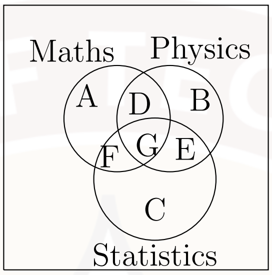
Week - 1

Solutions for Practice Assignment

Mathematics for Data Science - 1

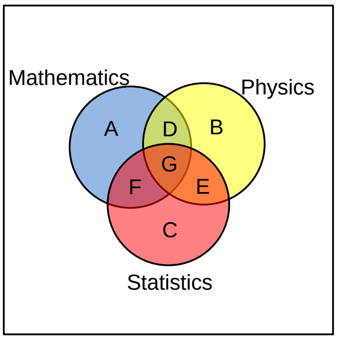
1. Given below is a Venn diagram for sets of students who take $Maths$, $Physics$, and $Statistics$. Which of the option(s) is(are) correct? [Notation: For sets P and Q, $P \setminus Q$ denotes the set of elements in P which are not in Q.]



- ○ D is the set of students who take both $Maths$ and $Statistics$.
- ○ $D \cup E \cup F \cup G$ is the set of all students who take at least two subjects.
- ○ E is a subset of the set of the students who have not taken $Maths$.
- ○ $Maths \setminus D$ is the set of all students who have taken only $Maths$.
- ○ $Physics \setminus (D \cup G \cup E)$ is the set of all students who have taken only $Physics$.

Solution: According to Figure 1, D is the set of students who take both $Maths$ and $Physics$. Hence the first statement is not valid.

The second option - $D \cup E \cup F \cup G$ is the set of all students who take at least two subjects - is correct. This is because D is the set of students who take both $Maths$ and $Physics$, E is the set of students who take both $Physics$ and $Statistics$, F is the set of students who take both $Maths$ and $Statistics$ and G is the set of students who take all three subjects.



PS-1.1: Figure for Question 1

Third option - E is a subset of the set of the students who have not taken $Maths$ - is also correct. E is the set of students who take both $Physics$ and $Statistics$ and G is the set of students who take $Maths$ in addition to $Physics$ and $Statistics$. $(B \cup E \cup C)$ is the set of students who have not taken $Maths$. Clearly, E is a subset of this set. As E and G are two different sets, this option is correct.

Fourth option - $Maths \setminus D$ is the set of all students who have taken only $Maths$ - is not correct. $Maths \setminus D$ represents the students of $Maths$ who have not taken $Physics$ and may or may not have taken $Statistics$. This implies that students who take only $Maths$ (set A), or the students who take both $Maths$ and $Statistics$ (set F) or the students who take all three subjects (set G) are also included in $Maths \setminus D$ set. Hence this option is not correct.

Fifth option - $Physics \setminus (D \cup G \cup E)$ is the set of all students who have taken only $Physics$ - is correct. $(D \cup G \cup E)$ represents the students who take only $Maths$ and $Physics$ or all three subjects or $Physics$ and $Statistics$. $Physics \setminus (D \cup G \cup E)$ represents B, which is the set of students who only take $Physics$. Hence this option is correct.

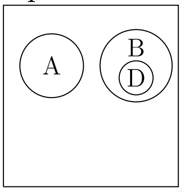
2. Let A be the set of natural numbers less than 6 and whose greatest common divisor with 6 is 1. Let B be the set of divisors of 6. What are the cardinalities of $A, B, A \cup B,$ and $A \cap B$?

   - ◯ (1,5,6,0)
   - ◯ (1,4,5,0)
   - ◯ (2,4,5,1)
   - ◯ (2,4,6,1)

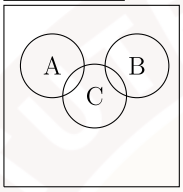
   Solution: We have set A={1, 5}, B={1, 2, 3, 6}, $A \cup B$ ={1, 2, 3, 5, 6} and $A \cap B$={1}. It follows that the cardinalities (i.e. number of elements) of A, B, $A \cup B$ and $A \cap B$ are respectively 2, 4, 5 and 1. Hence, the third option - {2, 4, 5, 1} - is correct.

3. Let A be the set of all even natural numbers (including zero), B be the set of all odd natural numbers, C be the set of all natural numbers which divide 100, and D be the set of all prime numbers less than 100. Which of the following is(are) correct representation of these sets? [Note: A region represented in a Venn diagram could be empty. Take the set of real numbers to be the universal set.]

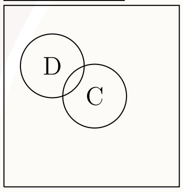
   ◯ Option 1

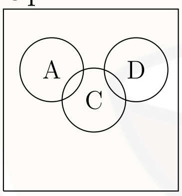
   ◯ Option 2

   ◯ Option 3

   ◯ Option 4

   Solution: By definition, $A = \{0, 2, 4, 6, 8, \ldots\}$, $B = \{1, 3, 5, 7, \ldots\}$, $C = \{1, 2, 4, 5, 10, \ldots, 100\}$ and $D = \{2, 3, 5, 7, 11, .., 97\}$.
   Option 1 shows D as a subset of all odd natural numbers. But D contains element 2, whereas B does not. Hence, this option is wrong.
   Option 2 has overlap between A and C and overlap between B and C, but no overlap between A and B. A and B are sets of even and odd natural numbers which have no overlap. C is the set of natural numbers which divide 100. $A \cap C = \{2, 4, 10, 20, 50, 100\}$ and $B \cap C = \{1, 5, 25\}$. Hence, this option is correct.
   Option 3 represents C and D sets with an overlap between them. The overlapping area includes the set of all prime numbers which can divide 100. This is the set $\{2, 5\}$. Hence, option 3 is also correct.
   $A \cap D = \{2\}$, but there is no overlap between A and D in Option 4. Hence, this option is wrong.

4. Let A be the set of natural numbers which are multiples of 5 strictly less than 100, and B be the set of natural numbers which divide 100. What are the cardinalities of the following sets?

$B \setminus A$ (the set of elements in B but not in A), $A \cap B$, and B

- ◯ (2, 5, 7)
- ◯ (4, 5, 9)
- ◯ (3, 4 , 7)
- ◯ (3, 5, 8)

Solution: By definition, $A = \{5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85, 90, 95\}$, $B = \{1, 2, 4, 5, 10, 20, 25, 50, 100\}$, $B \setminus A = \{1, 2, 4, 100\}$ and $A \cap B = \{5, 10, 20, 25, 50\}$. It follows that the cardinalities of sets $B \setminus A$, $A \cap B$ and B are, respectively, 4, 5 and 9. Hence, option 2 is correct.

5. Suppose the cardinality of set A is 2 and the cardinality of set B is 3, what are the cardinalities of the cartesian product $A \times B$ and the power set of $A \times B$?

   - ◯ 6 and 36
   - ◯ 5 and 32
   - ◯ <u>6 and 64</u>
   - ◯ 5 and 25

   Solution: Let the cardinality of set A be $\mathrm{n}(A)$ and the cardinality of set B be $\mathrm{n}(B)$. Then, the cardinality of the cartesian product $(A \times B)$, $\mathrm{n}(A \times B) = \mathrm{n}(A) \times \mathrm{n}(B) = 3 \times 2 = 6$. If a set A has cardinality n, then the cardinality of power set of A is 2^n. It follows that the cardinality of the power set of $(A \times B)$ is $2^6 = 64$. Hence, the third option is correct.

6. In a survey, it is found that in a particular locality 64 houses buy English newspapers, 94 houses buy Tamil newspapers, and 26 houses buy both English and Tamil newspapers. How many houses buy newspapers of only one language?
Answer: 106
Solution: Number of houses which buy only English newspapers is (64 - 26) = 38.
Number of houses which buy only Tamil newspapers is (94 - 26) = 68.
Therefore, number of houses which buy either English or Tamil newspaper is
(68 + 38) = 106.

7. Which of the following numbers is(are) irrational?

   - ◯ $\underline{\sqrt{2+\sqrt{3}}}$
   - ◯ $(2+\sqrt{3})(2-\sqrt{3})$
   - ◯ $(2+\sqrt{3})+(2-\sqrt{3})$
   - ◯ $\underline{2\sqrt{3}+3\sqrt{2}}$

   Solution: Since $\sqrt{3}$ is an irrational number, it follows that $(2+\sqrt{3})$ and hence $\sqrt{(2+\sqrt{3})}$ are also irrational.
   In the second option, $(2+\sqrt{3})(2-\sqrt{3})$ = 4 - 3 = 1, which is a rational number.
   In the third option, $(2+\sqrt{3})+(2-\sqrt{3})$ = 4, which is also a rational number.
   Since both $\sqrt{3}$ and $\sqrt{2}$ are irrational numbers, we have $(2\sqrt{3}+3\sqrt{2})$ is an irrational number.

8. Which of the following is(are) true for the relation R given below?
$R = \{(a, b)|$ both a and b are even non-zero integers and $\frac{a}{b}$ is an integer $\}$

- ◯ R is a reflexive relation.
- ◯ R is a symmetric relation.
- ◯ R is a transitive relation.
- ◯ R is an equivalence relation.

Solution: A relation R on a set A is said to be reflexive if $(a, a) \in R$ for all $a \in A$. R is called symmetric if $(a, b) \in R$ implies $(b, a) \in R$, and R is called transitive if (a, b) and (b, c) is in R implies $(a, c) \in R$. If a relation R is reflexive, symmetric and transitive, then it is called equivalence relation.
For any non-zero even integer a, $\frac{a}{a} = 1$ is an integer. Hence, $(a, a) \in R$, which implies that R is reflexive.
Now, let a = 4, and b = 2. Then, $\frac{a}{b}= \frac{4}{2}= 2$ is an integer. Hence, $(a, b) \in R$. But $\frac{b}{a} = \frac{2}{4}= \frac{1}{2}$ is not an integer. Therefore, $(b, a) \notin R$. It follows that R is not symmetric.
Let $(a, b) \in R$ and $(b, c) \in R$. That is, both $\frac{a}{b}$ and $\frac{b}{c}$ are integers. Hence, their product $\frac{a}{b}.\frac{b}{c} = \frac{a}{c}$ is also an integer. It follows that $(a, c) \in R$. Therefore, R is transitive.
Although R is reflexive and transitive but not symmetric, it is not an equivalence relation.

9. Find the domain and range of the following real valued function.
$f(x) = \sqrt{3-x}$ (Note: $\sqrt{}$ denotes the positive square root)
   - ○ domain=$\{x \in \mathbb{R} \mid x \neq 3\}$
     range=$\{x \in \mathbb{R} | x \geq 3\}$
   - ○ domain=$\{x \in \mathbb{R} \mid x \geq 3\}$
     range=$\{x \in \mathbb{R} \mid x \geq 0\}$
   - ○ domain=$\{x \in \mathbb{R} \mid x \leq 3\}$
     range=$\{x \in \mathbb{R} \mid x \geq 0\}$
   - ○ domain=$\{x \in \mathbb{R} \mid x \leq 3\}$
     range=$\{x \in \mathbb{R} \mid x \leq 0\}$

Solution: The set of real numbers $\mathbb{R}$ includes all rational and irrational numbers.
$\sqrt{a}$ is real valued if $a \geq 0$. If f has to be real valued, then

$$3 - x \geq 0$$

$$\Rightarrow 3 \geq x$$

Hence, domain of the function f is $\{x \in \mathbb{R} \mid x \leq 3\}$.
Since $\sqrt{}$ denotes the positive square root (as given in the question statement), the range of function f is nothing but all the positive real numbers, i.e. $\{x \in \mathbb{R} \mid x \geq 0\}$.

10. Which of the following is(are) true for the given function?
$f : \mathbb{R} \to \mathbb{R}$
$f(x) = x^2 + 2$

- ◯ f is not injective.
- ◯ f is surjective.
- ◯ f is not surjective.
- ◯ f is bijective.

Solution: A function f is injective if $f(x_1) = f(x_2)$ implies $x_1 = x_2$, i.e. no two elements in the domain will have the same image. f is called surjective if for any element in the co-domain there is a pre-image in the domain, i.e. for any y in the co-domain, there exists an x in the domain such that $f(x) = y$. A function f is said to be bijective if it is both injective and surjective.
Since $f(x) = x^2 + 2$, we have $f(-1) = 3 = f(1)$. Hence, f is not injective. Now, the co-domain of the function is given as $\mathbb{R}$.
Now if f is surjective then codomain and the range should be same, that means every element in the codomain should have a preimage. Now let us try to find a preimage for 1 (observe that $1 \in \mathbb{R}$, as codomain of the function is given as $\mathbb{R}$). To find the preimage of 1, we have to find an element a from the domain for which $f(a) = 1$, i.e. $a^2 + 2 = 1$, i.e. $a^2 = -1$. Now we know that the square of any real number cannot be negative. Hence there cannot exist any real number a (in the domain) for which $f(a) = 1$. Hence 1 has no preimage. So codomain and range is not same. Hence f is not surjective. Also, $1 \in \mathbb{R}$. Let x be such that $x \in \mathbb{R}$, and $f(x) = 1$.
As the function is neither injective, nor surjective, therefore it is not bijective.

11. Find the domain of the following real valued function.
$f(x) = \frac{\sqrt{x+2}}{x^2-9}$

◯ $\underline{\{x \in \mathbb{R} \mid x \geq -2, x \neq 3\}}$

◯ $\{x \in \mathbb{R} \mid x \leq -2, x \geq 3\}$

◯ $\{x \in \mathbb{R} \mid x \neq -2, x \leq 3\}$

◯ $\{x \in \mathbb{R} \mid x \neq -2, x \neq 3\}$

Solution: $f(x) = \frac{\sqrt{x+2}}{x^2-9}$. For f to be a well-defined function, the denominator must be non-zero. That is,

$$x^2 - 9 \neq 0$$
$$\Rightarrow x \neq \pm 3$$

Further, if f has to be real valued, then $\sqrt{x+2}$ has to be real valued. Hence $x+2$ must be non-negative. That is,

$$x + 2 \geq 0$$
$$\Rightarrow x \geq -2$$

It follows that the domain of the function $f(x)$ is $\{x \in \mathbb{R} \mid x \geq -2, x \neq 3\}$.

12. Let S be the set {January, February, March, April, May, June, July, August, September, October, November, December} of months in a year. Define the following three relations:

- $R_1 := \{(a, b) \mid a, b \in S,\ a \text{ and } b \text{ have same last four letters. } \}$
- $R_2 := \{(a, b) \mid a, b \in S,\ a \text{ and } b \text{ have same number of days. } \}$
- $R_3 := \{(a, c) \mid a, c \in S,\ \text{for some } b \in S, (a, b) \in R_1, (b, c) \in R_2\}$

For example,(December, June)$\in R_3$ since (December, September)$\in R_1$, (September, June)$\in R_2$.

(a) Choose the correct option(s).

◯ R_3 is symmetric.

◯ R_3 is reflexive.

◯ R_3 is transitive.

◯ None of the above.

(b) What is the cardinality of R_3?

Answer: 85

Solution: For definitions of types of relations, please refer to solution of Question 8.
Every month has the same last four letters as itself (except *May* which has only three letters). In Table 1, the months whose name has been shown in red color have the same last four letters as each other. Similarly, the months whose name has been shown in blue color also have the same last four letters as each other.

Name of the months (Elements of S)
January
February
March
April
May
June
July
August
September
October
November
December

Table 1: Question 12 : R_1 relation

Hence $R_1 = \{$(Jan, Jan), (Jan, Feb), (Feb, Jan), (Feb, Feb), (Mar, Mar), (April, April), (June, June), (July, July), (Aug, Aug), (Oct, Oct), (Sept, Sept), (Sept, Nov), (Sept, Dec), (Nov, Sept), (Nov, Nov), (Nov, Dec), (Dec, Sept), (Dec, Nov), (Dec, Dec)$\}$

The relation R_2 consists of the pairs of months with the same number of days. In Table 2, the months whose name has been shown in red color have 31 days each. The months whose name has been shown in black color have 30 days each.

Name of the months
January
February
March
April
May
June
July
August
September
October
November
December

Table 2: Question 12: R_2 relation

Observe that it is a equivalence relation. The partition formed by this equivalence relation is as follows:

Class 1: Jan, Mar, May, July, Aug, Oct, Dec [Months with 31 days each]

Class 2: April, June, Sept, Nov [Months with 30 days each]

Class 3: Feb [Month with 28 or 29 days]

Now, R_3 is defined as follows:

$$R_3 = \{(a, c) \mid a, c \in S, \text{ for some } b \in S, (a, b) \in R_1, (b, c) \in R_2\}$$

If $(a, c) \in R_3$, then there must exist some pair $(a, b) \in R_1$.
Let us list out the number of elements of R_3 by listing out pairs starting with as shown below :

January: (Jan, Jan) $\in R_1$, Now we assume three partitions in the set S, formed by the relation R_2. These partitions are class 1, class 2, class 3. Hence from these classes, 7 pairs will be there in R_3 starting with January. These are {(Jan, Jan), (Jan, Mar), (Jan, May), (Jan, July), (Jan, Aug), (Jan, Oct), (Jan, Dec)}. Moreover, (Jan, Feb) is in R_1, and Feb is in another partition in S due to R_2. So there are total 8 pairs (adding (Jan, Feb) with previous 7 elements) in R_3 starting with Jan.

February: Since (Feb, Jan) is in R_1, then due to class 1 there will be 7 pairs : {(Feb, Jan), (Feb, Mar), (Feb, May), (Feb, July), (Feb, Aug), (Feb, Oct), (Feb, Dec)}. The element (Feb, Feb) will be in R_3 due to class 3. Hence 8 pairs are there in R_3 starting with Feb.

March: Due to class 1, seven pairs {(Mar, Jan), (Mar, Mar), (Mar, May), (Mar, July), (Mar, Aug), (Mar, Oct), (Mar, Dec)}.

April: Due to class 2, four pairs {(April, April), (April, June), (April, Sept), (April, Nov)}.

May: No pair will start with May as there is no pair in R_1 starting with May.

June: Due to class 2, 4 pairs: {(June, April), (June, June), (June, Sept), (June, Nov)}

July: Due to class 1, 7 pairs. {(July, Jan), (July, March), (July, July), (July, Aug), (July, Oct), (July, Dec)}

August: Due to class 1, 7 pairs. {(Aug, Jan), (Aug, Mar), (Aug, May), (Aug, July), (Aug, Aug), (Aug, Oct), (Aug, Dec)}

September: As (Sept, Dec) is a pair in R_1, it will pair up will all months in class 1, and as (Sept, Sept) is in R_1, it will pair up with all months with class 2. Hence there are total 11 pairs in R_3 starting with Sept : {(Sept, Jan), (Sept, Mar), (Sept, May), (Sept, July), (Sept, Aug), (Sept, Oct), (Sept, Dec), (Sept, April), (Sept, June), (Sept, Sept), (Sept, Nov)}

October: Due to class 1, 7 pairs are there : {(Oct, Jan), (Oct, Mar), (Oct, May), (Oct, July), (Oct, Aug), (Oct, Oct), (Oct, Dec)}

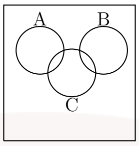
November: Due to both class 1 and class 2, 11 pairs : {(Nov, Jan), (Nov, Mar), (Nov, May), (Nov, July), (Nov, Aug), (Nov, Oct), (Nov, Dec), (Nov, April), (Nov, June), (Nov, Sept), (Nov, Nov)}

December: Due to both class 1 and class 2, 11 pairs: {(Dec, Jan), (Dec, Mar), (Dec, May), (Dec, July), (Dec, Aug), (Dec, Oct), (Dec, Dec), (Dec, April), (Dec, June), (Dec, Sept), (Dec, Nov)}

12. (b)
Hence cardinality of R_3 is $8+8+7+4+4+7+7+11+7+11+11 = 85$.
12. (a)

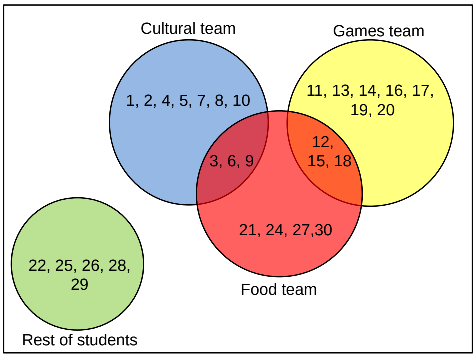
- (May, May) is not in R_3, hence R_3 is not reflexive.
- (Jan, May) is in R_3, but (May, Jan) is not in R_3, hence R_3 is not symmetric.
- (Mar, Dec) is in R_3, (Dec, Sept) is in R_3, but (Mar, Sept) is not in R_3. Hence R_3 is not transitive.

13. For a college event, thirty student volunteers were given id numbers from 1 to 30 such that each student gets a unique number. The students with id numbers from 1 to 10 are in Team 1 which coordinates the cultural program. The students with id numbers from 11 to 20 are in Team 2 which coordinates the games. The students whose roll numbers are multiples of 3 are in Team 3 which takes care of food. Now consider the following Venn diagram and choose the correct option(s).



- ◯ C, B , and A can represent Team 1, Team 2, and Team 3 respectively.
- ◯ A, B, and C can represent Team 1, Team 2, and Team 3 respectively.
- ◯ Roll number 15 has been assigned two jobs and is in both B and C.
- ◯ Roll number 25 is not in $A \cup B \cup C$.
- ◯ Roll number 10 is in both A and C.
- ◯ Cardinality of C is 20.

Solution:



PS-1.2: Venn diagram for Question 13

Figure PS-1.2 shows the Venn diagram corresponding to Question 13. Team 1, responsible for coordination of cultural programs, is represented by the blue circle. Team 2, responsible for game events, is represented by the yellow circle. Team 3, that takes care of food, is represented by the red circle. Rest of the students are represented using the green circle. Clearly, set A can correspond to the blue circle, B can denote the yellow circle and C can denote the red circle. That is, A, B, and C can represent Team 1, Team 2, and Team 3 respectively. Hence, option 2 is correct and option 1 is wrong. Roll number 15 is a common element between games team and food team, hence, option 3 is correct. Roll number 25 is located in the range of students with Roll number 21 to 30 but 25 is not divisible by 3. Hence, 25 does not belong to the set $A \cup B \cup C$ and so option 4 is correct. The number 10 is not divisible by 3, hence Roll number 10 is not in the set C. Therefore, option 5 is wrong. Further, since cardinality of C is 10, option 6 is also wrong.

Week - 2
Practice Assignment Solutions
Straight line - 1
Mathematics for Data Science - 1

NOTE: There are some questions which have functions with discrete valued domains (such as month or year). For simplicity, we treat them as continuous functions.

1 Multiple Choice Questions (MCQ):

1. If R is the set of all points which are 5 units away from the origin and are on the axes then R is:

   ◯ $R = \{(5,5),(-5,5),(-5,-5),(5,-5)\}$
   ◯ $R = \{(5,0),\ (5,-5),\ (5,5),\ (-5,0)\}$
   ◯ $R = \{(5,0),\ (0,5),\ (5,5),\ (0,-5)\}$
   ◯ $\boldsymbol{R = \{(5,0),\ (0,5),\ (-5,0),\ (0,-5)\}}$
   ◯ $R = \{(5,0),\ (0,5),\ (-5,0),\ (-5,5)\}$
   ◯ There is no such set.

   Solution:

   The points on the $x-$axis are represented by $(\pm a,\ 0)$, and on the $y-$axis are represented by $(0,\ \pm b)$, where a and b are the distances of the points $(\pm a,\ 0)$ and $(0,\ \pm b)$, respectively, from the origin. Therefore, the points (5, 0), (0, 5), (-5, 0), (0, -5) lie on the axes and are 5 units away from the origin. See Figure PS-2.1 for reference.

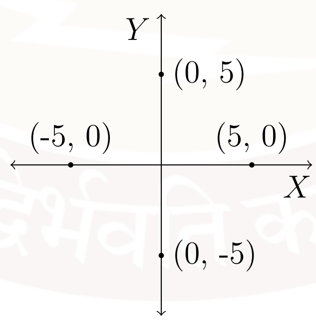


Figure PS-2.1

2. A point P divides the line segment MN such that $MP : PN = 2 : 1$. The coordinates of M and N are $(-2, 2)$ and $(1, -1)$ respectively. What will be the slope of the line passing through P and the point $(1, 1)$?

   - ◯ $\frac{4}{3}$
   - ◯ **1**
   - ◯ Inadequate information.
   - ◯ $-\frac{4}{3}$
   - ◯ $\tan(\frac{4}{3})$
   - ◯ None of the above.

Solution:

By the sectional formula, the coordinates of a point (x, y) that divides a line segment defined by two points $(x_1, y_1), (x_2, y_2)$ in the ratio $m : n$ is given by

$$x = \frac{m \times x_2 + n \times x_1}{m + n}$$

$$y = \frac{m \times y_2 + n \times y_1}{m + n}$$

Since point P divides the line segment formed by the points $M(-2, 2)$ and $N(1, -1)$ in the ratio 2:1, we obtain the coordinates of point P denoted by, say (x_p, y_p), using the sectional formula as follows.

$$x_p = \frac{2 \times 1 + 1 \times (-2)}{2 + 1} = 0$$

$$y_p = \frac{2 \times (-1) + 1 \times 2}{2 + 1} = 0$$

Hence point $P = (0, 0)$ denotes the origin as shown in Figure PS-2.2

Now, we compute the slope of the line passing through P and (1,1) as,

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 0}{1 - 0} = 1$$

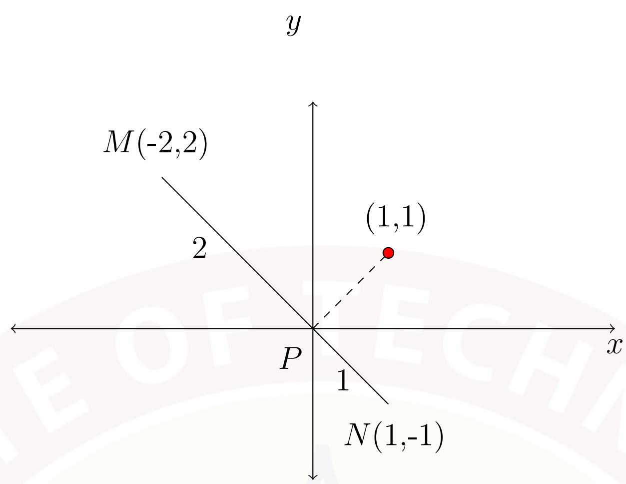


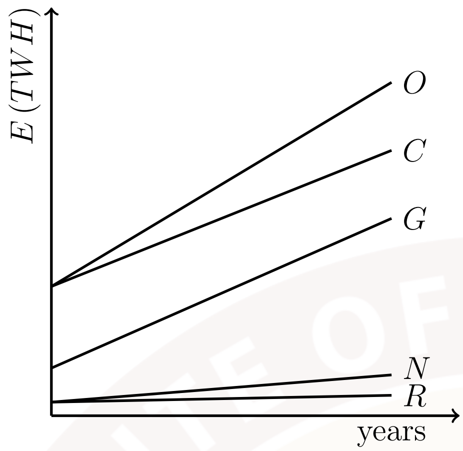
Figure PS-2.2

Use the following information to solve questions 3 and 4.
Table PS-2.1 shows the different types of energies consumed (approximate values) in years 1965 and 2015 across the world

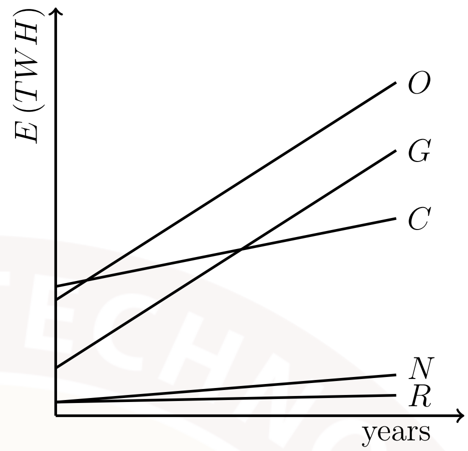
Energy type	Approximate energy used (TWH)	
	1965	2015
Oil (O)	19000	49000
Coal (C)	17000	38000
Gas (G)	7000	29000
Nuclear (N)	2000	6000
Renewable (R)	2000	3000

Table PS-2.1

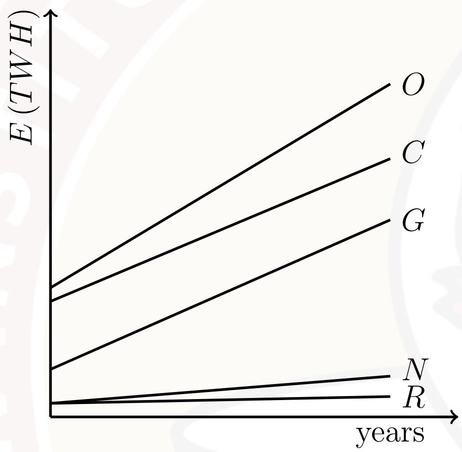
3. A student assumes a linear relationship between energy consumed (E) and the number of years after 1965. Choose the option which best represents the linear relationships assumed by the student (from 1965 to 2015). [Ans: Option C]



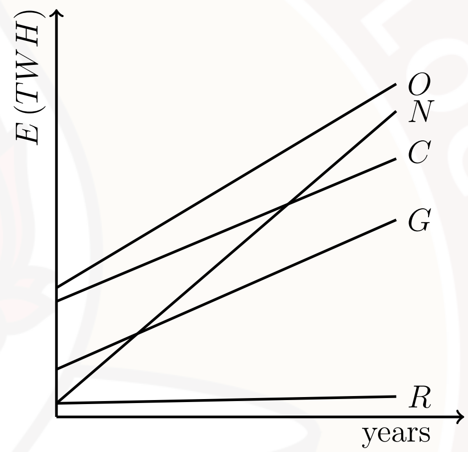
A.



