Matgeo Presentation - Bonus Problem

ee25btech11063 - Vejith

September 11, 2025

Question

Given 3 vectors $\mathbf{A}$,$\mathbf{B}$,$\mathbf{C}$ are coplanar then show $\det(\mathbf{M}) = 0$ where $\mathbf{M} = (\mathbf{A}\ \mathbf{B}\ \mathbf{C})$

Solution

Equation of plane through 3 coplanar points is

$$
\begin{aligned}
\mathbf{n}^T\mathbf{x} &= 0 && (0.1)\\
\implies \mathbf{n}^T\mathbf{A} = \mathbf{n}^T\mathbf{B} = \mathbf{n}^T\mathbf{C} &= 0 && (0.2)\\
\mathbf{M} &= \begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C}\end{pmatrix} && (0.3)\\
\implies \mathbf{n}^T\mathbf{M} &= \begin{pmatrix}\mathbf{n}^T\mathbf{A} & \mathbf{n}^T\mathbf{B} & \mathbf{n}^T\mathbf{C}\end{pmatrix} && (0.4)\\
\implies \mathbf{n}^T\mathbf{M} &= \begin{pmatrix}0 & 0 & 0\end{pmatrix} && (0.5)\\
\implies \mathbf{n}^T\mathbf{M} &= \mathbf{0} && (0.6)
\end{aligned}
$$

From (0.6) it means $\mathbf{M}$ has a non trivial vector in $it's$ null space

$$\implies rank(\mathbf{M}) < 3. \tag{0.7}$$

For a $3\times$ 3 square matrix like $\mathbf{M}$ if det$(\mathbf{M})\neq 0$ means $\mathbf{M}$ is invertible which means $\mathbf{M}$ is a full rank matrix

$\implies$ rank$(\mathbf{M})$=3.(if det$(\mathbf{M})\neq 0$)

Solution

From (0.7) rank($\mathbf{M}$)$<$3
$\implies$ $\mathbf{M}$ is not invertible
$\implies$ det($\mathbf{M}$)=0
proof 2:
3 vectors **A**,**B**,**C** are coplanar means they are linearly dependent.
let′s assume

$$
\begin{aligned}
\mathbf{C} &= \alpha\mathbf{A} + \beta\mathbf{B}. && (0.8)\\
\det(\mathbf{M}) &= \det(\begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C}\end{pmatrix}) && (0.9)\\
&= \det(\begin{pmatrix}\mathbf{A} & \mathbf{B} & \alpha\mathbf{A} + \beta\mathbf{B}\end{pmatrix}) && (0.10)\\
&= \alpha\det(\begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{A}\end{pmatrix}) + \beta\det(\begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{B}\end{pmatrix}) = 0 && (0.11)\\
&\implies \det(\mathbf{M}) = 0 && (0.12)
\end{aligned}
$$