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MODULE 2: CORRELATION & REGRESSION

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Correlation: Study of relation/ variation between two or more variables

Regression: prediction of one category variable by using values of one or many variables when they are correlated

Types of correlation

Nature of graph: 1) Linear
2) Non-linear

Direction of change: 1) Positive
2) Negative

Mean $(\bar{x}) = \frac{\Sigma x}{N}$

Standard deviation $(\sigma) = \sqrt{\frac{\Sigma(x-\bar{x})^2}{N}}$ (RMS)

Varience $(\sigma_x^2) = \frac{\Sigma(x-\bar{x})^2}{N}$

Covarience $(cov(x,y)) = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{N}$

Karl Pearson's coefficient $(r) = \frac{cov(x,y)}{\sigma x \, \sigma y}$

$$r = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{N \sigma x \, \sigma y}$$

$$r = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\sqrt{\Sigma(x-\bar{x})^2 \; \Sigma(y-\bar{y})^2}}$$

$$r = \frac{\Sigma xy - N\bar{x}\bar{y}}{\sqrt{(\Sigma x^2 - N\bar{x}^2) \cdot (\Sigma y^2 - N\bar{y}^2)}}$$

Properties: 1) $-1 \le r \le 1$

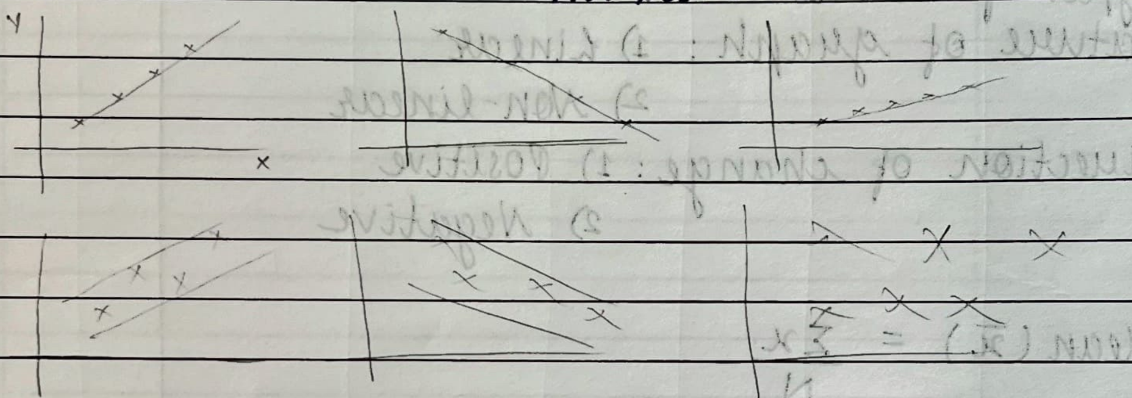
2) If x & y are independent variables they are

not correlated

3) correlation coefficient (r) is independent of change of origin and change of scale

Interpretation Table

$r > \pm 0.95$	Highly correlated
$\pm 0.75 < r < \pm 0.95$	correlated (with error)
$r < \pm 0.35$	Not related



From the following values of demand and price find the degree of correlation by using Karl Pearson

Demand : 65 66 67 67 68 69 70 72

Price : 67 68 65 68 72 72 69 71

Sr. No	values x	Y	$x-\bar{x}$	$y-\bar{y}$	$(x-\bar{x})^2$	$(y-\bar{y})^2$	$(x-\bar{x})(y-\bar{y})$
			$x-68$	$x-69$			
1	65	67	-3	-2	9	4	6
2	66	68	-2	-1	4	1	2
3	67	65	-1	-4	1	16	4
4	67	68	-1	-1	1	1	1
5	68	72	0	3	0	9	0
6	69	72	1	3	1	9	3
7	70	69	2	0	4	0	0
8	72	71	4	2	16	4	6
	544	552			36	44	22

$$\bar{x} = \frac{\Sigma x}{N} = \frac{544}{8} = 68 \Rightarrow \bar{x}$$

$$\bar{y} = \frac{\Sigma y}{N} = \frac{552}{8} = 69 \Rightarrow \bar{y}$$

$$r = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\sqrt{\Sigma(x-\bar{x})^2 \Sigma(y-\bar{y})^2}} = \frac{22}{\sqrt{36 \times 44}} = 0.552$$

