2.10.44

Bhoomika V - EE25BTECH11015

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Question

If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are unit vectors, then

$$\|\mathbf{a} - \mathbf{b}\|^2 + \|\mathbf{b} - \mathbf{c}\|^2 + \|\mathbf{a} - \mathbf{b}\|^2$$

does not exceed

- 4
- 9
- 8
- 6

Solution-Gram matrix

Let

$$x = \mathbf{a} \cdot \mathbf{b}, \quad y = \mathbf{a} \cdot \mathbf{c}, \quad z = \mathbf{b} \cdot \mathbf{c}.$$

Since $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are unit vectors, their Gram matrix is

$$G = \begin{pmatrix} \mathbf{a} \cdot \mathbf{a} & \mathbf{a} \cdot \mathbf{b} & \mathbf{a} \cdot \mathbf{c} \\ \mathbf{a} \cdot \mathbf{b} & \mathbf{b} \cdot \mathbf{b} & \mathbf{b} \cdot \mathbf{c} \\ \mathbf{a} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{c} & \mathbf{c} \cdot \mathbf{c} \end{pmatrix}.$$

$$= \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix}.$$

SOlution

Now consider

$$(1, 1, 1) \, G \, (1, 1, 1)^T = (\mathbf{a} + \mathbf{b} + \mathbf{c}) \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c}) \geq 0.$$

Expanding,

$$|\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2(x + y + z) = 3 + 2(x + y + z) \geq 0,$$

$$\implies \quad x + y + z \geq -\tfrac{3}{2}. \tag{1}$$

Solution

Now ,

$$|\mathbf{a} - \mathbf{b}|^2 + |\mathbf{b} - \mathbf{c}|^2 + |\mathbf{c} - \mathbf{a}|^2 = (2 - 2x) + (2 - 2z) + (2 - 2y).$$

So,

$$= 6 - 2(x + y + z).$$

From Equation (1)

$$6 - 2(x + y + z) \leq 6 - 2\left(-\tfrac{3}{2}\right) = 9.$$

Answer

Thus, $\|\mathbf{a} - \mathbf{b}\|^2 + \|\mathbf{b} - \mathbf{c}\|^2 + \|\mathbf{a} - \mathbf{b}\|^2$ does not exceed 9.

C Code - A function to find max value

```c
#include <math.h>

// Function to check inequality
// Returns 1 if inequality holds, 0 otherwise
int is_within_bound(float ax, float ay, float az,
                    float bx, float by, float bz,
                    float cx, float cy, float cz) {

    // Compute squared distances
    float ab = (ax - bx)*(ax - bx) + (ay - by)*(ay - by) + (az -
        bz)*(az - bz);
    float bc = (bx - cx)*(bx - cx) + (by - cy)*(by - cy) + (bz -
        cz)*(bz - cz);
    float ca = (cx - ax)*(cx - ax) + (cy - ay)*(cy - ay) + (cz -
        az)*(cz - az);
```

C Code - A function to find max value

```c
float sum = ab + bc + ca;

    if (sum <= 9.0f) {
        return 1; // Inequality holds
    } else {
        return 0; // Inequality violated
    }
}
```

Python Code

```python
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d.art3d import Poly3DCollection
import ctypes
import os

# --- Load the C library ---
try:
    c_lib = ctypes.CDLL('./code.so')
except OSError:
    print(" Error: 'code.so' not found. Compile with:")
    print(" gcc -shared -o code.so -fPIC inequality.c -lm")
    exit()

# Define argument and return types
c_lib.is_within_bound.argtypes = [
    ctypes.c_float, ctypes.c_float, ctypes.c_float,
    ctypes.c_float, ctypes.c_float, ctypes.c_float,
    ctypes.c_float, ctypes.c_float, ctypes.c_float]
```

Python Code

```
c_lib.is_within_bound.restype = ctypes.c_int

# --- Function to generate random unit vector ---
def random_unit_vector():
    vec = np.random.randn(3)
    return vec / np.linalg.norm(vec)

# --- Generate unit vectors a, b, c ---
a = random_unit_vector()
b = random_unit_vector()
c = random_unit_vector()

# --- Call C function ---
result = c_lib.is_within_bound(a[0], a[1], a[2],
                               b[0], b[1], b[2],
                               c[0], c[1], c[2])
```

Python Code

```python
if result == 1:
    print(" The inequality holds (sum  9).")
else:
    print(" The inequality is violated.")

# --- Plotting ---
fig = plt.figure(figsize=(8,6))
ax = fig.add_subplot(111, projection='3d')

# Plot points
ax.scatter(*a, color="red", s=50)
ax.scatter(*b, color="blue", s=50)
ax.scatter(*c, color="green", s=50)

# Draw triangle
triangle = np.array([a, b, c])
ax.add_collection3d(Poly3DCollection([triangle], alpha=0.2,
    facecolor='cyan'))
```

Python Code

```python
# Edges
ax.plot(*zip(a,b), color="black")
ax.plot(*zip(b,c), color="black")
ax.plot(*zip(c,a), color="black")

# Labels
ax.text(*a, "a", color="red")
ax.text(*b, "b", color="blue")
ax.text(*c, "c", color="green")

# Axes labels
ax.set_xlabel("X-axis")
ax.set_ylabel("Y-axis")
ax.set_zlabel("Z-axis")
ax.set_title("Triangle formed by unit vectors a, b, c")

plt.show()
```