10.6.8 - Eigenvector Method

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Question

Construct a pair of tangents to a circle of radius 4cm from a point P lying outside the circle at a distance of 6cm from the centre. (10, 2023)

Method: Using Eigenvector Decomposition

Problem Setup

Let the center of the circle be at origin. The equation is $x^2 + y^2 = 16$ and Point P is at distance 6 from center along x-axis.

$$O = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1}$$

$$P = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \tag{2}$$

$$\vec{x}^\top \vec{V} \vec{x} + 2\vec{u}^\top \vec{x} + f = 0 \tag{3}$$

where

$$\vec{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad f = -16 \tag{4}$$

Step 1: Eigenvalue Decomposition

The eigenvalue equation is:

$$\vec{V}\vec{p} = \lambda\vec{p} \tag{5}$$

The characteristic equation:

$$\det(\vec{V} - \lambda\vec{I}) = 0 \tag{6}$$

$$\det\begin{pmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} = 0 \tag{7}$$

$$(1-\lambda)^2 = 0 \tag{8}$$

Eigenvalues: $\lambda_1 = \lambda_2 = 1$

Step 2: Finding Eigenvectors

For $\lambda_1 = 1$:

$$(\vec{V} - \lambda_1 \vec{I})\vec{p}_1 = 0 \tag{9}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p_{11} \\ p_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{10}$$

Choose normalized eigenvector: $\vec{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

For $\lambda_2 = 1$, choose orthogonal eigenvector:

$$\vec{p}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{11}$$

Step 3: Eigenvector Matrix

The orthogonal eigenvector matrix is:

$$\vec{P} = \begin{pmatrix} \vec{p}_1 & \vec{p}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \vec{I} \tag{12}$$

Spectral Decomposition:

$$\vec{V} = \vec{P}\vec{D}\vec{P}^\top \tag{13}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \vec{I} \tag{14}$$

where $\vec{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Step 4: Principal Axes Transformation

Transform to principal coordinates:

$$\vec{y} = \vec{P}^{\top}(\vec{x} - \vec{c}) \tag{15}$$

where $\vec{c} = -\vec{V}^{-1}\vec{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

In principal axes, the conic equation becomes:

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 = -f \tag{16}$$

$$y_1^2 + y_2^2 = 16 \tag{17}$$

Step 5: Semi-axes from Eigenvalues

The radius along each eigenvector direction:

$$a = b = \sqrt{\frac{-f}{\lambda_1}} = \sqrt{\frac{16}{1}} = 4 \tag{18}$$

This confirms:

- Circle is symmetric in all directions
- Eigenvectors $\vec{p}_1$ and $\vec{p}_2$ form principal axes
- Radius $= 4$ cm along both axes

Step 6: Transform Point P

Transform external point P to principal coordinates:

$$\vec{y}_P = \vec{P}^\top (P - \vec{c}) = \vec{I} \begin{pmatrix} 6 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \tag{19}$$

For tangent from external point, contact point $\vec{q}$ must satisfy:

(a) $\vec{q}$ lies on circle: $\vec{q}^\top \vec{V} \vec{q} + f = 0$

(b) Tangent passes through P: $(\vec{V}\vec{q})^\top P + f = 0$

Step 7: Finding Contact Points

From condition (b) with $\vec{V} = \vec{I}$:

$$\vec{q}^{\top} P + f = 0 \tag{20}$$

$$\begin{pmatrix} q_1 & q_2 \end{pmatrix} \begin{pmatrix} 6 \\ 0 \end{pmatrix} = 16 \tag{21}$$

$$6q_1 = 16 \implies q_1 = \frac{8}{3} \tag{22}$$

From condition (a):

$$q_1^2 + q_2^2 = 16 \tag{23}$$

$$\left(\frac{8}{3}\right)^2 + q_2^2 = 16 \tag{24}$$

$$q_2^2 = \frac{80}{9} \implies q_2 = \pm\frac{4\sqrt{5}}{3} \tag{25}$$

Step 8: Express in Eigenvector Basis

The contact points expressed as linear combinations of eigenvectors:

