Problem 8.3.12

Find the equation of the set of all points the sum of whose distances from the points $\begin{pmatrix}3\\0\end{pmatrix}$ and $\begin{pmatrix}9\\0\end{pmatrix}$ is 12.

Input Variables

Variable	Value
$\mathbf{F}_1$	$\begin{pmatrix}3\\0\end{pmatrix}$
$\mathbf{F}_2$	$\begin{pmatrix}9\\0\end{pmatrix}$
$2a$	12

Table 1

Solution

Step 1: Center and axis data

$$\mathbf{c} = \frac{\mathbf{F}_1 + \mathbf{F}_2}{2} = \begin{pmatrix}6\\0\end{pmatrix}, \quad \mathbf{v} = \frac{\mathbf{F}_2 - \mathbf{F}_1}{\|\mathbf{F}_2 - \mathbf{F}_1\|} = \begin{pmatrix}1\\0\end{pmatrix}, \quad c_f = \frac{\|\mathbf{F}_2 - \mathbf{F}_1\|}{2} = 3, \quad a = 6. \tag{1}$$

Shift to the midpoint frame: $\mathbf{y} := \mathbf{x} - \mathbf{c}$.

Step 2: Start from the sum-of-distances definition

$$\|\mathbf{y} - c_f\mathbf{v}\| + \|\mathbf{y} + c_f\mathbf{v}\| = 2a. \tag{2}$$

Step 3: Eliminate square roots (squaring twice)

Let $r_\pm := \|\mathbf{y} \pm c_f\mathbf{v}\|$. From (2), $r_+ + r_- = 2a$.

$$r_+r_- = 2a^2 - \|\mathbf{y}\|^2 - c_f^2, \tag{3}$$

$$r_+ - r_- = \frac{2c_f}{a}\,\mathbf{v}^\top\mathbf{y} \;\Rightarrow\; r_+r_- = a^2 - \frac{c_f^2}{a^2}(\mathbf{v}^\top\mathbf{y})^2. \tag{4}$$

Equating the two expressions for $r_+ r_-$ yields

$$\|\mathbf{y}\|^2 - \frac{c_f^2}{a^2}(\mathbf{v}^\top \mathbf{y})^2 = a^2 - c_f^2 \ =: \ b^2. \tag{5}$$

Step 4: Principal directions and the matrix D

Choose an orthonormal basis of principal directions:

$$\mathbf{p}_1 = \mathbf{v}, \qquad \mathbf{p}_2 \perp \mathbf{p}_1, \qquad P := \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} \ \text{(orthonormal)}. \tag{6}$$

Decompose $\mathbf{y}$ as $\mathbf{y} = \alpha\,\mathbf{p}_1 + \beta\,\mathbf{p}_2$, where $\alpha = \mathbf{p}_1^\top \mathbf{y} = \mathbf{v}^\top \mathbf{y}$ and $\beta = \mathbf{p}_2^\top \mathbf{y}$. Then $\|\mathbf{y}\|^2 = \alpha^2 + \beta^2$. Substituting into (5) gives

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} = 1. \tag{7}$$

In matrix form this is

$$\mathbf{y}^\top \Big(P \operatorname{diag}\!\big(\tfrac{1}{a^2}, \tfrac{1}{b^2}\big)\, P^\top \Big) \mathbf{y} = 1. \tag{8}$$

Hence define

$$D := P \operatorname{diag}\!\Big(\tfrac{1}{a^2}, \tfrac{1}{b^2}\Big)\, P^\top, \qquad \text{so that} \qquad (\mathbf{x} - \mathbf{c})^\top D\,(\mathbf{x} - \mathbf{c}) = 1. \tag{9}$$

Step 5: Specialization to this data

Here $\mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{p}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, so $P = I$ and

$$b^2 = a^2 - c_f^2 = 36 - 9 = 27, \tag{10}$$

$$D = \operatorname{diag}\!\Big(\tfrac{1}{a^2}, \tfrac{1}{b^2}\Big) = \begin{pmatrix} \frac{1}{36} & 0 \\ 0 & \frac{1}{27} \end{pmatrix}. \tag{11}$$

