ARIMA Modelling and Forecasting

Introduction

- Describe the stationarity of the AR process
- Determine the mean and variance of the AR process
- Assess the importance of Box-Jenkins methodology
- Describe the various types of forecasts
- Evaluate the measures of forecast importance

Stationarity of the AR process

- If an AR model is not stationary, this implies that previous values of the error term will have a non-declining effect on the current value of the dependent variable.
- This implies that the coefficients on the MA process would not converge to zero as the lag length increases.
- For an AR model to be stationary, the coefficients on the corresponding MA process decline with lag length, converging on 0.

AR Process

- The test for stationarity in an AR model (with p lags)is that the roots of the characteristic equation lie outside the unit circle (i.e. > 1), where the characteristic equation is:

$$1-\phi_1 z-\phi_2 z^2-\ldots-\phi_p z^p=0$$

Unit Root

- When testing for stationarity for any variable, we describe it as testing for a ‘unit root’, this is based on this same idea.
- The most basic AR model is the AR(1) model, on which most tests for stationarity are based, such as the Dickey-Fuller test (covered later)

Unit Root Test

$$y_t = y_{t-1} + u_t$$

$$y_t = Ly_t + u_t$$

$$(1-L)y_t = u_t$$

$$(1-z) = 0$$

(characteristic equation)

Unit Root Test

- With the AR(1) model, the characteristic equation of $(1-z)= 0$, suggests that it has a root of $z = 1$. This lies on the unit circle, rather than outside it, so we conclude that it is non-stationary.
- As we increase the lags in the AR model, so the potential number of roots increases, so for 2 lags, we have a quadratic equation producing 2 roots, for the model to be stationary, they both need to lie outside the unit circle.

AR Process

- The (unconditional mean) for an AR(1) process, with a constant (μ) is given by:

$$y_t = \mu + \phi_1 y_{t-1} + u_t \rightarrow AR(1)$$

$$E(y_t) = \frac{\mu}{1-\phi_1}$$

AR Process

- The (unconditional) variance for an AR process of order 1 (excluding the constant) is:

$$\mathrm{var}(y_t) = \frac{\sigma^2}{(1-\phi_1^2)}$$

$$|\phi_1| < 1$$

Box-Jenkins Methodology

- This is a method for estimating ARIMA models, based on the ACF and PACF as a means of determining the stationarity of the variable in question and the lag lengths of the ARIMA model.
- Although the ACF and PACF methods for determining the lag length in an ARIMA model are commonly used, there are other methods termed information criteria which can also be used (covered later)

Box-Jenkins

- The Box-Jenkins approach typically comprises four parts:

  - Identification of the model

  - Estimation, usually OLS

  -Diagnostic checking (mostly for autocorrelation)

  - Forecasting

ACF and PACF

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
. |*. |	. |*. |	1	0.093	0.093	0.7580	0.384
.*| . |	.*| . |	2	-0.073	-0.082	1.2307	0.540
. | . |	. | . |	3	-0.023	-0.008	1.2772	0.735
.*| . |	.*| . |	4	-0.106	-0.111	2.3010	0.681
. | . |	. | . |	5	-0.003	0.016	2.3017	0.806
. |*. |	. |*. |	6	0.140	0.124	4.1334	0.659
. |*. |	. |*. |	7	0.162	0.141	6.6260	0.469
.*| . |	.*| . |	8	-0.103	-0.128	7.6356	0.470
. | . |	. | . |	9	-0.047	-0.003	7.8513	0.549
.*| . |	.*| . |	10	-0.124	-0.117	9.3624	0.498
. | . |	. |*. |	11	0.028	0.088	9.4434	0.581
. |** |	. |*. |	12	0.217	0.166	14.231	0.286

Identification

- Identification of the most appropriate model is the most important part of the process, where it becomes as much ‘art’ as ‘science’.
- The first step is to determine if the variable is stationary, this can be done with the correlogram. If it is not stationary it needs to be first-differenced. (it may need to be differenced again to induce stationarity)
- The next stage is to determine the *p* and *q* in the ARIMA (*p*, *I*, *q*) model (the *I* refers to how many times the data needs to be differenced to produce a stationary series)

Identification

- To determine the appropriate lag structure in the AR part of the model, the PACF or Partial correlogram is used, where the number of non-zero points of the PACF determine where the AR lags need to be included.
- To determine the MA lag structure, the ACF or correlogram is used, again the non-zero points suggest where the lags should be included.
- Seasonal dummy variables may also need to be included if the data exhibits seasonal effects.

Diagnostic Checks

- With this approach we only test for autocorrelation usually, using the Q or Ljung-Box statistic.
- If there is evidence of autocorrelation, we need to go back to the identification stage and respecify the model, by adding more lags.
- A criticism of this approach is that it fails to identify if the model is too big or over-parameterised, it only tells us if it is too small.

