K J Somaiya College of Engineering, Vidyavihar, Mumbai

(A Constituent College of SVU)

Engineering Mechanics Notes

Module 1 – System of Forces

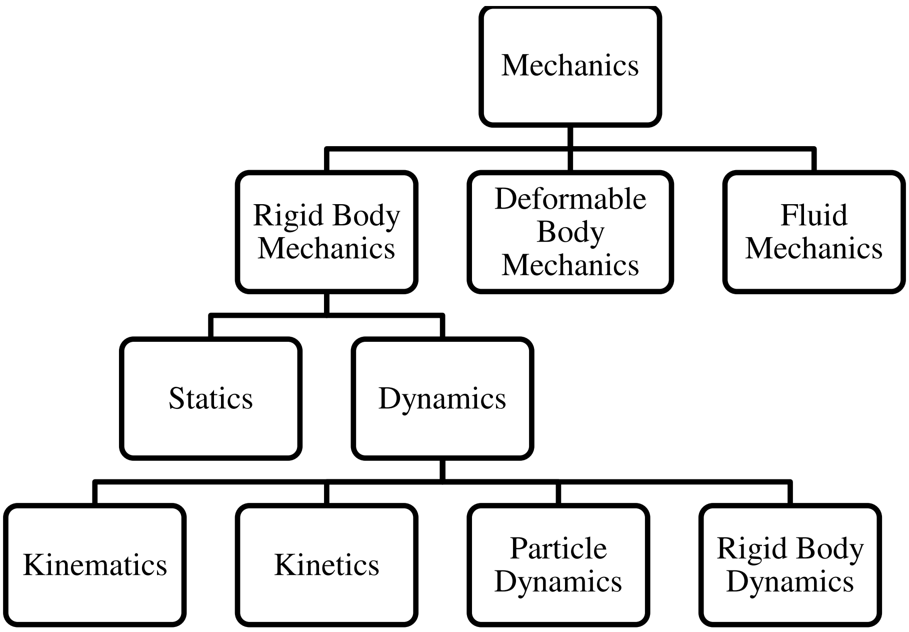
Module Section 1.1 – System of Coplanar Forces

Class: FY BTech Division: C3
Professor: Aniket S. Patil Date: 09/03/2023

References: Engineering Mechanics, by M. D. Dayal & Engineering Mechanics – Statics and Dynamics, by N. H. Dubey.

Mechanics is defined as the branch of physics which deals with the study of resultant effect of action of forces on bodies, which may be at rest or in motion.

Classification of Mechanics:



- Rigid Body Mechanics: In this, bodies are assumed to be perfectly rigid, i.e., there is no deformation of bodies under any load acting on them.
- Statics: It is the study of effect of force system acting on a particle or rigid body which is at rest.
- Dynamics: It is the study of effect of force system acting on a particle or rigid body which is in motion. Dynamics can be classified into two categories: "Kinematics and Kinetics", depending on whether forces acting on the body are considered; or "Particle Dynamics and Rigid Body Dynamics", depending on whether the body is considered to be an ideal particle or a rigid body.

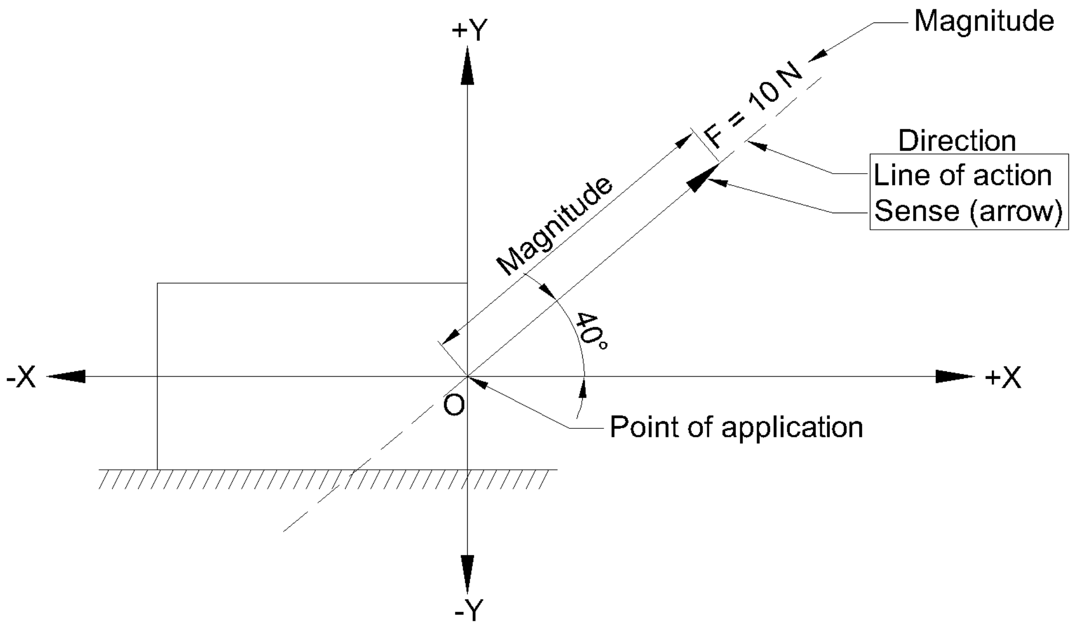
- o Kinematics is concerned only with the study of motion of the body without consideration of the forces causing the motion.
- o Kinetics relates the forces acting on the body to the motion of the body.
- o Particle dynamics is the motion analysis of a body considering the body as an idealized particle.
- o Rigid body dynamics is the motion analysis involving the shape and size of the body.

Force: A force is defined as an external agency or action which changes or tends to change the state of rest or of uniform motion of a body upon which it acts.

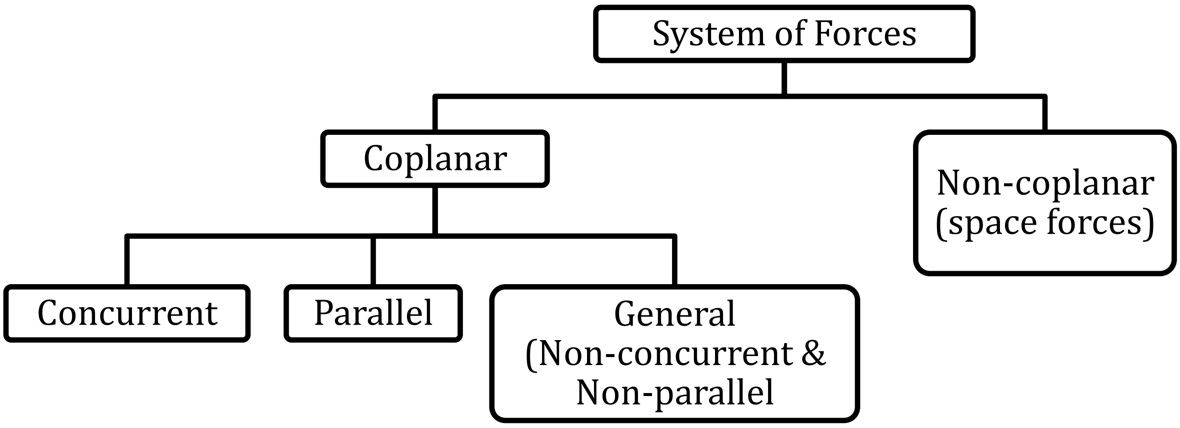
- It is a result of the action of one body on another. It could be due to direct action like physically lifting something or by remote action like gravitational force.
- It imparts motion or affects the motion of a body.
- It can accelerate, decelerate or stop a moving body depending on its direction.
- It may rotate a body.
- It may maintain the equilibrium condition of a body.
- It may be of push type or pull type.

Force is a vector quantity and is quantified by following factors:

a) Magnitude: It is the quantity of force measured in Newtons.
b) Direction (Line of action and sense): It is the orientation of line of action indicated by an angle with respect to the reference axes, along with a sense that indicates on which side the force acts.
c) Point of application (aka Location): It is the exact point at which the force acts and can be indicated by the distance from the origin.



System of Forces: When a number of forces act simultaneously on a body then they are said to form a system of forces or force system.

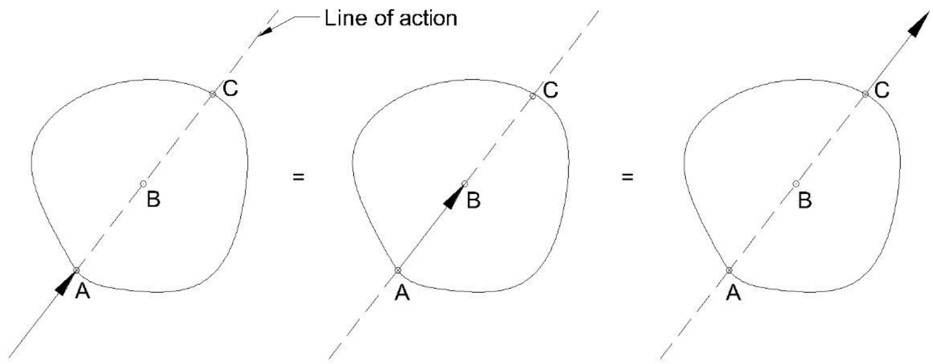


In Coplanar System of Forces, all forces lie in one plane; while in Non-coplanar System of Forces, all the forces in the system do not lie in a single plane.

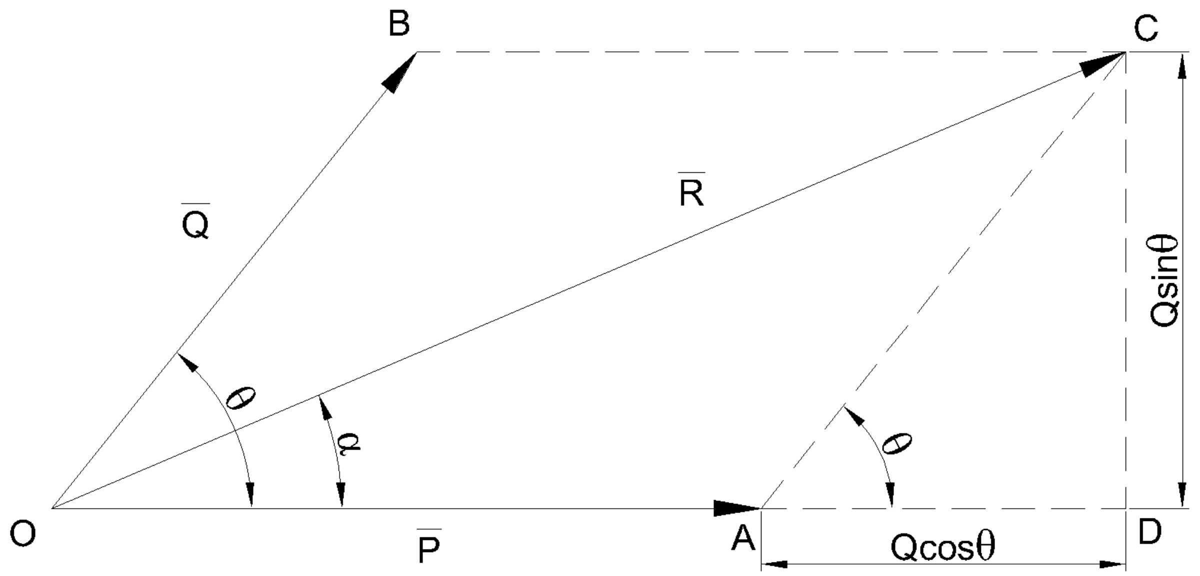
Both systems can be subdivided into:

a) Concurrent Force System: In this, the lines of action of all the forces in the system pass through the same point.
   e.g., lamp hanging from string, electric pole supporting heavy cables, forces on a tripod, etc.
b) Parallel Force System: In this, the lines of action of all the forces in the system are parallel to each other.
   e.g., weighing scale, things places on a table, people sitting on a bench, etc.
c) General Force System: In this, the lines of action of all the forces in the system are neither concurrent nor parallel to each other.
   e.g., a moving vehicle has engine power, friction due to road, wind resistance, weight of vehicle and passengers, etc. acting in various directions.

Principle of Transmissibility of Force: A force being a sliding vector will not affect the state of a rigid body (whether at rest or in motion) if the force acts from a different point along its line of action. E.g., in a train, the engine can be located at the front pulling the other cars with it or at the back pushing them forward.



Law of Parallelogram for Vectors: If two vectors acting simultaneously at a point are represented in magnitude and direction by two adjacent sides of a parallelogram, then their resultant is represented in magnitude and direction by the diagonal of the parallelogram which passes through the point of intersection of the two sides representing the vectors.

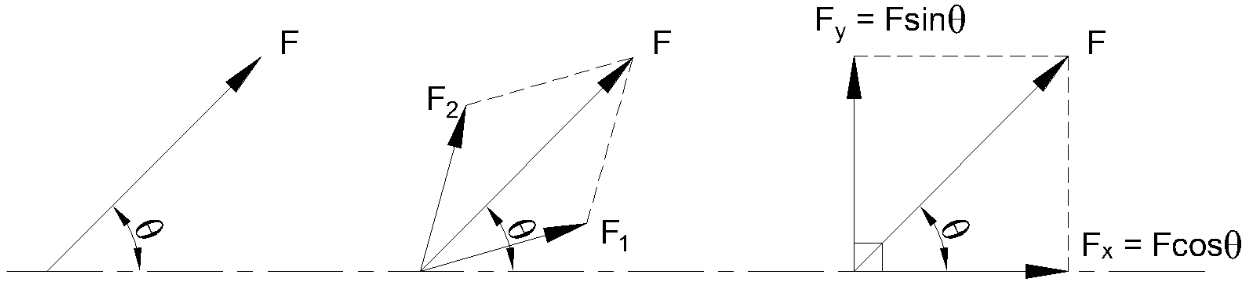


In ΔOCD,

$$OC^2 = OD^2 + CD^2 = (OA + AD)^2 + CD^2$$
$$R^2 = (P + Q\cos\theta)^2 + (Q\sin\theta)^2$$
$$R^2 = P^2 + 2PQ\cos\theta + Q^2\cos^2\theta + Q^2\sin^2\theta$$
$$R^2 = P^2 + Q^2 + 2PQ\cos\theta$$
$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$$

$$\tan\alpha = \frac{CD}{OD} = \frac{CD}{OA + AD}$$
$$\tan\alpha = \frac{Q\sin\theta}{P + Q\cos\theta}$$

Resolution of Forces: It is the process of breaking a force into components, such that the components combined together would have the same effect as the original force. There are various ways to resolve a force, but the most beneficial for calculations is the resolution of force into its rectangular (or perpendicular) components.



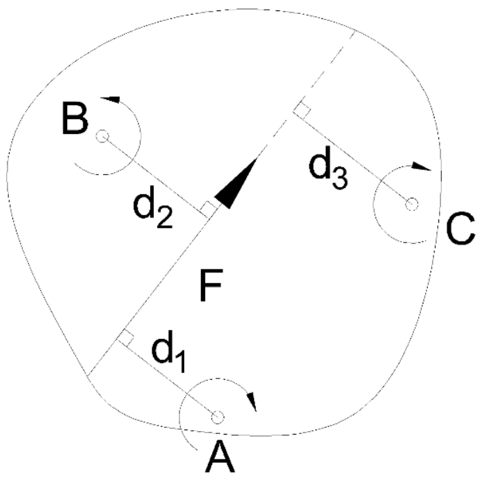
Note: In rectangular resolution, the component of force adjacent to the given angle is taken as $F\cos\theta$, and not always the x-direction component.

Moment of a Force: The rotational effect of a force is known as moment. Basically, the tendency of a force to rotate a rigid body about an axis or a point is measured by the moment of the force about that axis or point. E.g., door being opened or closed about a hinge, tightening or loosening of nut with a spanner, etc.

The point about which we calculate the moment is called the moment centre, and the rotational effect of the same force will vary from one moment centre to another.

The moment is measured by multiplying the magnitude of force and the perpendicular distance from the moment centre. This perpendicular distance is called as moment arm.

Moments of force F about moment centres A, B, C, with moment arms d_1, d_2, d_3:



$$M_A^F = F \times d_1 \; (\circlearrowright)$$
$$M_B^F = F \times d_2 \; (\circlearrowleft)$$
$$M_C^F = F \times d_3 \; (\circlearrowright)$$

Assuming anti-clockwise as positive, written as $\circlearrowleft$ +ve. We can also write the moments as,

$$M_A^F = -F \times d_1$$
$$M_B^F = +F \times d_2$$
$$M_C^F = -F \times d_3$$

Note: To find the direction of rotation, consider the moment centre as a fixed point or a hinge and based on the direction of the force, the body will tend to rotate in clockwise or anti-clockwise. That is the direction of the moment.

Varignon's Theorem: The algebraic sum of the moments of a system of coplanar forces about any point in the plane is equal to the moment of the resultant force of the system about the same point.

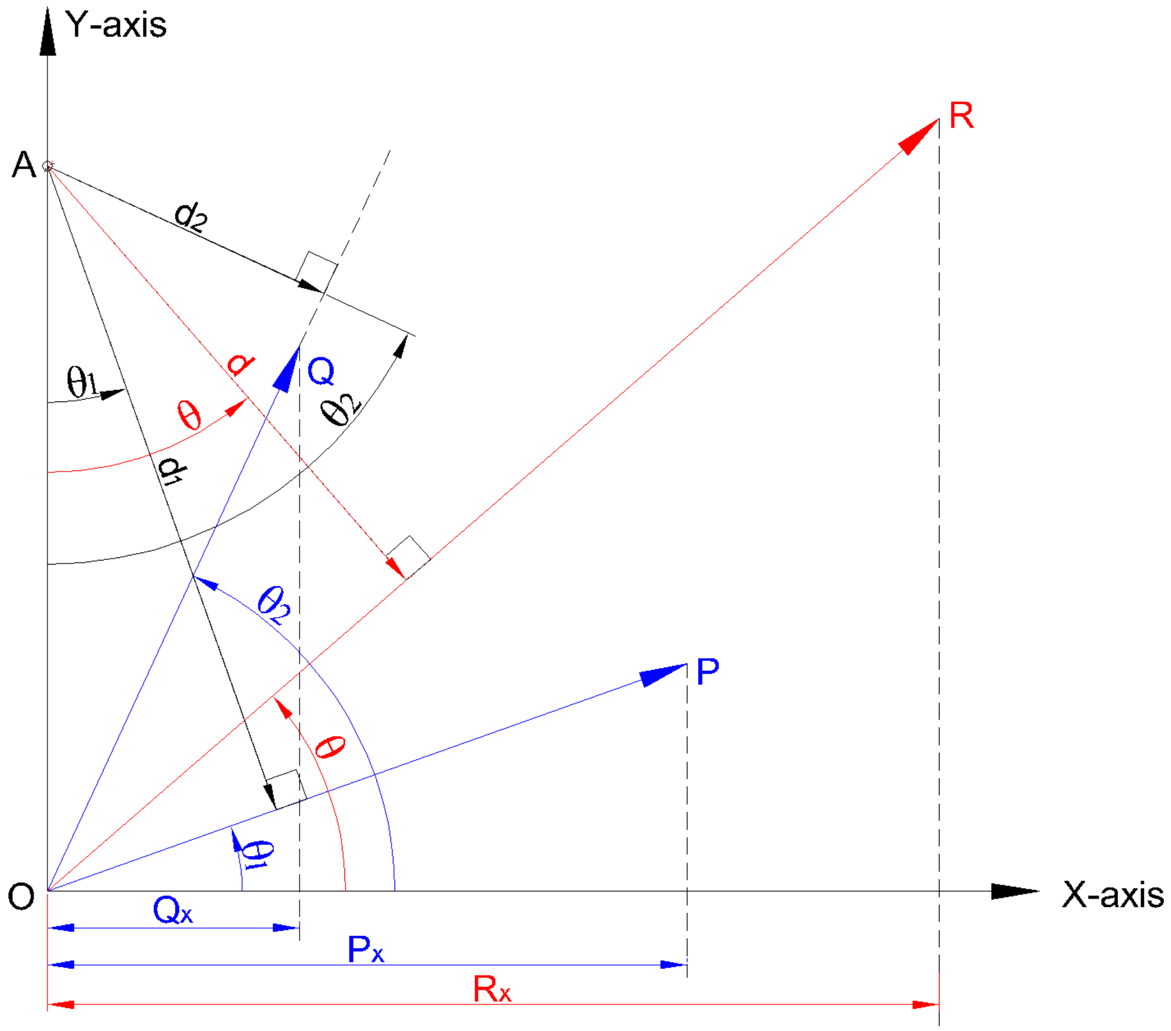
(Note: While the statement and proof is for coplanar forces, the theorem is applicable to a system of non-coplanar forces as well.)

<u>Proof</u>:

Let P & Q be 2 concurrent forces acting at O (origin), making angles θ_1 & θ_2 with x-axis. Let their resultant be R making angle θ with x-axis.

Let A be a point on the y-axis about which moments are to be taken. Let d_1, d_2, & d be the moment arms of P, Q & R respectively from moment centre A.

Let the components of the forces in x-direction be denoted by adding a 'subscript x'.



Moment due to force P about A,

$$M_A^P = +P \times d_1$$
$$M_A^P = P \times OA \cos \theta_1$$
$$M_A^P = P \cos \theta_1 \times OA$$
$$M_A^P = P_x OA$$

Similarly, moment due to Q about A, $M_A^Q = +Q \times d_2 = Q_x OA$

and, moment due to R about A, $M_A^R = +R \times d = R_x OA$

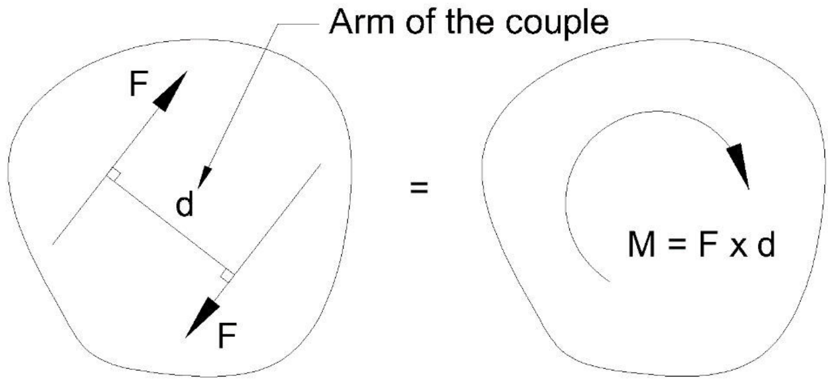
Now, the sum of moments of forces P & Q about A is given by,

$$\sum M_A^F = M_A^P + M_A^Q$$
$$\sum M_A^F = P_x OA + Q_x OA$$
$$\sum M_A^F = (P_x + Q_x) OA$$
$$\sum M_A^F = R_x OA$$
$$\sum M_A^F = M_A^R$$

This means that the sum of moments of the two forces about a point is equal to the moment of resultant force about that point.

Hence, the theorem is proved.

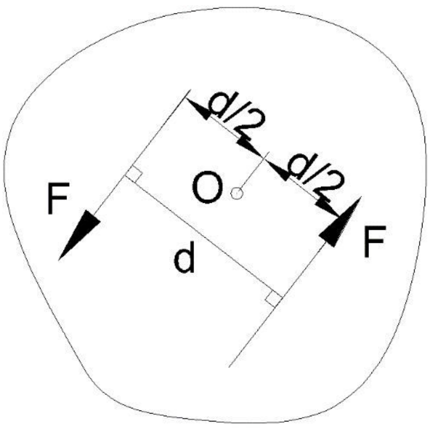
Couple: Two non-collinear parallel forces of equal magnitude but in opposite directions to each other form a couple. It causes rotation of the body. E.g., steering wheel of a vehicle, key rotation to lock or unlock, tap opening or closing, etc.

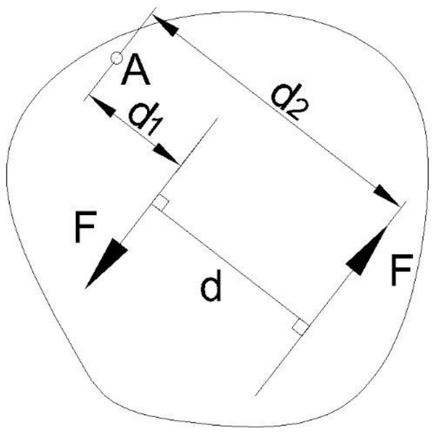


Properties of a couple:

1. Moment of a couple is equal to the product of one of the forces and the arm of the couple.
2. It tends to rotate the body about an axis perpendicular to the plane of the forces involved. It can only rotate and not translate the body.
3. The resultant force if a couple is zero.
4. Moment of a couple can be added algebraically as scalar quantity with proper sign convention.
5. A couple can be replaced by a couple only and not by a single force.
6. A couple is a free vector and does not have a moment centre, like moment of a force.

Couple is a free vector:



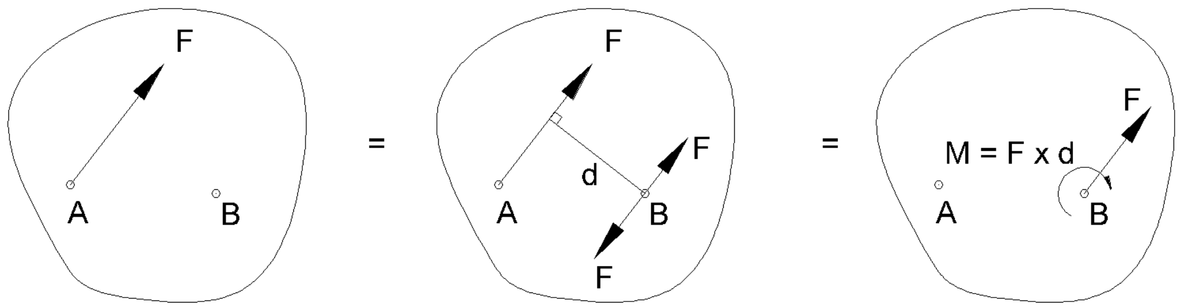


$$M_O^F = +F \times \frac{d}{2} + F \times \frac{d}{2}$$
$$M_O^F = F \times d$$

$$M_A^F = -F \times d_1 + F \times d_2$$
$$M_A^F = F(d_2 - d_1)$$
$$M_A^F = F \times d$$

Shift a force to a new parallel position:

To shift a force to a new parallel position, a couple is required to be added to the system. In below figure, force F acting at A is to be shifted to B. First step is to add two collinear forces of equal magnitude F & -F at B. The force F at A and force –F at B are parallel and form a couple with couple arm d. Thus, we have a single force F at B and a couple $M = F \times d$ in the system.



Force originally at A

Two collinear equal but opposite forces added at B

Force shifted to B with a couple due to two forces.

Composition of Forces or Resultant of Forces: Composition means to combine the forces acting in a system into a single force, which has the same effect as the number of forces acting together. This single force is known as the resultant of the system.

Types of Resultant:

a) Resultant Force: After composition of a number of forces results in a single force, it is called as a resultant force indicated by a magnitude and angle.
b) Resultant Couple: If a resultant force is zero, but the resultant moment is not zero, such a system reduces to a couple. It is possible in a Parallel system or General system, but not in a Concurrent system.
c) Resultant Force-Couple: When a resultant force is shifted to a new parallel position without change in direction, it introduces a couple in the system. This resultant consists of a single force and a single couple.

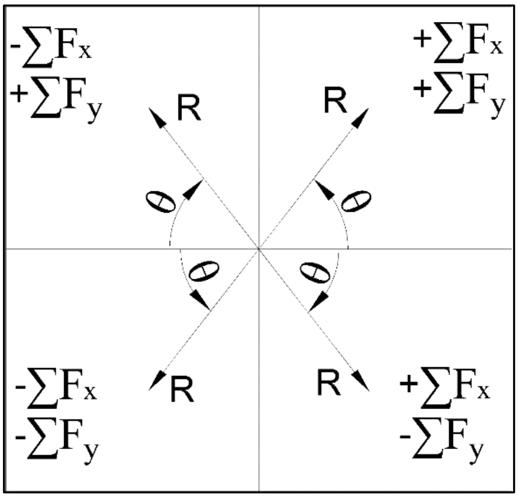
Types of Problems on Resultant of Forces:

1. Resultant of Concurrent Force System using Parallelogram Law of Vectors: Since force is a vector, two concurrent forces can be combined into a resultant using the law of parallelogram. This can be applied to multiple forces, two at a time, but that becomes cumbersome.
2. Resultant of Concurrent Force System using Method of Resolution: *Step 1*: Resolve all forces into their components along horizontal x-direction and vertical y-direction.
   Step 2: Find the algebraic sum of all components in x-direction to get $\sum F_x$, using → +ve; and all components in y-direction to get $\sum F_y$, using ↑ +ve.

Step 3: Find the magnitude if the resultant force, R and its direction considering the angle θ it makes with x-axis.

$$R = \sqrt{(\sum F_x)^2 + (\sum F_y)^2} \ \& \ \tan\theta = \frac{\sum F_y}{\sum F_x}$$

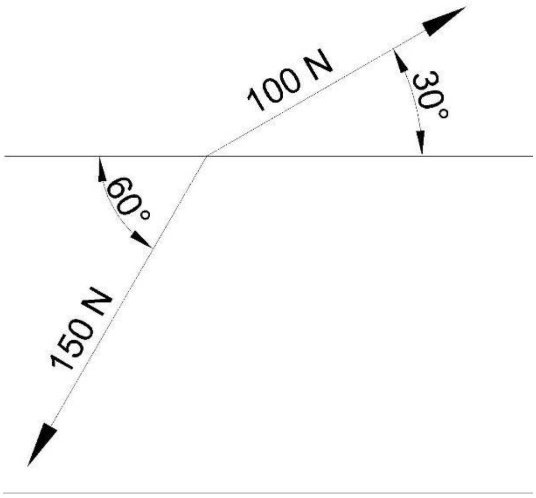
Step 4: Draw a simple diagram of resultant with proper direction considering the signs of $\sum F_x$ & $\sum F_y$ as shown below.



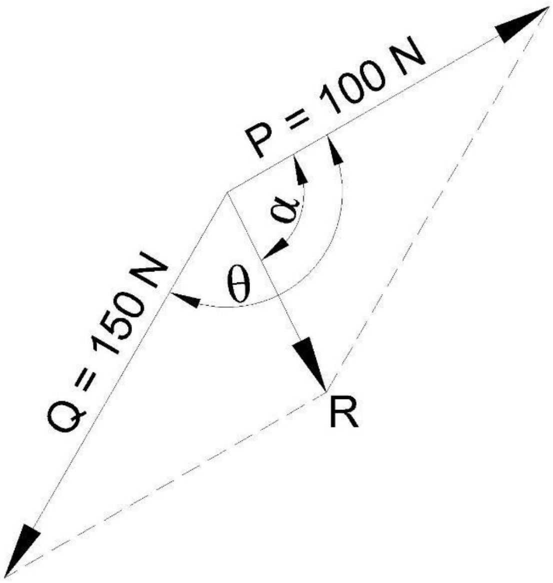
3. <u>Resultant of Parallel Force System using Varignon's Theorem</u>:
   Step 1: Since all forces are in one direction only, they can simply be added taking proper sign conventions. $R = \sum F$
   Step 2: Location of the resultant force (point of application) is found using Varignon's theorem. We assume the resultant to be acting at some perpendicular distance d to the right or left of the reference point.
   $$\sum M_A^F = M_A^R = R \times d$$
   If value of d is positive, the assumption of right or left is correct. If value of d is negative, resultant lies to the opposite side to what was assumed.
4. <u>Resultant of General Force System (Non-coplanar & Non-parallel)</u>:
   The steps for this are basically a combination of both the above systems.
   Step 1: Resolve all forces into their components along horizontal x-direction and vertical y-direction.
   Step 2: Find the algebraic sum of all components in x-direction to get $\sum F_x$, using → +ve; and all components in y-direction to get $\sum F_y$, using ↑ +ve.
   Step 3: Find the magnitude if the resultant force, R and its direction considering the angle θ it makes with x-axis.
   $$R = \sqrt{(\sum F_x)^2 + \left(\sum F_y\right)^2} \ \& \ \tan\theta = \frac{\sum F_y}{\sum F_x}$$
   Step 4: Find the perpendicular distance d where the resultant acts using Varignon's theorem.
   $$\sum M_A^F = M_A^R = R \times d$$
   Step 5: Draw a simple diagram of resultant with proper direction and distance labelled from reference point.

Numericals:

N1: Find the resultant of the given two forces.



Soln: Let, $P = 100\ N, Q = 150\ N, \theta = 150°$



Using Law of Parallelogram, the resultant force,

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$$

$$R = \sqrt{100^2 + 150^2 + 2 \times 100 \times 150 \times \cos 150°}$$

$$R = 80.74\ N$$

And, the angle between P & R,

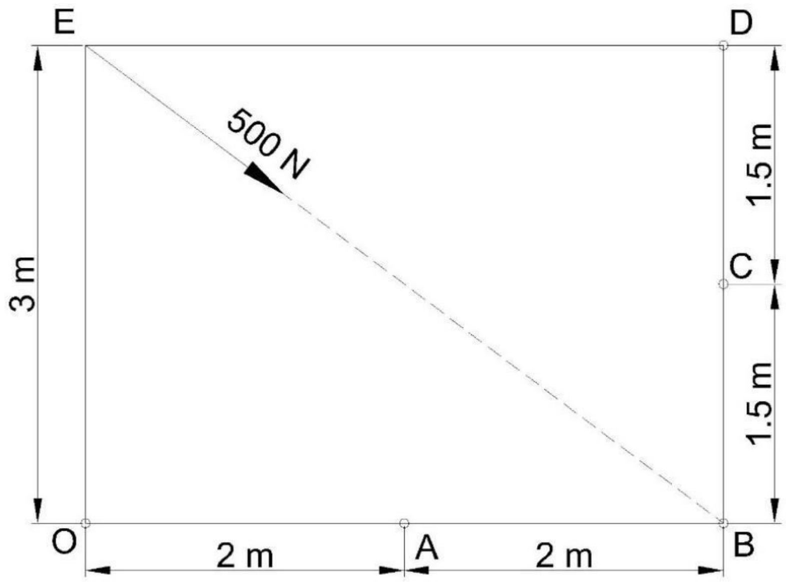
$$\tan\alpha = \frac{Q\sin\theta}{P + Q\cos\theta}$$

$$\tan\alpha = \frac{150 \times \sin 150°}{100 + 150 \times \cos 150°}$$

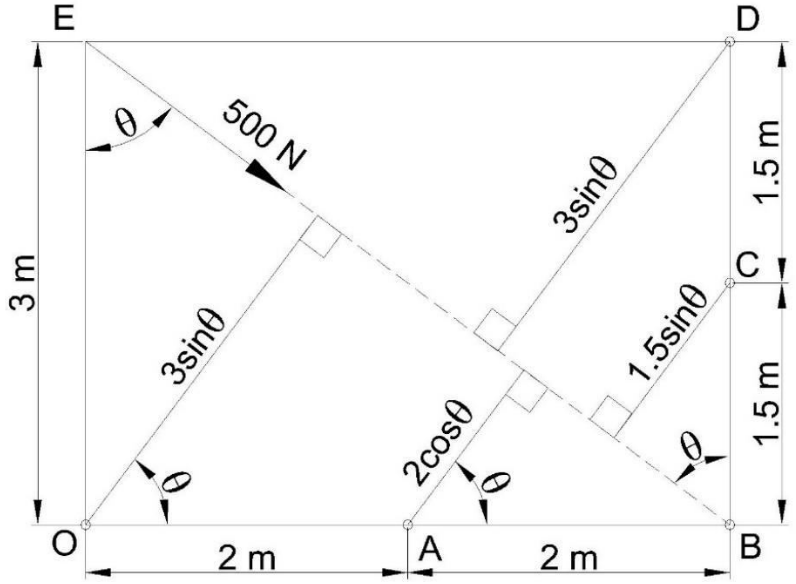
$$\tan\alpha = |-2.51|$$

$$\therefore \alpha = 68.26°$$

N2: Find the moment of the 500 N force about the points O, A, B, C & D.



Soln: Method 1: Perpendicular distances to the line of action



$$\theta = \tan^{-1}\frac{OB}{OE}$$

$$\theta = \tan^{-1}\frac{4}{3}$$

$$\therefore \sin\theta = \frac{4}{5} = 0.8$$

$$\& \cos\theta = \frac{3}{5} = 0.6$$

Moment about O, $M_O = -500 \times 3\sin\theta = -500 \times 3 \times 0.8 = -1200$ Nm

$$M_O = 1200 \text{ Nm } (\circlearrowright)$$

Moment about A, $M_A = -500 \times 2\cos\theta = -500 \times 2 \times 0.6 = -600$ Nm

$$M_A = 600 \text{ Nm } (\circlearrowright)$$

Moment about B, $M_B = 0$, as the point lies on the line of action of force.

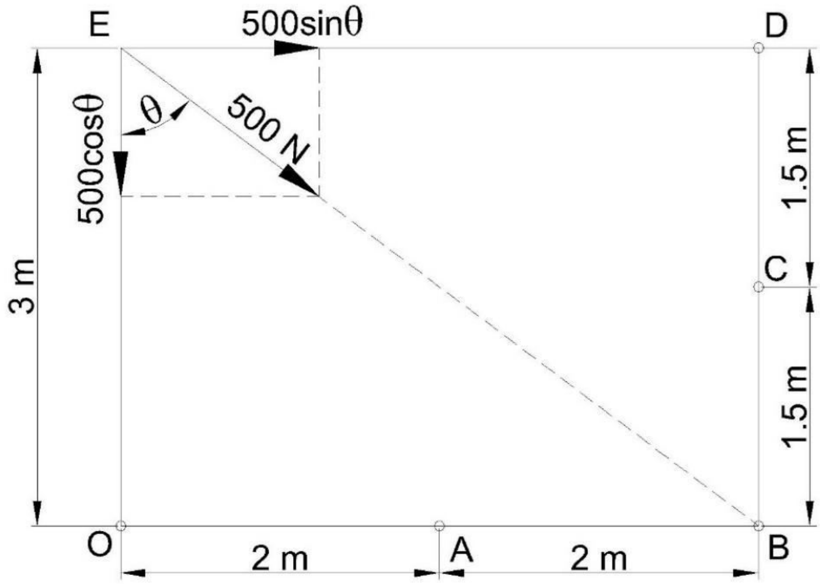
Moment about C, $M_C = +500 \times 1.5\sin\theta = +500 \times 1.5 \times 0.8 = +600$ Nm

$$M_C = 600 \text{ Nm } (\circlearrowleft)$$

Moment about D, $M_D = +500 \times 3\sin\theta = +500 \times 3 \times 0.8 = +1200$ Nm

$$M_D = 1200 \text{ Nm } (\circlearrowleft)$$

Method 2: Resolution of force into components along known distances



Moment about O,

$$\sum M_O = -500\sin\theta \times 3 + 500\cos\theta \times 0$$
$$\sum M_O = -500 \times 0.8 \times 3 + 0 = -1200 \text{ Nm}$$
$$\sum M_O = 1200 \text{ Nm } (\circlearrowright)$$

Moment about A,

$$\sum M_A = -500\sin\theta \times 3 + 500\cos\theta \times 2$$
$$\sum M_A = -500 \times 0.8 \times 3 + 500 \times 0.6 \times 2 = -600 \text{ Nm}$$
$$\sum M_A = 600 \text{ Nm } (\circlearrowright)$$

Moment about B,

$$\sum M_B = -500\sin\theta \times 3 + 500\cos\theta \times 4$$
$$\sum M_B = -500 \times 0.8 \times 3 + 500 \times 0.6 \times 4$$
$$\sum M_B = 0$$

Moment about C,

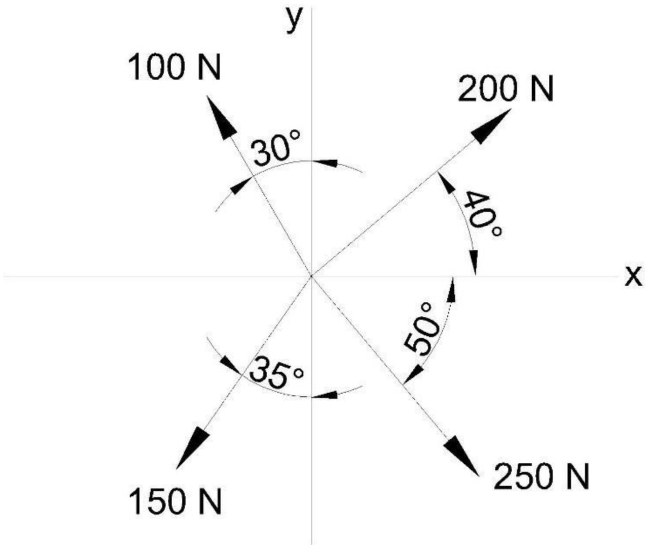
$$\sum M_C = -500\sin\theta \times 1.5 + 500\cos\theta \times 4$$
$$\sum M_C = -500 \times 0.8 \times 1.5 + 500 \times 0.6 \times 4 = +600 \text{ Nm}$$
$$\sum M_C = 600 \text{ Nm } (\circlearrowleft)$$

Moment about D,

$$\sum M_D = -500\sin\theta \times 0 + 500\cos\theta \times 4$$
$$\sum M_D = 0 + 500 \times 0.6 \times 4 = +1200 \text{ Nm}$$
$$\sum M_D = 1200 \text{ Nm } (\circlearrowleft)$$

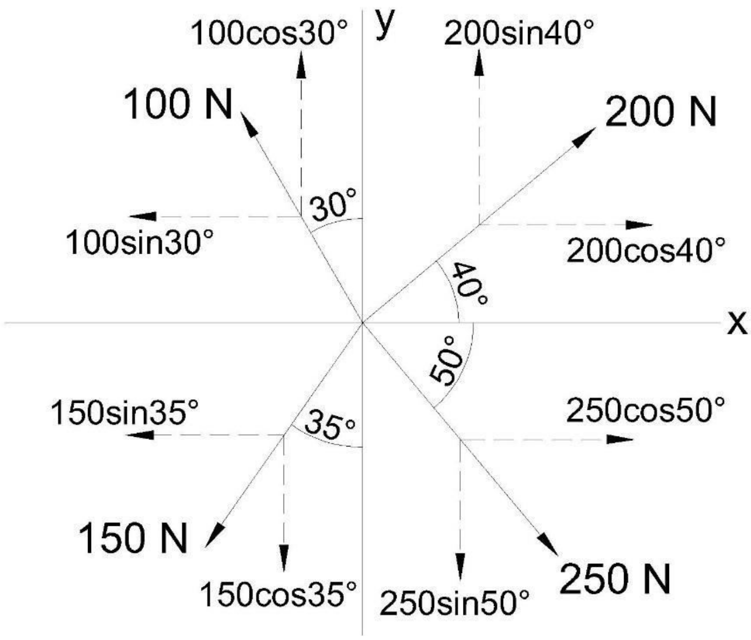
NOTE: Method 1 can become complicated as many perpendicular distances need to be calculated, hence Method 2 is recommended.

N3: Find the resultant of the following four forces as shown in the figure.



Soln: This is a concurrent force system.

Resolving all forces in their horizontal and vertical components:



$$\therefore \sum F_x = 200\cos 40° - 100\sin 30° - 150\sin 35° + 250\cos 50°$$
$$\Rightarrow \sum F_x = +177.869 \text{ N or } 177.869 \text{ N } (\rightarrow)$$

$$\therefore \sum F_y = 200\sin 40° + 100\cos 30° - 150\cos 35° - 250\sin 50°$$
$$\Rightarrow \sum F_y = -99.224 \text{ N or } 99.224 \text{ N } (\downarrow)$$

Hence, the magnitude of the resultant force,

$$\therefore R = \sqrt{(\sum F_x)^2 + \left(\sum F_y\right)^2}$$
$$\Rightarrow R = \sqrt{(177.869)^2 + (-99.224)^2}$$
$$\Rightarrow R = 203.673 \text{ N}$$

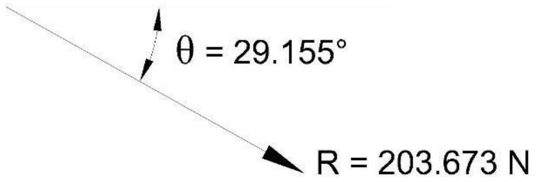
Also, the angle it makes with the x-axis,

$$\therefore \tan\theta = \frac{\sum F_y}{\sum F_x} = \frac{99.224}{177.869}$$
$$\theta = 29.155°$$

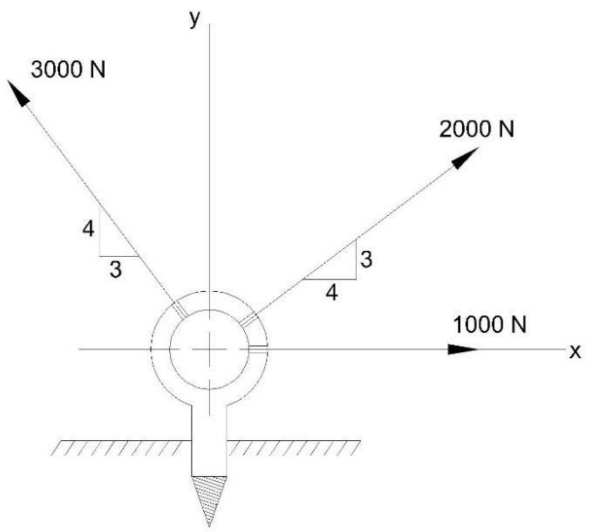
This gives the direction of the resultant.

Since, $\sum F_x$ is in → direction and $\sum F_y$ is in ↓ direction, the resultant R lies in the 4th quadrant (↘). Hence, the resultant of the given forces is given by,

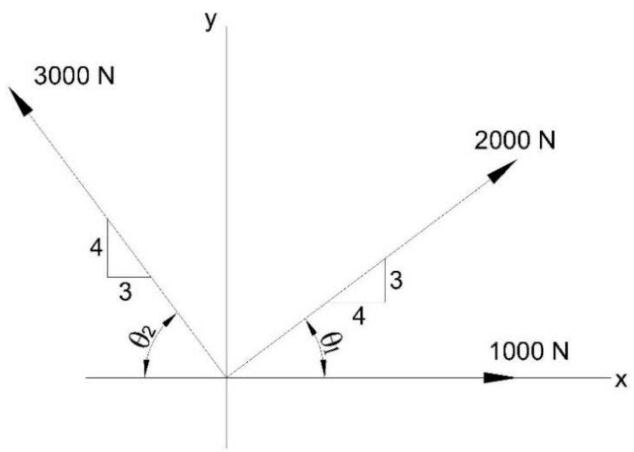
$$\boxed{\begin{gathered} R = 203.673\ \text{N} \\ \text{at } \theta = 29.155°\ (\searrow) \\ \text{acting at origin.} \end{gathered}}$$



N4: An eye bolt is being pulled from the ground by three forces as shown. Determine the resultant force acting on the eye bolt.



Soln: This is a concurrent force system, all forces acting at the centre of eye bolt.



$$\theta_1 = \tan^{-1}\frac{3}{4}\ \&\ \theta_2 = \tan^{-1}\frac{4}{3}$$
$$\therefore \sin\theta_1 = \frac{3}{5} = 0.6$$
$$\&\ \sin\theta_2 = \frac{4}{5} = 0.8$$
$$\therefore \cos\theta_1 = \frac{4}{5} = 0.8$$
$$\&\ \cos\theta_2 = \frac{3}{5} = 0.6$$

Resolving the forces in x & y directions and add accordingly, we get,

$$\therefore \sum F_x = 1000 + 2000\cos\theta_1 - 3000\cos\theta_2$$
$$\Rightarrow \sum F_x = 1000 + 2000 \times 0.8 - 3000 \times 0.6$$
$$\Rightarrow \sum F_x = +800\ \text{N or } 800\ \text{N}\ (\rightarrow)$$

$$\therefore \sum F_y = 2000\sin\theta_1 + 3000\sin\theta_2$$
$$\Rightarrow \sum F_y = 2000 \times 0.6 + 3000 \times 0.8$$
$$\Rightarrow \sum F_y = +3600\ \text{N or } 3600\ \text{N}\ (\uparrow)$$

Magnitude of the resultant,

$$R = \sqrt{(\sum F_x)^2 + (\sum F_y)^2}$$
$$R = \sqrt{(800)^2 + (3600)^2}$$
$$R = 3687.82\ \text{N}$$

Inclination of the resultant,

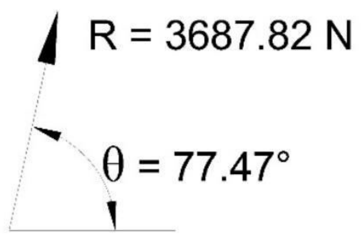
$$\theta = \tan^{-1}\frac{\sum F_y}{\sum F_x} = \tan^{-1}\frac{3600}{800}$$
$$\theta = 77.47°$$

Direction of the resultant,

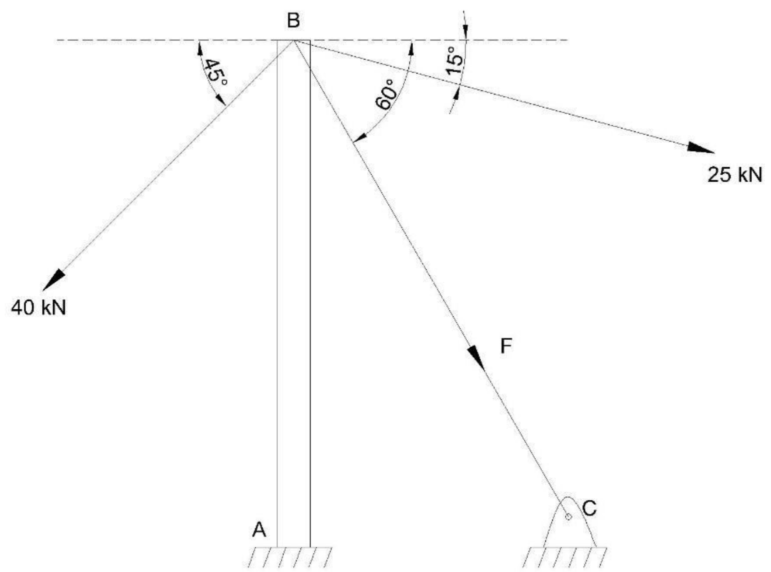
$\because \sum F_x$ is positive $(\rightarrow)$ and $\sum F_y$ is positive $(\uparrow)$

$\therefore$ R is in the 1st quadrant $(\nearrow)$

$$\boxed{\therefore\ R = 3687.82\ \text{N at } \theta = 77.47°}$$



N5: Determine the force F in the cable BC if the resultant of the 3 concurrent forces acting at B is vertical. Also determine the resultant.



This is a concurrent force system of 3 forces acting at B.

It is given that the resultant of the force system is vertical, it means that the algebraic sum of all the force components in the horizontal direction is zero.

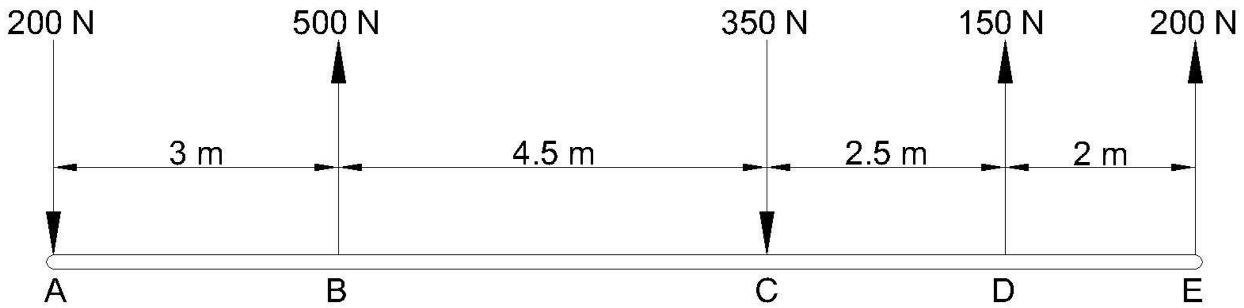
$$\therefore \sum F_x = 0$$
$$\Rightarrow 25\cos 15° - 40\cos 45° + F\cos 60° = 0$$
$$\boxed{\therefore F = 8.27\ \text{kN}}$$

Also, since resultant is vertical,

$$\therefore R = \sum F_y$$
$$\Rightarrow R = -25\sin 15° - 40\sin 45° - 8.27\sin 60°$$
$$\Rightarrow \therefore R = -41.92\ \text{kN}$$
$$\boxed{\therefore R = 41.92\ \text{kN}\ (\downarrow)}$$

N6: Figure shows four parallel forces acting on a beam ABCDE.

i) Determine the resultant of the system and its location from A.
ii) Replace the system by a single force and couple acting at a point B.
iii) Replace the system by a single force and couple acting at a point D.



Soln: This is a parallel system of forces; hence we can simply find the algebraic sum for the resultant.

i) Resultant and location from A

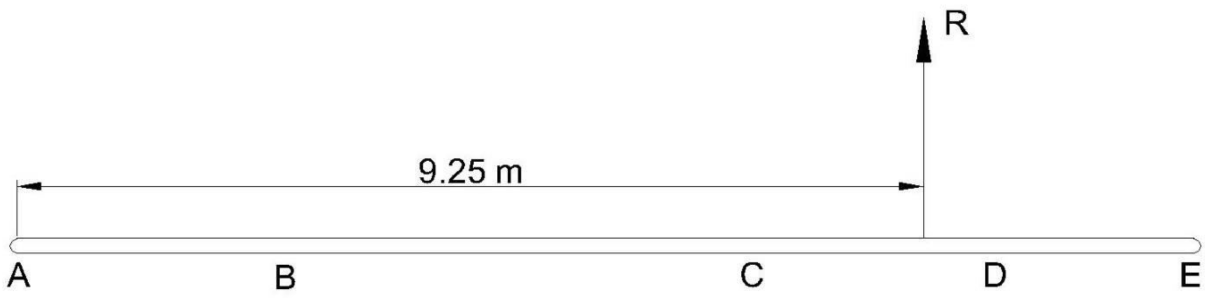
$$R = \sum F \ \uparrow +\text{ve}$$
$$R = -200 + 500 - 350 + 150 + 200 = 300\ \text{N}$$
$$R = 300\ \text{N}\ (\uparrow)$$

For the location of the resultant, let us assume that it is at a distance d to the right side of A. Using Varignon's theorem,

$$\sum M_A^F = M_A^R \ \circlearrowleft +\text{ve}$$
$$+500 \times 3 - 350 \times 7.5 + 150 \times 10 + 200 \times 12 = +300 \times d$$
$$\therefore d = 9.25\ \text{m}$$

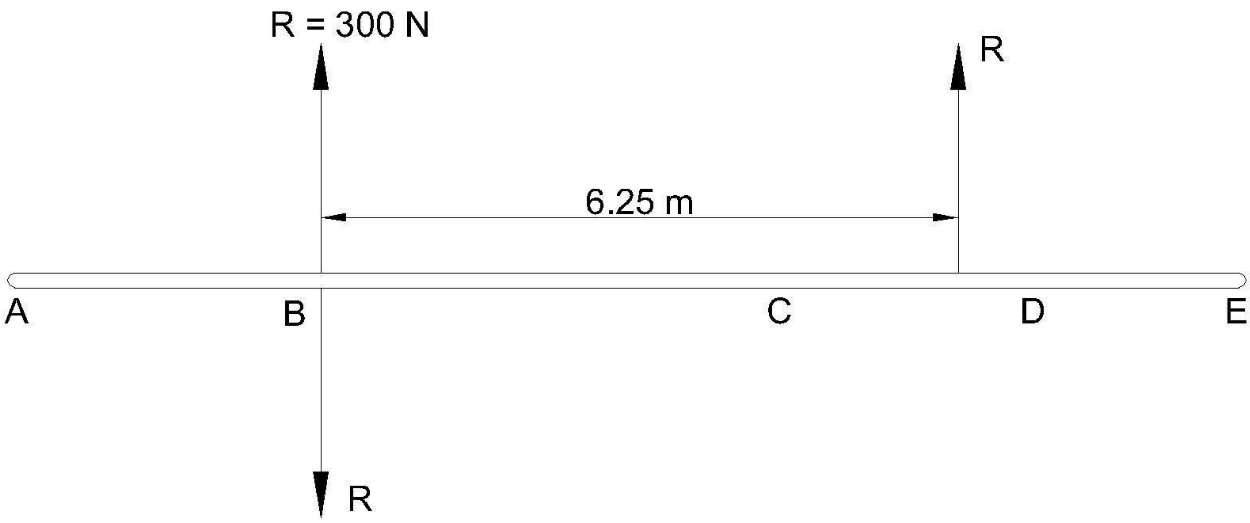
Since, d is +ve, our assumption of resultant being on the right side is correct.

R = 300 N (↑) acting at a distance of d = 9.25 m to the right of A.



ii) Single force and couple at B

<u>Method 1</u>: To shift a force to a new parallel position, a couple is required to be added to the system. Add two collinear forces of equal magnitude R & -R at B. The resultant R at B is as required and the remaining forces form a couple.



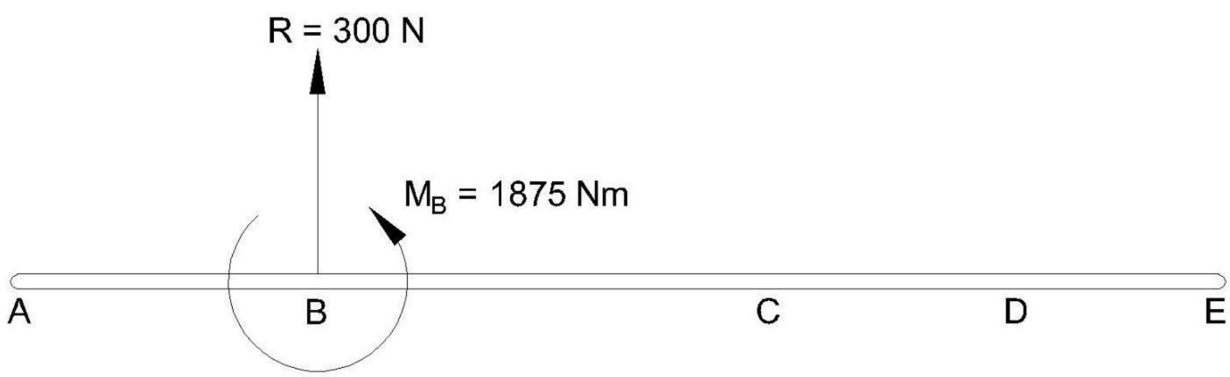
The single force acting at B is R = 300 N (↑)

And couple at B would be, $M_B = +R \times 6.25$ (↺ +ve)

$$M_B = +300 \times 6.25 = 1875 \text{ Nm}$$

$$\therefore M_B = 1875 \text{ Nm} ↺$$

Hence, the system can be replaced by a single force R = 300 N (↑) and a couple of 1875 Nm ↺ at B.



