Problem 12.318

Let V be the vector space of all real polynomials of degree at most 20. Define the subspaces

$$W_1 = \{p \in V : p(1) = p(\tfrac{1}{2}) = p(5) = p(7) = 0\}, \quad W_2 = \{p \in V : p(\tfrac{1}{2}) = p(3) = p(4) = p(7) = 0\}. \tag{1}$$

Find $\dim(W_1 \cap W_2)$.

Input Variables

Symbol	Description
$p(x)$	Polynomial of degree ≤ 20
c_i	Coefficients of $p(x)$
a	Point of evaluation (root condition)
A	Constraint matrix from evaluations

Table 1

Definitions

Vector Space: A set V together with two operations (vector addition and scalar multiplication) is called a vector space over the field $\mathbb{R}$ if for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and scalars $a, b \in \mathbb{R}$, the following conditions hold:

- Closure under addition: $\mathbf{u} + \mathbf{v} \in V$.
- Commutativity: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- Associativity: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
- Existence of zero vector: $\exists \mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
- Existence of additive inverse: $\forall \mathbf{u} \in V$, $\exists (-\mathbf{u}) \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
- Closure under scalar multiplication: $a\mathbf{u} \in V$.
- Distributivity: $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ and $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$.
- Compatibility: $a(b\mathbf{u}) = (ab)\mathbf{u}$.
- Identity: $1 \cdot \mathbf{u} = \mathbf{u}$.

Subspace: A subset $W \subseteq V$ is called a subspace of V if:

1. $\mathbf{0} \in W$ (contains the zero vector),
2. If $\mathbf{u}, \mathbf{v} \in W$, then $\mathbf{u} + \mathbf{v} \in W$ (closed under addition),
3. If $\mathbf{u} \in W$ and $\alpha \in \mathbb{R}$, then $\alpha\mathbf{u} \in W$ (closed under scalar multiplication).

Dimension of a Subspace: The dimension of a subspace W of V is the number of vectors in a basis of W, i.e.,

$$\dim(W) = \text{number of linearly independent vectors that span } W.$$

Solution

Step 1: Represent the polynomial

$$p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{20} x^{20}, \tag{2}$$

$$\mathbf{c} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{20} \end{pmatrix} \in \mathbb{R}^{21}. \tag{3}$$

Step 2: Each condition $p(a) = 0$ gives

$$p(a) = \begin{pmatrix} 1 & a & a^2 & \cdots & a^{20} \end{pmatrix} \mathbf{c} = 0. \tag{4}$$

Step 3: For the intersection $W_1 \cap W_2$, the polynomial must vanish at

$$\{1, \tfrac{1}{2}, 5, 7, 3, 4\}.$$

Thus we obtain the matrix equation

$$A\mathbf{c} = \mathbf{0}, \quad \text{where} \tag{5}$$

$$A = \begin{pmatrix} 1 & 1 & 1^2 & \cdots & 1^{20} \\ 1 & \frac{1}{2} & (\frac{1}{2})^2 & \cdots & (\frac{1}{2})^{20} \\ 1 & 5 & 5^2 & \cdots & 5^{20} \\ 1 & 7 & 7^2 & \cdots & 7^{20} \\ 1 & 3 & 3^2 & \cdots & 3^{20} \\ 1 & 4 & 4^2 & \cdots & 4^{20} \end{pmatrix}. \tag{6}$$

Step 4: The system $A\mathbf{c} = \mathbf{0}$ is a homogeneous system with 21 unknowns and 6 independent equations. Hence the number of free variables is

$$21 - 6 = 15. \tag{7}$$

Final Answer:

$$\dim(W_1 \cap W_2) = 15 \tag{8}$$

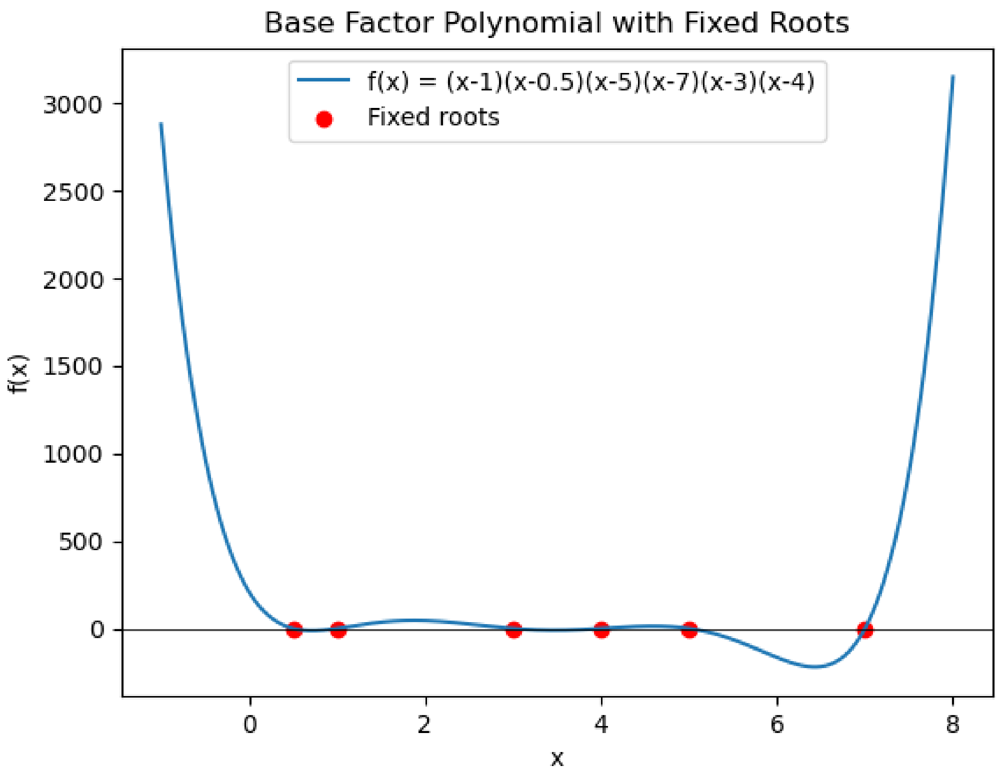

Figure 1