B.



C.



D.

Solution:

Let x-axis and y-axis represent the years and the energy consumption respectively. The energy consumption in 2015 is in the order $O > C > G > N > R$, which is represented correctly in options (A) and (C). However, option (A) shows the energy consumption of O and C being same in the year 1965, which is not true. Hence, option (A) is not correct. Therefore, the correct answer is option (C).

4. The student estimated the energy consumption in 2025 and created Table PS-2.2. Choose the correct option.

Energy type	Approximate energy used (TWH)		
	1965	2015	2025
Oil (O)	19000	49000	o
Coal (C)	17000	38000	c
Gas (G)	7000	29000	g
Nuclear (N)	2000	6000	n
Renewable (R)	2000	3000	r

Table PS-2.2

- ◯ $o = 64000$
- ◯ $c = 48500$
- ◯ $g = 38500$
- ◯ $n = 8000$
- ◯ $r = 3500$
- ◯ **None of the above**.

Solution:

As earlier, let x-axis and y-axis represent the years and the energy consumption respectively. Using the data provided for two years, we can find the equation of the line in two-point form. Equation for the energy type *oil* (O) will be

$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\implies y - 19000 = \frac{49000 - 19000}{2015 - 1965}(x - 1965)$$

On solving the above equation with $x = 2025$,

$$\implies y - 19000 = \frac{49000 - 19000}{2015 - 1965}(2025 - 1965)$$

$$y = 55000$$

Equation for the energy type *coal* (C) will be

$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\implies y - 17000 = \frac{38000 - 17000}{2015 - 1965}(x - 1965)$$

On solving the above equation with $x = 2025$:

$$\implies y - 17000 = \frac{38000 - 17000}{2015 - 1965}(2025 - 1965)$$

$$y = 42200$$

Equation for the energy type *gas* (G) will be

$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\implies y - 7000 = \frac{29000 - 7000}{2015 - 1965}(x - 1965)$$

On solving the above equation with $x = 2025$,

$$\implies y - 7000 = \frac{29000 - 7000}{2015 - 1965}(2025 - 1965)$$

$$y = 33400$$

Equation for the energy type *nuclear* (N) will be

$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\implies y - 2000 = \frac{6000 - 2000}{2015 - 1965}(x - 1965)$$

On solving the above equation with $x = 2025$,

$$\implies y - 2000 = \frac{6000 - 2000}{2015 - 1965}(2025 - 1965)$$

$$y = 6800$$

Equation for the energy type *renewable* (R) will be:

$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\implies y - 2000 = \frac{3000 - 2000}{2015 - 1965}(x - 1965)$$

On solving the above equation with $x = 2025$:

$$\implies y - 2000 = \frac{3000 - 2000}{2015 - 1965}(2025 - 1965)$$

$$y = 3200$$

Thus, none of the options given is correct.

2 Multiple Select Questions (MSQ):

1. The elements of a relation R are shown as points in the graph in Figure P-2.3. Choose the set of correct options:

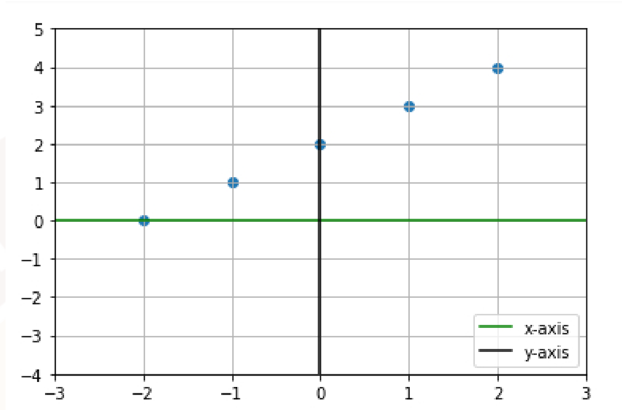


Figure PS-2.3

- ○ **$\boldsymbol{R}$ can be represented as $\boldsymbol{R = \{(-2,0),\ (-1,1),\ (0,2),\ (1,3),\ (2,4)\}}$.**
- ○ We can write R as $R = \{(a,b)|(a,b) \in X \times Y, b = a+2\}$, where X is the set of all values on the $x-axis$, and Y is the set of all values on the $y-axis$.
- ○ R cannot be a function because it is a finite set.
- ○ R is a symmetric relation.
- ○ **$\boldsymbol{R}$ is a function because it has only one output for one input.**
- ○ If R is a function then it is a bijective function on $X \times Y$, where X is the set of all values on the $x-axis$, and Y is the set of all values on the $y-axis$.
- ○ **We can write $\boldsymbol{R}$ as $\boldsymbol{R = \{(a,b)|(a,b) \in X \times Y, b = a+2\}}$, where $\boldsymbol{X = \{-2,-1,0,1,2\}}$ and $\boldsymbol{Y = \{0,1,2,3,4\}}$.**

Solution:

- Option (a) is correct since the coordinates of the points in the Figure P-2.3 are as is defined by the function.
- Option (b) is incorrect. We can write R as $\{R = (a,b) \mid (a,b)\ in\ X \times Y,\ b = a+2\}$, where $X = \{-2,-1,0,1,2\}$ and $Y = \{0,1,2,3,4\}$. Here R is a finite set so we can not write for all values of x-axis or y-axis.
- Option (c) is incorrect since R can be a function of a finite set.
- Option (d) is incorrect since R is not a symmetric relation. For example, corresponding to the element (-2, 0), there is no element (0, -2) in R.

- Option (e) is correct since for every value of X there is single corresponding value in Y.
- Option (f) is incorrect since R as a function is not defined for all values on the x-axis, and Y is not the set of all values on the y-axis, whereas $X = \{-2, -1, 0, 1, 2\}$ and $Y = \{0, 1, 2, 3, 4\}$.
- Option (g) is correct, and explained in accordance with definition of function.

2. Find the values of a for which the triangle ΔABC is an isosceles triangle, where A, B, and C have the coordinates $(-1,1)$, $(1,3)$, and $(3,a)$ respectively.

   ◯ **If $\boldsymbol{AB = BC}$, then $\boldsymbol{a = 1}$.**

   ◯ If $AB = BC$, then $a = -1$ or -5.

   ◯ **If $\boldsymbol{BC = CA}$, then $\boldsymbol{a = -1}$.**

   ◯ If $BC = CA$, then $a = 1$.

Solution:

As we know, for an isosceles triangle two of its sides are equal. According to the question the vertices of C is $(3, a)$ therefore, depending on the value of a we can have length of $AB = BC$ or $BC = CA$

Since the vertices of triangle are given, we can find the length of each side using distance formula.

Value of a when length of $AB = BC$:

Length of any side of triangle is given by

$$\sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

$$\implies \sqrt{(3-1)^2 + (1-(-1))^2} \;=\; \sqrt{(a-3)^2 + (3-1)^2}$$

$$\implies \sqrt{8} \;=\; \sqrt{4 + (a-3)^2}$$

Squaring them on both sides, we have

$$\implies (a-3)^2 \;=\; 4 \implies a - 3 \;=\; \pm 2$$

Therefore,

$$a \;=\; 5,\; 1$$

But, if $a = 5$ then the three points will be co-linear therefore,

$$\boldsymbol{a = 1}$$

Value of a when length of $BC = CA$:

$$\sqrt{(a-3)^2 + (3-1)^2} \;=\; \sqrt{(a-1)^2 + (3-(-1))^2}$$

$$\implies \sqrt{4 + (a-3)^2} \;=\; \sqrt{16 + (a-1)^2}$$

Squaring on both sides of the equation, we get

$$\implies 4 + (a-3)^2 \;=\; 16 + (a-1)^2 \implies (a-3)^2 - (a-1)^2 \;=\; 12$$

$$\implies (2a-4)(-2) \;=\; \implies 12 \implies a = -1$$

Therefore,

$$\boldsymbol{a \;=\; -1}$$

3. A plane begins to land when it is at a height of 1500 metre above the ground. It follows a straight line path and lands at a point which is at a horizontal distance of 2700 metre away. There are two towers which are at horizontal distances of 900 metre and 1800 metre away in the same direction as the landing point. Choose the correct option(s) regarding the plane's trajectory and safe landing.

   - ○ The trajectory of the path could be $\frac{y}{27} + \frac{x}{15} = 100$ if $x - axis$ and $y - axis$ are horizontal and vertical respectively.
   - ○ The maximum safe height of the towers are 1000 metre and 1500 metre respectively.
   - ○ **The trajectory of the path could be $\frac{y}{15} + \frac{x}{27} = 100$ if $x - axis$ and $y - axis$ are horizontal and vertical respectively.**
   - ○ The maximum safe height of the towers are 1500 metre and 500 metre respectively.
   - ○ **The maximum safe height of the towers are 1000 metre and 500 metre respectively.**
   - ○ None of the above.

Solution:

Let us consider the height of plane from ground as $y-$axis and horizontal distance on ground as $x-$axis as shown in Figure PS-2.4
Then, the point P(0,1500) represents the position of the airplane when it began its descent and point Q(2700,0) represents the point where the plane landed.
The two towers which are 900m and 1800m away from the $y-$axis are represented by A and B respectively.
The equation of a straight line path traced by plane from $P(0, 1500)$ to $Q(2700, 0)$ can be obtained using the intercept-form.

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$\implies \frac{x}{2700} + \frac{y}{1500} = 1$$

On rearranging:

$$\frac{y}{15} + \frac{x}{27} = 100$$

Now, to check the maximum safe height of towers:
For tower A at $X-$coordinate $= 900m$, the maximum safe height will be:

$$\frac{y}{15} + \frac{900}{27} = 100$$

$$\implies y = 1000m$$

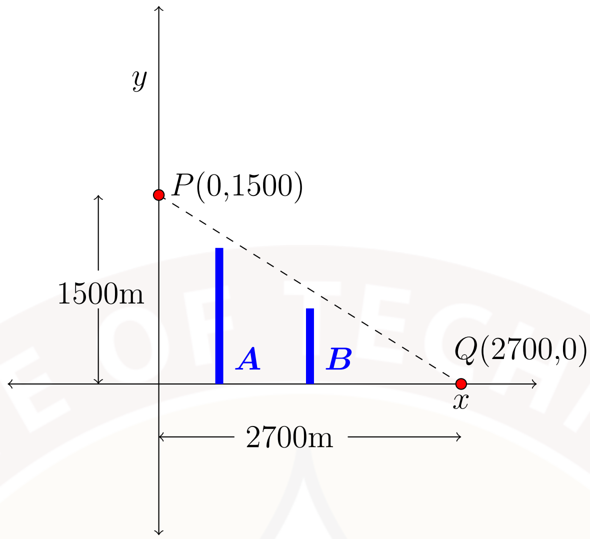


Figure PS-2.4

For tower B at $X-$coordinate $= 1800m$, the maximum safe height will be:

$$\frac{y}{15} + \frac{1800}{27} = 100$$

$$\implies y = 500m$$

3 Numerical Answer Type (NAT):

Use the following information to solve the question 1-2.
The coordinates of points A, B, C and E are shown in the figure PS-2.5 below. Points D and F are the midpoints of lines BC and AD respectively. Using the data given and Figure PS-2.5, answer the questions below.

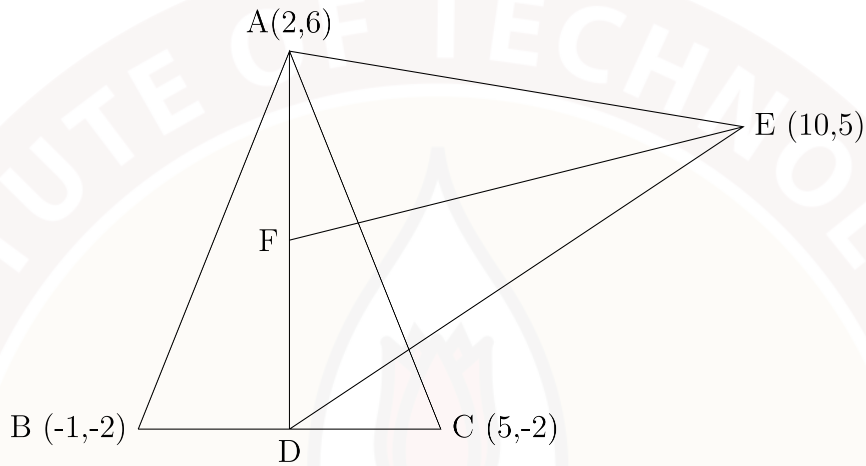


Figure PS-2.5

1. Find the area of triangle ADE. [Ans: 32]

   Solution:

   By the sectional formula, the coordinates of a point (x, y) that divides a line segment defined by two points $(x_1, y_1), (x_2, y_2)$ in the ratio $m : n$ is given by

$$x = \frac{m \times x_2 + n \times x_1}{m + n}$$

$$y = \frac{m \times y_2 + n \times y_1}{m + n}$$

   Since point D is the midpoint of the line segment BC formed by the points $B(-1, -2)$ and $C(5, -2)$ so they are in the ratio 1:1. Thus, we can obtain the coordinates of the point D denoted by, say (x_d, y_d), using the sectional formula as follows.

$$x_d = \frac{1 \times 5 + 1 \times (-1)}{1 + 1} = 2$$

$$y_d = \frac{1 \times (-2) + 1 \times (-2)}{1 + 1} = -2$$

Therefore,

$$\implies \quad D(2, -2)$$

Now, area of triangle ADE with vertices A(2, 6), D(2, -2) and E(10, 5) can be obtained as:

$$= \frac{1}{2} | \ x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \ |$$

$$= \frac{1}{2} | \ 2(-2 - 5) + 2(5 - 6) + 10(6 - (-2)) \ |$$

$$= 32$$

2. Let the slope of a line FG be 2 and the coordinate of the point G be (a, 9). Then, what is the value of a? [Ans: 5.5]

   Solution:

   As seen earlier, the point F is the midpoint of the line segment AD formed by the points $A(2, 6)$ and $D(2, -2)$ so they are in the ratio 1:1. Thus we can obtain the coordinates of the point F denoted by, say (x_f, y_f), using the sectional formula as follows.

$$x_f = \frac{1 \times 2 + 1 \times 2}{1 + 1} = 2$$

$$y_f = \frac{1 \times (-2) + 1 \times 6}{1 + 1} = 2$$

   Therefore,

$$\implies \quad F(2, 2)$$

   Now, the slope of FG will be $= \frac{9 \ - \ 2}{a \ - \ 2} = 2$
   On solving the above equation, we get $\boldsymbol{a \ = \ 5.5}$

3. Leo rents a motorcycle for 2 days. Hence, the rental company provides the motorcycle at Rs. 500 per day with 100 km free per day. The additional charges after 100 km are Rs. 2 per km Leo drives the motorcycle for a total of 500 km. How much (Rs.) will he have to pay to the rental company? [Ans: 1600]

   Solution:

   Leo has rented a motorcycle for 2 days, thus he has to pay Rs. 1,000 for free 200 km ride. Thereafter, he has to pay Rs. 2 per km. for rest of 300km, which accounts for Rs. 600. Thus, in total he has to pay Rs. 1,600.

Week - 3
Practice problems
Straight line
Mathematics for Data Science - 1

1 Multiple Choice Questions (MCQ):

1. A vehicle is travelling on a straight line path and it passes through the points $A(-4, 2)$, $B(-1, 3)$, and $C(2, \mu)$. The value of μ is:

   - ◯ 2
   - ◯ **4**
   - ◯ -2
   - ◯ 10

   Solution:

   Since the vehicle is travelling on a straight line path and passes through the points A, B, and C, it follows that A, B, and C are collinear. Hence the slope of the straight line path joining A and B will be equal to the slope of the straight line path joining B and C. Using the slope formula for two points, we have

$$\frac{3-2}{-1+4} = \frac{\mu - 3}{2+1}$$

$$\implies \boldsymbol{\mu = 4}.$$

2. Suppose two boats are starting their journey from the ferry ghat A (considered as the origin), one towards ferry ghat B along the straight line $y = -2x$ and the other towards the ferry ghat C along a straight line perpendicular to the path followed by B. The river is 1 km wide uniformly and parallel to the X-axis. Suppose Rahul wants to go to the exact opposite point of A along the river.

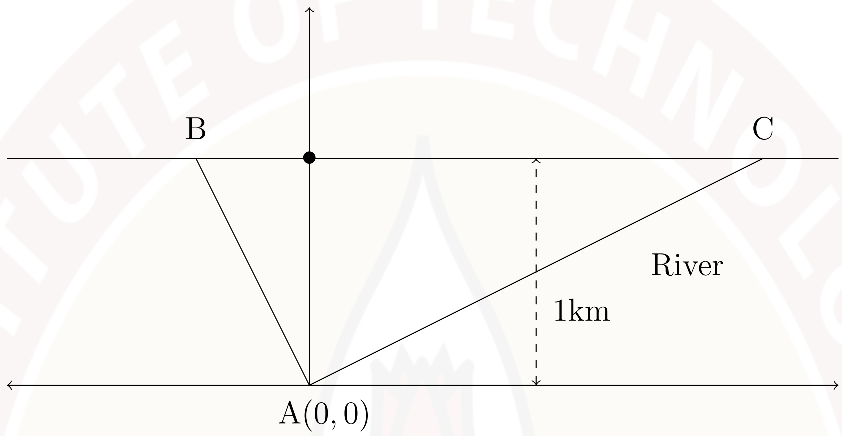


Figure PS-3.1

Then, answer the following questions.

(a) How much total distance does Rahul have to travel to reach his destination if he takes the boat towards ferry ghat B?

- ◯ $\sqrt{5}$
- ◯ $\sqrt{5} + 2$
- ◯ $\frac{\sqrt{5}}{2}$
- ◯ $\boldsymbol{\frac{\sqrt{5}+1}{2}}$

Solution:

See the Figure PS-3.2 for reference:

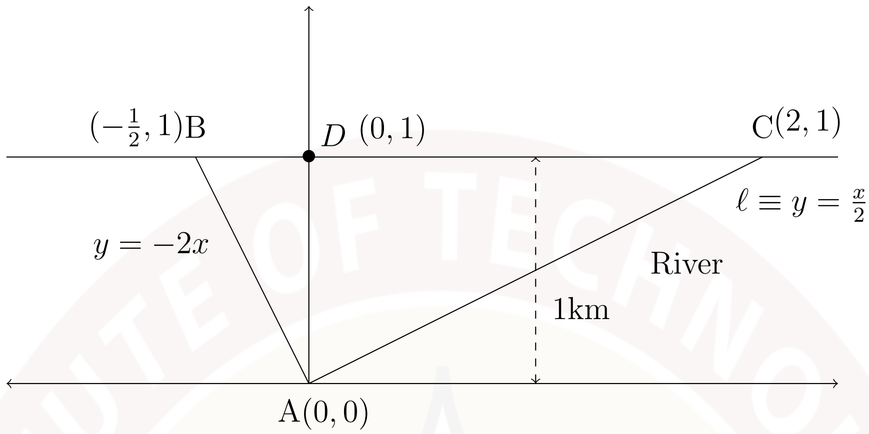


Figure PS-3.2

Since the point A is assumed to be the origin, the side of the river from which Rahul is starting his journey is considered to be the X-axis. The path towards Rahul's destination, which is perpendicular to the X-axis, is hence the Y-axis. Let D be Rahul's destination, which is 1 km away from the point A and is on the opposite side of the river. It follows that the point D is $(0, 1)$.

Hence, the equation of the line representing the opposite side of the river is $y = 1$. Solution of the equations $y = 1$ and $y = -2x$ gives the location of ferry ghat B which is the point $(-\frac{1}{2}, 1)$.

Using the distance formula between two points, the distance between ferry ghat A and ferry ghat B is given by

$$\sqrt{(-\frac{1}{2} - 0)^2 + (1 - 0)^2} = \frac{\sqrt{5}}{2}\text{units}$$

Similarly, the distance between ferry ghat B and the point D is $\frac{1}{2}$ units.

Hence, the total distance that Rahul has to travel to reach his destination D if he takes the boat toward ferry ghat B is given by

$$\frac{\sqrt{5}}{2} + \frac{1}{2} = \frac{\sqrt{5}+1}{2} \text{ units}$$

(b) How much total distance does Rahul have to travel to reach his destination if he takes the boat towards ferry ghat C?

- ○ $\sqrt{5}$
- ○ $\boldsymbol{\sqrt{5} + 2}$
- ○ $\frac{\sqrt{5}}{2}$
- ○ $\frac{\sqrt{5}+1}{2}$

Solution: Let ℓ denote the path towards ferry ghat C from A. The equation of path ℓ will be $y = mx$ since it passes through the origin. Since ℓ is perpendicular to the line $y = -2x$, which has a slope $m_1 = -2$, it follows that

$$m = -\frac{1}{m_1} = \frac{1}{2}$$

$\implies$ the equation of ℓ is $y = \frac{x}{2}$.

Solution of the equations $y = \frac{x}{2}$ and $y = 1$ gives the location of ferry ghat C which is (2,1).

Using the distance formula between two points, the distance between ferry ghat A and ferry ghat C is

$$\sqrt{(2-0)^2 + (1-0)^2} = \sqrt{5} \text{ units}$$

Similarly, the distance between ferry ghat C and the destination point D is 2 units.

Hence, the total distance that Rahul has to travel to reach his destination D if he takes the boat towards ferry ghat C is $\sqrt{5} + 2$ units.

3. Suppose a bird is flying along the straight line $4x - 5y = 20$ on the plane formed by the path of the flying bird and the line of eye point view of a person who shoots an arrow which passes through the origin and the point $(10, 8)$. What is the point on the co-ordinate plane where the arrow hits the bird?

   - ◯ $(20, 12)$
   - ◯ $(25, 16)$
   - ◯ **The arrow will miss the bird.**
   - ◯ Inadequate information.

   Solution:
   Using the two point form of a line, the equation of the path of arrow passing through the origin and the point (10,8) is

$$(y - 0) = \frac{8 - 0}{10 - 0}(x - 0) \implies 8x - 10y = 0$$

The slope intercept form of the above line is given by

$$y = \frac{8}{10}x$$

From the above line, we obtain the slope as

$$m_1 = \frac{8}{10} = \frac{4}{5}$$

Similarly, for the path of the bird along the straight line $4x - 5y = 20$, we get the slope

$$m_2 = \frac{4}{5}$$

Here, $m_1 = m_2$,
That is, the lines $8x - 10y = 0$ and $4x - 5y = 20$ have the same slope. Therefore, the path of flying bird and the path of the arrow are parallel to each other. Hence, the arrow will miss the bird.

4. We plot the displacement (S) versus time (t) for different velocities as it follows the equation $S = vt$, where v is the velocity. Identify the best possible straight lines in the Figure P-3.2 for the given set of velocities.

Table PS-3.1

v_1	v_2	v_3	v_4
1	-2	0.5	-1

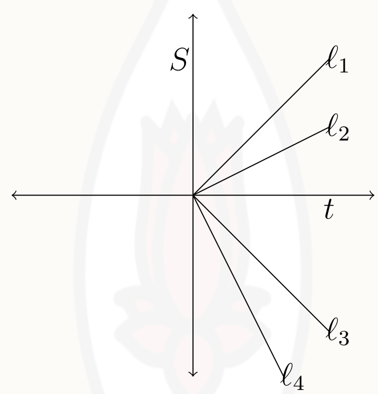


Figure PS-3.3

- ◯ $v_1 \to \ell_1$, $v_2 \to \ell_2$, $v_3 \to \ell_3$, and $v_4 \to \ell_4$.
- ◯ $v_1 \to \ell_1$, $v_2 \to \ell_4$, $v_3 \to \ell_3$, and $v_4 \to \ell_2$.
- ◯ $\boldsymbol{v_1 \to \ell_1, v_2 \to \ell_4, v_3 \to \ell_2}$**, and** $\boldsymbol{v_4 \to \ell_3}$**.**
- ◯ $v_1 \to \ell_2$, $v_2 \to \ell_4$, $v_3 \to \ell_1$, and $v_4 \to \ell_3$.

Solution:
From Figure PS-3.3, ℓ_1 and ℓ_2 have positive slope and the slope of ℓ_1 is greater than the slope of ℓ_2. Similarly the slopes of ℓ_3 and ℓ_4 are negative and the slope of line ℓ_3 is greater than the slope of line ℓ_4.
Substituting the value of v in equation $s = vt$, we get the equations

$$s = t, s = -2t, s = 0.5t, s = -t$$

By comparing the above equations of lines and the lines in Figure PS-3.3, we conclude that v_1 corresponds to the line ℓ_1, v_2 corresponds to the line ℓ_4, v_3 corresponds to the line ℓ_2, and v_4 corresponds to the line ℓ_3.

2 Multiple Select Questions (MSQ):

5. A constructor is asked to construct a road which is at a distance of $\sqrt{2}$ km from the municipality office and perpendicular to a road which can be defined by the equation of the straight line $x - y = 8$ (considering the municipality office to be the origin). Find out the possible equations of the straight lines to represent the new road to be constructed.

   - ◯ $x - y - 2 = 0$
   - ◯ $\boldsymbol{x + y + 2 = 0}$
   - ◯ $x - y + 2 = 0$
   - ◯ $\boldsymbol{x + y - 2 = 0}$

   Solution:

   See the Figure PS-3.4 for reference:

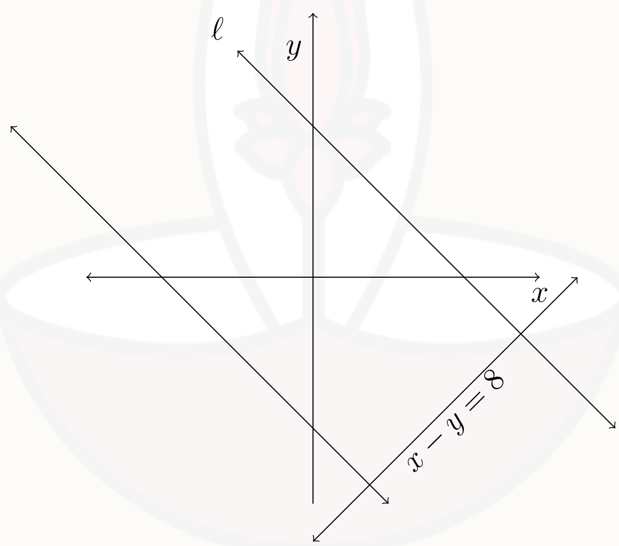


Figure PS-3.4

Let the new road constructed be denoted by ℓ. Given, ℓ is perpendicular to the straight line $x - y = 8$. That is, ℓ is perpendicular to the line $y = x - 8$ whose slope is $m_1 = 1$. Therefore, the slope of ℓ is

$$m_2 = -\frac{1}{m_1} = -1$$

By the slope intercept form, the equation of ℓ is

$$y = m_2\ x\ +\ c$$

$$\implies y = -x + c \text{ , where } c \text{ is a constant}$$

That is, ℓ is the line given by

$$x + y - c = 0$$

It is given that the distance of ℓ from the municipality office is $\sqrt{2}$.
The distance formula of a point (x_1, y_1) from a line $(Ax + By + C = 0)$ is given by $\frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}$. Substituting $x_1 = 0, y_1 = 0, A = 1, B = 1, C = -c$ in formula, we will get the distance of the point(0,0) from the line ℓ,

$$\frac{|1 \times 0 + 1 \times 0 - c|}{\sqrt{1^2 + 1^2}}$$

which is equal to $\sqrt{2}$. So,

$$\frac{|0 + 0 - c|}{\sqrt{1+1}} = \sqrt{2}$$

$$\implies \frac{|c|}{\sqrt{2}} = \sqrt{2}$$

$$\implies |c| = 2$$

$\implies c = +2$ or $c = -2$.
Hence, the equation of the new road ℓ is

$$x + y + 2 = 0$$

or

$$x + y - 2 = 0.$$

6. Suppose there are two roads perpendicular to each other which are both at the same distance from Priya's house (considered as the origin). The meeting point of the two roads is on the x-axis and at a distance of 5 units from Priya's house.
Choose the correct possible equations representing the roads.

   - ◯ Inadequate information.
   - ◯ $y = \frac{1}{2}x + 5, y = -2x - 5$
   - ◯ $\boldsymbol{y = -x - 5, y = x + 5}$
   - ◯ $y = 2x - 10, y = -2x - 10$
   - ◯ $y = 2x - 5, y = -\frac{1}{2}x - 5$
   - ◯ $\boldsymbol{y = -x + 5, y = x - 5}$
   - ◯ $x = 5, x = -5$

Solution:
Denote the two roads by ℓ_1 and ℓ_2. The meeting point of ℓ_1 and ℓ_2 are on the X-axis and at a distance of 5 units from Priya's house (origin) i.e x-intercepts of the roads are 5 or -5 and passing through the points (5,0) or (-5,0) respectively.

Case 1: when x-intercept is 5 and passes through (5,0)
Using intercept form of a line on the axes, the equation of line ℓ_1 is

$$\frac{x}{5} + \frac{y}{b} = 1$$

where b is a constant.

That is, ℓ_1 is

$$bx + 5y - 5b = 0 \tag{1}$$

See Figure PS-3.5 for reference.

