Fourier Series

- **Dirichlet's Conditions :**

Consider a single-valued function $f(x)$ in interval $(a, a+2l)$ which satisfies following conditions known as Dirichlet's Conditions.

1. $f(x)$ is defined in the interval $(a, a+2l)$ and $f(x) = f(x+2l)$
2. $f(x)$ is a continuous function **OR** has finite number of discontinuities in the interval $(a, a+2l)$
3. $f(x)$ has no maxima or minima **OR** has finite number of maxima or minima

If these 3 conditions are satisfied, $f(x)$ can be written as :

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cdot \cos\left(\frac{n\pi x}{l}\right) + b_n \cdot \sin\left(\frac{n\pi x}{l}\right)\right]}$$

Fourier series of $f(x)$ in interval $(a, a+2l)$

where $a_0 = \frac{1}{l}\int_a^{a+2l} f(x)\,dx$,

$$a_n = \frac{1}{l}\int_a^{a+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_a^{a+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Fourier Coefficients

→ Fourier series in interval $(0, 2\pi)$

Q. Obtain fourier series for $f(x) = \left(\frac{\pi - x}{2}\right)^2$, $0 \le x \le 2\pi$

and $f(x+2\pi) = f(x)$.

Deduce: (i) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$

(ii) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$

(iii) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$

(iv) $\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots$

Ans. $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{4\pi}\int_0^{2\pi} (\pi - x)^2\,dx$

$$= \frac{1}{4\pi}\left[\frac{(\pi-x)^3}{-3}\right]_0^{2\pi} = \frac{-1}{12\pi}\left(-\pi^3 - \pi^3\right)$$

$$\therefore a_0 = \frac{\pi^2}{6}$$

Remember!!!
when period $= (0, 2\pi)$ or $(-\pi, \pi)$
$\cos\left(\frac{n\pi x}{L}\right) \rightarrow \cos(nx)$
$\sin\left(\frac{n\pi x}{L}\right) \rightarrow \sin(nx)$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot\cos(nx)\,dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi} (\pi - x)^2 \cdot \cos(nx)\,dx$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2 \cdot \cancel{\left(\frac{\sin(nx)}{n}\right)} - 2(\pi - x)(-1)\cdot\left(\frac{-\cos(nx)}{n^2}\right) + 2\cdot\cancel{\left(\frac{-\sin(nx)}{n^3}\right)}\right]_0^{2\pi}$$

$$= \frac{-1}{2n^2\pi}\left[(\pi - x)\cdot\cos(nx)\right]_0^{2\pi} = \frac{-1}{2n^2\pi}(-\pi - \pi)$$

$$\therefore a_n = \frac{1}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi-x)^2}{4}\cdot \sin(nx)\,dx$$

$$\therefore b_n = \frac{1}{4\pi}\left[(\pi-x)^2\left(\frac{-\cos(nx)}{n}\right) - \cancel{(-2)(\pi-x)\left(\frac{-\sin(nx)}{n^2}\right)}^{\,0} + 2\left(\frac{\cos(nx)}{n^3}\right)\right]_0^{2\pi}$$

$$\therefore b_n = \frac{1}{4\pi n}\left[-(\pi-x)^2\cdot\cos(nx) + \frac{2\cos(nx)}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi n}\left[-\pi^2\cdot 1 + \frac{2}{n^2} - (-\pi^2)\cdot 1 - \frac{2}{n^2}\right]$$

$$= 0$$

$$\therefore \left(\frac{\pi-x}{2}\right)^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right]$$

$$= \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2}\cdot\cos(nx)$$

(i) Putting $x=0$,

$$\therefore \frac{\pi^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{12} + \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

$$\therefore \frac{\pi^2}{4} - \frac{\pi^2}{12} = \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \quad -①$$

Hence Proved.

(ii) Putting $x=\pi$,

$$0 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$\therefore \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots \quad -②$$ Hence Proved.

(iii) Adding ① and ② deductions,

$$\frac{\pi^2}{6} + \frac{\pi^2}{12} = \frac{1}{1^2} + \frac{1}{1^2} + \cancel{\frac{1}{2^2}} - \cancel{\frac{1}{2^2}} + \frac{1}{3^2} + \frac{1}{3^2} + \cancel{\frac{1}{4^2}} - \cancel{\frac{1}{4^2}} + \cdots$$

$$\frac{18\pi^2}{72} = \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \cdots \qquad \therefore \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Hence Proved.

(iii) Parseval's Identity :

$$\frac{1}{l}\int_{a}^{a+2l} f^2(x)\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

[Use whenever you need square of summation series]

(iv) $a_0 = \frac{\pi^2}{6}$, $a_n = \frac{1}{n^2}$, $b_n = 0$

∴ Using Parseval's Identity,

$$\frac{1}{16}\times\frac{1}{\pi}\int_{0}^{2\pi}(\pi-x)^4\,dx = \frac{\pi^4}{72} + \sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\therefore \frac{1}{16\pi}\left[\frac{(\pi-x)^5}{-5}\right]_{0}^{2\pi} = \frac{\pi^4}{72} + \sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\therefore \frac{\pi^4}{40} - \frac{\pi^4}{72} = \sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\therefore \frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \cdots$$

Hence Proved.

Q. If $f(x) = e^{-x}$ in $(0, 2\pi)$.
Find value of
(i) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1}$

(ii) $\operatorname{cosech}(\pi)$.

