5.13.34

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Question

For 3×3 matrices $\mathbf{M}$ and $\mathbf{N}$, which of the following statement(s) is (are) **NOT** correct?

(a) $\mathbf{N}^T\mathbf{M}\mathbf{N}$ is symmetric or skew symmetric, according as $\mathbf{M}$ is symmetric or skew symmetric.

(b) $\mathbf{M}\mathbf{N} - \mathbf{N}\mathbf{M}$ is skew symmetric for all matrices $\mathbf{M}$ and $\mathbf{N}$.

(c) $\mathbf{M}\mathbf{N}$ is symmetric for all symmetric matrices $\mathbf{M}$ and $\mathbf{N}$.

(d) $(\operatorname{adj}\mathbf{M})(\operatorname{adj}\mathbf{N}) = \operatorname{adj}(\mathbf{M}\mathbf{N})$ for all invertible matrices $\mathbf{M}$ and $\mathbf{N}$.

Solution

(a)

$$(\mathbf{N}^T\mathbf{M}\mathbf{N})^T = \mathbf{N}^T\mathbf{M}^T\mathbf{N} \tag{1}$$

If $\mathbf{M}^T = \mathbf{M}$,

$$(\mathbf{N}^T\mathbf{M}\mathbf{N})^T = \mathbf{N}^T\mathbf{M}\mathbf{N} \tag{2}$$

Hence symmetric.

If $\mathbf{M}^T = -\mathbf{M}$,

$$(\mathbf{N}^T\mathbf{M}\mathbf{N})^T = -\mathbf{N}^T\mathbf{M}\mathbf{N} \tag{3}$$

Hence skew symmetric.

Thus, (a) is **correct**.

(b)

Assume $\mathbf{MN} - \mathbf{NM}$ is skew symmetric $\forall\, \mathbf{M}, \mathbf{N}$.

Then,

$$(\mathbf{MN} - \mathbf{NM})^T = -(\mathbf{MN} - \mathbf{NM}) \tag{4}$$

$$\Rightarrow \mathbf{N}^T\mathbf{M}^T - \mathbf{M}^T\mathbf{N}^T = -\mathbf{MN} + \mathbf{NM} \tag{5}$$

Let $\mathbf{M}$ and $\mathbf{N}$ be symmetric:

$$\mathbf{M}^T = \mathbf{M}, \quad \mathbf{N}^T = \mathbf{N} \tag{6}$$

Then,

$$\mathbf{NM} - \mathbf{MN} = -\mathbf{MN} + \mathbf{NM} \tag{7}$$

$$\Rightarrow 2(\mathbf{NM} - \mathbf{MN}) = 0 \tag{8}$$

$$\Rightarrow \mathbf{MN} = \mathbf{NM} \tag{9}$$

Thus, this requires all symmetric matrices to commute, which is not true in general. Hence, (b) is **not correct**.

(c)

$$(\mathbf{MN})^T = \mathbf{N}^T\mathbf{M}^T \tag{10}$$

If $\mathbf{M}$ and $\mathbf{N}$ are symmetric,

$$(\mathbf{MN})^T = \mathbf{NM} \tag{11}$$

For $\mathbf{MN}$ to be symmetric,

$$\mathbf{MN} = \mathbf{NM} \tag{12}$$

Thus, $\mathbf{MN}$ is symmetric $\iff$ $\mathbf{M}$ and $\mathbf{N}$ commute. Since all symmetric matrices do not commute, (c) is **not correct**.

(d)

For any invertible matrix **A**,

$$\operatorname{adj}(\mathbf{A}) = \left|\mathbf{A}\right| \mathbf{A}^{-1} \tag{13}$$

Hence,

$$(\operatorname{adj} \mathbf{M})(\operatorname{adj} \mathbf{N}) = \left|\mathbf{M}\right| \left|\mathbf{N}\right| \mathbf{M}^{-1}\mathbf{N}^{-1}, \tag{14}$$

$$\operatorname{adj}(\mathbf{MN}) = \left|\mathbf{MN}\right| (\mathbf{MN})^{-1} = \left|\mathbf{M}\right| \left|\mathbf{N}\right| \mathbf{N}^{-1}\mathbf{M}^{-1}. \tag{15}$$

Equality holds only if

$$\mathbf{M}^{-1}\mathbf{N}^{-1} = \mathbf{N}^{-1}\mathbf{M}^{-1} \iff \mathbf{MN} = \mathbf{NM}. \tag{16}$$

Since this does not hold for all invertible matrices, (d) is **not correct**.

$$\boxed{\text{Statements (b), (c), and (d) are NOT correct.}}$$