2.10.85

EE25BTECH11057 - Rushil Shanmukha Srinivas

Problem: Let P be the plane 3x+2y+3z=16 and let S: $\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k}$,where $\alpha+\beta+\gamma = 7$ and the distance of (α,β,γ) from the plane is $2/\sqrt{22}$ .Let $\mathbf{u},\mathbf{v},\mathbf{w}$ be three distinct vectors in S such that $|\mathbf{u}-\mathbf{v}| = |\mathbf{v}-\mathbf{w}| = |\mathbf{w}-\mathbf{u}|$. Let V be the volume of the parallelopiped determined by vectors $\mathbf{u},\mathbf{v},\mathbf{w}$. Then the value of (80/3)V is

Solution:

$$P: \ \mathbf{n}^\top \mathbf{x} = c, \qquad \mathbf{n} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} \ c = 16, \mathbf{O} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{0.1}$$

The distance of point $\mathbf{P_0} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$ from the plane is

$$\text{dist}(\mathbf{P_0}, P) = \frac{|\mathbf{n}^\top \mathbf{P_0} - c|}{||\mathbf{n}||} \tag{0.2}$$

Given $\alpha+\beta+\gamma = 7$ and $\text{dist}(\mathbf{P_0}, P) = \frac{2}{\sqrt{22}}$, we have

$$\frac{|3\alpha + 2\beta + 3\gamma - 16|}{\sqrt{22}} = \frac{2}{\sqrt{22}} \quad \Longrightarrow \quad \mathbf{n}^\top \mathbf{P_0} = 18 \ \text{ or } \ \mathbf{n}^\top \mathbf{P_0} = 14. \tag{0.3}$$

Thus S lies on the intersections

$$\Pi: \ \mathbf{m}^\top \mathbf{x} = 7, \qquad \mathbf{m} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \tag{0.4}$$

with

$$P_+ : \mathbf{n}^\top \mathbf{x} = 18, \qquad P_- : \mathbf{n}^\top \mathbf{x} = 14. \tag{0.5}$$

(i) Intersection of $\mathbf{m}^\top \mathbf{x} = 7$ and $\mathbf{n}^\top \mathbf{x} = 18$.

Write the augmented system in matrix form:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 3 & 2 & 3 & 18 \end{array}\right) \xrightarrow{R_2 \longrightarrow R_2 - 3R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & -1 & 0 & -3 \end{array}\right). \tag{0.6}$$

From the second row we get $y = 3$. Substitute into $x+y+z = 7$:

$$x + 3 + z = 7 \quad \Longrightarrow \quad z = 4 - x. \tag{0.7}$$

So the line is

$$\mathbf{L_1} = \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} + k_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \mathbf{a} + k_1\mathbf{d} \tag{0.8}$$

(ii) Intersection of $\mathbf{m}^\top\mathbf{x} = 7$ and $\mathbf{n}^\top\mathbf{x} = 14$.
The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 3 & 2 & 3 & 14 \end{array}\right) \xrightarrow{R_2 \longrightarrow R_2 - 3R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & -1 & 0 & -7 \end{array}\right). \tag{0.9}$$

So $y = 7$. From $x + y + z = 7$ we get

$$x + 7 + z = 7 \quad \implies \quad z = -x. \tag{0.10}$$

$$\mathbf{L_2} = \begin{pmatrix} 0 \\ 7 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \mathbf{b} + k_2\mathbf{d} \tag{0.11}$$

hence the two lines are parallel.

