Problem 4.12.44

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Question

Question: Find the equation of the set of points which are equidistant from the points $\mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$.

Solution

Let $\mathbf{X} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be the position vector of any point equidistant from $\mathbf{A}$ and $\mathbf{B}$.

The condition for $\mathbf{X}$ to be equidistant is:

$$\|\mathbf{X} - \mathbf{A}\| = \|\mathbf{X} - \mathbf{B}\| \tag{2.1}$$

Squaring both sides we get:

$$(\mathbf{X} - \mathbf{A})^\top (\mathbf{X} - \mathbf{A}) = (\mathbf{X} - \mathbf{B})^\top (\mathbf{X} - \mathbf{B}) \tag{2.2}$$

Expanding,

$$\mathbf{X}^\top \mathbf{X} - 2\mathbf{A}^\top \mathbf{X} + \mathbf{A}^\top \mathbf{A} = \mathbf{X}^\top \mathbf{X} - 2\mathbf{B}^\top \mathbf{X} + \mathbf{B}^\top \mathbf{B} \tag{2.3}$$

Simplifying,

$$-2\mathbf{A}^\top \mathbf{X} + \mathbf{A}^\top \mathbf{A} = -2\mathbf{B}^\top \mathbf{X} + \mathbf{B}^\top \mathbf{B} \tag{2.4}$$

Rearranging,

$$2(\mathbf{B} - \mathbf{A})^\top \mathbf{X} = \mathbf{B}^\top \mathbf{B} - \mathbf{A}^\top \mathbf{A} \tag{2.5}$$

Solution

Calculate $\mathbf{B}-\mathbf{A}$:

$$\mathbf{B}-\mathbf{A} = \begin{pmatrix} 3-1 \\ 2-2 \\ -1-3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix} \tag{2.6}$$

Calculate $\mathbf{B}^\top\mathbf{B}$ and $\mathbf{A}^\top\mathbf{A}$:

$$\mathbf{B}^\top\mathbf{B} = 3^2 + 2^2 + (-1)^2 = 14, \quad \mathbf{A}^\top\mathbf{A} = 1^2 + 2^2 + 3^2 = 14 \tag{2.7}$$

Thus,

$$2\begin{pmatrix} 2 & 0 & -4 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 14 - 14 = 0 \tag{2.8}$$

Simplifying,

$$\begin{pmatrix} 4 & 0 & -8 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \tag{2.9}$$

Solution

This matrix equation represents the plane:

$$4a - 8c = 0 \tag{2.10}$$

or equivalently,

$$a - 2c = 0 \tag{2.11}$$

Final Answer: The set of points equidistant from **A** and **B** lies on the plane defined by

$$\boxed{\begin{pmatrix} 4 & 0 & -8 \end{pmatrix} \mathbf{x} = 0}$$

Graph

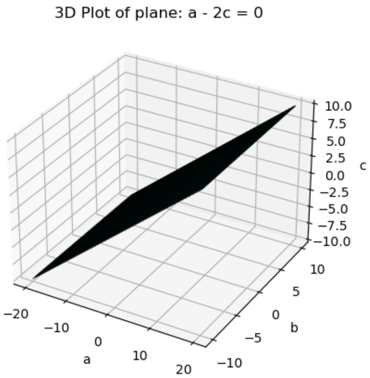

Figure: Vector Representation

C Code

```c
#include <stdio.h>
#include "matfun.h"

int main() {
    double A[3] = {1, 2, 3};
    double B[3] = {3, 2, -1};
    double BA[3];
    double rhs;

    // Compute B - A
    vector_subtract(B, A, BA, 3);

    // Compute dot products B.B and A.A
    double BTB = dot_product(B, B, 3);
    double ATA = dot_product(A, A, 3);

    rhs = BTB - ATA;
```

