Problem 5.13.18.

If the system of linear equations

$$x + ky + 3z = 0, \tag{1}$$
$$3x + ky - 2z = 0, \tag{2}$$
$$2x + 4y - 3z = 0 \tag{3}$$

has a non-zero solution (x, y, z), then $\frac{xz}{y^2}$ is equal to

$$\text{a) } 10 \quad \text{b) } -30 \quad \text{c) } 30 \quad \text{d) } -10 \tag{4}$$

Input Variables:

Variable	Description
x, y, z	Unknowns of the system
k	Parameter in the system

Solution:

Start with the augmented matrix:

$$\begin{pmatrix} 1 & k & 3 & 0 \\ 3 & k & -2 & 0 \\ 2 & 4 & -3 & 0 \end{pmatrix}. \tag{5}$$

Eliminating below the first pivot:

$$R_2 \to R_2 - 3R_1, \quad R_3 \to R_3 - 2R_1 \Rightarrow \begin{pmatrix} 1 & k & 3 & 0 \\ 0 & -2k & -11 & 0 \\ 0 & 4-2k & -9 & 0 \end{pmatrix}. \tag{6}$$

Next, remove the second entry in row 3:

$$R_3 \to R_3 + \left(\tfrac{2}{k} - 1\right) R_2 \Rightarrow \begin{pmatrix} 1 & k & 3 & 0 \\ 0 & -2k & -11 & 0 \\ 0 & 0 & \frac{2(k-11)}{k} & 0 \end{pmatrix}. \tag{7}$$

For a homogeneous system $A\mathbf{v} = 0$, a non-trivial solution exists only if $\operatorname{rank}(A) < 3$. Hence the last pivot must vanish:

$$\frac{2(k-11)}{k} = 0 \Rightarrow k = 11. \tag{8}$$

Substitute $k = 11$:

$$\begin{pmatrix} 1 & 11 & 3 & 0 \\ 3 & 11 & -2 & 0 \\ 2 & 4 & -3 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 11 & 3 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{9}$$

From row 2: $2y + z = 0 \Rightarrow z = -2y$.
From row 1: $x + 11y + 3z = 0 \Rightarrow x = -5y$.

$$\mathbf{v} = y \begin{pmatrix} -5 \\ 1 \\ -2 \end{pmatrix}, \quad y \neq 0. \tag{10}$$

Finally,

$$\frac{xz}{y^2} = \frac{(-5y)(-2y)}{y^2} = 10. \tag{11}$$

$$\boxed{10} \tag{12}$$

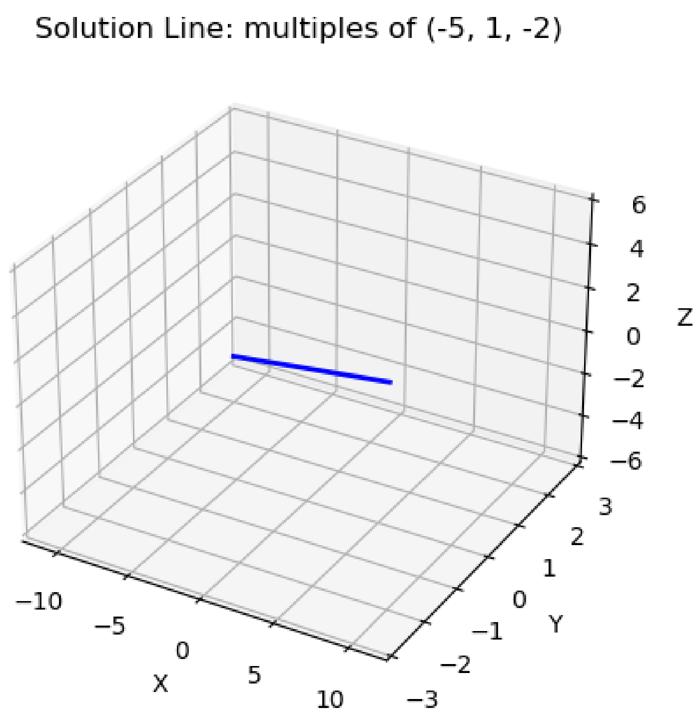

Figure 1