4.13.8

EE25BTECH11003 - Adharvan Kshathriya Bommagani

Question:
The orthocentre of the triangle formed by the lines $x+y=1$, $2x+3y=6$ and $4x-y+4=0$ lies in the quadrant number ______.

Solution:

The three lines, written in the vector normal form $\mathbf{n}^\top \mathbf{x} = c$, are:

$$L_1 : \begin{pmatrix} 1 \\ 1 \end{pmatrix}^\top \begin{pmatrix} x \\ y \end{pmatrix} = 1 \tag{1}$$

$$L_2 : \begin{pmatrix} 2 \\ 3 \end{pmatrix}^\top \begin{pmatrix} x \\ y \end{pmatrix} = 6 \tag{2}$$

$$L_3 : \begin{pmatrix} 4 \\ -1 \end{pmatrix}^\top \begin{pmatrix} x \\ y \end{pmatrix} = -4 \tag{3}$$

The vertices **A**, **B**, **C** are the intersection points of these lines. We solve for them using gaussian elimination(row reduction).

Vertex **A**: Intersection of L_1 and L_2 The system is: $x+y=1$ and $2x+3y=6$. Augmented matrix:

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 3 & 6 \end{array}\right) \tag{4}$$

Apply the row operation $R_2 \to R_2 - 2R_1$:

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 4 \end{array}\right) \tag{5}$$

From the second row, $y = 4$. Substituting into the first row $(x + y = 1)$, we get

$$x + 4 = 1 \implies x = -3. \tag{6}$$

$$\mathbf{A} = \begin{pmatrix} -3 \\ 4 \end{pmatrix} \tag{7}$$

Vertex **B**: Intersection of L_2 and L_3 The system is: $2x + 3y = 6$ and $4x - y = -4$. Augmented matrix:

$$\left(\begin{array}{cc|c} 2 & 3 & 6 \\ 4 & -1 & -4 \end{array}\right) \tag{8}$$

Apply the row operation $R_2 \to R_2 - 2R_1$:

$$\left(\begin{array}{cc|c} 2 & 3 & 6 \\ 0 & -7 & -16 \end{array}\right) \tag{9}$$

From the second row,

$$-7y = -16 \implies y = \frac{16}{7}. \tag{10}$$

Substituting into the first row $(2x + 3y = 6)$, we get

$$2x + 3(\frac{16}{7}) = 6 \implies 2x = 6 - \frac{48}{7} = -\frac{6}{7} \implies x = -\frac{3}{7}. \tag{11}$$

$$\mathbf{B} = \begin{pmatrix} -\frac{3}{7} \\ \frac{16}{7} \end{pmatrix} \tag{12}$$

Vertex **C** : Intersection of L_1 and L_3 The system is: $x + y = 1$ and $4x - y = -4$. Augmented matrix:

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 4 & -1 & -4 \end{array}\right) \tag{13}$$

Apply the row operation $R_2 \to R_2 - 4R_1$:

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -5 & -8 \end{array}\right) \tag{14}$$

From the second row,

$$-5y = -8 \implies y = \frac{8}{5} \tag{15}$$

Substituting into the first row $(x + y = 1)$, we get,

$$x + \frac{8}{5} = 1 \implies x = 1 - \frac{8}{5} = -\frac{3}{5} \tag{16}$$

$$\mathbf{C} = \begin{pmatrix} -\frac{3}{5} \\ \frac{8}{5} \end{pmatrix} \tag{17}$$

Altitude from Vertex A (AD):
This altitude passes through $\mathbf{A}=\begin{pmatrix} -3 \\ 4 \end{pmatrix}$ and is perpendicular to line L_3.
A normal vector for the altitude is therefore,

$$\begin{pmatrix} 1 \\ 4 \end{pmatrix}. \tag{18}$$

The equation is

$$x + 4y = k. \tag{19}$$

Since it passes through $\mathbf{A}=\begin{pmatrix}-3\\4\end{pmatrix}$:

$$(-3)+4(4)=k \implies k=13. \tag{20}$$

Equation of Altitude AD:

$$\begin{pmatrix}1\\4\end{pmatrix}^\top \begin{pmatrix}x\\y\end{pmatrix}=13. \tag{21}$$