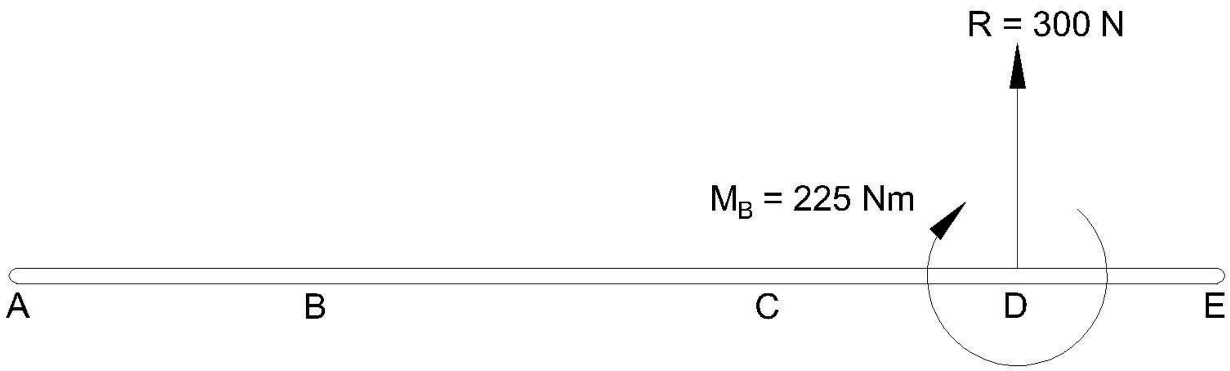
iii) Single force and couple at D

Method 2: The single force acting at D would be the same as $R = 300\text{ N } (\uparrow)$

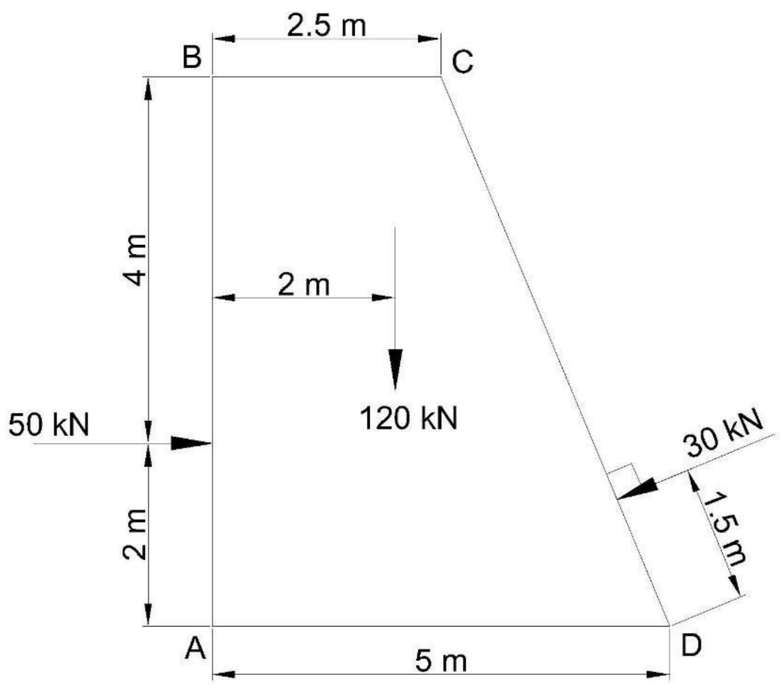
To find couple at D, take moments of all forces about D

$$\sum M_D^F \circlearrowleft +ve$$
$$\sum M_D^F = +200 \times 10 - 500 \times 7 + 350 \times 2.5 + 200 \times 2 = -225\text{ Nm}$$
$$\therefore \sum M_D^F = 225\text{ Nm} \circlearrowright$$

Hence, the system can be replaced by a single force $R = 300\text{ N } (\uparrow)$ and a couple of $225\text{ Nm} \circlearrowright$ at D.

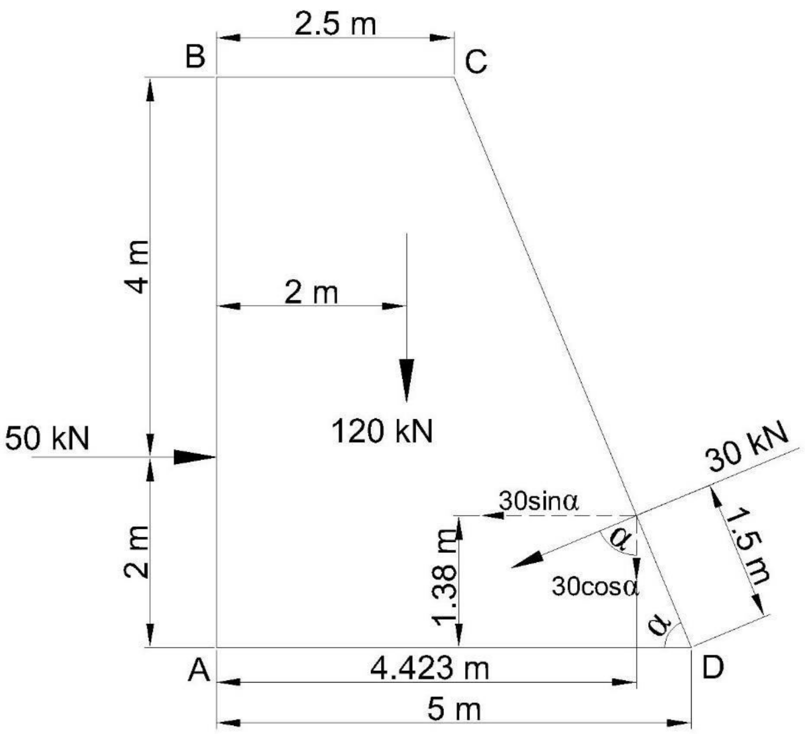


N7: A dam is subjected to three forces 50 kN on the upstream face AB, 30 kN force on the downstream inclined face and its own weight of 120 kN as shown. Determine the single force and locate its point of intersection with the base AD assuming all the forces to lie in a single plane.



Soln: This is a general force system of three coplanar forces acting on the dam.

Finding the angle of the inclined face and resolving the downstream force, we get,



$$\alpha = \tan^{-1}\frac{6}{2.5} = 67.4°$$

So, horizontal component of 30 kN is 30sinα, and vertical component is 30cosα.

Distance between 30sinα and horizontal base AD is

$$1.5\sin\alpha = 1.38 \text{ m}$$

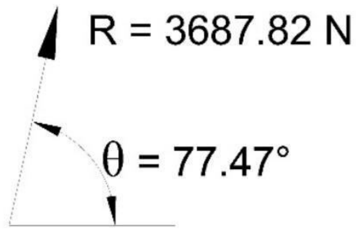
Distance between 30cosα and vertical face AB is

$$l(AD) - 1.5\cos\alpha$$
$$= 5 - 0.577 = 4.423\ m$$

Adding all the components of the forces in x & y directions, we get,

$$\therefore \sum F_x = +50 - 30\sin 67.4° = +22.3 \text{ kN}$$
$$\Rightarrow \sum F_x = 22.3 \text{ kN } (\rightarrow)$$

$$\therefore \sum F_y = -120 - 30\cos 67.4° = -131.5 \text{ kN}$$
$$\Rightarrow \sum F_y = 131.5 \text{ kN } (\downarrow)$$



Magnitude of the resultant,

$$R = \sqrt{(\sum F_x)^2 + \left(\sum F_y\right)^2}$$
$$R = \sqrt{(22.3)^2 + (131.5)^2}$$
$$R = 133.4 \text{ kN}$$

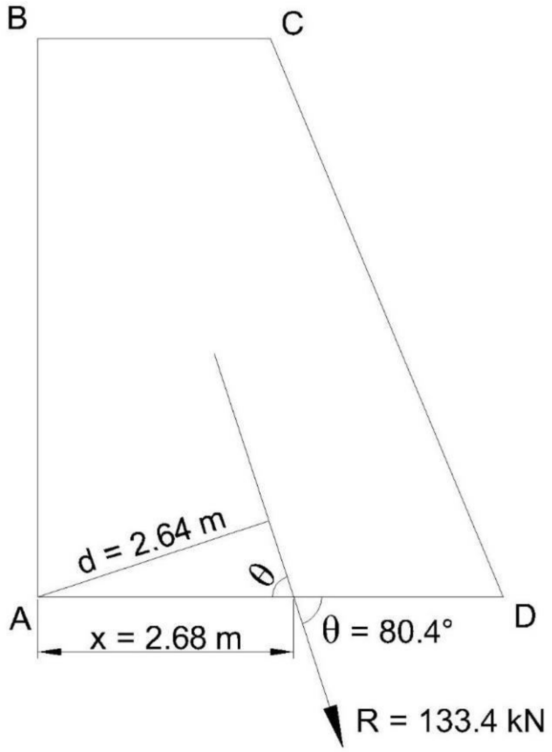
Direction of the resultant,

$$\theta = \tan^{-1}\frac{\sum F_y}{\sum F_x}$$
$$\theta = \tan^{-1}\frac{131.5}{22.3}$$
$$\theta = 80.4°$$

$\because \sum F_x$ is positive ($\rightarrow$) and $\sum F_y$ is negative ($\downarrow$), $\therefore$ R is in the 4th quadrant ($\searrow$).

Now for location of resultant, specifically, its point of intersection with base AD:

Let the resultant be at a perpendicular distance ‘d’ m to the right of A. Also, let it cut the base at a distance ‘x’ m from end A.



From Varignon’s theorem,

$$\sum M_A^F = M_A^R \ (\circlearrowleft +\text{ve})$$

$$-(50 \times 2) - (120 \times 2)$$

$$-(30 \cos 67.4° \times 4.423)$$

$$+(30 \sin 67.4° \times 1.38)$$

$$= -(133.4 \times d)$$

$$\therefore d = 2.64 \text{ m}$$

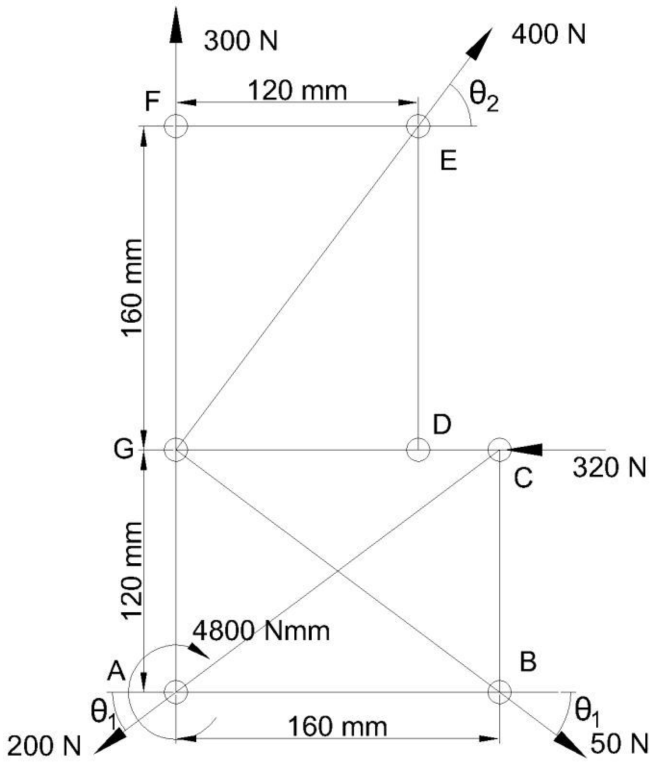
Also,

$$\sin 80.4° = \frac{d}{x} = \frac{2.64}{x}$$

$$\rightarrow x = 2.68 \text{ m}$$

Hence, the resultant is R = 133.4 kN at θ = 80.4° (↘) having its point of intersection with the base AD at x = 2.68 m from the right of A.

<u>N8</u>: Find the resultant of a coplanar forces system given in the figure below, located it on AB with due consideration to the applied moment.

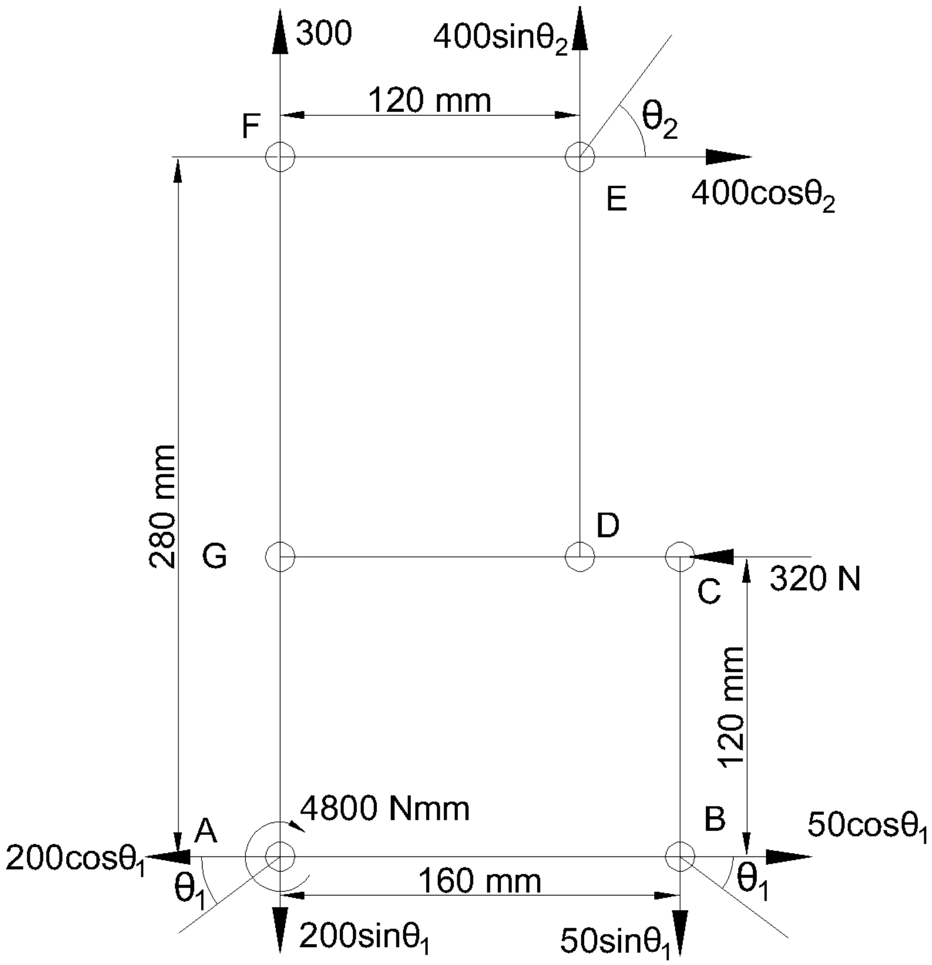


Soln: This is a general force system, since the lines of action of all forces don't meet at a single point.

Calculations of angles:

$$\theta_1 = \tan^{-1}\frac{BC}{BD} = \tan^{-1}\frac{120}{160} = \tan^{-1}\frac{3}{4}$$
$$\therefore \sin\theta_1 = 0.6 \;\&\; \cos\theta_1 = 0.8$$
$$\theta_2 = \tan^{-1}\frac{FG}{EF} = \tan^{-1}\frac{160}{120} = \tan^{-1}\frac{4}{3}$$
$$\therefore \sin\theta_2 = 0.8 \;\&\; \cos\theta_2 = 0.6$$

Resolving the various forces and adding their horizontal and vertical components,



$$\therefore \sum F_x = -200\cos\theta_1 + 50\cos\theta_1 - 320 + 400\cos\theta_2$$
$$\Rightarrow \sum F_x = -200 \times 0.8 + 50 \times 0.8 - 320 + 400 \times 0.6$$
$$\Rightarrow \sum F_x = -200 \text{ N}$$
$$\Rightarrow \sum F_x = 200 \text{ N } (\leftarrow)$$

$$\therefore \sum F_y = -200\sin\theta_1 - 50\sin\theta_1 + 400\sin\theta_2 + 300$$
$$\Rightarrow \sum F_y = -200 \times 0.6 - 50 \times 0.6 + 400 \times 0.8 + 300$$
$$\Rightarrow \sum F_y = +470 \text{ N}$$
$$\Rightarrow \sum F_y = 470 \text{ N } (\uparrow)$$

Magnitude of the resultant,

$$R = \sqrt{(\sum F_x)^2 + (\sum F_y)^2}$$
$$R = \sqrt{(200)^2 + (470)^2}$$
$$R = 510.78 \text{ N}$$

Direction of the resultant,

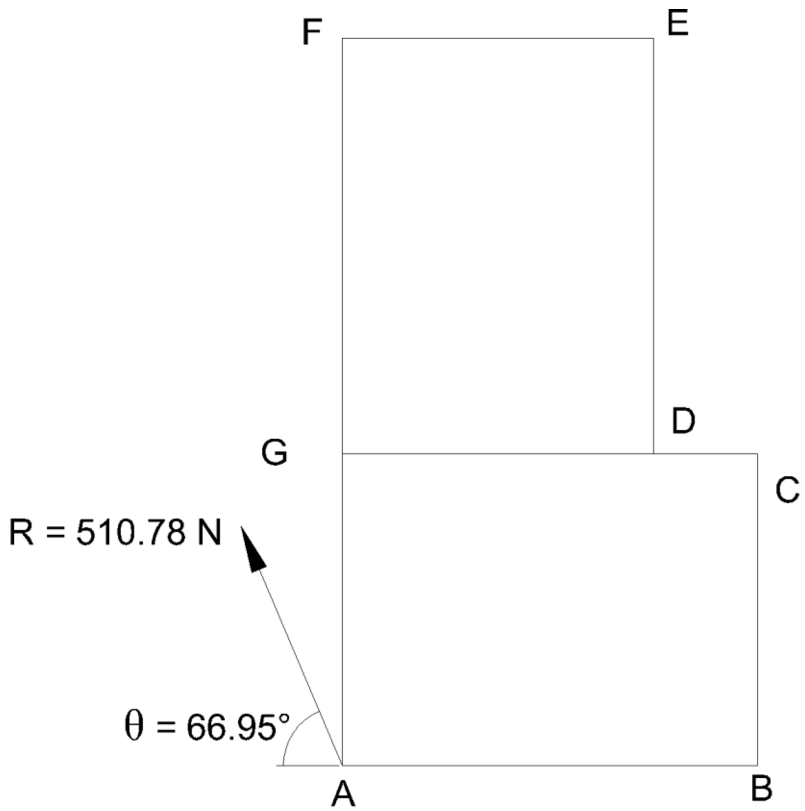
$$\theta = \tan^{-1}\frac{\sum F_y}{\sum F_x}$$
$$\theta = \tan^{-1}\frac{470}{200}$$
$$\theta = 66.95°$$

$\because \sum F_x$ is negative ($\leftarrow$) and $\sum F_y$ is positive ($\uparrow$), $\therefore$ R is in the 2nd quadrant ($\nwarrow$).

For the location of the resultant, let us assume that it is at a distance d to the right side of A. Using Varignon's theorem,

$$\sum M_A^F = M_A^R \quad \circlearrowleft +ve$$
$$-4800 - (50\sin\theta_1 \times 160) + 320 \times 120$$
$$-(400\cos\theta_2 \times 280) + (400\sin\theta_2 \times 120) = +510.78 \times d$$
$$0 = 510.78 \times d$$
$$\therefore d = 0$$

This implies that the resultant acts exactly at point A.



K J Somaiya College of Engineering, Vidyavihar, Mumbai

(A Constituent College of SVU)

Engineering Mechanics Notes

Module 1 – System of Forces

Module Section 1.2 – Forces in Space

Class: FY BTech Division: C3
Professor: Aniket S. Patil Date: 05/04/2023

References: Engineering Mechanics, by M. D. Dayal & Engineering Mechanics – Statics and Dynamics, by N. H. Dubey.

Vectors:

1. Basic Vector Operations:

$$\overline{P} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$$
$$\overline{Q} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$$

a) Dot Product

$$\overline{P} \cdot \overline{Q} = x_1x_2 + y_1y_2 + z_1z_2 \quad \text{or}$$
$$\overline{P} \cdot \overline{Q} = |\overline{P}||\overline{Q}| \cos\theta$$

b) Cross Product

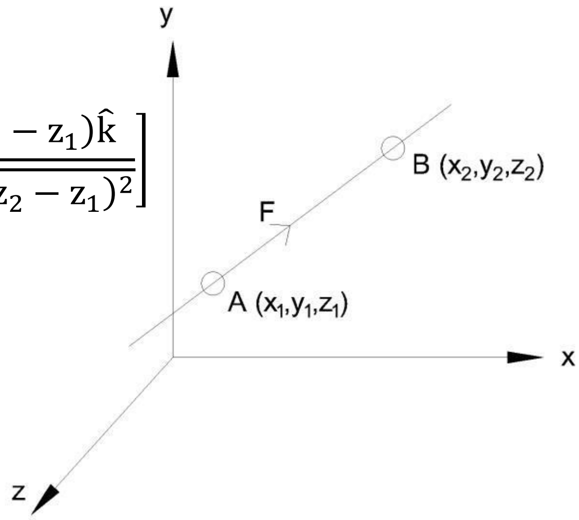
$$\overline{P} \times \overline{Q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} \quad \text{or}$$
$$\overline{P} \times \overline{Q} = |\overline{P}||\overline{Q}| \sin\theta\, \hat{n}$$

(where $\hat{n}$ is the unit vector normal to the plane of $\overline{P}$ & $\overline{Q}$)

2. Force Vector:

$$\overline{F} = (F)(\hat{e}_{AB})$$
$$\overline{F} = (F)\left[\frac{(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}\right]$$
$$\overline{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$$

(where $\hat{e}_{AB}$ is the unit vector in the direction of AB)



3. Magnitude of Force & Direction Angles:

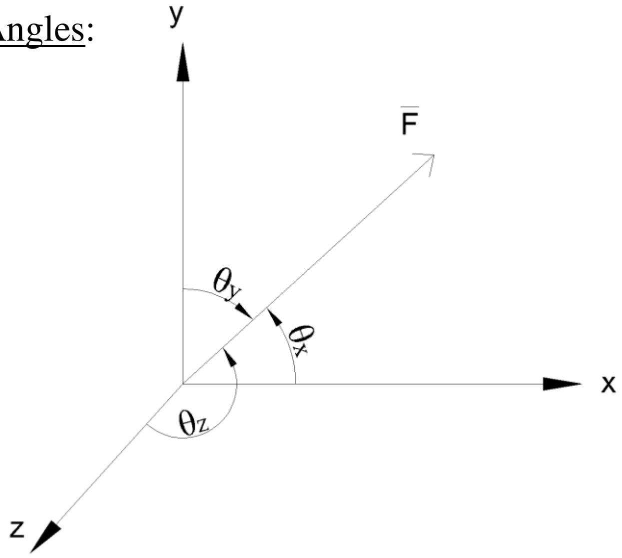
$$\bar{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$$

$$|\bar{F}| \text{ or } F = \sqrt{{F_x}^2 + {F_y}^2 + {F_z}^2}$$

$$F_x = F\cos\theta_x$$
$$F_y = F\cos\theta_y$$
$$F_z = F\cos\theta_z$$

By direction cosine rule,

$$\cos{\theta_x}^2 + \cos{\theta_y}^2 + \cos{\theta_z}^2 = 1$$



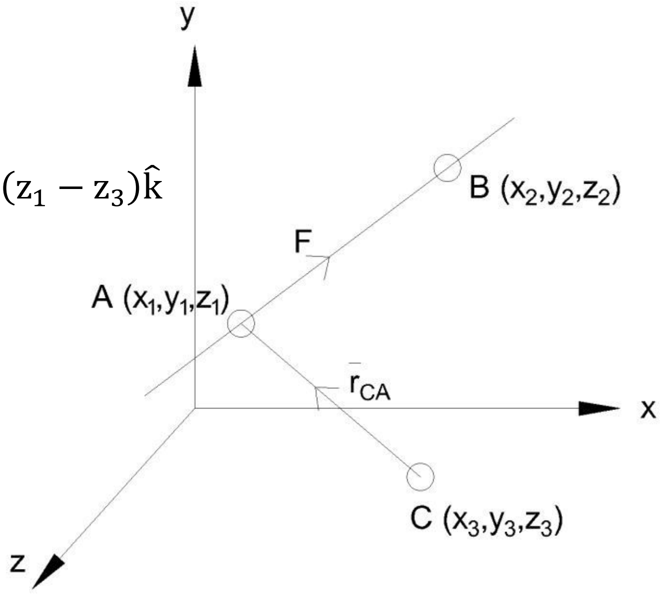
4. Moment Vector:

$$\bar{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$$

$$\bar{r}_{CA} = (x_1 - x_3)\hat{i} + (y_1 - y_3)\hat{j} + (z_1 - z_3)\hat{k}$$
$$\bar{r}_{CA} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\bar{M}_C = \bar{r}_{CA} \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

$$\bar{M}_C = M_x\hat{i} + M_y\hat{j} + M_z\hat{k}$$



5. <u>Vector Component of a Force along a given line</u>:

$$\bar{F} = (F)(\hat{e}_{AB})$$

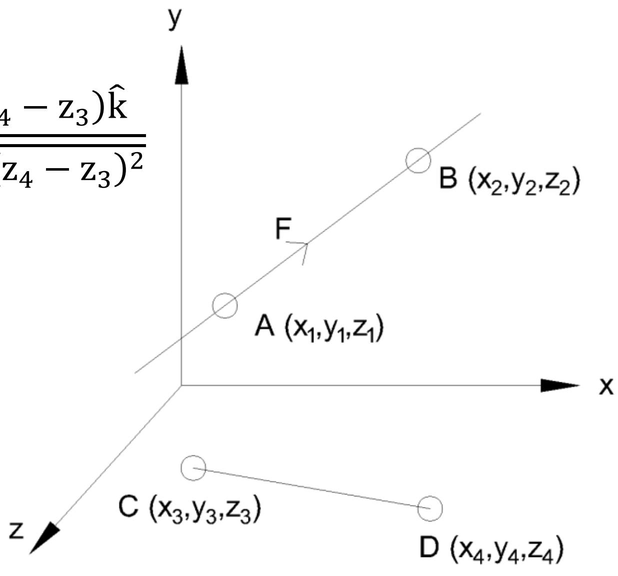
$$\hat{e}_{CD} = \frac{(x_4 - x_3)\hat{\imath} + (y_4 - y_3)\hat{\jmath} + (z_4 - z_3)\hat{k}}{\sqrt{(x_4 - x_3)^2 + (y_4 - y_3)^2 + (z_4 - z_3)^2}}$$

$$\hat{e}_{CD} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

$$F_{CD} = \bar{F} \cdot \hat{e}_{CD}$$

$$F_{CD} = F_x x + F_y y + F_z z$$

$$\bar{F}_{CD} = (F_{CD})(\hat{e}_{CD})$$



6. <u>Moment of a Force about a given line</u>:

$$\bar{M}_C = \bar{r}_{CA} \times \bar{F}$$

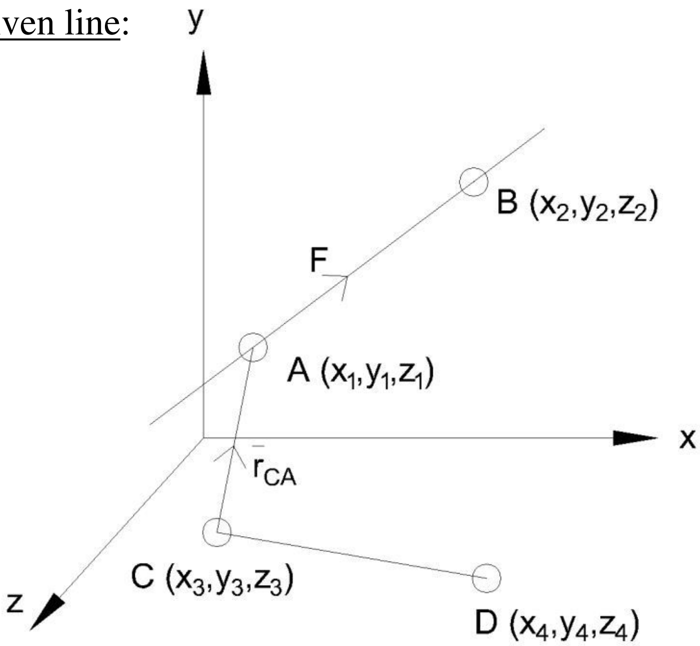
$$\bar{M}_C = M_x\hat{\imath} + M_y\hat{\jmath} + M_z\hat{k}$$

$$\hat{e}_{CD} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

$$M_{CD} = \bar{M}_C \cdot \hat{e}_{CD}$$

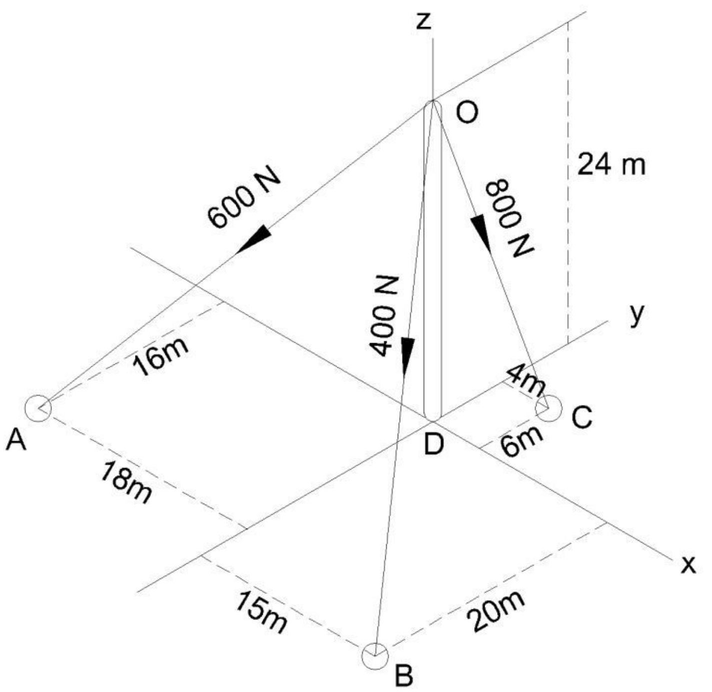
$$M_{CD} = M_x x + M_y y + M_z z$$

$$\bar{M}_{CD} = (M_{CD})(\hat{e}_{CD})$$



Numericals:

<u>N1</u>: A tower is being held in place by three cables. If the force of each cable acting on the tower is shown in figure, determine the resultant.

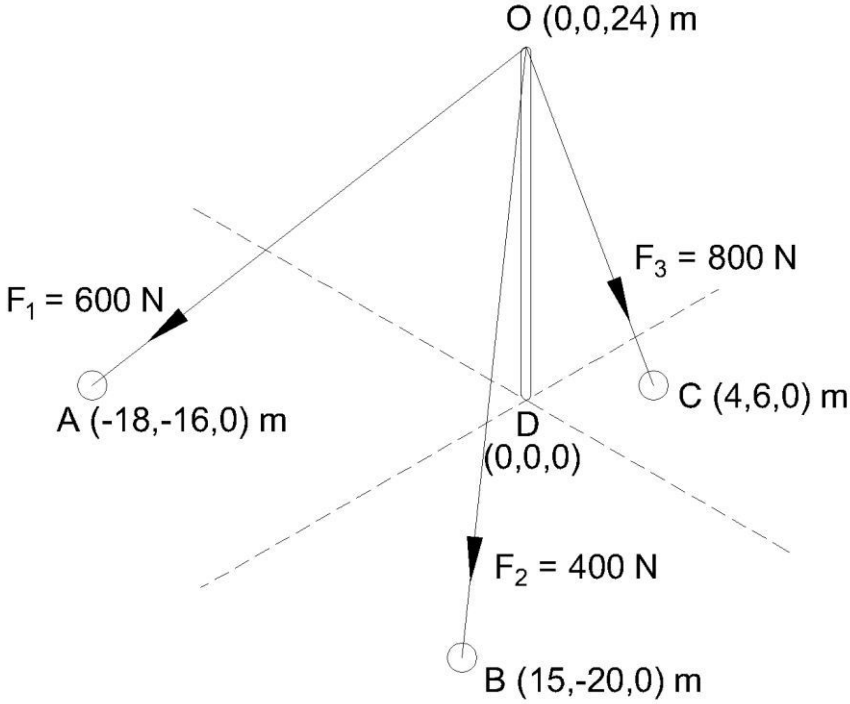


Soln: This is a concurrent space force system of 3 forces acting at O.

Let $\bar{F}_1$, $\bar{F}_2$, and $\bar{F}_3$ be the forces in the cables OA, OB and OC respectively.

$\therefore F_1 = 600\text{ N}, \qquad F_2 = 400\text{ N}, \qquad F_3 = 800\text{ N}$

And the co-ordinates of the points based on their distances from origin D are:
D (0,0,0) m
O (0,0,24) m
A (-18,-16,0) m
B (15,-20,0) m
C (4,6,0) m



$$\therefore \bar{F}_1 = (F_1)(\hat{e}_{OA}) = 600\left[\frac{(-18-0)\hat{\imath} + (-16-0)\hat{\jmath} + (0-24)\hat{k}}{\sqrt{(-18)^2 + (-16)^2 + (-24)^2}}\right]$$

$\bar{F}_1 = (-317.6\hat{\imath} - 282.4\hat{\jmath} - 423.5\hat{k})$ N

$$\therefore \bar{F}_2 = (F_2)(\hat{e}_{OB}) = 400\left[\frac{(15-0)\hat{\imath} + (-20-0)\hat{\jmath} + (0-24)\hat{k}}{\sqrt{(15)^2 + (-20)^2 + (-24)^2}}\right]$$

$\bar{F}_2 = (+173.1\hat{\imath} - 230.8\hat{\jmath} - 277\hat{k})$ N

$$\therefore \bar{F}_3 = (F_3)(\hat{e}_{OC}) = 800\left[\frac{(4-0)\hat{\imath} + (6-0)\hat{\jmath} + (0-24)\hat{k}}{\sqrt{(4)^2 + (6)^2 + (-24)^2}}\right]$$

$\bar{F}_3 = (+127.7\hat{\imath} + 191.5\hat{\jmath} - 766.2\hat{k})$ N

Resultant force in vector form is simply given by vector addition of the forces.

$$\begin{aligned}\therefore \bar{R} = \bar{F}_1 + \bar{F}_2 + \bar{F}_3 &= (-317.6\hat{\imath} - 282.4\hat{\jmath} - 423.5\hat{k}) \\ &+(+173.1\hat{\imath} - 230.8\hat{\jmath} - 277\hat{k}) \\ &+(+127.7\hat{\imath} + 191.5\hat{\jmath} - 766.2\hat{k}) \\ \mathbf{\bar{R}} &= (\mathbf{-16.8\hat{\imath} - 321.7\hat{\jmath} - 1466.7\hat{k}})\ \mathbf{N}\end{aligned}$$

N2: The lines of actions of three forces concurrent at origin O pass respectively through point A (-1,2,4), B (3,0,-3), C (2,-2,4). Force F_1 = 40 N passes through A, F_2 = 10 N passes through B, F_3 = 30 N passes through C. Find the magnitude and direction of their resultant.

Soln: In this concurrent space force system, putting the forces in vector form we get,

$$\bar{F}_1 = (F_1)(\hat{e}_{OA}) = 40\left[\frac{-1\hat{\imath} + 2\hat{\jmath} + 4\hat{k}}{\sqrt{1^2 + 2^2 + 4^2}}\right] = (-8.729\hat{\imath} + 17.457\hat{\jmath} + 34.915\hat{k})\ \text{N}$$

$$\bar{F}_2 = (F_2)(\hat{e}_{OB}) = 10\left[\frac{3\hat{\imath} + 0\hat{\jmath} - 3\hat{k}}{\sqrt{3^2 + 0^2 + 3^2}}\right] = (+7.071\hat{\imath} + 0\hat{\jmath} - 7.071\hat{k})\ \text{N}$$

$$\bar{F}_3 = (F_3)(\hat{e}_{OC}) = 30\left[\frac{2\hat{\imath} - 2\hat{\jmath} + 4\hat{k}}{\sqrt{2^2 + 2^2 + 4^2}}\right] = (+12.247\hat{\imath} - 12.247\hat{\jmath} + 24.495\hat{k})\ \text{N}$$

Resultant of these forces is,

$$\begin{aligned}\therefore \bar{R} &= (-8.729\hat{\imath} + 17.457\hat{\jmath} + 34.915\hat{k}) + (+7.071\hat{\imath} + 0\hat{\jmath} - 7.071\hat{k}) \\ &\quad + (+12.247\hat{\imath} - 12.247\hat{\jmath} + 24.495\hat{k})\end{aligned}$$

$\bar{R} = (+10.589\hat{\imath} + 5.27\hat{\jmath} + 52.339\hat{k})$ N

Magnitude of the resultant force,

$$\mathbf{R} = \sqrt{{R_x}^2 + {R_y}^2 + {R_z}^2} = \sqrt{10.589^2 + 5.27^2 + 52.339^2} = \mathbf{53.66\ N}$$

Direction of the resultant force is given by the angles θ_x, θ_y, and θ_z.

$R_x = R\cos\theta_x$

$\Rightarrow 10.589 = 53.66 \cos\theta_x$

$\Rightarrow \boldsymbol{\theta_x = 78.62°}$

$R_y = R\cos\theta_y$

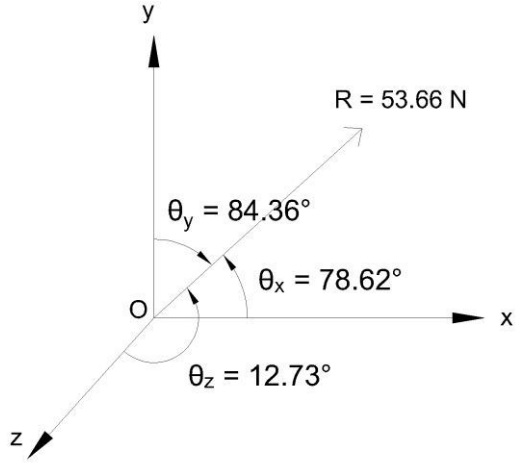
$\Rightarrow 5.27 = 53.66 \cos\theta_y$

$\Rightarrow \boldsymbol{\theta_y = 84.36°}$

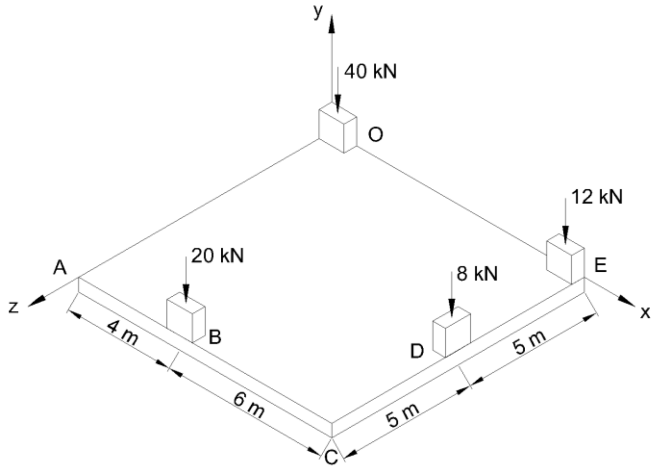
$R_z = R\cos\theta_z$

$\Rightarrow 52.339 = 53.66 \cos\theta_z$

$\Rightarrow \boldsymbol{\theta_z = 12.73°}$



N3: A square foundation mat supports the four columns as shown. Determine the magnitude and point of application of the resultant of the four loads.



Soln: This is a parallel space force system with 4 forces. The co-ordinates of the points through which the forces act are as follows,

O (0,0,0), B (4,0,10), D (10,0,5), E (10,0,0)

Let the forces 20 kN, 8 kN, 12 kN & 40 kN be F_1, F_2, F_3 & F_4 respectively.

All the forces are parallel to y axis and in the downward direction; hence all of them will have $-\hat{j}$ in their vector forms.

$\bar{F}_1 = -20\hat{j}$ kN; $\quad \bar{F}_2 = -80\hat{j}$ kN; $\quad \bar{F}_3 = -12\hat{j}$ kN; $\quad \bar{F}_4 = -40\hat{j}$ kN

Resultant, $\bar{R} = \bar{F}_1 + \bar{F}_2 + \bar{F}_3 + \bar{F}_4 = -20\hat{j} - 80\hat{j} - 12\hat{j} - 40\hat{j} = -80\hat{j}$ kN

For point of application, we first need to find out the moment of all forces about a point (let's take it from origin). So, the position vectors for each force will be,

$$\bar{r}_{OB} = (4-0)\hat{i} + (0-0)\hat{j} + (10-0)\hat{k} = \left(4\hat{i} + 10\hat{k}\right) \text{ m}$$

$$\bar{r}_{OD} = \left(10\hat{i} + 5\hat{k}\right) \text{ m}; \qquad \bar{r}_{OE} = \left(10\hat{i} + 0\hat{k}\right) = 10\hat{i} \text{ m}; \qquad \bar{r}_{OO} = 0 \text{ m}$$

Let resultant act at a point P (x,0,z) m. $\therefore \bar{r}_{OB} = \left(x\hat{i} + z\hat{k}\right)$ m

Now, the moment vectors of the forces about the origin,

$\overline{M}_O^{F_1} = \bar{r}_{OB} \times \bar{F}_1 = (4\hat{\imath} + 10\hat{k}) \times (-20\hat{\jmath}) = -80(\hat{\imath} \times \hat{\jmath}) - 200(\hat{k} \times \hat{\jmath})$
$\overline{M}_O^{F_1} = (200\hat{\imath} - 80\hat{k})\ \text{kNm} \qquad \{\because \hat{\imath} \times \hat{\jmath} = \hat{k}, \qquad \hat{k} \times \hat{\jmath} = -\hat{\imath}\}$

$\overline{M}_O^{F_2} = \bar{r}_{OD} \times \bar{F}_2 = (10\hat{\imath} + 5\hat{k}) \times (-8\hat{\jmath}) = (40\hat{\imath} - 80\hat{k})\ \text{kNm}$

$\overline{M}_O^{F_3} = \bar{r}_{OE} \times \bar{F}_3 = (10\hat{\imath}) \times (-12\hat{\jmath}) = (-120\hat{k})\ \text{kNm}$

$\overline{M}_O^{F_4} = 0\ \{\because \bar{F}_4$ passes through the origin$\}$

And the moment of resultant about the origin in terms of x and z,

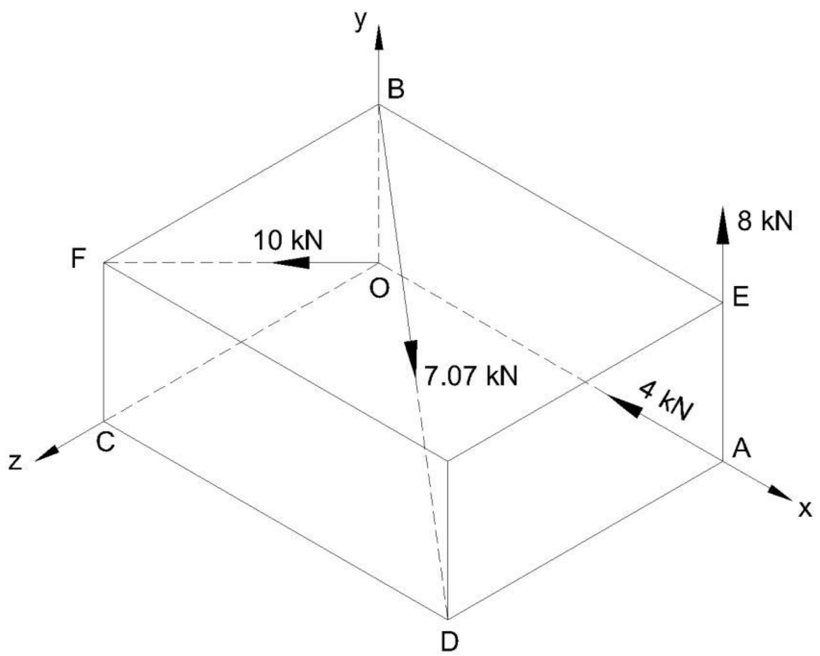
$\overline{M}_O^{R} = \bar{r}_{OP} \times \bar{R} = (x\hat{\imath} + z\hat{k}) \times (-40\hat{\jmath}) = [(80z)\hat{\imath} - (80x)\hat{k}]\ \text{kNm}$

From Varignon's theorem,

$\sum \overline{M}_O^{F} = \overline{M}_O^{R} \Rightarrow \overline{M}_O^{F_1} + \overline{M}_O^{F_2} + \overline{M}_O^{F_3} + \overline{M}_O^{F_4} = \overline{M}_O^{R}$
$\Rightarrow (200\hat{\imath} - 80\hat{k}) + (40\hat{\imath} - 80\hat{k}) + (-120\hat{k}) + 0 = (80z)\hat{\imath} - (80x)\hat{k}$
$\Rightarrow 240\hat{\imath} - 280\hat{k} = (80z)\hat{\imath} - (80x)\hat{k}$
$\Rightarrow 80z = 240\ \& \quad -80x = -280$
$\Rightarrow z = 3\ \text{m} \quad \& \quad x = 3.5\ \text{m}$

Hence, the magnitude of the resultant is $\mathbf{R = 80\ kN}$ and passes through point $\mathbf{P\ (3.5, 0, 3)\ m}$.

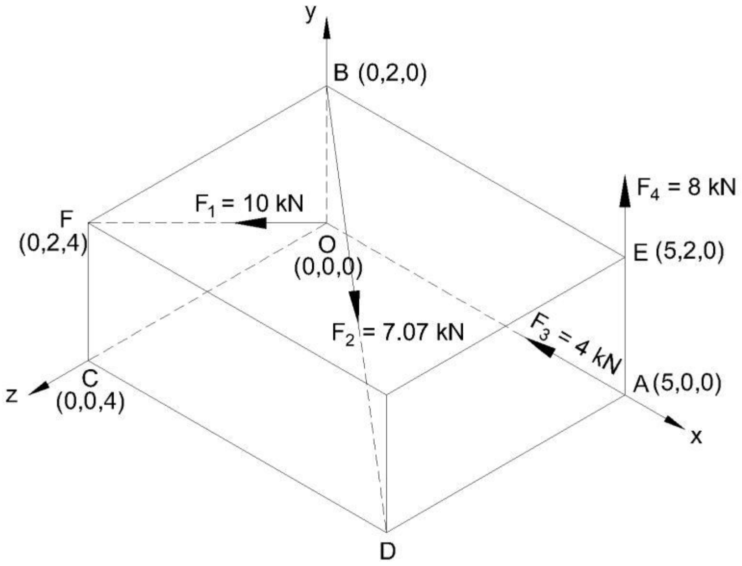
N4: A rectangle parallelepiped carries four forces as shown in the figure. Reduce the force system to a resultant force applied at the origin and moment around the origin. OA = 5 m, OB = 2 m, OC = 4 m.



Soln: The given system is a general space force system of 4 forces.

Let forces 10 kN, 7.07 kN, 4 kN & 8 kN be labelled as F_1, F_2, F_3 & F_4 respectively.

The co-ordinates of the various points through which the forces pass are:



Now, putting the forces in vector form,

$$\bar{F}_1 = (F_1)(\hat{e}_{OF}) = 10\left[\frac{0\hat{i} + 2\hat{j} + 4\hat{k}}{\sqrt{0^2 + 2^2 + 4^2}}\right] = \left(0\hat{i} + 4.472\hat{j} + 8.944\hat{k}\right)\,\text{kN}$$

$$\bar{F}_2 = (F_2)(\hat{e}_{BD}) = 7.07\left[\frac{5\hat{i} - 2\hat{j} + 4\hat{k}}{\sqrt{5^2 + 2^2 + 4^2}}\right] = \left(5.27\hat{i} - 2.108\hat{j} + 4.216\hat{k}\right)\,\text{kN}$$

$$\bar{F}_3 = (F_3)(\hat{e}_{AO}) = 4\,(-\hat{i}) = (-4\hat{i})\,\text{kN}\ \{\because \text{it is along x} - \text{axes towards origin}\}$$

$$\bar{F}_4 = (F_4)(\hat{e}_{AE}) = 8\,(\hat{j}) = (8\hat{j})\,\text{kN}\ \{\because \text{it is along y} - \text{axes upwards}\}$$

The resultant force, $\bar{R} = \bar{F}_1 + \bar{F}_2 + \bar{F}_3 + \bar{F}_4$

$$\bar{R} = \left(0\hat{i} + 4.472\hat{j} + 8.944\hat{k}\right) + \left(5.27\hat{i} - 2.108\hat{j} + 4.216\hat{k}\right) + (-4\hat{i}) + (8\hat{j})$$

$$\mathbf{\bar{R} = \left(1.27\hat{i} + 10.364\hat{j} + 13.16\hat{k}\right)\,kN}$$

Taking moments of all force about the origin,

$$\bar{M}_O^{F_1} = 0\ \ \{\because \bar{F}_1 \text{ passes through the origin}\}$$

$$\bar{M}_O^{F_2} = \bar{r}_{OB} \times \bar{F}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & 0 \\ 5.27 & -2.108 & 4.216 \end{vmatrix} = \left(8.432\hat{i} + 0\hat{j} - 10.54\hat{k}\right)\,\text{kNm}$$

$$\bar{M}_O^{F_3} = \bar{r}_{OA} \times \bar{F}_3 = 0\ \{\because \bar{r}_{OA}\,\&\,\bar{F}_3 \text{ are along the same directions}\}$$

$$\bar{M}_O^{F_4} = \bar{r}_{OA} \times \bar{F}_4 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 0 & 0 \\ 0 & 8 & 0 \end{vmatrix} = \left(40\hat{k}\right)\,\text{kNm}$$

The resultant moment about the origin, $\overline{M}_O = \overline{M}_O^{F_1} + \overline{M}_O^{F_2} + \overline{M}_O^{F_3} + \overline{M}_O^{F_4}$

$\overline{M}_O = 0 + (8.432\hat{i} + 0\hat{j} - 10.54\hat{k}) + 0 + (40\hat{k})$

$\mathbf{\overline{M}_O = (8.432\hat{i} + 29.46\hat{k}) \ kNm}$

Hence, the resultant force and moment at origin is,

$\mathbf{\overline{R} = (1.27\hat{i} + 10.364\hat{j} + 13.16\hat{k}) \ kN}$

$\mathbf{\overline{M}_O = (8.432\hat{i} + 29.46\hat{k}) \ kNm}$

K J Somaiya College of Engineering, Vidyavihar, Mumbai

(A Constituent College of SVU)

Engineering Mechanics Notes

Module 2 – Kinematics of Particles & Rigid Bodies

Module Section 2.1 – Kinematics of Particles

Class: FY BTech Division: C3
Professor: Aniket S. Patil Date: 06/04/2023

References: Engineering Mechanics, by M. D. Dayal & Engineering Mechanics – Statics and Dynamics, by N. H. Dubey.

Kinematics: It is concerned only with the study of motion of the body without consideration of the forces causing the motion.

Particle: In particle dynamics, we idealise the body being analysed as a particle. It does not mean that we are dealing with a very small object. But, it means that the size and shape of the body is not important for the analysis of the motion. For example, if a ship is traveling between two ports kilometres apart, then for a simple motion analysis the shape and size of the ship is not relevant in the calculations.

Whenever a body is treated as a particle, all the forces acting in the body are to be assumed to be concurrent at the mass centre of the body. Any rotation of the body is also neglected.

Reference Frame: For any motion analysis, we need to take a reference frame i.e. an origin with a set of co-ordinate axes, for the measurement of motion parameters. The reference frame could be fixed or moving. Newtonian frame of reference also known as inertial reference frame is a set of co-ordinates axes fixed or moving with uniform velocity. Newton's laws of motion are valid for these.

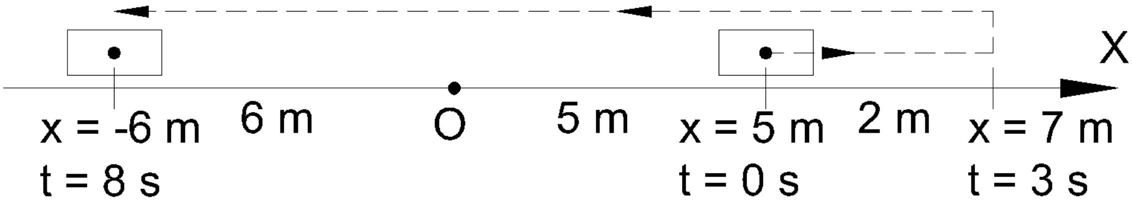
Rectilinear Motion: Motion of a particle in a straight line is known as a rectilinear motion. E.g., car moving on a straight highway, lift traveling in a vertical well, etc.

Position (x), Displacement (s), & Distance (d):

Position means the location of a particle with respect to a fixed reference point, usually called origin O. The position is taken as positive on one side of the origin and negative on the other. It is labelled by 'x' in S.I. unit of metre (m).

Displacement is a change in position of the particle. It is a vector quantity. It is a straight line vector connecting the initial position to the final position and has no relation with the actual distance travelled. It is labelled by 's' in S.I. unit of metre (m).

Distance is the actual length of the total path traced by the particle during the period of motion. It is a scalar quantity. It is labelled by 'd' in S.I. unit of metre (m).



E.g., let a particle at t = 0 s, occupy a position x = 5 m. Then it moves 2 m in the +ve X-direction and occupies position x = 7 m at t = 3 s. Then it reverses itself in the –ve direction, and occupies position x = 6 m at t = 8 s.

So here, displacement, $s = \Delta x = x_8 - x_0 = (-6) - (5) = -11 \text{ m or } 11 \text{ m} \leftarrow$

And the distance, $d = |x_3 - x_0| + |x_8 - x_3| = |7 - 5| + |-6 - 7| = 15 \text{ m}$

<u>Velocity</u> & <u>Speed</u>:

Velocity is the rate of change of displacement with respect to time. It is a vector quantity. It is labelled by 'v' in S.I. unit of metre/second (m/s).

$$\text{Average velocity}, v_{av} = \frac{\Delta x}{\Delta t}$$

$$\text{Instantaneous velocity}, v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

Speed is the rate of change of distance with respect to time. It is a scalar quantity. *The magnitude of velocity is also known as speed.*

$$\text{Average speed} = \frac{\text{Distance travelled}}{\text{Time interval}}$$

<u>Acceleration</u>:

Acceleration is the rate of change of velocity with respect to time. It is labelled by 'a' in S.I. unit of metre/second^2 (m/s^2).

$$\text{Average acceleration}, a_{av} = \frac{\Delta v}{\Delta t}$$

$$\text{Instantaneous acceleration}, a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

Positive acceleration is simply called acceleration and negative acceleration is called retardation or deceleration. Positive acceleration means magnitude of velocity increase with time and particle is moving in the positive direction. Negative acceleration means the particle moves slowly in the positive direction or moves faster in the negative direction.

Types of Rectilinear Motions:

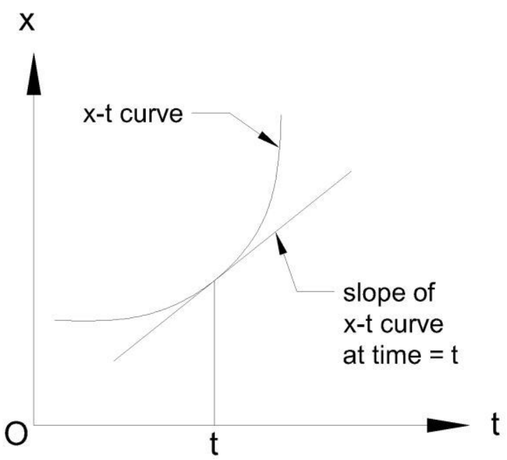
1. <u>Uniform Velocity Motion</u>
   If a particle's velocity remains constant throughout the motion, then it is said to be under motion with uniform velocity. E.g., motion of sound, package on a conveyor, etc.
   $$v = \frac{s}{t}$$

2. <u>Uniform Acceleration Motion</u>
   If a particle's velocity changes at a constant rate throughout the motion, then it is said to be under motion with uniform acceleration. This means that its acceleration is constant. This gives us the equations of kinematics.
   $$v = u + at$$
   $$s = ut + \frac{1}{2}at^2$$
   $$v^2 = u^2 + 2as$$
   Motion under gravity is a special case of this, with $a = g = 9.81\ m/s \downarrow$.

3. <u>Variable Acceleration Motion</u>
   If a particle's acceleration itself is changing throughout the motion, then it is said to be under the motion with variable acceleration. This motion is usually defined by acceleration written as a function of time or velocity or position. For the solution, we use calculus on the instantaneous relations between position, velocity, acceleration and time.
   $$v = \frac{dx}{dt} \ \& \ a = \frac{dv}{dt}$$
   $$\therefore \frac{a}{v} = \frac{dv/dt}{dx/dt} = \frac{dv}{dx} \rightarrow a = v\frac{dv}{dx}$$

Motion Curves: The motion of a particle along a straight line can be represented by motion curves. They are the graphical representation of position, displacement, velocity and acceleration with time.

1. <u>Position-Time (x-t) curve</u>
   Since,
   $$v = \frac{dx}{dt}$$
   at any instant of time,
   the slope of x-t curve gives
   the velocity of the particle
   at that instant.
   $$\therefore \boxed{v = [\text{slope x} - \text{t curve}]_{\text{at time = t}}}$$



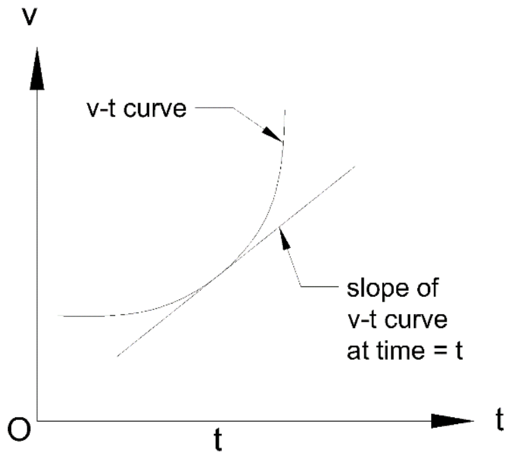
2. Velocity-Time (v-t) curve

Since,

$$a = \frac{dv}{dt}$$

at any instant of time,
the slope of v-t curve gives
the velocity of the particle
at that instant.

