10.7.4

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Question

Prove that $y = 2x + 2\sqrt{3}$ is commom tangent to the parabola $y^2 = 16\sqrt{3}x$ and the ellipse $2x^2 + y^2 = 4$

Solution

General Formulae of a conic

$$\mathbf{x}^T V\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{1}$$

The tangent condition for line $\mathbf{n}^T\mathbf{x} + c = 0$ at contact point $\mathbf{q}$ is

$$V\mathbf{q} + \mathbf{u} = \lambda\mathbf{n} \tag{2}$$

$$\mathbf{u}^T\mathbf{q} + f = \lambda c \tag{3}$$

Solution

for some scalar λ. For Parabola $y^2 = 16\sqrt{3}\,x$

$$V = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{4}$$

$$\mathbf{u} = \begin{pmatrix} -8\sqrt{3} \\ 0 \end{pmatrix} \tag{5}$$

$$f = 0. \tag{6}$$

Line: $2x - y + 2\sqrt{3} = 0$, so

$$\mathbf{n} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{7}$$

$$c = 2\sqrt{3} \tag{8}$$

Solution

First condition

$$V\mathbf{q}+\mathbf{u}=\begin{pmatrix}-8\sqrt{3}\\ y\end{pmatrix}=\lambda\begin{pmatrix}2\\ -1\end{pmatrix}. \tag{9}$$

Thus $\lambda=-4\sqrt{3}$ and $y=4\sqrt{3}$.
Second condition:

$$\mathbf{u}^T\mathbf{q}+f=-8\sqrt{3}\,x=\lambda c=(-4\sqrt{3})(2\sqrt{3})=-24, \tag{10}$$

giving $x=\sqrt{3}$.

Solution

$$\mathbf{q} = \begin{pmatrix} \sqrt{3} \\ 4\sqrt{3} \end{pmatrix} \tag{11}$$

So the line touches the parabola at $(\sqrt{3}, 4\sqrt{3})$.
For Circle $2x^2 + y^2 = 4$

$$V = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \tag{12}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{13}$$

$$f = -4. \tag{14}$$

Solution

First condition:

$$V\mathbf{q} = \lambda\mathbf{n} \tag{15}$$

we get

$$\mathbf{q} = \lambda\begin{pmatrix}1\\-1\end{pmatrix}. \tag{16}$$

Second condition:

$$\mathbf{u}^T\mathbf{q} + f = -4 = \lambda c = \lambda(2\sqrt{3}) \tag{17}$$

$$\Rightarrow \quad \lambda = -\frac{2}{\sqrt{3}}. \tag{18}$$

Solution

So

$$\mathbf{q} = -\frac{2}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{2\sqrt{3}}{3} \\ \frac{2\sqrt{3}}{3} \end{pmatrix}. \tag{19}$$

$$\mathbf{q} = \begin{pmatrix} -\frac{2\sqrt{3}}{3} \\ \frac{2\sqrt{3}}{3} \end{pmatrix} \tag{20}$$

So the line touches the circle at $\left(-\frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\right)$. Hence given line is common tangent to both the curves.

Python Code

```
    import numpy as np
import matplotlib.pyplot as plt
# Line: y = 2x + 23
def line(x):
    return 2*x + 2*np.sqrt(3)
# Parabola: y^2 = 163 x => x = y^2 / (163)
y_parabola = np.linspace(-10, 10, 400)
x_parabola = y_parabola**2 / (16 * np.sqrt(3))
# Ellipse: 2x^2 + y^2 = 4
theta = np.linspace(0, 2*np.pi, 400)
a = np.sqrt(2) # semi-major axis
b = 2 # semi-minor axis
```

Python Code

```python
    x_ellipse = a * np.cos(theta)
y_ellipse = b * np.sin(theta)
# Line domain (chosen to cover range of ellipse and parabola)
x_line = np.linspace(-2, 2, 400)
y_line = line(x_line)

# Plotting
plt.figure(figsize=(8, 8))
plt.plot(x_parabola, y_parabola, label=r'$y^2 = 16\sqrt{3}x$',
    color='green')
plt.plot(x_parabola, -y_parabola, color='green') # the other half
     of the parabola
```

Python Code

```
    plt.plot(x_ellipse, y_ellipse, label=r'$2x^2 + y^2 = 4$',
        color='blue')
plt.plot(x_line, y_line, label=r'$y = 2x + 2\sqrt{3}$', color='
    red', linestyle='--')

plt.axhline(0, color='black', linewidth=0.5)
plt.axvline(0, color='black', linewidth=0.5)
plt.grid(True)
plt.legend()
```

Python Code

```
    plt.title("Common Tangent to a Parabola and an Ellipse")
plt.xlabel("x-axis")
plt.ylabel("y-axis")
plt.axis('equal')
plt.show()
```

C Code

```c
   #include <stdio.h>
#include <math.h>

// Define square root of 3
#define SQRT3 1.73205080757

// Function to check if a line is tangent to a conic
int check_tangent() {
    double A, B, C, D;
    double discriminant;
```

C Code

```c
// For Parabola: y = 2x + 23 into y^2 = 163 x
A = 1;
B = -2 * SQRT3;
C = 3;
discriminant = B*B - 4*A*C;

if (fabs(discriminant) > 1e-6) return 0; // Not a tangent to
    parabola
```

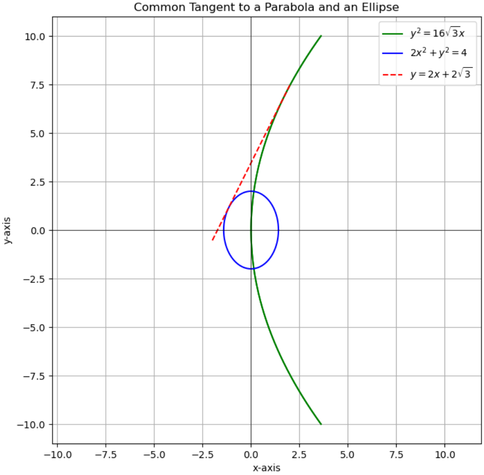
C Code

```
 // For Ellipse: y = 2x + 23 into 2x^2 + y^2 = 4
A = 6;
B = 8 * SQRT3;
C = 8;
discriminant = B*B - 4*A*C;

if (fabs(discriminant) > 1e-6) return 0; // Not a tangent to
    ellipse

return 1; // Common tangent
}
```

Python and C Code

```python
    import ctypes
# Load the shared object file
conic = ctypes.CDLL("./conic_tangent.so")
# Call the function
result = conic.check_tangent()
if result == 1:
    print("The line is a common tangent to both the parabola and
        the ellipse.")
else:
    print("The line is NOT a common tangent.")
```

plot


Common Tangent to a Parabola and an Ellipse
$y^2 = 16\sqrt{3}x$
$2x^2 + y^2 = 4$
$y = 2x + 2\sqrt{3}$
10.0
7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5
−10.0
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0
x-axis
y-axis