10.7.7

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Question
The slope of the line touching both the parabolas $y^2 = 4x$ and $x^2 = -32y$ is
Solution:
The equation of a parabola in Matrix form is

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$

For y^2=4x

$$\mathbf{V_1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2}$$

$$\mathbf{u_1} = -2\mathbf{e_1} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{3}$$

$$f_1 = 0 \tag{4}$$

$$\implies \mathbf{x}^\top \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} -2 \\ 0 \end{pmatrix} \mathbf{x} = 0 \tag{5}$$

For x^2=-32y

$$\mathbf{V_2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{6}$$

$$\mathbf{u_2} = 16\mathbf{e_2} = \begin{pmatrix} 0 \\ 16 \end{pmatrix} \tag{7}$$

$$f_2 = 0 \tag{8}$$

$$\implies \mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} 0 \\ 16 \end{pmatrix} \mathbf{x} = 0 \tag{9}$$

a line **x=h** +k**m** touches (1) if

$$\mathbf{m}^\top (\mathbf{V}\mathbf{q} + \mathbf{u}) = 0 \, (\text{where } \mathbf{q} \text{ is the point of contact}) \tag{10}$$

$$\mathbf{m}^\top (\mathbf{V_1}\mathbf{q_1} + \mathbf{u_1}) = 0 \tag{11}$$

$$\mathbf{m}^\top (\mathbf{V_2}\mathbf{q_2} + \mathbf{u_2}) = 0 \tag{12}$$

let

$$\mathbf{q_2} - \mathbf{q_1} = c\mathbf{m} \, (\text{for some scalar } c) \tag{13}$$

substitute (13) in (12)

$$\implies \mathbf{m}^\top (\mathbf{V_2}(\mathbf{q_1} + c\mathbf{m}) + \mathbf{u_2}) = 0 \tag{14}$$

$$\implies \mathbf{m}^\top \mathbf{V_2}\mathbf{q_1} + \mathbf{m}^\top \mathbf{V_2} c\mathbf{m} + \mathbf{m}^\top \mathbf{u_2} = 0 \tag{15}$$

$$\implies \begin{pmatrix} 1 & m \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{q_1} + \begin{pmatrix} 1 & m \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c \\ cm \end{pmatrix} + \begin{pmatrix} 1 & m \end{pmatrix} \begin{pmatrix} 0 \\ 16 \end{pmatrix} = 0 \tag{16}$$

$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{q_1} + \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} c \\ cm \end{pmatrix} + 16m = 0 \tag{17}$$

$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{q_1} = -c - 16m \tag{18}$$

on expanding (11)

$$\implies \mathbf{m}^\top \mathbf{V_1}\mathbf{q_1} + \mathbf{m}^\top \mathbf{u_1} = 0 \tag{19}$$

$$\implies \begin{pmatrix} 1 & m \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{q_1} + \begin{pmatrix} 1 & m \end{pmatrix} \begin{pmatrix} -2 \\ 0 \end{pmatrix} = 0 \tag{20}$$

$$\implies \begin{pmatrix} 0 & m \end{pmatrix} \mathbf{q_1} = 2 \tag{21}$$

Equations (18) and (21) can be written as

$$\begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix} \mathbf{q_1} = \begin{pmatrix} -c - 16m \\ 2 \end{pmatrix} \tag{22}$$

The augmented matrix can be written as

$$\implies \left(\begin{array}{cc|c} 1 & 0 & -c-16m \\ 0 & m & 2 \end{array}\right) \tag{23}$$

$$\implies \mathbf{q_1} = \begin{pmatrix} -c-16m \\ \frac{2}{m} \end{pmatrix} \tag{24}$$

From (13)

$$\mathbf{q_2} = \mathbf{q_1} + c\mathbf{m} \tag{25}$$

$$\implies \mathbf{q_2} = \begin{pmatrix} -16m \\ \frac{2}{m} + cm \end{pmatrix} \tag{26}$$

substitute $\mathbf{q_1}$ in (5)

$$\implies \frac{1}{m^2} + 16m = -c \tag{27}$$

substitute $\mathbf{q_2}$ in (9)

$$\implies 8m^2 + \frac{2}{m} = -cm \tag{28}$$

on solving (27) and (28) we get

$$m = \frac{1}{2} \tag{29}$$

$$\implies \mathbf{m} = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} \tag{30}$$

$\implies$ slope of the line touching both the parabolas $=\frac{1}{2}$

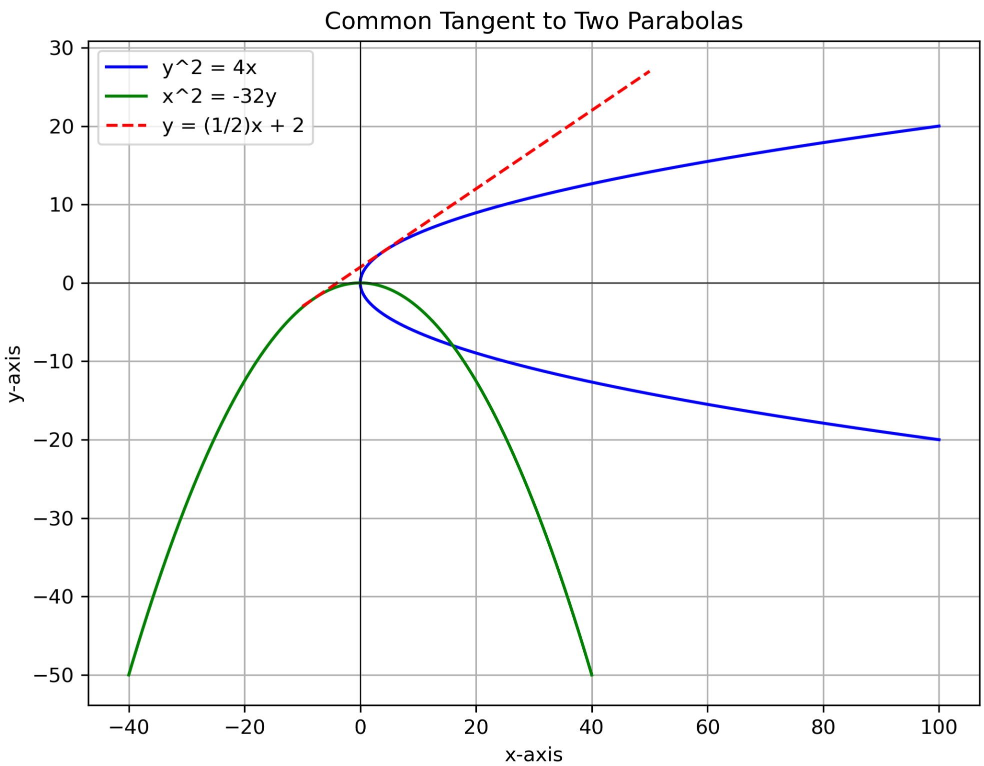

Fig. 0