7.4.33

Harsha-EE25BTECH11026

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Question

A circle C of radius 1 unit is inscribed in an equilateral triangle PQR. The points of contact of C with sides PQ, QR, RP are D, E, F respectively.The line PQ is given by the equation $\sqrt{3}x + y - 6 = 0$ and the point **D** is $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$. Further, it is giventhat the origin and the centre of C are on same side of line PQ. The equation of circle C is

1. $\left(x - 2\sqrt{3}\right)^2 + (y-1)^2 = 1$
2. $\left(x - 2\sqrt{3}\right)^2 + \left(y + \frac{1}{2}\right)^2 = 1$
3. $\left(x - \sqrt{3}\right)^2 + (y-1)^2 = 1$
4. $\left(x - \sqrt{3}\right)^2 + (y+1)^2 = 1$

Theoretical Solution

According to the question,

$$\text{Equation of tangent PQ} : \mathbf{n}^\top \mathbf{x} = c \tag{1}$$

where $\mathbf{n} = \begin{pmatrix} \sqrt{3} & 1 \end{pmatrix}^\top$ and $c = 6$

$$\text{Point of tangency}\,(\mathbf{D}) : \begin{pmatrix} \frac{3\sqrt{3}}{2} \\ \frac{3}{2} \end{pmatrix} \tag{2}$$

$$radius(r) = 1 \tag{3}$$

Theoretical Solution

As the point of tangency **D** and centre of circle **u** are along the direction of the vector **n**,

$$\therefore \mathbf{D} - \mathbf{u} = \lambda \mathbf{n} \text{ , for some scalar } \lambda \tag{4}$$

$$\implies \mathbf{u} = \mathbf{D} - \lambda \mathbf{n} \tag{5}$$

Also,

$$\frac{|\mathbf{n}^\top \mathbf{O} - c|}{\|\mathbf{n}\|} = r \tag{6}$$

$$|\mathbf{n}^\top \mathbf{u} - c| = r\|\mathbf{n}\| \tag{7}$$

$$\mathbf{n}^\top \mathbf{u} = c \pm r\|\mathbf{n}\| \tag{8}$$

Theoretical Solution

To decide the sign , we need to use the fact that the origin and centre of circle are on the same side of the line PQ.

$$\therefore \left(\mathbf{n}^\top \mathbf{u} - c\right)\left(\mathbf{n}^\top \begin{pmatrix} 0 \\ 0 \end{pmatrix} - c\right) > 0 \tag{9}$$

$$\implies \mathbf{n}^\top \mathbf{u} < c \tag{10}$$

$$\therefore \mathbf{n}^\top \mathbf{u} = c - r\|\mathbf{n}\| \tag{11}$$

Substituting value of **u**,

$$\mathbf{n}^\top \left(\mathbf{D} - \lambda \mathbf{n}\right) = c - r\|\mathbf{n}\| \tag{12}$$

$$\implies \lambda = \frac{\mathbf{n}^\top \mathbf{D} + r\|n\| - c}{\mathbf{n}^\top \mathbf{n}} \tag{13}$$

$$\mathbf{u} = \mathbf{D} - \frac{\mathbf{n}^\top \mathbf{D} + r\|n\| - c}{\mathbf{n}^\top \mathbf{n}} \mathbf{n} \tag{14}$$

Theoretical Solution

Substituting the values,

$$\therefore \mathbf{u} = \begin{pmatrix} \frac{3\sqrt{3}}{2} \\ \frac{3}{2} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \tag{15}$$

$$\therefore \text{Required equation of circle} : \|\mathbf{x}\|^2 - 2\begin{pmatrix} \sqrt{3} & 1 \end{pmatrix}\mathbf{x} + 3 = 0 \tag{16}$$

C Code -Finding the equation of circle

```c
#include <stdio.h>
#include <math.h>

// structure for 2D point
typedef struct {
    double x, y;
} Point;

// normalize vector
void normalize(double *x, double *y) {
    double norm = sqrt((*x)*(*x) + (*y)*(*y));
    if (norm > 1e-12) {
        *x /= norm;
        *y /= norm;
    }
}
```

C Code -Finding the equation of circle

```
// rotate by theta
Point rotate(Point v, double theta) {
    Point r;
    double c = cos(theta), s = sin(theta);
    r.x = c*v.x - s*v.y;
    r.y = s*v.x + c*v.y;
    return r;
}
// solve 2x2 system
Point intersect(double a1,double b1,double c1, double a2,double
    b2,double c2) {
    double det = a1*b2 - a2*b1;
    Point P;
    P.x = (b1*c2 - b2*c1)/det;
    P.y = (c1*a2 - c2*a1)/det;
    return P;
}
```

C Code -Finding the equation of circle

```c
// main solver: fills arrays with results
void solve_geometry(double *results) {
    // Given: line PQ: sqrt(3)x + y - 6 = 0
    double a = sqrt(3.0), b = 1.0, c = -6.0;
    Point D = {3*sqrt(3.0)/2.0, 1.5};
    double r = 1.0;
    // unit normal
    double nx=a, ny=b;
    normalize(&nx,&ny);
    // candidate centers
    Point C1 = {D.x + r*nx, D.y + r*ny};
    Point C2 = {D.x - r*nx, D.y - r*ny};
    // check side with origin
    double origin_side = a*0 + b*0 + c;
    double side1 = a*C1.x + b*C1.y + c;
    Point O = (side1*origin_side > 0) ? C1 : C2;
```

