9.5.9

Puni Aditya - EE25BTECH11046

Question:
Find the value of p for which one root of the quadratic equation

$$px^2 - 14x + 8 = 0$$

is 6 times the other.
Solution:

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0$$
$$\mathbf{x} = \mathbf{h} + \kappa\mathbf{m}$$

$$\kappa_{1,2} = \frac{-\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u}) \pm \sqrt{(\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u}))^2 - (\mathbf{m}^\top\mathbf{V}\mathbf{m})\,g(\mathbf{h})}}{\mathbf{m}^\top\mathbf{V}\mathbf{m}} \tag{1}$$

$$\text{Parabola: } px^2 - 14x - y + 8 = 0$$
$$\text{Line: } \mathbf{e_2}^\top\mathbf{x} = 0$$

$$\mathbf{V} = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}, \ \mathbf{u} = \begin{pmatrix} -7 \\ -1/2 \end{pmatrix}, \ f = 8 \tag{2}$$

$$\mathbf{x} = \kappa\mathbf{e_1} \implies \mathbf{h} = \mathbf{0}, \ \mathbf{m} = \mathbf{e_1} \tag{3}$$

$$\mathbf{m}^\top\mathbf{V}\mathbf{m} = \mathbf{e_1}^\top\mathbf{V}\mathbf{e_1} = p \tag{4}$$
$$\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u}) = \mathbf{e_1}^\top\mathbf{u} = -7 \tag{5}$$
$$g(\mathbf{h}) = g(\mathbf{0}) = 8 \tag{6}$$

$$\kappa_{1,2} = \frac{-(-7) \pm \sqrt{(-7)^2 - (p)(8)}}{p} = \frac{7 \pm \sqrt{49 - 8p}}{p} \tag{7}$$

Let the intersection points be $\mathbf{x_1} = \kappa_1\mathbf{e_1}$ and $\mathbf{x_2} = \kappa_2\mathbf{e_1}$. The condition that one root is 6 times the other means one intersection point's position vector is 6 times the other's.

$$\mathbf{x_2} = 6\mathbf{x_1} \tag{8}$$
$$\kappa_2\mathbf{e_1} = 6(\kappa_1\mathbf{e_1}) \tag{9}$$
$$(\kappa_2 - 6\kappa_1)\mathbf{e_1} = \mathbf{0} \implies \kappa_2 = 6\kappa_1 \tag{10}$$

$$\frac{7+\sqrt{49-8p}}{p} = 6\left(\frac{7-\sqrt{49-8p}}{p}\right) \tag{11}$$

$$7+\sqrt{49-8p} = 42-6\sqrt{49-8p} \tag{12}$$

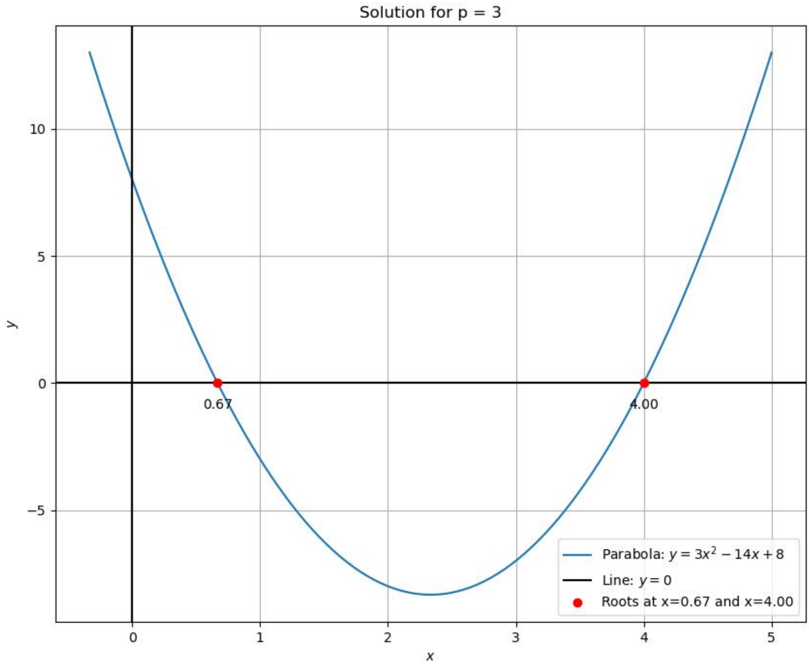
$$7\sqrt{49-8p} = 35 \tag{13}$$

$$\sqrt{49-8p} = 5 \tag{14}$$

$$49-8p = 25 \tag{15}$$

$$24 = 8p \tag{16}$$

$$p = 3 \tag{17}$$



Plot