Matrices in Geometry - 7.4.44

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Problem Statement

Let **P** be a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, 0 < b < a$. Let the line parallel to the X axis passing through **P** meet the circle $x^2 + y^2 = a^2$ at the point **Q** such that **P** and **Q** are on the same side of the X axis. For two positive real numbers r and s, find the locus of the point **R** on **PQ** such that $PR = r$ as **P** varies over the ellipse.

Solution

The given ellipse is

$$\mathbf{E} \;:\; \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \;:\; \mathbf{V} = \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \;,\; f = -a^2 b^2 \tag{1}$$

$$\implies \mathbf{E} \;: \mathbf{x}^\top \mathbf{V}\mathbf{x} + f = 0 \tag{2}$$

The line parallel to the X-axis and passing through a point **P** on the ellipse is

$$\mathbf{L} \;:\; \mathbf{n}^\top \mathbf{x} = c \;:\; \mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \;,\; c = y_P \tag{3}$$

P satisfies this line; therefore, $c = y_P$

Solution

R is a point on line **L** and at a distance r from **P**

$$\mathbf{R} - \mathbf{P} = r\mathbf{e_1} \implies \mathbf{P} = \mathbf{R} - r\mathbf{e_1} \; ; \; \mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4}$$

Since, **P** is a point on **E**

$$\mathbf{P}^\top \mathbf{V} \mathbf{P} + f = 0 \tag{5}$$

Substituting $\mathbf{P} = \mathbf{Q} - r\mathbf{e_1}$

$$\left(\mathbf{R} - r\mathbf{e_1}\right)^\top \mathbf{V} \left(\mathbf{R} - r\mathbf{e_1}\right) + f = 0 \tag{6}$$

$$\implies \mathbf{R}^\top \mathbf{V} \mathbf{R} - 2r\mathbf{R}^\top \mathbf{V} \mathbf{e_1} + r^2 \mathbf{e_1}^\top \mathbf{V} \mathbf{e_1} + f = 0 \tag{7}$$

$$\mathbf{R} = \begin{pmatrix} x \\ y \end{pmatrix} , \; \mathbf{V} = \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} , \; \mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \; f = -a^2 b^2 \tag{8}$$

Solution

Thus the locus of the point $\mathbf{R}$ is

$$\mathbf{x}^\top \mathbf{V}'\mathbf{x} + 2\mathbf{u}'^\top \mathbf{x} + f' = 0 \ : \ \mathbf{V}' = \mathbf{V} \ , \ \mathbf{u}' = \left(-\mathbf{V}\mathbf{e_1}\right) \ , \ f' = f + r^2 \mathbf{e_1}^\top \mathbf{V}\mathbf{e_1} \tag{9}$$

Simplifying this equation, we get

$$\mathbf{x}^\top \mathbf{V}'\mathbf{x} + 2\mathbf{u}'^\top \mathbf{x} + f' = 0 \tag{10}$$

$$\mathbf{V}' = \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \ , \ \mathbf{u}' = \begin{pmatrix} -b^2 r \\ 0 \end{pmatrix} \ , \ f' = b^2 r^2 - a^2 b^2 \tag{11}$$

This is the equation of locus of the point $\mathbf{R}$, which is an ellipse.

Solution

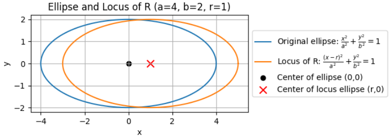

Figure: Figure for 8.4.40 for $a = 4, b = 2, r = 1$