C Code -Finding the equation of circle

```c
// tangency vectors
Point vD = {D.x - O.x, D.y - O.y};
Point vE = rotate(vD, 2*M_PI/3.0);
Point vF = rotate(vD, -2*M_PI/3.0);

Point E = {O.x + vE.x, O.y + vE.y};
Point F = {O.x + vF.x, O.y + vF.y};

// tangent lines: ax+by+c=0
double cPQ = -(vD.x*D.x + vD.y*D.y);
double cQR = -(vE.x*E.x + vE.y*E.y);
double cRP = -(vF.x*F.x + vF.y*F.y);

// vertices
Point P = intersect(vD.x, vD.y, cPQ, vF.x, vF.y, cRP);
Point Q = intersect(vD.x, vD.y, cPQ, vE.x, vE.y, cQR);
Point R = intersect(vE.x, vE.y, cQR, vF.x, vF.y, cRP);
```

```c
    // fill results array (order: O,D,E,F,P,Q,R)
    results[0]=O.x; results[1]=O.y;
    results[2]=D.x; results[3]=D.y;
    results[4]=E.x; results[5]=E.y;
    results[6]=F.x; results[7]=F.y;
    results[8]=P.x; results[9]=P.y;
    results[10]=Q.x; results[11]=Q.y;
    results[12]=R.x; results[13]=R.y;
}
```

Python+C code

```python
import ctypes
import numpy as np
import math as m
import matplotlib.pyplot as plt

lib = ctypes.CDLL("./libcircle_solver.so")
# prepare result array (14 doubles: O,D,E,F,P,Q,R)
results = (ctypes.c_double * 14)()
lib.solve_geometry(results)

vals = np.array(results)
O = vals[0:2]
D = vals[2:4]
E = vals[4:6]
F = vals[6:8]
P = vals[8:10]
Q = vals[10:12]
R = vals[12:14]
```

Python+C code

```
h,k,r= O[0],O[1],1
print("Equation of circle:")
print(f"(x-{h:.2f})^2+(y-{k:.2f})^2={r}")
r = 1.0
theta = np.linspace(0, 2*np.pi, 400)
xc = O[0] + r*np.cos(theta)
yc = O[1] + r*np.sin(theta)
plt.figure(figsize=(7,7))
plt.plot(xc, yc, 'b-', label="Incircle (r=1)")
plt.plot([P[0],Q[0],R[0],P[0]], [P[1],Q[1],R[1],P[1]], 'm--',
    label="Equilateral Triangle")
```

Python+C code

```
plt.scatter([D[0],E[0],F[0]], [D[1],E[1],F[1]], c=['red','orange'
    ,'purple'], label="Tangency D,E,F")
plt.scatter(O[0], O[1], c='black', s=40, label="Center O")
plt.scatter([P[0],Q[0],R[0]], [P[1],Q[1],R[1]], c='blue', s=30,
    label="Vertices P,Q,R")
plt.gca().set_aspect("equal")
plt.grid(True, linestyle='--', alpha=0.6)
plt.legend()
plt.title("Equilateral triangle PQR with incircle C (C computed
    in C)")
plt.show()
```

Python code

```python
import numpy as np
import matplotlib.pyplot as plt
import matplotlib as mp
mp.use("TkAgg")

a, b, c = np.sqrt(3), 1.0, -6.0
D = np.array([3*np.sqrt(3)/2.0, 3.0/2.0])

def unit(v):
    n = np.linalg.norm(v)
    return v / n if n != 0 else v

def line_from_normal_and_point(nvec, P):
    # a x + b y + c = 0 with normal nvec = [a,b]; c = -(a*Px + b*
        Py)
    a,b = nvec[0], nvec[1]
    c = -(a*P[0] + b*P[1])
    return np.array([a,b,c])
```

Python code

```python
def intersect(L1, L2):
    A = np.array([[L1[0], L1[1]], [L2[0], L2[1]]])
    B = -np.array([L1[2], L2[2]])
    return np.linalg.solve(A, B)

def rotate(v, theta):
    c,s = np.cos(theta), np.sin(theta)
    R = np.array([[c,-s],[s,c]])
    return R @ v

n = np.array([a,b])
n_unit = unit(n)

# two candidate centers:
C1 = D + r * n_unit
C2 = D - r * n_unit
```

Python code

```
# Choose centre on same side of PQ as origin
def side_sign(point):
    return np.sign(a*point[0] + b*point[1] + c)
origin_side = side_sign(np.array([0.0,0.0]))
if side_sign(C1) == origin_side:
    O = C1
else:
    O = C2

h,k = O[0], O[1]

vD = D - O # vector from centre to D
theta = 2*np.pi/3.0 # 120 degrees
vE = rotate(vD, theta)
vF = rotate(vD, -theta)

E = O + vE
F = O + vF
```