Ans. $$a_0 = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx$$

$$= \frac{-1}{\pi}\left[e^{-x}\right]_0^{2\pi} = \frac{1-e^{-2\pi}}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{n^2+1}\left(-\cos(nx) + n\cancel{\sin(nx)}^{\,0}\right)\right]_0^{2\pi}$$

$$= \frac{1}{(n^2+1)\pi}\left(e^{-2\pi}\cdot(-1) - 1(-1)\right) = \frac{1}{n^2+1}\cdot\left(\frac{1-e^{-2\pi}}{\pi}\right)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{n^2+1}\left(-\cancel{\sin(nx)}^{\,0} - n\cos(nx)\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi(n^2+1)}\left(e^{-2\pi}\cdot(-n) - 1(-n)\right)$$

$$b_n = \frac{n}{n^2+1}\left(\frac{1-e^{-2\pi}}{\pi}\right)$$

(continued) →

$$\therefore f(x) = \left(\frac{1-e^{-2\pi}}{2\pi}\right) + \frac{1-e^{-2\pi}}{\pi} \sum_{n=1}^{\infty} \left(\frac{\cos(nx)}{n^2+1} + \frac{n \cdot \sin(nx)}{n^2+1}\right)$$

(i) Putting $x = \pi$,

$$e^{-\pi} = \left(\frac{1-e^{-2\pi}}{2\pi}\right) + \left(\frac{1-e^{-2\pi}}{\pi}\right) \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$$

$$\therefore e^{-\pi} = \cancel{\left(\frac{1-e^{-2\pi}}{2\pi}\right)} + \cancel{\left(\frac{-1}{2}\frac{(1-e^{-2\pi})}{\pi}\right)} + \left(\frac{1-e^{-2\pi}}{\pi}\right) \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1}$$

$$\therefore \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1} = \frac{\pi \cdot e^{-\pi}}{1-e^{-2\pi}} = \frac{\pi}{e^{\pi}-e^{-\pi}}$$

(ii) $$\therefore \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1} = \frac{\pi}{2\left(\frac{e^{\pi}-e^{-\pi}}{2}\right)} = \frac{\pi}{2} \cdot \operatorname{cosech}(\pi)$$

$$\therefore \operatorname{cosech}(\pi) = \frac{2}{\pi} \sum_{n=2}^{\infty} \left(\frac{(-1)^n}{n^2+1}\right)$$

Q. $f(x) = \cos(px)$ in $(0, 2\pi)$ where p is not an integer.
Deduce that (i) $\pi \operatorname{cosec}(p\pi) = \frac{1}{p} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{1}{p+n} + \frac{1}{p-n}\right]$

(ii) $\pi \cot(2\pi p) = \frac{1}{2p} + p \sum_{n=1}^{\infty} \frac{1}{p^2-n^2}$

Ans. $a_0 = \frac{1}{\pi} \int_0^{2\pi} \cos(px)\, dx$

$= \frac{1}{\pi}\left[\frac{\sin(px)}{p}\right]_0^{2\pi} = \frac{1}{p\pi}\left[\sin(2p\pi) - \sin(0)\right]$

$a_0 = \frac{\sin(2p\pi)}{2p\pi}$ //

$a_n = \frac{1}{\pi} \int_0^{2\pi} \cos(px) \cdot \cos\left(\frac{n\pi x}{\pi}\right) dx$

$= \frac{1}{\pi} \int_0^{2\pi} \cos(px) \cdot \cos(nx)\, dx$

$\therefore a_n = \frac{1}{2\pi} \int_0^{2\pi} \left(\cos((p+n)x) + \cos((p-n)x)\right) dx$

$= \frac{1}{2\pi} \left[\frac{\sin((p+n)x)}{p+n} + \frac{\sin((p-n)x)}{p-n}\right]_0^{2\pi}$

$= \frac{1}{2\pi} \left(\frac{\sin((p+n)2\pi)}{p+n} + \frac{\sin((p-n)2\pi)}{p-n}\right)$

* [But, $\sin(p \pm n)2\pi = \sin(2p\pi) \cdot \cos(2n\pi) = \sin(2p\pi)$]

$\therefore a_n = \frac{\sin(2p\pi)}{2\pi} \left[\frac{1}{p+n} + \frac{1}{p-n}\right]$

$a_n = \frac{\sin(2p\pi)}{2\pi} \left(\frac{2p}{p^2-n^2}\right)$ //

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \cos(px)\cdot\sin(nx)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\sin((p+n)x) - \sin((p-n)x)\right)dx$$

$$= \frac{1}{2\pi}\left[\frac{\cos((p-n)x)}{p-n} - \frac{\cos((p+n)x)}{p+n}\right]$$

$$\boxed{\text{But } \cos(p\pm n)2\pi = \cos(2p\pi)}$$

$$\therefore b_n = \frac{1}{2\pi}\left[\frac{\cos(2p\pi)-1}{p-n} - \frac{\cos(2p\pi)-1}{p+n}\right]$$

$$b_n = \frac{\cos(2p\pi)-1}{2\pi}\left(\frac{2n}{p^2-n^2}\right) = \frac{n(\cos(2p\pi)-1)}{\pi(p^2-n^2)}$$

$$\therefore f(x) = \cos(px) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right]$$

$$= \frac{\sin(2p\pi)}{2p\pi} + \frac{p\sin(2p\pi)}{\pi}\sum_{n=1}^{\infty}\frac{\cos(nx)}{p^2-n^2} + \frac{(\cos(2p\pi)-1)}{\pi}\sum_{n=1}^{\infty}\frac{n\sin(nx)}{p^2-n^2}$$

(i) Putting $x=\pi$,

$$\cos(p\pi) = \frac{\sin(2p\pi)}{2p\pi} + \frac{p\sin(2p\pi)}{2\pi}\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{p+n} + \frac{1}{p-n}\right) + 0$$

$$\therefore \cancel{\cos(p\pi)} = \frac{\cancel{2}\sin(p\pi)\cancel{\cos(p\pi)}}{\cancel{2}p\pi} + \frac{p\,\cancel{2}\sin(p\pi)\cancel{\cos(p\pi)}}{\cancel{2}\pi}\sum_{n=1}^{\infty}(-1)^n\left[\frac{1}{p+n} + \frac{1}{p-n}\right]$$

$$\therefore \pi\operatorname{cosec}(p\pi) = \frac{1}{p} + \sum_{n=1}^{\infty}(-1)^n\left[\frac{1}{p+n} + \frac{1}{p-n}\right]$$

Hence Proved.

