# 4.13.6

EE25BTECH11001 - Aarush Dilawri

## Question:

Given the points  $\mathbf{A}(0, 4)$  and  $\mathbf{B}(0, -4)$ , the equation of the locus of the point  $\mathbf{p}(x, y)$ , such that  $(AP - BP)^2 = 6^2$

## Solution:

$$\mathbf{A}, \mathbf{B}, \mathbf{P} \in \mathbb{R}^n \quad (0.1)$$

Let the given scalar be  $\delta \geq 0$

$$(r_1 - r_2)^2 = \delta^2 \quad (0.2)$$

where  $r_1 = \|\mathbf{P} - \mathbf{A}\|$  and  $r_2 = \|\mathbf{P} - \mathbf{B}\|$ .

Taking square root,

$$|r_1 - r_2| = \pm \delta \quad (0.3)$$

Let's define

$$D = s\delta \quad (0.4)$$

where  $s \in \{+1, -1\}$  such that

$$r_1 - r_2 = s\delta = D \quad (0.5)$$

Let's find  $r_1^2 - r_2^2$

$$\|\mathbf{P} - \mathbf{A}\|^2 - \|\mathbf{P} - \mathbf{B}\|^2 = (\mathbf{P} - \mathbf{A})^\top (\mathbf{P} - \mathbf{A}) - (\mathbf{P} - \mathbf{B})^\top (\mathbf{P} - \mathbf{B}) \quad (0.6)$$

$$= -2\mathbf{P}^\top \mathbf{u} + \mathbf{A}^\top \mathbf{A} - \mathbf{B}^\top \mathbf{B} = -2\mathbf{P}^\top \mathbf{u} + \alpha. \quad (0.7)$$

where

$$\mathbf{u} = \mathbf{A} - \mathbf{B} \quad \text{and} \quad \alpha = \mathbf{A}^\top \mathbf{A} - \mathbf{B}^\top \mathbf{B}. \quad (0.8)$$

Use  $(r_1 - r_2)(r_1 + r_2) = r_1^2 - r_2^2$  and  $r_1 - r_2 = D$  to get

$$D(r_1 + r_2) = -2\mathbf{P}^\top \mathbf{u} + \alpha \implies r_1 + r_2 = \frac{-2\mathbf{P}^\top \mathbf{u} + \alpha}{D}. \quad (0.9)$$

Hence

$$r_1 = \frac{(r_1 - r_2) + (r_1 + r_2)}{2} = \frac{D}{2} + \frac{\alpha}{2D} - \frac{\mathbf{P}^\top \mathbf{u}}{D}. \quad (0.10)$$

Square this expression and equate to the explicit quadratic form for  $r_1^2$ :

$$\left(\frac{D}{2} + \frac{\alpha}{2D} - \frac{\mathbf{P}^\top \mathbf{u}}{D}\right)^2 = (\mathbf{P} - \mathbf{A})^\top (\mathbf{P} - \mathbf{A}) = \mathbf{P}^\top \mathbf{P} - 2\mathbf{P}^\top \mathbf{A} + \mathbf{A}^\top \mathbf{A}. \quad (0.11)$$

Multiply both sides by  $D^2$  and simplify. After collecting terms one obtains the general quadratic (conic) equation in the vector  $\mathbf{P}$ :

$$\mathbf{P}^\top (\mathbf{u}\mathbf{u}^\top - D^2 I) \mathbf{P} + (- (D^2 + \alpha) \mathbf{u} + 2D^2 \mathbf{A})^\top \mathbf{P} + \frac{(D^2 + \alpha)^2}{4} - D^2 \mathbf{A}^\top \mathbf{A} = 0. \quad (0.12)$$

Now substitute  $\mathbf{A} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$ ,  $\mathbf{B} = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$  and  $\delta = 6$ .

$$\mathbf{u} = \mathbf{A} - \mathbf{B} = \begin{pmatrix} 0 \\ 8 \end{pmatrix} \quad (0.13)$$

$$\alpha = \mathbf{A}^\top \mathbf{A} - \mathbf{B}^\top \mathbf{B} = 16 - 16 = 0 \quad (0.14)$$

$$D = s\delta = \pm 6 \quad \Rightarrow \quad D^2 = 36 \quad (0.15)$$

The quadratic matrix equation becomes

$$\mathbf{P}^\top (\mathbf{u}\mathbf{u}^\top - 36I) \mathbf{P} + (-D^2 \mathbf{u} + 2D^2 \mathbf{A})^\top \mathbf{P} + \frac{D^4}{4} - 36\mathbf{A}^\top \mathbf{A} = 0 \quad (0.16)$$

Now compute each term.

$$\mathbf{u}\mathbf{u}^\top = \begin{pmatrix} 0 & 0 \\ 0 & 64 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (0.17)$$

So

$$\mathbf{u}\mathbf{u}^\top - 36I = \begin{pmatrix} -36 & 0 \\ 0 & 28 \end{pmatrix} \quad (0.18)$$

Next, the linear coefficient:

$$-D^2 \mathbf{u} + 2D^2 \mathbf{A} = -36 \begin{pmatrix} 0 \\ 8 \end{pmatrix} + 72 \begin{pmatrix} 0 \\ 4 \end{pmatrix} \quad (0.19)$$

$$= \begin{pmatrix} 0 \\ -288 \end{pmatrix} + \begin{pmatrix} 0 \\ 288 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (0.20)$$

So there is no linear term.

Finally, the constant term:

$$\frac{D^4}{4} - 36\mathbf{A}^\top \mathbf{A} = \frac{1296}{4} - 36(16) \quad (0.21)$$

$$= 324 - 576 \quad (0.22)$$

$$= -252 \quad (0.23)$$

Therefore, the locus is given by

$$\mathbf{P}^\top \begin{pmatrix} -36 & 0 \\ 0 & 28 \end{pmatrix} \mathbf{P} - 252 = 0 \quad (0.24)$$

or equivalently

$$\mathbf{P}^\top \begin{pmatrix} -36 & 0 \\ 0 & 28 \end{pmatrix} \mathbf{P} = 252 \quad (0.25)$$

Expanding with  $\mathbf{P} = \begin{pmatrix} x \\ y \end{pmatrix}$ ,

$$-36x^2 + 28y^2 = 252 \quad (0.26)$$

Dividing through,

$$\frac{y^2}{9} - \frac{x^2}{7} = 1 \quad (0.27)$$

Thus the locus is a hyperbola centered at the origin with equation

$$\frac{y^2}{9} - \frac{x^2}{7} = 1 \quad (0.28)$$