Join Matrix :

$A = [a_{ij}]$, $B = [b_{ij}]$ be $m \times n$ zero-one matrices. Then, the Join of A and B denoted by $A \vee B$, is the $m \times n$ zero-one matrix with (i, j)th entry $a_{ij} \vee b_{ij}$

eg $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

$$A \vee B = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix}$$

$$\therefore A \vee B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Boolean product :

$A = [a_{ij}]$ be an $m \times k$ zero-one matrix & $B = [b_{ij}]$ be a $k \times n$ zero-one matrix. Then the boolean product of A and B denoted by

$$[A \odot B]_{m \times n} = c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \dots \vee ((a_{ik}) \wedge b_{kj})$$

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Connectivity Relation R^*.

Let R be a relation on the set A. The connectivity relation R^* consists of pairs (a, b) such that there is a path of length at least one from a to b in R.

Transitive closure and Connectivity:-

Theorem: The transitive closure of a relation R equals the connectivity relation R^*

Theorem:- Let M_R be the zero-one matrix of the relation R on a set with n elements. Then the zero-one-matrix of the transitive closure R^* is

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} - \vee M_R^{[n]}$$

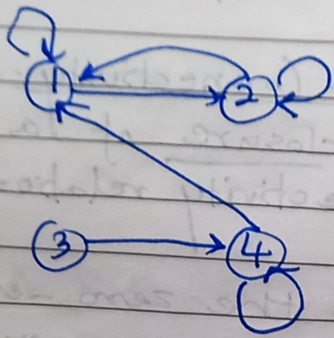
Example of Transitive closure,

step 1 :— $n = 4$

Let A be the set $\{1, 2, 3, 4\}$ and R be the relation

$R = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$

What is the transitive closure of R?



$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

step 2

From M_R (~~join Matrix~~) Boolean product

Calculate $M_R^{[2]}$ =

$$M_R^{[2]} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

$$M_R^{[3]} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}, \quad M_R^{[4]} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

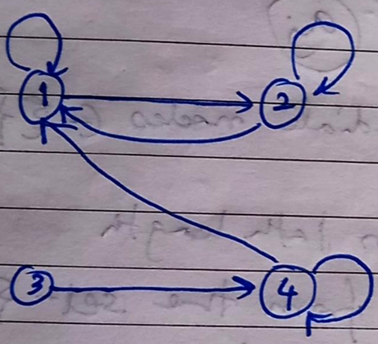
step 3 Join Matrix.

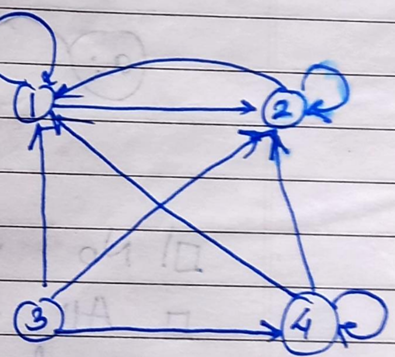
$$M^{*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee M_R^{[4]}$$

$$M_R^{*} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

step 4

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad M_R^{*} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$





$\therefore$ $R = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$

$R^{*} = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,2), (3,4), (4,1), (4,2), (4,4)\}$

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Ex. 1) Apply Warshall's Algorithm to find transitive closure of

$A = \{1, 2, 3, 4, 5\}$

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} = W_0$$

for W_1 Take first column & first row write all positions where 1 is present in column 1

write all ~~of~~ ordered pair

Take cross product

$C_1 \quad \{1, 2, 3, 5\}$

$R_1 \quad \{1, 2, 5\}$

$C_1 \times R_1 = \{(1,1), (1,2), (1,5), (2,1), (2,2), (2,5), (3,1), (3,2), (3,5), (5,1), (5,2), (5,5)\}$

$W_1 = W_0 \cup (C_1 \times R_1)$

$$W_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

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$$W_2 = \quad C_2 = \{1, 2, 3, 4, 5\}$$

$$R_2 = \{1, 2, 3, 4, 5\}$$

$$C_2 \times R_2 = \{(1,1), (1,2), (1,3), (1,4), (1,5), (2,1), (2,2), (2,3), (2,4), (2,5), (3,1), (3,2), (3,3), (3,4), (3,5), (4,1), (4,2), (4,3), (4,4), (4,5), (5,1), (5,2), (5,3), (5,4), (5,5)\}$$

$$W_2 = W_1 \cup (C_2 \times R_2)$$

$$W_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$W_3 = W_2 = W_4 = W_5 = M_R^\infty =$ all 1's.

