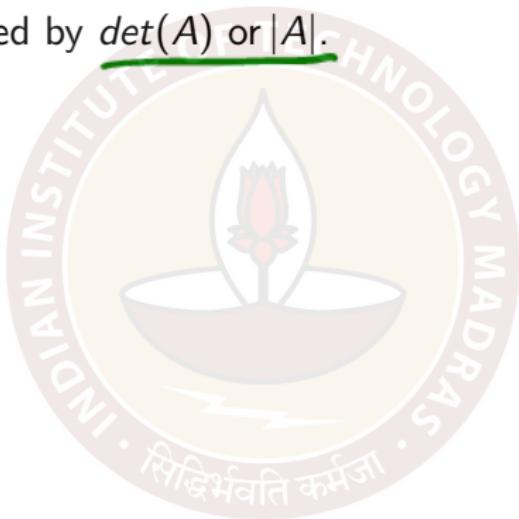


# Determinants (Part 1)

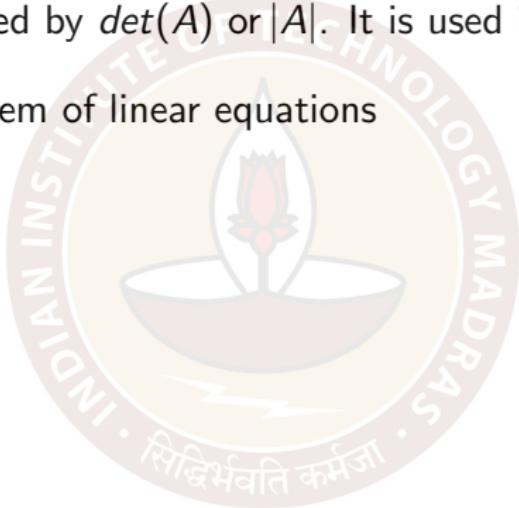
Sarang S. Sane

Every square matrix  $A$  has an associated number, called its determinant and denoted by  $\det(A)$  or  $|A|$ .



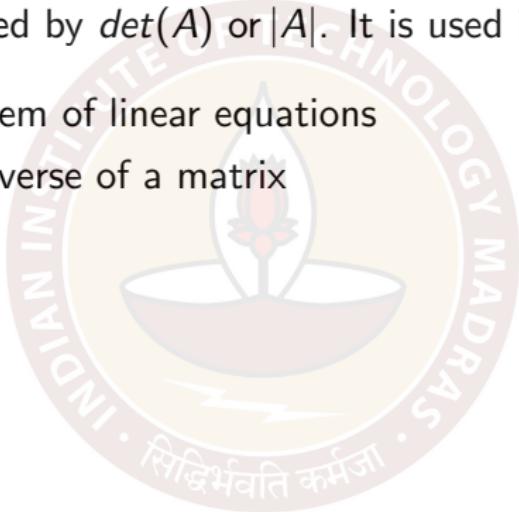
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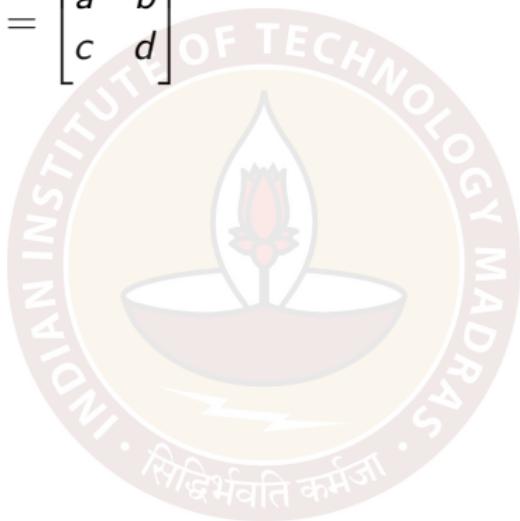
- ▶ solving a system of linear equations
- ▶ finding the inverse of a matrix
- ▶ calculus and more.

