4.7.22

EE25BTECH11031 - Sai Sreevallabh

Question:

In what direction should a line be drawn through the point $(1, 2)$ so that its point of intersection with the line $x + y = 4$ is at a distance $\sqrt{63}$

Solution:

The given point is $\mathbf{P} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

The given line can be represented as $\mathbf{n}^\top \mathbf{x} = c$, where

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad c = 4 \tag{0.1}$$

A parametric point on the line passing through the point $\mathbf{P}$ is given by

$$\mathbf{r} = \mathbf{P} + \lambda \mathbf{m} \tag{0.2}$$

where $\mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix}$

Plugging in the parametric form of the point in the line equation, we get:

$$\mathbf{n}^\top (\mathbf{P} + \lambda \mathbf{m}) = c \tag{0.3}$$

$$\lambda = \frac{c - \mathbf{n}^\top \mathbf{P}}{\mathbf{n}^\top \mathbf{m}} \tag{0.4}$$

Replacing this value of λ in the equation of the parametric point, we get it to be

$$\mathbf{r} = \mathbf{P} + \left(\frac{c - \mathbf{n}^\top \mathbf{P}}{\mathbf{n}^\top \mathbf{m}} \right) \mathbf{m} \tag{0.5}$$

Let the distance between this point and $\mathbf{P}$ be $d = \sqrt{63}$.

$$d = \|\mathbf{r} - \mathbf{P}\| \tag{0.6}$$

$$d = \left| \frac{c - \mathbf{n}^\top \mathbf{P}}{\mathbf{n}^\top \mathbf{m}} \right| \|\mathbf{m}\| \tag{0.7}$$

Substituting the values, we get

$$\sqrt{63} = \left| \frac{4-3}{1+m} \right| \sqrt{1+m^2} \tag{0.8}$$

Squaring on both sides:

$$63 = \frac{1+m^2}{(1+m)^2} \tag{0.9}$$

This is an equation in m. Upon solving this equation, we get:

$$m = \frac{-63 \pm 5\sqrt{5}}{62} \tag{0.10}$$

$\therefore$ The direction vector $\mathbf{m}$ can take two values:

$$\mathbf{m} = \begin{pmatrix} 1 \\ \frac{-63+5\sqrt{5}}{62} \end{pmatrix} \quad \text{or} \quad \mathbf{m} = \begin{pmatrix} 1 \\ \frac{-63-5\sqrt{5}}{62} \end{pmatrix}$$