Box Product

EE25BTECH11008 - Anirudh M Abhilash

October 5, 2025

Problem Statement

If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three non-coplanar vectors, then find the value of

$$(\mathbf{a} + \mathbf{b} + \mathbf{c}) \cdot [(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} + \mathbf{c})].$$

Solution

We know that the scalar triple product is defined as

$$[\mathbf{p}\ \mathbf{q}\ \mathbf{r}] = \mathbf{p} \cdot (\mathbf{q} \times \mathbf{r}).$$

Expanding the cross product,

$$\begin{align}
(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} + \mathbf{c}) &= \mathbf{a} \times \mathbf{a} + \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{a} + \mathbf{b} \times \mathbf{c} \tag{1} \\
&= 0 + \mathbf{a} \times \mathbf{c} - \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c}. \tag{2}
\end{align}$$

Solution (cont..)

Hence,

$$\begin{aligned}
(\mathbf{a}+\mathbf{b}+\mathbf{c}) \cdot [(\mathbf{a}+\mathbf{b}) \times (\mathbf{a}+\mathbf{c})] \\
&= (\mathbf{a}+\mathbf{b}+\mathbf{c}) \cdot \left(-\mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c} + \mathbf{b}\times\mathbf{c}\right) \\
&= -[\mathbf{a}\ \mathbf{a}\ \mathbf{b}] - [\mathbf{b}\ \mathbf{a}\ \mathbf{b}] - [\mathbf{c}\ \mathbf{a}\ \mathbf{b}] \\
&\quad + [\mathbf{a}\ \mathbf{a}\ \mathbf{c}] + [\mathbf{b}\ \mathbf{a}\ \mathbf{c}] + [\mathbf{c}\ \mathbf{a}\ \mathbf{c}] \\
&\quad + [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] + [\mathbf{b}\ \mathbf{b}\ \mathbf{c}] + [\mathbf{c}\ \mathbf{b}\ \mathbf{c}] \qquad (3)
\end{aligned}$$

Expanded using linearity of the scalar triple product.

Solution (cont..)

All terms containing repeated vectors vanish, so we have

$$= -[\mathbf{c}\ \mathbf{a}\ \mathbf{b}] + [\mathbf{b}\ \mathbf{a}\ \mathbf{c}] + [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]. \tag{4}$$

Now, using the properties of the scalar triple product:

$$[\mathbf{c}\ \mathbf{a}\ \mathbf{b}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}], \quad [\mathbf{b}\ \mathbf{a}\ \mathbf{c}] = -[\mathbf{a}\ \mathbf{b}\ \mathbf{c}].$$

Solution (cont..)

Hence,

$$-[\mathbf{c}\ \mathbf{a}\ \mathbf{b}] + [\mathbf{b}\ \mathbf{a}\ \mathbf{c}] + [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = -[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] - [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] + [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \quad (5)$$

$$= -[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]. \quad (6)$$

$$\boxed{(\mathbf{a} + \mathbf{b} + \mathbf{c}) \cdot [(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} + \mathbf{c})] = -[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$$

C Code (Computations)

https://github.com/Anirudh-EE25BTECH11008/ee1030-2025/tree/main/EE25BTECH11008/MATGEO/2.10.37/Codes/verify.c

Python Code (Verifying)

https://github.com/Anirudh-EE25BTECH11008/ee1030-2025/tree/main/EE25BTECH11008/MATGEO/2.10.37/Codes/verify.py