$\therefore$ $\boxed{a = [\text{slope } v - t \text{ curve}]_{\text{at time} = t}}$



Now,

$$v = \frac{dx}{dt} \rightarrow dx = vdt$$

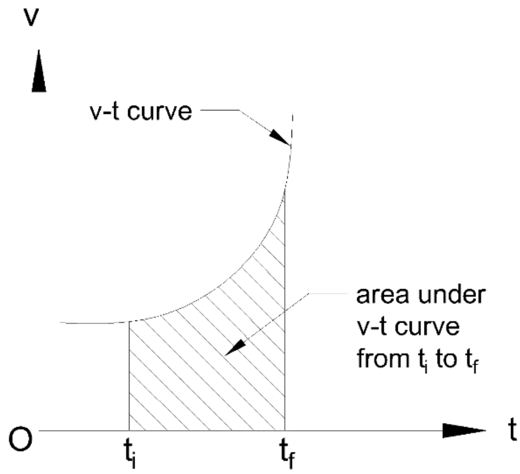
Integrating, $\int dx = \int vdt$

$\int vdt$ represents the area under
the v-t curve from t_i to t_f

$$\int_{x_i}^{x_f} dx = [AUC\ v - t]_{\text{from } t_i \text{ to } t_f}$$

$$x_f - x_i = [AUC\ v - t]_{t_i - t_f}$$

$\therefore$ $\boxed{x_f = x_i + [AUC\ v - t]_{t_i - t_f}}$



3. Acceleration-Time (a-t) curve

Since,

$$a = \frac{dv}{dt} \rightarrow dv = adt$$

$$\rightarrow \int dv = \int adt$$

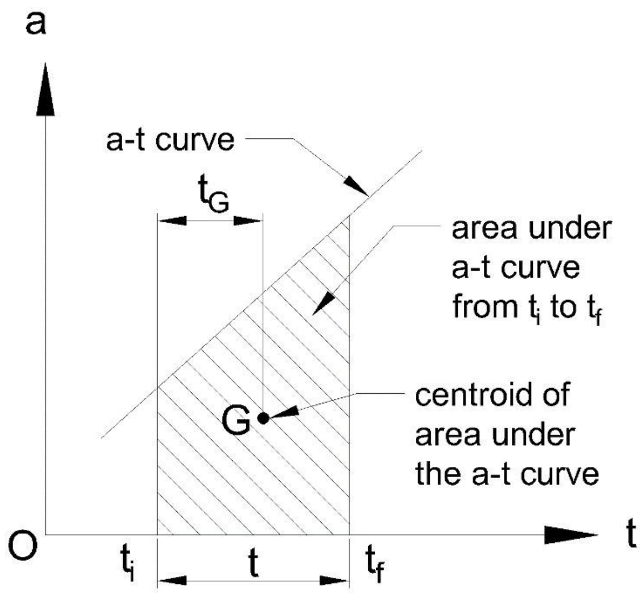
$\int vdt$ represents the area under
the a-t curve from t_i to t_f

$$\int_{v_i}^{v_f} dx = [AUC\ a - t]_{\text{from } t_i \text{ to } t_f}$$

$$v_f - v_i = [AUC\ a - t]_{t_i - t_f}$$

$\therefore$ $\boxed{v_f = v_i + [AUC\ a - t]_{t_i - t_f}}$

From a-t curve particle's position
can also be known at an instant,
using area moment method.

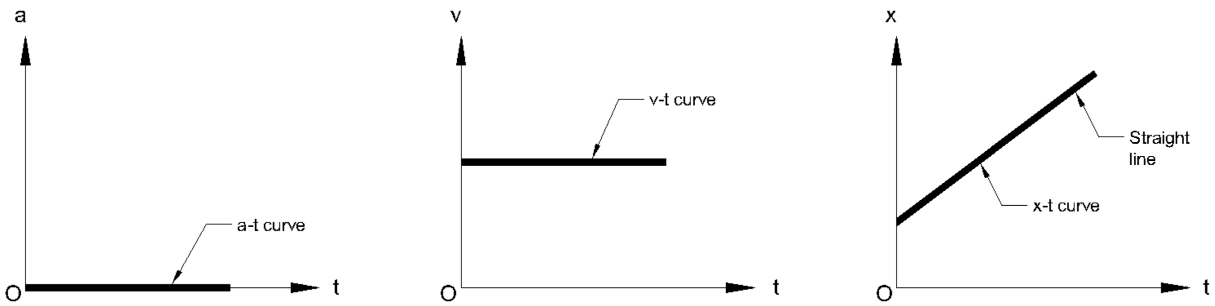


Here, $t = t_f - t_i$ and t_G is time from t_i to the centroid of AUC a-t.

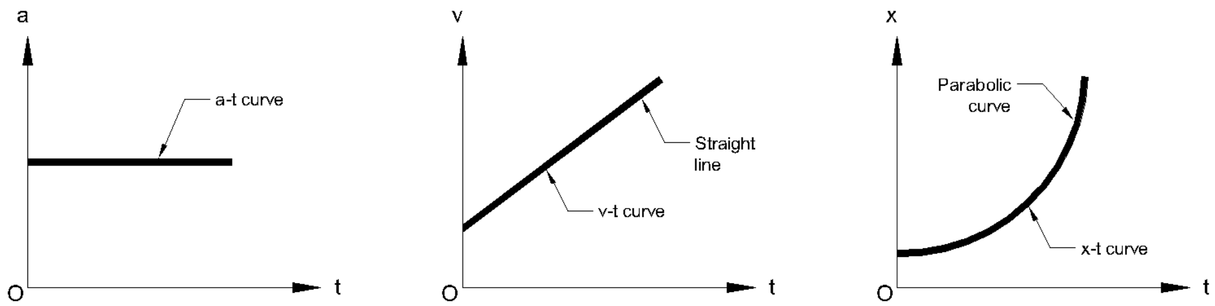
$$\boxed{x_f = x_i + v_i \times t + [AUC\ a - t]_{t_i - t_f} \times (t - t_G)}$$

Standard Motion Curves:

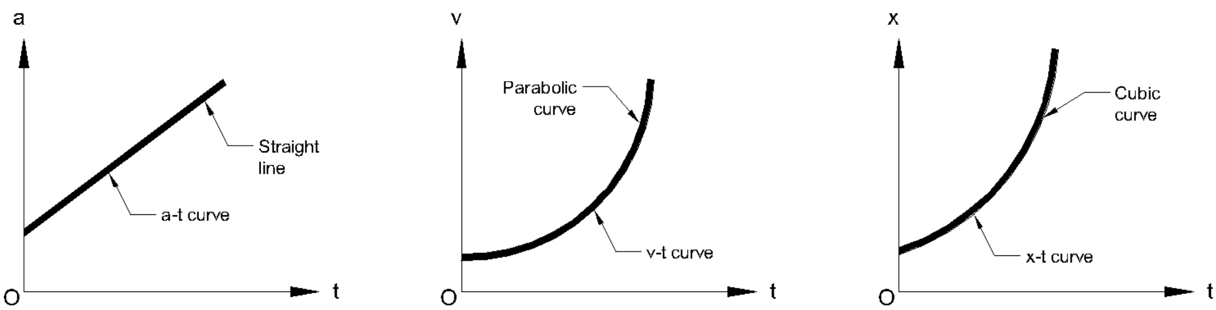
1. Uniform Velocity Motion Curves



2. Uniform Acceleration Motion Curves



3. Variable Acceleration (Linear Variation) Motion Curves



Curvilinear Motion: A particle which travels on a curved path is said to be performing curvilinear motion.

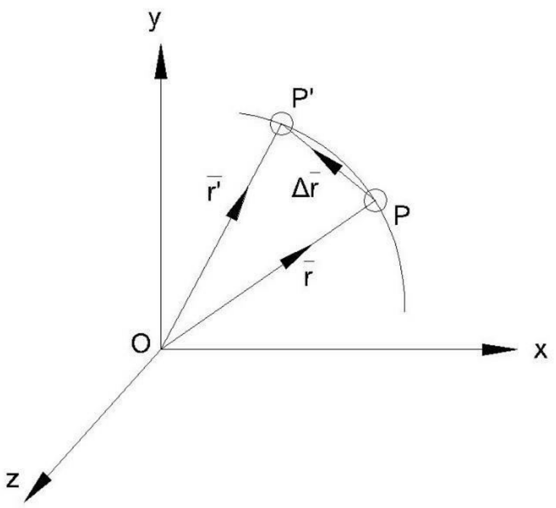
Position: It is represented by a position vector $\bar{r}$ with a starting point from the origin of the reference axis till the particle P. As the particle travels along the curved path, the value of $\bar{r}$ keeps changing.

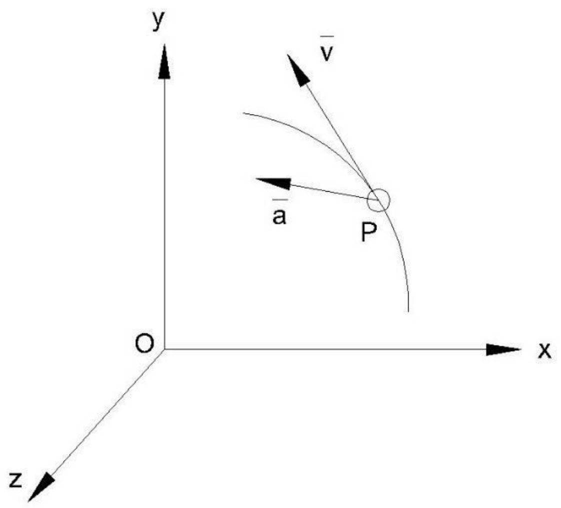
Velocity: Suppose a particle changes position from P to P', i.e., the position vectors from $\bar{r}$ to $\bar{r}'$, in a time interval of Δt.

$$\text{Average velocity}, v_{av} = \frac{\Delta \bar{r}}{\Delta t}$$

$$\text{Instantaneous velocity}, \bar{v} = \lim_{\Delta t \to 0} \frac{\Delta \bar{r}}{\Delta t} = \frac{d\bar{r}}{dt}$$

In curvilinear motion, instantaneous velocity of a particle is always tangent to the curved path at that instant.





Acceleration: As the direction of velocity continuously changes in a curvilinear motion, there exists acceleration at every instant of the motion.

$$\text{Average acceleration}, a_{av} = \frac{\Delta \bar{v}}{\Delta t}$$

$$\text{Instantaneous acceleration}, \bar{a} = \lim_{\Delta t \to 0} \frac{\Delta \bar{v}}{\Delta t} = \frac{d\bar{v}}{dt}$$

NOTE: In rectilinear motion, x, v, a are always along the path of the particle, whereas in curvilinear motion, $\bar{r}$, $\bar{v}$, $\bar{a}$ are always changing directions. Hence, we need to consider different component systems for its analysis.

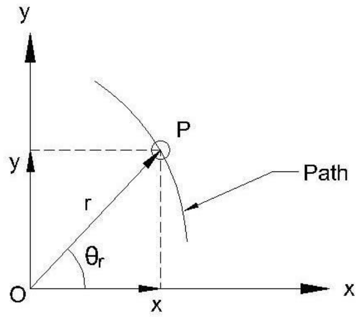
Curvilinear Motion by Rectangular Component System: Curvilinear motion can be split into motion along x, y, z directions, which can be independently considered as three rectilinear motions along those directions respectively.

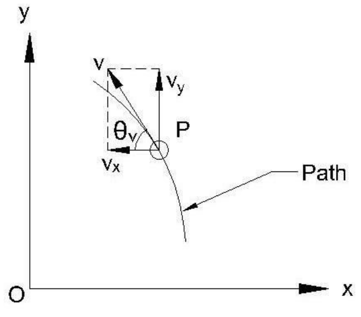
$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k} \qquad \& \quad r = \sqrt{x^2 + y^2 + z^2}$$

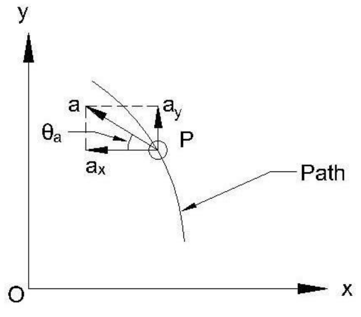
$$\bar{v} = \frac{d\bar{r}}{dt} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k} \quad \& \quad v = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

$$\bar{a} = \frac{d\bar{v}}{dt} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k} \quad \& \quad a = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

For a particle in xy plane, the rectangular components will be as shown below:







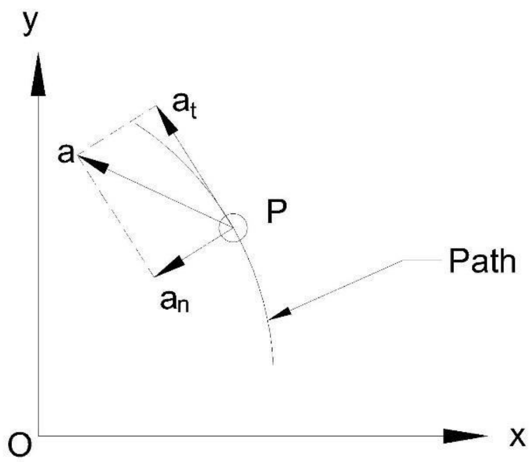
$$\tan\theta_r = \frac{y}{x}; \qquad \tan\theta_v = \frac{v_y}{v_x}; \qquad \tan\theta_a = \frac{a_y}{a_x}$$

Curvilinear Motion by Tangential & Normal Component System (N-T System): Curvilinear motion can also be studied by splitting the acceleration along the tangent to the path and normal to the path.

The velocity vector is always directed towards the tangential direction. But the net acceleration may be in any direction. So, it is convenient to express acceleration as *tangential acceleration* (a_t) and *normal acceleration* (a_n).

$$\bar{a} = a_n\hat{e}_n + a_t\hat{e}_t$$

$$a = \sqrt{a_n^2 + a_t^2}$$



a_n represents the change in direction and is always directed towards the centre of curvature.

$$a_n = \frac{v^2}{\rho}$$

Where ρ is the radius of curvature and v is the velocity at a particular instant.

a_t represents the change in velocity and is given by, $a_t = \frac{dv}{dt}$

For curves which are defined as y = f(x),

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

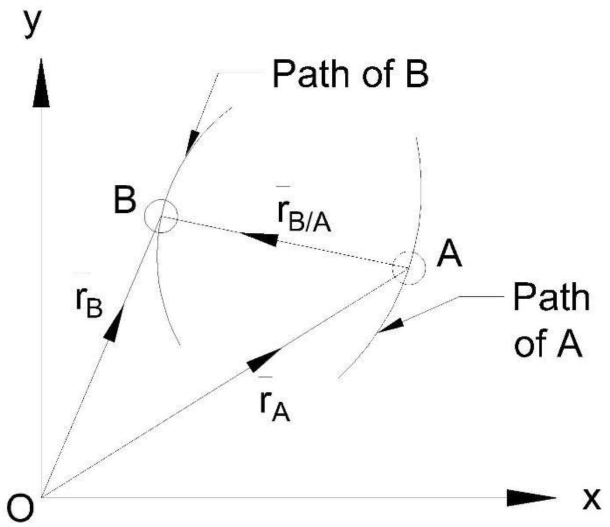
And, in terms of rectangular components,

$$\rho = \left|\frac{v^3}{a_x v_y - a_y v_x}\right|$$

Relative Motion: If motion analysis is done, not from a fixed reference frame, but from a moving reference, then such analysis comes under relative motion. E.g., person in a moving vehicle observing another moving vehicle, pilot of a fighter jet observing a moving target, etc.

In the figure, two particles move independent of each other, their position vectors measured from a fixed frame of reference xoy.

If now A observes B, then A will find B to be occupying the position $\bar{r}_{B/A}$. This is measured from the moving reference located at A.



Relative relations of B w.r.t A are:

$$\bar{r}_{B/A} = \bar{r}_B - \bar{r}_A$$

$$\bar{v}_{B/A} = \bar{v}_B - \bar{v}_A$$

$$\bar{a}_{B/A} = \bar{a}_B - \bar{a}_A$$

Numericals:

<u>Part I – Rectilinear Motion</u>:

<u>N1</u>: The velocity of a particle travelling in a straight line is given by the equation $v = (6t - 3t^2)$ m/s, where t is in seconds. If $s = 0$ when $t = 0$, determine the particle's deceleration and position when $t = 3$ s. How far has the particle travelled during the 3 second time interval and what is its average speed?

Soln: Given $v = 6t - 3t^2$

$$\therefore a = \frac{dv}{dt} = \frac{d(6t - 3t^2)}{dt} = (6 - 6t)\,m/s^2$$

$$\therefore \text{at } t = 3\text{ s}, \qquad a = 6 - 6(3) = -12\ m/s^2$$

$$\text{Now}, v = \frac{dx}{dt} \rightarrow dx = vdt$$

$$\therefore dx = (6t - 3t^2)dt$$

Integrating both sides going from $x = 0, t = 0$ (given) to unknown values x & t,

$$\int_0^x dx = \int_0^t (6t - 3t^2)dt$$

$$\therefore x = 3t^2 - t^3$$

$$\therefore \text{at } t = 3\text{ s}, \qquad x = 3(3)^2 - 3^3 = 0$$

$$\text{or } x_3 = 0$$

Now, to calculate distance, we must check whether the particle reverses its direction during the time interval of 3 seconds. For the particle to reverse, its velocity must become zero at the reversal point.

$$\therefore v = 0 \rightarrow 6t - 3t^2 = 0$$
$$3t(2 - t) = 0$$
$$\therefore t = 0 \text{ or } t = 2 \text{ s}$$

At $t = 0$ the particle has started from rest, and at $t = 2$ s the particle must have reversed its direction.

Hence, the positions at various key points is,

$$\text{at } t = 0 \text{ s}, \quad x_0 = 0 \text{ (given)}$$
$$\text{at } t = 2 \text{ s}, \quad x_2 = 3(2)^2 - 2^3 = 4 \text{ m}$$
$$\text{at } t = 3 \text{ s}, \quad x_3 = 0 \text{ (found)}$$

Hence, the total distance travelled in 3 seconds $= |x_2 - x_0| + |x_3 - x_2|$

$$d = 4 + 4 = 8 \text{ m}$$

So, the average speed,

$$v_{av} = \frac{d}{t} = \frac{8}{3} = 2.67 \text{ m/s}$$

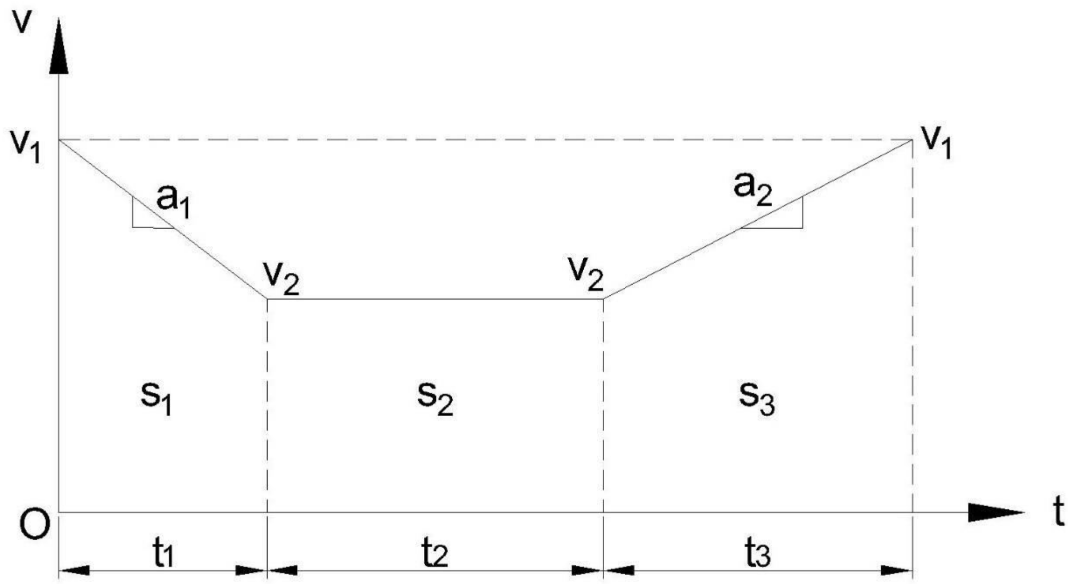
Hence, at $t = 3$ s, the particle's deceleration is 12 m/s^2 and its position is at origin. And, the particle has travelled 8 m during the 3 second time interval with an average speed of 2.67 m/s.

N2: A train travelling with a speed of 90 kmph slows down on account of work in progress, at a retardation of 1.8 kmph/s to 36 kmph. With this, it travels 600 m. Thereafter, it gains further speed with 0.9 kmph/s till getting the original speed. Find the delay caused.

Soln: Given data:

$$v_1 = 90 \text{ kmph} = 90 \times \frac{1000}{3600} \text{ m/s} = 25 \text{ m/s}$$
$$a_1 = -1.8 \text{ kmph/s} = -1.8 \times \frac{1000}{3600} \text{ m/s}^2 = -0.5 \text{ m/s}^2$$
$$v_2 = 36 \text{ kmph} = 36 \times \frac{1000}{3600} \text{ m/s} = 10 \text{ m/s}$$
$$a_2 = 0.9 \text{ kmph/s} = 0.9 \times \frac{1000}{3600} \text{ m/s}^2 = 0.25 \text{ m/s}^2$$
$$s_2 = 600 \text{ m}$$

We can solve this problem using the equations of kinematics or we can draw the v-t motion curve. Solving by v-t curve:



Section 1:

Acceleration is given by the slope of the v-t curve.

$$\therefore a_1 = [\text{Slope v} - \text{t}]_{t_1} = \frac{v_2 - v_1}{t_1}$$
$$\Rightarrow -0.5 = \frac{10 - 25}{t_1}$$
$$\therefore t_1 = \frac{10 - 25}{-0.5} = 30 \text{ s}$$

Distance is given by the area under the curve of the v-t curve.

$$\therefore s_1 = [\text{AUC v} - \text{t}]_{t_1} = \frac{1}{2} \times (v_1 + v_2) \times t_1$$
$$\Rightarrow s_1 = \frac{1}{2} \times (25 + 10) \times 30 = 525 \text{ m}$$

Section 2:

$$\because s_2 = [\text{AUC v} - \text{t}]_{t_2} = v_2 \times t_2$$
$$\Rightarrow 600 = 10 \times t_2$$
$$\therefore t_2 = \frac{600}{10} = 60 \text{ s}$$

Section 3:

$$\because a_2 = [\text{Slope v} - \text{t}]_{t_3} = \frac{v_1 - v_2}{t_3}$$
$$\Rightarrow 0.25 = \frac{25 - 10}{t_3}$$

$$\therefore t_3 = \frac{25 - 10}{0.25} = 60 \text{ s}$$

$$\because s_3 = [\text{AUC v} - \text{t}]_{t_3} = \frac{1}{2} \times (v_2 + v_1) \times t_3$$

$$\Rightarrow s_3 = \frac{1}{2} \times (10 + 25) \times 60 = 1050 \text{ m}$$

Hence, total distance, $d = s_1 + s_2 + s_3 = 525 + 600 + 1050 = 2175 \text{ m}$

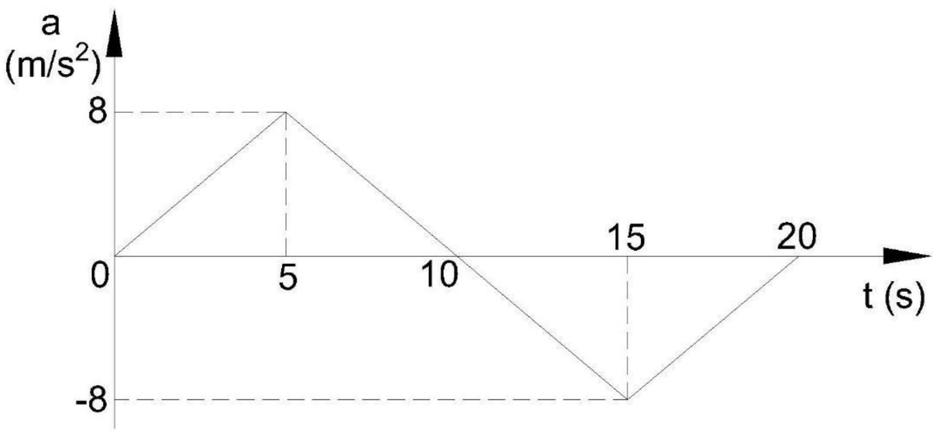
And, total time, $t = t_1 + t_2 + t_3 = 30 + 60 + 60 = 150 \text{ s}$

If there was no work being done, the train would have travelled the total distance at the constant speed of 25 m/s. The time required for such a scenario,

$$t' = \frac{d}{v_1} = \frac{2175}{25} = 87 \text{ s}$$

Hence, the delay caused $= t - t' = 150 - 87 = 63 \text{ s}$.

N3: The acceleration-time diagram for linear motion is shown below. Construct velocity-time and displacement-time diagrams for the motion assuming that the motion starts from the rest.



Soln: Velocity-Time Diagram:

$$v_f = v_i + [\text{AUC a} - \text{t}]_{t_i - t_f}$$

$$v_5 = v_0 + [\text{AUC a} - \text{t}]_{0-5} = 0 + \frac{1}{2} \times 5 \times 8 = 20 \text{ m/s}$$

$$v_{10} = v_5 + [\text{AUC a} - \text{t}]_{5-10} = 20 + \frac{1}{2} \times 5 \times 8 = 40 \text{ m/s}$$

$$v_{15} = v_{10} + [\text{AUC a} - \text{t}]_{10-15} = 40 - \frac{1}{2} \times 5 \times 8 = 20 \text{ m/s}$$

$$v_{20} = v_{15} + [\text{AUC a} - \text{t}]_{20-15} = 20 - \frac{1}{2} \times 5 \times 8 = 0 \text{ m/s}$$

Now, since a-t curve is a straight line with some slope, v-t curve will be parabolic.

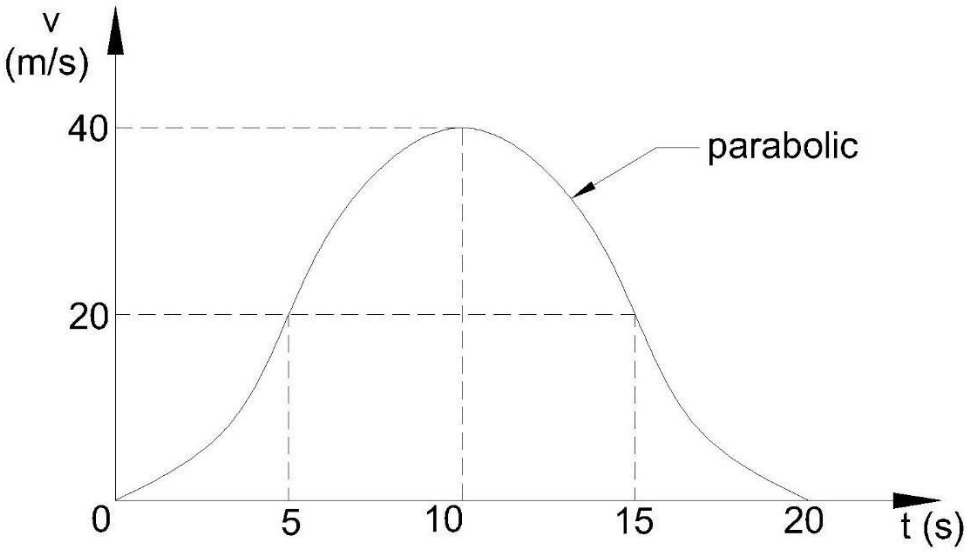
NOTE: Following statements are not required in exam, these are for reference.

{From 0-5 s, acceleration is positive and increasing, that means the velocity is increasing quickly. So, the parabolic curve will be concave up.

From 5-10 s, acceleration is positive but decreasing, that means the velocity is increasing slowly. So, the parabolic curve will be concave down.

From 10-15 s, acceleration is negative and decreasing, that means the velocity is decreasing quickly. So, the parabolic curve will be concave down.

From 15-20 s, acceleration is negative but increasing, that means the velocity is decreasing slowly. So, the parabolic curve will be concave up.}



Displacement-Time Diagram (Or Position-Time Diagram):

Method I: Area under the v-t curve

$$x_f = x_i + [\text{AUC v} - t]_{t_i - t_f}$$

$$x_5 = x_0 + [\text{AUC v} - t]_{0-5} = 0 + \frac{1}{3} \times 5 \times 20 = 33.33 \text{ m}$$

$$x_{10} = x_5 + [\text{AUC v} - t]_{5-10} = 33.33 + 5 \times 20 + \frac{2}{3} \times 5 \times 20 = 200 \text{ m}$$

$$x_{15} = x_{10} + [\text{AUC v} - t]_{10-15} = 200 + 5 \times 20 + \frac{2}{3} \times 5 \times 20 = 366.67 \text{ m}$$

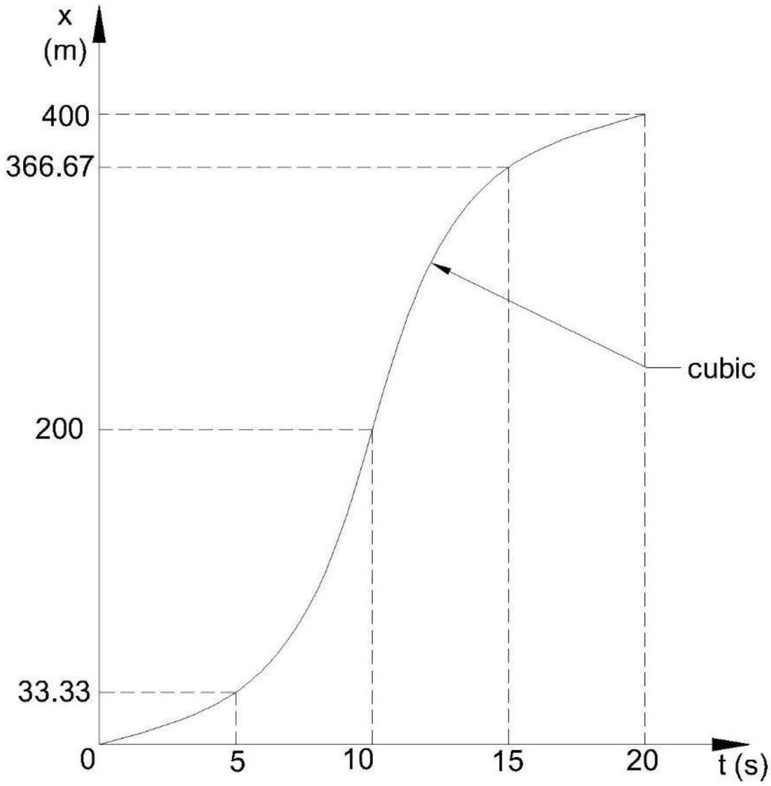
$$x_{20} = x_{15} + [\text{AUC v} - t]_{20-15} = 366.67 + \frac{1}{3} \times 5 \times 20 = 0 \text{ m}$$

[NOTE: Area under a concave up parabolic curve is given by $\frac{1}{3} \times$ base $\times$ height.

And for concave down parabolic curve is given by $\frac{2}{3} \times$ base $\times$ height.

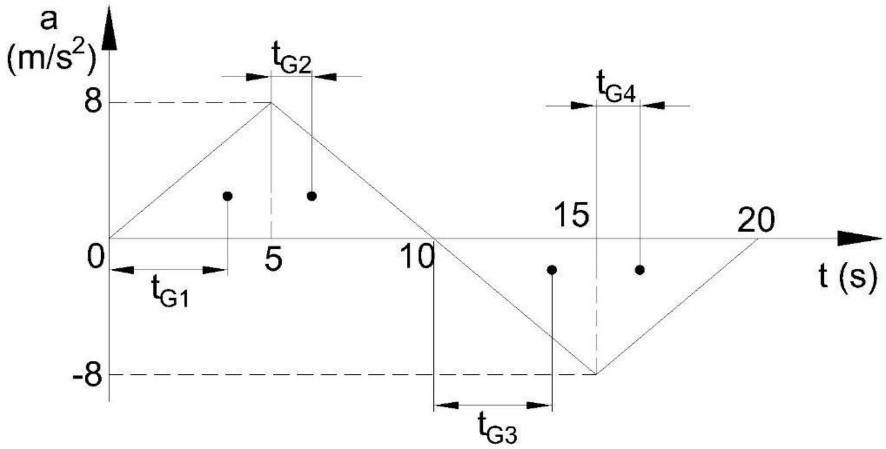
Don't forget to add the rectangular section below the parabolic area.]

Now, since v-t curve is a parabolic curve, x-t curve will be cubic in nature.



Method II: Moment-area under a-t curve

$$x_f = x_i + v_i \times t + [\text{AUC a} - \text{t}]_{t_i - t_f} \times (t - t_G)$$



[NOTE: For a right-angled triangle, the centre of gravity is 1/3rd the length of base and 1/3rd the length of height, from the vertex having right angle.]

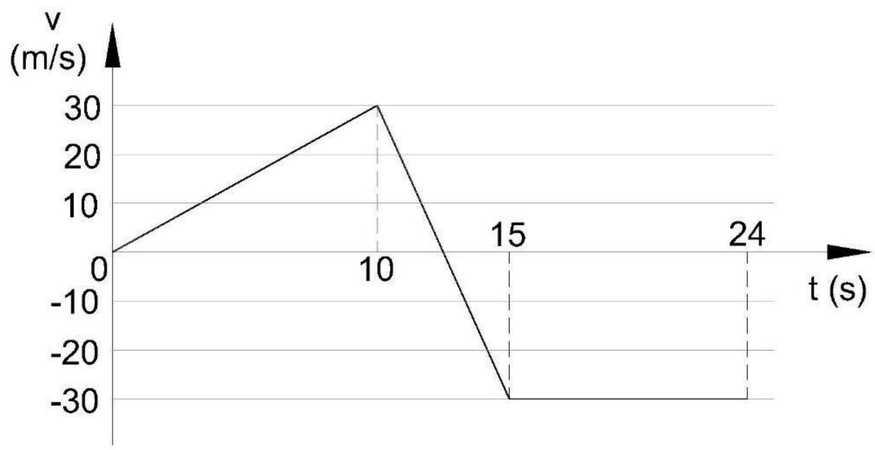
$$x_5 = x_0 + v_0 \times t + [\text{AUC a} - \text{t}]_{0-5} \times (t - t_{G1})$$
$$x_5 = 0 + 0 \times (5 - 0) + \frac{1}{2} \times 5 \times 8 \times \left(5 - \frac{2}{3} \times 5\right) = 33.33 \text{ m}$$

$$x_{10} = x_5 + v_5 \times t + [\text{AUC a} - \text{t}]_{5-10} \times (t - t_{G2})$$
$$x_{10} = 33.33 + 20 \times (10 - 5) + \frac{1}{2} \times 5 \times 8 \times \left(5 - \frac{1}{3} \times 5\right) = 200 \text{ m}$$

$$x_{15} = x_{10} + v_{10} \times t + [\text{AUC a} - \text{t}]_{10-15} \times (t - t_{G3})$$
$$x_{15} = 200 + 40 \times (15 - 10) - \frac{1}{2} \times 5 \times 8 \times \left(5 - \frac{2}{3} \times 5\right) = 366.67 \text{ m}$$

$$x_{20} = x_{15} + v_{15} \times t + [\text{AUC a} - \text{t}]_{15-20} \times (t - t_{G4})$$
$$x_{20} = 366.67 + 20 \times (20 - 15) - \frac{1}{2} \times 5 \times 8 \times \left(5 - \frac{1}{3} \times 5\right) = 400 \text{ m}$$

N4: A particle moves in a straight line with a velocity-time diagram as shown in figure. If $s = -25$ m at $t = 0$ s, draw displacement-time and acceleration-time diagrams for 0 to 24 seconds.



Soln: Position calculations: Given - $x_0 = -25$ m

$$x_f = x_i + [\text{AUC v} - \text{t}]_{t_i - t_f}$$

$$x_{10} = x_0 + [\text{AUC v} - \text{t}]_{0-10} = -25 + \frac{1}{2} \times 10 \times 30 = 125 \text{ m}$$

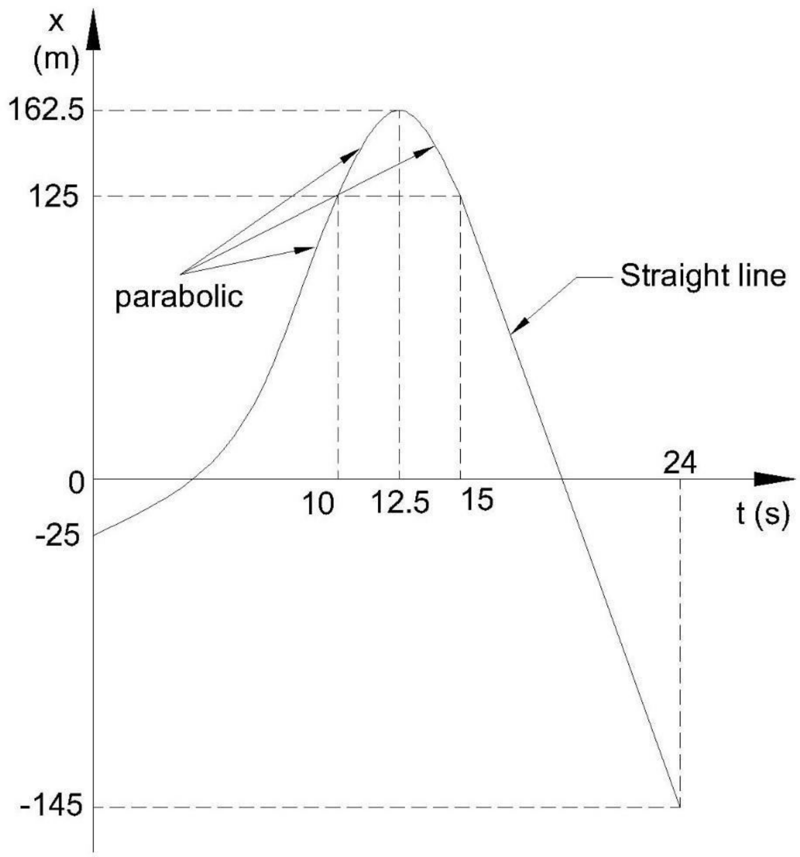
Somewhere between $t = 10$ s to $t = 15$ s, the particle has zero velocity and after that point it has negative velocity, which means at that point, particle reverses position. This point is at $t = 12.5$ s, considering the similarity of triangles.

$$x_{12.5} = x_{10} + [\text{AUC v} - \text{t}]_{10-12.5} = 125 + \frac{1}{2} \times 2.5 \times 30 = 162.5 \text{ m}$$

$$x_{15} = x_{12.5} + [\text{AUC v} - \text{t}]_{12.5-15} = 162.5 - \frac{1}{2} \times 2.5 \times 30 = 125 \text{ m}$$

$$x_{24} = x_{15} + [\text{AUC v} - \text{t}]_{15-24} = 125 - 9 \times 30 = -145 \text{ m}$$

{The x-t curves from 0 to 15 s will be parabolic since, v-t curves are straight lines with some slope. But from 15 to 24 s, x-t curve will be straight line since v-t curve is a horizontal line}



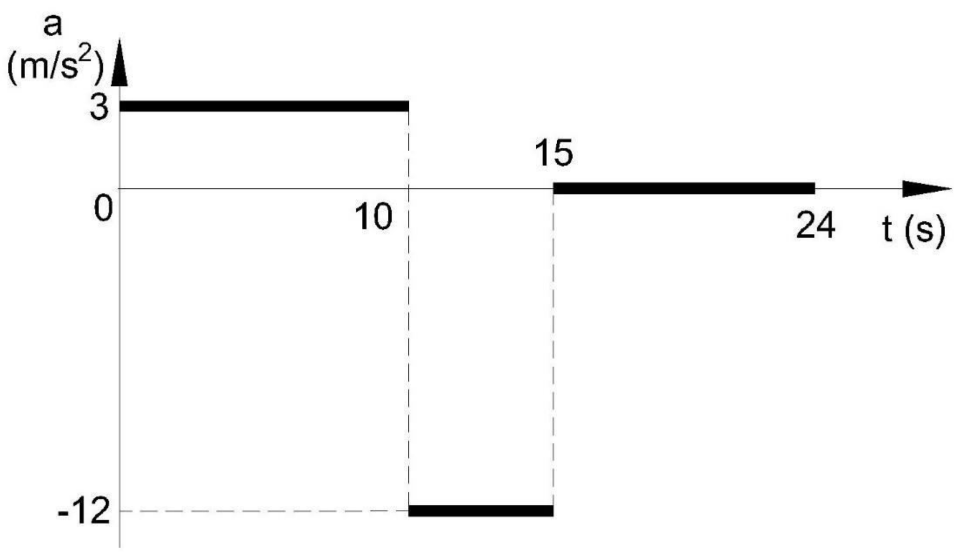
Acceleration calculations:

$$a = [\text{slope v} - \text{t curve}]_{\text{at time} = t}$$

$$a_{0-10} = \left[\frac{v_{10} - v_0}{\Delta t}\right]_{0-10} = \frac{30 - 0}{10 - 0} = 3\ \text{m/s}^2$$

$$a_{10-15} = \left[\frac{v_{15} - v_{10}}{\Delta t}\right]_{10-15} = \frac{-30 - 30}{15 - 10} = -12\ \text{m/s}^2$$

$$a_{15-24} = \left[\frac{v_{24} - v_{15}}{\Delta t}\right]_{15-24} = \frac{-30 - (-30)}{24 - 15} = 0$$



Part II – Curvilinear Motion:

N5: The position vector of a particle is given by $\bar{r} = \left(\frac{1}{4} t^3 \hat{\imath} + 3t^2 \hat{\jmath}\right)$ m.

Determine at t = 2 s,

a) the radius of curvature of the path,
b) N-T components of acceleration.

Soln:

$$\text{Position vector, } \bar{r} = \left(\frac{1}{4} t^3 \hat{\imath} + 3t^2 \hat{\jmath}\right) \text{ m}$$
$$\therefore \text{ Velocity vector, } \bar{v} = \frac{d\bar{r}}{dt} = \left(\frac{3}{4} t^2 \hat{\imath} + 6t\hat{\jmath}\right) \text{ m/s}$$
$$\therefore \text{ Acceleration vector, } \bar{a} = \frac{d\bar{v}}{dt} = \left(\frac{3}{2} t\hat{\imath} + 6\hat{\jmath}\right) \text{ m/s}^2$$
$$\therefore \text{ at t} = 2 \text{ s,}$$
$$\bar{v} = \left(\frac{3}{4} \times 2^2 \hat{\imath} + 6 \times 2\hat{\jmath}\right) = (3\hat{\imath} + 12\hat{\jmath}) \text{ m/s}$$
$$\Rightarrow v = 12.369 \text{ m/s}$$
$$\bar{a} = \left(\frac{3}{2} \times 2\hat{\imath} + 6\hat{\jmath}\right) = (3\hat{\imath} + 6\hat{\jmath}) \text{ m/s}^2$$
$$\Rightarrow a = 6.708 \text{ m/s}^2$$

Using the following equation for radius of curvature since rectangular components of velocity and acceleration are known,

$$\rho = \left|\frac{v^3}{a_x v_y - a_y v_x}\right|$$
$$\rho = \left|\frac{12.369^{\,3}}{3 \times 12 - 6 \times 3}\right|$$
$$\boldsymbol{\rho = 105.1 \text{ m}}$$

Now, for the N-T components of acceleration,

$$a_n = \frac{v^2}{\rho} = \frac{12.369^2}{105.1}$$
$$\mathbf{a_n = 1.456 \text{ m/s}^2}$$

From $a = \sqrt{a_n^2 + a_t^2}$, we get,

$$a_t = \sqrt{a^2 - a_n^2}$$
$$a_t = \sqrt{6.708^2 - 1.456^2}$$
$$\mathbf{a_t = 6.548 \text{ m/s}^2}$$

N6: The curvilinear motion of a particle is defined by $v_x = (25 - 8t)\ m/s$ and $y = (48 - 3t^2)$ m. Knowing at $t = 0, x = 0$, find at time $t = 4$ s, the position, velocity, and acceleration vectors. Also find corresponding magnitudes.

Soln: Considering the factors in x-direction:

$$\because v_x = (25 - 8t)\ m/s$$
$$\therefore v_x = \frac{dx}{dt} = (25 - 8t)$$
$$dx = (25 - 8t)dt$$

Integrating both sided from $x = 0, t = 0$ till some unknown x & t, we get,

$$\int_0^x dx = \int_0^t (25 - 8t)dt$$
$$x = (25t - 4t^2)\ m$$

Also, for acceleration in x-direction,

$$a_x = \frac{dv_x}{dt} = \frac{d(25 - 8t)}{dt}$$
$$a_x = -8\ m/s^2$$

Considering the factors in y-direction:

$$\because y = (48 - 3t^2)\ m$$
$$\therefore v_y = \frac{dy}{dt} = -6t\ m/s$$
$$\therefore a_y = \frac{dv_y}{dt} = -6\ m/s^2$$

Position vector at $t = 4$ s,

$$x = (25(4) - 4(4)^2) = 36\ m$$
$$y = (48 - 3(4)^2) = 0\ m$$
$$\mathbf{\bar{r} = (36\hat{i} + 0\hat{j})\ m \rightarrow r = 36\ m}$$

Velocity vector at $t = 4$ s,

$$v_x = \big(25 - 8(4)\big) = -7\ m/s$$
$$v_y = -6(4) = -24\ m/s$$
$$\mathbf{\bar{v} = (-7\hat{i} - 24\hat{j})\ m/s \rightarrow v = 25\ m/s}$$

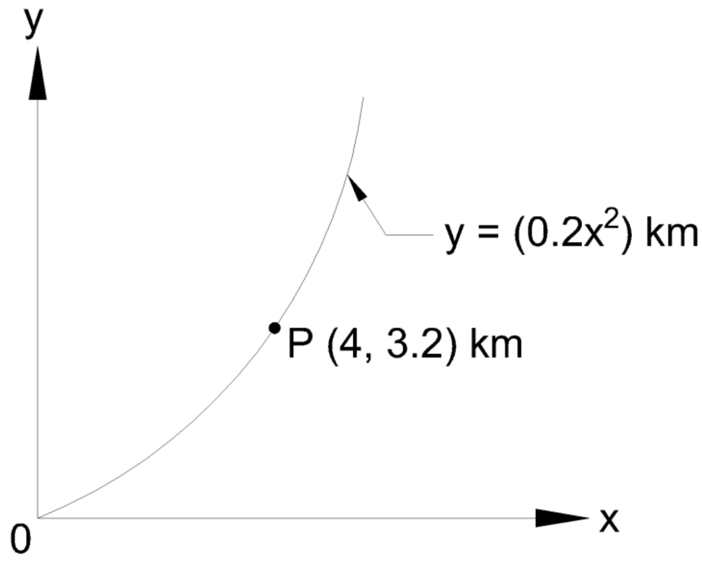
Acceleration vector at $t = 4$ s,

$$a_x = -8\ m/s^2$$
$$a_y = -6\ m/s^2$$
$$\mathbf{\bar{a} = (-8\hat{i} - 6\hat{j})\ m/s^2 \rightarrow a = 10\ m/s^2}$$

N7: An airplane travels on a curved path. At P it has speed of 360 kmph which is increasing at a rate of 0.5 m/s^2. Figure shows more details. Determine at P:

a) the magnitude of total acceleration

b) angle made by the acceleration vector with the positive x-axis.



Soln: Since speed (or velocity) given is along the path of the curve, it is tangential,

$$v = 360 \text{ kmph} = 360 \times \frac{1000}{3600} \text{ m/s} = 100 \text{ m/s}$$

Acceleration given is also along the direction of velocity, hence it is also tangential,

$$a_t = 0.5 \, m/s^2$$

Equation of the path of airplane is given as $y = 0.2x^2$

$$\therefore \frac{dy}{dx} = 0.4x \Rightarrow \left[\frac{dy}{dx}\right]_{x=4 \text{ km}} = 1.6$$

$$\therefore \frac{d^2y}{dx^2} = 0.4 \Rightarrow \left[\frac{d^2y}{dx^2}\right]_{x=4 \text{ km}} = 0.4$$

Now, using the following relation of radius of curvature in terms of derivatives,

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{[1 + (1.6)^2]^{\frac{3}{2}}}{0.4}$$

$$\rho = 16.792486 \text{ km} = 16792.486 \text{ m}$$

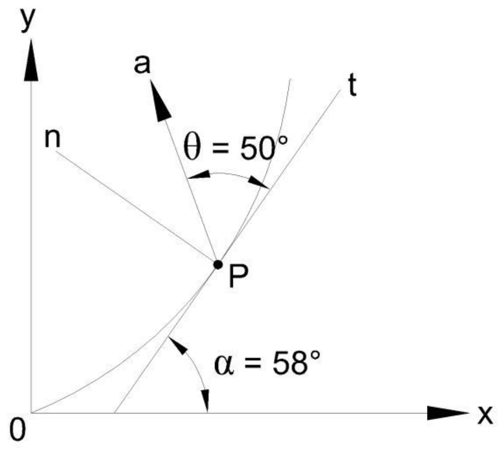
Now, using the relations for acceleration,

$$a_n = \frac{v^2}{\rho} = \frac{100^2}{16792.486} = 0.5955 \, m/s^2$$

$$\therefore a = \sqrt{a_n^2 + a_t^2} = \sqrt{0.5955^2 + 0.5^2}$$
$$\therefore \mathbf{a = 0.777\ m/s^2}$$

Let θ be the angle made by the acceleration vector with the tangent to the curved path at x = 4 km.

$$\tan\theta = \frac{a_n}{a_t} = \frac{0.5955}{0.5}$$
$$\therefore \theta = 49.98° \approx 50°$$



Let α be the angle made by the tangent of the path at x = 4 km with the x-axis.

$$\tan\alpha = \frac{dy}{dx} = 1.6$$
$$\therefore \alpha = 57.99° \approx 58°$$

Hence, the angle made by the acceleration vector with the x-axis is given by,

$$\theta + \alpha = 50° + 58° = \mathbf{108°}$$

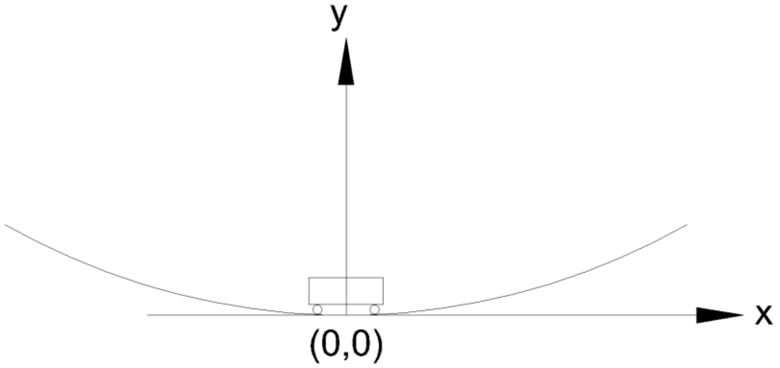
<u>N8</u>: A car travels along a vertical curve on a road, the equation of the curve being $x^2 = 200y$ (x-horizontal and y-vertical in m). The speed of the car is constant and equal to 72 kmph. (i) Find its acceleration when the car is at the deepest point on the curve, (ii) What is the radius of curvature of the curve at this point?

Soln: Given:

$$x^2 = 200y \rightarrow y = \frac{x^2}{200}$$
$$v = 72\ \text{kmph} = 20\ \ m/s$$

Since, the velocity is constant; the acceleration along the tangential direction is zero.

$$\therefore a_t = 0\,m/s^2$$



The deepest point of the curve will be at (0,0) since x-axis is to be taken as horizontal and y-axis is to be taken as vertical (given).

$$\therefore \frac{dy}{dx} = \frac{1}{200}(2x) = \frac{x}{100} \Rightarrow \left[\frac{dy}{dx}\right]_{x=0} = 0$$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{100} \Rightarrow \left[\frac{d^2y}{dx^2}\right]_{x=0} = \frac{1}{100}$$

Now, using the following relation of radius of curvature in terms of derivatives,

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{[1 + (0)^2]^{\frac{3}{2}}}{\frac{1}{100}}$$

$$\boldsymbol{\rho = 100\ m}$$

Now, using the relations for acceleration,

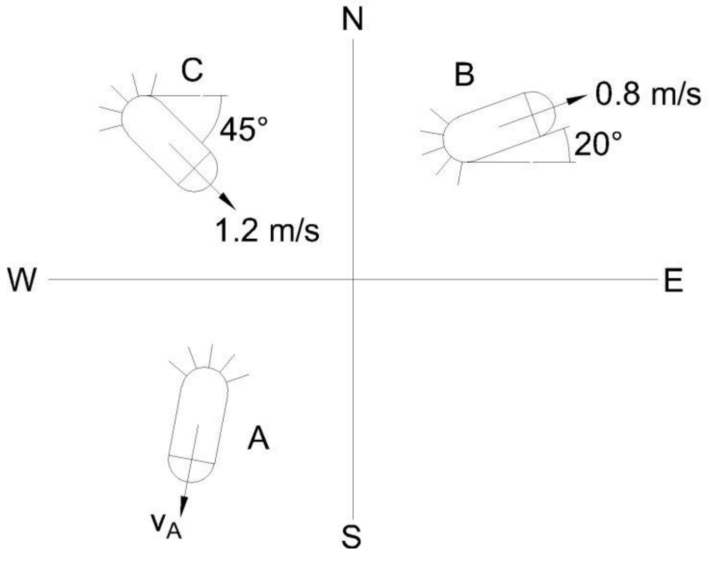
$$a_n = \frac{v^2}{\rho} = \frac{20^2}{100}$$

$$\mathbf{a_n = 4\ m/s^2}$$

Part III – Relative Motion:

N9: Three ships sail in different directions as shown. If the captain of ship C observes ship A, he finds ship A sailing at 3 m/s at $\theta = 60° \swarrow$. Find

a) True velocity of ship A,
b) Velocity of B as observed by A
c) Velocity of C as observed by B.



Soln: Given: Using cosine and sine for x & y components of velocity, we get,

$$v_B = 0.8\ m/s, \theta_B = 20° \nearrow \quad \Rightarrow \quad \bar{v}_B = (0.752\hat{i} + 0.274\hat{j})\ m/s$$

$$v_C = 1.2\ m/s, \theta_C = 45° \searrow \quad \Rightarrow \quad \bar{v}_c = (0.848\hat{i} - 0.848\hat{j})\ m/s$$

$$v_{A/C} = 3\ m/s, \theta_{A/C} = 60° \swarrow \quad \Rightarrow \quad \bar{v}_{A/C} = (-1.5\hat{i} - 2.6\hat{j})\ m/s$$

We know, relative velocity is given by,

$$\bar{v}_{A/C} = \bar{v}_A - \bar{v}_C$$

$$(-1.5\hat{\imath} - 2.6\hat{\jmath}) = \bar{v}_A - (0.848\hat{\imath} - 0.848\hat{\jmath})$$

$$\bar{v}_A = (-0.652\hat{\imath} - 3.45\hat{\jmath})\ \text{m/s}$$

$$\mathbf{v_A = 3.51\ m/s}, \boldsymbol{\theta_A = 79.3°} \swarrow$$

Now, for velocity of ship B w.r.t. A,

$$\bar{v}_{B/A} = \bar{v}_B - \bar{v}_A$$

$$\bar{v}_{B/A} = (0.752\hat{\imath} + 0.274\hat{\jmath}) - (-0.652\hat{\imath} - 3.45\hat{\jmath})$$

$$\bar{v}_{B/A} = (1.404\hat{\imath} + 3.724\hat{\jmath})\ \text{m/s}$$

$$\mathbf{v_{B/A} = 3.98\ m/s}, \boldsymbol{\theta_{B/A} = 69.34°} \nearrow$$

Similarly, for velocity of ship C w.r.t. B,

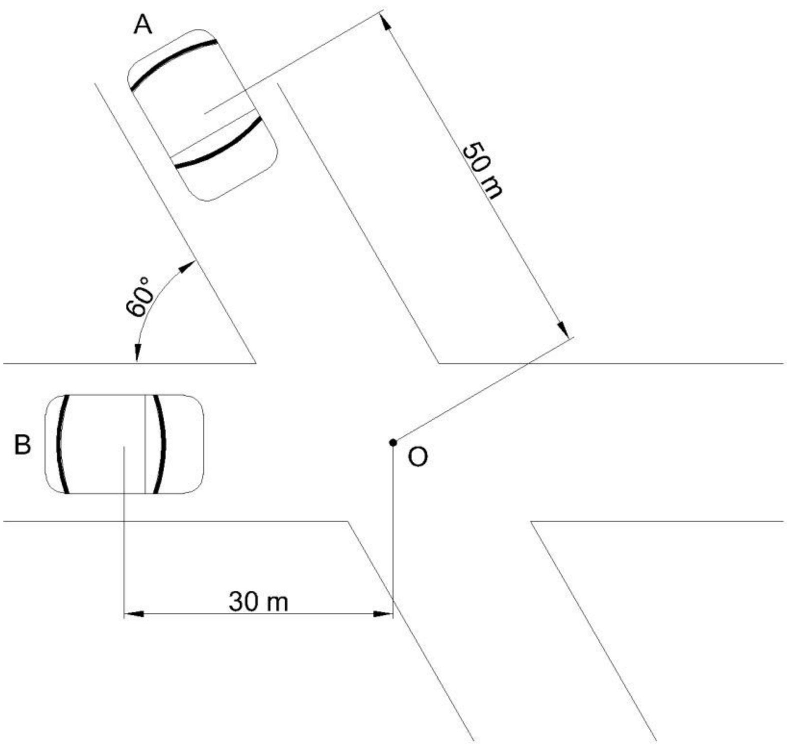
$$\bar{v}_{C/B} = \bar{v}_C - \bar{v}_B$$

$$\bar{v}_{C/B} = (0.848\hat{\imath} - 0.848\hat{\jmath}) - (0.752\hat{\imath} + 0.274\hat{\jmath})$$

$$\bar{v}_{C/B} = (0.096\hat{\imath} - 1.122\hat{\jmath})\ \text{m/s}$$

$$\mathbf{v_{C/B} = 1.126\ m/s}, \boldsymbol{\theta_{C/B} = 85.1°} \searrow$$

N10: Figure shows the location of cars A and B at t = 0. Car A starts from rest and travels towards the intersection at a uniform rate of 2 m/s^2. Car B travels towards the intersection at a constant speed of 8 m/s. Determine the relative velocity and acceleration of car B w.r.t. car A at t = 6 s.



