Matgeo-q.2.10.25

AI25BTECH11036-SNEHAMRUDULA

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Q 2.10.25. In $\triangle PQR$, let $\mathbf{a} = \overrightarrow{QR}$, $\mathbf{b} = \overrightarrow{RP}$ and $\mathbf{c} = \overrightarrow{PQ}$. If $\|\mathbf{a}\| = 12$, $\|\mathbf{b}\| = 4\sqrt{3}$, and $\mathbf{b} \cdot \mathbf{c} = 24$, then which of the following is/are true?

1. $\frac{\|\mathbf{c}\|^2}{2} - \|\mathbf{a}\| = 12$
2. $\frac{\|\mathbf{c}\|^2}{2} + \|\mathbf{a}\| = 30$
3. $\| \mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{a} \| = 48\sqrt{3}$
4. $\mathbf{a} \cdot \mathbf{b} = -72$

solution

Given $\|\mathbf{a}\| = 12, \quad \|\mathbf{b}\| = 4\sqrt{3}, \quad \mathbf{b} \cdot \mathbf{c} = 24,$ $\qquad(1)$

$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0} \ \Rightarrow \ \mathbf{c} = -(\mathbf{a} + \mathbf{b}).$ $\qquad(2)$

(d) $\quad \mathbf{b} \cdot \mathbf{c} = \mathbf{b} \cdot (-\mathbf{a} - \mathbf{b}) = -\mathbf{a} \cdot \mathbf{b} - \|\mathbf{b}\|^2 = 24$ $\qquad(3)$

$\Rightarrow \ \mathbf{a} \cdot \mathbf{b} = -(24 + \|\mathbf{b}\|^2) = -(24 + 48) = -72.$ $\qquad(4)$

(a),(b) $\quad \|\mathbf{c}\|^2 = \|\mathbf{a} + \mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + 2\,\mathbf{a} \cdot \mathbf{b} = 48.$ $\qquad(5)$

$\dfrac{\|\mathbf{c}\|^2}{2} - \|\mathbf{a}\| = 12 \ \text{(true)}, \quad \dfrac{\|\mathbf{c}\|^2}{2} + \|\mathbf{a}\| = 36 \ \text{(false)}.$ $\qquad(6)$

(c) $\quad \mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{a} = \mathbf{a} \times \mathbf{b} + (-\mathbf{a} - \mathbf{b}) \times \mathbf{a} = \mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{a} - \mathbf{b} \times \mathbf{a}$ $\qquad(7)$

$= \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{b} = 2\,(\mathbf{a} \times \mathbf{b}).$ $\qquad(8)$

$\qquad(9)$

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \, \|\mathbf{b}\|} = \frac{-72}{12 \cdot 4\sqrt{3}} = -\frac{\sqrt{3}}{2} \Rightarrow \sin\theta = \frac{1}{2}. \tag{10}$$

$$\therefore \ \|\mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{a}\| = 2\,\|\mathbf{a} \times \mathbf{b}\| = 2\,\|\mathbf{a}\|\,\|\mathbf{b}\|\sin\theta = 48\sqrt{3} \ \text{ (true)}. \tag{11}$$

Graphical Representation

Triangle PQR with vectors a, b, c