GATE 2007 MA

AI25BTECH11012 - UNNATHI GARIGE

Q.1-Q.20 carry one mark each.

1) Consider $\mathbb{R}^2$ with the usual topology. Let

$$S = \{(x, y) \in \mathbb{R}^2 : x \text{ is an integer}\}$$

Then S is

GATE MA 2007

a) open but NOT closed
b) both open and closed
c) neither open nor closed
d) closed but NOT open

2) Suppose $X = \{\alpha, \beta, \delta\}$. Let

$$\mathcal{T}_1 = \{\emptyset, X, \{\alpha\}, \{\alpha, \beta\}\} \quad \text{and} \quad \mathcal{T}_2 = \{\emptyset, X, \{\alpha\}, \{\beta, \delta\}\}.$$

Then GATE MA 2007

a) both $\mathcal{T}_1 \cap \mathcal{T}_2$ and $\mathcal{T}_1 \cup \mathcal{T}_2$ are topologies
b) neither $\mathcal{T}_1 \cap \mathcal{T}_2$ nor $\mathcal{T}_1 \cup \mathcal{T}_2$ is a topology
c) $\mathcal{T}_1 \cup \mathcal{T}_2$ is a topology but $\mathcal{T}_1 \cap \mathcal{T}_2$ is NOT a topology
d) $\mathcal{T}_1 \cap \mathcal{T}_2$ is a topology but $\mathcal{T}_1 \cup \mathcal{T}_2$ is NOT a topology

3) For a positive integer n, let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} \frac{1}{4n+5} & \text{if } 0 \le x \le n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{f_n(x)\}$ converges to zero GATE MA 2007

a) uniformly but NOT in L^1 norm
b) uniformly and also in L^1 norm
c) pointwise but NOT uniformly
d) in L^1 norm but NOT pointwise

4) Let P_1 and P_2 be two projection operators on a vector space. Then GATE MA 2007

a) $P_1 + P_2$ is a projection if $P_1P_2 = P_2P_1 = 0$
b) $P_1 - P_2$ is a projection if $P_1P_2 = P_2P_1 = 0$
c) $P_1 + P_2$ is a projection
d) $P_1 - P_2$ is a projection

5) Consider the system of linear equations GATE MA 2007

$$x + y + z = 3,$$
$$x - y - z = 4,$$
$$-5y + kz = 6$$

Then the value of k for which this system has an infinite number of solutions is

a) $k = -5$
b) $k = 0$
c) $k = 1$
d) $k = 3$

6) Let GATE MA 2007

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ x & y & z \end{bmatrix}$$

and let $V = \{(x, y, z) \in \mathbb{R}^3 : \det(A) = 0\}$. Then the dimension of V equals:

a) 0 b) 1 c) 2 d) 3

7) Let GATE MA 2007

$$S = \{0\} \cup \left\{ \frac{1}{4n + 7} : n = 1, 2, \ldots \right\}$$

Then the number of analytic functions which vanish only on S is:

a) infinite b) 0 c) 1 d) 2

8) It is given that $\sum_{n=0}^{\infty} a_n z^n$ converges at $z = 3 + i4$. Then the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ is: GATE MA 2007

a) ≤ 5 b) ≥ 5 c) < 5 d) > 5

9) The value of α for which $G = \langle \alpha, 1, 3, 9, 19, 27 \rangle$ is a cyclic group under multiplication modulo 56 is: GATE MA 2007

a) 5 b) 15 c) 25 d) 35

10) Consider $\mathbb{Z}_{24}$ as the additive group modulo 24. Then the number of elements of order 8 in the group $\mathbb{Z}_{24}$ is: GATE MA 2007

a) 1 b) 2 c) 3 d) 4

11) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} 1, & \text{if } xy = 0, \\ 2, & \text{otherwise.} \end{cases}$$

If $S = \{(x, y) : f \text{ is continuous at the point } (x, y)\}$, then: GATE MA 2007

a) S is open
b) S is connected
c) $S = \emptyset$
d) S is closed

12) Consider the linear programming problem

$$\begin{aligned}
&\text{Maximize } z = c_1x_1 + c_2x_2, \quad c_1, c_2 > 0,\\
&\text{subject to}\\
&x_1 + x_2 \le 3\\
&2x_1 + 3x_2 \le 4\\
&x_i \ge 0
\end{aligned}$$

Then: GATE MA 2007

a) the primal has an optimal solution but the dual does NOT have an optimal solution
b) both the primal and the dual have optimal solutions
c) the dual has an optimal solution but the primal does NOT have an optimal solution
d) neither the primal nor the dual have optimal solutions

