4.9.5

EE25BTECH11003 - Adharvan Kshathriya Bommagani

Question:
Equations of the lines through the point (3,2) and making an angle of 40°with the line $x - 2y = 3$ are.

Solution:

First, we express the given point and line using column vectors.
The line passes through the point $(3, 2)$. We can represent this with a position vector $\mathbf{h}$:

$$\mathbf{h} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

The given line is $x - 2y = 3$. From the formula $\mathbf{n}^\top \mathbf{x} = c$, we can identify the **normal vector** to this line, which we'll call $\mathbf{n_1}$:

$$\mathbf{n_1} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

The **direction vector** of a line, $\mathbf{m_1}$, is orthogonal to its normal vector, meaning $\mathbf{m_1}^\top \mathbf{n_1} = 0$. A simple choice is:

$$\mathbf{m_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

We need to find the direction vectors, $\mathbf{m_2}$ and $\mathbf{m_3}$, for the new lines by rotating the known direction vector $\mathbf{m_1}$ by both $+40°$ and $-40°$. The rotation matrix $R(\theta)$ is:

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

For the first line (rotation by +40°):

$$\begin{aligned} \mathbf{m_2} = R(40°)\mathbf{m_1} &= \begin{pmatrix} \cos(40°) & -\sin(40°) \\ \sin(40°) & \cos(40°) \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2\cos(40°) - \sin(40°) \\ 2\sin(40°) + \cos(40°) \end{pmatrix} \end{aligned}$$

For the second line (rotation by -40°):

$$\begin{aligned} \mathbf{m_3} = R(-40°)\mathbf{m_1} &= \begin{pmatrix} \cos(40°) & \sin(40°) \\ -\sin(40°) & \cos(40°) \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2\cos(40°) + \sin(40°) \\ -2\sin(40°) + \cos(40°) \end{pmatrix} \end{aligned}$$

We use the vector equation of a line, $\mathbf{x} = \mathbf{h} + \kappa\mathbf{m}$, and convert it to Cartesian form.

The normal form is $\mathbf{n}^\top \mathbf{x} = c$, where $c = \mathbf{n}^\top \mathbf{h}$. A normal vector $\mathbf{n}$ can be obtained from a direction vector $\mathbf{m} = \begin{pmatrix} u \\ v \end{pmatrix}$ as $\mathbf{n} = \begin{pmatrix} -v \\ u \end{pmatrix}$.

First Line in Normal Form:
The normal vector $\mathbf{n_2}$ from $\mathbf{m_2}$ is:

$$\mathbf{n_2} = \begin{pmatrix} -(2\sin(40^\circ) + \cos(40^\circ)) \\ 2\cos(40^\circ) - \sin(40^\circ) \end{pmatrix}$$

The constant $c_2 = \mathbf{n_2}^\top \mathbf{h}$ is:

$$\begin{aligned} c_2 &= [-(2\sin(40^\circ) + \cos(40^\circ)), \quad 2\cos(40^\circ) - \sin(40^\circ)] \begin{pmatrix} 3 \\ 2 \end{pmatrix} \\ &= -3(2\sin(40^\circ) + \cos(40^\circ)) + 2(2\cos(40^\circ) - \sin(40^\circ)) \\ &= \cos(40^\circ) - 8\sin(40^\circ) \end{aligned}$$

The equation is:

$$\begin{pmatrix} -(2\sin(40^\circ) + \cos(40^\circ)) \\ 2\cos(40^\circ) - \sin(40^\circ) \end{pmatrix}^\top \begin{pmatrix} x \\ y \end{pmatrix} = \cos(40^\circ) - 8\sin(40^\circ)$$

Second Line in Normal Form:
The normal vector $\mathbf{n_3}$ from $\mathbf{m_3}$ is:

$$\mathbf{n_3} = \begin{pmatrix} -(-2\sin(40^\circ) + \cos(40^\circ)) \\ 2\cos(40^\circ) + \sin(40^\circ) \end{pmatrix} = \begin{pmatrix} 2\sin(40^\circ) - \cos(40^\circ) \\ 2\cos(40^\circ) + \sin(40^\circ) \end{pmatrix}$$

The constant $c_3 = \mathbf{n_3}^\top \mathbf{h}$ is:

$$\begin{aligned} c_3 &= [2\sin(40^\circ) - \cos(40^\circ), \quad 2\cos(40^\circ) + \sin(40^\circ)] \begin{pmatrix} 3 \\ 2 \end{pmatrix} \\ &= 3(2\sin(40^\circ) - \cos(40^\circ)) + 2(2\cos(40^\circ) + \sin(40^\circ)) \\ &= \cos(40^\circ) + 8\sin(40^\circ) \end{aligned}$$

The equation is:

$$\begin{pmatrix} 2\sin(40^\circ) - \cos(40^\circ) \\ 2\cos(40^\circ) + \sin(40^\circ) \end{pmatrix}^\top \begin{pmatrix} x \\ y \end{pmatrix} = \cos(40^\circ) + 8\sin(40^\circ)$$

Plot of the Lines:

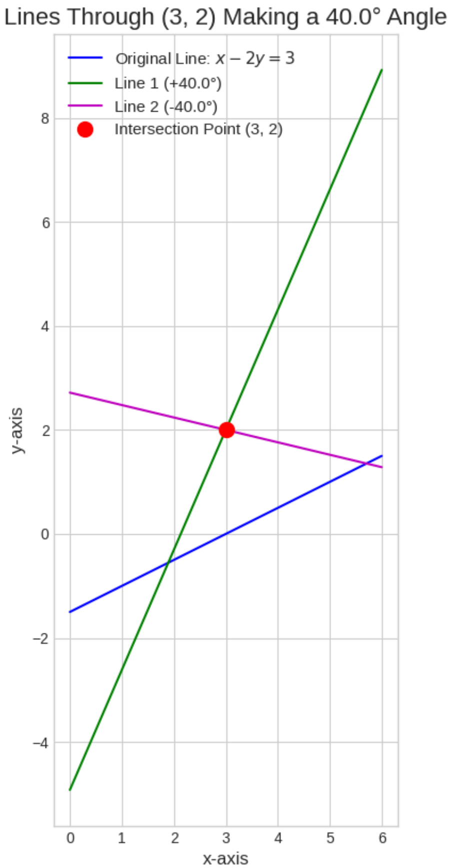

Fig. 0: Figure for 4.9.5