Problem Statement

Find the equation of the plane passing through the intersection of the planes

$$\mathbf{n}_1^\top \mathbf{x} - c_1 = 0, \qquad \mathbf{n}_2^\top \mathbf{x} - c_2 = 0 \tag{1}$$

which is parallel to the x-axis, and compute the perpendicular distance of this plane from the x-axis.

Input Data

Quantity	Value
$\mathbf{n}_1$	$\begin{pmatrix}1\\1\\1\end{pmatrix}$
c_1	1
$\mathbf{n}_2$	$\begin{pmatrix}2\\3\\-1\end{pmatrix}$
c_2	-4
$\mathbf{e}_1$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$

Table 1: Input data for the problem

Solution

Step 1. General plane through the intersection

$$\left(\mathbf{n}_1 + k\mathbf{n}_2\right)^\top \mathbf{x} - \left(c_1 + kc_2\right) = 0 \tag{2}$$

$$\mathbf{n} = \mathbf{n}_1 + k\mathbf{n}_2 \tag{3}$$

$$c = c_1 + kc_2 \tag{4}$$

Step 2. Condition for parallelism with x-axis

$$\mathbf{e}_1^\top \mathbf{n} = 0 \tag{5}$$

$$\mathbf{e}_1^\top \mathbf{n}_1 + k\,\mathbf{e}_1^\top \mathbf{n}_2 = 0 \tag{6}$$

$$k = -\frac{\mathbf{e}_1^\top \mathbf{n}_1}{\mathbf{e}_1^\top \mathbf{n}_2} \tag{7}$$

Step 3. Required plane

$$\mathbf{n} = \mathbf{n}_1 - \frac{\mathbf{e}_1^\top \mathbf{n}_1}{\mathbf{e}_1^\top \mathbf{n}_2}\,\mathbf{n}_2 \tag{8}$$

$$c = c_1 - \frac{\mathbf{e}_1^\top \mathbf{n}_1}{\mathbf{e}_1^\top \mathbf{n}_2}\,c_2 \tag{9}$$

$$\mathbf{n}^\top \mathbf{x} = c \tag{10}$$

Step 4. Distance from the x-axis

Let a point on the x-axis be

$$\mathbf{P} = t\mathbf{e}_1 \tag{11}$$

The perpendicular distance is

$$d = \frac{\left|\mathbf{n}^\top \mathbf{P} - c\right|}{\|\mathbf{n}\|} \tag{12}$$

$$= \frac{|c|}{\|\mathbf{n}\|}, \quad \text{since } \mathbf{n}^\top \mathbf{e}_1 = 0 \tag{13}$$

Step 5. Substitution of values

$$\mathbf{e}_1^\top \mathbf{n}_1 = 1, \qquad \mathbf{e}_1^\top \mathbf{n}_2 = 2 \tag{14}$$

$$k = -\tfrac{1}{2} \tag{15}$$

$$\mathbf{n} = \mathbf{n}_1 - \tfrac{1}{2}\mathbf{n}_2 = \begin{pmatrix} 0 \\ -\frac{1}{2} \\ \frac{3}{2} \end{pmatrix} \tag{16}$$

$$c = c_1 - \tfrac{1}{2}c_2 = 3 \tag{17}$$

Norm:

$$\|\mathbf{n}\|^2 = \mathbf{n}_1^\top \mathbf{n}_1 - \mathbf{n}_1^\top \mathbf{n}_2 + \tfrac{1}{4}\mathbf{n}_2^\top \mathbf{n}_2 \tag{18}$$

$$= 3 - 4 + \tfrac{1}{4}(14) = \tfrac{5}{2} \tag{19}$$

$$\|\mathbf{n}\| = \tfrac{\sqrt{10}}{2} \tag{20}$$

Final Results

$$\text{Equation of Plane:} \quad \left(\mathbf{n}_1 - \tfrac{1}{2}\mathbf{n}_2\right)^\top \mathbf{x} = c_1 - \tfrac{1}{2}c_2 \tag{21}$$

$$\implies \begin{pmatrix} 0 & -\frac{1}{2} & \frac{3}{2} \end{pmatrix} \mathbf{x} = 3 \tag{22}$$

$$\text{Distance from } x\text{-axis:} \quad d = \frac{|3|}{\sqrt{10}/2} = \frac{6}{\sqrt{10}} = \frac{3\sqrt{10}}{5} \tag{23}$$

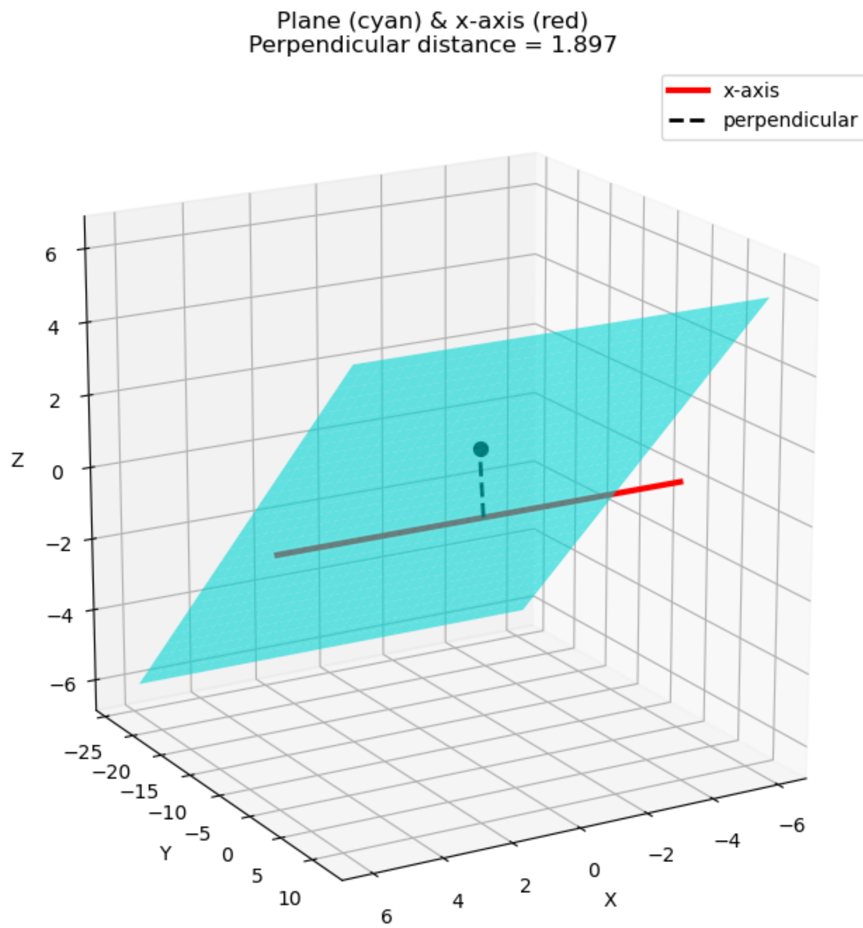
Plane (cyan) & x-axis (red)
Perpendicular distance = 1.897
x-axis
perpendicular
Z
6
4
2
0
−2
−4
−6
−25
−20
−15
−10
−5
0
5
10
Y
6
4
2
0
−2
−4
−6
X

Figure 1