# Matrices in Geometry 8.4.40

EE25BTECH11037 - Divyansh

**Question:** Let $\mathbf{P}$ be a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, 0 < b < a$. Let the line parallel to the X axis passing through $\mathbf{P}$ meet the circle $x^2 + y^2 = a^2$ at the point $\mathbf{Q}$ such that $\mathbf{P}$ and $\mathbf{Q}$ are on the same side of the X axis. For two positive real numbers $r$ and $s$, find the locus of the point $\mathbf{R}$ on $\mathbf{PQ}$ such that $PR = r$ as $\mathbf{P}$ varies over the ellipse.

**Solution:**
The given ellipse is

$$\mathbf{E} \ : \ \mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \ : \ \mathbf{V} = \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ , \ f = -a^2b^2 \tag{1}$$

$$\implies \mathbf{E} \ : \mathbf{x}^\top \mathbf{V} \mathbf{x} + f = 0 \tag{2}$$

The line parallel to the X-axis and passing through a point $\mathbf{P}$ on the ellipse is

$$\mathbf{L} \ : \ \mathbf{n}^\top \mathbf{x} = c \ : \ \mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ , \ c = y_P \tag{3}$$

$\mathbf{P}$ satisfies this line; therefore, $c = y_P$

$\mathbf{R}$ is a point on line $\mathbf{L}$ and at a distance $r$ from $\mathbf{P}$

$$\mathbf{R} - \mathbf{P} = r\mathbf{e_1} \implies \mathbf{P} = \mathbf{R} - r\mathbf{e_1} \ ; \ \mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4}$$

Since, $\mathbf{P}$ is a point on $\mathbf{E}$

$$\mathbf{P}^\top \mathbf{V} \mathbf{P} + f = 0 \tag{5}$$

Substituting $\mathbf{P} = \mathbf{Q} - r\mathbf{e_1}$

$$(\mathbf{R} - r\mathbf{e_1})^\top \mathbf{V} (\mathbf{R} - r\mathbf{e_1}) + f = 0 \implies \mathbf{R}^\top \mathbf{V} \mathbf{R} - 2r\mathbf{R}^\top \mathbf{V} \mathbf{e_1} + r^2 \mathbf{e_1}^\top \mathbf{V} \mathbf{e_1} + f = 0 \tag{6}$$

$$\mathbf{R} = \begin{pmatrix} x \\ y \end{pmatrix} \ , \ \mathbf{V} = \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \ , \ \mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ , \ f = -a^2b^2 \tag{7}$$

Thus the locus of the point $\mathbf{R}$ is

$$\mathbf{x}^\top \mathbf{V}' \mathbf{x} + 2\mathbf{u}'^\top \mathbf{x} + f' = 0 \ : \ \mathbf{V}' = \mathbf{V} \ , \ \mathbf{u}' = (-\mathbf{V}\mathbf{e_1}) \ , \ f' = f + r^2 \mathbf{e_1}^\top \mathbf{V} \mathbf{e_1} \tag{8}$$

Simplifying this equation, we get

$$b^2x^2 + a^2y^2 - 2b^2xr + b^2r^2 - a^2b^2 = 0 \tag{9}$$

$$b^2\left(x^2 - 2xr + r^2\right) + a^2y^2 = a^2b^2 \tag{10}$$

$$\frac{(x-r)^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{11}$$

This is the equation of locus of the point $\mathbf{R}$, which is an ellipse. Let us try to draw the locus for $a = 4, b = 2, r = 1$

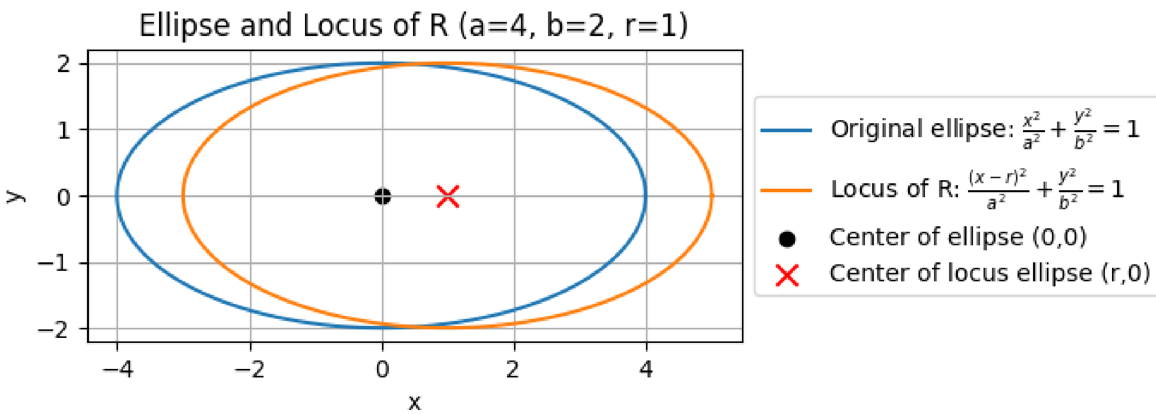


Fig. 1: Figure for 8.4.40 for $a = 4, b = 2, r = 1$