Chapter -3 Greedy Method

- ✓ **General Method**
- ✓ **Knapsack Problem**
- ✓ **Minimum Cost Spanning Tree –Kruskal and primal Algo**
- ✓ **Single Source Shorted Path**

General Method :

- An algorithm which always takes the best immediate, or local, solution while finding an answer. Greedy algorithms will always find the overall, or globally, *optimal solution* for some *optimization problems*, but may find less-than-optimal solutions for some instances of other problems.

Example of Greedy Method

- *Prim's algorithm* and *Kruskal's algorithm* are greedy algorithms which find the globally optimal solution, a *minimum spanning tree*. In contrast, any known greedy algorithm to find an *Euler cycle* might not find the shortest path, that is, a solution to the *traveling salesman* problem.

- *Dijkstra's algorithm* for finding *shortest paths* is another example of a greedy algorithm which finds an optimal solution.

Features

- Start with a solution to a small sub problem
- Build up to a solution to the whole problem
- Make choices that look good in the short term
- Disadvantage: Greedy algorithms don't always work (Short term solutions can be disastrous in the long term). Hard to prove correct
- Advantage: Greedy algorithm work fast when they work. Simple algorithm, easy to implement

Greedy Algorithm

```
Procedure GREEDY(A,n)
// A(1:n) contains the n inputs//
        solution  ← ϕ //initialize the solution to empty//
for i ← 1 to n do
        x ← SELECT(A)
        if FEASIBLE(solution,x)
                then solution ← UNION(solution,x)
        end if
repeat
        return(solution)
end GREEDY
```

Knapsack Problem

Greedy method is best suited to solve more complex problems such as a knapsack problem. In a knapsack problem there is a knapsack or a container of capacity M n items where, each item I is of weight wi and is associated with a profit pi. The problem of knapsack is to fill the available items into the knapsack so that the knapsack gets filled up and yields a maximum profit. If a fraction xi of object i is placed into the knapsack, then a profit pi *xi is earned. The constrain is that all chosen objects should sum up to M.
OR

- **Problem definition**
  - Given n objects and a knapsack where object i has a weight w_i and the knapsack has a capacity m
  - If a fraction x_i of object i placed into knapsack, a profit $p_i x_i$ is earned
    The objective is to obtain a filling of knapsack maximizing the total profit
- **Problem formulation (Formula 4.1-4.3)**

$$maximize \sum_{1 \le i \le n} p_i x_i \qquad (4.1)$$

$$subject\ to \sum_{1 \le i \le n} w_i x_i \le m \qquad (4.2)$$

$$and\ 0 \le x_i \le 1, \quad 1 \le i \le n \qquad (4.3)$$

-
- **A *feasible solution* is any set satisfying (4.2) and (4.3)**
- **An *optimal solution* is a feasible solution for which (4.1) is maximized**
- Greedy selection policy: three natural possibilities
- Policy 1: Choose the lightest remaining item, and take as much of it as can fit.
- Policy 2: Choose the most profitable remaining item, and take as much of it as can fit.
- Policy 3: Choose the item with the highest price per unit weight (P[i]/W[i]), and take as much of it as can fit. =➔Policy 3 always gives an optimal solution.

Illustration

Consider a knapsack problem of finding the optimal solution where, M=15, (p1,p2,p3…p7) = (10, 5, 15, 7, 6, 18, 3) and (w1,w2, …., w7) = (2, 3, 5, 7, 1, 4, 1).In order to find the solution, one can follow three different strategies.

Strategy 1: non-increasing profit values(Largest Profit)

Let (a,b,c,d,e,f,g) represent the items with profit (10,5,15,7,6,18,3) then the sequence of objects with non increasing profit is (f,c,a,d,e,b,g).

Item chosen for inclusion	Quantity of item included	Remaining space in M	P_iX_i
f	1 full unit	15-4=11	18*1=18
C	1 full unit	11-5=6	15*1=15
A	1 full unit	6-2=4	10*1=10
d	4/7 unit	4-4=0	4/7*7=04

Profit= 47 units   The solution set is (1, 0, 1, 4/7, 0, 1, 0).

Strategy 2: non-decreasing weights(Smallest Wight)

The sequence of objects with non-decreasing weights is (e,g,a,b,f,c,d).

