Matrices in Geometry 9.5.1

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Question: Find the roots of

$$x^2 + 3x - 10 = 0 \tag{1}$$

Solution: Expressing the given equation as parabola

$$y = x^2 + 3x - 10 \tag{2}$$

Representing this equation as a conic section

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \ , \ \ \mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ , \ \ \mathbf{u} = \begin{pmatrix} 3/2 \\ -1/2 \end{pmatrix} \ , \ \ f = -10 \tag{3}$$

We need to find intersection points with $y = 0$, that is, the X-axis.

$$\mathbf{x} = \mathbf{h} + k\mathbf{m} \ , \ \ \mathbf{h} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ , \ \ \mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4}$$

Substituting $\mathbf{x} = k\mathbf{m}$

$$k^2 \mathbf{m}^\top \mathbf{V} \mathbf{m} + 2k\mathbf{u}^\top \mathbf{m} + f = 0 \tag{5}$$

$$\implies k = \frac{1}{2}\left[-2\mathbf{u}^\top \mathbf{m} \ \pm \ \sqrt{4\,(\mathbf{u}^\top \mathbf{m})^2 - 4f\mathbf{m}^\top \mathbf{V}\mathbf{m}} \right] \tag{6}$$

$$\implies k = -\mathbf{u}^\top \mathbf{m} \ \pm \ \sqrt{(\mathbf{u}^\top \mathbf{m})^2 - f\mathbf{m}^\top \mathbf{V}\mathbf{m}} \tag{7}$$

$$\mathbf{u}^\top \mathbf{m} = \begin{pmatrix} 3/2 & -1/2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 3/2 \tag{8}$$

$$\mathbf{m}^\top \mathbf{V} \mathbf{m} = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \tag{9}$$

$$k = -\frac{3}{2} \ \pm \ \sqrt{\frac{9}{4} - (-10)\,1} = -\frac{3}{2} \ \pm \ \sqrt{\frac{49}{4}} \tag{10}$$

$$\implies k = -\frac{3}{2} \ \pm \ \frac{7}{2} \ \implies \ \boxed{k = 2 \text{ OR } k = -5} \tag{11}$$

Substituting k into $\mathbf{x}$, we get

$$\mathbf{x} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \text{ OR } \mathbf{x} = \begin{pmatrix} -5 \\ 0 \end{pmatrix} \tag{12}$$

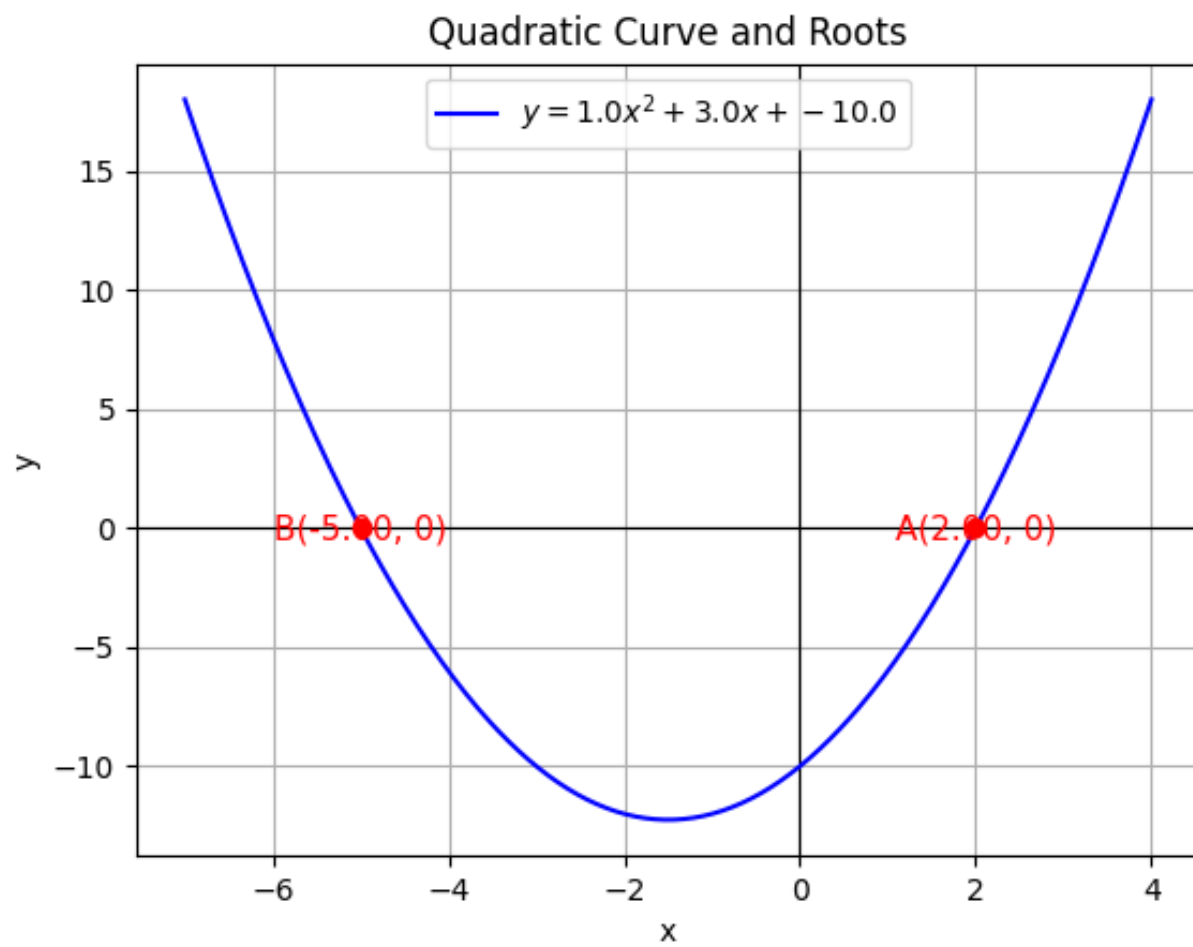This implies that the roots of $x^2 + 3x - 10 = 0$ are 2 and -5.
Let us plot this

Fig. 1: Graph for 9.5.1