### Determinant of a $1 \times 1$ matrix :

If  $A = [a]$ , a  $1 \times 1$  matrix then  $\det(A) = a$

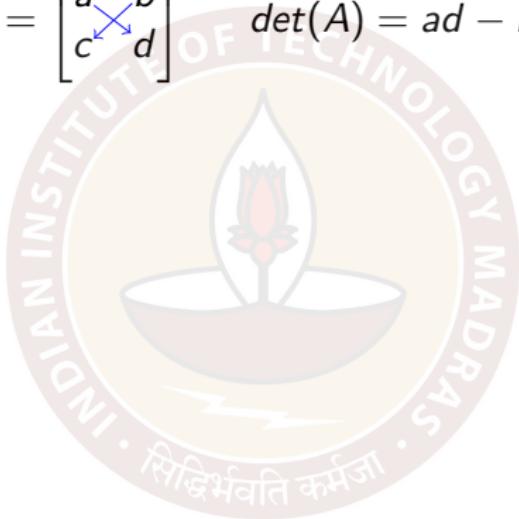
# Determinant of a $2 \times 2$ matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$



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Example

$$A = \begin{bmatrix} 2 & 3 \\ 6 & 10 \end{bmatrix}$$

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$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \det(A) = ad - bc$$

Example

$$A = \begin{bmatrix} 2 & 3 \\ 6 & 10 \end{bmatrix} \quad \det(A) = 20 - 18 = 2$$

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$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{det}(A) = ad - bc$$

Example

$$A = \begin{bmatrix} 2 & 3 \\ 6 & 10 \end{bmatrix} \quad \text{det}(A) = 20 - 18 = 2$$

Example

$$A = \begin{bmatrix} 5 & 2/3 \\ 6 & 3/7 \end{bmatrix}$$

# Determinant of a $2 \times 2$ matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{det}(A) = ad - bc$$

Example

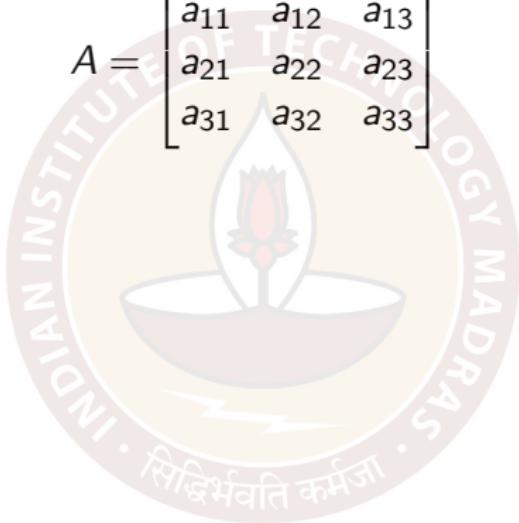
$$A = \begin{bmatrix} 2 & 3 \\ 6 & 10 \end{bmatrix} \quad \text{det}(A) = 20 - 18 = 2$$

Example

$$A = \begin{bmatrix} 5 & 2/3 \\ 6 & 3/7 \end{bmatrix} \quad \text{det}(A) = 15/7 - 4 = -13/7$$

# Determinant of a $3 \times 3$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$



# Determinant of a $3 \times 3$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

We will obtain the determinant by expanding with respect to the 1st row :

