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Question

Consider a circle with its centre lying on focus of the parabola $y^2 = 2px$ such that it touches the directrix of the parabola. Then a point of intersection of the circle and the parabola is

1. $(\frac{p}{2}, p)$ or $(\frac{p}{2}, -p)$
2. $(\frac{p}{2}, -\frac{p}{2})$
3. $(-\frac{p}{2}, p)$
4. $(-\frac{p}{2}, -\frac{p}{2})$

Solution

General Conic Form

Any conic can be represented as:

$$\mathbf{x}^T A\mathbf{x} + \mathbf{b}^T\mathbf{x} + c = 0 \tag{1}$$

Parabola: $y^2 = 2px$

Rewriting:

$$y^2 - 2px = 0 \tag{2}$$

Matrix representation:

$$\mathbf{A}_p = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{b}_p = \begin{pmatrix} -2p \\ 0 \end{pmatrix}, \quad c_p = 0 \tag{3}$$

So the parabola becomes:

$$\mathbf{x}^T A_p\mathbf{x} + \mathbf{b}_p^T\mathbf{x} = 0 \tag{4}$$

Circle: Center at $(\frac{p}{2}, 0)$, Radius p

Circle equation:

$$(x - \frac{p}{2})^2 + y^2 = p^2 \Rightarrow x^2 - px + \frac{p^2}{4} + y^2 = p^2 \Rightarrow x^2 + y^2 - px - \frac{3p^2}{4} = 0 \tag{5}$$

Matrix representation:

$$\mathbf{A}_c = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{b}_c = \begin{pmatrix} -p \\ 0 \end{pmatrix}, \quad c_c = -\frac{3p^2}{4} \tag{6}$$

So the circle becomes:

$$\mathbf{x}^T A_c\mathbf{x} + \mathbf{b}_c^T\mathbf{x} + c_c = 0 \tag{7}$$

Solving the System

From the parabola:

$$y^2 = 2px \tag{8}$$

Substitute into the circle:

$$x^2 + y^2 - px - \frac{3p^2}{4} = 0 \Rightarrow x^2 + 2px - px - \frac{3p^2}{4} = 0 \Rightarrow x^2 + px - \frac{3p^2}{4} = 0 \tag{9}$$

Solve the quadratic:

$$x = \frac{-p \pm \sqrt{p^2 + 4 \cdot \frac{3p^2}{4}}}{2} = \frac{-p \pm \sqrt{4p^2}}{2} = \frac{-p \pm 2p}{2} \Rightarrow x = \frac{p}{2}, -\frac{3p}{2} \tag{10}$$

Now find y using $y^2 = 2px$:
For $x = \frac{p}{2}$:

$$y^2 = p^2 \Rightarrow y = \pm p \tag{11}$$

Final Answer

Intersection points:

$$(\frac{p}{2}, p), \quad (\frac{p}{2}, -p) \tag{12}$$

Correct Option: (a)

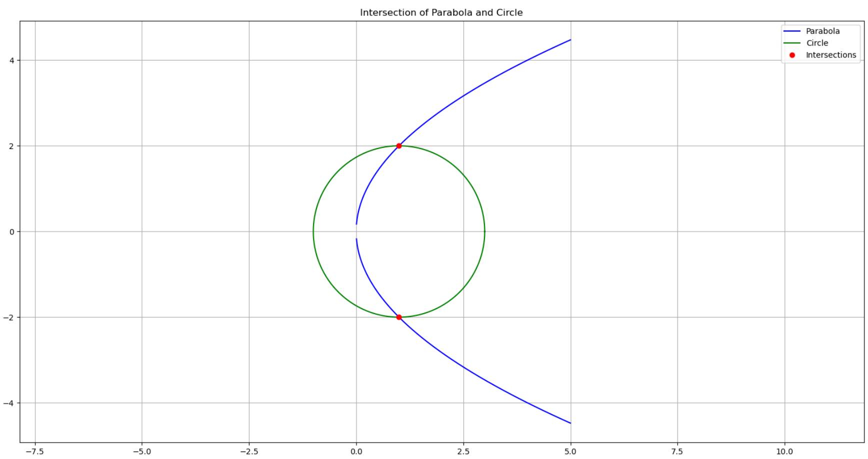

Figure 1: Caption