4.9.3

EE25BTECH11001 - Aarush Dilawri

Question:
Find the equations of the two lines passing through the origin which intersect the line $\frac{x-3}{2} = \frac{y-3}{1} = \frac{z}{1}$ at angles of $\frac{\pi}{3}$ each.
Solution:
The given line can be expressed as

$$\mathbf{x} = \mathbf{h} + \kappa\mathbf{m} \tag{0.1}$$

$$\text{where} \quad \mathbf{h} = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{m} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \tag{0.2}$$

Any point $\mathbf{P}$ on this line can be given as

$$\mathbf{P} = \mathbf{h} + \kappa\mathbf{m} \tag{0.3}$$

The line through the origin and $\mathbf{P}$ will have direction vector $\mathbf{P}$.

Since the angle between $\mathbf{m}$ and $\mathbf{P}$ is $\frac{\pi}{3}$,

$$\cos\theta = \frac{\mathbf{m}^\top \mathbf{P}}{\|\mathbf{m}\| \, \|\mathbf{P}\|} \tag{0.4}$$

$$\implies \left(\mathbf{m}^\top \mathbf{P}\right)^2 = \cos^2\theta \, (\mathbf{m}^\top \mathbf{m})(\mathbf{P}^\top \mathbf{P}). \tag{0.5}$$

Substituting $\mathbf{P} = \mathbf{h} + \kappa\mathbf{m}$ and solving, we get a quadratic equation in κ

$$\kappa^2 \left(\mathbf{m}^\top \mathbf{m}\right)^2 \sin^2\theta + 2\kappa \left(\mathbf{m}^\top \mathbf{m}\right)\left(\mathbf{m}^\top \mathbf{h}\right) \sin^2\theta + \left(\mathbf{m}^\top \mathbf{h}\right)^2 - \mathbf{m}^\top \mathbf{m} \cos^2\theta \mathbf{h}^\top \mathbf{h} = 0 \tag{0.6}$$

Plugging in the values,

$$27\kappa^2 + 81\kappa + 54 = 0 \tag{0.7}$$

$$\kappa^2 + 3\kappa + 2 = 0 \tag{0.8}$$

$$\implies \kappa = -1, -2 \tag{0.9}$$

Therefore, the direction vectors of the lines are

$$\mathbf{x} = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} - 2\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \tag{0.10}$$

Therefore, the equations of the lines are

$$\mathbf{x} = \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \mu \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \tag{0.11}$$

See Figure,