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

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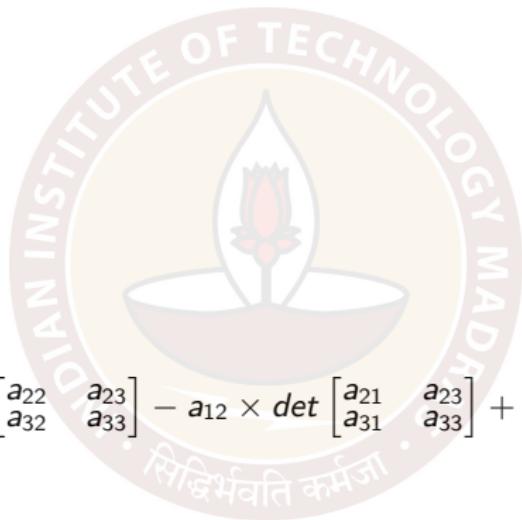
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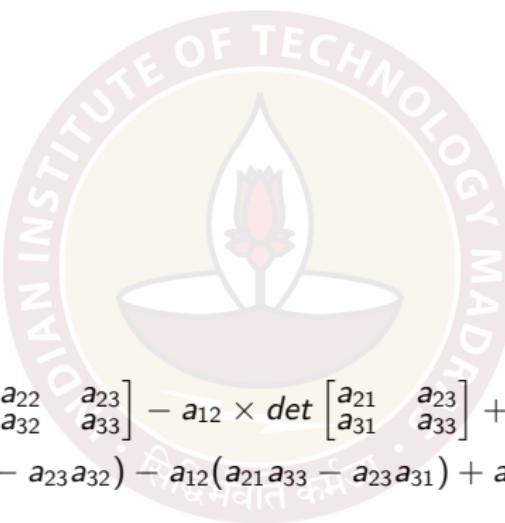
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

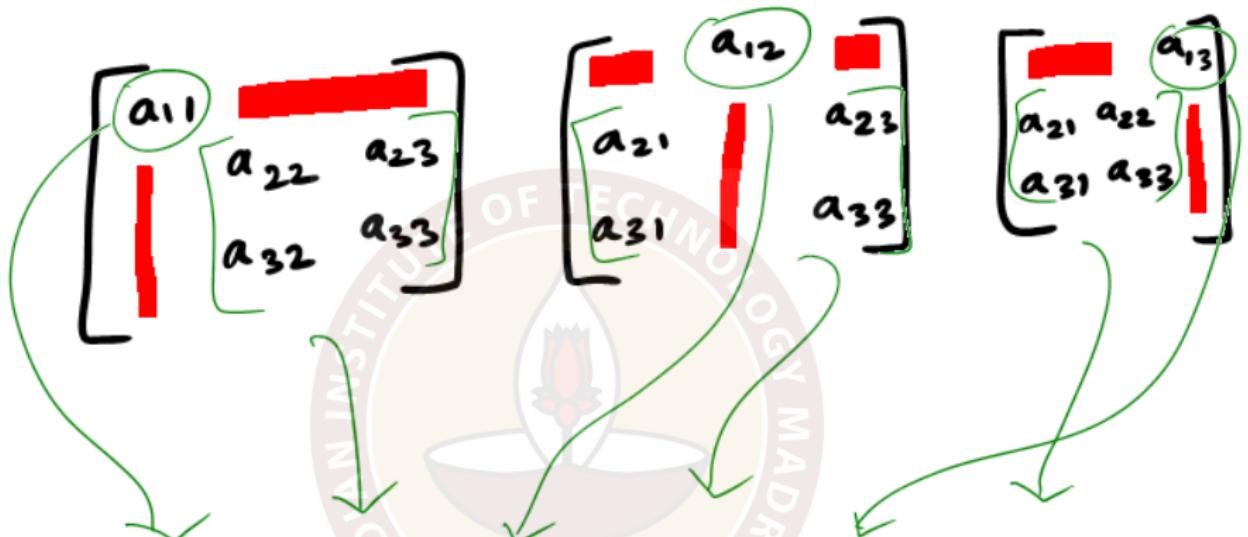
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det(A) = a_{11} \times \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \times \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \times \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$




$$\begin{aligned} \det(A) &= a_{11} \times \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \times \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \times \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}$$



$$\begin{aligned}
 \det(A) &= a_{11} \times \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \times \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \times \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \\
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 &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}
 \end{aligned}$$

