8.4.16

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Question

Let the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be reciprocal to that of the ellipse $x^2 + 4y^2 = 4$. If the hyperbola passes through a focus of the ellipse, then

1. the equation of the hyperbola is $\frac{x^2}{3} - \frac{y^2}{2} = 1$
2. a focus of the hyperbola is $(2, 0)$
3. the eccentricity of the hyperbola is $\sqrt{\frac{5}{3}}$
4. the equation of the hyperbola is $x^2 - 3y^2 = 3$

Solution

The general equation of the conic can be written as:

$$\mathbf{x}^{\mathbf{T}}\mathbf{V}\mathbf{x} + 2\mathbf{u}^{\mathbf{T}}\mathbf{x} + f = 0 \tag{1}$$

For the given ellipse, we get

$$\mathbf{V} = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{pmatrix}, f = -1, \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2}$$

Since the major axis of the ellipse is X-axis, $\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Using the formula:

$$\mathbf{V} = \left\|n\right\|^2 \mathbf{I} - e^2\mathbf{n}\mathbf{n}^{\mathbf{T}} \tag{3}$$

For both ellipse and hyperbola, we get:

$$\mathbf{V} = \begin{pmatrix} 1 - e^2 & 0 \\ 0 & 1 \end{pmatrix} \tag{4}$$

Comparing with $\mathbf{V}$ obtained for an ellipse, we get $e_E = \frac{\sqrt{3}}{2}$

Solution

Given that $e_H \cdot e_E = 1$
Thus, the eccentricity of the hyperbola is $e_H = \frac{2}{\sqrt{3}}$

Substituting $e_H = \frac{2}{\sqrt{3}}$ in the hyperbola equation, (f=1)

$$\mathbf{x}^{\mathbf{T}} \begin{pmatrix} -\frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 1 = 0 \implies \frac{x^2}{3} - y^2 = 1 \tag{5}$$

To find the focal length of the hyperbola, we use:

$$c = \sqrt{\frac{|\lambda_1 - \lambda_2|}{\|\mathbf{V}\|}} \tag{6}$$

The eigenvalues of a diagonal matrix are the diagonal elements of matrix **V** for the hyperbola

$$\lambda_1 = -\frac{1}{3}, \lambda_2 = 1 \tag{7}$$

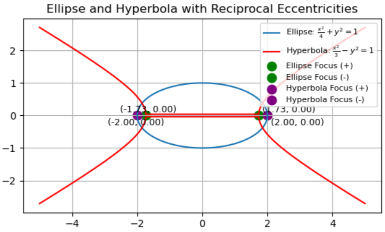
Solution

$$\implies c = 2 \tag{8}$$

Thus, the focus of the hyperbola is $\begin{pmatrix} \pm 2 \\ 0 \end{pmatrix}$

Option (2) and (4) are correct

Plot



Ellipse and Hyperbola with Reciprocal Eccentricities
Ellipse: $\frac{x^2}{4}+y^2=1$
Hyperbola: $\frac{x^2}{3}-y^2=1$
Ellipse Focus (+)
Ellipse Focus (-)
Hyperbola Focus (+)
Hyperbola Focus (-)
(-1.73, 0.00)
(-2.00, 0.00)
(1.73, 0.00)
(2.00, 0.00)

C Code

```c
#include <math.h>
double ellipse_eccentricity(double a, double b) {
    return sqrt(1.0 - (b*b) / (a*a));
}
double hyperbola_eccentricity(double a, double b) {
    return sqrt(1.0 + (b*b) / (a*a));
}
double ellipse_focus(double a, double b) {
    return sqrt(a*a - b*b);
}
double hyperbola_focus(double a, double b) {
    return sqrt(a*a + b*b);
}
```

