2.10.50

EE25BTECH11021 - Dhanush Sagar

Question

A variable plane at a distance of one unit from the origin cuts the coordinate axes at A, B and C.

If the centroid $D(x, y, z)$ of triangle ABC satisfies the relation

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = k,$$

then the value of k is :

1) 3
2) 1
3) $\frac{1}{3}$
4) 9

Solution

We write the plane in vector form as

$$\mathbf{n}^\top \mathbf{x} = 1, \tag{4.1}$$

and introduce the axis intercepts

$$\mathbf{A} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}, \qquad \mathbf{C} = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}. \tag{4.2}$$

Define

$$\mathbf{e} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{M} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}. \tag{4.3}$$

(Thus the columns of $\mathbf{M}$ are $\mathbf{A}, \mathbf{B}, \mathbf{C}$.)

Since $\mathbf{A}, \mathbf{B}, \mathbf{C}$ lie on the plane we have

$$\mathbf{n}^\top \mathbf{A} = 1, \qquad \mathbf{n}^\top \mathbf{B} = 1, \qquad \mathbf{n}^\top \mathbf{C} = 1. \tag{4.4}$$

These three equations combine into the single matrix relation

$$\mathbf{n}^\top \mathbf{M} = \mathbf{e}^\top. \tag{4.5}$$

Transposing yields

$$\mathbf{M}^\top \mathbf{n} = \mathbf{e}. \tag{4.6}$$

Because $\mathbf{M}$ is diagonal (hence $\mathbf{M} = \mathbf{M}^\top$) and invertible,

$$\mathbf{n} = \mathbf{M}^{-1}\mathbf{e}. \tag{4.7}$$

The centroid $\mathbf{D}$ of triangle ABC is

$$\mathbf{D} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} = \tfrac{1}{3}\mathbf{M}\mathbf{e}. \tag{4.8}$$

Introduce the diagonal matrix of centroid coordinates

$$\mathbf{X} = \mathrm{diag}(x, y, z), \tag{4.9}$$

and substitute the given relation

$$\mathbf{D} = \mathbf{X}\mathbf{e}. \tag{4.10}$$

Comparing with $\mathbf{D} = \frac{1}{3}\mathbf{M}\mathbf{e}$ we obtain the matrix identity

$$\mathbf{M} = 3\mathbf{X}. \tag{4.11}$$

Using $\mathbf{n} = \mathbf{M}^{-1}\mathbf{e}$ and $\mathbf{M} = 3\mathbf{X}$ we get

$$\mathbf{n} = (3\mathbf{X})^{-1}\mathbf{e} = \tfrac{1}{3}\mathbf{X}^{-1}\mathbf{e}. \tag{4.12}$$

The perpendicular distance d from the origin to the plane $\mathbf{n}^\top\mathbf{x} = 1$ is

$$d = \frac{1}{\|\mathbf{n}\|} = \frac{1}{\sqrt{\mathbf{n}^\top\mathbf{n}}}. \tag{4.13}$$

Compute $\mathbf{n}^\top\mathbf{n}$ in purely matrix form:

$$\mathbf{n}^\top\mathbf{n} = \left(\tfrac{1}{3}\mathbf{X}^{-1}\mathbf{e}\right)^\top\left(\tfrac{1}{3}\mathbf{X}^{-1}\mathbf{e}\right) \tag{4.14}$$

$$= \tfrac{1}{9}\,\mathbf{e}^\top\mathbf{X}^{-2}\mathbf{e}. \tag{4.15}$$

Hence

$$\frac{1}{d^2} = \mathbf{n}^\top\mathbf{n} = \tfrac{1}{9}\,\mathbf{e}^\top\mathbf{X}^{-2}\mathbf{e}. \tag{4.16}$$

Proof of expansion

Since $\mathbf{X} = \mathrm{diag}(x, y, z)$, we have

$$\mathbf{X}^{-2} = \mathrm{diag}\left(\tfrac{1}{x^2}, \tfrac{1}{y^2}, \tfrac{1}{z^2}\right). \tag{4.17}$$

Thus

$$\mathbf{e}^\top\mathbf{X}^{-2}\mathbf{e} = \begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{x^2} & 0 & 0\\ 0 & \frac{1}{y^2} & 0\\ 0 & 0 & \frac{1}{z^2}\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} \tag{4.18}$$

$$= \tfrac{1}{x^2} + \tfrac{1}{y^2} + \tfrac{1}{z^2}. \tag{4.19}$$

Final result

Therefore,

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{9}{d^2}. \tag{4.20}$$

For the given problem $d = 1$,

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = 9, \tag{4.21}$$

so the required constant is

$$\boxed{k = 9}.$$

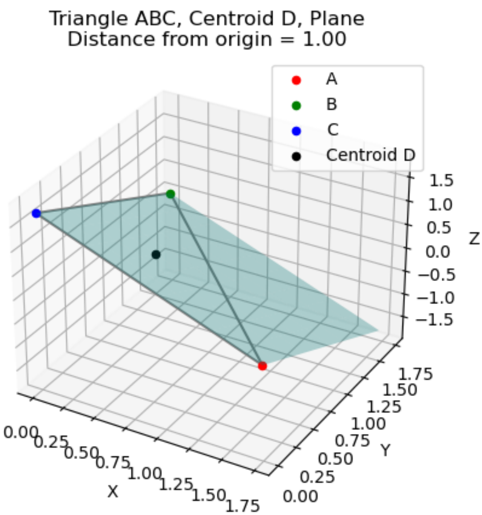

Fig. 4.1