Extra

EE25BTECH11023 - Venkata Sai

Question:
Find the equations of tangents drawn from origin to the circle $x^2+y^2-2rx-2hy+h^2=0$,are

1) $x = 0$
2) $y = 0$
3) $\left(h^2 - r^2\right) x - 2rhy = 0$
4) $\left(h^2 - r^2\right) x + 2rhy = 0$

Solution:
Equation of a circle is given by

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0, \tag{1}$$

The parametric equation of this line is

$$\mathbf{x} = \mathbf{h} + k\mathbf{m}. \tag{2}$$

$$(\mathbf{h} + k\mathbf{m})^T \mathbf{V} (\mathbf{h} + k\mathbf{m}) + 2\mathbf{u}^T (\mathbf{h} + k\mathbf{m}) + f \tag{3}$$

$$\mathbf{h}^\top \mathbf{V}\mathbf{h} + k\left(\mathbf{m}^\top \mathbf{V}\mathbf{h} + \mathbf{h}^\top \mathbf{V}\mathbf{m}\right) + k^2\mathbf{m}^\top \mathbf{V}\mathbf{m} + 2\mathbf{u}^\top \mathbf{h} + 2k\,\mathbf{u}^\top \mathbf{m} + f \tag{4}$$

$$\left(\mathbf{m}^\top \mathbf{V}\mathbf{m}\right)k^2 + 2\left(\mathbf{m}^\top \mathbf{V}\mathbf{h} + \mathbf{u}^\top \mathbf{m}\right) k + \left(\mathbf{h}^\top \mathbf{V}\mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f\right) \tag{5}$$

Hence the quadratic in k is

$$Ak^2 + Bk + C = 0, \tag{6}$$

where

$$A = \mathbf{m}^\top \mathbf{V}\mathbf{m}, \tag{7}$$

$$B = 2\left(\mathbf{m}^\top \mathbf{V}\mathbf{h} + \mathbf{u}^\top \mathbf{m}\right), \tag{8}$$

$$C = \left(\mathbf{h}^\top \mathbf{V}\mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f\right) \tag{9}$$

Discriminant is 0 as tangent intersects the circle at only one point

$$B^2 - 4AC = 0 \tag{10}$$

$$A = \mathbf{m}^\top \mathbf{V}\mathbf{m}, \tag{11}$$

$$B = 2\left(\mathbf{m}^\top \mathbf{V}\mathbf{h} + \mathbf{u}^\top \mathbf{m}\right), \tag{12}$$

$$C = \left(\mathbf{h}^\top \mathbf{V}\mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f\right) \tag{13}$$

Discriminant is 0 as tangent intersects the circle at only one point

$$B^2 - 4AC = 0 \tag{14}$$

Substituting the expressions for A, B, and C and expanding:

$$\left(2\left(\mathbf{m}^\top\mathbf{V}\mathbf{h}+\mathbf{u}^\top\mathbf{m}\right)\right)^2-4\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)\left(\mathbf{h}^\top\mathbf{V}\mathbf{h}+2\mathbf{u}^\top\mathbf{h}+f\right)=0. \tag{15}$$

$$\left(\mathbf{m}^\top\mathbf{V}\mathbf{h}+\mathbf{u}^\top\mathbf{m}\right)\left(\mathbf{m}^\top\mathbf{V}\mathbf{h}+\mathbf{u}^\top\mathbf{m}\right)^\top-\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)\left(\mathbf{h}^\top\mathbf{V}\mathbf{h}+2\mathbf{u}^\top\mathbf{h}+f\right)=0 \tag{16}$$

$$\left(\mathbf{m}^\top\mathbf{V}\mathbf{h}+\mathbf{u}^\top\mathbf{m}\right)\left(\mathbf{h}^\top\mathbf{V}\mathbf{m}+\mathbf{m}^\top\mathbf{u}\right)-\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)\left(\mathbf{h}^\top\mathbf{V}\mathbf{h}+2\mathbf{u}^\top\mathbf{h}+f\right)=0 \tag{17}$$

$$\begin{aligned}\left(\mathbf{m}^\top\mathbf{V}\mathbf{h}\right)\left(\mathbf{h}^\top\mathbf{V}\mathbf{m}\right)+\left(\mathbf{m}^\top\mathbf{V}\mathbf{h}\right)\left(\mathbf{m}^\top\mathbf{u}\right)+\left(\mathbf{u}^\top\mathbf{m}\right)\left(\mathbf{h}^\top\mathbf{V}\mathbf{m}\right)+\left(\mathbf{u}^\top\mathbf{m}\right)\left(\mathbf{m}^\top\mathbf{u}\right)\\-\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)\left(\mathbf{h}^\top\mathbf{V}\mathbf{h}\right)+2\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)\left(\mathbf{u}^\top\mathbf{h}\right)+f\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)=0\end{aligned} \tag{18}$$

$$\begin{aligned}\left(\mathbf{m}^\top\mathbf{V}\mathbf{h}\right)\left(\mathbf{h}^\top\mathbf{V}\mathbf{m}\right)+\left(\mathbf{m}^\top\mathbf{V}\mathbf{h}\right)\left(\mathbf{m}^\top\mathbf{u}\right)+\left(\mathbf{u}^\top\mathbf{m}\right)\left(\mathbf{h}^\top\mathbf{V}\mathbf{m}\right)+\left(\mathbf{u}^\top\mathbf{m}\right)\left(\mathbf{m}^\top\mathbf{u}\right)\\-\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)\left(\mathbf{h}^\top\mathbf{V}\mathbf{h}\right)-2\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)\left(\mathbf{u}^\top\mathbf{h}\right)-f\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)=0\end{aligned} \tag{19}$$

