10.7.4

EE25BTECH11060 - V.Namaswi

Question

Prove that $y = 2x + 2\sqrt{3}$ is commom tangent to the parabola $y^2 = 16\sqrt{3}x$ and the ellipse $2x^2 + y^2 = 4$

Solution

General Formulae of a conic

$$\mathbf{x}^T V\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{1}$$

The tangent condition for line $\mathbf{n}^T\mathbf{x} + c = 0$ at contact point $\mathbf{q}$ is

$$V\mathbf{q} + \mathbf{u} = \lambda\mathbf{n} \tag{2}$$

$$\mathbf{u}^T\mathbf{q} + f = \lambda c \tag{3}$$

for some scalar λ. For Parabola $y^2 = 16\sqrt{3}\,x$

$$V = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{4}$$

$$\mathbf{u} = \begin{pmatrix} -8\sqrt{3} \\ 0 \end{pmatrix} \tag{5}$$

$$f = 0. \tag{6}$$

Line: $2x - y + 2\sqrt{3} = 0$, so

$$\mathbf{n} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{7}$$

$$c = 2\sqrt{3} \tag{8}$$

First condition

$$V\mathbf{q} + \mathbf{u} = \begin{pmatrix} -8\sqrt{3} \\ y \end{pmatrix} = \lambda\begin{pmatrix} 2 \\ -1 \end{pmatrix}. \tag{9}$$

Thus $\lambda = -4\sqrt{3}$ and $y = 4\sqrt{3}$.

Second condition:

$$\mathbf{u}^T\mathbf{q} + f = -8\sqrt{3}\,x = \lambda c = (-4\sqrt{3})(2\sqrt{3}) = -24, \tag{10}$$

giving $x = \sqrt{3}$.

$$\mathbf{q} = \begin{pmatrix} \sqrt{3} \\ 4\sqrt{3} \end{pmatrix} \tag{11}$$

So the line touches the parabola at $(\sqrt{3}, 4\sqrt{3})$.

For Circle $2x^2 + y^2 = 4$

$$V = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \tag{12}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{13}$$

$$f = -4. \tag{14}$$

First condition:

$$V\mathbf{q} = \lambda \mathbf{n} \tag{15}$$

we get

$$\mathbf{q} = \lambda \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{16}$$

Second condition:

$$\mathbf{u}^T \mathbf{q} + f = -4 = \lambda c = \lambda(2\sqrt{3}) \tag{17}$$

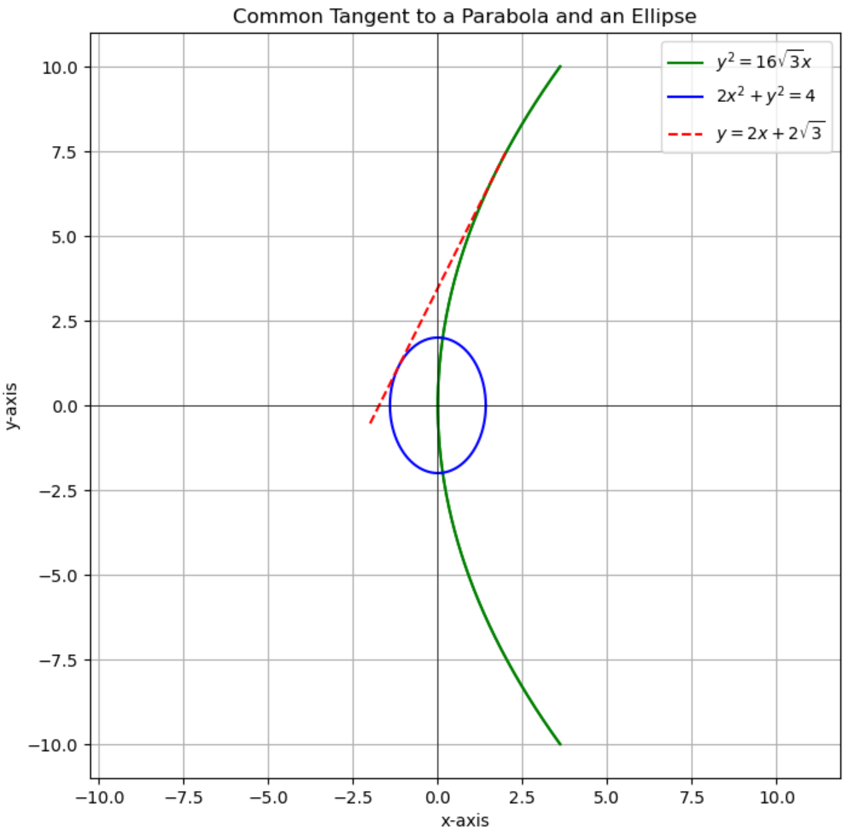
$$\implies \quad \lambda = -\frac{2}{\sqrt{3}}. \tag{18}$$

So

$$\mathbf{q} = -\frac{2}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{2\sqrt{3}}{3} \\ \frac{2\sqrt{3}}{3} \end{pmatrix}. \tag{19}$$

$$\mathbf{q} = \begin{pmatrix} -\frac{2\sqrt{3}}{3} \\ \frac{2\sqrt{3}}{3} \end{pmatrix} \tag{20}$$

So the line touches the circle at $\left(-\frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\right)$. Hence given line is common tangent to both the curves.



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