

# 2.8.31

EE25BTECH11044 - Sai Hasini Pappula

**Question.** Given  $A(1, -2)$ ,  $B(2, 3)$ ,  $C(a, 2)$ , and  $D(-4, -3)$  which form a parallelogram. Find the value of  $a$  and the height of the parallelogram when  $AB$  is taken as the base. Use only vector projections and norms.

**Solution.** Represent the points as vectors:

$$\mathbf{A} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} a \\ 2 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}. \quad (0.1)$$

Since the diagonals of a parallelogram bisect each other,

$$\mathbf{A} + \mathbf{C} = \mathbf{B} + \mathbf{D}. \quad (0.2)$$

That is,

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} a \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -4 \\ -3 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}. \quad (0.3)$$

Equating components gives

$$1 + a = -2 \quad \implies \quad a = -3. \quad (0.4)$$

Thus

$$\mathbf{C} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}. \quad (0.5)$$

Now define the base and side vectors:

$$\mathbf{AB} = \mathbf{B} - \mathbf{A} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \quad \mathbf{AD} = \mathbf{D} - \mathbf{A} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}. \quad (0.6)$$

The projection  $\mathbf{AP}$  of  $\mathbf{AD}$  on  $\mathbf{AB}$  is

$$\mathbf{AP} = \frac{\mathbf{AB}^\top \mathbf{AD}}{\|\mathbf{AB}\|^2} \mathbf{AB} \quad (0.7)$$

Compute:

$$\mathbf{AB}^\top \mathbf{AD} = 1(-5) + 5(-1) = -10, \quad \|\mathbf{AB}\|^2 = 1^2 + 5^2 = 26. \quad (0.8)$$

So

$$\mathbf{AP} = \frac{-10}{26} \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} -\frac{5}{13} \\ -\frac{25}{13} \end{pmatrix}. \quad (0.9)$$

Subtracting, the perpendicular component is

$$\mathbf{r} = \mathbf{AD} - \mathbf{AP} = \begin{pmatrix} -5 \\ -1 \end{pmatrix} - \begin{pmatrix} -\frac{5}{13} \\ -\frac{25}{13} \end{pmatrix} = \begin{pmatrix} -\frac{60}{13} \\ \frac{12}{13} \end{pmatrix}. \quad (0.10)$$

The required height is the norm of  $\mathbf{r}$ :

$$h = \|\mathbf{r}\| = \sqrt{\left(-\frac{60}{13}\right)^2 + \left(\frac{12}{13}\right)^2}. \quad (0.11)$$

$$h = \frac{\sqrt{3744}}{13} = \frac{12\sqrt{26}}{13} = \frac{24}{\sqrt{26}}. \quad (0.12)$$

**Final Answer:**

$$a = -3, \quad h = \frac{12\sqrt{26}}{13} = \frac{24}{\sqrt{26}} \quad (0.13)$$

