2.8.19

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Problem Statement

If

$$\mathbf{r} \cdot \mathbf{a} = 0, \quad \mathbf{r} \cdot \mathbf{b} = 0, \quad \mathbf{r} \cdot \mathbf{c} = 0$$

for some non-zero vector $\mathbf{r}$, find

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

Solution: Step 1 – Orthogonality

From

$$\mathbf{r} \cdot \mathbf{a} = 0, \tag{1}$$
$$\mathbf{r} \cdot \mathbf{b} = 0, \tag{2}$$
$$\mathbf{r} \cdot \mathbf{c} = 0, \tag{3}$$

we see that $\mathbf{a}, \mathbf{b}, \mathbf{c}$ lie in the subspace orthogonal to $\mathbf{r}$:

$$\mathbf{a}, \mathbf{b}, \mathbf{c} \in \text{span}\{\mathbf{r}\}^{\perp}. \tag{4}$$

Step 2 – Dimension Argument

Since $\mathbf{r} \neq \mathbf{0}$, the orthogonal subspace has dimension at most 2:

$$\dim(\text{span}\{\mathbf{r}\}^{\perp}) = 2. \tag{5}$$

Thus, any three vectors in this atmost 2D subspace are linearly dependent:

$$\mathbf{c} = \lambda_1 \mathbf{a} + \lambda_2 \mathbf{b}, \quad \text{for some scalars } \lambda_1, \lambda_2. \tag{6}$$

Step 3 – Scalar Triple Product

The scalar triple product of linearly dependent vectors is zero:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \tag{7}$$

$$= 0. \tag{8}$$

Final Answer

$$\boxed{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0}$$

Reasoning: The vectors lie in a plane perpendicular to **r** and are therefore linearly dependent, making the scalar triple product vanish.

C Code

```c
#include <stdio.h>

double scalar_triple(double a[3], double b[3], double c[3]) {
    return a[0]*(b[1]*c[2] - b[2]*c[1])
         - a[1]*(b[0]*c[2] - b[2]*c[0])
         + a[2]*(b[0]*c[1] - b[1]*c[0]);
}
```

Python code through shared output

```
import ctypes
import matplotlib.pyplot as plt
import numpy as np
from mpl_toolkits.mplot3d import Axes3D
# Load the shared library
lib = ctypes.CDLL(./libstp.so)
lib.scalar_triple.restype = ctypes.c_double
# Define vectors
a = (ctypes.c_double * 3)(1, 0, 0)
b = (ctypes.c_double * 3)(0, 1, 0)
c = (ctypes.c_double * 3)(1, 1, 0)
# Call the C function
result = lib.scalar_triple(a, b, c)
print(Scalar triple product =, result)
# Convert to numpy arrays for plotting
a_vec = np.array([a[0], a[1], a[2]])
b_vec = np.array([b[0], b[1], b[2]])
c_vec = np.array([c[0], c[1], c[2]])
```

Python code through shared output

```
# Plot vectors in 3D
fig = plt.figure()
ax = fig.add_subplot(111, projection='3d')
origin = np.array([0, 0, 0])
ax.quiver(*origin, *a_vec, color='r', label='a')
ax.quiver(*origin, *b_vec, color='g', label='b')
ax.quiver(*origin, *c_vec, color='b', label='c')
ax.set_xlim([0, 1.5])
ax.set_ylim([0, 1.5])
ax.set_zlim([0, 1.5])
ax.set_xlabel(X)
ax.set_ylabel(Y)
ax.set_zlabel(Z)
ax.set_title(fScalar Triple Product = {result})
ax.legend()
plt.show()
```

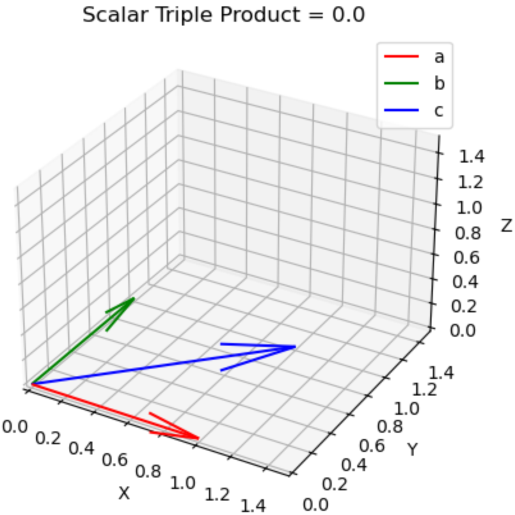
Only Python code

```
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D
# Define 3 coplanar vectors (lying in xy-plane)
a = np.array([1, 1, 0])
b = np.array([1, 0, 0])
c = np.array([0, 1, 0])
# Cross product b x c
b_cross_c = np.cross(b, c)
# Scalar triple product a . (b x c)
scalar_triple = np.dot(a, b_cross_c)
# Print the scalar triple product (should be 0)
print(fScalar triple product: {scalar_triple:.2f})
# Plotting
fig = plt.figure()
ax = fig.add_subplot(111, projection='3d')
```

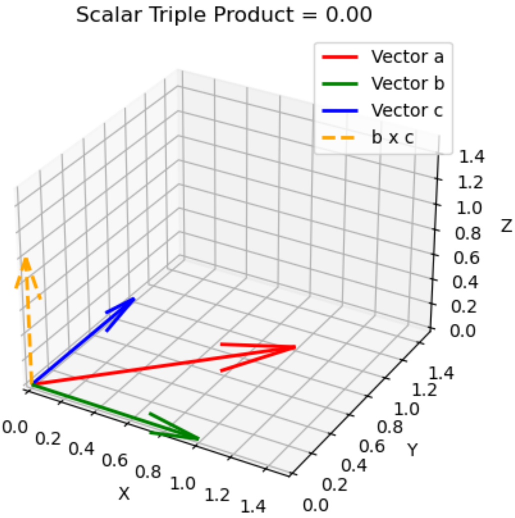
Only Python code

```
# Plot origin
origin = np.array([0, 0, 0])
# Plot vectors
ax.quiver(*origin, *a, color='r', label='Vector a', linewidth=2)
ax.quiver(*origin, *b, color='g', label='Vector b', linewidth=2)
ax.quiver(*origin, *c, color='b', label='Vector c', linewidth=2)
# Plot b x c
ax.quiver(*origin, *b_cross_c, color='orange', linestyle='dashed'
    , label='b x c', linewidth=2)
ax.set_xlim([0, 1.5])
ax.set_ylim([0, 1.5])
ax.set_zlim([0, 1.5])
ax.set_xlabel('X')
ax.set_ylabel('Y')
ax.set_zlabel('Z')
ax.set_title(f'Scalar Triple Product = {scalar_triple:.2f}')
ax.legend()
ax.grid(True)
plt.show()
```

PLOTS


Scalar Triple Product = 0.0
a
b
c
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
Z
0.0 0.2 0.4 0.6 0.8 1.0 1.2 1.4
X
0.0 0.2 0.4 0.6 0.8 1.0 1.2 1.4
Y

PLOTS



Scalar Triple Product = 0.00
Vector a
Vector b
Vector c
b x c
X
Y
Z
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4