Soln: Uniform Acceleration Motion of Car A:

Given, $u = 0, a = 2\ m/s^2, t = 6\ s$

$$v = u + at = 0 + 2 \times 6 = 12\ \ m/s$$

$$s = ut + \frac{1}{2}at^2 = 0 + \frac{1}{2} \times 2 \times 6^2 = 36\ m$$

Uniform Velocity Motion of Car B:

Given, $v = 8\ m/s, t = 6\ s$

$$\text{From } v = \frac{s}{t} \rightarrow s = v \times t = 8 \times 6 = 48\ m$$

Car A has travelled 36 m, that means it is $50 - 36 = 14$ m away from intersection.

$$r_A = 14\ m, \qquad \theta_r = 60° \nwarrow \quad \Rightarrow \quad \bar{r}_A = (-7\hat{i} - 12.12\hat{j})\ m$$

Car B has travelled 48 m, that means it is $48 - 30 = 18$ m past the intersection.

$$r_B = 18\ m, \qquad \theta_r = 0° \longrightarrow \quad \Rightarrow \quad \bar{r}_A = (18\hat{i})\ m$$

Now, relative position of B w.r.t A at t = 6 s is given by,

$$\bar{r}_{B/A} = \bar{r}_B - \bar{r}_A = (18\hat{i}) - (-7\hat{i} - 12.12\hat{j})$$
$$\bar{r}_{B/A} = (25\hat{i} - 12.12\hat{j})\ m$$
$$\mathbf{r_{B/A} = 27.78\ m, \theta_r = 25.86°} \searrow$$

Velocity vector of A is, $v_A = 12\ \ m/s,\ \theta_v = 60° \searrow \quad \Rightarrow \quad \bar{v}_A = (6\hat{i} - 10.39\hat{j})\ \ m/s$

Velocity vector of B is, $v_B = 8\ \ m/s,\ \theta_v = 0° \longrightarrow \quad \Rightarrow \quad \bar{v}_B = (8\hat{i})\ \ m/s$

Now, relative velocity of B w.r.t A at t = 6 s is given by,

$$\bar{v}_{B/A} = \bar{v}_B - \bar{v}_A$$
$$\bar{v}_{B/A} = (8\hat{i}) - (6\hat{i} - 10.39\hat{j})$$
$$\bar{v}_{B/A} = (2\hat{i} + 10.39\hat{j})\ \ m/s$$
$$\mathbf{v_{B/A} = 10.58\ \ m/s, \theta_v = 79.1°} \nearrow$$

Acceleration vector of A, $a_A = 2\ \ m/s^2,\ \theta_a = 60° \searrow \Rightarrow \bar{a}_A = (1\hat{i} - 1.732\hat{j})\ \ m/s^2$

Acceleration vector of B, $a_B = 0 \rightarrow \bar{a}_B = 0$

Now, relative acceleration of B w.r.t A at t = 6 s is given by,

$$\bar{a}_{B/A} = \bar{a}_B - \bar{a}_A$$
$$\bar{a}_{B/A} = 0 - (1\hat{i} - 1.732\hat{j})$$
$$\bar{a}_{B/A} = (-1\hat{i} + 1.732\hat{j})\ \ m/s^2$$
$$\mathbf{a_{B/A} = 2\ \ m/s^2, \theta_a = 60°} \nwarrow$$

K J Somaiya College of Engineering, Vidyavihar, Mumbai

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Engineering Mechanics Notes

Module 2 – Kinematics of Particles & Rigid Bodies

Module Section 2.2 – Kinematics of Rigid Bodies

Class: FY BTech Division: C3
Professor: Aniket S. Patil Date: 09/04/2023

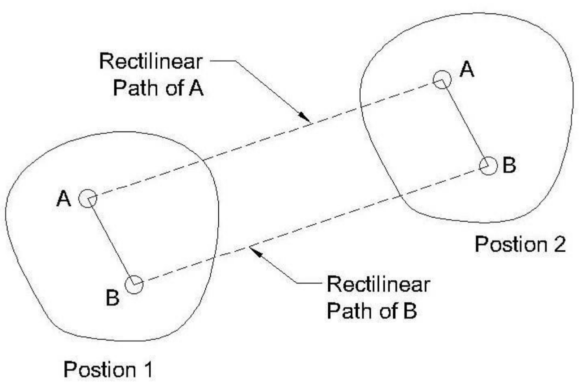
References: Engineering Mechanics, by M. D. Dayal & Engineering Mechanics – Statics and Dynamics, by N. H. Dubey.

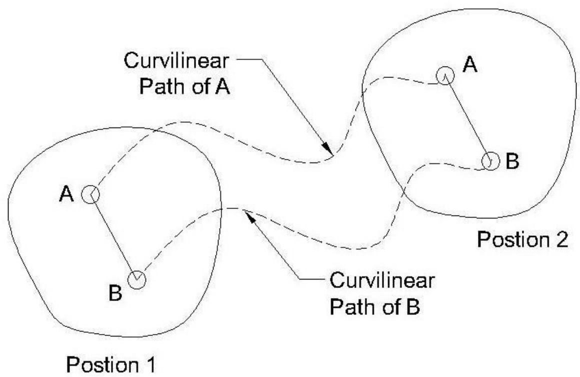
Types of Rigid Body Motion:

1. Translation Motion
2. Rotation about a fixed axis
3. General Plane Motion (More important for numericals)
4. Motion about a fixed point (not in syllabus)
5. General Motion (not in syllabus)

Translation Motion: In this, all the particles forming the body travel along parallel paths, and the orientation on the body does not change during the motion. The motion may be rectilinear or curvilinear.

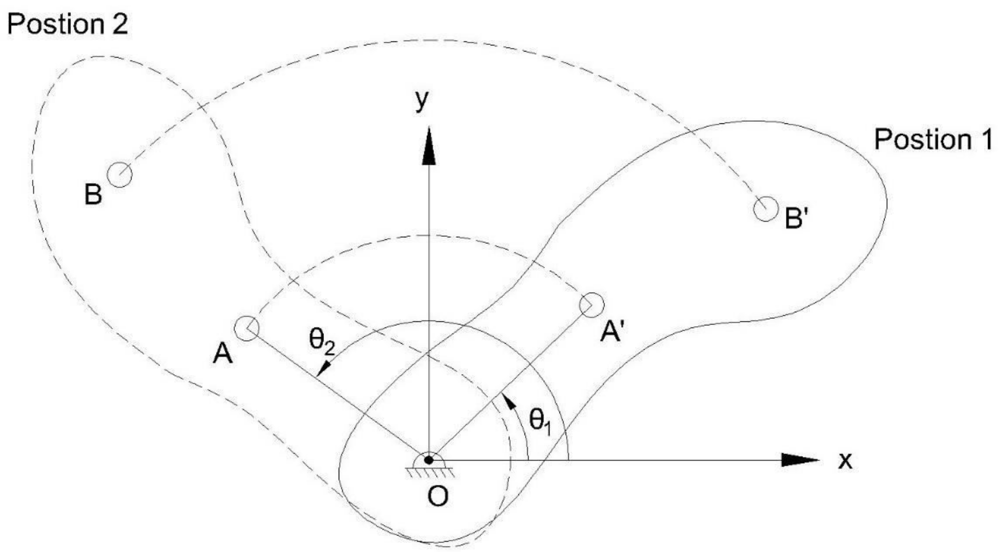
Let a body move from position 1 to position 2, with two points A & B labelled for reference. The line joining A & B maintains the same direction orientation in both positions. The path travelled by A is parallel to the path travelled by B, be it straight or curved.





At any given instant, in a translational motion, all particles of the body have the same displacement, same velocity and same acceleration. Hence, at its centre of gravity G, a rigid body is similar to a particle in translation motion.

Rotation about Fixed Axis: In this, all the particles of the body travel along concentric circular paths about a common centre of rotation. The axis of rotation is perpendicular to the plane of motion.



Angular Position θ is measured in anticlockwise direction from x-axis in radians.

Angular Displacement is the change in angular position. It is also labelled with θ and measured in radians (rad) given by, θ or $\Delta\theta = \theta_2 - \theta_1$.

$$1 \text{ revolution} = 2\pi \text{ radians} = 360°$$

Angular Velocity is the rate of change of angular position with respect to time measured in radians per second (rad/s).

$$\omega = \frac{d\theta}{dt} \ \circlearrowleft +\text{ve} \qquad\qquad 1 \text{ rpm} = \frac{2\pi}{60} \text{ rad/s}$$

Angular Acceleration is the rate of change of angular velocity with respect to time measured in radians per second squared (rad/s^2).

$$\alpha = \frac{d\omega}{dt}$$

Types of Rotational about Fixed Axis:

1. Uniform Angular Velocity Motion

$$\omega = \frac{\theta}{t}$$

2. Uniform Angular Acceleration Motion

$$\omega = \omega_0 + \alpha t$$
$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$
$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

3. Variable Angular Acceleration Motion

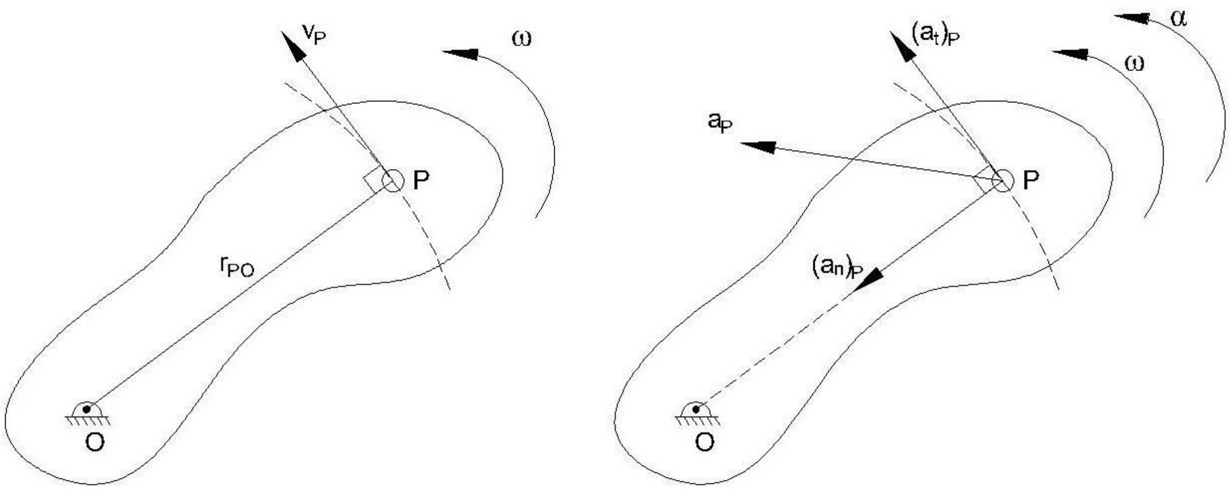
$$\omega = \frac{d\theta}{dt} \quad \& \quad \alpha = \frac{d\omega}{dt} \quad \rightarrow \quad \alpha = \omega \frac{d\omega}{d\theta}$$

Relations between Linear and Angular Parameters:

All particles in a rotating body will have the same angular velocity but different linear velocities. For a point P, if v_P is the linear velocity and r_{PO} is the radial distance from P to O, then $v_P = r_{PO} \times \omega$.

In general, for any particle with linear velocity v located at a radial distance of r from the axis of rotation with the body having an angular velocity of ω,

$$\mathbf{v = r\omega}$$



If the particle has a linear acceleration of a_P which can be resolved into normal component $(a_n)_P$ and tangential component $(a_t)_P$, and the body has an angular acceleration of α, then,

$$\because a_n = \frac{v^2}{\rho} \quad \Rightarrow \quad (a_n)_P = \frac{(v_P)^2}{r_{PO}} = \frac{(r_{PO} \times \omega)^2}{r_{PO}} = \frac{(r_{PO})^2(\omega)^2}{r_{PO}} = r_{PO} \times \omega^2$$

$$\because a_t = \frac{dv}{dt} \quad \Rightarrow \quad (a_t)_P = \frac{dv_P}{dt} = \frac{d(r_{PO} \times \omega)}{dt} = r_{PO}\frac{d\omega}{dt} = r_{PO} \times \alpha$$

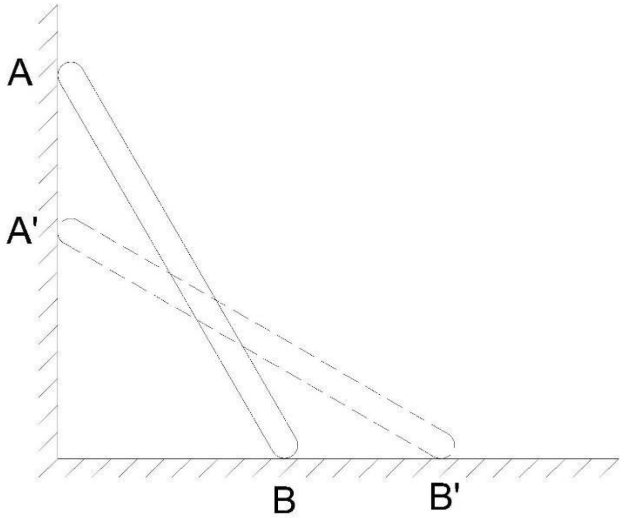
In general, for any particle with linear acceleration a located at a radial distance of r from the axis of rotation with the body having an angular velocity of ω,

$$\mathbf{a_n = r\omega^2} \quad \& \quad \mathbf{a_t = r\alpha}$$

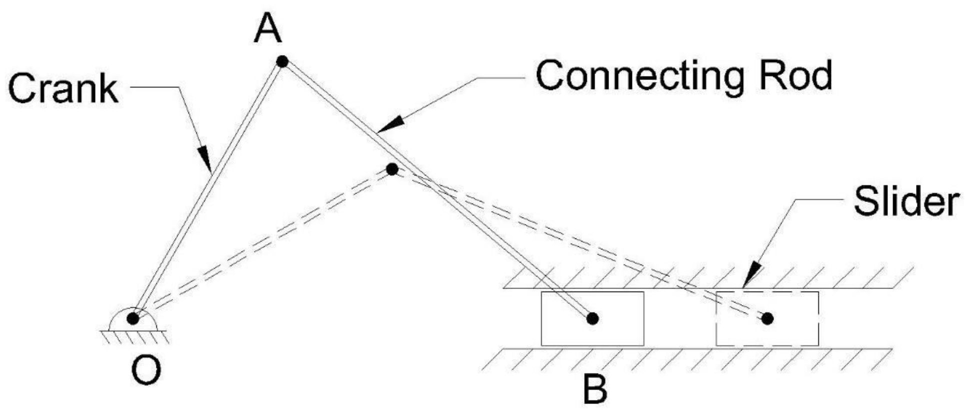
General Plane Motion: It is a combination of translation motion and rotational motion happening at the same time.

Example 1: Consider a ladder AB having the top end A on a vertical wall and bottom end B on the floor and its sliding. Hence, the velocity of A will be vertically down and that of B will be horizontally towards right.

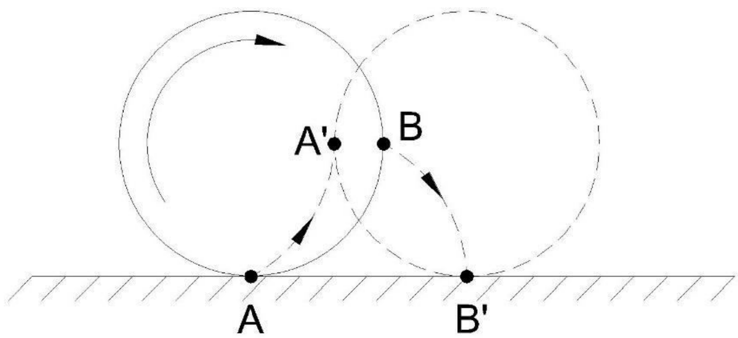
Here, since A and B are not moving in the same direction, it is not a translation motion, but it is not strictly rotation either even though there is some rotation involved. Hence, it is a combination of both, i.e., general plane (GP) motion.



Example 2: In a slider-crank mechanism, shown below, the crank undergoes rotational motion about a fixed hinge support and the slider undergoes back and forth translation motion. The connecting rod linking the crank and slider undergoes GP motion.



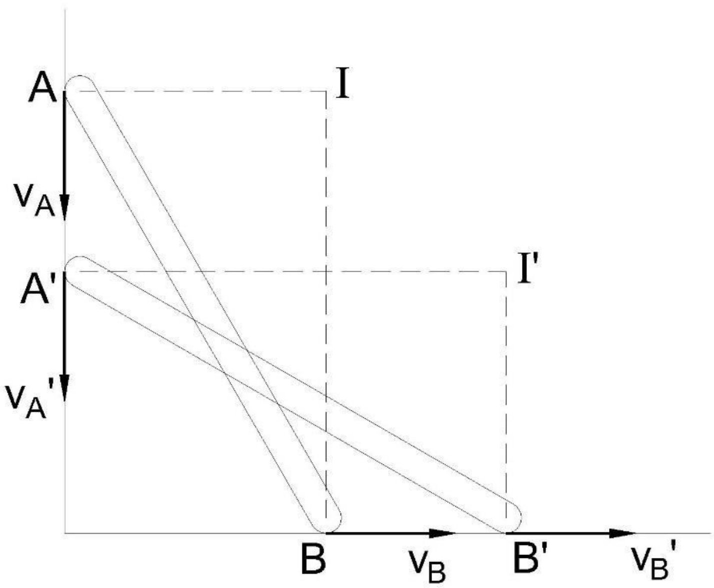
Example 3: When a wheel rolls without slipping on the ground, the wheel rotates as well as translates. Hence, it undergoes GP motion.

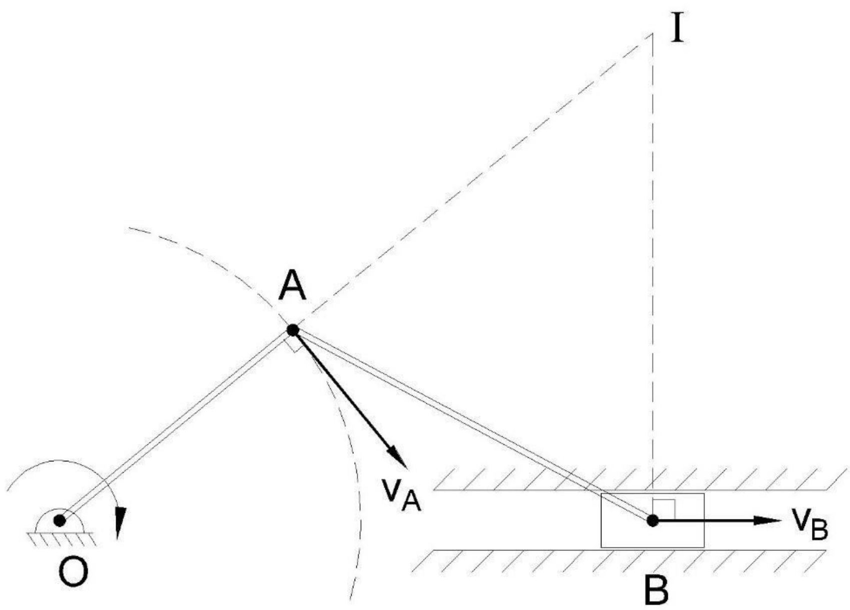


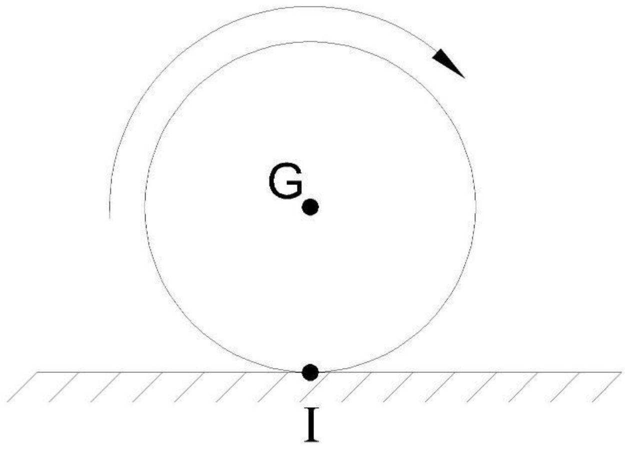
Instantaneous Centre of Rotation (I.C.R.): For general plane motion, at a particular instant, the body can be said to be rotating about a specific point. This point keeps changing as the body moves through the plane. This is called the instantaneous centre of rotation.

It is defined as the point about which a general plane moving body rotates at any given instant. The locus of the ICR's throughout the motion is known as centrode. ICR's are usually denoted by the letter I.

Instantaneous Centre Method: To find the angular velocity of a GP body, we use this method. Find the points on a GP body whose velocity is known. If we draw perpendiculars to the direction of velocity of those points, they will intersect at a certain point. This point is the I.C.R. and we can find the radial lengths to those points. Depending on the known quantities, we can use the relation v = rω to find the unknown quantities in a given problem.



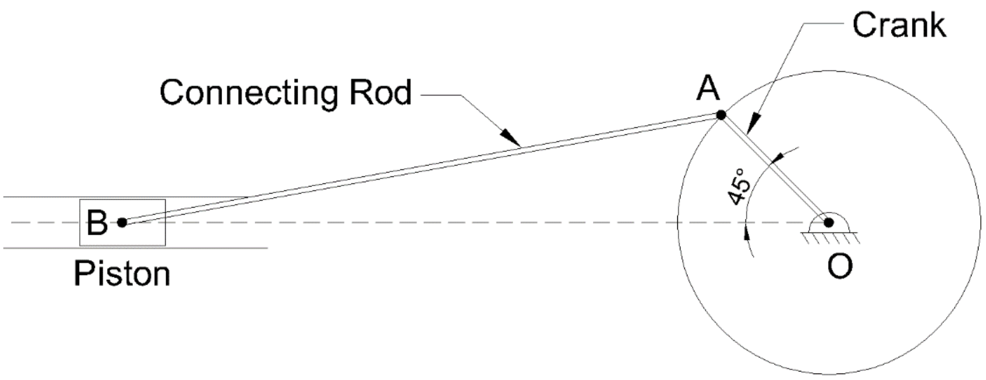




For a wheel which rolls without slipping (performing a general plane motion), the Instantaneous Centre of Rotation is the point of contact with the ground. This is because the centre of rotation should have zero velocity and the point in contact with the ground has velocity of the ground, which is zero.

Numericals:

N1: In a crank and connected rod mechanism, the length of crank and connecting rod are 300 mm and 1200 mm respectively. The crank is rotating at 180 rpm anticlockwise. Find the velocity of piston, when the crank is at an angle of 45° with the horizontal.



Soln: The crank OA performs rotation motion about fixed axis at O, the connecting rod AB performs General Plane Motion, while piston B performs translation motion.

Given:

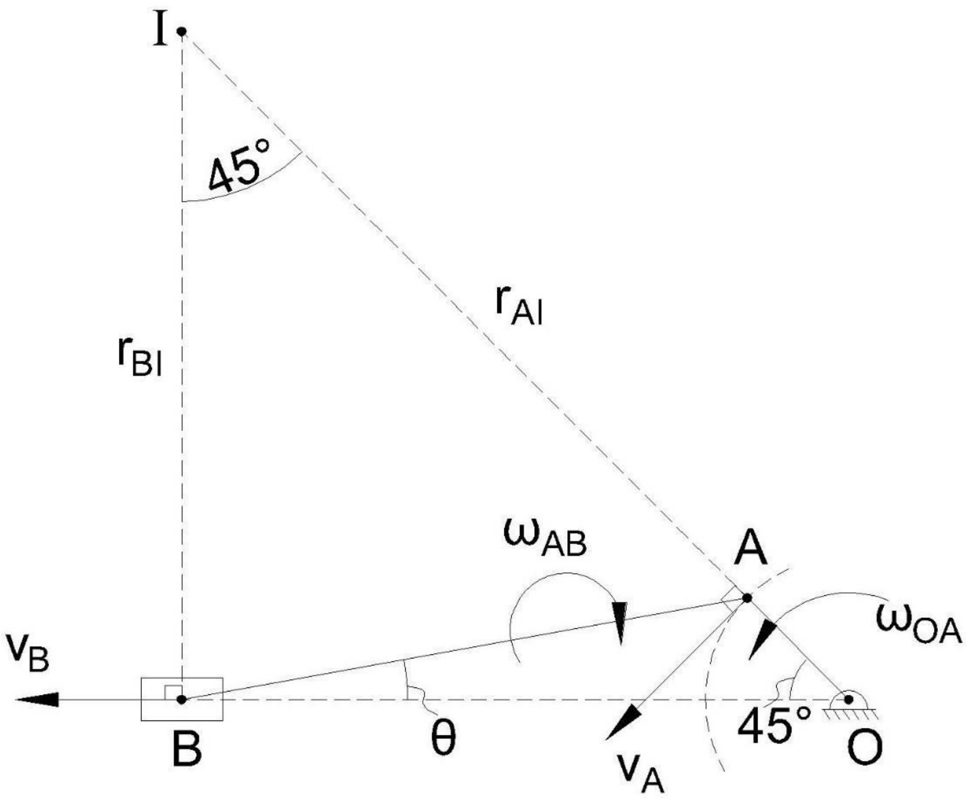
$$OA = 300\ \text{mm} = 0.3\ \text{m}$$

$$AB = 1200\ \text{mm} = 1.2\ \text{m}$$

$$\omega_{OA} = 180\ \text{rpm}\ \circlearrowleft = 180 \times \frac{2\pi}{60}\ \text{rad/s}$$

$$\omega_{OA} = 18.849\ \text{rad/s}$$

$$\angle AOB = 45°$$



$$\because v = r\omega \rightarrow v_A = r_{OA} \times \omega_{OA}$$
$$v_A = 0.3 \times 18.849 = 5.655\ \text{m/s} \ \swarrow$$

Drawing perpendiculars from the velocity of A (↙) and the velocity of B (←), we can locate their intersection point I.

In $\Delta OBI, \because \angle BOI = 45° \ \& \ \angle IBO = 90° \Rightarrow \angle BIO = 180° - 90° - 45° = 45°$

In $\Delta OAB, \angle AOB = 45° \ \ \& \ \text{ let } \angle ABO = \theta$

$\therefore$ using sine rule, $\dfrac{0.3}{\sin\theta} = \dfrac{1.2}{\sin 45°} \Rightarrow \sin\theta = \dfrac{0.3 \times 0.707}{1.2} \Rightarrow \theta = 10.18°$

$\therefore$ In $\Delta ABI, \qquad \angle ABI = 90° - \theta = 90° - 10.18° = 79.82°$

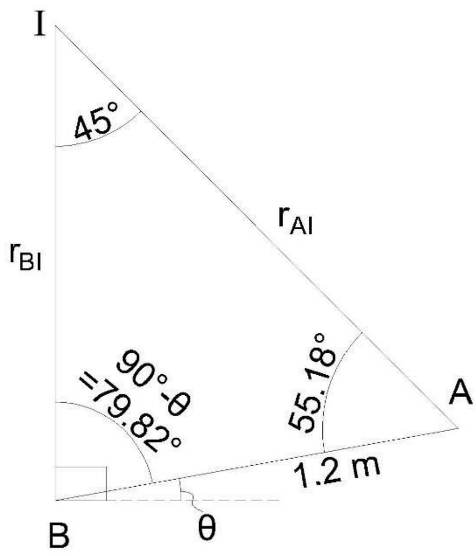
$\Rightarrow \angle IAB = 180° - 79.82° - 45° = 55.18°$

$\therefore$ using sine rule in ΔABI,

$$\frac{1.2}{\sin 45°} = \frac{r_{AI}}{\sin 79.82°} = \frac{r_{BI}}{\sin 55.18°}$$

$$\therefore r_{AI} = \frac{1.2 \times \sin 79.82°}{\sin 45°} = 1.6703\ \text{m}$$

$$\therefore r_{BI} = \frac{1.2 \times \sin 55.18°}{\sin 45°} = 1.39\ \text{m}$$



"I" is the instantaneous centre of rotation for connecting rod AB which is undergoing GP Motion. If AB is rotating about I with an angular velocity of ω_{AB},

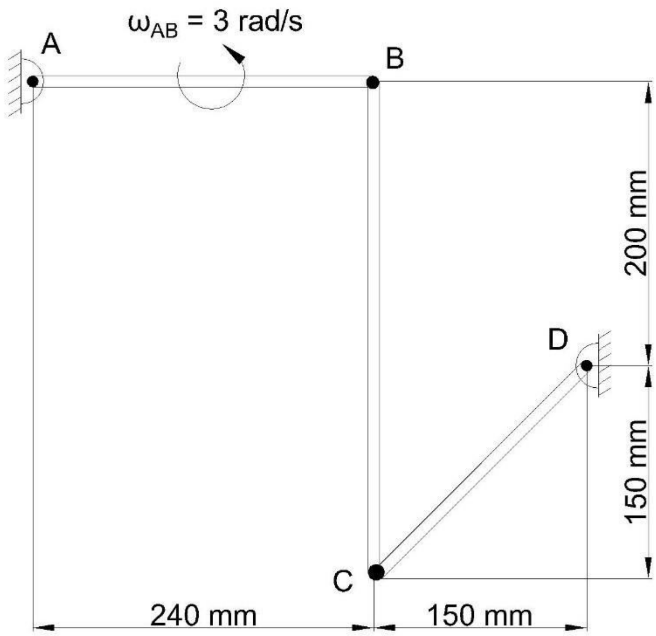
For point A, $v_A = r_{AI} \times \omega_{AB} \Rightarrow 5.655 = 1.6703 \times \omega_{AB}$

$$\therefore \omega_{AB} = 3.392 \text{ rad/s} \circlearrowright$$

For point B, $v_B = r_{BI} \times \omega_{AB} = 1.39 \times 3.392$

$$\therefore \mathbf{v_B = 4.714\ m/s} \leftarrow$$

<u>N2</u>: In the position shown, bar AB has a constant angular velocity of 3 rad/s anticlockwise. Determine the angular velocity of bar CD.



Soln: Rods AB and CD perform rotational motion and rod BC performs GP motion.

Given:

$$AB = 240 \text{ mm} = 0.24 \text{ m}$$

$$BC = 350 \text{ mm} = 0.35 \text{ m}$$

$$CD = \sqrt{150^2 + 150^2} = 212.132 \text{ mm} = 0.212 \text{ m}$$

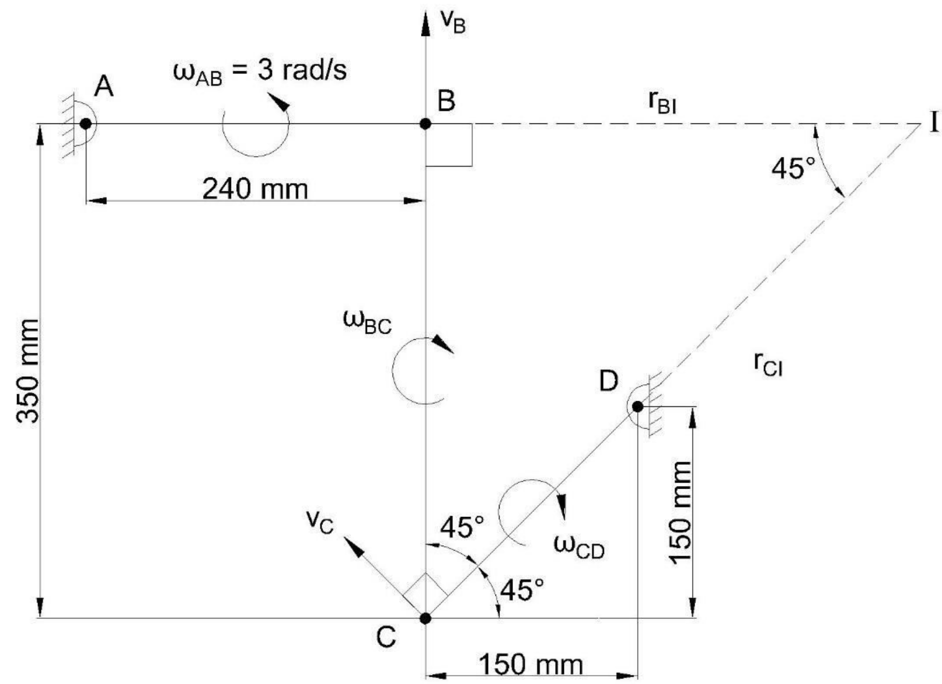
$$\omega_{AB} = 3 \text{ rad/s} \circlearrowleft$$

Velocity of B will be perpendicular to the rod AB in the upward direction.

$$v_B = r_{AB} \times \omega_{AB} = 0.24 \times 3$$
$$\therefore v_B = 0.72 \text{ m/s} \uparrow$$

Velocity of C will be perpendicular to the rod CD, up and to the left (↖).

We draw perpendiculars from the velocity of B (↑) and the velocity of C (↖), to find their intersection point I, which is the instantaneous centre of rotation for rod BC.



ΔBCI is a right-angles isosceles triangle.

$\therefore \mathrm{r_{BI}} = \mathrm{BC} = 0.35\ \mathrm{m} \quad \& \quad \mathrm{r_{CI}} = \sqrt{0.35^2 + 0.35^2} = 0.495\ \mathrm{m}$

The rod BC performs GP motion about centre of rotation I.

For point B, $\mathrm{v_B} = \mathrm{r_{BI}} \times \omega_{\mathrm{BC}} \Rightarrow 0.72 = 0.35 \times \omega_{\mathrm{BC}}$

$$\therefore \omega_{\mathrm{BC}} = 2.057\ \mathrm{rad/s} \circlearrowright$$

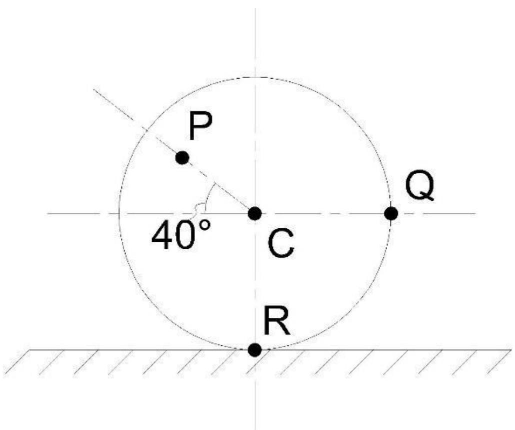
For point C, $\mathrm{v_C} = \mathrm{r_{CI}} \times \omega_{\mathrm{BC}} = 0.495 \times 2.057$

$$\therefore \mathrm{v_C} = 1.018\ \mathrm{m/s} \nwarrow$$

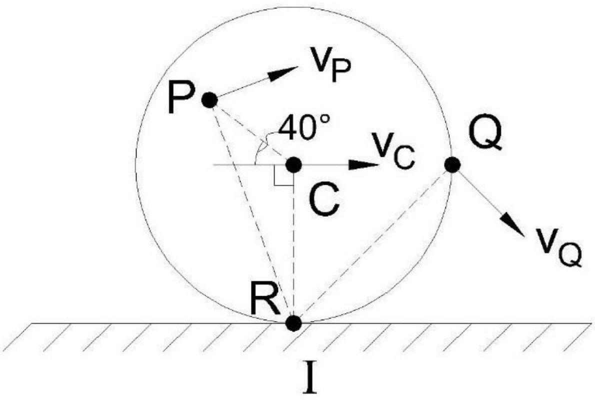
Rod CD is rotating about D, $\mathrm{v_C} = \mathrm{r_{CD}} \times \omega_{\mathrm{CD}} \Rightarrow 1.018 = 0.212 \times \omega_{\mathrm{BC}}$

$$\therefore \boldsymbol{\omega_{\mathrm{CD}} = 4.8\ \mathrm{rad/s}} \circlearrowright$$

N3: A 0.4 m diameter wheel rolls on a horizontal plane without slip, such that its centre has a velocity of 10 m/s towards right. Find angular velocity of the wheel and also velocities of points P, Q, and R shown on the wheel. Given l(CP) = 0.15 m.



Soln: For a wheel which rolls without slipping (performing a general plane motion), the Instantaneous Centre of Rotation is the point of contact with the ground. This is because the centre of rotation should have zero velocity and the point in contact with the ground has velocity of the ground, which is zero. Hence, point R is the instantaneous centre of rotation I.

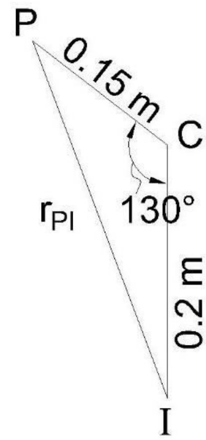


Given:

$$\text{Diameter} = 0.4\ \text{m} \Rightarrow \text{CI} = 0.2\ \text{m}$$
$$v_C = 10\ \text{m/s}$$
$$\text{CP} = 0.15\ \text{m}$$

Let ω be the angular velocity of the wheel (all points will have the same ω as it is one single body moving together)

For point C, $v_C = r_{CI} \times \omega \Rightarrow 10 = 0.2 \times \omega$



$$\therefore \omega = 50\ \text{rad/s} \circlearrowright$$

$\therefore$ In ΔCPI, $\angle PCI = 90° + 40° = 130°$

$\therefore$ using cosine rule,

$r_{PI}^2 = 0.15^2 + 0.2^2 - 2(0.15)(0.2)\cos 130°$

$r_{PI} = 0.3179$ m

For point P, $v_P = r_{PI} \times \omega = 0.3179 \times 50 \Rightarrow$

$$\therefore \mathbf{v_P = 15.895\ m/s} \nearrow$$

$\therefore$ In right angled ΔICQ, $r_{PI} = \sqrt{0.2^2 + 0.2^2} = 0.2828$ m

For point Q, $v_Q = r_{QI} \times \omega = 0.2828 \times 50 \Rightarrow$

$$\therefore \mathbf{v_Q = 14.14\ m/s} \searrow$$

Since, R coincides with the instantaneous centre I, velocity of point R is zero because ICR has zero velocity. $\therefore \mathbf{v_R = 0}$

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Engineering Mechanics Notes

Module 3 – Centroid of Wires, Laminas and Solids

Class: FY BTech Division: C3
Professor: Aniket S. Patil Date: 05/05/2023

Centre of Gravity is a point where the whole weight of the body is assumed to act; or, it is a point where the entire distribution of gravitational force (weight) is supposed to be concentrated. It is usually denoted by the letter G.

Centroid is the geometrical centre of an object, where the entire length of a wire, or area of a plane lamina, or volume of a solid is supposed to be concentrated.

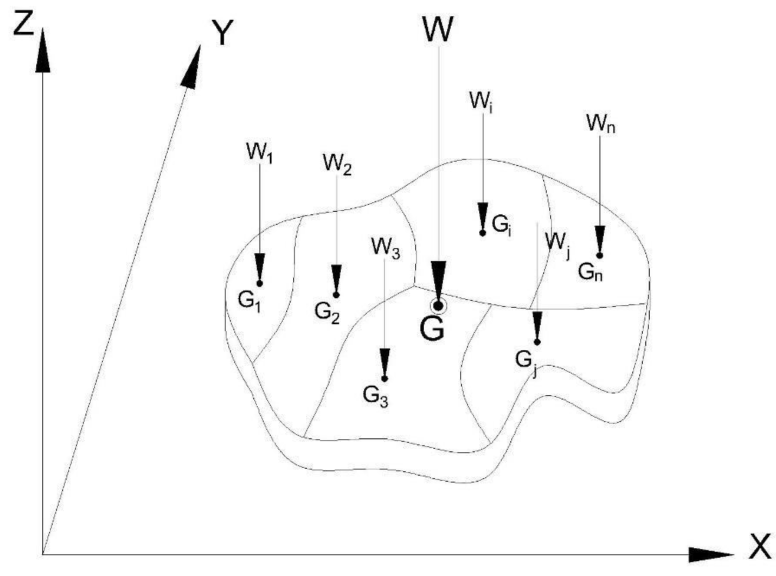
NOTE: For a uniformly distributed object (homogenous material), centre of gravity and centroid are the same point. But if an object is made of different composite materials, these points will be different.

Centre of Mass is a point where the entire mass is supposed to be concentrated.

NOTE: At the surface of the earth, centre of mass and centre of gravity will be the same point. But if the body is too large compared to the earth, then gravitational force will act differently at different parts of the body while the mass will remain as it is. Then, these points will be different.

Relation for Centre of Gravity:

Consider a body of weight W whose centre of gravity is located at $G\ (\bar{x}, \bar{y})$. If the body is split in "n" parts, each part will have its elemental weight W_i acting through its centre of gravity at $G_i\ (\bar{x}_i, \bar{y}_i)$.



The individual weights $W_1, W_2, W_3, \ldots, W_i, \ldots W_n$ form a system of parallel forces.

$$W = W_1 + W_2 + \cdots + W_i + \cdots + W_n$$
$$W = \sum W_i$$

Using Varignon's theorem to find the location of resultant weight force W:

Taking moments about y-axis:

$$W \times \bar{x} = W_1 \times x_1 + W_2 \times x_2 + \cdots + W_i \times x_i + \cdots + W_n \times x_n$$
$$W \times \bar{x} = \sum W_i x_i$$

$$\bar{x} = \frac{\sum W_i x_i}{W} = \frac{\sum W_i x_i}{\sum W_i}$$

Similarly, taking moments about x-axis, we will get:

$$\bar{y} = \frac{\sum W_i y_i}{W} = \frac{\sum W_i y_i}{\sum W_i}$$

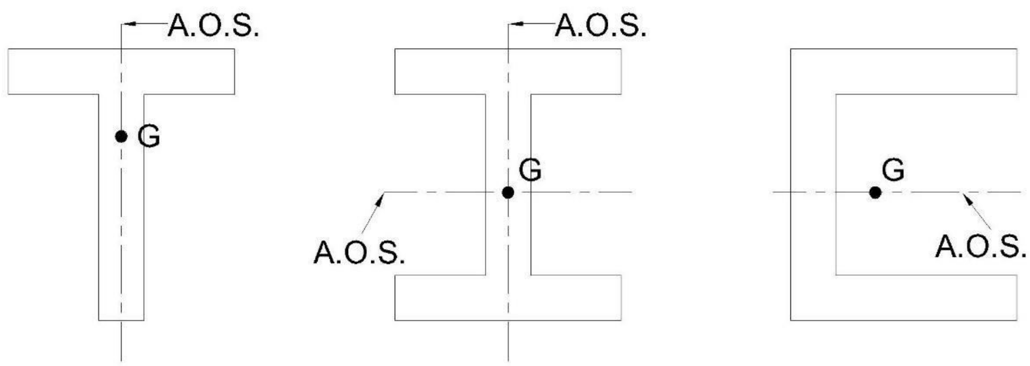
Relation for Centroid of Plane Laminas or Areas:

$$\text{Weight} = \text{mass} \times g$$
$$W = (\text{density} \times \text{volume}) \times g$$
$$W = (\text{density} \times \text{area} \times \text{thickness}) \times g$$
$$W = \rho \times A \times t \times g = (\rho \times t \times g) \times A$$

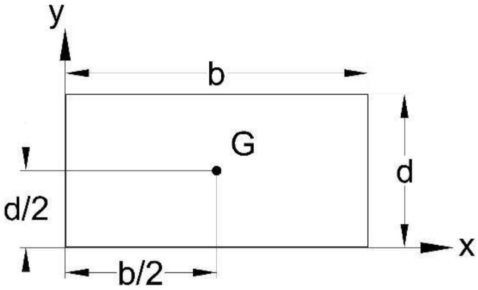
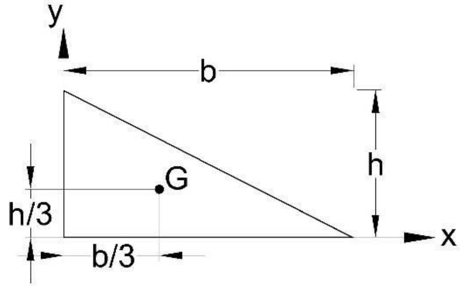
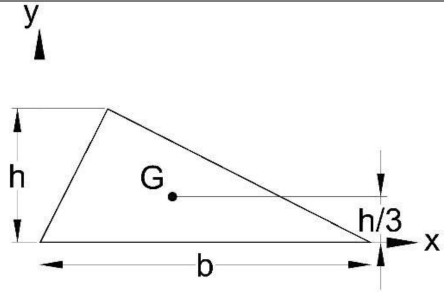
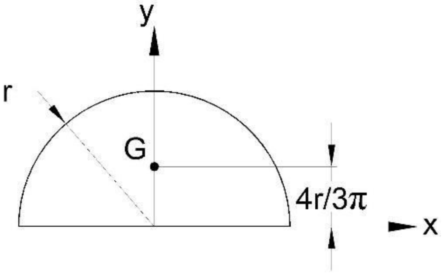
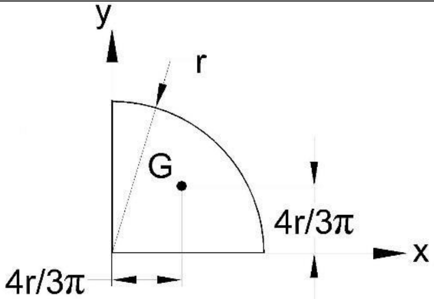
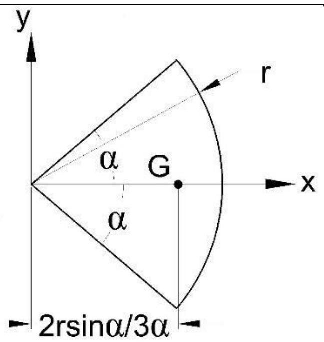
For uniform bodies (same density and thickness throughout), the centroid of the body is given by,

$$\bar{x} = \frac{\sum (\rho \times t \times g) A_i x_i}{\sum (\rho \times t \times g) A_i} = \frac{\sum A_i x_i}{\sum A_i}$$
$$\bar{y} = \frac{\sum (\rho \times t \times g) A_i y_i}{\sum (\rho \times t \times g) A_i} = \frac{\sum A_i y_i}{\sum A_i}$$

Axis of Symmetry: It is defined as the line which divides the figure into two equal parts such that each part is a mirror image of the other. If a body is symmetrical, then its centroid will lie on the axis of symmetry. If it has more than one axis of symmetry, the centroid will lie on the intersection of the axes of symmetry.



Centroids of Regular Plane Laminas:

Shape	**Figure**	**Area**	$\bar{x}$	$\bar{y}$
Rectangle		$b \times d$	$\frac{b}{2}$	$\frac{d}{2}$
Right Angled Triangle		$\frac{1}{2} \times b \times h$	$\frac{b}{3}$	$\frac{h}{3}$
Any Triangle		$\frac{1}{2} \times b \times h$	–	$\frac{h}{3}$
Semi-Circle		$\frac{\pi r^2}{2}$	0	$\frac{4r}{3\pi}$
Quarter-Circle		$\frac{\pi r^2}{4}$	$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$
Sector of a Circle		$r^2\alpha$	$\frac{2r\sin\alpha}{3\alpha}$ α in radians	0

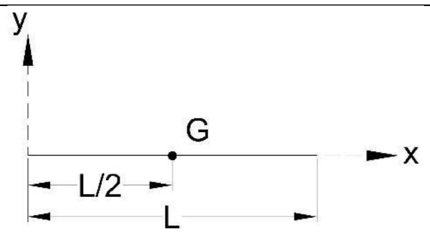
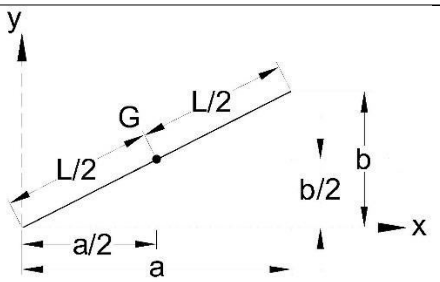
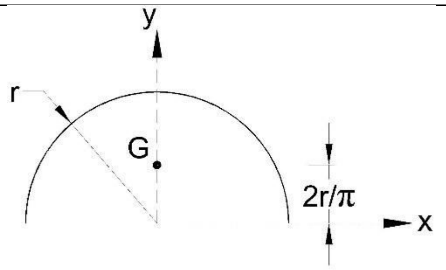
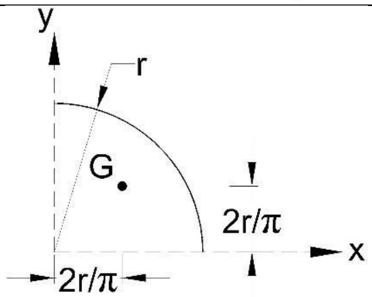
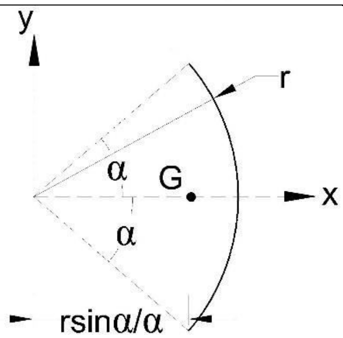
Relation for Centroid of Lines or Wires:

Lines are 1-D figures whose length is more prominent than its thickness, with the thickness being uniform throughout its length.

$$\bar{x} = \frac{\sum A_i x_i}{\sum A_i} = \frac{\sum L_i \times b \times x_i}{\sum L_i \times b} = \frac{\sum L_i x_i}{\sum L_i} \text{ \& Similarly, } \bar{y} = \frac{\sum L_i y_i}{\sum L_i}$$

The physical bodies which are equivalent to lines are bent up wires, pipe lines, etc.

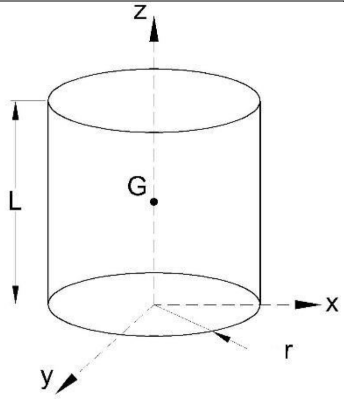
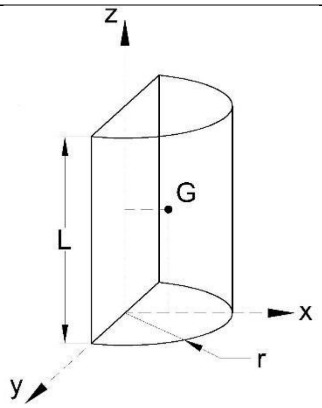
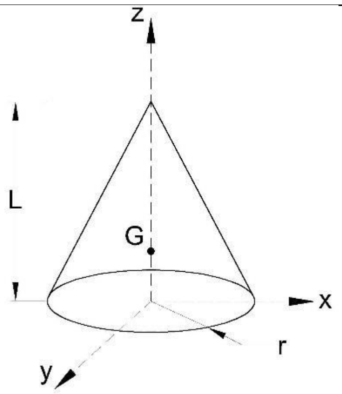
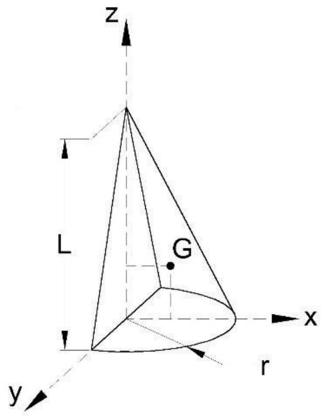
Centroids of Regular Lines:

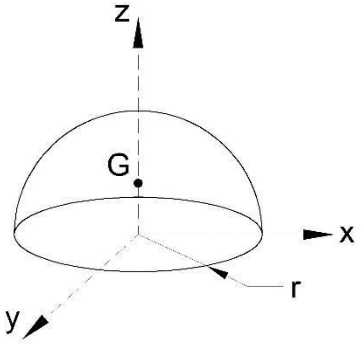
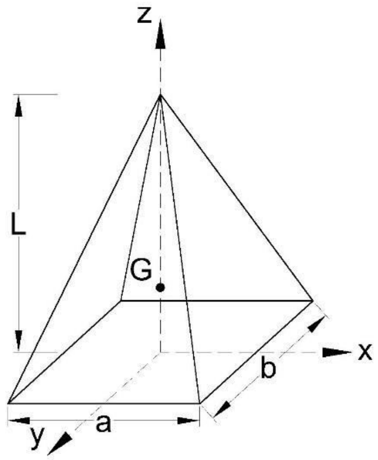
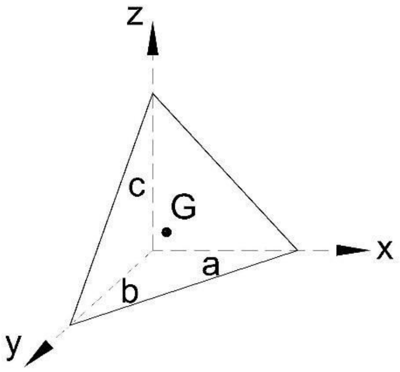
Shape	Figure	Length	$\bar{x}$	$\bar{y}$
Straight Horizontal Line		L	$\frac{L}{2}$	0
Straight Inclined Line		L	$\frac{a}{2}$	$\frac{b}{2}$
Semi-Circular Arc		πr	0	$\frac{2r}{\pi}$
Quarter-Circular Arc		$\frac{\pi r}{2}$	$\frac{2r}{\pi}$	$\frac{2r}{\pi}$
Circular Arc		$2r\alpha$	$\frac{r\sin\alpha}{\alpha}$ α in radians	0

Relation for Centroid of Solids:

$$\bar{x} = \frac{\sum W_i x_i}{\sum W_i} = \frac{\sum V_i \times \rho \times g \times x_i}{\sum V_i \times \rho \times g} = \frac{\sum V_i x_i}{\sum V_i} \text{ \& Similarly, } \bar{y} = \frac{\sum V_i y_i}{\sum V_i}, \bar{z} = \frac{\sum V_i z_i}{\sum V_i}$$

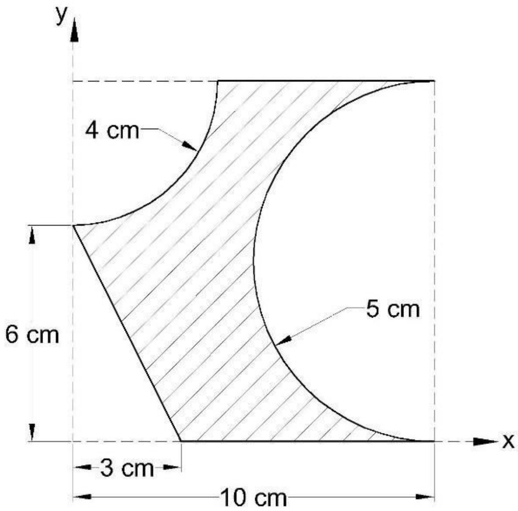
Centroids of Regular Solids:

Shape	Figure	Volume	$\bar{x}$	$\bar{y}$	$\bar{z}$
Cylinder		$\pi r^2 L$	0	0	$\frac{L}{2}$
Semi-Cylinder		$\frac{\pi r^2 L}{2}$	$\frac{4r}{3\pi}$	0	$\frac{L}{2}$
Cone		$\frac{1}{3}\pi r^2 L$	0	0	$\frac{L}{4}$
Half Cone		$\frac{\pi r^2 L}{6}$	$\frac{r}{\pi}$	0	$\frac{L}{4}$

Hemi-Sphere		$\frac{2}{3}\pi r^3$	0	0	$\frac{3r}{8}$
Square Pyramid		$\frac{1}{3}abL$	0	0	$\frac{L}{4}$
Regular Tetrahedron		$\frac{1}{6}abc$	$\frac{a}{4}$	$\frac{b}{4}$	$\frac{c}{4}$

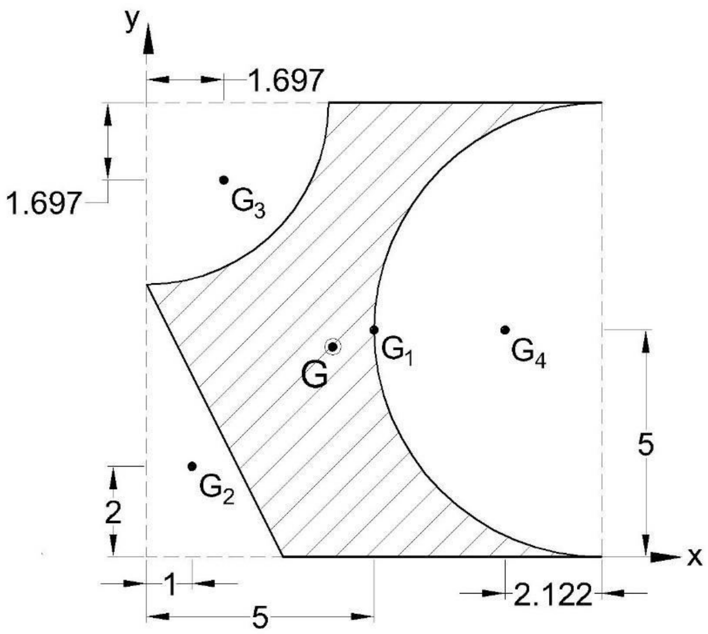
Numericals:

N1: Find the centroid of the shaded area as shown in the given figure.



Soln: The shaded region can be obtained by taking an entire square of 10 cm x 10 cm and subtracting a right triangle of 6 cm x 3 cm, a quarter-circle of radius 4 cm, and a semi-circle of radius 5 cm.

