2.10.56

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Question

Let two non-collinear unit vectors $\hat{a}$ and $\hat{b}$ form an acute angle. A point **P** moves so that at any time t the position vector **P** (where **O** is the origin) is given by $\mathbf{a}\cos t + \mathbf{b}\sin t$. When **P** is farthest from origin **O**, let M be the length of **P** and $\hat{\mathbf{u}}$ be the unit vector along **P**. Then,

1. $\hat{\mathbf{u}} = \frac{\mathbf{a}+\mathbf{b}}{|\mathbf{a}+\mathbf{b}|}$ and $M = (1 + \mathbf{a}\cdot\mathbf{b})^{\frac{1}{2}}$
2. $\hat{\mathbf{u}} = \frac{\mathbf{a}-\mathbf{b}}{|\mathbf{a}-\mathbf{b}|}$ and $M = (1 + \mathbf{a}\cdot\mathbf{b})^{\frac{1}{2}}$
3. $\hat{\mathbf{u}} = \frac{\mathbf{a}+\mathbf{b}}{|\mathbf{a}+\mathbf{b}|}$ and $M = (1 + 2\mathbf{a}\cdot\mathbf{b})^{\frac{1}{2}}$
4. $\hat{\mathbf{u}} = \frac{\mathbf{a}-\mathbf{b}}{|\mathbf{a}-\mathbf{b}|}$ and $M = (1 + 2\mathbf{a}\cdot\mathbf{b})^{\frac{1}{2}}$

Equation

Given equation:

$$\mathbf{P} = \mathbf{a}\cos t + \mathbf{b}\sin t \tag{1}$$

Which can be written as :

$$\mathbf{P} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \tag{2}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \mathbf{x} \tag{3}$$

Let

$$\mathbf{x} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \quad \textit{and} \quad \mathbf{G} = \begin{pmatrix} 1 & (\mathbf{a})^T(\mathbf{b}) \\ (\mathbf{a})^T(\mathbf{b}) & 1 \end{pmatrix} \tag{4}$$

From given if $\mathbf{P}$ is farthest from origin , then length of $\mathbf{P}$ is given as M.From this we can say that

$$M = \max \|\mathbf{P}\| \tag{5}$$

Now,

$$\|\mathbf{P}\| = \sqrt{(\mathbf{P})^T(\mathbf{P})} \tag{6}$$

$$\|\mathbf{P}\| = \sqrt{\left(\begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix}\mathbf{x}\right)^T\left(\begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix}\mathbf{x}\right)} \tag{7}$$

$$\|\mathbf{P}\| = \sqrt{\mathbf{x}^T\begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix}^T\begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix}\mathbf{x}} \tag{8}$$

Theoretical solution

Let **G** be a gram matrix:

$$\mathbf{G} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix}^T \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} = \begin{pmatrix} 1 & (\mathbf{a})^T(\mathbf{b}) \\ (\mathbf{a})^T(\mathbf{b}) & 1 \end{pmatrix} \tag{9}$$

$$\|\mathbf{P}\|^2 = \mathbf{x}^T \begin{pmatrix} 1 & (\mathbf{a})^T(\mathbf{b}) \\ (\mathbf{a})^T(\mathbf{b}) & 1 \end{pmatrix} \mathbf{x} \tag{10}$$

$$\|\mathbf{P}\|^2 = \mathbf{x}^T\mathbf{G}\mathbf{x} \tag{11}$$

Now we should find the maximum value of $x^T Gx$ such that $\|x\| = 1$

By **Rayleigh-Ritz theorem**, the maximum value of the quadratic form if **x** is a unit vector will be the largest eigenvalue (λ_{max}) of the matrix G. So,

$$max\,\|\mathbf{P}\| = \sqrt{\lambda_{max}} \tag{12}$$

Now we will calculate the Eigen value for the matrix G:

$$\left|\mathbf{G} - \lambda\mathbf{I}\right| = 0 \tag{13}$$

$$\left|\begin{pmatrix} 1-\lambda & (\mathbf{a})^T(\mathbf{b}) \\ (\mathbf{a})^T(\mathbf{b}) & 1-\lambda \end{pmatrix}\right| = 0 \tag{14}$$

$$(1-\lambda)^2 - ((\mathbf{a})^T(\mathbf{b}))^2 = 0 \tag{15}$$

$$1-\lambda = (\mathbf{a})^T(\mathbf{b}) \;\; or \;\; 1-\lambda = -(\mathbf{a})^T(\mathbf{b}) \tag{16}$$

$$\lambda = 1 + (\mathbf{a})^T(\mathbf{b}) \;\; or \;\; \lambda = 1 - (\mathbf{a})^T(\mathbf{b}) \tag{17}$$

