

Matrices in Geometry 5.13.63

EE25BTECH11037 - Divyansh

Question: Let $\mathbf{M} = \begin{pmatrix} \sin^4(\theta) & -1 - \sin^2(\theta) \\ 1 + \cos^2(\theta) & \cos^4(\theta) \end{pmatrix} = \alpha \mathbf{I} + \beta \mathbf{M}^{-1}$

Where $\alpha = \alpha(\theta)$ and $\beta = \beta(\theta)$ are real numbers, and \mathbf{I} is the 2×2 identity matrix. If α^* is the minimum of the set $(\alpha(\theta) : \theta \in [0, 2\pi))$ and β^* is the minimum of the set $(\beta(\theta) : \theta \in [0, 2\pi))$. Then the value of $\alpha^* + \beta^*$ is

1) $-\frac{31}{16}$

2) $-\frac{17}{16}$

3) $-\frac{37}{16}$

4) $-\frac{29}{16}$

Solution: Using the Cayley-Hamilton Theorem,

$$\mathbf{M}^2 - \text{tr}(\mathbf{M})\mathbf{M} + \det(\mathbf{M})\mathbf{I} = 0 \quad (1)$$

$$\implies \mathbf{M} - \text{tr}(\mathbf{M})\mathbf{I} + \det(\mathbf{M})\mathbf{M}^{-1} = 0 \quad (2)$$

The given expression is

$$\mathbf{M} - \alpha \mathbf{I} - \beta \mathbf{M}^{-1} = 0 \quad (3)$$

On comparing, we get

$$\alpha = \text{tr}(\mathbf{M}), \beta = -\det(\mathbf{M}) \quad (4)$$

$$\alpha(\theta) = \sin^4(\theta) + \cos^4(\theta) = 1 - 2\sin^2(\theta)\cos^2(\theta) \quad (5)$$

$$\implies \alpha = 1 - \sin^2(2\theta)/2 \quad (6)$$

$$\alpha^* = \min(\alpha(\theta)) = 1 - 1/2 = \frac{1}{2}, \quad (7)$$

(\because for minimizing α , $\sin^2(2\theta)$ should be maximum)

$$\beta(\theta) = -\det(\mathbf{M}) = -(\sin^4(\theta)\cos^4(\theta) + \sin^2(\theta)\cos^2(\theta) + 2) \quad (8)$$

$$\implies \beta = -((\sin(2\theta)/2)^4 + (\sin(2\theta)/2)^2 + 2) \quad (9)$$

$$\beta^* = -((1/2)^4 + (1/2)^2 + 2) = -\frac{37}{16} \quad (10)$$

(\because for minimizing β , $\sin^2(2\theta)$ should be maximum)

Now,

$$\alpha^* + \beta^* = \frac{1}{2} - \frac{37}{16} = -\frac{29}{16} \quad (11)$$

Thus, the correct option is option 4)