Bayesian Machine Learning, MLE, MAP - I

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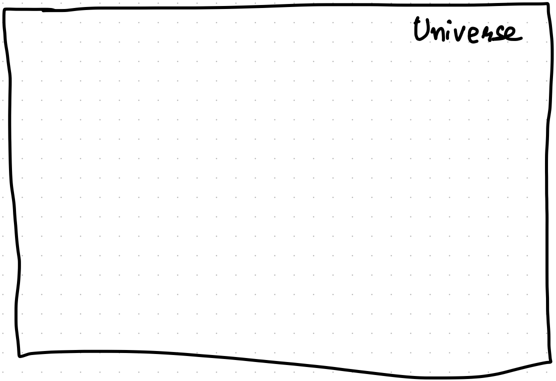
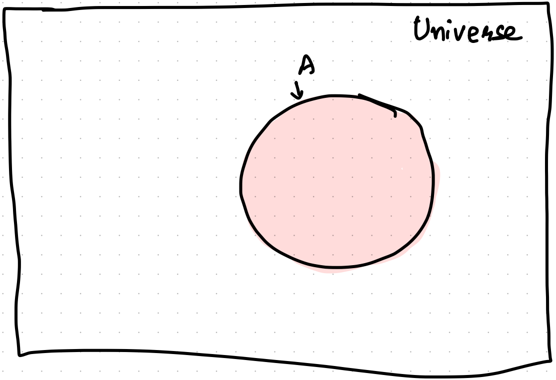
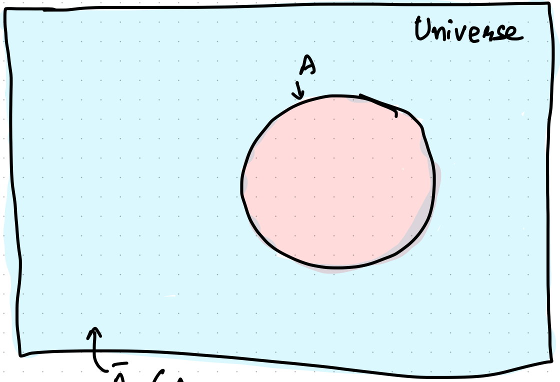
- Particularly useful when we do not have a large amount of data - use what *we* know about the model than depend on the data.
- Also allows us to predict with confidence quantified typically using variance.


Universe



Universe :
All people

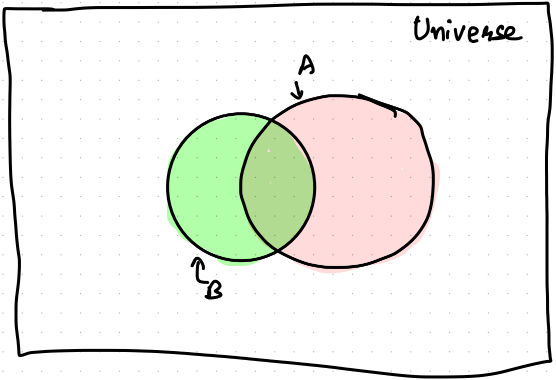
A :
People having
disease



Universe:
All people

A:
People having disease

$\bar{A}$ (A complement or Not A or $\neg A$)
People not having disease



Universe:
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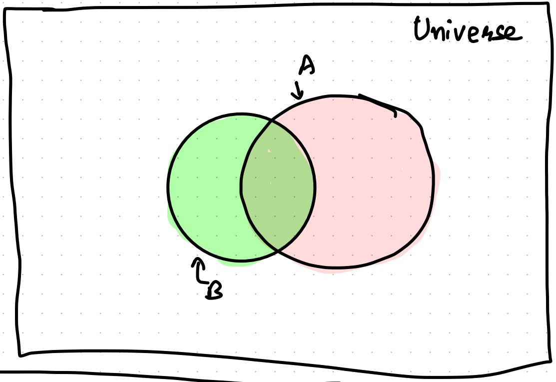
People having disease

B:
People tested + for disease

$\neg B$ or $\overline{B}$: People tested -ve for disease



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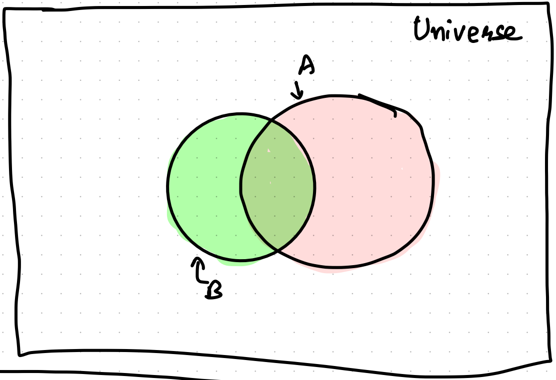