# Examples

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 8 & 7 \\ 5 & 6 & 9 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 8 & 7 \\ 5 & 6 & 0 \end{bmatrix}$$
$$\det(A) = 2 \times \det \begin{bmatrix} 8 & 7 \\ 6 & 9 \end{bmatrix} - 4 \times \det \begin{bmatrix} 3 & 7 \\ 5 & 9 \end{bmatrix} + 1 \times \det \begin{bmatrix} 3 & 8 \\ 5 & 6 \end{bmatrix}$$

## Examples

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 8 & 7 \\ 5 & 6 & 9 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= 2 \times \det \begin{bmatrix} 8 & 7 \\ 6 & 9 \end{bmatrix} - 4 \times \det \begin{bmatrix} 3 & 7 \\ 5 & 9 \end{bmatrix} + 1 \times \det \begin{bmatrix} 3 & 8 \\ 5 & 6 \end{bmatrix} \\ &= 2(72 - 42) - 4(27 - 35) + 1(18 - 40) \end{aligned}$$

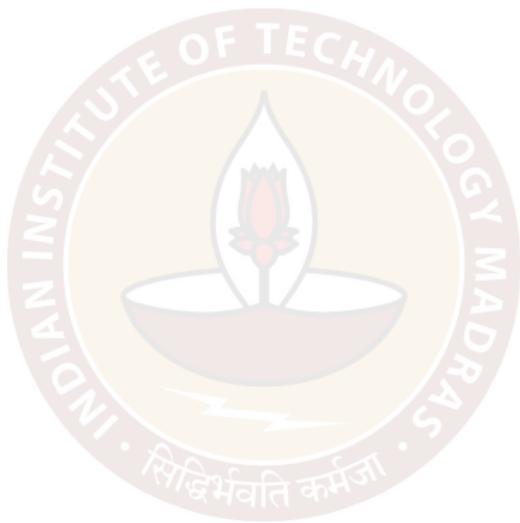
## Examples

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 8 & 7 \\ 5 & 6 & 9 \end{bmatrix}$$

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## Determinant of the Identity matrix

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \det(I_2) = 1 - 0 = 1$$

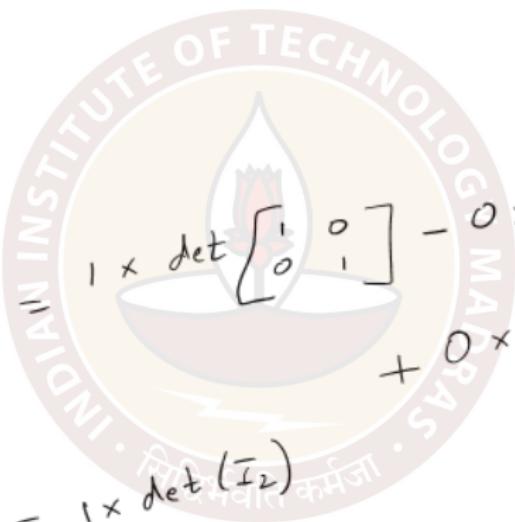


$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\det(I_2) = 1 - 0 = 1$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

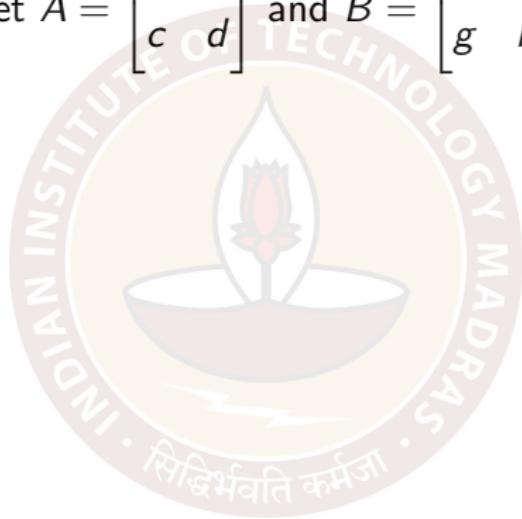
$$\det(I_3)$$



$$\begin{aligned}
 & 1 \times \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 0 \times \det \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 & + 0 \times \det \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\
 & = 1 \times \det(I_2) \\
 & = 1 \times 1 = 1.
 \end{aligned}$$

