2.4.13

EE25BTECH11019 - Darji Vivek M.

Question:
$\mathbf{B} - \mathbf{A} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{D} - \mathbf{C} = -3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ are two vectors. The position vectors of the points $\mathbf{A}$ and $\mathbf{C}$ are $6\mathbf{i} + 7\mathbf{j} + 4\mathbf{k}$ and $-9\mathbf{j} + 2\mathbf{k}$, respectively. Find the position vectors of a point $\mathbf{P}$ on the line $\mathbf{AB}$ and a point $\mathbf{Q}$ on the line $\mathbf{CD}$ such that $\mathbf{Q} - \mathbf{P}$ is perpendicular to both $\mathbf{B} - \mathbf{A}$ and $\mathbf{D} - \mathbf{C}$.

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Solution:

Symbol	Meaning
$\mathbf{a}$	$\begin{pmatrix} 6 \\ 7 \\ 4 \end{pmatrix}$ (position vector of $\mathbf{A}$)
$\mathbf{c}$	$\begin{pmatrix} 0 \\ -9 \\ 2 \end{pmatrix}$ (position vector of $\mathbf{C}$)
$\mathbf{d}_1$	$\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$ (direction of $\mathbf{B} - \mathbf{A}$)
$\mathbf{d}_2$	$\begin{pmatrix} -3 \\ 2 \\ 4 \end{pmatrix}$ (direction of $\mathbf{D} - \mathbf{C}$)
λ, μ	Parameters for $\mathbf{P}, \mathbf{Q}$ on $\mathbf{AB}, \mathbf{CD}$ respectively

TABLE 0: Variables Used

Matrix Method:
Points on the lines can be written as

$$\mathbf{P} = \mathbf{a} + \lambda \mathbf{d}_1, \qquad \mathbf{Q} = \mathbf{c} + \mu \mathbf{d}_2. \tag{1}$$

Then

$$\mathbf{Q} - \mathbf{P} = (\mathbf{c} - \mathbf{a}) + \mu \mathbf{d}_2 - \lambda \mathbf{d}_1. \tag{2}$$

Perpendicularity to both $\mathbf{d}_1$ and $\mathbf{d}_2$ gives

$$\mathbf{d}_1^\top (\mathbf{Q} - \mathbf{P}) = 0, \qquad \mathbf{d}_2^\top (\mathbf{Q} - \mathbf{P}) = 0. \tag{3}$$

This yields the 2×2 linear system

$$\underbrace{\begin{pmatrix} \mathbf{d}_1^\top \mathbf{d}_1 & \mathbf{d}_1^\top \mathbf{d}_2 \\ \mathbf{d}_2^\top \mathbf{d}_1 & \mathbf{d}_2^\top \mathbf{d}_2 \end{pmatrix}}_{\text{Gram matrix } G} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = - \begin{pmatrix} \mathbf{d}_1^\top (\mathbf{c} - \mathbf{a}) \\ \mathbf{d}_2^\top (\mathbf{c} - \mathbf{a}) \end{pmatrix}. \tag{4}$$

Compute the required dot products:

$$\mathbf{d}_1 = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{d}_2 = \begin{pmatrix} -3 \\ 2 \\ 4 \end{pmatrix}, \quad \mathbf{c} - \mathbf{a} = \begin{pmatrix} -6 \\ -16 \\ -2 \end{pmatrix}. \tag{5}$$

$$\mathbf{d}_1^\top \mathbf{d}_1 = 11, \quad \mathbf{d}_2^\top \mathbf{d}_2 = 29, \quad \mathbf{d}_1^\top \mathbf{d}_2 = \mathbf{d}_2^\top \mathbf{d}_1 = -7,$$

$$\mathbf{d}_1^\top (\mathbf{c} - \mathbf{a}) = -4, \qquad \mathbf{d}_2^\top (\mathbf{c} - \mathbf{a}) = -22.$$

Hence, from (**??**),

$$\begin{pmatrix} 11 & -7 \\ -7 & 29 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 4 \\ 22 \end{pmatrix}. \tag{6}$$

Solving,

$$\lambda = -1, \qquad \mu = 1. \tag{7}$$

Therefore,

$$\mathbf{P} = \mathbf{a} + \lambda \mathbf{d}_1 = \begin{pmatrix} 6 \\ 7 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} 3 \\ 8 \\ 3 \end{pmatrix}}, \tag{8}$$

$$\mathbf{Q} = \mathbf{c} + \mu \mathbf{d}_2 = \begin{pmatrix} 0 \\ -9 \\ 2 \end{pmatrix} + \begin{pmatrix} -3 \\ 2 \\ 4 \end{pmatrix} = \boxed{\begin{pmatrix} -3 \\ -7 \\ 6 \end{pmatrix}}. \tag{9}$$

Verification:

$$\mathbf{Q} - \mathbf{P} = \begin{pmatrix} -6 \\ -15 \\ 3 \end{pmatrix}, \quad \mathbf{d}_1^\top (\mathbf{Q} - \mathbf{P}) = 0, \quad \mathbf{d}_2^\top (\mathbf{Q} - \mathbf{P}) = 0, \tag{10}$$

confirming $\mathbf{Q} - \mathbf{P} \perp \mathbf{B} - \mathbf{A}$ and $\mathbf{Q} - \mathbf{P} \perp \mathbf{D} - \mathbf{C}$.

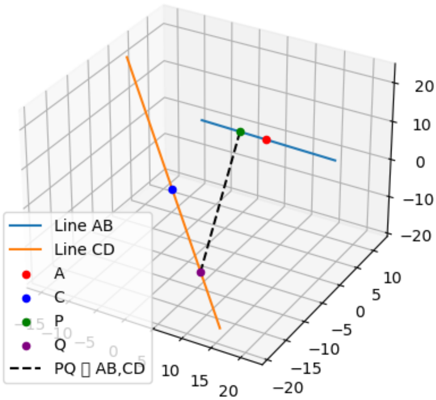

Fig. 0.1: plot