ARIMA Example

- Following the Box-Jenkins methodology, the following ARIMA(2,1,1) model was produced:

$$\Delta\hat{y}_t = 0.7 + 0.6\Delta y_{t-1} + 0.3\Delta y_{t-2} + 0.1u_{t-1}$$

$$(0.1) \quad (0.2) \quad (0.1) \quad (0.01)$$

$$\overline{\mathrm{R}}^2 = 0.12, LB(2) = 3.21$$

Parsimonious Model

- The aim of this type of modelling is to produce a model that is parsimonious, or as small as possible, whilst passing the diagnostic checks.
- A parsimonious model is desirable because including irrelevant lags in the model increases the coefficient standard errors and therefore reduces the t-statistics.
- Models that incorporate large numbers of lags, tend not to forecast well, as they fit data specific features, explaining much of the noise or random features in the data.

Forecasting

- One of the most important tests of how well a model performs is how well it forecasts.
- One of the most useful models for forecasting is the ARIMA model.
- To produce dynamic forecasts the model needs to include lags of either the variables or error terms.

Types of Forecast

- Forecasts can be either in-sample or out-of-sample forecasts.
- In general the out-of sample forecasts are a better test of how well the model works, as the forecast uses data not included in the estimation of the model.
- To conduct out-of-sample forecasts, we need to leave some observations at the end of our sample for this purpose

Types of Forecasts

- A one-step-ahead is a forecast for the next observation only.
- A multi-step-ahead forecast is for 1,2,3,…s steps ahead.
- A recursive window for the forecast means that the initial estimation date is fixed but the additional observations are added one by one to the estimation time span.
- A rolling window is where the estimation time period is fixed but the start and end dates successively increase by 1.

Conditional Forecasting

- A conditional expectation is one that is taken for time t + 1, conditional upon or given all information available up to and including time t (this is important later). It is written as:

$$E(y_{t+1} \mid \Omega_t)$$

$$\Omega_t - \text{all information at time } t$$

Measuring Forecast Accuracy

- To determine how accurate a forecast is, the simplest method is to plot the forecast against the actual values as a direct comparison
- In addition it may be worthwhile to compare the turning points, this is particularly important in finance.
- There are a number of methods to determine accuracy of the forecast, often more than one is included in a set of results.

Tests of Forecast Accuracy

- Tests of forecast accuracy are based on the difference between the forecast of the variables value at time *t* and the actual value at time *t*. The closer the two are together and the smaller the forecast error, the better the forecast.
- There are a variety of statistics measuring this accuracy, mostly based on an average of the errors between the actual and forecast values at time *t*.
- However these statistics provide little information in isolation, as they are unbounded from above and their value depends on the units of the variables being forecast.

Mean Squared Error (MSE)

- The MSE statistic can be defined as:

$$MSE = \frac{1}{T-(T_1-1)} \sum_{t=T_1}^{T} (y_{t+s} - f_{t,s})^2$$

T – total sample size

T_1 – first out-of-sample forecast observation

$f_{t,s}$ – s-step-ahead forecasts made at time t

y_{t+s} – the actual value at time t

MSE Example

Steps Ahead	Forecast	Actual	Squared Error
1	0.1	0.15	0.0025
2	0.25	0.20	0.0025
3	0.5	0.40	0.01

$$MSE = (0.0025 + 0.0025 + 0.01)/3 = 0.005$$

Forecast Accuracy

- There are a number of other measures used and these include:
  - Mean Absolute Error
  - Mean Average Prediction Error
  - Chows test for predictive failure
  - Theil’s U-statistic (where the forecast is compared to that of a benchmark model)
  - Root Mean Square Error (the square root of the MSE)

Theil's U-statistic

$$u = \sqrt{\sum_{t=T_1}^{T} \frac{\left(\frac{y_{t+s} - f_{t,s}}{x_{t+s}}\right)^2}{\left(\frac{y_{t+s} - fb_{t,s}}{x_{t+s}}\right)}}$$

$fb_{t,s}$ – forecast from a benchmark model

$\mathrm{U} = 1 \rightarrow$ both models forecast with equal accuracy

$\mathrm{U} < 1 \rightarrow$ model forecast is best

$\mathrm{U} > 1 \rightarrow$ benchmark model forecasts best

Financial Loss Functions

- When using financial data, it is not always the accuracy of the forecast that matters, very often it is the accuracy of forecasts relating to the sign of future returns or turning points that is important, as this decides if a profit is made.
- It is possible to estimate statistics that give the % of correct sign predictions and correct direction change predictions
- These statistics work on the principle that if the value of the forecast and actual value are the same, when multiplied together they produce a positive sign. Similarly with the change in forecast and actual values giving the correct direction change.

Correct Sign Predictions

- The formula for the % of correct sign predictions is as follows:

$$\%\ \text{correct signs} = \frac{1}{\mathrm{T} - (\mathrm{T}_1 - 1)} \sum_{t=T_1}^{T} z_{t+s}$$

$$Z_{t+s} = 1 if\,(y_{t+s} * f_{t,s}) > 0$$

$$Z_{t+s} = 0 \text{ otherwise}$$

Conclusion

- When using AR models, whether the series is stationary of not determine how stable it is.
- Box-Jenkins methodology is part art and part science.
- Forecasting of time series is an important measure of how well a model works
- There are many measures of how accurate a forecast is, usually a number are calculated to determine if the forecast is acceptable, although all have faults.