8.4.5

EE25BTECH11002 - Achat Parth Kalpesh

Question:
An ellipse is drawn by taking a diameter of the circle $(x-1)^2 + y^2 = 1$ as its semi minor axis and a diameter of the circle $x^2 + (y-2)^2 = 4$ as semi-major axis. If the centre of the ellipse is at the origin and its axes are the coordinate axes, then the equation of the ellipse is

1) $4x^2 + y^2 = 4$
2) $x^2 + 4y^2 = 8$
3) $4x^2 + y^2 = 8$
4) $x^2 + 4y^2 = 1$

Solution:
The standard equation of a circle is given as

$$(\mathbf{x}-\mathbf{c})^\top(\mathbf{x}-\mathbf{c}) = r^2 \tag{4.1}$$

Given two circles are

$$(\mathbf{x}-\mathbf{c_1})^\top(\mathbf{x}-\mathbf{c_1}) = 1 \tag{4.2}$$
$$(\mathbf{x}-\mathbf{c_2})^\top(\mathbf{x}-\mathbf{c_2}) = 4 \tag{4.3}$$

The centers and radii are

$$\mathbf{c_1} = \begin{pmatrix}1\\0\end{pmatrix}, \quad r_1 = 1 \tag{4.4}$$
$$\mathbf{c_2} = \begin{pmatrix}0\\2\end{pmatrix}, \quad r_2 = 2 \tag{4.5}$$

Verifing that the origin lies on both circles:

$$(\mathbf{0}-\mathbf{c_1})^\top(\mathbf{0}-\mathbf{c_1}) = 1 = r_1^2 \tag{4.6}$$
$$(\mathbf{0}-\mathbf{c_2})^\top(\mathbf{0}-\mathbf{c_2}) = 4 = r_2^2 \tag{4.7}$$

Thus, the diameters of both circles passing through the origin are along the directions

$$\mathbf{c_1} = \begin{pmatrix}1\\0\end{pmatrix} \text{ (along X-axis)} \tag{4.8}$$
$$\mathbf{c_2} = \begin{pmatrix}0\\2\end{pmatrix} \text{ (along Y-axis)} \tag{4.9}$$

Each circle's diameter length is $2r$. Therefore, the ellipse's semi-axes are equal to the respective radii:

$$b = r_1 = 1 \quad \text{(semi-minor axis)} \tag{4.10}$$
$$a = r_2 = 2 \quad \text{(semi-major axis)} \tag{4.11}$$

The standard equation of an ellipse centered at the origin with coordinate axes as its axes is

$$\mathbf{x}^\top A\mathbf{x} = 1 \tag{4.12}$$

where

$$A = \begin{pmatrix} \frac{1}{b^2} & 0 \\ 0 & \frac{1}{a^2} \end{pmatrix} \tag{4.13}$$

Substituting $a = 2$, $b = 1$,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \tag{4.14}$$

Hence,

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \mathbf{x} = 1 \tag{4.15}$$

Multiplying throughout by 4 gives

$$\mathbf{x}^\top \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = 4 \tag{4.16}$$

or equivalently,

$$4x^2 + y^2 = 4 \tag{4.17}$$

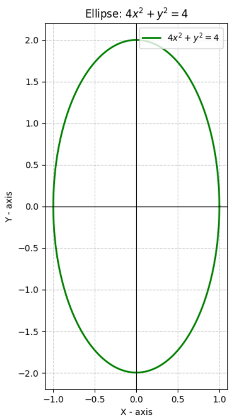

Fig. 4.1: Ellipse