4.13.41

EE25BTECH11036 - M Chanakya Srinivas

Problem

Find the area of the parallelogram formed by the lines

$$y = mx, \quad y = mx + 1, \quad y = nx, \quad y = nx + 1$$

Solution

Step 1: Express the lines in normal form

The general form of a line is

$$\mathbf{n}^\top \mathbf{x} = c \tag{1}$$

where $\mathbf{n}$ is the normal vector and c is the intercept on the normal.
For the given lines,

$$y - mx = 0 \implies \mathbf{n}_1^\top \mathbf{x} = 0, \tag{2}$$
$$y - mx - 1 = 0 \implies \mathbf{n}_2^\top \mathbf{x} = 1, \tag{3}$$
$$y - nx = 0 \implies \mathbf{n}_3^\top \mathbf{x} = 0, \tag{4}$$
$$y - nx - 1 = 0 \implies \mathbf{n}_4^\top \mathbf{x} = 1, \tag{5}$$

where

$$\mathbf{n}_1 = \mathbf{n}_2 = \begin{pmatrix} -m \\ 1 \end{pmatrix}, \quad \mathbf{n}_3 = \mathbf{n}_4 = \begin{pmatrix} -n \\ 1 \end{pmatrix}. \tag{6}$$

Step 2: Formula for intersection of two lines

For two lines $\mathbf{n}_1^\top \mathbf{x} = c_1$ and $\mathbf{n}_2^\top \mathbf{x} = c_2$, their intersection point satisfies

$$\begin{pmatrix} \mathbf{n}_1^\top \\ \mathbf{n}_2^\top \end{pmatrix} \mathbf{x} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \tag{7}$$

Hence,

$$\mathbf{x} = \begin{pmatrix} \mathbf{n}_1^\top \\ \mathbf{n}_2^\top \end{pmatrix}^{-1} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \tag{8}$$

Step 3: Construct the matrix and its inverse

$$\mathbf{N} = \begin{pmatrix} -m & 1 \\ -n & 1 \end{pmatrix}, \qquad \mathbf{N}^{-1} = \frac{1}{n - m} \begin{pmatrix} 1 & -1 \\ n & -m \end{pmatrix}. \tag{9}$$

Step 4: Compute intersection points

(i) Intersection of $y = mx$ and $y = nx$:

$$\mathbf{A} = \mathbf{N}^{-1}\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}. \tag{10}$$

(ii) Intersection of $y = mx + 1$ and $y = nx$:

$$\mathbf{B} = \mathbf{N}^{-1}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{n-m}\begin{pmatrix}1\\n\end{pmatrix}. \tag{11}$$

(iii) Intersection of $y = mx + 1$ and $y = nx + 1$:

$$\mathbf{C} = \mathbf{N}^{-1}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{n-m}\begin{pmatrix}0\\n-m\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}. \tag{12}$$

(iv) Intersection of $y = mx$ and $y = nx + 1$:

$$\mathbf{D} = \mathbf{N}^{-1}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{n-m}\begin{pmatrix}-1\\-m\end{pmatrix}. \tag{13}$$

Thus, the four vertices are

$$\mathbf{A} = \begin{pmatrix}0\\0\end{pmatrix}, \quad \mathbf{B} = \frac{1}{n-m}\begin{pmatrix}1\\n\end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix}0\\1\end{pmatrix}, \quad \mathbf{D} = \frac{1}{n-m}\begin{pmatrix}-1\\-m\end{pmatrix}. \tag{14}$$

Step 5: Find area of the parallelogram

Two adjacent sides are:

$$\mathbf{B} - \mathbf{A} = \frac{1}{n-m}\begin{pmatrix}1\\n\end{pmatrix}, \tag{15}$$

$$\mathbf{D} - \mathbf{A} = \frac{1}{n-m}\begin{pmatrix}-1\\-m\end{pmatrix}. \tag{16}$$