It is already given that $(\mathbf{a})^T(\mathbf{b}) > 0$(**a** and **b** form an acute angle) . so,

$$\lambda_{max} = 1 + (\mathbf{a})^T(\mathbf{b}) \tag{18}$$

From Eq.12

$$max \left\| \mathbf{P} \right\| = \sqrt{1 + (\mathbf{a})^T (\mathbf{b})} \tag{19}$$

The above equation can be written as

$$max \left\| \mathbf{P} \right\| = \sqrt{1 + \mathbf{a}.\mathbf{b}} \tag{20}$$

From Eq.5:

$$M = \sqrt{1 + \mathbf{a}.\mathbf{b}} \tag{21}$$

Now let us find the value of t for which $\left\| \mathbf{P} \right\|$ is max
With eigenvalue equation,We'll use matrix G and largest eigenvalue λ_{max} such that,

$$\left(\mathbf{G} - \lambda \mathbf{I} \right) \mathbf{x} = 0 \tag{22}$$

$$\left(\begin{pmatrix} 1 & (\mathbf{a})^T (\mathbf{b}) \\ (\mathbf{a})^T (\mathbf{b}) & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right) \mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{23}$$

$$\begin{pmatrix} 1-\lambda & (a)^T(b) \\ (a)^T(b) & 1-\lambda \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{24}$$

By substituting $\lambda = 1 + (\mathbf{a})^T(\mathbf{b})$. We get:

$$\begin{pmatrix} -(\mathbf{a})^T(\mathbf{b}) & (\mathbf{a})^T(\mathbf{b}) \\ (\mathbf{a})^T(\mathbf{b}) & -(\mathbf{a})^T(\mathbf{b}) \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{25}$$

$$\begin{pmatrix} -(\mathbf{a})^T(\mathbf{b}) & (\mathbf{a})^T(\mathbf{b}) \\ (\mathbf{a})^T(\mathbf{b}) & -(\mathbf{a})^T(\mathbf{b}) \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{26}$$

$$\begin{pmatrix} -(\mathbf{a})^T(\mathbf{b})\cos t + (\mathbf{a})^T(\mathbf{b})\sin t \\ (\mathbf{a})^T(\mathbf{b})\cos t - (\mathbf{a})^T(\mathbf{b})\sin t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{27}$$

$$-(\mathbf{a})^T(\mathbf{b})\cos t + (\mathbf{a})^T(\mathbf{b})\sin t = 0 \tag{28}$$

Theoretical Solution

$$(\mathbf{a})^T(\mathbf{b})\cos t = (\mathbf{a})^T(\mathbf{b})\sin t \tag{29}$$

$$\cos t = \sin t \tag{30}$$

We already know that:

$$\sin^2 t + \cos^2 t = 1 \tag{31}$$

So,

$$\sin t = \frac{1}{\sqrt{2}} \quad and \quad \cos t = \frac{1}{\sqrt{2}} \tag{32}$$

From above result

$$t = \frac{\pi}{4} \tag{33}$$

Now unit vector $\mathbf{u}$ along $\mathbf{P}$ is given by:

$$\mathbf{u} = \frac{\mathbf{P}}{\|\mathbf{P}\|} \tag{34}$$

$$\mathbf{u} = \frac{\mathbf{a}\cos t + \mathbf{b}\sin t}{\|\mathbf{a}\cos t + \mathbf{b}\sin t\|} \tag{35}$$

