Samyak Gondane-AI25BTECH11029

Question

If $\mathbf{a} = \hat{i} + \hat{j} + \hat{k}$, $\mathbf{b} = 4\hat{i} + 3\hat{j} + 4\hat{k}$ and $\mathbf{c} = \hat{i} + \alpha\hat{j} + \beta\hat{k}$ are linearly dependant vectors and $|c| = \sqrt{3}$, then

1. $\alpha = 1, \beta = -1$
2. $\alpha = 1, \beta = \pm 1$
3. $\alpha = -1, \beta = -1$
4. $\alpha = \pm 1, \beta = 1$

Solution

Linear Dependence of Vectors via Matrix Theory

Given three vectors in $\mathbb{R}^3$:

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 4 \\ 3 \\ 4 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix} \tag{1}$$

and the condition that $\|\mathbf{c}\| = \sqrt{3}$, we aim to find values of α and β such that the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are linearly dependent.

The magnitude of vector $\mathbf{c}$ is:

$$\|\mathbf{c}\|^2 = 1^2 + \alpha^2 + \beta^2 = 3 \quad \Rightarrow \quad \alpha^2 + \beta^2 = 2 \tag{1}$$

Construct a matrix with the vectors as columns:

$$M = \begin{pmatrix} 1 & 4 & 1 \\ 1 & 3 & \alpha \\ 1 & 4 & \beta \end{pmatrix} \tag{2}$$

Compute the determinant:

$$\begin{aligned} \det(M) &= 1 \cdot (3\beta - 4\alpha) - 4 \cdot (\beta - \alpha) + 1 \cdot (4 - 3) \\ &= 3\beta - 4\alpha - 4\beta + 4\alpha + 1 \\ &= -\beta + 1 \end{aligned}$$

Set $\det(M) = 0$ for linear dependence:

$$-2\beta + 2 = 0 \quad \Rightarrow \quad \beta = 1 \tag{2}$$

Substitute Equation (2) into Equation (1):

$$\alpha^2 + 1 = 2 \quad \Rightarrow \quad \alpha^2 = 1 \quad \Rightarrow \quad \alpha = \pm 1 \tag{3}$$

Final Answer

The values of α and β that satisfy both conditions are:

$$\boxed{\alpha = \pm 1, \quad \beta = 1} \tag{4}$$