Bayesian Machine Learning, MLE, MAP - I

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- Also allows us to predict with confidence quantified typically using variance.

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- $P(Test = +ve|Disease = True) = 0.99$
- $P(Test = -ve|Disease = False) = 0.99$
- Also, the disease is a rare one. Only one in 10,000 has it.
- Given the result of test is positive, what is the probability that someone has the disease?

Bayes Rule

- $P(T|D) = 0.99$
- $P(\bar{T}|\bar{D}) = 0.99$
- $P(T|\bar{D}) = 0.01$
- $P(\bar{T}|D) = 0.01$
- $P(D) = 10^{-4}$
- $P(\bar{D}) = 1 - 10^{-4}$

Given the above, calculate $P(D|T)$.

Problem

$$P(D|T) = \frac{P(T|D)P(D)}{P(T)} \tag{1}$$

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$$
\begin{aligned}
P(D|T) &= \frac{P(T|D)P(D)}{P(T)} \\
&= \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|\bar{D})P(\bar{D})}
\end{aligned} \tag{2}
$$

$$
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$$= \frac{(.99)\left(10^{-4}\right)}{(.99)\left(10^{-4}\right) + (.01)\left(1 - 10^{-4}\right)} = 0.09 << 0.99$$

Bayes Rule

- Notation: Let θ denote the parameters of the model and let $\mathcal{D}$ denote observed data. From Bayes Rule, we have

$$P(\theta|\mathcal{D}) = \frac{P(\mathcal{D}|\theta)P(\theta)}{P(\mathcal{D})}$$

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- In the above equation $P(\theta|\mathcal{D})$ is called the posterior, $P(\mathcal{D}|\theta)$ is called the likelihood, $P(\theta)$ is called the prior and $P(\mathcal{D})$ is called the evidence.

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- Posterior $\propto$ Likelihood $\times$ Prior

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- Similarly, for timestamp n, we will have $P(\theta|\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \ldots \mathcal{D}_{n-1})$ acting as the prior knowledge before we observe $\mathcal{D}_n$.

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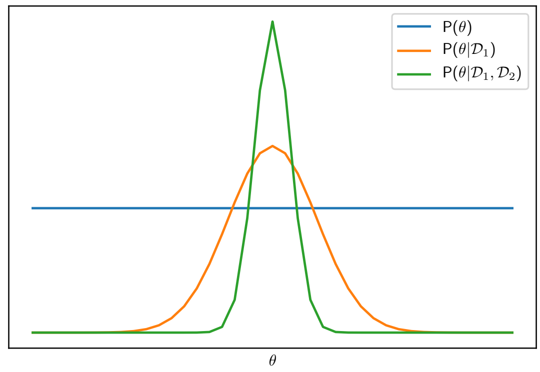

Figure 1: Online Learning: Variation of Prior as more data points arrive.

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- Idea find MLE estimate for θ

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- Log-likelihood $= \mathcal{LL}(\theta) = n_h \log(\theta) + n_t \log(1-\theta)$
- $\frac{\partial \mathcal{LL}(\theta)}{\partial \theta} = 0 \implies \frac{n_h}{\theta} + \frac{n_t}{1-\theta} = 0 \implies \theta_{MLE} = \frac{n_h}{n_h + n_t}$

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Any issues with maximum likelihood estimate or MLE?

Maximum A Posteriori estimate (MAP)

- **MLE does not handle prior knowledge**: What if we know that our coin is biased towards head?
- **MLE can overfit**: What is the probability of heads when we have observed 6 heads and 0 tails?

Maximum A Posteriori estimate (MAP)

Goal: Maximize the Posterior

$$\hat{\theta}_{MAP} = \arg\max_{\theta} P(\theta|\mathcal{D}) \tag{4}$$

$$\hat{\theta}_{MAP} = \arg\max_{\theta} P(\mathcal{D}|\theta)P(\theta) \tag{5}$$