2.10.55

EE25BTECH11026-Harsha

Question:
The edges of a parallelopiped are of unit length and are parallel to non-coplanar unit vectors $\hat{a}, \hat{b}, \hat{c}$ such that $\hat{a}.\hat{b} = \hat{b}.\hat{c} = \hat{c}.\hat{a} = \frac{1}{2}$. Then, the volume of the parallelopiped is

1) $\frac{1}{\sqrt{2}}$
2) $\frac{1}{2\sqrt{2}}$
3) $\frac{\sqrt{3}}{2}$
4) $\frac{1}{\sqrt{3}}$

Solution:
Let us solve the given equation theoretically and then verify the solution computationally.

According to the question,the edges of the parallelopiped are parallel to the unit vectors $\hat{a}, \hat{b}, \hat{c}$ and

$$\hat{a}.\hat{b} = \hat{b}.\hat{c} = \hat{c}.\hat{a} = \frac{1}{2}$$

As we know that the volume of parallelopiped is given by

$$V = \|[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\|$$

and

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}][\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^T = G$$

where G is the Gram Matrix.

$$\therefore G = \begin{pmatrix} \hat{a}.\hat{a} & \hat{a}.\hat{b} & \hat{a}.\hat{c} \\ \hat{b}.\hat{a} & \hat{b}.\hat{b} & \hat{b}.\hat{c} \\ \hat{c}.\hat{a} & \hat{c}.\hat{b} & \hat{c}.\hat{c} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}$$

For calculating the det(G), we can use the concept of eigen values.

Eigen values are those scalars which satisfies the following condition, For any non-zero eigen-vector $\mathbf{v}$ and coefficient matrix A,

$$M\mathbf{v} = \lambda\mathbf{v}\ , where\ \lambda\ is\ an\ eigen\ value.$$

$$G = (1-\rho)I\ + \rho\ \mathbf{1}\mathbf{1}^T,\ where\, \rho = \frac{1}{2}\ and\ \mathbf{1} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

Let $\mathbf{1}\mathbf{1}^T = J$. As we could see that the eigen-vector of J is $\mathbf{1}$ and by the rule,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = 3\mathbf{1}$$

So, 3 is a eigen value of J. Also we can observe that any vector orthogonal to J has eigen value 0 and since the eigen vector has only two degrees of freedom,

$$\therefore eigen\ values\ of\ J\ are\ \{3, 0, 0\}$$

Modifying the above equation on G,

$$\therefore G\mathbf{v} = \frac{1}{2}I\mathbf{v} + \frac{1}{2}J\mathbf{v}$$

$$\implies G\mathbf{v} = \frac{(1+\mu)}{2}\mathbf{v}$$

where μ is the eigen value of J. Here the eigen value of G is $\frac{1+\mu}{2}$ and substituting the obtained eigen values of J in this equation, we get the eigen values of G to be $\{2, \frac{1}{2}, \frac{1}{2}\}$

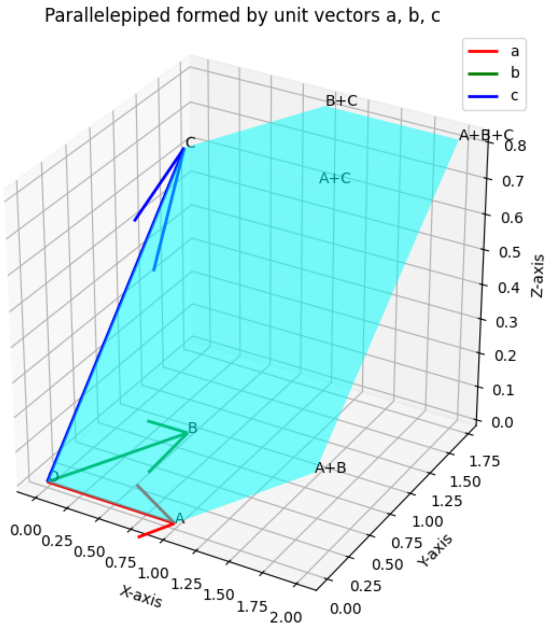
As we know that for eigen values of G being $\{\mu_1, \mu_2, \mu_3\}$

$$det(G) = \mu_1 \mu_2 \mu_3$$

$$\therefore det(G) = 2 \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}$$

$$\implies V = \sqrt{det(G)} = \frac{1}{\sqrt{2}}\ units$$

From the figure, taking an example of vectors **a** and **b** ,it is clearly verified that the theoretical solution matches with the computational solution.


Parallelepiped formed by unit vectors a, b, c
a
b
c
B+C
A+B+C
C
A+C
B
A+B
A
O
X-axis
Y-axis
Z-axis
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8