Section	**Area (A_i) cm^2**	**x_i cm**	**y_i cm**	**$A_i x_i$ cm^3**	**$A_i y_i$ cm^3**
Square	$10 \times 10 = 100$	$\frac{10}{2} = 5$	$\frac{10}{2} = 5$	500	500
Triangle	$-\frac{1}{2} \times 3 \times 6 = -9$	$\frac{3}{3} = 1$	$\frac{6}{3} = 2$	-9	-18
Quarter-Circle	$-\frac{\pi(4)^2}{4} = -12.57$	$\frac{4 \times 4}{3\pi} = 1.697$	$10 - \frac{4 \times 4}{3\pi} = 8.302$	-21.32	-104.33
Semi-Circle	$-\frac{\pi(5)^2}{2} = -39.27$	$10 - \frac{4 \times 5}{3\pi} = 7.878$	5	-309.37	-196.35
Total	$\boldsymbol{\sum A_i = 39.16}$			$\boldsymbol{\sum A_i x_i = 160.31}$	$\boldsymbol{\sum A_i y_i = 181.32}$



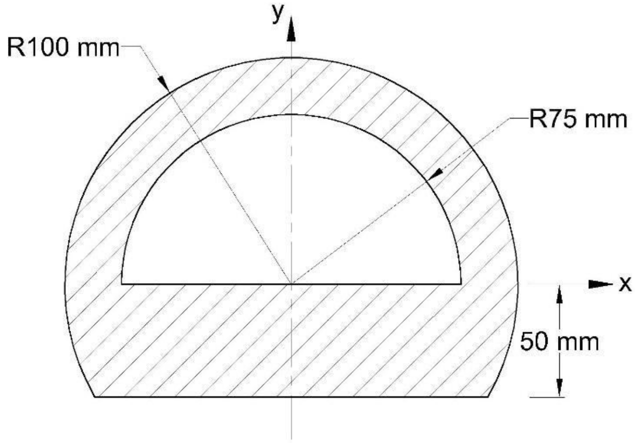
The co-ordinates of the centroid are given by,

$$\bar{x} = \frac{\sum A_i x_i}{\sum A_i} = \frac{160.31}{39.16} = 4.09 \text{ cm}$$

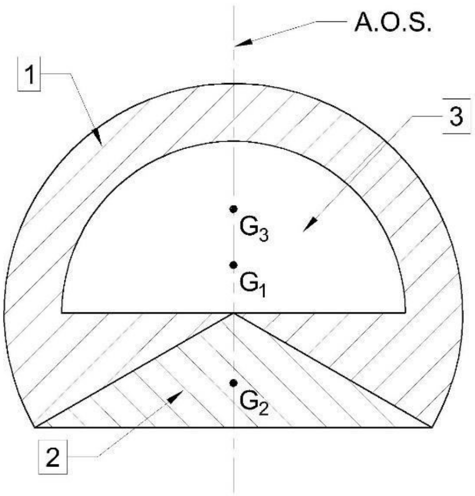
$$\bar{y} = \frac{\sum A_i y_i}{\sum A_i} = \frac{181.32}{39.16} = 4.63 \text{ cm}$$

Hence, the centroid of the shaded area is G (4.09, 4.63) cm.

N2: A semi-circular section is removed from the plane area as shown. Find the centroid of the shaded area.

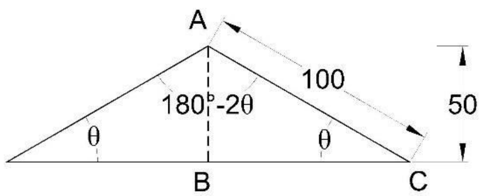


Soln: The shaded area can be obtained by adding a sector and a triangle and subtracting a semi-circle. All three areas are symmetrical about the y-axis; therefore, their centroids are going to lie on the y-axis, meaning x-coordinates are 0.



Let us first consider the triangular section. Within that consider the ΔABC:

$$\sin\theta = \frac{AB}{AC} = \frac{50}{100} \Rightarrow \theta = \sin^{-1} 0.5 = 30°$$

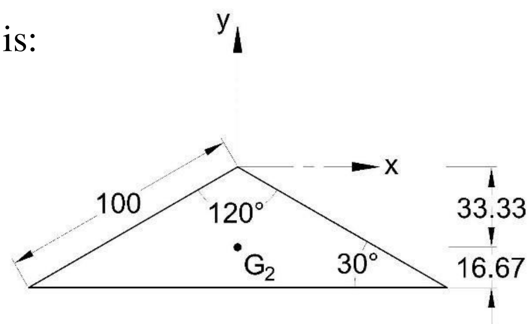


$$\text{And, } BC = \sqrt{100^2 - 50^2} = 86.6025 \text{ mm}$$

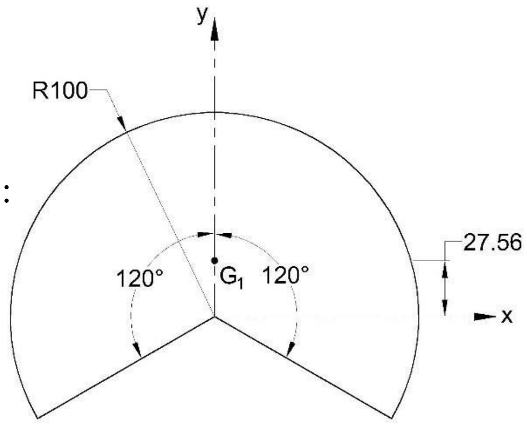
Hence, the area and y-coordinate for the triangle is:

$$A_2 = \frac{1}{2} \times (2 \times 86.6025) \times 50 = 4330.13 \text{ mm}^2$$

$$y_2 = 50 - \frac{1}{3} \times 50 = 50 - 16.67 = 33.33 \text{ mm}$$



The α for the sector will be 120° considering the angles found from the triangle, which in radians is 2.094395.

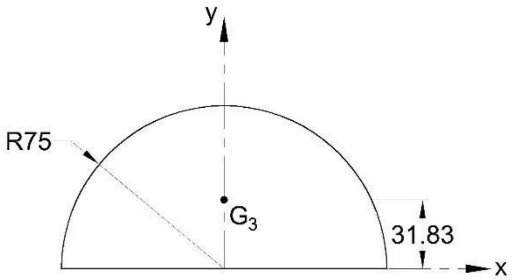


Hence, the area and y-coordinate for the sector is:

$$A_1 = 100^2 \times 2.094395 = 20943.95 \text{ mm}^2$$

$$y_1 = \frac{2}{3} \times \frac{100 \sin 2.094395}{2.094395} = 27.57 \text{ mm}$$

The area and y-coordinate for the semi-circle is:



$$A_1 = \frac{\pi}{2} \times 75^2 = 8835.73 \text{ mm}^2$$

$$y_1 = \frac{4}{3} \times \frac{75}{\pi} = 31.83 \text{ mm}$$

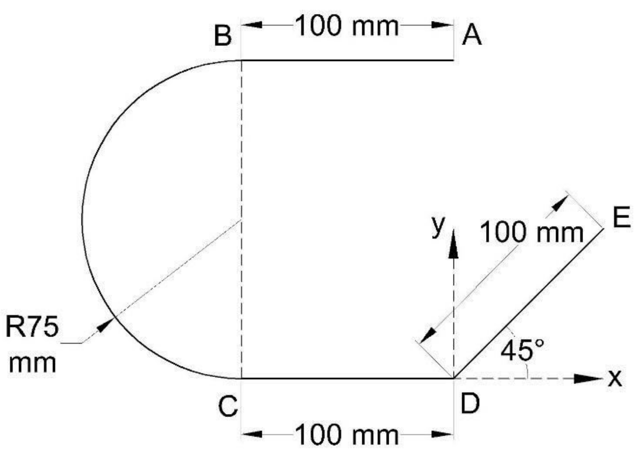
Section	**Area (A_i) mm²**	**y_i mm**	**A_iy_i mm³**
Sector	20943.95	27.57	577424.70
Triangle	4330.13	−33.33	−144323.23
Semi-Circle	−8835.73	31.83	−281241.29
Total	$\sum A_i = \mathbf{16438.35}$		$\sum A_i y_i = \mathbf{151860.18}$

The y-coordinate of the centroid,

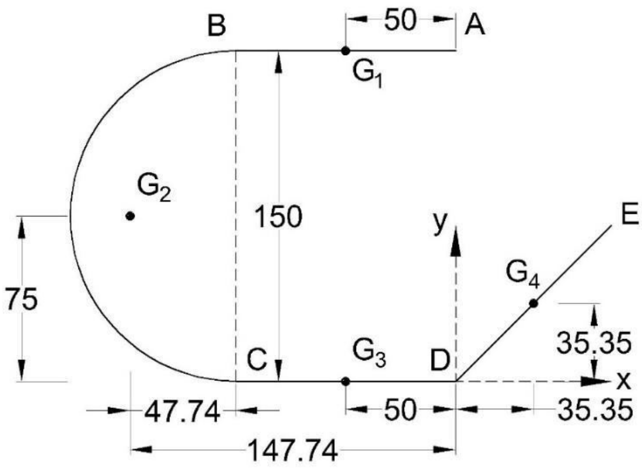
$$\bar{y} = \frac{\sum A_i y_i}{\sum A_i} = \frac{151860.18}{16438.35} = 9.24 \text{ mm}$$

Hence, the centroid of the shaded area is G (0, 9.24) mm.

<u>N3</u>: A uniform wire is bent into shape as shown. Calculate the C.G. of the wire.



Soln: The bent-up wire is made up of 2 straight horizontal portions AB and CD, a semi-circular portion BC and a straight inclined portion DE.



Section	Length (L_i) mm	x_i mm	y_i mm	L_ix_i mm^2	L_iy_i mm^2
AB	100	−50	150	−5000	15000
BC	$\pi \times 75 = 235.62$	−147.74	75	−34812	17671
CD	100	−50	0	−5000	0
DE	100	35.35	35.35	3535	3535
Total	$\sum L_i = \mathbf{535.62}$			$\sum L_ix_i = \mathbf{-41277}$	$\sum L_iy_i = \mathbf{36206}$

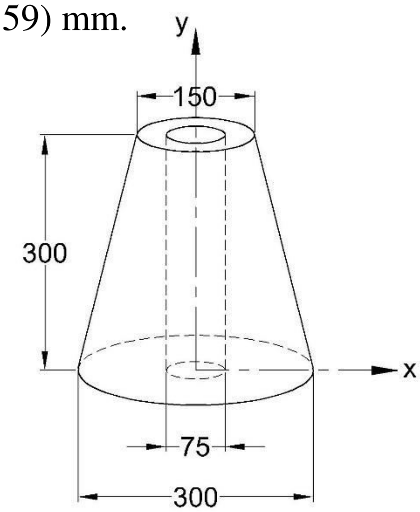
The co-ordinates of the centroid are given by,

$$\bar{x} = \frac{\sum L_i x_i}{\sum L_i} = \frac{-41277}{535.62} = -77.06 \text{ mm}$$

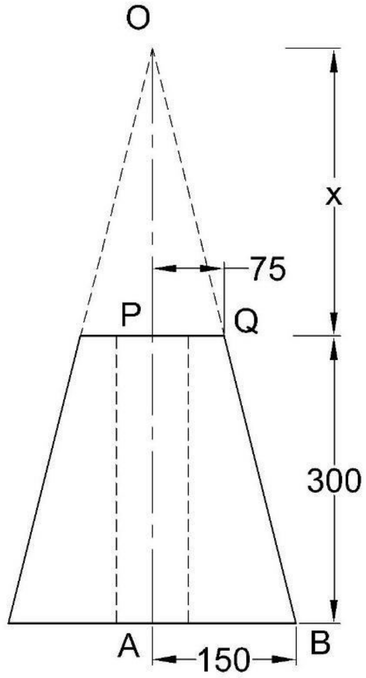
$$\bar{y} = \frac{\sum L_i y_i}{\sum L_i} = \frac{36206}{535.62} = 67.59 \text{ mm}$$

Hence, the centroid of the bent-up wire is G (-77.06, 67.59) mm.

N4: The frustum of a solid circular cone of smaller diameter 150 mm and larger diameter 300 mm with height 300 mm has an axial hole of 75 mm diameter. Determine the C.G. of the body.



Soln: The frustum of a cone can be considered as a bigger cone subtracted by a smaller cone. Considering the front projections of the cone as shown below:



In ΔOAB, AB is the radius of the larger cone.

$$\therefore AB = \frac{1}{2} \times 300 = 150 \text{ mm}$$

In ΔOPQ, PQ is the radius of the smaller cone.

$$\therefore PQ = \frac{1}{2} \times 150 = 75 \text{ mm}$$

Also, let OP, the height of smaller cone, be x mm.

$$\because \Delta OAB \sim \Delta OPQ, \therefore \frac{OA}{AB} = \frac{OP}{PQ}$$

$$\therefore \frac{x + 300}{150} = \frac{x}{75} \Rightarrow x = 300 \text{ mm}$$

Thus, the given body can be considered as a larger cone with height 600 mm and radius 150 mm subtracted by smaller cone with height 300 mm and radius 75 mm and subtracted by a cylinder with height 300 mm and radius 37.5 mm.

Also, the body is symmetric about y-axis, hence the C.G. will lie on it.

Section	**Volume (V_i) mm^3**	**y_i mm**	**V_iy_i mm^4**
Larger Cone	$\frac{1}{3}\pi(150^2)(600) = 4500000\pi$	$\frac{600}{4} = 150$	675000000π
Smaller Cone	$-\frac{1}{3}\pi(75^2)(300) = -562500\pi$	$300 + \frac{300}{4} = 375$	-210937500π
Cylinder	$-\pi(37.5^2)(300) = -421875\pi$	$\frac{300}{2} = 150$	-63281250π
Total	$\boldsymbol{\sum V_i = 3515625\pi}$		$\boldsymbol{\sum V_iy_i = 400781250\pi}$

The y-coordinate of the centroid,

$$\bar{y} = \frac{\sum V_i y_i}{\sum V_i} = \frac{400781250\pi}{3515625\pi} = 114 \text{ mm}$$

Hence, the centroid of the shaded area is G (0, 114) mm.

K J Somaiya College of Engineering, Vidyavihar, Mumbai

(A Constituent College of SVU)

Engineering Mechanics Notes

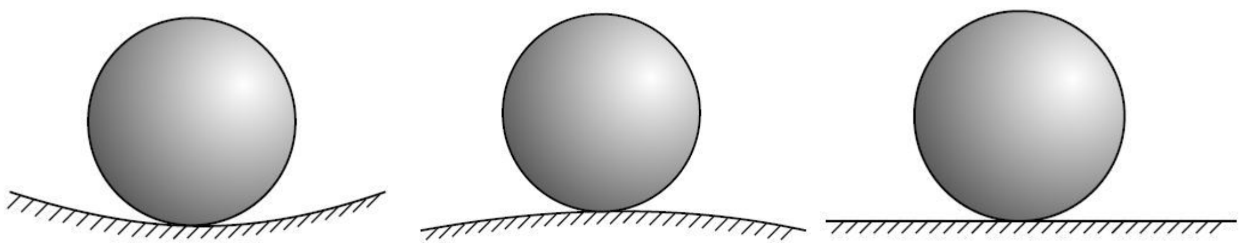
Module 4 – Equilibrium of Force System and Friction

Module Section 4.1 – Equilibrium of Force System

Class: FY BTech Faculty: Aniket S. Patil Date: 09/06/23

References: Engineering Mechanics, by M. D. Dayal & Engineering Mechanics – Statics and Dynamics, by N. H. Dubey.

Equilibrium: A body is said to be in equilibrium when it is in a state of rest or uniform motion. We can say there are three types of equilibria: Stable, Unstable and Neutral, as illustrated by following figures.



Conditions of Equilibrium (COE): From Newton's second law of motion, for a body to be in equilibrium the resultant of the system has to be zero. This implies that the sum of all forces should be zero, i.e., $\sum \bar{F} = 0$; and the sum of all moments should also be zero $\sum \bar{M} = 0$.

For a coplanar system of forces, the COE are: $\sum F_x = 0$, $\sum F_y = 0$ & $\sum M = 0$

COE for Various Force System:

1. Concurrent Force System: One of the following sets of equations can be used
    a. $\sum F_x = 0$ & $\sum F_y = 0$
    b. $\sum F_x = 0$ & $\sum M_A = 0$ (A should not lie on y-axis)
    c. $\sum F_y = 0$ & $\sum M_B = 0$ (B should not lie on x-axis)
2. Parallel Force System: One of the following sets of equations can be used
    a. $\sum F = 0$ & $\sum M = 0$
    b. $\sum M_A = 0$ & $\sum M_B = 0$ (line AB should not be parallel to the forces)
3. General Force System: One of the following sets of equations can be used
    a. $\sum F_x = 0$, $\sum F_y = 0$ & $\sum M = 0$
    b. $\sum F_x = 0$, $\sum M_A = 0$ & $\sum M_B = 0$
    (line AB should not be perpendicular to the x-axis)
    c. $\sum M_A = 0$, $\sum M_B = 0$ & $\sum M_C = 0$ (A, B, & C should not be collinear)

Free Body Diagram (FBD):

The Free Body Diagram (FBD) is a sketch of the body showing all active and reactive forces that acts on it after removing all supports with consideration of geometrical angles and distance given.

To investigate the equilibrium of a body, we remove the supports and replace them by the reactions which they exert on the body. The first step in equilibrium analysis is to identify all the forces that act on the body, which is represented by a free body diagram. Therefore, the free body diagram is the most important step in the solution of problems in mechanics.

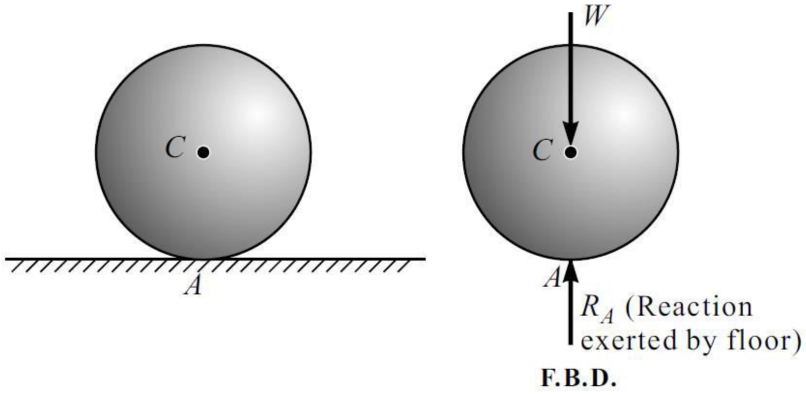
Importance of FBD:

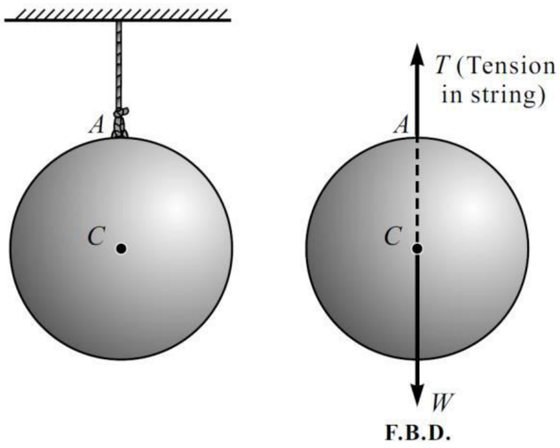
1. The sketch of FBD is the key step that translates a physical problem into a form that can be analysed mathematically.
2. The FBD is the sketch of a body, a portion of a body or two or more connected bodies completely isolated or free from all other bodies, showing the force exerted by all other bodies on the one being considered.
3. FBD represents all active (applied) forces and reactive (reactions) forces. Forces acting on the body that are not provided by the supports are called active force (weight of the body and applied forces). Reactive forces are those that are exerted on a body by the supports to which it is attached.
4. FBD helps in identifying known and unknown forces acting on a body.
5. FBD helps in identifying which type of force system is acting on the body so by applying appropriate condition of equilibrium, the required unknowns are calculated.

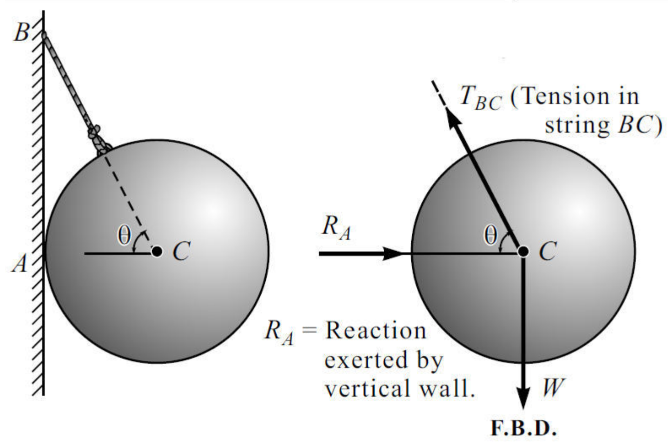
Procedure for Drawing an FBD:

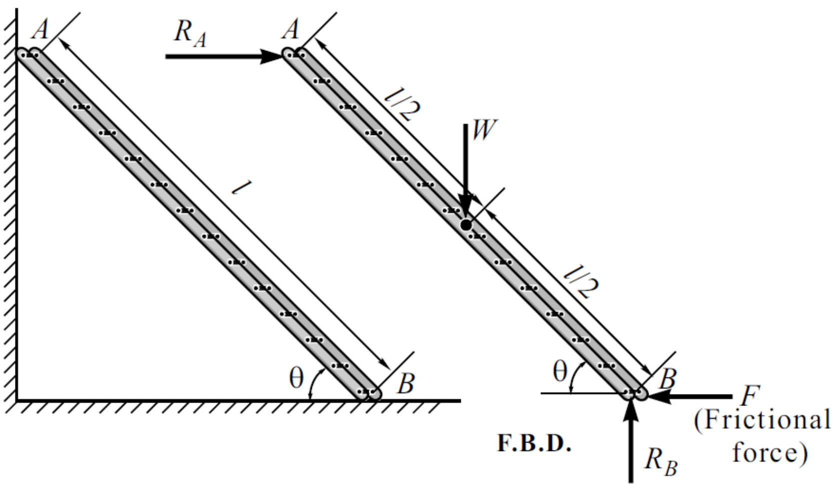
1. Draw a neat sketch of the body assuming that all supports are removed.
2. FBD may consist of an entire assembled structure or any combination or part of it.
3. Show all the relevant dimensions and angles on the sketch.
4. Show all the active forces on corresponding point of application and insert their magnitude and direction, if known.
5. Show all the reactive forces due to each support.
6. The FBD should be legible and neatly drawn, and of sufficient size, to show dimensions, since this may be needed in computation of moments of forces.
7. **If the sense of reaction is unknown, it should be assumed. The solution will determine the correct sense. A positive result indicates that the assumed sense is correct, whereas a negative result means the assumed sense is incorrect, so the correct sense is opposite to the assumed sense.**
8. Use principle of transmissibility wherever convenient.

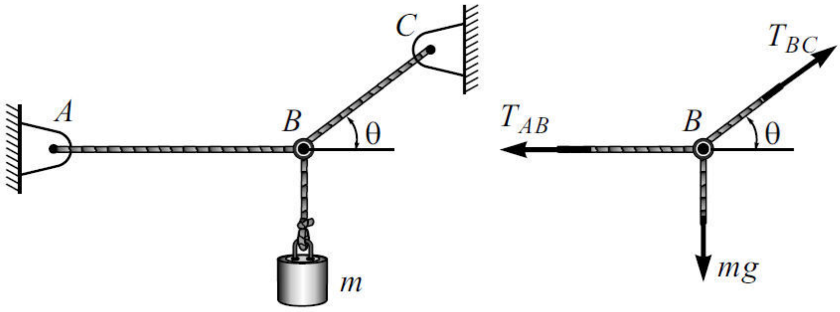
Examples:


C
A
W
C
A
R_A (Reaction exerted by floor)
F.B.D.

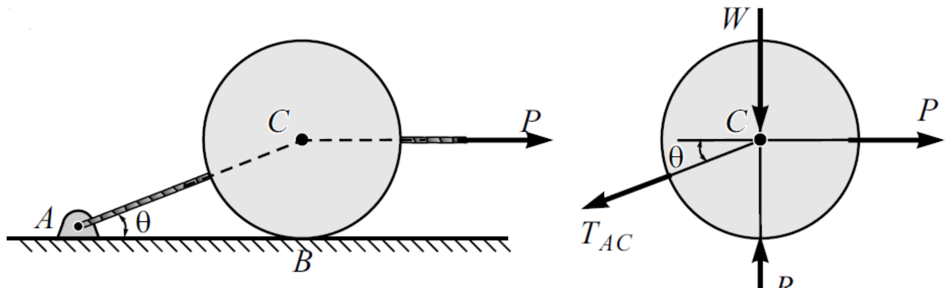

A
C
T (Tension in string)
A
C
W
F.B.D.


B
θ
C
A
T_{BC} (Tension in string BC)
R_A
θ
C
R_A = Reaction exerted by vertical wall.
W
F.B.D.


A
l
θ
B
R_A
A
l/2
W
l/2
θ
B
F (Frictional force)
R_B
F.B.D.



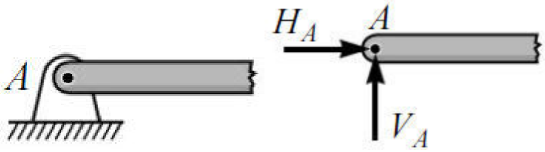
F.B.D.

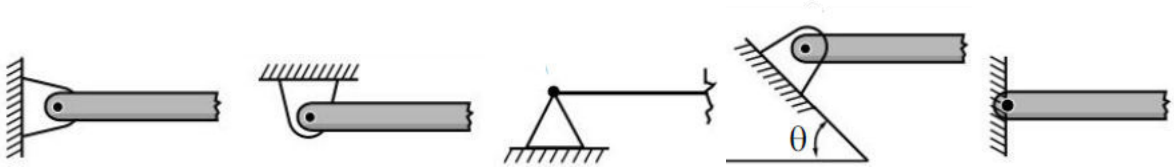


F.B.D.

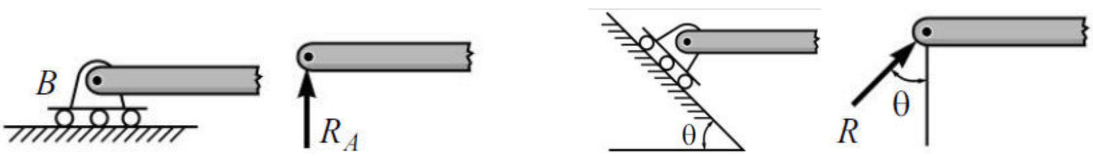
Types of Support:

1. Hinge (Pin) Support: The hinge support allows free rotation about the pin end but it does not allow linear displacement of that end. Since linear displacements are restricted in horizontal and vertical directions, the reactions offered at hinge support are H_A and V_A. E.g., doors on hinges, laptops, suitcases, etc.

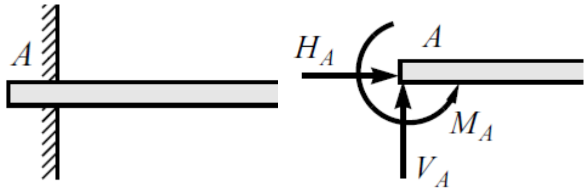




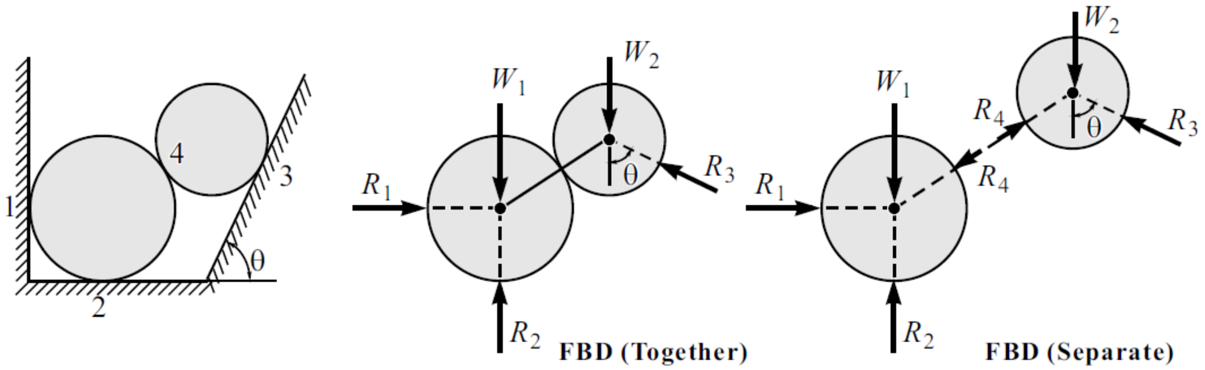
2. Roller Support: A roller support is equivalent to a frictionless surface. It can only exert a force that is perpendicular to the supporting surface. The roller support is free to roll along the surface on which it rests. Since the linear displacement in normal direction to surface of roller is restricted, it offers a reaction in normal direction to surface of roller (R_A). E.g., sliding doors, drawers, etc. Collar or slider free to move along smooth guides are also similar to roller support since they can support force normal to guide only.



3. Fixed Support: When the end of a beam is fixed then that support is said to be fixed support. Fixed support neither allows linear displacement nor rotation of a beam. Due to these restrictions, the reactions offered at fixed supports are horizontal component H_A, vertical component V_A and couple component M_A.



4. Smooth Surface Contact: When a body is in contact with a smooth (frictionless) surface at only one point, the reaction is a force normal to the surface, acting at the point of contact.

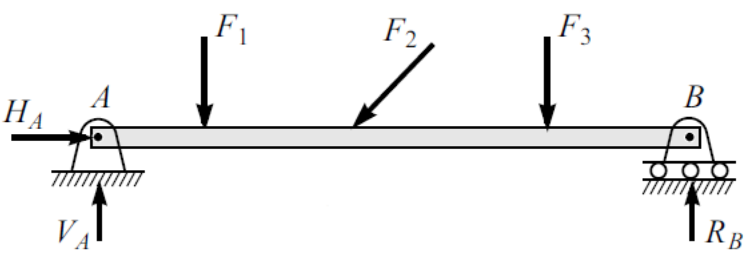


5. Inextensible String, Cable, Belt Rope, Cord, Chain or Wire: The force developed in rope is always a tension away from the body in the direction of rope. When one end of a rope is connected to a body, then the rope is not to be considered as a part of the system and it is replaced by tension in FBD.

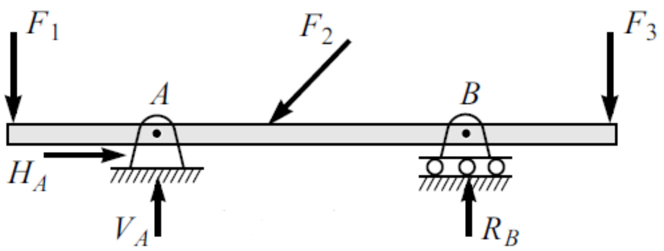
Types of Beams:

A horizontal member which takes transverse load (perpendicular to the length of the member) in addition to other loading is called beam. It is capable to take all types of loads, i.e., transverse load, tensile load, compressive load, twisting load, etc.

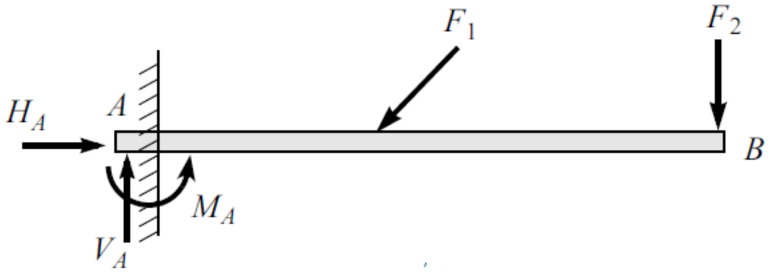
1. Simply Supported Beam: As the name indicates, it is the simplest of all beams which is supported by a hinge at one end and a roller at the other end.



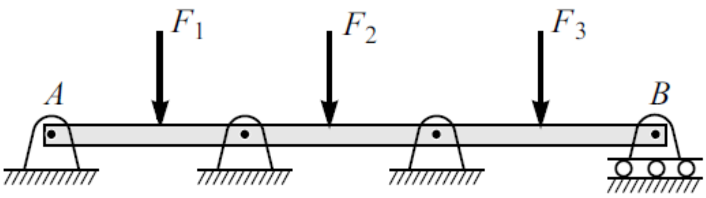
2. Overhang Beam: Here, one end or both the ends of simply supported beam is projected beyond the supports, which means that the portion of beam extends beyond the hinge and roller supports.



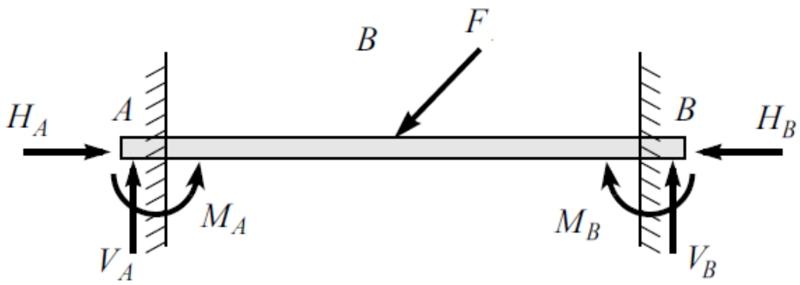
3. Cantilever Beam: A beam which is fixed at one end and free at the other end is called a cantilever beam. E.g., wall bracket, projected balconies, etc. One end of the beam is cast in concrete and is nailed, bolted, riveted, or welded.



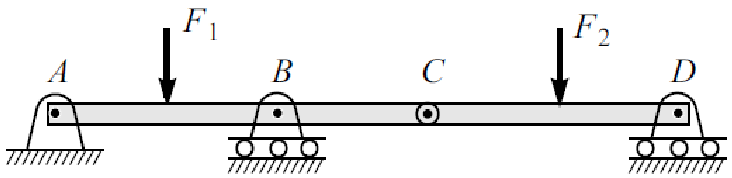
4. Continuous Beam: A beam which has more than two support is said to be a continuous beam. The extreme left and right supports are the end supports of the beam. Such beams are also called statically indeterminate beams because the reactions cannot be obtained by the equation of equilibrium.



5. Fixed Beam: A beam which is fixed at both the ends is called a fixed beam. E.g., supporting column in a building.

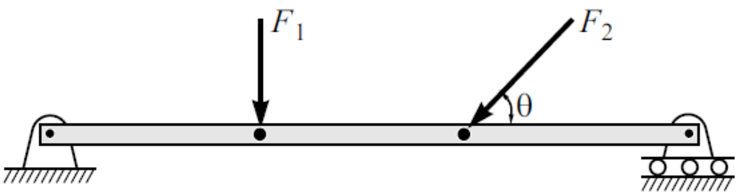


6. Beams Linked with Internal Hinges: Here two or more beams are connected to each other by pin joint and continuous beam is formed. Such joints are called internal hinges. Internal hinges allow us to draw FBD of beam at its joint, if required.



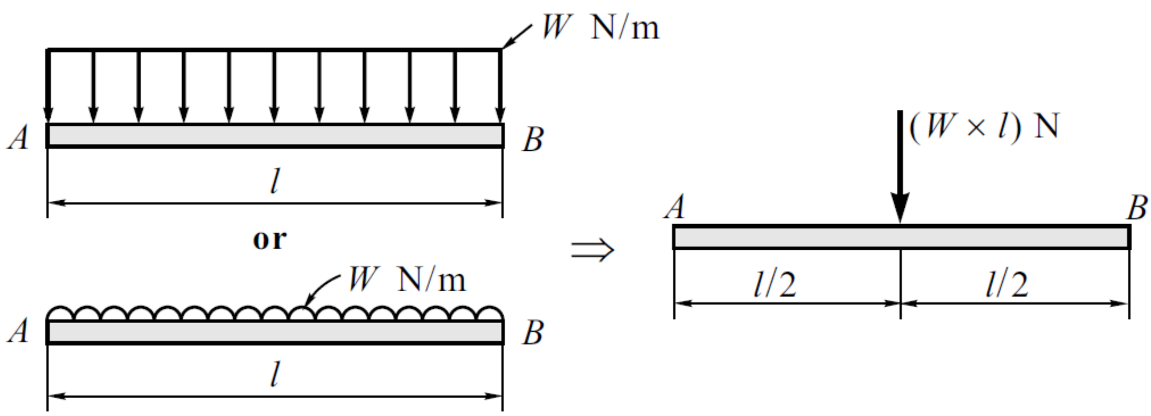
Types of Loads:

1. Point Load: If the whole intensity of load is assumed to be concentrated at a point, then it is known as point load.

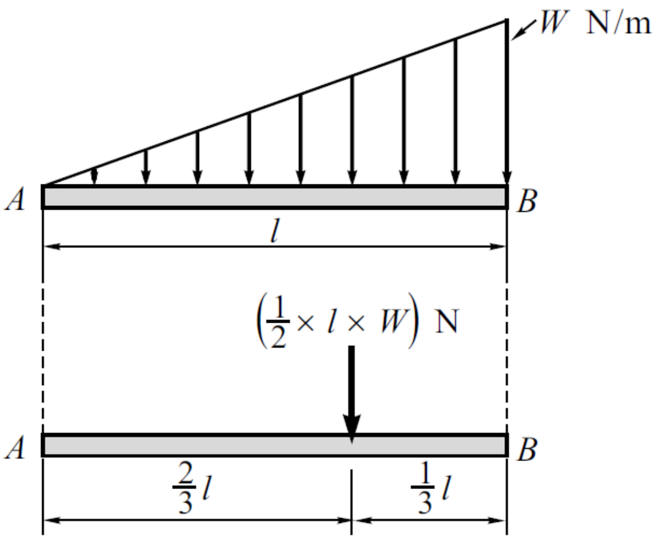


2. Distributed Load: When a load acts throughout the length of a beam or body in varying degree, it can be called as distributed load. This load may consist of the weight of materials supported directly or indirectly by the beam or it may be caused by wind or hydraulic pressure.
   A distributed load on a beam can be replaced by a concentrated point load. The magnitude of this equivalent point load is equal to the area under loading diagram and it acts through the centroid.

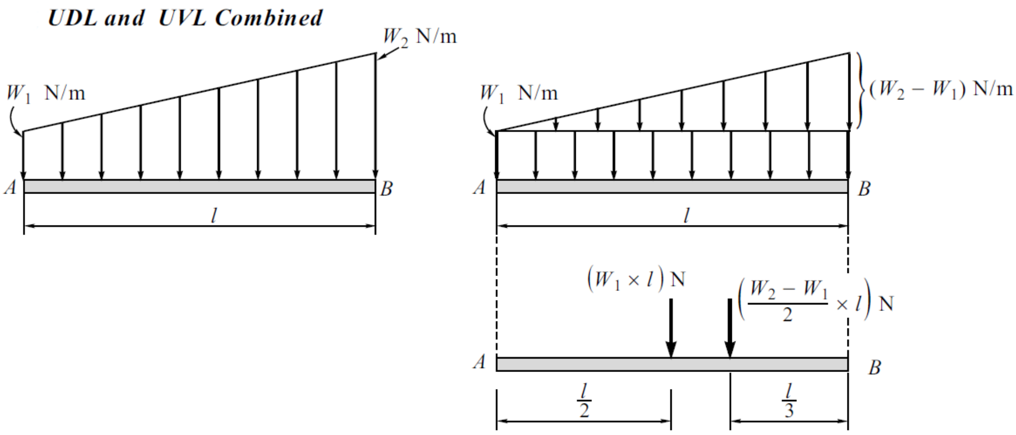
a) Uniformly Distributed Load (UDL): If the whole intensity of load is distributed uniformly along the length of loading, then it is called uniformly distributed load. E.g., weight of a slab of a building flooring.



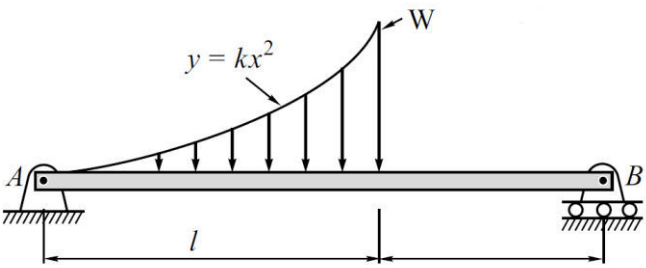
b) Uniformly Varying Load (UVL): If the whole intensity of load is distributed uniformly at varying rate along the length of loading, then, it is known as uniformly varying load. E.g., in a dam the hydraulic pressure varies linearly with the depth.



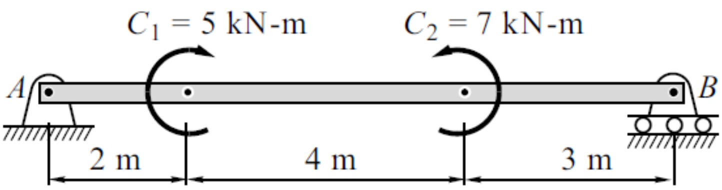
c) <u>Trapezoidal Load (UDL + UVL)</u>: If the whole intensity of load is distributed uniformly at varying rate along the length of loading from some lower intensity at on end to a higher intensity at the other end, then, it is known as trapezoidal load.



d) <u>Varying Load</u>: The varying load is given by some relation.



3. <u>Couple</u>: A couple load acting on a body tends to cause rotation of the body. Its location on the body is of no significance because couples are free vectors.

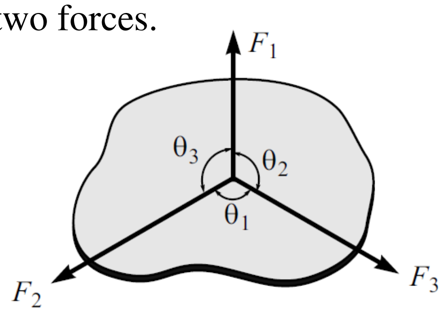


Equilibrium of Two Force System: If a body is in equilibrium and is acted by upon by only two forces, then these forces must be equal in magnitude, opposite in direction and collinear.

Equilibrium of Three Force System: If a body is in equilibrium and is acted by upon by three coplanar forces, then these forces must be form either a concurrent system or parallel system.

Lami's Theorem: If three concurrent coplanar forces acting on a body having same nature (i.e., pulling or pushing) are in equilibrium, then each force is proportional to the sine of angle included between the other two forces.

$$\frac{F_1}{\sin\theta_1} = \frac{F_2}{\sin\theta_2} = \frac{F_3}{\sin\theta_3}$$

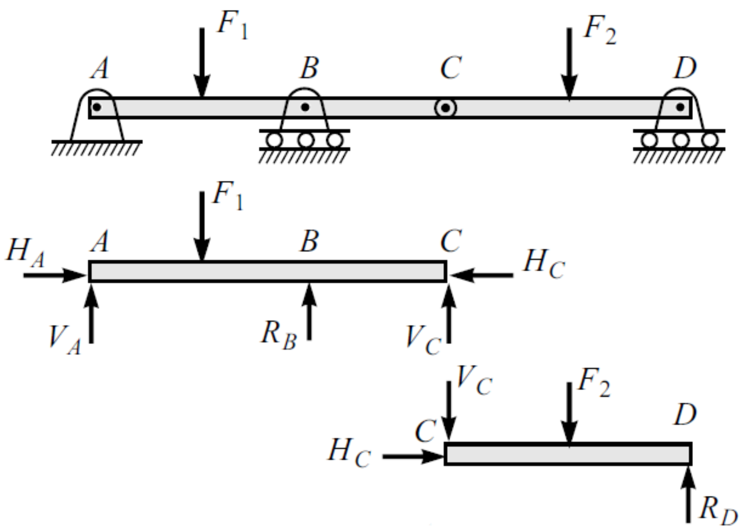


It is applicable to three non-parallel coplanar concurrent forces only. Nature of three forces must be same (i.e., pulling or pushing). If any force is in the opposite sense, then simply placing a negative sign with it, Lami's theorem can be applied.

Equilibrium of Connected Bodies:

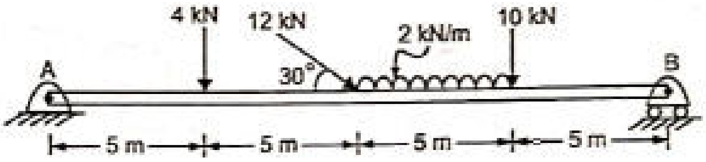
When two or more rigid bodies are connected to each other, they form a system of connected bodies. COE can be applied to the entire system or individual bodies can be isolated from internal connections and COE can be applied to them too. This is useful when the number of unknowns is more than 3.

At the internal connection (hinge, roller, smooth surface, etc.), the reactions of first body on the second are assumed in some direction and the reactions of second body on the first are assumed in the opposite sense, with equal magnitude. The reactions will depend upon the type of connection. (NOTE: Internal hinge is not in syllabus.)

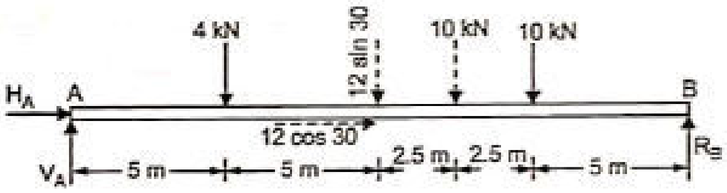


Numericals:

Ex. 3.1 A beam AB is hinged at end A and roller supported at end B. It is acted upon by loads as shown. Find the support reactions.



Solution:



FBD - Beam AB

Figure shows the FBD of the beam AB. Hinge at A gives reaction R_A having components H_A and V_A. Roller at B gives a vertical reaction R_B.

The u.d.l has been converted into a point load of 2 kN/m × 5 m = 10 kN acting at the center of u.d.l. The 12 kN inclined load has been resolved into components.

Applying Conditions of Equilibrium (COE) to the beam AB

$\Sigma M_A = 0$ ↺ +ve

$-(4 \times 5) - (12 \sin 30 \times 10) - (10 \times 12.5) - (10 \times 15) + (R_B \times 20) = 0$

$R_B = 17.75$ kN

$R_B = 17.75$ kN ↑ .... **Ans.**

$\Sigma F_X = 0$ → +ve

$H_A + 12 \cos 30 = 0$

$H_A = -10.39$ kN

$H_A = 10.39$ kN ←

$\Sigma F_Y = 0$ ↑ +ve

$V_A - 4 - 12 \sin 30 - 10 - 10 + 17.75 = 0$

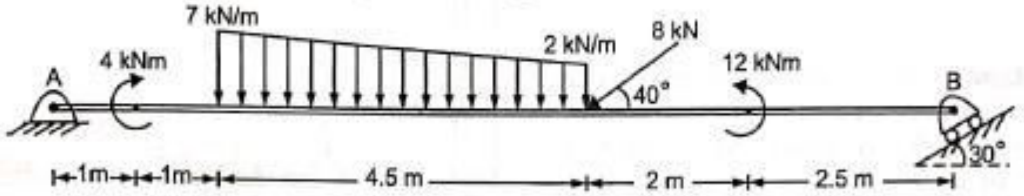
$V_A = 12.25$ kN ↑

Adding vectorially the components H_A and V_A, the reaction

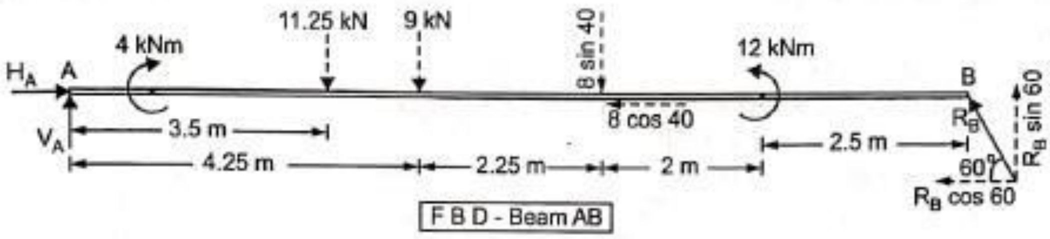
$R_A = 16.06$ kN $\theta = 49.69°$ ↖ .... **Ans.**

Note : Hinge reaction answers may also be written as $H_A = 10.39$ kN ←, $V_A = 12.25$ kN ↑

Ex. 3.2 The beam AB is loaded by forces and couples as shown. Find the reaction force offered by the supports to keep the system in equilibrium. ***(VJTI Apr 17)***



Solution:



F B D - Beam AB

Figure shows the FBD of the beam AB. The hinge at A offers reaction R_A having components H_A and V_A.

The roller at B offers reaction R_B normal to the surface on which the roller is supported.

The trapezoidal load has been replaced by two equivalent point loads.

The inclined 8 kN force has been resolved into components.

The beam is loaded with two couples viz. 4 kNm clockwise and 12 kNm anti-clockwise couples.

Applying COE to the beam AB.

$\Sigma M_A = 0$ ↺ + ve

$-4-(11.25\times3.5)-(9\times4.25)-(8\sin40\times6.5)+12+(R_B\sin60\times11)=0$

$R_B = 10.82$ kN

$\therefore\quad R_B = 10.82$ kN $\quad\theta = 60°$ .... **Ans.**

$\Sigma F_X = 0 \rightarrow$ + ve

$H_A - 8\cos40 - 10.82\cos60 = 0$

$H_A = 11.54$ kN

$H_A = 11.54$ kN $\rightarrow$ .... **Ans.**

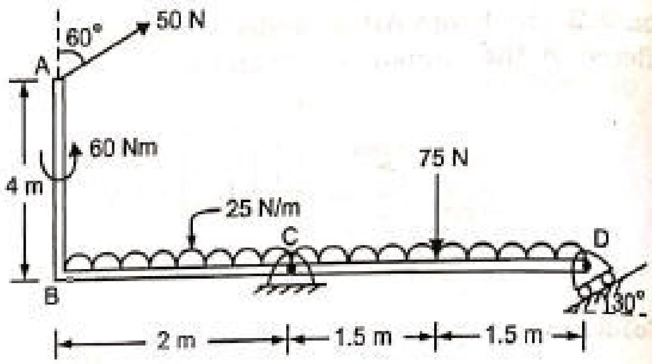
$\Sigma F_Y = 0 \uparrow$ + ve

$V_A - 11.25 - 9 - 8\sin40 + 10.82\sin60 = 0$

$V_A = 16.02$ kN

$V_A = 16.02$ kN $\uparrow$ .... **Ans.**

Ex. 3.3 A L shaped beam is loaded as shown. Find support reactions.



Solution: The system consists of a single L shaped beam externally supported by a hinge at C and a roller at D.

Figure shows the FBD

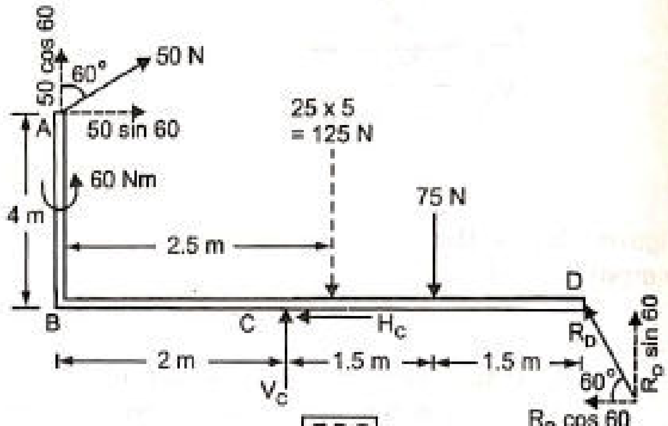
Applying COE

$\Sigma M_C = 0$ ↺ + ve

$+ 60 - (50\sin 60 \times 4) - (50\cos 60 \times 2)$
$- (125 \times 0.5) - (75 \times 1.5)$
$+ (R_D \sin 60 \times 3) = 0$

$\therefore \quad R_D = 130.17$ N, $\theta = 60°$

.... **Ans.**



FBD

$\Sigma F_X = 0 \quad \rightarrow + $ ve

$50 \sin 60 - H_C - 130.17 \cos 60 = 0$

$\therefore \quad H_C = -21.78$ N

or $\quad H_C = 21.78$ N $\rightarrow$

(– ve value indicates assumption about the sense of unknown force is incorrect) .... **Ans.**

$\Sigma F_Y = 0 \quad \uparrow + $ ve

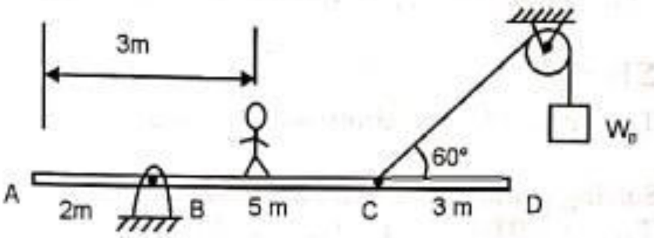
$50 \cos 60 + V_C - 125 - 75 + 130.17 \sin 60 = 0$

$\therefore \quad V_C = 62.27$ N

or $\quad V_C = 62.27$ N $\uparrow$

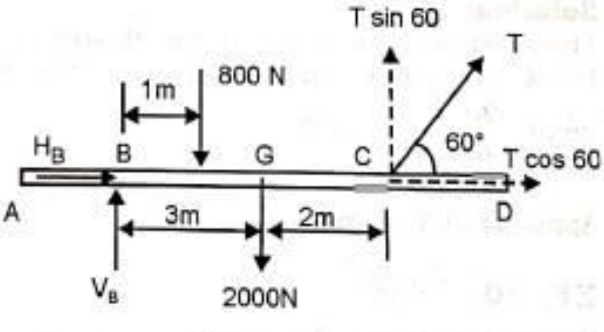
(+ ve value indicates assumption about the sense of unknown force is correct) .... **Ans.**

Ex. 3.5 A man of 800 N weight stands on a 10 m long uniform beam ABCD of self weight 2000 N. The beam is supported by hinge at B and a rope whose one end is attached at C and the other end carries a counterweight W_B. Find the value of W_B needed to keep the beam in a horizontal position as shown and also the hinge reactions.



Solution: The beam is supported by a hinge at B giving reactions H_B and V_B as shown and a rope. Since the rope passes over a smooth pulley the tension T in it is equal to the counterweight W_B. The weight 2000 N of the beam acts through its centre of gravity G i.e. its mid point.

Applying C O E



F B D - Beam AD

$\Sigma M_B = 0$ ↺ +ve

$-(800 \times 1) - (2000 \times 3) + (T\sin 60 \times 5) = 0$

$\therefore \quad T = 1570.4$ N

$\therefore \quad W_B = T = 1570.4$ N .... **Ans.**

$\Sigma F_x = 0 \quad \rightarrow \; +ve$

$H_B + T\cos 60 = 0$

$\therefore \; H_B = -T\cos 60$

$= -1570.4 \cos 60$

$= -785.19$

or $H_B = 785.19$ N $\leftarrow$ .... **Ans.**

$\Sigma F_y = 0 \quad \uparrow \; +ve$

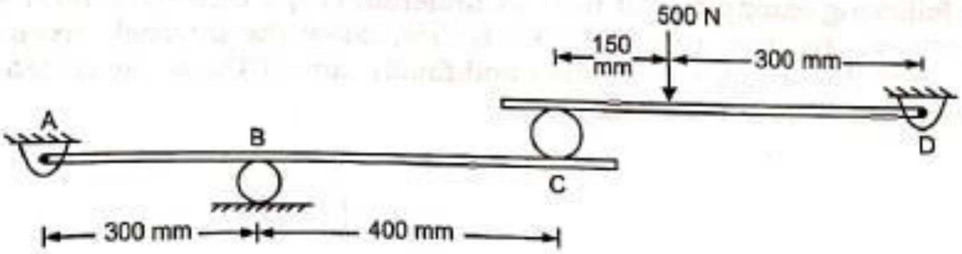
$V_B - 800 - 2000 + T\sin 60 = 0$

$\therefore \; V_B - 2800 + 1570.4 \sin 60 = 0$

$\therefore \; V_B = 1450$ N

or $V_B = 1450$ N $\uparrow$ .... **Ans.**

Ex. 3.10 For a lever system shown, find the support reactions.

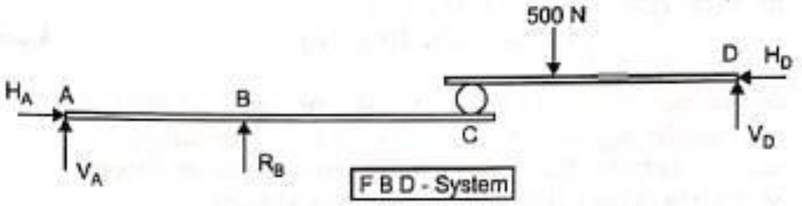


Solution: The system contains two bodies AC and CD. The external supports are;

1) Hinge at A giving reaction R_A. Let H_A and V_A be the components of R_A.
2) Roller at B giving reaction R_B.
3) Hinge at D giving reaction R_D. Let H_D and V_D be the components of R_D

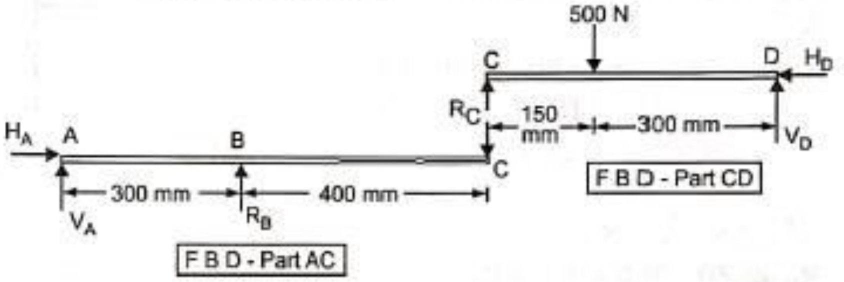
The bodies are internally connected by a roller at C.

Figure shows the FBD of the system of two connected bodies. There are in all five unknowns viz., H_A, V_A, H_D, V_D, and R_B and we have three COE for the system.



We are therefore not in a position to find the unknowns.

Let us therefore isolate the two bodies and apply COE to each of them. Refer figure. Note that the internal force R_C occurs in pair, of same magnitude, collinear and opposite in sense.



Applying COE to body CD.

$M_D = 0$ ↺ + ve

$+ (500 \times 300) - (R_C \times 450) = 0$

$R_C = 333.33$ N $\qquad \therefore \qquad R_C = 333.33$ N ↑ on body CD.

