

2.10.33

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Question

Let α, β, γ be distinct real numbers. The points with position vectors $\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$, $\beta\hat{i} + \gamma\hat{j} + \alpha\hat{k}$, $\gamma\hat{i} + \alpha\hat{j} + \beta\hat{k}$

1. are collinear
2. form an equilateral triangle
3. form a scalene triangle
4. form a right angled triangle

Solution

Let \mathbf{A} be $\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$, \mathbf{B} be $\begin{pmatrix} \beta \\ \gamma \\ \alpha \end{pmatrix}$, and \mathbf{C} be $\begin{pmatrix} \gamma \\ \alpha \\ \beta \end{pmatrix}$.

First, we need to check when the three points are collinear. We can do this using the collinearity matrix:

$$(\mathbf{C} - \mathbf{A} \quad \mathbf{B} - \mathbf{A})^T \quad (1)$$

If the rank of the matrix is 1, then the points are collinear.

$$\begin{pmatrix} \gamma - \alpha & \alpha - \beta & \beta - \gamma \\ \beta - \alpha & \gamma - \beta & \alpha - \gamma \end{pmatrix} \quad (2)$$

The rank of this matrix will be 1 only when all the elements in the bottom row of the matrix are equal to 0. This occurs only when $\alpha = \beta = \gamma$, which contradicts

the fact that α, β, γ are distinct.

Therefore the points must be non-collinear and form a triangle.

The sides of the triangle are $\mathbf{A} - \mathbf{B}$, $\mathbf{B} - \mathbf{C}$, $\mathbf{C} - \mathbf{A}$.

$$\mathbf{A} - \mathbf{B} \text{ is } \begin{pmatrix} \alpha - \beta \\ \beta - \gamma \\ \gamma - \alpha \end{pmatrix} \quad (3)$$

$$\mathbf{B} - \mathbf{C} \text{ is } \begin{pmatrix} \beta - \gamma \\ \gamma - \alpha \\ \alpha - \beta \end{pmatrix} \quad (4)$$

$$\mathbf{C} - \mathbf{A} \text{ is } \begin{pmatrix} \gamma - \alpha \\ \alpha - \beta \\ \beta - \gamma \end{pmatrix} \quad (5)$$

$\|\mathbf{A} - \mathbf{B}\|, \|\mathbf{B} - \mathbf{C}\|, \|\mathbf{C} - \mathbf{A}\|$ are all equal, and equal to

$$\sqrt{(\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2}$$

The three points therefore form an **equilateral triangle**, so option (2) is correct.

For example, let us take $\alpha = 2, \beta = 1, \gamma = 3$. We get an equilateral triangle as shown below:

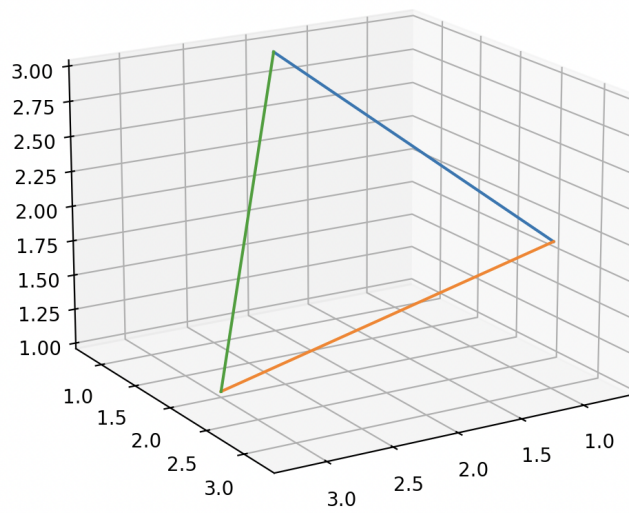


Figure 1: Equilateral Triangle