12.101

AI25BTECH11023 - Pratik R

Question

$\mathbf{A}$ is 2×2 with $\text{tr}(\mathbf{A}) = 5$, $\det(\mathbf{A}) = 6$. Let the characteristic polynomial of $(\mathbf{A} + \mathbf{I_2})^{-1}$ be $x^2 - bx + c$. Find $b/c =$ (integer).

Solution

Given:

$$\text{tr}(\mathbf{A}) = 5 \tag{0.1}$$
$$\det(\mathbf{A}) = 6 \tag{0.2}$$

Let the eigenvalues of $\mathbf{A}$ be λ_1 and λ_2:

$$\lambda_1 + \lambda_2 = 5 \tag{0.3}$$
$$\lambda_1 \lambda_2 = 6 \tag{0.4}$$

Eigenvalues of $\mathbf{A} + \mathbf{I_2}$ are:

$$\lambda_1 + 1, \quad \lambda_2 + 1 \tag{0.5}$$

Eigenvalues of $(\mathbf{A} + \mathbf{I_2})^{-1}$ are:

$$\frac{1}{\lambda_1 + 1}, \quad \frac{1}{\lambda_2 + 1} \tag{0.6}$$

Thus, its characteristic polynomial is:

$$x^2 - \left(\frac{1}{\lambda_1 + 1} + \frac{1}{\lambda_2 + 1} \right) x + \frac{1}{(\lambda_1 + 1)(\lambda_2 + 1)} \tag{0.7}$$

Calculate:

$$(\lambda_1 + 1)(\lambda_2 + 1) = \lambda_1 \lambda_2 + (\lambda_1 + \lambda_2) + 1 = 6 + 5 + 1 = 12 \tag{0.8}$$
$$\frac{1}{\lambda_1 + 1} + \frac{1}{\lambda_2 + 1} = \frac{(\lambda_1 + 1) + (\lambda_2 + 1)}{(\lambda_1 + 1)(\lambda_2 + 1)} = \frac{5 + 2}{12} = \frac{7}{12} \tag{0.9}$$

Therefore:

$$b = \frac{7}{12}, \quad c = \frac{1}{12} \tag{0.10}$$
$$\frac{b}{c} = \frac{\frac{7}{12}}{\frac{1}{12}} = 7 \tag{0.11}$$

Hence, the answer is 7.