2.8.17

EE25BTECH11034 - Kishora Karthik

Question:
Show that the straight lines whose direction cosines (l, m, n) are given by the equations $2l + 2m - n = 0$ and $mn + nl + lm = 0$ are at right angles.
Solution:
Let the direction cosines be represented by the column vector,

$$\mathbf{v} = \begin{pmatrix} l \\ m \\ n \end{pmatrix} \tag{1}$$

The linear equation $2l + 2m - n = 0$ can be written as,

$$\mathbf{c}^\top \mathbf{v} = 0 \tag{2}$$

Where,

$$\mathbf{c} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \tag{3}$$

The quadratic equation $mn + nl + lm = 0$ is equivalent to $2mn + 2nl + 2lm = 0$. This can be written as a quadratic form $\mathbf{v}^T A\mathbf{v} = 0$, where A is the symmetric matrix:

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{4}$$

For a system defined by $\mathbf{c}^\top \mathbf{v} = 0$ and $\mathbf{v}^\top A\mathbf{v} = 0$, the two solution vectors are orthogonal if and only if the following algebraic condition is met:

$$\mathbf{c}^\top \left(\text{Tr}(A)I - A\right)\mathbf{c} = 0 \tag{5}$$

$$\text{Tr}(A) = 0 + 0 + 0 = 0 \tag{6}$$

Substituting this into the condition, it simplifies to:

$$\mathbf{c}^T(0 \cdot I - A)\mathbf{c} = -\mathbf{c}^\top A\mathbf{c} = 0 \tag{7}$$

We therefore only need to verify that $\mathbf{c}^\top A\mathbf{c} = 0$.

$$\mathbf{c}^\top A\mathbf{c} = \begin{pmatrix} 2 & 2 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \tag{8}$$

$$\mathbf{c}^\top A\mathbf{c} = \begin{pmatrix}(2\cdot 0+2\cdot 1-1\cdot 1) & (2\cdot 1+2\cdot 0-1\cdot 1) & (2\cdot 1+2\cdot 1-1\cdot 0)\end{pmatrix}\begin{pmatrix}2\\2\\-1\end{pmatrix} \tag{9}$$

$$\mathbf{c}^\top A\mathbf{c} = \begin{pmatrix}1 & 1 & 4\end{pmatrix}\begin{pmatrix}2\\2\\-1\end{pmatrix} \tag{10}$$

$$\mathbf{c}^\top A\mathbf{c} = (1)(2) + (1)(2) + (4)(-1) \tag{11}$$

$$\mathbf{c}^\top A\mathbf{c} = 2 + 2 - 4 \tag{12}$$

$$\mathbf{c}^\top A\mathbf{c} = 0 \tag{13}$$

Since the condition is satisfied, the given lines are at right angles.