Item chosen for inclusion	Quantity of item included	Remaining space in M	P_iX_I
E	1 full unit	15-1=14	6*1=6
G	1 full unit	14-1=13	3*1=3
A	1 full unit	13-2=11	10*1=10
b	1 full unit	11-3=8	5*1=05
f	1 full unit	8-4=4	18*1=18
c	4/5 unit	4-4=0	4/5*15=12

Profit= 54 units   The solution set is (1,1,4/5,0,1,1,1).

Strategy 3: maximum profit per unit of capacity used (This means that the objects are considered in decreasing order of the ratio Pi/wI)
a: P1/w1 =10/2 = 5 b: P2/w2 =5/3=1.66 c: P3/w3 =15/5 = 3 d: P4/w4 =7/7=1 e: P5/w5 =6/1=6 f: P6/w6 =18/4 = 4.5 g: P7/w7 =3/1=3
Hence, the sequence is (e, a, f, c, g, b, d)

Item chosen for inclusion	Quantity of item included	Remaining space in M	P_iX_I
E	1 full unit	15-1=14	6*1=6
A	1 full unit	14-2=12	10*1=10
F	1 full unit	12-4=8	18*1=18
C	1 full unit	8-5=3	15*1=15
g	1 full unit	3-1=2	3*1=3
b	2/3 unit	2-2=0	2/3*5=3.33

Profit= 55.33 units   The solution set is (1,2/3,1,0,1,1,1).
Example2. n = 3, M = 20, $(p_1, p_2, p_3) = (25, 24, 15)$ $(w_1, w_2, w_3) = (18, 15, 10)$

Sol: $p_1/w_1 = 25/18 = 1.32$

$p_2/w_2 = 24/15 = 1.6$

$p_3/w_3 = 15/10 = 1.5$

Optimal solution: $x_1 = 0$, $x_2 = 1$, $x_3 = 1/2$

total profit = 24 + 7.5 = 31.5

Algorithm GREEDY_KNAPSACK (P,W,M,X,n)

//P(1:n) and W(1:n) contain the profit and weights respectively of the n objects ordered so that P(i)/W(i) >=P(i+1)/W(i+1).M is the knapsack size and X(1:n) is the solution vector

```
Real P(1:n),W(1:n),X(1:n) ,M, cu;
Integer I,n;
X← 0                //initialize solution to Zero
Cu←M            // cu is remaining knapsack capacity
for  i← 1 to n do
      if(W(i)  >cu )  then  exit  endif
       X(i)←1
      Cu← cu-W(i)
Repeat

If (i<=n ) then  X(i) ← cu/W(i) endif

End
```

Time complexity

- Sorting: O(n log n) using fast sorting algorithm like merge sort
- GreedyKnapsack: O(n)
- So, total time is O(n log n)

Minimum Cost Spanning Tree –Kruskal and Prim's Algo

Tree:

– A tree is a graph with the followingproperties:
• The graph is connected (can go from anywhere to anywhere)
• There are no cycle

Spanning Tree

• A spanning tree is a tree that spans all the nodes Thus, if there are n nodes in the network, a tree spanning this network will have n-1 arcs that go through all the nodes.

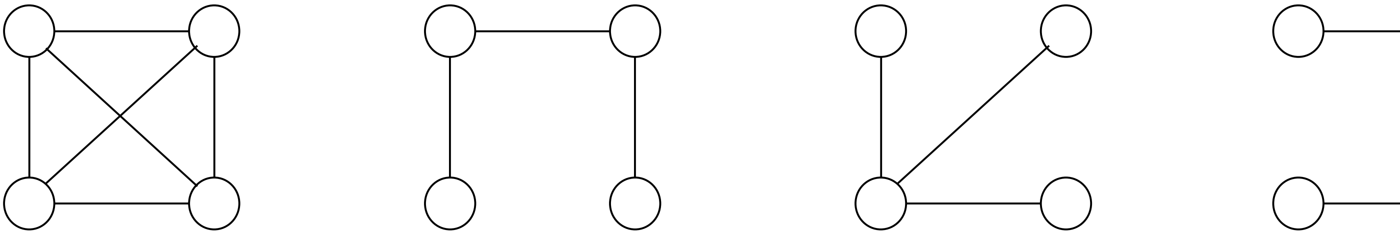
Minimum Spanning tree

• It is the shortest spanning tree (length of a tree is equal to the sum of the length of the arcs on the tree).
• Very important
– Practice (eg. communication)
– Theory (eg. basis)
– Algorithms (as a sub problem)

A **minimum spanning tree** (MST) or **minimum weight spanning tree** is then a spanning tree with weight less than or equal to the weight of every other spanning tree.