X : 100 200 300 400 500

Y : 30 40 50 60 70

Sr. No.	Values X	Y	$\hat{x} = \frac{x-300}{100}$	$\hat{y} = \frac{y-50}{10}$	$\hat{x} - \bar{\hat{x}}$	$\hat{y} - \bar{\hat{y}}$	$(\hat{x}-\bar{\hat{x}})^2$	$(\hat{y}-\bar{\hat{y}})^2$	$(\hat{x}-\bar{\hat{x}})(\hat{y}-\bar{\hat{y}})$
1	100	30	-2	-2	-2	-2	4	4	4
2	200	40	-1	-1	-1	-1	1	1	1
3	300	50	0	0	0	0	0	0	0
4	400	60	1	1	1	1	1	1	1
5	500	70	2	2	2	2	4	4	4
							10	10	10

$$r = \frac{\Sigma(\hat{x} - \bar{\hat{x}})(\hat{y} - \bar{\hat{y}})}{\sqrt{\Sigma(\hat{x}-\bar{\hat{x}})^2 \Sigma(\hat{y}-\bar{\hat{y}})^2}}$$

$$= \frac{10}{\sqrt{10 \times 10}}$$

$$= 1$$

Formula: $r = \dfrac{\Sigma dx dy - N\bar{dx}\,\bar{dy}}{\sqrt{(\Sigma dx^2 - N(\bar{dx})^2)(\Sigma dy^2 - N(\bar{dy})^2)}}$

$dx = \dfrac{x_i - a}{h}$ h, k - common difference, if no common difference $h = k = 1$

$dy = \dfrac{y_i - b}{k}$ a = nearest int to $\bar{x}$

b = nearest int to $\bar{y}$

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1) Find Karl Pearson's coefficient (r) for the following data

X: 10 12 14 15 16 17 18 10 14 15

Y: 17 16 15 12 10 9 8 15 13 12

Sr No	Values X	Y	$dx = \frac{x_i - a}{h}$	$dy = \frac{y_i - b}{k}$	d^2x	d^2y	$dxdy$
1	10	17	-4	4	16	16	-16
2	12	16	-2	3	4	9	-6
3	14	15	0	2	0	4	0
4	15	12	1	-1	1	1	-1
5	16	10	2	-3	4	9	-6
6	17	9	3	-4	9	16	-12
7	18	8	4	-5	16	25	-20
8	10	15	-4	2	16	4	-8
9	14	13	0	0	0	0	0
10	15	12	1	-1	1	1	1
	141	127	1	-3	67	85	-70

$\bar{x} = \frac{\Sigma x}{N} = \frac{141}{10} = 14.1$ $a = 14$ {No common diff ∴ $h = k = 1$}

$\bar{y} = \frac{\Sigma y}{N} = \frac{127}{10} = 12.7$ ($b = 13$)

$\bar{dx} = \frac{\Sigma dx}{N} = \frac{1}{10} = 0.1$

$\bar{dy} = \frac{\Sigma dy}{N} = \frac{-3}{10} = -0.3$

$$r = \frac{\Sigma dxdy - N\bar{dx}\,\bar{dy}}{\sqrt{(\Sigma d^2x - N(\bar{dx})^2)(\Sigma d^2y - N(\bar{dy})^2)}}$$

$$= \frac{(-70) - (10)(0.1)(-0.3)}{\sqrt{(67 - 10(0.1)^2)(85 - 10(-0.3)^2)}}$$

$$= \frac{-69.7}{\sqrt{[67 - 0.1][85 - 0.9]}}$$

$$= -0.9292$$

This is negative strong correlation

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2)

Sr No	Values X	Y	$dx = \frac{x_i - a}{h}$	$dy = \frac{y_i - b}{k}$	d^2x	d^2y	$dxdy$
1	28	23	-8	-8	64	64	64
2	45	34	9	3	81	9	27
3	40	33	4	2	16	4	8
4	38	34	2	3	4	9	6
5	35	30	-1	-1	1	1	1
6	33	26	-3	-5	9	25	15
7	40	28	4	-3	16	9	-12
8	32	31	-4	0	16	0	0
9	36	36	0	5	0	25	0
10	33	35	-3	4	9	16	-12
	360	310	0	0	216	162	97

$$\bar{x} = \frac{\Sigma x}{N} = \frac{360}{10} = 36 = a$$

$$\bar{y} = \frac{\Sigma y}{N} = \frac{310}{10} = 31 = b$$

$$r = \frac{\Sigma dxdy}{\sqrt{\Sigma d^2x \Sigma d^2y}} = \frac{97}{\sqrt{216 \times 162}} = 0.5185$$

