8.2.56

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Question

Given the ellipse with equation $9x^2 + 25y^2 = 225$, find the eccentricity and foci.

Solution:

Step 1: Represent the Ellipse in Matrix Form

$$\text{The given equation of the ellipse is } 9x^2 + 25y^2 = 225 \tag{1}$$

$$\text{The general form of conic is} g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{2}$$

By rearranging the terms:

$$9x^2 + 25y^2 - 225 = 0 \tag{3}$$

By comparing the equation to the general form, we identify the matrices and vectors:

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} 9 & 0 \\ 0 & 25 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad f = -225 \tag{4}$$

Theoretical Solution

Step 2: Find the Eccentricity

The eccentricity e is given by the formula $e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}}$, where λ_1 and λ_2 are the eigenvalues of the matrix $\mathbf{V}$. For our diagonal matrix $\mathbf{V}$, the eigenvalues are the diagonal entries: $\lambda_1 = 9 \quad \text{and} \quad \lambda_2 = 25$

$$\text{Using the formula:} e = \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{25-9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5} \tag{5}$$

The eccentricity of the ellipse is $\frac{4}{5}$.
Step 3: Find the Foci

The foci lie on the major axis, and their location depends on the center and the distance ae.
The center $\mathbf{c}$ of the conic is given by the formula $\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u}$. Since $\mathbf{u}$ is the zero vector, the center is at the origin: $\mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Foci Location

The major axis of the ellipse corresponds to the eigenvector of the smaller eigenvalue of $\mathbf{V}$, which is $\lambda_1 = 9$. The eigenvector for $\lambda_1 = 9$ is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which lies along the x-axis.
The distance from the center to each focus is ae.

$$ae = \left(\sqrt{\frac{f_0}{|\lambda_1|}} \right) e \tag{6}$$

$$\text{where} f_0 = \mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} - f = 0 - (-225) = 225. \tag{7}$$

Theoretical Solution

$$ae = \left(\sqrt{\frac{225}{9}}\right)\left(\frac{4}{5}\right) = \sqrt{25} \times \frac{4}{5} = 5 \times \frac{4}{5} = 4 \tag{8}$$

Since the center is at the origin and the major axis is on the x-axis, the foci are at $(\pm 4, 0)$.
The foci of the ellipse are at $(4, 0)$ and $(-4, 0)$.

C Code

```c
#include <stdio.h>
#include <math.h>

int main() {
    // --- Step 1: Represent the Ellipse in Matrix Form ---
    // The equation is 9x^2 + 25y^2 = 225
    // In matrix form: x^T * V * x + 2u^T * x + f = 0
    // V = [[9, 0], [0, 25]]
    // u = [0, 0]
    // f = -225

    double V[2][2] = {{9.0, 0.0}, {0.0, 25.0}};
    double u[2] = {0.0, 0.0};
    double f = -225.0;

    // --- Step 2: Find the Eccentricity ---
    // The eigenvalues of V are the diagonal elements.
```

C Code

```
double lambda1 = V[0][0]; // 9
   double lambda2 = V[1][1]; // 25

   // The eccentricity formula is e = sqrt(1 - lambda1/lambda2)
   double eccentricity = sqrt(1.0 - (lambda1 / lambda2));

   printf("The eccentricity of the ellipse is: %.2f\n\n",
       eccentricity);

   // --- Step 3: Find the Foci ---
   // The center is c = -V^-1 * u. Since u is the zero vector,
       the center is at (0, 0).
   double center_x = 0.0;
   double center_y = 0.0;
```

C Code

```c
// The major axis length is 2*a, where a = sqrt(f0 / |lambda_min|)
    // f0 = u^T*V^-1*u - f = 0 - f = -f
    double f0 = -f; // 225

    // The semi-major axis 'a' corresponds to the smaller eigenvalue (9).
    double semi_major_axis = sqrt(f0 / lambda1); // sqrt(225/9) = sqrt(25) = 5

    // The distance from the center to each focus is 'ae'
    double foci_distance = semi_major_axis * eccentricity;
```

C Code

```
    // The foci are located on the major axis (the x-axis,
        corresponding to lambda1)
    printf("The foci are located at (+/- ae, 0):\n");
    printf("Foci: (%.2f, %.2f) and (%.2f, %.2f)\n", foci_distance
        , 0.0, -foci_distance, 0.0);

    return 0;
}
```

Python Code

```
import numpy as np
import matplotlib.pyplot as plt

def plot_ellipse_solution():
    # Given equation: 9x^2 + 25y^2 = 225

    # 1. Find a, b, e, and the foci
    a = np.sqrt(25)
    b = np.sqrt(9)

    eccentricity = np.sqrt(1 - (b**2 / a**2))

    # Distance from center to foci
    c_foci = a * eccentricity

    foci_coords = [(-c_foci, 0), (c_foci, 0)]

    print(f"Semi-major axis (a): {a}")
    print(f"Semi-minor axis (b): {b}")
```

Python Code

```
print(f"Eccentricity (e): {eccentricity:.2f}")
  print(f"Foci: ({foci_coords[0][0]:.2f}, {foci_coords[0][1]:.2f}) and ({foci_coords[1][0]:.2f}, {foci_coords[1][1]:.2f})")

  # 2. Plotting
  theta = np.linspace(0, 2 * np.pi, 100)
  x = a * np.cos(theta)
  y = b * np.sin(theta)

  fig, ax = plt.subplots(figsize=(8, 8))
  ax.plot(x, y, label=r'Ellipse $9x^2 + 25y^2 = 225$')

  # Plot the foci
  ax.plot(foci_coords[0][0], foci_coords[0][1], 'ro', label=f'Foci')
  ax.plot(foci_coords[1][0], foci_coords[1][1], 'ro')
```

Python Code

```python
# Annotate the foci and center
    ax.annotate(f'({foci_coords[0][0]:.0f}, {foci_coords[0][1]:.0f})', foci_coords[0], textcoords="offset points", xytext=(-15, 10), ha='center')
    ax.annotate(f'({foci_coords[1][0]:.0f}, {foci_coords[1][1]:.0f})', foci_coords[1], textcoords="offset points", xytext=(15, 10), ha='center')
    ax.plot(0, 0, 'go', label='Center')

    # Add eccentricity label to the plot title
    ax.set_title(f'Ellipse with Eccentricity e = {eccentricity:.2f}')
    ax.set_xlabel('x')
    ax.set_ylabel('y')
```

Python Code

```
     ax.grid(True, linestyle='--')
    ax.axhline(0, color='black', linewidth=0.5)
    ax.axvline(0, color='black', linewidth=0.5)
    ax.set_aspect('equal', adjustable='box')
    ax.legend()
    plt.show()

plot_ellipse_solution()
```

Plot

Beamer/figs/ellipse.png