- **Definition Let *G=(V, E*) be at undirected connected graph. A subgraph *t=(V, E')* of *G* is a *spanning tree* of *G* iff t is a tree.**

Example



<u>Algorithm for a Spanning Tree</u>

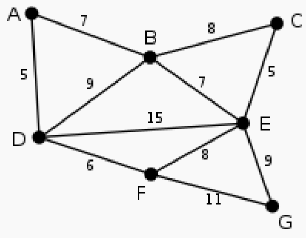
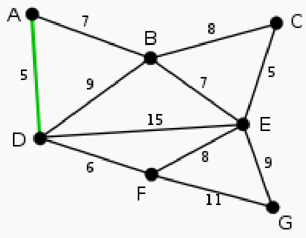
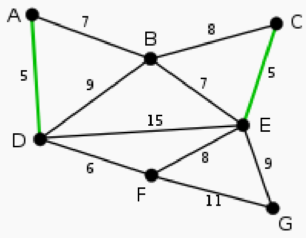
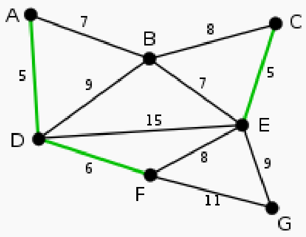
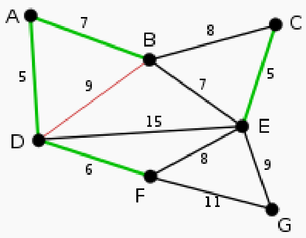
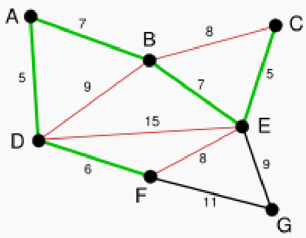
• Two basic algorithms exists – Kruskal (by arc) – Prim (by sub-tree)
• Both are greedy
• May have different complexity (efficiency) depending on the topology (eg. density) of the network

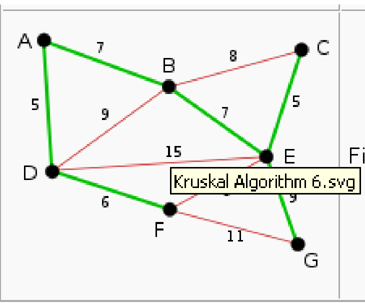
<u>Kruskal Algorithm</u>

• Append new arcs to the tree in increasing order of the arc length (making sure cycles are not created)

Kruskal's algorithm is an algorithm in graph theory that finds a minimum spanning tree for a connected weighted graph. This means it finds a subset of the edges that forms a tree that includes every vertex, where the total weight of all the edges in the tree is minimized. If the graph is not connected, then it finds a minimum spanning forest (a minimum spanning tree for each connected component). Kruskal's algorithm is an example of a greedy algorithm.

<u>Example</u>

Image	Description
	This is our original graph. The numbers near the arcs indicate their weight. None of the arcs are highlighted.
	**AD** and **CE** are the shortest arcs, with length 5, and **AD** has been arbitrarily chosen, so it is highlighted.
	**CE** is now the shortest arc that does not form a cycle, with length 5, so it is highlighted as the second arc.
	The next arc, **DF** with length 6, is highlighted using much the same method.
	The next-shortest arcs are **AB** and **BE**, both with length 7. **AB** is chosen arbitrarily, and is highlighted. The arc **BD** has been highlighted in red, because there already exists a path (in green) between **B** and **D**, so it would form a cycle (**ABD**) if it were chosen.
	The process continues to highlight the next-smallest arc, **BE** with length 7. Many more arcs are highlighted in red at this stage: **BC** because it would form the loop **BCE**, **DE** because it would form the loop **DEBA**, and **FE** because it would form **FEBAD**.



