

4.11.5

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Question. Find the equation of the plane passing through the points $(2, 5, -3)$, $(-2, -3, 5)$, and $(5, 3, -3)$. Also, find the point of intersection of this plane with the line passing through points $(3, 1, 5)$ and $(-1, -3, -1)$.

Solution:

Let us solve the given equation theoretically and then verify the solution computationally.

Let the given points be:

$$\mathbf{A} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -2 \\ -3 \\ 5 \end{pmatrix} \text{ and } \mathbf{C} = \begin{pmatrix} 5 \\ 3 \\ -3 \end{pmatrix} \quad (1)$$

For equation of plane:

$$(\mathbf{A} \quad \mathbf{B} \quad \mathbf{C})^T \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (2)$$

$$\begin{pmatrix} 2 & -2 & 5 \\ 5 & -3 & 3 \\ -3 & 5 & -3 \end{pmatrix}^T \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (3)$$

$$\begin{pmatrix} 2 & 5 & -3 \\ -2 & -3 & 5 \\ 5 & 3 & -3 \end{pmatrix} \mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 5 & -3 & | & 1 \\ -2 & -3 & 5 & | & 1 \\ 5 & 3 & -3 & | & 1 \end{pmatrix} \quad (4)$$

$$\begin{pmatrix} 2 & 5 & -3 & | & 1 \\ -2 & -3 & 5 & | & 1 \\ 5 & 3 & -3 & | & 1 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2 + R_1]{R_3 \leftarrow R_3 - \frac{5}{2}R_1} \begin{pmatrix} 2 & 5 & -3 & | & 1 \\ 0 & 2 & 2 & | & 2 \\ 0 & -\frac{19}{2} & \frac{9}{2} & | & -\frac{3}{2} \end{pmatrix} \quad (5)$$

$$\begin{pmatrix} 2 & 5 & -3 & | & 1 \\ 0 & 2 & 2 & | & 2 \\ 0 & -\frac{19}{2} & \frac{9}{2} & | & -\frac{3}{2} \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{1}{2}R_2} \begin{pmatrix} 2 & 5 & -3 & | & 1 \\ 0 & 1 & 1 & | & 1 \\ 0 & -\frac{19}{2} & \frac{9}{2} & | & -\frac{3}{2} \end{pmatrix} \quad (6)$$

$$\begin{pmatrix} 2 & 5 & -3 & | & 1 \\ 0 & 1 & 1 & | & 1 \\ 0 & -\frac{19}{2} & \frac{9}{2} & | & -\frac{3}{2} \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + \frac{19}{2}R_2} \begin{pmatrix} 2 & 5 & -3 & | & 1 \\ 0 & 1 & 1 & | & 1 \\ 0 & 0 & 14 & | & 8 \end{pmatrix} \quad (7)$$

$$\left(\begin{array}{ccc|c} 2 & 5 & -3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 14 & 8 \end{array} \right) \xrightarrow{R_3 \leftarrow \frac{R_3}{14}} \left(\begin{array}{ccc|c} 2 & 5 & -3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & \frac{4}{7} \end{array} \right) \quad (8)$$

$$\left(\begin{array}{ccc|c} 2 & 5 & -3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & \frac{4}{7} \end{array} \right) \xrightarrow[R_2 \leftarrow R_2 - R_3]{R_1 \leftarrow R_1 + 3R_3} \left(\begin{array}{ccc|c} 2 & 5 & 0 & \frac{19}{7} \\ 0 & 1 & 0 & \frac{3}{7} \\ 0 & 0 & 1 & \frac{4}{7} \end{array} \right) \quad (9)$$

$$\left(\begin{array}{ccc|c} 2 & 5 & 0 & \frac{19}{7} \\ 0 & 1 & 0 & \frac{3}{7} \\ 0 & 0 & 1 & \frac{4}{7} \end{array} \right) \xrightarrow{R_1 \leftarrow R_1 - 5R_2} \left(\begin{array}{ccc|c} 2 & 0 & 0 & \frac{4}{7} \\ 0 & 1 & 0 & \frac{3}{7} \\ 0 & 0 & 1 & \frac{4}{7} \end{array} \right) \quad (10)$$

$$\left(\begin{array}{ccc|c} 2 & 0 & 0 & \frac{4}{7} \\ 0 & 1 & 0 & \frac{3}{7} \\ 0 & 0 & 1 & \frac{4}{7} \end{array} \right) \xrightarrow{R_1 \leftarrow \frac{R_1}{2}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{2}{7} \\ 0 & 1 & 0 & \frac{3}{7} \\ 0 & 0 & 1 & \frac{4}{7} \end{array} \right) \quad (11)$$

The equation of plane can be given as:

$$\mathbf{n}^T \mathbf{x} = c \quad (12)$$

Therefore the equation of plane can be given as:

$$\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}^T \mathbf{x} = 7 \quad (13)$$

Where

$$n = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \text{ and } c = 7 \quad (14)$$

The equation of line passing through the points (3, 1, 5) and (-1, -3, -1)

$$\mathbf{x} = \mathbf{h} + k\mathbf{m} \quad (15)$$

Multiply with \mathbf{n}^T on both sides:

$$\mathbf{n}^T \mathbf{x} = \mathbf{n}^T \mathbf{h} + k\mathbf{n}^T \mathbf{m} \quad (16)$$

From Eq.12

$$c = \mathbf{n}^T \mathbf{h} + k\mathbf{n}^T \mathbf{m} \quad (17)$$

$$\frac{c - \mathbf{n}^T \mathbf{h}}{\mathbf{n}^T \mathbf{m}} = k \quad (18)$$

$$\mathbf{m} = \begin{pmatrix} 3 - (-1) \\ 1 - (-3) \\ 5 - (-1) \end{pmatrix} \quad (19)$$

$$\mathbf{m} = \begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix} \quad (20)$$

Now substitute the corresponding values in Eq.18

$$\frac{7 - \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}^T \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}}{\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}^T \begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix}} = k \quad (21)$$

$$\frac{7 - 29}{44} = k \quad (22)$$

Solving this we get

$$k = \frac{-1}{2} \quad (23)$$

Now substitute the value of k in Eq.15

$$\mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix} \quad (24)$$

$$\mathbf{x} = \begin{pmatrix} 3 - 2 \\ 1 - 2 \\ 5 - 3 \end{pmatrix} \quad (25)$$

Therefore the point of intersection is:

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \quad (26)$$

From the figure it is clearly verified that the theoretical solution matches with the computational solution.

