Graduate Level Z-Test and T-Test Questions with Detailed Solutions

Question 1: One-Sample Z-Test (Population Standard Deviation Known)

Question: A pharmaceutical company claims their new pain medication reduces pain scores by an average of 15 points on a 100-point scale. Historical data shows pain score reductions follow a normal distribution with $\sigma = 4.2$. You test 64 patients and find a sample mean reduction of 13.8 points. At $\alpha = 0.01$, test the company's claim.

Solution: We use a one-sample z-test because the population standard deviation is known ($\sigma = 4.2$) and we have a large sample size ($n = 64$). The sampling distribution of the mean will be approximately normal due to the Central Limit Theorem.

Table to use: Standard Normal (Z) Table

Step-by-step solution:

- H_0: $\mu = 15$ (company's claim is true)
- H_1: $\mu \neq 15$ (two-tailed test)
- $\alpha = 0.01$, so critical values are ± 2.576
- Test statistic: $z = (\bar{x} - \mu_0)/(\sigma/\sqrt{n}) = (13.8 - 15)/(4.2/\sqrt{64}) = -1.2/0.525 = -2.286$
- $|z| = 2.286 < 2.576$, so we fail to reject H_0
- **Conclusion:** There is insufficient evidence to reject the company's claim at $\alpha = 0.01$.

Question 2: One-Sample T-Test (Population Standard Deviation Unknown, Small Sample)

Question: A psychology researcher wants to test if a new therapy reduces anxiety scores. The population mean anxiety score is known to be 45. She tests 12 patients and finds: $\bar{x} = 41.3$, $s = 6.8$. Test at $\alpha = 0.05$ if the therapy is effective (reduces anxiety).

Solution: We use a one-sample t-test because the population standard deviation is unknown and we must estimate it from the sample, and our sample size is small ($n = 12 < 30$). The t-distribution accounts for the additional uncertainty from estimating σ.

Table to use: T-Distribution Table with $df = n - 1 = 11$

Step-by-step solution:

- H_0: $\mu = 45$
- H_1: $\mu < 45$ (one-tailed test, therapy should reduce scores)

- $\alpha = 0.05$, df = 11, critical value = -1.796 (one-tailed)
- Test statistic: $t = (\bar{x} - \mu_0)/(s/\sqrt{n}) = (41.3 - 45)/(6.8/\sqrt{12}) = -3.7/1.963 = -1.885$
- $t = -1.885 < -1.796$, so we reject H_0
- **Conclusion:** The therapy significantly reduces anxiety scores at $\alpha = 0.05$.

Question 3: Two-Sample Z-Test (Independent Samples, Known Population Variances)

Question: Two manufacturing plants produce light bulbs. Plant A has historically produced bulbs with a lifetime standard deviation of 120 hours, while Plant B has $\sigma = 95$ hours. Random samples show: Plant A ($n_1 = 50$, $\bar{x}_1 = 1850$ hours), Plant B ($n_2 = 45$, $\bar{x}_2 = 1920$ hours). Test at $\alpha = 0.02$ if there's a significant difference in mean lifetimes.

Solution: We use a two-sample z-test because both population standard deviations are known and we have large sample sizes. The sampling distribution of the difference in means will be approximately normal.

Table to use: Standard Normal (Z) Table

Step-by-step solution:

- $H_0: \mu_1 - \mu_2 = 0$
- $H_1: \mu_1 - \mu_2 \neq 0$ (two-tailed test)
- $\alpha = 0.02$, critical values = ± 2.326
- Standard error: $SE = \sqrt{[(\sigma_1^2/n_1) + (\sigma_2^2/n_2)]} = \sqrt{[(120^2/50) + (95^2/45)]} = \sqrt{[288 + 200.56]} = 22.09$
- Test statistic: $z = (\bar{x}_1 - \bar{x}_2)/SE = (1850 - 1920)/22.09 = -70/22.09 = -3.168$
- $|z| = 3.168 > 2.326$, so we reject H_0
- **Conclusion:** There is a significant difference in mean lifetimes between the two plants.

