2.8.5

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Problem Statement

A line makes angles α, β, γ and δ with the diagonals of a cube, prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3} \tag{2.8.5.1}$$

Solution:

Symbol	Value	Description
$\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3, \mathbf{D}_4$	$\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\1\end{pmatrix}, \dots$	Column vectors for the four cube diagonals
$\mathbf{L}$	$\begin{pmatrix}l\\m\\n\end{pmatrix}$	Line's unit direction vector, where $\mathbf{L}^\top\mathbf{L} = 1$

For angle θ_i between the line $\mathbf{L}$ and a diagonal $\mathbf{D}_i$,

$$\cos\theta_i = \frac{\mathbf{L}^\top\mathbf{D}_i}{\|\mathbf{L}\|\|\mathbf{D}_i\|} = \frac{\mathbf{L}^\top\mathbf{D}_i}{\sqrt{3}} \tag{2.8.5.2}$$

Since $\mathbf{L}^\top\mathbf{D}_i$ is a scalar, it equals its own transpose, so

$$(\mathbf{L}^\top\mathbf{D}_i)^2 = (\mathbf{L}^\top\mathbf{D}_i)(\mathbf{D}_i^\top\mathbf{L}). \tag{2.8.5.3}$$

using (2.8.5.2) and (2.8.5.3) to find S i.e sum of squares,

$$S = \sum_{i=1}^{4}\cos^2\theta_i = \sum_{i=1}^{4}\frac{(\mathbf{L}^\top\mathbf{D}_i)(\mathbf{D}_i^\top\mathbf{L})}{3} = \frac{1}{3}\mathbf{L}^\top\left(\sum_{i=1}^{4}\mathbf{D}_i\mathbf{D}_i^\top\right)\mathbf{L} \tag{2.8.5.4}$$

The expression $\mathbf{D}_i\mathbf{D}_i^\top$ is the **outer product** of the vector with itself. Let's calculate the matrix $M = \sum_{i=1}^{4}\mathbf{D}_i\mathbf{D}_i^\top$.

$$\mathbf{D}_1\mathbf{D}_1^\top = \begin{pmatrix}1\\1\\1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}^\top = \begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} \quad \mathbf{D}_2\mathbf{D}_2^\top = \begin{pmatrix}-1\\1\\1\end{pmatrix}\begin{pmatrix}-1\\1\\1\end{pmatrix}^\top = \begin{pmatrix}1&-1&-1\\-1&1&1\\-1&1&1\end{pmatrix}$$

$$\mathbf{D}_3\mathbf{D}_3^\top = \begin{pmatrix}1\\-1\\1\end{pmatrix}\begin{pmatrix}1\\-1\\1\end{pmatrix}^\top = \begin{pmatrix}1&-1&1\\-1&1&-1\\1&-1&1\end{pmatrix} \quad \mathbf{D}_4\mathbf{D}_4^\top = \begin{pmatrix}1\\1\\-1\end{pmatrix}\begin{pmatrix}1\\1\\-1\end{pmatrix}^\top = \begin{pmatrix}1&1&-1\\1&1&-1\\-1&-1&1\end{pmatrix}$$

By adding these four matrices we get,

$$M = \sum_{i=1}^{4} \mathbf{D}_i \mathbf{D}_i^\top = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = 4I \tag{2.8.5.5}$$

Substituting(2.8.5.5) in (2.8.5.4) we get,

$$S = \frac{1}{3}\mathbf{L}^\top (4I)\mathbf{L} = \frac{4}{3}\mathbf{L}^\top I\mathbf{L} = \frac{4}{3}\mathbf{L}^\top \mathbf{L} \tag{2.8.5.6}$$

Since $\mathbf{L}$ is a unit vector, $\mathbf{L}^\top \mathbf{L} = \|\mathbf{L}\|^2 = 1$.

$$S = \frac{4}{3}(1) = \frac{4}{3} \tag{2.8.5.7}$$

Thus, it is proven that

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + \cos^2 \delta = \frac{4}{3}.$$

See Figure 2.8.5.1.

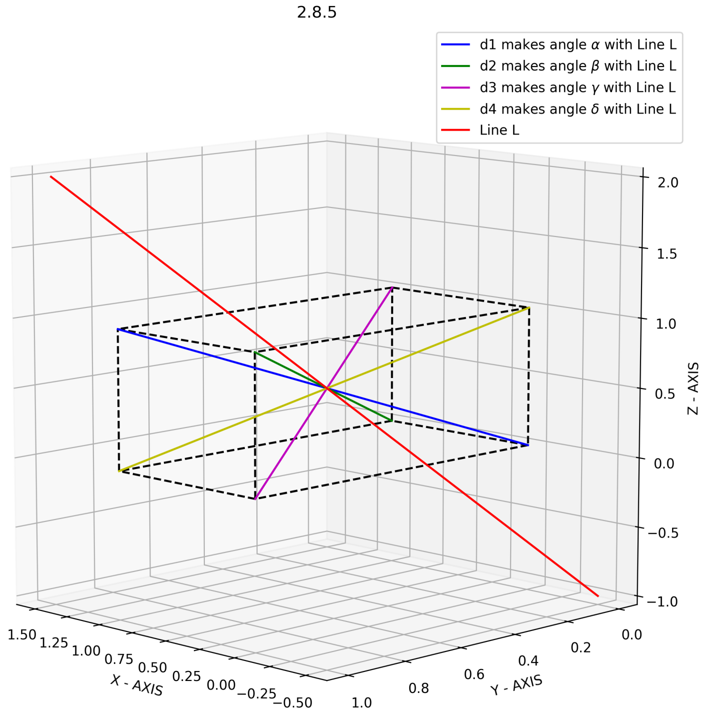
2.8.5
d1 makes angle α with Line L
d2 makes angle β with Line L
d3 makes angle γ with Line L
d4 makes angle δ with Line L
Line L
X - AXIS
Y - AXIS
Z - AXIS
1.50
1.25
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
1.0
0.8
0.6
0.4
0.2
0.0
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0

Fig. 2.8.5.1