Altitude from Vertex C (CE):
This altitude passes through $\mathbf{C}=\begin{pmatrix}-\frac{3}{5}\\\frac{8}{5}\end{pmatrix}$ and is perpendicular to line L_2. A normal vector for the altitude is therefore,

$$\begin{pmatrix}3\\-2\end{pmatrix} \tag{22}$$

The equation is

$$3x-2y=k \tag{23}$$

Since it passes through $\mathbf{C}=\begin{pmatrix}-\frac{3}{5}\\\frac{8}{5}\end{pmatrix}$:

$$3(-\frac{3}{5})-2(\frac{8}{5})=k \implies k=-\frac{9}{5}-\frac{16}{5}=-5 \tag{24}$$

Equation of Altitude CE:

$$\begin{pmatrix}3\\-2\end{pmatrix}^\top \begin{pmatrix}x\\y\end{pmatrix}=-5 \tag{25}$$

We solve the system for the two altitudes: $x+4y=13$ and $3x-2y=-5$. Augmented matrix:

$$\left(\begin{array}{cc|c}1 & 4 & 13\\3 & -2 & -5\end{array}\right) \tag{26}$$

Apply the row operation $R_2 \to R_2 - 3R_1$:

$$\left(\begin{array}{cc|c}1 & 4 & 13\\0 & -14 & -44\end{array}\right) \tag{27}$$

From the second row,

$$-14y=-44 \implies y=\frac{44}{14}=\frac{22}{7} \tag{28}$$

Substituting into the first row ($x+4y=13$):

$$x+4(\tfrac{22}{7})=13 \implies x=13-\frac{88}{7}=\frac{91-88}{7}=\frac{3}{7} \tag{29}$$

The orthocentre is

$$\mathbf{H}=\left(\tfrac{3}{7},\tfrac{22}{7}\right) \tag{30}$$

The coordinates of the orthocentre are $(\frac{3}{7}, \frac{22}{7})$. Since both the x-coordinate and y-coordinate are positive, the orthocentre lies in the first quadrant.

Plot of the Lines and Orthocentre:

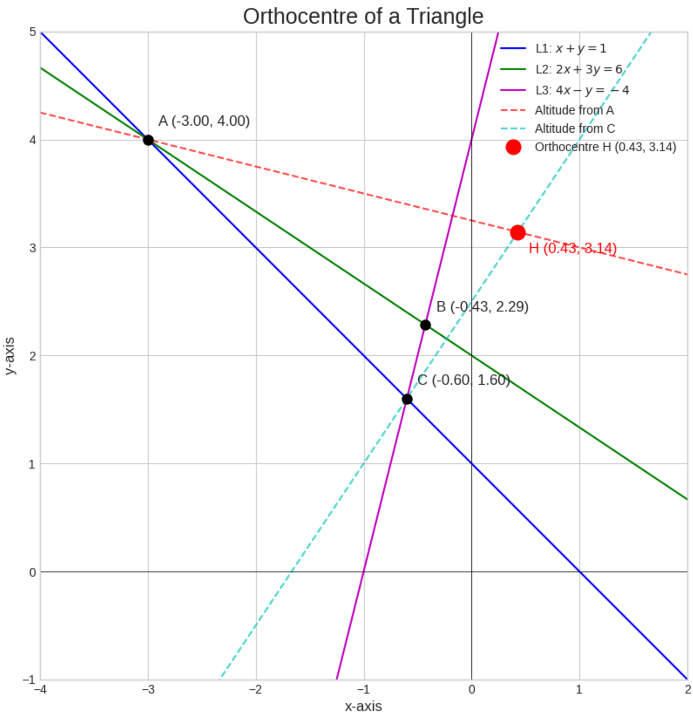

Fig. 0: Figure for 4.13.8