9.2.6

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Question:

Area of the region in the first quadrant enclosed by the x-axis, the line y = x and the circle $x^2 + y^2 = 32$ is ____________.

Solution:

Let the conic section be $g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0$. Let the line be $\mathbf{x} = \mathbf{h} + \kappa\mathbf{m}$. To find the points of intersection, we substitute the line equation into the conic equation.

$$g(\mathbf{h} + \kappa\mathbf{m}) = (\mathbf{h} + \kappa\mathbf{m})^\top \mathbf{V}(\mathbf{h} + \kappa\mathbf{m}) + 2\mathbf{u}^\top(\mathbf{h} + \kappa\mathbf{m}) + f = 0 \tag{1}$$

$$= \left(\mathbf{h}^\top + \kappa\mathbf{m}^\top\right)\mathbf{V}(\mathbf{h} + \kappa\mathbf{m}) + 2\mathbf{u}^\top\mathbf{h} + 2\kappa\mathbf{u}^\top\mathbf{m} + f = 0 \tag{2}$$

$$= \mathbf{h}^\top\mathbf{V}\mathbf{h} + 2\kappa\mathbf{m}^\top\mathbf{V}\mathbf{h} + \kappa^2\mathbf{m}^\top\mathbf{V}\mathbf{m} + 2\mathbf{u}^\top\mathbf{h} + 2\kappa\mathbf{u}^\top\mathbf{m} + f = 0 \tag{3}$$

$$= \left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)\kappa^2 + 2\left(\mathbf{m}^\top\mathbf{V}\mathbf{h} + \mathbf{m}^\top\mathbf{u}\right)\kappa + \left(\mathbf{h}^\top\mathbf{V}\mathbf{h} + 2\mathbf{u}^\top\mathbf{h} + f\right) = 0 \tag{4}$$

$$= \left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)\kappa^2 + 2\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u})\kappa + g(\mathbf{h}) = 0 \tag{5}$$

This is a quadratic equation in κ.

$$\kappa_{1,2} = \frac{-2\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u}) \pm \sqrt{4\left(\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u})\right)^2 - 4\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)g(\mathbf{h})}}{2\mathbf{m}^\top\mathbf{V}\mathbf{m}} \tag{6}$$

$$\kappa_{1,2} = \frac{-\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u}) \pm \sqrt{\left(\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u})\right)^2 - \left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)g(\mathbf{h})}}{\mathbf{m}^\top\mathbf{V}\mathbf{m}} \tag{7}$$

Using (7) to find the intersection points that define the boundaries of the area,

$$\text{Circle: } x^2 + y^2 - 32 = 0 \implies \mathbf{V} = \mathbf{I}, \mathbf{u} = \mathbf{0}, f = -32$$

$$\text{Lines: } \mathbf{x} = \kappa\begin{pmatrix}1\\0\end{pmatrix}, \text{ so } \mathbf{h_1} = \mathbf{0}, \mathbf{m_1} = \begin{pmatrix}1\\0\end{pmatrix} \text{ and } \mathbf{x} = \kappa\begin{pmatrix}1\\1\end{pmatrix}, \text{ so } \mathbf{h_2} = \mathbf{0}, \mathbf{m_2} = \begin{pmatrix}1\\1\end{pmatrix}$$

$$g(\mathbf{h_1}) = g(\mathbf{0}) = -32 \tag{8}$$

$$\mathbf{m_1}^\top\mathbf{V}\mathbf{m_1} = \begin{pmatrix}1 & 0\end{pmatrix}\mathbf{I}\begin{pmatrix}1\\0\end{pmatrix} = 1 \tag{9}$$

$$\mathbf{m_1}^\top(\mathbf{V}\mathbf{h_1} + \mathbf{u}) = \begin{pmatrix}1 & 0\end{pmatrix}(\mathbf{I}\mathbf{0} + \mathbf{0}) = 0 \tag{10}$$

$$\kappa = \frac{0 \pm \sqrt{0^2 - (1)(-32)}}{1} = \pm\frac{\sqrt{32}}{1} = \pm 4\sqrt{2} \tag{11}$$

$$g(\mathbf{h_2}) = g(\mathbf{0}) = -32 \tag{12}$$

$$\mathbf{m_2}^\top \mathbf{V} \mathbf{m_2} = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{I} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 \tag{13}$$

$$\mathbf{m_2}^\top (\mathbf{V}\mathbf{h_2} + \mathbf{u}) = \begin{pmatrix} 1 & 1 \end{pmatrix} (\mathbf{I}\mathbf{0} + \mathbf{0}) = 0 \tag{14}$$

$$\kappa = \frac{0 \pm \sqrt{0^2 - (2)(-32)}}{2} = \pm\frac{\sqrt{64}}{2} = \pm 4 \tag{15}$$