13) Let

$$f(x) = x^{10} + x - 1, x \in \mathbb{R}$$

and let $x_k = k$, $k = 0, 1, 2, \dots, 10$. Then the value of the divided difference

$$f[x_0, x_1, x_2, \dots, x_{10}]$$

is: GATE MA 2007

a) -1 b) 0 c) 1 d) 10

14) Let X, Y be jointly distributed random variables having the joint probability density function

$$f(x, y) = \begin{cases} 1, & \text{if } 0 < x + y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $P(Y \ge \max(X, 1 - X))$ is GATE MA 2007

a) $\frac{1}{2}$ b) 1 c) $\frac{1}{4}$ d) $\frac{1}{6}$

15) Let $X_1, X_2, \dots$ be a sequence of independent and identically distributed chi-square random variables, each having 4 degrees of freedom. Define

$$S_n = \sum_{i=1}^{n} X_i$$

If $\frac{S_n}{n} \to \mu$ as $n \to \infty$, then $\mu =$ GATE MA 2007

a) 8 b) 16 c) 24 d) 32

16) Let $\{E_n : n = 1, 2, \dots\}$ be a decreasing sequence of Lebesgue measurable sets on $\mathbb{R}$ and let F be a Lebesgue measurable set on $\mathbb{R}$ such that $E_n \cap F = \emptyset$. Suppose that F has Lebesgue measure 2 and the Lebesgue measure of E_n equals

$$\frac{2n+2}{3n+1}, n = 1, 2, \dots$$

Then the Lebesgue measure of the set $(\bigcap_{n=1}^{\infty} E_n) \cup F$ equals GATE MA 2007

a) $\frac{5}{3}$ b) 2 c) $\frac{7}{3}$ d) $\frac{8}{3}$

17) The extremum for the variational problem

$$\int_0^{\frac{\pi}{8}} ((y')^2 + 2yy' - 16y^2)\, dx, \quad y(0) = 0,\ y\left(\frac{\pi}{8}\right) = 1,$$

occurs for the curve GATE MA 2007

a) $y = \sin(4x)$
b) $y = \sqrt{2}\sin(2x)$
c) $y = 1 - \cos(4x)$
d) $y = \dfrac{1-\cos(8x)}{2}$

18) Suppose $y_p(x) = x\cos(2x)$ is a particular solution of

$$y'' + \alpha y = \sin(2x).$$

Then the constant α equals GATE MA 2007

a) -4 b) -2 c) 2 d) 4

19) If $F(s) = \tan^{-1}(s) + k$ is the Laplace transform of some function $f(t)$, $t \geq 0$, then $k =$ GATE MA 2007

a) $-\pi$ b) $\frac{\pi}{2}$ c) 0 d) $\frac{\pi}{2}$

20) Let

$$S = \{(0, 1, 1), (1, 0, 1), (-1, 2, 1)\} \subseteq \mathbb{R}^3.$$

Suppose $\mathbb{R}^3$ is endowed with the standard inner product . Define

$$M = \{x \in \mathbb{R}^3 : \langle x, y \rangle = 0 \text{ for all } y \in S\}$$

Then the dimension of M equals GATE MA 2007

a) 0 b) 1 c) 2 d) 3

Q.21-Q.75 carry one mark each.

21) Let X be an uncountable set and let

$$\tau = \{U \subseteq X : X \setminus U \text{ is countable or } X \setminus U \text{ is finite}\}.$$

Then the topological space (X, τ) GATE MA 2007

a) is separable
b) is Hausdorff
c) has a countable basis
d) has a countable basis at each point

22) Suppose (X, τ) is a topological space. Let $\{S_\alpha\}_{\alpha \in A}$ be a sequence of subsets of X. Then GATE MA 2007

a) $(S_1 \bigcup S_2)'' = S_1'' \bigcup S_2''$
b) $(\bigcap S_{''})'' = \bigcap S_{''}$
c) $\overline{\bigcup S_{''}} = \bigcup_\alpha \overline{S_{''}}$
d) $\overline{S_1 \bigcup S_2} = \overline{S_1} \cup \overline{S_2}$

23) Let (X, d) be a metric space. Consider the metric ρ on X defined by

$$\rho(x, y) = \min(d(x, y), 1), \quad x, y \in X.$$

Suppose τ and τ_1 are topologies on X defined by d and ρ respectively. Then GATE MA 2007

a) τ_1 is a proper subset of τ_2
b) τ_2 is a proper subset of τ_1
c) neither τ_2 nor τ_1 is a subset of the other
d) $\tau_1 = \tau_2$

24) A basis of the vector space

$$W = \{(x, y, z, w) \in \mathbb{R}^4 : x + y + z = 0, y + z + w = 0, 2x + y - z + w = 0\}$$

is GATE MA 2007

a) $\{(1, 1, 1, 1), (2, 1, 1, 1)\}$
b) $\{(1, -1, 0, 1), (0, 1, -1, 0)\}$
c) $\{(1, 0, -1, 0), (2, 1, 1, 1)\}$
d) $\{(1, 0, -1, 0), (0, 1, -1, 0)\}$