The slope of the road ℓ_1 is $m_1 = -\frac{b}{5}$.
Since the road ℓ_2 is perpendicular to ℓ_1, the slope of road ℓ_2 is

$$m_2 = -\frac{1}{m_1} = \frac{5}{b}$$

Using the slope intercept form, the equation of the road ℓ_2 is

$$y = \frac{5}{b}x + c \Longrightarrow by - 5x - bc = 0 \text{ where } b \text{ and } c \text{ are constant}$$

The roads ℓ_1 and ℓ_2 are at the same distance from Priya's house (origin).
Using distance formula of a line from a point, we get

$$\frac{|-5b|}{\sqrt{b^2+25}} = \frac{|-bc|}{\sqrt{b^2+25}} \Longrightarrow |c| = |5| \Longrightarrow c = 5 \quad or - 5$$

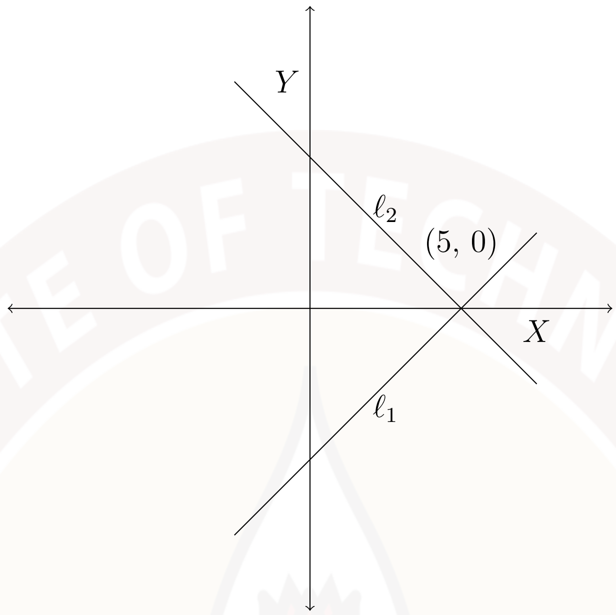


Figure PS-3.5

When $c = 5$, the equation of road ℓ_2 becomes $by - 5x - 5b = 0$. Since ℓ_2 passes through $(5, 0)$, we get $b = -5$.
Therefore, the equation of the road ℓ_2 is $y = -x + 5$.
Substituting $b = -5$ in Equation (1), we will get the equation of the road ℓ_1 as $y = x - 5$.
When $c = -5$, we will get the same equation alternatively.

Case 2: when x-intercept is -5 and passing through (-5,0)
We follow the same process as in Case 1 and we get the equation of the road ℓ_2 as $y = x + 5$ and the equation of the road ℓ_1 as $y = -x - 5$.
See Figure PS-3.6 for reference.

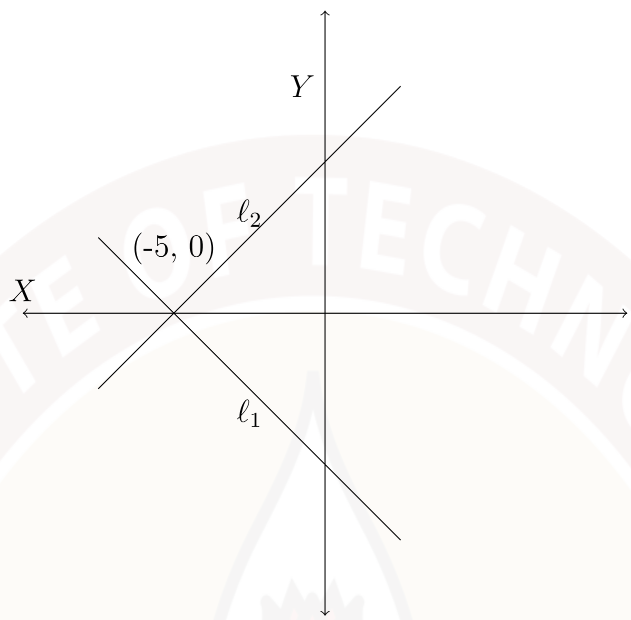


Figure PS-3.6

7. Consider the following two diagrams.

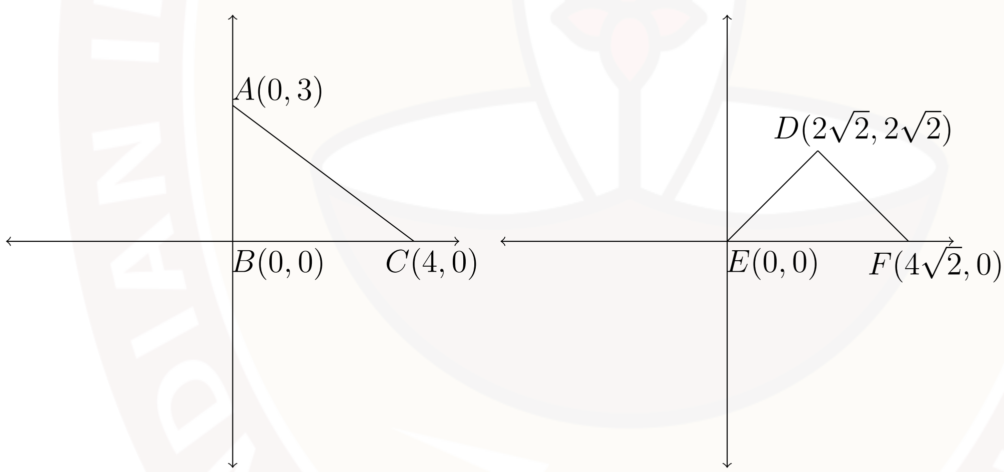


Figure PS-3.7

Which of the following option(s) is(are) true about the triangles ΔABC and ΔDEF given in Figure PS-3.7?

- ◯ Only ΔABC is a right angled triangle while ΔDEF is not.
- ◯ **Both ΔABC and ΔDEF are right angled triangles.**
- ◯ The area of ΔABC is greater than the area of ΔDEF.
- ◯ Both the triangles have the same area.

◯ **The area of ΔDEF is 8 sq.unit.**

Solution:

In Figure PS-3.7, vertices A and C are on $Y-$axis and $X-$ axis respectively and the vertex B is at the origin itself.
Therefore, $\triangle ABC$ is a right angle triangle.
The distance formula between two points $(x_1, y_1), (x_2, y_2)$ is given by

$$\sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

Using the above formula, in $\triangle DEF$, the length of side DE is

$$\sqrt{(2\sqrt{2} - 0)^2 + (2\sqrt{2} - 0)^2} = \sqrt{(2\sqrt{2})^2 + (2\sqrt{2})^2} = 4$$

Similarly, the length of side DF is

$$\sqrt{(4\sqrt{2} - 2\sqrt{2})^2 + (0 - 2\sqrt{2})^2} = 4$$

The length of side EF is $4\sqrt{2}$. We have

$$DE^2 + DF^2 = 16 + 16 = 32 = (4\sqrt{2})^2 = EF^2$$

Hence, by the Pythagoras theorem, $\triangle DEF$ is also a right angled triangle.
Area of the right angled $\triangle ABC = \frac{1}{2} \times 4 \times 3 = 6$ sq. unit.
Area of the right angled $\triangle DEF = \frac{1}{2} \times 4 \times 4 = 8$ sq. unit.

8. Let the diagonals of a quadrilateral with one vertex at $(0,0)$ bisect each other perpendicularly at the point $(1,2)$. Further, let one of the diagonals be on the straight line $y = 2x$. Then, which of the following is (are) correct statements?

   - ◯ **The diagonally opposite vertex of $(0,0)$ is $(2,4)$.**
   - ◯ The other diagonal is on the straight line $y = -\frac{1}{2}x$.
   - ◯ **The other diagonal is on the straight line $y = -\frac{1}{2}x + \frac{5}{2}$.**
   - ◯ The diagonally opposite vertex of $(0,0)$ is $(\frac{3}{2}, 3)$.

   Solution:

Figure PS-3.8 shows a sketch of the quadrilateral.

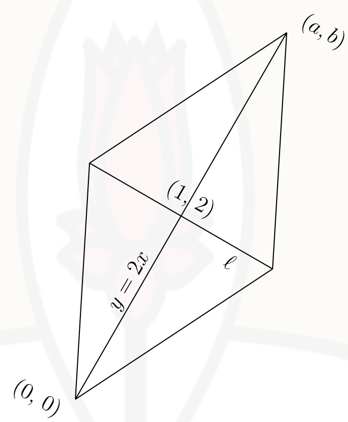


Figure PS-3.8

The diagonal $y = 2x$ has slope $m_1 = 2$.
Let the other diagonal, perpendicular to the line $y = 2x$, be on the line ℓ.
So, the slope of the line ℓ is

$$m_2 = -\frac{1}{m_1} = -\frac{1}{2}$$

From the slope intercept form, the equation of the line ℓ is $y = -\frac{x}{2} + c$, where c is a constant.
Since both the diagonals intersect at the point (1,2) and one diagonal is on line ℓ, the point (1,2) belongs to ℓ and hence $c = \frac{5}{2}$.
Hence, the equation of the line ℓ is $\boldsymbol{y = -\frac{1}{2}x + \frac{5}{2}}$.

Let the opposite vertex of $(0,0)$ be (a,b).
Since the point (1, 2) is the bisection point of the both diagonals, it follows that the point $(1,2)$ is mid-point of the line segment joining the points $(0,0)$ and (a,b).
Using the section formula of a line segment,

$$\frac{a}{2} = 1 \implies a = 2$$

$$\frac{b}{2} = 2 \implies b = 4$$

Hence the diagonally opposite vertex of $(0,0)$ is $(2,4)$.

9. A woman is reported missing in a locality. The police department finds a human femur bone during their investigation. They estimate the height H of a female adult (in cm) using the relationship $H = 1.8f + 70$, where f is the length (in cm) of the femur bone. The length of the femur found is 35 cm, and the missing woman is known to be 130 cm tall. In the particular locality, maximum height of a female is 195 cm and the minimum length of a female femur bone is 15 cm. Based on the given data answer the following questions.

   (a) Choose the set of correct options.

   ◯ If an error of 1 cm is allowed, bone could belong to missing female.

   ◯ **If an error of 3 cm is allowed, bone could belong to missing female.**

   ◯ If the height as a function of femur length is known to be accurate, the range of the function is [70, 195].

   ◯ **If the height as a function of femur length is known to be accurate, the range of the function is [97, 195].**

   ◯ **If the height as a function of femur length is known to be accurate, the domain of the function is [15, $\frac{625}{9}$].**

Solution:

The relationship between height of a woman H and the length of her femur bone f is given by

$$H = 1.8f + 70. \tag{2}$$

Since the length of the femur bone found during the investigation is 35 cm, we have

$$H = 1.8 \times 35 + 70 = 133 \text{ cm}$$

The height of missing woman is known to be 130cm. Since $133 - 130 = 3 \leq 3$ and by our assumption, an error of 3 cm is allowed, it is possible that the femur bone found during the investigation belongs to the missing woman.

Given that the maximum height of a female in that location is 195 cm.
Substituting $H = 195$ in Equation (2) , we get the maximum length of female femur bone in that location i.e maximum $f = \frac{625}{9}$ cm.
Since the minimum length of of femur bone known in that location is 15 cm and if height as a function of femur length is known to be accurate then the domain of the function is [15, $\frac{625}{9}$].

Given that the minimum length of the female femur bone in that location is 15 cm.
The minimum height of a female in that location is $H = 1.8 \times 15 + 70 = 97$ cm.
Since the maximum height of a female in that location is 195 cm, the range of the height function is $[97, 195]$

(b) A new detective agency came up with a relationship $H = mf + 70$, where H is the height of a male adult (in cm) and f is the length (in cm) of the femur bone. They have used the following sample set given below in the Table P-3.2 , such that the sum squared error is minimum.

height(H) (in cm)	150	160	170	180
length of femur bone(f) (in cm)	40	42	48	56

Table PS-3.2

Choose the correct option (only one option is correct).

- ◯ $m = 1$
- ◯ $m = 1.5$
- ◯ $\boldsymbol{m = 2}$
- ◯ $m = 2.5$

Solution:

From Table PS-3.3, we can see that the minimum SSE is for $m = 2$.

H (in cm)	f (in cm)	$(H - mf - 70)^2$			
		$m = 1$	$m = 2$	$m = 1.5$	$m = 2.5$
150	40	1600	0	400	400
160	42	2304	36	729	225
170	48	2704	16	784	400
180	56	2916	4	676	900
SSE		$\sum = 9524$	$\sum = 56$	$\sum = 2589$	$\sum = 1925$

Table PS-3.3

3 Numerical Answer Type (NAT):

10. What will be the slopes of the straight lines perpendicular to the following lines?
a) $2x + 5y - 9 = 0$

Answer:2.5
Solution:
Using the slope intercept form, the slope of the line $2x + 5y - 9 = 0$ is $m_1 = -\frac{2}{5}$.
Let the slope of the perpendicular line be m_2. Then

$$m_1.m_2 = -1$$

$$\implies m_2 = \frac{5}{2} = 2.5$$

b) $-5x + 25y + 28 = 0$

Answer:-5
Solution:
Using the slope intercept form, the slope of the line $-5x + 25y + 28 = 0$ is $m_1 = \frac{1}{5}$.
Let the slope of the perpendicular line be m_2, then

$$m_1.m_2 = -1$$

$$\implies m_2 = -5.$$

Mathematics for Data Science - 1

Practice Assignment Solutions

Week-4

1. Multiple Choice Questions (MCQ):

1. What will be the equation of the tangent to the curve $f(x) = 2x^2 + 9x + 20$ at point (-3,11)?

   - ◯ $y = 3x$
   - ◯ $\boldsymbol{y = -3x + 2}$
   - ◯ $y = -3x + 20$
   - ◯ $y = -\frac{x}{3} + 2$
   - ◯ $y = \frac{x}{3} + 20$
   - ◯ $y = -\frac{x}{3}$

Solution:

A rough diagram is given in the Figure PS-4.1 .

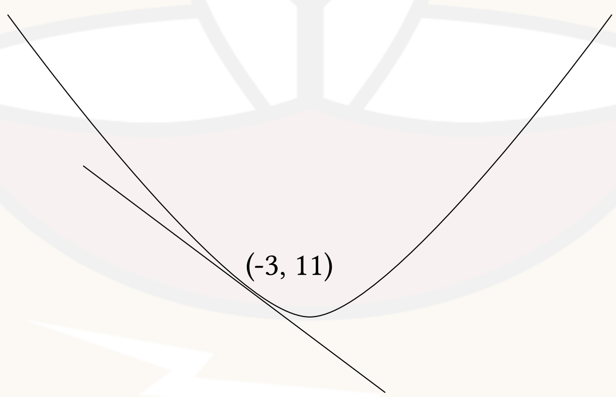


Figure PS-4.1

Let the equation of the tangent be $y = mx + c$, where m is the slope of the tangent line.
Note that m is also the slope of f at (-3, 11).
The slope of any quadratic function $g(x) = ax^2 + bx + c$, where $a \neq 0$ at x will be $2ax + b$.

Therefore, at $x = -3$,

$$m = 2ax + b \implies m = 2 \times 2 \times (-3) + 9 \implies m = -3$$

Since the tangent passes through the point (-3, 11), it should satisfy the equation of the tangent.

$$y = mx + c \implies 11 = -3 \times (-3) + c \implies c = 2.$$

So, the equation of the tangent will be $\boldsymbol{y = -3x + 2}$.

2. Find the length of the line segment on the straight line $y = 2$ bounded by the curve $y = 4x^2$.

   - ◯ $\frac{1}{\sqrt{2}}$
   - ◯ $\boldsymbol{\sqrt{2}}$
   - ◯ $1 + \sqrt{2}$
   - ◯ $1 + \frac{1}{\sqrt{2}}$

Solution:

Given $y = 4x^2$. Observe that, on comparing the above with the the general form of a quadratic function $f(x) = ax^2 + bx + c$, we have $b = 0$ which means Y-axis is the axis of symmetry. Also $c = 0 \implies$ the curve represented by this function will pass through the origin.
$-b/2a = 0$ and at $x = 0 \implies y = 0$ which means the vertex is at the origin and $a > 0 \implies$ the parabola is upward opened.
$y = 2$ is a constant function and it will pass through the point (0, 2). A rough diagram is given in the Figure PS-4.2

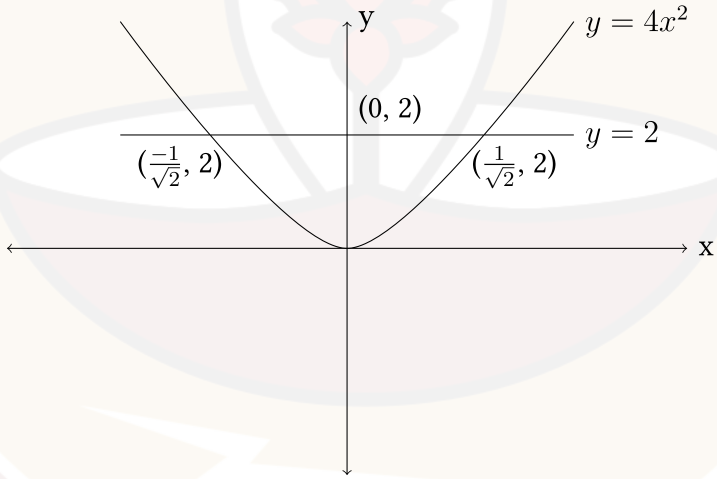
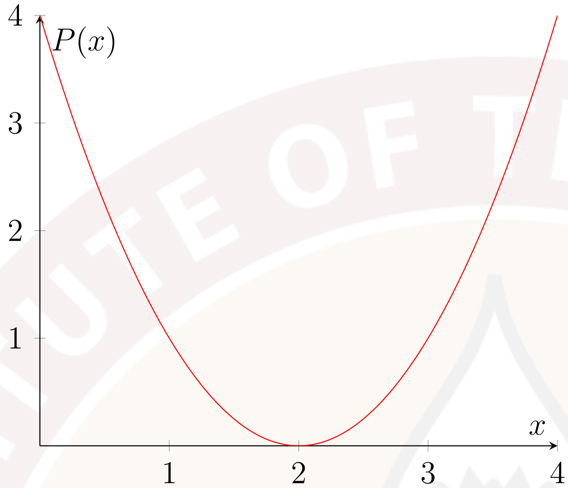
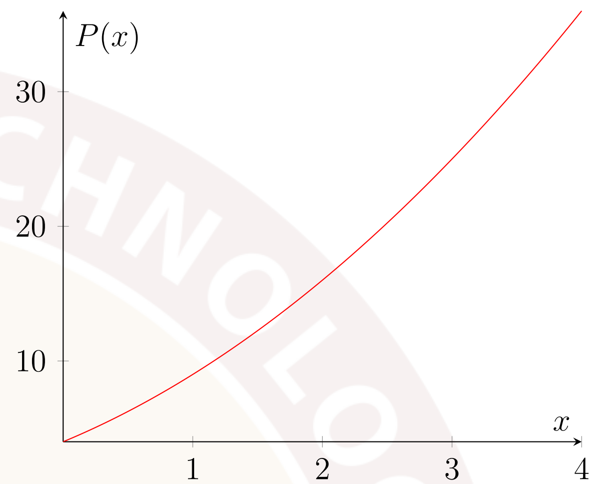


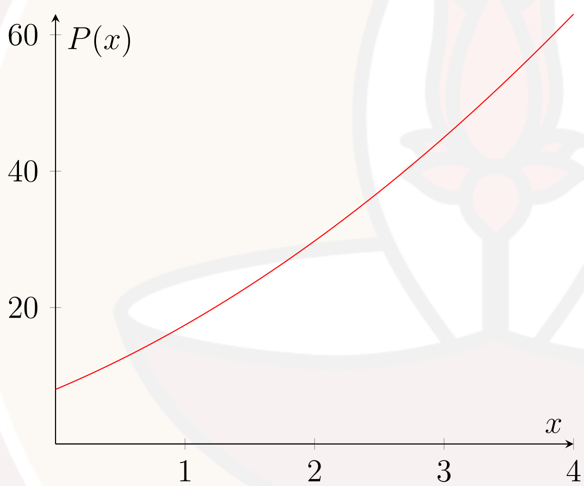
Figure PS-4.2

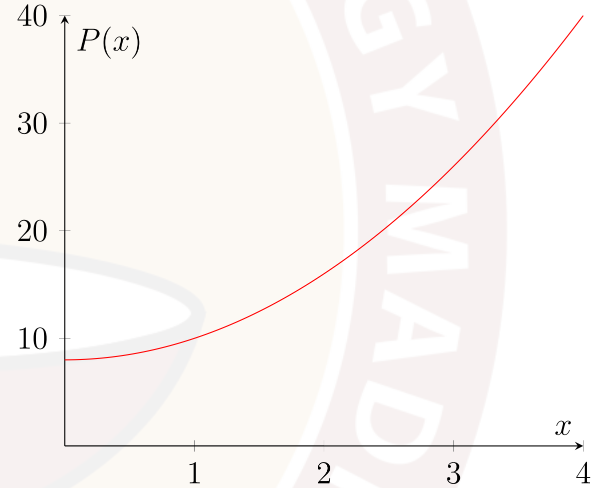
At the intersection points, $4x^2 = 2 \implies x = \pm\frac{1}{\sqrt{2}}$ which means the intersection points will be $(\frac{-1}{\sqrt{2}}, 2)$ and $(\frac{1}{\sqrt{2}}, 2)$.
Observe that these intersecting points will be the end points of the required line segment on the straight line $y = 2$.
Therefore, the length of the line segment on the straight line $y = 2$ bounded by the curve $y = 4x^2$ will be $\sqrt{(2-2)^2 + (\frac{1}{\sqrt{2}} - (\frac{-1}{\sqrt{2}})^2} = \sqrt{0 + (\frac{2}{\sqrt{2}})^2} = \sqrt{0 + (\sqrt{2})^2} = \boldsymbol{\sqrt{2}}$.

3. Mr. Mehta has two sons. Both sons send money to their father each month separately as $M_1(x) = (x-2)^2$ and $M_2(x) = (x+2)^2$ respectively. If x denotes the month, then choose the curve which best represents the total amount (P(x)) received by Mr. Mehta every month.









Solution:

Given,

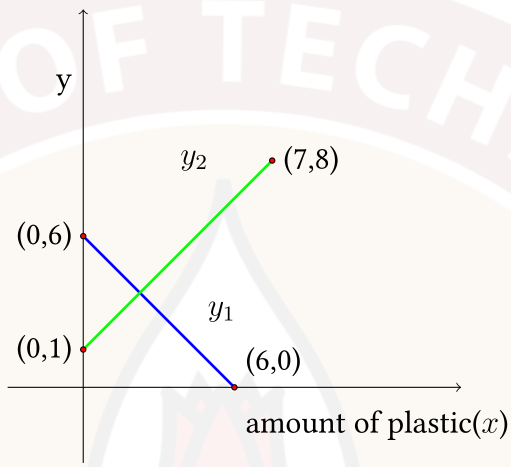
$$M_1(x) = (x-2)^2$$
$$M_2(x) = (x+2)^2.$$

So, the total amount received by Mr. Mehta is:

$$P(x) = M_1(x) + M_2(x) = (x-2)^2 + (x+2)^2 = x^2 - 4x + 4 + x^2 + 4x + 4$$
$$\Rightarrow P(x) = 2x^2 + 8.$$

In $P(x),\ b = 0$ which means Y-axis will be the axis of symmetry of the curve $p(x)$.
Now, the curve shown in the first option is not symmetric about the line $x = 0$. So, option 1 is incorrect.
The curve in the second option, passes through the origin but that is not the case for $P(x)$ as $x = 0 \implies P(x) = 8$. So, option 2 is incorrect.
The curve in the third option, does not pass through (4, 40). So, option 3 is also incorrect.
Now, the curve in the last option will pass through the points (0, 8), (1, 10), and (4,40). So, the curve in the fourth option will be the best curve that represents the total amount received by Mr.Mehta every month.

4. A civil engineer found that the durability d of the road she is laying depends on two functions y_1 and y_2 as follows: $d = ay_1y_2$ where $a > 0$. Functions y_1 and y_2 depend on the amount of plastic (x) mixed in bitumen, and their variations are shown in the graph given below. Find the values of functions y_1 and y_2 such that the durability of the road is maximum.



Solution:

Given, the durability of the road $d = ay_1y_2$.
From the given graph, the equations of the lines:

$$y_1 = 6 - x$$
$$y_2 = x + 1$$
$$\implies d = ay_1y_2 = a(6-x)(x+1) = -ax^2 + 5ax + 6a$$

Here $a > 0 \implies -a < 0$ which means the curve represented by d is open downward and the durability d of the road is the maximum at $x = \frac{-b}{2a} = \frac{-5a}{2(-a)} = \frac{5}{2}$.
Therefore, the value of $y_1 = 6 - x = 6 - \frac{5}{2} = \frac{7}{2}$ and the value of $y_2 = x + 1 = \frac{5}{2} + 1 = \frac{7}{2}$.

5. Let A be the set of all points on the curve defined by the function $f_1(x) = x^2 - x - 42$ and let B be the set of all points on the curve f_2 defined by the reflection of the curve f_1 with respect to X axis. If C is the set of all points on the axes then choose the correct option regarding the cardinality of set D where $D = (A \cap B) \cup (A \cap C) \cup (B \cap C)$.

   - ◯ infinite.
   - ◯ 8
   - ◯ **4**
   - ◯ 6
   - ◯ 2
   - ◯ zero.

Solution:

For the function $f_1(x) = x^2 - x - 42$, $a > 0 \implies$ opening upward, $-\frac{b}{2a} = \frac{1}{2} \implies x = \frac{1}{2}$ is the axis of symmetry.
$x = 0 \implies f_1(0) = -42$ so, it will pass through the point (0, -42).
The reflection of $f_1(x)$ with respect to $X-$ axis i.e. $f_2(x)$ will pass through the point (0, 42).
For intersection points of both curves:
Both the curves will be intersecting on same place on $X-$ axis as they are mirror image of each other around $X-$ axis. A rough diagram is given in the Figure PS-4.3

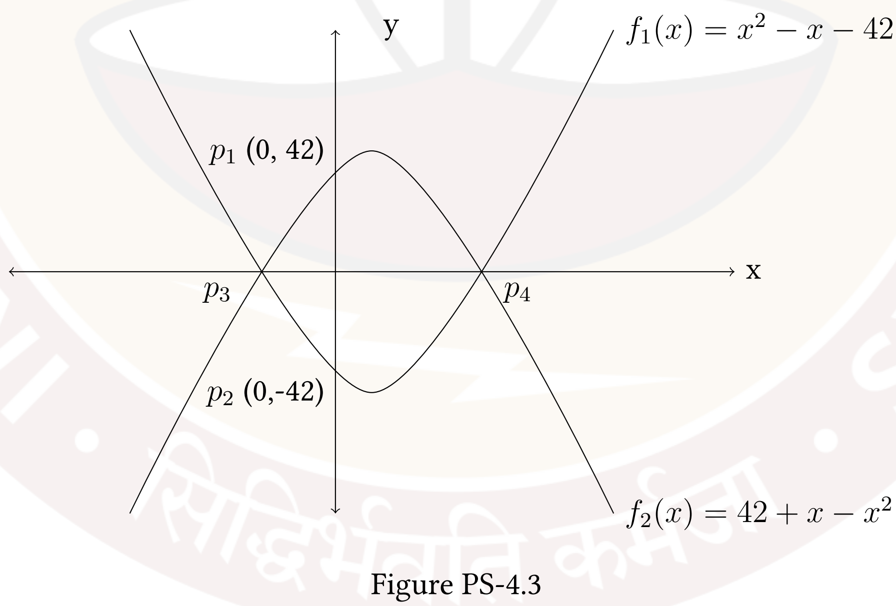


Figure PS-4.3

Since A is the set of all points on the curve f_1, B will be the set of all points on the curve f_2 and C will be the set of all points on the X-axis or Y-axis.

From Figure PS-4.3,
$A \cap\ B$ is the set of all points which are on f_1 and f_2. Therefore, $A \cap\ B$ = $\{p_3, p_4\}$.
$A \cap C$ is the set of all points which are on the curve f_1 and on the X-axis or Y-axis. Therefore, $A \cap\ C$ = $\{p_3, p_4, p_2\}$.
$B \cap\ C$ is the set of all points which are on the curve f_2 and on the $X-$axis or $Y-$axis. Therefore, $B \cap\ C$ = $\{p_3, p_1, p_4\}$.
Now, $D = (A \cap\ B) \cup\ (A \cap\ C) \cup\ (B \cap\ C) = \{p_1, p_1, p_1, p_4\}$ and therefore, the cardinality of D is 4.

6. Let $f_1(x) = x^2 - 25$. Let A be the set of all points inside the region by the curves representing $f_1(x)$ and its reflection $f_2(x)$ with respect to $X-$ axis (excluding the points on curve). Choose the correct option.

   - ◯ The cardinality of A is 2.
   - ◯ The cardinality of A is 4.
   - ◯ **$\boldsymbol{Y-}$ coordinates of the points in set A belong to the interval $\boldsymbol{(-25, 25)}$.**
   - ◯ $Y-$ coordinates of the points in set A belong to the interval $[-25, 25]$.
   - ◯ $X-$ coordinates of the points in set A belong to the interval $[-5, 5]$.
   - ◯ $X-$ coordinates of the points in set A will be all real numbers because f_1 is a quadratic function.

