Week 10 SWI

2. (1 point) Joint pmf of two random variables X and Y are given in Table

$x \backslash y$	1	2	3	$f_X(x)$
1	0.05	0	a_1	0.15
2	0.1	0.2	a_3	a_2
3	a_4	0.2	a_5	0.45
$f_Y(y)$	0.3	0.4	a_6	

Find the value of $f_{Y|X=2}(2)$ i.e $(P(Y = 3|X = 3))$

Solution:

$$\sum f_{XY}(x,y) = 1 \text{ .... } (i)$$

$$f_X(x) = \sum_{y \in R_y} f_{XY}(x,y) \text{ ....}(ii)$$

$$f_Y(y) = \sum_{x \in R_X} f_{XY}(x,y) \text{ ....}(iii)$$

Hence, $a_1 = 0.10$, $a_2 = 0.40$, $a_3 = 0.1$, $a_4 = 0.15$, $a_5 = 0.1$, $a_6 = 0.3$

$$f_{Y|X=3}(1) = \frac{f_{XY}(3,3)}{f_X(3)} = \frac{0.1}{0.45} = 0.22$$

3. (1 point) **(Multiple Select)** Which of the following options is/are correct?

   A. If $Cov[X, Y] = 0$, then X and Y are independent random variables.

   B. $Cov[X, X] = Var(X)$

   C. If X and Y are two independent random variables and $Z = X + Y$ then $f_Z(z) = \sum_x f_X(x) \times f_Y(z - x)$

   D. If X and Y are two independent random variables and $Z = X + Y$ then $f_Z(z) = \sum_y f_X(x) \times f_Y(z - x)$

Solution:

Option B

$Cov[X, X]$ is the covariance between X and X i.e $Var(X)$

Option C is correct from its definition.

5. Let X be a discrete random variable taking even integer values from 0 to 10. The cumulative distribution function of X is as follows:

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 0.250 & \text{if } 0 \leq x < 2 \\ 0.898 & \text{if } 2 \leq x < 4 \\ 0.955 & \text{if } 4 \leq x < 6 \\ 0.981 & \text{if } 6 \leq x < 8 \\ 0.995 & \text{if } 8 \leq x < 10 \\ 1.000 & \text{if } x \geq 10 \end{cases}$$

Calculate $P(X \leq 4 \mid X > 0)$.

1. 0.955
2. 0.705
3. 0.976
4. 0.94

Solution:

$$P(X \leq 4 \mid X > 0) = \frac{P((X \leq 4) \cap (X > 0))}{P(X > 0)}$$

Since X is taking discrete even integer values from 0 to 10,

$P(X = 0) = 0.250$
$P(X = 2) = 0.898 - 0.250 = 0.648$
$P(X = 4) = 0.955 - 0.898 = 0.057$
$P(X = 6) = 0.981 - 0.955 = 0.026$
$P(X = 8) = 0.995 - 0.981 = 0.014$
$P(X = 10) = 1 - 0.995 = 0.005$

$$\Rightarrow P(X \leq 4 \mid X > 0) = \frac{P(X = 2) + P(X = 4)}{P(X > 0)} = \frac{0.648 + 0.057}{1 - P(X \leq 0)}$$

$$\Rightarrow P(X \leq 4 \mid X > 0) = \frac{0.705}{1 - 0.250} = 0.94$$

Therefore, the correct option is (d).

6. Which of the following statement(s) is/are always true? [3 marks]

   1. $E(X^2) = [E(X)]^2$
   2. $E(-X) = E(X)$
   3. $E[(2X+1)^2] = 4E(X^2) + 4E(X) + 1$
   4. $E[(2X+1)^2] = 4[E(X)]^2 + 4E(X) + 1$

Solution:
Let the random variable X takes the values 1 and -1 with probabilities $\frac{1}{2}$ and $\frac{1}{2}$ respectively.

(a) $E(X^2) = (1^2 \times \frac{1}{2}) + ((-1)^2 \times \frac{1}{2}) = 1$
$E(X) = (1 \times \frac{1}{2}) + (-1 \times \frac{1}{2}) = 0$
$[E(X)]^2 = 0$
Therefore, option (a) is incorrect.

(b) $E(-X) = (-1 \times \frac{1}{2}) + (1 \times \frac{1}{2}) = 0$
$E(X) = (1 \times \frac{1}{2}) + (-1 \times \frac{1}{2}) = 0$
Now let the random variable X is taking the values 1, 2 with probabilities $\frac{1}{2}$ and $\frac{1}{2}$ respectively.
$E(-X) = (-1 \times \frac{1}{2}) + (-2 \times \frac{1}{2}) = \frac{-3}{2}$
$E(X) = (1 \times \frac{1}{2}) + (2 \times \frac{1}{2}) = \frac{3}{2}$
Therefore, option (b) is incorrect.

(c) $E[(2X+1)^2] = E(4X^2+1+4X)$

Using the properties of expectation, $E[(2X+1)^2] = 4E(X^2) + 4E(X) + 1$

Therefore, option (c) is correct and option (d) is incorrect.

7. (1 point) Let X and Y be two random variables with joint PMF $f_{XY}(x, y)$ given in Table 10.3.

$x \backslash y$	0	1	2
0	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$
1	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$

Table 10.3: Joint PMF of X and Y.

Calculate $P(X = 0, Y \leq 1)$

Solution:

$$P(X = 0, Y \leq 1) = P(X = 0, Y = 0) + P(X = 0, Y = 1)$$

$$\frac{1}{6} + \frac{1}{4}$$
$$\frac{5}{12}$$

8. (1 point) A discrete random variables X has the cumulative distribution function is defined as follows.

$$F_X(x) = \begin{cases} \frac{x^2 + k}{40}, & \text{for } x = 1, 2, 3 \end{cases}$$

The value of k equals.

Solution:

For k

$F_X(3) = 1$

$$\frac{x^3 + k}{40} = 1$$

Solving above equation to get $k = 13$

9. (1 point) Two random variables X and Y are jointly distributed with joint pmf

$$f_{XY}(x,y) = \begin{cases} ax + \frac{y}{4}, & \text{for } x, y \in \{0,1\} \\ 0, & \text{otherwise} \end{cases}$$

Calculate the value of a

Answer: 0.25

Solution:

$\sum f_{XY}(x, y) = 1$

While solving above equation we get $a = 0.25$

10. (1 point) A discrete random variables X has the probability function as given in table 10.4.

x	1	2	3	4	5	6
$P(X)$	a	a	a	b	b	0.3

Table 2: Table 10.4: Probability distribution

If $E(X) = 4.2$, then evaluate $a + 3b$

Answer: 0.3

Answer: 0.7

$\sum P(X=x)=1$

$3a+2b=0.7$

$E(X)=\sum P(X=x_i)\times x_i$

$6a+9b=2.4$

Solving both equations, we get $a=0.1$ and $b=0.2$

then, $a+3b=.1+.2\times 3=0.7$