2.10.2

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Question

Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be vectors of lengths 3, 4, and 5 respectively such that $\mathbf{A} \perp \mathbf{B} + \mathbf{C}$, $\mathbf{B} \perp \mathbf{C} + \mathbf{A}$, and $\mathbf{C} \perp \mathbf{A} + \mathbf{B}$. Find the length of the vector $\mathbf{A} + \mathbf{B} + \mathbf{C}$.

Theoretical Solution

Given the orthogonality conditions:

$$\mathbf{A} \perp \mathbf{B} + \mathbf{C} \implies \mathbf{A}^T(\mathbf{B} + \mathbf{C}) = 0 \tag{1}$$

$$\mathbf{B} \perp \mathbf{C} + \mathbf{A} \implies \mathbf{B}^T(\mathbf{C} + \mathbf{A}) = 0 \tag{2}$$

$$\mathbf{C} \perp \mathbf{A} + \mathbf{B} \implies \mathbf{C}^T(\mathbf{A} + \mathbf{B}) = 0 \tag{3}$$

Expanding, we get:

$$\mathbf{A}^T\mathbf{B} + \mathbf{A}^T\mathbf{C} = 0 \tag{4}$$

$$\mathbf{B}^T\mathbf{C} + \mathbf{B}^T\mathbf{A} = 0 \tag{5}$$

$$\mathbf{C}^T\mathbf{A} + \mathbf{C}^T\mathbf{B} = 0 \tag{6}$$

Theoretical Solution

Adding :

$$(\mathbf{A}^T\mathbf{B} + \mathbf{A}^T\mathbf{C}) + (\mathbf{B}^T\mathbf{C} + \mathbf{B}^T\mathbf{A}) + (\mathbf{C}^T\mathbf{A} + \mathbf{C}^T\mathbf{B}) = 0 \tag{7}$$

Grouping like terms and noting dot products are symmetric:

$$2\left(\mathbf{A}^T\mathbf{B} + \mathbf{B}^T\mathbf{C} + \mathbf{C}^T\mathbf{A}\right) = 0 \implies \mathbf{A}^T\mathbf{B} + \mathbf{B}^T\mathbf{C} + \mathbf{C}^T\mathbf{A} = 0 \tag{8}$$

Theoretical Solution

Now compute the squared length of $\mathbf{A}+\mathbf{B}+\mathbf{C}$:

$$\begin{align}
||\mathbf{A}+\mathbf{B}+\mathbf{C}||^2 &= (\mathbf{A}+\mathbf{B}+\mathbf{C})^T(\mathbf{A}+\mathbf{B}+\mathbf{C}) \tag{9}\\
&= \mathbf{A}^T\mathbf{A}+\mathbf{B}^T\mathbf{B}+\mathbf{C}^T\mathbf{C}+2(\mathbf{A}^T\mathbf{B}+\mathbf{B}^T\mathbf{C}+\mathbf{C}^T\mathbf{A}) \tag{10}\\
&= ||\mathbf{A}||^2+||\mathbf{B}||^2+||\mathbf{C}||^2+2(\mathbf{A}^T\mathbf{B}+\mathbf{B}^T\mathbf{C}+\mathbf{C}^T\mathbf{A}) \tag{11}\\
&= 3^2+4^2+5^2+2\times 0 \tag{12}\\
&= 9+16+25 \tag{13}\\
&= 50 \tag{14}
\end{align}$$

Therefore,

$$||\mathbf{A}+\mathbf{B}+\mathbf{C}|| = \sqrt{50} = 5\sqrt{2} \tag{15}$$

Python Code

```
import numpy as np
import numpy.linalg as la
import math
a=3
b=4
c=5
#x=a.b,y=b.c,z=c.a
#x+y=0,y+z=0,x+z=0
B=np.array([0,0,0])
A=np.array([[1,1,0],[0,1,1],[1,0,1]])
X=la.solve(A,B)
d=a*a+b*b+c*c+2*np.sum(X)
print(math.sqrt(d))
```