4.11.14

EE25BTECH11036 - M Chanakya Srinivas

Question

Find the value of λ for which the following lines are perpendicular to each other. Hence determine whether the lines intersect or not.

$$\frac{x-5}{5\lambda+2} = \frac{2-y}{5} = \frac{1-z}{-1}, \tag{1}$$

$$\frac{x}{1} = \frac{y+\frac{1}{2}}{2\lambda} = \frac{z-1}{3}. \tag{2}$$

Solution

Step 1: Write lines in vector form

Choose points and direction vectors:

$$\mathbf{A}_1 = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}, \qquad \mathbf{m}_1 = \begin{pmatrix} 5\lambda+2 \\ -5 \\ 1 \end{pmatrix}, \tag{3}$$

$$\mathbf{A}_2 = \begin{pmatrix} 0 \\ -1/2 \\ 1 \end{pmatrix}, \qquad \mathbf{m}_2 = \begin{pmatrix} 1 \\ 2\lambda \\ 3 \end{pmatrix}. \tag{4}$$

Then the lines are

$$\mathbf{r}_1 = \mathbf{A}_1 + \kappa_1 \mathbf{m}_1, \tag{5}$$

$$\mathbf{r}_2 = \mathbf{A}_2 + \kappa_2 \mathbf{m}_2. \tag{6}$$

Step 2: Perpendicularity condition

Lines are perpendicular if

$$\mathbf{m}_1^\top \mathbf{m}_2 = 0.$$

Compute the dot product:

$$\mathbf{m}_1^\top \mathbf{m}_2 = \begin{pmatrix} 5\lambda+2 & -5 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2\lambda \\ 3 \end{pmatrix} \tag{7}$$

$$= (5\lambda+2)(1) + (-5)(2\lambda) + (1)(3) \tag{8}$$

$$= 5\lambda + 2 - 10\lambda + 3 \tag{9}$$

$$= -5\lambda + 5. \tag{10}$$

Hence

$$-5\lambda + 5 = 0 \implies \boxed{\lambda = 1}.$$

Step 3: Intersection condition

Lines intersect if

$$\mathbf{r}_1 = \mathbf{r}_2 \quad \Longrightarrow \quad \kappa_1 \mathbf{m}_1 - \kappa_2 \mathbf{m}_2 = \mathbf{A}_2 - \mathbf{A}_1.$$

Define

$$M(\lambda) = \begin{pmatrix} \mathbf{m}_1 & -\mathbf{m}_2 \end{pmatrix} = \begin{pmatrix} 5\lambda + 2 & -1 \\ -5 & -2\lambda \\ 1 & -3 \end{pmatrix}, \tag{11}$$

$$\mathbf{z} = \begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix}, \qquad \mathbf{b} = \mathbf{A}_2 - \mathbf{A}_1 = \begin{pmatrix} -5 \\ -5/2 \\ 0 \end{pmatrix}. \tag{12}$$

Then

$$M(\lambda)\mathbf{z} = \mathbf{b}.$$

Step 4: Form augmented matrix and do row reduction

$$\left[\begin{array}{cc|c} 5\lambda + 2 & -1 & -5 \\ -5 & -2\lambda & -5/2 \\ 1 & -3 & 0 \end{array}\right]$$

Use row 3 to eliminate κ_1:

$$R_3 : 1 \cdot \kappa_1 - 3 \cdot \kappa_2 = 0 \quad \Rightarrow \quad \kappa_1 = 3\kappa_2.$$

Substitute $\kappa_1 = 3\kappa_2$ into row 1 and row 2:

$$\text{Row 1: } (5\lambda + 2)(3\kappa_2) - 1 \cdot \kappa_2 = -5 \quad \Rightarrow \quad (15\lambda + 5)\kappa_2 = -5, \tag{13}$$

$$\text{Row 2: } -5(3\kappa_2) - 2\lambda\kappa_2 = -5/2 \quad \Rightarrow \quad (-15 - 2\lambda)\kappa_2 = -5/2. \tag{14}$$

