10.7.50

EE25BTECH11001 - Aarush Dilawri

Question:
Consider the family of circles $x^2 + y^2 = r^2, 2 < r < 5$. If in the first quadrant, the common tangent to a circle of this family and the ellipse $4x^2 + 25y^2 = 100$ meets the coordinate axes at **A** and **B**, then find the equation of the locus of the midpoint of AB.

Solution:

$$\text{The family of circles is } \mathbf{X}^\top \mathbf{X} = r^2, \qquad 2 < r < 5, \tag{1}$$

$$\text{and the ellipse is } \mathbf{X}^\top \mathbf{V}\mathbf{X} = 100, \quad \mathbf{V} = \begin{pmatrix} 4 & 0 \\ 0 & 25 \end{pmatrix}. \tag{2}$$

Let the common tangent meet the coordinate axes at

$$\mathbf{A} = a\mathbf{e}_1, \qquad \mathbf{B} = b\mathbf{e}_2, \qquad a, b > 0. \tag{3}$$

The equation of the line passing through **A** and **B** can be written as

$$\frac{\mathbf{e}_1^\top \mathbf{X}}{a} + \frac{\mathbf{e}_2^\top \mathbf{X}}{b} = 1. \tag{4}$$

This is of the form $\mathbf{n}^\top \mathbf{X} = c$, with

$$\mathbf{n} = \begin{pmatrix} \frac{1}{a} \\ \frac{1}{b} \end{pmatrix}, \quad c = 1. \tag{5}$$

Let the midpoint of **A** and **B** be

$$\mathbf{m} = \frac{\mathbf{A} + \mathbf{B}}{2}. \tag{6}$$

From this,

$$a = 2\,\mathbf{e}_1^\top \mathbf{m}, \quad b = 2\,\mathbf{e}_2^\top \mathbf{m}. \tag{7}$$

The ellipse is

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} = 100. \tag{8}$$

The line is

$$\mathbf{n}^\top \mathbf{x} = c. \tag{9}$$

Suppose $\mathbf{x}_0 = \alpha\mathbf{n}$ is a solution. Then

$$\mathbf{n}^\top\mathbf{x}_0 = \alpha\mathbf{n}^\top\mathbf{n} = c \implies \alpha = \frac{c}{\mathbf{n}^\top\mathbf{n}}. \tag{10}$$

So a particular solution is

$$\mathbf{x}_0 = \frac{c}{\mathbf{n}^\top\mathbf{n}}\mathbf{n}. \tag{11}$$

Any point on the line can be written as

$$\mathbf{x} = \mathbf{x}_0 + \mu\mathbf{m}, \tag{12}$$

where $\mathbf{m}$ is a direction vector satisfying

$$\mathbf{n}^\top\mathbf{m} = 0. \tag{13}$$

Substitute into $\mathbf{x}^\top\mathbf{V}\mathbf{x} = 100$:

$$(\mathbf{x}_0 + \mu\mathbf{m})^\top \mathbf{V}(\mathbf{x}_0 + \mu\mathbf{m}) = 100. \tag{14}$$

Expanding,

$$\mathbf{x}_0^\top\mathbf{V}\mathbf{x}_0 + 2\mu\mathbf{m}^\top\mathbf{V}\mathbf{x}_0 + \mu^2\mathbf{m}^\top\mathbf{V}\mathbf{m} = 100. \tag{15}$$

This is a quadratic in μ

For tangency, discriminant = 0: That is,

$$\left(2\mathbf{m}^\top\mathbf{V}\mathbf{x}_0\right)^2 - 4\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)\left(\mathbf{x}_0^\top\mathbf{V}\mathbf{x}_0 - 100\right) = 0. \tag{16}$$

After simplification using

$$\mathbf{x}_0 = \frac{c}{\mathbf{n}^\top\mathbf{n}}\mathbf{n}, \qquad \mathbf{n}^\top\mathbf{m} = 0, \tag{17}$$

the condition reduces to

$$c^2 = 100\,\mathbf{n}^\top\mathbf{V}^{-1}\mathbf{n}. \tag{18}$$

$\therefore$ For the line $\mathbf{n}^\top\mathbf{X} = c$ to be tangent to $\mathbf{X}^\top\mathbf{V}\mathbf{X} = 100,$ the condition is $c^2 = 100\,\mathbf{n}^\top\mathbf{V}^{-1}\mathbf{n}$.

$$\text{Here } c = 1, \quad \mathbf{V}^{-1} = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{25} \end{pmatrix}. \text{ Substituting gives } 1 = 100\Big(\frac{1}{4a^2} + \frac{1}{25b^2}\Big). \tag{19}$$

$$\Rightarrow \frac{25}{a^2} + \frac{4}{b^2} = 1. \tag{20}$$

Also, for the line $\mathbf{n}^\top\mathbf{X} = c$ to be tangent to the circle $\mathbf{X}^\top\mathbf{X} = r^2$, the distance from the origin must equal r.

$$\frac{|c|}{\|\mathbf{n}\|} = r. \tag{21}$$

$$\text{With } c = 1 \text{ this gives } \|\mathbf{n}\|^2 = \frac{1}{r^2}. \text{ So } \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{r^2}. \tag{22}$$

We now express the ellipse tangency condition in terms of $\mathbf{m}$. Substitute $a = 2\mathbf{e}_1^\top\mathbf{m},\ b = 2\mathbf{e}_2^\top\mathbf{m}$:

$$\frac{25}{4\left(\mathbf{e}_1^\top\mathbf{m}\right)^2} + \frac{4}{4\left(\mathbf{e}_2^\top\mathbf{m}\right)^2} = 1. \tag{23}$$

$$\implies \left(4\left(\mathbf{e}_1^\top\mathbf{m}\right)^2 - 25\right)\left(\mathbf{e}_2^\top\mathbf{m}\right)^2 = 4\left(\mathbf{e}_1^\top\mathbf{m}\right)^2. \tag{24}$$

$$\text{Or equivalently, } 4\left(\mathbf{e}_1^\top\mathbf{m}\right)^2\left(\mathbf{e}_2^\top\mathbf{m}\right)^2 - 4\left(\mathbf{e}_1^\top\mathbf{m}\right)^2 - 25\left(\mathbf{e}_2^\top\mathbf{m}\right)^2 = 0. \tag{25}$$

$$\text{Finally, let } \mathbf{m} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{e}_1^\top\mathbf{m} = x, \quad \mathbf{e}_2^\top\mathbf{m} = y. \tag{26}$$

$$\text{Substituting gives the locus equation } 4x^2y^2 - 4x^2 - 25y^2 = 0. \tag{27}$$

$$\text{Required locus: } 4x^2y^2 - 4x^2 - 25y^2 = 0. \tag{28}$$

See Fig. 0 ,

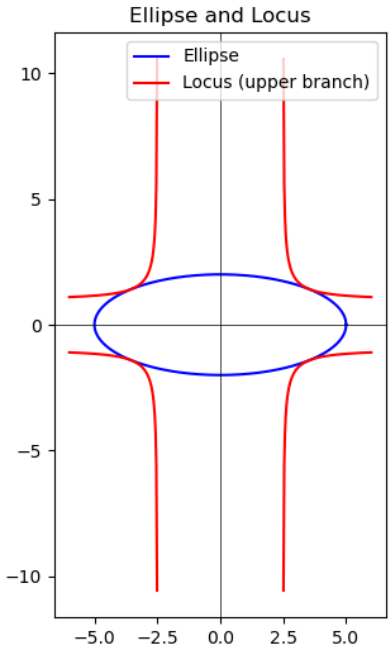

Fig. 0