10.7.94

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Question:

A circle touches the X axis and also touches the circle with centre at $(0, 3)$ and radius 2. The locus of the centre of the circle is

1) an ellipse
2) a circle
3) a hyperbola
4) a parabola

Solution:

Let the center of the moving circle be $\mathbf{c}$ and its radius be r. The circle touches the X-axis (the line $\mathbf{e_1}^\top \mathbf{x} = 0$), so its radius is the y-coordinate of its center.

$$r = \mathbf{e_2}^\top \mathbf{c} \ \left(\text{assuming } \mathbf{e_2}^\top \mathbf{c} > 0\right) \tag{1}$$

The fixed circle has center $\mathbf{c}_f$ and radius r_f. The distance between the centers of two externally touching circles is the sum of their radii.

$$\left\|\mathbf{c} - \mathbf{c}_f\right\| = r + r_f \tag{2}$$

$$\left\|\mathbf{c} - \mathbf{c}_f\right\| = \mathbf{e_2}^\top \mathbf{c} + r_f \tag{3}$$

Squaring both sides,

$$\left(\mathbf{c} - \mathbf{c}_f\right)^\top \left(\mathbf{c} - \mathbf{c}_f\right) = \left(\mathbf{e_2}^\top \mathbf{c} + r_f\right)^2 \tag{4}$$

$$\mathbf{c}^\top \mathbf{c} - 2\mathbf{c}_f^\top \mathbf{c} + \mathbf{c}_f^\top \mathbf{c}_f = \left(\mathbf{e_2}^\top \mathbf{c}\right)^2 + 2r_f \left(\mathbf{e_2}^\top \mathbf{c}\right) + r_f^2 \tag{5}$$

Rearranging to the matrix quadratic form $\mathbf{c}^\top \mathbf{V} \mathbf{c} + 2\mathbf{u}^\top \mathbf{c} + f = 0$:

$$\mathbf{c}^\top \mathbf{c} - \left(\mathbf{c}^\top \mathbf{e_2}\right)\left(\mathbf{e_2}^\top \mathbf{c}\right) - 2\mathbf{c}_f^\top \mathbf{c} - 2r_f \mathbf{e_2}^\top \mathbf{c} + \mathbf{c}_f^\top \mathbf{c}_f - r_f^2 = 0 \tag{6}$$

$$\mathbf{c}^\top \left(\mathbf{I} - \mathbf{e_2}\mathbf{e_2}^\top\right) \mathbf{c} + 2\left(-\mathbf{c}_f - r_f \mathbf{e_2}\right)^\top \mathbf{c} + \left(\mathbf{c}_f^\top \mathbf{c}_f - r_f^2\right) = 0 \tag{7}$$

The given values are $\mathbf{c}_f = 3\mathbf{e_2}$ and $r_f = 2$.

$$\mathbf{V} = \mathbf{I} - \mathbf{e_2}\mathbf{e_2}^\top = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{8}$$

$$\mathbf{u} = -\mathbf{c}_f - r_f \mathbf{e_2} = -3\mathbf{e_2} - 2\mathbf{e_2} = -5\mathbf{e_2} = \begin{pmatrix} 0 \\ -5 \end{pmatrix} \tag{9}$$

$$f = \mathbf{c}_f^\top \mathbf{c}_f - r_f^2 = (3\mathbf{e_2})^\top (3\mathbf{e_2}) - 2^2 = 9 - 4 = 5 \tag{10}$$

The locus in the standard form of the conic is

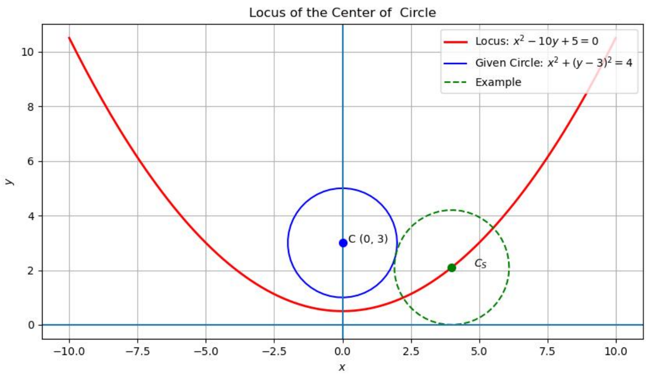
$$\mathbf{c}^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{c} + 2\begin{pmatrix} 0 & -5 \end{pmatrix} \mathbf{c} + 5 = 0 \tag{11}$$

The type of conic section is determined by the eigenvalues of $\mathbf{V}$. For a diagonal matrix, the eigenvalues are the diagonal entries.

$$\lambda_1 = 1, \ \lambda_2 = 0 \tag{12}$$

$$\left|\mathbf{V}\right| = \lambda_1 \lambda_2 = 1 \cdot 0 = 0 \tag{13}$$

Since one of the eigenvalues is zero, the locus is a parabola.
The correct option is **4) a parabola**.



Plot