2.7.33

EE25BTECH11021 - Dhanush Sagar

Question:
Find the value of p if

$$(2\hat{i} + 6\hat{j} + 27\hat{k}) \times (\hat{i} + 3\hat{j} + p\hat{k}) = 0.$$

Solution:
The given vectors are

$$\mathbf{A} = \begin{pmatrix} 2 \\ 6 \\ 27 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 1 \\ 3 \\ p \end{pmatrix}. \tag{0.1}$$

Form the 2×3 matrix with these rows:

$$M = \begin{pmatrix} 2 & 6 & 27 \\ 1 & 3 & p \end{pmatrix}. \tag{0.2}$$

If $\mathbf{A} \times \mathbf{B} = \mathbf{0}$, then $\mathbf{A}$ and $\mathbf{B}$ are linearly dependent, so $\text{rank}(M) < 2$. We perform row-reduction to find the rank and the condition on p.

Begin with M:

$$M = \begin{pmatrix} 2 & 6 & 27 \\ 1 & 3 & p \end{pmatrix}. \tag{0.3}$$

Eliminate the leading entry of the second row by replacing R_2 with $R_2 - \frac{1}{2}R_1$:

$$R_2 \leftarrow R_2 - \tfrac{1}{2}R_1$$

$$\begin{pmatrix} 2 & 6 & 27 \\ 1 & 3 & p \end{pmatrix} \xrightarrow{R_2 - \frac{1}{2}R_1} \begin{pmatrix} 2 & 6 & 27 \\ 0 & 0 & p - \frac{27}{2} \end{pmatrix}. \tag{0.4}$$

Now scale the first row to make a leading 1: $R_1 \leftarrow \frac{1}{2}R_1$:

$$\begin{pmatrix} 2 & 6 & 27 \\ 0 & 0 & p - \frac{27}{2} \end{pmatrix} \xrightarrow{R_1 \div 2} \begin{pmatrix} 1 & 3 & \frac{27}{2} \\ 0 & 0 & p - \frac{27}{2} \end{pmatrix}. \tag{0.5}$$

This is the RREF form (up to the final optional normalization of the second row). The rank is the number of nonzero rows in RREF. Thus

$$\text{rank}(M) = \begin{cases} 2, & \text{if } p - \frac{27}{2} \neq 0, \\ 1, & \text{if } p - \frac{27}{2} = 0. \end{cases}$$

For $\mathbf{A} \times \mathbf{B} = \mathbf{0}$ we need $\text{rank}(M) < 2$, hence

$$p - \frac{27}{2} = 0 \quad \Rightarrow \quad p = \frac{27}{2}. \tag{0.6}$$

Final answer:

$$\boxed{p = \frac{27}{2}} \tag{0.7}$$

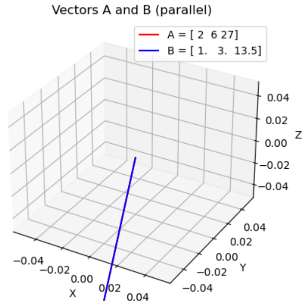

Fig. 0.1