$\Sigma F_Y = 0$ ↑ + ve

$333.33 - 500 + V_D = 0$

$V_D = 166.67$ N $\qquad \therefore \qquad V_D = 166.67$ N ↑ .... **Ans.**

$\Sigma F_X = 0$ → + ve

$H_D = 0$ .... **Ans.**

Applying COE to body AC

using $R_C = 333.33$ N ↓ on body AC

$\Sigma M_A = 0$ ↺ + ve

$-(333.33 \times 700) + (R_B \times 300) = 0$

$R_B = 777.7$ N $\qquad \therefore \qquad R_B = 777.7$ N ↑ .... **Ans.**

$\Sigma F_Y = 0$ ↑ +ve

$V_A + 777.7 - 333.3 = 0$

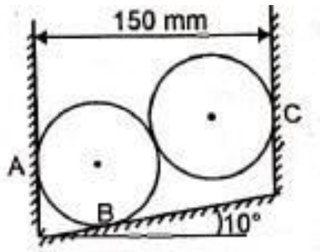
$V_A = -444.4 \qquad \therefore \qquad V_A = 444.4$ N ↓ .... **Ans.**

$\Sigma F_X = 0$ → + ve

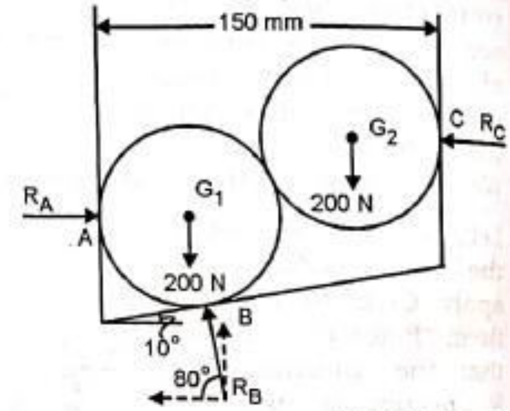
$H_A = 0$ .... **Ans.**

Ex. 3.11 Two cylinders each of diameter 100 mm and each weighing 200 N are placed as shown in figure. Assuming that all the contact surfaces are smooth find the reactions at A, B and C.

(MU Dec 09, May 13)



Solution: The system consists of two cylinders supported against three smooth surfaces at A, B and C. Let R_A, R_B and R_C be the reactions at three supports. The FBD of the system is shown.



F B D - System

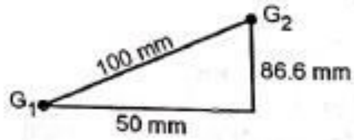
Applying COE to the system

$\Sigma M_{G1} = 0$ ↺ +ve

$$-(200 \times 50) + (R_C \times 86.6) = 0$$

$$\therefore \quad R_C = 115.47\ \text{N}$$

$$\text{or} \quad R_C = 115.47\ \text{N} \leftarrow$$ .... **Ans.**



$\Sigma F_y = 0 \quad \uparrow +\text{ve}$

$$R_B \sin 80 - 200 - 200 = 0$$

$$\therefore \quad R_B = 406.17\ \text{N}$$

$$\text{or} \quad R_B = 406.17\ \text{N}\ ,\ \theta = 80^\circ$$ .... **Ans.**

$\Sigma F_x = 0$

$$R_A - R_B \cos 80 - R_C = 0$$

$$R_A - 406.17 \cos 80 - 115.47 = 0$$

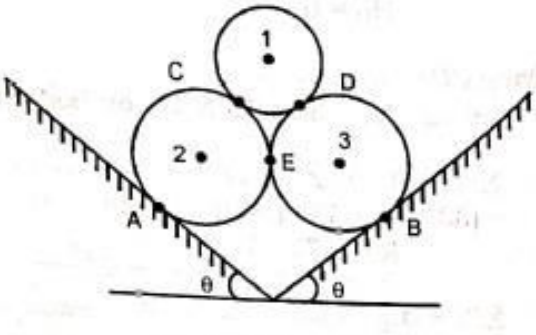
$$\therefore \quad R_A = 186\ \text{N}$$

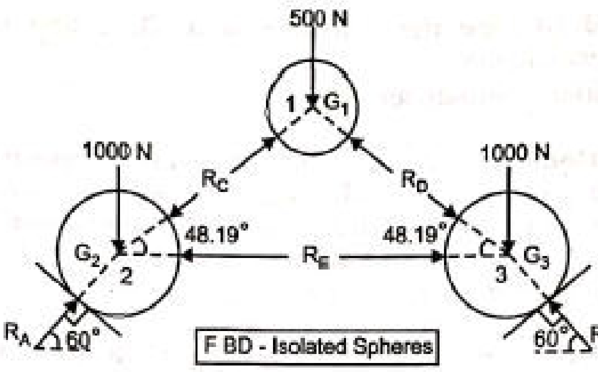
$$\text{or} \quad R_A = 186\ \text{N} \rightarrow$$ .... **Ans.**

Ex. 3.12 Three smooth spheres rest against two inclined smooth planes as shown. Determine

a) The reaction force at contact points when $\theta = 30^\circ$

b) The minimum angle θ for which the spheres remain in equilibrium.
Take for sphere 1 weight = 500 N and radius = 0.2 m
for spheres 2 and 3 weight = 1000 N and radius = 0.4 m





Solution: a) Given: $\theta = 30°$

Let us isolate the bodies as shown in figure. Since the external supports at A and B and internal supports at C, D and E are smooth surfaces, these offer a reaction force normal to the smooth surface.

Applying COE to sphere 1

$\Sigma F_X = 0 \rightarrow +ve$

$R_C \cos 48.19 - R_D \cos 48.19 = 0$

$\therefore \quad R_C = R_D$ .... (1)

$\Sigma F_Y = 0 \uparrow +ve$

$-500 + R_C \sin 48.19 + R_D \sin 48.19 = 0$ .... (2)

Solving equations (1) and (2) we get, $R_C = R_D = 335.4$ N

Applying COE to sphere 2

$\Sigma F_Y = 0 \uparrow +ve$

$R_A \sin 60 - R_C \sin 48.19 - 1000 = 0$ .... (3)

Substituting $R_C = 335.4$ N, we get, $R_A = 1443.3$ N .... **Ans.**

$\Sigma F_X = 0 \rightarrow +ve$

$R_A \cos 60 - R_C \cos 48.19 - R_E = 0$ .... (4)

Substituting $R_A = 1443.3$ N, and $R_C = 335.4$ N, we get $R_E = 498.1$ N .... **Ans.**

By symmetry of loading and symmetry of supports we can say,

$R_B = R_A = 1443.3$ N .... **Ans.**

b) To find minimum angle θ

As the angle θ is slowly reduced, a stage will be reached when the pyramid of spheres will collapse.

At the minimum angle θ when the system is about to collapse, the reaction at E becomes zero. i.e. $R_E = 0$

By analysis of sphere (1) as done earlier, the reactions $R_C = R_D = 335.4$ N remain the same.

Let us now apply COE and analyse sphere (2) to find the minimum angle θ

$\Sigma F_X = 0 \rightarrow +ve$

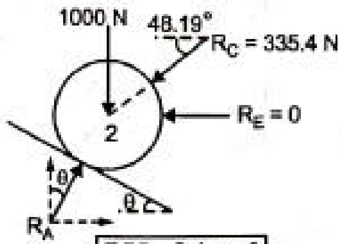
$R_A \sin \theta - 335.4 \cos 48.19 = 0$

$\therefore \quad R_A \sin \theta = 223.6$ .... (5)

$\Sigma F_Y = 0 \uparrow +ve$

$R_A \cos \theta - 335.4 \sin 48.19 - 1000 = 0$

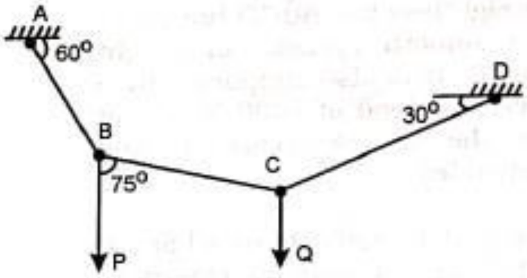
$\therefore \quad R_A \cos \theta = 1250$ .... (6)



F BD - Sphere 2

Solving equations (5) and (6) we get, $\theta = 10.14°$ ... **Ans.**

Ex. 3.15 A string ABCD carries two loads P and Q. If P = 50 kN, find force Q and tensions in different portions of the string.

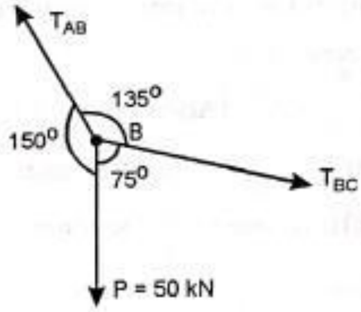


Solution: Isolating joint B of the string. Let T_{AB} and T_{BC} be the tensions in the string portions AB and BC respectively.
Using Lami's equation

$$\frac{T_{AB}}{\sin 75} = \frac{T_{BC}}{\sin 150} = \frac{50}{\sin 135}$$

$\therefore$ $T_{AB} = 68.3$ kN .... **Ans.**

$T_{BC} = 35.35$ kN ....**Ans.**



F B D - Joint B

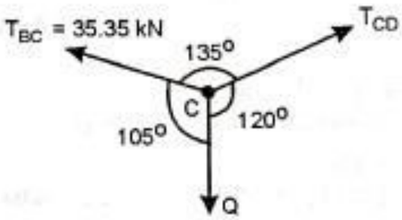
Now isolating joint C.
Let T_{CD} be the tension in portion CD.

Using Lami's equation

$$\frac{35.35}{\sin 120} = \frac{T_{CD}}{\sin 105} = \frac{Q}{\sin 135}$$

$\therefore$ $T_{CD} = 39.43$ kN .... **Ans.**

Q = 28.86 kN .... **Ans.**



F B D - Joint C

We have solved the problem using Lami's equation. As an Exercise solve the problem applying COE.

Problem 7

A roller of weight W = 1000 N rests on a smooth inclined plane. It is kept from rolling down the plane by string AC as shown in Fig. 3.7(a). Find the tension in the string and reaction at the point of contact D.

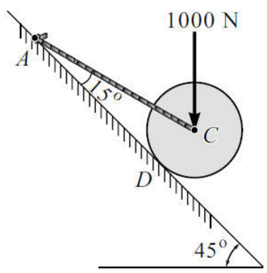


Fig. 3.7(a)

Solution

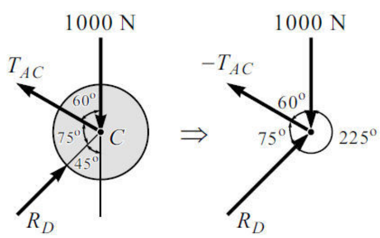
(i) Draw the F.B.D. of the roller.

(ii) By Lami's theorem,

$$\frac{1000}{\sin 75^\circ} = \frac{R_D}{\sin 60^\circ} = \frac{-T_{AC}}{\sin 225^\circ}$$

$\therefore R_D = 896.58$ N $\left(\angle 45^\circ\right)$

$\therefore T_{AC} = 732$ N



Problem 8

A cylinder of 50 kg mass is resting on a smooth surface which are inclined at 30° and 60° to horizontal as shown in Fig. 3.8(a). Determine the reaction at contact A and B.

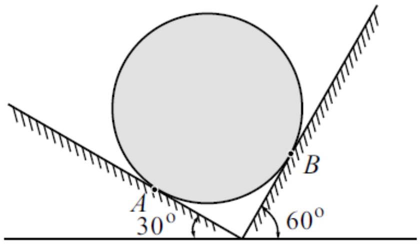


Fig. 3.8(a)

Solution

(i) Consider the F.B.D. of the cylinder.

(ii) By Lami's theorem, we have

$$\frac{50 \times 9.81}{\sin 90^\circ} = \frac{R_A}{\sin 120^\circ} = \frac{R_B}{\sin 150^\circ}$$

$$R_A = 424.79 \text{ N}$$

$$R_B = 245.25 \text{ N}$$

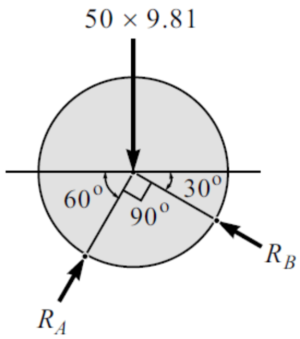


Fig. 3.8(b) : F.B.D. of Cylinder

Problem 15

Two spheres A and B are resting in a smooth through as shown in Fig. 3.15(a). Draw the free body diagrams of A and B showing all the forces acting on them, both in magnitude and direction. Radii of spheres A and B are 250 mm and 200 mm, respectively.

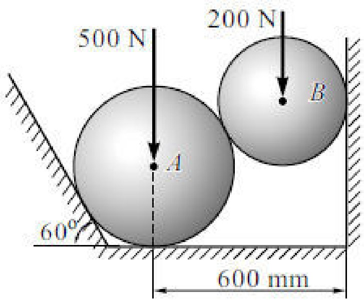


Fig. 3.15(a)

Solution

(i) From Fig. 3.15(b). $AB = 450$ mm and $AC = 400$ mm

$$\cos\theta = \frac{AC}{AB} = \frac{400}{450} \quad \therefore \theta = 27.27^\circ$$

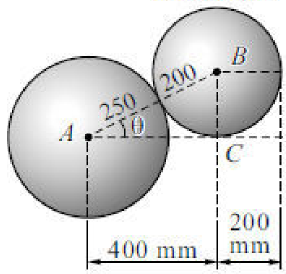


Fig. 3.15(b)

(ii) Consider the F.B.D. of sphere B [Fig. 3.15(c)]

By Lami's theorem,

$$\frac{200}{\sin 152.73^\circ} = \frac{R_1}{\sin 117.27^\circ} = \frac{R_2}{\sin 90^\circ}$$

$$\therefore R_1 = 388 \text{ N } (\leftarrow) \text{ and}$$

$$R_2 = 436.51 \text{ N } (\measuredangle 27.27^\circ)$$

(iii) Consider the F.B.D. of sphere A [Fig. 3.15(d)]

$$\Sigma F_x = 0$$

$$R_4 \cos 30^\circ - 436.51 \cos 27.27^\circ = 0$$

$$R_4 = 448 \text{ N } (\measuredangle 30^\circ)$$

$$\Sigma F_y = 0$$

$$-500 + R_3 - 436.51 \sin 27.27^\circ + 448 \sin 30^\circ = 0$$

$$R_3 = 476 \text{ N } (\uparrow)$$

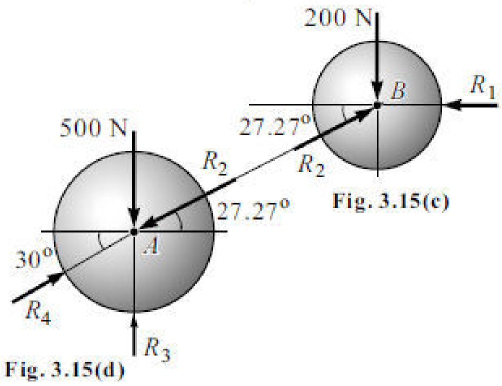


Fig. 3.15(c)

Fig. 3.15(d)

Problem 21

Three cylinders are piled up in a rectangular channel as shown in Fig. 3.21(a). Determine the reactions at point 6 between the cylinder *A* and the vertical wall of the channel.

(Cylinder *A* : radius = 4 cm, m = 15 kg,
Cylinder *B* : radius = 6 cm, m = 40 kg,
Cylinder *C* : radius = 5 cm, m = 20 kg).

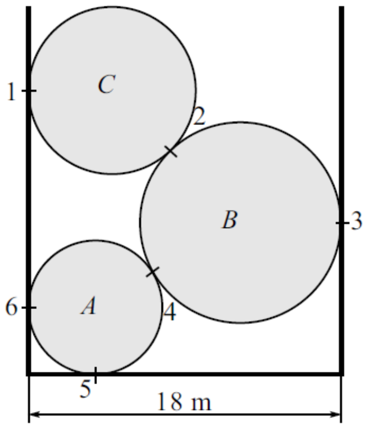


Fig. 3.21(a)

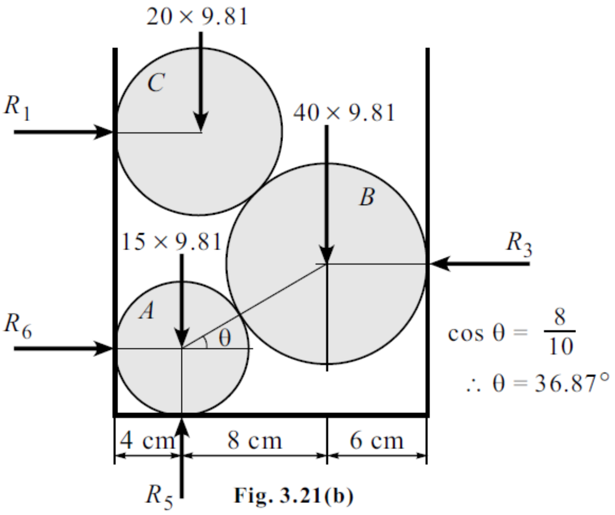


Fig. 3.21(b)

Solution

(i) Consider F.B.D. of entire system as shown in Fig. 3.21(b).

$$\Sigma F_y = 0$$

$$R_5 - 20 \times 9.81 - 40 \times 9.81 - 15 \times 9.81 = 0$$

$$R_5 = 735.75 \text{ N}$$

(ii) Consider the F.B.D. of cylinder *A* [Refer to Fig. 3.21(c)].

$$\Sigma F_y = 0$$

$$735.75 - 15 \times 9.81 - R_4 \sin 36.87^\circ = 0$$

$$R_4 = 981 \text{ N}$$

$$\Sigma F_x = 0$$

$$R_6 - R_4 \cos 36.87^\circ = 0$$

$$R_6 = 784.8 \text{ N } (\rightarrow)$$

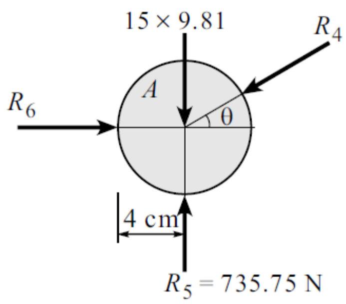


Fig. 3.21(c)

Problem 35

Find the support reactions at A and B for the beam loaded as shown in Fig. 3.35(a).

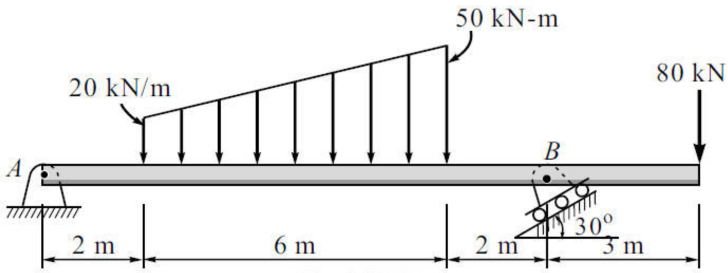


Fig. 3.35(a)

Solution

(i) Consider the F.B.D. of beam AB [Fig. 3.35(b)].

(ii) $\Sigma M_A = 0$

$R_B \sin 60^\circ \times 10 - 120 \times 5 - 90 \times 6 - 80 \times 13 = 0$

$R_B = 251.73$ kN (60°)

(iii) $\Sigma F_x = 0$

$H_A - 251.73 \cos 60^\circ = 0$

$H_A = 125.87$ kN $(\rightarrow)$

(iv) $\Sigma F_y = 0$

$V_A - 120 - 90 + 251.73 \sin 60^\circ - 80 = 0$

$V_A = 72$ kN $(\uparrow)$

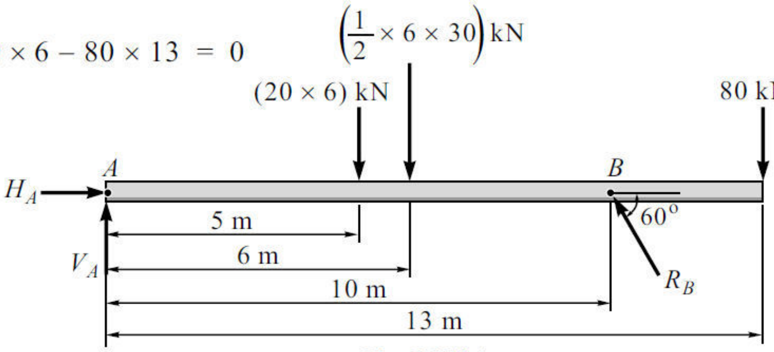


Fig. 3.35(b)

Problem 36

Find analytically the support reaction at B and the load P, for the beam shown in Fig. 3.36(a), if the reaction of support A is zero.

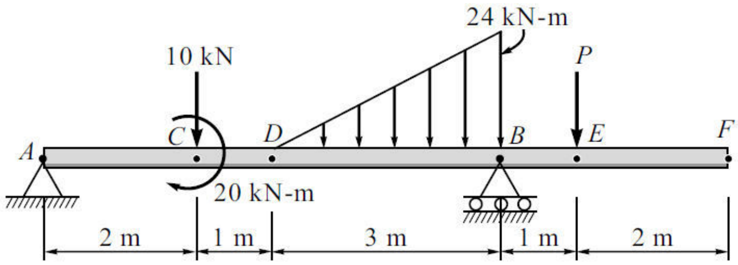


Fig. 3.36(a)

Solution

(i) Consider the F.B.D. of beam AF.

(ii) $\Sigma F_y = 0$

$V_A + R_B - 10 - 36 - P = 0$ ($V_A = 0$ given)

$R_B - P = 46$ ...(I)

(iii) $\Sigma M_A = 0$

$R_B \times 6 - 10 \times 2 - 20 - 36 \times 5 - P \times 7 = 0$

$6 R_B - 7P = 220$ ...(II)

(iv) Solving Eqs. (I) and (II)

$R_B = 102$ kN $(\uparrow)$

(v) From Eq. (I)

$P = 102 - 46$

$P = 56$ kN $(\downarrow)$

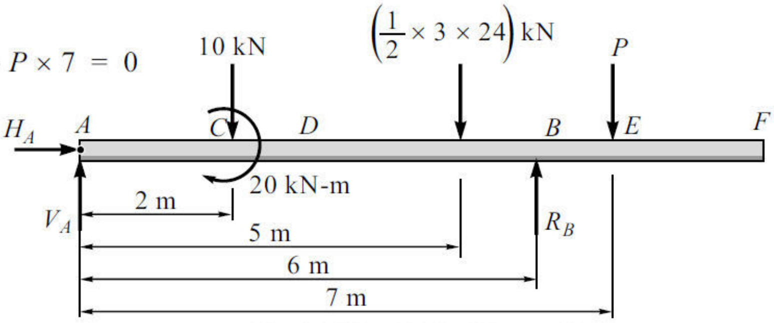


Fig. 3.36(b) : F.B.D of Beam AF

Problem 37

Find the support reactions at A and F for the given Fig. 3.37(a).

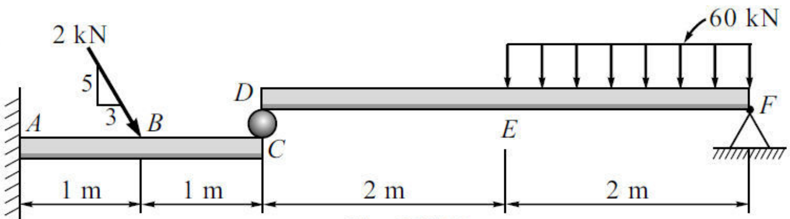


Fig. 3.37(a)

Solution

(i) Consider the F.B.D. of beam DF [Fig. 3.37(b)].

$\Sigma M_F = 0$

$120 \times 1 - R_D \times 4 = 0 \quad \therefore \; R_D = 30 \text{ kN}$

$\Sigma F_x = 0 \qquad \therefore \; H_F = 0$

$\Sigma F_y = 0$

$R_D + V_F - 120 = 0$

$V_F = 90 \text{ kN } (\uparrow)$

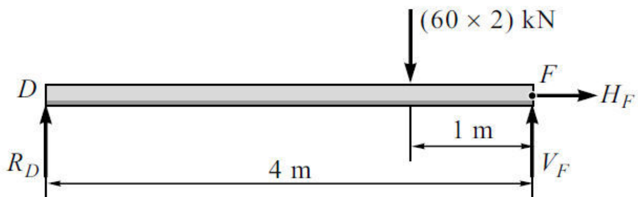


Fig. 3.37(b)

(ii) Consider the F.B.D. of beam AC [Fig. 3.37(c)].

$\Sigma M_A = 0$

$M_A - 2 \sin 59.04^\circ \times 1 - 30 \times 2 = 0$

$M_A = 61.72 \text{ kN-m } (\circlearrowleft)$

$\Sigma F_x = 0$

$2 \cos 59.04^\circ - H_A = 0$

$H_A = 1.03 \text{ kN } (\leftarrow)$

$\Sigma F_y = 0$

$V_A - 2 \sin 59.04^\circ - 30 = 0$

$V_A = 31.72 \text{ kN } (\uparrow)$

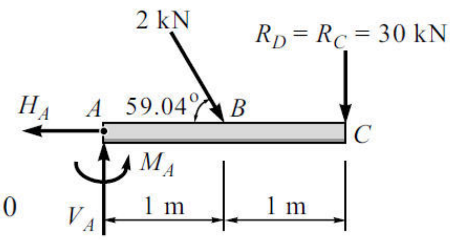


Fig. 3.37(c)

Problem 38

Two beams AB and CD are arranged as shown in Fig. 3.38(a). Find the support reactions at D.

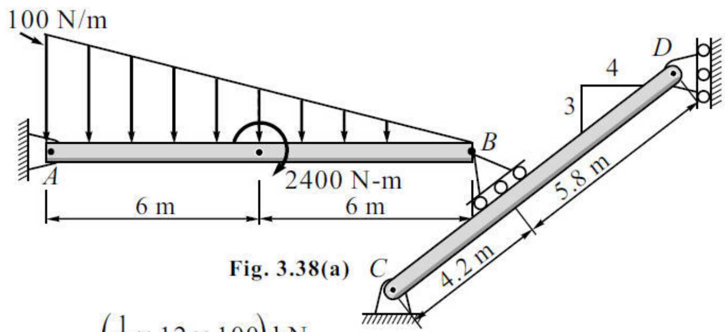


Fig. 3.38(a)

Solution

(i) Consider the F.B.D. of beam AB.

$\Sigma M_A = 0$

$R_B \sin 53.13^\circ \times 12 - 600 \times 4 - 2400 = 0$

$R_B = 500 \text{ N}$

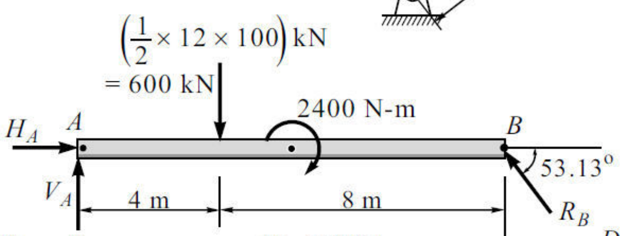


Fig. 3.38(b)

(ii) Consider the F.B.D. of beam CD.

$\Sigma M_C = 0$

$R_D \sin 36.87^\circ \times 10 - 500 \times 4.2 = 0$

$R_D = 350 \text{ N } (\leftarrow)$

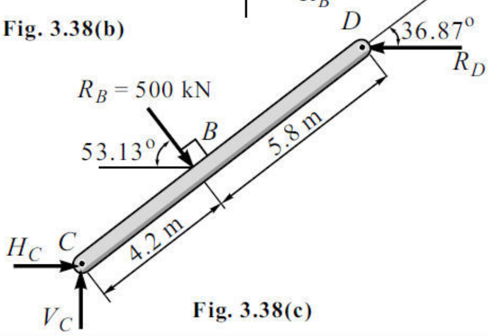


Fig. 3.38(c)

K J Somaiya College of Engineering, Vidyavihar, Mumbai

(A Constituent College of SVU)

Engineering Mechanics Notes

Module 4 – Equilibrium of Force System and Friction

Module Section 4.2 – Friction

Class: FY BTech Faculty: Aniket S. Patil Date: 09/06/23

References: Engineering Mechanics, by M. D. Dayal & Engineering Mechanics – Statics and Dynamics, by N. H. Dubey.

Friction: When the surface of a body slides along/moves over or tends to slide along/move over the surface of another body, a force opposing the motion develops tangentially between the contact surfaces. This force which opposes the movement or tendency of movement is called a frictional force or simply friction.

Friction is due to the resistance offered to motion by minutely projecting portions at the contact surfaces. These microscopic projections get interlocked. To a small extent, the material of two bodies in contact also produces resistance to motion due to intramolecular force of attraction, i.e., adhesive properties.

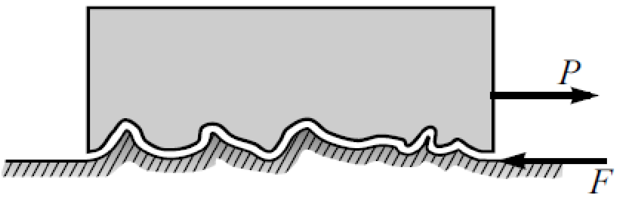


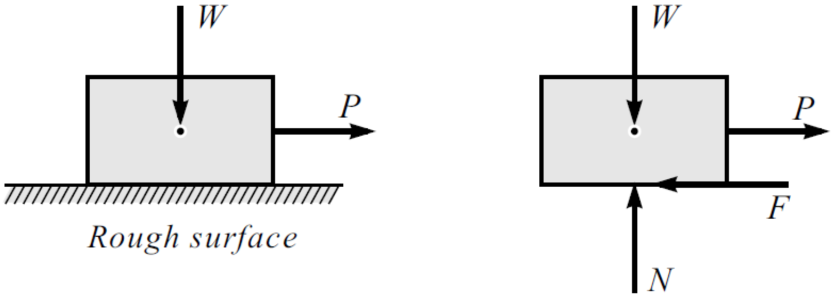
Fig. : Magnified Microscopic View of Rough Surface

Dry Friction: Dry friction develops when the unlubricated surface of two solids are in contact under a condition of sliding or a tendency to slide. A frictional force tangent to the surfaces of contact is developed both during the interval leading up to impending slippage and while slippage takes place. The direction of the frictional force always opposes the relative motion or impending motion. This type of friction is also known as Coulomb friction.

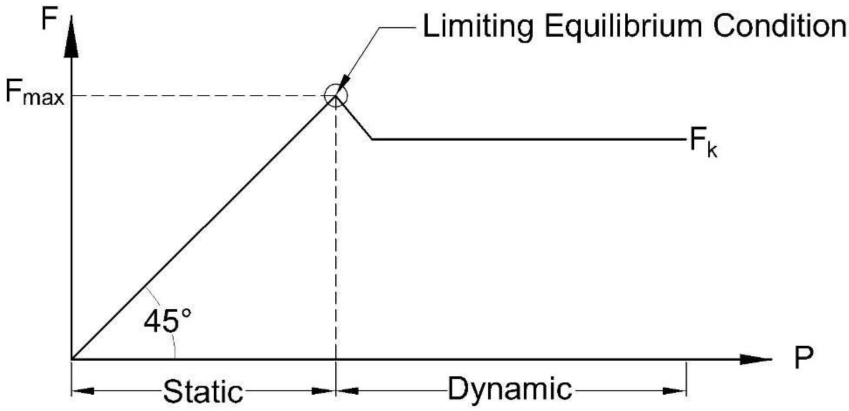
Fluid Friction: Fluid friction is developed when adjacent layers in a fluid (liquid or gas) are moving at different velocities. This motion gives rise to frictional forces between fluid elements and these forces depend on the relative velocity between layers. Fluid frictions depends not only on the velocity gradients within the fluid but also on viscosity of fluid, which is a measure of its resistance to shearing action between fluid layers. Fluid friction is treated in the study of fluid mechanics and is beyond the scope of this text. So, we are going to deal with dry friction only.

Limiting Equilibrium Condition:

Consider a block of weight W resting on a horizontal surface. The contacting surface possesses a certain amount of roughness. Let P be the horizontal force applied which will vary continuously from zero to a value sufficient to just move the block and then to maintain the motion. The free body diagram of the block shows active forces (i.e., Applied force P and weight of block W) and reactive forces (i.e., normal reaction N and tangential frictional force F).



As applied force P increases, the frictional force F is equal in magnitude and opposite in direction. However, there is a limit beyond which the magnitude of the frictional force cannot be increased. If the applied force is more than this maximum frictional force (F_{max}), there will be movement of one body over the other. Once the body begins to move, there is decrease in frictional force F from maximum value observed under static condition. The frictional force between the two surfaces when the body is in motion is called kinetic or dynamic friction (F_k).



<u>Limiting Frictional Force (F_{max})</u>: It is the maximum frictional force developed at the surface max when the block is at the verge of motion (impending motion).

<u>Coefficient of Friction</u>: By experimental evidence it is proved that limiting frictional force is directly proportional to normal reaction.

<u>Coefficient of Static Friction</u>: The ratio of limiting frictional force (F_{max}) and normal reaction (N) is called the coefficient of static friction (μ_s).

$$F_{max} \propto N \Rightarrow F_{max} = \mu_s N \Rightarrow \mu_s = \frac{F_{max}}{N}$$

<u>Coefficient of Kinetic Friction</u>: The ratio of kinetic frictional force (F_k) and normal reaction (N) is called the coefficient of static friction (μ_k).

$$F_k \propto N \Rightarrow F_k = \mu_k N \Rightarrow \mu_k = \frac{F_k}{N}$$

Kinetic friction is always less than limiting friction. And μ is always less than 1.

Laws of Friction:

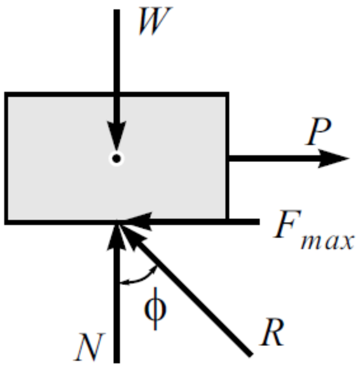
1. The frictional force is always tangential to the contact surface and acts in a direction opposite to that in which the body tends to move.
2. The magnitude of frictional force is self-adjusting to the applied force till the limiting frictional force is reached and at the limiting frictional force the body will have impending motion.
3. Limiting frictional force F_{max} is directly proportional to normal reaction ($F_{max} = \mu_s N$).
4. For a body in motion, kinetic frictional force F_k developed is less than that of limiting frictional force F_{max} and the relation $F_k = \mu_k N$ is applicable.
5. Frictional force depends upon the roughness of the surface and the material in contact.
6. Frictional force is independent of the area of contact between the two surfaces.
7. Frictional force is independent of the speed of the body.
8. Coefficient of static friction μ_s is always greater than the coefficient of kinetic friction μ_s.

Angle of Friction: It is the angle made by the "resultant of the limiting frictional force F_{max} and the normal reaction N" with the "normal reaction".

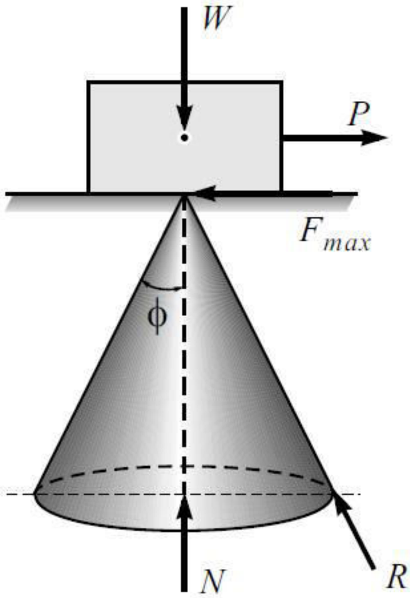
$$R = \sqrt{F_{max}^2 + N^2}$$

$$\tan\phi = \frac{F_{max}}{N} = \mu_s$$

$$\phi = \tan^{-1}\mu_s$$



Cone of Friction: When the applied force P is just sufficient to produce the impending motion of given body, angle of friction ϕ is obtained. If the direction of applied force P is gradually changed through 360°, the resultant R generates a right circular cone with semi vertex angle equal to ϕ. This is called the cone of friction.



Angle of Repose: It is the minimum angle of inclination of a plane with the horizontal at which the body so kept will just begin to slide down on it without the application of any external force (due to self-weight).

$\sum F_x = 0 \Rightarrow$

$F_{max} - W \sin\alpha = 0$

$\mu_s N - W \sin\alpha = 0$

$W \sin\alpha = \mu_s N - \text{(i)}$

$\sum F_y = 0 \Rightarrow$

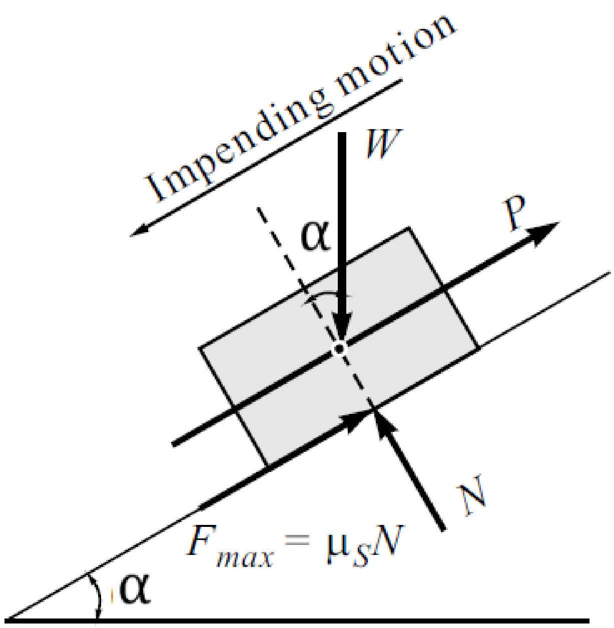
$N - W \cos\alpha = 0$

$W \cos\alpha = N - \text{(ii)}$

$\text{(i)} \div \text{(ii)} \Rightarrow \dfrac{W \sin\alpha}{W \cos\alpha} = \dfrac{\mu_s N}{N}$

$\tan\alpha = \mu_s = \tan\phi$

$\therefore \alpha = \phi$



Hence, the angle of repose is equal to the angle of friction.

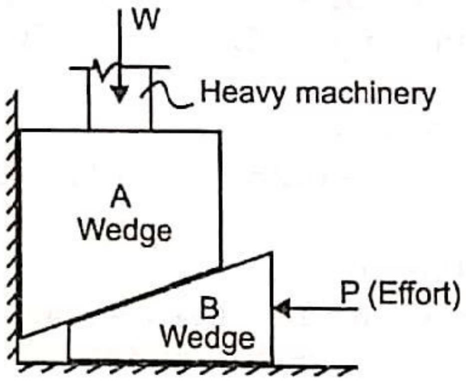
The above relation also shows that the angle of repose is independent of the weight of the body and it depends on μ.

Applications of Friction:

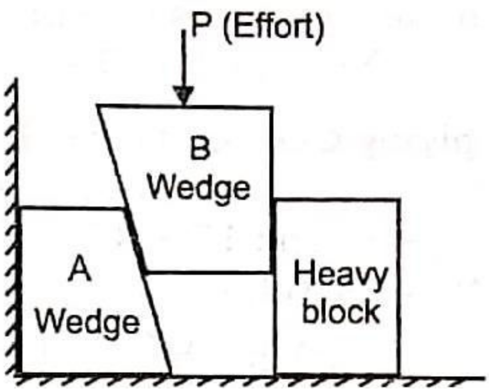
1. The running vehicle is controlled by applying brake to its tyre because of friction. The vehicle moves with better grip and does not slip due to appropriate friction between the tyre and the road.
2. One can walk comfortably on the floor because of proper gripping between floor and the sole of the shoes. It is difficult to walk on oily or soapy floor.
3. Belt and pulley arrangement permits loading and unloading of load effectively because of suitable friction.
4. Lift moves smoothly without slipping due to proper rope and pulley friction combination.
5. A simple lifting machine like screw jack functions effectively based on principle of wedge friction.

Wedges: A tapper shaped block (with very less angle of inclination) which are used for lifting or shifting or holding the heavy block by very less effort is called a wedge. The lifting or shifting of the distance is very small. While installing heavy machinery horizontal levelling is required with zero error. It is possible to adjust the small height by inserting wedge as a packing. Sometimes, combination of wedge is also used to push or shift heavy bodies by little distance. Simple lifting machine such as screw jack is based on principle of wedge which is used to raise or lower the heavy load by small effort.

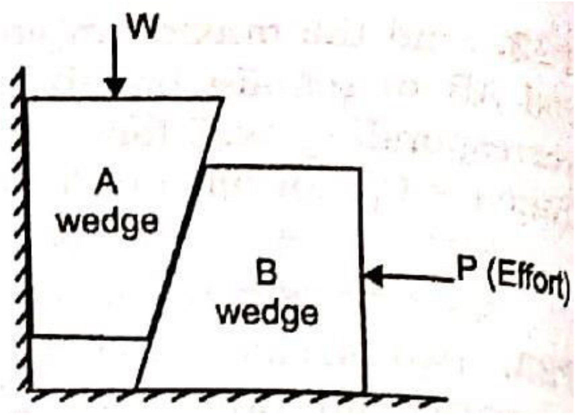
To lift heavy loads:



To slide heavy load:



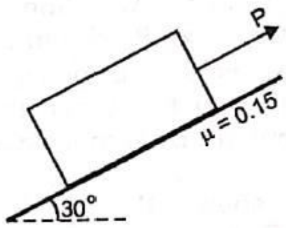
To hold the system in equilibrium:



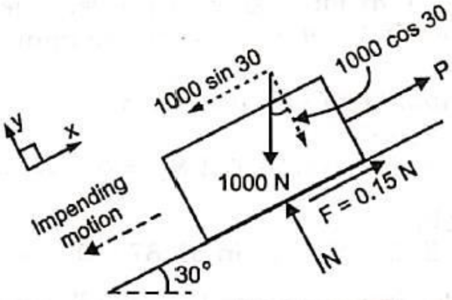
Ladders: Many a times, we come across the uses of ladder for reaching the higher height. Ladders are used by painters and carpenters who want to peg a nail in the wall for mounting a photo frame. We observe that care is taken to place the ladder at an appropriate angle with respect to ground and wall. We try to adjust the friction offered by the ground and wall in contact with ladder.

The forces acting on the ladder are normal reactions; frictional forces between the ground, the wall and the ladder, weight of the ladder and the weight of the man climbing the ladder. Considering the free body diagram of ladder, we get general force system. In ladder problems we need to use all 3 COEs.

Ex. 4.2 A block of weight 1000 N is kept on a rough inclined surface. A force P is applied parallel to plane to keep the block in equilibrium. Determine range of values of P for which the block will be in equilibrium. ***(MU May 13)***



Solution: Since we have to find range of values of P for equilibrium, let us first find P_{min}, which would be just sufficient to prevent the block from moving down the plane. The friction force therefore acts up the plane. Taking the axis as shown



F B D - Block

Applying COE

$$\Sigma F_y = 0$$

$$N - 1000 \cos 30 = 0$$

$$\therefore \quad N = 866 \text{ N}$$

$$\Sigma F_x = 0$$

$$P - 1000 \sin 30 + 0.15 \text{ N} = 0$$

$$\therefore \quad P - 1000 \sin 30 + 0.15\,(866) = 0$$

$$\text{or} \quad P_{min} = 370.1 \text{ N}$$

When P is maximum for equilibrium of the block, it tends to just cause the block to move up the plane, thereby the friction force acts down the plane.

Applying COE

$\Sigma F_y = 0$

$N - 1000 \cos 30 = 0$

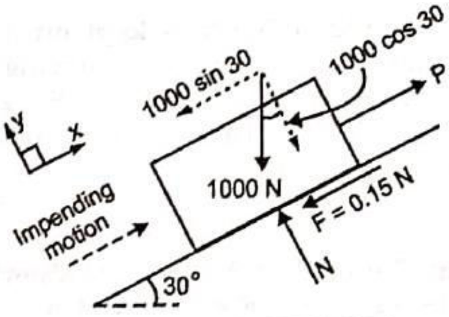
$\therefore \quad N = 866$ N

$\Sigma F_x = 0$

$P - 1000 \sin 30 - 0.15\,N = 0$

$\therefore \quad P - 1000 \sin 30 - 0.15\,(866) = 0$

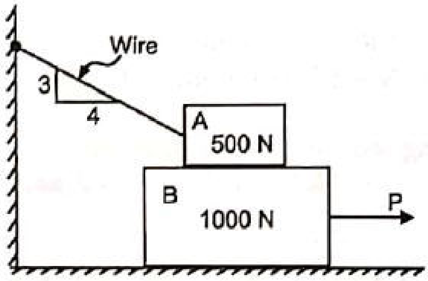
or $\quad P_{max} = 629.9$ N



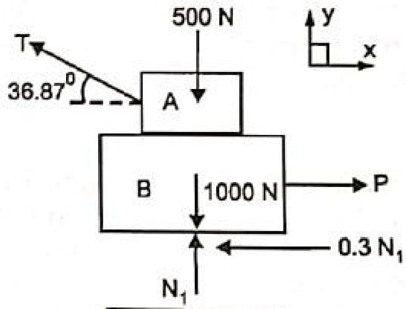
F B D - Block

The block is in equilibrium within the range $370.1 \text{ N} \le P \le 629.9$ N .... **Ans.**

Ex. 4.3 The upper block is tied to a vertical wall by a wire. Determine the horizontal force P required to just pull the lower block. Coefficient of friction for all surfaces is 0.3



Solution: Figure shows the FBD of the entire system. We find there are three unknowns viz. P, N_1 and T, and we have only two equations of equilibrium $\Sigma F_x = 0$ and $\Sigma F_y = 0$ for dimensionless blocks. We will therefore have to isolate the two blocks.



F B D - System

Figure shows the blocks isolated. Since block B tends to move to the right the friction force acts to the left. Hence for the block A, friction acts to the right.

Applying COE to block A

$\Sigma F_x = 0$

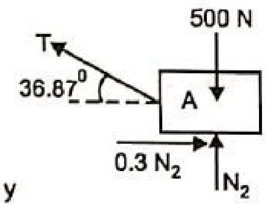
$-T \cos 36.87 + 0.3\,N_2 = 0$ -------- (1)

$\Sigma F_y = 0$

$-500 + N_2 + T \sin 36.87 = 0$ -------- (2)

Solving equations (1) and (2), we get

$N_2 = 408.2$ N



F B D - Block A

Applying COE to block B

$\Sigma F_y = 0$

$N_1 - N_2 - 1000 = 0$

$\therefore \quad N_1 - 408.2 - 1000 = 0$

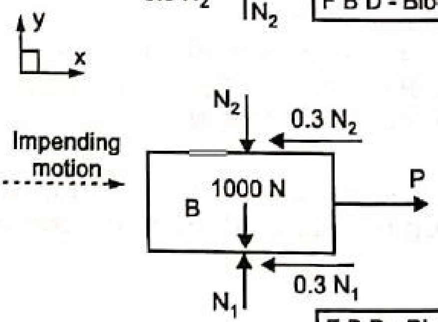
or $\quad N_1 = 1408.2$ N

$\Sigma F_x = 0$

$P - 0.3\,N_2 - 0.3\,N_1 = 0$

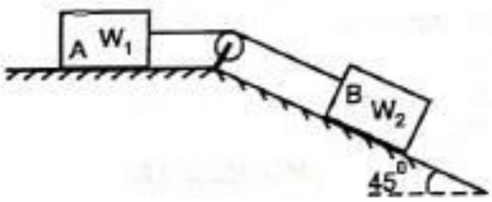
$\therefore \quad P - 0.3\,(408.2) - 0.3\,(1408.2) = 0$

or $\quad P = 544.9$ N -------- **Ans.**

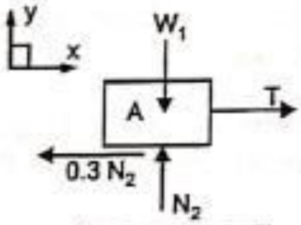


F B D - Block B

Ex. 4.4 Two blocks weighing W_1 and W_2 are connected by a string passing over a small smooth pulley as shown. If μ = 0.3 for both the planes, find the minimum ratio W_1/W_2 required to maintain equilibrium.



Solution: The component of weight of block B is responsible for causing motion of the system to impend down the plane.

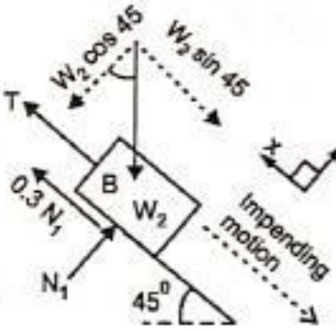


F B D - Block A

Isolating the two blocks

Taking different axes for A and B as shown

Applying COE to block B



F B D - Block B

$$\Sigma F_y = 0$$
$$N_1 - W_2 \cos 45 = 0$$
$$\therefore \quad N_1 = 0.707\, W_2 \qquad \text{--------(1)}$$

$$\Sigma F_x = 0$$
$$T + 0.3\, N_1 - W_2 \sin 45 = 0$$
$$\therefore \quad T + 0.3\,(0.707\, W_2) - W_2 \sin 45 = 0$$
$$\text{or} \quad T = 0.4949\, W_2 \qquad \text{--------(2)}$$

Applying COE to block A

$$\Sigma F_y = 0$$
$$N_2 - W_1 = 0$$
$$\therefore \quad N_2 = W_1 \qquad \text{--------(3)}$$

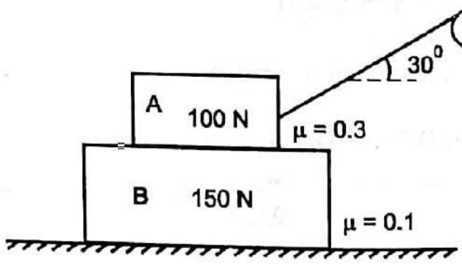
$$\Sigma F_x = 0$$
$$T - 0.3\, N_2 = 0$$

Substituting values of T and N_2

$$0.4949\, W_2 - 0.3\, W_1 = 0$$
$$\therefore \quad \frac{W_1}{W_2} = 1.65 \qquad \text{----------} \ \textbf{Ans.}$$

Ex. 4.6 Blocks A and B are resting on ground as shown. μ between ground and block is 0.1 and that between A and B is 0.3. Find the minimum value of P in the pan so that motion starts.

(MU May 13)



Solution: There are two possibilities. One is that block A moves over block B, while the other possibility is that both blocks A and B move together over the ground.

1st Possibility: Let block A move over block B

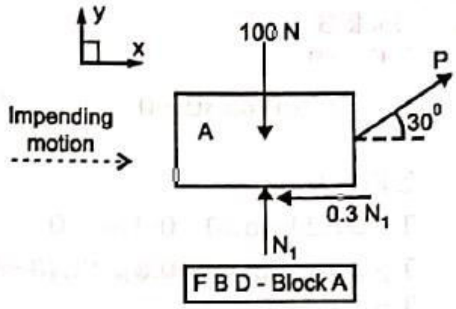
Applying COE

$\Sigma F_x = 0$

$P \cos 30 - 0.3\ N_1 = 0$ --------- (1)

$\Sigma F_y = 0$

$N_1 - 100 + P \sin 30 = 0$ --------- (2)



Solving equations (1) and (2)

$P = 29.53$ N

2nd Possibility: Both blocks A and B move together over the ground

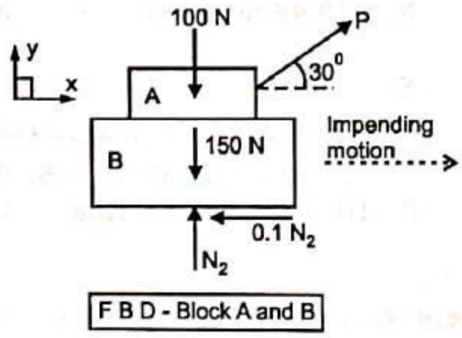
Applying COE

$\Sigma F_x = 0$

$P \cos 30 - 0.1\ N_2 = 0$ ----(3)

$\Sigma F_y = 0$

$N_2 - 100 - 150 + P \sin 30 = 0$ ----(4)

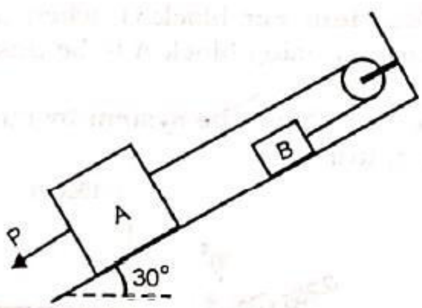


Solving equations (3) and (4)

$P = 27.29$ N

Since P required to move A and B together over the ground is less than P required for A to move over B, the system is set in motion at P = 27.29 N with both blocks moving together. .... **Ans.**

Ex. 4.7 Determine the force P to cause motion to impend. Take masses A and B as 8 kg and 4 kg respectively and coefficient of static friction as 0.3. The force P and rope are parallel to the inclined plane. Assume smooth pulley. ***(MU Dec 10)***



Solution: This is a system of two blocks connected to each other by a rope. As the force P applied to block A, impends to pull it down the plane, the block B impends to travel up the plane.

Isolating Block B
COE – Block B

$$\Sigma F_y = 0$$

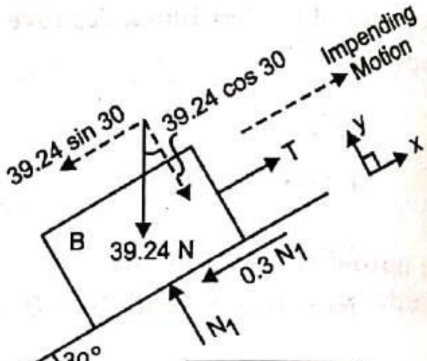
$$N_1 - 39.24\cos 30 = 0 \qquad \therefore \qquad N_1 = 33.98\,N$$

$$\Sigma F_x = 0$$

$$T - 39.24\sin 30 - 0.3\,N_1 = 0$$

$$T - 39.24\sin 30 - 0.3 \times 33.98 = 0$$

$$\therefore \qquad T = 29.81\,N$$



F B D - Block B

Isolating Block A
COE – Block A

$$\Sigma F_y = 0$$

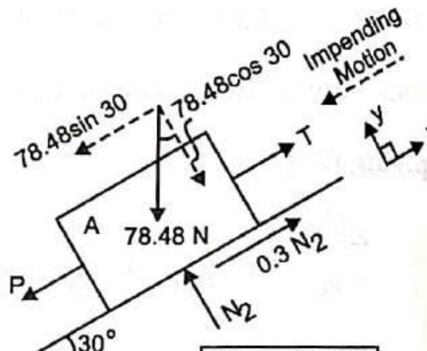
$$N_2 - 78.48\cos 30 = 0 \qquad \therefore \qquad N_2 = 67.96\,N$$

$$\Sigma F_x = 0$$

$$-P + T + 0.3\,N_2 - 78.48\sin 30 = 0$$

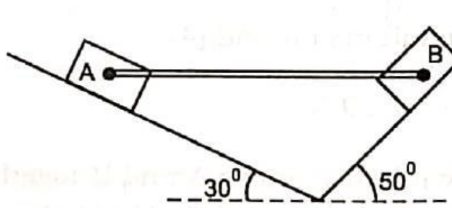
$$-P + 29.81 + 0.3 \times 67.96 - 78.48\sin 30 = 0$$

$$\therefore \qquad P = 10.96\,N \qquad \text{......... } \textbf{Ans.}$$



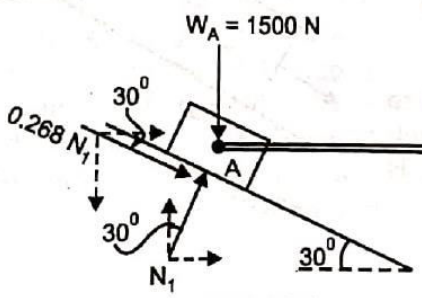
F B D - Block A

Ex. 4.8 Two blocks A and B are pin connected to a rod as shown. If block A weighs 1500 N, determine the maximum value of weight of block B for which equilibrium of the system is maintained. Angle of friction for all surfaces of contact is 15°.

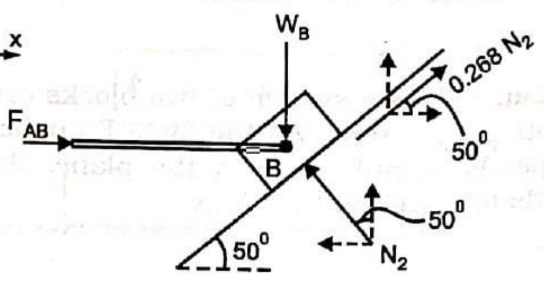


Solution: For block B, when its weight W_B is maximum, it will tend to slide down the slope causing block A to be pushed up its plane.

Let us isolate the system by cutting the rod. Assume force F_{AB} in the rod is compressive in nature.



F B D - Block A



F B D - Block B

Applying COE to Block A

$\Sigma F_Y = 0$

$N_1 \cos 30 - 0.268\ N_1 \sin 30 - 1500 = 0$

$\therefore \quad N_1 = 2049\ N$

$\Sigma F_X = 0$

$- F_{AB} + 0.268\ N_1 \cos 30 + N_1 \sin 30 = 0$

Substituting $N_1 = 2049$ N, we get

$F_{AB} = 1500\ N$

(+ ve value of F_{AB} indicates that force F_{AB} in the rod is compressive as assumed)

Applying COE to Block B

$\Sigma F_X = 0$

$F_{AB} + 0.268\ N_2 \cos 50 - N_2 \sin 50 = 0$

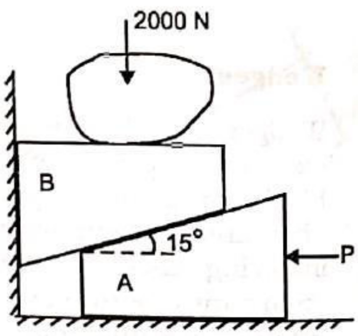
Substituting $F_{AB} = 1500$, we get, $\quad N_2 = 2526.2\ N$

$\Sigma F_Y = 0$

$- W_B + N_2 \cos 50 + 0.268\ N_2 \sin 50 = 0$

Substituting $N_2 = 2526.2$ N, we get, $W_{B(maximum)} = 2142.4\ N$ .... **Ans.**

Ex. 4.9 To raise a heavy stone block weighing 2000 N, the arrangement shown is used. What horizontal force P is necessary to be applied to the wedge in order to raise the block. $\mu = 0.25$. Neglect the weight of the wedges.



