10.7.53

EE25BTECH11004 - Aditya Appana

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Question

Let C_1 and C_2 be respectively, the parabolas $x^2 = y - 1$ and $y^2 = x - 1$. Let $\mathbf{P}$ be any point on C_1 and $\mathbf{Q}$ be any point on C_2. Let $\mathbf{P}_i$ and $\mathbf{Q}_i$ be the reflections of $\mathbf{P}$ and $\mathbf{Q}$ respectively with respect to the line $y = x$. Prove that $\mathbf{P}_i$ lies on C_2, $\mathbf{Q}_1$ lies on C_1, and $\mathbf{PQ} \geq min(\mathbf{PP_1}, \mathbf{QQ_1})$. Hence or otherwise determine points $\mathbf{P}_0$ and $\mathbf{Q}_0$ on the parabolas C_1 and C_2 respectively such that $\mathbf{P}_0\mathbf{Q}_0 \leq \mathbf{PQ}$ for all pairs of points $(\mathbf{P}, \mathbf{Q})$ with $\mathbf{P}$ on C_1 and $\mathbf{Q}$ on C_2.

Solution

The representation of a conic in vector form is

$$\mathbf{x^T V x} + 2\mathbf{u^T x} + f = 0 \tag{1}$$

$x^2 = y - 1$ represented in this form is:

$$\mathbf{x^T} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}^T \mathbf{x} + 1 = 0 \tag{2}$$

$y^2 = x - 1$ represented in this form is:

$$\mathbf{x^T} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix}^T \mathbf{x} + 1 = 0 \tag{3}$$

Given $\mathbf{P}$ lies on (1), and given $\mathbf{P_i}$ is the mirror image of $\mathbf{P}$ with respect to line $y = x$. The mirror image $\mathbf{P_i}$ can therefore be represented as:

$$\mathbf{P_i} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{P} \tag{4}$$

If we want to prove that $\mathbf{P_i}$ lies on C_2, we need to show that $\mathbf{P_i}$ satisfies C_2.

$$\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{P}\right)^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{P} + 2 \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{P} + 1 = 0 \tag{5}$$

$$\mathbf{P}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{P} + 2 \begin{pmatrix} -\frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{P} + 1 = 0 \tag{6}$$

$$\mathbf{P}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{P} + 2 \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix}^T \mathbf{P} + 1 = 0 \tag{7}$$

(7) is the same as C_1, so $\mathbf{P_i}$ satisfies C_2. In a similar manner, it can be proved that $\mathbf{Q_i}$ lies on C_1.

We now need to prove $\mathbf{PQ} \geq min(\mathbf{PP_1}, \mathbf{QQ_1})$. Take $\mathbf{P}(1, 2)$ and $\mathbf{Q}(5, 2)$.

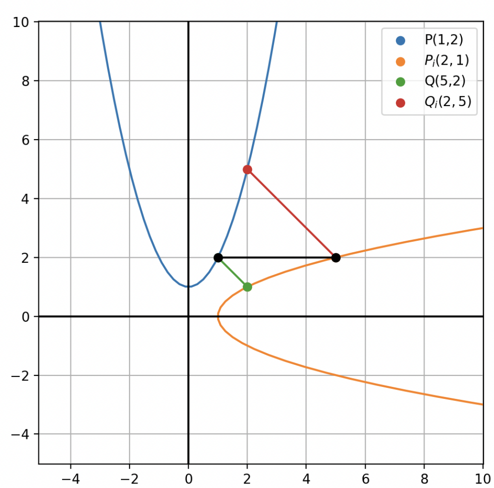

Figure 1: Plot

$$min(\mathbf{PP_1}, \mathbf{QQ_1}) = \mathbf{PP_i} \tag{8}$$

$$\|\mathbf{PP_i}\| = \|\mathbf{P} - \mathbf{P_i}\| = \left\| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{P} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{P} \right\| = \tag{9}$$

$$\sqrt{2} \left\| \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \mathbf{P} \right\| \tag{10}$$

Since the matrix is orthogonal, this equals:

$$\sqrt{2}\, \|\mathbf{P}\| = \tag{11}$$

$$\|\mathbf{PP_i}\| = \sqrt{2} \times \sqrt{1^2 + 2^2} = \sqrt{10} \tag{12}$$

$$\|\mathbf{PQ}\| = \sqrt{(5-1)^2 + (2-2)^2} = 4 \tag{13}$$

$$\mathbf{PQ} \geq \mathbf{PP_i} \tag{14}$$

We now need to find $\mathbf{P}_0$ and $\mathbf{Q}_0$ on the parabolas C_1 and C_2 respectively such that $\mathbf{P}_0\mathbf{Q}_0 \leq \mathbf{PQ}$ for all pairs of points $(\mathbf{P}, \mathbf{Q})$ with $\mathbf{P}$ on C_1 and $\mathbf{Q}$ on C_2.

The line of shortest distance will be normal to both parabolas, and it will be perpendicular to the line $y = x$. Therefore the tangents at the point of intersection of line of shortest distance and the parabola will have same slope as $y = x$, $m = 1$.

The equation of tangent to a conic in vector form can be expressed as:

$$\mathbf{m}^T(\mathbf{Vq} + \mathbf{u}) = 0 \tag{15}$$

Where $\mathbf{m}$ is slope of tangent, and $\mathbf{q} = \begin{pmatrix} x \\ y \end{pmatrix}$ is the point of contact. Substituting the values from (2), we get:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{q} + \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} \right) = 0 \tag{16}$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{q} + \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} = 0 \tag{17}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \frac{1}{2} = 0 \tag{18}$$

$$x - \frac{1}{2} = 0 \tag{19}$$

$$x = \frac{1}{2} \tag{20}$$

Substituting the value of x in (2), we get $y = \frac{5}{4}$. Therefore:

$$\mathbf{P_0} = \begin{pmatrix} \frac{1}{2} \\ \frac{5}{4} \end{pmatrix} \tag{21}$$

$$\mathbf{Q_0} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{P_0} = \begin{pmatrix} \frac{5}{4} \\ \frac{1}{2} \end{pmatrix} \tag{22}$$