12.349

EE25BTECH11023 - Venkata Sai

Question:
Let $T_1, T_2 : \mathbb{R}_5 \to \mathbb{R}_3$ be linear transformations such that rank$(T_1) = 3$ and nullity$(T_2) = 3$. Let $T_3 : \mathbb{R}_3 \to \mathbb{R}_3$ be a linear transformation such that $T_3 \circ T_1 = T_2$. Then rank(T_3) is . . . (MA 2014)

Solution:
According to Rank-Nullity theorem,
For a linear transformation $T : \mathbb{R}_m \to \mathbb{R}_n$

$$\text{rank}\,(T) + \text{nullity}\,(T) = \dim\,(\text{domain}) \tag{1}$$

where $\dim \mathbb{R}_m$ is the dimension of the domain i.e vector space $\mathbb{R}_m$

Given $T_2 : \mathbb{R}_5 \to \mathbb{R}_3$ and nullity(T_2)=3

$$\text{rank}\,(T_2) + \text{nullity}\,(T_2) = \dim \mathbb{R}_5 \tag{2}$$

$$\text{rank}\,(T_2) + 3 = 5 \tag{3}$$

$$\text{rank}\,(T_2) = 2 \tag{4}$$

Given $T_1 : \mathbb{R}_5 \to \mathbb{R}_3$ and rank(T_1)=3

$$\dim\,(\text{Co-domain}) = 3 \tag{5}$$

$$\text{rank}\,(T_1) = \dim\,(\text{Co-domain}) \tag{6}$$

It is onto and hence

$$\dim\,(\text{Im}\,(T_1)) = \dim\,(\text{Co-domain}) \tag{7}$$

$$\dim\,(\text{Im}\,(T_1)) = 3 \implies \text{Im}\,(T_1) = \mathbb{R}_3 \tag{8}$$

where $\text{Im}\,(T)$ is the Image space of the linear transformation T
Given $T_3 : \mathbb{R}_3 \to \mathbb{R}_3$

$$T_3 \circ T_1 = T_2 \tag{9}$$

$$(T_3 \circ T_1)\,(\mathbb{R}_5) = \text{Im}\,(T_2) \tag{10}$$

$$T_3\,(T_1\,(R_5)) = \text{Im}\,(T_2) \tag{11}$$

$$T_3\,(\text{Im}T_1) = \text{Im}\,(T_2) \tag{12}$$

$$T_3\,(\mathbb{R}_3) = \text{Im}\,(T_2) \tag{13}$$

$$\text{Im}\,(T_3) = \text{Im}\,(T_2) \tag{14}$$

$$\implies \text{rank}(T_3) = \text{rank}(T_2) \tag{15}$$

From (4)

$$\text{rank}(T_3) = 2 \tag{16}$$