4.13.70

AI25BTECH11001 - ABHISEK MOHAPATRA

Question: If $\begin{vmatrix} a & a^2 & 1+a3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$ and the vectors $\mathbf{A} = \begin{pmatrix} 1 \\ a \\ a^2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 \\ b \\ b^2 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 1 \\ c \\ c^2 \end{pmatrix}$ are co-planar, then the product abc = ______.

Solution: Let equation of the plane be $\mathbf{n}^\top \mathbf{x} = 0$.

so,

$$\mathbf{n}^\top \mathbf{A} = 0, \mathbf{n}^\top \mathbf{B} = 0, \mathbf{n}^\top \mathbf{C} = 0 \tag{1}$$

so ,

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}^\top \mathbf{n} = 0, \tag{2}$$

$$\begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix} \mathbf{n} = 0, \tag{3}$$

$$\begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix} \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{pmatrix} \tag{4}$$

$$\xrightarrow{R_3 \leftarrow R_3 - \frac{c-a}{b-a} R_2} \begin{pmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & 0 & (c-a)(c-b) \end{pmatrix} \tag{5}$$

Product of the eigen values is $(b-a)(c-a)(c-b)$. And,for the no non-trivial solution of $\mathbf{n}$ to exist,

$$(b-a)(c-a)(c-b) \neq 0 \tag{6}$$

Reducing the given determinant to ref,

$$\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} \xrightarrow[R_3 \leftarrow R_3 - \frac{c}{a} R_1]{R_2 \leftarrow R_2 - \frac{b}{a} R_1} \begin{vmatrix} a & a^2 & 1+a^3 \\ 0 & b^2-ab & 1-\frac{b}{a}+b^3-a^2b \\ 0 & c^2-ca & 1-\frac{c}{a}+c^3-a^2c \end{vmatrix} \tag{7}$$

$$\xrightarrow{R_3 \leftarrow R_3 - \frac{c^2-ca}{b^2-ba} R_2} \begin{vmatrix} a & a^2 & 1+a^3 \\ 0 & b^2-ab & 1-\frac{b}{a}+b^3-a^2b \\ 0 & 0 & (1-\frac{c}{a})(1-\frac{c}{b})+c(c-a)(c-b) \end{vmatrix} \tag{8}$$

Product of the eigen values

$$(b-a)(a-c)(b-c) + abc(b-a)(c-a)(c-b) = 0 \tag{9}$$

$$(1+abc)(b-a)(c-a)(c-b) = 0 \tag{10}$$

From eq 7

$$1 + abc = 0 \Rightarrow abc = -1 \tag{11}$$