

Matrices in Geometry 4.13.42

EE25BTECH11037 - Divyansh

Question: Let $0 < \alpha < \frac{\pi}{2}$ be a fixed angle. If $\mathbf{P} = (\cos \theta, \sin \theta)$ and $\mathbf{Q} = (\cos(\alpha - \theta), \sin(\alpha - \theta))$, then \mathbf{Q} can be obtained from \mathbf{P} by

- (a) clockwise rotation around the origin through an angle α
- (b) anticlockwise rotation around the origin through an angle α
- (c) reflection in the line through origin with slope $\tan \alpha$
- (d) reflection in the line through origin with slope $\tan\left(\frac{\alpha}{2}\right)$

Solution:

We know that $\mathbf{Q} = \begin{pmatrix} \cos(\alpha - \theta) \\ \sin(\alpha - \theta) \end{pmatrix}$ and $\mathbf{P} = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$

We also know that the rotation matrix $\mathbf{R} = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}$, where α is anticlockwise.

We can obtain \mathbf{Q} by

$$\mathbf{Q} = \mathbf{R}\mathbf{P} \implies \mathbf{Q} = \begin{pmatrix} \cos \theta \cos \alpha - \sin \theta \sin \alpha \\ \sin \theta \cos \alpha + \cos \theta \sin \alpha \end{pmatrix} \quad (1)$$

We know from trigonometric identities that

$$\cos(\alpha - \theta) = \cos \theta \cos \alpha + \sin \theta \sin \alpha \quad (2)$$

$$\sin(\alpha - \theta) = \cos \theta \sin \alpha - \sin \theta \cos \alpha \quad (3)$$

If we take α clockwise, that is, exchange it with $-\alpha$, we will get the rotation matrix as

$$\mathbf{R} = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \quad (4)$$

Thus, the required rotation matrix is \mathbf{R} using which \mathbf{Q} can be obtained from \mathbf{P} by clockwise rotation around the origin through an angle α , therefore the correct option is (a)

Let us plot a graph for $\theta = 45^\circ$ and $\alpha = 30^\circ$

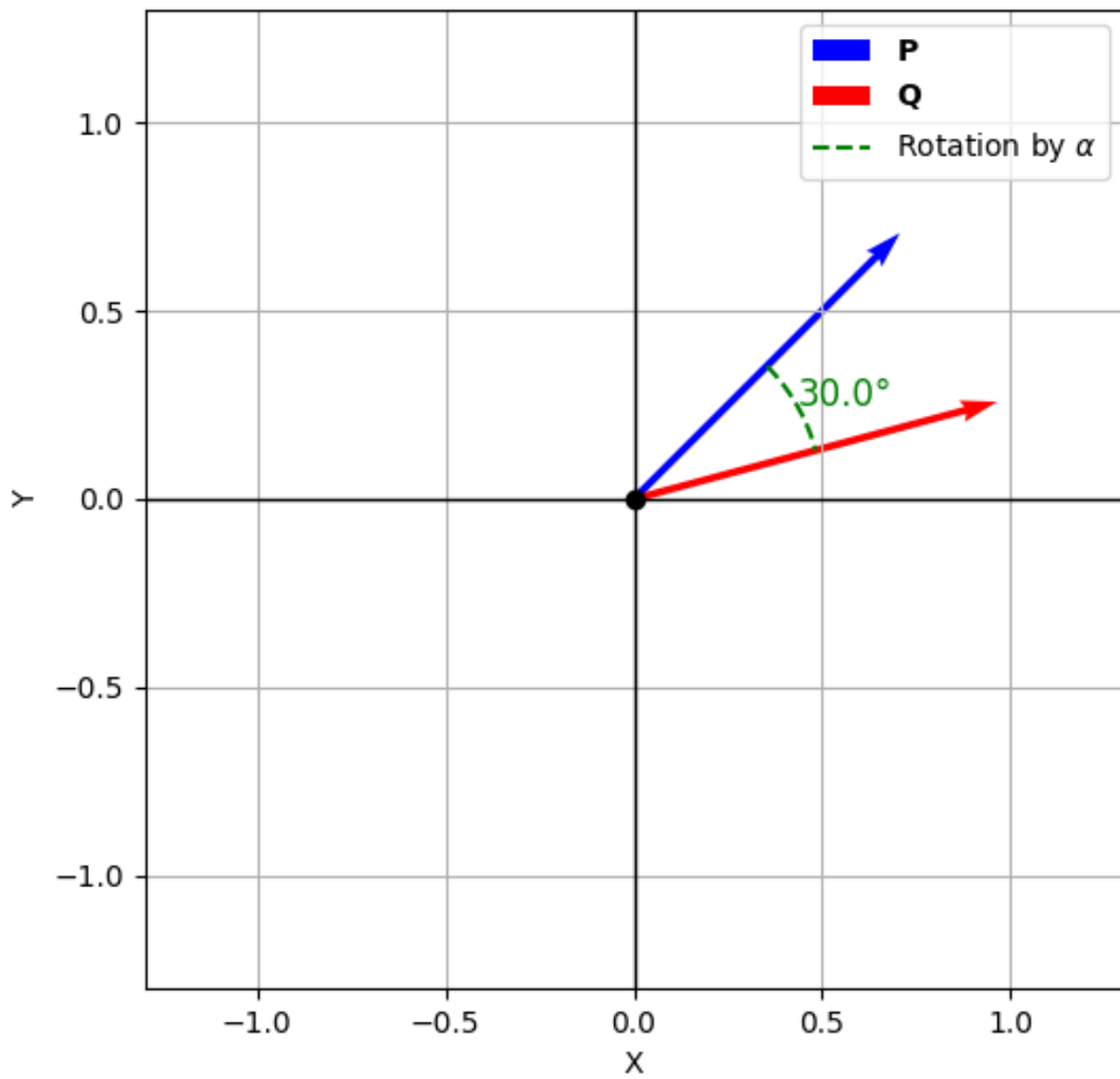


Fig. 1: Graph for 4.13.42, where $\theta = 45^\circ$ and $\alpha = 30^\circ$