Linear Regression II

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$$\begin{array}{r} 30 = \theta_0 + 6\theta_1 + 30\theta_2 \\ 40 = \theta_0 + 5\theta_1 + 20\theta_2 \\ \hline -10 = -1\theta_1 - 10\theta_2 \end{array} \tag{1}$$

The above equation can have infinitely many solutions.

Under-determined system: $\epsilon_i = 0$ for all i

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Then it is an over determined system. So, the sum of squared residuals > 0.

Class Exercise

Solve the linear system below using normal equation method

x_1	x_2	y
1	2	4
2	4	6
3	6	8

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The matrix X is not full rank.

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- Avoid dummy variable trap

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Can we use the direct encoding?
Then this implies that S>W>E>N

Dummy Variables

N-1 Variable encoding

	Is it N?	Is it E?	Is it W?
N	1	0	0
E	0	1	0
W	0	0	1
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Dummy Variables

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Is it S $=$ 1 - (Is it N $+$ Is it W $+$ Is it E)

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This introduces dependencies between them, and this can confusion in classifiers.

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F	...
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θ_1 is chosen based on 5-5.9, 5.2-5.9, 5.4-5.9 $\theta_1 =$ Avg. female height (5+5.2+5.4)/3 - Avg. male height(5.9)

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$$y_i = \theta_0 + \theta_1 x_i + \epsilon_i = \begin{cases} \theta_0 + \theta_1 + \epsilon_i & \text{if } i \text{ th person is female} \\ \theta_0 - \theta_1 + \epsilon_i & \text{if } i \text{ th person is male.} \end{cases}$$

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Now, θ_0 can be interpreted as average person height. θ_1 as the amount that female height is above average and male height is below average.

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Now, we compute the derivative of it with all the θ_j. Let us solve for x being a scalar.

Alternative parameter estimation

$$\begin{aligned}
\frac{\partial}{\partial \theta_0} \sum \epsilon_i^2 &= 2 \sum (y_i - \theta_0 - \theta_1 x_i)(-1) = 0 \\
0 &= \sum y_i - N\theta_0 - \sum \theta_1 x_i \\
\theta_0 &= \frac{\sum y_i - \theta_1 \sum x_i}{N}
\end{aligned} \tag{3}$$

$$\boxed{\theta_0 = \bar{y} - \theta_1 \bar{x}}$$

Alternative parameter estimation

$$\frac{\partial}{\partial \theta_1} \sum \epsilon_i^2 = 0$$

$$\implies 2\sum_{i=1}^{N}(y_i - \theta_0 - \theta_1 x_i)(-x_i) = 0$$

$$\implies \sum_{i=1}^{N}(x_i y_i - \theta_0 x_i - \theta_1 x_i^2) = 0$$

$$\implies \sum \theta_1 x_i^2 = \sum x_i y_i - \sum \theta_0 x_i$$

$$\implies \sum \theta_1 x_i^2 = \sum x_i y_i - \sum (\bar{y} - \theta_1 \bar{x}) x_i$$

Alternative parameter estimation

$$\implies \sum \theta_1 x_i^2 = \sum x_i y_i - \bar{y} \sum x_i + \theta_1 \bar{x} \sum x_i$$

$$\implies \sum x_i y_i - \sum x_i y = \theta_1 (-\bar{x} \sum x_i + \sum x_i^2)$$

$$\theta_1 = \frac{x_i y_i - \sum x_i y}{\sum x_i^2 - \bar{x} \sum x_i}$$

Alternative parameter estimation

$$\theta_1 = \frac{\frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y})}{\frac{1}{N}(x_i - \bar{x})^2}$$

$$\boxed{\theta_1 = \frac{Cov(x, y)}{variance(x)}}$$