Solution:

A rough diagram is shown in the Figure PS-4.4

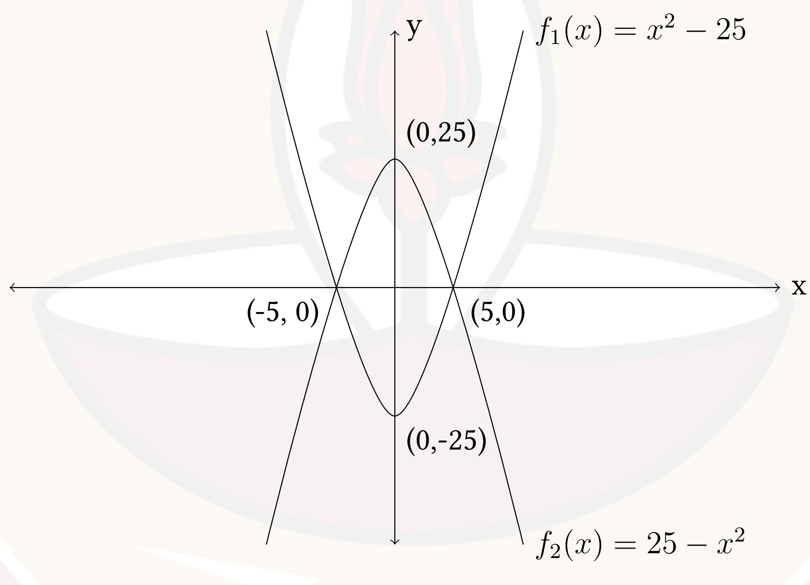


Figure PS-4.4

From the Figure PS-4.4, observe that the set A is infinite, because the region between the two curves f_1 and f_2 has infinitely many points. Therefore, the cardinality of A is not finite. So, options 1 and 2 are wrong.
Also, the region is in between the lines y = +25 and y = -25. Therefore, Y-coordinates of all the points in set A lie between -25 and +25 (-25 and +25 are excluded because they are points on the curves). So, option 3 is correct and option 4 is incorrect because -25 and +25 are included.

Also, the points in A are in between the lines x = -5 and x = +5 (-5 and +5 are excluded because they are points on the curves). Therefore, the X-coordinates of the points in set A belong to the interval (-5, 5). So, options 5 and 6 are incorrect.

2. Multiple Select Questions (MSQ):

1. **Choose the correct set of options regarding the function $\mathbf{f(x) = x^2 + 6x + 8}$**

   - ◯ $y = -3$ is the axis of symmetry.
   - ◯ **-2 and -4 are the zeroes of the above function**.
   - ◯ The maximum value of the above function is -1.
   - ◯ **Slope of the function at $(-3, -1)$ is zero**.
   - ◯ **$2x + 6$ is the slope of this curve at any given x**.
   - ◯ The function is symmetric around $x = 3$.

Solution:

Given, $f(x) = x^2 + 6x + 8$.
The axis of symmetry of $f(x)$ is $x = \frac{-b}{2a} = \frac{-6}{2} = -3$.
Therefore, $x = -3$ is the axis of symmetry of curve $f(x)$. So, options 1 and 6 are incorrect.

For zeros:

$$f(-2) = (-2)^2 + 6(-2) + 8 = 4 - 12 + 8 = 0$$
$$f(-4) = (-4)^2 + 6(-4) + 8 = 16 - 24 + 8 = 0$$

Hence, -2 and -4 are the zeros of the given function. So, option 2 is correct.

As $f(x)$ is an upward parabola, the maximum value of the function is + ∞ at $x = +\infty$. So, option 3 is incorrect.

Now, at $x = -3$, $f(x) = f(-3) = (-3)^2 + 6(-3) + 8 = 9 - 18 + 8 = -1$.
Therefore, the point (-3, -1) is the vertex of the given function. Also, the slope of the function at vertex is always 0. So, option 4 is correct.

We know that the slope of any given quadratic function $g(x) = ax^2 + bx + c; a, b, c \in \mathbb{R}$ at point $(x, g(x))$ is $2ax + b$. Here, $a = 1, b = 6$ and $c = 8$
Therefore, the slope of $f(x)$ is $2x + 6$ at any given x. So, option 5 is correct.

2. A quadratic function f is such that its value decreases over the interval $(-\infty, -2)$ and increases over the interval $(-2, \infty)$, and $f(0) = f(-4) = 23$. Then, f can be

   ○ $-3x^2 - 12x + 23$

   ○ $\boldsymbol{3x^2 + 12x + 23}$

   ○ $5(x-2)^2 + 3$

   ○ $\boldsymbol{5(x+2)^2 + 3}$

   ○ $\boldsymbol{ax^2 + 4ax + 23,\ a > 0}$

   ○ $ax^2 + 4ax + 23,\ a < 0$

Solution:

Given, the values of f decreases over $(-\infty, -2)$ and increases over interval $(-2, \infty)$. Also, $f(0) = f(-4) = 23$.
The curve f is roughly shown in the Figure PS-4.5.

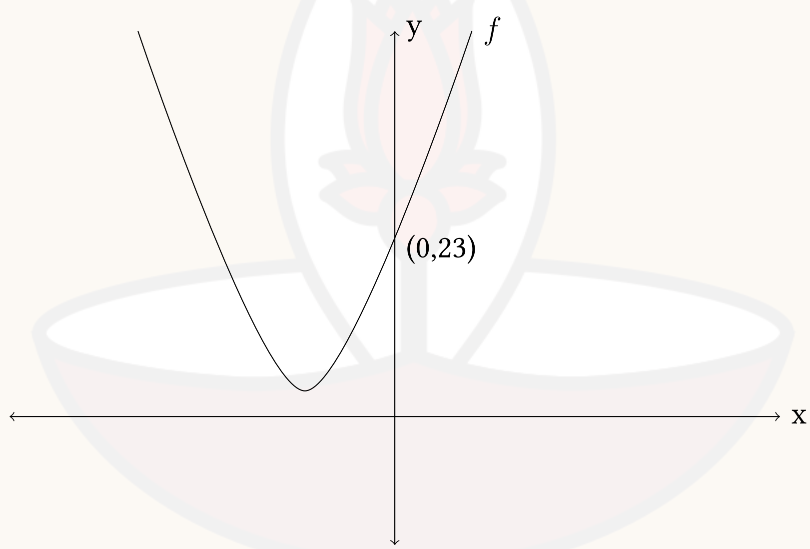


Figure PS-4.5

Suppose $f(x) = ax^2 + bx + c$, for any $a, b, c \in \mathbb{R}$.
We have $f(0) = 23 = a(0)^2 + b(0) + c = c \Rightarrow c = 23$.
Now, $f(-4) = 23 = a(-4)^2 + b(-4) + 23 \Rightarrow 16a - 4b = 0 \Rightarrow b = 4a$.
As the curve f which is shown in the Figure PS-4.5 is an upward parabola, the value of a should be positive.

Therefore, the quadratic function that satisfies the given conditions will be of the form $f(x) = ax^2 + 4ax + 23$, for all $a > 0$. So, option 5 is correct and option 6 is incorrect.

If $a = 3$, then f can be $3x^2 + 12x + 23$. So, option 2 is correct.

If $a = 5$, then f can be $5x^2 + 20x + 23 = 5(x + 2)^2 + 3$. So, option 4 is correct.

In option 1, the leading coefficient of the given function is $-3 = a < 0$. So, it is incorrect.

3. Suppose one root of a quadratic equation of the form $ax^2 + bx + c = 0$, with $a, b, c \in \mathbb{R}$, is $2 + \sqrt{3}$. Then choose the correct set of options.

   - ◯ **There can be infinitely many such quadratic equations.**
   - ◯ There is no such quadratic equation.
   - ◯ There is a unique quadratic equation satisfying the properties.
   - ◯ $\boldsymbol{x^2 - 4x + 1 = 0}$ **is one such quadratic equation**.
   - ◯ $x^2 - 2x - 3 = 0$ is one such quadratic equation.

Solution:

Given, $2 + \sqrt{3}$ is a root of $ax^2 + bx + c = 0$. One root of the quadratic equation is known. The other root can be any real number k.
For each value of k we will have a different quadratic equation. Therefore, there can be infinitely many quadratic equations that have $2 + \sqrt{3}$ as a root. So, option 1 is correct and options 2,3 are incorrect.

Now, option 4 is correct because the function value (at $x = 2 + \sqrt{3}$) is

$$(2+\sqrt{3})^2 - 4(2+\sqrt{3}) + 1 = 4 + 4\sqrt{3} + 3 - 8 - 4\sqrt{3} + 1 = 0$$
$$\Rightarrow 2 + \sqrt{3} \text{ is a root of } x^2 - 4x + 1 = 0.$$

Option 5 is incorrect because the function value (at $x = 2 + \sqrt{3}$) is

$$(2+\sqrt{3})^2 - 2(2+\sqrt{3}) - 3 = 4 + 4\sqrt{3} + 3 - 4 - 2\sqrt{3} - 3 = 2\sqrt{3} \neq 0$$
$$\Rightarrow 2 + \sqrt{3} \text{ is not a root of } x^2 - 2x - 3 = 0.$$

4. A company's profits are known to be dependent on the months of a year. The profit pattern (in lakhs of Rupees) from January to December is $P(x) = -2x^2 + 25x$. Here, x represents the month number, starting from 1 (for January) and ending at 12 (for December). On this basis, choose the correct option.

   - ◯ **The maximum profit in a month is Rs.78 lakhs.**
   - ◯ The maximum profit in a month is Rs.78.125 lakhs.
   - ◯ The maximum profit in a month is Rs.77 lakhs.
   - ◯ **The maximum profit is recorded in June.**
   - ◯ The profit in December is 144 lakhs.
   - ◯ None of the above.

Solution:

The profit of the company is given as $P(x) = -2x^2 + 25x$. Observe $P(x)$ is downward open. So, the maximum profit will be recorded at vertex.
The X-Coordinate of the vertex is $x = \frac{-b}{2a} = \frac{-25}{2(-2)} = 6.25$
So, the vertex lies between the lines $x = 6$ and $x = 7$
Therefore, the maximum profit will be recorded in the month of June($x = 6$) or July($x = 7$).
The profit(in lakhs of Rupees) in June is

$$P(6) = -2(6)^2 + 25(6) = -72 + 150 = 78$$

and profit(in lakhs of Rupees) in July is

$$P(7) = -2(7)^2 + 25(7) = -98 + 175 = 77$$

Therefore, the maximum profit of Rs.78 lakhs is recorded in the month of June. So, options 1 and 4 are correct.

The profit (in lakhs of Rupees) in December is

$$P(12) = -2(12)^2 + 25(12) = -288 + 300 = 12$$

So, option 5 is incorrect.

5. Raghav sells 2000 packets of bread for Rs. 20000 each day, and makes a profit of Rs. 4,000 per day. He finds that if the cost price increases by Rs. x per packet, he can increase the selling price by Rs. $2x$ per packet. However, when this price increase happens, he loses $200x$ of his customers. Choose the correct options.

   - ◯ **For the maximum profit per day, cost price is Rs. 12 per packet**.
   - ◯ For the maximum profit per day, cost price is Rs. 4 per packet.
   - ◯ For the maximum profit per day, the sale price increases by Rs. 4 per packet.
   - ◯ For the maximum profit per day, Raghav will lose 400 customers.
   - ◯ **The maximum difference in profit per day could be Rs. 3200**.
   - ◯ The maximum difference in profit per day could be Rs. 7200.

Solution:

The selling price of bread $\frac{20000}{2000}$ = 10 Rupees per packet.
We know that, selling price - cost price = profit $\Rightarrow$ 20000 - cost price = 4000 $\Rightarrow$ cost price per day= 16000.
Therefore, the cost price is = $\frac{16000}{2000}$ = 8 Rupees per packet.

Now, if the cost price of each packet increases to $8 + x$ and the selling price of each packet is increased to $10 + 2x$, then the customers left will be $2000 - 200x$.

So, the total profit (say P) in terms of x:

profit = (selling price of each packet- cost price of each packet) x (number of customers)

$$\implies P(x) = \{(10 + 2x) - (8 + x)\} \times (2000 - 200x)$$
$$\implies P(x) = (2 + x)(2000 - 200x)$$
$$\implies P(x) = 4000 + 1600x - 200x^2.$$

The maximum profit occurs at $x = -\frac{b}{2a}$ = $-\frac{1600}{2(-200)}$ = 4.

Hence, for the maximum profit per day:

cost price per packet = $8 + x$ = 8 + 4 = **12**.
sale price per packet = $10 + 2x$ = 10 + 8 = **18**
The customers he loses = $200x$ = 200(4) = **800**.
Maximum profit = $4000 + 1600x - 200x^2$ = $4000 + 1600(4) - 200(4)^2$ = **7200**

Therefore, maximum difference in profit = 7200 - 4000 = **3200 Rupees**.
So, the options 1 and 5 are correct.

3. Numerical answer type(NAT):

1. A farmer has a wire of length 576 metres. He uses it to fence his rectangular field to protect it from animals. If he fences his field with four rounds of wire, and the field has the maximum area possible to accommodate such a fencing, what is the area (in square metres) of the field?

   Solution:

   Suppose, the length of the rectangular field is 'l' metres and breadth of the rectangular field is 'm' metres. So, the perimeter of the rectangular field will be $2(l+m)$.
   Now, as he fences his field with four rounds of wire, we have four times the perimeter of the field which, in turn, is equal to the length of the wire. i.e,

$$4(2(l+m)) = 576$$
$$\Rightarrow l+m = \frac{576}{8} = 72$$
$$\Rightarrow m = 72 - l$$

Area of field (A) = lm

$$\Rightarrow A = l(72-l) = 72l - l^2$$

The maximum area of the field (A_{max})= $-\frac{b^2}{4a} + c$

$$A_{max} = -\frac{72^2}{4\times(-1)} + 0$$

$$\Rightarrow \boldsymbol{A_{max} = 1296} \textbf{ square metres}$$

2. Consider the quadratic function $f(x) = x^2 - 2x - 8$. Two points P and Q are chosen on this curve such that they are 2 units away from the axis of symmetry. R is the point of intersection of axis of symmetry and the $X - axis$. And S is the vertex of the curve. Based on this information, answer the following:
   (a) What is the height of $\triangle PQR$ taking PQ as the base?
   (b) What is the height of $\triangle PQS$ taking PQ as the base?

Solution:

The axis of symmetry of $f(x)$ is $x = \frac{-b}{2a} = \frac{-(-2)}{2(1)} = 1$ and two units away points will be $x = 1 + 2 = 3$ and $x = 1 - 2 = -1$.
At $x = 3 \Rightarrow f(x) = -5$ and at $x = -1 \Rightarrow f(x) = -5$. Also, the vertex of the curve is (1, -9).
A rough diagram can be drawn with this information as shown in Figure PS-4.6.

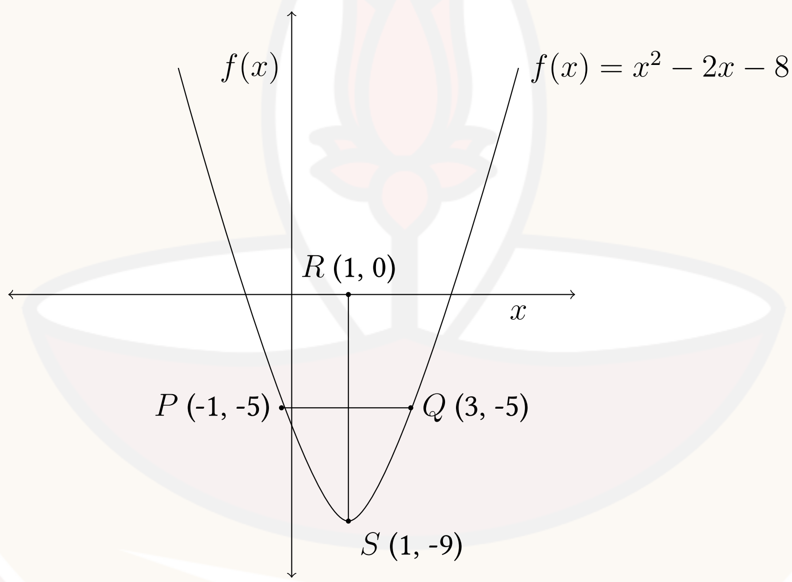


Figure PS-4.6

(a) From the above figure, the height of $\triangle PQR$ taking PQ as the base will be the distance between lines $y = 0$ and $y = -5$ and that is equal to 0 - (-5) = 5 units.

(b) From the above figure, the height of $\triangle PQS$ taking PQ as the base will be the distance between lines $y = -5$ and $y = -9$ and that is equal to (-5) - (-9) = 4 units.

Week - 5
Practice Assignment Solution
Quadratic Equations
Mathematics for Data Science - 1

NOTE:

- There are some questions which have functions with discrete-valued domains (such as month or year). For simplicity, we treat them as continuous functions.
- For a given quadratic equation $ax^2 + bx + c = 0, \ \ a \neq 0$:
  - ◯ Sum of roots $= -\frac{b}{a}$.
  - ◯ Product of roots $= \frac{c}{a}$.

1 Multiple Choice Questions (MCQ):

1. What will be the value of parameter k, if the discriminant of equation $4x^2+9x+10k = 0$ is 1?

   ◯ $\frac{82}{80}$
   ◯ $\frac{41}{80}$
   ◯ $\boldsymbol{\frac{1}{2}}$
   ◯ $\frac{41}{160}$
   ◯ 1
   ◯ None of the above.

   Solutions:
   Comparing the given equation $4x^2 + 9x + 10k = 0$ with the standard quadratic equation $ax^2 + bx + c = 0$:

$$a = 4, \ b = 9, \text{ and } c = 10k$$
$$\text{Discriminant } (d) = b^2 - 4ac$$
$$d = 9^2 - 4 \times 4 \times 10k$$
$$1 = 81 - 160k$$
$$\boldsymbol{k = \frac{1}{2}}$$

2. A boat has a speed of 30 km/hr in still water. In flowing water, it covers a distance of 50 km in the direction of flow and comes back in the opposite direction. If it covers this total of 100 km in 10 hours, then what is the speed of flow of the water (in km/hr)?

   - ◯ $5 - 5\sqrt{37}$
   - ◯ $-10\sqrt{6}$
   - ◯ $\mathbf{10\sqrt{6}}$
   - ◯ $20\sqrt{3}$
   - ◯ $-20\sqrt{3}$
   - ◯ 2

Solutions:
Total time taken by the boat = time taken by the boat in the direction of flow + time taken by the boat in the opposite direction of flow.
We know that:

$$time(t) = \frac{distance}{net\ speed}$$

Considering the direction of flow of water to be positive:
The net speed in the direction of flow (v_f) = speed of the boat in still water + speed of flow.
The net speed in the opposite direction of flow (v_b) = speed of the boat in still water - speed of flow.
Let the speed of flow be x then,

$$10 = \frac{50}{v_f} + \frac{50}{v_b}$$

$$10 = \frac{50}{30+x} + \frac{50}{30-x}$$

$$1 = \frac{5}{30+x} + \frac{5}{30-x}$$

$$1 = \frac{5(30-x+30+x)}{(30+x)(30-x)}$$

$$(30+x)(30-x) = 300$$

$$30^2 - x^2 = 300$$

$$x^2 = 600x = \pm 10\sqrt{6}$$

Speed of flow can not be negative therefore, the correct answer is $\mathbf{10\sqrt{6}}$.

3. A stunt man performs a bike stunt between two houses of the same height as shown in Figure 1. His bike (lowest part of the bike) makes an angle of θ at house A with the horizontal at the beginning of the stunt, follows a parabolic path and lands at house B with an angle of $(180 - \theta)$ with the horizontal.

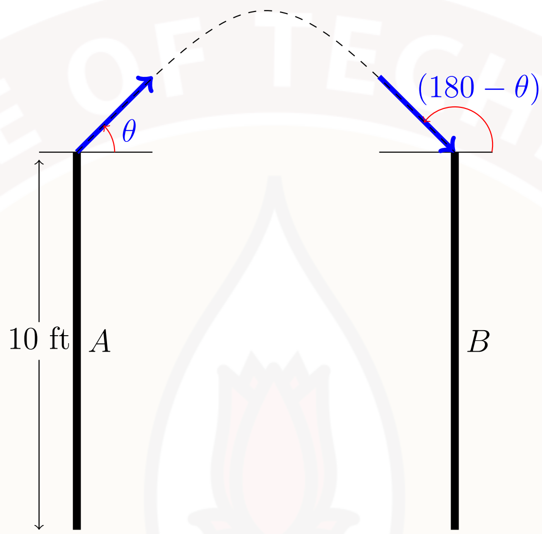


Figure PA-5.1

If the maximum height achieved by the bike is 12.5 ft from the ground and $tan\ \theta = 1$, then find the distance between the two houses.

- ◯ 1 ft
- ◯ 2.5 ft
- ◯ 5 ft
- ◯ **10 ft**
- ◯ 15 ft
- ◯ 20 ft

Solution:
Assuming the top of the house A to be origin, the horizontal direction as $X-$ axis, and the vertical direction as $Y-$ axis.

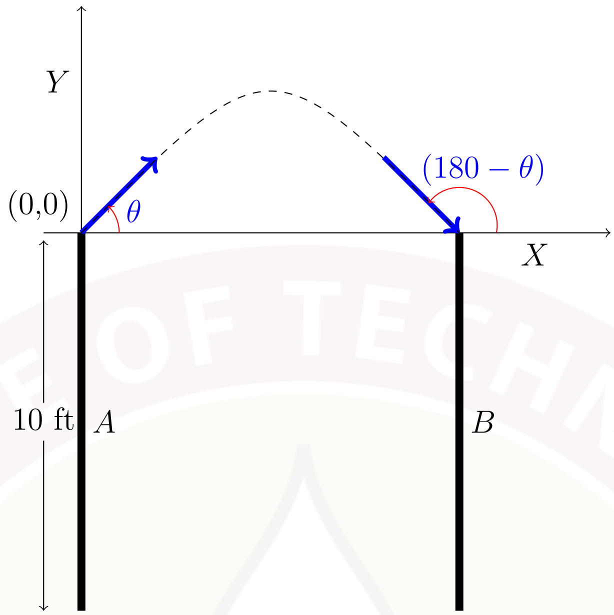


Figure M1W5PAS-3.1

Let the quadratic function representing the above curve be $f(x) = ax^2 + bx + c$.
Since the curve passes through the origin, we have $c = 0$.
The curve is making an angle θ with respect to positive $X-$ axis which means the slope of the tangent at the curve is $\tan\theta$.
We also know that the slope of the curve represented by quadratic function at $x = x$ is $2ax + b$. Therefore,

$$\begin{aligned} 2ax + b &= \tan\theta \\ 2a \times 0 + b &= 1 \\ b &= 1 \end{aligned}$$

The maximum height achieved by the bike is 12.5 ft which means the $y-$ coordinate of the vertex is $12.5 - 10 = 2.5$.
The $x-$ coordinate of the vertex for a curve represented by function $ax^2 + bx + c$ is

$$-\frac{b}{2a} = -\frac{1}{2a}$$

Therefore,

$$
\begin{aligned}
f(x) &= ax^2 + bx + c \\
f(-\frac{1}{2a}) &= 2.5 \\
a \times (-\frac{1}{2a})^2 + 1 \times (-\frac{1}{2a}) + 0 &= 2.5 \\
\frac{1}{4a} - \frac{1}{2a} &= 2.5 \\
-\frac{1}{4a} &= 2.5 \\
a &= -\frac{1}{10}
\end{aligned}
$$

Axis of symmetry,

$$x = -\frac{b}{2a} = -\frac{1}{2 \times (-1/10)} = 5$$

Because of symmetricity, the coordinate of landing point will be (10, 0).
Therefore two houses A and B are 10 ft apart.

2 Multiple Select Question (MSQ):

4. Given that $f_1(x) = -x^2 - 6x$ and $f_2(x) = x^2 + 6x + 10$. Let $f(x)$ be a function such that the domain of $f(x)$ is $[\alpha,\ \beta]$, where $f_1(\alpha) = f_2(\alpha)$ and $f_1(\beta) = f_2(\beta)$, then choose the set of correct options.

   - ○ Range of $f(x)$ is $[-1, 3]$.
   - ○ Range of $f(x)$ is $[0, 5]$.
   - ○ Domain of $f(x)$ is $[-5, 5]$.
   - ○ **Domain of $\boldsymbol{f(x)}$ is $\boldsymbol{[-5, -1]}$.**
   - ○ **Inadequate information provided for finding the range of $\boldsymbol{f(x)}$.**
   - ○ Inadequate information provided for finding the domain of $f(x)$.

Solution:
Since $f_1(\alpha) = f_2(\alpha)$ and $f_1(\beta) = f_2(\beta)$, we have α and β are the abscissa of intersection points of both the curves.
To find the intersection points of the curves represented by $f_1(x)$ and $f_2(x)$:

$$f_1(x) = f_2(x)$$
$$-x^2 - 6x = x^2 + 6x + 10$$
$$2x^2 + 12x + 10 = 0$$

Here,

$$a = 2, b = 12, \text{ and } c = 10$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-12 \pm \sqrt{12^2 - 4 \times 2 \times 10}}{2 \times 2}$$

$$x = \frac{-12 \pm 8}{4} = -1, -5$$

Therefore,

$$\alpha = -5 \text{ and } \beta = -1.$$

Since the Domain of $\boldsymbol{f(x)}$ is $\boldsymbol{[\alpha, \beta]}$ domain of $\boldsymbol{f(x)}$ is $\boldsymbol{[-5, -1]}$.

The figure below gives a rough pictorial representation of $f_1(x)$ and $f_2(x)$ (drawn with smooth lines).
$f(x)$ can have any shape. An example is shown in the figure (drawn with dashed lines) for $f(x)$.

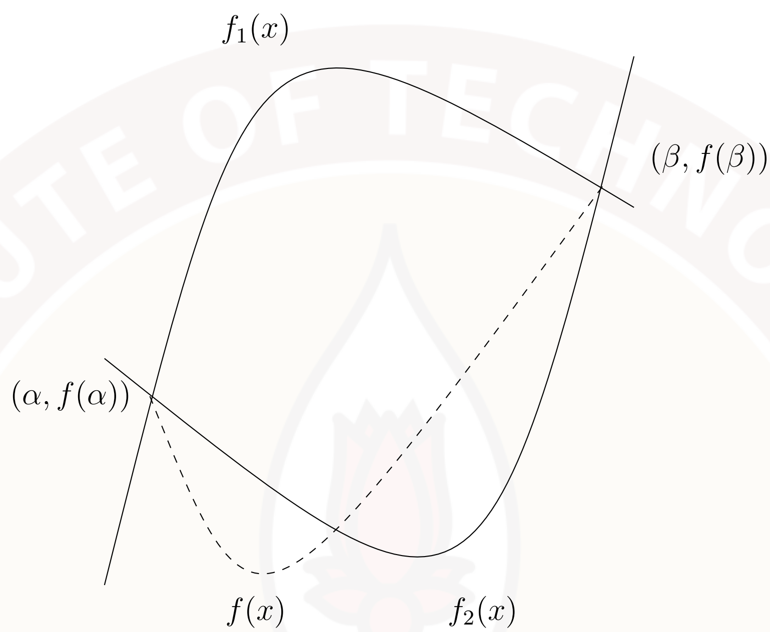


Figure M1W5PAS-4.1

As it is clear from figure that we do not know the minimum and maximum value of $f(x)$, we do not have proper data to comment on the range.

5. If $f(x) = 2x^2 + (5+k)x + 7$, $g(x) = 5x^2 + (3+k)x + 1$, $h_1(x) = f(x) - g(x)$, and $h_2(x) = g(x) - f(x)$, then choose the set of correct options.

- ◯ **Roots for $\boldsymbol{h_1(x) = 0}$ and roots for $\boldsymbol{h_2(x) = 0}$ are real, distinct, and the roots are the same for $\boldsymbol{h_1(x) = 0}$ and $\boldsymbol{h_2(x) = 0}$.**
- ◯ Roots for $h_1(x) = 0$ and roots for $h_2(x) = 0$ are real and distinct but the roots are not the same for $h_1(x) = 0$ and $h_2(x) = 0$.
- ◯ **Sum of roots of quadratic equation $\boldsymbol{h_1(x) = 0}$ will be $\boldsymbol{\frac{2}{3}}$.**
- ◯ **Product of roots of quadratic equation $\boldsymbol{h_2(x) = 0}$ will be -2.**
- ◯ **Axis of symmetry for both the functions $\boldsymbol{h_1(x)}$ and $\boldsymbol{h_2(x)}$ will be the same.**
- ◯ Vertex for both the functions $h_1(x)$ and $h_2(x)$ will be the same.

Solution:
Given that,

$$h_1(x) = f(x) - g(x)$$
$$h_1(x) = -(g(x) - f(x))$$
$$h_1(x) = -h_2(x)$$

Negative sign before any function does not make any changes on zeros of the function. Therefore, roots of $h_1(x) = 0$ and roots of $h_2(x) = 0$ will be same.

Now, for the properties of $h_1(x)$:

$$h_1(x) = f(x) - g(x) = 2x^2 + (5+k)x + 7 - (5x^2 + (3+k)x + 1)$$
$$h_1(x) = -3x^2 + 2x + 6$$
$$d = 2^2 - 4(-3) \times 6 > 0$$

It means the roots of $h_1(x)$ are real and distinct.
The roots of $h_1(x) = 0$ has the same as the roots of $h_2(x) = 0$, which means the roots for $h_2(x) = 0$ will also be real and distinct.

Sum of the roots of $h_1(x) = -3x^2 + 2x + 6$ will be $-\frac{b}{a} = -\frac{2}{(-3)} = \frac{2}{3}$.

Product of the roots of $h_1(x) = -3x^2 + 2x + 6$ will be $\frac{c}{a} = \frac{6}{(-3)} = -2$.

Multiplying a quadratic function by the minus sign does not make any changes in the

axis of symmetry.
The answer to all the above questions can be seen in the given figure.

