4.10.8

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Question

Show that the area of the triangle formed by the lines $y = m_1 x + c_1$, $y = m_2 x + c_2$ and $x = 0$ is

$$\frac{(c_1 - c_2)^2}{2|m_1 - m_2|}.$$

Solution

Vertices:

$$\mathbf{A} = \begin{pmatrix} 0 \\ c_1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 \\ c_2 \end{pmatrix}.$$

Intersection of the two lines (RREF):

$$\left(\begin{array}{cc|c} m_1 & -1 & -c_1 \\ m_2 & -1 & -c_2 \end{array}\right) \xrightarrow{R_2 \leftarrow R_2 - R_1} \left(\begin{array}{cc|c} m_1 & -1 & -c_1 \\ m_2 - m_1 & 0 & -(c_2 - c_1) \end{array}\right) \tag{1}$$

$$\Rightarrow (m_2 - m_1)x = -(c_2 - c_1)$$

$$\Rightarrow x^* = \frac{c_2 - c_1}{m_1 - m_2}, \quad y^* = m_1 x^* + c_1. \tag{2}$$

Vectors:

$$\mathbf{u} = \mathbf{AB} = \begin{pmatrix} 0 \\ c_2 - c_1 \end{pmatrix}, \quad \mathbf{v} = \mathbf{AC} = \begin{pmatrix} x^* \\ y^* - c_1 \end{pmatrix}.$$

Observe that $y^* - c_1 = m_1 x^*$, hence

$$\mathbf{v} = x^* \begin{pmatrix} 1 \\ m_1 \end{pmatrix}.$$

Compute norms and dot product:

$$\begin{aligned} \|\mathbf{u}\|^2 &= (c_2 - c_1)^2, && (3) \\ \|\mathbf{v}\|^2 &= x^{*2}(1 + m_1^2), && (4) \\ \mathbf{u} \cdot \mathbf{v} &= (c_2 - c_1)(m_1 x^*). && (5) \end{aligned}$$

Using $\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$:

$$\begin{aligned} \|\mathbf{u} \times \mathbf{v}\|^2 &= (c_2 - c_1)^2 x^{*2}(1 + m_1^2) - (c_2 - c_1)^2 m_1^2 x^{*2} && (6) \\ &= (c_2 - c_1)^2 x^{*2}. && (7) \end{aligned}$$

Thus,

$$\begin{aligned} \|\mathbf{u} \times \mathbf{v}\| &= |c_2 - c_1|\,|x^*| = |c_2 - c_1| \left| \frac{c_2 - c_1}{m_1 - m_2} \right| && (8) \\ &= \frac{(c_2 - c_1)^2}{|m_1 - m_2|}. && (9) \end{aligned}$$

Area:

$$\text{Area} = \tfrac{1}{2}\|\mathbf{u} \times \mathbf{v}\| = \frac{(c_1 - c_2)^2}{2|m_1 - m_2|}. \qquad (10)$$

$$\boxed{\frac{(c_1 - c_2)^2}{2|m_1 - m_2|}}$$

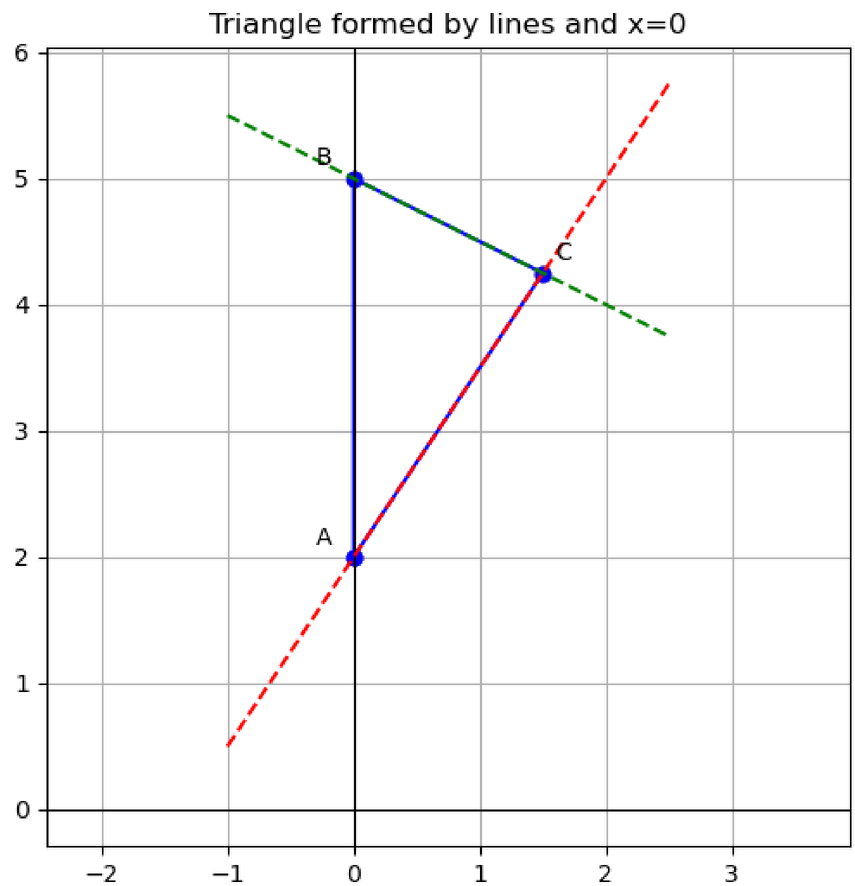

Figure 1: Triangle formed by the lines and $x = 0$