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# Question

Consider a circle with its centre lying on focus of the parabola  $y^2 = 2px$  such that it touches the directrix of the parabola. Then a point of intersection of the circle and the parabola is

1  $(\frac{p}{2}, p)$  or  $(\frac{p}{2}, -p)$

2  $(\frac{p}{2}, -\frac{p}{2})$

3  $(-\frac{p}{2}, p)$

4  $(-\frac{p}{2}, -\frac{p}{2})$

## Conic Representation

Any conic can be expressed as:

$$\mathbf{x}^T V \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad \text{where } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (1)$$

**Parabola:**  $x_2^2 = 2px_1$

Matrix form:

$$V_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{u}_1 = \begin{pmatrix} -p \\ 0 \end{pmatrix}, \quad f_1 = 0 \quad (2)$$

# Solution

**Circle: Center  $(\frac{p}{2}, 0)$ , Radius  $p$**

Expanded form:

$$(x_1 - \frac{p}{2})^2 + x_2^2 = p^2 \Rightarrow x_1^2 + x_2^2 - px_1 - \frac{3p^2}{4} = 0 \quad (3)$$

Matrix form:

$$V_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} -\frac{p}{2} \\ 0 \end{pmatrix}, \quad f_2 = -\frac{3p^2}{4} \quad (4)$$

## Parametric Line of Intersection

Using the parametric form of the chord of intersection between the parabola and the circle:

$$\mathbf{x}(\mu) = \mathbf{h} + \mu \mathbf{m} \quad (5)$$

Here,  $\mathbf{h} = \begin{pmatrix} \frac{p}{2} \\ 0 \end{pmatrix}$  is the center of the circle (also the focus of the parabola),  
and  $\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  is the direction vector of the vertical chord.

# Solution

Let the line be:

$$\mathbf{x}(\mu) = \mathbf{h} + \mu \mathbf{m} \quad \text{where } \mathbf{h} = \begin{pmatrix} \frac{p}{2} \\ 0 \end{pmatrix}, \quad \mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (6)$$

This gives:

$$\mathbf{x}(\mu) = \begin{pmatrix} \frac{p}{2} \\ \mu \end{pmatrix} \quad (7)$$

## Substitute into Parabola Equation

We evaluate:

$$\mathbf{x}(\mu)^T V_1 \mathbf{x}(\mu) + 2\mathbf{u}_1^T \mathbf{x}(\mu) + f_1 = 0 \quad (8)$$

Compute:

$$\mathbf{x}(\mu)^T V_1 \mathbf{x}(\mu) = \mu^2, \quad 2\mathbf{u}_1^T \mathbf{x}(\mu) = 2(-p)\left(\frac{p}{2}\right) = -p^2 \quad (9)$$

So:

$$\mu^2 - p^2 = 0 \Rightarrow \mu = \pm p \quad (10)$$

## Final Intersection Points

Substitute back:

$$\mathbf{x}(\mu) = \begin{pmatrix} \frac{p}{2} \\ \pm p \end{pmatrix} \quad (11)$$

Intersection points:

$$\mathbf{a}_1 = \begin{pmatrix} \frac{p}{2} \\ p \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} \frac{p}{2} \\ -p \end{pmatrix} \quad (12)$$



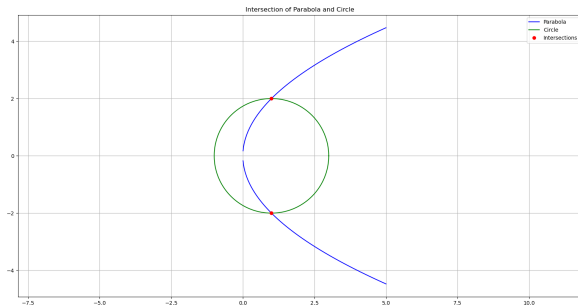


Figure: Caption