Universe:
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$$P(\text{Disease}) = \frac{|A|}{|\text{Universe}|} = P(A)$$

$$P(\text{Test}) = \frac{|B|}{|\text{Universe}|} = P(B)$$

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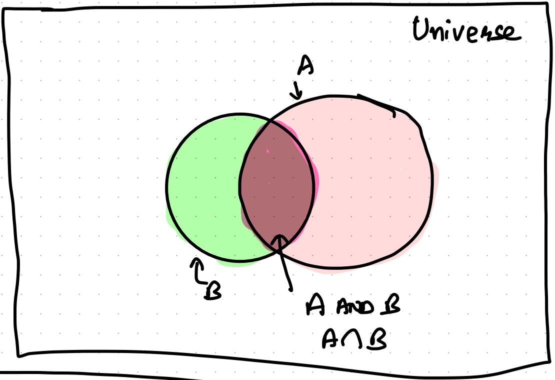

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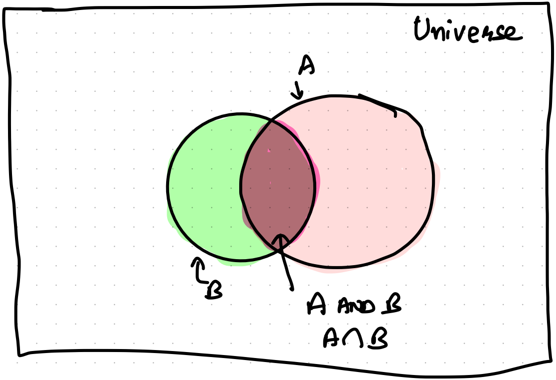
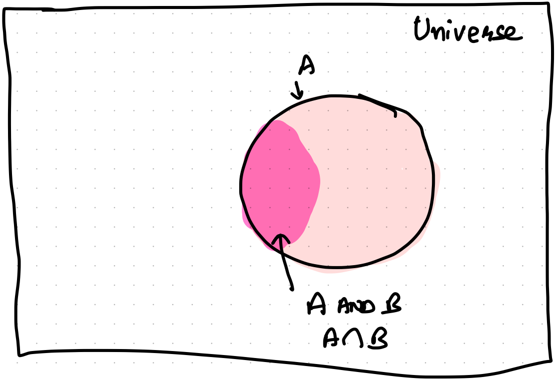
$$P(\text{Test}) = \frac{|B|}{|\text{Universe}|} = P(B)$$

$$P(\underset{(=T)}{\text{Disease}} \text{ AND } \underset{=(T)}{\text{TEST}}) = P(A \text{ AND } B) = \frac{|A \cap B|}{|\text{Universe}|}$$



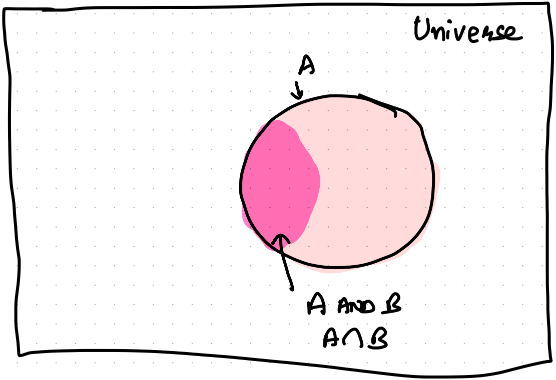
P(Test(=T) GIVEN DISEASE(=T)



P(Test(=T) GIVEN DISEASE(=T))

$$= P(B \mid A)$$

$$= \frac{|A \cap B|}{|A|} = \frac{\frac{|A \cap B|}{|\text{Universe}|}}{\frac{|A|}{|\text{Universe}|}} = \frac{P(AB)}{P(A)}$$



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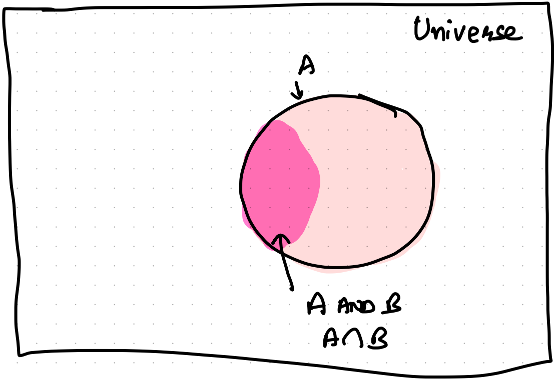

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$$\boxed{P(A \mid B)\,P(B) = P(B \mid A)\,P(A)}$$

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- $P(Test = +ve|Disease = True) = 0.99$
- $P(Test = -ve|Disease = False) = 0.99$
- Also, the disease is a rare one. Only one in 10,000 has it.
- Given the result of test is positive, what is the probability that someone has the disease?