Finally, the process finishes with the arc **EG** of length 9, and the minimum spanning tree is found

Algorithm :

KRUSKAL (E,Cost,n,T,mincost)
//E is the set of edges in G.
//G has n vertices.
//Cost(U,v) is the cost of edge (u,v).
//T is the set of edes in the minimum spanning tree and mincost is its cost

1. Real min cost, cost(1:n,1:n)
2. Integer PARENT(1:n) ,T(1:n-1,2) ,n construct a heap out of the edge costs using HEAPIFY
3. Parent← 1 // each vertex is in a different set
4. I←mincost ← 0
5. While i←n-1 and heap not empty do
6. Delete a minimum cost edge (u,v) from the heap and reheapify using ADJUST
7. J← FIND (u) ;K← FIND(V)
8. If j != k then i← i+1
9. T(I,1) ← u ;T(I,2) ← v
10. Mincost ←mincost +cost (u,v)
11. Endif
12. Repeat
13. If i< > n-2 then print (“no spanning tree) end if
14. Return
15. End Kruskal

How to implement -

Two functions should be considered

- **Determining an edge with minimum cost**
- **Deleting this edge**

Analysis of Algorithm

Where *E* is the number of edges in the graph and *V* is the number of vertices, Kruskal's algorithm can be shown to run in $O(E \log E)$ time, or equivalently, $O(E \log V)$ time, all with simple data structures. These running times are equivalent because:

- E is at most V^2 and $\log V^2 = 2\log V$ is $O(\log V)$.
- If we ignore isolated vertices, which will each be their own component of the minimum spanning forest, $V \leq E+1$, so $\log V$ is $O(\log E)$.

We can achieve this bound as follows: first sort the edges by weight using a comparison sort in $O(E \log E)$ time; this allows the step "remove an edge with minimum weight from S" to operate in constant time. Next, we use a disjoint-set data structure to keep track of which vertices are in which components. We need to perform $O(E)$ operations, two 'find' operations and possibly one union for each edge. Even a simple disjoint-set data structure such as disjoint-set forests with union by rank can perform $O(E)$ operations in $O(E \log V)$ time. Thus the total time is $O(E \log E) = O(E \log V)$.

Or

Edge set E.
Operations are:

- Is E empty?
- Select and remove a least-cost edge.

Use a min heap of edges.

- Initialize. O(e) time.
- Remove and return least-cost edge. O(log e) time.

Set of selected edges T.
Operations are:

- Does T have n - 1 edges?
- Does the addition of an edge (u, v) to T result in a cycle?

Add an edge to T.

Use an array linear list for the edges of T.

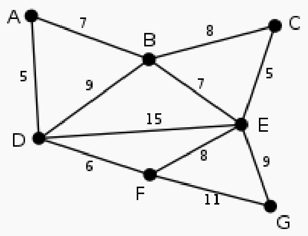
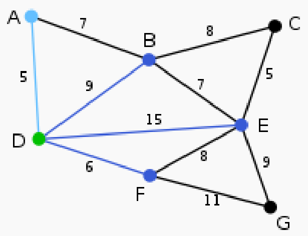
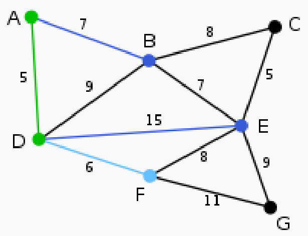
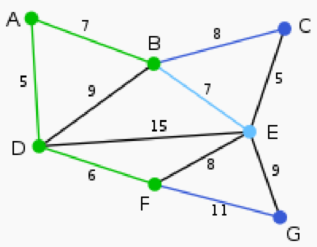
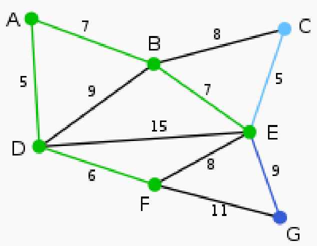
- Does T have n - 1 edges?
  - Check size of linear list. O(1) time.
- Does the addition of an edge (u, v) to T result in a cycle?
  - Not easy.
- Add an edge to T.
  - Add at right end of linear list. O(1) time.