Python + C Code

```
import ctypes
import numpy as np
import matplotlib.pyplot as plt

lib = ctypes.CDLL("./libconic.so")

lib.ellipse_eccentricity.restype = ctypes.c_double
lib.ellipse_eccentricity.argtypes = [ctypes.c_double, ctypes.
    c_double]

lib.hyperbola_eccentricity.restype = ctypes.c_double
lib.hyperbola_eccentricity.argtypes = [ctypes.c_double, ctypes.
    c_double]

lib.ellipse_focus.restype = ctypes.c_double
lib.ellipse_focus.argtypes = [ctypes.c_double, ctypes.c_double]

lib.hyperbola_focus.restype = ctypes.c_double
lib.hyperbola_focus.argtypes = [ctypes.c_double, ctypes.c_double]
```

```python
a_e, b_e = 2, 1
a_h, b_h = np.sqrt(3), 1

e_e = lib.ellipse_eccentricity(a_e, b_e)
e_h = lib.hyperbola_eccentricity(a_h, b_h)
c_e = lib.ellipse_focus(a_e, b_e)
c_h = lib.hyperbola_focus(a_h, b_h)

print(f"Eccentricities: e_E={e_e:.3f}, e_H={e_h:.3f}, Product={
    e_e*e_h:.3f}")
print(f"Ellipse focus distance: {c_e:.3f}, Hyperbola focus
    distance: {c_h:.3f}")

theta = np.linspace(0, 2*np.pi, 400)
x_ellipse = a_e * np.cos(theta)
y_ellipse = b_e * np.sin(theta)
focus1 = (c_e, 0)
focus2 = (-c_e, 0)
```

```python
x_h = np.linspace(-5, 5, 2000)
x_h = x_h[np.abs(x_h) >= a_h]
y_h = np.sqrt((x_h**2 / a_h**2 - 1) * b_h**2)
focus_h1 = (c_h, 0)
focus_h2 = (-c_h, 0)
x_f, y_f = focus1
lhs = x_f**2 / a_h**2 - y_f**2 / b_h**2
print("Hyperbola LHS at ellipse focus =", lhs)
plt.plot(x_ellipse, y_ellipse, label=r'Ellipse: $\frac{x^2}{4} +
    y^2 = 1$')
plt.plot(x_h, y_h, 'r', label=r'Hyperbola: $\frac{x^2}{3} - y^2 =
     1$')
plt.plot(x_h, -y_h, 'r')
plt.scatter(*focus1, color='green', s=80, label='Ellipse Focus
    (+)')
plt.scatter(*focus2, color='green', s=80, label='Ellipse Focus
    (-)')
plt.scatter(*focus_h1, color='purple', s=80, label='Hyperbola
    Focus (+)')
```

```
plt.scatter(*focus_h2, color='purple', s=80, label='Hyperbola
    Focus (-)')
plt.gca().set_aspect('equal')
plt.legend(fontsize=8, loc="upper right")
plt.title(fr"Ellipse & Hyperbola ($e_E={e_e:.3f}, e_H={e_h:.3f}$)
    ")
plt.grid(True)
plt.savefig("/mnt/c/Users/bharg/Documents/backupmatrix/
    ee25btech11013/matgeo/8.4.16/figs/Figure_1.png")
plt.show()
```

Python Code

```
import numpy as np
import matplotlib.pyplot as plt

# Ellipse: x^2/4 + y^2 = 1

a_e, b_e = 2, 1
h_e, k_e = 0, 0

theta = np.linspace(0, 2*np.pi, 400)
x_ellipse = h_e + a_e * np.cos(theta)
y_ellipse = k_e + b_e * np.sin(theta)

# Ellipse foci
c_e = np.sqrt(a_e**2 - b_e**2)
focus1 = (h_e + c_e, k_e)
focus2 = (h_e - c_e, k_e)

# Hyperbola: x^2/3 - y^2 = 1
a_h = np.sqrt(3)
```

Python Code

```
x_h = np.linspace(-5, 5, 2000)
x_h = x_h[np.abs(x_h) >= a_h] # valid domain
y_h = np.sqrt((x_h**2 / a_h**2 - 1) * b_h**2)

# Hyperbola foci (2, 0)
c_h = 2
focus_h1 = (h_h + c_h, k_h)
focus_h2 = (h_h - c_h, k_h)

# Check if hyperbola passes through ellipse focus
x_f, y_f = focus1
lhs = x_f**2 / a_h**2 - y_f**2 / b_h**2
print("Hyperbola LHS at ellipse focus =", lhs)

plt.plot(x_ellipse, y_ellipse, label=r'Ellipse: $\frac{x^2}{4} +
    y^2 = 1$')
plt.plot(x_h, y_h, 'r', label=r'Hyperbola: $\frac{x^2}{3} - y^2 =
     1$')
plt.plot(x_h, -y_h, 'r')
```

Python Code

```python
plt.scatter(*focus_h1, color='purple', s=80, label='Hyperbola
    Focus (+)')
plt.scatter(*focus_h2, color='purple', s=80, label='Hyperbola
    Focus (-)')

plt.text(focus1[0]+0.1, focus1[1]+0.1, f'({focus1[0]:.2f}, {
    focus1[1]:.2f})', fontsize=9)
plt.text(focus2[0]-0.8, focus2[1]+0.1, f'({focus2[0]:.2f}, {
    focus2[1]:.2f})', fontsize=9)
plt.text(focus_h1[0]+0.1, focus_h1[1]-0.3, f'({focus_h1[0]:.2f},
    {focus_h1[1]:.2f})', fontsize=9)
plt.text(focus_h2[0]-0.9, focus_h2[1]-0.3, f'({focus_h2[0]:.2f},
    {focus_h2[1]:.2f})', fontsize=9)
plt.gca().set_aspect('equal')
plt.legend(loc = "upper right", fontsize=8)
plt.grid(True)
plt.title("Ellipse and Hyperbola with Reciprocal Eccentricities")
plt.savefig("/mnt/c/Users/bharg/Documents/backupmatrix/
    ee25btech11013/matgeo/8.4.16/figs/Figure_1.png")
```