12.37

EE25BTECH11023 - Venkata Sai

Question:

Let $\mathcal{M}$ be the set of 3×3 real symmetric positive definite matrices. Consider $S = \left\{\mathbf{A} \in \mathcal{M} : \mathbf{A}^{50} - \mathbf{A}^{48} = 0\right\}$. The number of elements in S equals

Solution:

If a matrix is symmetric then it is diagonalizable. Hence

$$\mathbf{A} = \mathbf{PDP}^{-1} \tag{1}$$

$$\mathbf{A}^2 = \left(\mathbf{PDP}^{-1}\right)^2 \tag{2}$$

$$= \mathbf{PDP}^{-1}\mathbf{PDP}^{-1} \tag{3}$$

$$= \mathbf{PDIDP}^{-1} \tag{4}$$

$$= \mathbf{PD}^2\mathbf{P}^{-1} \tag{5}$$

$$\mathbf{A}^k = \mathbf{PD}^k\mathbf{P}^{-1} \tag{6}$$

$$\mathbf{A}^{50} = \mathbf{PD}^{50}\mathbf{P}^{-1} \tag{7}$$

$$\mathbf{A}^{48} = \mathbf{PD}^{48}\mathbf{P}^{-1} \tag{8}$$

Given

$$\mathbf{A}^{50} - \mathbf{A}^{48} = 0 \tag{9}$$

$$\mathbf{PD}^{50}\mathbf{P}^{-1} - \mathbf{PD}^{48}\mathbf{P}^{-1} = 0 \tag{10}$$

$$\mathbf{P}\left(\mathbf{D}^{50} - \mathbf{D}^{48}\right)\mathbf{P}^{-1} = 0 \tag{11}$$

$$\implies \left(\mathbf{D}^{50} - \mathbf{D}^{48}\right) = 0 \tag{12}$$

$$\implies \left(\lambda^{50} - \lambda^{48}\right) = 0 \tag{13}$$

where λ are the eigen values

$$\lambda^{48}\left(\lambda^2 - 1\right) = 0 \implies \lambda^{48} = 0 \text{ or } \lambda^2 - 1 = 0 \tag{14}$$

$$\lambda = 0 \text{ or } \lambda = \pm 1 \tag{15}$$

For a positive definite matrix, eigen values must be greater than 0. Hence

$$\lambda = 1 \implies \mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I} \tag{16}$$

$$\mathbf{A} = \mathbf{PIP}^{-1} = \mathbf{PP}^{-1} = \mathbf{I} \tag{17}$$

Hence Number of elements in S is 1