25) Consider $\mathbb{R}^3$ with the standard inner product. Let

$$S = \{(1, 1, 1), (2, -1, 2), (-1, 2, 1)\}$$

For a subset W of $\mathbb{R}^3$, let $L(W)$ denote the linear span of W in $\mathbb{R}^3$. Then an orthonormal set T with $L(S) = L(T)$ is GATE MA 2007

a) $\left\{\frac{1}{\sqrt{3}}(1, 1, 1), \frac{1}{\sqrt{6}}(1, 0, -2), \frac{1}{\sqrt{2}}(1, -1, 0)\right\}$
b) $\{(0, 0, 0), (0, 1, 0), (0, 0, 1)\}$
c) $\left\{\frac{1}{\sqrt{3}}(1, 1, 1), \frac{1}{\sqrt{2}}(1, 0, -1)\right\}$
d) $\left\{\frac{1}{\sqrt{3}}(1, 1, 1), \frac{1}{\sqrt{2}}(1, -1, 0)\right\}$

26) Let A be a 3×3 matrix. Suppose that the eigenvalues of A are $-1, 0, 1$ with respective eigenvectors $(1,-1,0)^T$, $(1,1,-2)^T$ and $(1,1,1)^T$.
Then $6A$ equals GATE MA 2007

a) $\begin{bmatrix} -1 & 5 & 2 \\ 5 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$
c) $\begin{bmatrix} 1 & 5 & 3 \\ 5 & 1 & 3 \\ 3 & 3 & 3 \end{bmatrix}$
b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
d) $\begin{bmatrix} -3 & 9 & 0 \\ 9 & -3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

27) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by

$$T((x,y,z)) = (x+y-z, x+y+z, y-z).$$

Then the matrix of the linear transformation T with respect to the ordered basis

$$B = \{(0,1,0),(0,0,1),(1,0,0)\} of \mathbb{R}^3$$

is GATE MA 2007

a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$
c) $\begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$
b) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$
d) $\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

28) Let $Y(x) = (y_1(x), y_2(x))^T$ and let

$$A = \begin{bmatrix} -3 & 1 \\ k & -1 \end{bmatrix}.$$

Further, let S be the set of values of k for which all the solutions of the system of equations $Y'(x) = AY(x)$ tend to zero as $x \to \infty$.
Then S is given by GATE MA 2007

a) $\{k : k \le -1\}$
c) $\{k : k < -1\}$
b) $\{k : k \le 3\}$
d) $\{k : k < 3\}$

29) Let

$$u(x,y) = f(xe^y) + g(y^2 \cos y),$$

where f and g are infinitely differentiable functions. Then the partial differential equation of minimum order satisfied by u is GATE MA 2007

a) $u_x + xu_{xx} = u_y$
b) $u_y + xu_{xx} = xu_y$
c) $u_y - xu_{xx} = u_x$
d) $u_y - xu_{xx} = xu_y$

30) Let C be the boundary of the triangle formed by the points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. Then the value of the line integral GATE MA 2007

$$\oint_C -2y\,dx + (3x - 4y^2)\,dy + (z^2 + 3y)\,dz$$

is

a) 0
b) 1
c) 2
d) 4

31) Let X be a complete metric space and let $E \subset X$.
Consider the following statements: GATE MA 2007

a) E is compact,
b) E is closed and bounded,
c) E is closed and totally bounded,
d) Every sequence in E has a subsequence converging in E.

Which one of the above statements does **NOT** imply all the other statements?

a) a
b) b
c) c
d) d

32) Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \sin(nx).$$

Then the series GATE MA 2007

a) converges uniformly on $\mathbb{R}$
b) converges pointwise but NOT uniformly on $\mathbb{R}$
c) converges in L^1 norm to an integrable function on $[0, 2\pi]$ but does NOT converge uniformly on $\mathbb{R}$
d) does NOT converge pointwise

33) Let $f(z)$ be an analytic function. Then the value of GATE MA 2007

$$\int_0^{2\pi} f(e^{it}) \cos(t)\,dt$$

equals

a) 0
b) $2\pi f(0)$
c) $2\pi f'(0)$
d) $\pi f'(0)$

34) Let G_1 and G_2 be the images of the disc $\{z \in \mathbb{C} : |z+1| < 1\}$ under the transformations

$$w = \frac{(1-i)z + 2}{(1+i)z + 2} \quad \text{and} \quad w = \frac{(1+i)z + 2}{(1-i)z + 2}$$

respectively. Then

a) $G_1 = \{w \in \mathbb{C} : \text{Im}(w) < 0\}$ and $G_2 = \{w \in \mathbb{C} : \text{Im}(w) > 0\}$
b) $G_1 = \{w \in \mathbb{C} : \text{Im}(w) > 0\}$ and $G_2 = \{w \in \mathbb{C} : \text{Im}(w) < 0\}$
c) $G_1 = \{w \in \mathbb{C} : |w| > 2\}$ and $G_2 = \{w \in \mathbb{C} : |w| < 2\}$
d) $G_1 = \{w \in \mathbb{C} : |w| < 2\}$ and $G_2 = \{w \in \mathbb{C} : |w| > 2\}$

