8.2.11

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Question:

Find the coordinates of the focus, vertex, eccentricity, axis of the conic section, the equation of the directrix and the length of the latus rectum.

$$\frac{x^2}{100} + \frac{y^2}{400} = 1$$

Solution:

We use an affine transformation to convert the conic equation to its standard form.

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0$$

The symmetric matrix $\mathbf{V}$ is spectrally decomposed to align axes with eigenvectors.

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^\top, \ \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \ \mathbf{P}^\top \mathbf{P} = \mathbf{I} \tag{1}$$

Substituting the decomposition into the conic equation.

$$\mathbf{x}^\top \mathbf{P}\mathbf{D}\mathbf{P}^\top \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{2}$$

A rotation

$$\mathbf{x_r} = \mathbf{P}^\top \mathbf{x} \tag{3}$$

aligns the conic with the coordinate axes.

$$\mathbf{x} = \mathbf{P}\mathbf{x_r} \tag{4}$$

Applying the rotation to the conic equation.

$$(\mathbf{P}\mathbf{x_r})^\top \mathbf{P}\mathbf{D}\mathbf{P}^\top (\mathbf{P}\mathbf{x_r}) + 2\mathbf{u}^\top (\mathbf{P}\mathbf{x_r}) + f = 0 \tag{5}$$

$$\mathbf{x_r}^\top \mathbf{P}^\top \mathbf{P}\mathbf{D}\mathbf{P}^\top \mathbf{P}\mathbf{x_r} + 2\left(\mathbf{P}^\top \mathbf{u}\right)^\top \mathbf{x_r} + f = 0 \tag{6}$$

$$\mathbf{x_r}^\top \mathbf{D}\mathbf{x_r} + 2\mathbf{u_r}^\top \mathbf{x_r} + f = 0 \tag{7}$$

A translation

$$\mathbf{x_c} = \mathbf{x_r} + \mathbf{D}^{-1}\mathbf{u_r} \tag{8}$$

moves the conic's center to the origin.

$$f_c = f - \mathbf{u_r}^\top \mathbf{D}^{-1}\mathbf{u_r} \tag{9}$$

The center of the conic in the original coordinates is

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{10}$$

$$\mathbf{c} = -\left(\mathbf{P}\mathbf{D}\mathbf{P}^\top\right)^{-1} \mathbf{u} = -\mathbf{P}\mathbf{D}^{-1}\mathbf{P}^\top \mathbf{u} = -\mathbf{P}\mathbf{D}^{-1}\mathbf{u_r} \tag{11}$$

The complete transformation from original to centered coordinates is

$$\mathbf{x_c} = \mathbf{P}^\top (\mathbf{x} - \mathbf{c}) \tag{12}$$

$$\mathbf{x_c} = \mathbf{P}^\top \mathbf{x} + \mathbf{D}^{-1}\mathbf{u_r} = \mathbf{P}^\top \mathbf{x} - \mathbf{P}^\top \mathbf{c} = \mathbf{P}^\top (\mathbf{x} - \mathbf{c}) \tag{13}$$

$$\implies \mathbf{x} = \mathbf{P}\mathbf{x_c} + \mathbf{c} \tag{14}$$

The given conic equation

$$\frac{x^2}{100} + \frac{y^2}{400} - 1 = 0 \tag{15}$$

$$\mathbf{V} = \begin{pmatrix} \frac{1}{100} & 0 \\ 0 & \frac{1}{400} \end{pmatrix}, \ \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ f = -1 \tag{16}$$

The major axis corresponds to smaller eigenvalue.

$$\lambda_1 = \frac{1}{400},\ \lambda_2 = \frac{1}{100},\ \mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{17}$$

Applying the rotation to find the canonical coordinates.

$$\mathbf{x_c} = \mathbf{P}^\top \mathbf{x} \implies \begin{pmatrix} x_c \\ y_c \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix} \tag{18}$$

The standard form of the ellipse in canonical coordinates.

$$\frac{x_c^2}{-f/\lambda_1} + \frac{y_c^2}{-f/\lambda_2} = 1 \tag{19}$$

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} = \sqrt{1 - \frac{1/400}{1/100}} = \frac{\sqrt{3}}{2} \tag{20}$$

$$\mathbf{f_c} = \pm\sqrt{\frac{f(\lambda_1 - \lambda_2)}{\lambda_1\lambda_2}}\mathbf{e_1} = \pm 10\sqrt{3}\mathbf{e_1} \tag{21}$$

$$\mathbf{v_c} = \pm\sqrt{\frac{-f}{\lambda_1}}\mathbf{e_1} = \pm 20\mathbf{e_1} \tag{22}$$

$$\mathbf{d_c} : \mathbf{e_1}^\top \mathbf{x_c} = \pm\sqrt{\frac{-f\lambda_2}{\lambda_1(\lambda_2 - \lambda_1)}} = \pm\frac{40}{\sqrt{3}} \tag{23}$$

$$L = \frac{-2f}{\lambda_2}\sqrt{\frac{\lambda_1}{-f}} = 10 \tag{24}$$

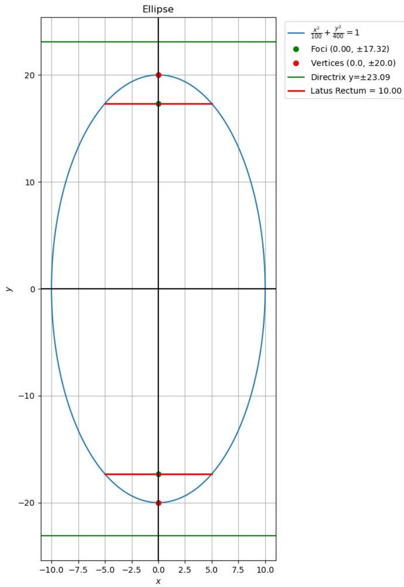
Transforming properties back to the original coordinate system using (14)

$$\mathbf{f} = \mathbf{P}\left(\pm 10\sqrt{3}\mathbf{e_1}\right) = \pm 10\sqrt{3}\mathbf{e_2} \tag{25}$$

$$\mathbf{v} = \mathbf{P}(\pm 20\mathbf{e_1}) = \pm 20\mathbf{e_2} \tag{26}$$

$$\mathbf{d} : \mathbf{e_2}^\top \mathbf{x} = \pm\frac{40}{\sqrt{3}} \implies \begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x} = \pm\frac{40}{\sqrt{3}} \tag{27}$$

Property	Value
Eccentricity	$\frac{\sqrt{3}}{2}$
Axis	$x = 0$
Vertices	$(0, \pm 20)$
Foci	$\left(0, \pm 10\sqrt{3}\right)$
Directrices	$y = \pm \frac{40}{\sqrt{3}}$
Latus Rectum	10



Plot