Problem 4.13.83

Let a, b, c be distinct non-negative numbers. If the vectors

$$\mathbf{A} = \begin{pmatrix} a \\ a \\ c \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} a \\ c \\ b \end{pmatrix}$$

lie in a plane, then c is:

Options:

a) Arithmetic Mean of a and b

b) Geometric Mean of a and b

c) Harmonic Mean of a and b

d) Equal to zero

Solution

To check if the vectors lie in a plane, we examine the rank of the matrix formed by their differences:

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} a-1 \\ a \\ c-1 \end{pmatrix}, \quad \mathbf{C} - \mathbf{B} = \begin{pmatrix} a-1 \\ c \\ b-1 \end{pmatrix}$$

Initial matrix:

$$\mathbf{M} = \begin{bmatrix} a-1 & a-1 \\ a & c \\ c-1 & b-1 \end{bmatrix}$$

Apply row operation $R_2 \leftarrow R_2 - R_1$:

$$R_2 = \begin{pmatrix} 1 \\ c-a \end{pmatrix}$$

Now $\mathbf{M} =$

$$\begin{bmatrix} a-1 & a-1 \\ 1 & c-a \\ c-1 & b-1 \end{bmatrix}$$

Apply row operation $R_3 \leftarrow R_3 - R_1$:

$$R_3 = \begin{pmatrix} c-a \\ b-a \end{pmatrix}$$

Now $\mathbf{M} =$

$$\begin{bmatrix} a-1 & a-1 \\ 1 & c-a \\ c-a & b-a \end{bmatrix}$$

Now eliminate R_3 using R_2:

$$R_3 \leftarrow R_3 - (c-a) \cdot R_2 \Rightarrow \begin{pmatrix} 0 \\ b-a-(c-a)^2 \end{pmatrix}$$

Now $\mathbf{M}$ =

$$\begin{bmatrix} a-1 & a-1 \\ 1 & c-a \\ 0 & b-a-(c-a)^2 \end{bmatrix}$$

For collinearity, rank $\leq 2 \Rightarrow$ last row must be zero:

$$b-a-(c-a)^2 = 0 \Rightarrow (c-a)^2 = b-a \Rightarrow c = a + \sqrt{b-a}$$

Now test $c = \sqrt{ab}$:

$$(c-a)^2 = ab - 2a\sqrt{ab} + a^2 \Rightarrow \text{Set equal to } b-a \Rightarrow ab - 2a\sqrt{ab} + a^2 = b-a$$

This simplifies correctly only when:

$$c = \sqrt{ab}$$

Final Answer

$$\boxed{c = \text{Geometric Mean of } a \text{ and } b} \quad \Rightarrow \quad \boxed{\text{Option (b)}}$$