4.10.23

EE25BTECH11021 - Dhanush Sagar

Question

A straight the through a fixed point (2, 3) intersects the coordinate axes at distinct points P and Q. If O is the origin and the rectangle OPRQ is completed, then the locus of R is

Solution

Equation of a line with normal vector $\mathbf{n}$ through $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$:

$$\mathbf{n}^T \mathbf{x} = \mathbf{n}^T \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{0.1}$$

The x-intercept is $\mathbf{P} = \begin{pmatrix} p \\ 0 \end{pmatrix}$. The y-intercept is $\mathbf{Q} = \begin{pmatrix} 0 \\ q \end{pmatrix}$.

The origin is

$$\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{0.2}$$

If the intercepts are $\mathbf{P} = \begin{pmatrix} p \\ 0 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} 0 \\ q \end{pmatrix}$, then normal vector is

$$\mathbf{n} = \begin{pmatrix} \frac{1}{p} \\ \frac{1}{q} \end{pmatrix} \tag{0.3}$$

The opposite vertex of rectangle OPRQ is then

$$\mathbf{R} = \begin{pmatrix} p \\ q \end{pmatrix} \tag{0.4}$$

Write $\mathbf{n}$ in terms of $\mathbf{R}$ by

$$\mathbf{n} = \begin{pmatrix} \frac{1}{p} \\ \frac{1}{q} \end{pmatrix} = \begin{pmatrix} \frac{1}{x} \\ \frac{1}{y} \end{pmatrix} \text{ where } \mathbf{R} = \begin{pmatrix} x \\ y \end{pmatrix} : \tag{0.5}$$

$$\begin{pmatrix} \frac{1}{x} & \frac{1}{y} \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = 1 \tag{0.6}$$

Multiply out the left-hand side:

$$\frac{2}{x} + \frac{3}{y} = 1 \tag{0.7}$$

Clear denominators by multiplying both sides by xy:

$$2y + 3x = xy \tag{0.8}$$

Rearrange to standard quadratic form:

$$xy - 3x - 2y = 0 \tag{0.9}$$

A conic in matrix form is

$$\mathbf{x}^T \mathbf{V}\mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0, \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}. \tag{0.10}$$

Here, the matrix corresponding to the xy term is symmetric:

$$\mathbf{V} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}, \quad f = 0 \tag{0.11}$$

$$\mathbf{x}^T \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -3 \\ -2 \end{pmatrix}^T \mathbf{x} = 0 \tag{0.12}$$

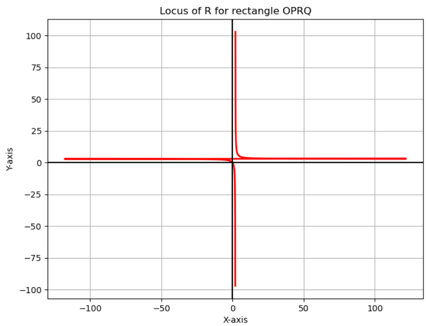

Fig. 0.1