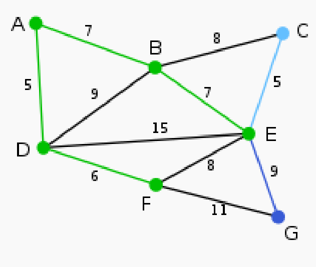
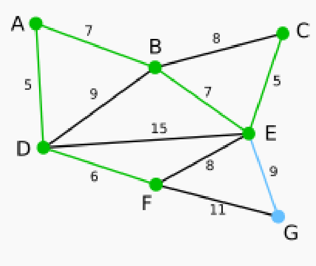
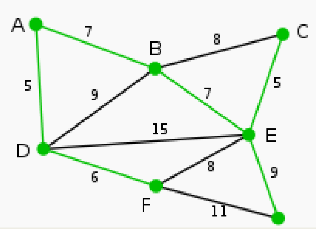
Just use an array rather than ArrayLinearList

- Use FastUnionFind.
- Initialize.
  - O(n) time.
- At most 2e finds and n-1 unions.
  - Very close to O(n + e).
- Min heap operations to get edges in increasing order of cost take O(e log e).
- Overall complexity of Kruskal’s method is O(n + e log e).

Prim’s Algorithm

• Similar, except that we always have a sub-tree as a partial solution: the new arc we add connects a node in the existing sub-tree to a node not yet in the sub-tree.
Example

Image	Description
	This is our original weighted graph. The numbers near the arcs indicate their weight.
	Vertex **D** has been arbitrarily chosen as a starting point. Vertices **A**, **B**, **E** and **F** are connected to **D** through a single edge. **A** is the vertex nearest to **D** and will be chosen as the second vertex along with the edge **AD**.
	The next vertex chosen is the vertex nearest to *either* **D** or **A**. **B** is 9 away from **D** and 7 away from **A**, **E** is 15, and **F** is 6. **F** is the smallest distance away, so we highlight the vertex **F** and the arc **DF**.
	In this case, we can choose between **C**, **E**, and **G**. **C** is 8 away from **B**, **E** is 7 away from **B**, and **G** 11 away from **F**. **E** is nearest, so we highlight the vertex **E** and the arc **BE**.
	Here, the only vertices available are **C** and **G**. **C** is 5 away from **E**, and **G** is 9 away from **E**. **C** is chosen, so it is highlighted along with the arc **EC**.

	Here, the only vertices available are **C** and **G**. **C** is 5 away from **E**, and **G** is 9 away from **E**. **C** is chosen, so it is highlighted along with the arc **EC**.
	Vertex **G** is the only remaining vertex. It is 11 away from **F**, and 9 away from **E**. **E** is nearer, so we highlight it and the arc **EG**.
	Now all the vertices have been selected and the minimum spanning tree is shown in green. In this case, it has weight 39.

<u>Algorithm</u>

<u>PRIME(E,COST,nT,mincost)</u>

//E is the set of edges in G
//COST (n,n) is the cost adjacency matrix of an n vertex graph such that COST(I,j) is either a positive real number + infinity. If no edge exists. A minimum spanning tree is computed and stored as set of edges in the array T(1:n-1,2). T(1,1)T(I,2) is an edge in the min-cost spanning tree .The final cost is assigned to mincost

1. real COST(n,n) ,mincost ;
2. integer NEAR(n) ,n,I,j,k,l ,T(1:n-1,2);
3. (K,l) ← edge with minimum cost
4. mincost = cost(k,l);
5. T(1,1),T(1,2) ←(k,l)
6. for I ←1 to n do // Initialize near.
7. If COST(i ,l)< COST (i, k) then NEAR (i) ← l
8. Else NEAR(i) ← k endif
9. Repeat
10. NEAR (k) ← NEAR (l) ← 0
11. for I ← 2 to n-1 do //find n-2 additional edges for T.
12. // Let j be an index such that NEAR (J)!= 0 and COST (j ,NEAR(j)) is minimum
13. (T(i,1) ,T(i,2)) ← (J ,NEAR (j))
14. mincost ←mincost +COST (j,NEAR(j))
15. NEAR (j) ← 0
16. for K ←1 to n do //update NEAR
17. if NEAR(K) != 0 and COST (K ,NEAR (k)) > COST(K,j) then

```
        NEAR(K) ← j
End if
Repeat
Repeat
If mincost >= infinity  then print  (“no spanning tree )
End PRIM
```

<u>Time complexity</u>

Minimum edge weight data structure	Time complexity (total)
adjacency matrix, searching	O(VE)