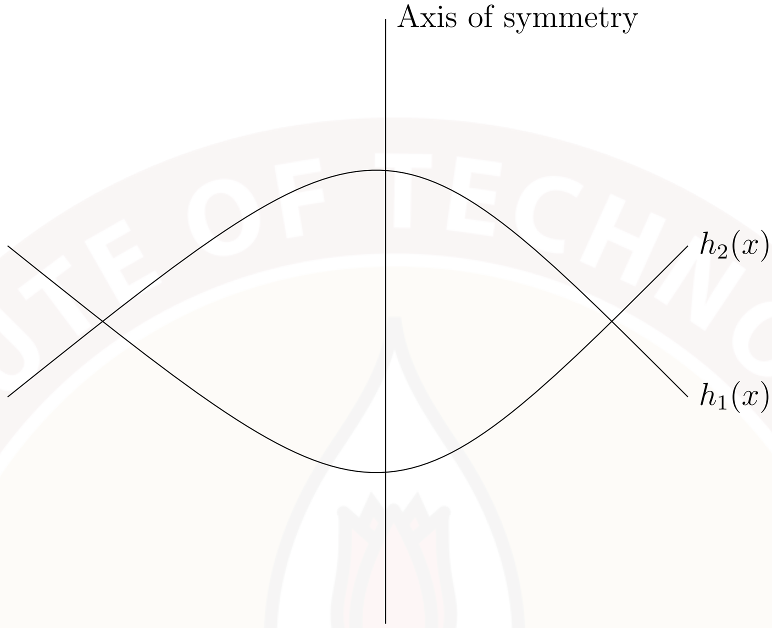


Figure M1W5PAS-5.1

Use following information for questions 6-8.

Vaishali wants to set up a small plate making machine in her village. Table P-5.1 shows the different costs involved in making the plates. Figure 5 shows her survey regarding the demand (number of packets of the plate) versus selling price of plate per packet (in ₹) per day.

Cost type	Cost
Electricity	₹1.5 per packet
Miscellaneous	₹6.5 per packet
Raw material	₹10 per packet

Table P-5.1

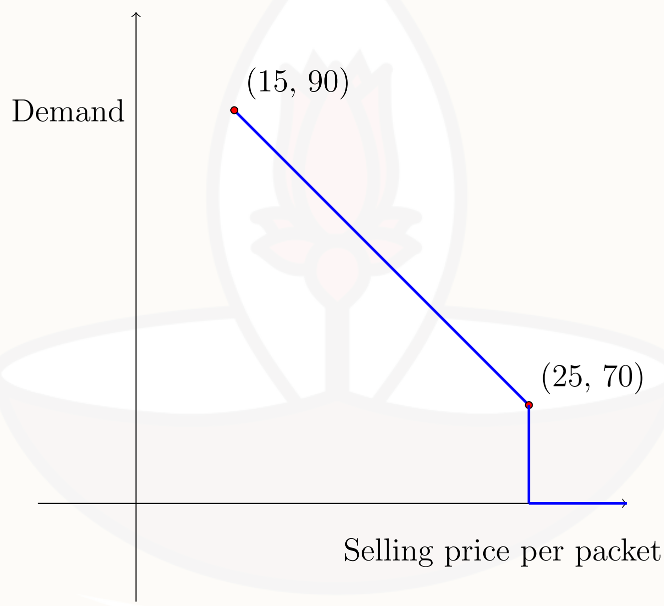


Figure PA-5.2

6. Choose the correct option which shows the profit obtained by Vaishali per day. Here, x is the selling price per packet.

   - ◯ $2(60 - x)$
   - ◯ $x(x - 18)$
   - ◯ $\mathbf{2(x - 18)(60 - x)}$
   - ◯ $2(x + 18)(60 - x)$
   - ◯ Inadequate information.

Solution:
From the figure, it is clear that the demand is dependent on the selling price of plates. Let y be the demand of the numbers of packets, then from two-points form of a line,

$$y - 90 = \frac{70 - 90}{25 - 15}(x - 15)$$
$$y - 90 = -2(x - 15)$$
$$y = -2x + 120$$

From the table, total cost per packet (in ₹)$= 1.5 + 6.5 + 10 = 18$

Per day profit = Demand per day$\times$(Selling price per packet - Cost per packet)

$$Profit = y(x - 18)$$
$$Profit = (-2x + 120)(x - 18)$$
$$Profit = 2(x - 18)(60 - x)$$

7. Choose the set of correct options.

   - ◯ **Vaishali should sell a packet with a minimum price of ₹18 so as not to incur any loss.**
   - ◯ Vaishali should sell a packet with a minimum price of ₹12 so as not to incur any loss.
   - ◯ To make maximum profit per day, the selling price per packet should be ₹39.
   - ◯ **To make maximum profit per day, the selling price per packet should be ₹25.**
   - ◯ Vaishali should sell a packet with maximum price of ₹60 so as not to incur any loss.
   - ◯ **Vaishali should sell a packet with a maximum price of ₹25 so as not to incur any loss.**

Solution:
From question 6,

$$Profit = 2(x-18)(60-x)$$

$$Profit = -2x^2 + 156x - 2160 \tag{1}$$

To get minimum selling price with no loss, profit should be zero. Therefore,

$$2(x-18)(60-x) = 0$$
$$x = 18 \text{ or } 60$$

From the graph given in question, it is clear that we can not sell a packet at ₹60, because the demand will be zero.
Therefore, the minimum selling price will be ₹18 per packet.

Since the profit is a quadratic function of the selling price (x) in equation (1) with negative coefficient of x^2.
Therefore, the maximum profit will occur at

$$x = -\frac{b}{2a} = -\frac{156}{2 \times (-2)} = 39$$

A rough pictorial representation is shown in Figure below,

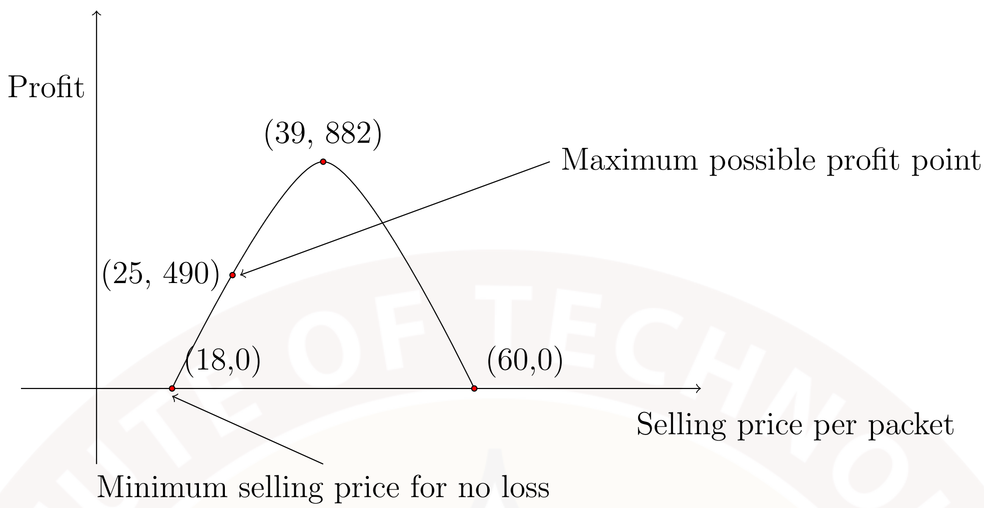


Figure M1W5PAS-7.1

The increase in selling price will result in profit increment till 39. But the maximum acceptable selling price is ₹25, therefore the maximum profit will occur at a selling price of ₹25.

So, from the figure it is clear that the maximum selling price for no loss is ₹60 but we can not increase the price beyond ₹25. Therefore, the maximum profit to incur any loss will be ₹25.

3 Numerical Answer type (NAT):

8. What should be the price of plate per packet (₹) to make a profit of ₹490 per day?
[Hint: $(x-53)$ a factor of $2(-x^2+78x-1325)$.] [Ans: 25]

Solution:
From equation (1) $Profit = -2x^2 + 156x - 2160$

$$-2x^2 + 156x - 2160 = 490$$
$$-2x^2 + 156x - 2650 = 0$$
$$2(-x^2 + 78x - 1325) = 0$$

It is given that $(x-53)$ a factor of $2(-x^2+78x-1325)$. So dividing $2(-x^2+78x-1325)$ by $(x-53)$ we will get $-2x+50$.
Therefore,

$$2(-x^2 + 78x - 1325) = 0$$

$$(x-53)(2x-50) = 0$$

If

$$x - 53 = 0$$

$$x = 53$$

But selling price can not go beyond 25.
Now if,

$$2x - 50 = 0$$
$$x = \frac{50}{2}$$
$$x = 25$$

Therefore, the selling price of plate should be ₹25 .

9. What will be the value of $m+n$ if the sum of the roots and the product of the roots of equation $(5m+5)x^2 - (4n+3)x + 10 = 0$ are 3 and 2 respectively?

Solution:

We know that the sum of the roots of an equation $ax^2+bx+c=0$ is $\frac{-b}{a}$ and the product of its roots is $\frac{c}{a}$.

Here, $a = 5m+5, b = -(4n+3), c = 10$. Substituting these values we get,
The product of the roots of the given equation

$$\frac{c}{a} = \frac{10}{5m+5} = 2$$
$$5m+5 = 5$$
$$m+1 = 1$$
$$\boldsymbol{m = 0}$$

The sum of the roots as

$$\frac{-b}{a} = \frac{-(-(4n+3))}{5m+5} = 3$$
$$4n+3 = 3(5m+5)$$

For $m = 0$

$$4n+3 = 3\times 5$$
$$4n = 12$$
$$\boldsymbol{n = 3}$$

Therefore,

$$m+n = 0+3 = 3.$$

10. What will the sum of two positive integers be if the sum of their squares is 369 and the difference between them is 3?. **Solution:**

Let a and b be the two positive integers. Given that

$$a^2 + b^2 = 369 \tag{2}$$
$$a - b = 3 \tag{3}$$

Squaring equation (3) on both sides, we get

$$\begin{aligned}(a-b)^2 &= 3^2\\ a^2 - 2ab + b^2 &= 9\\ 369 - 2ab &= 9\\ 2ab &= 369 - 9\end{aligned}$$

$$2ab = 360$$

Now, to find the sum of the integers

$$\begin{aligned}(a+b)^2 &= a^2 + 2ab + b^2\\ (a+b)^2 &= 369 + 360\\ (a+b)^2 &= 729\\ a+b &= \pm\sqrt{729} = \pm 27\end{aligned}$$

As a and b are positive integers, their sum should also be a positive integer.
Therefore, $\boldsymbol{a + b = 27}$.

Week - 6
Practice Assignment
Algebra of polynomials
Mathematics for Data Science - 1

Syllabus covered:

- Addition
- Subtraction
- Multiplication
- Division

1 Multiple Choice Questions (MCQ):

1. Let x be the number of years since the year 2000 (i.e., $x = 0$ denotes the year 2000). The total amount of profit (in ₹) on books in a shop is given by the function $T(x) = 5x^3 + 3x + 1$. The shop sells books of four languages English, Bengali, Hindi, and Tamil. The profits from selling English and Bengali books are given by $E(x) = 3x^3 - 5x^2 + x$ and $B(x) = x^2 + 4x + 5$ respectively. The profit from selling Hindi and Tamil books are found to be the same.

(a) Which of the following polynomial functions represents the profit from selling Tamil books?

○ $2x^3 + 4x^2 - 2x - 4$
○ $x^3 - 2x^2 - x + 2$
○ $\boldsymbol{x^3 + 2x^2 - x - 2}$
○ $2x^3 - 4x^2 - 2x + 4$

(b) In which year was the profit from Hindi books zero?

○ **2001**
○ 2002
○ 2004
○ 2010

Solution:

(a) The total profit from selling English and Bengali books is= $E(x) + B(x) = (3x^3 - 5x^2 + x) + (x^2 + 4x + 5) = 3x^3 - 4x^2 + 5x + 5$. Hence the total profit from selling Hindi and Tamil books is= $T(x) - (3x^3 - 4x^2 + 5x + 5) = 5x^3 + 3x + 1 - 3x^3 + 4x^2 - 5x - 5 = 2x^3 + 4x^2 - 2x - 4$.
As the profit from selling Hindi and Tamil books are found to be the same, the profit from selling Tamil books is= $\frac{1}{2}(2x^3 + 4x^2 - 2x - 4) = x^3 + 2x^2 - x - 2$

(b) Profit from selling Hindi books (which is same as the profit from selling Tamil books) is $x^3 + 2x^2 - x - 2$.

$$x^3 + 2x^2 - x - 2 = x^2(x+2) - 1(x+2) = (x+2)(x^2-1) = (x+2)(x+1)(x-1)$$

So the profit will be zero if $(x+2)(x+1)(x-1) = 0$, i.e., at $x = -2, -1, 1$ the profit can be 0. But in this context, x cannot be negative. So $x = 1$ is the only possibility. Hence in the year 2001 the profit from Hindi books was zero.

2. Find the quadratic polynomial which when divided by x, $x-1$, and $x+1$ gives the remainders 7, 14, and 8 respectively.

   - ◯ $4x^2 - 3x + 7$
   - ◯ $x^2 + 7x + 7$
   - ◯ $7x^2 + x + 7$
   - ◯ $\mathbf{4x^2 + 3x + 7}$

   Solution: Let the quadratic polynomial which is satisfying the given condition be $p(x) = ax^2 + bx + c$.
   When it is divided by x the remainder is 7. It implies that if we substitute $x = 0$ in $p(x)$ we will get 7, i.e., $p(0) = 7$. Similarly we have $p(1) = 14$ and $p(-1) = 8$.
   Hence we have the following equations:

$$\begin{aligned} p(0) &= a(0)^2 + b(0) + c \\ &= c \\ &= 7 \\ p(1) &= a.(1)^2 + b.1 + c \\ &= a + b + c \\ &= 14 \\ p(-1) &= a(-1)^2 + b(-1) + c \\ &= a - b + c \\ &= 8 \end{aligned}$$

   So, we have $c = 7$, and substituting c in the second and third equation we get, $a + b = 7$, and $a - b = 1$. By solving these two equations we get $a = 4$ and $b = 3$.
   Hence the quadratic polynomial is $4x^2 + 3x + 7$.

3. Box A has length x unit, breadth $(x+1)$ unit, and height $(x+2)$ unit. Box B has length $(x+1)$ unit, breadth $(x+1)$ unit, and height $(x+2)$ unit. There are two more boxes C and D of cubic shape with side x unit. The total volume of A and B is y cubic unit more than the total volume of C and D. Find y in terms of x.

   - ◯ $x^3 + 7x^2 + 7x + 2$
   - ◯ $\mathbf{7x^2 + 7x + 2}$
   - ◯ $7x^2 - 7x - 2$
   - ◯ $x^3 + 7x^2 - 7x - 2$

   Solution: The volume of box A is $x(x+1)(x+2) = x^3 + 3x^2 + 2x$ cubic unit.
   The volume of box B is $(x+1)(x+1)(x+2) = (x^2+2x+1)(x+2) = x^3+4x^2+5x+2$ cubic unit.
   The volume of box C and D is x^3 cubic unit each. So the total volume of A and B is $2x^3 + 7x^2 + 7x + 2$ and the total volume of C and D is $2x^3$.
   Hence $y = (2x^3 + 7x^2 + 7x + 2) - 2x^3 = 7x^2 + 7x + 2$.

4. The population of a bacteria culture in laboratory conditions is known to be a function of time of the form $p(t) = at^5 + bt^2 + c$, where p represents the population (in lakhs) and t represents the time (in minutes). Suppose a student conducts an experiment to determine the coefficients a, b, and c in the formula and obtains the following data:

   - $p(0) = 3$
   - $p(1) = 5$
   - $p(2) = 39$

   Which of the following options is correct?

   - ○ $p(t) = 3t^5 - t^2 + 3$
   - ○ $p(t) = 4t^5 - 2t^2 + 3$
   - ○ $\boldsymbol{p(t) = t^5 + t^2 + 3}$
   - ○ $p(t) = 39t^5 + 5t^2 + 3$

   Solution: Given that, $p(t) = at^5 + bt^2 + c$.
   $p(0) = c = 3$
   $p(1) = a + b + c = 5$, putting $c = 3$, we get $a + b = 2$.
   $p(2) = a(2)^5 + b(2)^2 + c = 32a + 4b + c = 39$, substituting $c = 3$, we get $32a + 4b = 36$, implies, $8a + b = 9$ (cancelling 4 from both sides)
   By solving these two equations we get $a = 1$, and $b = 1$.
   Hence, $p(t) = t^5 + t^2 + 3$.

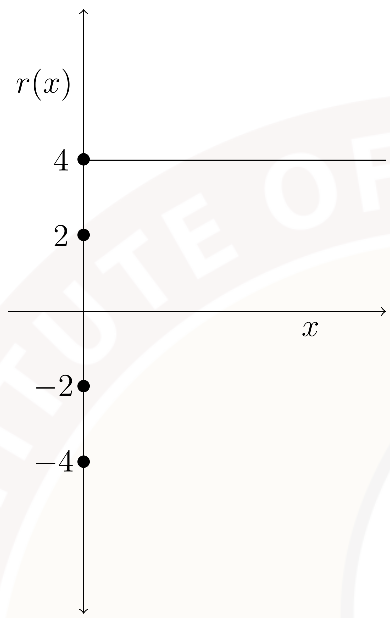
5. If the polynomials $x^3 + ax^2 + 5x + 7$ and $x^3 + 2x^2 + 3x + 2a$ leave the same remainder when divided by $(x - 2)$, then the value of a is:

- ◯ $\frac{3}{2}$
- ◯ $-\boldsymbol{\frac{3}{2}}$
- ◯ $\frac{5}{2}$
- ◯ $-\frac{5}{2}$

Solution: Given that both the polynomials leave same remainder when divided by $(x - 2)$. By substituting $x = 2$ both the polynomial should have same value.
By substituting $x = 2$ in $x^3 + ax^2 + 5x + 7$, we get $8 + 4a + 10 + 7 = 4a + 25$.
By substituting $x = 2$ in $x^3 + 2x^2 + 3x + 2a$, we get $8 + 8 + 6 + 2a = 2a + 22$.
So we have,

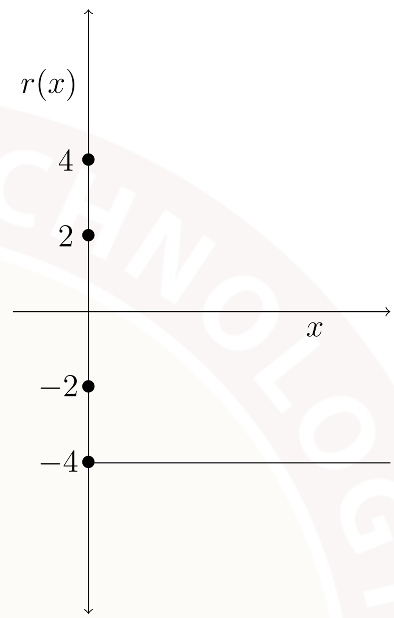
$$4a + 25 = 2a + 22$$
$$2a = -3$$
$$a = -\frac{3}{2}$$

6. Let $r(x)$ be a polynomial function which is obtained as the remainder after dividing the polynomial $2x^3 + x^2 - 5$ by the polynomial $2x - 3$. Choose the correct option which represents the polynomial $r(x)$ most appropriately.



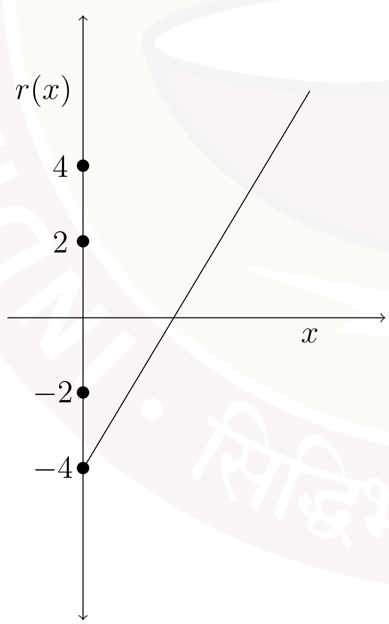
◯ **Option A**

Fig P-6.2



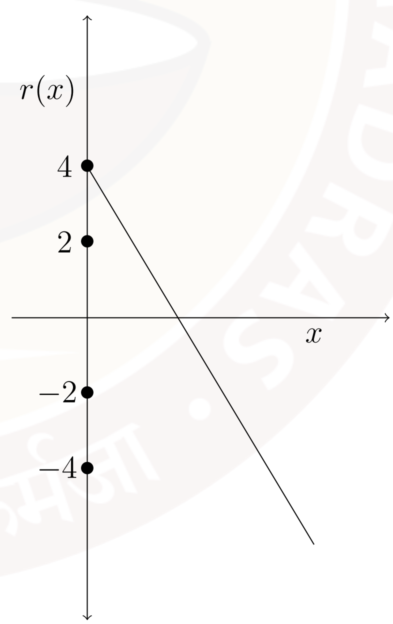
◯ Option B

Fig P-6.3



◯ Option C

Fig P-6.4



◯ Option D

Fig P-6.5

Solution We get 4 as the remainder if $2x^3 + x^2 - 5$ is divided by the polynomial $2x - 3$.

$$2x^3 + x^2 - 5 = (2x - 3)(x^2 + 2x + 3) + 4$$

Hence $r(x) = 4$, which is a constant polynomial. Hence, the first option is the correct.

2 Multiple Select Questions (MSQ):

7. By dividing a polynomial $p(x)$ with another polynomial $q(x)$ we get $h(x)$ as the quotient and $r(x)$ as the remainder.

(a) The maximum degree of $r(x)$ can be,

- ◯ $deg\ p(x)$
- ◯ $deg\ (p(x)) - 1$
- ◯ $deg\ q(x)$
- ◯ $\boldsymbol{deg\ (q(x)) - 1}$

(b) If $deg\ p(x) < deg\ q(x)$, then choose the set of correct answers:

- ◯ $\boldsymbol{h(x) = 0}$
- ◯ $deg\ h(x) = deg\ q(x)$
- ◯ $deg\ r(x) = deg\ q(x)$
- ◯ $\boldsymbol{deg\ r(x) = deg\ p(x)}$

Solution:

(a) The degree of the remainder $r(x)$ should be strictly less than the degree of the polynomial $q(x)$. So the maximum degree of $r(x)$ is $deg\ (q(x)) - 1$.

(b) If $deg\ p(x) < deg\ q(x)$, then quotient will be zero polynomial, hence $deg\ h(x) = 0$. The remainder will be $p(x)$ itself. So $deg\ r(x) = deg\ p(x)$.

3 Numerical Answer Type (NAT):

8. An open box can be made from a piece of cardboard of length $7x$ unit and breadth $5x$ unit, by cutting squares of side x unit out of the corners of the rectangular cardboard, then folding up the sides as shown in the Figure P-6.1.

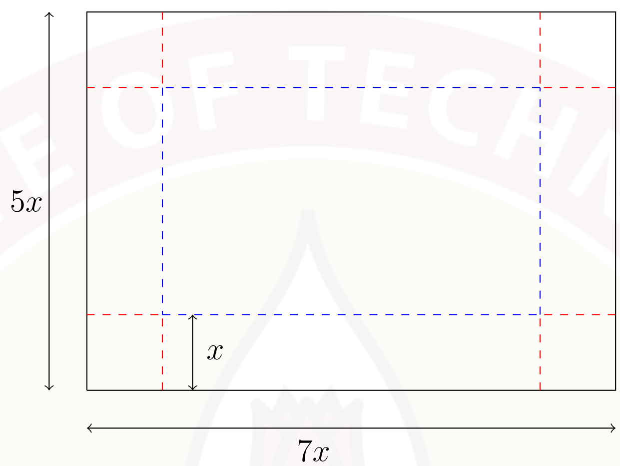


Figure P-6.1

(a) What will be the coefficient of x^3 in the polynomial representing the volume of the box? [Answer:15]

(b) What will be the coefficient of x^2 in the polynomial representing the volume of the box? [Answer:0]

Solution: As the sides of the piece of the cardboard has been cut out, the length of the box made will be $7x - (x + x) = 5x$ unit and the breadth of the box made will be $5x - (x + x) = 3x$ unit, and the height will be x unit.
Hence the volume of the box will be $5x \times 3x \times x = 15x^3$ cubic unit.

(a) The coefficient of x^3 in the polynomial representing the volume of the box is 15.

(b) The coefficient of x^2 in the polynomial representing the volume of the box is 0.

Week - 7
Practice Assignment
Graphs of polynomials
Mathematics for Data Science - 1

NOTE:
There are some questions which have functions with discrete valued domains (such as month or year). For simplicity, we treat them as continuous functions.

Syllabus Covered:

- Graphs of Polynomials: Identification and Characterization
- Zeroes of Polynomial Functions
- Graphs of Polynomials: Multiplicities
- Graphs of Polynomials: Behavior at X-intercepts
- Graphs of Polynomials: End behavior
- Graphs of Polynomials: Turning points
- Graphs of Polynomials: Graphing & Polynomial creation

1 Multiple Select Questions (MSQ):

1. Figure: M1W7PA-7.1 shows the graph of polynomial $p(x)$. Choose the set of correct options.

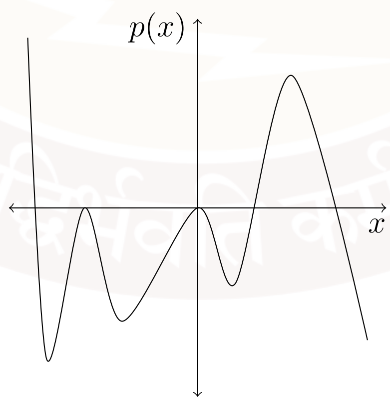


Figure: M1W7PA-7.1

- ◯ The degree of $p(x)$ is minimum 5.
- ◯ **The degree of $p(x)$ is minimum 7**.
- ◯ **x^4 could be a factor of $p(x)$**.
- ◯ $p(x)$ is an odd function.
- ◯ Multiplicity of a positive root of $p(x)$ can be even.
- ◯ **Multiplicities of zero and at least one negative root could be the same**.

Solution:

Option (b): Correct
Let a_1, a_2, a_3, a_4, and a_5 be the points at which the value of $p(x) = 0$ are as shown in Figure: M1W7PAS-7.1. At points a_1, a_4, a_5, the curve crosses in a linear fashion hence the degree should be 1, which accounts for total 3 degrees. At points a_2 and a_3, the curve bounces back, therefore it can have at least 2 degrees each, which accounts for 4 degrees together.
Therefore all together the degree of $p(x)$ is minimum 7.

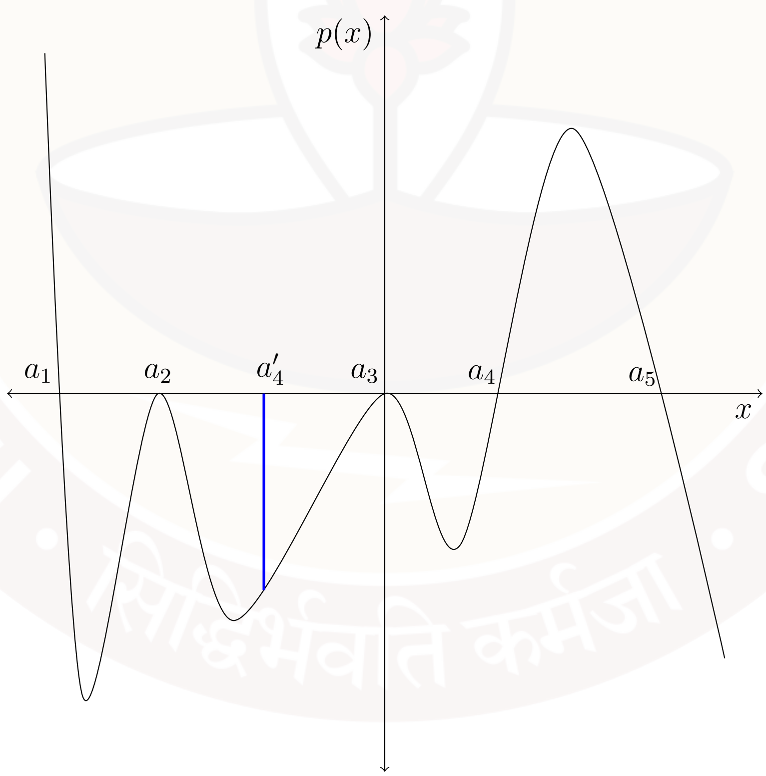


Figure: M1W7PAS-7.1

Option (c): Correct
Since at $x = 0$, the curve bounces back therefore, at point $(0, 0)$ the factor will be of the form x^n where n is an even number. Hence when $n = 4$, x^4 could be a factor of $p(x)$.

Option (d): Incorrect
A function is odd when $f(-x) = -f(x)$ for all $x \in \mathbb{R}$ which means the graph is symmetric with respect to the origin. But it is not the case as in Figure: M1W7PAS-7.1. Therefore, $p(x)$ is not an odd function.

Option (e): Incorrect
Multiplicities of a positive root of $p(x)$ cannot be even because at points a_4 and a_5 the curve crosses in a linear fashion hence the multiplicity should be 1.

Option (f): Correct
At a_2 the root is negative and at a_3 it has zero root and the curve bounces back at both point. Therefore at those points the factor will be of the form x^n where n is an even natural number and they can be same.