Therefore the **centered matrix equation of the locus** is exactly (9) with

$$\boxed{(\mathbf{x} - \begin{pmatrix} 6 \\ 0 \end{pmatrix})^\top \begin{pmatrix} \frac{1}{36} & 0 \\ 0 & \frac{1}{27} \end{pmatrix} (\mathbf{x} - \begin{pmatrix} 6 \\ 0 \end{pmatrix}) = 1} \tag{12}$$

Step 6: General quadratic (matrix) form

Expanding (9) gives $\mathbf{x}^\top V \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0$ with

$$V = D, \qquad \mathbf{u} = -V\mathbf{c}, \qquad f = \mathbf{c}^\top V \mathbf{c} - 1. \tag{13}$$

Numerically,

$$\boxed{V = \begin{pmatrix} \frac{1}{36} & 0 \\ 0 & \frac{1}{27} \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} -\frac{1}{6} \\ 0 \end{pmatrix}, \quad f = 0}. \tag{14}$$

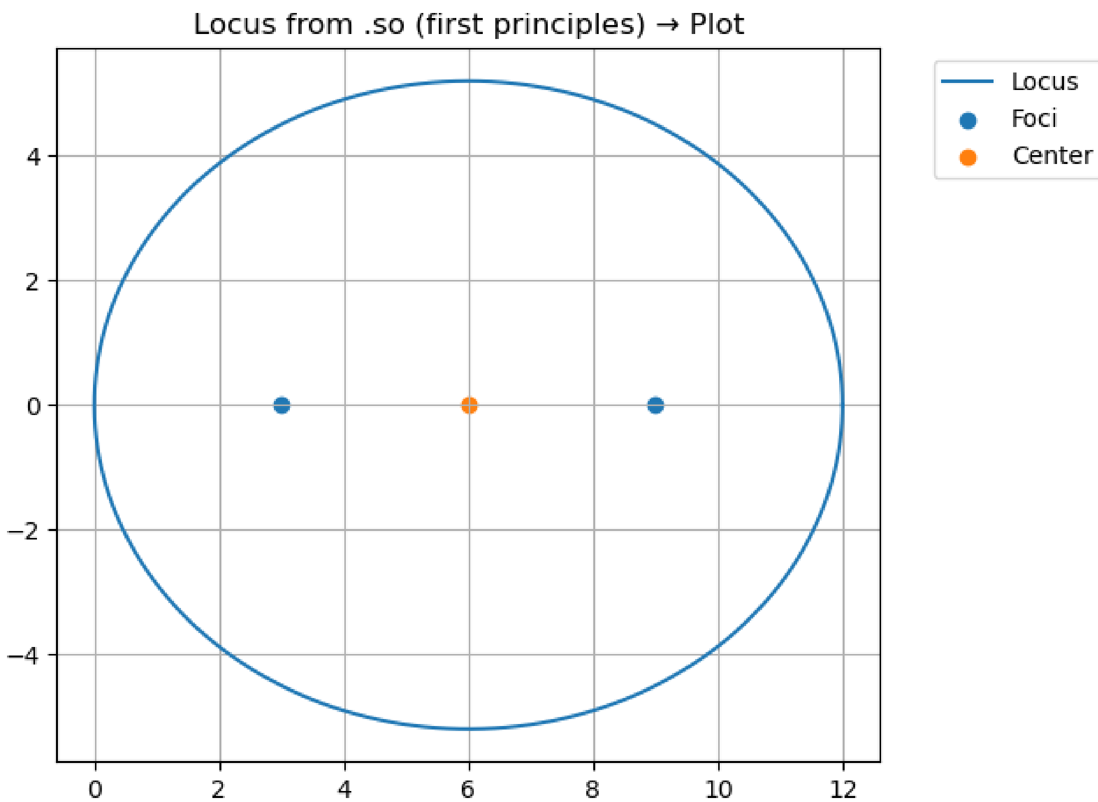

Figure 1