Angle Between Vectors Using Gram Matrix

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Problem Statement

Find the acute angle between the planes

$$\begin{aligned} x - 2y - 2z &= 5 \\ 3x - 6y + 2z &= 7 \end{aligned}$$

Solution

The angle between two planes is the angle between their normals. Let

$$\mathbf{n}_1 = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}, \quad \mathbf{n}_2 = \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix}.$$

The dot product is

$$\mathbf{n}_1^\top \mathbf{n}_2 = 1 \cdot 3 + (-2)(-6) + (-2)(2) = 11. \tag{1}$$

The norms are

$$\|\mathbf{n}_1\| = \sqrt{1^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3, \tag{2}$$

$$\|\mathbf{n}_2\| = \sqrt{3^2 + (-6)^2 + 2^2} = \sqrt{49} = 7. \tag{3}$$

Solution (cont..)

Hence,

$$\cos\theta = \frac{\mathbf{n}_1^\top \mathbf{n}_2}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|} \tag{4}$$

$$= \frac{11}{3 \cdot 7} \tag{5}$$

$$= \frac{11}{21}. \tag{6}$$

Therefore, the acute angle between the planes is

$$\boxed{\theta = \arccos\left(\tfrac{11}{21}\right) \approx 58.41^\circ}$$

Python Code (Plotting Normals)

```
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

u = np.array([1, −2, −2])
v = np.array([3, −6, 2])
origin = np.zeros(3)

fig = plt.figure()
ax = fig.add_subplot(111, projection=’3d’)

ax.quiver(*origin, *u, color=’r’, arrow_length_ratio=0.1)
ax.text(u[0]*1.1, u[1]*1.1, u[2]*1.1, ”u”, color=’r’)
```

Python Code (cont..)

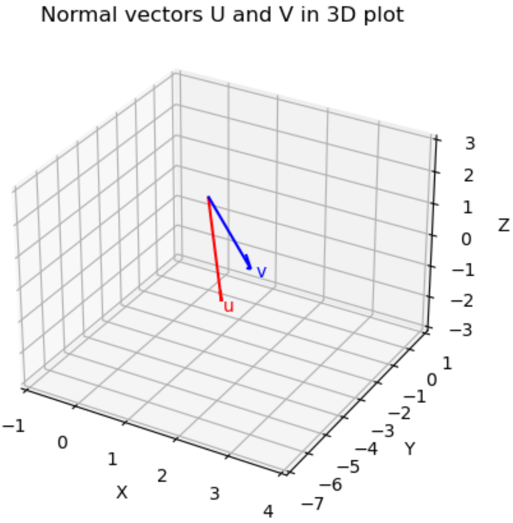
```
ax.quiver(*origin, *v, color='b', arrow_length_ratio=0.1)
ax.text(v[0]*1.1, v[1]*1.1, v[2]*1.1, "v", color='b')

all_points = np.vstack([origin, u, v])
ax.set_xlim([all_points[:,0].min()-1, all_points[:,0].max()+1])
ax.set_ylim([all_points[:,1].min()-1, all_points[:,1].max()+1])
ax.set_zlim([all_points[:,2].min()-1, all_points[:,2].max()+1])

ax.set_xlabel("X")
ax.set_ylabel("Y")
ax.set_zlabel("Z")
ax.set_title("Normal-vectors-U-and-V-in-3D-plot")

plt.show()
```

Plot (Python)


Normal vectors U and V in 3D plot
u
v
X
−1
0
1
2
3
4
Y
1
0
−1
−2
−3
−4
−5
−6
−7
Z
3
2
1
0
−1
−2
−3

C Code (Finding Angle)

```c
#include <stdio.h>
#include <math.h>

double find_angle(int A[3], int B[3]) {
    int dot = 0;
    double a_mag = 0;
    double b_mag = 0;
    for (int i=0; i<3; i++) {
        dot += A[i]*B[i];
        a_mag += A[i]*A[i];
        b_mag += B[i]*B[i];
    }
```

C Code (cont..)

```
    a_mag = pow(a_mag, 0.5);
    b_mag = pow(b_mag, 0.5);
    double cos_theta = dot/(a_mag*b_mag);
    if (cos_theta > 1.0) {
        cos_theta = 1.0;
    }
    if (cos_theta < -1.0) {
        cos_theta = -1.0;
    }
    return acos(cos_theta);
}
```

Python Code (Calling C)

```
import ctypes
import numpy

lib = ctypes.CDLL(”./computations.so”)

lib.find_angle.argtypes = [
    ctypes.POINTER((ctypes.c_int * 3)),
    ctypes.POINTER((ctypes.c_int * 3))
]
lib.find_angle.restype = ctypes.c_double
```

Python Code (cont..)

```
def compute_angle(u, v):
    u_arr = (ctypes.c_int * 3)(*u)
    v_arr = (ctypes.c_int * 3)(*v)
    theta = lib.find_angle(u_arr, v_arr)
    return theta

u = [1, −2, −2]
v = [3, −6, 2]
theta = compute_angle(u, v)

print(f"Angle (radians): {theta}")
print(f"Angle (degrees): {theta * 180 / numpy.pi}")
```