The intersection points are

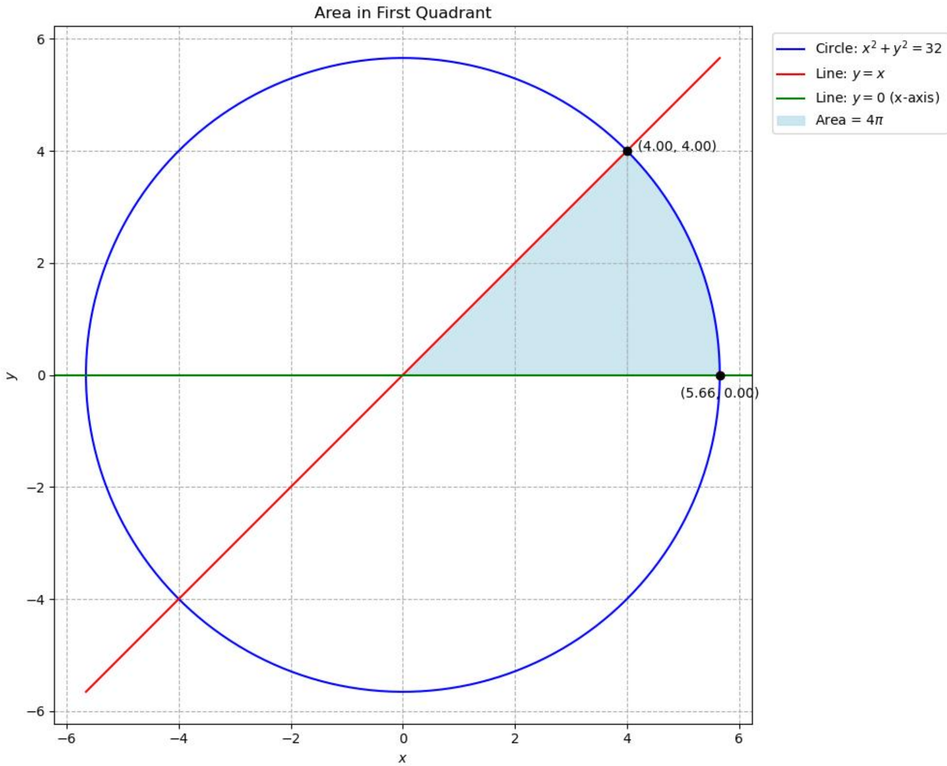
$$\mathbf{x_i} = \kappa \mathbf{m_i} \tag{16}$$

In the first quadrant, the intersection points defining the region are:

$$\mathbf{x_1} = \begin{pmatrix} 4\sqrt{2} \\ 0 \end{pmatrix}, \ \mathbf{x_2} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{17}$$

The area is the sum of two integrals, split at the x-coordinate of $\mathbf{x_2}$.

$$\begin{aligned}
A &= \int_0^4 x\,dx + \int_4^{4\sqrt{2}} \sqrt{32 - x^2}\,dx && (18)\\
&= \left[\frac{x^2}{2}\right]_0^4 + \left[\frac{x}{2}\sqrt{32 - x^2} + 16\sin^{-1}\left(\frac{x}{4\sqrt{2}}\right)\right]_4^{4\sqrt{2}} && (19)\\
&= \frac{16}{2} + \left[0 + 16\sin^{-1}(1)\right] - \left[\frac{4}{2}\sqrt{16} + 16\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right] && (20)\\
&= 8 + 16\left(\frac{\pi}{2}\right) - \left[8 + 16\left(\frac{\pi}{4}\right)\right] && (21)\\
&= 8 + 8\pi - 8 - 4\pi = 4\pi && (22)
\end{aligned}$$


Area in First Quadrant
Circle: $x^2 + y^2 = 32$
Line: $y = x$
Line: $y = 0$ (x-axis)
Area = 4π
(4.00, 4.00)
(5.66, 0.00)
x
y
−6
−4
−2
0
2
4
6

Plot