

Assignment 9: 4.13.59

EE25BTECH11055 - Subhodeep Chakraborty

Question:

Determine all values of α for which the point (α, α^2) lies inside the triangle formed by the lines. (1992)

$$2x + 3y - 1 = 0$$

$$x + 2y - 3 = 0$$

$$5x - 6y - 1 = 0$$

Solution:

Given:

$$\mathbf{n}_1^\top \mathbf{x} = c_1 \qquad \mathbf{n}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} c_1 = 1 \qquad (1)$$

$$\mathbf{n}_2^\top \mathbf{x} = c_2 \qquad \mathbf{n}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} c_2 = 3 \qquad (2)$$

$$\mathbf{n}_3^\top \mathbf{x} = c_3 \qquad \mathbf{n}_3 = \begin{pmatrix} 5 \\ -6 \end{pmatrix} c_3 = 1 \qquad (3)$$

$$\mathbf{P} = \begin{pmatrix} \alpha \\ \alpha^2 \end{pmatrix} \qquad (4)$$

For finding vertices:

$$\begin{pmatrix} n_1 & n_2 \end{pmatrix}^\top \mathbf{V}_3 = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \qquad (5)$$

$$\begin{pmatrix} n_3 & n_1 \end{pmatrix}^\top \mathbf{V}_2 = \begin{pmatrix} c_3 \\ c_1 \end{pmatrix} \qquad (6)$$

$$\begin{pmatrix} n_2 & n_3 \end{pmatrix}^\top \mathbf{V}_1 = \begin{pmatrix} c_2 \\ c_3 \end{pmatrix} \qquad (7)$$

Let us define $d_i = \mathbf{n}_i^\top \mathbf{V}_i - c_i$ as the sign denoting which side of the line the vertex opposite to it lies on.

Also define matrix $\mathbf{D} = \text{diag}(d_1, d_2, d_3)$

For point to lie inside triangle, we need $d_i \cdot (\mathbf{n}_i^\top \mathbf{P} - c_i) > 0$. In matrix form, this is written as:

$$\mathbf{D} = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \qquad (8)$$

$$\mathbf{D} \begin{pmatrix} \mathbf{n}_1^\top \mathbf{P} - c_1 \\ \mathbf{n}_2^\top \mathbf{P} - c_2 \\ \mathbf{n}_3^\top \mathbf{P} - c_3 \end{pmatrix} > \mathbf{0} \qquad (9)$$

Let

$$\mathbf{N} = \begin{pmatrix} n_1 & n_2 & n_3 \end{pmatrix}^\top \qquad (10)$$

$$\mathbf{C} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \qquad (11)$$

Thus representing everything in terms of matrices,

$$\mathbf{D}(\mathbf{NP} - \mathbf{C}) > \mathbf{0} \quad (12)$$

is the required inequality. On substituting values, we get

$$\alpha \in [-3/2, -1] \cup [1/3, 1] \quad (13)$$