Question 4: Two-Sample T-Test (Independent Samples, Equal Variances Assumed)

Question: A nutrition study compares protein levels in two diets. Diet A: $n_1 = 15$, $\bar{x}_1 = 28.4$g, $s_1 = 4.2$g. Diet B: $n_2 = 18$, $\bar{x}_2 = 31.7$g, $s_2 = 4.8$g. Assume equal population variances. Test at $\alpha = 0.05$ if Diet B has significantly higher protein.

Solution: We use a two-sample t-test with pooled variance because population standard deviations are unknown and we're assuming equal variances with small sample sizes. The t-distribution accounts for uncertainty in estimating the common variance.

Table to use: T-Distribution Table with $df = n_1 + n_2 - 2 = 31$

Step-by-step solution:

- H_0: $\mu_2 - \mu_1 = 0$
- H_1: $\mu_2 - \mu_1 > 0$ (one-tailed test)
- $\alpha = 0.05$, df = 31, critical value = 1.696
- Pooled variance: $s^2_p = [(n_1-1)s_1^2 + (n_2-1)s_2^2]/(n_1+n_2-2) = [(14)(17.64) + (17)(23.04)]/31 = 20.52$
- Standard error: $SE = s_p\sqrt{(1/n_1 + 1/n_2)} = \sqrt{20.52} \times \sqrt{(1/15 + 1/18)} = 4.53 \times 0.365 = 1.653$
- Test statistic: $t = (\bar{x}_2 - \bar{x}_1)/SE = (31.7 - 28.4)/1.653 = 3.3/1.653 = 1.996$
- $t = 1.996 > 1.696$, so we reject H_0
- **Conclusion:** Diet B has significantly higher protein content.

Question 5: Paired T-Test (Dependent Samples)

Question: A fitness program claims to reduce body fat percentage. 10 participants were measured before and after: Differences (After - Before): -2.1, -1.8, -0.3, -2.9, -1.2, -2.5, -0.8, -3.1, -1.9, -2.2 Test at $\alpha = 0.01$ if the program significantly reduces body fat.

Solution: We use a paired t-test because we have dependent samples (same individuals measured twice), and we're analyzing the differences within pairs. The population standard deviation of differences is unknown, requiring the t-distribution.

Table to use: T-Distribution Table with df = n - 1 = 9

Step-by-step solution:

- H_0: $\mu d = 0$ (no change in body fat)
- H_1: $\mu d < 0$ (reduction in body fat)
- $\alpha = 0.01$, df = 9, critical value = -2.821
- Calculate: $\bar{d} = -1.88$, $sd = 0.893$
- Test statistic: $t = \bar{d}/(sd/\sqrt{n}) = -1.88/(0.893/\sqrt{10}) = -1.88/0.282 = -6.667$
- $t = -6.667 < -2.821$, so we reject H_0
- **Conclusion:** The fitness program significantly reduces body fat percentage.

Question 6: Two-Sample T-Test (Unequal Variances - Welch's T-Test)

Question: Compare reaction times between two age groups. Young adults: $n_1 = 25$, $\bar{x}_1 = 245$ms, $s_1 = 18$ms. Older adults: $n_2 = 20$, $\bar{x}_2 = 278$ms, $s_2 = 35$ms. The variances appear unequal (F-test p-value = 0.003). Test at $\alpha = 0.05$ if older adults have significantly slower reaction times.

Solution: We use Welch's t-test (unequal variances t-test) because the F-test indicates significantly different variances, and population standard deviations are unknown. This test doesn't pool variances and uses adjusted degrees of freedom.