Spearman Rank correlation (R)

$$R = 1 - \left(\frac{6 \Sigma di^2}{N^3 - N}\right)$$ N = No. of data

Final rank of x => R1
Final rank of y => R2
Difference betn ranks => R1 − R2 = di $-1 \le R \le 1$

Find the rank correlation for the following data

X: 18 20 34 52 12
Y: 39 23 33 18 46

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1)

Sr. No.	X	R1	Y	R2	di=R1-R2	di^2
1	18	2	39	4	-2	4
2	20	3	23	2	1	1
3	34	4	35	3	1	1
4	52	5	18	1	4	16
5	12	1	46	5	-4	16
						38

$$R = 1 - \left(\frac{6(\Sigma di^2)}{N^3 - N}\right)$$

$$= 1 - \left(\frac{6(38)^2}{125-5}\right)$$

$$= 0.9$$

Equal rank - give average rank as per position & adjust $\frac{1}{12}(m_i^3 - m_i)$ term in formula

Find rank correlation

Sr. No.	X	R1	Y	R2	di=R1-R2	di^2
1	10	1	12	1	0	0
2	12	2	18	2	0	0
3	18	4.5	25	4	0.5	0.25
4	18	4.5	25	4	0.5	0.25
5	15	3	50	6	-3	9
6	40	6	25	4	2	4
						13.5

in category X: Rank 4&5 have same value 18

Rank given is $\frac{4+5}{2} = 4.5$

$m_1 = 2$

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in category X: Rank 3, 4 & 5 have same value 25

Rank given is $\frac{3+4+5}{3} = 4$

$N = 6$, $\Sigma di^2 = 13.5$, $m_2 = 3$

$$R = 1 - \frac{6\left[\Sigma di^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2)\right]}{N^3 - N}$$

$$= 1 - \frac{6\left[(13.5)^2 + \frac{1}{12}(2^3-2) + \frac{1}{12}(3^3-3)\right]}{6^3 - 6}$$

$$= 1 - \frac{96}{210} = 0.5428$$

3) In a musical competition 10 competitors were ranked by 3 judges X, Y, Z in the following data. By using correlation method discuss which pair has the nearest approach for evaluation of music.

Sr. No.	R_1 X	R_2 Y	R_3 Z	$d_{xy} = R_1 - R_2$	d_{xy}^2	$d_{yz} = R_2 - R_3$	d_{yz}^2	$d_{xz} = R_1 - R_3$	d_{xz}^2
1	1	3	6	-2	4	-3	9	-5	25
2	6	5	4	1	1	1	1	2	4
3	5	8	9	-3	9	-1	1	-4	16
4	10	4	8	6	36	-4	16	2	4
5	3	7	1	-4	16	6	36	2	4
6	2	10	2	-8	64	8	64	0	0
7	4	2	3	2	4	-1	1	1	1
8	9	1	10	8	64	-9	81	-1	1
9	7	6	5	1	1	1	1	2	4
10	8	9	7	-1	1	2	4	1	1

$\Sigma d_{xy}^2 = 200$ $\Sigma d_{yz}^2 = 214$ $\Sigma d_{xz}^2 = 60$

$N = 10$, $\Sigma d_{xy}^2 = 200$

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$$R_{xy} = 1 - \frac{6(\Sigma dxy)^2}{N^3-N} = 1 - \frac{6\times 200}{1000-10} = -0.2121$$

$N = 10$, $\Sigma dyz^2 = 214$

$$R_{yz} = 1 - \frac{6(\Sigma dyz)^2}{N^3-N} = 1 - \frac{6\times 214}{1000-10} = -0.2969$$

$N = 10$, $\Sigma dxz^2 = 60$

$$R_{xz} = 1 - \frac{6\ \Sigma dxz^2}{N^3-N} = 1 - \frac{6\times 60}{1000-10} = 0.6363$$

4) If Karl Pearson coefficient of the data is 0.4. Covariance between x and Y is 1.6. Variance of Y is given to be 25. Then find variance of x

Given : $r = 0.4$ Find: σx

$cov(x,y) = 1.6$

$\sigma y^2 = 25$

$\sigma y = 5$

Formula: $r = \dfrac{cov(x,y)}{\sigma x\,\sigma y}$

$$\sigma x = \frac{cov(x,y)}{r\,\sigma y}$$

$$= \frac{1.6}{(0.4)5}$$

$\sigma x = 0.8$

$\therefore$ Variance of $x = \sigma x^2 = 0.64$

5) If the rank correlation between x & y is given to be 0.143 and sum of squares of difference between ranks is 48 then find N.

