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Question

Consider a circle with its centre lying on focus of the parabola $y^2 = 2px$ such that it touches the directrix of the parabola. Then a point of intersection of the circle and the parabola is

1. $(\frac{p}{2}, p)$ or $(\frac{p}{2}, -p)$
2. $(\frac{p}{2}, -\frac{p}{2})$
3. $(-\frac{p}{2}, p)$
4. $(-\frac{p}{2}, -\frac{p}{2})$

Solution

Conic Representation

Any conic can be expressed as:

$$\mathbf{x}^T V\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \quad \text{where } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{1}$$

Parabola: $x_2^2 = 2px_1$

Matrix form:

$$V_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{u}_1 = \begin{pmatrix} -p \\ 0 \end{pmatrix}, \quad f_1 = 0 \tag{2}$$

Circle: Center $(\frac{p}{2}, 0)$, Radius p

Expanded form:

$$(x_1 - \frac{p}{2})^2 + x_2^2 = p^2 \Rightarrow x_1^2 + x_2^2 - px_1 - \frac{3p^2}{4} = 0 \tag{3}$$

Matrix form:

$$V_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} -\frac{p}{2} \\ 0 \end{pmatrix}, \quad f_2 = -\frac{3p^2}{4} \tag{4}$$

Parametric Line of Intersection

Using the parametric form of the chord of intersection between the parabola and the circle:

$$\mathbf{x}(\mu) = \mathbf{h} + \mu\mathbf{m} \tag{5}$$

Here, $\mathbf{h} = \begin{pmatrix} \frac{p}{2} \\ 0 \end{pmatrix}$ is the center of the circle (also the focus of the parabola), and $\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is the direction vector of the vertical chord.

Let the line be:

$$\mathbf{x}(\mu) = \mathbf{h} + \mu\mathbf{m} \quad \text{where } \mathbf{h} = \begin{pmatrix} \frac{p}{2} \\ 0 \end{pmatrix}, \quad \mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{6}$$

This gives:

$$\mathbf{x}(\mu) = \begin{pmatrix} \frac{p}{2} \\ \mu \end{pmatrix} \tag{7}$$

Substitute into Parabola Equation

We evaluate:

$$\mathbf{x}(\mu)^T V_1 \mathbf{x}(\mu) + 2\mathbf{u}_1^T \mathbf{x}(\mu) + f_1 = 0 \tag{8}$$

Compute:

$$\mathbf{x}(\mu)^T V_1 \mathbf{x}(\mu) = \mu^2, \quad 2\mathbf{u}_1^T \mathbf{x}(\mu) = 2(-p)(\frac{p}{2}) = -p^2 \tag{9}$$

So:

$$\mu^2 - p^2 = 0 \Rightarrow \mu = \pm p \tag{10}$$

Final Intersection Points

Substitute back:

$$\mathbf{x}(\mu) = \begin{pmatrix} \frac{p}{2} \\ \pm p \end{pmatrix} \tag{11}$$

Intersection points:

$$\mathbf{a}_1 = \begin{pmatrix} \frac{p}{2} \\ p \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} \frac{p}{2} \\ -p \end{pmatrix} \tag{12}$$

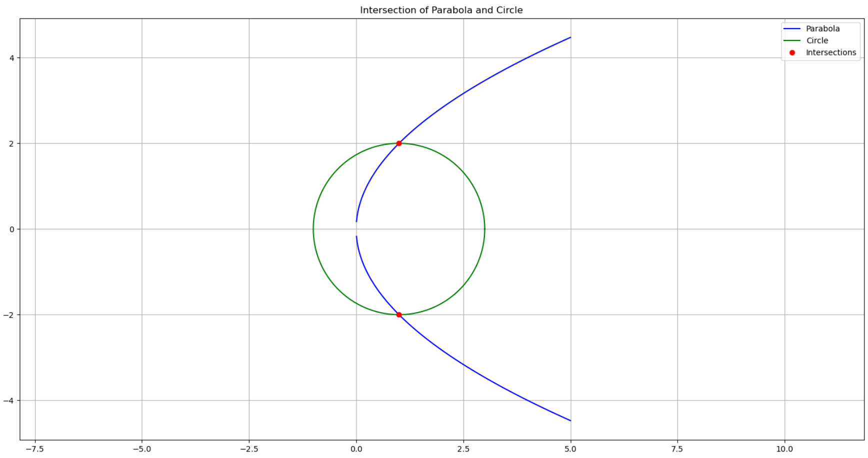

Figure 1: Caption