

Extra

EE25BTECH11023 - Venkata Sai

Question:

Find the equations of tangents drawn from origin to the circle $x^2 + y^2 - 2rx - 2hy + h^2 = 0$, are

1) $x = 0$

3) $(h^2 - r^2)x - 2rhy = 0$

2) $y = 0$

4) $(h^2 - r^2)x + 2rhy = 0$

Solution:

Equation of a circle is given by

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0, \quad (1)$$

The parametric equation of this line is

$$\mathbf{x} = \mathbf{h} + k\mathbf{m}. \quad (2)$$

$$(\mathbf{h} + k\mathbf{m})^T \mathbf{V} (\mathbf{h} + k\mathbf{m}) + 2\mathbf{u}^T (\mathbf{h} + k\mathbf{m}) + f = 0 \quad (3)$$

$$\mathbf{h}^T \mathbf{V} \mathbf{h} + k(\mathbf{m}^T \mathbf{V} \mathbf{h} + \mathbf{h}^T \mathbf{V} \mathbf{m}) + k^2 \mathbf{m}^T \mathbf{V} \mathbf{m} + 2\mathbf{u}^T \mathbf{h} + 2k\mathbf{u}^T \mathbf{m} + f = 0 \quad (4)$$

$$(\mathbf{m}^T \mathbf{V} \mathbf{m})k^2 + 2(\mathbf{m}^T \mathbf{V} \mathbf{h} + \mathbf{u}^T \mathbf{m})k + (\mathbf{h}^T \mathbf{V} \mathbf{h} + 2\mathbf{u}^T \mathbf{h} + f) = 0 \quad (5)$$

Hence the quadratic in k is

$$Ak^2 + Bk + C = 0, \quad (6)$$

where

$$A = \mathbf{m}^T \mathbf{V} \mathbf{m}, \quad (7)$$

$$B = 2(\mathbf{m}^T \mathbf{V} \mathbf{h} + \mathbf{u}^T \mathbf{m}), \quad (8)$$

$$C = (\mathbf{h}^T \mathbf{V} \mathbf{h} + 2\mathbf{u}^T \mathbf{h} + f) \quad (9)$$

Discriminant is 0 as tangent intersects the circle at only one point

$$B^2 - 4AC = 0 \quad (10)$$

$$A = \mathbf{m}^T \mathbf{V} \mathbf{m}, \quad (11)$$

$$B = 2(\mathbf{m}^T \mathbf{V} \mathbf{h} + \mathbf{u}^T \mathbf{m}), \quad (12)$$

$$C = (\mathbf{h}^T \mathbf{V} \mathbf{h} + 2\mathbf{u}^T \mathbf{h} + f) \quad (13)$$

Discriminant is 0 as tangent intersects the circle at only one point

$$B^2 - 4AC = 0 \quad (14)$$

Substituting the expressions for A, B, and C and expanding:

$$\left(2(\mathbf{m}^\top \mathbf{Vh} + \mathbf{u}^\top \mathbf{m})\right)^2 - 4(\mathbf{m}^\top \mathbf{Vm})(\mathbf{h}^\top \mathbf{Vh} + 2\mathbf{u}^\top \mathbf{h} + f) = 0. \quad (15)$$

$$(\mathbf{m}^\top \mathbf{Vh} + \mathbf{u}^\top \mathbf{m})(\mathbf{m}^\top \mathbf{Vh} + \mathbf{u}^\top \mathbf{m})^\top - (\mathbf{m}^\top \mathbf{Vm})(\mathbf{h}^\top \mathbf{Vh} + 2\mathbf{u}^\top \mathbf{h} + f) = 0 \quad (16)$$

$$(\mathbf{m}^\top \mathbf{Vh} + \mathbf{u}^\top \mathbf{m})(\mathbf{h}^\top \mathbf{Vm} + \mathbf{m}^\top \mathbf{u}) - (\mathbf{m}^\top \mathbf{Vm})(\mathbf{h}^\top \mathbf{Vh} + 2\mathbf{u}^\top \mathbf{h} + f) = 0 \quad (17)$$

$$\begin{aligned} &(\mathbf{m}^\top \mathbf{Vh})(\mathbf{h}^\top \mathbf{Vm}) + (\mathbf{m}^\top \mathbf{Vh})(\mathbf{m}^\top \mathbf{u}) + (\mathbf{u}^\top \mathbf{m})(\mathbf{h}^\top \mathbf{Vm}) + (\mathbf{u}^\top \mathbf{m})(\mathbf{m}^\top \mathbf{u}) \\ &- (\mathbf{m}^\top \mathbf{Vm})(\mathbf{h}^\top \mathbf{Vh}) + 2(\mathbf{m}^\top \mathbf{Vm})(\mathbf{u}^\top \mathbf{h}) + f(\mathbf{m}^\top \mathbf{Vm}) = 0 \end{aligned} \quad (18)$$

