12.670

EE25BTECH11032 - Kartik Lahoti

Question:

Consider the linear transformation $\mathbf{T}\colon \mathbb{C}^3 \to \mathbb{C}^3$ defined by

$$\mathbf{T}(x, y, z) = \left(x, \frac{\sqrt{3}}{2}y - \frac{1}{2}z, \frac{1}{2}y + \frac{\sqrt{3}}{2}z\right)$$

where $\mathbb{C}$ is the set of all complex numbers and $\mathbb{C}^3 = \mathbb{C}\times\mathbb{C}\times\mathbb{C}$. Which of the following statements is TRUE?

1) There exists a non-zero vector $\mathbf{X}$ such that $\mathbf{T}(\mathbf{X}) = -\mathbf{X}$
2) There exists a non-zero vector $\mathbf{Y}$ and a real number $\lambda \neq 1$ such that $\mathbf{T}(\mathbf{Y}) = \lambda\mathbf{Y}$
3) $\mathbf{T}$ is diagonalizable
4) $\mathbf{T}^2 = \mathbf{I}_3$, where $\mathbf{I}_3$ us the 3×3 identity matrix

Solution:

Let this function be written as

$$\mathbf{y} = \mathbf{T}\mathbf{x} \tag{4.1}$$

where , $\mathbf{x}$ and $\mathbf{y}$ are complex vector in $\mathbb{C}^3$

Now,

$$\mathbf{T}\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{4.2}$$

$$\mathbf{T}\mathbf{e_2} = \begin{pmatrix} 0 \\ \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} \tag{4.3}$$

$$\mathbf{T}\mathbf{e_3} = \begin{pmatrix} 0 \\ \frac{-1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \tag{4.4}$$

$$\therefore \mathbf{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \tag{4.5}$$

Now,

$$\left|\lambda\mathbf{I}-\mathbf{T}\right| = \begin{vmatrix} \lambda-1 & 0 & 0 \\ 0 & \lambda-\frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & \frac{-1}{2} & \lambda-\frac{\sqrt{3}}{2} \end{vmatrix} \tag{4.6}$$

$$= (\lambda-1)\left(\left(\lambda-\frac{\sqrt{3}}{2}\right)^2+\frac{1}{4}\right) \tag{4.7}$$

This gives eigen values as

$$\lambda_1 = 1, \lambda_2 = \frac{\sqrt{3}}{2}+i\frac{1}{2}, \lambda_3 = \frac{\sqrt{3}}{2}-i\frac{1}{2} \tag{4.8}$$

Option 1 : INCORRECT . $\because$ No Eigen Value = -1
Option 2 : INCORRECT . $\because$ No Eigen Value other than 1 is real.
Option 3 : CORRECT. $\because$ All eigen values are distinct!
Hence , Answer : Option (3)