4.7.8

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Problem Statement
Find the shortest distance between the lines

$$\mathbf{l_1} : \mathbf{r}_1 = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \lambda(2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}), \quad \mathbf{l_2} : \mathbf{r}_2 = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}} + \mu(3\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) \tag{4.7.8.1}$$

Solution:
In this case, the given lines are

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \kappa_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \tag{4.7.8.2}$$

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \kappa_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \tag{4.7.8.3}$$

with

$$\mathbf{A} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} \tag{4.7.8.4}$$

Then,

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{4.7.8.5}$$

Calculating the rank of matrix $\begin{pmatrix} \mathbf{M} & \mathbf{B} - \mathbf{A} \end{pmatrix}$,

$$\begin{pmatrix} 2 & 3 & 1 \\ -1 & -5 & 0 \\ 1 & 2 & -1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 1 & 2 & -1 \\ -1 & -5 & 0 \\ 2 & 3 & 1 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 + R_1} \begin{pmatrix} 1 & 2 & -1 \\ 0 & -3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 0 & -3 & -1 \end{pmatrix} \xrightarrow{R_2 \to -R_2} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -3 \\ 0 & -3 & -1 \end{pmatrix} \xrightarrow[R_3 \to R_3 + 3R_2]{R_1 \to R_1 - 2R_2} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & -10 \end{pmatrix}$$

$$\xrightarrow{R_3 \to -\frac{1}{10} R_3} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_2 \to R_2 + 3R_3]{R_1 \to R_1 - 5R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Clearly, the rank of this matrix is 3, and therefore, the lines are skew.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 2 & -1 & 1 \\ 3 & -5 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 13 \\ 13 & 38 \end{pmatrix} \tag{4.7.8.6}$$

$$\mathbf{M}^T(\mathbf{B} - \mathbf{A}) = \begin{pmatrix} 2 & -1 & 1 \\ 3 & -5 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4.7.8.7}$$

We solve the least squares solution $\mathbf{M}^\top\mathbf{M}\begin{pmatrix} \kappa_1 & -\kappa_2 \end{pmatrix} = \mathbf{M}^\top(\mathbf{B}-\mathbf{A})$ using augmented matrix,

$$\begin{pmatrix} 6 & 13 & 1 \\ 13 & 38 & 1 \end{pmatrix} \xrightarrow{R_2 \to 6R_2 - 13R_1} \begin{pmatrix} 6 & 13 & 1 \\ 0 & 59 & -7 \end{pmatrix} \xrightarrow{R_1 \to 59R_1 - 13R_2} \begin{pmatrix} 354 & 0 & 150 \\ 0 & 59 & -7 \end{pmatrix}$$
$$\xrightarrow[R_2 \to R_2/59]{R_1 \to R_1/354} \begin{pmatrix} 1 & 0 & 150/354 \\ 0 & 1 & -7/59 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 25/59 \\ 0 & 1 & -7/59 \end{pmatrix}$$

yielding

$$\begin{pmatrix} \kappa_1 \\ -\kappa_2 \end{pmatrix} = \frac{1}{59}\begin{pmatrix} 25 \\ -7 \end{pmatrix} \implies \kappa_1 = \frac{25}{59}, \quad \kappa_2 = \frac{7}{59} \tag{4.7.8.8}$$

Substituting the above in (4.7.8.2) and (4.7.8.3),

$$\mathbf{x_1} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \frac{25}{59}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{59}\begin{pmatrix} 109 \\ 34 \\ 25 \end{pmatrix} \tag{4.7.8.9}$$

$$\mathbf{x_2} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \frac{7}{59}\begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = \frac{1}{59}\begin{pmatrix} 139 \\ 24 \\ -45 \end{pmatrix} \tag{4.7.8.10}$$

Thus, the required distance is

$$\|\mathbf{x_2} - \mathbf{x_1}\| = \left\| \frac{1}{59}\begin{pmatrix} 30 \\ -10 \\ -70 \end{pmatrix} \right\| = \frac{\sqrt{30^2 + (-10)^2 + (-70)^2}}{59} = \frac{\sqrt{5900}}{59} = \frac{10\sqrt{59}}{59} = \frac{10}{\sqrt{59}} \tag{4.7.8.11}$$

Shortest distance between the given lines is:

$$d = \frac{10}{\sqrt{59}} \tag{4.7.8.12}$$

See Figure 4.7.8.1.

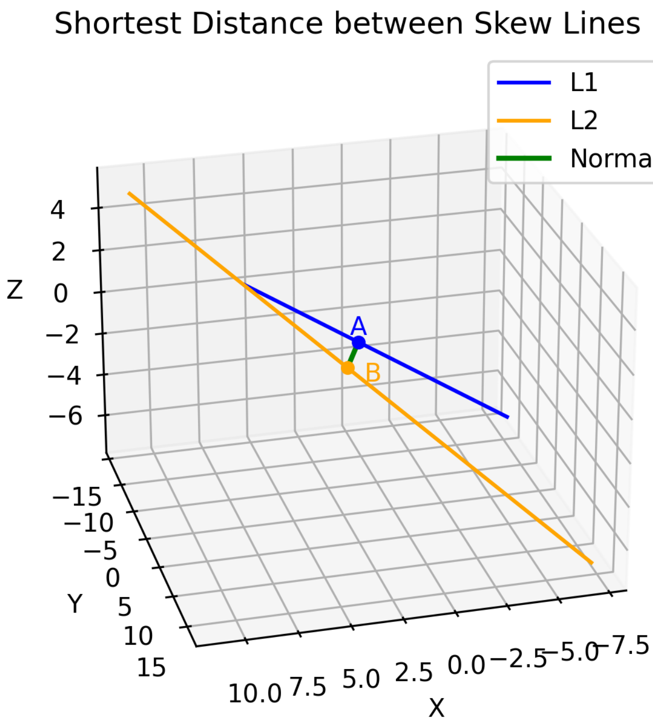
Shortest Distance between Skew Lines
L1
L2
Normal
A
B
Z
4
2
0
−2
−4
−6
Y
−15
−10
−5
0
5
10
15
X
10.0
7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5

Fig. 4.7.8.1