7.4.30

EE25BTECH11023 - Venkata Sai

Question:
A circle is given by $x^2 + (y-1)^2 = 1$, another circle C touches it externally and also the X axis, then the locus of its centre is

1) $\{(x, y) : x^2 = 4y\} \cup \{(x, y) : y \leq 0\}$
2) $\{(x, y) : x^2 + (y-1)^2 = 4\} \cup \{(x, y) : y \leq 0\}$
3) $\{(x, y) : x^2 = 4y\} \cup \{(0, y) : y \leq 0\}$
4) $\{(x, y) : x^2 = 4y\} \cup \{(0, y) : y \leq 0\}$

Solution:
As the circle touches $X-$axis , Distance of a point from x-axis is given by

$$r = |\mathbf{n}^\top \mathbf{c}| \tag{1}$$

where $\mathbf{n}$ is the unit vector normal to x-axis
For the given circle with radius r_1 and center c_1

$$x^2 + (y-1)^2 = 1 \tag{2}$$

$$\mathbf{p} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathbf{n} \text{ and } r_1 = 1 \tag{3}$$

Distance between their centers equal to sum of their radius

$$||\mathbf{c} - \mathbf{p}|| = r + r_1 \tag{4}$$

$$||\mathbf{c} - \mathbf{n}|| = |\mathbf{n}^\top \mathbf{c}| + 1 \tag{5}$$

$$||\mathbf{c} - \mathbf{n}||^2 = \left(|\mathbf{n}^\top \mathbf{c}| + 1\right)^2 \tag{6}$$

$$(\mathbf{c} - \mathbf{n})\left(\mathbf{c}^\top - \mathbf{n}^\top\right) = \left(|\mathbf{n}^\top \mathbf{c}| + 1\right)^2 \tag{7}$$

$$\mathbf{c}^\top \mathbf{c} + \mathbf{n}\mathbf{n}^\top - \mathbf{c}^\top \mathbf{n} - \mathbf{n}^\top \mathbf{c} = \left(|\mathbf{n}^\top \mathbf{c}|\right)^2 + 2|\mathbf{n}^\top \mathbf{c}| + 1 \tag{8}$$

$$\mathbf{c}^\top \mathbf{c} + \mathbf{n}\mathbf{n}^\top - \mathbf{c}^\top \mathbf{n} - \mathbf{n}^\top \mathbf{c} = \left(\mathbf{n}^\top \mathbf{c}\right)^\top \left(\mathbf{n}^\top \mathbf{c}\right) + 2|\mathbf{n}^\top \mathbf{c}| + 1 \tag{9}$$

$$\mathbf{c}^\top \mathbf{c} + ||\mathbf{n}||^2 - 2\mathbf{n}^\top \mathbf{c} = \left(\mathbf{c}^\top \mathbf{n}\mathbf{n}^\top \mathbf{c}\right) + 2|\mathbf{n}^\top \mathbf{c}| + 1 \tag{10}$$

$$\mathbf{c}^\top \mathbf{c} + 1 = \left(\mathbf{c}^\top \mathbf{n}\mathbf{n}^\top \mathbf{c}\right) + 2\mathbf{n}^\top \mathbf{c} + 2|\mathbf{n}^\top \mathbf{c}| + 1 \tag{11}$$

$$\mathbf{c}^\top \mathbf{c} - \left(\mathbf{c}^\top \mathbf{n}\mathbf{n}^\top \mathbf{c}\right) = 2\mathbf{n}^\top \mathbf{c} + 2|\mathbf{n}^\top \mathbf{c}| \tag{12}$$

$$\mathbf{c}^\top \left(\mathbf{I} - \mathbf{n}\mathbf{n}^\top\right) \mathbf{c} = 2\mathbf{n}^\top \mathbf{c} + 2|\mathbf{n}^\top \mathbf{c}| \tag{13}$$

$$\mathbf{c}^\top \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \right) \mathbf{c} = 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}^\top \mathbf{c} + 2 \left| \begin{pmatrix} 0 \\ 1 \end{pmatrix}^\top \mathbf{c} \right| \tag{14}$$

$$\mathbf{c}^\top \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) \mathbf{c} = 2 \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{c} + 2 \left| \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{c} \right| \tag{15}$$

$$\begin{pmatrix} x & y \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \pm 2 \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{16}$$

$$\begin{pmatrix} x & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 4 \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \ (\textbf{or}) \ \begin{pmatrix} x & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - 2 \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{17}$$

$$\begin{pmatrix} x & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 4y \ \ (\textbf{or}) \ \ \begin{pmatrix} x & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \tag{18}$$

$$x^2 = 4y \ \ (\text{or}) \ \ x^2 = 0 \implies x = 0 \tag{19}$$

Case (1)

$$x^2 = 4y \implies y \geq 0 \tag{20}$$

Case (2)

$$x = 0 \tag{21}$$

$$\mathbf{n}^\top \mathbf{c} \leq 0 \implies \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq 0 \implies y \leq 0 \tag{22}$$

Hence from Case (1) and Case (2)

$$\left\{ (x, y) : x^2 = 4y \right\} \bigcup \{ (x, y) : x = 0 \text{ AND } y \leq 0 \} \tag{23}$$

$$\left\{ (x, y) : x^2 = 4y \right\} \bigcup \{ (0, y) : y \leq 0 \} \tag{24}$$

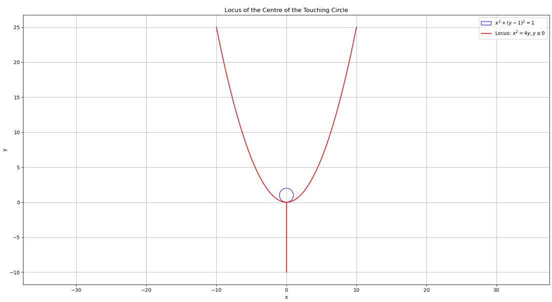

Fig. 4.1