

12.267

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Question:

Two matrices **A** and **B** are said to be similar if

$$\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

for some invertible matrix **P**. Which of the following statements is NOT TRUE?

- 1) $\det \mathbf{A} = \det \mathbf{B}$
- 2) Trace of **A** = Trace of **B**
- 3) **A** and **B** have the same eigenvectors
- 4) **A** and **B** have the same eigenvalues

Solution:

Let **A** and **B** be similar matrices, such that

$$\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} \quad (1)$$

for an invertible matrix **P**.

For the determinant in 1),

$$|\mathbf{B}| = |\mathbf{P}^{-1}\mathbf{A}\mathbf{P}| \quad (2)$$

$$= |\mathbf{P}^{-1}| |\mathbf{A}| |\mathbf{P}| \quad (3)$$

$$= \frac{1}{|\mathbf{P}|} |\mathbf{A}| |\mathbf{P}| = |\mathbf{A}| \quad (4)$$

The statement is true.

For the trace in 2),

The cyclic property of trace of matrices states

$$\text{Tr}(\mathbf{XYZ}) = \text{Tr}(\mathbf{ZXY}) \quad (5)$$

Using (5)

$$\text{Tr}(\mathbf{B}) = \text{Tr}(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}) \quad (6)$$

$$= \text{Tr}(\mathbf{A}\mathbf{P}\mathbf{P}^{-1}) = \text{Tr}(\mathbf{A}) \quad (7)$$

The statement is true.

For the eigenvalues in 4), we examine the characteristic polynomial.

$$|\mathbf{B} - \lambda\mathbf{I}| = |\mathbf{P}^{-1}\mathbf{A}\mathbf{P} - \lambda\mathbf{I}| \quad (8)$$

$$= |\mathbf{P}^{-1}\mathbf{A}\mathbf{P} - \lambda\mathbf{P}^{-1}\mathbf{I}\mathbf{P}| \quad (9)$$

$$= |\mathbf{P}^{-1}(\mathbf{A} - \lambda\mathbf{I})\mathbf{P}| \quad (10)$$

$$= |\mathbf{P}^{-1}| |\mathbf{A} - \lambda\mathbf{I}| |\mathbf{P}| \quad (11)$$

$$= |\mathbf{A} - \lambda\mathbf{I}| \quad (12)$$

Since the characteristic polynomials are identical, the eigenvalues are the same. The statement is true.

For the eigenvectors in 3), let \mathbf{v} be an eigenvector of \mathbf{A} with eigenvalue λ , so that

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \quad (13)$$

$$\mathbf{B}(\mathbf{P}^{-1}\mathbf{v}) = (\mathbf{P}^{-1}\mathbf{A}\mathbf{P})(\mathbf{P}^{-1}\mathbf{v}) \quad (14)$$

$$= \mathbf{P}^{-1}\mathbf{A}(\mathbf{P}\mathbf{P}^{-1})\mathbf{v} \quad (15)$$

$$= \mathbf{P}^{-1}\mathbf{A}\mathbf{v} = \mathbf{P}^{-1}(\lambda\mathbf{v}) \quad (16)$$

$$= \lambda(\mathbf{P}^{-1}\mathbf{v}) \quad (17)$$

This shows that if \mathbf{v} is an eigenvector of \mathbf{A} , then $\mathbf{P}^{-1}\mathbf{v}$ is the eigenvector of \mathbf{B} for the same eigenvalue. Since $\mathbf{v} \neq \mathbf{P}^{-1}\mathbf{v}$ in general, the eigenvectors are not the same. The statement is not true.

The statement that is NOT TRUE is 3) **A and B have the same eigenvectors.**