5.10.3

EE25BTECH11065 - Yoshita J

Question:
Balance the following chemical equation.

$$Fe + H_2O \rightarrow Fe_3O_4 + H_2$$

Solution:
Let the balanced version of the equation be

$$x_1Fe + x_2H_2O \rightarrow x_3Fe_3O_4 + x_4H_2 \tag{1}$$

which results in the following equations based on the conservation of each element:

$$\text{For Fe: } x_1 - 3x_3 = 0 \tag{2}$$
$$\text{For H: } 2x_2 - 2x_4 = 0 \implies x_2 - x_4 = 0 \tag{3}$$
$$\text{For O: } x_2 - 4x_3 = 0 \tag{4}$$

This can be expressed as a homogeneous system of linear equations:

$$x_1 + 0x_2 - 3x_3 + 0x_4 = 0 \tag{5}$$
$$0x_1 + x_2 + 0x_3 - x_4 = 0 \tag{6}$$
$$0x_1 + x_2 - 4x_3 + 0x_4 = 0 \tag{7}$$

This results in the matrix equation $A\mathbf{x} = \mathbf{0}$, where:

$$\begin{pmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & -4 & 0 \end{pmatrix} \mathbf{x} = \mathbf{0}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \tag{8}$$

The coefficient matrix can be reduced as follows using Gaussian elimination to find the null space:

$$\begin{pmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & -4 & 0 \end{pmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{pmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -4 & 1 \end{pmatrix} \xrightarrow{R_3 \to -\frac{1}{4}R_3} \begin{pmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1/4 \end{pmatrix}$$

$$\xrightarrow{R_1 \to R_1 + 3R_3} \begin{pmatrix} 1 & 0 & 0 & -3/4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1/4 \end{pmatrix}$$

From the reduced row echelon form, we get the solutions in terms of the free variable x_4:

$$x_1 = \frac{3}{4}x_4, \quad x_2 = x_4, \quad x_3 = \frac{1}{4}x_4 \tag{9}$$

Thus,

$$\mathbf{x} = x_4 \begin{pmatrix} 3/4 \\ 1 \\ 1/4 \\ 1 \end{pmatrix} \tag{10}$$

By substituting $x_4 = 4$, the simplest integer solution is found. Hence,

$$\mathbf{x} = 4 \begin{pmatrix} 3/4 \\ 1 \\ 1/4 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 1 \\ 4 \end{pmatrix} \tag{11}$$

This gives $x_1 = 3, x_2 = 4, x_3 = 1$, and $x_4 = 4$. Hence, the balanced equation finally becomes:

$$\boxed{3Fe + 4H_2O \rightarrow Fe_3O_4 + 4H_2} \tag{12}$$