Solution: In this wedge problem we need to find force P required to just lift the stone.

Isolating wedges A and B as shown.

Applying COE to wedge B

$\Sigma F_x = 0$

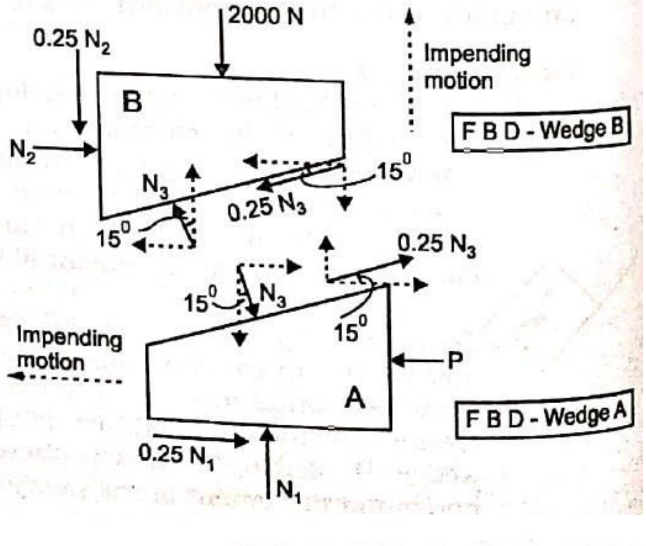
$N_2 - N_3 \sin 15$
$- 0.25\ N_3 \cos 15 = 0 \quad$ -------- (1)

$\Sigma F_y = 0$

$N_3 \cos 15 - 0.25\ N_3 \sin 15$
$- 0.25\ N_2 - 2000 = 0 \quad$ -------- (2)

Solving equations (1) and (2)

$N_3 = 2576.6\ N$



Applying COE to wedge A

$\Sigma F_y = 0$

$N_1 - N_3 \cos 15 + 0.25\ N_3 \sin 15 = 0$

$\therefore N_1 - 2576.6 \cos 15$
$+ 0.25\ (2576.6) \sin 15 = 0$

Or $\quad N_1 = 2322\ N$

$\Sigma F_x = 0$

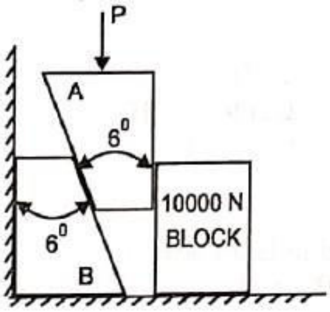
$- P + 0.25\ N_1 + N_3 \sin 15 + 0.25\ N_3 \cos 15 = 0$

$\therefore \quad - P + 0.25\ (2322) + 2576.6 \sin 15 + 0.25\ (2576.6) \cos 15 = 0$

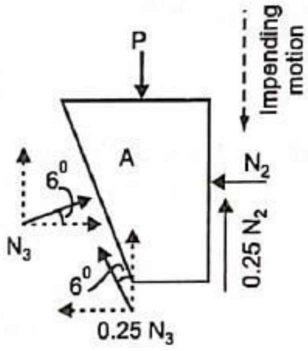
or $\quad P = 1869.6\ N$ .... **Ans.**

Ex. 4.10 Two 6° wedges are used to push the block horizontally as shown. Calculate the minimum force P required to push the block of weight 10000 N. $\mu = 0.25$ for all surfaces.

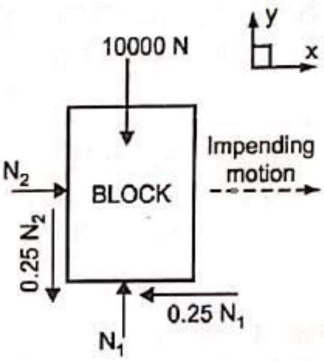
(MU Dec 08)



Solution: Figure below shows the FBD of the isolated wedge A and the block.



F B D - Wedge A



F B D - Block

Applying COE to block

$\Sigma F_x = 0$

$N_2 - 0.25\ N_1 = 0$ -------- (1)

$\Sigma F_y = 0$

$N_1 - 10000 - 0.25\ N_2 = 0$ -------- (2)

Solving equations (1) and (2)

$N_2 = 2666.7$ N

Applying COE to wedge A

$\Sigma F_x = 0$

$N_3 \cos 6 - 0.25\ N_3 \sin 6 - N_2 = 0$

$\therefore\ N_3 \cos 6 - 0.25\ N_3 \sin 6 - 2666.7 = 0$

or $N_3 = 2753.7$ N

$\Sigma F_y = 0$

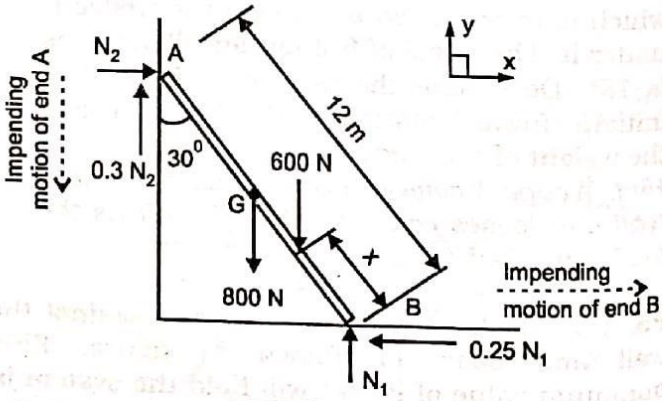
$-P + N_3 \sin 6 + 0.25\ N_3 \cos 6 + 0.25\ N_2 = 0$

$\therefore\ -P + 2753.7 \sin 6 + 0.25\ (2753.7) \cos 6 + 0.25\ (2666.7) = 0$

or $P = 1639.2$ N........... **Ans.**

Ex. 4.13 A 12 m ladder is resting against a vertical wall making 30° angle with the wall. Static friction between wall and ladder is 0.3 and that between ground and ladder is 0.25. A 600 N man ascends the ladder. How high will he be able to go before the ladder slips? Assume the weight of the ladder to be 800 N.

Solution: Figure shows the FBD of ladder AB resting against the wall and floor. Let the person climb the distance x on the ladder when the ladder is on the verge of slipping. The weight of the ladder acts through its C.G.



F B D - Ladder

Applying COE to the ladder

$\Sigma F_x = 0$

$N_2 - 0.25\ N_1 = 0$ ------ (1)

$$\Sigma F_y = 0$$

$$0.3\,N_2 + N_1 - 800 - 600 = 0 \qquad \text{------ (2)}$$

Solving equations (1) and (2)

$$N_1 = 1302.4\text{ N}, \qquad N_2 = 325.6\text{ N}$$

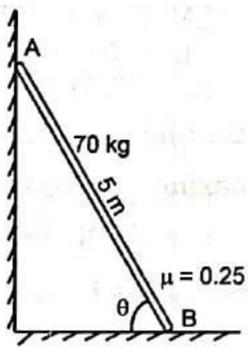
$$\Sigma M_B = 0 \quad \circlearrowleft +\text{ve}$$

$$-(N_2 \times 12\cos 30) - (0.3\,N_2 \times 12\sin 30) + (800 \times 6\sin 30) + (600 \times x\sin 30) = 0$$

$$\therefore \quad x = 5.23\text{ m} \qquad \text{............ **Ans.**}$$

Ex. 4.14 The rod AB of length 5 m and mass 70 kg is leaning against a wall. Find minimum θ for equilibrium. Take coefficient of friction = 0.25.

(KJS Nov 15)



Solution: For minimum value of θ, the ladder impends to slip down and away from the wall.

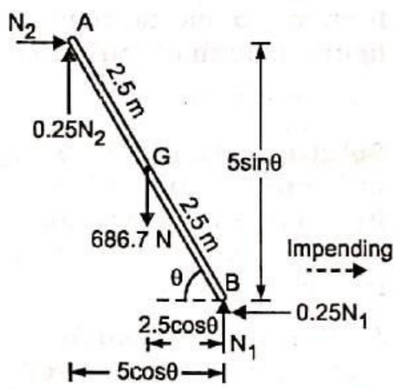
COE – ladder

$$\Sigma F_x = 0 \quad \rightarrow +\text{ve}$$

$$-0.25\,N_1 + N_2 = 0 \qquad \text{......... (1)}$$

$$\Sigma F_y = 0 \quad \uparrow +\text{ve}$$

$$N_1 + 0.25\,N_2 - 686.7 = 0 \qquad \text{......... (2)}$$



F B D - Ladder

Solving equation (1) and (2), we get

$$N_1 = 646.3\text{ N} \quad \text{and} \quad N_2 = 161.6\text{ N}$$

$$\Sigma M_B = 0 \quad \circlearrowleft +\text{ve}$$

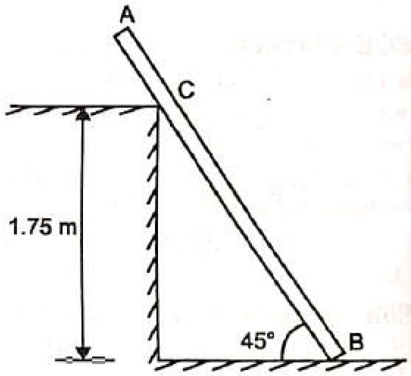
$$-(N_2 \times 5\sin\theta) - (0.25\,N_2 \times 5\cos\theta) + (686.7 \times 2.5\cos\theta) = 0$$

$$\therefore \quad -(161.6 \times 5\sin\theta) - (0.25 \times 161.6 \times 5\cos\theta) + 1716.65\cos\theta = 0$$

$$\therefore \quad 1514.75\cos\theta - 808\sin\theta = 0$$

$$\therefore \quad \tan\theta = \frac{1514.75}{808} \quad \text{or} \quad \theta = 61.92° \qquad \text{......... **Ans.**}$$

Ex.4.16 Determine minimum value of coefficient of friction so as to maintain the position shown in figure. Length of rod AB is 3.5 m and it weighs 250 N. ***(MU Dec 07)***



Solution: The rod AB is supported by a rough surface at B and a rough edge at C. Since the ladder looses its equilibrium position by slipping to the right, frictional forces at B and C have to shown opposite to impending motion.

Applying COE to rod AC

$\Sigma M_B = 0$ + ve

$+ (250 \times 1.237) - (N_2 \times 2.475) = 0$

$\therefore \quad N_2 = 125$ N

$\Sigma F_x = 0 \rightarrow +ve$

$N_2 \cos 45 - \mu\, N_2 \sin 45 - \mu N_1 = 0$

$\therefore \quad 125 \cos 45 - \mu \times 125 \sin 45 - \mu\, N_1 = 0$

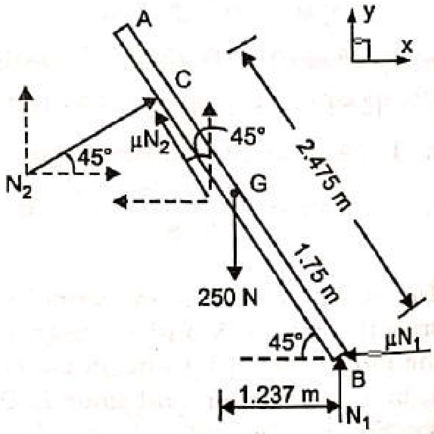
or $\quad \mu\,(N_1 + 88.39) = 88.39 \quad \ldots\ldots\ldots (1)$

$\Sigma F_y = 0 \uparrow + ve$

$N_2 \sin 45 + \mu\, N_2 \cos 45 + N_1 - 250 = 0$

$\therefore \quad 125 \sin 45 + \mu \times 125 \cos 45 + N_1 - 250 = 0$

or $\quad N_1 = 161.61 - 88.39\,\mu \quad \ldots\ldots\ldots (2)$



F B D - Rod

Substituting value of N_1 from equation (2) in (1)

$\mu\,[(161.61 - 88.39\,\mu) + 88.39] = 88.39$

or $\quad -88.39\,\mu^2 + 250\,\mu - 88.39 = 0$

Solving the above quadratic equation , we get $\mu = 0.414$ or 2.414

since μ cannot be greater than 1 selecting the feasible value, we get $\mu = 0.414$...... **Ans.**

K J Somaiya College of Engineering, Vidyavihar, Mumbai

(A Constituent College of SVU)

Engineering Mechanics Notes

Module 5 – Kinetics of Particle

Module Section 5.1 – Kinetics – Newton's Second Law

Class: FY BTech   Faculty: Aniket S. Patil   Date: 11/06/23

References: Engineering Mechanics, by M. D. Dayal & Engineering Mechanics – Statics and Dynamics, by N. H. Dubey.

Newton's Second Law of Motion: It states that "*the rate of change of momentum of a body is directly proportional to the resultant force and takes place in the direction of the force*".

Momentum is the quantity of motion possessed by a body, calculated by the product of its mass and velocity.

$$\frac{d}{dt}(m\bar{v}) \propto \sum\bar{F}$$

$$m\frac{d\bar{v}}{dt} = k\sum\bar{F} \text{ (for constant mass)}$$

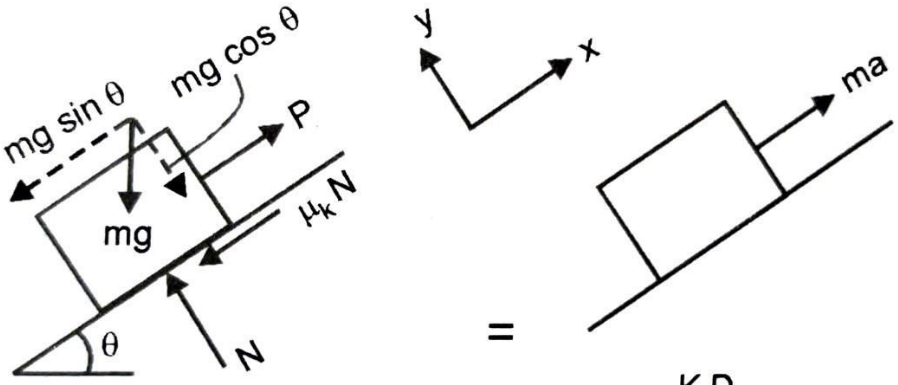
$$\Rightarrow m\bar{a} = k\sum\bar{F}$$

$$\therefore \sum\bar{F} = m\bar{a} \ (\because k = 1)$$

For rectilinear motion, or quantities in non-vector forms,

$$\sum F_x = ma_x; \sum F_y = ma_y; \sum F_z = ma_z$$

This results in another statement for the law, "*if the resultant force acting on a particle is not zero, the particle will have an acceleration proportional to the magnitude of the resultant and in the direction of the resultant force*".



F.B.D.   K.D.

For a block of mass m being pulled on an incline by a force P on a surface with kinetic coefficient of friction as μ_k, a free body diagram (FBD) is shown on the left and the corresponding kinetic diagram (KD) is shown on the right. This can be considered a visual representation of the Newton's second law.

Applying the NSL, in x and y directions as shown in the diagram, we get,

$$\sum F_x = ma_x$$

$$\Rightarrow P - mg\sin\theta - \mu_k N = ma$$

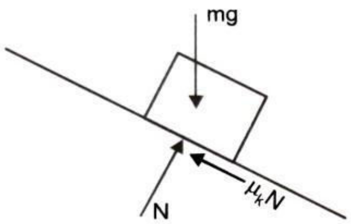
$$\sum F_y = ma_y$$

$$\Rightarrow N - mg\cos\theta = 0 \ (\because \text{no acceleration in y direction})$$

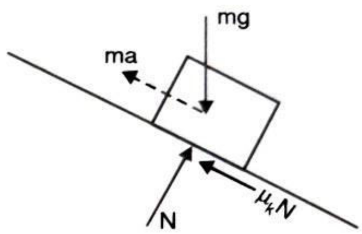
Solving the above two equations, we can find any unknowns required by the analysis, such as the acceleration.

D'Alembert's Principle: If the equation $\sum F = ma$ is rearranged as $\sum F - ma = 0$, treating the "– ma" as an inertia force, the system can be considered to be in dynamic equilibrium.

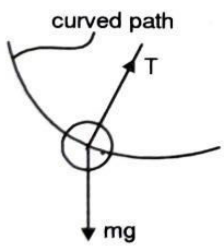
This is useful only in that COE can be used just like in static equilibrium situations, but it is not a realistic analysis.



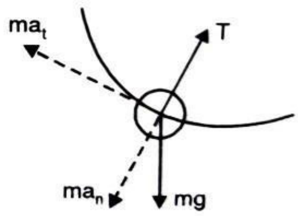
Actual forces acting on the block



Actual forces + Inertia force creates a state of dynamic equilibrium.



Actual forces acting on the pendulum



Actual forces + Inertia forces create a state of dynamic equilibrium

Applying the D'Alembert's Principle, we get a very similar equation as in NSL,

$$\sum F_x - ma_x = 0$$

$$\Rightarrow P - mg\sin\theta - \mu_k N - ma = 0$$

Ex. 10.2 A 500 N crate kept on the top of a 15° sloping surface is pushed down the plane with an initial velocity of 20 m/s. If $\mu_s = 0.5$ and $\mu_k = 0.4$, determine the distance traveled by the block and the time it will take as it comes to rest. ***(MU Dec 17)***

Solution: Applying equation of Newton's Second Law

$$\sum F_y = ma_y$$

$$N - 500\cos 15 = 0$$

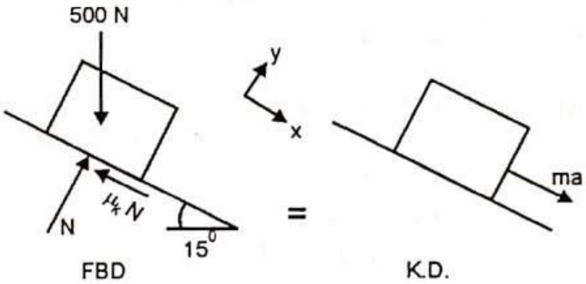
$$\therefore \quad N = 482.96 \text{ Newton}$$

$$\sum F_x = ma_x$$

$$500\sin 15 - \mu_k N = ma$$

$$500\sin 15 - 0.4\,(482.96) = \left(\frac{500}{9.81}\right)a$$

$$\therefore \quad a = -1.25 \text{ m/s}^2$$



The block travels down the slope with a –ve acceleration i.e deceleration of 1.25 m/s².

Kinematics

This is a case of Rectilinear motion – Uniform acceleration

u = 20 m/s, v = 0, s = ?, a = – 1.25 m/s², t = ?

Using $v^2 = u^2 + 2as$

$$0 = (20)^2 + 2 \times (-1.25) \times s$$

$$\therefore \quad s = 160 \text{ m} \qquad \text{......... Ans.}$$

using $v = u + at$

$$0 = 20 - 1.25 \times t$$

$$t = 16 \text{ sec} \qquad \text{........ Ans.}$$

Ex. 10.3 An elevator with a person inside, of total mass 500 kg starts moving upwards at a constant acceleration and attains a velocity of 3 m/s after traveling a distance of 3m. i) Determine the tension in the cable ii) If after attaining the velocity of 3 m/s the elevator stops in 2.5 seconds. Find the pressure exerted by the elevator to the person weighing 600N in the elevator. ***(VJTI Nov 09)***

Solution:

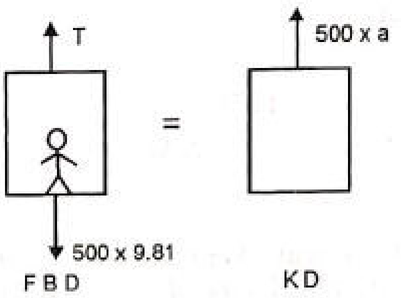
Kinematics: Motion of elevator stage (1) is rectilinear motion with uniform acceleration.

$u = 0$, $v = 3$ m/s, $s = 3$ m, $a = ?$, $t = -$

Using $v^2 = u^2 + 2as$

$$3^2 = 0 + 2 \times a \times s$$

$$\therefore \quad a = 1.5 \text{ m/s}^2$$



Kinetics of elevator

Let T be the tension entire cable

Applying NSL

$$\Sigma F_y = m a_y \quad \uparrow +ve$$

$$T - 500 \times 9.81 = 500 \times a$$

$$T - 4905 = 500 \times 1.5$$

or $\quad T = 5655\text{ N}$ .... **Ans.**

Kinematics: Motion of elevator stage (2)

$u = 3$ m/s, $v = 0$, $s = -$, $a = ?$, $t = 2.5$ sec.

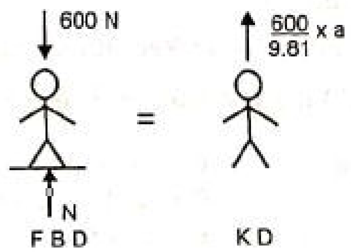
Using $v = u + at$

$$0 = 3 + a \times 2.5$$

or $\quad a = -1.2 \text{ m/s}^2$ .... **Ans.**

Kinetics of person inside the elevator

Let us isolate the person. Let N be the normal reaction the person receives from the floor of the elevator.



Applying N S L

$$\Sigma F_y = m a_y \quad \uparrow +ve$$

$$N - 600 = \frac{600}{9.81} \times a$$

$$N - 600 = 61.62 \times (-1.2)$$

or $\quad N = 526.6$ N .... **Ans.**

Problem 4

Two blocks A (10 kg mass), B (28 kg mass) are separated by 12 m as shown in Fig. 13.4(a). If the blocks start moving, find the time 't' when the blocks collide. Assume $\mu = 0.25$ for block A and plane and $\mu = 0.10$ for block B and plane.

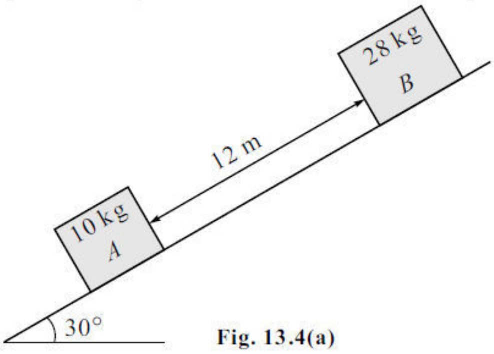


Fig. 13.4(a)

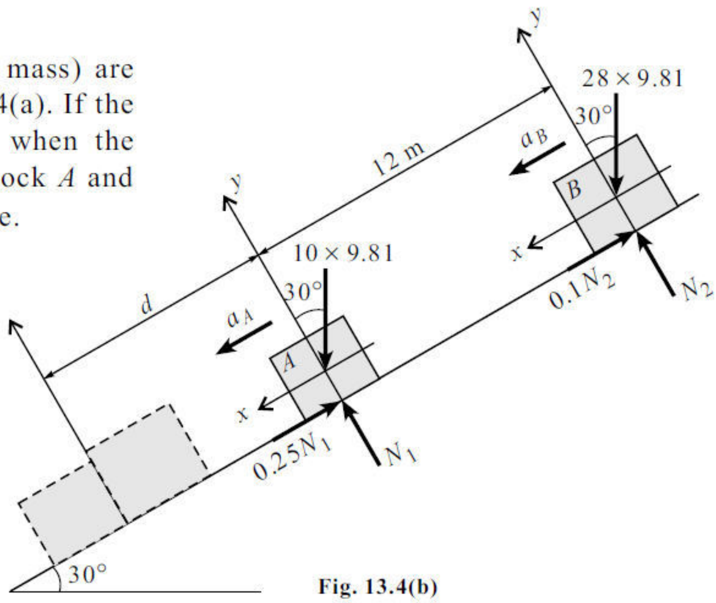


Fig. 13.4(b)

Solution

Refer to Fig. 13.4(b).

(i) Consider the F.B.D. of block A

By Newton's second law, we have

$$\sum F_x = ma_x$$

$$10 \times 9.81 \sin 30^\circ - 0.25 \times 10 \times 9.81 \cos 30^\circ = 10a_A$$

$$a_A = 2.781 \text{ m/s}^2 \ (30^\circ)$$

(ii) Consider the F.B.D. of block B

By Newton's second law, we have

$$\sum F_x = ma_x$$

$$29 \times 9.81 \sin 30^\circ - 0.1 \times 28 \times 9.81 \cos 30^\circ = 28a_B$$

$$a_B = 4.055 \text{ m/s}^2 \ (30^\circ)$$

(iii) Motion of block A

$$d = 0 + \frac{1}{2}a_A t^2 \quad \text{... (I)}$$

(iv) Motion of block B

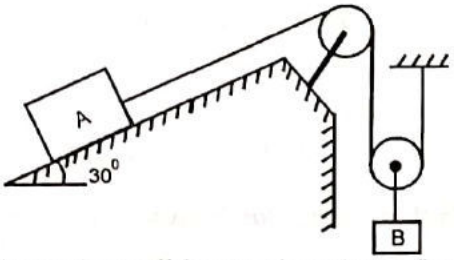
$$d + 12 = 0 + \frac{1}{2}a_B t^2 \quad \text{... (II)}$$

(v) From Eqs. (I) and (II), we get

$$\frac{1}{2} \times 2.781 \times t^2 + 12 = \frac{1}{2} \times 4.055 \times t^2$$

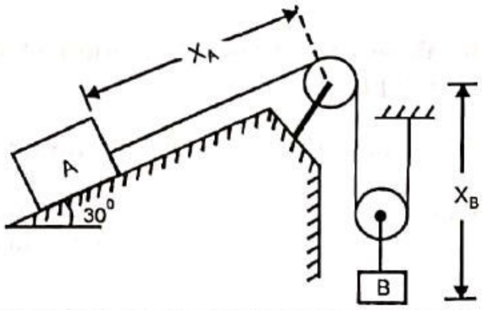
$\therefore \ t = 4.34$ s (Time when the blocks collide)

Ex. 10.5 A package A of mass 25 kg is being pulled up the incline by a load B of mass 60 kg connected to it by an inextensible rope passing over frictionless pulleys. Determine the accelerations of the two blocks and the tension in the connecting rope. Take $\mu_s = 0.4$ and $\mu_k = 0.3$ between the incline and A.



Solution: Downward movement of load B causes package A to slide up the plane. Let us develop the relation between the accelerations of A and B using Constant String Length Method (CSLM).

Let variables x_A and x_B define the positions of A and B. As x_B increases, x_A would decrease. If L is the length of string, then the length L is the sum of string portions in terms of x_A and x_B and plus/minus constants (constants are the string portions which don't change during motion like the length of cord wrapped over the pulleys).



$\therefore$ $L = (2\,x_B) + (-\,x_A) \pm$ constants (x_A is – ve because it reduces with increase in x_B)

Differentiating w.r.t time

$$0 = 2\,v_B - v_A$$

Differentiating again w.r.t time

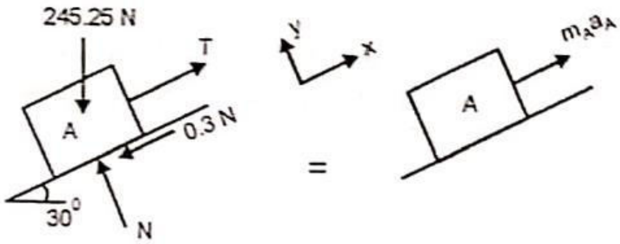
$$0 = 2\,a_B - a_A \quad \text{or} \quad a_A = 2\,a_B \quad \text{.......... (1)}$$

Let us isolate A and B and perform kinetic analysis of each of them

Kinetics of package A

Applying equations of Newton's second law to A

$$\sum F_y = ma_y$$
$$N - 245.25\cos 30 = 0$$
$$N = 212.39 \text{ Newton}$$
$$\sum F_x = ma_x$$
$$T - 245.25\sin 30 - 0.3\,N = m_A a_A$$
$$T - 245.25\sin 30 - 0.3(212.39) = 25\,a_A$$
$$T - 186.34 = 25\,a_A \quad \text{.......(2)}$$

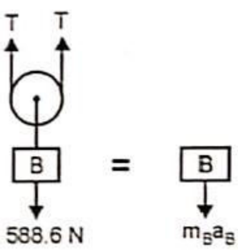


F.B.D. K.D.

Kinetics of load B

Applying equations of Newton's Second Law to B

$$\sum F_y = ma_y \uparrow \quad +\text{ve}$$
$$2T - 588.6 = -\,m_B a_B$$
$$2T - 588.6 = -\,60\,a_B \quad \text{.........(3)}$$



FBD KD

Solving equations (1), (2) and (3), we get

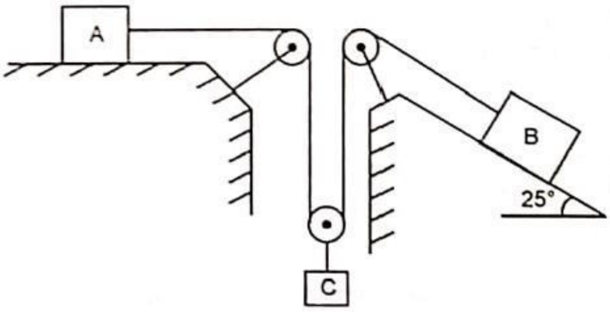
$a_A = 2.7\ m/s^2$ .... **Ans.**

$a_B = 1.35\ m/s^2$ .... **Ans.**

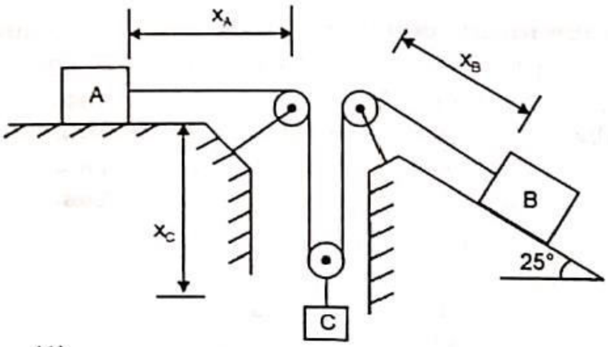
$T = 253.84$ N .... **Ans.**

Ex. 10.6 Find acceleration of block A, B and C shown in figure when the system is released from rest. Mass of blocks A, B and C is 5 kg, 10 kg and 50 kg respectively. Coefficient of friction for blocks A and B is 0.3. Neglect weight of pulley and rope friction.

(MU Dec 07)



Solution: Blocks A, B and C are connected to each other by a string and perform dependent motion. We need to first find the relation between the acceleration of the three blocks using CSLM.



Let x_A, x_B and x_C be the variable positions of blocks A, B and C.

Applying CSLM

Total length of string

$$L = x_A + x_B + 2x_C \quad \pm \text{ constant} \quad .... (1)$$

If C moves down then A moves to the left and B moves up the slope, causing variable x_C to increase with time and variables x_A and x_B to decrease with time.

Therefore correcting the above equation (1) we get

$$L = -x_A - x_B + 2x_C \quad \pm \text{ constants} \quad .... (2)$$

Differentiating equation (2) twice w.r.t. time, we get the acceleration relation as

$$0 = -a_A - a_B + 2a_C \quad .... (3)$$

Let us now isolate the blocks A, B and C and perform kinetic analysis using NSL to each of them separately. Let T be the tension in the string.

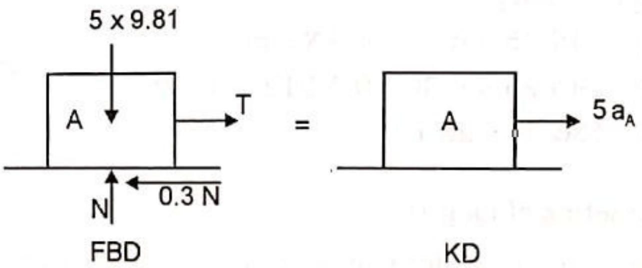
<u>Kinetics of block A</u>

Applying NSL

$\Sigma F_y = m.a_y \uparrow +\text{ve}$

$N - 5 \times 9.81 = 0$

$\therefore \quad N = 49.05$ N



$\Sigma F_x = m.a_x \rightarrow +\text{ve}$

$T - 0.3\ N = 5\ a_A$

$T - 0.3\ (49.05) = 5\ a_A$

$\therefore \quad a_A = 0.2\ T - 2.943 \quad .... (4)$

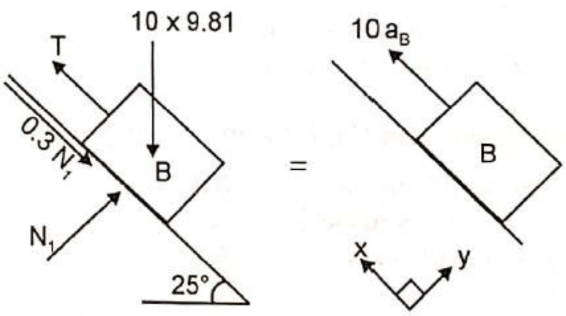
<u>Kinetics of block B</u>

Applying NSL

$\Sigma F_y = m.a_y$

$N_1 - 10 \times 9.81 \cos 25 = 0$

$\therefore\ N_1 = 88.91$ N



$\Sigma F_x = m.a_x$

$T - 0.3\ N_1 - 10 \times 9.81 \sin 25 = 10\ a_B$

$T - 0.3 \times 88.91 - 41.459 = 10\ a_B$

$\therefore\ a_B = 0.1\ T - 6.813 \quad .... (5)$

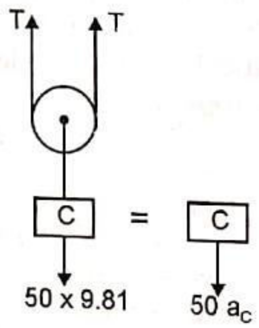
Kinetics of block C

Applying NSL

$\Sigma F_y = m.a_y$ ↑ + ve

$T + T - 50 \times 9.81 = -50\, a_C$

$\therefore \quad a_C = -0.04\,T + 9.81$ .... (6)



Substituting equations (4), (5) and (6) in equation (3)

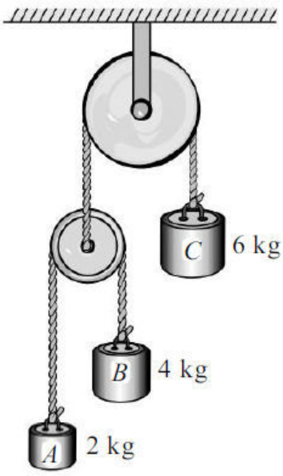
$0 = -(0.2\,T - 2.943) - (0.1\,T - 6.813) + 2(-0.04\,T + 9.81)$

$\therefore \quad T = 77.3$ N .... **Ans.**

also $\quad a_A = 12.517\ m/s^2$ .... **Ans.**

$a_B = 0.917\ m/s^2$ .... **Ans.**

$a_C = 6.718\ m/s^2$ .... **Ans.**



Problem 12

In the system of pulleys, the pulleys are massless and the strings are inextensible. Mass of A = 2 kg, mass of B = 4 kg and mass = 6 kg as shown in Fig. 13.12(a). If the system is released from rest, find **(i)** tension in each of the three string, and **(ii)** acceleration of each of the three masses.

Solution

Assume the direction of motion of all block as above.

(i) Kinematic relation [Fig. 13.12(b)]

$$Tx_A + Tx_B + 2Tx_C = 0$$

$$x_A + x_B + 2x_C = 0$$

Differentiating w.r.t. t

$$v_A + v_B + 2v_C = 0$$

Differentiating w.r.t. t again,

$$a_A + a_B + 2a_C = 0 \qquad \text{..... (I)}$$

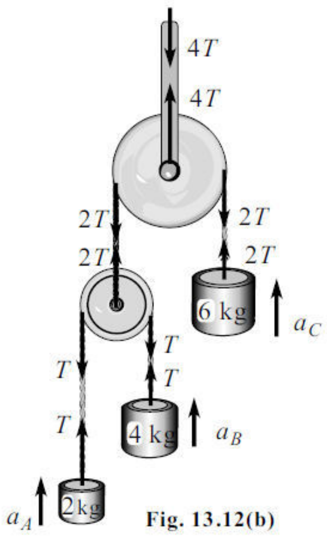


Fig. 13.12(b)

(ii) Consider the F.B.D. of block A [Fig. 13.12(c)]

$$\sum F_y = ma_y$$

$$T = 2 \times 9.81 = 2a_A$$

$$a_A = 0.5T - 9.81 \qquad \text{..... (II)}$$

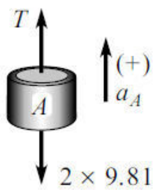


Fig. 13.12(c) : F.B.D. of Block A

(iii) Consider the F.B.D. of block B [Fig. 13.12(d)]

$$\sum F_y = ma_y$$

$$T - 4 \times 9.81 = 4a_B$$

$$a_B = 0.25T - 9.81 \qquad \text{..... (III)}$$

(iv) Consider the F.B.D. of block C [Fig. 13.12(e)]

$$\sum F_y = ma_y$$

$$2T - 6 \times 9.81 = 6a_C$$

$$a_C = 0.33T - 9.81 \quad \text{..... (IV)}$$

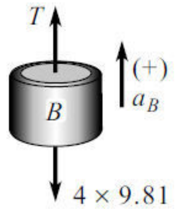


Fig. 13.12(d) : F.B.D. of Block B

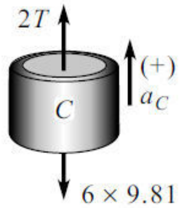


Fig. 13.12(e) : F.B.D. of Block C

(v) Putting Eqs. (II), (III) and (IV) in Eq. (I),

$$a_A + a_B + a_C = 0$$

$$(0.5T - 9.81) + (0.25T - 9.81) + (0.33T - 9.81) = 0$$

$$0.5T + 0.25T + 0.33T - 9.81 - 9.81 - 9.81 = 0$$

$$1.08T - 29.43 = 0$$

$$T = 27.25 \text{ N}$$

(vi) From Eq. (I),

$$a_A = 0.5 \times 27.25 - 9.81$$

$$a_A = 3.82 \text{ m/s}^2 \ (\uparrow)$$

(vii) From Eq. (II),

$$a_B = 0.25 \times 27.25 - 9.81$$

$$a_B = -3 \text{ m/s}^2 \text{ (Wrong assumed direction)}$$

$$a_B = 3 \text{ m/s}^2 \ (\downarrow)$$

(viii) From Eq. (III),

$$a_C = 0.33 \times 27.25 - 9.81$$

$$a_C = -0.82 \text{ m/s}^2 \text{ (Wrong assumed direction)}$$

$$a_C = 0.82 \text{ m/s}^2 \ (\downarrow)$$

K J Somaiya College of Engineering, Vidyavihar, Mumbai

(A Constituent College of SVU)

Engineering Mechanics Notes

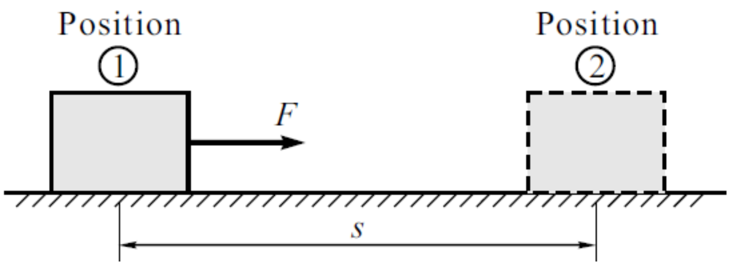
Module 5 – Kinetics of Particle

Module Section 5.2 – Kinetics – Work Energy Principle

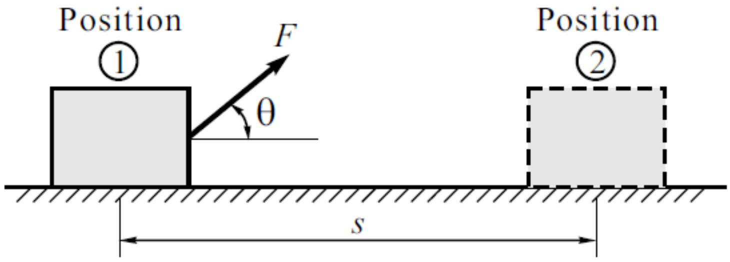
Class: FY BTech Faculty: Aniket S. Patil Date: 11/06/23

References: Engineering Mechanics, by M. D. Dayal & Engineering Mechanics – Statics and Dynamics, by N. H. Dubey.

Work: It is defined as the product of the displacement and the force in the direction of the displacement.



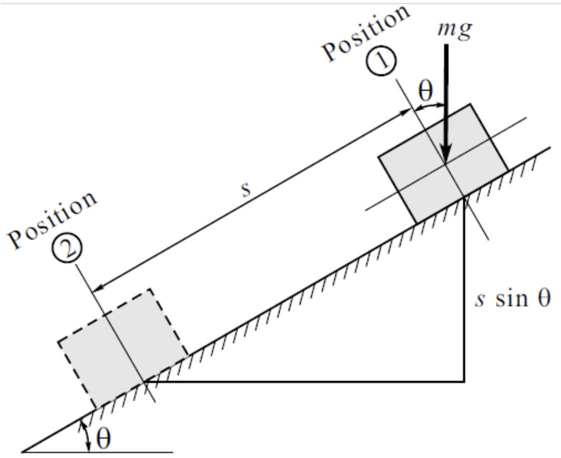
Work done = Force × Displacement $\Rightarrow U = F \times s$



If a particle is subjected to a force F at an angle θ with horizontal and the particle is displaced by s from position 1 to position 2 then work done U is the product of force component in the direction of displacement and displacement. $\Rightarrow U = F\cos\theta \times s$

1. Work done by a force is positive if the directions of force and displacement are in same direction. E.g., Work done by force of gravity is positive when a body moves from an upper position to lower position.
2. Work done by a force is negative if the directions of force displacement both are in opposite direction. E.g., Work done by force of gravity is negative when a body moves from a lower position to a higher position.
3. Work done is zero if either the displacement is zero or the force acts normal to the displacement. E.g., Gravity does not work when body moves horizontally.
4. Work is a scalar quantity.
5. Unit of work is Nm or Joule (J).

Work Done by Weight Force:



Work done = Component of weight in the direction of displacement × Displacement

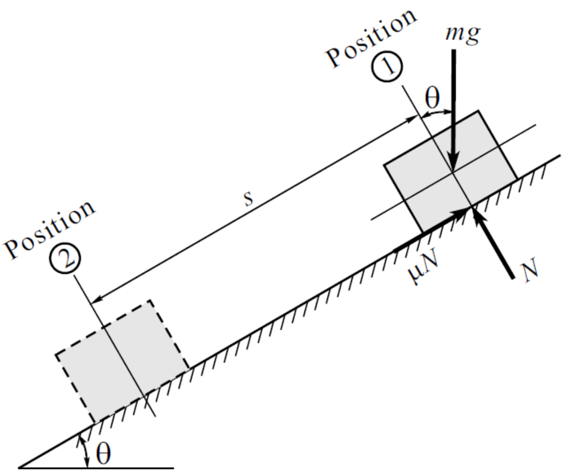
$U = mg \sin\theta \times s$

OR

Work done = Weight force × Displacement in the direction of weight force

$U = mg \times s \sin\theta$

Work Done by Frictional Force:



Work done = – Friction × Displacement

$U = \mu N \times s$

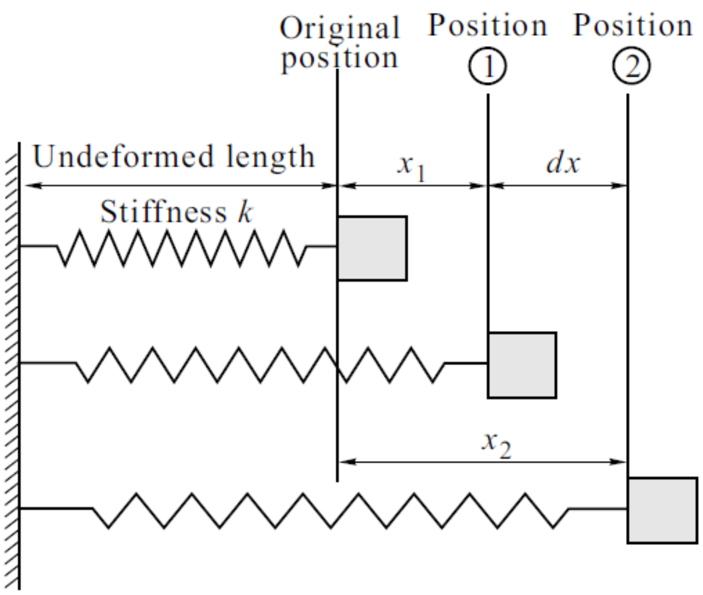
Work done by friction force is negative because direction of frictional force and displacement is opposite.

Work done by normal reaction (N) and component of weight force perpendicular to inclined plane (mg cos θ) is zero.

Work Done by Spring Force:

Consider a spring of stiffness k as shown in the figure, with some undeformed (free/original) length. Let x_1 & x_2 be deformations of spring at positions 1 & 2.

∴ Spring force $F = -k \times x$



where k is the spring stiffness (N/m)
x is the deformation of spring (m)
– ve sign indicates direction of spring force acting towards original position.

Work done = Spring force × Deformation

$$U = \int_{x_1}^{x_2} -kx\,dx$$

$$U = -\frac{1}{2}k(x_2^2 - x_1^2) \quad \text{OR} \quad U = \frac{1}{2}k(x_1^2 - x_2^2)$$

Kinetic Energy: It is the energy possessed by a particle by virtue of its motion.

$$\text{K.E.} = \frac{1}{2}\text{mv}^2$$

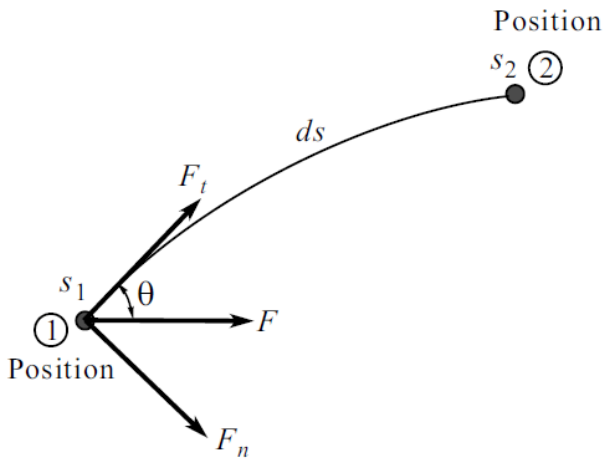
Potential Energy: It is the energy possessed by a particle by virtue of its position.

$$\text{P.E.} = +\text{mgh}, \text{if displacement is downwards}$$

$$\text{P.E.} = -\text{mgh}, \text{if displacement is upwards}$$

Work Energy Principle:

Work done by the forces acting on a particle during some displacement is equal to the change in kinetic energy during that displacement.



Consider the particle having mass m is acted upon by a force F and moving along a path as shown. Let v_1 and v_2 be the velocities of the particle at position 1 and position 2 and the corresponding displacement s_1 and s_2 respectively.

By Newton's second law in the tangential direction, we have,

$$\sum F_t = ma_t$$

$$F\cos\theta = ma_t = m\frac{dv}{dt} = m\frac{dv}{ds}\times\frac{ds}{dt} = mv\frac{dv}{ds}$$

$$F\cos\theta\, ds = mv\, dv$$

Integrating both sides, we have,

$$\int_{s_1}^{s_2} F\cos\theta\, ds = \int_{v_1}^{v_2} mv\, dv$$

$$U_{1-2} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

$$\therefore \text{Work done} = \Delta\text{Kinetic Energy}$$

Principle of Conservation of Energy:

Conservative Forces: If the work of a force is moving a particle between two positions is independent of the path followed by the particle and can be expressed as a change in its potential energy, then such forces is called as conservative forces. E.g., weight force of particle (gravity force), spring force and elastic force.

Non-Conservative Forces: The forces in which the work is dependent upon the path followed by the particles is known as non-conservative forces. E.g., frictional force

When a particle is moving from position 1 to position 2 under the action of only conservative forces (i.e., when frictional force does not exist) then by energy conservation principle we say that the total energy remains constant.

Total energy = Kinetic energy + Potential energy + Strain energy of spring

$$\text{Total energy} = \frac{1}{2}mv^2 \pm mgh + \frac{1}{2}mx^2$$

OR Total energy at position 1 = Total energy at position 2

$$KE_1 + PE_1 + SE_1 = KE_2 + PE_2 + SE_2$$

Ex. 11.2 A block is pushed with an initial velocity on a horizontal surface such that it travels 1500 mm before coming to rest. If $\mu_s = 0.25$ and $\mu_k = 0.2$ find the time of travel.

Solution: Applying Work Energy Principle from position (1) to position (2)

$T_1 = ½ mv^2 = 0.5 mv^2$ J

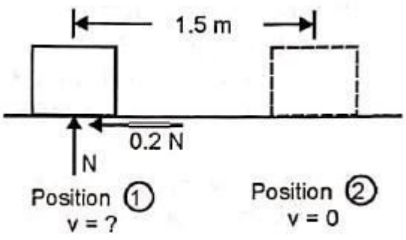
$T_2 = 0$

U_{1-2} by frictional force

$U = -\mu_k N.s$

$= -0.2 (m \times 9.81) \times 1.5$

$= -2.943$ m J



using $T_1 + \sum U_{1-2} = T_2$

$0.5 mv^2 - 2.943 m = 0$

$\therefore$ $v = 2.426$ m/s .... Initial velocity of block

Kinematics

Block performs rectilinear motion with uniform acceleration (since forces remain constant)

$u = 2.426$ m/s, $v = 0$, $s = 1.5$ m, $a = ?$, $t = t$ sec.

using $v^2 = u^2 + 2as$

$0 = (2.426)^2 + 2a \times 1.5$

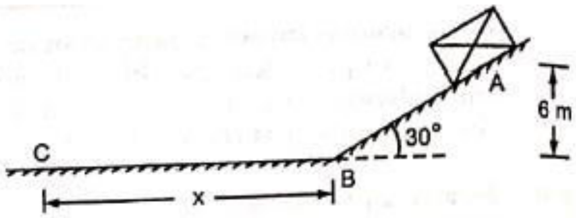
$\therefore$ $a = -1.962$ m/s^2

using $v = u + at$

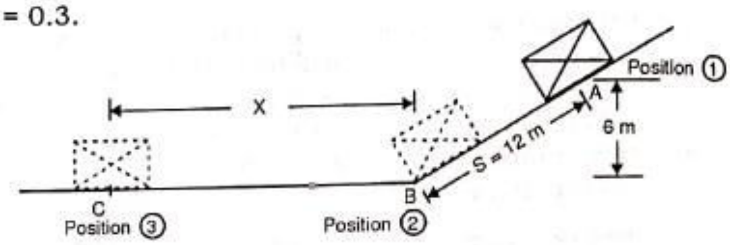
$0 = 2.426 - 1.962\,t$

$\therefore$ $t = 1.236$ sec .... **Ans.**

Ex. 11.1 A 20 kg crate is released from rest on the top of incline at A. It travels on the incline and finally comes to rest on the horizontal surface at C. Find the distance x it travels on the horizontal surface and also the maximum velocity it attains during the motion. Take $\mu_k = 0.3$.



Solution: The crate acquires maximum velocity at the lower most point B on the incline.
Applying Work Energy Principle from position (1) to (2).



$T_1 = 0$ ....since block starts from rest.
$T_2 = ½ mv^2 = ½ \times 20 \times v^2 = 10 v^2$ J
U_{1-2} 1) Work by weight force
$U = m g h$
$= 20 \times 9.81 \times 6 = 1177.2$ J
2) Work by frictional force
$U = -\mu_k N.s$
$\therefore \quad U = -0.3 \times 169.9 \times 12$
$= -611.64$ J

here, for the inclined surface, normal reaction,
$N = W \cos 30 = 20 \times 9.81 \cos 30 = 169.9$ N
and distance traveled by block,
$s = \dfrac{6}{\sin 30} = 12$ m

Using $\quad T_1 + \sum U_{1-2} = T_2$
$0 + [\,1177.2 - 611.64\,] = 10 v^2$
$\therefore \quad v = 7.52$ m/s
i.e $\quad v_{max} = 7.52$ m/s at B .... **Ans.**

To find the distance x traveled on the horizontal surface, we will apply work energy principle from position (2) to position (3)

$T_2 = ½ mv^2 = ½ \times 20 \times (7.52)^2 = 565.56$ J
$T_3 = 0$
U_{2-3} 1) only by frictional force
$= -u_k N.s$
$\therefore \quad U_{2-3} = -0.3 \times 196.2 \times x$
$= -58.86\, x$ J

For the horizontal surface,
$N = W = 20 \times 9.81 = 196.2$ N
Also the distance traveled by block,
s = x meters.

Using $\quad T_2 + \Sigma U_{2-3} = T_3$
$565.56 + [-58.86\, x] = 0$
$\therefore \quad x = 9.608$ m .... **Ans.**

Problem 16

A 20 N block is released from rest. It slides down the inclined having $\mu = 0.2$ as shown in Fig. 14.16(a). Determine the maximum compression of the spring and the distance moved by the block when the energy is released from compressed spring. Springs constant $k = 1000$ N/m.

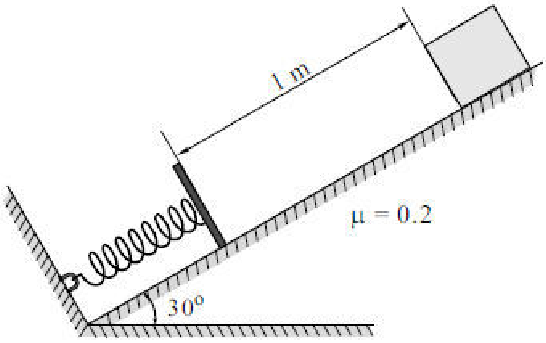


Fig. 14.16(a)

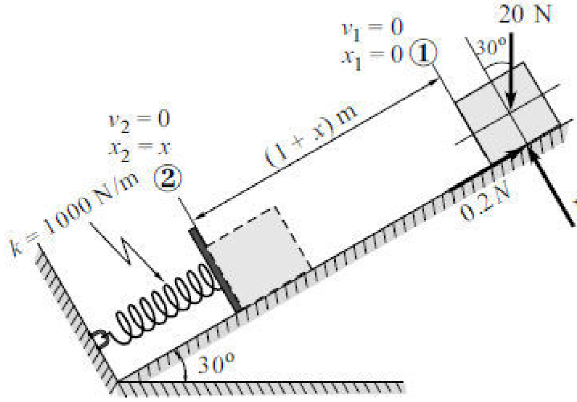


Fig. 14.16(b)

Solution

Part (i) Maximum compression of the spring

Let x be the maximum deformation of spring at position ② where the block comes to rest ($v_2 = 0$).

By work - energy principle, we have

Work done = Change in kinetic energy

$$\frac{1}{2} \times 1000(0^2 - x^2) + 20 \sin 30^\circ (1 + x) - 0.2 \times 20 \cos 30^\circ (1 + x) = 0 - 0$$

$\therefore \; x = 0.121$ m

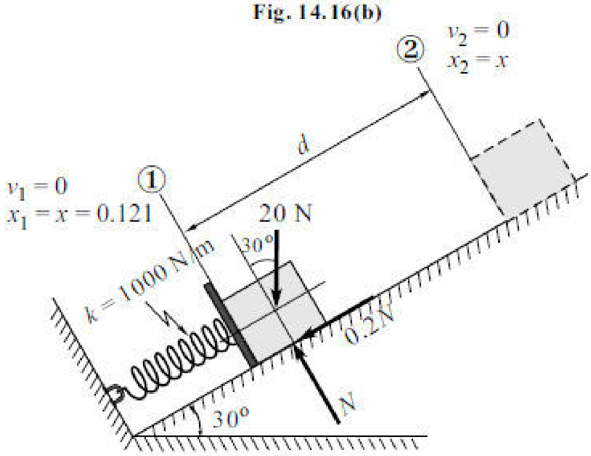


Fig. 14.16(c)

Part (ii) Distance moved by the block

By work - energy principle, we have

Work done = Change in kinetic energy

$$\frac{1}{2} \times 1000(0.121^2 - 0^2) - 20 \sin 30^\circ \times d - 0.2 \times 20 \cos 30^\circ \times d = 0 - 0$$

$\therefore \; d = 0.5437$ m

Problem 13

A 1 kg collar is attached to a spring and slides without friction along a circular rod which lies in horizontal plane as shown in Fig. 14.E13. The spring has a constant $k = 250$ N/m and is undeformed when collar is at B. Knowing that collar passes through point D with a speed of 1.8 m/s, determine the speed of the collar when it passes through point C and point B.

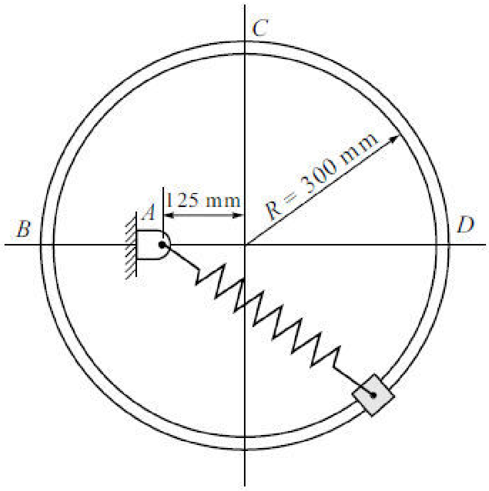


Fig. 14.13(a)

Solution

Undeformed length of spring is at B,

$$= 300 - 125 = 175 \text{ mm}$$
$$= 0.175 \text{ m}$$

Deformation of spring at position D,

$$= 125 + 300 - 175 = 250 \text{ mm}$$
$$= 0.25 \text{ m}$$

Deformation of spring at position C,

$$= \sqrt{125^2 + 300^2} - 175$$
$$= 325 - 175 = 150 \text{ mm}$$
$$= 0.15 \text{ m}$$

By principle of conservation of energy, we have total energy at any position remains constant.

P.E. throughout the ring is zero because it is at same level (horizontal).

