Problem

Find the coordinates of the foot of the perpendicular $\mathbf{Q}$ drawn from $\mathbf{P} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ to the plane $\mathbf{N}^T\mathbf{x} = 1$, where $\mathbf{N} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$. Also, find the distance $\|\mathbf{P} - \mathbf{Q}\|$ and the image of the point $\mathbf{P}$ treating the plane as a mirror.

Solution

Let the point be:

$$\mathbf{P} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

The plane is defined by:

$$\mathbf{N}^T\mathbf{x} = 1 \quad \text{where} \quad \mathbf{N} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

To find the foot of the perpendicular from $\mathbf{P}$ to the plane, we use:

$$\mathbf{Q} = \mathbf{P} - \frac{\mathbf{N}^T\mathbf{P} - 1}{\mathbf{N}^T\mathbf{N}}\mathbf{N}$$

Compute:

$$\mathbf{N}^T\mathbf{P} = \begin{pmatrix} 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 6 - 2 + 1 = 5 \quad \Rightarrow \quad \mathbf{N}^T\mathbf{P} - 1 = 4$$

$$\mathbf{N}^T\mathbf{N} = \begin{pmatrix} 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 4 + 1 + 1 = 6$$

So:

$$\mathbf{Q} = \mathbf{P} - \frac{4}{6}\mathbf{N} = \mathbf{P} - \frac{2}{3}\mathbf{N} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 - \frac{4}{3} \\ 2 + \frac{2}{3} \\ 1 - \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ \frac{8}{3} \\ \frac{1}{3} \end{pmatrix}$$

The distance is:

$$\|\mathbf{P} - \mathbf{Q}\| = \left\|\frac{2}{3}\mathbf{N}\right\| = \frac{2}{3}\sqrt{6}$$

The image of $\mathbf{P}$ reflected across the plane is:

$$\mathbf{R} = \mathbf{P} - 2 \cdot \frac{\mathbf{N}^T\mathbf{P} - 1}{\mathbf{N}^T\mathbf{N}}\mathbf{N} = \mathbf{P} - \frac{4}{3}\mathbf{N} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} - \frac{4}{3}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 - \frac{8}{3} \\ 2 + \frac{4}{3} \\ 1 - \frac{4}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{10}{3} \\ -\frac{1}{3} \end{pmatrix}$$

Final Answer

Foot of perpendicular: $\mathbf{Q} = \begin{pmatrix} \frac{5}{3} \\ \frac{8}{3} \\ \frac{1}{3} \end{pmatrix}$

Distance $\|\mathbf{P} - \mathbf{Q}\| = \frac{2}{3}\sqrt{6}$

Image of $\mathbf{P}$: $\begin{pmatrix} \frac{1}{3} \\ \frac{10}{3} \\ -\frac{1}{3} \end{pmatrix}$

Plot

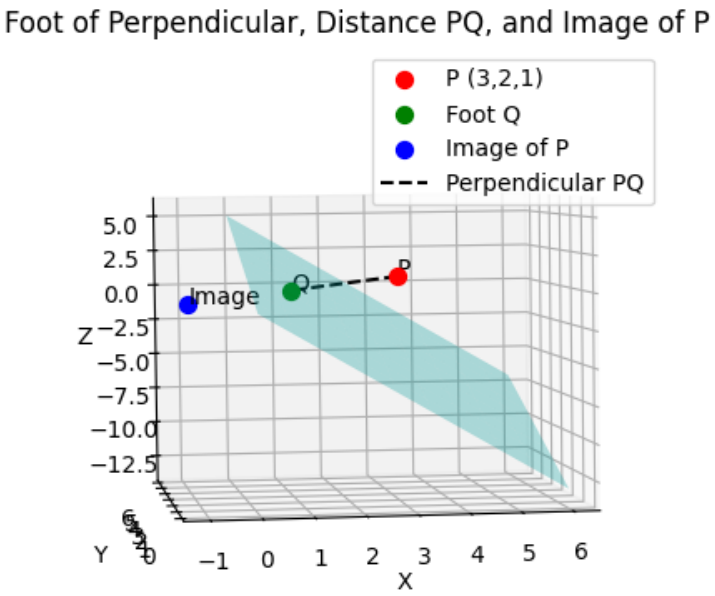

Fig. 1: Point **P** on the line and dividing **AB**