8.4.16

EE25BTECH11013 - Bhargav

Question:
Let the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be reciprocal to that of the ellipse $x^2 + 4y^2 = 4$. If the hyperbola passes through a focus of the ellipse, then

1) the equation of the hyperbola is $\frac{x^2}{3} - \frac{y^2}{2} = 1$
2) a focus of the hyperbola is $(2, 0)$
3) the eccentricity of the hyperbola is $\sqrt{\frac{5}{3}}$
4) the equation of the hyperbola is $x^2 - 3y^2 = 3$

Solution:
The general equation of the conic can be written as:

$$\mathbf{x^T V x} + 2\mathbf{u^T x} + f = 0 \tag{4.1}$$

For the given ellipse, we get

$$\mathbf{V} = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{pmatrix}, f = -1, \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.2}$$

Since the major axis of the ellipse is X-axis, $\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
Using the formula:

$$\mathbf{V} = \|n\|^2 \mathbf{I} - e^2 \mathbf{nn^T} \tag{4.3}$$

For both ellipse and hyperbola, we get:

$$\mathbf{V} = \begin{pmatrix} 1 - e^2 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.4}$$

Comparing with $\mathbf{V}$ obtained for an ellipse, we get $e_E = \frac{\sqrt{3}}{2}$
Given that $e_H \cdot e_E = 1$
Thus, the eccentricity of the hyperbola is $e_H = \frac{2}{\sqrt{3}}$

Substituting $e_H = \frac{2}{\sqrt{3}}$ in the hyperbola equation, (f=1)

$$\mathbf{x^T} \begin{pmatrix} -\frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 1 = 0 \implies \frac{x^2}{3} - y^2 = 1 \tag{4.5}$$

To find the focal length of the hyperbola, we use:

$$c = \sqrt{\frac{|\lambda_1 - \lambda_2|}{\|\mathbf{V}\|}} \tag{4.6}$$

The eigenvalues of a diagonal matrix are the diagonal elements of matrix $\mathbf{V}$ for the hyperbola

$$\lambda_1 = -\frac{1}{3}, \lambda_2 = 1 \tag{4.7}$$

$$\implies c = 2 \tag{4.8}$$

Thus, the focus of the hyperbola is $\begin{pmatrix} \pm 2 \\ 0 \end{pmatrix}$

Option (2) and (4) are correct

The theoretical solution can be verified graphically