2.10.2

Gautham-AI25BTECH11013

September 30, 2025

Question

Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be vectors of lengths 3, 4, and 5 respectively such that $\mathbf{A} \perp \mathbf{B} + \mathbf{C}$, $\mathbf{B} \perp \mathbf{C} + \mathbf{A}$, and $\mathbf{C} \perp \mathbf{A} + \mathbf{B}$. Find the length of the vector $\mathbf{A} + \mathbf{B} + \mathbf{C}$.

Theoretical Solution

Let the Gram matrix G for the vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be:

$$G = \begin{pmatrix} \mathbf{A}^T\mathbf{A} & \mathbf{A}^T\mathbf{B} & \mathbf{A}^T\mathbf{C} \\ \mathbf{B}^T\mathbf{A} & \mathbf{B}^T\mathbf{B} & \mathbf{B}^T\mathbf{C} \\ \mathbf{C}^T\mathbf{A} & \mathbf{C}^T\mathbf{B} & \mathbf{C}^T\mathbf{C} \end{pmatrix} = \begin{pmatrix} 9 & a & b \\ a & 16 & c \\ b & c & 25 \end{pmatrix} \tag{1}$$

where $a = \mathbf{A}^T\mathbf{B}$, $b = \mathbf{A}^T\mathbf{C}$, and $c = \mathbf{B}^T\mathbf{C}$.
Given the orthogonality conditions:

$$\mathbf{A} \perp \mathbf{B} + \mathbf{C} \implies \mathbf{A}^T(\mathbf{B} + \mathbf{C}) = 0 \implies a + b = 0, \tag{2}$$
$$\mathbf{B} \perp \mathbf{C} + \mathbf{A} \implies \mathbf{B}^T(\mathbf{C} + \mathbf{A}) = 0 \implies c + a = 0, \tag{3}$$
$$\mathbf{C} \perp \mathbf{A} + \mathbf{B} \implies \mathbf{C}^T(\mathbf{A} + \mathbf{B}) = 0 \implies b + c = 0. \tag{4}$$

Theoretical Solution

This system can be written as:

$$a + b = 0 \tag{5}$$
$$c + a = 0 \tag{6}$$
$$b + c = 0. \tag{7}$$

In matrix form:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{8}$$

Convert the coefficient matrix to upper triangular form by row operations:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \tag{9}$$

Theoretical Solution

From the last row:

$$2c = 0 \implies c = 0 \tag{10}$$

From the second row:

$$-b + c = 0 \implies b = 0 \tag{11}$$

From the first row:

$$a + b = 0 \implies a = 0 \tag{12}$$

Thus, the Gram matrix is:

$$G = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 25 \end{pmatrix} \tag{13}$$

Theoretical Solution

Let $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Now, the squared length of $\mathbf{A} + \mathbf{B} + \mathbf{C}$ is:

$$||\mathbf{A} + \mathbf{B} + \mathbf{C}||^2 = \mathbf{u}^T \mathbf{G} \mathbf{u} \tag{14}$$

Expanding using the Gram matrix:

$$||\mathbf{A} + \mathbf{B} + \mathbf{C}||^2 = 50 \tag{15}$$

Therefore,

$$||\mathbf{A} + \mathbf{B} + \mathbf{C}|| = \sqrt{50} = 5\sqrt{2} \tag{16}$$

Python Code

```
import numpy as np
import numpy.linalg as la
import math
a=3
b=4
c=5
#x=a.b,y=b.c,z=c.a
#x+y=0,y+z=0,x+z=0
B=np.array([0,0,0])
A=np.array([[1,1,0],[0,1,1],[1,0,1]])
X=la.solve(A,B)
d=a*a+b*b+c*c+2*np.sum(X)
print(math.sqrt(d))
```