4.7.62

AI25BTECH11001 - ABHISEK MOHAPATRA

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Question: If $\begin{vmatrix} a & a^2 & 1+a3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$ and the vectors $\mathbf{A} = \begin{pmatrix} 1 \\ a \\ a^2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 \\ b \\ b^2 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 1 \\ c \\ c^2 \end{pmatrix}$ are co-planar, then the product abc = _____.

Solution: Let equation of the plane be $\mathbf{n}^\top \mathbf{x} = 0$.

so,

$$\mathbf{n}^\top \mathbf{A} = 0, \mathbf{n}^\top \mathbf{B} = 0, \mathbf{n}^\top \mathbf{C} = 0 \tag{0.1}$$

so ,

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}^\top \mathbf{n} = 0, \tag{0.2}$$

$$\begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix} \mathbf{n} = 0, \tag{0.3}$$

$$\begin{pmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{pmatrix} \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - R_1} \begin{pmatrix}1 & a & a^2\\ 0 & b-a & b^2-a^2\\ 0 & c-a & c^2-a^2\end{pmatrix} \tag{0.4}$$

$$\xrightarrow{R_3 \leftarrow R_3 - \frac{c-a}{b-a}R_2} \begin{pmatrix}1 & a & a^2\\ 0 & b-a & b^2-a^2\\ 0 & 0 & (c-a)(c-b)\end{pmatrix} \tag{0.5}$$

Product of the eigen values is $(b-a)(c-a)(c-b)$. And,for the no non-trivial solution of $\mathbf{n}$ to exist,

$$(b-a)(c-a)(c-b) \neq 0 \tag{0.6}$$

Reducing the given determinant to ref,

$$\begin{vmatrix}a & a^2 & 1+a^3\\ b & b^2 & 1+b^3\\ c & c^2 & 1+c^3\end{vmatrix} \xrightarrow[R_3 \leftarrow R_3 - \frac{c}{a}R_1]{R_2 \leftarrow R_2 - \frac{b}{a}R_1} \begin{vmatrix}a & a^2 & 1+a^3\\ 0 & b^2-ab & 1-\frac{b}{a}+b^3-a^2b\\ 0 & c^2-ca & 1-\frac{c}{a}+c^3-a^2c\end{vmatrix} \tag{0.7}$$

$$\xrightarrow{R_3 \leftarrow R_3 - \frac{c^2-ca}{b^2-ba}R_2} \begin{vmatrix}a & a^2 & 1+a^3\\ 0 & b^2-ab & 1-\frac{b}{a}+b^3-a^2b\\ 0 & 0 & (1-\frac{c}{a})(1-\frac{c}{b})+c(c-a)(c-b)\end{vmatrix} \tag{0.8}$$

Product of the eigen values

$$(b-a)(a-c)(b-c)+abc(b-a)(c-a)(c-b)=0 \tag{0.9}$$

$$(1+abc)(b-a)(c-a)(c-b)=0 \tag{0.10}$$

From eq 7

$$1+abc=0 \Rightarrow abc=-1 \tag{0.11}$$