4.13.70

AI25BTECH11001 - ABHISEK MOHAPATRA

Question: If $\begin{pmatrix} a & a^2 & 1+a3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{pmatrix} = 0$ and the vectors $\mathbf{A} = \begin{pmatrix} 1 & a & a^2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 \\ b \\ b^2 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 1 & c & c^2 \end{pmatrix}$ are co-planar, then the product abc = _____.

Solution: Let equation of the plane be $\mathbf{n}^\top \mathbf{x} = 0$.

so,

$$\mathbf{n}^\top \mathbf{A} = 0, \mathbf{n}^\top \mathbf{B} = 0, \mathbf{n}^\top \mathbf{C} = 0 \tag{1}$$

so ,

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}^\top \mathbf{n} = 0, \tag{2}$$

so for a unique plane to exist the rank of the matrix at left must be 3.Or,

$$det \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \neq 0 \tag{3}$$

$$\Rightarrow \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0 \tag{4}$$

solving the given determinant,

$$\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0 \tag{5}$$

$$\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} + \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = 0 \tag{6}$$

$$\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} + \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0 \tag{7}$$

$$abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0 \tag{8}$$

$$(abc+1) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0 \tag{9}$$

so,

$$abc + 1 = 0 \Rightarrow abc = -1 \tag{10}$$