7.4.32

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Question

$ABCD$ is a square of side length 2 units. C_1 is the circle touching all the sides of the square $ABCD$ and C_2 is the circumcircle of square $ABCD$. L is a fixed line in same plane and **R** is a fixed point.

1. If **P** is any point of C_1 and **Q** is another point on C_2, then $\frac{PA^2+PB^2+PC^2+PD^2}{QA^2+QB^2+QC^2+QD^2}$
    1. 0.75
    2. 1.25
    3. 1
    4. 0.5

1. If a circle is such that it touches the line L and the circle C_1 externally, such that both the circles are on the same side of the line, then locus of centre of the circle
    1. ellipse
    2. hyperbola
    3. parabola
    4. circle
2. A L' through **A** is drawn parallel to BD. Point S moves such that its distances from the line BD and the vertex **A** are equal. If locus of S cuts L' at T_2 and T_3 and AC at T_1, then area of $\triangle T_1 T_2 T_3$ is
    1. $1/2$ sq.units
    2. $2/3$ sq.units
    3. 1 sq.units
    4. 2 sq.units

Let

Let:

The centre of incircle and circumcircle be $\mathbf{O}$.

The radius of incircle be r_1 and that of circumcircle be r_2.

Given:

$$r_1 = 1 \tag{1}$$

$$r_2 = \sqrt{2} \tag{2}$$

Let **P** be any point on incircle and **Q** be any point on circumcircle.
$X \epsilon \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$

$$|\mathbf{X} - \mathbf{P}|^2 = |\mathbf{X}|^2 + |\mathbf{P}|^2 - 2\mathbf{P}.\mathbf{X} \tag{3}$$

$$\tag{4}$$

Summation over all $\mathbf{X} | \mathbf{X} \epsilon \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$:

$$\sum |\mathbf{X} - \mathbf{P}|^2 = \sum |\mathbf{X}|^2 + 4.|\mathbf{P}|^2 - 2\mathbf{P} \sum \mathbf{X} \tag{5}$$

$$\tag{6}$$

For, $\mathbf{P} = \mathbf{P}$

$$\sum |\mathbf{X} - \mathbf{P}|^2 = \sum |\mathbf{X}|^2 + 4.|\mathbf{P}|^2 - 2\mathbf{P}\sum \mathbf{X} \tag{7}$$

$$4\left(1^2 + 1^2\right) + 4\,(1) - 2\mathbf{P}\left(\begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}1\\-1\end{pmatrix} + \begin{pmatrix}-1\\1\end{pmatrix} + \begin{pmatrix}-1\\-1\end{pmatrix}\right) \tag{8}$$

$$\therefore |\mathbf{A} - \mathbf{P}|^2 + |\mathbf{B} - \mathbf{P}|^2 + |\mathbf{C} - \mathbf{P}|^2 + |\mathbf{D} - \mathbf{P}|^2 = 12 \tag{9}$$

1st question

For, $\mathbf{P} = \mathbf{Q}$

$$\sum |\mathbf{X} - \mathbf{Q}|^2 = \sum |\mathbf{X}|^2 + 4.|\mathbf{Q}|^2 - 2\mathbf{Q} \sum \mathbf{X} \tag{10}$$

$$4\left(1^2 + 1^2\right) + 4\left(2\right) - 2\mathbf{Q}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix}\right) \tag{11}$$

$$\therefore |\mathbf{A} - \mathbf{Q}|^2 + |\mathbf{B} - \mathbf{Q}|^2 + |\mathbf{C} - \mathbf{Q}|^2 + |\mathbf{D} - \mathbf{Q}|^2 = 12 \tag{12}$$

conclusion:

$$\frac{12}{16} = 0.75 \tag{13}$$

Hence, option(a) is correct.

Let the radius of the moving circle be r, the centre of the circle be $\mathbf{X}$ and the line equation be $\hat{\mathbf{n}}^\top \mathbf{X} = c$,

2nd question

$$|\hat{\mathbf{n}}^\top \mathbf{X} - c| = r \tag{14}$$

$$||\mathbf{X}|| = r + 1 \tag{15}$$

$$||\mathbf{X}|| = |\hat{\mathbf{n}}\mathbf{X} - (c-1)\,| \tag{16}$$

$$||\mathbf{X}||^2 = |\hat{\mathbf{n}}\mathbf{X} - (c-1)\,|^2 \tag{17}$$

$$\mathbf{X}^\top \mathbf{X} = \left(\hat{\mathbf{n}}^\top \mathbf{X}\right)^2 + (c-1)^2 - 2\hat{\mathbf{n}}^\top \mathbf{X}\,(c-1) \tag{18}$$

$$\mathbf{X}^\top \mathbf{X} - (\hat{\mathbf{n}}\mathbf{X})^2 + 2\hat{\mathbf{n}}^\top \mathbf{X}\,(c-1) - (c-1)^2 \tag{19}$$

$$\mathbf{X}^\top \left(I - \hat{\mathbf{n}}\hat{\mathbf{n}}^\top\right) \mathbf{X} + 2\,(c-1)\,\hat{\mathbf{n}}^\top \mathbf{X} - (c-1)^2 \tag{20}$$

2nd question

Equation (20) is the equation of parabola.
Hence, correct option is (c).

