7.4.39

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October 1, 2025

Question

If $\left(m_i, \frac{1}{m_i}\right)$, $m_i > 0, i = 1, 2, 3, 4$ are four distinct points on a circle, then show that $m_1 m_2 m_3 m_4 = 1$

Theoretical Solution

Let the circle equation be

$$\|\mathbf{x}\|^2 + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$

where, $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ with a and b as constants.
Let $\mathbf{P} = \begin{pmatrix} m \\ \frac{1}{m} \end{pmatrix}$ be a arbitrary vector in space.
Putting $\mathbf{P}$ in the circle , we get

$$\|\mathbf{P}\|^2 + 2\mathbf{u}^\top \mathbf{P} + f = 0 \tag{2}$$

Theoretical Solution

$$m^2 + \frac{1}{m^2} + 2am + \frac{2b}{m} + f = 0 \tag{3}$$

$$m^4 + 2am^3 + fm^2 + 2bm + 1 = 0 \tag{4}$$

Theoretical Solution

A general polynomial of degree n, has companion matrix as

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & 0 & \cdots & 0 & -c_2 \\ 0 & 0 & 1 & \cdots & 0 & -c_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -c_{n-1} \end{pmatrix} \tag{5}$$

The eigen values of the Companion Matrix $\mathbf{C}$ are the roots of the polynomial.

Theoretical Solution

For the question ,

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -2b \\ 0 & 1 & 0 & -f \\ 0 & 0 & 1 & -2a \end{pmatrix} \tag{6}$$

Here Eigen values of $\mathbf{C}$ are m_i where $i \in \{1, 2, 3, 4\}$

Theoretical Solution

Introducing Reversal Matrix

$$\mathbf{J} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{7}$$

$$\mathbf{J}^2 = \mathbf{I} \tag{8}$$

The Matrix **J** flips Rows when pre-multiplied and flips column when post-multiplied.

Theoretical Solution

Thus , if $\mathbf{C}$ represents $p\left(m\right)$
Then,

$$\mathbf{JCJ} = m^4 p\left(\frac{1}{m}\right) \tag{9}$$

Since $p\left(\frac{1}{m}\right)$ has eigen values $1/m_i$, we can say

$$\mathbf{JCJ} = \mathbf{C}^{-1} \tag{10}$$

$$\tag{11}$$

Theoretical Solution

Taking determinant, using 8

$$\left|\mathbf{C}\right| = \left|\mathbf{C}^{-1}\right| \tag{12}$$

$$\left|\mathbf{C}\right|^2 = 1 \tag{13}$$

Theoretical Solution

Since **C** is a real companion matrix of a monic quartic whose constant term is 1 ,

$$\left|C\right| = (-1)^4 \, 1 \tag{14}$$

Also,

$$\left|C\right| = m_1 m_2 m_3 m_4 \tag{15}$$

$$\therefore \, m_1 m_2 m_3 m_4 = 1 \tag{16}$$

Hence Proved

C Code - To find Solution of Circle and Hyperbola

```c
#include <math.h>
double calc(double *X , double *Y , double c, double r)
{
    double temp1 , temp2 ,prod;
    temp1 = (r*r + sqrt(pow(r,4) - 4 * pow(c,4)))/2;
    temp2 = (r*r - sqrt(pow(r,4) - 4 * pow(c,4)))/2;
    X[0] = sqrt(temp1);
    X[1] = -sqrt(temp1);
    X[2] = sqrt(temp2);
    X[3] = -sqrt(temp2);
    prod = pow(sqrt(temp1),2) * pow(sqrt(temp2),2);
    for(int i = 0 ; i< 4 ; i++)
        Y[i] = c*c/X[i] ;
    return prod;
}
```

C Code - Generate Circle

```c
void circle_gen(double *X , double *Y , double *P, int n , double
     r)
{

    for (int i = 0 ; i < n ; i++ )
    {
        double theta = 2.0 * M_PI * i / n ;
        X[i] = P[0] + r * cos(theta);
        Y[i] = P[1] + r * sin (theta);
    }
}
```

C Code - To generate Hyperbola

```c
void points_gen (double *X , double a , double b , int n )
{
    double temp = (b - a )/(double)n ;
    for (int i = 0 ; i <= n ; i++ )
    {
        X[i] = a + temp * i ;
    }
}

void hyper_gen (double *X , double *Y , double c , int n )
{
    for(int i = 0; i < n ;i++)
    {
        Y[i] = c*c/X[i];
    }
}
```

