10.3.12

EE25BTECH11013 - Bhargav

Question:
If the line $y = \sqrt{3}x + K$ touches the parabola $x^2 = 16y$, then find the value of K.

Solution:

The equation of the conic (*parabola*) can be written as

$$\mathbf{x}^{\mathbf{T}}\mathbf{V}\mathbf{x} + 2\mathbf{u}^{\mathbf{T}}\mathbf{x} + f = 0 \tag{0.1}$$

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -8 \end{pmatrix}, f = 0, \mathbf{m}^{\mathbf{T}} = \begin{pmatrix} 1 & \sqrt{3} \end{pmatrix} \tag{0.2}$$

$$\mathbf{x} = \mathbf{h} + k_i\mathbf{m} \tag{0.3}$$

The value of k_i can be found out by solving the line and conic equation

$$(\mathbf{h} + k_i\mathbf{m})^\top \mathbf{V}(\mathbf{h} + k_i\mathbf{m}) + 2\mathbf{u}^\top(\mathbf{h} + k_i\mathbf{m}) + f = 0 \tag{0.4}$$

$$\implies k_i^2\mathbf{m}^\top\mathbf{V}\mathbf{m} + 2k_i\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u}) + \mathbf{h}^\top\mathbf{V}\mathbf{h} + 2\mathbf{u}^\top\mathbf{h} + f = 0 \tag{0.5}$$

$$\text{or, } k_i^2\mathbf{m}^\top\mathbf{V}\mathbf{m} + 2k_i\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u}) + g(\mathbf{h}) = 0 \tag{0.6}$$

Solving the above quadratic gives the equation

$$k_i = \frac{1}{\mathbf{m}^\top\mathbf{V}\mathbf{m}}\left(-\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u}) \pm \sqrt{[\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u})]^2 - g(\mathbf{h})(\mathbf{m}^\top\mathbf{V}\mathbf{m})}\right) \tag{0.7}$$

Since the tangent passes through one point of the conic, and $g(\mathbf{q}) = 0$

$$\mathbf{m}^{\mathbf{T}}(\mathbf{V}\mathbf{q} + \mathbf{u}) = 0 \tag{0.8}$$

$$\mathbf{m}^{\mathbf{T}}\mathbf{V}\mathbf{q} = -\mathbf{m}^{\mathbf{T}}\mathbf{u} \tag{0.9}$$

$$\mathbf{q} = -\frac{\left(\mathbf{m}^{\mathbf{T}}\mathbf{V}\right)^T \mathbf{m}^{\mathbf{T}}\mathbf{u}}{\left\|\mathbf{m}^{\mathbf{T}}\mathbf{V}\right\|^2} \tag{0.10}$$

On solving, we get

$$\mathbf{q} = \begin{pmatrix} 8\sqrt{3} \\ t \end{pmatrix}, t \in \mathbf{R} \tag{0.11}$$

Since $\mathbf{q}$ lies on the conic,

$$g(\mathbf{q}) = 0 \tag{0.12}$$

$$\implies \mathbf{q}^{\mathbf{T}}\mathbf{V}\mathbf{q} + 2\mathbf{u}^{\mathbf{T}}\mathbf{q} + f = 0 \tag{0.13}$$

Substituting and solving gives $t = 12$

$$\therefore \mathbf{q} = \begin{pmatrix} 8\sqrt{3} \\ 12 \end{pmatrix} \tag{0.14}$$

Therefore $k = -12$