2.10.70

EE25BTECH11042 - Nipun Dasari

Question:
In a $\triangle$ABC, D and E are points on BC and AC respectively, such that $BD = 2DC$ and $AE = 3EC$. Let P be the point of intersection of AD and BE. Find $\frac{BP}{PE}$ using vector methods.
Solution:

Let vertex A be the origin. The position vectors are:

$$\mathbf{a} = \mathbf{0}, \quad \mathbf{b} = \text{Position vector of B}, \quad \mathbf{c} = \text{Position vector of C} \tag{0.1}$$

Position vector of point D, which divides BC in the ratio 2 : 1:

$$\mathbf{d} = \frac{1\mathbf{b} + 2\mathbf{c}}{2+1} = \frac{\mathbf{b} + 2\mathbf{c}}{3} \tag{0.2}$$

Position vector of point E, which divides AC in the ratio 3 : 1:

$$\mathbf{e} = \frac{1\mathbf{a} + 3\mathbf{c}}{3+1} = \frac{3\mathbf{c}}{4} \tag{0.3}$$

Let P divide AD in the ratio $AP : PD = \lambda : 1$. The position vector of P is:

$$\mathbf{p} = \frac{1\mathbf{a} + \lambda\mathbf{d}}{\lambda + 1} = \frac{\lambda}{\lambda + 1}\mathbf{d} \tag{0.4}$$

Substituting for $\mathbf{d}$:

$$\mathbf{p} = \left(\frac{\lambda}{\lambda + 1}\right)\left(\frac{\mathbf{b} + 2\mathbf{c}}{3}\right) = \frac{\lambda}{3(\lambda + 1)}\mathbf{b} + \frac{2\lambda}{3(\lambda + 1)}\mathbf{c} \tag{0.5}$$

Let P divide BE in the ratio $BP : PE = \mu : 1$. The position vector of P is:

$$\mathbf{p} = \frac{1\mathbf{b} + \mu\mathbf{e}}{\mu + 1} \tag{0.6}$$

Substituting for $\mathbf{e}$:

$$\mathbf{p} = \frac{1}{\mu + 1}\mathbf{b} + \frac{\mu}{\mu + 1}\mathbf{e} = \frac{1}{\mu + 1}\mathbf{b} + \frac{\mu}{\mu + 1}\left(\frac{3\mathbf{c}}{4}\right) = \frac{1}{\mu + 1}\mathbf{b} + \frac{3\mu}{4(\mu + 1)}\mathbf{c} \tag{0.7}$$

Since $\mathbf{b}$ and $\mathbf{c}$ are non-collinear, we equate their coefficients from (0.5) and (0.7).
Coefficients of $\mathbf{b}$:

$$\frac{\lambda}{3(\lambda + 1)} = \frac{1}{\mu + 1} \tag{0.8}$$

Coefficients of $\mathbf{c}$:

$$\frac{2\lambda}{3(\lambda + 1)} = \frac{3\mu}{4(\mu + 1)} \tag{0.9}$$

From (0.8) and (0.9), we can see that the LHS of (0.9) is twice the LHS of (0.8).

$$2\left(\frac{1}{\mu+1}\right) = \frac{3\mu}{4(\mu+1)} \tag{0.10}$$

Multiplying both sides by $4(\mu+1)$:

$$8 = 3\mu \implies \mu = \frac{8}{3} \tag{0.11}$$

The required ratio is $BP : PE = \mu : 1$.

$$\therefore \frac{BP}{PE} = \mu = \frac{8}{3} \tag{0.12}$$