Question 2.10.29

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Question:

The volume of the parallelopiped whose sides are given by $OA = 2\mathbf{i} - 2\mathbf{j}$, $OB = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $OC = 3\mathbf{i} - \mathbf{k}$, is

Solution:

To find the volume of the parallelopiped, we can use the Gram matrix formula:

$$V = \sqrt{\det(\mathbf{G})} \tag{1}$$

$$\mathbf{OA} = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}, \quad \mathbf{OB} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{OC} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \tag{2}$$

$$\tag{3}$$

We define the Gram matrix $\mathbf{G}$ as:

$$\mathbf{G} = \begin{pmatrix} \mathbf{OA}^{\mathrm{T}}\mathbf{OA} & \mathbf{OA}^{\mathrm{T}}\mathbf{OB} & \mathbf{OA}^{\mathrm{T}}\mathbf{OC} \\ \mathbf{OB}^{\mathrm{T}}\mathbf{OA} & \mathbf{OB}^{\mathrm{T}}\mathbf{OB} & \mathbf{OB}^{\mathrm{T}}\mathbf{OC} \\ \mathbf{OC}^{\mathrm{T}}\mathbf{OA} & \mathbf{OC}^{\mathrm{T}}\mathbf{OB} & \mathbf{OC}^{\mathrm{T}}\mathbf{OC} \end{pmatrix} \tag{4}$$

Calculating the dot products:

$$\mathbf{OA}^{\mathrm{T}}\mathbf{OA} = 8, \quad \mathbf{OA}^{\mathrm{T}}\mathbf{OB} = 0, \quad \mathbf{OA}^{\mathrm{T}}\mathbf{OC} = 6 \tag{5}$$

$$\mathbf{OB}^{\mathrm{T}}\mathbf{OB} = 3, \quad \mathbf{OB}^{\mathrm{T}}\mathbf{OC} = 4 \tag{6}$$

$$\mathbf{OC}^{\mathrm{T}}\mathbf{OC} = 10 \tag{7}$$

The Gram matrix $\mathbf{G}$ becomes:

$$\mathbf{G} = \begin{pmatrix} 8 & 0 & 6 \\ 0 & 3 & 4 \\ 6 & 4 & 10 \end{pmatrix} \tag{8}$$

Therefore, $V = \sqrt{|\det(\mathbf{G})|} = \sqrt{4} = 2$.

Plot:

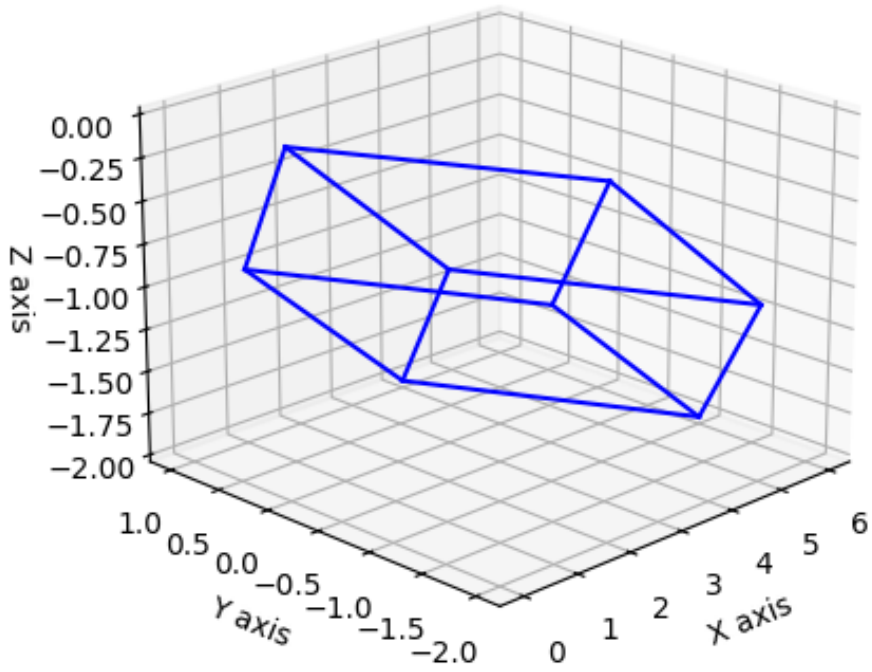

Figure: Parallelopiped formed by vectors **OA**, **OB** and **OC**