Area of parallelogram is

$$\text{Area} = |(\mathbf{B} - \mathbf{A}) \times (\mathbf{D} - \mathbf{A})| \tag{17}$$

$$= \left| \frac{1}{(n-m)^2} \det\begin{pmatrix}1 & -1\\ n & -m\end{pmatrix} \right| \tag{18}$$

$$= \frac{|m-n|}{(n-m)^2} = \frac{1}{|m-n|}. \tag{19}$$

Final Answer

$$\boxed{\text{Area of the parallelogram} = \frac{1}{|m-n|}} \tag{20}$$

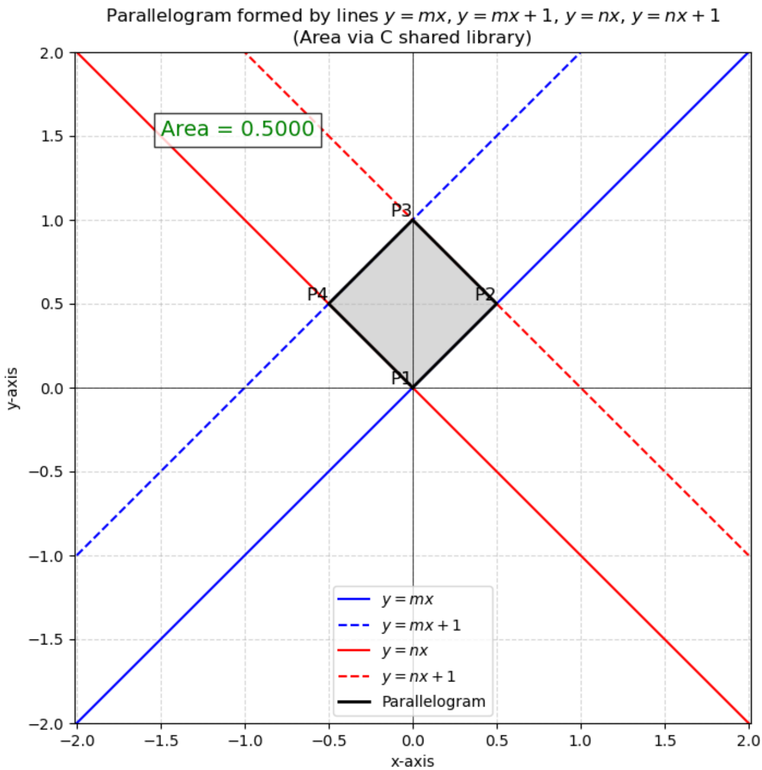

Fig. 1: Parallelogram formed by the given lines.

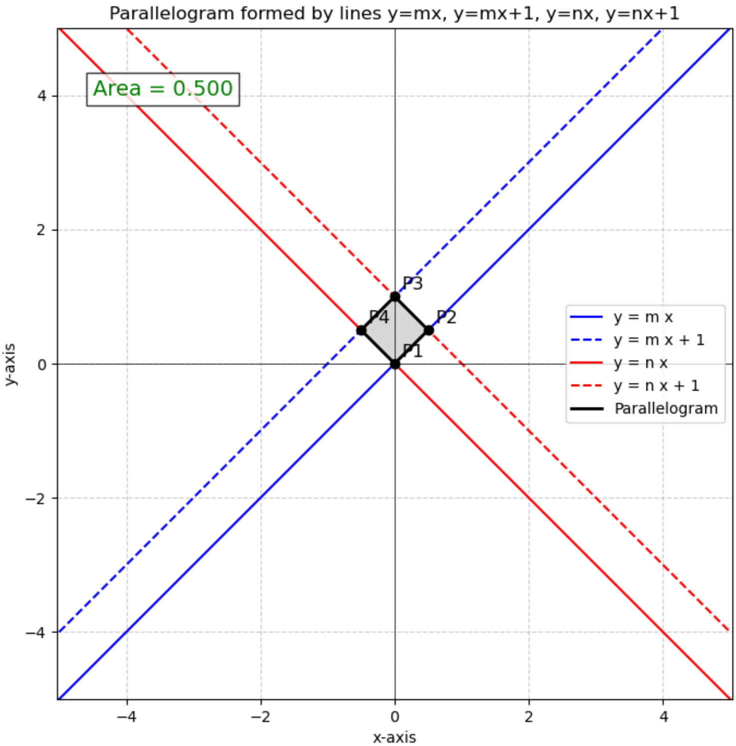

Fig. 2: Verification of intersection points.