# Determinant of a product of matrices

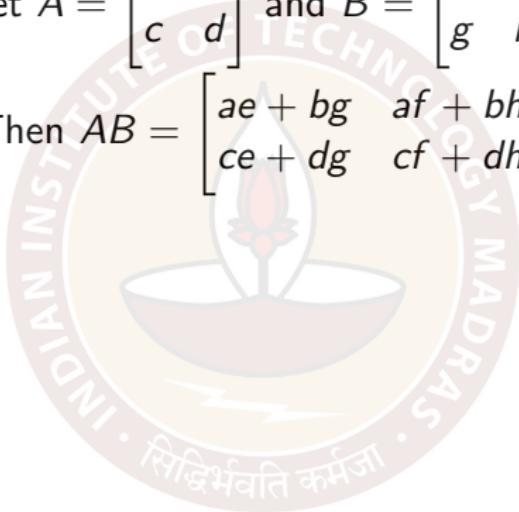
Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  and  $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$



# Determinant of a product of matrices

$$\text{Let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$\text{Then } AB = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$



$$\text{Let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$\text{Then } AB = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

$$\det(AB) = (ae + bg)(cf + dh) - (af + bh)(ce + dg)$$

# Determinant of a product of matrices

Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  and  $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

Then  $AB = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$

$$\begin{aligned} \det(AB) &= (ae + bg)(cf + dh) - (af + bh)(ce + dg) \\ &= aecf + bgcf + aedh + bgdh - afce - bhce - afdg - bhdg \end{aligned}$$

$$\text{Let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$\text{Then } AB = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

$$\begin{aligned} \det(AB) &= (ae + bg)(cf + dh) - (af + bh)(ce + dg) \\ &= ae\cancel{f} + bg\cancel{cf} + aedh + bg\cancel{dh} - \cancel{af}e - bh\cancel{ce} - afdg - \cancel{bh}\cancel{dg} \\ &= bgcf + aedh - bhce - afdg \end{aligned}$$

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Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  and  $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

Then  $AB = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$

$$\begin{aligned} \det(AB) &= (ae + bg)(cf + dh) - (af + bh)(ce + dg) \\ &= aecf + bgcf + aedh + bgdh - afce - bhce - afdg - bhdg \\ &= bgcf + aedh - bhce - afdg \\ &= \textcolor{blue}{bcfg} + \textcolor{red}{adeh} - \textcolor{blue}{bceh} - \textcolor{red}{adfg} \end{aligned}$$

# Determinant of a product of matrices

Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  and  $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

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$$\begin{aligned} \det(AB) &= (ae + bg)(cf + dh) - (af + bh)(ce + dg) \\ &= aecf + bgcf + aedh + bgdh - afce - bhce - afdg - bhdg \\ &= bgcf + aedh - bhce - afdg \\ &= \textcolor{blue}{bcfg} + \textcolor{blue}{adeh} - \textcolor{blue}{bceh} - \textcolor{blue}{adfg} \\ &= \textcolor{red}{(ad - bc)(eh - fg)} \end{aligned}$$

# Determinant of a product of matrices

Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  and  $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

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$$\begin{aligned} \det(AB) &= (ae + bg)(cf + dh) - (af + bh)(ce + dg) \\ &= aecf + bgcf + aedh + bgdh - afce - bhce - afdg - bhdg \\ &= bgcf + aedh - bhce - afdg \\ &= \textcolor{blue}{bcfg} + \textcolor{red}{adeh} - \textcolor{blue}{bceh} - \textcolor{red}{adfg} \\ &= \textcolor{red}{(ad - bc)(eh - fg)} \\ &= \det(A)\det(B). \end{aligned}$$

