2.4.16

EE25BTECH11021 - Dhanush Sagar

Question
given points are

$$\mathbf{A} = \begin{pmatrix} 0 \\ 7 \\ -10 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 1 \\ 6 \\ -6 \end{pmatrix}, \qquad \mathbf{C} = \begin{pmatrix} 4 \\ 9 \\ -6 \end{pmatrix}.$$

a) prove the given points forms isosceles triangle
b) prove the given points forms right angled triangle
solution
We consider the vectors

$$\mathbf{A} = \begin{pmatrix} 0 \\ 7 \\ -10 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 1 \\ 6 \\ -6 \end{pmatrix}, \qquad \mathbf{C} = \begin{pmatrix} 4 \\ 9 \\ -6 \end{pmatrix}.$$

PROOF OF: **A**, **B**, **C** ARE NOT COLLINEAR (RANK METHOD)

Form the difference vectors $\mathbf{B} - \mathbf{A}$ and $\mathbf{C} - \mathbf{A}$.

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} 1 - 0 \\ 6 - 7 \\ -6 - (-10) \end{pmatrix} \tag{0.1}$$

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} \tag{0.2}$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} 4 - 0 \\ 9 - 7 \\ -6 - (-10) \end{pmatrix} \tag{0.3}$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix} \tag{0.4}$$

Place these as columns in the 3×2 matrix M.

$$M = \begin{pmatrix} \mathbf{B} - \mathbf{A} & \mathbf{C} - \mathbf{A} \end{pmatrix} \tag{0.5}$$

$$M = \begin{pmatrix} 1 & 4 \\ -1 & 2 \\ 4 & 4 \end{pmatrix} \tag{0.6}$$

Compute the 2×2 minor using rows 1 and 2.

$$\det \begin{pmatrix} 1 & 4 \\ -1 & 2 \end{pmatrix} = 1 \cdot 2 - 4 \cdot (-1) \tag{0.7}$$

$$\det \begin{pmatrix} 1 & 4 \\ -1 & 2 \end{pmatrix} = 2 + 4 = 6 \neq 0 \tag{0.8}$$

Hence $\operatorname{rank}(M) = 2$, so $\mathbf{B} - \mathbf{A}$ and $\mathbf{C} - \mathbf{A}$ are linearly independent. Therefore $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are not collinear and determine a triangle.

A) VERIFICATION FOR ISOSCELES TRIANGLES

$$\mathbf{AB} = \mathbf{B} - \mathbf{A} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}, \tag{0.9}$$

$$\mathbf{BC} = \mathbf{C} - \mathbf{B} = \begin{pmatrix} 4 - 1 \\ 9 - 6 \\ -6 - (-6) \end{pmatrix} \tag{0.10}$$

$$\mathbf{BC} = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}, \tag{0.11}$$

$$\mathbf{CA} = \mathbf{A} - \mathbf{C} = \begin{pmatrix} 0 - 4 \\ 7 - 9 \\ -10 - (-6) \end{pmatrix} \tag{0.12}$$

$$\mathbf{CA} = \begin{pmatrix} -4 \\ -2 \\ -4 \end{pmatrix}. \tag{0.13}$$

$$||\mathbf{AB}||^2 = (\mathbf{B} - \mathbf{A})^T (\mathbf{B} - \mathbf{A}) \tag{0.14}$$

$$||\mathbf{AB}||^2 = 1^2 + (-1)^2 + 4^2 \tag{0.15}$$

$$||\mathbf{AB}||^2 = 18 \tag{0.16}$$

$$||\mathbf{BC}||^2 = (\mathbf{C} - \mathbf{B})^T(\mathbf{C} - \mathbf{B}) \tag{0.17}$$

$$||\mathbf{BC}||^2 = 3^2 + 3^2 + 0^2 \tag{0.18}$$

$$||\mathbf{BC}||^2 = 18 \tag{0.19}$$

$$||\mathbf{CA}||^2 = (\mathbf{A} - \mathbf{C})^T(\mathbf{A} - \mathbf{C}) \tag{0.20}$$

$$||\mathbf{CA}||^2 = (-4)^2 + (-2)^2 + (-4)^2 \tag{0.21}$$

$$||\mathbf{CA}||^2 = 36 \tag{0.22}$$

$$||\mathbf{AB}|| = ||\mathbf{BC}|| = 3\sqrt{2}, \tag{0.23}$$

$$||\mathbf{CA}|| = 6 \tag{0.24}$$

Therefore the non-collinear vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ determine a triangle, and since two sides are equal, that triangle is **isosceles** (with equal sides **AB** and **BC**).

B) VEIFICATION FOR RIGHT ANGLED TRIANGLE (MATRIX / INNER-PRODUCT TEST)

To show the triangle is right-angled, compute the inner product of two adjacent side vectors **AB** and **BC**.

$$(\mathbf{AB})^T(\mathbf{BC}) = \begin{pmatrix} 1 & -1 & 4 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} \tag{0.25}$$

$$(\mathbf{AB})^T(\mathbf{BC}) = 1 \cdot 3 + (-1) \cdot 3 + 4 \cdot 0 \tag{0.26}$$

$$(\mathbf{AB})^T(\mathbf{BC}) = 3 - 3 + 0 = 0. \tag{0.27}$$

Since the inner product is zero, $\mathbf{AB} \perp \mathbf{BC}$ and therefore the angle $\angle ABC$ is a right angle; the triangle is **right-angled at** B.

Final statement: The non-collinear vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ determine a triangle which is both **isosceles** (with $\|\mathbf{AB}\| = \|\mathbf{BC}\|$) and **right-angled** (with $\mathbf{AB} \perp \mathbf{BC}$); hence the triangle is a *right isosceles* triangle with the right angle at vertex B.

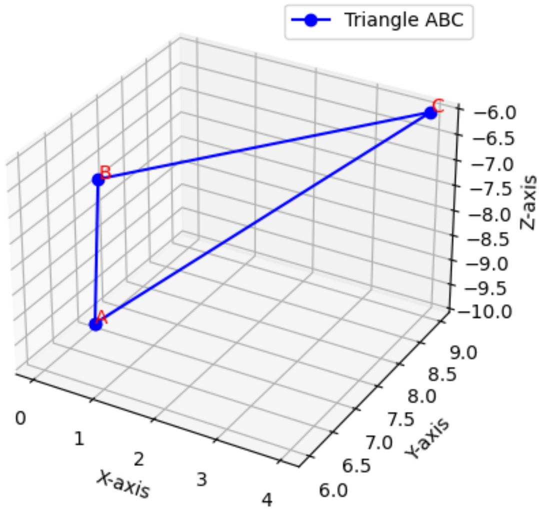

Fig. 0.1