

12.349

EE25BTECH11023 - Venkata Sai

Question:

Let $T_1, T_2 : \mathbb{R}_5 \rightarrow \mathbb{R}_3$ be linear transformations such that $\text{rank}(T_1) = 3$ and $\text{nullity}(T_2) = 3$. Let $T_3 : \mathbb{R}_3 \rightarrow \mathbb{R}_3$ be a linear transformation such that $T_3 \circ T_1 = T_2$. Then $\text{rank}(T_3)$ is ...
(MA 2014)

Solution:

According to Rank-Nullity theorem,

For a linear transformation $T : \mathbb{R}_m \rightarrow \mathbb{R}_n$

$$\text{rank}(T) + \text{nullity}(T) = \dim(\text{domain}) \quad (1)$$

where $\dim \mathbb{R}_m$ is the dimension of the domain i.e vector space \mathbb{R}_m

Given $T_2 : \mathbb{R}_5 \rightarrow \mathbb{R}_3$ and $\text{nullity}(T_2)=3$

$$\text{rank}(T_2) + \text{nullity}(T_2) = \dim \mathbb{R}^5 \quad (2)$$

$$\text{rank}(T_2) + 3 = 5 \quad (3)$$

$$\text{rank}(T_2) = 2 \quad (4)$$

Given $T_1 : \mathbb{R}_5 \rightarrow \mathbb{R}_3$ and $\text{rank}(T_1)=3$

$$\dim(\text{Co-domain}) = 3 \quad (5)$$

$$\text{rank}(T_1) = \dim(\text{Co-domain}) \quad (6)$$

It is onto and hence

$$\dim(\text{Im}(T_1)) = \dim(\text{Co-domain}) \quad (7)$$

$$\dim(\text{Im}(T_1)) = 3 \implies \text{Im}(T_1) = \mathbb{R}_3 \quad (8)$$

where $\text{Im}(T)$ is the Image space of the linear transformation T

Given $T_3 : \mathbb{R}_3 \rightarrow \mathbb{R}_3$

$$T_3 \circ T_1 = T_2 \quad (9)$$

$$(T_3 \circ T_1)(\mathbb{R}_5) = \text{Im}(T_2) \quad (10)$$

$$T_3(T_1(\mathbb{R}_5)) = \text{Im}(T_2) \quad (11)$$

$$T_3(\text{Im}T_1) = \text{Im}(T_2) \quad (12)$$

$$T_3(\mathbb{R}_3) = \text{Im}(T_2) \quad (13)$$

$$\text{Im}(T_3) = \text{Im}(T_2) \quad (14)$$

$$\implies \text{rank}(T_3) = \text{rank}(T_2) \quad (15)$$

From (4)

$$\text{rank}(T_3) = 2 \quad (16)$$