Python Code - 1

```python
import ctypes as ct
import numpy as np
import matplotlib.pyplot as plt

handc1 = ct.CDLL("./generator.so")

handc1.circle_gen.argtypes = [
    ct.POINTER(ct.c_double),
    ct.POINTER(ct.c_double),
    ct.POINTER(ct.c_double),
    ct.c_int,
    ct.c_double
]

handc1.circle_gen.restype = None
```

Python Code - 1

```
O = np.zeros(2 , dtype = np.float64).reshape(-1,1)
X_cic = np.zeros(200, dtype = np.float64).reshape(-1,1)
Y_cic = np.zeros(200, dtype = np.float64).reshape(-1,1)

handc1.circle_gen(
    X_cic.ctypes.data_as(ct.POINTER(ct.c_double)),
    Y_cic.ctypes.data_as(ct.POINTER(ct.c_double)),
    O.ctypes.data_as(ct.POINTER(ct.c_double)),
    200 , 3 )
handc1.points_gen.argtypes = [
    ct.POINTER(ct.c_double),
    ct.c_double,
    ct.c_double,
    ct.c_int]
```

Python Code - 1

```
handc1.points_gen.restype = None


pt = 400
X_hyper_p = np.zeros(pt,dtype=np.float64).reshape(-1,1)
Y_hyper_p = np.zeros(pt,dtype=np.float64).reshape(-1,1)

handc1.points_gen(
    X_hyper_p.ctypes.data_as(ct.POINTER(ct.c_double)),
    0.1,
    30,
    pt
)
```

Python Code - 1

```
X_hyper_n = np.zeros(pt,dtype=np.float64).reshape(-1,1)
Y_hyper_n = np.zeros(pt,dtype=np.float64).reshape(-1,1)

handc1.points_gen(
    X_hyper_n.ctypes.data_as(ct.POINTER(ct.c_double)),
    -30,-0.1,pt
)
handc1.hyper_gen.argtypes = [
    ct.POINTER(ct.c_double),
    ct.POINTER(ct.c_double),
    ct.c_double,
    ct.c_int
]
```

Python Code - 1

```
handc1.hyper_gen.restype = None

handc1.hyper_gen(
    X_hyper_p.ctypes.data_as(ct.POINTER(ct.c_double)),
    Y_hyper_p.ctypes.data_as(ct.POINTER(ct.c_double)),
    1,pt
)

handc1.hyper_gen(
    X_hyper_n.ctypes.data_as(ct.POINTER(ct.c_double)),
    Y_hyper_n.ctypes.data_as(ct.POINTER(ct.c_double)),
    1,pt
)
```

Python Code - 1

```
plt.figure()
plt.plot(X_cic,Y_cic,"blue",label= "Circle")
plt.plot(X_hyper_p,Y_hyper_p,"red",label="Rectangular Hyperbola")
plt.plot(X_hyper_n,Y_hyper_n,"red")

handc2 = ct.CDLL("./func.so")

handc2.calc.argtypes = [
    ct.POINTER(ct.c_double),
    ct.POINTER(ct.c_double),
    ct.c_double,
    ct.c_double
]
```

Python Code - 1

```
handc2.calc.restype = ct.c_double

x = np.zeros(4,dtype=np.float64)
y = np.zeros(4,dtype=np.float64)

prod = handc2.calc(
    x.ctypes.data_as(ct.POINTER(ct.c_double)),
    y.ctypes.data_as(ct.POINTER(ct.c_double)),
    1 , 3
)
print("m_1m_2m_3m_4 = ",prod)
plt.scatter(x,y)
vert_labels = [r'$m_1$',r'$m_2$',r'$m_3$',r'$m_4$']
```

Python Code - 1

```python
for i, txt in enumerate(vert_labels):
    plt.annotate(f'({txt},1/{txt})',
                 (x[i], y[i]),
                 textcoords="offset points",
                 xytext=(10,-15),
                 ha='center')

plt.xlabel('$x$')
plt.ylabel('$y$')
```

Python Code - 1

```
plt.legend(loc='best')
plt.grid()
plt.axis('equal')
plt.xlim([-10/2,10/2])
plt.ylim([-10/2,10/2])
plt.title("7.4.39")
plt.axhline(0, color='black', linewidth=0.7)
plt.axvline(0, color='black', linewidth=0.7)
plt.savefig("../figs/graph1.png")
plt.show()
```

