10.6.8

EE25BTECH11052 - Shriyansh Kalpesh Chawda

Question

Construct a pair of tangents to a circle of radius 4cm from a point P lying outside the circle at a distance of 6cm from the centre. (10, 2023)

Solution

Let the center of the circle be at origin,The equation is $x^2 + y^2 = 16$ and Point P (at distance 6 from center along x-axis)

$$O = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1}$$

$$P = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \tag{2}$$

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{3}$$

where

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad f = -16 \tag{4}$$

The center and radius are:

$$\mathbf{c} = -\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad r = \sqrt{||\mathbf{u}||^2 - f} = \sqrt{0 + 16} = 4 \tag{5}$$

The equation of a tangent line at a point of contact $\mathbf{q}$ on the circle is given by:

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^\top \mathbf{x} + \mathbf{u}^\top \mathbf{q} + f = 0 \tag{6}$$

For this tangent to pass through the external point $P = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$, the equation must hold when $\mathbf{x} = P$. Since $\mathbf{V} = \mathbf{I}$ and $\mathbf{u} = \mathbf{0}$:

$$(\mathbf{I}\mathbf{q})^\top P + f = 0 \tag{7}$$

$$\mathbf{q}^\top P - 16 = 0 \tag{8}$$

Let $\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ be a point of contact. It must satisfy two conditions:

(a) $\mathbf{q}$ lies on the circle: $q_1^2 + q_2^2 = 16$

(b) The tangent at $\mathbf{q}$ passes through P: $\mathbf{q}^\top P = 16$

From condition (a) & (b):

$$\begin{pmatrix} q_1 & q_2 \end{pmatrix}\begin{pmatrix} 6 \\ 0 \end{pmatrix} = 16 \implies 6q_1 = 16 \implies q_1 = \frac{8}{3} \tag{9}$$

$$\left(\frac{8}{3}\right)^2 + q_2^2 = 16 \implies \frac{64}{9} + q_2^2 = 16 \implies q_2^2 = \frac{80}{9} \tag{10}$$

$$q_2 = \pm\frac{4\sqrt{5}}{3} \tag{11}$$

The two points of contact are:

$$\mathbf{q}_1 = \begin{pmatrix} \frac{8}{3} \\ \frac{4\sqrt{5}}{3} \end{pmatrix}, \quad \mathbf{q}_2 = \begin{pmatrix} \frac{8}{3} \\ -\frac{4\sqrt{5}}{3} \end{pmatrix} \tag{12}$$

The equation of the tangent at a point $\mathbf{q}$ is $(\mathbf{V}\mathbf{q} + \mathbf{u})^\top \mathbf{x} + \mathbf{u}^\top \mathbf{q} + f = 0$.
Tangent 1 at $\mathbf{q}_1$:

$$\mathbf{V}\mathbf{q}_1 = \begin{pmatrix} \frac{8}{3} \\ \frac{4\sqrt{5}}{3} \end{pmatrix} \tag{13}$$

$$\begin{pmatrix} \frac{8}{3} & \frac{4\sqrt{5}}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - 16 = 0 \tag{14}$$

$$\frac{8}{3}x + \frac{4\sqrt{5}}{3}y = 16 \tag{15}$$

$$2x + \sqrt{5}y = 12 \tag{16}$$

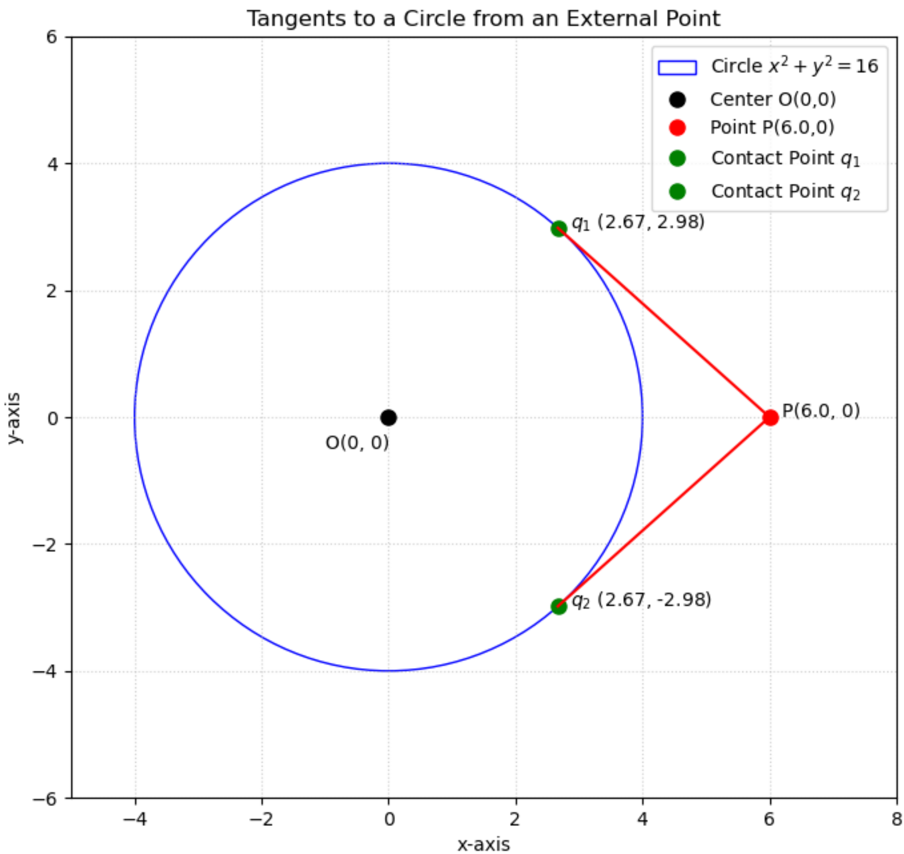
Tangent 2 at $\mathbf{q}_2$:

$$\begin{pmatrix} \frac{8}{3} & -\frac{4\sqrt{5}}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - 16 = 0 \tag{17}$$

$$2x - \sqrt{5}y = 12 \tag{18}$$

The equations of tangents are:

$$\boxed{2x + \sqrt{5}y = 12 \quad \text{and} \quad 2x - \sqrt{5}y = 12} \tag{19}$$



Tangents to a Circle from an External Point
Circle $x^2 + y^2 = 16$
Center O(0,0)
Point P(6.0,0)
Contact Point q_1
Contact Point q_2
q_1 (2.67, 2.98)
q_2 (2.67, -2.98)
P(6.0, 0)
O(0, 0)
y-axis
x-axis