4.13.47

EE25BTECH11043 - Nishid Khandagre

Question: The ends $\mathbf{A}$, $\mathbf{B}$ of a straight line segment of constant length c slide upon the fixed rectangular axes OX, OY respectively. If the rectangle $OAPB$ be completed, then show that the locus of the foot of perpendicular drawn from $\mathbf{P}$ to $\mathbf{AB}$ is

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$$

Solution:
Given

$$\mathbf{A} = \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{0.1}$$

$$\mathbf{B} = \begin{pmatrix} 0 \\ b \end{pmatrix} \tag{0.2}$$

Since $OAPB$ is a rectangle, the opposite corner $\mathbf{P}$ is:

$$\mathbf{P} = \mathbf{A} + \mathbf{B} \tag{0.3}$$

$$= \begin{pmatrix} a \\ b \end{pmatrix} \tag{0.4}$$

$\mathbf{B} - \mathbf{A}$ has fixed length of c

$$\|\mathbf{B} - \mathbf{A}\|^2 = \left(\mathbf{B} - \mathbf{A}\right)^\top \left(\mathbf{B} - \mathbf{A}\right) \tag{0.5}$$

$$c^2 = a^2 + b^2 \tag{0.6}$$

Let $\mathbf{H}$ be the foot of the perpendicular from $\mathbf{P}$ to the line through $\mathbf{A}$ in the direction $\mathbf{B} - \mathbf{A}$.

$$\mathbf{H} = \mathbf{A} + \lambda\left(\mathbf{B} - \mathbf{A}\right) \tag{0.7}$$

$$\lambda = \frac{\left(\mathbf{P} - \mathbf{A}\right)^\top \left(\mathbf{B} - \mathbf{A}\right)}{\left(\mathbf{B} - \mathbf{A}\right)^\top \left(\mathbf{B} - \mathbf{A}\right)} \tag{0.8}$$

$$\mathbf{P} - \mathbf{A} = \left(\mathbf{A} + \mathbf{B}\right) - \mathbf{A} \tag{0.9}$$

$$= \mathbf{B} \tag{0.10}$$

So,

$$\lambda = \frac{\mathbf{B}^\top \left(\mathbf{B} - \mathbf{A}\right)}{\left(\mathbf{B} - \mathbf{A}\right)^\top \left(\mathbf{B} - \mathbf{A}\right)} \tag{0.11}$$

$$= \frac{\mathbf{B}^\top \mathbf{B} - \mathbf{B}^\top \mathbf{A}}{a^2 + b^2} \tag{0.12}$$

We know

$$\mathbf{B}^\top \mathbf{A} = 0 \tag{0.13}$$

$$\mathbf{B}^\top \mathbf{B} = b^2 \tag{0.14}$$

$$\lambda = \frac{b^2}{a^2 + b^2} \tag{0.15}$$

Now compute $\mathbf{H}$:

$$\mathbf{H} = \mathbf{A} + \frac{b^2}{a^2 + b^2} \left(\mathbf{B} - \mathbf{A}\right) \tag{0.16}$$

$$= \begin{pmatrix} a \\ 0 \end{pmatrix} + \frac{b^2}{a^2 + b^2} \begin{pmatrix} -a \\ b \end{pmatrix} \tag{0.17}$$

$$= \begin{pmatrix} a - \frac{ab^2}{a^2+b^2} \\ \frac{b^3}{a^2+b^2} \end{pmatrix} \tag{0.18}$$

$$= \begin{pmatrix} \frac{a(a^2+b^2) - ab^2}{a^2+b^2} \\ \frac{b^3}{a^2+b^2} \end{pmatrix} \tag{0.19}$$

$$= \begin{pmatrix} \frac{a^3}{a^2+b^2} \\ \frac{b^3}{a^2+b^2} \end{pmatrix} \tag{0.20}$$

Let $\mathbf{H} = \begin{pmatrix} x \\ y \end{pmatrix}$. Then,

$$x = \frac{a^3}{a^2 + b^2} \tag{0.21}$$

$$y = \frac{b^3}{a^2 + b^2} \tag{0.22}$$

Using the constraint $a^2 + b^2 = c^2$:

$$a^3 = x(a^2 + b^2) = xc^2 \tag{0.23}$$

$$b^3 = y(a^2 + b^2) = yc^2 \tag{0.24}$$

Thus,

$$a = (xc^2)^{1/3} = c^{2/3} x^{1/3} \tag{0.25}$$

$$b = (yc^2)^{1/3} = c^{2/3} y^{1/3} \tag{0.26}$$

Substitute these into $a^2 + b^2 = c^2$:

$$(c^{2/3} x^{1/3})^2 + (c^{2/3} y^{1/3})^2 = c^2 \tag{0.27}$$

$$c^{4/3} x^{2/3} + c^{4/3} y^{2/3} = c^2 \tag{0.28}$$

$$c^{4/3} (x^{2/3} + y^{2/3}) = c^2 \tag{0.29}$$

The locus is:

$$x^{2/3} + y^{2/3} = c^{2/3} \tag{0.30}$$

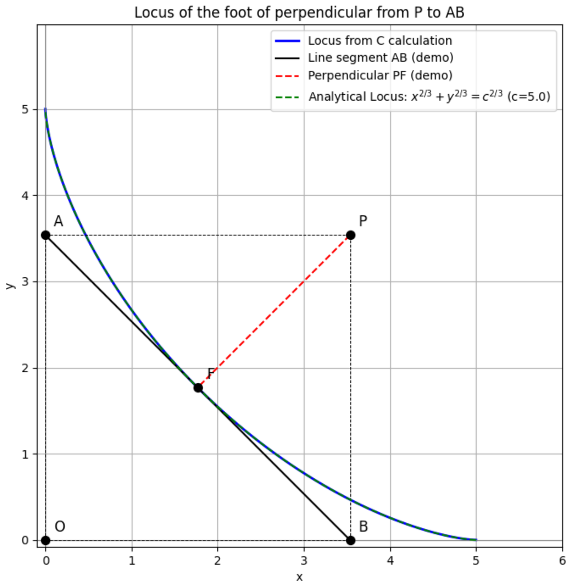

Fig. 0.1