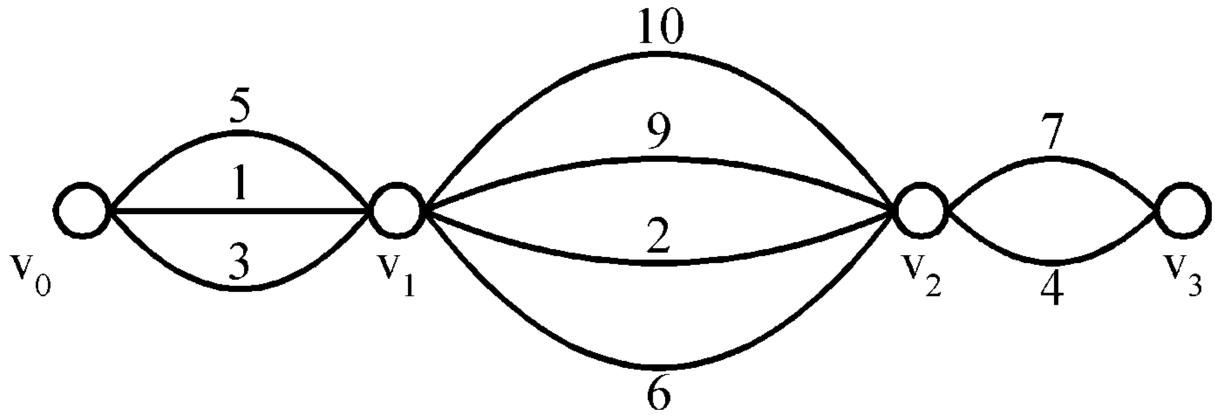
<u>Shortest Path :</u>

In graph theory, the **shortest path problem** is the problem of finding a path between two vertices (or nodes) such that the sum of the weights of its constituent edges is minimized

- Directed weighted graph.
- Path length is sum of weights of edges on path.
- The vertex at which the path begins is the source vertex.
- The vertex at which the path ends is the destination vertex.

<u>Example</u>

Finding the quickest way to get from one location to another on a road map; in this case, the vertices represent locations and the edges represent segments of road and are weighted by the time needed to travel that segment.



- Problem: Find a shortest path from v_0 to v_3.
- The greedy method can solve this problem.
- The shortest path: 1 + 2 + 4 = 7.

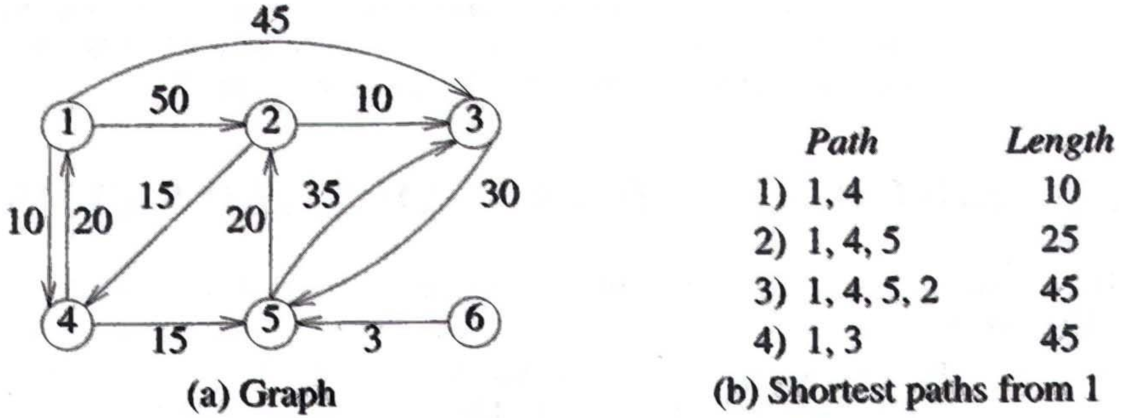
The problem is also sometimes called the **single-pair shortest path problem**, to distinguish it from the following generalizations:

- The **single-source shortest path problem**, in which we have to find shortest paths from a source vertex *v* to all other vertices in the graph.
- The **single-destination shortest path problem**, in which we have to find shortest paths from all vertices in the graph to a single destination vertex *v*. This can be reduced to the single-source shortest path problem by reversing the edges in the graph.
- The **all-pairs shortest path problem**, in which we have to find shortest paths between every pair of vertices *v*, *v'* in the graph.

