2.8.25

EE25BTECH11043 - Nishid Khandagre

Question: If $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are mutually perpendicular vectors of equal magnitudes, show that $\mathbf{A}+\mathbf{B}+\mathbf{C}$ is equally inclined to $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$.

Solution: Given:

$$\mathbf{A}^\top \mathbf{B} = 0 \tag{0.1}$$
$$\mathbf{B}^\top \mathbf{C} = 0 \tag{0.2}$$
$$\mathbf{C}^\top \mathbf{A} = 0 \tag{0.3}$$

$$\|\mathbf{A}\| = \|\mathbf{B}\| = \|\mathbf{C}\| = k \tag{0.4}$$

This implies:

$$\mathbf{A}^\top \mathbf{A} = \|\mathbf{A}\|^2 = k^2 \tag{0.5}$$
$$\mathbf{B}^\top \mathbf{B} = \|\mathbf{B}\|^2 = k^2 \tag{0.6}$$
$$\mathbf{C}^\top \mathbf{C} = \|\mathbf{C}\|^2 = k^2 \tag{0.7}$$

Let

$$\mathbf{R} = \left(\mathbf{A}+\mathbf{B}+\mathbf{C}\right) \tag{0.8}$$

The cosine of the angle θ between two vectors $\mathbf{X}$ and $\mathbf{Y}$ is given by

$$\cos\theta = \frac{\mathbf{X}^\top \mathbf{Y}}{\|\mathbf{X}\|\,\|\mathbf{Y}\|} \tag{0.9}$$

$$\begin{aligned}
\|\mathbf{R}\|^2 &= \mathbf{R}^\top \mathbf{R} && (0.10)\\
&= \left(\mathbf{A}+\mathbf{B}+\mathbf{C}\right)^\top \left(\mathbf{A}+\mathbf{B}+\mathbf{C}\right) && (0.11)\\
&= \left(\mathbf{A}^\top+\mathbf{B}^\top+\mathbf{C}^\top\right)\left(\mathbf{A}+\mathbf{B}+\mathbf{C}\right) && (0.12)\\
&= \mathbf{A}^\top\mathbf{A}+\mathbf{A}^\top\mathbf{B}+\mathbf{A}^\top\mathbf{C}+\mathbf{B}^\top\mathbf{A}+\mathbf{B}^\top\mathbf{B}+\mathbf{B}^\top\mathbf{C}+\mathbf{C}^\top\mathbf{A}+\mathbf{C}^\top\mathbf{B}+\mathbf{C}^\top\mathbf{C} && (0.13)\\
&= \|\mathbf{A}\|^2+0+0+0+\|\mathbf{B}\|^2+0+0+0+\|\mathbf{C}\|^2 && (0.14)\\
&= k^2+k^2+k^2 && (0.15)\\
&= 3k^2 && (0.16)
\end{aligned}$$

Therefore, $\|\mathbf{R}\| = \sqrt{3}k$.

Now, let α be the angle between $\mathbf{R}$ and $\mathbf{A}$.

$$\cos\alpha = \frac{\mathbf{R}^\top \mathbf{A}}{||\mathbf{R}||\,||\mathbf{A}||}(\textbf{??}) \tag{0.17}$$
$$= \frac{\left(\mathbf{A}+\mathbf{B}+\mathbf{C}\right)^\top \mathbf{A}}{||\mathbf{R}||\,||\mathbf{A}||} \tag{0.18}$$
$$= \frac{\mathbf{A}^\top\mathbf{A}+\mathbf{B}^\top\mathbf{A}+\mathbf{C}^\top\mathbf{A}}{||\mathbf{R}||\,||\mathbf{A}||} \tag{0.19}$$
$$= \frac{||\mathbf{A}||^2+0+0}{||\mathbf{R}||\,||\mathbf{A}||} \tag{0.20}$$
$$= \frac{k^2}{(\sqrt{3}k)(k)} \tag{0.21}$$
$$= \frac{k^2}{\sqrt{3}k^2} \tag{0.22}$$
$$= \frac{1}{\sqrt{3}} \tag{0.23}$$

