4.12.26

AI25BTECH11021 - Abhiram Reddy N

Question

Prove that the locus of the foot of the perpendicular from the origin O on the line $\frac{x}{a} + \frac{y}{b} = 1$, which satisfies the condition $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$ (where c is a constant), is given by the circle $x^2 + y^2 = c^2$, using vector algebra with matrix notation and transpose.

Steps to Solve

1. Vector and Matrix Representation of the Line and Points

The equation of the line L in intercept form is:

$$\frac{x}{a} + \frac{y}{b} = 1$$

We can express the line's equation in the form $\mathbf{r}^T\mathbf{n} = 1$, where $\mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{n} = \begin{pmatrix} 1/a \\ 1/b \end{pmatrix}$:

$$L : \begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} 1/a \\ 1/b \end{pmatrix} = 1 \quad \text{(Equation 1)}$$

Let $P(x_0, y_0)$ be the foot of the perpendicular from the origin $O(0, 0)$. The position vector of P is $\mathbf{p}$:

$$\mathbf{p} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \quad \text{(Equation 2)}$$

2. Condition of Perpendicularity and Point on the Line

OP is Parallel to the Normal **n** The vector **p** (which is **OP**) is perpendicular to the line L, hence it must be parallel to the line's normal vector **n**:

$$\mathbf{p} = \lambda\mathbf{n} \quad \text{for some scalar } \lambda \quad \text{(Equation 3)}$$

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \lambda \begin{pmatrix} 1/a \\ 1/b \end{pmatrix} \implies x_0 = \frac{\lambda}{a}, \quad y_0 = \frac{\lambda}{b}$$

From these, we isolate the components of the normal vector:

$$\frac{1}{a} = \frac{x_0}{\lambda} \quad \text{and} \quad \frac{1}{b} = \frac{y_0}{\lambda} \quad \text{(Equation 4)}$$

P lies on L Since P is on the line L, it must satisfy the line's equation (Equation 1):

$$\mathbf{p}^T\mathbf{n} = 1 \quad \text{(Equation 5)}$$

Substituting the scalar components:

$$\frac{x_0}{a} + \frac{y_0}{b} = 1$$

Now, substitute the expressions for $\frac{1}{a}$ and $\frac{1}{b}$ from (Equation 4):

$$x_0\left(\frac{x_0}{\lambda}\right) + y_0\left(\frac{y_0}{\lambda}\right) = 1$$

$$\frac{x_0^2 + y_0^2}{\lambda} = 1$$

$$x_0^2 + y_0^2 = \lambda \quad \text{(Equation 6)}$$

In matrix notation, the square of the distance from the origin to P is:

$$\mathbf{p}^T\mathbf{p} = \lambda \quad \text{(Equation 7)}$$

3. Using the Given Locus Condition

The line satisfies the condition:

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2} \quad \text{(Equation 8)}$$

Substitute the expressions for $\frac{1}{a}$ and $\frac{1}{b}$ from (Equation 4) into (Equation 8):

$$\left(\frac{x_0}{\lambda}\right)^2 + \left(\frac{y_0}{\lambda}\right)^2 = \frac{1}{c^2}$$

$$\frac{x_0^2 + y_0^2}{\lambda^2} = \frac{1}{c^2} \quad \text{(Equation 9)}$$

Now, substitute $x_0^2 + y_0^2 = \lambda$ (from Equation 6) into (Equation 9):

$$\frac{\lambda}{\lambda^2} = \frac{1}{c^2}$$

$$\frac{1}{\lambda} = \frac{1}{c^2} \implies \lambda = c^2 \quad \text{(Equation 10)}$$

4. Final Locus Equation

Substitute $\lambda = c^2$ (Equation 10) back into the expression for P's coordinates (Equation 6) or (Equation 7):

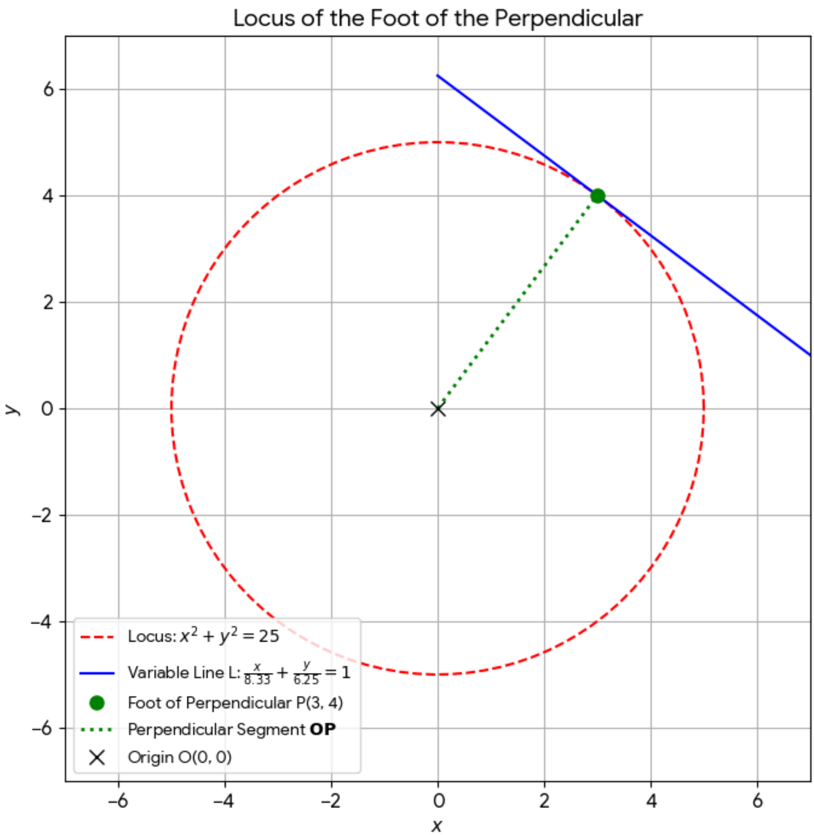
$$x_0^2 + y_0^2 = c^2$$

$$\mathbf{p}^T\mathbf{p} = c^2 \quad \text{(Equation 11)}$$

The locus of the foot of the perpendicular $P(x_0, y_0)$ is therefore:

$$x^2 + y^2 = c^2 \quad \text{(Equation 12)}$$

This is the equation of a circle centered at the origin with radius c.



Plot of the curves
Fig1