9.2.6

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Question

Area of the region in the first quadrant enclosed by the x-axis, the line y $=$ x and the circle $x^2 + y^2 = 32$ is ____________.

Theoretical Solution

Let the conic section be $g\left(\mathbf{x}\right) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0$. Let the line be $\mathbf{x} = \mathbf{h} + \kappa\mathbf{m}$. To find the points of intersection, we substitute the line equation into the conic equation.

$$
\begin{aligned}
g\left(\mathbf{h} + \kappa\mathbf{m}\right) &= \left(\mathbf{h} + \kappa\mathbf{m}\right)^\top \mathbf{V}\left(\mathbf{h} + \kappa\mathbf{m}\right) + 2\mathbf{u}^\top\left(\mathbf{h} + \kappa\mathbf{m}\right) + f = 0 && (1)\\
&= \left(\mathbf{h}^\top + \kappa\mathbf{m}^\top\right)\mathbf{V}\left(\mathbf{h} + \kappa\mathbf{m}\right) + 2\mathbf{u}^\top\mathbf{h} + 2\kappa\mathbf{u}^\top\mathbf{m} + f = 0 && (2)\\
&= \mathbf{h}^\top\mathbf{V}\mathbf{h} + 2\kappa\mathbf{m}^\top\mathbf{V}\mathbf{h} + \kappa^2\mathbf{m}^\top\mathbf{V}\mathbf{m} + 2\mathbf{u}^\top\mathbf{h} + 2\kappa\mathbf{u}^\top\mathbf{m} + f = 0 && (3)\\
&= \left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)\kappa^2 + 2\left(\mathbf{m}^\top\mathbf{V}\mathbf{h} + \mathbf{m}^\top\mathbf{u}\right)\kappa + \left(\mathbf{h}^\top\mathbf{V}\mathbf{h} + 2\mathbf{u}^\top\mathbf{h} + f\right) && (4)\\
&= \left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)\kappa^2 + 2\mathbf{m}^\top\left(\mathbf{V}\mathbf{h} + \mathbf{u}\right)\kappa + g\left(\mathbf{h}\right) = 0 && (5)
\end{aligned}
$$

This is a quadratic equation in κ.

$$\kappa_{1,2} = \frac{-2\mathbf{m}^\top\left(\mathbf{Vh}+\mathbf{u}\right) \pm \sqrt{4\left(\mathbf{m}^\top\left(\mathbf{Vh}+\mathbf{u}\right)\right)^2 - 4\left(\mathbf{m}^\top\mathbf{Vm}\right)g\left(\mathbf{h}\right)}}{2\mathbf{m}^\top\mathbf{Vm}} \tag{6}$$

$$\kappa_{1,2} = \frac{-\mathbf{m}^\top\left(\mathbf{Vh}+\mathbf{u}\right) \pm \sqrt{\left(\mathbf{m}^\top\left(\mathbf{Vh}+\mathbf{u}\right)\right)^2 - \left(\mathbf{m}^\top\mathbf{Vm}\right)g\left(\mathbf{h}\right)}}{\mathbf{m}^\top\mathbf{Vm}} \tag{7}$$

Using (7) to find the intersection points that define the boundaries of the area,

$$\text{Circle: } x^2+y^2-32=0 \implies \mathbf{V}=\mathbf{I}, \mathbf{u}=\mathbf{0}, f=-32$$

$$\text{Lines: } \mathbf{x}=\kappa\begin{pmatrix}1\\0\end{pmatrix}, \text{ so } \mathbf{h_1}=\mathbf{0}, \mathbf{m_1}=\begin{pmatrix}1\\0\end{pmatrix} \text{ and}$$

$$\mathbf{x}=\kappa\begin{pmatrix}1\\1\end{pmatrix}, \text{ so } \mathbf{h_2}=\mathbf{0}, \mathbf{m_2}=\begin{pmatrix}1\\1\end{pmatrix}$$

Theoretical Solution

$$g\left(\mathbf{h_1}\right) = g\left(\mathbf{0}\right) = -32 \tag{8}$$

$$\mathbf{m_1}^\top \mathbf{V}\mathbf{m_1} = \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{I} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \tag{9}$$

$$\mathbf{m_1}^\top \left(\mathbf{V}\mathbf{h_1} + \mathbf{u}\right) = \begin{pmatrix} 1 & 0 \end{pmatrix} \left(\mathbf{I}\mathbf{0} + \mathbf{0}\right) = 0 \tag{10}$$

$$\kappa = \frac{0 \pm \sqrt{0^2 - (1)(-32)}}{1} = \pm\frac{\sqrt{32}}{1} = \pm 4\sqrt{2} \tag{11}$$

Theoretical Solution

$$g\left(\mathbf{h_2}\right) = g\left(\mathbf{0}\right) = -32 \tag{12}$$

$$\mathbf{m_2}^\top \mathbf{V} \mathbf{m_2} = \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{I} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 \tag{13}$$

$$\mathbf{m_2}^\top \left(\mathbf{V}\mathbf{h_2} + \mathbf{u}\right) = \begin{pmatrix} 1 & 1 \end{pmatrix} \left(\mathbf{I}\mathbf{0} + \mathbf{0}\right) = 0 \tag{14}$$

$$\kappa = \frac{0 \pm \sqrt{0^2 - (2)(-32)}}{2} = \pm \frac{\sqrt{64}}{2} = \pm 4 \tag{15}$$

Theoretical Solution

The intersection points are

$$\mathbf{x_i} = \kappa \mathbf{m_i} \tag{16}$$

In the first quadrant, the intersection points defining the region are:

$$\mathbf{x_1} = \begin{pmatrix} 4\sqrt{2} \\ 0 \end{pmatrix}, \; \mathbf{x_2} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{17}$$

Theoretical Solution

The area is the sum of two integrals, split at the x-coordinate of $\mathbf{x_2}$.

$$A = \int_0^4 x\,dx + \int_4^{4\sqrt{2}} \sqrt{32 - x^2}\,dx \tag{18}$$

$$= \left[\frac{x^2}{2}\right]_0^4 + \left[\frac{x}{2}\sqrt{32 - x^2} + 16\sin^{-1}\left(\frac{x}{4\sqrt{2}}\right)\right]_4^{4\sqrt{2}} \tag{19}$$

$$= \frac{16}{2} + \left[0 + 16\sin^{-1}(1)\right] - \left[\frac{4}{2}\sqrt{16} + 16\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right] \tag{20}$$

$$= 8 + 16\left(\frac{\pi}{2}\right) - \left[8 + 16\left(\frac{\pi}{4}\right)\right] \tag{21}$$

$$= 8 + 8\pi - 8 - 4\pi = 4\pi \tag{22}$$

Plot

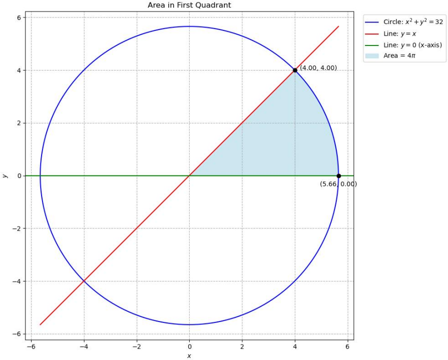

Figure: Plot