2. Choose the correct options.

- ◯ Every function must be either an odd function or an even function.
- ◯ **A function is an even function if $f(x) = f(|x|)$.**
- ◯ **$f(x) = 0$, for all $x \in \mathbb{R}$, is an even function.**
- ◯ Every even degree polynomial is an even function.

Solution:

Option (a): Incorrect
Some functions could be neither odd nor even. For example, $f(x) = x^3 + x^2$ then $f(-x) = -x^3 + x^2$.
It is not an even function because $f(-x) \neq f(x)$ and not an odd function because $f(-x) \neq -f(x)$.

Option (b): Correct
Given, $f(x) = f(|x|) \implies f(x) = f(x)$ or $f(x) = f(-x)$.
Function is an even function when $f(x) = f(-x)$.

Option (c): Correct
As $f(x) = 0$, $f(-x) = 0$ and $f(x) = f(-x)$ therefore it is an even function.

Option (d): Incorrect
Because it may have other terms which will influence the nature of the function.
Example:$f(x) = x^4 + x^3$
It is an example of even degree polynomial but it is not even function because $f(-x) \neq f(x)$ where $f(-x) = x^4 - x^3$

3. The polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_0$ has the following properties:

- $p(x)$ is an even degree polynomial.
- $p(x)$ has at least one positive real root and at least one negative real root.
- $(x-2)^n,\ max(n) = 2$ is a factor of $p(x)$.
- $p(0) \neq 0$

From the options given, choose the the possible representations of $p(x)$.
[Ans: Options C, E]

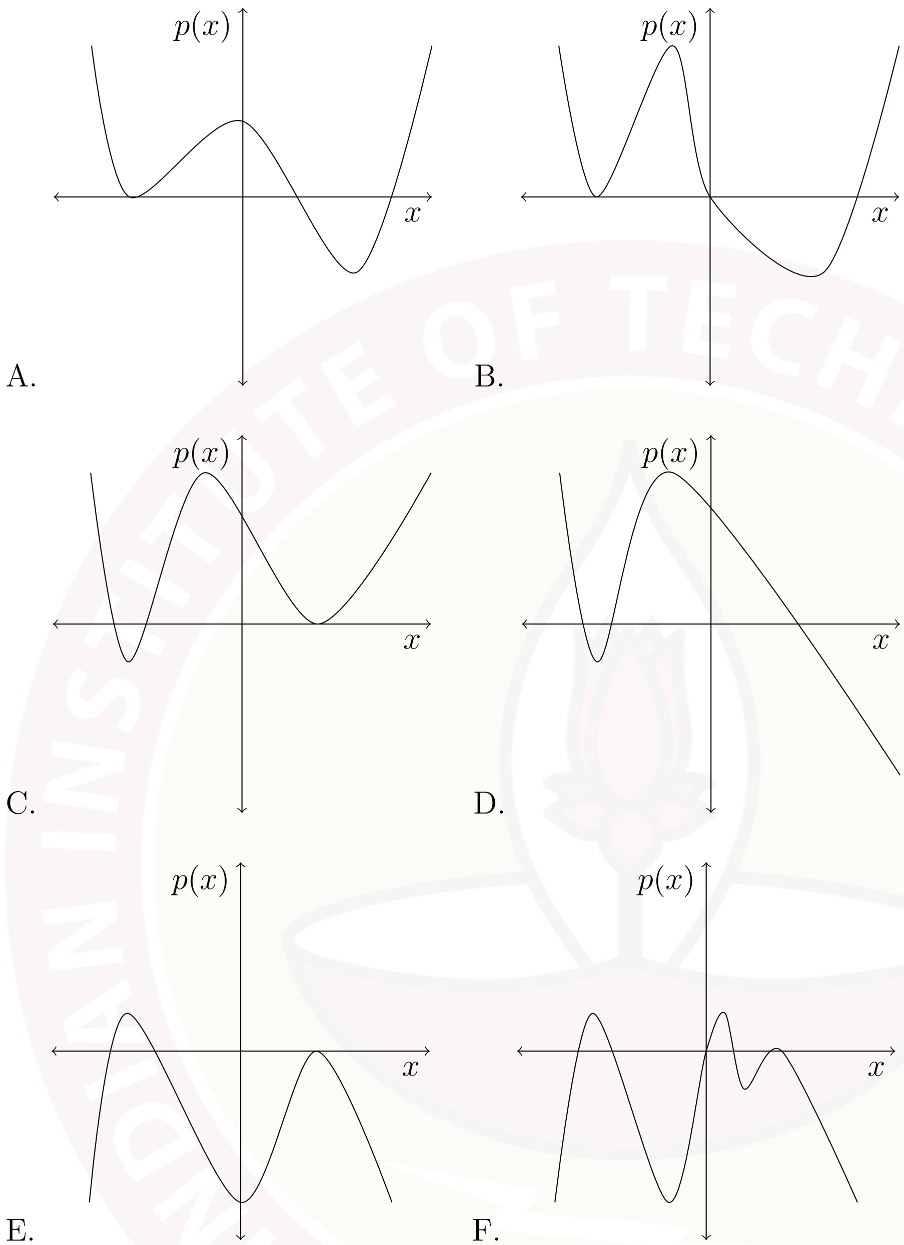


Figure: M1W7PA-7.2

Solution:

According to the condition $(x-2)^n$, $max(n) = 2$, options C, E and F qualify to be right options. But option F does not fulfil the condition $p(0) \neq 0$ condition. Hence options C and E are correct.

2 Multiple Choice Questions (MCQ):

4. Suppose a cubic polynomial f intersects the X-axis at $x = 1$ and $x = -2$. Moreover, $f(x) < 0$ when $x \in (0, 1)$, and $f(x) > 0$ when $x \in (-2, 0)$. Find out the equation of the polynomial.

   ◯ Inadequate information.

   ◯ $a(x^3 - x^2 - 2x),\ a > 0$

   ◯ $\boldsymbol{a(x^3 + x^2 - 2x),\ a > 0}$

   ◯ $a(x^3 + 3x^2 - x - 3),\ a < 0$

   Solution:

   A cubic polynomial can have at most three roots. It is given that f intersects the X-axis at $x = 1$ and $x = -2$ which accounts for the two factors $(x - 1)$ and $(x - (-2))$. Therefore, the equation is of the form $f(x) = a(x - b)(x - 1)(x + 2)$ where a and b are constants.
   Based on the end behavior of $f(x)$ two possible rough diagrams are shown in Figure: M1W7PAS-7.2.

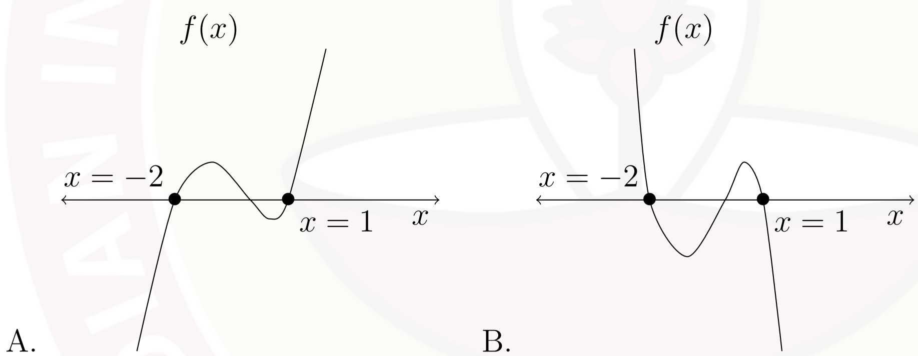


Figure: M1W7PAS-7.2

   Also $f(x) < 0$ when $x \in (0, 1)$, $f(x) > 0$ when $x \in (-2, 0)$ so only A in Figure: M1W7PAS-7.2 can represent the function. When $x \to \infty$, $f(x) = \infty$, and $x \to -\infty$, $f(x) = -\infty$, shows that $a > 0$.

   Clearly, the function is changing the sign at $x = 0$ which means $(x - 0)$ is a factor of $f(x)$ which means $b = 0$. Therefore,
   $f(x) = a(x)(x - 1)(x + 2) \implies \boldsymbol{f(x) = a(x^3 + x^2 - 2x).}$

5. The volume of a box V, varies with some variable x as $V(x) = x^3 + 12x^2 + 39x + 28$ cubic metres. If $(x + a)$ metre is the measurement of one side of the box, then choose the correct option for a.

   - ◯ $a = -1, 5, 3$
   - ◯ $a = 1, 5, 3$
   - ◯ $a = -7, -4, 1$
   - ◯ $a = 7, 2, 2$
   - ◯ $\boldsymbol{a = 7, 1, 4}$
   - ◯ $a = 28, 1, 1$

Solution:

We know that the volume of a box is determined by multiplying the lengths of sides of the box. If $(x + a)$ is the measurement of one side, then it will be a factor of the volume polynomial.
By hit and trial method, one of the roots of $V(x) = x^3 + 12x^2 + 39x + 28$ is -1. Hence $(x + 1)$ is one of the factors of the cubic polynomial $V(x)$. On dividing $V(x)$ by $(x + 1)$, we get $x^2 + 11x + 28$ which on factorization gives the other factors $(x + 7)$ and $(x + 4)$. Thus the possible values of a are 1, 4, and 7.

Use the following information for questions 6 and 7.
Ankita has to travel to various locations for advertising her company's products. The company reimburses her expenses such as accommodation, food etc. The company also blacklists an employee whenever the employee's expenditure in a given month exceeds ₹ 9000. The accounts department fits the data of Ankita's monthly expenditure to a polynomial $E(x)$ (in ₹) where x is the number of months since her joining the company. The polynomial fit is known to be applicable for a period of two years.

6. If $E(x) - 9000 = a(x-4.5)(x-12)(x-20),\ a > 0$, then how many times has Ankita been black listed in two years?

   - ◯ 7
   - ◯ 4
   - ◯ **11**
   - ◯ 3
   - ◯ 15
   - ◯ 9

   Solution:

   Company blacklists an employee whenever employee's expenditure in given month is more than 9000 which means if $E(x) > 9000 \implies E(x) - 9000 > 0$.
   On solving,

$$E(x) - 9000 > 0$$
$$a(x-4.5)(x-12)(x-20) > 0$$

   Clearly the zeros of the polynomial are 4.5, 12, and 20. The above polynomial is a cubic polynomial (odd degree polynomial) and $a > 0$, then using end behavior a rough plot is shown in Figure: M1W7PAS-7.3.

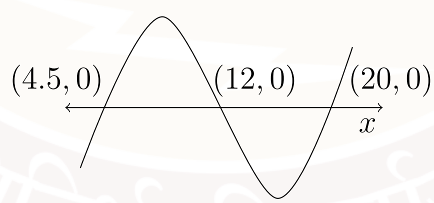


Figure: M1W7PAS-7.3

   From the Figure: M1W7PAS-7.3 clearly $E(x) - 9000 > 0$ when $x \in (4.5, 12)$ and $x \in (20, 24]$ i.e, when the number of months are $x = 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 21,\ 22,\ 23,$ and 24. Therefore, Ankita will be black listed 11 times in two years.

7. Choose the information that is not required to solve question number 6.

   - ◯ End behavior of $E(x)$.
   - ◯ Zeroes of the function.
   - ◯ Degree of the polynomial $E(x)$.
   - ◯ **Exact value of $\boldsymbol{a}$.**

   Solution:

   The end behavior of a polynomial function is the behavior of the graph of $E(x)$ as x approaches $+\infty$ or $-\infty$. So, the knowledge of sign of the leading coefficient is used to predict the end behavior of the function. In question number 6, the fact that $a > 0$ was crucial to solve the problem.

   Degree of the polynomial $E(x)$, is used to determine the maximum number of solutions it can have and also the number of times it will cross the $X-$axis when graphed. In question number 6 the degree is 3.

   Zeroes of the function are critical to determine where the function touches or crosses $X-$axis. In question number 6, three zeroes were given.

   Exact value of a in the question number 6 will simply increase the $y-$coordinate of the vertex value, which is not of concern in the above problem when $a > 0$. Thus this information is not required to solve question number 6.

8. An equipment shows the reading $y(x)$ upon applying load x (in tonnes). Starting from $x = 0$ tonne, the load is steadily increased and thus the reading $y(x)$ is also observed to increase. The first stage of failure is observed at a certain load x_1 after which increasing the load results in a decrease in the reading. The load is continually increased after the first stage of failure and the second stage of failure occurs at load x_2 where the reading reaches 80 and the equipment stops working.

   Use the information provided below and find the maximum load x_2 (in tonnes) that can be applied to this equipment so that it does not stop working.
   Useful information:

   (a) $y(x) - 80 = ax(bx^2 + cx + d)(x + 1)(x - 4)$

   (b) $c^2 - 4bd < 0$

   (c) $ab < 0$

   Choose the correct option.

   - ◯ 0
   - ◯ -1
   - ◯ **4**.
   - ◯ None of the above.

Solution:

As the reading y is a dependent function of x, it can not be less than zero as load x cannot be negative. Initially the equipment reading $y(x)$ increases as the load x increases but it starts to decrease as load increases after the 1^{st} failure i.e, when load $x = x_1$ and stops working when $y(x) = 80$. (see Figure: M1W7PAS-7.4).

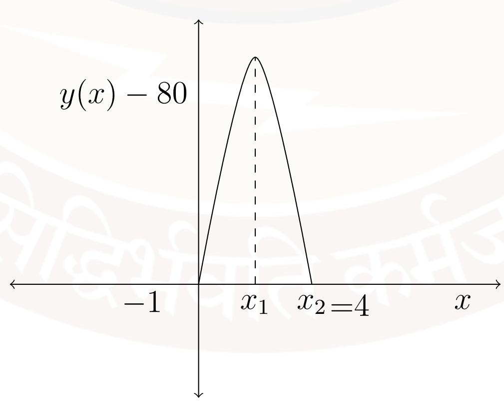


Figure: M1W7PAS-7.4

So the equipment works when $y(x) \geq 80 \implies y(x) - 80 \geq 0$.
On solving,

$$y(x) - 80 \geq 0$$
$$ax(bx^2 + cx + d)(x + 1)(x - 4) \geq 0$$

Since the polynomial is of degree 5, it could have at most 5 roots.
Finding the roots,

$$x = 0$$
$$x = -1$$
$$x = 4$$

For other two roots,

$$bx^2 + cx + d = 0$$

Given that $c^2 - 4bd < 0$ which indicates $bx^2 + cx + d$ has no real root. The curve represented by $y(x) - 80$ is shown in the Figure: M1W7PAS-7.4. Clearly, $y(x) - 80 \geq 0$ when $x \leq 4$. Thus maximum load will be $x_2 = 4$ (in tonnes) that can be applied to this equipment so that it does not stop working.

3 Numerical Type Questions (NAT):

9. Let A be the interval $[\alpha, \beta]$, where α and β are the smallest and the largest roots respectively of $f_1(x) = x^4 - 3x^3 - 9x^2 - 3x - 10$. If B is the largest proper subset of A such that elements of B are integers, then what is the cardinality of B? [Ans: 8]

Solution:

By hit and trial method, one of the roots is -2, thus the factor $(x + 2)$ when divides $f_1(x)$ gives $x^3 - 5x^2 + x - 5$. Now we can write it as $x^2(x - 5) + 1(x - 5)$, thus we get $(x - 5)(x^2 + 1)$.
$(x^2 + 1)$ has no real root, therefore the interval of A is all real values in $[-2,\ 5]$. Since B is the largest proper subset of A and it contains only integers, thus the elements of B will be -2, -1, 0, 1, 2, 3, 4, 5 and its cardinality is 8.

10. A train follows a path along the curve $y = x^3 + 12x^2 + 3x$ and Riya is travelling on a path $y = 0$. How many places can Riya catch the train? [Ans: 3]

Solution:

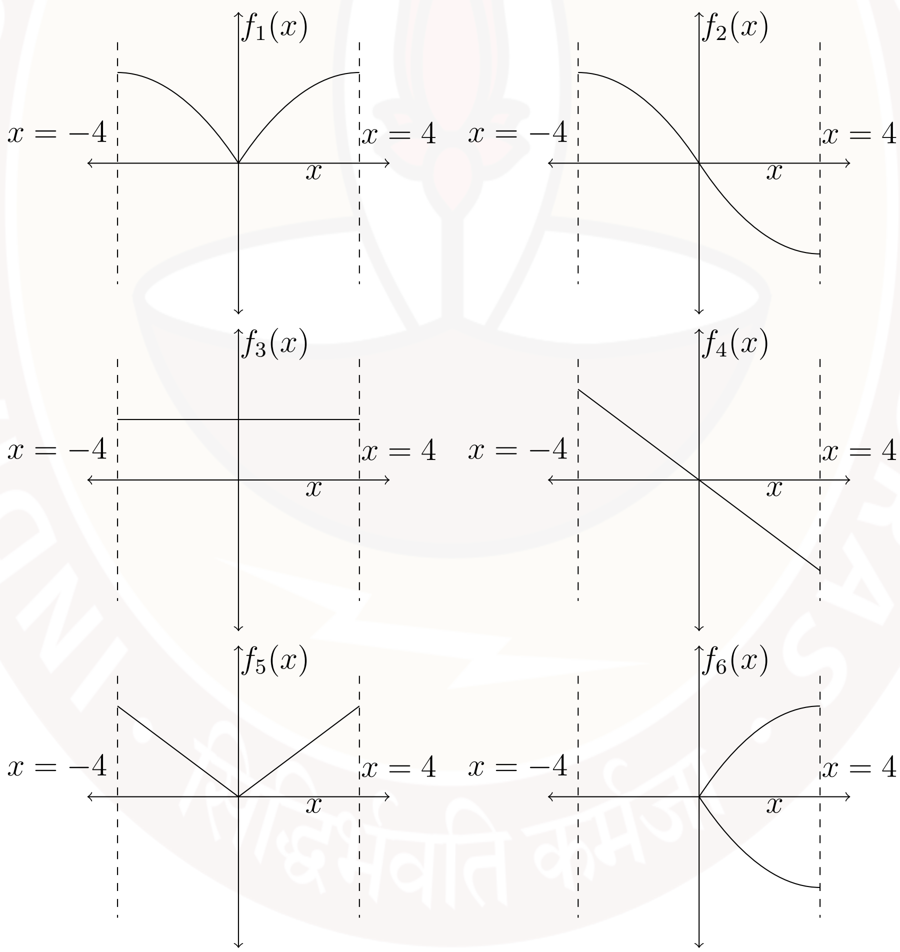
Riya takes the path $y = 0$ which means she is travelling along the $X-$axis. So, Riya can catch the train at $x-$intercepts of the curve $y = x^3 + 12x^2 + 3x$. On factorizing we get $x(x^2 + 12x + 3)$. So x is one factor and other factors can be obtained from $x^2 + 12x + 3$. The discriminant of $x^2 + 12x + 3$ is given by $b^2 - 4ac = 12 \times 12 - 4 \times 1 \times 3 = 132 > 0$, thus it will have 2 real and distinct roots different from 0. So altogether Riya can catch the train at 3 places.

Week - 8
Practice assignment Solution
Exponential Functions
Mathematics for Data Science - 1

NOTE:
There are some questions which have functions with discrete valued domains (such as month or year). For simplicity, we treat them as continuous functions.

1 Multiple Choice Questions (MCQ):

Answer the questions 1, 2, and 3 based on the given graphs.



Domain for each one is $[-4, 4]$.

1. Choose the correct option.
   - ◯ f_3 is not a function.
   - ◯ **f_6 is not a function.**
   - ◯ f_5 is not a function.
   - ◯ All of the above are functions.

   Solution:

   Vertical line test fails only for f_6 and therefore $f_6(x)$ is not a function.

2. Choose the correct option.
   - ◯ f_1 and f_3 are one-one functions in the given domain.
   - ◯ **f_2 and f_4 are one-one functions in the given domain.**
   - ◯ f_3 and f_5 are one-one functions in the given domain.
   - ◯ f_5 is one-one function in the given domain.

   Solution:

   The function f_2 and f_4 are strictly decreasing function in the domain [-4, 4], therefore these are one to one functions.

   Or

   The functions f_2 and f_4 are the only functions which satisfy the conditions of horizontal and vertical line tests in the domain [-4, 4], therefore these are one to one functions.

3. Choose the correct option.
   - ◯ f_1 and f_5 are strictly increasing functions in the given domain.
   - ◯ **f_2 and f_4 are strictly decreasing functions in the given domain.**
   - ◯ f_4 and f_5 are strictly decreasing functions in the given domain.
   - ◯ f_5 is strictly increasing function in the given domain.

   Solution:

   A function $f(x)$ is said to be strictly decreasing on a given interval if $f(b) < f(a)$ for all $b > a$, where $a,\ b$ belong to the domain. On the other hand, if $f(b) <= f(a)$ for all $b > a$, then the function is said to be simply decreasing function.
   Clearly from the given graph, f_2 and f_4 are strictly decreasing functions in the domain [-4, 4].

Use the following information for the questions 4 and 5.
Let N_0 be the number of atoms of a radioactive material at the initial stage i.e., at time $t = 0$, and $N(t)$ be the number of atoms of the same radioactive material at a given time t, which is given by the equation $N(t) = N_0 e^{-\lambda t}$, where λ is the decay constant.

4. If at time t_1, the number of atoms reduces to the half of N_0 and at the time t_2 the number of atoms reduces to the one fourth of N_0, then which one of the following equations is correct?
   - ◯ $e^{\frac{t_1}{t_2}} = 2$
   - ◯ $e^{\frac{t_2}{t_1}} = 2$
   - ◯ $\boldsymbol{e^{\lambda(t_2-t_1)} = 2}$
   - ◯ $e^{\lambda(t_1-t_2)} = 2$

   Solution:

   According to the question, at t_1,
   $$N(t) = \frac{N_0}{2}$$
   According to the equation,
   $$N(t) = N_0\ e^{-\lambda t}$$
   Therefore for $t = t_1$,
   $$\frac{1}{2} \times N_0 = N_0\ e^{-\lambda t_1}$$
   $$\frac{1}{2} = e^{-\lambda t_1} \tag{1}$$
   It is also given that at t_2, $N = \frac{N_0}{4}$
   $$\frac{1}{4} \times N_0 = N_0\ e^{-\lambda t_2}$$
   $$\frac{1}{4} = e^{-\lambda t_2} \tag{2}$$
   On dividing (1) by (2) we get,
   $$e^{\lambda(t_2-t_1)} = 2$$

5. If $N_{\frac{1}{\lambda}}$ is the number of atoms at time $t = \frac{1}{\lambda}$, then what is the ratio of N_0 to $N_{\frac{1}{\lambda}}$?
   - ◯ $1 : e$
   - ◯ $\boldsymbol{e : 1}$
   - ◯ $1 : e^{-\lambda}$

○ $1 : e^{\lambda}$

Solution:

It is given that at $t = \frac{1}{\lambda}$, $N = N'$

$$N' = N_0 \; e^{-\frac{\lambda}{\lambda}}$$

$$N' = \frac{N_0}{e}$$

$$\frac{N_0}{N'} = \frac{e}{1}$$

Therefore,

$N_0 : N' = e : 1$

2 Multiple Select Questions (MSQ):

6. Selvi deposits ₹P in a bank A which provides an interest rate of 10% per year. After 10 years, she withdraws the whole amount from bank A and deposits it in another bank B for n years which provides an interest rate of 12.5% per year. $M_A(x)$ represents the amount in Selvi's account after x years of depositing in bank A. $M_B(y)$ represents the amount in Selvi's account after y years of depositing in bank B. If the interests are compounded yearly, then choose the set of correct options.

   - ○ **$M_A(x)$ is an one-one function of x, for $x \in (0, 10)$.**
   - ○ **$M_B(y)$ is an one-one function of y.**
   - ○ $M_A(12) = P \times 1.1^{12}$
   - ○ **$M_A(12) = 0$**
   - ○ **$M_A(x)$ is a strictly increasing function of x, for $x \in (0, 10)$.**
   - ○ $M_B(y)$ is a decreasing function of y.
   - ○ **$M_B(n) = (P \times 1.1^{10}) \times (1.125)^n$**
   - ○ $M_B(n) = (P \times 1.1^n) \times (1.125)^{10}$

Solution:

When the principal amount P is compounded annually, the amount M after q years is given by

$$M = P \times \left(1 + \frac{\text{Interest rate}}{100}\right)^q$$

Amount $M_A(x)$ after x years in bank A will be

$$M_A(x) = P \times \left(1 + \frac{10}{100}\right)^x$$

So after 10 years the amount $M_A(10)$ will be

$$M_A(10) = P \times (1.1)^{10}$$

As Selvi has withdrawn all the amounts from bank A after 10 years so amount left in bank A after 12 years will be $M_A(12) = 0$.

After 10 years the new principal amount $P \times (1.1)^{10}$ is deposited in another bank B, so for any years y the amount will be $M_B(y)$ which is given by

$$M_B(y) = P \times (1.1)^{10} \times \left(1 + \frac{12.5}{100}\right)^y$$

So for n years

$$M_B(n) = P \times (1.1)^{10} \times (1.125)^n$$

Clearly $M_A(x)$ and $M_B(y)$ are strictly increasing functions therefore both are one-to-one functions of x and y respectively.

Use the following information for questions 7 and 8.

There are two offers in a shop. In the first offer, the discount in total payable amount is $M(n)\%$ if the number of products bought at a time is n. The second offer involves a discount of ₹1000 on the total payable amount. If Geeta shops of ₹15,000, then answer the following questions.

7. If the total payable amounts after applying the first and second offers (one at a time) are represented by the functions $f(n)$ and $g(n)$ respectively and the total payable amount after applying both the offers together is represented by $T(n)$, then choose the set of correct options.

   ◯ $f(n) = (100 - M(n)) \times 15000$ and $g(n) = 14000$

   ◯ $f(n) = (100 - M(n)) \times 1500$ and $g(n) = (100 - M(n)) \times 15000 - 1000$

   ◯ $\boldsymbol{f(n) = (100 - M(n)) \times 150}$ and $\boldsymbol{g(n) = 14000}$

   ◯ $T(n) = (100 - M(n)) \times 15000$ is the total payable amount when the first offer is applied after the second.

   ◯ **$\boldsymbol{T(n) = (100 - M(n)) \times 140}$ is the total payable amount when the first offer is applied after the second.**

   ◯ **$\boldsymbol{T(n) = (100 - M(n)) \times 150 - 1000}$ is the total payable amount when the second offer is applied after the first.**

Solution:

It is given that total payable amount without any offer is ₹15,000. Then, total payable amount after first offer is

$$f(n) = \frac{(100 - M(n))}{100} \times 15,000 = (100 - M(n)) \times 150$$

And total payable amount if second offer is applied will be

$$g(n) = 15,000 - 1000 = ₹14,000.$$

Now, total payable amount when the first offer is applied after the second will be

$$T(n) = \frac{100 - M(n)}{100} \times g(n)$$

$$T(n) = \frac{(100 - M(n))}{100} \times 14000 = (100 - M(n)) \times 140$$

And total payable amount when the second offer is applied after the first will be

$$T(n) = f(n) - 1000$$

$$T(n) = \frac{(100 - M(n))}{100} \times 15000 - 1000 = (100 - M(n)) \times 150 - 1000$$

8. If Geeta is allowed to use the offer in any sequence and $M(n) = -n^2 + 18n - 72$, where $n \in \{6, 7, 8, 9\}$, then choose the set of correct options which minimizes the total payable amount.

   - ◯ Total payable amount is same irrespective of the order in which the offers are applied.
   - ◯ **She should choose offer one and then offer two i.e., $\boldsymbol{fog(M(n))}$.**
   - ◯ She should choose offer two and then offer one i.e. $gof(M(n))$.
   - ◯ **If she chooses offer one and then offer two, the minimum payable amount will be ₹12650.**

Solution:

Total payable amount when she choose offer one and then offer two is

$$T_1(n) = (100 - M(n)) \times 150 - 1000$$

It is given that $M(n) = -n^2 + 18n - 72$, so

$$T_1(n) = (100 - (-n^2 + 18n - 72)) \times 150 - 1000$$

On solving we get,

$$T_1(n) = 150n^2 - 2700n + 24800$$

And total payable amount when she chooses offer two and then offer one is

$$T_2(n) = (100 - M(n)) \times 140$$

On substituting $M(n)$ and solving we get,

$$T_2(n) = 140n^2 - 2520n + 24080$$

Since the coefficient of n^2 is positive for both $T_1(n)$ and $T_2(n)$ therefore minimum value i.e., minimum payable amount of these function can be calculated as follows
For $T_1(n)$

$$\text{Vertex}(n) = \frac{-b}{2a} = \frac{-(-2700)}{2 \times 150} = 9$$

The minimum payable amount will be

$$T_1(9) = 150(9)^2 - 2700(9) + 24800 = ₹12{,}650$$

For $T_2(n)$

$$\text{Vertex}(n) = \frac{-b}{2a} = \frac{-(-2520)}{2 \times 140} = 9$$

The minimum payable amount will be

$$T_2(9) = 140(9)^2 - 2520(9) + 24080 = ₹12{,}740$$

Thus if she chooses offer one and then offer two, the minimum payable amount will be ₹12,650.

n	$T_1(n)$ ₹	$T_2(n)$ ₹
6	14000	14000
7	13250	13300
8	12800	12880
9	12650	12740

Table: M1W8PAS-1

From Table: M1W8PAS-1, it is clear that for all the values of n the total payable amount is lower for $T_1(n)$ as compared to $T_2(n)$ therefore she should choose offer one and then offer two.
Note: This can be also identified by plotting the graph for $T_1(n)$ and $T_2(n)$.

3 Numerical Answer Type (NAT):

Use the following information for questions 9-15.
Given two real valued functions $f(x) = \frac{5x+9}{2x}$, $g(y) = \sqrt{y^2 - 9}$. If $h(x) = f(g(x))$, then answer the following questions.