(iii) Perpendicular distance between the two parallel lines: $\mathbf{P} = \mathbf{a} + p_1\mathbf{d}$ *and* $\mathbf{Q} = \mathbf{b} + p_2\mathbf{d}$ *be* 2*points on* $\mathbf{L_1}, \mathbf{L_2}$.

$$perpendicular\ distance = \text{dist}(\mathbf{P}, \mathbf{Q}), \mathbf{M} = \begin{pmatrix} \mathbf{d} & \mathbf{d} \end{pmatrix} \tag{0.12}$$

$$\mathbf{M}^\top(a - b) + \mathbf{M}^\top\mathbf{M}\begin{pmatrix} p_1 \\ -p_2 \end{pmatrix} = 0, \tag{0.13}$$

$$\begin{pmatrix} \mathbf{d} \\ \mathbf{d} \end{pmatrix}(a - b) + \begin{pmatrix} \mathbf{d} \\ \mathbf{d} \end{pmatrix}\begin{pmatrix} \mathbf{d} & \mathbf{d} \end{pmatrix}\begin{pmatrix} p_1 \\ -p_2 \end{pmatrix} = 0 \tag{0.14}$$

On solving we get ,

$$p_1 - p_2 = 2\ take\ p_1 = 2\ and\ p_2 = 0 \tag{0.15}$$

Points are $\mathbf{P} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 0 \\ 7 \\ 0 \end{pmatrix}, \mathbf{P} - \mathbf{Q} = \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix}$

$$Distance = D = \|\mathbf{P} - \mathbf{Q}\| = \sqrt{24} \tag{0.16}$$

(iv) Area of the equilateral triangle formed by $\mathbf{u}, \mathbf{v}, \mathbf{w}$: As the two lines are parallel and let s = length of side of triangle

$$D = \frac{\sqrt{3}s}{2} \implies s = 4\sqrt{2} \tag{0.17}$$

$$Area\ of\ the\ equilateral\ triangle = A = \frac{\sqrt{3}s^2}{4} = 8\sqrt{3} \tag{0.18}$$

(v) Volume of the parallelepiped determined by three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$: $Volume\ of\ Parallelepiped = 6(Volume\ of\ Tetrahedron) = 2 \times base\ area \times height$
Height=h=Perpendicular distance from origin to plane containing $\mathbf{u}, \mathbf{v}, \mathbf{w}$

$$h = \frac{|\mathbf{m}^\top \mathbf{O} - c|}{||\mathbf{m}||} = \frac{|0-7|}{\sqrt{3}} = \frac{7}{\sqrt{3}} \tag{0.19}$$

$$Volume = 2 \times 8\sqrt{3} \times \frac{7}{\sqrt{3}} = 112 \tag{0.20}$$

$$\frac{80}{3}V = \frac{80}{3} \times 112 = \frac{8960}{3}. \tag{0.21}$$

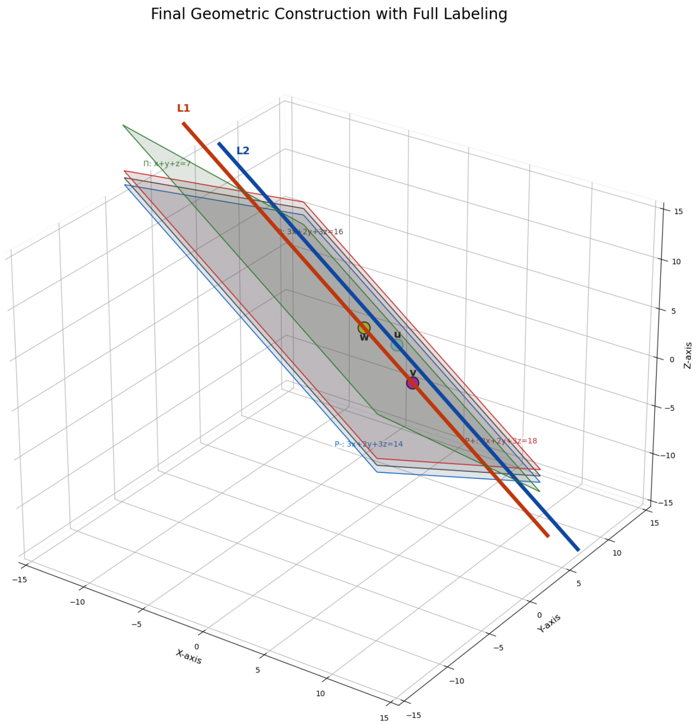

Fig: Representation of Planes and vectors