1.3.4

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Question 1.3.4
If $A(1,3)$, $B(4,2)$, $C(x,5)$, and $D(x,4)$ are the vertices of a parallelogram $ABCD$, then the value of x is _____. (10, 2012)

Solution:

In a parallelogram, opposite sides are equal and parallel. Since $ABCD$ is a parallelogram, vectors **AB** and **CD** must be equal.

$$\mathbf{AB} = \mathbf{B} - \mathbf{A} = \begin{pmatrix} 4-1, 2-3 \end{pmatrix} = \begin{pmatrix} 3, -1 \end{pmatrix} \tag{0.1}$$

$$\mathbf{CD} = \mathbf{D} - \mathbf{C} = \begin{pmatrix} x-x, 4-5 \end{pmatrix} = \begin{pmatrix} 0, -1 \end{pmatrix} \tag{0.2}$$

Clearly, $\mathbf{AB} \neq \mathbf{CD}$, so let's try using diagonals. In a parallelogram, the diagonals bisect each other.

Midpoint of diagonal AC:

$$\mathbf{M_{AC}} = \begin{pmatrix} \frac{1+x}{2} \\ \frac{3+5}{2} \end{pmatrix} = \begin{pmatrix} \frac{1+x}{2} \\ 4 \end{pmatrix} \tag{0.3}$$

Midpoint of diagonal BD:

$$\mathbf{M_{BD}} = \begin{pmatrix} \frac{4+x}{2} \\ \frac{2+4}{2} \end{pmatrix} = \begin{pmatrix} \frac{4+x}{2} \\ 3 \end{pmatrix} \tag{0.4}$$

Equating midpoints:

$$\frac{1+x}{2} = \frac{4+x}{2} \quad \text{and} \quad 4 = 3 \tag{0.5}$$

The second equation is false, so diagonals do not bisect each other. Let's try using opposite sides again, but this time equating **AD** and **BC**:

$$\mathbf{AD} = \mathbf{D} - \mathbf{A} = \begin{pmatrix} x-1 \\ 4-3 \end{pmatrix} = \begin{pmatrix} x-1 \\ 1 \end{pmatrix} \tag{0.6}$$

$$\mathbf{BC} = \mathbf{C} - \mathbf{B} = \begin{pmatrix} x-4 \\ 5-2 \end{pmatrix} = \begin{pmatrix} x-4 \\ 3 \end{pmatrix} \tag{0.7}$$

Equating vectors:

$$x - 1 = x - 4 \quad \text{and} \quad 1 = 3 \tag{0.8}$$

Again, contradiction. So let's try using the property that opposite sides are equal in length.

$$\text{Length of } \mathbf{AD}: \quad |AD| = \sqrt{(x-1)^2 + (4-3)^2} = \sqrt{(x-1)^2 + 1} \tag{0.9}$$

$$\text{Length of } \mathbf{BC}: \quad |BC| = \sqrt{(x-4)^2 + (5-2)^2} = \sqrt{(x-4)^2 + 9} \tag{0.10}$$

Equating lengths:

$$\sqrt{(x-1)^2 + 1} = \sqrt{(x-4)^2 + 9} \tag{0.11}$$

Squaring both sides:

$$(x-1)^2 + 1 = (x-4)^2 + 9 \tag{0.12}$$

$$x^2 - 2x + 1 + 1 = x^2 - 8x + 16 + 9 \tag{0.13}$$

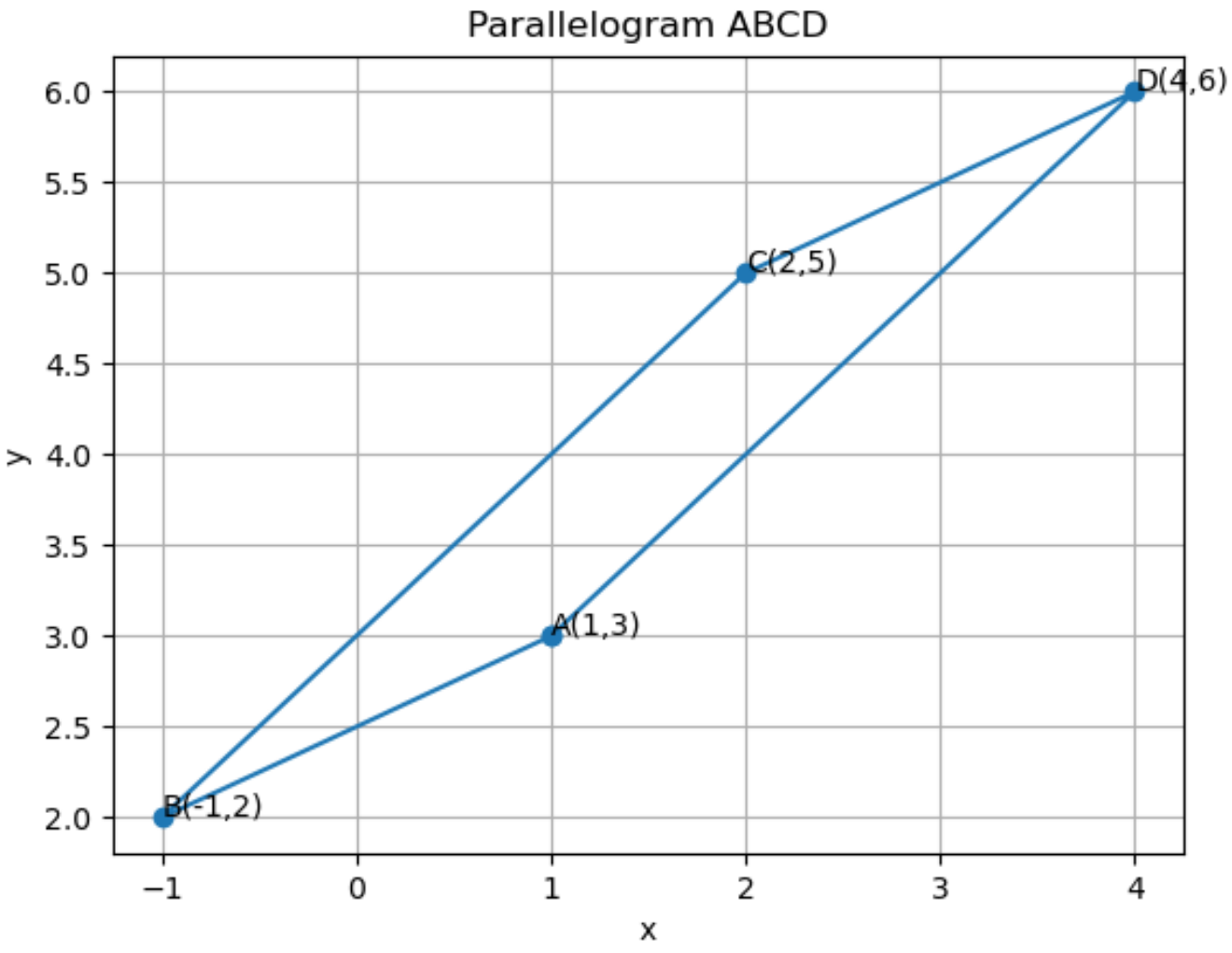
$$x^2 - 2x + 2 = x^2 - 8x + 25 \tag{0.14}$$

Subtract x^2 from both sides:

$$-2x + 2 = -8x + 25 \tag{0.15}$$

$$6x = 23 \Rightarrow x = \frac{23}{6} \tag{0.16}$$

Answer: $\boxed{\frac{23}{6}}$


Parallelogram ABCD
D(4,6)
C(2,5)
A(1,3)
B(-1,2)
6.0
5.5
5.0
4.5
4.0
3.5
3.0
2.5
2.0
y
−1
0
1
2
3
4
x