35) Let $f(z) = 2^z - 2^{-z}$. Then the maximum value of $|f(z)|$ on the unit disc

$$D = \{z \in \mathbb{C} : |z| \leq 1\}$$

equals GATE MA 2007

a) 1 b) 2 c) 3 d) 4

36) Let GATE MA 2007

$$f(z) = \frac{1}{z^2 - 3z + 2}$$

Then the coefficient of $\frac{1}{z}$ in the Laurent series expansion of $f(z)$ for $|z| > 2$ is

a) 0 b) 1 c) 3 d) 5

37) Let $f : \mathbb{C} \to \mathbb{C}$ be an arbitrary analytic function satisfying $f(0) = 0$ and $f(1) = 2$. Then GATE MA 2007

a) there exists a sequence $\{z_n\}$ such that $|z_n| > n$ and $|f(z_n)| > n$
b) there exists a sequence $\{z_n\}$ such that $|z_n| > n$ and $|f(z_n)| < n$
c) there exists a bounded sequence $\{z_n\}$ such that $|f(z_n)| > n$
d) there exists a sequence $\{z_n\}$ such that $z_n \to 0$ and $f(z_n) \to 2$

38) Define $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = \begin{cases} 0, & \text{if } \text{Re}(z) = 0 \text{ or } \text{Im}(z) = 0, \\ \frac{1}{z}, & \text{otherwise.} \end{cases}$$

Then the set of points where f is analytic is GATE MA 2007

a) $\{z : \text{Re}(z) \neq 0 \text{ and } \text{Im}(z) \neq 0\}$
b) $\{z : \text{Re}(z) \neq 0\}$
c) $\{z : \text{Re}(z) \neq 0 \text{ or } \text{Im}(z) \neq 0\}$
d) $\{z : \text{Im}(z) \neq 0\}$

39) Let $U(n)$ be the set of all positive integers less than n and relatively prime to n. Then $U(n)$ is a group under multiplication modulo n. For $n = 248$, the number of elements in $U(n)$ is GATE MA 2007

a) 60 b) 120 c) 180 d) 240

40) Let $\mathbb{R}[x]$ be the polynomial ring in x with real coefficients and let $I = \langle x^2 + 1 \rangle$ be the ideal generated by the polynomial $x^2 + 1$ in $\mathbb{R}[x]$. Then GATE MA 2007

a) I is a maximal ideal
b) I is a prime ideal but NOT a maximal ideal

c) I is NOT a prime ideal
d) $\mathbb{R}[x]/I$ has zero divisors

41) Consider $\mathbb{Z}5$ and $\mathbb{Z}20$ as rings modulo 5 and 20, respectively. Then the number of homomorphisms $\varphi : \mathbb{Z}5 \to \mathbb{Z}20$ is GATE MA 2007

a) 1 b) 2 c) 4 d) 5

42) Let $\mathbb{Q}$ be the field of rational numbers and consider $\mathbb{Z}_2$ as a field modulo 2. Let

$$f(x) = x^3 - 9x^2 + 9x + 3.$$

Then $f(x)$ is GATE MA 2007

a) irreducible over $\mathbb{Q}$ but reducible over $\mathbb{Z}_2$
b) irreducible over both $\mathbb{Q}$ and $\mathbb{Z}_2$
c) reducible over $\mathbb{Q}$ but irreducible over $\mathbb{Z}_2$
d) educible over both $\mathbb{Q}$ and $\mathbb{Z}_2$

43) Let $\mathbb{Q}$ be the field of rational numbers and consider $\mathbb{Z}_2$ as a field modulo 2. Let

$$f(x) = x^3 - 9x^2 + 9x + 3.$$

Then $f(x)$ is GATE MA 2007

a) irreducible over $\mathbb{Q}$ but reducible over $\mathbb{Z}_2$
b) irreducible over both $\mathbb{Q}$ and $\mathbb{Z}_2$
c) reducible over $\mathbb{Q}$ but irreducible over $\mathbb{Z}_2$
d) reducible over both $\mathbb{Q}$ and $\mathbb{Z}_2$

44) Consider $\mathbb{Z}_5$ as a field modulo 5 and let GATE MA 2007

$$f(x) = x^4 + 4x^3 + 4x^2 + 4x + 1.$$

Then the zeros of $f(x)$ over $\mathbb{Z}_5$ are 1 and 3 with respective multiplicity

a) 1 and 4 b) 2 and 3 c) 2 and 2 d) 1 and 2

45) Consider the Hilbert space GATE MA 2007

$$\ell^2 = \left\{x = \{x_n\};\ x_n \in \mathbb{R},\ \sum x_n^2 < \infty\right\}.$$

