frame=single, breaklines=true, columns=fullflexible

4.13.84

ai25btech11015 – M Sai Rithik

Question

Find the value of k such that the line

$$\frac{x-4}{1} = \frac{y-2}{1} = \frac{z-k}{2}$$

lies in the plane

$$2x - 4y + z = 7.$$

Solution

Write the line in the standard parametric matrix form using a general position vector $\mathbf{p}$ and direction vector $\mathbf{v}$:

$$\mathbf{r}(t) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{p} + t\,\mathbf{v}, \tag{1}$$

where $\mathbf{p}$ is any point on the line and $\mathbf{v}$ is the direction vector.

The plane can be written using its normal vector $\mathbf{n}$ as a dot-product condition:

$$\mathbf{n}^T\,\mathbf{r} = 7. \tag{2}$$

Substitute (**??**) into (**??**) and use linearity:

$$\mathbf{n}^T(\mathbf{p} + t\mathbf{v}) = 7. \tag{3}$$

Distribute the transpose-dot product:

$$\mathbf{n}^T\mathbf{p} \; + \; t\,\mathbf{n}^T\mathbf{v} \; = \; 7. \tag{4}$$

For the entire line to lie in the plane the identity (**??**) must hold for all real t. Hence the coefficient of t must be zero and the constant term must equal 7. Thus we obtain the two matrix equations:

$$\mathbf{n}^T\mathbf{v} = 0, \tag{5}$$

$$\mathbf{n}^T\mathbf{p} = 7. \tag{6}$$

Now substitute the specific normal, position and direction vectors from the problem:

$$\mathbf{n} = \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}, \tag{7}$$

$$\mathbf{p} = \begin{pmatrix} 4 \\ 2 \\ k \end{pmatrix}, \tag{8}$$

$$\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}. \tag{9}$$

Evaluate the t-coefficient condition (**??**) by matrix multiplication:

$$\mathbf{n}^T\mathbf{v} = \begin{bmatrix} 2 & -4 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 2 \cdot 1 + (-4) \cdot 1 + 1 \cdot 2 = 0. \tag{10}$$

So (**??**) is satisfied (no restriction on k from this equation).

Evaluate the constant condition (**??**):

$$\mathbf{n}^T\mathbf{p} = \begin{bmatrix} 2 & -4 & 1 \end{bmatrix} \begin{pmatrix} 4 \\ 2 \\ k \end{pmatrix} = 2 \cdot 4 + (-4) \cdot 2 + 1 \cdot k = k. \tag{11}$$

Set (**??**) equal to 7 and solve for k:

$$k = 7. \tag{12}$$

Answer

$$\boxed{k = 7} \tag{13}$$

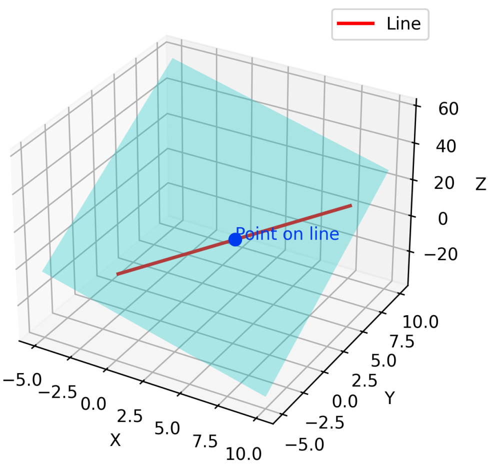

Figure 1: (Sketch: plane through A, B, C).