(ii) Putting $x = 2\pi$,

$$\cos(2p\pi) = \frac{\sin(2p\pi)}{2p\pi} + \frac{p\sin(2p\pi)}{\pi}\sum_{n=1}^{\infty}\frac{1}{p^2-n^2}$$

$$\therefore \pi\cot(2p\pi) = \frac{1}{2p} + p\sum_{n=1}^{\infty}\frac{1}{p^2-n^2}$$

Hence Proved.

Q. Find the Fourier series in the interval $(-\pi, \pi)$ where

$$f(x) = \begin{cases} 0, & -\pi \le x < 0 \\ \sin(x), & 0 \le x \le \pi \end{cases}$$

Deduce: (i) $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} \ldots = \frac{1}{2}$

(ii) $\frac{(\pi-2)}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7}$

Ans. $f(-x) = \begin{cases} 0 & 0 < x \le \pi \\ \sin(-x) & -\pi \le x \le 0 \end{cases}$

$\therefore f(x)$ is neither even nor odd.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin(x)\,dx = \frac{-1}{\pi}\left[\cos(x)\right]_0^{\pi} = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin(x)\cos(nx)\,dx = \frac{1}{2\pi}\int_0^{\pi}\left(\sin((n+1)x) - \sin((n-1)x)\right)dx$$

$$\therefore a_n = \frac{1}{2\pi}\left[\frac{\cos((n-1)x)}{n-1} - \frac{\cos((n+1)x)}{n+1}\right]_0^{\pi}$$

$\left[\text{But } \cos((n\pm 1)\pi) = \cos(n\pi)\cdot\cos(\pi) \mp \cancel{\sin(n\pi)}\cdot\cancel{\sin(\pi)}\right]$

$$a_n = \frac{1}{2\pi}\left[\frac{(-1)^{n-1}}{n-1} - \frac{(-1)^{n+1}}{n+1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{((-1)^{n+1} - 1)}{2\pi}\left[\frac{1}{n-1} - \frac{1}{n+1}\right]$$

$$a_n = \frac{((-1)^{n+1} - 1)}{\pi(n^2-1)}$$

(continued) →

$$\Rightarrow \cancel{} = \frac{1}{\pi(n^2-1)} \begin{cases} -2 & , \text{ if } n = 2k \\ 0 & , \text{ if } n = 2k+1 \end{cases} \quad (n \neq 1)$$

$$\therefore a_n = \begin{cases} \dfrac{-2}{\pi(n^2-1)} & , \text{ if } n = 2k \quad (n \neq 1) \\ 0 & , \text{ if } n = 2k+1 \end{cases}$$

(Since at $k=1, n=3$, so a_1 term $\nexists$)

$$a_1 = \frac{1}{\pi}\int_0^{\pi} \sin(x)\cdot\cos(x)\,dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} \sin(2x)\,dx = \frac{1}{2\pi}\left[\frac{-\cos(2x)}{2}\right]_0^{\pi}$$

$$a_1 = 0$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin(x)\sin(nx)\,dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} \left(\cos((n-1)x) - \cos((n+1)x)\right)dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin((n-1)x)}{n-1} - \frac{\sin((n+1)x)}{n+1}\right]_0^{\pi}$$

$$= 0 \quad \forall\ (n \neq 1)$$

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \sin(x)\cdot\sin(x)\,dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} \left[1 - \cos(2x)\right]dx$$

$$= \frac{1}{2\pi}\left[x - \frac{\sin(2x)}{2}\right]_0^{\pi} = \frac{1}{2\pi} \times \pi$$

$$b_1 = \frac{1}{2}$$

(continued) →

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right)$$

$$= \frac{1}{\pi} - \frac{2}{\pi}\sum_{k=1}^{\infty}\left(\frac{1}{4k^2-1}\cos(2kx)\right) + \frac{\sin(x)}{2}$$

(i) Putting $x=0$,

$$f(0) = 0 = \frac{1}{\pi} - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)(2k+1)} + 0$$

$$\therefore 1 = 2\sum_{k=1}^{\infty}\frac{1}{(2k-1)(2k+1)}$$

$$\therefore \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots = \frac{1}{2}$$

Hence Proved

(ii) Putting $x = \pi/2$,

$$f(\pi/2) = 1 = \frac{1}{\pi} - \frac{2}{\pi}\sum_{k=1}^{\infty}\left(\frac{\cos(k\pi)}{4k^2-1}\right) + \frac{1}{2}$$

$$\therefore \left(1 - \frac{1}{\pi} - \frac{1}{2}\right)\times\frac{\pi}{2} = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{4k^2-1}$$

$$\therefore \frac{\pi-2}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \cdots$$

Hence Proved

[ALWAYS CHECK EVEN/ODD IF $(-\pi,\pi)$!!!]

Q. $f(x) = |\cos x|$ in $(-\pi, \pi)$

hence, find Fourier series for $|\sin(x)|$.