$$\mathbf{m}^\top\left(\mathbf{V}\mathbf{h}\mathbf{h}^\top\mathbf{V}+\mathbf{V}\mathbf{h}\mathbf{u}^\top+\mathbf{u}\mathbf{h}^\top\mathbf{V}+\mathbf{u}\mathbf{u}^\top-\left(\mathbf{h}^\top\mathbf{V}\mathbf{h}\right)\mathbf{V}-2\left(\mathbf{u}^\top\mathbf{h}\right)\mathbf{V}-f\mathbf{V}\right)\mathbf{m}=0 \tag{20}$$

$$\mathbf{m}^\top\left(\mathbf{V}\mathbf{h}\left(\mathbf{V}\mathbf{h}\right)^\top+\mathbf{V}\mathbf{h}\mathbf{u}^\top+\mathbf{u}\left(\mathbf{V}\mathbf{h}\right)^\top+\mathbf{u}\mathbf{u}^\top-\left(\mathbf{h}^\top\mathbf{V}\mathbf{h}\right)\mathbf{V}-2\left(\mathbf{u}^\top\mathbf{h}\right)\mathbf{V}-f\mathbf{V}\right)\mathbf{m}=0 \tag{21}$$

$$\mathbf{m}^\top\left(\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right)\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right)^\top-\left(\mathbf{h}^\top\mathbf{V}\mathbf{h}+2\mathbf{u}^\top\mathbf{h}+f\right)\mathbf{V}\right)\mathbf{m}=0 \tag{22}$$

$$\mathbf{m}^\top\left(\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right)\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right)^\top-g\left(\mathbf{h}\right)\right)\mathbf{m}=0 \tag{23}$$

where

$$g\left(\mathbf{h}\right)=\left(\mathbf{h}^\top\mathbf{V}\mathbf{h}+2\mathbf{u}^\top\mathbf{h}+f\right)\mathbf{V} \tag{24}$$

On comparing the given circle with matrix equation

$$x^2+y^2=\mathbf{x}^\top\begin{pmatrix}1&0\\0&1\end{pmatrix}\mathbf{x}\implies\mathbf{V}=\begin{pmatrix}1&0\\0&1\end{pmatrix} \tag{25}$$

$$-2rx-2hy=2\mathbf{u}^\top\mathbf{x}\implies\mathbf{u}^\top=\begin{pmatrix}-r&-h\end{pmatrix}\implies\mathbf{u}=\begin{pmatrix}-r\\-h\end{pmatrix} \tag{26}$$

$$h^2=f \tag{27}$$

Given tangents are drawn from the origin

$$\mathbf{h}=\begin{pmatrix}0\\0\end{pmatrix} \tag{28}$$

$$\mathbf{V}\mathbf{h}+\mathbf{u}=\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix}+\begin{pmatrix}-r\\-h\end{pmatrix}=\begin{pmatrix}-r\\-h\end{pmatrix} \tag{29}$$

$$\mathbf{V}\mathbf{h}=0\implies\mathbf{h}^\top\mathbf{V}\mathbf{h}=0 \tag{30}$$

$$g(\mathbf{h}) = \left(\mathbf{h}^\top \mathbf{V}\mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f\right)\mathbf{V} = \left(0 + 2\begin{pmatrix}-r\\-h\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix} + h^2\right)\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \tag{31}$$

$$g(\mathbf{h}) = h^2 \tag{32}$$

$$\mathbf{m}^\top \sum \mathbf{m} \implies \sum = \begin{pmatrix}-r\\-h\end{pmatrix}\begin{pmatrix}-r & -h\end{pmatrix} - h^2\mathbf{I} \tag{33}$$

$$\sum = \begin{pmatrix}r^2 & rh\\rh & h^2\end{pmatrix} - \begin{pmatrix}h^2 & 0\\0 & h^2\end{pmatrix} = \begin{pmatrix}r^2 - h^2 & rh\\rh & 0\end{pmatrix} \tag{34}$$

$$\left|\sum - \lambda\mathbf{I}\right| = 0 \tag{35}$$

$$\left|\begin{pmatrix}r^2 - h^2 - \lambda & rh\\rh & -\lambda\end{pmatrix}\right| = 0 \tag{36}$$

$$\left(r^2 - h^2 - \lambda\right)(-\lambda) - r^2h^2 = 0 \tag{37}$$

$$\lambda^2 - \left(r^2 - h^2\right)\lambda - r^2h^2 = 0 \tag{38}$$

$$\left(\lambda - r^2\right)\left(\lambda + h^2\right) = 0 \tag{39}$$

the eigen vectors are

$$\lambda = r^2 \text{ or } \lambda = -h^2 \tag{40}$$

Because the tangent passes through origin

$$\mathbf{m} = \mathbf{x} \tag{41}$$

$$\begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}r^2 - h^2 & rh\\rh & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = 0 \tag{42}$$

$$\begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}\left(r^2 - h^2\right)x + (rh)y\\rhx\end{pmatrix} = 0 \tag{43}$$

$$x\left(\left(r^2 - h^2\right)x + rhy\right) + y(rhx) = 0 \tag{44}$$

$$x^2\left(r^2 - h^2\right) + 2rhxy = 0 \tag{45}$$

$$x\left(\left(r^2 - h^2\right)x + 2rhy\right) = 0 \tag{46}$$

$$x = 0 \text{ and } \left(r^2 - h^2\right)x + 2rhy = 0 \tag{47}$$