<u>Single Source Shortest Path :</u>

- **<u>Design of greedy algorithm</u>**

Building the shortest paths one by one, in non decreasing order of path lengths
e.g., in Figure 4.15
1→4: 10
1→4→5: 25
…
We need to determine 1) the next vertex to which a shortest path must be generated and 2) a shortest path to this vertex
Three observations
If the next shortest path is to vertex *u*, then the path begins at v_0, ends at *u*, and goes through only those vertices that are in *S*.
The destination of the next path generated must be that of vertex *u* which has the minimum distance, *dist(u)*, among all vertices not in *S*.
Having selected a vertex *u* as in observation 2 and generated the shortest v_0 to *u* path, vertex *u* becomes a member of *S*.



	Path	Length
1)	1, 4	10
2)	1, 4, 5	25
3)	1, 4, 5, 2	45
4)	1, 3	45

(a) Graph　　(b) Shortest paths from 1

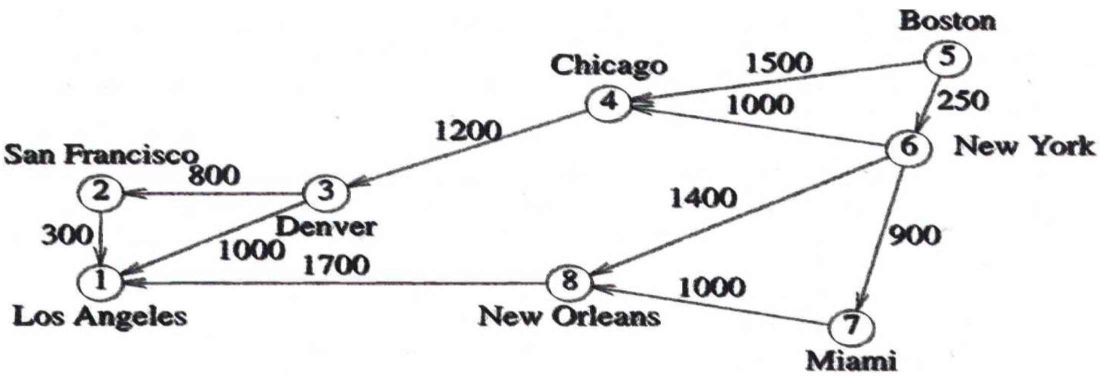
<u>Algorithm   :Greedy algorithm (Dijkstra's algorithm)</u>

ShortestPaths(v, cost, dist, n)

//DIST(j) ,1<=j<=n is set to the length of the shortest path from vertex v to vertex j in a diagraph G with n vertices. DIST(v) is set to zero. G is represented by its cost adjacency matrix ,COST(n,n)

```
Boolean S(1:n); real COST(1:n,1:n) DIST(1:n)
Integer u,v, n,num ,I,w
For I ← to n do  //initialize set S to empty
  S(i)←0 ; DIST(i) ← COST(V,i)
Repeat
S(V) ←1 ;DIST(v) ← 0  //put vertex v in set  S
For  num ← 2 to n-1 do    //determine n-1 paths from vertex v //
     Choose u such that DIST (u) =min {DIST(w)}
    S(w)=0
    S(u)←1    //put vertex u in set s
    For all W with S(w) =0 do
        DIST(w) ← min(DIST(w) ,DIST( u) +COST(u,w))
    Repeat
Repeat
End SHORTEST-PATHS.
```

Time Complexity   :- O(n^2)



(a) Digraph

	1	2	3	4	5	6	7	8
1	0							
2	300	0						
3	100	800	0					
4			1200	0				
5				1500	0	250		
6				1000		0	900	1400
7							0	1000
8	1700							0

(b) Length-adjacency matrix

Iteration	S	Vertex selected	Distance							
			LA [1]	SF [2]	DEN [3]	CHI [4]	BOST [5]	NY [6]	MIA [7]	NO [8]
Initial	--	----	+∞	+∞	+∞	1500	0	250	+∞	+∞
1	{5}	6	+∞	+∞	+∞	1250	0	250	1150	1650
2	{5,6}	7	+∞	+∞	+∞	1250	0	250	1150	1650
3	{5,6,7}	4	+∞	+∞	2450	1250	0	250	1150	1650
4	{5,6,7,4}	8	3350	+∞	2450	1250	0	250	1150	1650
5	{5,6,7,4,8}	3	3350	3250	2450	1250	0	250	1150	1650
6	{5,6,7,4,8,3}	2	3350	3250	2450	1250	0	250	1150	1650
	{5,6,7,4,8,3,2}									