Python code

```
assert np.allclose(np.linalg.norm(vD), r, atol=1e-8)
assert np.allclose(np.linalg.norm(vE), r, atol=1e-8)
assert np.allclose(np.linalg.norm(vF), r, atol=1e-8)

L_PQ_from_tangent = line_from_normal_and_point(vD, D)
scale_given = np.sqrt(a*a + b*b)
L_PQ_normed = L_PQ_from_tangent / scale_given

L_QR = line_from_normal_and_point(vE, E)
L_RP = line_from_normal_and_point(vF, F)
# side names: PQ (tangent at D), QR (tangent at E), RP (tangent at F)
P = intersect(L_PQ_from_tangent, L_RP) # P = PQ  RP
Q = intersect(L_PQ_from_tangent, L_QR) # Q = PQ  QR
R = intersect(L_QR, L_RP) # R = QR  RP

u_vec = np.array([h,k])
f_const = h*h + k*k - r*r
```

Python code

```
print("\nExpanded scalar form:")
print(f"x^2 + y^2 - 2*{h:.2g}*x - 2*{k:.2g}*y + {f_const:.2g} = 0
    ")
# Print the three tangent lines in normalized readable form
def pretty_line(L):
    # scale to unit normal for readability
    nrm = np.linalg.norm(L[:2])
    La = L / nrm
    a_s,b_s,c_s = La[0], La[1], La[2]
    return f"{a_s:.12g} x + {b_s:.12g} y + {c_s:.12g} = 0"
```

Python code

```
#plot
theta_vals = np.linspace(0, 2*np.pi, 600)
xc = O[0] + r * np.cos(theta_vals)
yc = O[1] + r * np.sin(theta_vals)
plt.figure(figsize=(7,7))
plt.plot(xc, yc, label="Incircle (r=1)")
# triangle edges
tri_x = [P[0], Q[0], R[0], P[0]]
tri_y = [P[1], Q[1], R[1], P[1]]
plt.plot(tri_x, tri_y, 'm--', linewidth=1.8, label="Equilateral
    Triangle")
# tangency lines (extended for visibility)
xlim_min = min(P[0],Q[0],R[0], O[0]) - 3
xlim_max = max(P[0],Q[0],R[0], O[0]) + 3
xvals = np.linspace(xlim_min, xlim_max, 400)
# PQ from given coeffs
yvals_pq = -(a*xvals + c)/b
plt.plot(xvals, yvals_pq, 'g-', label="Given side PQ")
```

Python code

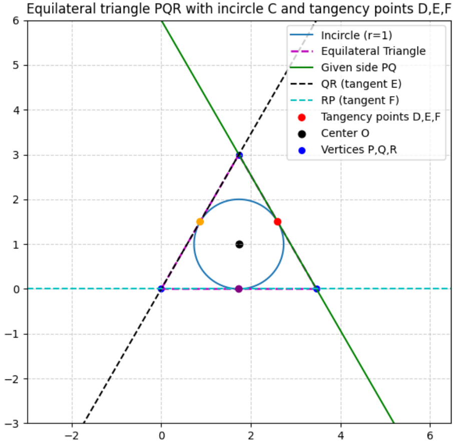
```python
# tangent lines via their normals
def plot_line(L, style, label):
    aL,bL,cL = L
    # avoid vertical division by zero; param in x
    if abs(bL) > 1e-8:
        y = -(aL*xvals + cL)/bL
        plt.plot(xvals, y, style, label=label)
    else:
        xconst = -cL/aL
        plt.axvline(x=xconst, linestyle=style, label=label)

plot_line(L_QR, 'k--', 'QR (tangent E)')
plot_line(L_RP, 'c--', 'RP (tangent F)')
```

Python code

```python
# points
plt.scatter([D[0], E[0], F[0]], [D[1], E[1], F[1]], c=['red','orange','purple'], zorder=5,
            label='Tangency points D,E,F')
plt.scatter(O[0], O[1], c='black', s=40, label='Center O')
plt.scatter([P[0],Q[0],R[0]],[P[1],Q[1],R[1]], c='blue', s=30, label='Vertices P,Q,R')
plt.xlim(xlim_min, xlim_max)
plt.ylim(min(P[1],Q[1],R[1], O[1]) - 3, max(P[1],Q[1],R[1], O[1]) + 3)
plt.gca().set_aspect("equal", adjustable="box")
plt.grid(True, linestyle='--', alpha=0.6)
plt.legend()
plt.title("Equilateral triangle PQR with incircle C and tangency points D,E,F")
plt.savefig("/home/user/Matrix/Matgeo_assignments/7.4.33/figs/Figure_1.png")
plt.show()
```

Plot



Equilateral triangle PQR with incircle C and tangency points D,E,F
Incircle (r=1)
Equilateral Triangle
Given side PQ
QR (tangent E)
RP (tangent F)
Tangency points D,E,F
Center O
Vertices P,Q,R
6
5
4
3
2
1
0
−1
−2
−3
−2
0
2
4
6