Let

$$E = \left\{x = \{x_n\} \mid |x_n| < \frac{1}{n} \text{ for all } n\right\}$$

be a subset of ℓ^2. Then

a) $E^\circ = \left\{x \mid |x_n| < \frac{1}{n} \text{ for all } n\right\}$
b) $E^\circ = E$
c) $E^\circ = \left\{x \mid |x_n| < \frac{1}{n} \text{ for all but finitely many } n\right\}$
d) $E^\circ = \emptyset$

46) Let X be a normed linear space and let $E_1, E_2 \subseteq X$. Define

$$E_1 + E_2 = \{x + y : x \in E_1, y \in E_2\}.$$

Then $E_1 + E_2$ is: GATE MA 2007

a) open if E_1 or E_2 is open
b) NOT open unless both E_1 and E_2 are open
c) closed if E_1 or E_2 is closed
d) closed if both E_1 and E_2 are closed

47) For each $a \in \mathbb{R}$, consider the linear programming problem:

Max. $z = x_1 + 2x_2 + 3x_3 + 4x_4$
subject to GATE MA 2007

$$\begin{aligned} ax_1 + 2x_2 &\le 1 \\ x_1 + 2x_2 + 3x_3 &\le 2 \\ x_1, x_2, x_3, x_4 &\ge 0 \end{aligned}$$

Let $S = \{a \in \mathbb{R}$: the given LP problem has a basic feasible solution$\}$. Then:

a) $S = \emptyset$
b) $S = \mathbb{R}$
c) $S = (0, \infty)$
d) $S = (-\infty, 0)$

48) Consider the linear programming problem:
Max. $z = x_1 + 5x_2 + 3x_3$ subject to

$$\begin{aligned} 2x_1 - 3x_2 + 5x_3 &\le 3 \\ x_1 - x_2 &\le 5 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

Then the dual of this LP problem: GATE MA 2007

a) has a feasible solution but does NOT have a basic feasible solution
b) has a basic feasible solution
c) has infinite number of feasible solutions
d) has no feasible solution

49) Consider a transportation problem with two warehouses and two markets. The warehouse capacities are $a_1 = 2$ and $a_2 = 4$, and the market demands are $b_1 = 3$ and $b_2 = 3$. Let x_{ij} be the quantity shipped from warehouse i to market j, and c_{ij} be the corresponding unit cost. Suppose that $c_{11} = 1$, $c_{21} = 1$, and $c_{22} = 2$. Then $(x_{11}, x_{12}, x_{21}, x_{22}) = (2, 0, 1, 3)$ is optimal for every: GATE MA 2007

a) $c_{12} \in [1, 2]$
b) $c_{12} \in [0, 3]$
c) $c_{12} \in [1, 3]$
d) $c_{12} \in [2, 4]$

50) The smallest degree of the polynomial that interpolates the data
is: GATE MA 2007

x	-2	-1	0	1	2	3
$f(x)$	-58	-21	-12	-13	-6	27

TABLE 50

a) 3 b) 4 c) 5 d) 6

51) Suppose that x_n is sufficiently close to 3. Which of the following iterations $x_{n+1} = g(x_n)$ will converge to the fixed point $x = 3$? GATE MA 2007

a) $x_{n+1} = -16 + 6x_n + \dfrac{3}{x_n}$
b) $x_{n+1} = \sqrt{3 + 2x_n}$
c) $x_{n+1} = \dfrac{3}{x_n} - \dfrac{x_n}{2}$
d) $x_{n+1} = \dfrac{x_n^2 - 3}{2}$

52) Consider the quadrature formula:

$$\int_{x_1}^{x_2} f(x)\,dx \approx \frac{1}{2}\left[f(x_1) + f(x_2)\right],$$

where x_1 and x_2 are quadrature points. Then the highest degree of the polynomial for which the above formula is exact equals: GATE MA 2007

a) 1 b) 2 c) 3 d) 4

53) Let A, B and C be three events such that:

$P(A) = 0.4$, $P(B) = 0.5$, $P(A \cup B) = 0.6$, $P(C) = 0.6$, and $P(A \cap B \cap C^c) = 0.1$.

Then $P(A \cap B \cap C) =$ GATE MA 2007

a) $\dfrac{1}{2}$ b) $\dfrac{1}{3}$ c) $\dfrac{1}{4}$ d) $\dfrac{1}{5}$

54) Consider two identical boxes B_1 and B_2, where the box B_i $(i = 1, 2)$ contains $i+1$ red and $5-i+1$ white balls. A fair die is cast. Let the number of dots shown on the top face of the die be N. If N is even or 5, then two balls are drawn with replacement from the box B_1; otherwise, two balls are drawn with replacement from the box B_2. The probability that the two drawn balls are of different colours is: GATE MA 2007