Ans. $f(-x) = f(x)$

$\therefore f$ is even function $\Rightarrow b_n = 0$

[If $f(x) = \text{even}$, $b_n = 0$, $a_0, a_n =$ T.F.
$f(x) = \text{odd}$, $b_n =$ T.F., $a_0, a_n = 0$]

$$\therefore a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |\cos x|\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |\cos x|\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos(x)\,dx - \int_{\pi/2}^{\pi} \cos(x)\,dx\right]$$

$$= \frac{2}{\pi}\left(\left[\sin(x)\right]_0^{\pi/2} - \left[\sin(x)\right]_{\pi/2}^{\pi}\right)$$

$$\therefore a_0 = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} |\cos(x)| \cos(nx)\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos(x)\cos(nx)\,dx - \int_{\pi/2}^{\pi} \cos(x)\cos(nx)\,dx\right]$$

$$a_n = \frac{1}{\pi}\left(\int_0^{\pi/2} \left(\cos(n+1)x + \cos(n-1)x\right)dx - \int_{\pi/2}^{\pi} \left(\cos(n+1)x + \cos(n-1)x\right)dx\right)$$

$$= \frac{1}{\pi}\left(\left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right]_0^{\pi/2} - \left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right]_{\pi/2}^{\pi}\right) \quad (n \neq 1)$$

$$\left[\sin(n \pm 1)\tfrac{\pi}{2} = \pm\cos(n\pi/2)\right]$$

$$\therefore a_n = \frac{-4}{\pi(n^2-1)} \cdot \cos\left(\frac{n\pi}{2}\right), \quad (n \neq 1)$$

Finding, a_1,

$$a_1 = \frac{2}{\pi}\left(\int_0^{\pi/2} \cos^2(x)\,dx - \int_{\pi/2}^{\pi} \cos^2(x)\,dx\right)$$

$$= \frac{1}{\pi}\left(\int_0^{\pi/2} (\cos(2x)+1)\,dx - \int_{\pi/2}^{\pi} (\cos(2x)+1)\,dx\right)$$

$$= \frac{1}{\pi}\left(\left[\frac{\sin(2x)}{2}+x\right]_0^{\pi/2} - \left[\frac{\sin(2x)}{2}+x\right]_{\pi/2}^{\pi}\right)$$

$$a_1 = 0$$

But, $\cos(n\pi/2) = \begin{cases} (-1)^k, & \text{if } n=2k \\ 0, & \text{if } n=2k+1 \end{cases}$

$$\therefore a_n = \begin{cases} \dfrac{-4(-1)^k}{\pi(n^2-1)}, & \text{if } n=2k \\ 0, & \text{if } n=2k+1 \end{cases}$$

$$\therefore f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^k}{4k^2-1}\cos(2kx)$$

Replacing x by $x+\pi/2$,

$$\left|\cos(\pi/2+x)\right| = |-\sin x| = |\sin x|$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^k}{4k^2-1}\cos(2k(x+\pi/2))$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^k(-1)^k}{4k^2-1}\cdot\cos(2kx)$$

$$\therefore |\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\cos(2kx)}{4k^2-1}$$

Q. $f(x) = \begin{cases} x+\pi/2 , & -\pi < x < 0 \\ \pi/2 - x , & 0 < x < \pi \end{cases}$

Deduce : (i) $\pi^2/8 = 1/1^2 + 1/3^2 + 1/5^2 + \dots$

(ii) $\pi^4/96 = 1/1^4 + 1/3^4 + 1/5^4$

Ans. $f(-x) = \begin{cases} \pi/2 - x , & 0 < x < \pi \\ x+\pi/2 , & -\pi < x < 0 \end{cases}$

$= f(x)$

$\therefore f(x) =$ even , $\underline{\underline{b_n = 0}}$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$ (since even)

$\therefore \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{2}-x\right)dx = \frac{2}{\pi}\left[\frac{\pi x}{2} - \frac{x^2}{2}\right]_0^{\pi}$

$\underline{\underline{a_0 = 0}}$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{2}-x\right)\cos(nx)\,dx$

$a_n = \frac{2}{\pi}\left[\left(\frac{\pi}{2}-x\right)\left(\frac{\sin(nx)}{n}\right) - (-1)\left(\frac{-\cos(nx)}{n^2}\right)\right]_0^{\pi}$

$= \frac{-2}{\pi n^2}\left(\cos(n\pi) - \cos(0)\right)$

$a_n = \frac{-2}{\pi n^2}\left((-1)^n - 1\right)$

$\therefore a_n = \begin{cases} \frac{4}{\pi n^2} , & \text{if } n = 2k-1 \\ 0 , & \text{if } n = 2k \end{cases}$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + 0$$

$$= \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos((2k-1)x)}{(2k-1)^2}$$

Put $x = 0$

(i) $$f(0) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$$

Since $f(x)$ is discontinuous at 0,

$$f(0) = \frac{1}{2}\left(\lim_{x\to 0^-} f(x) + \lim_{x\to 0^+} f(x)\right)$$

$$= \frac{1}{2}\left[\lim_{x\to 0^-}\left(x+\frac{\pi}{2}\right) + \lim_{x\to 0^+}\left(\frac{\pi}{2}-x\right)\right]$$

$$= \frac{1}{2}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \frac{\pi}{2}$$

$$\frac{\pi}{2} = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$$

$$\therefore \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$ Hence Proved.

(ii) Using Parseval's Identity,

$$\frac{2}{\pi}\int_0^{\pi} f^2(x)\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 \quad \text{(for even)}$$

$$\therefore \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{2}-x\right)^2 dx = 0 + \sum_{k=1}^{\infty} \frac{16}{\pi^2(2k-1)^4}$$

↓ (solving integral)

$$\therefore \frac{\pi^2}{6} = \frac{16}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^4}$$

$$\therefore \frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots$$

Hence Proved.