# Determinant of a product of matrices

$$\text{Let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

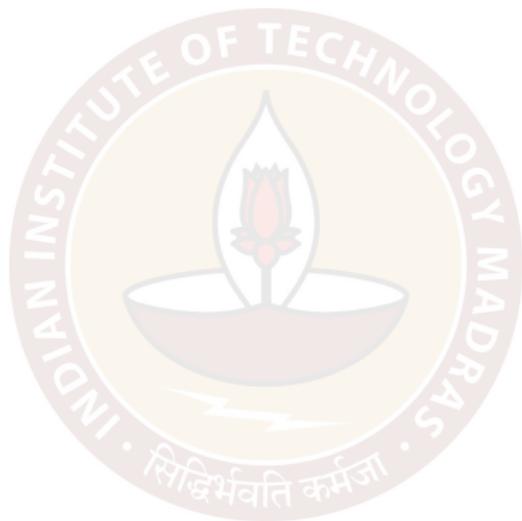
$$\text{Then } AB = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

$$\begin{aligned} \det(AB) &= (ae + bg)(cf + dh) - (af + bh)(ce + dg) \\ &= aecf + bgcf + aedh + bgdh - afce - bhce - afdg - bhdg \\ &= bgcf + aedh - bhce - afdg \\ &= \cancel{bcfg} + \cancel{adeg} - \cancel{bcef} - \cancel{adfg} \\ &= \cancel{(ad - bc)}(\cancel{eh} - \cancel{fg}) \\ &= \det(A)\det(B). \end{aligned}$$

It can be checked that for  $3 \times 3$  matrices this equality holds.

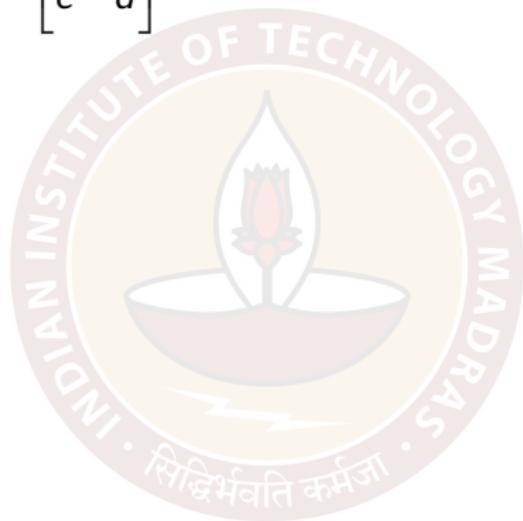
$$A A^{-1} = I = A^{-1} A$$
$$\det(A A^{-1}) = \det(I)$$
$$\det(A) \det(A^{-1}) = 1$$
$$\Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}.$$

# Properties : Switching two rows.

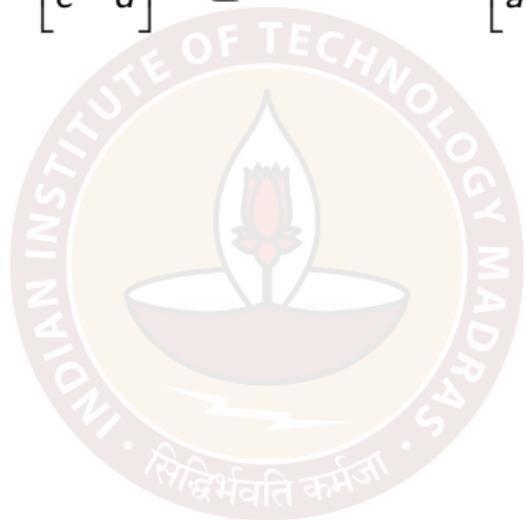


## Properties : Switching two rows.

Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ .



Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Define  $\tilde{A} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$ .



$$\text{Let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \quad \text{Define } \tilde{A} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}.$$

$$\det(\tilde{A}) = cb - da = -(ad - bc) = -\det(A).$$

Switching two columns

$$\tilde{\tilde{A}} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

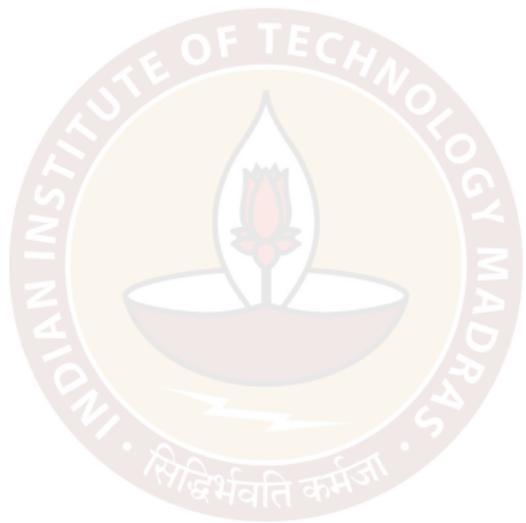
$$\det(\tilde{\tilde{A}}) = bc - ad$$

$$= - (ad - bc) = -\det(A)$$

also true for  $3 \times 3$  matrices.

This is

# Properties : Adding multiples of a row to another row.



## Properties : Adding multiples of a row to another row.

Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ .



Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Define  $\tilde{A} = \begin{bmatrix} a + tc & b + td \\ c & d \end{bmatrix}$ .

$$\begin{bmatrix} a & b \\ a+tc & b+td \end{bmatrix} = \begin{bmatrix} a & b \\ a+tc & b+td \end{bmatrix}$$



$$\text{Let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \quad \text{Define } \tilde{A} = \begin{bmatrix} a + tc & b + td \\ c & d \end{bmatrix}.$$

$$\det(\tilde{A}) = (a + tc)d - (b + td)c = ad + tcd - bc - tdc = ad - bc = \det(A).$$

$$\tilde{A} = \begin{bmatrix} a+tb & b \\ c+td & d \end{bmatrix}$$

$$\begin{aligned} \det(\tilde{A}) &= (a+tb)d - b(c+td) \\ &= ad + tbd - bc - btd \\ &= ad - bc = \det(A) \end{aligned}$$

Check this for  $3 \times 3$  matrices.

# Properties : Scalar multiplication of a row by a constant $t$ .



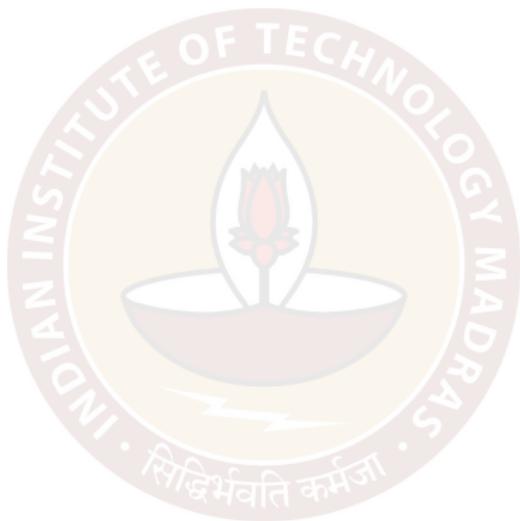
Properties : Scalar multiplication of a row by a constant  $t$ .

Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ .



## Properties : Scalar multiplication of a row by a constant $t$ .

Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Define  $\tilde{A} = \begin{bmatrix} a & tb \\ c & td \end{bmatrix}$ .



$$\text{Let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \quad \text{Define } \tilde{A} = \begin{bmatrix} a & tb \\ c & td \end{bmatrix}.$$

$$\det(\tilde{A}) = atd - tbc = t(ad - bc) = t\det(A).$$

$$\tilde{\tilde{A}} = \begin{bmatrix} ta & tb \\ c & d \end{bmatrix}$$

$$\det(\tilde{\tilde{A}}) = tad - tb^2$$

$$= t(ad - bc) = t \det(A).$$

Same thing for  $3 \times 3$  matrices.

# Thank you

