Matrices in Geometry 4.11.13

EE25BTECH11035 - Kushal B N

Question:
Find the equation of the plane passing through the points (2,5,-3), (-2,-3,5), and (5,3,-3). Also find the point of intersection of this plane with the line passing through points (3,1,5) and (-1,-3,-1).

Given:
The points $\mathbf{A}\begin{pmatrix}2\\5\\-3\end{pmatrix}$, $\mathbf{B}\begin{pmatrix}-2\\-3\\5\end{pmatrix}$ and $\mathbf{C}\begin{pmatrix}5\\3\\-3\end{pmatrix}$ which pass through a plane.
The points $\mathbf{D}\begin{pmatrix}3\\1\\5\end{pmatrix}$ and $\mathbf{E}\begin{pmatrix}-1\\-3\\-1\end{pmatrix}$ which pass through a line.

Solution:
If $\mathbf{n}$ is the normal vector to the plane, then by the plane equation we can write

$$\mathbf{n}^\top\mathbf{A} = c \tag{1}$$

$$\mathbf{n}^\top\mathbf{B} = c \tag{2}$$

$$\mathbf{n}^\top\mathbf{C} = c \tag{3}$$

which can be writeen as

$$\begin{pmatrix}A & B & C\end{pmatrix}^\top \mathbf{n} = \begin{pmatrix}1\\1\\1\end{pmatrix} \tag{4}$$

Forming the augmented matrix for this

$$\left(\begin{array}{ccc|c}2 & 5 & -3 & 1\\ -2 & -3 & 5 & 1\\ 5 & 3 & -3 & 1\end{array}\right) \tag{5}$$

$$\left(\begin{array}{ccc|c}2 & 5 & -3 & 1\\ -2 & -3 & 5 & 1\\ 5 & 3 & -3 & 1\end{array}\right) \xleftrightarrow{R_1\leftarrow\frac{R_1}{2}} \left(\begin{array}{ccc|c}1 & \frac{5}{2} & \frac{-3}{2} & \frac{1}{2}\\ -2 & -3 & 5 & 1\\ 5 & 3 & -3 & 1\end{array}\right) \tag{6}$$

$$\xleftrightarrow{R_2\leftarrow\frac{R_2}{2}} \left(\begin{array}{ccc|c}1 & \frac{5}{2} & \frac{-3}{2} & \frac{1}{2}\\ 0 & 1 & 1 & 1\\ 0 & \frac{-19}{2} & \frac{9}{2} & \frac{-3}{2}\end{array}\right) \xleftrightarrow[R_1\leftarrow R_1-\frac{5}{2}R_2]{R_3\leftarrow R_3+\frac{19}{2}R_2} \left(\begin{array}{ccc|c}1 & 0 & -4 & -2\\ 0 & 1 & 1 & 1\\ 0 & 0 & 14 & 8\end{array}\right) \tag{7}$$

$$\xleftrightarrow{R_3\leftarrow\frac{1}{14}R_3} \left(\begin{array}{ccc|c}1 & 0 & -4 & -2\\ 0 & 1 & 1 & 1\\ 0 & 0 & 1 & \frac{4}{7}\end{array}\right) \xleftrightarrow[R_2\leftarrow R_2-R_3]{R_1\leftarrow R_1+4R_3} \left(\begin{array}{ccc|c}1 & 0 & 0 & \frac{2}{7}\\ 0 & 1 & 0 & \frac{3}{7}\\ 0 & 0 & 1 & \frac{4}{7}\end{array}\right) \tag{8}$$

Thus, multiplying by 7, the plane equation can be expressed as

$$\begin{pmatrix}2 & 3 & 4\end{pmatrix}\mathbf{x} = 7 \tag{9}$$

Now, the line passing through the two given points in parametric form

$$\mathbf{x} = \mathbf{P} + \lambda\mathbf{m} \tag{10}$$

where $\mathbf{P} = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$ and $\mathbf{m} = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} - \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix} \equiv \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$,

Now substituting the parametric form in the plane equation $\mathbf{n}^\top \mathbf{x} = c$ where here $\mathbf{n} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ and $c = 7$,

$$\mathbf{n}^\top (\mathbf{P} + \lambda \mathbf{m}) = c \tag{11}$$

$$\mathbf{n}^\top \mathbf{P} + \lambda \mathbf{n}^\top \mathbf{m} = c \tag{12}$$

$$\implies \lambda = \frac{c - \mathbf{n}^\top \mathbf{P}}{\mathbf{n}^\top \mathbf{m}} \tag{13}$$

So that,

$$\mathbf{x} = \mathbf{P} + \left(\frac{c - \mathbf{n}^\top \mathbf{P}}{\mathbf{n}^\top \mathbf{m}} \right) \mathbf{m} \tag{14}$$

Substituting the values in equation (14) we get the intersection point as,

$$\implies \boxed{\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}} \tag{15}$$

Final Answer:

$\therefore$ The equation of the plane is $\begin{pmatrix} 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 7$ and the point of intersection of this plane with the line through the given two points is $\mathbf{P} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

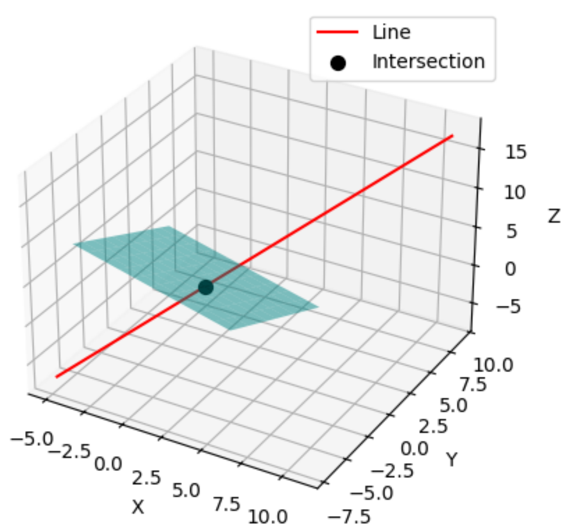

Fig. 1