9.2.33

EE25BTECH11065-Yoshita J

September 14,2025

Question

Find the area of the region enclosed by the parabola $x^2 = y$ and the line $y = x + 2$, using the matrix formulation of conics and the intersection-of-line-with-conic formula.

Theoretical Solution

The given ellipse can be expressed as conics with parameters,

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0, \qquad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

where,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{u} = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}, \qquad f = 0. \tag{1}$$

The line parameters are

$$\mathbf{x} = \mathbf{h} + \kappa\,\mathbf{m}, \;\; \kappa \in \mathbb{R}.$$

where,

$$\mathbf{h} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \qquad \mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{2}$$

Theoretical Solution

Substituting the given parameters to find the intersection point,

$$\kappa = \frac{1}{\mathbf{m}^\top \mathbf{V}\mathbf{m}}\left(- \mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u}) \pm \sqrt{\left(\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u})\right)^2 - g(\mathbf{h})\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)}\right), \tag{3}$$

where

$$g(\mathbf{h}) = \mathbf{h}^\top\mathbf{V}\mathbf{h} + 2\mathbf{u}^\top\mathbf{h} + f. \tag{4}$$

Solving,

$$\mathbf{m}^\top\mathbf{V}\mathbf{m} = \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 1. \tag{5}$$

$$\mathbf{V}\mathbf{h}+\mathbf{u} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}0\\2\end{pmatrix} + \begin{pmatrix}0\\-\frac{1}{2}\end{pmatrix} = \begin{pmatrix}0\\-\frac{1}{2}\end{pmatrix}. \tag{6}$$

Theoretical Solution

$$\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u}) = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} = -\frac{1}{2}. \tag{7}$$

$$g(\mathbf{h}) = \mathbf{h}^\top\mathbf{V}\mathbf{h} + 2\mathbf{u}^\top\mathbf{h} = 0 + 2\begin{pmatrix} 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = -2. \tag{8}$$

Now the discriminant,

$$\left(\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u})\right)^2 - g(\mathbf{h})\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right) = \left(-\frac{1}{2}\right)^2 - (-2)\cdot 1 = \frac{1}{4} + 2 = \frac{9}{4}, \tag{9}$$

so

$$\sqrt{\cdot} = \frac{3}{2}. \tag{10}$$

Theoretical Solution

Hence

$$\kappa = -(-\tfrac{1}{2}) \pm \tfrac{3}{2} = \tfrac{1}{2} \pm \tfrac{3}{2} \implies \kappa_1 = 2,\ \kappa_2 = -1. \tag{11}$$

Points of intersection,

$$\mathbf{x}_i = \mathbf{h} + \kappa_i \mathbf{m} \tag{12}$$

$$\mathbf{x}_1 = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \qquad \mathbf{x}_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix} - 1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \tag{13}$$

Thus the intersection points are

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 \\ 4 \end{pmatrix}. \tag{14}$$

Theoretical Solution

Area of the enclosed region,

$$A = \int_{-1}^{2} \left[(x+2) - x^2\right] dx = \frac{9}{2}. \tag{15}$$

Therefore the area of the region enclosed by $x^2 = y$ and $y = x + 2$ is

$$\boxed{\frac{9}{2}}$$

C Code

```
#include <stdio.h>

double calculate_area() {
    double upper_bound_integral = (2.0 * 2.0 / 2.0) + (2.0 * 2.0)
         - (2.0 * 2.0 * 2.0 / 3.0);
    double lower_bound_integral = (-1.0 * -1.0 / 2.0) + (2.0 *
        -1.0) - (-1.0 * -1.0 * -1.0 / 3.0);
    return upper_bound_integral - lower_bound_integral;
}
```

Python Code

```
import numpy as np
import matplotlib.pyplot as plt

def parabola(x):
    return x**2

def line(x):
    return x + 2

x_range = np.linspace(-3, 4, 400)
y_parabola = parabola(x_range)
y_line = line(x_range)

fig, ax = plt.subplots(figsize=(8, 6))
```

Python Code

```python
ax.plot(x_range, y_parabola, 'b-', label='$y = x^2$')
ax.plot(x_range, y_line, 'r-', label='$y = x + 2$')

x_fill = np.linspace(-1, 2, 100)
ax.fill_between(x_fill, parabola(x_fill), line(x_fill), color='
    gray', alpha=0.3, label='Area = 9/2')

intersection_points_x = [-1, 2]
intersection_points_y = [1, 4]
ax.plot(intersection_points_x, intersection_points_y, 'ko')

ax.text(-1, 1, ' (-1, 1)', verticalalignment='bottom',
    horizontalalignment='right')
ax.text(2, 4, ' (2, 4)', verticalalignment='bottom',
    horizontalalignment='left')
```

Python Code

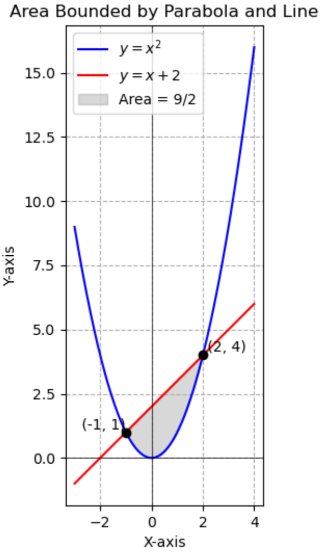
```
ax.set_title('Area Bounded by Parabola and Line')
ax.set_xlabel('X-axis')
ax.set_ylabel('Y-axis')
ax.grid(True, linestyle='--')
ax.legend()

ax.axhline(0, color='black', linewidth=0.5)
ax.axvline(0, color='black', linewidth=0.5)

ax.set_aspect('equal', adjustable='box')

plt.show()
```

Plot



Area Bounded by Parabola and Line
$y = x^2$
$y = x + 2$
Area = 9/2
(2, 4)
(-1, 1)
Y-axis
X-axis
15.0
12.5
10.0
7.5
5.0
2.5
0.0
−2
0
2
4