a) $\dfrac{7}{25}$ b) $\dfrac{9}{25}$ c) $\dfrac{12}{25}$ d) $\dfrac{16}{25}$

55) Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with

$$P(X_i = 1) = P(X_i = -1) = \frac{1}{2}.$$

Suppose for the standard normal random variable Z, $P(-0.1 < Z \le 0.1) = 0.08$. If $S_n = \sum_{i=1}^{n} X_i$, then

$$\lim P\left(\frac{S_n}{\sqrt{n}} > \frac{n}{10}\right) =$$

a) 0.42 b) 0.46 c) 0.5 d) 0.54

56) Let $X_1, X_2, \ldots, X_5$ be a random sample of size 5 from a population having standard normal distribution. Let

$$\bar{X} = \frac{1}{5}\sum_{i=1}^{5} X_i \quad \text{and} \quad T = \sum_{i=1}^{5}(X_i - \bar{X})^2.$$

Then $E(T^2\bar{X}^2) =$ GATE MA 2007

a) 3 b) 3.6 c) 4.8 d) 5.2

57) Let $x_1 = 3.5$, $x_2 = 7.5$ and $x_3 = 5.2$ be observed values of a random sample of size three from a population having uniform distribution over the interval $(\theta, \theta + 5)$, where $\theta \in (0, \infty)$ is unknown and is to be estimated. Then which of the following is NOT a maximum likelihood estimate of θ? GATE MA 2007

a) 2.4 b) 2.7 c) 3 d) 3.3

58) The value of

$$\int_0^1 \int_y^1 x^2 e^{x^2}\, dx\, dy$$

equals GATE MA 2007

a) $\frac{1}{4}$ b) $\frac{1}{3}$ c) $\frac{1}{2}$ d) 1

59)

$$\lim_{n\to\infty}\left[(n+1)\int_0^1 x^n \ln(1+x)\, dx\right] =$$

GATE MA 2007

a) 0 b) $\ln 2$ c) $\ln 3$ d) ∞

60) Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^4, & \text{if } x \text{ is rational,} \\ 2x^4 - 1, & \text{if } x \text{ is irrational.} \end{cases}$$

Let S be the set of points where f is continuous. Then GATE MA 2007

a) $S = \{1\}$ b) $S = \{-1\}$ c) $S = \{-1, 1\}$ d) $S = \emptyset$

61) For a positive real number p, let $\{f_n : n = 1, 2, \dots\}$ be a sequence of functions defined on $[0, 1]$ by

$$f_n(x) = \begin{cases} n^{p+1}x, & 0 \le x \le \frac{1}{n} \\ \frac{1}{n^p}, & \frac{1}{n} < x \le 1. \end{cases}$$

Let $f(x) = \lim_{n\to\infty} f_n(x)$, $x \in [0, 1]$. Then, on $[0, 1]$, GATE MA 2007

a) f is Riemann integrable
b) the improper integral $\int_0^1 f(x)dx$ converges for $p \ge 1$
c) the improper integral $\int_0^1 f(x)dx$ converges for $p < 1$
d) f_n converges uniformly

62) Which of the following inequality is NOT true for $x \in \left[\frac{1}{4}, \frac{3}{4}\right]$ GATE MA 2007

a) $e^{-x} > \sum_{j=0}^{\infty} \frac{(-x)^j}{j!}$
b) $e^{-x} < \sum_{j=0}^{\infty} \frac{(-x)^j}{j!}$
c) $e^{-x} = \sum_{j=0}^{\infty} \frac{(-x)^j}{j!}$
d) $e^{-x} > \sum_{j=0}^{10} \frac{(-x)^j}{j!}$

63) Let $u(x, y)$ be the solution to the Cauchy problem

$$xu_x + u_y = 1, \quad u(x, 0) = 2\ln(x), \ x > 1.$$

Then $u(e, 1) =$ GATE MA 2007

a) -1
b) 0
c) 1
d) e

64) Suppose

$$y(x) = \lambda \int_0^{2\pi} y(t)\sin(x + t)\, dt, \ x \in [0, 2\pi]$$

has eigenvalues $\lambda = \frac{1}{\pi}$ and $\lambda = -\frac{1}{\pi}$ with corresponding eigenfunctions $y_1(x) = \sin(x) + \cos(x)$ and $y_2(x) = \sin(x) - \cos(x)$, respectively. Then the integral equation

$$y(x) = f(x) + \frac{1}{\pi}\int_0^{2\pi} y(t)\sin(x + t)\, dt, \ x \in [0, 2\pi]$$

has a solution when $f(x) =$ GATE MA 2007

a) 1
b) $\cos(x)$
c) $\sin(x)$
d) $1+\sin(x)+\cos(x)$

65) Consider the Neumann problem

$$u_{xx} + u_{yy} = 0, \quad 0 < x < \pi, \ -1 < y < 1,$$

$$u_y(0, y) = u_y(\pi, y) = 0,$$

$$u_y(x, -1) = 0, \quad u_y(x, 1) = \alpha + \beta\sin(x).$$

The problem admits solution for GATE MA 2007

a) $\alpha = 0,\ \beta = 1$
b) $\alpha = -1,\ \beta = \dfrac{\pi}{2}$
c) $\alpha = 1,\ \beta = \dfrac{\pi}{2}$
d) $\alpha = 1,\ \beta = -\pi$

