Matrices in Geometry 2.10.64

EE25BTECH11035 - Kushal B N

Question: The position vectors of the points **A**, **B**, **C** and **D** are $\left(3\hat{i} - 2\hat{j} - \hat{k}\right), \left(2\hat{i} + 3\hat{j} - 4\hat{k}\right), \left(-\hat{i} + \hat{j} + 2\hat{k}\right)$ and $\left(4\hat{i} + 5\hat{j} + \lambda\hat{k}\right)$ respectively. If the points **A**, **B**, **C** and **D** lie on a plane, find the value of λ.

Given:
$\mathbf{A}\begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$, $\mathbf{B}\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$, $\mathbf{C}\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{D}\begin{pmatrix} 4 \\ 5 \\ \lambda \end{pmatrix}$.

Solution:

The general equation for a plane with normal vector $\mathbf{n}$ passing through point P is

$$\mathbf{n}^\top \mathbf{P} = d \tag{1}$$

So,

$$\mathbf{n}^\top \mathbf{A} = d \tag{2}$$

$$\mathbf{n}^\top \mathbf{B} = d \tag{3}$$

$$\mathbf{n}^\top \mathbf{C} = d \tag{4}$$

$$\mathbf{n}^\top \mathbf{D} = d \tag{5}$$

Forming direction vectors in the plane,

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} -1 \\ 5 \\ -3 \end{pmatrix} \tag{6}$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} -4 \\ 3 \\ 3 \end{pmatrix} \tag{7}$$

Now, the normal vector will be orthogonal to both of these direction vectors, so that

$$(\mathbf{B} - \mathbf{A})^\top \mathbf{n} = 0 \tag{8}$$

$$(\mathbf{C} - \mathbf{A})^\top \mathbf{n} = 0 \tag{9}$$

Combining the above two equations,

$$\begin{pmatrix} (\mathbf{B} - \mathbf{A})^\top \\ (\mathbf{C} - \mathbf{A})^\top \end{pmatrix} \mathbf{n} = 0 \tag{10}$$

$$\begin{pmatrix} -1 & 5 & -3 \\ -4 & 3 & 3 \end{pmatrix} \mathbf{n} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{11}$$

Augmented matrix:

$$\implies \begin{pmatrix} -1 & 5 & -3 & | & 0 \\ -4 & 3 & 3 & | & 0 \end{pmatrix} \tag{12}$$

$$\begin{pmatrix} -1 & 5 & -3 & | & 0 \\ -4 & 3 & 3 & | & 0 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 4R_1} \begin{pmatrix} -1 & 5 & -3 & | & 0 \\ 0 & -17 & 15 & | & 0 \end{pmatrix} \tag{13}$$

$$\begin{pmatrix} -1 & 5 & -3 & | & 0 \\ 0 & -17 & 15 & | & 0 \end{pmatrix} \xrightarrow{R_2 \to \frac{-1}{17} R_2} \begin{pmatrix} -1 & 5 & -3 & | & 0 \\ 0 & 1 & \frac{-15}{17} & | & 0 \end{pmatrix} \tag{14}$$

$$\begin{pmatrix} -1 & 5 & -3 & | & 0 \\ 0 & 1 & \frac{-15}{17} & | & 0 \end{pmatrix} \xrightarrow{R_1 \to R_1 - 5R_2} \begin{pmatrix} -1 & 0 & \frac{24}{17} & | & 0 \\ 0 & 1 & \frac{-15}{17} & | & 0 \end{pmatrix} \tag{15}$$

$$\implies \mathbf{n} = \begin{pmatrix} \frac{24}{17} \\ \frac{15}{17} \\ 1 \end{pmatrix} t \tag{16}$$

So we can take $t = 17$ in order to get integer coefficients,

$$\mathbf{n} = \begin{pmatrix} 24 \\ 15 \\ 17 \end{pmatrix} \tag{17}$$

Substituting this in (2),

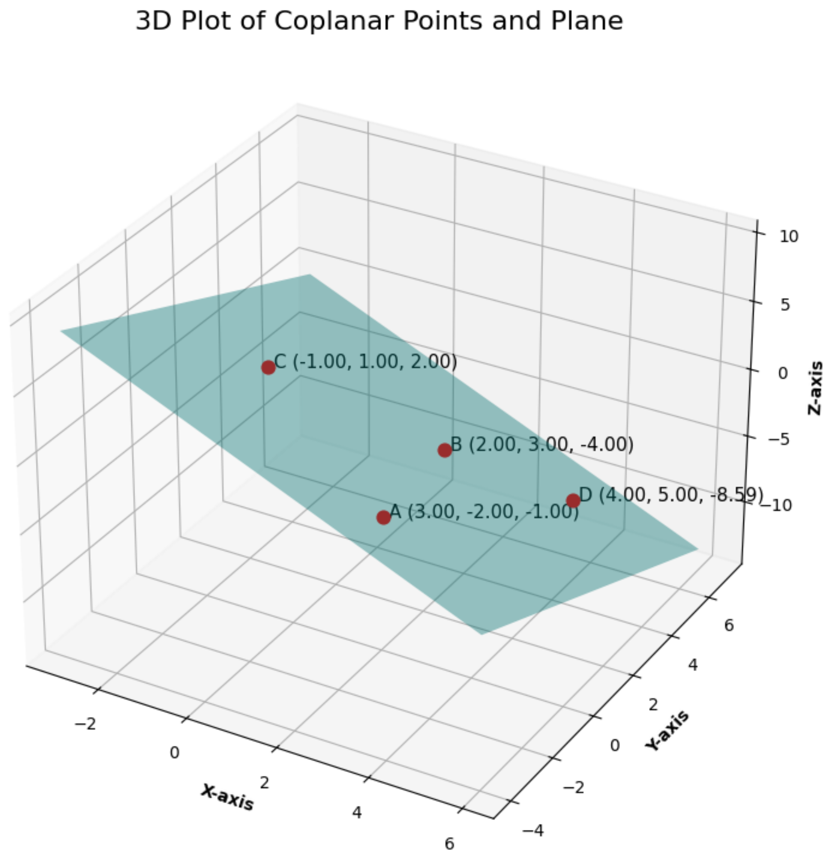
$$d = \begin{pmatrix} 24 & 15 & 17 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} = 25 \tag{18}$$

So substituting for $\mathbf{D}$ and d in the equation (5), we have

$$\begin{pmatrix} 24 & 15 & 17 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ \lambda \end{pmatrix} = 25 \tag{19}$$

$$\implies \boxed{\lambda = \frac{-146}{17}} \tag{20}$$

Final Answer:
The value of λ is $\frac{-146}{17}$.


3D Plot of Coplanar Points and Plane
C (-1.00, 1.00, 2.00)
B (2.00, 3.00, -4.00)
D (4.00, 5.00, -8.59)
A (3.00, -2.00, -1.00)
X-axis
Y-axis
Z-axis