Step 5: Solve for consistency

The system is consistent if

$$\frac{-5}{15\lambda + 5} = \frac{-5/2}{-15 - 2\lambda} \quad \Longrightarrow \quad -\frac{1}{3\lambda + 1} = \frac{5}{30 + 4\lambda}.$$

Solve:

$$-(30 + 4\lambda) = 5(3\lambda + 1) \tag{15}$$

$$-30 - 4\lambda = 15\lambda + 5 \tag{16}$$

$$19\lambda = -35 \tag{17}$$

$$\boxed{\lambda} = -\frac{35}{19}. \tag{18}$$

Step 6: conclusions

- Perpendicularity occurs at $\lambda = 1$. At this value, $\operatorname{rank}[M(1) \mid \mathbf{b}] > \operatorname{rank}(M(1))$, so lines are *skew*.

- Intersection occurs at $\lambda = -35/19$. At this value, $\text{rank}[M(\lambda) \mid \mathbf{b}] = \text{rank}(M(\lambda))$, so lines intersect (but are not perpendicular).

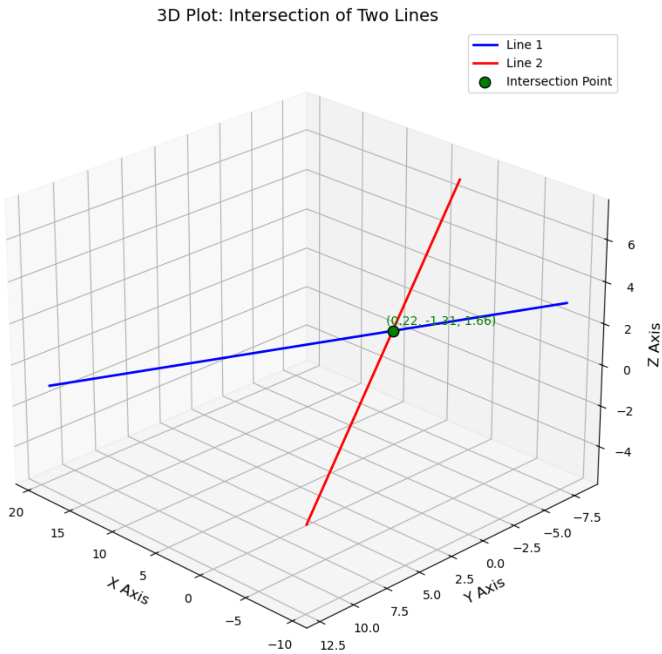

Fig. 1

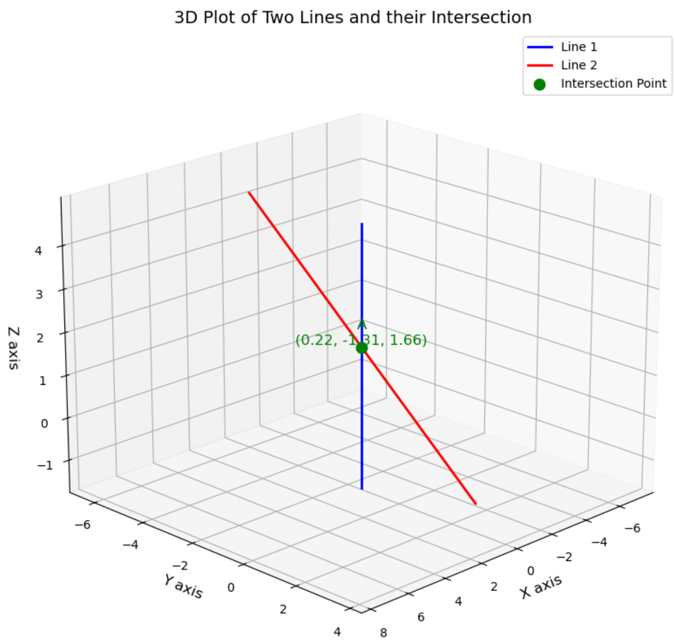
3D Plot of Two Lines and their Intersection
Line 1
Line 2
Intersection Point
(0.22, -1.31, 1.66)
Z axis
Y axis
X axis
4
3
2
1
0
−1
−6
−4
−2
0
2
4
8
6
4
2
0
−2
−4
−6

Fig. 2