9. If domain of $f(x)$ and $g(x)$ are $(-\infty, m) \cup (m, \infty)$ and $\mathbb{R} \setminus (-n, n)$ respectively, then find the value of $m + n$. [Ans: 3]
Solution:

At $x = 0$ the function $f(x) \to \infty$ or the function is undefined at $x = 0$ thus the domain of $f(x)$ is $\mathbb{R} \setminus 0$.
We can also write the domain as $(-\infty, 0) \cup (0, \infty)$ therefore, $m = 0$.

It is given that $g(y) = \sqrt{y^2 - 9}$ on changing the variable in terms of x we get $g(x) = \sqrt{x^2 - 9}$.

$g(x)$ will be defined when $x^2 - 9 \geq 0$. On solving

$$x^2 \geq 9$$

$$x \geq 3$$

or

$$x \leq -3$$

Thus the domain will be $\mathbb{R} \setminus (-3, 3)$, hence $n = 3$. So, $m + n = 0 + 3 = 3$

10. If range of $f(x)$ and $g(x)$ are $(-\infty, m) \cup (m, \infty)$ and $[n, \infty)$ respectively, then find the value of $2(m + n)$. [Ans: 5]

Solution:
As $f(x)$ is defined everywhere except 0, therefore there will be an asymptote at $x = 0$.
If we draw a graph of $f(x)$:
End behaviour:
As $x \to \infty$, $f(x) \to \frac{5}{2}$.
As $x \to -\infty$, $f(x) \to \frac{5}{2}$.
The end behaviours show that the function has another asymptote at $f(x) = y = \frac{5}{2}$.

Intercept:

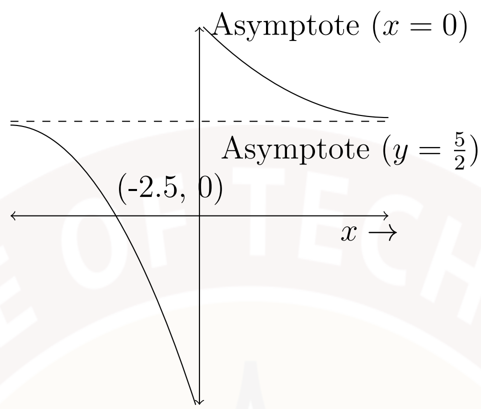
$$f(x) = 0 \implies \frac{5x + 9}{2x} = 0$$

$$x = -\frac{9}{5}$$

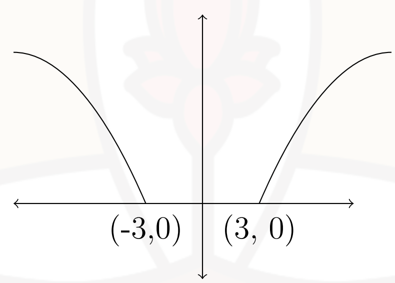
It means $f(x)$ might change the sign at $x = -\frac{9}{5}$.
For $-\infty < x < 0$, $f(x)$ will have value from $-\infty$ to $\frac{5}{2}$.

Similarly for $0 < x < \infty$, $f(x)$ will have value from $\frac{5}{2}$ to ∞.

Therefore the range of $f(x)$ is $(-\infty, \frac{5}{2}) \cup (\frac{5}{2}, \infty)$. A rough diagram of $f(x)$ is shown below.



As $g(x) = \sqrt{x^2 - 9}$ is a positive square root function so it will have only the positive values including zero at $x = 3$ and $x = -3$.
A rough diagram is created using the facts that the $g(x)$ is not defined from (-3, 3) and at $x = 3$ the function gives the value zero. At ∞ the function provides the value ∞. As the quadratic function involved and the $b = 0$ the function will be symmetric about $y-$axis.



Therefore the range will be $[0, \infty)$. Thus $m = 2.5$ and $n = 0$, so,

$$2(m + n) = 2(2.5 + 0) = 5$$

11. If domain of $h(x)$ is $(-\infty, -3) \cup (m, \infty)$, then find the value of m. [Ans: 3]

Solution:
Given,

$$h(x) = f(g(x))$$
$$h(x) = f(\sqrt{x^2 - 9})$$
$$= 2.5 + \frac{4.5}{\sqrt{x^2 - 9}}$$

There are two possibilities when the function is undefined. Firstly when the denominator is zero and secondly when the function in square root provides negative value. It means

$\sqrt{x^2-9} \neq 0$ and $x^2 - 9 \geq 0$.
Combining both the conditions we can say the function is defined only when

$$x^2 - 9 > 0$$

$$x^2 > 9 \implies -3 < x < 3$$

Thus the domain will be $(-\infty, -3) \cup (3, \infty)$, hence $m = 3$.

12. If domain of $f^{-1}(x)$ is $(-\infty, m) \cup (m, \infty)$, then find the value of $2m$. [Ans: 5]

Solution:
Given that $f(x) = \frac{5x+9}{2x}$ let us say $f(x) = y$ so $y = \frac{5x+9}{2x}$ on rearranging,

$$y = \frac{5}{2} + \frac{9}{2x}$$
$$\frac{2y-5}{2} = \frac{9}{2x}$$
$$x = \frac{9}{2y-5}$$

Therefore $f^{-1}(x) = \frac{9}{2x-5}$. This function will be defined when

$$2x - 5 \neq 0$$

$$x \neq \frac{5}{2}$$

The domain of this function is $(-\infty, 2.5) \cup (2.5, \infty)$ thus $m = 2.5$ therefore $2m = 5$

13. If $f^{-1}(5) = 9/m$, then find the value of m. [Ans: 5]

Solution:
$f^{-1}(5) = \frac{9}{2\times 5-5} = \frac{9}{5}$, thus $m = 5$.

Week - 9
Prcactice Assignment
Mathematics for Data Science - 1

1 Multiple Select Question (MSQ)

1. If $b > 0$ and $\boxed{4\log_x b + 9\log_{b^5x} b = 1,}$ then the possible value(s) of x is(are)
   - (a) $\boldsymbol{b^{10}}$
   - (b) b^9
   - (c) $\boldsymbol{b^{-2}}$
   - (d) b^5
   - (e) b^4

Soln:

$$4\log_x b + 9\log_{b^5x} b = 1$$

$$\Rightarrow \frac{4}{\log_b x} + \frac{9}{\log_b(b^5x)} = 1$$

$$\Rightarrow \frac{4}{\log_b x} + \frac{9}{\log_b b^5 + \log_b x} = 1$$

$$\Rightarrow \frac{4}{\log_b x} + \frac{9}{5\cancel{\log_b b}^{1} + \log_b x} = 1$$

Let $p = \log_b x$

$$\Rightarrow \frac{4}{p} + \frac{9}{5+p} = 1$$

$$\Rightarrow \frac{4(5+p)+9p}{p(5+p)} = 1 \Longrightarrow 20 + 4p + 9p = 5p + p^2$$

$$\Rightarrow \boxed{p^2 - 8p - 20 = 0}$$

$\Rightarrow p^2 - 8p - 20 = 0$

$\Rightarrow p^2 - 10p + 2p - 20 = 0$

$\Rightarrow p(p-10) + 2(p-10) = 0$

$\Rightarrow (p+2)(p-10) = 0$

$\boxed{p = -2, 10}$

We know that $\boxed{p = \log_b x}$

If $p = -2$

$-2 = \log_b x$

$\boxed{x = b^{-2}}$

If $p = 10$

$10 = \log_b x$

$\boxed{x = b^{10}}$

Note:-

$$\boxed{\log_{a^b} c = \frac{1}{b} \log_a c}$$

✓ * Can be used later

Proof:- $LHS = \log_{a^b} c$

$$= \frac{1}{\log_c a^b} = \frac{1}{b \log_c a} = \frac{1}{b} \times \frac{1}{\log_c a} = \frac{1}{b} \log_a c = RHS$$

$\boxed{LHS = RHS}$

2. George deposits ₹5L in a bank that compounded quarterly at the rate of 20% per year. How long will it take to increase his money to 16 times the principal amount(in year)?

   (a) $\frac{\ln 16}{4}$

   (b) $\mathbf{\frac{\ln 16}{4 \ln \frac{21}{20}}}$

   (c) $\mathbf{\frac{\ln 2}{\ln \frac{21}{20}}}$

   (d) $\mathbf{\log_{\frac{21}{20}} 2}$

   (e) $\frac{\ln 2^{\frac{1}{4}}}{\ln \frac{21}{20}}$

Soln Formula for compound interest

$$\boxed{A = P\left(1 + \frac{R}{100}\right)^{t}}$$

$$\Rightarrow \boxed{A = P\left(1 + \frac{R}{n \times 100}\right)^{nt}}$$

where

t = time period (years)

R = Interest rate per year

P = Principal or initial deposits

A = Amount after t years

n = No. of times it compounded in a year.

$n = 4$

$$A = P\left(1 + \frac{20}{400}\right)^{4t}$$

$$\boxed{A = 16P}$$

$$\Rightarrow 16\cancel{P} = \cancel{P}\left(1 + \frac{1}{20}\right)^{4t}$$

$$\Rightarrow 16 = \left(\frac{21}{20}\right)^{4t}$$

formula: $\ln a^b = b \ln a$

$$\Rightarrow \ln 16 = 4t \ln\left(\frac{21}{20}\right)$$

$$\boxed{t = \frac{1}{4}\left(\frac{\ln 16}{\ln\left(\frac{21}{20}\right)}\right)}$$

$$t = \frac{\ln(16)^{1/4}}{\ln\left(\frac{21}{20}\right)} = \frac{\ln(2^4)^{1/4}}{\ln\left(\frac{21}{20}\right)} = \boxed{\frac{\ln 2}{\ln\left(\frac{21}{20}\right)}}$$

We have

$$\boxed{t = \frac{\ln 2}{\ln \frac{21}{20}}}$$

$$\boxed{\text{Formula : } \log_b a = \frac{\log a}{\log b}}$$

Using change of base formula, we get,

$$\boxed{t = \log_{\frac{21}{20}} 2}$$

3. Choose the set of correct options.

   (a) ~~$\log_5 2$ is a rational number~~

   (b) ~~If $0 < b < 1$ and $0 < x < 1$ then $\log_b x < 0$~~

   (c) **If $\log_3(\log_5 x) = 1$ then $x = 125$**

   (d) ~~If $0 < b < 1, 0 < x < 1$ and $x > b$ then $\log_b x > 1$~~

   (e) **If $0 < b < 1$ and $0 < x < y$ then $\log_b x > \log_b y$**

Soln

(a) Let $\log_5 2$ be rational number, thus it can be written in $\frac{p}{q}$ form,

$\log_5 2 = p/q$

$\Rightarrow 2 = 5^{\frac{p}{q}}$

$$\boxed{2^q = 5^p}$$

2 & 5 are co-primes & 2 cannot divide 5

thus assumption is wrong.

So it should irrational

(b) Given:- $0 < b < 1$ & $0 < x < 1$ then $\boxed{\log_b x < 0}$

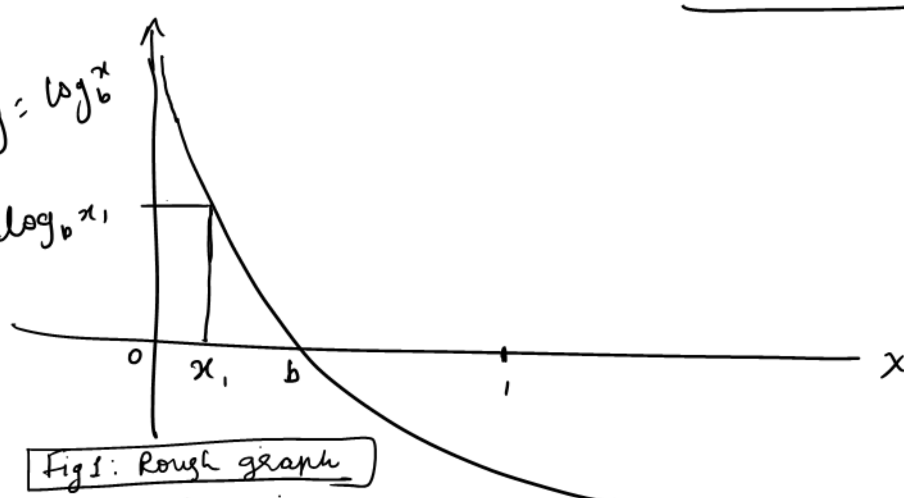


Fig 1: Rough graph

Let x_1 be in

$\boxed{0 < x_1 < b < 1}$; then from above graph (Fig 1)

$\boxed{\log_b x_1 > 0}$ $\therefore$ statement of option(b) is wrong.

✓ option (c): Given that,

$$\boxed{\log_3 \log_5 (x) = 1}$$ then $x = ?$

$\log_5 x = 3^1 = 3$

$x = 5^3 = 125$

$$\boxed{x = 125}$$

Formula:

$\log_a x = b$

$\boxed{x = a^b}$

option (d) Given that:-

$0 < b < 1$, $0 < x < 1$, & $\boxed{x > b}$ then $\boxed{\log_b x > 1}$

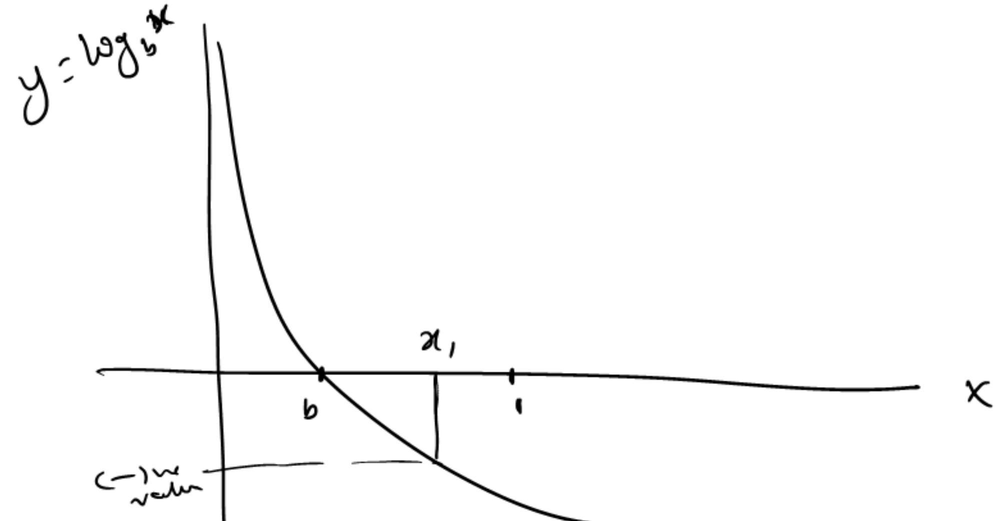


Fig 2: Rough graph

Let $0 < b < x_1 < 1$ where $x_1 \in x$

then,

$\log_b x_1 < 1$

Thus the statement of option (d) is wrong.

option (c) Given that:

$0 < b < 1$ & $0 < x < y$ then $\boxed{\log_b x > \log_b y}$

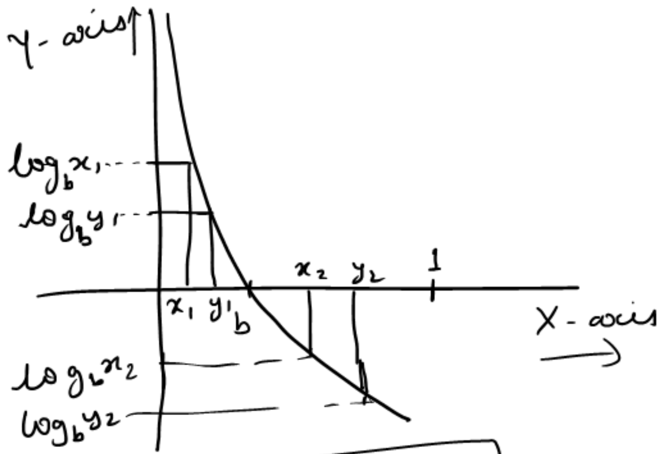


Fig 3: Rough graph

Case I: when $x_1 < b$ & $y_1 < b$

also; $x_1 < y_1$

From graph (Fig 3)

$\log_b x_1 > \log_b y_1$

Thus in case I, the given statement is right

Case II: $b < x_2 < 1$ & $b < y_2 < 1$

also $x_2 < y_2$

From graph (Fig 3)

$\log_b x_2 > \log_b y_2$

Thus in case II, the given statement is right again

Thus overall the given statement is right, thus the option (c) is correct.

4. Suppose that two types of insects are found in a pond. Their growth in number after t seconds is given by the equations $f(t) = 5^{3t-2}$ and $h(t) = 3^{2t+1} (t \neq 0)$. For what value of t will both insects be of same number in the pond?

   (a) $\frac{\ln 3+2\ln 5}{3\ln 5-2\ln 3}$

   (b) $\frac{\ln 75}{\ln \frac{125}{9}}$

   (c) $\log_{\frac{125}{9}} 75$

   (d) $\frac{\ln 5+2\ln 3}{3\ln 3-2\ln 5}$

Soln:-

Insects number will be same when

$$f(t) = h(t)$$

$$5^{3t-2} = 3^{2t+1}$$

$$\Rightarrow (3t-2)\ln 5 = (2t+1)\ln 3$$

$$\Rightarrow (3\ln 5)t - 2\ln 5 = (2\ln 3)t + \ln 3$$

$$(3\ln 5)t - (2\ln 3)t = \ln 3 + 2\ln 5$$

$$\Rightarrow t(3\ln 5 - 2\ln 3) = \ln 3 + 2\ln 5$$

$$\Rightarrow \boxed{t = \frac{\ln 3 + 2\ln 5}{3\ln 5 - 2\ln 3}}$$

formula:

① $\ln a^b = b\ln a$

② $\ln(ab) = \ln a + \ln b$

③ $\ln\frac{a}{b} = \ln a - \ln b$

$$\Rightarrow t = \frac{\ln 3 + \ln 5^2}{\ln 5^3 - \ln 3^2}$$

$$\Rightarrow = \frac{\ln 3 \times 25}{\ln\left(\frac{125}{9}\right)}$$

$$\Rightarrow \boxed{t = \frac{\ln 75}{\ln\left(\frac{125}{9}\right)}}$$

Formula:

$$\log_b a = \frac{\log a}{\log b}$$

Using change of base formula

$$\boxed{t = \log_{\frac{125}{9}} 75}$$

2 Multiple Choice Question (MCQ)

5. If $\log_{\sqrt{2}}(x+4) - \log_2\left(\frac{1}{2}x+2\right) = 1$ then x is
   (a) **−3**
   (b) 1
   (c) −4
   (d) 5

Soln:-

Given that:

Using derived formula

$\log_{a^b} c = \frac{1}{b}\log_a c$

$\log_{2^{1/2}}(x+4) - \log_2\left(\frac{1}{2}x+2\right) = 1$

$\Rightarrow \frac{1}{2}\log_2(x+4) - \log_2\left(\frac{x}{2}+2\right) = 1$

$\Rightarrow \log_2(x+4)^{1/2} - \log_2\left(\frac{x+4}{2}\right) = 1$

$\Rightarrow \log_2\left(\frac{(x+4)^{1/2}}{\left(\frac{x+4}{2}\right)}\right) = 1$

formula:

$\log_b x = a$

$x = b^a$

$\Rightarrow \frac{(x+4)^{1/2}}{\left(\frac{x+4}{2}\right)} = 2$

$\Rightarrow (x+4)^{1/2} = \cancel{2}\left(\frac{x+4}{\cancel{2}}\right)$

Squaring on b.s

$(x+4) = (x+4)^2$

$(x+4)^2 - (x+4) = 0$

$(x+4)\left((x+4)-1\right) = 0$

we get

$x+4 = 0$ or $x+4-1 = 0$

$\boxed{x = -4}$ $\boxed{x = -3}$

From Question we have

$\log_{\sqrt{2}}(x+4) - \log_2\left(\frac{x}{2}+2\right) = 1$

$\boxed{\text{when } x = -4}$

$\log_{\sqrt{2}}(-4+4) - \log_2\left(\frac{-4}{2}+2\right) = 1$

Notice: $-4+4 = 0$

Thus -4 is out of domain of log function.

$\boxed{\text{Now when } x=-3}$

$\cancel{\log(-3+4)}^{0} - \log_2\left(\frac{-3}{2}+2\right) = 1$

$-\log_2\left(\frac{1}{2}\right) = 1$

$-\left[\cancel{\log_2 1}^{0} - \cancel{\log_2 2}^{1}\right] = 1$

$1 = 1$

Thus $x = -3$ is the right option.

6. Seismologists use the Richter scale to measure and report the magnitude of earthquake as given by the equation $\boxed{R = \ln I - \ln I_0,}$ where I is the intensity of an earthquake with respect to a minimal or reference intensity I_0(i.e $I = cI_0$, where c is a constant). The reference intensity is the smallest earth movement that can be recorded on a seismograph. If an earthquake in city A recorded of magnitude 8.0 in Richter scale and intensity of the earthquake in city B is the reference intensity, then what is the ratio of intensity of earthquake in city A with respect to city B?

   (a) $e^0 : 1$

   (b) $e^1 : 2$

   (c) $\mathbf{e^8 : 1}$

   (d) $e^5 : 1$

   (e) $e^8 : 2$

Soln Using the given equation

To find: $\frac{I}{I_0} = ?$

$$\boxed{R = \ln I - \ln I_0}$$

$$\Rightarrow R = \ln \frac{I}{I_0}$$

$$\Rightarrow 8 = \ln \frac{I}{I_0}$$

$$\Rightarrow e^8 = \frac{I}{I_0}$$

$$\Rightarrow \frac{I}{I_0} = \frac{e^8}{1} \Rightarrow e^8 : 1$$

$\therefore$ The ratio of intensity of earthquake in city A w.r.t city B is $e^8 : 1$

.

7. Suppose that the number of bacteria present in a loaf of rotten bread after t minutes is given by the equation $G(t) = G_0 3^{kt}$, where G_0 represents the number of bacteria at $t = 0$, k is a constant (Given $\ln 730 = 6.59$ and $\ln 3 = 1.09$). If the initial number of bacteria is 1000 and it takes 1 min to increase to 9000 then how long(in minutes) would it take for the bacteria count to grow to 730000(integer value of t)?

(a) 2

(b) 1

(c) **3**

(d) 6

Soln :-

Given :- $G_0 = 1000$

At, $t = 1$ min $\quad G(t) = 9000$

To find :-

At what time (t), $G(t) = 7,30,000$

Solve :-

$G(t) = G_0 3^{kt}$ —— ①

$\Rightarrow \dfrac{G(t)}{G_0} = 3^{kt}$

At $t = 1$ min

$\Rightarrow \dfrac{9\cancel{000}}{1\cancel{000}} = 3^{kt}$

$\Rightarrow \ln 3^2 = kt \ln 3$

$\Rightarrow 2 \cancel{\ln 3} = kt \cancel{\ln 3}$

$\Rightarrow \boxed{k = 2}$

On substituting the values of K & G_0, equation ① becomes

$G(t) = 1000\ 3^{2t}$; when $G(t) = 7,30,000$, then

$\Rightarrow \dfrac{730\cancel{000}}{1\cancel{000}} = 3^{2t}$

$\Rightarrow 730 = 3^{2t}$

$\Rightarrow \ln 730 = 2t \ln 3$

$\Rightarrow t = \dfrac{\ln 730}{2 \ln 3} = \dfrac{6.59}{2 \times 1.09}$

$$\boxed{t = 3 \text{ min}}$$

Thus at $t = 3$ min (integer value) bacteria count would be 7,30,000.

Let c_A and c_B be the luminosity(luminous efficacy) of the bulbs A and B respectively. The bulb A is $f(x)$ times brighter than the B, if $f(x) = 3^{x^2+1}$(i.e $c_A = f(x) \times c_B$), where x is the difference of the magnitude of supply voltage between the bulb A and the bulb B. Answer the questions 8 and 9 based on above information.

8. If the bulb A is 10 times brighter than the bulb B, then the difference of the magnitude of supply voltage between the two bulbs is
   (a) $\sqrt{\log_3 5 - 1}$
   (b) $\sqrt{\log_3 10}$
   (c) $\sqrt{\frac{\ln 10}{\ln 3}}$
   (d) $\sqrt{\mathbf{\log_3 \frac{10}{3}}}$

Soln:-

given: $C_A = 10\, C_B \quad \text{—} (1)$

Luminosity is the measure of brightness

$$C_A = C_B \times f(x)$$

$$C_A = C_B \times 3^{x^2+1} \quad \text{—} (2)$$

solve

$$10 \cancel{C_B} = \cancel{C_B}\, 3^{x^2+1}$$

$$\Rightarrow \log_3 10 = (x^2+1) \cancel{\log_3 3}$$

$$\Rightarrow \log_3 10 = x^2+1$$

$$\Rightarrow x^2 = \log_3 10 - 1$$

$$\Rightarrow x = \sqrt{\log_3 10 - 1}$$

$$\Rightarrow x = \sqrt{\frac{\log 10}{\log 3} - 1}$$

$$\Rightarrow \quad x = \sqrt{\frac{\log 10 - \log 3}{\log 3}}$$

$$\Rightarrow \quad x = \sqrt{\frac{\log \frac{10}{3}}{\log 3}}$$

$$\boxed{\text{formula: } \log_b a = \frac{\log a}{\log b}}$$

$$\boxed{x = \sqrt{\log_3 \frac{10}{3}}}$$

The difference b/w the magnitude of 2 bulbs is $\sqrt{\log_3 \frac{10}{3}}$

3 Numerical Answer Type (NAT)

9. If 4 voltage and 3 voltage are the supply voltages for the bulbs A and B respectively then how many times the bulb A is brighter than the bulb B?
   Ans : 9

Soln:- Since x is the difference b/w the supply voltage of A & B, thus

$x = 4 - 3 = 1$

We know that

$$C_A = C_B \times f(x)$$

We have to find $f(x)$

$f(x) = 3^{x^2+1}$

$= 3^{1+1} = 3^2 = 9$

$f(x) = 9$

$$\boxed{C_A = 9\, C_B}$$

∴ 9 times brighter

10. Find the number of values of x satisfying the equation $(5x)^{\log_{(5x)^{\frac{1}{5}}}(6x^3-36x^2+66x-35)^{\frac{1}{5}}} = 1.$
Ans: 3

Soln:- Given that:

$(5x)^{\log_{(5x)^{1/5}}(6x^3-36x^2+66x-35)^{1/5}} = 1$

$\Rightarrow (5x)^{5\log_{5x}(6x^3-36x^2+66x-35)^{1/5}} = 1$

$\Rightarrow (5x)^{\log_{5x}(6x^3-36x^2+66x-35)^{\cancel{5/5}}} = 1$

$\Rightarrow 6x^3-36x^2+66x-35 = 1$

$\Rightarrow 6x^3-36x^2+66x-36 = 0$

$\Rightarrow 6(x^3-6x^2+11x-6) = 0$

$\Rightarrow x^3-6x^2+11x-6 = 0$

Using derived formula

$\boxed{\log_{a^b}c = \frac{1}{b}\log_a c}$

Formula:

$\boxed{a^{\log_a x} = x}$

Hit & trial method

when $x=1$

$1-6+11-6=0$

$0=0$

Synthetic division to find other roots

$$
\begin{array}{r|l}
 & x^2-5x+6 \\
x-1 & x^3-6x^2+11x-6 \\
 & \cancel{x^3}-x^2 \\
 & (\cancel{-})\quad(+) \\
\hline
 & -\cancel{5x^2}+11x-6 \\
 & -\cancel{5x^2}+5x \\
 & (\cancel{+})\quad(-) \\
\hline
 & 6x-6 \\
 & 6x-6 \\
 & (-)\quad(+) \\
\hline
 & 0
\end{array}
$$

Factorizing to get other roots

$\Rightarrow x^2 - 5x + 6 = 0$

$\Rightarrow x^2 - 2x - 3x + 6 = 0$

$\Rightarrow x(x-2) - 3(x-2)$

$\Rightarrow (x-3)(x-2) = 0$

$x - 3 = 0$ or $x - 2 = 0$

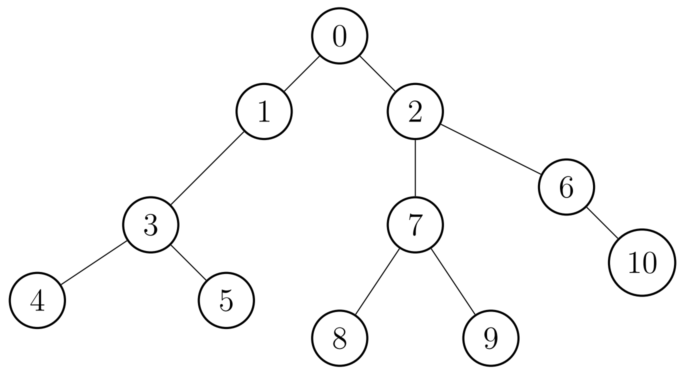
$x = 2, 3, 1$

We got 3 values all together.

Mathematics for Data Science - 1
Practice Assignment
Week 10

1 MULTIPLE CHOICE QUESTIONS:

1. Suppose we obtain the following DFS tree rooted at node 0 for an undirected graph with vertices $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.



Which of the following cannot be an edge in the original graph? [Ans: c]

(a) $(1, 4)$
(b) $(0, 4)$
(c) $(7, 10)$
(d) $(2, 9)$

Soln: DFS tree for an undirected graph can have edges only in the same branch because if there is an edge between two vertices in G which are in different branches of DFS tree, then the neighbours of vertex 'u' must be visited in DFS in order to remove it from the stack.

