12.547

EE25BTECH11013 - Bhargav

Question:
Consider $\mathbf{R^3}$ with the usual inner product. If d is the distance from (1,1,1) to the subspace span $\{(1,1,0),(0,1,1)\}$ of $\mathbf{R^3}$, then $3d^2$ is
Solution:
Let $\mathbf{W} = \text{span}\{u_1, u_2\}$
Where $\mathbf{U} = \begin{pmatrix} u_1 & u_2 \end{pmatrix}$

The distance from $\mathbf{P} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ to the subspace span $\mathbf{W}$ can be found by finding the projection of $\mathbf{P}$ onto $\mathbf{W}$.

Let $\mathbf{Ux}$ be the projection of $\mathbf{P}$ on the span $\mathbf{W}$
Where $\mathbf{x}$ is the column vector containing the coefficients that scale the basis vectors of the subspace to give the projection point.

$$\mathbf{U^T}\left(\mathbf{P} - \mathbf{Ux}\right) = 0 \tag{0.1}$$

(since $\mathbf{U}$ is perpendicular to $\mathbf{P} - \mathbf{Ux}$)

$$\implies \mathbf{U^T U x} = \mathbf{U^T P} \tag{0.2}$$

Since the columns of $\mathbf{U}$ are Linearly independent, so are the columns of $\mathbf{U^T U}$ and hence $\mathbf{U^T U}$ is invertible

$$\mathbf{x} = \left(\mathbf{U^T U}\right)^{-1} \mathbf{U^T P} \tag{0.3}$$

Hence the projection of $\mathbf{P}$ on the span $\mathbf{W}$ is

$$\mathbf{Ux} = \mathbf{U}\left(\mathbf{U^T U}\right)^{-1} \mathbf{U^T P} \tag{0.4}$$

The distance of $\mathbf{P}$ from the span $\mathbf{W}$ is:

$$d = ||\mathbf{P} - \mathbf{Ux}|| \tag{0.5}$$

$$d = \left\|\mathbf{P} - \mathbf{U}\left(\mathbf{U^T U}\right)^{-1} \mathbf{U^T P}\right\| \tag{0.6}$$

$$\mathbf{P} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{U} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{0.7}$$

Substituting the values in (0.6):

$$d = \frac{1}{\sqrt{3}} \tag{0.8}$$

$\boxed{3d^2 = 1}$