2.4.16

EE25BTECH11021 - Dhanush Sagar

Question
given points are

$$\mathbf{A} = \begin{pmatrix} 0 \\ 7 \\ -10 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 1 \\ 6 \\ -6 \end{pmatrix}, \qquad \mathbf{C} = \begin{pmatrix} 4 \\ 9 \\ -6 \end{pmatrix}.$$

solution
We consider the vectors

$$\mathbf{A} = \begin{pmatrix} 0 \\ 7 \\ -10 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 1 \\ 6 \\ -6 \end{pmatrix}, \qquad \mathbf{C} = \begin{pmatrix} 4 \\ 9 \\ -6 \end{pmatrix}.$$

PROOF OF: $\mathbf{A}, \mathbf{B}, \mathbf{C}$ ARE NOT COLLINEAR (RANK METHOD)

Form the difference vectors $\mathbf{B} - \mathbf{A}$ and $\mathbf{C} - \mathbf{A}$.

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} 1 - 0 \\ 6 - 7 \\ -6 - (-10) \end{pmatrix} \tag{0.1}$$

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} \tag{0.2}$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} 4 - 0 \\ 9 - 7 \\ -6 - (-10) \end{pmatrix} \tag{0.3}$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix} \tag{0.4}$$

Place these as columns in the 3×2 matrix M.

$$M = \begin{pmatrix} \mathbf{B} - \mathbf{A} & \mathbf{C} - \mathbf{A} \end{pmatrix} \tag{0.5}$$

$$M = \begin{pmatrix} 1 & 4 \\ -1 & 2 \\ 4 & 4 \end{pmatrix} \tag{0.6}$$

Compute the 2×2 minor using rows 1 and 2.

$$\det \begin{pmatrix} 1 & 4 \\ -1 & 2 \end{pmatrix} = 1 \cdot 2 - 4 \cdot (-1) \tag{0.7}$$

$$\det \begin{pmatrix} 1 & 4 \\ -1 & 2 \end{pmatrix} = 2 + 4 = 6 \neq 0 \tag{0.8}$$

Hence $\operatorname{rank}(M) = 2$, so $\mathbf{B} - \mathbf{A}$ and $\mathbf{C} - \mathbf{A}$ are linearly independent. Therefore $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are not collinear and determine a triangle.

A) Verification for isosceles triangles

Isosceles via perpendicular bisector:

$$\text{Midpoint } \mathbf{M} \text{ of } \overline{AC} = \frac{\mathbf{A} + \mathbf{C}}{2} = \frac{1}{2} \begin{pmatrix} 0 + 4 \\ 7 + 9 \\ -10 + (-6) \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ -8 \end{pmatrix}, \tag{0.9}$$

$$\mathbf{MB} = \mathbf{B} - \mathbf{M} = \begin{pmatrix} 1 - 2 \\ 6 - 8 \\ -6 - (-8) \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}, \tag{0.10}$$

$$\mathbf{AC} = \mathbf{C} - \mathbf{A} = \begin{pmatrix} 4 - 0 \\ 9 - 7 \\ -6 - (-10) \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix}. \tag{0.11}$$

$$\mathbf{MB}^\top \mathbf{AC} = \begin{pmatrix} -1 & -2 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix} = (-1) \cdot 4 + (-2) \cdot 2 + 2 \cdot 4 = -4 - 4 + 8 = 0. \tag{0.12}$$

Thus $\mathbf{MB} \perp \mathbf{AC}$ and $\mathbf{M}$ is the midpoint of AC, so the line through $\mathbf{M}$ in the direction of $\mathbf{MB}$ is the *perpendicular bisector* of AC and passes through $\mathbf{B}$. By the perpendicular-bisector property, points on it are equidistant from A and C, hence

$$\|\mathbf{AB}\| = \|\mathbf{BC}\|. \tag{0.13}$$

Therefore, the triangle is *isosceles* with equal sides AB and BC.

B)Veification for right angled triangle (matrix / inner-product test)

To show the triangle is right-angled, compute the inner product of two adjacent side vectors $\mathbf{AB}$ and $\mathbf{BC}$.

$$\mathbf{AB} = \mathbf{B} - \mathbf{A} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}, \tag{0.14}$$

$$\mathbf{BC} = \mathbf{C} - \mathbf{B} = \begin{pmatrix} 4 - 1 \\ 9 - 6 \\ -6 - (-6) \end{pmatrix} \tag{0.15}$$

$$\mathbf{BC} = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}, \tag{0.16}$$

$$(\mathbf{AB})^T(\mathbf{BC}) = \begin{pmatrix} 1 & -1 & 4 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} \tag{0.17}$$

$$(\mathbf{AB})^T(\mathbf{BC}) = 1 \cdot 3 + (-1) \cdot 3 + 4 \cdot 0 \tag{0.18}$$

$$(\mathbf{AB})^T(\mathbf{BC}) = 3 - 3 + 0 = 0. \tag{0.19}$$

Since the inner product is zero, $\mathbf{AB} \perp \mathbf{BC}$ and therefore the angle $\angle ABC$ is a right angle; the triangle is **right-angled at** B.

Final statement: The non-collinear vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ determine a triangle which is both **isosceles** (with $||\mathbf{AB}|| = ||\mathbf{BC}||$) and **right-angled** (with $\mathbf{AB} \perp \mathbf{BC}$); hence the triangle is a *right isosceles* triangle with the right angle at vertex B.

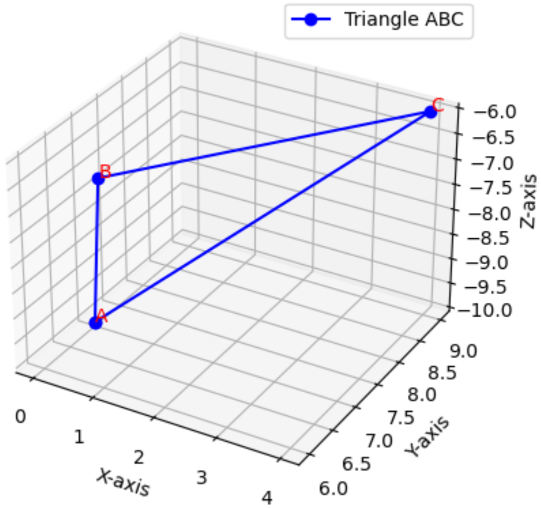
Triangle ABC
A
B
C
X-axis
Y-axis
Z-axis
0
1
2
3
4
6.0
6.5
7.0
7.5
8.0
8.5
9.0
−6.0
−6.5
−7.0
−7.5
−8.0
−8.5
−9.0
−9.5
−10.0

Fig. 0.1