Now subtituiting the value of t in above equation:

$$\mathbf{u} = \frac{\mathbf{a}\frac{1}{\sqrt{2}} + \mathbf{b}\frac{1}{\sqrt{2}}}{\left\|\mathbf{a}\frac{1}{\sqrt{2}} + \mathbf{b}\frac{1}{\sqrt{2}}\right\|} \tag{36}$$

$$\mathbf{u} = \frac{\mathbf{a} + \mathbf{b}}{\|\mathbf{a} + \mathbf{b}\|} \tag{37}$$

From Eq.21 and Eq.37 (a) is correct

C Code

```c
#include <stdio.h>
#include <math.h>

// Dot product of two 2D vectors
double dot(double a[], double b[]) {
    return a[0]*b[0] + a[1]*b[1];
}

// Magnitude of a 2D vector
double magnitude(double a[]) {
    return sqrt(dot(a, a));
}

// Compute max length M and unit vector u using matrix method
void compute(double a[], double b[], double *M, double u[]) {
    double c = dot(a, b); // a  b
    *M = sqrt(1 + c); // largest eigenvalue's sqrt

```

C Code

```c


    // Direction = a + b
    double temp[2] = {a[0] + b[0], a[1] + b[1]};
    double norm = magnitude(temp);
    u[0] = temp[0] / norm;
    u[1] = temp[1] / norm;
}
```

Python Code

```python
import ctypes
import numpy as np
import matplotlib.pyplot as plt

# Load shared library
lib = ctypes.CDLL(./vec.so) # use vec.dll on Windows

# Define argument & return types
lib.compute.argtypes = [
    np.ctypeslib.ndpointer(dtype=np.double, ndim=1, flags=C_CONTIGUOUS),
    np.ctypeslib.ndpointer(dtype=np.double, ndim=1, flags=C_CONTIGUOUS),
    ctypes.POINTER(ctypes.c_double),
    np.ctypeslib.ndpointer(dtype=np.double, ndim=1, flags=C_CONTIGUOUS)
]
```

Python Code

```python
# Example vectors
a = np.array([1.0, 0.0], dtype=np.double)
b = np.array([0.6, 0.8], dtype=np.double)

M = ctypes.c_double()
u = np.zeros(2, dtype=np.double)

# Call C function
lib.compute(a, b, ctypes.byref(M), u)

print(From C library:)
print(M =, M.value)
print(u =, u)

# Plot in same style as attachment
O = np.array([0.0, 0.0])
P = u * M.value
```

Python Code

```
plt.plot([O[0], P[0]], [O[1], P[1]], 'b-', label=Vector OP)
plt.scatter(*O, color=red, s=100, label=O(0,0))
plt.scatter(*P, color=green, s=100, label=fP({P[0]:.2f},{P[1]:.2f}))
plt.scatter(u, color=purple, marker=, s=200, label=fu({u[0]:.2f},{u[1]:.2f}))
plt.axhline(0, color='black')
plt.axvline(0, color='black')
plt.legend()
plt.title(Figure)
plt.savefig(/media/indhiresh-s/New Volume/Matrix/ee1030-2025/ee25btech11027/MATGEO/2.10.56/figs/figure1.png)
plt.show()
```

Plot

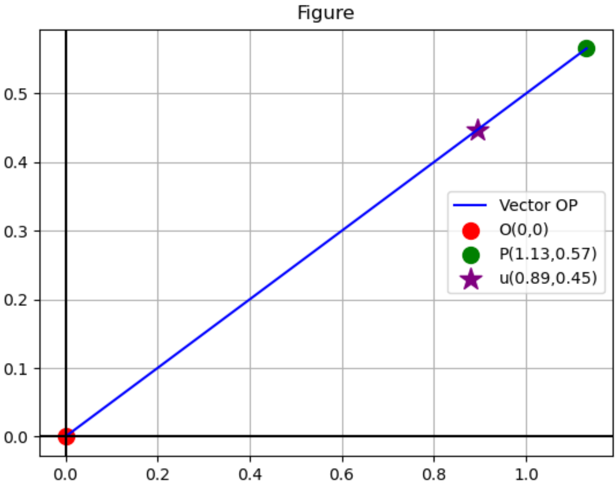
Figure
0.5
0.4
0.3
0.2
0.1
0.0
0.0
0.2
0.4
0.6
0.8
1.0
Vector OP
O(0,0)
P(1.13,0.57)
u(0.89,0.45)