66) The functional

$$\int_0^1 (1+x)(y')^2\,dx, \quad y(0) = 0,\ y(1) = 1,$$

possesses GATE MA 2007

a) strong maxima
b) strong minima
c) weak maxima but NOT a strong maxima
d) weak minima but NOT a strong minima

67) The value of α for which the integral equation

$$u(x) = \alpha \int_0^1 e^{xt} u(t)\,dt,$$

has a non-trivial solution is GATE MA 2007

a) -2 b) -1 c) 1 d) 2

68) Let $P_n(x)$ be the Legendre polynomial of degree n and let

$$P_{n+1}(0) = -\frac{m}{m+1} P_{n-1}(0), \quad m = 1, 2, \ldots$$

If $P_2(0) = -\dfrac{5}{16}$ then $\int_{-1}^{1} \left[P_2^2(x)\right] dx =$ GATE MA 2007

a) $\dfrac{2}{13}$ b) $\dfrac{2}{9}$ c) $\dfrac{5}{16}$ d) $\dfrac{2}{5}$

69) For which of the following pair of functions $y_1(x)$ and $y_2(x)$, continuous functions $p(x)$ and $q(x)$ can be determined on $[-1, 1]$ such that $y_1(x)$ and $y_2(x)$ give two linearly independent solutions of GATE MA 2007

$$y'' + p(x)y' + q(x)y = 0, \quad x \in [-1, 1].$$

a) $y_1(x) = x\sin(x),\ y_2(x) = \cos(x)$
b) $y_1(x) = xe^x,\ y_2(x) = \sin(x)$
c) $y_1(x) = e^{-x},\ y_2(x) = e^{-1}$
d) $y_1(x) = x^2,\ y_2(x) = \cos(x)$

70) Let $J_0(s)$ and $J_1(s)$ be the Bessel functions of the first kind of orders zero and one, respectively. If

$$\mathcal{L}(J_0)(s) = \frac{1}{\sqrt{s^2+1}},$$

then $\mathcal{L}(J_1)(s) =$ GATE MA 2007

a) $\dfrac{s}{\sqrt{s^2+1}}$

b) $\dfrac{1}{\sqrt{s^2+1}}$

c) $1-\dfrac{1}{\sqrt{s^2+1}}$

d) $\dfrac{1}{\sqrt{s^2+1}}-1$

Common Data Questions

Common Data for Questions 71, 72, 73:

Let $P[0,1] = \{p : p \text{ is a polynomial function on } [0,1]\}$. For $p \in P[0,1]$, define

$$\|p\| = \sup\{|p(x)| : 0 \le x \le 1\}.$$

Consider the map $T : P[0,1] \to P[0,1]$ defined by

$$(Tp)(x) = \frac{d}{dx}(p(x)).$$

Then $P[0,1]$ is a normed linear space and T is a linear map. The map T is said to be closed if the set $G = \{(p, Tp) : p \in P[0,1]\}$ is a closed subset of $P[0,1] \times P[0,1]$.

71) The linear map T is GATE MA 2007

a) one to one and onto

b) one to one but NOT onto

c) onto but NOT one to one

d) neither one to one nor onto

72) The normed linear space $P[0,1]$ is GATE MA 2007

a) a finite dimensional normed linear space which is NOT a Banach space

b) a finite dimensional Banach space

c) an infinite dimensional normed linear space which is NOT a Banach space

d) an infinite dimensional Banach space

73) The map T is GATE MA 2007

a) closed and continuous

b) neither continuous nor closed

c) continuous but NOT closed

d) closed but NOT continuous

Common Data for Questions 74, 75:

Let X and Y be jointly distributed random variables such that the conditional distribution of Y, given $X = x$, is uniform on the interval $(x-1, x+1)$. Suppose $\mathbb{E}(X) = 1$ and $\text{Var}(X) = \frac{5}{3}$.

74) The mean of the random variable Y is GATE MA 2007

a) $\frac{1}{2}$

b) 1

c) $\frac{3}{2}$

d) 2

75) The variance of the random variable Y is GATE MA 2007

a) $\frac{1}{2}$ b) $\frac{2}{3}$ c) 1 d) 2

Linked Answer Questions: Q.76 to Q.85 carry two marks each.