Table to use: T-Distribution Table with adjusted df (Welch-Satterthwaite equation)

Step-by-step solution:

- H_0: $\mu_2 - \mu_1 = 0$
- H_1: $\mu_2 - \mu_1 > 0$ (one-tailed)
- Adjusted df = $[(s_1^2/n_1 + s_2^2/n_2)^2]/[(s_1^2/n_1)^2/(n_1-1) + (s_2^2/n_2)^2/(n_2-1)]$
- df = $[(18^2/25 + 35^2/20)^2]/[(18^2/25)^2/24 + (35^2/20)^2/19] = [74.29^2]/[0.44 + 31.36] = 169.7$
- Use df = 30 (conservative), critical value = 1.697
- SE = $\sqrt{(s_1^2/n_1 + s_2^2/n_2)} = \sqrt{(324/25 + 1225/20)} = \sqrt{74.29} = 8.62$
- Test statistic: $t = (278 - 245)/8.62 = 33/8.62 = 3.827$
- $t = 3.827 > 1.697$, so we reject H_0
- **Conclusion:** Older adults have significantly slower reaction times.

Question 7: One-Sample Z-Test (Proportion)

Question: A political poll claims 52% of voters support a candidate. In a random sample of 400 voters, 190 support the candidate. Test at $\alpha = 0.05$ if the actual support differs from the claimed 52%.

Solution: We use a one-sample z-test for proportions because we have a large sample size (n = 400) and we're testing a population proportion. The sampling distribution of the sample proportion is approximately normal when $np \geq 5$ and $n(1-p) \geq 5$.

Table to use: Standard Normal (Z) Table

Step-by-step solution:

- H_0: $p = 0.52$
- H_1: $p \neq 0.52$ (two-tailed)
- Check conditions: $np_0 = 400(0.52) = 208 \geq 5$ ✓, $n(1-p_0) = 400(0.48) = 192 \geq 5$ ✓
- Sample proportion: $\hat{p} = 190/400 = 0.475$
- Standard error: SE = $\sqrt{[p_0(1-p_0)/n]} = \sqrt{[0.52(0.48)/400]} = 0.025$
- Test statistic: $z = (\hat{p} - p_0)/SE = (0.475 - 0.52)/0.025 = -1.8$
- $\alpha = 0.05$, critical values = ± 1.96
- $|z| = 1.8 < 1.96$, so we fail to reject H_0
- **Conclusion:** There is insufficient evidence that actual support differs from 52%.

Question 8: Two-Sample Z-Test (Proportions)

Question: Compare graduation rates between two schools. School A: 285 graduates out of 350 students. School B: 310 graduates out of 400 students. Test at $\alpha = 0.01$ if there's a significant difference in graduation rates.

Solution: We use a two-sample z-test for proportions because we have large sample sizes and we're comparing two population proportions. The sampling distribution of the difference in sample proportions is approximately normal under the null hypothesis.

Table to use: Standard Normal (Z) Table

Step-by-Step solution:

- H_0: $p_1 - p_2 = 0$
- H_1: $p_1 - p_2 \neq 0$ (two-tailed)
- Sample proportions: $\hat{p}_1 = 285/350 = 0.814$, $\hat{p}_2 = 310/400 = 0.775$
- Pooled proportion: $\hat{p} = (285 + 310)/(350 + 400) = 595/750 = 0.793$
- Standard error: $SE = \sqrt{[\hat{p}(1-\hat{p})(1/n_1 + 1/n_2)]} = \sqrt{[0.793(0.207)(1/350 + 1/400)]} = 0.0297$
- Test statistic: $z = (\hat{p}_1 - \hat{p}_2)/SE = (0.814 - 0.775)/0.0297 = 1.313$
- $\alpha = 0.01$, critical values = ± 2.576
- $|z| = 1.313 < 2.576$, so we fail to reject H_0
- **Conclusion:** No significant difference in graduation rates between schools.

Question 9: One-Sample T-Test (Borderline Case - n = 30)

Question: A coffee shop claims their average service time is 3.5 minutes. You observe 30 customers with $\bar{x} = 3.8$ minutes and $s = 1.2$ minutes. The distribution appears slightly skewed. Test at $\alpha = 0.10$ if service time is significantly longer than claimed.

