8.2.56

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Question:

Given the ellipse with equation $9x^2 + 25y^2 = 225$, find the eccentricity and foci.
Solution:

Step 1: Represent the Ellipse in Matrix Form

$$\text{The given equation of the ellipse is } 9x^2 + 25y^2 = 225 \tag{0.1}$$

$$\text{The general form of conic is} \quad g(\mathbf{x}) = \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{0.2}$$

By rearranging the terms:

$$9x^2 + 25y^2 - 225 = 0 \tag{0.3}$$

By comparing the equation to the general form, we identify the matrices and vectors:

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} 9 & 0 \\ 0 & 25 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad f = -225 \tag{0.4}$$

Step 2: Find the Eccentricity

The eccentricity e is given by the formula $e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}}$, where λ_1 and λ_2 are the eigenvalues of the matrix $\mathbf{V}$. For our diagonal matrix $\mathbf{V}$, the eigenvalues are the diagonal entries: $\lambda_1 = 9 \quad \text{and} \quad \lambda_2 = 25$

$$\text{Using the formula:} e = \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{25-9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5} \tag{0.5}$$

The eccentricity of the ellipse is $\frac{4}{5}$.
Step 3: Find the Foci

The foci lie on the major axis, and their location depends on the center and the distance ae.
The center $\mathbf{c}$ of the conic is given by the formula $\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u}$. Since $\mathbf{u}$ is the zero vector, the center is at the origin: $\mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ Foci Location

The major axis of the ellipse corresponds to the eigenvector of the smaller eigenvalue of $\mathbf{V}$, which is $\lambda_1 = 9$. The eigenvector for $\lambda_1 = 9$ is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which lies along the x-axis.

The distance from the center to each focus is ae.

$$ae = \left(\sqrt{\frac{f_0}{|\lambda_1|}}\right)e \tag{0.6}$$

$$\text{where} f_0 = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 0 - (-225) = 225. \tag{0.7}$$

$$ae = \left(\sqrt{\frac{225}{9}}\right)\left(\frac{4}{5}\right) = \sqrt{25} \times \frac{4}{5} = 5 \times \frac{4}{5} = 4 \tag{0.8}$$

Since the center is at the origin and the major axis is on the x-axis, the foci are at $(\pm 4, 0)$.

The foci of the ellipse are at $(4, 0)$ and $(-4, 0)$.

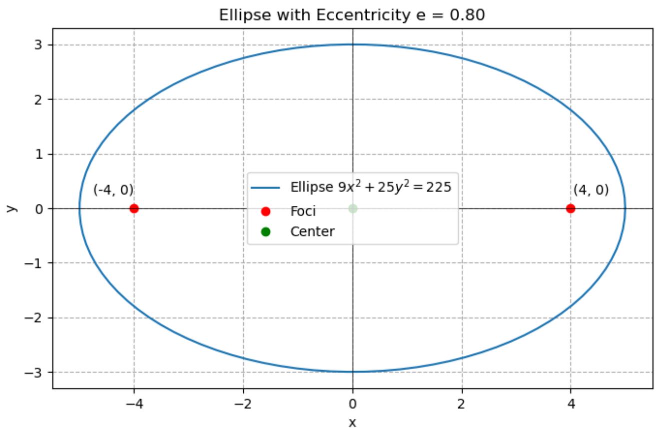

Fig. 0.1