4.7.41

EE25BTECH11044 - Sai Hasini Pappula

Question

Find the distance of the point

$$\mathbf{P} = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}$$

from the line

$$\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9}. \tag{0.1}$$

Solution

$$\text{Line:} \quad \mathbf{r}(t) = \mathbf{a} + t\mathbf{m}, \qquad \mathbf{a} = \begin{bmatrix} -5 \\ -3 \\ 6 \end{bmatrix}, \quad \mathbf{m} = \begin{bmatrix} 1 \\ 4 \\ -9 \end{bmatrix}.$$

Derivation using the projection matrix

For any nonzero vector $\mathbf{m}$ the orthogonal projection matrix onto $\mathbf{m}$ is

$$P = \frac{\mathbf{m}\mathbf{m}^T}{\mathbf{m}^T\mathbf{m}}, \tag{0.2}$$

so the component of a vector $\mathbf{v}$ perpendicular to $\mathbf{m}$ is $(I - P)\mathbf{v}$. Hence the distance from a point $\mathbf{p}$ to the line through $\mathbf{a}$ with direction $\mathbf{m}$ is

$$d = \left\| \left(I - \frac{\mathbf{m}\mathbf{m}^T}{\mathbf{m}^T\mathbf{m}} \right)(\mathbf{p} - \mathbf{a}) \right\|. \tag{0.3}$$

Equivalent scalar form

Let $\mathbf{w} = \mathbf{p} - \mathbf{a}$. Using $P = \dfrac{\mathbf{m}\mathbf{m}^T}{\mathbf{m}^T\mathbf{m}}$ we get

$$d^2 = \mathbf{w}^T(I - P)\mathbf{w} = \mathbf{w}^T\mathbf{w} - \mathbf{w}^T\frac{\mathbf{m}\mathbf{m}^T}{\mathbf{m}^T\mathbf{m}}\mathbf{w} = \|\mathbf{w}\|^2 - \frac{(\mathbf{m}^T\mathbf{w})^2}{\mathbf{m}^T\mathbf{m}}.$$

This form is often quicker for computation.

Substitute the given vectors

$$\mathbf{p} = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}, \qquad \mathbf{p} - \mathbf{a} = \mathbf{w} = \begin{bmatrix} 7 \\ 7 \\ -7 \end{bmatrix}.$$

Compute the denominator:

$$\mathbf{m}^T\mathbf{m} = 1^2 + 4^2 + (-9)^2 = 1 + 16 + 81 = 98.$$

Compute the inner product $\mathbf{m}^T\mathbf{w}$:

$$\mathbf{m}^T\mathbf{w} = 1 \cdot 7 + 4 \cdot 7 + (-9) \cdot (-7) = 7 + 28 + 63 = 98.$$

Now use the scalar form:

$$\|\mathbf{w}\|^2 = 7^2 + 7^2 + (-7)^2 = 49 + 49 + 49 = 147,$$

so

$$d^2 = 147 - \frac{98^2}{98} = 147 - 98 = 49, \qquad d = \sqrt{49} = 7.$$

Final Answer

$$\boxed{d = 7}$$