Convex Functions

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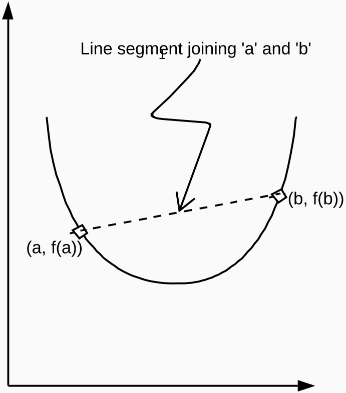
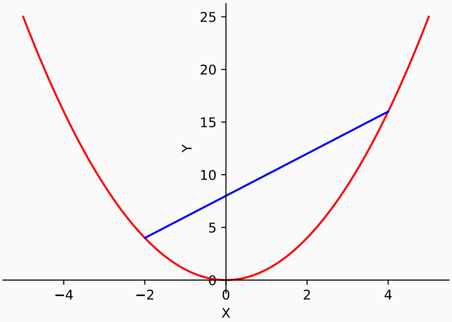
Definition

- Convexity is defined on an interval $[\alpha, \beta]$
- The line segment joining $(a, f(a)) and (b, f(b))$ should be *above or on* the function f for all points in interval $[\alpha, \beta]$.



Example: $y = x^2$

Convex on the entire real line i.e. $(-\infty, \infty)$



Example: $y = |x|$

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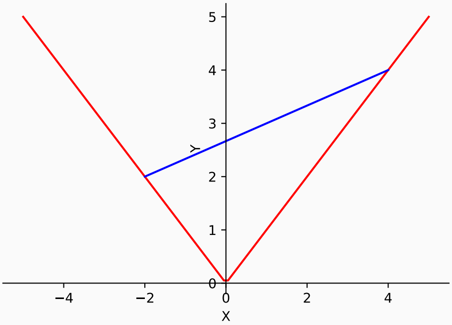
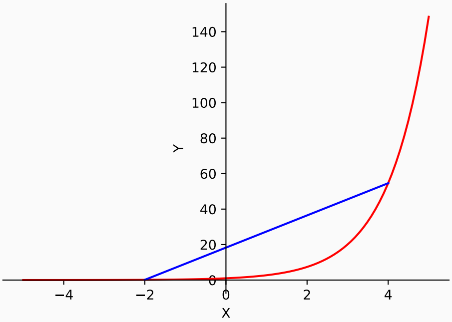


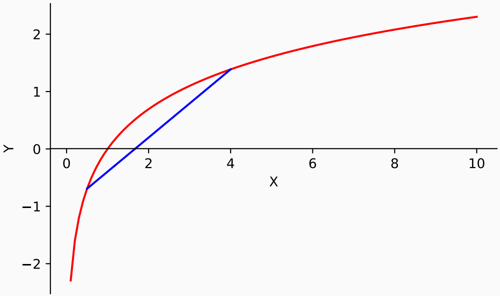
Example: $y = e^x$

Convex on the entire real line i.e. $(-\infty, \infty)$



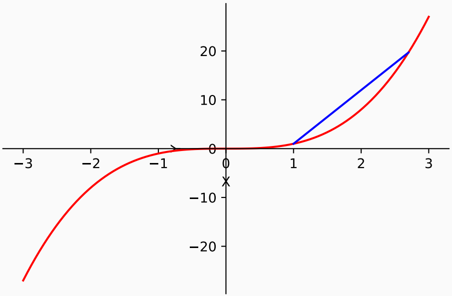
Example: $y = log_e x$

Not convex on the entire real line i.e. $(-\infty, \infty)$



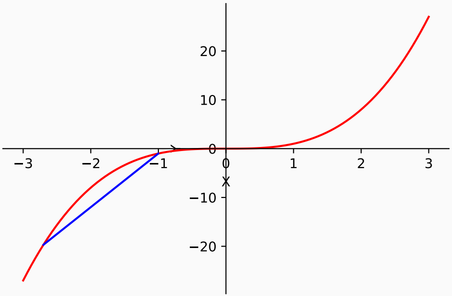
Example: $y = x^3$

It is convex for the interval $[0, \infty)$



Example: $y = x^3$

It is concave for the interval $(-\infty, 0]$



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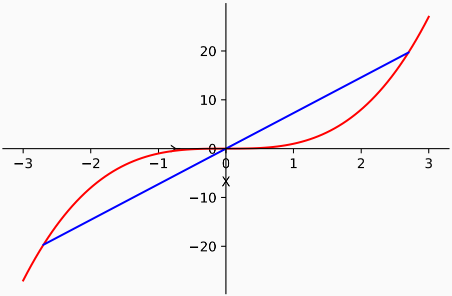
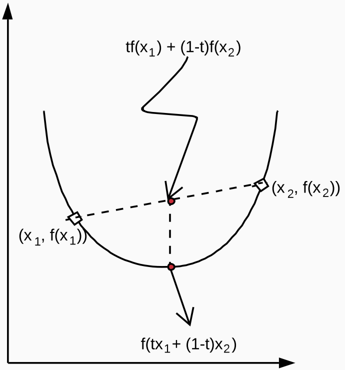
But, it is not convex for the interval $(-\infty, \infty)$



Mathematical Formulation

Function f is convex on set X, if $\forall x_1, x_2 \in X$ and $\forall t \in [0,1]$

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$



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Here,

$$\begin{aligned} \text{LHS - RHS} &= (t^2 - t)x_1^2 + [(1-t)^2 - (1-t)]x_2^2 + 2t(1-t)x_1x_2 \\ &= (t^2 - t)x_1^2 + (t^2 - t)x_2^2 - 2(t^2 - t)x_1x_2 \\ &= (t^2 - t)(x_1 - x_2)^2 \end{aligned}$$

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Here, $(t^2 - t) \leq 0$ since $t \in [0, 1]$ and $(x_1 - x_2)^2 \geq 0$

Hence, LHS -RHS ≤ 0

Hence LHS $\leq$ RHS

Hence proved.

Alternative ways to prove convexity

The Double-Derivative Test

If f''(x) > 0, the function is convex.

For example,

$$\frac{\partial^2(x^2)}{\partial x^2} = 2 > 0 \Rightarrow x^2 \text{ is a Convex function.}$$

Alternative ways to prove convexity

The double derivate test for multi-parameter function is equal to using the Hessian Matrix

A function $f(x_1, x_2, \ldots, x_n)$ is convex iff its $n \times n$ Hessian Matrix is positive semidefinite for all posible values of $(x_1, x_2, \ldots, x_n)$

$$\mathrm{H} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

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Eigen Values of H are 2 and $2 > 0 \Rightarrow$ H is positive semi-definite.
Hence, $f(x_1, x_2) = x_1^2 + x_2^2$ is convex.

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$X^T X$ is positive semi-definite for any $X \in \mathbb{R}^{m \times n}$.
Hence, linear least squares function is convex.

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Using this we can say that,

- $(y - x\theta)^T(y - x\theta) + \theta^T\theta$ is convex
- $(y - x\theta)^T(y - x\theta) + |\theta|$ is convex