Q. Interval $(0,2l)$, $f(x) = 2x - x^2$ $(0 \le x \le 3)$. [Period = 3]

Ans. $2l = 3$

$\therefore l = 3/2$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \frac{2}{3}\int_0^3 (2x - x^2)\,dx$$

$$= \frac{2}{3}\left[x^2 - \frac{x^3}{3}\right]_0^3 = \frac{2}{3}\left[9 - 9\right]$$

$$a_0 = 0$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{3}\int_0^3 (2x - x^2)\cos\left(\frac{2n\pi x}{3}\right)dx$$

$$\therefore a_n = \frac{2}{3}\left[(2x - x^2)\cdot\cancel{\left(\frac{\sin\left(\frac{2n\pi x}{3}\right)}{2n\pi/3}\right)} - (2-2x)\left(\frac{-\cos\left(\frac{2n\pi x}{3}\right)}{4n^2\pi^2/9}\right) + (-2)\left(\frac{-\cancel{\sin\left(\frac{2n\pi x}{3}\right)}}{8n^3\pi^3/27}\right)\right]_0^3$$

$$= \frac{3}{2n^2\pi^2}\left[(2-2x)\left(\cos\left(\frac{2n\pi x}{3}\right)\right)\right]_0^3$$

$$= \frac{3}{2n^2\pi^2}(-6)$$

$$\therefore a_n = \frac{-9}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{3}\int_0^3 (2x - x^2)\sin\left(\frac{2n\pi x}{3}\right)dx$$

$$b_n = \frac{2}{3}\left[(2x-x^2)\left(\frac{-\cos\left(\frac{2n\pi x}{3}\right)}{\frac{2n\pi}{3}}\right) - (2-2x)\left(\frac{-\sin\left(\frac{2n\pi x}{3}\right)}{4n^2\pi^2/9}\right)^{0} + (-2)\left(\frac{\cos\left(\frac{2n\pi x}{3}\right)}{8n^3\pi^3/27}\right)\right]_0^3$$

$$\therefore b_n = -\left[(2x-x^2)\left(\cos\left(\frac{2n\pi x}{3}\right)\right) + 2\left(\frac{\cos\left(\frac{2n\pi x}{3}\right)}{4n^2\pi^2/9}\right)\right]_0^3 \times \frac{1}{n\pi}$$

$$= \frac{-1}{n\pi}\left[(-3)(1) + (2)\cancel{\left(\frac{9}{4n^2\pi^2}\right)} - 0 - (2)\cancel{\left(\frac{9}{4n^2\pi^2}\right)}\right]$$

$$b_n = \frac{3}{n\pi}$$

$$\therefore f(x) = 2x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\therefore 2x - x^2 = \frac{-9}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{\cos\left(2n\pi x/3\right)}{n^2}\right) + \frac{3}{\pi}\sum_{n=1}^{\infty}\left(\frac{\sin\left(2n\pi x/3\right)}{n}\right)$$

Q. Find Fourier Series of

$$f(x) = \begin{cases} 0, & -2<x<-1 \\ 1+x, & -1<x<0 \\ 1-x, & 0<x<1 \\ 0, & 1<x<2 \end{cases}$$

Ans. $\therefore f(-x) = \begin{cases} 0, & -2<x<-1 \\ 1-x, & 0<x<1 \\ 1+x, & -1<x<0 \\ 0, & 1<x<2 \end{cases}$

$f(x) = f(-x)$ $\therefore$ $f(x)$ is even, $b_n = 0$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2}\int_{-2}^{2} f(x)\,dx$$

$$= \frac{2}{2}\int_{0}^{2} f(x)\,dx \text{ (since even)}$$

$$a_0 = \int_0^1 (1-x)\,dx = \left[x - \frac{x^2}{2}\right]_0^1 = 1 - \frac{1}{2}$$

$\therefore a_0 = 1/2$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$\therefore a_n = \int_0^1 (1-x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \left[\underbrace{\cancel{(1-x)\frac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}}}_{0 \text{ (at } x=1),\ 0 \text{ (at } x=0)} - \frac{(+1)\left(+\cos\left(\frac{n\pi x}{2}\right)\right)}{\left(\frac{n\pi}{2}\right)^2}\right]_0^1$$

$$\therefore a_n = \frac{-4}{n^2\pi^2}\left[\cos\left(\frac{n\pi x}{2}\right)\right]_0^1$$

$$a_n = \frac{-4}{n^2\pi^2}\left(\cos\left(\frac{n\pi}{2}\right) - 1\right)$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$= \frac{1}{4} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{\cos\left(\frac{n\pi x}{2}\right)}{n^2} \times \left(\cos\left(\frac{n\pi}{2}\right) - 1\right)\right]$$

→ Parseval's Identity as per functions

For Even:

$$\boxed{\frac{2}{L}\int_0^L f^2(x)\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2)}$$

For odd:

$$\boxed{\frac{2}{L}\int_0^L f^2(x)\,dx = \sum_{n=1}^{\infty}(b_n^2)}$$

• **Half-Range Series**

→ **Half-Range Sine Series:**
Half-Range ~~Sign~~ sine series for the function $f(x)$ in the interval $(0, L)$ is given by:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where, $b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$

→ **Half-Range Cosine Series:**
Half-Range Cosine Series for the function $f(x)$ in the interval $(0, L)$ is given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where, $a_0 = \frac{2}{L}\int_0^L f(x)\, dx$

$$a_n = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

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Q. Find half-range cosine series for $f(x) = x$ in $(0,2)$ and deduce that $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots = \frac{\pi^4}{96}$

Ans. $a_0 = \frac{2}{2} \int_0^2 x\,dx = \left[\frac{x^2}{2}\right]_0^2$

$\therefore a_0 = 2$

$a_n = \frac{2}{2} \int_0^2 x \cos\left(\frac{n\pi x}{2}\right) dx$

$= \left[\frac{x \sin(n\pi x/2)}{n\pi/2} + \frac{(1)(+\cos(\frac{n\pi x}{2}))}{n^2\pi^2/4} \right]_0^2$

$\therefore a_n = \frac{4}{n^2\pi^2}((-1)^n - 1)$

$\therefore a_n = \begin{cases} 0, & \text{if } n = 2k \\ \frac{-8}{n^2\pi^2}, & \text{if } n = 2k-1 \end{cases}$

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right)$

$= 1 - \frac{8}{\pi^2} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{(2k-1)\pi x}{2}\right)}{(2k-1)^2}$

By Parseval's Identity:

$\frac{2}{l} \int_0^l f^2(x)\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$

$\therefore \int_0^2 x^2\,dx = 2 + \frac{64}{\pi^4} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^4}$

$\therefore 2/3 = \frac{64}{\pi^4} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^4}$

$\therefore \frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots$ Hence Proved.

