AI25BTECH11034 - SUJAL CHAUHAN

4.8.30

Question:
Find the equation of a line passing through the point (2,3,2) and parallel to the line $\mathbf{r} = (-2\hat{i} + 3\hat{j}) + \lambda(2\hat{i} - 3\hat{j} + 6\hat{k})$. Also, find the distance beteween these two lines.

Theory:

Consider two parallel lines in 3D:

$$\mathbf{r}_1 = \mathbf{a}_1 + \lambda\mathbf{b}, \quad \lambda \in \mathbb{R}, \tag{1}$$

$$\mathbf{r}_2 = \mathbf{a}_2 + \mu\mathbf{b}, \quad \mu \in \mathbb{R}, \tag{2}$$

where $\mathbf{a}_1, \mathbf{a}_2$ are points on the respective lines and $\mathbf{b}$ is the common direction vector.

The vector $\mathbf{a}_2 - \mathbf{a}_1$ lies in the plane spanned by $\{\mathbf{a}_2 - \mathbf{a}_1, \mathbf{b}\}$. To find the shortest distance between the lines, we first determine a vector $\mathbf{n}$ that is orthogonal to both:

$$\mathbf{n}^T \begin{pmatrix} \mathbf{a}_2 - \mathbf{a}_1 & \mathbf{b} \end{pmatrix} = \mathbf{0}. \tag{3}$$

Solving this system yields an orthogonal vector $\mathbf{n}$. Then, the shortest distance d between the two parallel lines is the orthogonal projection of $(\mathbf{a}_2 - \mathbf{a}_1)$ onto the direction of $\mathbf{n}$:

$$d = \frac{(\mathbf{a}_2 - \mathbf{a}_1)^T\mathbf{n}}{\|\mathbf{n}\|}. \tag{4}$$

Solution:

The direction vector of the given parallel lines is

$$\mathbf{b} = \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}. \tag{5}$$

The first line is given by

$$\mathbf{r}_1 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}, \quad \mu \in \mathbb{R}. \tag{6}$$

The second line is

$$\mathbf{r}_2 = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}, \quad \lambda \in \mathbb{R}. \tag{7}$$

Now, the difference between the two given points on the lines is

$$\mathbf{a}_2 - \mathbf{a}_1 = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}. \tag{8}$$

To find the shortest distance, we first compute a vector $\mathbf{n}$ orthogonal to both $\mathbf{b}$ and $\mathbf{a}_2 - \mathbf{a}_1$. This requires solving

$$\mathbf{n}^T \begin{pmatrix} 2 & 4 \\ -3 & 0 \\ 6 & 2 \end{pmatrix} = \mathbf{0}. \tag{9}$$

On solving, we obtain

$$\mathbf{n} = k \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix}, \quad k \in \mathbb{R}. \tag{10}$$

Finally, the shortest distance between the two parallel lines is

$$d = \frac{\left|(\mathbf{a}_2 - \mathbf{a}_1)^T \mathbf{n}\right|}{\|\mathbf{n}\|} \tag{11}$$

$$= \frac{\left| \begin{pmatrix} 4 & 0 & 2 \end{pmatrix} \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix} \right|}{\sqrt{(-3)^2 + 2^2 + 2^2}} \tag{12}$$

$$= \frac{|-12 + 0 + 4|}{\sqrt{17}} \tag{13}$$

$$= \frac{8}{\sqrt{17}}. \tag{14}$$

Thus, the distance between the two parallel lines is

$$\boxed{\frac{8}{\sqrt{17}}}.$$

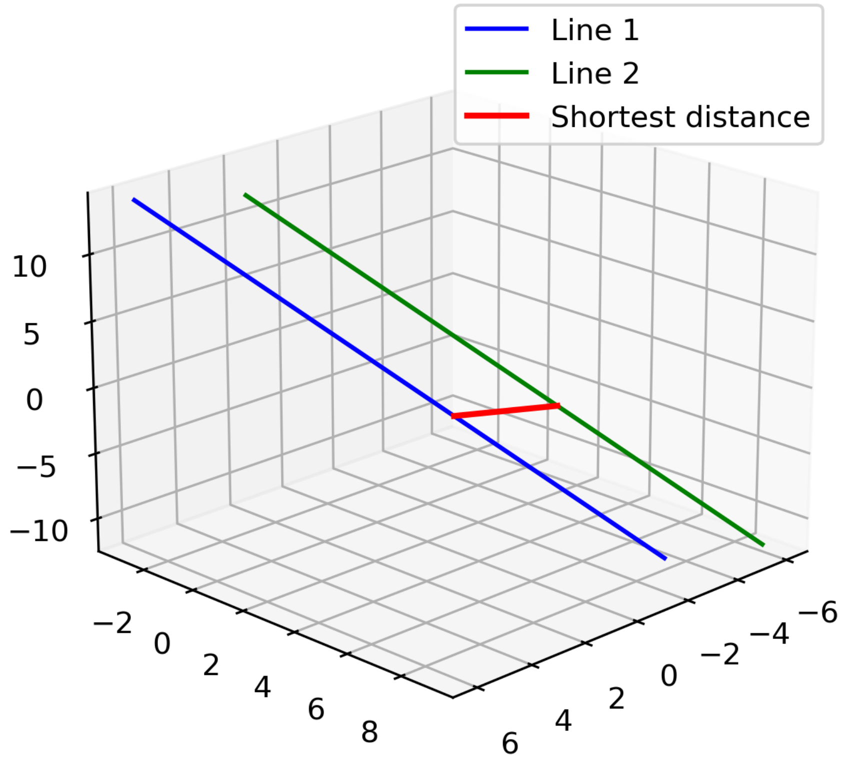

Figure 1: Shortest distance between two parallel lines