5.13.27

EE25BTECH11001 - Aarush Dilawri

Question: Let $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, a, b \in \mathbb{N}$.

(a) there cannot exist any $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{BA}$
(b) there exist more than one but finite number of $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{BA}$
(c) there exists exactly one $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{BA}$
(d) there exist infinitely many $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{BA}$

Solution:

$$\text{Let } \mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad a, b \in \mathbb{N}. \tag{4.1}$$

We compute $\mathbf{AB}$:

$$\mathbf{AB} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \tag{4.2}$$

$$= \begin{pmatrix} a & 2b \\ 3a & 4b \end{pmatrix}. \tag{4.3}$$

Similarly, compute $\mathbf{BA}$:

$$\mathbf{BA} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \tag{4.4}$$

$$= \begin{pmatrix} a & 2a \\ 3b & 4b \end{pmatrix}. \tag{4.5}$$

For $\mathbf{AB} = \mathbf{BA}$, we must have:

$$\begin{pmatrix} a & 2b \\ 3a & 4b \end{pmatrix} = \begin{pmatrix} a & 2a \\ 3b & 4b \end{pmatrix}. \tag{4.6}$$

Equating the corresponding entries gives:

$$2b = 2a \quad \implies \quad b = a, \tag{4.7}$$

$$3a = 3b \quad \implies \quad a = b. \tag{4.8}$$

Hence,

$$\mathbf{B} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = a\mathbf{I}. \tag{4.9}$$

Since $a \in \mathbb{N}$, there are infinitely many such $\mathbf{B}$.
Therefore, the answer is (d) there exist infinitely many $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{BA}$.