3.3.15

AI25BTECH110031
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Question(3.3.15) Construct a triangle ABC in which $BC = 7cm$, and median $AD = 5cm$, $\angle A = 60^\circ$. Write the steps of construction also.

Solution:

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \mathbf{C} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 3.5 \\ 0 \end{pmatrix}. \tag{0.1}$$

Since $AD = 5$, point $\mathbf{A}$ lies on the circle with center $\mathbf{D}$ and radius 5. We parametrize:

$$\mathbf{A} = \mathbf{D} + 5\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} 3.5 + 5\cos\theta \\ 5\sin\theta \end{pmatrix}. \tag{0.2}$$

Define the vectors

$$\mathbf{c} = \mathbf{AB} = \mathbf{B} - \mathbf{A} = \begin{pmatrix} -3.5 - 5\cos\theta \\ -5\sin\theta \end{pmatrix}, \tag{0.3}$$

$$\mathbf{b} = \mathbf{AC} = \mathbf{C} - \mathbf{A} = \begin{pmatrix} 3.5 - 5\cos\theta \\ -5\sin\theta \end{pmatrix}. \tag{0.4}$$

Angle condition:

$$\mathbf{c}^\top \mathbf{b} = \|c\|\,\|b\| \cos 60^\circ = \frac{1}{2}\|c\|\,\|b\| \tag{0.5}$$

Compute the dot product:

$$\mathbf{c}^\top \mathbf{b} = (-3.5 - 5\cos\theta)(3.5 - 5\cos\theta) + (-5\sin\theta)(-5\sin\theta) = \frac{51}{4}. \tag{0.6}$$

Hence

$$\|b\|\,\|c\| = \frac{51}{2}. \tag{0.7}$$

Now,

$$\|c\|^2 = \frac{149}{4} + 35\cos\theta, \qquad \|b\|^2 = \frac{149}{4} - 35\cos\theta. \tag{0.8}$$

Therefore,

$$(\|c\|\,\|b\|)^2 = \left(\frac{149}{4}\right)^2 - (35\cos\theta)^2. \tag{0.9}$$

Substituting $\|c\|\ \|b\| = \dfrac{51}{2}$,

$$\left(\frac{51}{2}\right)^2 = \left(\frac{149}{4}\right)^2 - (35\cos\theta)^2, \tag{0.10}$$

$$\cos^2\theta = \frac{11797}{19600}. \tag{0.11}$$

Thus,

$$\cos\theta = \pm\frac{\sqrt{11797}}{140}, \qquad \sin\theta = \pm\frac{\sqrt{7803}}{140}. \tag{0.12}$$

Finally, coordinates of A are

$$\mathbf{A} = \begin{pmatrix} \frac{7}{2} \pm \frac{\sqrt{11797}}{28} \\ \pm\frac{\sqrt{7803}}{28} \end{pmatrix}. \tag{0.13}$$

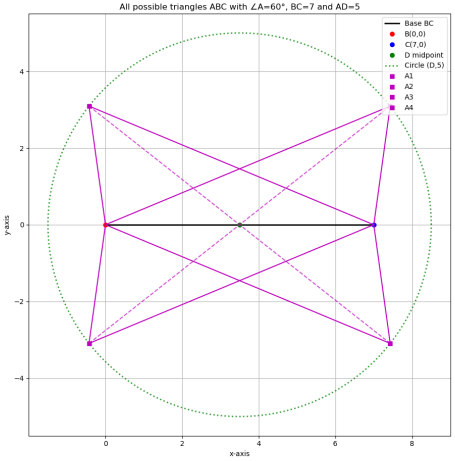

Fig. 0.1