6.4.12

AI25BTECH11012 - GARIGE UNNATHI

Question:
Find the shortest distance between the lines

$$\mathbf{r} = \hat{i} + 2\hat{j} + \hat{k} + \lambda(\hat{i} - \hat{j} + \hat{k})$$

$$\mathbf{r} = 2\hat{i} - 1\hat{j} - 1\hat{k} + \mu(2\hat{i} - \hat{j} + 2\hat{k})$$

Solution:
The given lines can be written in vector form as

$$\mathbf{x_1} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \tag{0.1}$$

$$\mathbf{x_2} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \tag{0.2}$$

$$\mathbf{M} = \begin{pmatrix} 1 & 2 \\ -1 & -1 \\ 1 & 2 \end{pmatrix} \tag{0.3}$$

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} \tag{0.4}$$

$$\begin{pmatrix} \mathbf{M} & \mathbf{B} - \mathbf{A} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -1 & -3 \\ 1 & 2 & -2 \end{pmatrix} \tag{0.5}$$

$$R_3 = R_3 + R_2 \tag{0.6}$$

$$= \begin{pmatrix} 1 & 2 & 1 \\ -1 & -1 & -3 \\ 0 & 1 & -5 \end{pmatrix} \tag{0.7}$$

$$R_2 = R_2 + R_1 \tag{0.8}$$

$$= \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -3 \\ 0 & 1 & -5 \end{pmatrix} \tag{0.9}$$

$$R_3 = R_3 - R_2 \tag{0.10}$$

$$= \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & -2 \end{pmatrix} \tag{0.11}$$

The rank of the matrix is 3 . So the given lines are skew

$$\begin{pmatrix}1 & -1 & 1\\2 & -1 & 2\end{pmatrix}\begin{pmatrix}1 & 2\\-1 & -1\\1 & 2\end{pmatrix}\kappa = \begin{pmatrix}1 & -1 & 1\\2 & -1 & 2\end{pmatrix}\begin{pmatrix}1\\-3\\-2\end{pmatrix} \tag{0.12}$$

$$\begin{pmatrix}3 & 5\\5 & 9\end{pmatrix}\kappa = \begin{pmatrix}2\\1\end{pmatrix} \tag{0.13}$$

The argumented matrix of the above matrix is

$$\begin{pmatrix}3 & 5 & 2\\5 & 9 & 1\end{pmatrix} \tag{0.14}$$

$$R_2 = R_2 - \frac{5}{3}R_1 \tag{0.15}$$

$$\begin{pmatrix}3 & 5 & 2\\0 & \frac{2}{3} & -\frac{7}{3}\end{pmatrix} \tag{0.16}$$

$$R_1 = R_1 - \frac{15}{2}R_2 \tag{0.17}$$

$$\begin{pmatrix}3 & 0 & \frac{39}{2}\\0 & \frac{2}{3} & -\frac{7}{3}\end{pmatrix} \tag{0.18}$$

yeilding

$$\begin{pmatrix}\lambda\\-\mu\end{pmatrix} = \begin{pmatrix}\frac{13}{2}\\-\frac{7}{2}\end{pmatrix} \tag{0.19}$$

$$\mathbf{x_1} = \frac{1}{2}\begin{pmatrix}15\\-9\\15\end{pmatrix}, \mathbf{x_2} = \frac{1}{2}\begin{pmatrix}-10\\-1\\-4\end{pmatrix} \tag{0.20}$$

The minimum distance between the lines is given by

$$\|\mathbf{x_2} - \mathbf{x_1}\| = \|\frac{1}{2}\begin{pmatrix}25\\-8\\19\end{pmatrix}\| \tag{0.21}$$

$$= \frac{5\sqrt{42}}{2} \tag{0.22}$$

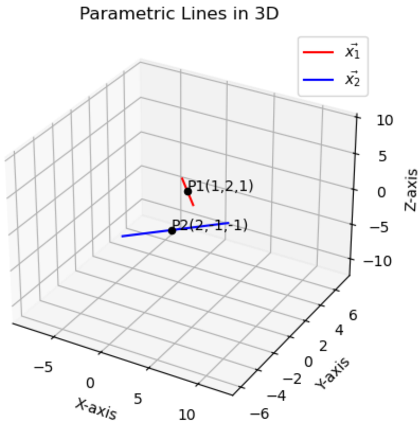
Parametric Lines in 3D
$\vec{x_1}$
$\vec{x_2}$
P1(1,2,1)
P2(2,-1,-1)
X-axis
Y-axis
Z-axis
−5
0
5
10
−6
−4
−2
0
2
4
6
−10
−5
0
5
10

Fig. 0.1