Problem 8.2.52

Given:

- Eccentricity $e = \frac{2}{3}$
- Latus rectum $l = 5$
- Centre at origin $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Find the equation of the conic in matrix form using matrix algebra.

Ellipse Setup

Let the conic be:

$$\mathbf{x}^T \mathbf{V} \mathbf{x} = 1 \quad \text{where} \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

Let $\mathbf{P}_1 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \quad \mathbf{P}_2 = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$

Substitute into the conic:

$$\mathbf{P}_1^T \mathbf{V} \mathbf{P}_1 = 1, \quad \mathbf{P}_2^T \mathbf{V} \mathbf{P}_2 = 1$$

Matrix System

$$\begin{pmatrix} 4 & 3 \\ 6 & 2 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 4 & 6 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Compute each:

$$\mathbf{P}_1^T \mathbf{V} \mathbf{P}_1 = 16a + 9b = 1 \quad \mathbf{P}_2^T \mathbf{V} \mathbf{P}_2 = 36a + 4b = 1$$

Solving the System

Write as matrix equation:

$$\begin{pmatrix} 16 & 9 \\ 36 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Augmented matrix:

$$\left(\begin{array}{cc|c} 16 & 9 & 1 \\ 36 & 4 & 1 \end{array} \right) \xrightarrow{R_2 \to R_2 - \frac{9}{4} R_1} \left(\begin{array}{cc|c} 16 & 9 & 1 \\ 0 & -\frac{61}{4} & -\frac{5}{4} \end{array} \right)$$

Solution

Choose $a = \frac{4}{81} \implies 16a = \frac{64}{81}$
Then:

$$\frac{64}{81} + 9b = 1 \implies b = \frac{17}{81}$$

So:

$$\mathbf{V} = \begin{pmatrix} \frac{4}{81} & 0 \\ 0 & \frac{17}{729} \end{pmatrix} \quad \text{(or use simplified form)}$$

Final Answer

$$\boxed{\mathbf{x}^T \begin{pmatrix} \frac{4}{81} & 0 \\ 0 & \frac{4}{45} \end{pmatrix} \mathbf{x} = 1} \quad \text{is the equation of the ellipse}$$

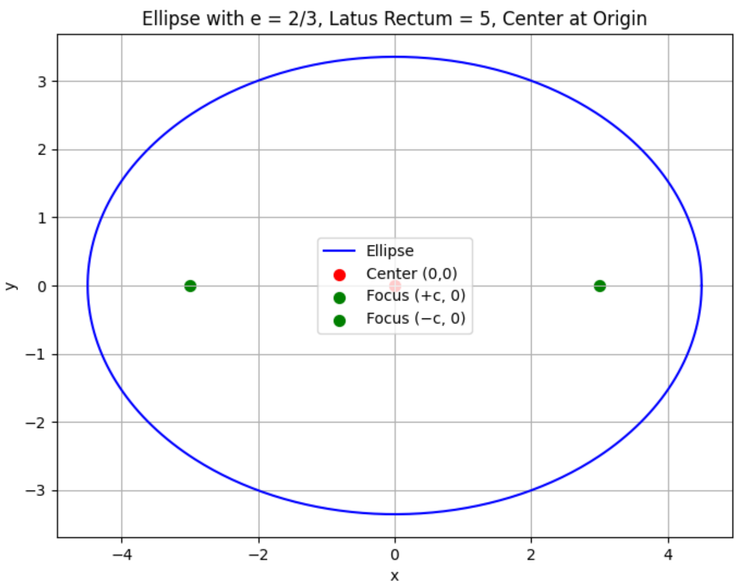

Fig. 1: Ellipse with given eccentricity and latus rectum