Let the point moving point be S and the line equation be $\mathbf{n}^\top \mathbf{S} = 0$.

$$\frac{|\mathbf{n}^\top \mathbf{S}|}{||\mathbf{n}||} = ||\mathbf{S} - \mathbf{A}|| \tag{21}$$

$$\frac{|\mathbf{n}^\top \mathbf{S}|^2}{||\mathbf{n}||^2} = ||\mathbf{S} - \mathbf{A}||^2 \tag{22}$$

$$\frac{|\mathbf{S}^\top \mathbf{n}\mathbf{n}^\top \mathbf{S}|}{||\mathbf{n}||} = (\mathbf{S} - \mathbf{A})^\top (\mathbf{S} - \mathbf{A}) \tag{23}$$

$$\mathbf{S}^\top \left(I - \hat{\mathbf{n}}\hat{\mathbf{n}}^\top \right) \mathbf{S} - 2\mathbf{A}^\top \mathbf{S} + \mathbf{A}^\top \mathbf{A} = 0 \tag{24}$$

3rd question

Equation 24 is the locus of the moving point.
Let:

$$\mathbf{A} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \mathbf{B} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{25}$$

$$\mathbf{C} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}; \mathbf{D} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{26}$$

$\mathbf{m_1}$ be the direction vector of line AC, $\mathbf{m_2}$ be the direction vector of line L'.

$$\mathbf{m_1} = \begin{pmatrix} -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{27}$$

$$\mathbf{m_2} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \tag{28}$$

3rd question

The equation of line L'

$$\mathbf{S} = \mathbf{A} + t\mathbf{m_2} \tag{29}$$

The equation of the line AC

$$\mathbf{S} = \lambda\mathbf{m_1} \tag{30}$$

Substituting equation (30) in (24) we get $\lambda = \frac{1}{2}$:

$$\mathbf{T_1} = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} \tag{31}$$

Substituting (29) in (24) we get $t = \frac{-1}{2}, \frac{1}{2}$

$$\mathbf{T_2} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{32}$$

$$\mathbf{T_3} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{33}$$

Now, finding area of the triangle:

$$\triangle T_1 T_2 T_3 = \frac{1}{2} \left| \begin{pmatrix} \mathbf{T_2} - \mathbf{T_1} & \mathbf{T_3} - \mathbf{T_2} \end{pmatrix} \right| \tag{34}$$

$$\triangle T_1 T_2 T_3 = 1 \tag{35}$$

Option (c) is correct.

C Code

```c
#include <stdio.h>
#include <math.h>
void get_results(double *out_data) {
    double side = 2.0;
    double r1 = side / 2.0;
    double r2 = side / sqrt(2.0);
    double ratio = (r1 * r1 + r1 * r1 + r1 * r1 + r1 * r1) /
                   (r2 * r2 + r2 * r2 + r2 * r2 + r2 * r2);
    double k = 2.0;
    double focus_x = 0.0;
    double focus_y = k + r1;
    double directrix_y = k - r1;
    double area = 2.0 / 3.0;
```

C code

```c
double a = 1;
double b = -1;
out_data[0] = ratio;
out_data[1] = focus_x;
out_data[2] = focus_y;
out_data[3] = directrix_y;
out_data[4] = area;
out_data[5] = a; //Ax
out_data[6] = a; //Ay
out_data[7] = b; //Bx
out_data[8] = a; //By
out_data[9] = b; //Cx
out_data[10] = b; //Cy
out_data[11] = a; //Dx
out_data[12] = b; //Dy
out_data[13] = 0.5; //T1x
out_data[14] = 0.5; //T1y;
out_data[15] = 0; //T2x;
out_data[16] = 2; //T2y
```