$$\vec{q}_1 = \frac{8}{3}\vec{p}_1 + \frac{4\sqrt{5}}{3}\vec{p}_2 \quad (26)$$

$$= \frac{8}{3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{4\sqrt{5}}{3}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (27)$$

$$= \begin{pmatrix} \frac{8}{3} \\ \frac{4\sqrt{5}}{3} \end{pmatrix} \quad (28)$$

$$\vec{q}_2 = \frac{8}{3}\vec{p}_1 - \frac{4\sqrt{5}}{3}\vec{p}_2 \quad (29)$$

$$= \begin{pmatrix} \frac{8}{3} \\ -\frac{4\sqrt{5}}{3} \end{pmatrix} \quad (30)$$

Step 9: Tangent Equations

The tangent at $\vec{q}$ is: $(\vec{V}\vec{q})^\top \vec{x} + f = 0$

Tangent 1 at $\vec{q}_1$:

$$\vec{V}\vec{q}_1 = \begin{pmatrix} \frac{8}{3} \\ \frac{4\sqrt{5}}{3} \end{pmatrix} \tag{31}$$

$$\begin{pmatrix} \frac{8}{3} & \frac{4\sqrt{5}}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - 16 = 0 \tag{32}$$

$$\frac{8}{3}x + \frac{4\sqrt{5}}{3}y = 16 \tag{33}$$

$$2x + \sqrt{5}y = 12 \tag{34}$$

Step 10: Second Tangent

Tangent 2 at $\vec{q}_2$:

$$\begin{pmatrix} \frac{8}{3} & -\frac{4\sqrt{5}}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - 16 = 0 \tag{35}$$

$$2x - \sqrt{5}y = 12 \tag{36}$$

Final Answer:

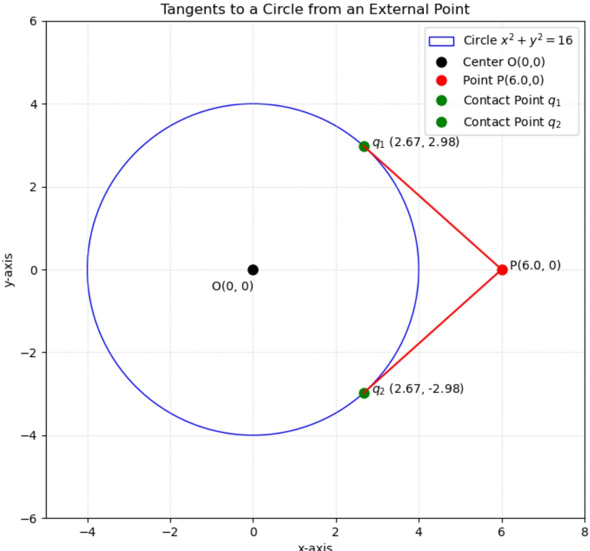
$$\boxed{2x + \sqrt{5}y = 12 \quad \text{and} \quad 2x - \sqrt{5}y = 12} \tag{37}$$

Summary of Eigenvector Method

1. Found eigenvalues: $\lambda_1 = \lambda_2 = 1$
2. Computed eigenvectors: $\vec{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{p}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
3. Spectral decomposition: $\vec{V} = \vec{P}\vec{D}\vec{P}^\top$
4. Transformed to principal axes
5. Calculated semi-axes using: $a = \sqrt{-f/\lambda_1}$
6. Found contact points in eigenvector basis
7. Derived tangent equations

Key Insight: Eigenvectors define the natural coordinate system for the conic!

Plot


Tangents to a Circle from an External Point
Circle $x^2 + y^2 = 16$
Center O(0,0)
Point P(6.0,0)
Contact Point q_1
Contact Point q_2
q_1 (2.67, 2.98)
q_2 (2.67, -2.98)
P(6.0, 0)
O(0, 0)
y-axis
x-axis
6
4
2
0
−2
−4
−6
−4
−2
0
2
4
6
8