4.7.43

Puni Aditya - EE25BTECH11046

Question:

Show that the points $\left(\hat{i}-\hat{j}+3\hat{k}\right)$ and $3\left(\hat{i}+\hat{j}+\hat{k}\right)$ are equidistant from the plane $\mathbf{r}\cdot\left(5\hat{i}+2\hat{j}-7\hat{k}\right)+9=0$ and lie on opposite sides of it.

Solution:

Let the given points be $\mathbf{P_1}=\begin{pmatrix}1\\-1\\3\end{pmatrix}$ and $\mathbf{P_2}=\begin{pmatrix}3\\3\\3\end{pmatrix}$. The equation of the given plane is

$$\begin{pmatrix}5 & 2 & -7\end{pmatrix}\mathbf{x}+9=0 \tag{1}$$

This can be written in the standard form $\mathbf{n}^\top\mathbf{x}=k$. Here, $\mathbf{n}=\begin{pmatrix}5\\2\\-7\end{pmatrix}$ and $k=-9$.

$$\begin{pmatrix}5 & 2 & -7\end{pmatrix}\mathbf{x}=-9 \tag{2}$$

The perpendicular distance of a point with position vector $\mathbf{P}$ from the plane $\mathbf{n}^\top\mathbf{x}=k$ is given by the formula

$$D=\frac{\left|\mathbf{n}^\top\mathbf{P}-k\right|}{\|\mathbf{n}\|} \tag{3}$$

$$\|\mathbf{n}\| = \sqrt{5^2+2^2+(-7)^2} \tag{4}$$

$$= \sqrt{25+4+49} = \sqrt{78} \tag{5}$$

Distance D_1 of the point $\mathbf{P_1}$ from the plane is

$$D_1 = \frac{\left|\begin{pmatrix}5 & 2 & -7\end{pmatrix}\begin{pmatrix}1\\-1\\3\end{pmatrix}-(-9)\right|}{\sqrt{78}} \tag{6}$$

$$= \frac{|(5)(1)+(2)(-1)+(-7)(3)+9|}{\sqrt{78}} \tag{7}$$

$$= \frac{|5-2-21+9|}{\sqrt{78}} \tag{8}$$

$$= \frac{|-9|}{\sqrt{78}} = \frac{9}{\sqrt{78}} \tag{9}$$

Distance D_2 of the point $\mathbf{P_2}$ from the plane is

$$D_2 = \frac{\left|\begin{pmatrix} 5 & 2 & -7 \end{pmatrix}\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} - (-9)\right|}{\sqrt{78}} \tag{10}$$

$$= \frac{|(5)(3) + (2)(3) + (-7)(3) + 9|}{\sqrt{78}} \tag{11}$$

$$= \frac{|15 + 6 - 21 + 9|}{\sqrt{78}} \tag{12}$$

$$= \frac{|9|}{\sqrt{78}} = \frac{9}{\sqrt{78}} \tag{13}$$

From (9) and (13), $D_1 = D_2$. Thus, the points are equidistant from the plane.

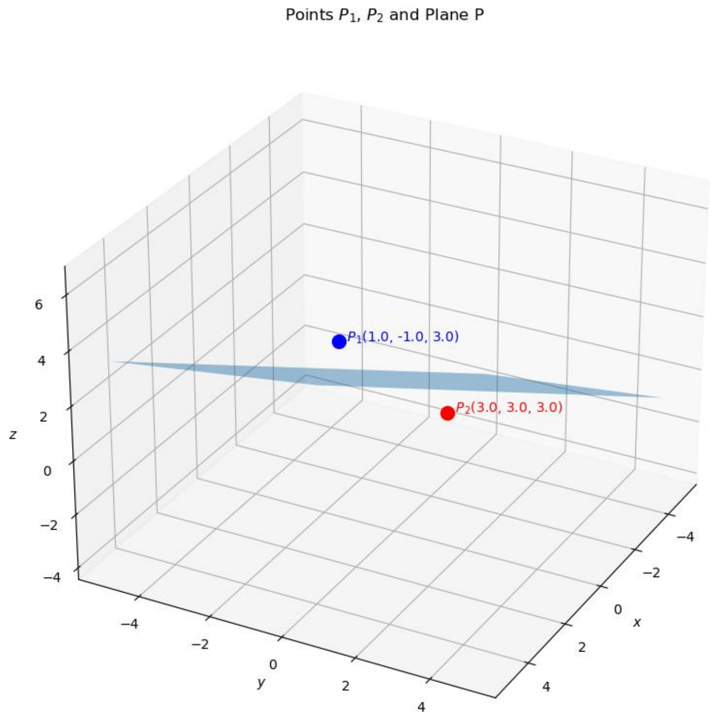
$$\mathbf{n}^\top \mathbf{P_1} - k = \begin{pmatrix} 5 & 2 & -7 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} - (-9) = -9 \tag{14}$$

$$\mathbf{n}^\top \mathbf{P_2} - k = \begin{pmatrix} 5 & 2 & -7 \end{pmatrix}\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} - (-9) = 9 \tag{15}$$

From (14) and (15),

$$\left(\mathbf{n}^\top \mathbf{P_1} - k\right)\left(\mathbf{n}^\top \mathbf{P_2} - k\right) = -81 < 0 \tag{16}$$

$\because$ $(\mathbf{n}^\top \mathbf{P_1} - k)(\mathbf{n}^\top \mathbf{P_2} - k) < 0$, the points $\mathbf{P_1}$ and $\mathbf{P_2}$ lie on opposite sides of the plane.



Plot