$$\begin{aligned} &(\mathbf{m}^\top \mathbf{Vh})(\mathbf{h}^\top \mathbf{Vm}) + (\mathbf{m}^\top \mathbf{Vh})(\mathbf{m}^\top \mathbf{u}) + (\mathbf{u}^\top \mathbf{m})(\mathbf{h}^\top \mathbf{Vm}) + (\mathbf{u}^\top \mathbf{m})(\mathbf{m}^\top \mathbf{u}) \\ &- (\mathbf{m}^\top \mathbf{Vm})(\mathbf{h}^\top \mathbf{Vh}) - 2(\mathbf{m}^\top \mathbf{Vm})(\mathbf{u}^\top \mathbf{h}) - f(\mathbf{m}^\top \mathbf{Vm}) = 0 \end{aligned} \quad (19)$$

$$\mathbf{m}^\top (\mathbf{Vh} \mathbf{h}^\top \mathbf{V} + \mathbf{Vh} \mathbf{u}^\top + \mathbf{u} \mathbf{h}^\top \mathbf{V} + \mathbf{u} \mathbf{u}^\top - (\mathbf{h}^\top \mathbf{Vh}) \mathbf{V} - 2(\mathbf{u}^\top \mathbf{h}) \mathbf{V} - f \mathbf{V}) \mathbf{m} = 0 \quad (20)$$

$$\mathbf{m}^\top (\mathbf{Vh} (\mathbf{Vh})^\top + \mathbf{Vh} \mathbf{u}^\top + \mathbf{u} (\mathbf{Vh})^\top + \mathbf{u} \mathbf{u}^\top - (\mathbf{h}^\top \mathbf{Vh}) \mathbf{V} - 2(\mathbf{u}^\top \mathbf{h}) \mathbf{V} - f \mathbf{V}) \mathbf{m} = 0 \quad (21)$$

$$\mathbf{m}^\top ((\mathbf{Vh} + \mathbf{u})(\mathbf{Vh} + \mathbf{u})^\top - (\mathbf{h}^\top \mathbf{Vh} + 2\mathbf{u}^\top \mathbf{h} + f) \mathbf{V}) \mathbf{m} = 0 \quad (22)$$

$$\mathbf{m}^\top ((\mathbf{Vh} + \mathbf{u})(\mathbf{Vh} + \mathbf{u})^\top - g(\mathbf{h})) \mathbf{m} = 0 \quad (23)$$

where

$$g(\mathbf{h}) = (\mathbf{h}^\top \mathbf{Vh} + 2\mathbf{u}^\top \mathbf{h} + f) \mathbf{V} \quad (24)$$

On comparing the given circle with matrix equation

$$x^2 + y^2 = \mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} \implies \mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (25)$$

$$-2rx - 2hy = 2\mathbf{u}^\top \mathbf{x} \implies \mathbf{u}^\top = \begin{pmatrix} -r & -h \end{pmatrix} \implies \mathbf{u} = \begin{pmatrix} -r \\ -h \end{pmatrix} \quad (26)$$

$$h^2 = f \quad (27)$$

Given tangents are drawn from the origin

$$\mathbf{h} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (28)$$

$$\mathbf{Vh} + \mathbf{u} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -r \\ -h \end{pmatrix} = \begin{pmatrix} -r \\ -h \end{pmatrix} \quad (29)$$

$$\mathbf{Vh} = 0 \implies \mathbf{h}^\top \mathbf{Vh} = 0 \quad (30)$$

$$g(\mathbf{h}) = (\mathbf{h}^\top \mathbf{V} \mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f) \mathbf{V} = \left(0 + 2 \begin{pmatrix} -r \\ -h \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} + h^2\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (31)$$

$$g(\mathbf{h}) = h^2 \quad (32)$$

$$\mathbf{m}^\top \sum \mathbf{m} \Rightarrow \sum = \begin{pmatrix} -r \\ -h \end{pmatrix} \begin{pmatrix} -r & -h \end{pmatrix} - h^2 \mathbf{I} \quad (33)$$

$$\sum = \begin{pmatrix} r^2 & rh \\ rh & h^2 \end{pmatrix} - \begin{pmatrix} h^2 & 0 \\ 0 & h^2 \end{pmatrix} = \begin{pmatrix} r^2 - h^2 & rh \\ rh & 0 \end{pmatrix} \quad (34)$$

$$|\sum - \lambda \mathbf{I}| = 0 \quad (35)$$

$$\left| \begin{pmatrix} r^2 - h^2 - \lambda & rh \\ rh & -\lambda \end{pmatrix} \right| = 0 \quad (36)$$

$$(r^2 - h^2 - \lambda)(-\lambda) - r^2 h^2 = 0 \quad (37)$$

$$\lambda^2 - (r^2 - h^2)\lambda - r^2 h^2 = 0 \quad (38)$$

$$(\lambda - r^2)(\lambda + h^2) = 0 \quad (39)$$

the eigen values are

$$\lambda = r^2 \text{ or } \lambda = -h^2 \quad (40)$$

Because the tangent passes through origin

$$\mathbf{m} = \mathbf{x} \quad (41)$$

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} r^2 - h^2 & rh \\ rh & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad (42)$$

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} (r^2 - h^2)x + (rh)y \\ rhx \end{pmatrix} = 0 \quad (43)$$

$$x((r^2 - h^2)x + rhy) + y(rhx) = 0 \quad (44)$$

$$x^2(r^2 - h^2) + 2rhxy = 0 \quad (45)$$

$$x((r^2 - h^2)x + 2rhy) = 0 \quad (46)$$

$$x = 0 \text{ and } (r^2 - h^2)x + 2rhy = 0 \quad (47)$$