Given : $R = 0.143$

$\Sigma di^2 = 48$

Find : N

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Formula: $R = 1 - \frac{6 \Sigma di^2}{N^3 - N}$

$\therefore 0.143 = 1 - \frac{6 \times 48}{N^3 - N}$

$\therefore \frac{6 \times 48}{N^3 - N} = 1 - 0.143$

$\therefore \frac{288}{N^3 - N} = 0.857$

$\therefore \frac{288}{0.857} = N^3 - N$

$\therefore 336.056 = N^3 - N$

$\therefore N^3 - N - 336.056 = 0$

$\therefore N = 7.000$

6) The coefficient of rank correlation of marks of 10 students in Physics and chemistry was found to be 0.5 After it was discovered that the difference of rank of 1 student was taken as 3 instead of 5 then find the correct coefficient of rank correlation

Given: wrong $R = 0.5$

$N = 10$

wrong 3 → correct 7

Formula: $R = 1 - \left(\frac{6 \Sigma di^2}{N^3 - N} \right)$

wrong $R = 1 - \frac{6 \left(\text{wrong } \Sigma di^2 \right)}{N^3 - N}$

$0.5 = 1 - \frac{6 \text{ wrong } \Sigma di^2}{1000 - 10}$

$\frac{6 \times \text{wrong } \Sigma di^2}{990} = 0.5$

wrong $\Sigma di^2 = 82.5$

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correct Σd_i^2 = wrong $\Sigma d_i^2 - 3^2 + 7^2$
$= 82.5 - 9 + 49$
$= 122.5$

correct R $= 1 - \frac{6 \times 122.5}{990}$

$= 0.2575$

7) A sample of 25 pairs has the following result
$\Sigma x = 127$, $\Sigma y = 100$ $\Sigma xy = 500$
$\Sigma x^2 = 760$, $\Sigma y^2 = 449$
later it was found that 2 pairs (8, 14) (8, 6) were taken instead of (8, 12) (6, 8)

given: wrong $\Sigma x = 127$
wrong $\Sigma y = 100$
wrong $\Sigma x^2 = 760$
wrong $\Sigma y^2 = 449$

correct Σx = wrong $\Sigma x - 8 + 8 - 8 + 6$
$= 127 - 2$
$\Sigma x = 125$

correct Σy = wrong $\Sigma y - 14 - 6 + 12 + 8$
$= 100$

correct Σx^2 = wrong $\Sigma x^2 - 8^2 + 8^2 - 8^2 + 6^2$
$= 760 - 28$
$= 732$

correct Σy^2 = wrong $\Sigma y^2 - 14^2 - 6^2 + 12^2 + 8^2$
$= 449 - 24$
$= 425$

correct Σxy = correct $\Sigma xy - (8 \times 14) - (8 \times 6) + (8 \times 12) + (6 \times 8)$
$= 500 - 112 - 48 + 96 + 48$
$= 484$

$$r = \frac{\Sigma xy - N\bar{x}\bar{y}}{\sqrt{[\Sigma x^2 - N(\bar{x})^2][\Sigma y^2 - N(\bar{y})^2]}}$$

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correct $\bar{x} = \frac{125}{25} = 5$

correct $\bar{y} = \frac{100}{25} = 4$

$$r = \frac{484 - (25 \times 5 \times 4)}{\sqrt{[732 - (25 \times 25)]\,[425 - (25 \times 16)]}}$$

$$= \frac{-16}{5\sqrt{107}}$$

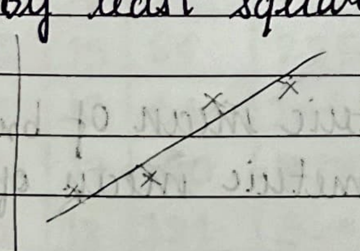
$$= -0.3093$$

Regression - method of estimating value of one variable when value of other is known and they are correlated

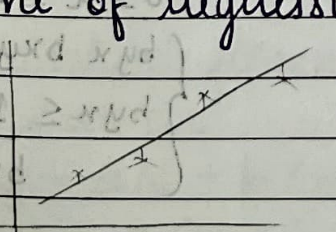
linear regression - in two variables

lines of regression:

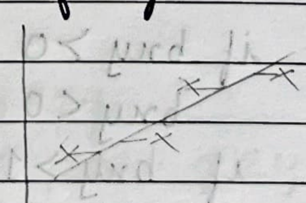
By least square method



line of regression y on x



line of regression x on y



For evaluation of data problems

eq^n of line of regression y on x is : $y = a + bx$	eq^n of line of regression x on y : $x = p + qy$
To find a + b: use normalised eq^n $\Sigma y = aN + b\Sigma x$	To find p + q use normalised eq^n $\Sigma x = pN + q\Sigma y$
$\Sigma xy = a\Sigma x + b\Sigma x^2$	$\Sigma xy = p\Sigma y + q\Sigma y^2$

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For coefficients of regression

Let regression coefficient of y on x is given by b_{yx}

Then line of regression is

$(y-\bar{y}) = b_{yx}(x-\bar{x})$

where $b_{yx} = r\dfrac{\sigma y}{\sigma x}$

$$b_{yx} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2}$$

& using $d_x = \dfrac{x-a}{h}$, $d_y = \dfrac{y-b}{k}$

$$b_{yx} = \frac{\Sigma d_x d_y - N(\bar{d_x})(\bar{d_y})}{\Sigma d^2_x - N(\bar{d_x})^2}$$

$$b_{yx} = \frac{\Sigma xy - N\bar{x}\bar{y}}{\Sigma x^2 - N(\bar{x})^2}$$

Let regression coefficient of x on y is given by b_{xy}

Then line of regression is

$(x-\bar{x}) = b_{xy}(y-\bar{y})$

where $b_{xy} = r\dfrac{\sigma x}{\sigma y}$

$$b_{xy} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(y-\bar{y})^2}$$

& using $d_x = \dfrac{x-a}{h}$ $d_y = \dfrac{y-b}{k}$

$$b_{xy} = \frac{\Sigma d_x d_y - N(\bar{d_x})(\bar{d_y})}{\Sigma d^2 y - N(\bar{d_y})^2}$$

$$b_{xy} = \frac{\Sigma xy - N\bar{x}\bar{y}}{\Sigma y^2 - N(\bar{y})^2}$$

relation

1> $b_{yx}\, b_{xy} = r^2$ (r is the geometric mean of b_{xy} & b_{yx})
coeff of correlation is the geometric mean of coeff of regression

2> $\dfrac{b_{yx} + b_{yx}}{2} \geq r$

$\{0 \leq r^2 \leq 1\}$

$\left\{ \begin{array}{l} b_{yx}\, b_{xy} \leq 1 \\ b_{yx} \leq \dfrac{1}{b_{xy}} \end{array} \right\}$

3> if $b_{xy} > 0$ then $b_{yx} > 0$
$b_{xy} < 0$ then $b_{yx} < 0$

4> if $b_{xy} > 1$ then $b_{yx} < 1$
$b_{xy} < 1$ then $b_{yx} > 1$

5> If $r = \pm 1$
$b_{xy} = \dfrac{1}{b_{yx}}$

6> Angle betn lines of regression

$(y-\bar{y}) = \underbrace{r\dfrac{\sigma y}{\sigma x}}_{\to m_1}(x-\bar{x}) \Rightarrow y = m_1 x$

$(x-\bar{x}) = r\dfrac{\sigma x}{\sigma y}(y-\bar{y})$

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$$(y-\bar{y}) = \frac{\sigma_y}{r\sigma_x}(x-\bar{x})$$

$$\tan\theta = \frac{m_1-m_2}{1+m_1m_2} = \frac{1-r^2}{r}\left(\frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}\right)$$

$r=0$ | $\tan\theta = \infty \Rightarrow \theta = \frac{\pi}{2}$

$r=\pm 1$ | $\tan\theta = 0 \Rightarrow \theta = 0$

1) Find the eq^n of lines of regression and their correlation coefficient for the following data and estimate y for x = 13 and estimate x for y = 20.

