2.10.65

EE25BTECH11036 - M Chanakya Srinivas

Problem Statement

Let $OACB$ be a parallelogram with O at the origin and OC a diagonal. Let D be the midpoint of OA. Using vector methods, prove that BD and CO intersect in the same ratio. Determine this ratio.

Solution

Let the position vectors of the vertices be:

$$\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{1}$$

$$\mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \tag{2}$$

$$\mathbf{B} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}, \tag{3}$$

$$\mathbf{C} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \tag{4}$$

Since D is the midpoint of OA, we have:

$$\mathbf{D} = \frac{\mathbf{O} + \mathbf{A}}{2} = \frac{1}{2}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{5}$$

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Step 1: Represent the lines in vector form
Line BD:

$$\mathbf{R_1} = \mathbf{B} + \lambda(\mathbf{D} - \mathbf{B}) \tag{6}$$

$$= \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix} + \lambda\left(\frac{1}{2}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} - \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}\right) \tag{7}$$

$$= \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix} - \lambda\left(\frac{1}{2}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) \tag{8}$$

Line CO:

$$\begin{aligned}
\mathbf{R_2} &= \mathbf{C} + \mu(\mathbf{O} - \mathbf{C}) && (9)\\
&= \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + \mu\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) && (10)\\
&= (1-\mu)\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} && (11)
\end{aligned}$$

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Step 2: Find the intersection by equating lines

$$\begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix} - \lambda\left(\frac{1}{2}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) = (1-\mu)\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{12}$$

Equating coefficients:

$$\begin{aligned}
\text{For } \mathbf{a}: &\quad 1 - \frac{\lambda}{2} = 0 \quad \Rightarrow \quad \lambda = 2 && (13)\\
\text{For } \mathbf{b}: &\quad 1 - \lambda = 1 - \mu \quad \Rightarrow \quad \mu = 2 && (14)
\end{aligned}$$

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Step 3: Interpret the ratio

- On BD, $\lambda = 2$ implies the intersection divides BD in the ratio $2:1$. - On CO, $\mu = 2$ implies the intersection divides CO in the ratio $2:1$.

$$\boxed{\text{The lines } BD \text{ and } CO \text{ intersect in the ratio } 2:1.} \tag{15}$$

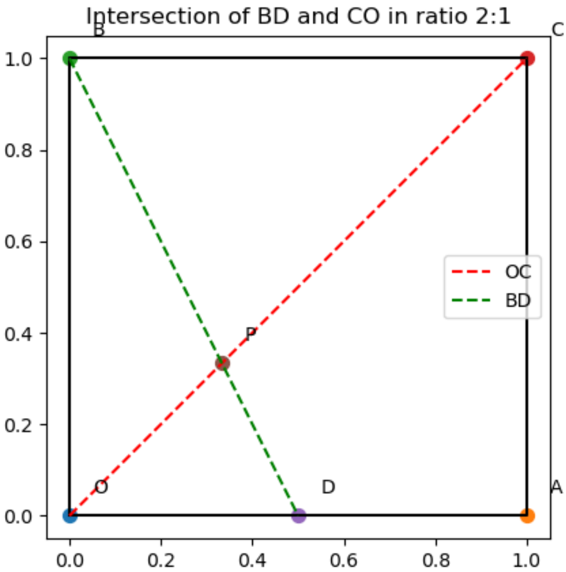
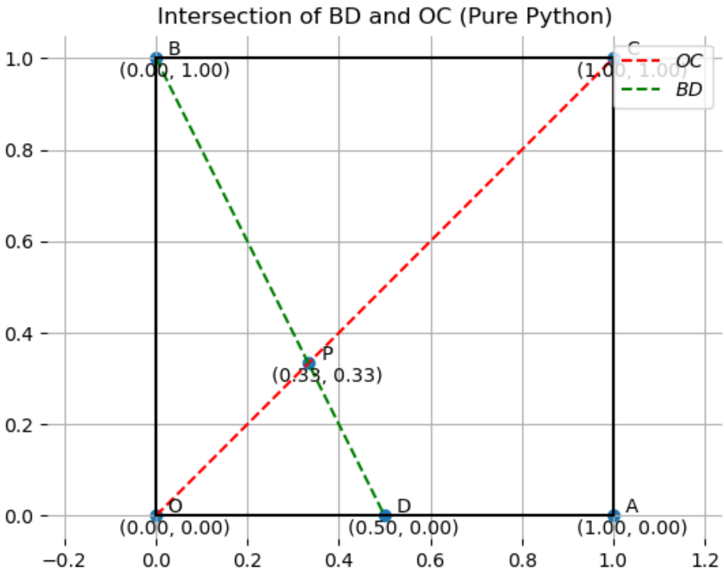

Intersection of BD and CO in ratio 2:1
B
C
P
O
D
A
OC
BD
0.0
0.2
0.4
0.6
0.8
1.0

Fig. 1



Intersection of BD and OC (Pure Python)
B
(0.00, 1.00)
C
(1.00, 1.00)
OC
BD
P
(0.33, 0.33)
O
(0.00, 0.00)
D
(0.50, 0.00)
A
(1.00, 0.00)
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
0.0
0.2
0.4
0.6
0.8
1.0
1.2

Fig. 2