5.8.19

EE25BTECH11021 - Dhanush sagar

Question:
If we add 1 to the numerator and subtract 1 from the denominator, a fraction reduces to 1. It becomes 1/2 if we only add 1 to the denominator. What is the fraction?

Solution:

Let the unknown fraction be represented as

$$\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \frac{x}{y}. \tag{1}$$

Affine transformations are written as a vector plus a translation.

case 1:add 1 to numerator, subtract 1 from denominator

$$T_1(\mathbf{u}) = \mathbf{u} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{2}$$

case 2:add 1 to denominator

$$T_2(\mathbf{u}) = \mathbf{u} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{3}$$

Condition for a fraction: For any vector $\begin{pmatrix} a \\ b \end{pmatrix}$, the requirement $\frac{a}{b} = k$ is equivalent to the linear equation

$$\mathbf{r}_k \begin{pmatrix} a \\ b \end{pmatrix} = 0, \qquad \mathbf{r}_k = \begin{pmatrix} 1 & -k \end{pmatrix}, \tag{4}$$

since $\mathbf{r}_k \begin{pmatrix} a \\ b \end{pmatrix} = a - kb = 0 \iff \frac{a}{b} = k$.

Case 1: $T_1(\mathbf{u})$ must yield fraction 1.use $\mathbf{r}_1 = \begin{pmatrix} 1 & -1 \end{pmatrix}$

$$\mathbf{r}_1(T_1(\mathbf{u})) = 0 \tag{5}$$

$$\implies \mathbf{r}_1\mathbf{u} + \mathbf{r}_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0 \tag{6}$$

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{u} + 2 = 0 \tag{7}$$

This gives the first equation.

Case 2: $T_2(\mathbf{u})$ must yield fraction $\frac{1}{2}$. To avoid fractions, multiply the functional by 2, i.e., use $\mathbf{r}_2 = \begin{pmatrix} 2 & -1 \end{pmatrix}$.

$$\mathbf{r}_2(T_2(\mathbf{u})) = 0 \tag{8}$$

$$\implies \mathbf{r}_2\mathbf{u} + \mathbf{r}_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \tag{9}$$

$$\begin{pmatrix} 2 & -1 \end{pmatrix}\mathbf{u} - 1 = 0 \tag{10}$$

This gives the second equation.

System of equations: Both conditions together form the system

$$\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}\mathbf{u} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}. \tag{11}$$

Gaussian elimination: Form the augmented matrix

$$\left[\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \middle| \begin{pmatrix} -2 \\ 1 \end{pmatrix}\right]. \tag{12}$$

Eliminate the entry below the pivot:

$$R_2 \leftarrow R_2 - 2R_1 \quad \Rightarrow \quad \left[\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \middle| \begin{pmatrix} -2 \\ 5 \end{pmatrix}\right]. \tag{13}$$

Now eliminate above the pivot:

$$R_1 \leftarrow R_1 + R_2 \quad \Rightarrow \quad \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \middle| \begin{pmatrix} 3 \\ 5 \end{pmatrix}\right]. \tag{14}$$

Final solution:

$$\mathbf{u} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}, \tag{15}$$

$$\frac{x}{y} = \frac{3}{5}. \tag{16}$$

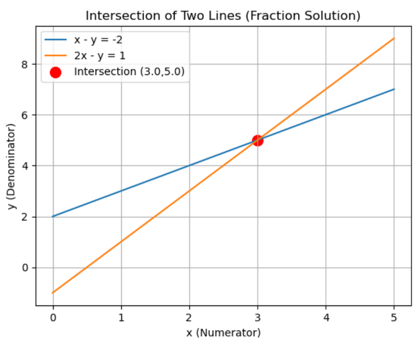

Fig. 0.1