Bayes Rule

- $P(T|D) = 0.99$
- $P(\bar{T}|\bar{D}) = 0.99$
- $P(T|\bar{D}) = 0.01$
- $P(\bar{T}|D) = 0.01$
- $P(D) = 10^{-4}$
- $P(\bar{D}) = 1 - 10^{-4}$

Given the above, calculate $P(D|T)$.

Problem

$$P(D|T) = \frac{P(T|D)P(D)}{P(T)} \quad (1)$$

Problem

$$
\begin{aligned}
P(D|T) &= \frac{P(T|D)P(D)}{P(T)} \\
&= \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|\bar{D})P(\bar{D})}
\end{aligned}
\tag{2}
$$

$$
= \frac{(.99)\left(10^{-4}\right)}{(.99)\left(10^{-4}\right) + (.01)\left(1 - 10^{-4}\right)}
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$$= \frac{(.99)\left(10^{-4}\right)}{(.99)\left(10^{-4}\right) + (.01)\left(1 - 10^{-4}\right)} = 0.09 << 0.99$$

Bayes Rule

- Notation: Let θ denote the parameters of the model and let $\mathcal{D}$ denote observed data. From Bayes Rule, we have

$$P(\theta|\mathcal{D}) = \frac{P(\mathcal{D}|\theta)P(\theta)}{P(\mathcal{D})}$$

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$$P(\theta|\mathcal{D}) = \frac{P(\mathcal{D}|\theta)P(\theta)}{P(\mathcal{D})}$$

- In the above equation $P(\theta|\mathcal{D})$ is called the posterior, $P(\mathcal{D}|\theta)$ is called the likelihood, $P(\theta)$ is called the prior and $P(\mathcal{D})$ is called the evidence.

Likelihood, Prior and Posterior

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- Posterior $\propto$ Likelihood $\times$ Prior

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- Similarly, for timestamp n, we will have $P(\theta|\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \dots \mathcal{D}_{n-1})$ acting as the prior knowledge before we observe $\mathcal{D}_n$.

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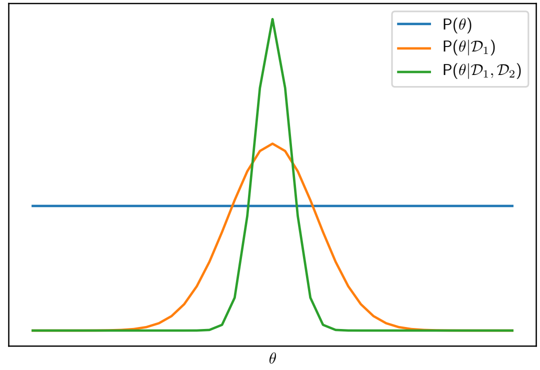

Figure 1: Online Learning: Variation of Prior as more data points arrive.

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- Idea find MLE estimate for θ

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- Log-likelihood $= \mathcal{LL}(\theta) = n_h \log(\theta) + n_t \log(1-\theta)$
- $\frac{\partial \mathcal{LL}(\theta)}{\partial \theta} = 0 \implies \frac{n_h}{\theta} + \frac{n_t}{1-\theta} = 0 \implies \theta_{MLE} = \frac{n_h}{n_h + n_t}$

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Any issues with maximum likelihood estimate or MLE?

Maximum A Posteriori estimate (MAP)

- **MLE does not handle prior knowledge**: What if we know that our coin is biased towards head?
- **MLE can overfit**: What is the probability of heads when we have observed 6 heads and 0 tails?

Maximum A Posteriori estimate (MAP)

Goal: Maximize the Posterior

$$\hat{\theta}_{MAP} = \arg\max_{\theta} P(\theta|\mathcal{D}) \tag{4}$$

$$\hat{\theta}_{MAP} = \arg\max_{\theta} P(\mathcal{D}|\theta)P(\theta) \tag{5}$$