

10.6.1

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Question:

Draw a circle of radius 2.5cm. Take a point P outside the circle at a distance of 7cm from the center. Then construct a pair of tangents to the circle from point P.

Solution:

The tangent directions \mathbf{m} from an external point \mathbf{h} to the circle $g(\mathbf{x}) = \mathbf{x}^\top \mathbf{x} - r^2 = 0$ satisfy $\mathbf{m}^\top \mathbf{\Sigma} \mathbf{m} = 0$, where

$$\mathbf{\Sigma} = \mathbf{h}\mathbf{h}^\top - g(\mathbf{h})\mathbf{I} \quad (1)$$

With the point $\mathbf{h} = d\mathbf{e}_1$, we have $g(\mathbf{h}) = d^2 - r^2$. From (1),

$$\begin{aligned} \mathbf{\Sigma} &= \begin{pmatrix} d^2 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} d^2 - r^2 & 0 \\ 0 & d^2 - r^2 \end{pmatrix} \\ &= \begin{pmatrix} r^2 & 0 \\ 0 & -(d^2 - r^2) \end{pmatrix} \end{aligned} \quad (2)$$

Since $\mathbf{\Sigma}$ is a diagonal matrix, its eigenvalues are the diagonal entries, $\lambda_1 = r^2$ and $\lambda_2 = -(d^2 - r^2)$. The matrix of orthonormal eigenvectors is $\mathbf{P} = \mathbf{I}$.

The unit direction vectors of the tangents are given by the formula:

$$\mathbf{m} = \frac{1}{\sqrt{\lambda_1 - \lambda_2}} \mathbf{P} \begin{pmatrix} \sqrt{-\lambda_2} \\ \pm \sqrt{\lambda_1} \end{pmatrix} \quad (3)$$

$$\lambda_1 - \lambda_2 = r^2 - (-(d^2 - r^2)) = d^2 \quad (4)$$

$$-\lambda_2 = d^2 - r^2 \quad (5)$$

Substituting these into (3):

$$\mathbf{m} = \frac{1}{\sqrt{d^2}} \mathbf{I} \begin{pmatrix} \sqrt{d^2 - r^2} \\ \pm \sqrt{r^2} \end{pmatrix} = \frac{1}{d} \begin{pmatrix} \sqrt{d^2 - r^2} \\ \pm r \end{pmatrix} \quad (6)$$

The points of contact are $\mathbf{q} = \mathbf{h} + \kappa \mathbf{m}$. The parameter κ is found by substituting the line equation into the circle equation:

$$g(\mathbf{h} + \kappa \mathbf{m}) = (\mathbf{h} + \kappa \mathbf{m})^\top (\mathbf{h} + \kappa \mathbf{m}) - r^2 = 0 \quad (7)$$

$$\kappa^2 (\mathbf{m}^\top \mathbf{m}) + 2\kappa (\mathbf{h}^\top \mathbf{m}) + g(\mathbf{h}) = 0 \quad (8)$$

For a tangent, this quadratic in κ has a single repeated root. Since \mathbf{m} is a unit vector,

$$\mathbf{m}^\top \mathbf{m} = 1 \quad (9)$$

The value of κ for the point of contact is:

$$\kappa = \frac{-2(\mathbf{h}^\top \mathbf{m})}{2(1)} = -\mathbf{h}^\top \mathbf{m} \quad (10)$$

We require $\kappa < 0$ for the tangent to point from \mathbf{h} to the circle, which implies $\mathbf{h}^\top \mathbf{m} > 0$.

$$\mathbf{h}^\top \mathbf{m} = (d\mathbf{e}_1)^\top \frac{1}{d} \begin{pmatrix} \sqrt{d^2 - r^2} \\ \pm r \end{pmatrix} = \sqrt{d^2 - r^2} > 0 \quad (11)$$

The condition is satisfied, and so $\kappa = -\sqrt{d^2 - r^2}$.

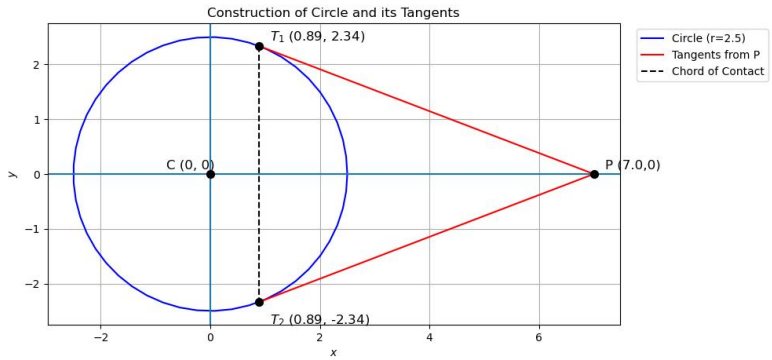
$$\mathbf{q} = d\mathbf{e}_1 - \sqrt{d^2 - r^2} \left(\frac{1}{d} \begin{pmatrix} \sqrt{d^2 - r^2} \\ \pm r \end{pmatrix} \right) \quad (12)$$

$$= \begin{pmatrix} \frac{r^2}{d} \\ \mp r \sqrt{d^2 - r^2} \end{pmatrix} \quad (13)$$

Substituting $r = 2.5$ and $d = 7$:

$$\mathbf{q} = \left(\pm \frac{\frac{(2.5)^2}{7}}{2.5 \sqrt{7^2 - (2.5)^2}} \right) \quad (14)$$

$$= \left(\pm \frac{\frac{6.25}{7}}{2.5 \sqrt{42.75}} \right) = \left(\pm \frac{\frac{25}{28}}{2.5 \sqrt{42.75}} \right) \quad (15)$$



Plot