5.13.9

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Question

Let $\mathbf{P}$ and $\mathbf{Q}$ be 3×3 matrices $\mathbf{P} \neq \mathbf{Q}$. If $\mathbf{P}^3 = \mathbf{Q}^3$ and $\mathbf{P}^2\mathbf{Q} = \mathbf{Q}^2\mathbf{P}$ then determinant of $(\mathbf{P}^2 + \mathbf{Q}^2)$ is equal to

Solution:

Given

$$\mathbf{P} \neq \mathbf{Q} \tag{0.1}$$
$$\mathbf{P}^3 = \mathbf{Q}^3 \tag{0.2}$$
$$\mathbf{P}^2\mathbf{Q} = \mathbf{Q}^2\mathbf{P} \tag{0.3}$$

let us solve for $(\mathbf{P}^2 + \mathbf{Q}^2)(\mathbf{P} - \mathbf{Q})$

$$(\mathbf{P}^2 + \mathbf{Q}^2)(\mathbf{P} - \mathbf{Q}) = \mathbf{P}^3 - \mathbf{P}^2\mathbf{Q} + \mathbf{Q}^2\mathbf{P} - \mathbf{Q}^3 \tag{0.4}$$

from equation **(0.2)** and **(0.3)**

$$(\mathbf{P}^2 + \mathbf{Q}^2)(\mathbf{P} - \mathbf{Q}) = \mathbf{0} \tag{0.5}$$

Let us assume $\det(\mathbf{P}^2 + \mathbf{Q}^2) \neq 0$
then $(\mathbf{P}^2 + \mathbf{Q}^2)$ is invertible and hence $(\mathbf{P}^2 + \mathbf{Q}^2)^{-1}$ exists

$$\therefore \mathbf{P} - \mathbf{Q} = \mathbf{0} \tag{0.6}$$
$$\implies \mathbf{P} = \mathbf{Q} \tag{0.7}$$

which contradicts equation **(0.1)**
Hence

$$\det(\mathbf{P}^2 + \mathbf{Q}^2) = 0 \tag{0.8}$$