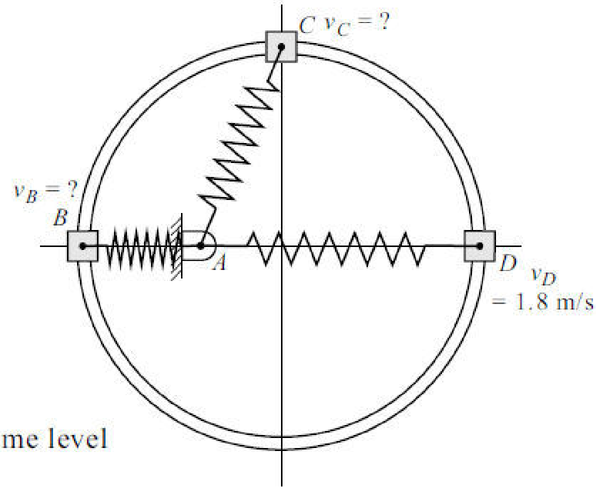


Fig. 14.13(b)

Total energy at position D = Total energy at position B

(K.E. + P.E. + S.E.) at D = (K.E. + P.E. + Spring energy) at B

$$\frac{1}{2} \times 1 \times 1.8^2 + 0 + \frac{1}{2} \times 250 \times 0.25^2 = \frac{1}{2} \times 1 \times v_B^2 + 0 + \frac{1}{2} \times 250 \times 0^2$$

$$9.43 = 0.5 v_B^2$$

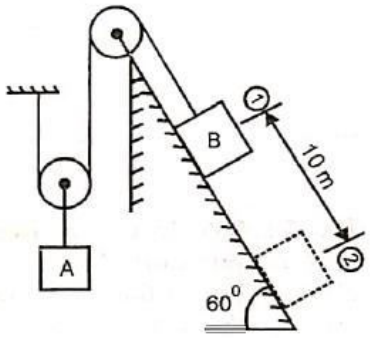
$$v_B = 4.343 \text{ m/s}$$

Total energy at position D = Total energy at position C

$$9.43 = \frac{1}{2} \times 1 \times v_C^2 + 0 + \frac{1}{2} \times 250 \times 0.15^2$$

$$9.43 = 0.5 v_C^2 + 2.8125$$

$$\therefore \; v_C = 3.638 \text{ m/s}$$

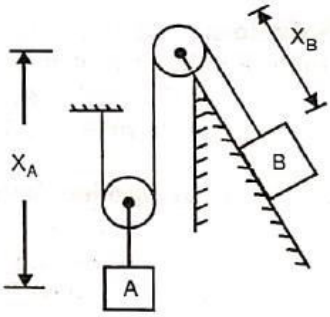


Ex. 11.5 Block B of weight 3000 N having a speed of 2 m/s in position (1) travels 10 m along and down the slope. Block A of weight 1000 N is connected to it by an inextensible string. Find the velocities of the blocks in the new position. Take $\mu_s = 0.35$ and $\mu_k = 0.3$ at the inclined surface.

Solution: We shall first find the relation between the velocities and the distance traveled by the two connected blocks.

Using constant string length method (CSLM)

If x_A and x_B are the variable positions of A and B measured from a fixed reference point, we have the length L of string in terms of x_A and x_B as



$L = (-2\,x_A) + x_B \pm$ constants

[x_A is – ve since with increase in x_B, x_A would decrease]

Differentiating w. r. to time

$0 = -2\,v_A + v_B$

or $v_B = 2\,v_A$ ...relation between the velocities

since the time interval is the same, we have

$x_B = 2\,x_A$ ....relation between distance traveled

Applying Work Energy Principle to the system of A and B from position (1) to (2).

$$
\begin{aligned}
T_1 &= \tfrac{1}{2} m_A v_A^2 + \tfrac{1}{2} m_B v_B^2 \\
&= \tfrac{1}{2}(101.94) \times (0.5\, v_B)^2 + \tfrac{1}{2}(305.81)\, v_B^2 \\
&= 165.65\, v_B^2 \\
&= 165.65\,(2)^2 = 662.6 \text{ J}
\end{aligned}
$$

$$
\begin{aligned}
T_2 &= \tfrac{1}{2} m_A v_A^2 + \tfrac{1}{2} m_B v_B^2 \\
&= 165.65\, v_B^2 \text{ J}
\end{aligned}
$$

U_{1-2}

1) by weight of block B

$U = +\,m\,g\,h = 3000 \times 8.66$

$= 25981$ J

$= 25981$ J

(+ ve since displacement is downwards)
Block B moves vertically down by
$h = 10 \sin 60 = 8.66$ m

2) by weight of block A

$U = -\,m\,g\,h = -\,1000 \times 5$

$= -\,5000$ J

(– ve since displacement is upwards)
since $x_A = 0.5\, x_B$, displacement of block A $= 0.5 \times 10 = 5$ m

3) by frictional force at the inclined surface

$U = -\,\mu k N \times s$

$U = -\,0.3 \times 1500 \times 10$

$= -\,4500$ J

block B travels a distance $s = 10$ m Normal reaction on the inclined surface $N = W \cos\theta$
$= 3000 \cos 60$
$= 1500$ Newton

Using $T_1 + \sum U_{1-2} = T_2$

$662.6 + [25981 - 5000 - 4500] = 165.65\, v_B^2$

$\therefore\quad v_B = 10.17$ m/s .... **Ans.**

also $v_A = 0.5\, v_B$

$\therefore\quad v_A = 0.5 \times 10.17 = 5.086$ m/s .... **Ans.**

Problem 22

Two springs each having stiffness of 0.5 N/cm are connected to ball *B* having a mass of 5 kg in a horizontal position producing initial tension of 1.5 m in each spring as shown in Fig. 14.E22. If the ball is allowed to fall from rest what will be its velocity after it has fallen through a height of 15 cm.

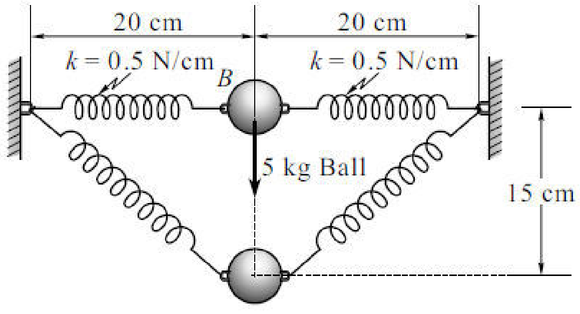


Fig. 14.22(a)

Solution

Method I

Initial position tension = 1.5 N

$T = kx$

$1.5 = (0.5)(x)$

$x = 3$ cm (Deformation in initial position)

$\therefore$ Free length of spring = 20 − 3 = 17 cm

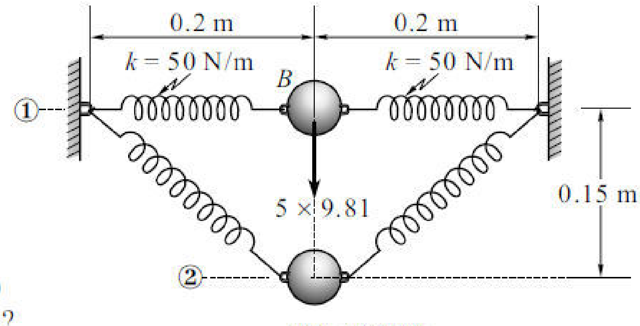


Fig. 14.22(b)

At position ①

$v_1 = 0$

$x_1 = 3$ cm

$\therefore\ x_1 = 0.03$ m

Displacement $h = 15$ cm

$\therefore\ h = 0.15$ m

At position ②

$v_2 = ?$

$x_2 = (25 - 17) = 8$ cm

$x_2 = 0.08$ m

Spring constant $k = 0.5$ N/cm

$\therefore\ k = 50$ N/m

By principle of work - energy, we have

Work done = Change in kinetic energy

$$5 \times 9.81 \times 0.15 + \left[\frac{1}{2} \times 50(0.03^2 - 0.08^2)\right] \times 2 = \frac{1}{2} \times 5 \times v_2^2 - 0$$

$v_2 = 1.68$ m/s

Method II

By principle of conservation of energy

Total energy = K.E. + P.E. + S.E.

Total energy remains constant at any position.

Total energy at position ① = Total energy at position ②

(K.E. + P.E. + S.E.) at position ① = (K.E. + P.E. + S.E.) at position ②

$$\frac{1}{2} \times 5 \times 0^2 + 5 \times 9.81 \times 0 + \frac{1}{2} \times 50 \times 0.03^2 = \frac{1}{2} \times 5 \times v_2^2 - 5 \times 9.81 \times 0.15 + \frac{1}{2} \times 50 \times 0.08^2$$

$$0.0225 = 2.5v_2^2 - 7.3575 + 0.16$$

$v_2 = 1.69$ m/s

K J Somaiya College of Engineering, Vidyavihar, Mumbai

(A Constituent College of SVU)

Engineering Mechanics Notes

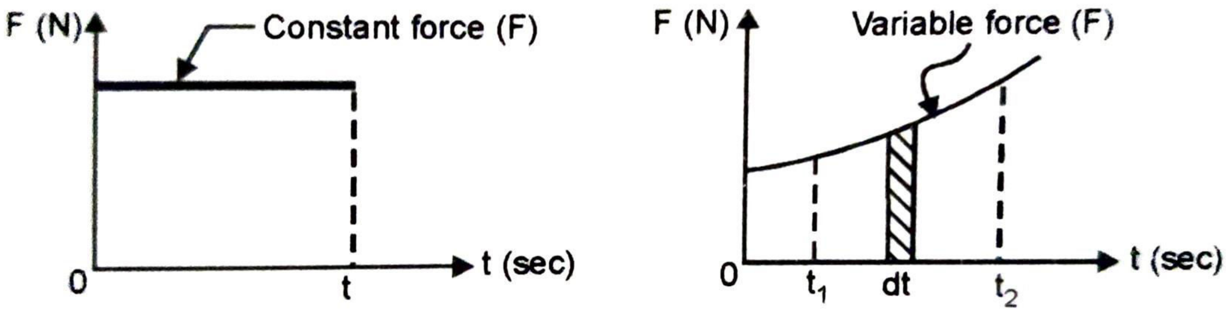
Module 5 – Kinetics of Particle

Module Section 5.3 – Kinetics – Impulse Momentum Equation

Class: FY BTech   Faculty: Aniket S. Patil   Date: 11/06/23

References: Engineering Mechanics, by M. D. Dayal & Engineering Mechanics – Statics and Dynamics, by N. H. Dubey.

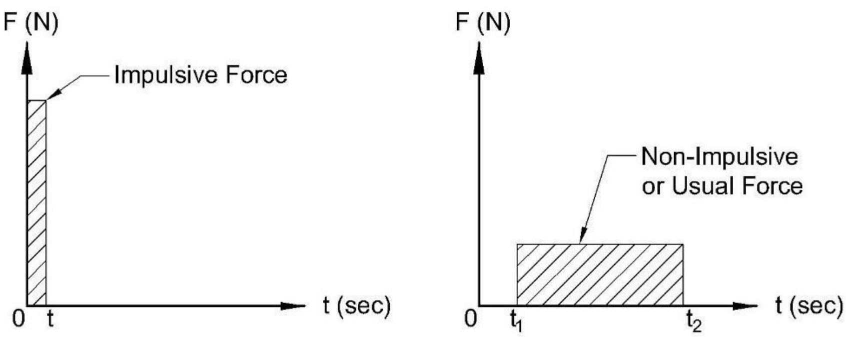
Impulse: For a particle acted upon by a force F for a duration of time t, the force is said to impart an impulse on the particle and the magnitude of this impulse is the product of the force and the duration for which it acts.



If the force F is constant during the time it acts, Impulse $= F \times t$

If the force F is variable from time t_1 to t_2, Impulse $= \int_{t_1}^{t_2} F\,dt$

Impulsive Force: A large force when acts for a very small time and which causes a considerable change in a particle's momentum is called an impulsive force. E.g., bats hitting balls, two bodies colliding, hammering nails, guns firing, etc.



Impulsive forces are different from usual forces because the impulse generated is mainly due to the large force, and the time is less important; whereas usual forces also generate impulse, where the time, an equally important parameter, is large and equally contributes to the impulse generated.

Impulse Momentum Equation:

From Newton's Second Law, we have,

$$F = \frac{d(mv)}{dt} \Rightarrow F\,dt = d(mv)$$

$$\int_{t_1}^{t_2} F\,dt = \int_{v_1}^{v_2} d(mv) = mv_2 - mv_1$$

$$\therefore \text{Impulse}_{1-2} = \Delta\text{Momentum}$$

$$mv_1 + \text{Impulse}_{1-2} = mv_2$$

Or $\qquad$ Initial Momentum + Impulse Imparted = Final Momentum

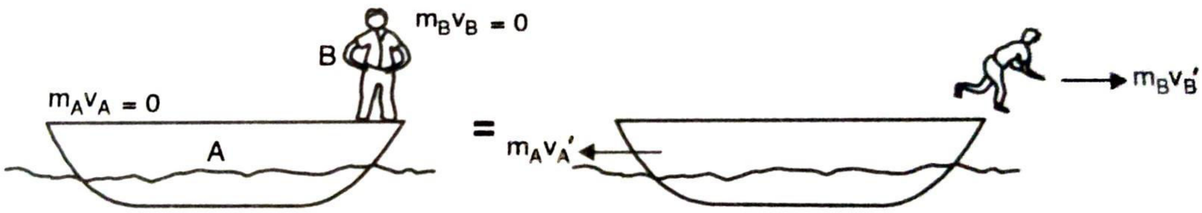
This gives rise to the Principle of Impulse Momentum which states that, "*for a particle or a system of particles acted upon by forces during a time interval, the total impulse acting on the system is equal to the difference between the final momentum and initial momentum during that period*".

Conservation of Momentum Equation:

In a system, if the resultant force is zero, the impulse momentum equation reduces to final momentum equal to initial momentum. Such situation arises in many cases because the force system consists of only action and reaction on the elements of the system. The resultant force is zero, only when entire system is considered, but not when the free body of each element of the system is considered.

If a man jumps off a boat, the action of the man is equal and opposite to the reaction of the boat. Hence, the resultant is zero in the system. Similar equation holds good when we consider the system of a gun and shell.

Initial momentum = Final momentum



$$mv_1 + \text{Impulse}_{1-2} = mv_2$$

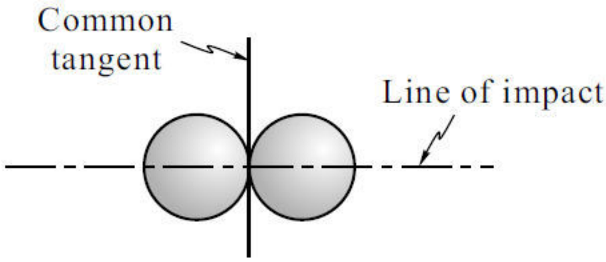
$$m_A v_A + m_B v_B = m_A v_A' + m_B v_B' \; (\because \text{Impulse}_{1-2} = 0)$$

$$m_A v_A' = -m_B v_B' \; (\because m_A v_A = 0, m_B v_B = 0)$$

"*For dynamic situations involving a system of particles, if the net impulse is zero, the momentum of the system is conserved.*"

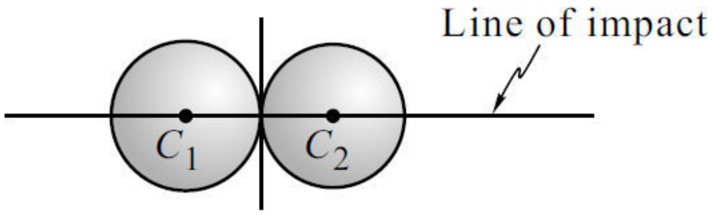
Impact: A collision of two bodies, which occurs for a very small interval of time and during which the two bodies exert relatively very large forces on each other, is called an impact.

<u>Line of Impact</u>: The common normal to the surfaces of two bodies in contact during the impact is called line of impact, and is perpendicular to the common tangent.

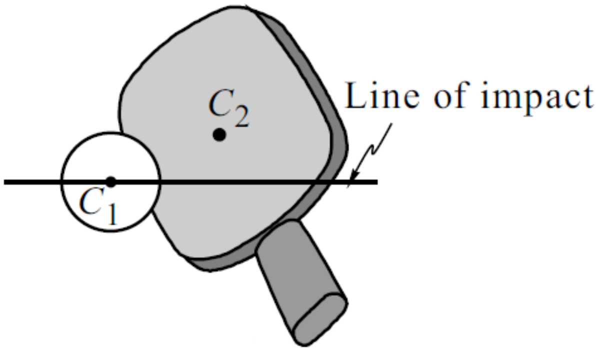


<u>Types of Impact</u>:

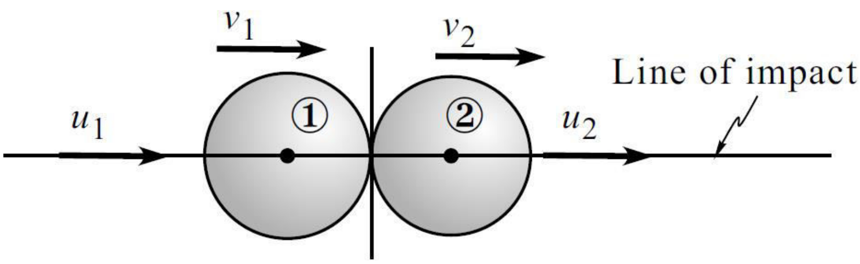
Central Impact: When the mass centres C_1 and C_2 of the colliding bodies lie on the line of impact, it is called central impact.



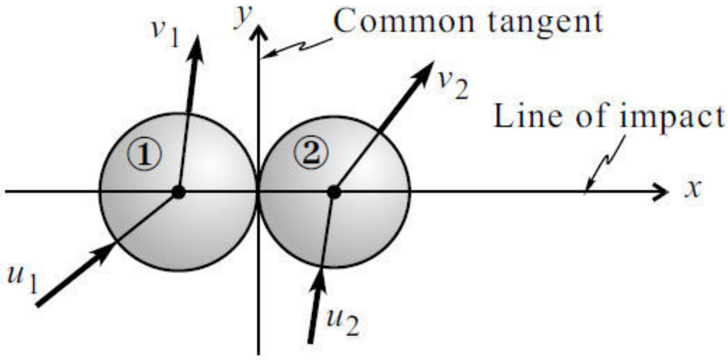
Non-Central Impact: When the mass centres C_1 and C_2 of the colliding bodies do not lie on the line of impact, it is called non-central or eccentric impact.



Direct Central Impact: When the direction of motion of the mass centres of two colliding bodies is along the line of impact then we say it is direct central impact. Here, the velocities of two bodies collision are collinear with the line of impact.

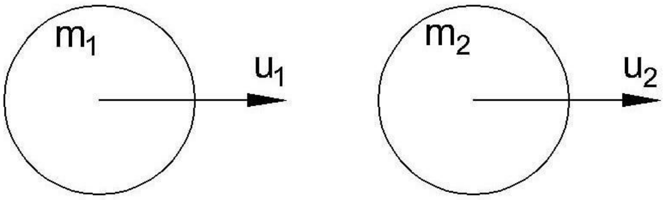


Oblique Central Impact: When the direction of motion of the mass centres of one or two colliding bodies is not along the line of impact (i.e., at the same angle with the line of impact) then we say it is oblique central impact. Here the velocities of two bodies collision are not collinear with the line of impact.

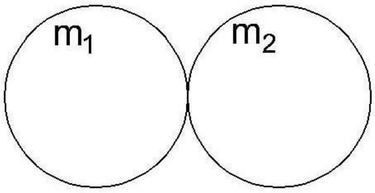


Direct Central Impact:

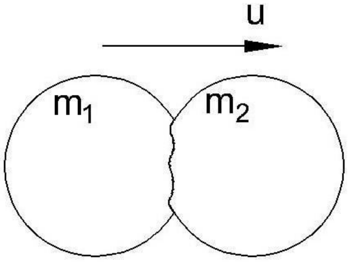
1. Two particles with masses m_1 & m_2 are traveling at velocities u_1 & u_2. If u_1 is greater than u_2, impact will occur.



2. When impact takes place, the period of impact is made of period of deformation and the period of restitution (regaining the original shape).

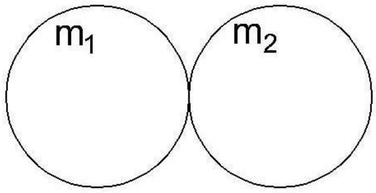


3. During the period of deformation, the particles exert large impulsive forces on each other. The deformation of both particles continues till maximum deformation, when both particles are momentarily united and are moving together with common velocity u.

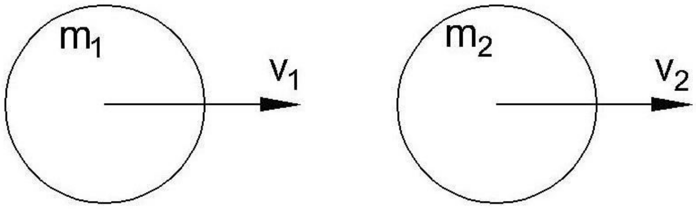


4. Now the period of restitution begins, during which the particles restore their shape. The particles may restore their shape completely, partially, or not at all, depending upon their properties. During this period also some impulsive

force is exerted by the particles on each other. At the end of this period the particles separate from each other.

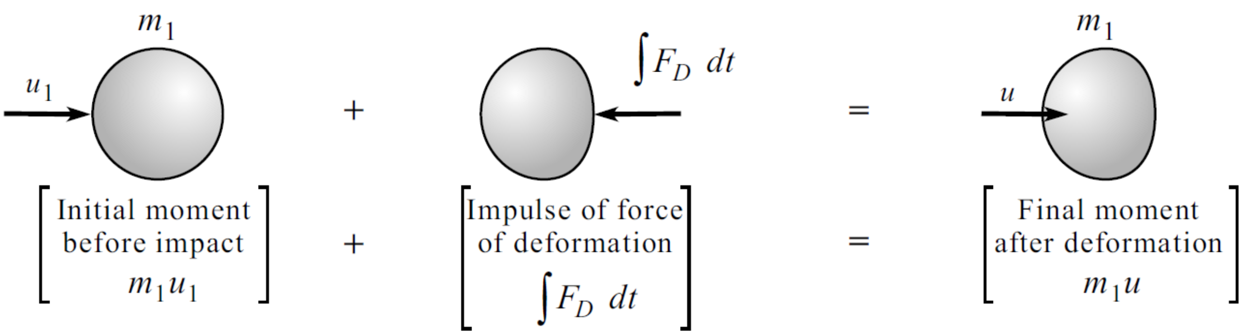


5. The particles now move independently with new velocities v_1 and v_2.



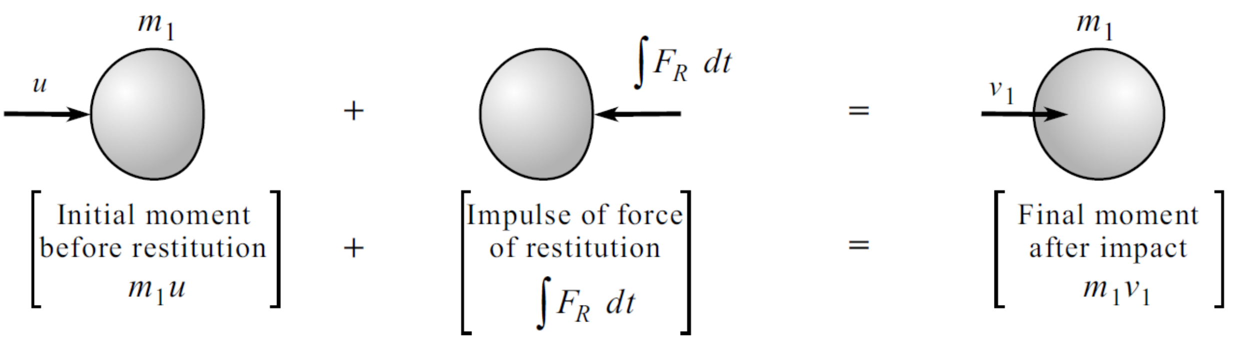
Coefficient of Restitution: *It is the ratio of the impulse exerted between the colliding particles during the period of restitution to the impulse exerted during the period of deformation*. This indicates the fraction of the shape that is regained by the particles, that was deformed during the collision.

During period of deformation, from beginning of impact till maximum deformation,



$$m_1 u_1 - \int F_D \, dt = m_1 u$$

During period of restitution, from maximum deformation till end of impact,



$$m_1 u - \int F_R \, dt = m_1 v_1$$

Hence, the deformation impulse and restitution impulse can be written as,

$$\int F_D\,dt = m_1u_1 - m_1u \quad \& \quad \int F_R\,dt = m_1u - m_1v_1$$

Therefore, coefficient of restitution, e, is given by,

$$\frac{\int F_R\,dt}{\int F_D\,dt} = \frac{m_1u - m_1v_1}{m_1u_1 - m_1u} = \frac{u - v_1}{u_1 - u} = e$$

Similarly, for particle 2, using the same impulse momentum equations,

$$\frac{v_2 - u}{u - u_2} = e$$

Using the addendo property from mathematics,

$$e = \frac{u - v_1 + v_2 - u}{u_1 - u + u - u_2} = \frac{v_2 - v_1}{u_1 - u_2}$$

Hence, another definition of coefficient of restitution for direct central impact is "*the ratio of velocity of separation to the velocity of approach*".

<u>Equations to solve Direct Central Impact problems</u>:

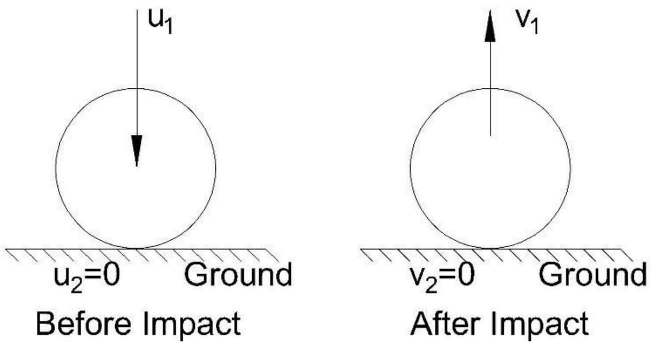
1. Total momentum of system is conserved along the line of impact.

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

2. Coefficient of restitution equation,

$$v_2 - v_1 = e(u_1 - u_2)$$

<u>Special Case of Direct Central Impact</u>**:** When a ball hits a wall or ground, which can be considered as having infinite mass as compared to the ball.



$$v_2 - v_1 = e(u_1 - u_2)$$

$$0 - v_1 = e(u_1 - 0)$$

$$v_1 = -eu_1$$

Only considering the magnitude,

$$v_1 = eu_1$$

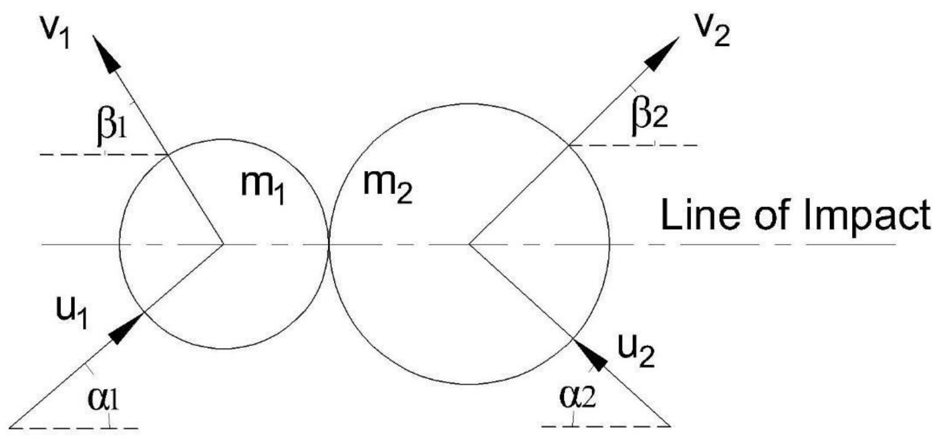
Let a ball be dropped from a height h on the ground and it reaches height h_n after n bounces, and the coefficient of restitution between ball and ground is e.

$$e = \frac{v_1}{u_1} = \frac{\sqrt{2gh_1}}{\sqrt{2gh}} = \left(\frac{h_1}{h}\right)^{\frac{1}{2}} \Rightarrow \text{For n bounces, } e = \left(\frac{h_n}{h}\right)^{\frac{1}{2n}}$$

Types of Impact Based on Coefficient of Restitution:

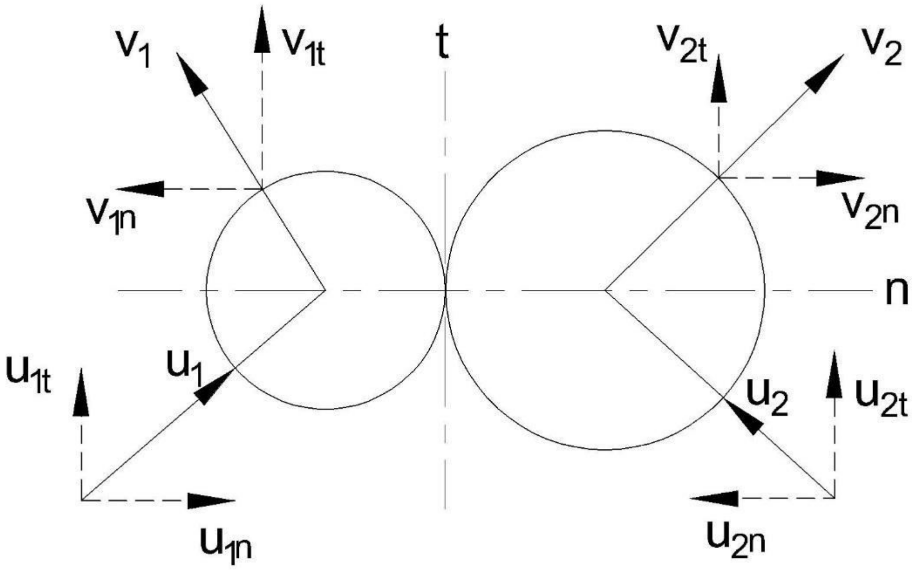
1. Perfectly Elastic Impact (e = 1)
   a. Momentum is conserved along the line of impact
   $$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$
   b. KE is conserved. No loss of KE.
   $$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$
2. Perfectly Plastic Impact (e = 0)
   a. After impact both the bodies collide and move together, and Momentum is conserved.
   $$m_1u_1 + m_2u_2 = (m_1 + m_2)v$$
   b. There is loss of KE during impact. Thus, KE is not conserved.
   $$KE_{loss} = \left(\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2\right) - \left(\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2\right)$$
3. Semi-elastic Impact (0 < e < 1)
   a. Momentum is conserved along the line of impact
   $$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$
   b. Coefficient of restitution value is to be used.
   $$v_2 - v_1 = e(u_1 - u_2)$$
   c. There is loss of KE during impact.
   $$KE_{loss} = \left(\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2\right) - \left(\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2\right)$$

Oblique Central Impact:



In oblique central impact, the impulsive force acts along the line of impact (common normal). Thus, the velocity changes occur only along the line of impact and no change in velocity takes place in a direction perpendicular to the line of impact (common tangent).

Here, not only the magnitudes of velocities after impact are unknown, but also the new directions of travel are unknown.



$$u_{1n} = u_1 \cos\alpha_1 \; ; \; u_{1t} = u_1 \sin\alpha_1$$

$$u_{2n} = u_2 \cos\alpha_2 \; ; \; u_{2t} = u_2 \sin\alpha_2$$

$$v_{1n} = v_1 \cos\beta_1 \; ; \; v_{1t} = v_1 \sin\beta_1$$

$$v_{2n} = v_2 \cos\beta_2 \; ; \; v_{2t} = v_2 \sin\beta_2$$

Equations to solve Oblique Central Impact problems:

1. The component of the total momentum of the two bodies along the line of impact is conserved.

$$m_1u_{1n} + m_2u_{2n} = m_1v_{1n} + m_2v_{2n}$$

2. Coefficient of restitution relation along the line of impact is,

$$e = \frac{v_{2n} - v_{1n}}{u_{1n} - u_{2n}}$$

3. Component of the momentum along the common tangent is conserved, which means the component of velocities along the tangent remains unchanged.

$$u_{1t} = v_{1t} \;\; \& \;\; u_{2t} = v_{2t}$$

4. Magnitudes of velocities after impact are given by,

$$v_1 = \sqrt{(v_{1n})^2 + (v_{1t})^2} \;\; \& \;\; v_2 = \sqrt{(v_{2n})^2 + (v_{2t})^2}$$

5. Directions of velocities after impact are given by,

$$\beta_1 = \tan^{-1}\frac{v_{1t}}{v_{1n}} \;\; \& \;\; \beta_2 = \tan^{-1}\frac{v_{2t}}{v_{2n}}$$

Problem 33

A cannon gun is nested by three springs each of 250 kN/cm stiffness as shown in Fig. 15.33(a). The gun fires a 500 kg shell with a muzzle velocity of 1000 m/s. Calculate the total recoil and the maximum force developed in each spring if the gun has a mass of 80,000 kg.

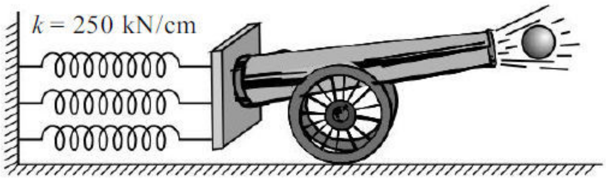


Fig. 15.33(a)

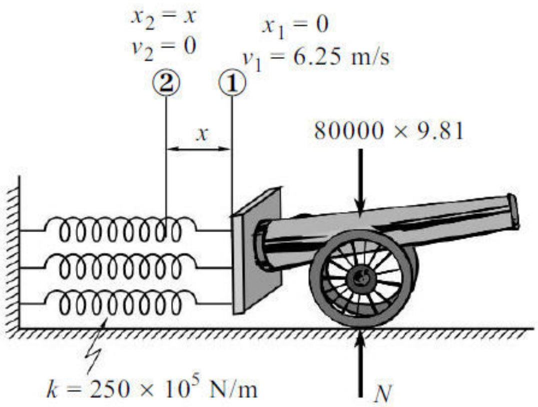


Fig. 15.33(b)

Solution

(i) By law of conservation of momentum, we have

Initial momentum = Final momentum

$$0 = 500 \times 1000 + 80000 \times v_{gun}$$

$$v_{gun} = -6.25 \text{ m/s}$$

$$\therefore \; v_{gun} = 6.25 \text{ m/s } (\leftarrow)$$

(iii) By work - energy principle, we have

Work done = Change in kinetic energy

$$3\left[\frac{1}{2} \times 250 \times 10^5 (0^2 - x^2)\right] = 0 - \frac{1}{2} \times 80000 \times 6.25^2$$

$$x = 0.204 \text{ m (maximum compression of spring)}$$

Spring force $F = kx$

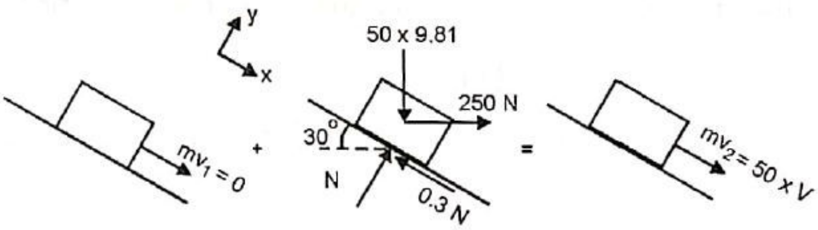
$$\therefore \; F = 250 \times 10^5 \times 0.204$$

$$\therefore \; F = 51.025 \times 10^5 \text{ N}$$

W

Ex. 12.3 A block of mass of 50 kg is placed on a plane inclined at 30° with the horizontal. A horizontal force of 250 N acts on the block tending to move the block down the plane. Determine its velocity 4 sec after starting from rest. Take $\mu_k = 0.3$.

Solution: We shall apply the Impulse Momentum Equation to the block for the first 4 sec of its motion.



Applying Impulse Momentum Equation in the y direction

$$mv_1 + \text{Impulse}_{1\text{-}2} = mv_2$$

Forces in the y direction are the weight component, the component of 250 N and the normal reaction.

$$\therefore \quad 0 + [-50 \times 9.81 \cos 30 + 250 \sin 30 + N] \times 4 = 0$$

or $\quad N = 300$ Newton

Applying Impulse Momentum Equation in the x direction

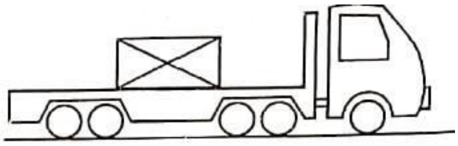
$$mv_1 + \text{Impulse}_{1\text{-}2} = mv_2$$

Forces in the x direction are the weight component, the component of 250 N and the frictional force.

$$\therefore \quad 0 + [50 \times 9.81 \sin 30 + 250 \cos 30 - 0.3 \times 300] \times 4 = 50 \text{ v}$$

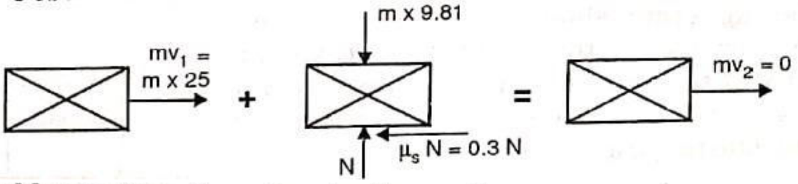
or $\quad v = 29.74$ m/s .... **Ans.**

Ex. 12.2 A truck traveling at a constant speed of 90 kmph on a straight highway carries a package on its flat bed trailor. $\mu_s = 0.3$ and $\mu_k = 0.2$ between the package and the flat bed. If the truck suddenly wants to come to a halt determine the minimum time in which it can do so without the package slipping on the flat bed.



Solution: As the truck driver applies the brakes, the package kept on it tends to slip forward. However the static frictional force prevents the package from slipping. Since the truck has to come to a halt in a minimum possible time implies that the static frictional force reaches its maximum value i.e. $\mu_s N$.

Let us analyse the kinetics of only the package. We shall draw three figures of the package. The L.H.S. and R.H.S. figures represent the initial and final momentum, while the central figure represents the FBD and is use to calculate the impulse, since Impulse = Force × time



Applying Impulse Momentum Equation in the x direction $\rightarrow$ + ve

$$mv_1 + \text{Impulse}_{1-2} = mv_2$$

$$m \times 25 + [-0.3 \times (m \times 9.81) \times t] = 0$$

or $t = 8.495$ sec .... **Ans.**

Force in the x direction is 0.3 N

Impulse = Force × time
$= (0.3\ N) \times t$
$= 0.3 \times (m \times 9.81) \times t$

Problem 36

A boy of 60 kg mass and a girl of 50 kg mass dive off the end of a boat of mass 160 kg with a horizontal velocity of 2 m/s relative to the boat as shown in Fig. 15.36. Considering the boat to be initially at rest, find its velocity just after **(i)** both the boy and girls dive off simultaneously, and **(ii)** the boy dives first followed by the girl.

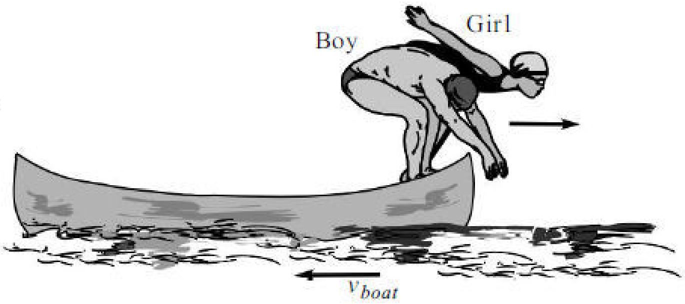


Fig. 15.36

Solution

(i) Both boy and girl dive off simultaneously

When boy and girl will jump together towards right the boat will move in opposite direction, i.e., towards left.

Here, velocity of boy and girl is 2 m/s relative to the boat.

$\therefore\ v_{boy/boat} = v_{boy} - v_{boat}$

$2 = v_{boy} - (-v_{boat})$

$v_{boy} = 2 - v_{boat}$

and $v_{girl/boat} = v_{girl} - v_{boat}$

$2 = v_{girl} - (-v_{boat})$

$v_{girl} = 2 - v_{boat}$

By conservation of momentum principle to the system of boy, girl and boat.

Initial momentum = Final momentum

$$0 = (\text{mass} \times \text{velocity})_{boy} + (\text{mass} \times \text{velocity})_{girl} + (\text{mass} \times \text{velocity})_{boat}$$

$$0 = 60(2 - v_{boat}) + 50(2 - v_{boat}) + 160(-v_{boat})$$

$$0 = 120 - 60v_{boat} + 100 - 50v_{boat} - 160v_{boat}$$

$$-220 = -270v_{boat}$$

$$v_{boat} = 1.227 \text{ m/s } (\leftarrow)$$

(ii) The boy dives first followed by the girl

Here, boy is jumping first and girl is still on boat.

By conservation of momentum principle,

Initial momentum = Final momentum

$$0 = (\text{Mass} \times \text{Velocity})_{boy} + (\text{Mass} \times \text{Velocity})_{boat}$$

$$0 = 60(2 - v_{boat}) + 160(-v_{boat})$$

$$0 = 120 - 60v_{boat} - 160v_{boat}$$

$$-220v_{boat} = -120$$

$$v_{boat} = 0.5455 \text{ m/s } (\leftarrow)$$

Later the girl jumps from the boat when the boat is moving back with a velocity of 0.5455 m/s.

By conservation of momentum principle.

Initial momentum = Final momentum

$(\text{mass} \times \text{velocity})_{boat} = (\text{mass} \times \text{velocity})_{girl} + (\text{mass} \times \text{velocity})_{boat}$

$160(-0.5455) = 50(2 - v_{boat}) + 160(-v_{boat})$

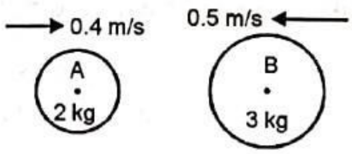
$-87.28 = 100 - 50v_{boat} - 160v_{boat}$

$-210v_{boat} = -187.28$

$v_{boat} = 0.8918 \text{ m/s} (\leftarrow)$

Ex. 12.8 A 2 kg ball moving with 0.4 m/s towards right, collides head on with another ball of mass 3 kg, moving with 0.5 m/s towards left. Determine the velocities of the balls after impact and the corresponding percentage loss of kinetic energy, when

i) the impact is perfectly elastic
ii) the impact is perfectly plastic
iii) the impact is such that e = 0.7



Solution: This is a case of Direct Central Impact.

i) Impact is perfectly elastic i.e. e = 1

Using Conservation of Momentum Equation $\rightarrow +ve$

$$m_A v_A + m_B v_B = m_A v_A' + m_B v_B'$$
$$2 \times 0.4 + 3 \times (-0.5) = 2 v_A' + 3 v_B'$$
$$-0.7 = 2 v_A' + 3 v_B' \qquad \text{............ (1)}$$

Using Coefficient of Restitution Equation $\rightarrow +ve$

$$v_B' - v_A' = e[v_A - v_B]$$
$$v_B' - v_A' = 1[0.4 - (-0.5)]$$
$$v_B' = 0.9 + v_A' \qquad \text{............ (2)}$$

Solving equations (1) and (2), we get

$v_A' = -0.68 \text{ m/s} = 0.68 \text{ m/s} \leftarrow$ .... **Ans.**

$v_B' = 0.22 \text{ m/s} = 0.22 \text{ m/s} \rightarrow$ .... **Ans.**

Since impact is perfectly elastic, implies that the energy is conserved i.e. there will be no loss of kinetic energy.

ii) Impact is perfectly plastic i.e. e = 0

In this case, the particles move together with a common velocity after impact,
i.e $v_A' = v_B' = v'$

Using Conservation of Momentum Equation

$$m_A v_A + m_B v_B = m_A v_A' + m_B v_B' \quad \rightarrow +ve$$
$$2 \times 0.4 + 3 \times (-0.5) = 2v' + 3v'$$
$$\therefore \quad v' = -0.14 \text{ m/s}$$

i.e. $v_A' = v_B' = 0.14 \text{ m/s} \leftarrow$ .... **Ans.**

Kinetic energy of the system before impact

$$= \frac{1}{2} m v_A^2 + \frac{1}{2} m v_B^2$$

$$= ½ \times 2 \times (0.4)^2 + ½ \times 3 \times (0.5)^2 = 0.535 \text{ J}$$

Kinetic energy of the system after impact

$$= ½ \times 2 \times (0.14)^2 + ½ \times 3 \times (0.14)^2 = 0.049 \text{ J}$$

$\therefore$ Percentage loss of kinetic energy $= \frac{0.535 - 0.049}{0.535} \times 100 = 90.84$ .... **Ans.**

iii) Impact when $e = 0.7$

$$m_A v_A + m_B v_B = m_A v_A' + m_B v_B' \quad \rightarrow +ve$$

$$2 \times 0.4 + 3 \times (-0.5) = 2 v_A' + 3 v_B'$$

$$-0.7 = 2 v_A' + 3 v_B' \quad \text{........... (1)}$$

Using Coefficient of Restitution Equation

$$v_B' - v_A' = e [v_A - v_B]$$

$$v_B' - v_A' = 0.7 [0.4 - (-0.5)]$$

$$v_B' = 0.63 + v_A' \quad \text{........... (2)}$$

Solving equations (1) and (2)

$v_A' = -0.518$ m/s $= 0.518$ m/s $\leftarrow$ .... **Ans.**

$v_B' = 0.112$ m/s $= 0.112$ m/s $\rightarrow$ .... **Ans.**

Kinetic energy of the system after impact

$$= ½ \times 2 \times (0.518)^2 + ½ \times 3 \times (0.112)^2 = 0.287 \text{ J}$$

$\therefore$ Percentage loss of kinetic energy $= \frac{0.535 - 0.287}{0.535} \times 100 = 46.33$ .... **Ans.**

Problem 4

Two balls having 20 kg and 30 kg masses are moving towards each other with velocities of 10 m/s and 5 m/s respectively as shown in Fig.15.4. If after impact the ball having 30 kg mass is moving with 6 m/s velocity to the right then determine the coefficient of restitution between the two balls.

Solution

(i) By law of conservation of momentum, we have

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

$$20 \times 10 + 30 \times (-5) = 20v_1 + 30 \times 6$$

$$v_1 = -6.5 \text{ m/s}$$

$$\therefore \; v_1 = 6.5 \text{ m/s } (\leftarrow)$$

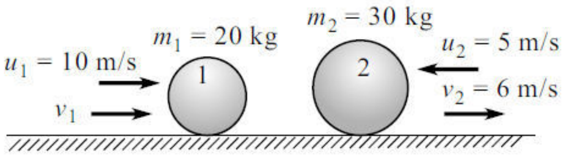


Fig. 15.4

(ii) For coefficient of restitution, we have the relation

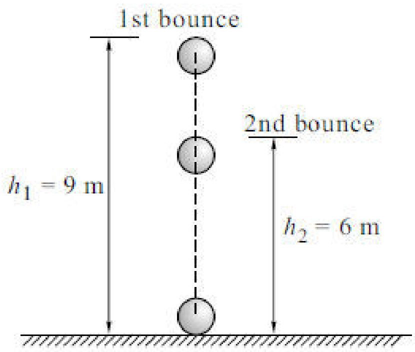
$$e = -\left[\frac{v_2 - v_1}{u_2 - u_1}\right]$$

$$e = -\left[\frac{6 - (-6.5)}{-5 - 10}\right]$$

$$e = 0.8333$$

Problem 7

A glass ball is dropped onto a smooth horizontal floor from which it bounces to a height of 9 m as shown in Fig.15.7(a). On the second bounce, it attains a height of 6 m. What is the coefficient of restitution between the glass and the floor? Also determine the height from where the glass ball was dropped.



$u_2 = v_2 = 0$ (velocity of horizontal floor)

Fig. 15.7(a)

Solution

(i) $u_1 = \sqrt{2gh_1}$ $(\downarrow)$ (velocity before impact)

$v_1 = \sqrt{2gh_2}$ $(\uparrow)$ (velocity after impact)

(ii) Coefficient of restitution gives

$$e = -\left[\frac{v_2 - v_1}{u_2 - u_1}\right] = -\left[\frac{0 - v_1}{0 - u_1}\right] = -\frac{v_1}{u_1}$$

$$e = -\frac{\sqrt{2gh_2}}{-\sqrt{2gh_1}} = \frac{\sqrt{h_2}}{\sqrt{h_1}} = \sqrt{\frac{6}{9}}$$

$$e = 0.816$$

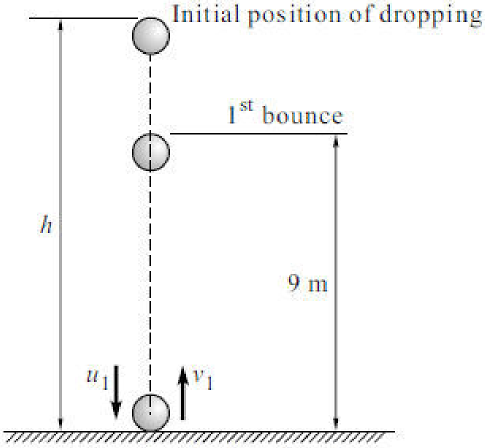
(iii) Coefficient of restitution gives

$$e = -\left[\frac{v_2 - v_1}{u_2 - u_1}\right]$$

$$0.816 = -\left[\frac{0 - \sqrt{2 \times 9.81 \times 9}}{0 - \left(-\sqrt{2 \times 9.81 \times h}\right)}\right]$$

$$0.816 = \sqrt{\frac{9}{h}}$$

$$\therefore \; h = 13.52 \text{ m}$$



$u_2 = v_2 = 0$ (velocity of surface)

Fig. 15.7(b)

Ex. 12.10 A ball drops from the ceiling of a room. After rebounding twice from the floor it reaches a height equal to half that of ceiling, find the coefficient of restitution.

(MU Dec 08)

Solution: Let e be the coefficient of restitution between the ball and the floor.

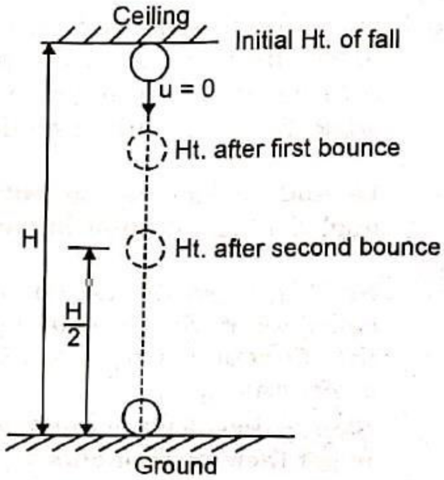
Using standard relation $e = \left(\frac{h'}{h}\right)^{\frac{1}{2n}}$

Where n = no. of bounces

h = Initial height of fall

h′ = height of rebound after nth bounce.

Let the ball fall from initial height of H and after 2 bounces it reaches a height of $\frac{H}{2}$ as given.



$$\therefore \quad e = \left(\frac{\frac{H}{2}}{H}\right)^{\frac{1}{2\times 2}} \quad \text{or} \quad e = \left(\frac{1}{2}\right)^{\frac{1}{4}} = \sqrt[4]{\frac{1}{2}} = 0.841 \ \dots \ \textbf{Ans.}$$

Problem 12

Two smooth spheres ① and ② having masses of 2 kg and 4 kg respectively collide with initial velocities as shown in Fig. 15.12(a). If the coefficient of restitution for the spheres is $e = 0.8$, determine the velocities of each sphere after collision.

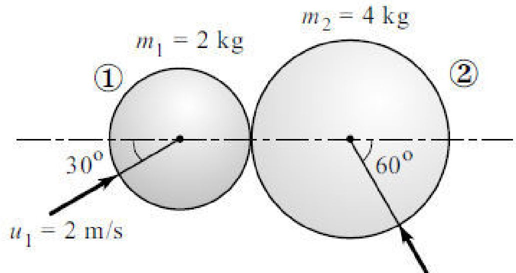


Fig. 15.12(a)

Solution

(i) By law of conservation of momentum along line of impact, we have

$$m_1 u_{1x} + m_2 u_{2x} = m_1 v_{1x} + m_2 v_{2x}$$

$$2 \times 2 \cos 30^\circ + 4 \times (-4 \cos 60^\circ) = 2(-v_{1x}) + 2v_{2x}$$

$$-v_{1x} + 2v_{2x} = -2.268 \qquad \dots \text{(I)}$$

Coefficient of restitution along the line of impact gives

$$e = -\left[\frac{v_{2x} - v_{1x}}{u_{2x} - u_{1x}}\right]$$

$$0.8 = -\left[\frac{v_{2x} - (-v_{1x})}{-4\cos 60^\circ - 2\cos 30^\circ}\right]$$

$$v_{2x} + v_{1x} = 2.986 \text{ m/s} \qquad \dots \text{(II)}$$

Solving Eqs. (I) and (II), we get

$$v_{1x} = 2.747 \text{ m/s } (\leftarrow) \quad \text{and} \quad v_{2x} = 0.239 \text{ m/s } (\rightarrow)$$

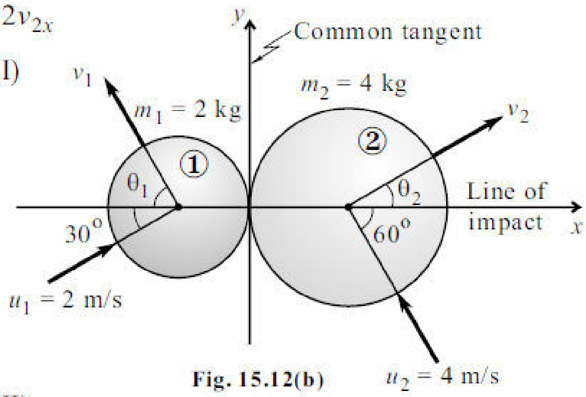


Fig. 15.12(b)

(ii) Component of velocity before and after impact along a common tangent is conserved.

$$v_{1y} = 2 \sin 30^\circ \qquad v_{2y} = 4 \sin 60^\circ$$

$$v_{1y} = 1 \text{ m/s } (\uparrow) \qquad v_{2y} = 3.464 \text{ m/s } (\uparrow)$$

For v_1, we have

$$\tan \theta_1 = \frac{v_{1y}}{v_{1x}} = \frac{1}{2.747}$$

$$\theta_1 = 20^\circ$$

$$v_1 = \sqrt{v_{1x}^2 + v_{1y}^2} = \sqrt{2.747^2 + 1^2}$$

$$v_1 = 2.923 \text{ m/s } (\theta_1)$$

Velocity of sphere ①

For v_2, we have

$$\tan \theta_2 = \frac{v_{2y}}{v_{2x}} = \frac{3.464}{0.239}$$

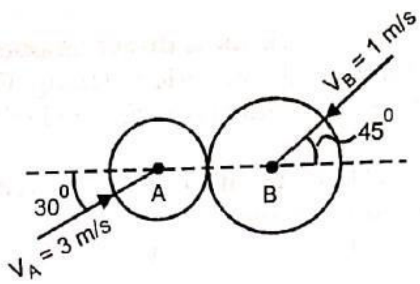
$$\theta_2 = 86.05^\circ$$

$$v_2 = \sqrt{v_{2x}^2 + v_{2y}^2} = \sqrt{0.239^2 + 3.464^2}$$

$$v_2 = 3.472 \text{ m/s } (\theta_2)$$

Velocity of sphere ②

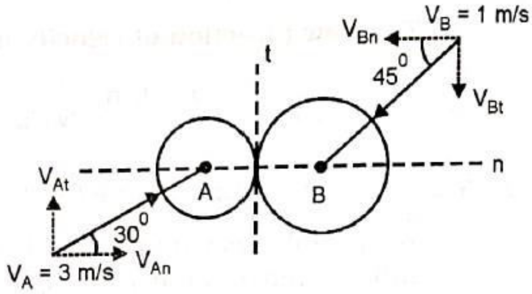
Ex. 12.12 Two smooth balls collide as shown. Find the velocities after impact.
Take $m_A = 1$ kg, $m_B = 2$ kg and $e = 0.75$



Solution: This is a case of Oblique Central Impact
Let the line of impact be the n direction and a perpendicular to it be the t direction.
Resolving the velocities along n and t direction.

$v_{An} = 2.6$ m/s $\rightarrow$, $\quad v_{At} = 1.5$ m/s $\uparrow$

$v_{Bn} = 0.707$ m/s $\leftarrow$, $\quad v_{Bt} = 0.707$ m/s $\downarrow$



Working in n direction

Using Conservation of Momentum Eqn. $\rightarrow$ + ve

$$m_A v_{An} + m_B v_{Bn} = m_A v_A{}'_n + m_B v_B{}'_n$$

$$1 \times 2.6 + 2 \times (-0.707) = 1 \times v_A{}'_n + 2 v_B{}'_n$$

$$1.186 = v_A{}'_n + 2 v_B{}'_n \quad \text{............ (1)}$$

Using Coefficient of Restitution Equation $\rightarrow$ + ve

$$v_B{}'_n - v_A{}'_n = e[v_{An} - v_{Bn}]$$

$$v_B{}'_n - v_A{}'_n = 0.75[2.6 - (-0.707)]$$

$$v_B{}'_n = v_A{}'_n + 2.48 \quad \text{............ (2)}$$

Solving equations (1) and (2), we get

$$v_{A'n} = -\ 1.26\ \text{m/s} = 1.26\ \text{m/s} \leftarrow$$
$$v_{B'n} = \ 1.22\ \text{m/s} = 1.22\ \text{m/s} \rightarrow$$

Working in t direction

Since velocities don't change in t direction

$$v_{A'_t} = v_{At} = 1.5\ \text{m/s} \uparrow$$
$$v_{B'_t} = v_{Bt} = 0.707\ \text{m/s} \downarrow$$

$\therefore$ Total velocity $v_A' = \sqrt{(v_{A'n})^2 + (v_{A't})^2} = \sqrt{(1.26)^2 + (1.5)^2} = 1.96\ \text{m/s}$

at angle $\alpha' = \tan^{-1}\left(\frac{v_{A't}}{v_{A'n}}\right) = \tan^{-1}\left(\frac{1.5}{1.26}\right) = 50^\circ$

$\therefore$ $v_A' = 1.96\ \text{m/s},\ \alpha' = 50^\circ$ .... **Ans.**

Similarly total velocity $v_B' = \sqrt{(v_{B'n})^2 + (v_{B't})^2} = \sqrt{(1.22)^2 + (0.707)^2} = 1.41\ \text{m/s}$

at angle $\beta' = \tan^{-1}\left(\frac{v_{B't}}{v_{B'n}}\right) = \tan^{-1}\left(\frac{0.707}{1.22}\right) = 30.1^\circ$

$\therefore$ $v_B' = 1.41\ \text{m/s},\ \beta' = 30.1^\circ$ .... **Ans.**