Python Code - 2

```
import math
import sys
sys.path.insert(0, '/home/kartik-lahoti/matgeo/codes/CoordGeo')
import numpy as np
import numpy.linalg as LA
import matplotlib.pyplot as plt
import matplotlib.image as mpimg

from conics.funcs import *
```

Python Code - 2

```
def intersect_hyperbola_circle(c, r):
    #Find intersection points of xy = c^2 and x^2 + y^2 = r^2
    # Coefficients for quadratic in X = x^2
    # X^2 - r^2*X + c^4 = 0
    coeffs = [1, -r**2, c**4]
    roots = np.roots(coeffs)

    points = []
    for X in roots:
        if np.isreal(X) and X >= 0: # valid x^2
            x_vals = [np.sqrt(X).real, -np.sqrt(X).real]
            for x in x_vals:
                if x != 0: # avoid division by zero
                    y = c**2 / x
                    points.append((x, y))
    return points
```

Python Code - 2

```
lt = intersect_hyperbola_circle(1,3)
inter1 = np.array([lt[0][0] , lt[0][1]]).reshape(-1,1)
inter2 = np.array([lt[1][0] , lt[1][1]]).reshape(-1,1)
inter3 = np.array([lt[2][0] , lt[2][1]]).reshape(-1,1)
inter4 = np.array([lt[3][0] , lt[3][1]]).reshape(-1,1)
prod = lt[0][0] * lt[1][0] * lt[2][0] * lt[3][0]
print('m_1m_2m_3m_4 = ',prod)
# xy = c^2
y_n = np.linspace(-30, -0.1 , 400)
y_n = y_n[y_n!=0]
x_n = 1**2/y_n

y_p = np.linspace(0.1, 30 , 400)
y_p = y_p[y_p!=0]
x_p = 1**2/y_p
```

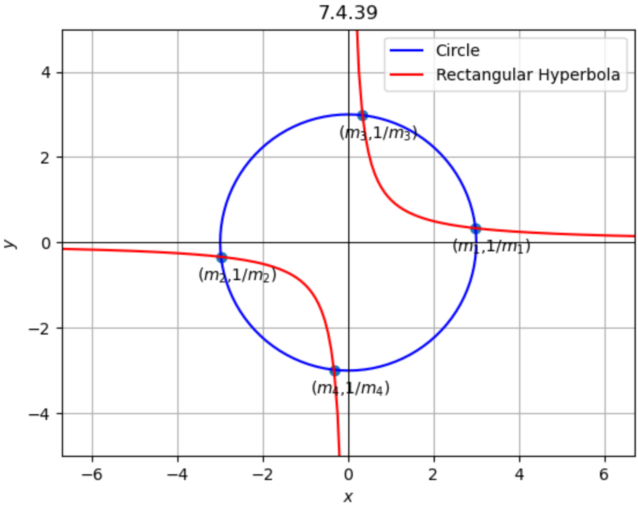
Python Code - 2

```
plt.figure()
plt.plot(x_p,y_p,"r-",label="Rectangular Hyperbola")
plt.plot(x_n,y_n,"r-")
O = np.zeros(2,dtype=np.float64).reshape(-1,1)
x_circ = circ_gen(O,3)
plt.plot(x_circ[0,:],x_circ[1,:],"blue",label="Circle")

coords = np.block([[inter1,inter2,inter3,inter4]])
plt.scatter(coords[0,:],coords[1,:])
vert_labels = [r'$m_1$',r'$m_2$',r'$m_3$',r'$m_4$']
```

Python Code - 2

```python
for i, txt in enumerate(vert_labels):
    plt.annotate(f'({txt},1/{txt})',(coords[0,i], coords[1,i]),
        textcoords="offset points",xytext=(10,-15),ha='center')

plt.xlabel('$x$')
plt.ylabel('$y$')
plt.legend(loc='best')
plt.grid()
plt.axis('equal')
plt.xlim([-10/2,10/2])
plt.ylim([-10/2,10/2])
plt.title("7.4.39")
plt.axhline(0, color='black', linewidth=0.7)
plt.axvline(0, color='black', linewidth=0.7)
plt.savefig("../figs/graph2.png")
plt.show()
```

Plot



7.4.39
Circle
Rectangular Hyperbola
(m3,1/m3)
(m1,1/m1)
(m2,1/m2)
(m4,1/m4)
y
x
4
2
0
−2
−4
−6
6