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Q. Find half-range cosine series of period 2π to represent $f(x) = \sin x$ in the interval $(0, \pi)$. Deduce that:

(i) $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots = \frac{1}{2}$

(ii) $\frac{1}{1^2\cdot 3^2} + \frac{1}{3^2\cdot 5^2} + \frac{1}{5^2\cdot 7^2} + \cdots = \frac{\pi^2 - 8}{16}$

(iii) $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \cdots$

Ans. $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} \sin x\,dx$

$= \frac{2}{\pi}\left[-\cos x\right]_0^{\pi}$ $\quad \therefore a_0 = \frac{4}{\pi}$

$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cdot \cos\left(\frac{n\pi x}{\pi}\right) dx$

$= \frac{-1}{\pi}\int_0^{\pi} \left(\sin(n+1)x + \sin(n-1)x\right) dx$

$= \frac{1}{\pi}\left[\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$

$\therefore a_n = \frac{-4}{\pi(n^2-1)}$, for $n = 2k$ $\quad (n \neq 1)$

Checking a_1 ?
(Check!)

$a_1 = 0$

$$= \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2kx)}{(4k^2-1)}$$

(i) $\therefore$ Putting $x=0$,

$$0 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)(2k+1)}$$

$$\therefore \frac{\pi^2}{8} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7}$$ // Hence Proved.

(ii) Using Parseval's Identity ? (complete!)

(Use according to which half-range series!)

(iii) Putting $x = \pi/2$,

$$1 = \frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{2}\left[\frac{1}{2k-1} - \frac{1}{2k+1}\right](-1)^{k+1}$$

$$\left(1 - \frac{2}{\pi}\right) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{2}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right)(-1)^{k+1}$$

$$\therefore \frac{\pi - 2}{4} = \frac{1}{2}\left[\left(1 - \frac{1}{3}\right) - \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) - \dots\right]$$

$$\therefore \frac{\pi}{4} = \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

$$\therefore \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Hence Proved.

Q. Prove that in the interval $(0,\pi)$,

$$\frac{e^{ax}-e^{-ax}}{e^{a\pi}-e^{-a\pi}} = \frac{2}{\pi}\left[\frac{\sin x}{a^2+1} - \frac{2\sin(2x)}{a^2+4} + \frac{3\sin(3x)}{a^2+9} - \dots\right]$$

Ans. We need to find half-range sine series for the function $e^{ax}-e^{-ax} = f(x)$ in the interval $(0,\pi)$

$$\therefore b_n = \frac{2}{\pi}\int_0^{\pi}(e^{ax}-e^{-ax})\sin(nx)\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi} e^{ax}\sin(nx)\,dx - \int_0^{\pi} e^{-ax}\sin(nx)\,dx\right]$$

$$\left[\text{We know, } \int e^{ax}\sin(nx) = \frac{e^{ax}}{a^2+n^2}\left(a\sin(nx) - n\cos(nx)\right)\right]$$

$$\therefore b_n = \frac{2}{\pi}\left[\left[\frac{e^{ax}}{a^2+n^2}(a\sin(nx) - n\cos(nx)) - \frac{e^{-ax}}{a^2+n^2}(-a\sin(nx) - n\cos(nx))\right]_0^{\pi}\right]$$

$$b_n = \frac{2}{(a^2+n^2)\pi}\left[e^{a\pi}(-n)\cos(n\pi) + e^{-a\pi}n\cos(n\pi) + \cancel{n} - \cancel{n}\right]$$

$$b_n = \frac{-2n(-1)^n}{\pi(a^2+n^2)}\left(e^{a\pi}-e^{-a\pi}\right) = \frac{2n(-1)^{n+1}}{\pi(a^2+n^2)}\left(e^{a\pi}-e^{-a\pi}\right)$$

//

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n\sin(nx) = \frac{2(e^{a\pi}-e^{-a\pi})}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot\sin(nx)\cdot n}{a^2+n^2}$$

$$\therefore \frac{e^{ax}-e^{-ax}}{e^{a\pi}-e^{-a\pi}} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot\sin(nx)\cdot n}{a^2+n^2}$$

$$\therefore \frac{e^{ax}-e^{-ax}}{e^{a\pi}-e^{-a\pi}} = \frac{2}{\pi}\left[\frac{\sin(x)}{a^2+1} - \frac{2\sin(2x)}{a^2+4} + \frac{3\sin(3x)}{a^2+9} - \dots\right]$$

//

Hence Proved //

- Complex form of Fourier Series :-

Complex form of Fourier Series ; for the function f(x) in the interval $(a, a+2l)$ is given by :

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}}}$$

where $C_n = \frac{1}{2}(a_n - ib_n)$

$$= \frac{1}{2l}\int_a^{a+2l} f(x) \cdot e^{-in\pi x/l}\, dx$$

Also, $C_0 = \frac{a_0}{2}$

$$C_{-n} = \frac{1}{2}(a_n + ib_n)$$

Q. Obtain complex form of Fourier series for ~~f(x)~~

$$f(x) = \begin{cases} a, & \text{if } 0 < x < l \\ -a, & \text{if } l < x < 2l \end{cases}$$

a is a positive constant. Hence deduce corresponding trigonometric Fourier series

Ans.