Let β be the angle between $\mathbf{R}$ and $\mathbf{B}$.

$$\cos\beta = \frac{\mathbf{R}^\top \mathbf{B}}{||\mathbf{R}||\,||\mathbf{B}||}(\textbf{??}) \tag{0.24}$$
$$= \frac{\left(\mathbf{A}+\mathbf{B}+\mathbf{C}\right)^\top \mathbf{B}}{||\mathbf{R}||\,||\mathbf{B}||} \tag{0.25}$$
$$= \frac{\mathbf{A}^\top\mathbf{B}+\mathbf{B}^\top\mathbf{B}+\mathbf{C}^\top\mathbf{B}}{||\mathbf{R}||\,||\mathbf{B}||} \tag{0.26}$$
$$= \frac{0+||\mathbf{B}||^2+0}{||\mathbf{R}||\,||\mathbf{B}||} \tag{0.27}$$
$$= \frac{k^2}{(\sqrt{3}k)(k)} \tag{0.28}$$
$$= \frac{k^2}{\sqrt{3}k^2} \tag{0.29}$$
$$= \frac{1}{\sqrt{3}} \tag{0.30}$$

Let γ be the angle between $\mathbf{R}$ and $\mathbf{C}$.

$$\cos\gamma = \frac{\mathbf{R}^\top \mathbf{C}}{\|\mathbf{R}\| \, \|\mathbf{C}\|} (\textbf{??}) \tag{0.31}$$

$$= \frac{\left(\mathbf{A} + \mathbf{B} + \mathbf{C}\right)^\top \mathbf{C}}{\|\mathbf{R}\| \, \|\mathbf{C}\|} \tag{0.32}$$

$$= \frac{\mathbf{A}^\top \mathbf{C} + \mathbf{B}^\top \mathbf{C} + \mathbf{C}^\top \mathbf{C}}{\|\mathbf{R}\| \, \|\mathbf{C}\|} \tag{0.33}$$

$$= \frac{0 + 0 + \|\mathbf{C}\|^2}{\|\mathbf{R}\| \, \|\mathbf{C}\|} \tag{0.34}$$

$$= \frac{k^2}{(\sqrt{3}k)(k)} \tag{0.35}$$

$$= \frac{k^2}{\sqrt{3}k^2} \tag{0.36}$$

$$= \frac{1}{\sqrt{3}} \tag{0.37}$$

Since $\cos\alpha = \cos\beta = \cos\gamma = \frac{1}{\sqrt{3}}$, it implies $\alpha = \beta = \gamma$.

Thus, $\mathbf{A} + \mathbf{B} + \mathbf{C}$ is equally inclined to $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$.

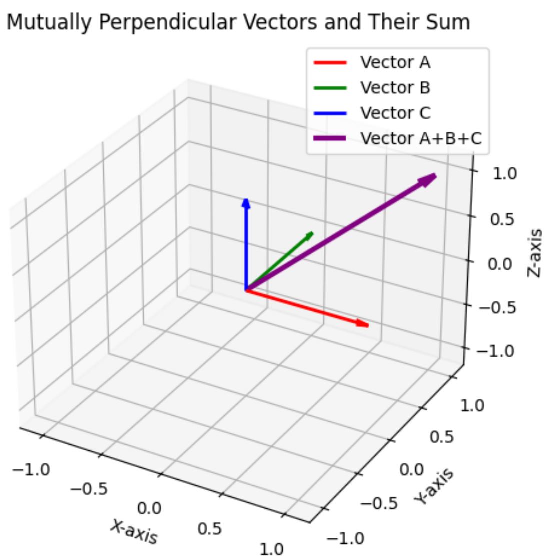

Fig. 0.1