

Problem 12.37

ee25btech11023-Venkata Sai

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Problem

Let \mathcal{M} be the set of 3×3 real symmetric positive definite matrices. Consider $S = \{\mathbf{A} \in \mathcal{M} : \mathbf{A}^{50} - \mathbf{A}^{48} = 0\}$. The number of elements in S equals

Condition

If a matrix is symmetric then it is diagonalizable. Hence

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \quad (3.1)$$

$$\mathbf{A}^2 = (\mathbf{P}\mathbf{D}\mathbf{P}^{-1})^2 \quad (3.2)$$

$$= \mathbf{P}\mathbf{D}\mathbf{P}^{-1}\mathbf{P}\mathbf{D}\mathbf{P}^{-1} \quad (3.3)$$

$$= \mathbf{P}\mathbf{D}\mathbf{I}\mathbf{D}\mathbf{P}^{-1} \quad (3.4)$$

$$= \mathbf{P}\mathbf{D}^2\mathbf{P}^{-1} \quad (3.5)$$

$$\mathbf{A}^k = \mathbf{P}\mathbf{D}^k\mathbf{P}^{-1} \quad (3.6)$$

$$\mathbf{A}^{50} = \mathbf{P}\mathbf{D}^{50}\mathbf{P}^{-1} \quad (3.7)$$

$$\mathbf{A}^{48} = \mathbf{P}\mathbf{D}^{48}\mathbf{P}^{-1} \quad (3.8)$$

Given

$$\mathbf{A}^{50} - \mathbf{A}^{48} = \mathbf{0} \quad (3.9)$$

$$\mathbf{P}\mathbf{D}^{50}\mathbf{P}^{-1} - \mathbf{P}\mathbf{D}^{48}\mathbf{P}^{-1} = \mathbf{0} \quad (3.10)$$

Conclusion

$$\mathbf{P} (\mathbf{D}^{50} - \mathbf{D}^{48}) \mathbf{P}^{-1} = \mathbf{0} \quad (3.11)$$

$$\implies (\mathbf{D}^{50} - \mathbf{D}^{48}) = \mathbf{0} \quad (3.12)$$

$$\implies (\lambda^{50} - \lambda^{48}) = 0 \quad (3.13)$$

where λ are the eigen values

$$\lambda^{48} (\lambda^2 - 1) = 0 \implies \lambda^{48} = 0 \text{ or } \lambda^2 - 1 = 0 \quad (3.14)$$

$$\lambda = 0 \text{ or } \lambda = \pm 1 \quad (3.15)$$

For a positive definite matrix, eigen values must be greater than 0. Hence

$$\lambda = 1 \implies \mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I} \quad (3.16)$$

$$\mathbf{A} = \mathbf{P}\mathbf{I}\mathbf{P}^{-1} = \mathbf{P}\mathbf{P}^{-1} = \mathbf{I} \quad (3.17)$$

Hence Number of elements in S is 1