$$\therefore C_n = \frac{1}{2l} \int_0^{2l} f(x) \cdot e^{-in\pi x/l} \, dx$$

$$= \frac{1}{2l} \left[a \int_0^l e^{-in\pi x/l} dx - a \int_l^{2l} e^{-in\pi x/l} dx \right]$$

$$= \frac{a}{2l} \left[\left[\frac{e^{-in\pi x/l}}{-in\pi/l} \right]_0^l - \left[\frac{e^{-in\pi x/l}}{-in\pi/l} \right]_l^{2l} \right]$$

$$C_n = \frac{a}{2l} \left(\frac{-l}{in\pi} \right) \left[\left(e^{-in\pi} - 1 \right) - \left(e^{-i2n\pi} - e^{-in\pi} \right) \right]$$

[we know, $e^{i\theta} = \cos\theta + i\sin\theta$, $e^{-i\theta} = \cos\theta - i\sin\theta$]

$$\therefore C_n = \frac{ai}{2n\pi} \left[(-1)^n - 1 - 1 + (-1)^n \right] = \frac{ai}{n\pi} \left[(-1)^n - 1 \right] \quad (n \neq 0)$$

$$\therefore C_0 = \frac{a_0}{2} = \frac{1}{2l} \int_0^{2l} f(x) dx$$

$$= \frac{a}{2l} \left[\int_0^l dx - \int_l^{2l} dx \right] = 0$$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/l}$$

$$f(x) = \frac{ai}{\pi} \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \frac{\left[(-1)^n - 1 \right]}{n} \cdot e^{in\pi x/l}$$

(continued) →

we know, $C_n = \frac{1}{2}(a_n - ib_n)$

$$C_{-n} = \frac{1}{2}(a_n + ib_n)$$

$$\therefore C_n + C_{-n} = a_n$$

$$\therefore a_n = \frac{ai}{n\pi}\left[(-1)^n - 1\right] - \frac{ai}{n\pi}\left[(-1)^n - 1\right]$$

$$\therefore a_n = 0$$

Also, $C_n - C_{-n} = -ib_n$

$$\therefore b_n = i(C_n - C_{-n})$$

$$= i\left[\frac{ai}{n\pi}\left[(-1)^n - 1\right] + \frac{ai}{n\pi}\left[(-1)^n - 1\right]\right]$$

$$b_n = \frac{-2a}{n\pi}\left[(-1)^n - 1\right]$$

$$a_0 = 2C_0 = 0$$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$= \frac{-2a}{\pi}\sum_{n=1}^{\infty}\frac{\left[(-1)^n - 1\right]}{n}\cdot\sin(nx)$$

Q: Find complex form of Fourier Series for $f(x) = \sinh(2x) + \cosh(2x)$ in $(-5, 5)$.

Ans. $\therefore f(x) = \left(\frac{e^{2x} - e^{-2x}}{2}\right) + \left(\frac{e^{2x} + e^{-2x}}{2}\right)$

$= e^{2x}$

$$C_n = \frac{1}{2l} \int_{a}^{a+2l} f(x)\, e^{-in\pi x/l}\, dx$$

$$= \frac{1}{10} \int_{-5}^{5} e^{2x} \cdot e^{-in\pi x/5}\, dx = \frac{1}{10} \int_{-5}^{5} e^{x\left(2 - \frac{in\pi}{5}\right)} dx$$

$$= \frac{1}{10} \left[\frac{e^{\left(\frac{10 - in\pi}{5}\right)x}}{\frac{10 - in\pi}{5}} \right]_{-5}^{5}$$

$$\therefore C_n = \frac{1}{10} \times \frac{5}{10 - in\pi} \left[e^{(10 - in\pi)} - e^{-(10 - in\pi)} \right]$$

$$\therefore C_n = \frac{\sinh(10 - in\pi)}{10 - in\pi}$$

Multiplying and dividing with conjugate, $(10 + in\pi)$

$$C_n = \frac{(10 + in\pi) \times \sinh(10 - in\pi)}{100 + n^2\pi^2}$$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} C_n \cdot e^{in\pi x/l}$$

$$= \sum_{n=-\infty}^{\infty} \frac{(10 + in\pi)}{100 + n^2\pi^2} \times \sinh(10 - in\pi) \times e^{in\pi x/5}$$

Q. Find complex form of Fourier Series.

for $f(x) = \begin{cases} 0, & 0<x<l \\ a, & l<x<2l \end{cases}$

Ans. $c_n = \frac{1}{2l}\int_0^{2l} f(x) e^{-in\pi x/l} dx$

$= \frac{1}{2l}\left[\int_0^{l} 0\, dx + \int_l^{2l} a\cdot e^{-in\pi x/l} dx\right]$

$= \frac{1}{2l}\left[\frac{a\cdot e^{-in\pi x/l}}{-in\pi/l}\right]_l^{2l}$

$= \frac{-a}{2l} \times \frac{l}{in\pi} \times \left[e^{-i2n\pi} - e^{-in\pi}\right]$

$= \frac{ai}{2n\pi}\left[\cos(2n\pi) - i\sin(2n\pi) - \cos(n\pi) + i\sin(n\pi)\right]$

$c_n = \frac{ai}{2n\pi}\left[1-(-1)^n\right]$, ($n \neq 0$, since n in denominator)

$\therefore$ Finding C_0, $\therefore C_0 = a_0/2$

$\therefore C_0 = \frac{1}{2l}\int_0^{2l} f(x) dx$

$= \frac{1}{2l}\left[ax\right]_l^{2l} = \frac{a}{2}$ $\therefore C_0 = \frac{a}{2}$

$\therefore f(x) = \frac{a}{2} + \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \frac{ai}{2n\pi}\left[1-(-1)^n\right]\cdot e^{in\pi x/l}$

- **Fourier Integral**

If $f(x)$ satisfies Dirichlet's conditions in each finite interval $-L \le x \le L$ and if $f(x)$ is integrable in the interval $(-\infty, \infty)$ then Fourier Integral Theorem states that:

$$\boxed{f(x) = \frac{1}{\pi}\int_{w=0}^{\infty}\int_{s=-\infty}^{\infty} f(s)\cos(w(s-x))\,dw\,ds} \leftarrow \text{Fourier Integral}$$