Solution: We use a one-sample t-test because the population standard deviation is unknown, even though $n = 30$ is the borderline between "small" and "large" samples. The t-test is more conservative and appropriate when σ is unknown, regardless of sample size.

Table to use: T-Distribution Table with df = 29

Step-by-step solution:

- H_0: $\mu = 3.5$
- H_1: $\mu > 3.5$ (one-tailed)
- $\alpha = 0.10$, df = 29, critical value = 1.311

- Test statistic: $t = (\bar{x} - \mu_0)/(s/\sqrt{n}) = (3.8 - 3.5)/(1.2/\sqrt{30}) = 0.3/0.219 = 1.370$
- $t = 1.370 > 1.311$, so we reject H_0
- **Conclusion:** Service time is significantly longer than claimed.

Question 10: Complex Scenario - Multiple Testing Considerations

Question: A pharmaceutical company tests a new drug's effectiveness across three dosage groups and a control group. They want to compare each dosage group to the control. Control group: $n_0 = 25$, $\bar{x}_0 = 4.2$, $s_0 = 1.8$. Low dose: $n_1 = 20$, $\bar{x}_1 = 5.1$, $s_1 = 2.1$. Medium dose: $n_2 = 22$, $\bar{x}_2 = 6.3$, $s_2 = 1.9$. High dose: $n_3 = 18$, $\bar{x}_3 = 7.8$, $s_3 = 2.3$. For the comparison between high dose and control only, test at $\alpha = 0.05$ assuming equal variances.

Solution: We use a two-sample t-test with pooled variance because we're comparing two independent groups with unknown population standard deviations and small sample sizes, assuming equal variances. Although this is part of a multiple comparison scenario, we're only analyzing one specific comparison as requested.

Table to use: T-Distribution Table with $df = n_1 + n_2 - 2 = 41$

Step-by-step solution:

- H_0: $\mu_3 - \mu_0 = 0$
- H_1: $\mu_3 - \mu_0 > 0$ (expect high dose to be more effective)
- $n_3 = 18$, $n_0 = 25$, so $df = 41$
- Pooled variance: $s^2_p = [(17)(2.3)^2 + (24)(1.8)^2]/41 = [90.13 + 77.76]/41 = 4.094$
- Standard error: $SE = \sqrt{4.094} \times \sqrt{(1/18 + 1/25)} = 2.023 \times 0.242 = 0.490$
- Test statistic: $t = (7.8 - 4.2)/0.490 = 3.6/0.490 = 7.347$
- $\alpha = 0.05$, $df = 41$, critical value ≈ 1.683
- $t = 7.347 > 1.683$, so we reject H_0
- **Conclusion:** The high dose is significantly more effective than the control.

Note: In practice, multiple comparisons would require Bonferroni correction or other adjustments to control family-wise error rate.

Summary of When to Use Each Test:

Z-Tests:

- **One-sample z-test:** Known σ, large n, or normal population
- **Two-sample z-test:** Known σ_1 and σ_2, large samples
- **Proportion z-tests:** Large samples, $np \geq 5$ and $n(1-p) \geq 5$

T-Tests:

- **One-sample t-test:** Unknown σ, any sample size
- **Two-sample t-test (pooled):** Unknown σ, equal variances assumed
- **Welch's t-test:** Unknown σ, unequal variances
- **Paired t-test:** Dependent samples, unknown σ of differences

Key Decision Points:

1. **Known vs. Unknown σ:** Known σ → z-test, Unknown σ → t-test
2. **Sample size:** Large samples can use z-test even with unknown σ (controversial)
3. **Independence:** Related samples → paired t-test
4. **Equal variances:** Test with F-test or use Welch's t-test when in doubt
5. **Proportions:** Use z-tests for proportions with large samples