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EX2

By using Warshall's algorithm, find the transitive closure of the relation

$R = \{(2,1), (2,3), (3,1), (3,4), (4,1), (4,3)\}$
on set $A = \{1, 2, 3, 4\}$.

Soln. represent M of R

$$M_R = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 & 1 \\ 4 & 1 & 0 & 1 & 0 \end{array}$$

$\underline{4 \times 4}$

step 1

$W_0 = M_R$

step 2 $\quad W_1 = W_0 \cup (C_1 \times R_1)$

all position where "1" is present

$C_1 = \{2, 3, 4\}$, $R_1 = \{ \}$.

$C_1 \times R_1 = \phi$.

$$W_1 = W_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

step 3

$W_2 = W_1 \cup (C_2 \times R_2)$

$C_2 = \{ \}$, $R_2 = \{1, 3\}$

$C_2 \times R_2 = \phi$

$W_2 = W_1 = W_0$

$$w_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

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Step 4 $\quad W_3 = W_2 \cup (C_3 \times R_3)$

$C_3 = \{2, 4\}$, $R_3 = \{1, 4\}$

$C_3 \times R_3 = \{(2,1), (2,4), (4,1), (4,4)\}$

$\therefore W_3 = W_2 \cup (C_3 \times R_3)$

$$\therefore W_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Step 5

$W_4 = W_3 \cup (C_4 \times R_4)$

$C_4 = \{2, 3, 4\}$, $R_4 = \{1, 3, 4\}$

$C_4 \times R_4 = \{(2,1), (2,3), (2,4), (4,1), (4,3), (4,4), (3,1), (3,3), (3,4)\}$

$\therefore W_4 = W_3 \cup (C_4 \times R_4)$

$$W_4 = M_R^{\infty} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

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4.8 (21)

Ex1(5) Let $A = \{1, 2, 3, 4\}$ and let R and S be the relations on A described by

$$M_R = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

& $$M_S = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Use Warshall's algorithm to compute the transitive closure of $R \cup S$

$$M_{R \cup S} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} = W_0$$

4×4.

Now step 1

$\therefore M_{R \cup S} = W_0$.

step 2

$$W_1 = W_0 \cup (C_1 \times R_1)$$

$C_1 \rightarrow$ all positions in column 1 where 1 is present in column1

$C_1 = \{1\}$,

$R_1 \rightarrow$ all positions in row 1, where 1 is present in row 1

$R_1 = \{1, 2, 3\}$

$\therefore C_1 \times R_1$ = cross product $= \{(1,1), (1,2), (1,3)\}$

$\therefore W_1 = W_0 \cup (C_1 \times R_1)$.

$$\therefore W_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

step 3 : $W_2 = W_1 \cup (C_2 \times R_2)$.

$C_2 = \{1, 2, 3, 4\}$

$R_2 = \{2\}$

$\therefore C_2 \times R_2 = \{(1,2), (2,2), (3,2), (4,2)\}$

$\therefore W_2 = W_1 \cup C_2 \times R_2$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

step 4

$W_3 = W_2 \cup (C_3 \times R_3)$

$C_3 = \{1, 3, 4\}$

$R_3 = \{2, 3\}$

$\therefore C_3 \times R_3 = \{(1,2), (1,3), (3,2), (3,3), (4,2), (4,3)\}$

$W_3 = W_2 \cup (C_3 \times R_3)$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

step 5

$$W_4 = W_3 \cup (C_4 \times R_4)$$

$$C_4 = \{1, 4\}$$

$$R_4 = \{2, 3, 4\}$$

$$\therefore C_4 \times R_4 = \{(1,2), (4,3), (1,4), (4,2), (4,3), (4,4)\}$$

$$\therefore W_4 = W_3 \cup (C_4 \times R_4) = M_R^{\infty}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$