C Code

```c
    // Multiply BA by 2
    scalar_multiply(BA, BA, 2, 3);

    printf("Vector (2(B - A)) is: [%.2f, %.2f, %.2f]\n", BA[0],
        BA[1], BA[2]);
    printf("Right-hand side value (B.B - A.A): %.2f\n", rhs);

    printf("Equation of the plane in vector form: [%.2f %.2f %.2f
        ] \\cdot X = %.2f\n", BA[0], BA[1], BA[2], rhs/2);

    return 0;
}
```

Python Code for Plotting

```python
import matplotlib.pyplot as plt
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

# Create grid for b and c
b_vals = np.linspace(-10, 10, 100)
c_vals = np.linspace(-10, 10, 100)
b_grid, c_grid = np.meshgrid(b_vals, c_vals)

# Calculate corresponding a using the plane equation a = 2c
a_grid = 2 * c_grid

fig = plt.figure()
ax = fig.add_subplot(111, projection='3d')
```

Python Code for Plotting

```
# Plot the plane
ax.plot_surface(a_grid, b_grid, c_grid, alpha=0.5, color='cyan',
    edgecolor='k')

ax.set_xlabel('a')
ax.set_ylabel('b')
ax.set_zlabel('c')
ax.set_title('3D Plot of plane: a - 2c = 0')

plt.show()
```

Python Code - Using Shared Object

```python
import ctypes
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

# Load the shared library
matfun = ctypes.CDLL('./matfun.so')

# Define argument and return types
matfun.vector_subtract.argtypes = [
    np.ctypeslib.ndpointer(dtype=np.float64),
    np.ctypeslib.ndpointer(dtype=np.float64),
    np.ctypeslib.ndpointer(dtype=np.float64),
    ctypes.c_int
]
matfun.dot_product.argtypes = [
    np.ctypeslib.ndpointer(dtype=np.float64),
    np.ctypeslib.ndpointer(dtype=np.float64),
```

Python Code - Using Shared Object

```
    ctypes.c_int
]
matfun.dot_product.restype = ctypes.c_double
matfun.scalar_multiply.argtypes = [
    np.ctypeslib.ndpointer(dtype=np.float64),
    np.ctypeslib.ndpointer(dtype=np.float64),
    ctypes.c_double,
    ctypes.c_int
]

# Define points A and B
A = np.array([1.0, 2.0, 3.0])
B = np.array([3.0, 2.0, -1.0])

BA = np.zeros(3)
rhs = 0.0
```

Python Code - Using Shared Object

```python
# Compute B - A using shared lib
matfun.vector_subtract(B, A, BA, 3)

# Compute dot products B.B and A.A using shared lib
rhs = matfun.dot_product(B, B, 3) - matfun.dot_product(A, A, 3)

# Calculate 2*(B - A)
BA2 = np.zeros(3)
matfun.scalar_multiply(BA, BA2, 2, 3)

print(f"Vector 2(B - A): {BA2}")
print(f"RHS (B.B - A.A): {rhs}")

# Create grid for b and c
b_vals = np.linspace(-10, 10, 100)
c_vals = np.linspace(-10, 10, 100)
b_grid, c_grid = np.meshgrid(b_vals, c_vals)
```

Python Code - Using Shared Object

```
# From plane equation 2(B - A)^T X = rhs
# Which is BA2^T * X = rhs
# BA2 = [4, 0, -8], so the equation is 4a - 8c = rhs = 0
# => a = 2c

# Calculate corresponding a using a = 2c
a_grid = 2 * c_grid

fig = plt.figure()
ax = fig.add_subplot(111, projection='3d')

# Plot the plane surface
ax.plot_surface(a_grid, b_grid, c_grid, alpha=0.5, color='cyan',
    edgecolor='k')
```

Python Code - Using Shared Object

```
ax.set_xlabel('a')
ax.set_ylabel('b')
ax.set_zlabel('c')
ax.set_title('3D Plot of plane: a - 2c = 0 (from shared library)'
    )

plt.show()
```

Plot-Using Both C and Python