Statement for Linked Answer Questions 76 & 77:
Suppose the equation

$$x^2y'' - xy' + (1 + x^2)y = 0$$

has a solution of the form

$$y = x^r \sum_{n=0}^{\infty} c_n x^n, \quad c_0 \neq 0.$$

76) The indicial equation for r is GATE MA 2007

a) $r^2 - 1 = 0$
b) $(r - 1)^2 = 0$
c) $(r + 1)^2 = 0$
d) $r^2 + 1 = 0$

77) For $n \geq 2$, the coefficients c_n will satisfy the relation GATE MA 2007

a) $n^2c_n - c_{n-2} = 0$
b) $c_n - n^2c_{n-2} = 0$
c) $c_n - n^2c_{n-2} = 0$
d) $c_n + n^2c_{n-2} = 0$

Statement for Linked Answer Questions 78 & 79:
A particle of mass m slides down without friction along a curve $z = 1 + \frac{x^2}{2}$ in the xz-plane under the action of constant gravity. Suppose the z-axis points vertically upwards. Let $\dot{x}$ and $\ddot{x}$ denote $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ respectively.

78) The Lagrangian of the motion is GATE MA 2007

a) $\frac{1}{2}m\dot{x}^2(1 + x^2) - mg\left(1 + \frac{x^2}{2}\right)$
b) $\frac{1}{2}m\dot{x}^2(1 + x^2) + mg\left(1 + \frac{x^2}{2}\right)$
c) $\frac{1}{2}mx^2\dot{x}^2 - mg\left(1 + \frac{x^2}{2}\right)$
d) $\frac{1}{2}m\dot{x}^2(1 - x^2) - mg\left(1 + \frac{x^2}{2}\right)$

79) The Lagrangian equation of motion is GATE MA 2007

a) $\ddot{x}(1 + x^2) = -x(g + \dot{x}^2)$
b) $\ddot{x}(1 + x^2) = x(g - \dot{x}^2)$
c) $\ddot{x} = -gx$
d) $\ddot{x}(1 - x^2) = -x(g - \dot{x}^2)$

Statement for Linked Answer Questions 80 & 81:
Let $u(x,t)$ be the solution of the one dimensional wave equation

$$u_{tt} = 4u_{xx}, \quad -\infty < x < \infty,\ t > 0,$$

$$u(x,0) = \begin{cases} 16 - x^2, & |x| \leq 4, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad u_t(x,0) = \begin{cases} 1, & |x| \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

80) For $1 < t < 3$, $u(2,t) =$ GATE MA 2007

a) $\left[16 - (2-2t)^2\right]^+ + \frac{1}{2}\left[1 - \min\{1, t-1\}\right]$
b) $\left[32 - (2-2t)^2 - (2+2t)^2\right]^+ + t$
c) $\left[32 - (2-2t)^2 - (2+2t)^2\right]^+ + 1$
d) $\left[16 - (2-2t)^2\right]^+ + \frac{1}{2}\left[1 - \max\{1, t-1\}\right]$

81) The value of $u(2,2)$ GATE MA 2007

a) equals 15
b) equals 16
c) equals 0
d) does NOT exist

Statement for Linked Answer Questions 82 & 83: Suppose $E = \{(x,y) : xy \neq 0\}$.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 0, & \text{if } xy = 0, \\ y \sin\left(\frac{1}{x}\right) + x \sin\left(\frac{1}{y}\right), & \text{otherwise.} \end{cases}$$

Let S_1 be the set of points in $\mathbb{R}^2$ where f_x exists and S_2 be the set of points in $\mathbb{R}^2$ where f_y exists. Also, let E_1 be the set of points where f_x is continuous and E_2 be the set of points where f_y is continuous.

82) S_1 and S_2 are given by GATE MA 2007

a) $S_1 = E \cup \{(x,y) : y = 0\}, \quad S_2 = E \cup \{(x,y) : x = 0\}$
b) $S_1 = E \cup \{(x,y) : x = 0\}, \quad S_2 = E \cup \{(x,y) : y = 0\}$
c) $S_1 = S_2 = \mathbb{R}^2$
d) $S_1 = S_2 = E \cup \{(0,0)\}$

83) E_1 and E_2 are given by GATE MA 2007

a) $E_1 = S_1, \quad E_2 = S_1 \cap S_2$
b) $E_1 = S_1 \cap S_2 \setminus \{(0,0)\}, \quad E_2 = S_1$
c) $E_1 = S_2, \quad E_2 = S_1$
d) $E_1 = S_2, \quad E_2 = S_2$

Statement for Linked Answer Questions 84 & 85:
Let

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 2 \\ 0 & 2 & 6 \end{bmatrix}$$

and let $\lambda_1 \geq \lambda_2 \geq \lambda_3$ be the eigenvalues of A.

84) The triple $(\lambda_1, \lambda_2, \lambda_3)$ equals GATE MA 2007

a) $(9, 4, 2)$
b) $(8, 4, 3)$
c) $(9, 3, 3)$
d) $(7, 5, 3)$

85) The matrix P such that GATE MA 2007

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

is

a) $\begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$

b) $\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$

c) $\begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix}$

d) $\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix}$

END OF THE QUESTION PAPER