→ **Fourier Sine and Cosine Integral**

Equation of Fourier Integral can be written as,

$$f(x) = \frac{1}{\pi}\int_{w=0}^{\infty}\int_{s=-\infty}^{\infty} f(s)\cos(ws)\cos(wx)\,dw\,ds$$

$$+ \frac{1}{\pi}\int_{w=0}^{\infty}\int_{s=-\infty}^{\infty} f(s)\sin(ws)\sin(wx)\,dw\,ds$$

- **Fourier Cosine Integral**

When $f(x)$ is an even function,

then $f(s)$ is even.

then $f(s)\sin(ws)$ is odd

★ and $f(s)\cdot\cos(ws)$ is even

$$\therefore \boxed{f(x) = \frac{2}{\pi}\int_{w=0}^{\infty}\cos(wx)\left[\int_{s=0}^{\infty} f(s)\cos(ws)\,ds\right]dw} \leftarrow \text{Fourier Cosine Integral (even)}$$

- **Fourier Sine Integral**

If $f(x)$ is odd, then $f(s)$ is odd

★ then $f(s)\cdot\sin(ws)$ is even, and $f(s)\cdot\cos(ws)$ is odd.

$$\therefore \boxed{f(x) = \frac{2}{\pi}\int_{w=0}^{\infty}\sin(wx)\left[\int_{s=0}^{\infty} f(s)\sin(ws)\,ds\right]dw} \leftarrow \text{Fourier Sine Integral (odd)}$$

Q. Express the following function $f(x) = \begin{cases} 1 & ; \text{ for } |x|<1 \\ 0 & , \text{ for } |x|>1 \end{cases}$

as Fourier Integral. Hence evaluate $\int_0^\infty \frac{\sin(w)\cos(wx)}{w}\, dw$

Ans. Clearly, $f(x)$ is even, $\therefore f(s)$ is even.

$$\therefore f(x) = \frac{2}{\pi}\int_{w=0}^{\infty} \cos(wx) \int_{s=0}^{\infty} f(s)\cos(ws)\, ds\, dw$$

$$= \frac{2}{\pi}\int_{w=0}^{\infty} \cos(wx) \left[\int_{s=0}^{1} (1) \cos(ws)\, ds\right] dw \qquad \left(\text{since } 1\le x\le\infty,\ f(x)=0\right)$$

$$= \frac{2}{\pi}\int_{w=0}^{\infty} \cos(wx) \left[\frac{\sin(ws)}{w}\right]_{s=0}^{1} dw$$

$$f(x) = \frac{2}{\pi}\int_{w=0}^{\infty} \frac{\cos(wx)\cdot\sin(w)}{w}\, dw$$

$$\therefore \frac{\pi}{2}\cdot f(x) = \int_0^\infty \frac{\sin(w)\cos(wx)}{w}\, dw$$

$$\therefore \int_0^\infty \frac{\sin(w)\cos(wx)}{w}\, dw = \begin{cases} \frac{\pi}{2} & , \text{ for } |x|<1 \\ 0 & , \text{ for } |x|>1 \\ \pi/4 & , \text{ for } |x|=1 \end{cases}$$

At $|x|=1$ (i.e. at $x=\pm1$)
$f(x)$ is discontinuous.

$$\therefore f(1) = \frac{1}{2}\left[\lim_{x\to1^-} f(x) + \lim_{x\to1^+} f(x)\right]$$

$$= \frac{1}{2}(1+0)$$

$$\therefore f(1) = 1/2 = f(-1)$$

Remember!

$$\boxed{\int_0^\infty \frac{\sin(w)}{w}\, dw = \frac{\pi}{2}}$$

H.W

Q. Find Fourier Integral Representation for ~~f(x)~~

$$f(x) = \begin{cases} 1-x^2, & \text{for } |x| \le 1 \\ 0, & \text{for } |x| > 1 \end{cases}$$

Q. Find Fourier Sine Integral representation for $f(x) = e^{-ax}/x$

Ans. $f(x) = \frac{2}{\pi} \int_{w=0}^{\infty} \sin(wx) \left[\int_{s=0}^{\infty} f(s) \sin(ws)\, ds \right] dw$ ← Fourier Sine Integral

Let $I(w) = \int_{s=0}^{\infty} \frac{e^{-as}}{s} \sin(ws)\, ds$ * [using DUIS concept ahead] (can even use Laplace)

$\therefore \frac{dI}{dw} = \frac{d}{dw} \int_{s=0}^{\infty} \frac{e^{-as}}{s} \cdot \sin(ws)\, ds$

$= \int_{s=0}^{\infty} \frac{\partial}{\partial w} \left(\frac{e^{-as}}{s} \cdot \sin(ws) \right) ds$

$= \int_{s=0}^{\infty} e^{-as} \cdot \cos(ws)\, ds$

$= \left[\frac{e^{-as}}{a^2+w^2} \cdot (-a\cos(ws) + w\sin(ws)) \right]_0^{\infty}$

$\frac{dI}{dw} = \frac{a}{a^2+w^2}$

$\therefore I(w) = \int \frac{a}{a^2+w^2}\, dw$

$I(w) = \frac{a}{a} \tan^{-1}\left(\frac{w}{a}\right) + c \quad = \tan^{-1}\left(\frac{w}{a}\right) + c$

At $w=0$, $I(w)=0$ $\quad \therefore 0 = \tan^{-1}(0) + c$

$\therefore c = 0$

$\therefore I(w) = \tan^{-1}\left(\frac{w}{a}\right)$

$\therefore f(x) = \frac{2}{\pi} \int_{w=0}^{\infty} \sin(wx) \tan^{-1}\left(\frac{w}{a}\right) dw$