12.27

EE25BTECH11013 - Bhargav

Question:
1200 men and 500 women can build a bridge in 2 weeks. 900 men and 250 women will take 3 weeks to build the same bridge. How many men will be needed to build the bridge in one week?
Solution:
Let one man complete work in X weeks and one woman complete work in Y weeks
In one week a man can complete $\frac{1}{X}$ work and woman can complete $\frac{1}{Y}$

$$\frac{1200}{X}+\frac{500}{Y}=\frac{1}{2} \implies XY-1000X-2400Y=0 \tag{0.1}$$

$$\frac{900}{X}+\frac{250}{Y}=\frac{1}{3} \implies XY-750X-2700Y=0 \tag{0.2}$$

Rotate the axis by 45° to remove the XY term in the equations

$$\mathbf{X}=\mathbf{Q}\mathbf{x} \tag{0.3}$$

(where $\mathbf{Q}$ is the rotation matrix)

$$\begin{pmatrix} X \\ Y \end{pmatrix}=\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \tag{0.4}$$

$$\implies XY=\frac{x^2-y^2}{2} \tag{0.5}$$

The conic equations become:

$$x^2-y^2-3400\sqrt{2}x+1400\sqrt{2}y=0 \tag{0.6}$$

$$x^2-y^2-3450\sqrt{2}x+1950\sqrt{2}y=0 \tag{0.7}$$

These can be represented as general conic equations:

$$\mathbf{x}^\top\mathbf{V}\mathbf{x}+2\mathbf{u}^\top\mathbf{x}+f=0 \tag{0.8}$$

For the conics: $\mathbf{V}=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\mathbf{u_1}=\begin{pmatrix} -1700\sqrt{2} \\ 700\sqrt{2} \end{pmatrix}$, $\mathbf{u_2}=\begin{pmatrix} -1725\sqrt{2} \\ 975\sqrt{2} \end{pmatrix}$, $f=0$

In homogeneous coordinates, using the form $\mathbf{x}^\top C\mathbf{x}=0$, where $\mathbf{x}=\begin{pmatrix} x & y & 1 \end{pmatrix}^T$, the matrices for the conics are:

$$\mathbf{C} = \begin{pmatrix} \mathbf{V} & \mathbf{u^T} \\ \mathbf{u} & f \end{pmatrix} \tag{0.9}$$

$$\implies \mathbf{C_1} = \begin{pmatrix} 1 & 0 & -1700\sqrt{2} \\ 0 & -1 & 700\sqrt{2} \\ -1700\sqrt{2} & 700\sqrt{2} & 0 \end{pmatrix} \tag{0.10}$$

$$\implies \mathbf{C_2} = \begin{pmatrix} 1 & 0 & -1725\sqrt{2} \\ 0 & -1 & 975\sqrt{2} \\ -1725\sqrt{2} & 975\sqrt{2} & 0 \end{pmatrix} \tag{0.11}$$

The intersection point of both the conics lies on the conic formed by their individual linear combination $\mathbf{C}(\mu) = \mathbf{C_1} + \mu\mathbf{C_2}$. We must find the value of μ that makes the determinant of the conic's matrix as 0.

$$\mathbf{C}(\mu) = \begin{pmatrix} 1+\mu & 0 & -1700\sqrt{2} - 1725\sqrt{2}\mu \\ 0 & -\mu - 1 & 700\sqrt{2} + 975\sqrt{2}\mu \\ -1700\sqrt{2} - 1725\sqrt{2}\mu & 700\sqrt{2} + 975\sqrt{2}\mu & 0 \end{pmatrix} \tag{0.12}$$

On solving $\det(\mathbf{C}(\mu)) = 0$, the simplest value of $\mu = -1$

$$\left(\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u_1}^\top \mathbf{x} + f\right) + (-1)\left(\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u_2}^\top \mathbf{x} + f\right) = 0 \tag{0.13}$$

The chord of intersection of the 2 hyperbolas is:

$$\left(\mathbf{u_1^T} - \mathbf{u_2^T}\right)\mathbf{x} = 0 \implies \begin{pmatrix} 1 & -11 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \tag{0.14}$$

The point of intersection of the common chord and the first hyperbola can be found out by solving them

$$\mathbf{x} = \mathbf{h} + k_i\mathbf{m}, \mathbf{h} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 11 \\ 1 \end{pmatrix} \tag{0.15}$$

$$(\mathbf{h} + k_i\mathbf{m})^\top \mathbf{V}(\mathbf{h} + k_i\mathbf{m}) + 2\mathbf{u}^\top(\mathbf{h} + k_i\mathbf{m}) + f = 0 \tag{0.16}$$

$$\implies k_i^2\mathbf{m}^\top\mathbf{V}\mathbf{m} + 2k_i\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u}) + \mathbf{h}^\top\mathbf{V}\mathbf{h} + 2\mathbf{u}^\top\mathbf{h} + f = 0 \tag{0.17}$$

$$\text{or, } k_i^2\mathbf{m}^\top\mathbf{V}\mathbf{m} + 2k_i\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u}) + g(\mathbf{h}) = 0 \tag{0.18}$$

Solving the above quadratic gives the equation

$$k_i = \frac{1}{\mathbf{m}^\top\mathbf{V}\mathbf{m}}\left(-\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u}) \pm \sqrt{[\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u})]^2 - g(\mathbf{h})(\mathbf{m}^\top\mathbf{V}\mathbf{m})}\right) \tag{0.19}$$

Solving we get:

$$k_1 = 0, k_2 = 300\sqrt{2} \tag{0.20}$$

The point of intersection:

$$\mathbf{x_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{x_2} = \begin{pmatrix} 3300\sqrt{2} \\ 300\sqrt{2} \end{pmatrix} \tag{0.21}$$

The point $\mathbf{x_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is not possible because it causes division by 0.

Substituting $\mathbf{x_2}$ in the rotation matrix equation:

$$\mathbf{X} = \begin{pmatrix} 3600 \\ 3000 \end{pmatrix} \tag{0.22}$$

A man can complete the work in 3600 weeks, a woman can complete the work in 3000 weeks