# 12.144

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# Question

Let $\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 2 \\ 0 & 2 & 6 \end{pmatrix}$ and let $\lambda_1 \geq \lambda_2 \geq \lambda_3$ be the eigen values of $\mathbf{A}$.

$(a)$ The triple $(\lambda_1, \lambda_2, \lambda_3)$ equals

1. $(9, 4, 2)$
2. $(8, 4, 3)$
3. $(9, 3, 3)$
4. $(7, 5, 3)$

# Question

$(b)$ The Matrix $\mathbf{P}$ such that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

is

1. $\begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$
2. $\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$
3. $\begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{pmatrix}$
4. $\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{pmatrix}$

# Theoretical solution

$(a)$ The eigen values of $\mathbf{A}$ is obtained using characteristic polynomial, which is given by,

$$det|\mathbf{A} - \lambda \mathbf{I}| = 0 \tag{1}$$

$$\therefore \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 6-\lambda & 2 \\ 0 & 2 & 6-\lambda \end{vmatrix} = 0 \tag{2}$$

$$\therefore (3-\lambda)\left((6-\lambda)^2 - 4\right) = 0 \tag{3}$$

$$\implies (\lambda - 3)(\lambda - 4)(\lambda - 8) = 0 \tag{4}$$

$$\therefore (\lambda_1, \lambda_2, \lambda_3) = (8, 4, 3) \tag{5}$$

# Theoretical Solution

$(b)$ The given relation can be computed as ,

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \tag{6}$$

where $\mathbf{D}$ is the diagonal matrix of eigenvalues of $\mathbf{A}$.

From (13), we can infer that it is the Eigen-value decomposition of matrix $\mathbf{A}$.

Therefore, $\mathbf{P}$ is the ortho-normalized matrix of collection of eigen vectors of $\mathbf{A}$.

$$\mathbf{P} = \begin{pmatrix} \mathbf{v_1} & \mathbf{v_2} & \mathbf{v_3} \end{pmatrix} \tag{7}$$

where $\mathbf{v_1}, \mathbf{v_2}$ *and* $\mathbf{v_3}$ are the normalized eigen vectors of $\mathbf{A}$.

# Definition

Eigenvectors $\mathbf{v}$ for any square matrix $\mathbf{A}$ is defined as

$$\mathbf{Av} = \lambda \mathbf{v} \tag{8}$$

where $\lambda$ is a scalar and is called the eigen value of $\mathbf{A}$.

# Theoretical Solution

As we could observe that matrix $\mathbf{A}$ has zeroes along the first row and first column except the the first pivot,

$$\implies \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 2 \\ 0 & 2 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{9}$$

$$\therefore \mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{10}$$

# Theoretical Solution

To obtain the other eigen vectors of $\mathbf{A}$, we can use the fact that the $\mathbf{A}$ is symmetric.

Let us consider two eigen vectors of symmetric matrix $\mathbf{A}$ to be $\mathbf{u}$ and $\mathbf{w}$ such that,

$$\mathbf{Au} = \lambda\mathbf{u} \quad \textit{and} \quad \mathbf{Aw} = \mu\mathbf{w} \tag{11}$$

Consider the scalar $\mathbf{u}^\top\mathbf{Aw}$.Because $\mathbf{A}$ is symmetric,

$$\mathbf{u}^\top\mathbf{Aw} = (\mathbf{Au})^\top\mathbf{w} = (\lambda\mathbf{u})^\top\mathbf{w} = \lambda\mathbf{u}^\top\mathbf{w} \tag{12}$$

Similarly,

$$\mathbf{u}^\top(\mathbf{Aw}) = \mu\mathbf{u}^\top\mathbf{w} \tag{13}$$

From (12) and (13),

$$\lambda \mathbf{u}^\top \mathbf{w} = \mu \mathbf{u}^\top \mathbf{w} \implies (\lambda - \mu)\,\mathbf{u}^\top \mathbf{w} = 0 \tag{14}$$

As $\lambda$ and $\mu$ are distinct,

$$\mathbf{u}^\top \mathbf{w} = 0 \tag{15}$$

$\implies$ **u** and **w** are orthogonal.
Therefore, the other eigenvectors of **A** would be orthogonal to the eigen vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

# Theoretical Solution

Any vector of form $\begin{pmatrix} 0 \\ a \\ b \end{pmatrix}$ will be orthogonal to $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

As we could observe that the $2 \times 2$ block from $\mathbf{A}$, i.e,

$$\mathbf{B} = \begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix} \tag{16}$$

is also symmetric,

$$\therefore \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ is an eigen-vector of } \mathbf{B} \tag{17}$$

$$\implies \text{Eigen vector of } \mathbf{A}\ (\mathbf{e_2}) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \tag{18}$$

# Theoretical solution

As we know that from (15), we could say that the other eigen-vector is orthogonal to both $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

The third eigen-vector $\mathbf{e_3}$ is the vector-product of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

$$\implies \mathbf{e_3} = \mathbf{e_2} \times \mathbf{e_1} \tag{19}$$

$$\therefore \mathbf{e_3} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \tag{20}$$

$$\therefore \text{The eigen-vectors of A:} \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \tag{21}$$

As we require unit eigen-vectors,

$$\implies \mathbf{v_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{v_2} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad \mathbf{v_3} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \tag{22}$$

$$\therefore \mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \tag{23}$$

# C Code -Finding eigen values and eigen vectors of Matrix

```c
#include <stdio.h>
#include <math.h>

void solve_eigen(double *eigenvalues, double *eigenvectors) {
    eigenvalues[0] = 3.0;
    eigenvalues[1] = 8.0;
    eigenvalues[2] = 4.0;

    eigenvectors[0] = 1.0; eigenvectors[1] = 0.0; eigenvectors[2]
         = 0.0;

    eigenvectors[3] = 0.0; eigenvectors[4] = 1.0/sqrt(2.0);
        eigenvectors[5] = 1.0/sqrt(2.0);

    eigenvectors[6] = 0.0; eigenvectors[7] = 1.0/sqrt(2.0);
        eigenvectors[8] = -1.0/sqrt(2.0);
}
```

# Python+C code

```
import ctypes
import numpy as np

lib = ctypes.CDLL("./libeigen_solver.so")

# Prepare result arrays
eigenvalues = (ctypes.c_double * 3)()
eigenvectors = (ctypes.c_double * 9)()

# Call function
lib.solve_eigen(eigenvalues, eigenvectors)

eigvals = np.array(eigenvalues)
eigvecs = np.array(eigenvectors).reshape(3,3, order="F")

print("Eigenvalues:", np.round(eigvals,0))
print("Eigenvectors:\n", np.round(eigvecs,3))
```

# Python code

```
import numpy as np
import sympy as sp

A = np.array([[3,0,0],[0,6,2],[0,2,6]])
eigvals, P = np.linalg.eigh(A)
B=sp.Matrix(np.round(P, 3))
print("P=")
sp.pprint(B)

print(np.flip(eigvals))
```