call.py

```
import ctypes as ct
import numpy as np
def get_results():
    lib = ct.CDLL("./problem.so")
    lib.get_results.argtypes = [ct.POINTER(ct.c_double)]
    data_type = ct.c_double * 19
    data = data_type()
    lib.get_results(data)
    arr = np.array(list(data))
    ratio, focus_x, focus_y, directrix_y, area, Ax, Ay, Bx, By,
        Cx, Cy, Dx, Dy, T1x, T1y, T2x, T2y, T3x, T3y= arr
    return ratio, np.array([focus_x, focus_y]), directrix_y, area
        , Ax, Ay, Bx, By, Cx, Cy, Dx, Dy, T1x, T1y, T2x, T2y, T3x
        , T3y
```

```python
import numpy as np
import matplotlib.pyplot as plt
from call import get_results
ratio, focus, directrix_y, area, Ax, Ay, Bx, By, Cx, Cy, Dx, Dy,
    T1x, T1y, T2x, T2y, T3x, T3y = get_results()
theta = np.linspace(0, 2*np.pi, 200)
A = ([Ax, Bx, Cx, Dx, Ax, Bx])
B = ([Ay, By, Cy, Dy, Ay, By])
plt.plot(A, B, color='black')
plt.text(Ax+0.1, Ay+0.1, "A", fontsize = 10, color = 'black')
plt.text(Bx+0.1, By+0.1, "B", fontsize = 10, color = 'black')
```

```
plt.text(Cx+0.1, Cy+0.1, "C", fontsize = 10, color = 'black')
plt.text(Dx+0.1, Dy+0.1, "D", fontsize = 10, color = 'black')
plt.plot(np.cos(theta), np.sin(theta), label="Inner Circle C1")
plt.plot(np.sqrt(2)*np.cos(theta), np.sqrt(2)*np.sin(theta),
    label="Outer Circle C2")
plt.axis("equal")
plt.grid(True)
plt.savefig("../figs/plot1.png")
plt.show()
```

```
import numpy as np
import matplotlib.pyplot as plt
from call import get_results
ratio, focus, directrix_y, area, Ax, Ay, Bx, By, Cx, Cy, Dx, Dy,
    T1x, T1y, T2x, T2y, T3x, T3y = get_results()
A = ([Ax, Bx, Cx, Dx, Ax, Bx])
B = ([Ay, By, Cy, Dy, Ay, By])
x = np.linspace(-5, 5, 300)
p = focus[1] - directrix_y
y = (x**2) / (4*p)
plt.plot(x, y, label="Locus (Parabola)")
plt.axhline(directrix_y, color='r', linestyle='--', label='
    Directrix')
plt.plot(focus[0], focus[1], 'go')
```

```
plt.text(Ax+0.1, Ay+0.1, "A", fontsize = 10, color = 'black')
plt.text(Bx+0.1, By+0.1, "B", fontsize = 10, color = 'black')
plt.text(Cx+0.1, Cy+0.1, "C", fontsize = 10, color = 'black')
plt.text(Dx+0.1, Dy+0.1, "D", fontsize = 10, color = 'black')
plt.text(focus[0]+0.1, focus[1]+0.1, 'focus', fontsize = 10,
    color = 'black')
plt.axis("equal")
plt.grid(True)
plt.savefig("../figs/plot2.png")
plt.show()
```

plot3.py

```
import numpy as np
import matplotlib.pyplot as plt
from call import get_results
ratio, focus, directrix_y, area, Ax, Ay, Bx, By, Cx, Cy, Dx, Dy,
    T1x, T1y, T2x, T2y, T3x, T3y = get_results()
A = ([Ax, Bx, Cx, Dx, Ax, Bx])
B = ([Ay, By, Cy, Dy, Ay, By])
C = ([T1x, T2x, T3x, T1x, T2x])
D = ([T1y, T2y, T3y, T1y, T2y])
triangle = np.array([[0,0], [2,0], [1, np.sqrt(3)/2]])
plt.plot(A, B, color='black')
```

```
plt.text(Ax+0.1, Ay+0.1, "A", fontsize = 10, color = 'black')
plt.text(Bx+0.1, By+0.1, "B", fontsize = 10, color = 'black')
plt.text(Cx+0.1, Cy+0.1, "C", fontsize = 10, color = 'black')
plt.text(Dx+0.1, Dy+0.1, "D", fontsize = 10, color = 'black')
plt.plot(C, D, color='black')
plt.text(T1x+0.1, T1y+0.1, "T1", fontsize = 10, color = 'black')
plt.text(T2x+0.1, T2y+0.1, "T2", fontsize = 10, color = 'black')
plt.text(T3x+0.1, T3y+0.1, "T3", fontsize = 10, color = 'black')
plt.axis("equal")
plt.grid(True)
plt.savefig("../figs/plot3.png")
plt.show()
```

Plot

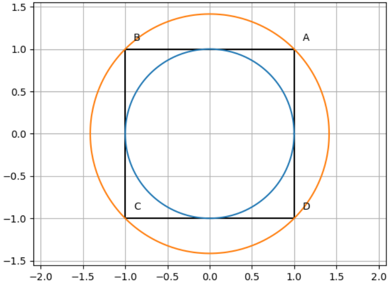

Figure: Plot of the given square and circles

Plot

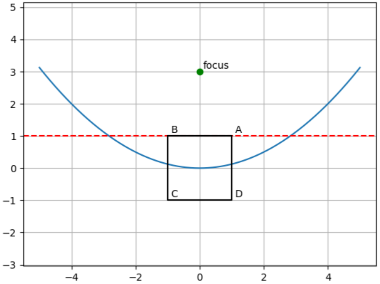

Figure: Plot of the given circles, square and locus of the point

Plot

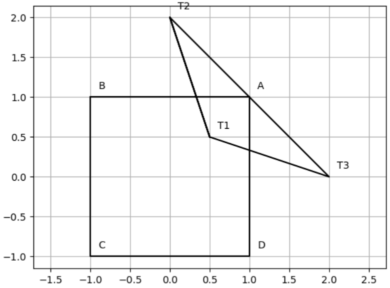

Figure: Plot of the given circles, square and locus of the point