5.4.35

EE25BTECH11041 - Naman Kumar

Question:
Let p be an odd prime number and $\mathbf{T_p}$ be the following set of 2×2 matrices

$$\mathbf{T_p} = \left\{ \mathbf{A} = \begin{pmatrix} a & b \\ c & a \end{pmatrix} : a, b, c \in \{0, 1, 2, \ldots, p-1\} \right\} \tag{1}$$

a) The number of $\mathbf{A}$ in $\mathbf{T_p}$ such that $\mathbf{A}$ is either symmetric or skew-symmetric or both, and det(A) divisible by p is

Solution:

Case 1: Symmentric

$$\mathbf{A} = \mathbf{A}^T \tag{2}$$

$$\begin{pmatrix} a & b \\ c & a \end{pmatrix} = \begin{pmatrix} a & c \\ b & a \end{pmatrix} \tag{3}$$

$$b = c \tag{4}$$

$$\mathbf{A} = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \tag{5}$$

Determinant divisible by p

$$|\mathbf{A}| = \begin{vmatrix} a & b \\ b & a \end{vmatrix} \tag{6}$$

$$= a^2 - b^2 \tag{7}$$

$$= (a-b)(a+b) \tag{8}$$

$$|\mathbf{A}| \mod p = 0 \tag{9}$$

$$(a-b)(a+b) \mod p = 0 \tag{10}$$

Since p is a prime number,

i) $a - b \mod p = 0$ only at $a - b = 0$:

$$a - b = 0 \tag{11}$$

$$a = b \tag{12}$$

So their are p pairs for (a,b) at a-b=0

ii)

$$a + b \mod p = 0 \tag{13}$$

$$\tag{14}$$

at a=1, b=p-1, a+b=p
at a=2, b=p-2, a+b=p

$$\vdots$$

so,similarly for all a, their is b in pair(a,b)
therefore their are p pairs

Total number of such $\mathbf{A}$ in case 1 $= i + ii - i \cap ii$
Total = p + p -1 (case of elements = 0)
=2p-1

Case 2:Skew-Symmetric

$$\mathbf{A} + \mathbf{A}^T = 0 \tag{15}$$

$$\begin{pmatrix} a & b \\ c & a \end{pmatrix} + \begin{pmatrix} a & c \\ b & a \end{pmatrix} = 0 \tag{16}$$

$$\begin{pmatrix} 2a & b+c \\ c+b & 2a \end{pmatrix} = 0 \tag{17}$$

$$a = 0, b = -c \tag{18}$$

$$\mathbf{A} = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \tag{19}$$

Now, for mod

$$|\mathbf{A}| \mod p = 0 \tag{20}$$

$$(0 + b^2) \mod p = 0 \tag{21}$$

$$b^2 \mod p = 0 \tag{22}$$

Only when b=0, so

$$\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{23}$$

Already included in case 1 therefore, final answer = 2p-1

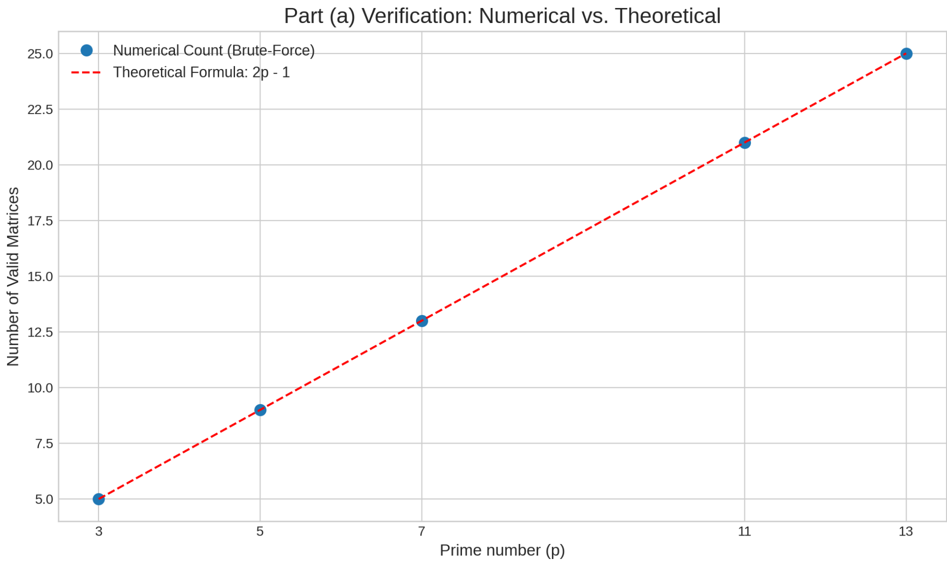
Part (a) Verification: Numerical vs. Theoretical
Numerical Count (Brute-Force)
Theoretical Formula: 2p - 1
Number of Valid Matrices
25.0
22.5
20.0
17.5
15.0
12.5
10.0
7.5
5.0
3
5
7
11
13
Prime number (p)

Fig. 1