Bonus Question

EE25BTECH11043 - Nishid Khandagre

Question: Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be unit vectors such that $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$. Which of the following are correct?

1) $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a} = \mathbf{0}$
2) $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a} \neq \mathbf{0}$
3) $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{c} \neq \mathbf{0}$
4) $\mathbf{a} \times \mathbf{b}, \mathbf{b} \times \mathbf{c}, \mathbf{c} \times \mathbf{a}$ are mutually perpendicular.

Solution: Given that $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are unit vectors.

$$\mathbf{a}^T\mathbf{a} = \mathbf{b}^T\mathbf{b} = \mathbf{c}^T\mathbf{c} = 1 \tag{1}$$

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0} \tag{2}$$

Given

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \mathbf{0} \tag{3}$$

$$\mathbf{A} = \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{pmatrix} \tag{4}$$

$$\mathbf{A} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \mathbf{0} \tag{5}$$

$$\mathbf{A}^T\mathbf{A} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \mathbf{0} \tag{6}$$

$$\mathbf{G} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \mathbf{0} \tag{7}$$

Gram matrix $\mathbf{G}$:

$$\mathbf{G} = \begin{pmatrix} \mathbf{a}^T\mathbf{a} & \mathbf{a}^T\mathbf{b} & \mathbf{a}^T\mathbf{c} \\ \mathbf{b}^T\mathbf{a} & \mathbf{b}^T\mathbf{b} & \mathbf{b}^T\mathbf{c} \\ \mathbf{c}^T\mathbf{a} & \mathbf{c}^T\mathbf{b} & \mathbf{c}^T\mathbf{c} \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{a}^T\mathbf{b} & \mathbf{a}^T\mathbf{c} \\ \mathbf{b}^T\mathbf{a} & 1 & \mathbf{b}^T\mathbf{c} \\ \mathbf{c}^T\mathbf{a} & \mathbf{c}^T\mathbf{b} & 1 \end{pmatrix} \tag{8}$$

$$\mathbf{G}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1+\mathbf{a}^T\mathbf{b}+\mathbf{a}^T\mathbf{c}\\ \mathbf{b}^T\mathbf{a}+1+\mathbf{b}^T\mathbf{c}\\ \mathbf{c}^T\mathbf{a}+\mathbf{c}^T\mathbf{b}+1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \tag{9}$$

$$1+\mathbf{a}^T\mathbf{b}+\mathbf{a}^T\mathbf{c} = 0 \tag{10}$$

$$\mathbf{b}^T\mathbf{a}+1+\mathbf{b}^T\mathbf{c} = 0 \tag{11}$$

$$\mathbf{c}^T\mathbf{a}+\mathbf{c}^T\mathbf{b}+1 = 0 \tag{12}$$

from this we get

$$\mathbf{a}^T\mathbf{b} = \mathbf{b}^T\mathbf{c} = \mathbf{c}^T\mathbf{a} = k \tag{13}$$

Now

$$\left(\mathbf{a}+\mathbf{b}+\mathbf{c}\right) = \mathbf{0} \tag{14}$$

$$\tag{15}$$

$$\left(\mathbf{a}+\mathbf{b}+\mathbf{c}\right)^T\left(\mathbf{a}+\mathbf{b}+\mathbf{c}\right) = \mathbf{0}^T\mathbf{0} = 0 \tag{16}$$

$$\mathbf{a}^T\mathbf{a}+\mathbf{b}^T\mathbf{b}+\mathbf{c}^T\mathbf{c}+2\left(\mathbf{a}^T\mathbf{b}+\mathbf{b}^T\mathbf{c}+\mathbf{c}^T\mathbf{a}\right) = 0 \tag{17}$$

Substitute $\mathbf{a}^T\mathbf{a} = \mathbf{b}^T\mathbf{b} = \mathbf{c}^T\mathbf{c} = 1$

$$1+1+1+2\left(\mathbf{a}^T\mathbf{b}+\mathbf{b}^T\mathbf{c}+\mathbf{c}^T\mathbf{a}\right) = 0 \tag{18}$$

$$3+2\left(\mathbf{a}^T\mathbf{b}+\mathbf{b}^T\mathbf{c}+\mathbf{c}^T\mathbf{a}\right) = 0 \tag{19}$$

Hence

$$\mathbf{a}^T\mathbf{b}+\mathbf{b}^T\mathbf{c}+\mathbf{c}^T\mathbf{a} = -\frac{3}{2} \tag{20}$$

$$k+k+k = -\frac{3}{2} \tag{21}$$

$$3k = -\frac{3}{2} \tag{22}$$

$$k = -\frac{1}{2} \tag{23}$$

Thus,

$$\mathbf{a}^T\mathbf{b} = \mathbf{b}^T\mathbf{c} = \mathbf{c}^T\mathbf{a} = -\frac{1}{2} \tag{24}$$

Given

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0} \tag{25}$$

$$\mathbf{a} \times (\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{0} \tag{26}$$

$$\mathbf{a} \times \mathbf{a} + \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} = \mathbf{0} \tag{27}$$

Since $\mathbf{a} \times \mathbf{a} = \mathbf{0}$:

$$\mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} = \mathbf{0} \tag{28}$$

$$\mathbf{a} \times \mathbf{b} = -\mathbf{a} \times \mathbf{c} \tag{29}$$

$$\mathbf{a} \times \mathbf{b} = \mathbf{c} \times \mathbf{a} \tag{30}$$

$$\mathbf{b} \times (\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{b} \times \mathbf{a} + \mathbf{b} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} = \mathbf{0} \tag{31}$$

Since $\mathbf{b} \times \mathbf{b} = \mathbf{0}$:

$$\mathbf{b} \times \mathbf{a} + \mathbf{b} \times \mathbf{c} = \mathbf{0} \tag{32}$$

$$-\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} = \mathbf{0} \tag{33}$$

$$\mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{b} \tag{34}$$

Combining these, :

$$\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a} \tag{35}$$

Now

$$||\mathbf{a} \times \mathbf{b}||^2 = ||\mathbf{a}||^2 \, ||\mathbf{b}||^2 - (\mathbf{a}^T \mathbf{b})^2 \tag{36}$$

Substitute the values:

$$||\mathbf{a} \times \mathbf{b}||^2 = (1)^2(1)^2 - \left(-\frac{1}{2}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4} \tag{37}$$

Therefore,

$$||\mathbf{a} \times \mathbf{b}|| = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} \tag{38}$$

Since $||\mathbf{a} \times \mathbf{b}|| = \frac{\sqrt{3}}{2} \neq 0$, therefore $\mathbf{a} \times \mathbf{b} \neq \mathbf{0}$. Thus,

$$\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a} \neq \mathbf{0} \tag{39}$$