12.579

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Question:

Let V be the vector space of all 3×3 matrices with complex entries over the real field. If

$$W_1 = \left\{ \mathbf{A} \in V : \mathbf{A} = \bar{\mathbf{A}}^\top \right\}$$
$$W_2 = \{ \mathbf{A} \in V : \text{trace}(\mathbf{A}) = 0 \}$$

then the dimension of $W_1 + W_2$ is equal to ________.

Solution:

A basis for a vector space V over a field F is a set $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ where for any $\mathbf{v} \in V$, there exist unique scalars $c_i \in F$ such that:

$$\mathbf{v} = \sum_{i=1}^{n} c_i \mathbf{v}_i \tag{1}$$

This uniqueness implies linear independence, where the only solution to the following equation is $c_1 = \cdots = c_n = 0$.

$$\sum_{i=1}^{n} c_i \mathbf{v}_i = \mathbf{0} \tag{2}$$

Let V be the space of $n \times n$ complex matrices over $\mathbb{R}$. Let $\mathbf{E}_{jk}$ be the matrix with 1 at position (j, k) and 0 elsewhere. Any $\mathbf{A} \in V$ can be written as:

$$\mathbf{A} = \sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk} \mathbf{E}_{jk} = \sum_{j,k} \left(x_{jk} + i y_{jk} \right) \mathbf{E}_{jk} = \sum_{j,k} x_{jk} \mathbf{E}_{jk} + \sum_{j,k} y_{jk} \left(i \mathbf{E}_{jk} \right) \tag{3}$$

The set

$$\mathcal{B}_V = \{\mathbf{E}_{11}, \ldots, \mathbf{E}_{nn}, i\mathbf{E}_{11}, \ldots, i\mathbf{E}_{nn}\} \tag{4}$$

spans V. Linear independence over $\mathbb{R}$ is shown by:

$$\sum_{j,k} x_{jk}\mathbf{E}_{jk} + \sum_{j,k} y_{jk}\left(i\mathbf{E}_{jk}\right) = \mathbf{0} \implies \sum_{j,k}\left(x_{jk} + iy_{jk}\right)\mathbf{E}_{jk} = \mathbf{0} \tag{5}$$

This implies

$$x_{jk} + iy_{jk} = 0, \text{ thus } x_{jk} = 0 \text{ and } y_{jk} = 0 \text{ for all } j, k \tag{6}$$

$\mathcal{B}_V$ is a basis.

$$\dim(V) = n^2 + n^2 = 2n^2 \tag{7}$$

Let $\mathcal{B}_{1\cap 2}$ be a basis for $W_1 \cap W_2$. Extending it to bases $\mathcal{B}_1$ for W_1 and $\mathcal{B}_2$ for W_2. The set

$$\mathcal{B}_{1+2} = \mathcal{B}_1 \cup \mathcal{B}_2$$

spans $W_1 + W_2$. For linear independence:

$$\sum a_i\mathbf{u}_i + \sum b_j\mathbf{v}_j + \sum c_l\mathbf{w}_l = \mathbf{0} \implies \sum a_i\mathbf{u}_i + \sum b_j\mathbf{v}_j = -\sum c_l\mathbf{w}_l \tag{8}$$

The vector is in $W_1 \cap W_2$, so $-\sum c_l\mathbf{w}_l = \sum d_i\mathbf{u}_i$. Since $\mathcal{B}_2$ is a basis, all $c_l = 0, d_i = 0$, which implies all $a_i = 0, b_j = 0$.

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) \tag{9}$$

For W_1, the Hermitian condition

$$a_{jk} = \bar{a}_{kj} \tag{10}$$

From (10),

$$x_{jk} + iy_{jk} = x_{kj} - iy_{kj} \implies x_{jk} = x_{kj} \text{ and } y_{jk} = -y_{kj} \tag{11}$$

For diagonal elements,

$$y_{jj} = -y_{jj} \implies y_{jj} = 0 \tag{12}$$

This gives n real parameters. For off-diagonal elements, there are

$$\frac{n^2 - n}{2} \tag{13}$$

pairs. Each pair is determined by one complex number $\left(x_{jk}, y_{jk}\right)$, giving

$$2 \times \frac{n^2 - n}{2} = n^2 - n \tag{14}$$

real parameters.

$$\dim\left(W_1\right) = n + \left(n^2 - n\right) = n^2 \tag{15}$$

For W_2, the trace condition imposes two independent real constraints on the $2n^2$ parameters of V:

$$\operatorname{trace}\left(\mathbf{A}\right) = \sum x_{jj} + i \sum y_{jj} = 0 \implies \sum x_{jj} = 0 \text{ and } \sum y_{jj} = 0 \tag{16}$$

$$\dim\left(W_2\right) = 2n^2 - 2 \tag{17}$$

For $W_1 \cap W_2$, matrices are Hermitian, so $y_{jj} = 0$. The trace condition becomes one real constraint on the n^2 parameters of W_1.

$$\dim\left(W_1 \cap W_2\right) = n^2 - 1 \tag{18}$$

For this problem,

$$n = 3 \tag{19}$$

$$\dim\left(W_1\right) = 3^2 = 9 \tag{20}$$

$$\dim\left(W_2\right) = 2\left(3^2\right) - 2 = 16 \tag{21}$$

$$\dim\left(W_1 \cap W_2\right) = 3^2 - 1 = 8 \tag{22}$$

$$\therefore \dim\left(W_1 + W_2\right) = 9 + 16 - 8 = 17 \tag{23}$$