

Problem 12.349

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Problem

Let $T_1, T_2 : \mathbb{R}_5 \rightarrow \mathbb{R}_3$ be linear transformations such that $\text{rank}(T_1) = 3$ and $\text{nullity}(T_2) = 3$. Let $T_3 : \mathbb{R}_3 \rightarrow \mathbb{R}_3$ be a linear transformation such that $T_3 \circ T_1 = T_2$. Then $\text{rank}(T_3)$ is ... (MA 2014)

Given

According to Rank-Nullity theorem,

For a linear transformation $T : \mathbb{R}_m \rightarrow \mathbb{R}_n$

$$\text{rank}(T) + \text{nullity}(T) = \dim(\text{domain}) \quad (3.1)$$

where $\dim \mathbb{R}_m$ is the dimension of the domain i.e vector space \mathbb{R}_m

Given $T_2 : \mathbb{R}_5 \rightarrow \mathbb{R}_3$ and $\text{nullity}(T_2)=3$

$$\text{rank}(T_2) + \text{nullity}(T_2) = \dim \mathbb{R}^5 \quad (3.2)$$

$$\text{rank}(T_2) + 3 = 5 \quad (3.3)$$

$$\text{rank}(T_2) = 2 \quad (3.4)$$

Given $T_1 : \mathbb{R}_5 \rightarrow \mathbb{R}_3$ and $\text{rank}(T_1)=3$

$$\dim(\text{Co-domain}) = 3 \quad (3.5)$$

$$\text{rank}(T_1) = \dim(\text{Co-domain}) \quad (3.6)$$

Conclusion

It is onto and hence

$$\dim(\operatorname{Im}(T_1)) = \dim(\text{Co-domain}) \quad (3.7)$$

$$\dim(\operatorname{Im}(T_1)) = 3 \implies \operatorname{Im}(T_1) = \mathbb{R}_3 \quad (3.8)$$

where $\operatorname{Im}(T)$ is the Image space of the linear transformation T

Given $T_3 : \mathbb{R}_3 \rightarrow \mathbb{R}_3$

$$T_3 \circ T_1 = T_2 \quad (3.9)$$

$$(T_3 \circ T_1)(\mathbb{R}_5) = \operatorname{Im}(T_2) \quad (3.10)$$

$$T_3(T_1(\mathbb{R}_5)) = \operatorname{Im}(T_2) \quad (3.11)$$

$$T_3(\operatorname{Im} T_1) = \operatorname{Im}(T_2) \quad (3.12)$$

$$T_3(\mathbb{R}_3) = \operatorname{Im}(T_2) \quad (3.13)$$

$$\operatorname{Im}(T_3) = \operatorname{Im}(T_2) \quad (3.14)$$

$$\implies \operatorname{rank}(T_3) = \operatorname{rank}(T_2) \quad (3.15)$$

From (4)

$$\operatorname{rank}(T_3) = 2 \quad (3.16)$$