4.6.9

EE25BTECH11021 - Dhanush Sagar

Question

Find the equation of the plane containing the two parallel lines $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z}{3}$ and $\frac{x}{4} = \frac{y-2}{-2} = \frac{z+1}{6}$. Also, determine whether the plane thus obtained contains the line $\frac{x-2}{3} = \frac{y-1}{1} = \frac{z-2}{5}$.

Solution

The general equation of a plane is

$$\mathbf{n}^\top \mathbf{X} = c \tag{0.1}$$

where $\mathbf{n}$ is the normal vector.

$$\text{Line 1: } \frac{x-1}{2} = \frac{y+1}{-1} = \frac{z}{3} \tag{0.2}$$

$$\text{Line 2: } \frac{x}{4} = \frac{y-2}{-2} = \frac{z+1}{6}. \tag{0.3}$$

From the two given parallel lines we extract:

$$\mathbf{v} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \qquad \text{(common direction vector)} \tag{0.4}$$

$$\mathbf{p}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{p}_2 = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \qquad \text{(points on each line)} \tag{0.5}$$

$$\mathbf{d} = \mathbf{p}_2 - \mathbf{p}_1 = \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} \qquad \text{(difference of points)} \tag{0.6}$$

The normal $\mathbf{n}$ must be orthogonal to both $\mathbf{v}$ and $\mathbf{d}$:

$$\mathbf{n}^\top \mathbf{v} = 0 \tag{0.7}$$

$$\mathbf{n}^\top \mathbf{d} = 0 \tag{0.8}$$

To fix the scale of $\mathbf{n}$, we impose a normalization condition using point $\mathbf{p}_1$:

$$\mathbf{n}^\top \mathbf{p}_1 = 1 \tag{0.9}$$

Put these column vectors into rows of new matrix $\mathbf{A}$

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 3 \\ -1 & 3 & -1 \\ 1 & -1 & 0 \end{pmatrix} \tag{0.10}$$

from the equations 0.7,0.8,0.9 we can write

$$\begin{pmatrix} 2 & -1 & 3 \\ -1 & 3 & -1 \\ 1 & -1 & 0 \end{pmatrix} \mathbf{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{0.11}$$

This augmented matrix form is

$$\begin{pmatrix} 2 & -1 & 3 & 0 \\ -1 & 3 & -1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix} \tag{0.12}$$

Row-reducing:

$$\begin{pmatrix} 2 & -1 & 3 & 0 \\ -1 & 3 & -1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \to 2R_2 + R_1} \begin{pmatrix} 2 & -1 & 3 & 0 \\ 0 & 5 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix} \tag{0.13}$$

$$\xrightarrow{R_3 \to R_3 - \frac{1}{2} R_1} \begin{pmatrix} 2 & -1 & 3 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & -\frac{1}{2} & -\frac{3}{2} & 1 \end{pmatrix} \tag{0.14}$$

$$\xrightarrow{R_3 \to 5R_3 - R_2} \begin{pmatrix} 2 & -1 & 3 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & -7 & 5 \end{pmatrix} \tag{0.15}$$

From the last row, the third entry of $\mathbf{n}$ is $-\frac{5}{7}$. Back-substitution gives

$$\mathbf{n} = \begin{pmatrix} \frac{8}{7} \\ \frac{1}{7} \\ -\frac{5}{7} \end{pmatrix} \tag{0.16}$$

Therefore, the plane equation is

$$\mathbf{n}^\top \mathbf{X} = 1 \tag{0.17}$$

Equivalently, multiplying throughout by 7 gives

$$\begin{pmatrix} 8 & 1 & -5 \end{pmatrix} \mathbf{X} = 7 \tag{0.18}$$

This is the required plane passing through the given parallel lines.

Check if the third line $\frac{x-2}{3} = \frac{y-1}{1} = \frac{z-2}{5}$. lies in the plane by verifying the point and direction:

$$\mathbf{P}_3 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{d}_3 = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} \tag{0.19}$$

$$\mathbf{n}^T \mathbf{P}_3 = \begin{pmatrix} 8 & 1 & -5 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 7 \tag{0.20}$$

$$\mathbf{n}^T \mathbf{d}_3 = \begin{pmatrix} 8 & 1 & -5 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} = 0 \tag{0.21}$$

Therefore, the plane containing the first two lines has the matrix form:

$$\begin{pmatrix} 8 & 1 & -5 \end{pmatrix} \mathbf{r} = 7$$

and it also contains the third line.

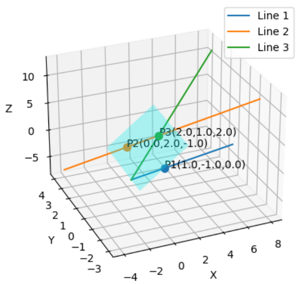

Fig. 0.1