Sr. No.	X	Y	X^2	Y^2	XY
1	5	11	25	121	55
2	6	14	36	196	84
3	7	14	49	196	98
4	8	15	64	225	120
5	9	12	81	144	108
6	10	17	100	289	170
7	11	18	121	324	198
Σ	56	101	476	1495	833

$\bar{x} = \frac{\Sigma x}{N} = \frac{56}{7} = 8$

$\bar{y} = \frac{\Sigma y}{N} = \frac{101}{7} = 14.4285$

Normalised formula

$y = a + bx$

$\Sigma y = aN + b\Sigma x$

$\Sigma xy = a\Sigma x + b\Sigma x^2$

$\therefore 101 = a(7) + b(56)$

$\therefore 833 = a(56) + b(476)$

$a = 7.2857$

$b = 0.8928$

$y = 7.2857 + 0.8928x$

line of regression y on x

$x = p + qy$

$\Sigma x = pN + q\Sigma y$

$\Sigma xy = p\Sigma y + q\Sigma y^2$

$\therefore 56 = p(7) + q(101)$

$\therefore 833 = p(101) + q(1495)$

$p = -1.5643$

$q = 0.6628$

$x = -1.5643 + 0.6628y$

line of regression x on y

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$$b_{yx} = \frac{\Sigma xy - N\bar{x}\bar{y}}{\Sigma x^2 - N(\bar{x})^2}$$
$$= \frac{833 - 7(8)(14.4285)}{476 - 7(8)^2}$$
$$= 0.8928$$

$$b_{xy} = \frac{\Sigma xy - N\bar{x}\bar{y}}{\Sigma y^2 - N(\bar{y})^2}$$
$$= \frac{833 - 7(8)(14.4285)}{1495 - 7(14.4285)^2}$$
$$= 0.6628$$

$r = \sqrt{b_{xy}\, b_{yx}} = \sqrt{0.66 \times 0.89} = 0.769$

put $y = 20$

$x = -1.5643 + 0.6628(20) = 11.6917$

put $x = 13$

$y = 7.2857 + 0.8928(13) = 18.8921$

Regression

1 Find the angle between line of regression using data

$\sigma x = 4 \quad \sigma y = 5 \quad r_{xy} = 0.6$

Let θ be angle between lines of regression

$$\tan\theta = \left(\frac{1 - r^2}{r}\right)\left(\frac{\sigma x\, \sigma y}{\sigma x^2 + \sigma y^2}\right)$$
$$= \left(\frac{1 - (0.6)^2}{0.6}\right)\left(\frac{4 \times 5}{16 + 25}\right)$$
$$= \frac{0.64}{0.6} \times \frac{20}{41}$$
$$= 1.06 \times 0.48$$
$$= 0.52$$

Given the following weight x & height y of 1000 people

$\bar{x} = 150$ lbs $\quad \sigma x = 20$ lbs

$\bar{y} = 68$ inches $\quad \sigma y = 2.5$ inchs

$r = 0.6$

John's weight is 200 lbs. Find height

Smith 5 ft tall Find weight
30 inch

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To find height using weight:

$(y-\bar{y}) = \frac{r\sigma_y}{\sigma_x}$

$(y-68) = 0.075\,(200-150)$

$y-68 = 0.075 \times 50$

$y-68 = 3.75$

$y = 71.75$

To find weight using height of regression x on y

$(x-\bar{x}) = r\frac{\sigma_x}{\sigma_y}(y-\bar{y})$

$(x-150) = 0.6\ \frac{20}{2.5}\ (y-68)$

$(x-150) = 0.6\left(\frac{20}{2.5}\right)(71.75-68)$

$x-150 = 0.6 \times 8 \times 3.25$

$x = 150+18$

$x = 150-38.4$

$x = 111.6$

$\sigma_x = \sigma_y = \sigma$ & angle between lines of regression is $\tan^{-1} 3$ then find coefficient of correlation

lines of regression

Given: $6y = 5x+90$ $\quad 15x = 8y+130 \quad \sigma_x^2 = 16$

To find: 1) $\bar{x}$ & $\bar{y}$ $\quad$ 2) r $\quad$ 3) σ_y

solve equation of lines of regression simultaneous

point of intersection $\quad \bar{x} = 30,\ \bar{y} = 40$

$5x-6y+90 = 0$

$15x-8y-130 = 0$

Assume $6y = 5x+90$ in line regression x on y

$5x = 6y-90$

$x = \frac{6}{5}y - 90$

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$$b_{xy} = \frac{6}{5}$$

Assume $15x = 8y + 130$ is line of regression

$$8y = 15x - 130$$

$$y = \frac{15x - 130}{8}$$

$$b_{yx} = \frac{15}{8}$$

$$r = 1.5$$

$\sigma y = 5x + 90$ as line of regression y on x

$$y = \frac{5}{6} + \frac{90}{6}$$

$$b_{yx} = \frac{5}{6}$$

take $15x = 8y + 130$ as line of regression x on y

$$x = \frac{18}{5}y + \frac{130}{15}$$

$$b_{xy} = \frac{8}{15} = 0.66$$