Clearly from Figure 1, we have five branches (b1, b2, b3, b4, b5)

✓ Option (a) :- Vertices (1, 4) ∈ branch(b1)

✓ Option (b) :- Vertices (0, 4) ∈ branch(b1)

✗ Option (c) :- Vertex (7) ∈ branch(b3) & vertex (10) ∈ branch(b5) which cannot form a possible edge in original unrooted DFS tree.

✓ Option (d) :- Vertex (2, 9) ∈ branch(b4)

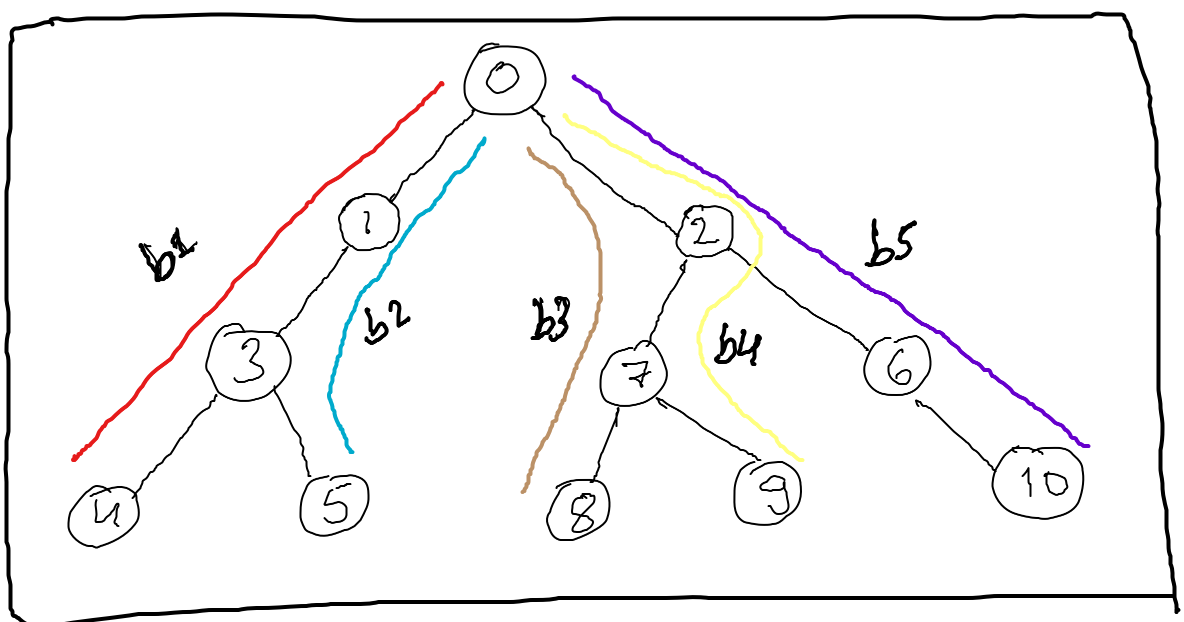
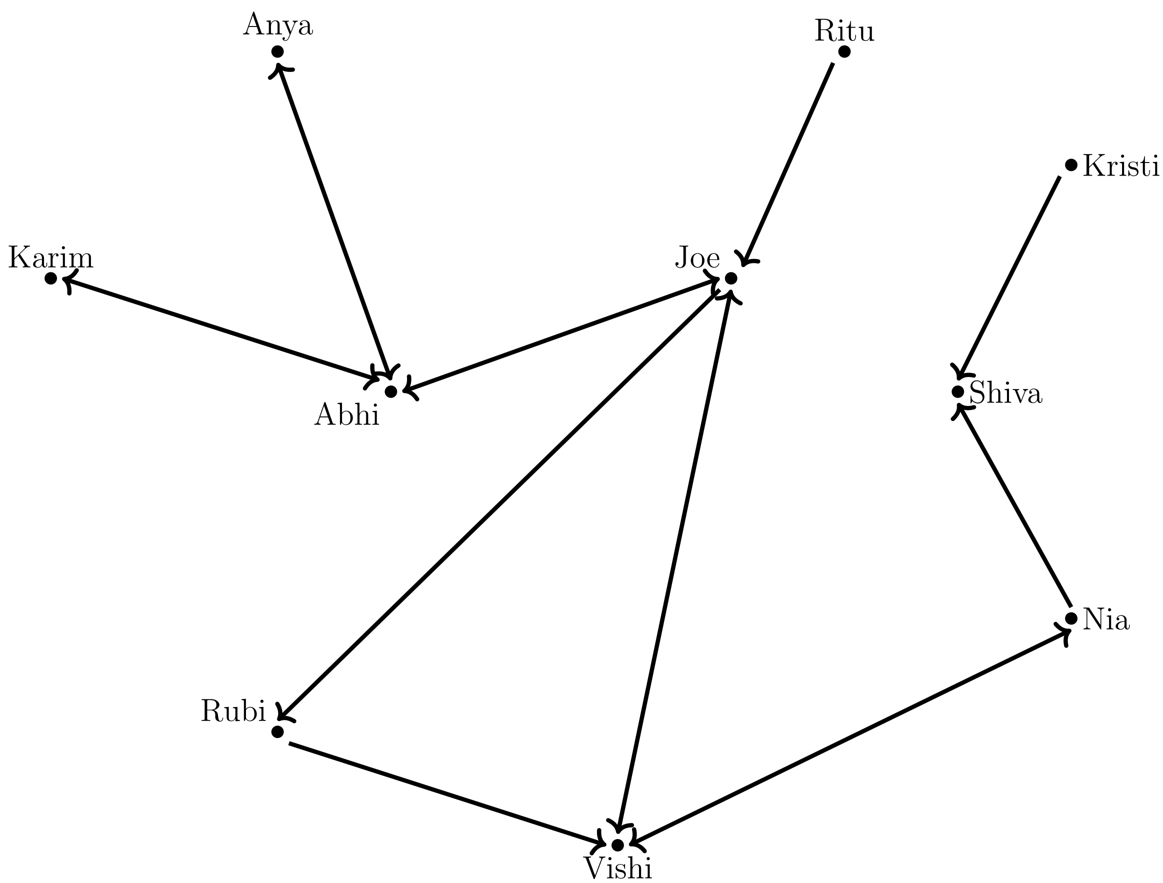


Figure 1

Use the following information for the questions 2-4:
Ten friends in a college decided to have a night party at the home of one of them. Unfortunately at D-day the government closes many of the routs of the city. The below graph shows the location of their homes and the open routs as the graph $G = (V, E)$, where V is the set of nodes and E is the edges representing the open routs.



2. The possible place for the party is. [Ans: a]
   (a) Shiva's house.
   (b) Abhi's house.
   (c) Joe's house.
   (d) Vishi's house.

Soln:-

Given that $G = (V, E)$ where V is set of nodes representing a person's house & E is the edge representing the open routes. Let us consider the rechability of each node

Note: Vertex (v) is rechable from vertex (u) if there is a path from u to v, where u & $v \in V$

(i) Kristi's & Ritu's house are not recheable by anyone

(ii) Anya's, Karim's, Abhi's, Joe's, Rubi's, Vishi's and Nia's house are not recheable by Kristi and Shiva

(iii) Shiva's house is recheable by everyone so the best possible place for the party is Shiva's house as this node is recheable from every other nodes.

3. Let $V_1 = \{\text{Kristi, Shiva, Nia}\}$ and $E_1 = E \cap (V_1 \times V_1)$, that is, E_1 is the subset of edges of G with both end points in V_1. Choose the correct option. [Ans: d]

   (a) $G_1 = (V_1, E_1)$ is an undirected graph.
   (b) $G_1 = (V_1, E_1)$ is a cyclic graph.
   (c) $G_1 = (V_1, E_1)$ will not be a graph.
   (d) $G_1 = (V_1, E_1)$ is a directed graph.

Soln

Given that: $V_1 = \{\text{Kristi, Shiva, Nia}\}$

$V_1 \times V_1 = \{(\text{Kristi, Shiva}), (\text{Shiva, Kristi}), (\text{Kristi, Nia}), (\text{Nia, Kristi}), (\text{Shiva, Nia}), (\text{Nia, Kristi})\}$

Note: $V_1 \times V_1$ represents the interconnected path in V_1

$E = \{(\text{Kristi, Shiva}), (\text{Nia, Shiva}), \ldots\}$

$E_1 = E \cap (V_1 \times V_1) = \{(\text{Kristi, Shiva}), (\text{Nia, Shiva})\}$

So, clearly $G_1 = (V_1, E_1)$ is a directed graph as E_1 is directed. Figure 1.1 shows the subgraph G_1.

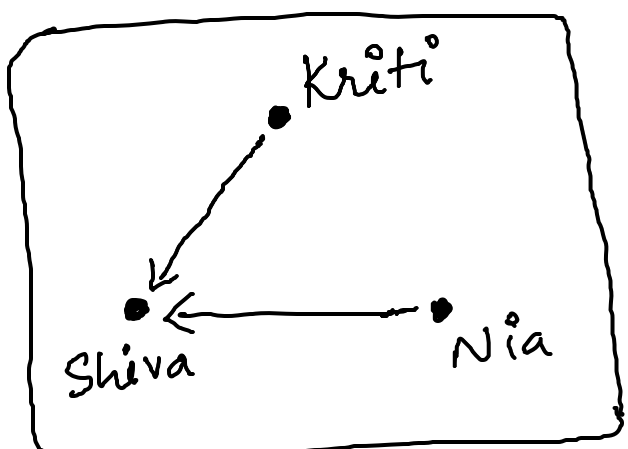


Figure 1-1: $G_1 = (V_1, E_1)$

4. If Joe wants to have the party on his home, then at most how many members can join the party. [Ans: d]

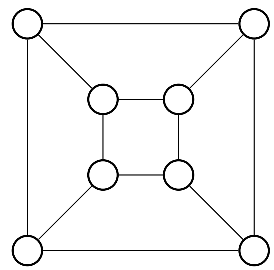
    (a) 5.
    (b) 6.
    (c) 7.
    (d) 8.

Soln

As seen earlier Joe's home was not reachable by kristi and Shiva. & there are 10 members altogether. Thus 8 members in total will join the party

2 MUTIPLE SELECT QUESTIONS:

5. Suppose G be a graph (shown in the below figure). Let V be the set of vertices of G, V_i be the maximum independent set and V_c be the minimum vertex cover. Which of the followings is(are) true? [Ans: a,d]



(a) Cardinality of V_i is 4.

(b) Cardinality of V_c is 3.

(c) Cardinality of V_i is 5.

(d) Cardinality of V_c is 4.

Soln Vertex cover:

In a graph G, vertex cover is the set of vertices that includes at least one end point of every edge of the graph.

So, minimum vertex cover (V_c) is a vertex cover having smallest possible number of vertices.

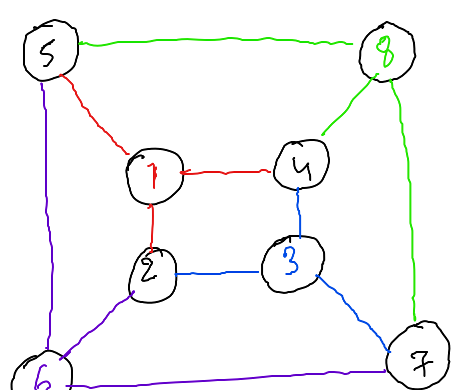


Figure 2

From Figure 2, one of the possible minimum vertex cover $(V_c) = \{1, 3, 6, 8\}$

Thus, cardinality of $(V_c) = 4$

<u>Independent set:</u>

Given a graph $G = (V, E)$, where V is vertex & E is edges, $F \subseteq V$ is an independent set if there are no edges between vertices in F.

Maximum independent set (V_i) is said to be maximal if no vertex of 'G' can be added to F

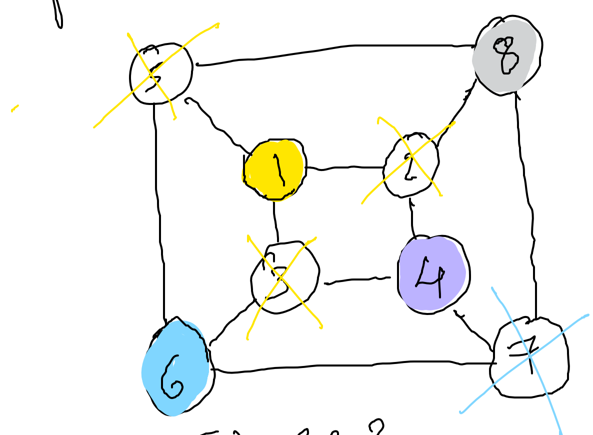
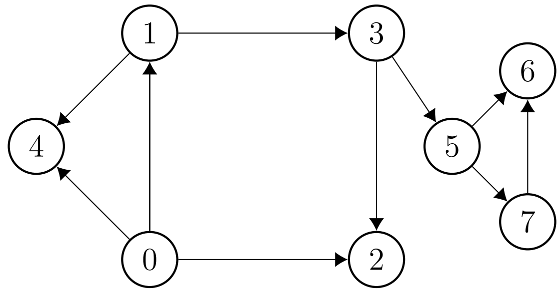


Figure 3

From Figure 3, one of the possible values of $V_i = \{1, 4, 6, 8\}$

Thus, Cardinality of $V_i = 4$

6. Consider the graph given below.



Suppose we perform BFS/DFS so that when we visit a vertex, we explore its unvisited neighbours in a random order. Which of the following options are correct? [a,c,d]

(a) If we perform Breadth First Search at node 0, then one of the possible order in which the nodes will be visited is 01423567.

(b) If we perform Depth First Search at node 0, then one of the possible order in which the nodes will be visited is 04123576

(c) If we perform Breadth First Search at node 0, then one of the possible order in which the nodes will be visited is 01423576.

(d) If we perform Depth First Search at node 0, then one of the possible order in which the nodes will be visited is 04132567.

Soln

BFS tree from node 0, for the given graph could be as follows

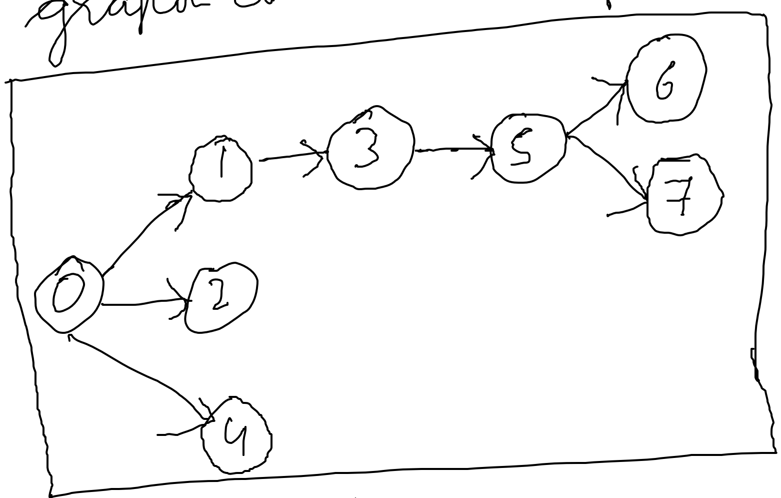


Figure 4

Explore 0, visit 1, 4, 2
Explore 1, visit 3
Explore 4
Explore 2
Explore 3, visit 5
Explore 5, visit 6, 7
Explore 6
Explore 7

$\boxed{0\ 1\ 4\ 2\ 3\ 5\ 6\ 7}$

Explore 0, visit 1, 4, 2
Explore 1, visit 3
Explore 4,
Explore 2
Explore 3, visit 5
Explore 5, visit 7, 6
Explore 7
Explore 6

$\boxed{0\ 4\ 1\ 2\ 3\ 5\ 7\ 6}$

These are the two possible orders however more orders are possible too.

DFS tree from the node 0, for the given graph could be as follows.

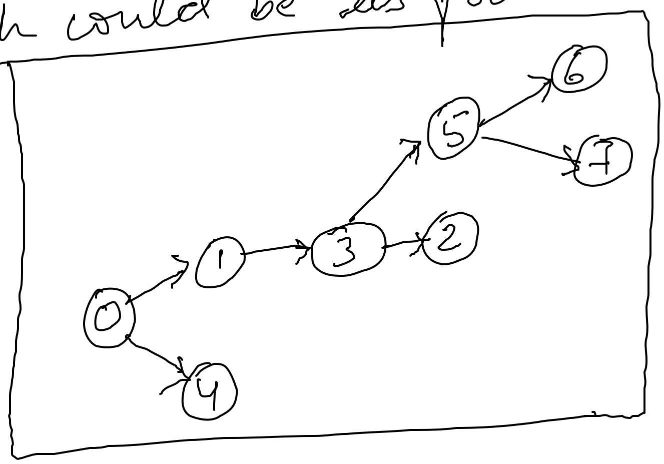


Figure 5

One of the possible order is 0 4 1 3 2 5 6 7.

Thus option (a), (c), (d) are right options

7. Which of the followings options are correct? [Ans: a,b,c,d]

   (a) In Depth First search of a directed graph only back edges generate cycles.
   (b) If the maximum independent set of a graph G contains only 1 element, then the graph must be connected.
   (c) If we add an edge to a tree T, then the resulting graph becomes cyclic.
   (d) In a connected graph G having n vertices, at least two vertices have same degree.

Soln

(a) In a DFS, the vertices $v_0, v_1, v_2, \ldots v_{n-1}$ are connected by outward edge.
Suppose there is a backward edge (v_i, v_j) where $j < i$ for some $i, j \in \{0, 1, \ldots n-1\}$ then $v_j \to v_{j+1} \to v_{j+2} \to \cdots v_{i-1} \to v_i$ is a cycle.

(b) Consider a graph G, which is disconnected, then atleast the graph has 2 components of connected graph. Then, the maximum idependent set of a graph G contains more than 2 elements in independent set.

Thus, for a graph with 1 element in maximum independent set must be a connected graph.

(c) Suppose in a tree there exists a path from vertex i to vertex j, for all $i, j \in V$

Now if we add an edge 'x' connecting vertex i to vertex j then there exists a path from vertex i to vertex j as shown below in Figure 6 which forms cycle

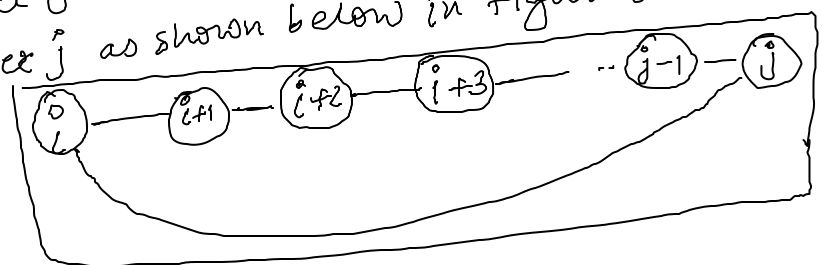
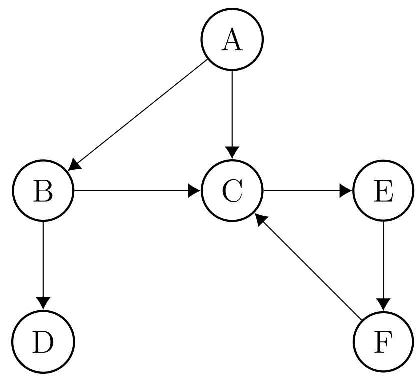


Figure 6

Thus if we add an edge to a tree T, then the resulting graph becomes cyclic.

(d) Let $V_1, V_2, V_3 \cdots V_n$ be the vertices and let V_1 has degree 1, V_2 has degree 2, V_3 has degree 3, $\cdots V_{n-1}$ has degree $n-1$. Now the degree of vertex V_n should choose from $\{1, 2, 3, \cdots n-1\}$ which is one of the degree of the above vertices i.e., if the degree of the vertex V_n is 'k', then V_k and V_n has the same degree 'k'.

8. Consider the following directed graph.



Suppose DFS of this graph is performed from node A, such that when we visit a vertex, we explore its unvisited neighbours in alphabetical order.

Which of the following options are correct? [Ans: a,d]

(a) AC is a forward edge.

(b) CE is a backward edge.

(c) BD is a forward edge.

(d) FC is a backward edge.

Soln

DFS tree of the given graph when performed from node A is shown in Figure 7

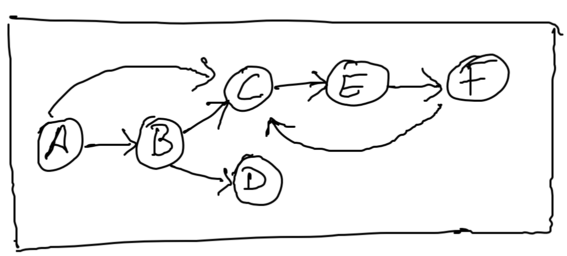


Figure 7

The only forward edge is A → C } Refer Figure
and only backward edge is F → C }

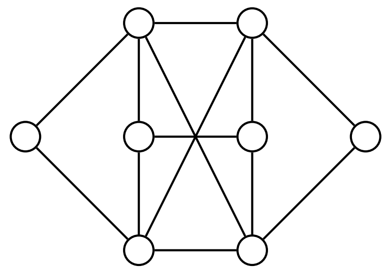
Rest all edges are the normal edge of DFS tree.

Therefore, AC forms forward edge & FC forms backward edge respectively.

.

3 NUMERICAL ANSWER TYPE:

9. The cardinality of the maximum independent set of the graph given below is [ans: 4]



Soln

One of the possible way to find the maximum independent set is shown in Figure 8.

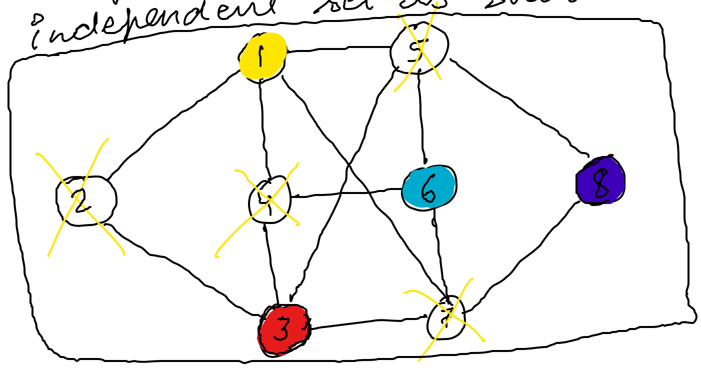
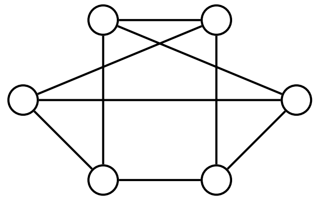


Figure 8

$V_i = \{1, 3, 6, 8\}$

Cardinality of $V_i = 4$

10. The minimum colouring of the below graph is [Answer: 2]



Soln We know that, graph $G = (V, E)$, set of colors C coloring is a function $c: V \to C$ such that $(u,v) \in E \Rightarrow c(u) \neq c(v)$

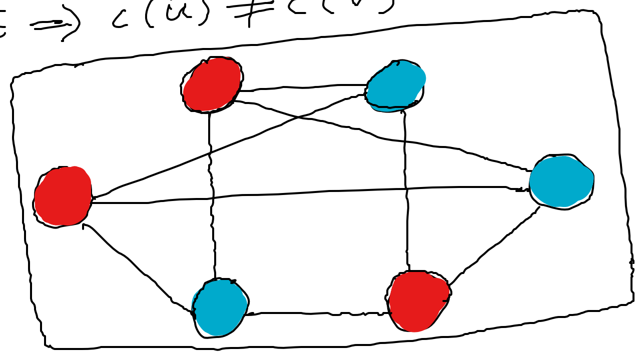


Figure 9

In simplest form, it is a way of coloring the vertices of a graph such that no two adjacent vertices are of the same color

From Figure 9 the minimum coloring required is 2.

Mathematics for Data Science - 1
Practice Assignment
Week 11

1 MULTIPLE CHOICE QUESTIONS:

1. Suppose $R = \{(1,3),(3,4),(4,5)\}$ is a relation on the set $\{1,3,4,5,7\}$. Which of the following represents the transitive closure of R? [Ans: c]

   (a) $\{(1,3),(3,4),(1,4),(4,5),(3,5),(5,1)\}$
   (b) $\{(1,3),(3,4),(4,5),(3,5),(1,5),(4,3)\}$
   (c) $\{(1,3),(3,4),(4,5),(3,5),(1,5),(1,4)\}$
   (d) $\{(1,3),(3,1),(3,4),(4,3),(4,5),(5,4)\}$

Given $\Rightarrow$ $R = \{(1,3),(3,4),(4,5)\}$

$\Rightarrow$ for Transitive clouser if $(1,3)$ and $(3,4)$ are in relation $(1,4)$ should also be in Relation.

$\Rightarrow$ similarily $(3,4)$ and $(4,5) \in R$
$(3,5) \in R$

$\Rightarrow$ $R = \{(1,3)(3,4),(4,5),(1,4),(3,5)\}$

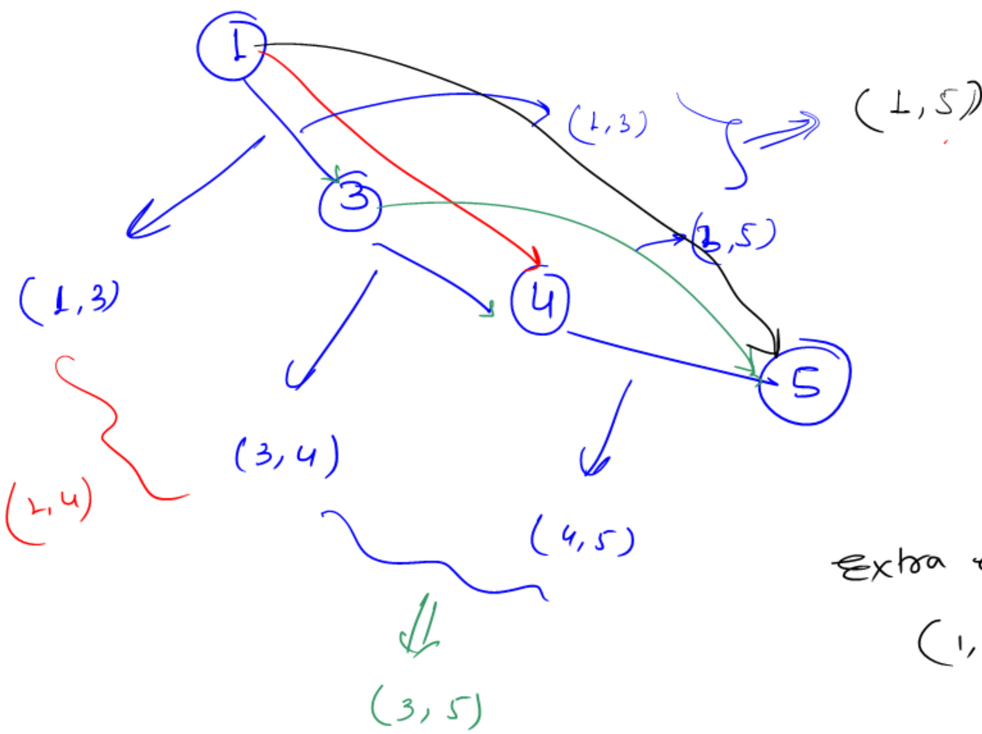
Now $(1,4)$ and $(4,5) \in R$
$(1,5) \in R$

Therefore Transitive clouser of R will be:

$\{(1,3),(3,5),(4,5),(1,4),(3,5),(1,5)\}$

Second method: If we solve using graphs.

(1, 3), (3, 4), (4, 5)



Extra edges:

(1, 4), (3, 5), (1, 5)

2. An undirected graph G has 31 vertices. The sum of the degrees of all the vertices in G is M. The number of vertices of odd degree in G is N. Which of these values are possible for M and N? [Ans: c]

(a) $M = 98$, $N = 11$

(b) $M = 103$, $N = 10$

(c) $M = 98$, $N = 10$

(d) $M = 103$, $N = 11$

⇒ M is sum of degrees of all vertices therefore M should be even.

⇒ Let E be the sum of degrees of all vertices having even degrees, so E also be even.

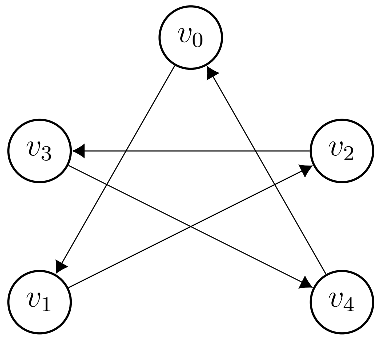
⇒ Let N′ be the sum of all vertices having odd degrees then we can write as

$$\underbrace{M}_{\text{even}} = \underbrace{N'}_{\text{must be even}} + \underbrace{E}_{\text{even}}$$

Now if N′ is even which is sum of odd degree vertices therefore N (No. of odd degree vertices) should be even.

Only option (c) satisfies the condition.

3. Consider the graph given below



If A is the adjacency matrix of G, then which of the following represents A^2? [Ans: b]

$\Rightarrow$ we can find first A and then A^2

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$A^2 = A \times A.$

(a)

	v_0	v_1	v_2	v_3	v_4
v_0	0	1	0	0	0
v_1	0	0	1	0	0
v_2	0	0	0	1	0
v_3	0	0	0	0	1
v_4	1	0	0	0	0

(b)

	v_0	v_1	v_2	v_3	v_4
v_0	0	0	1	0	0
v_1	0	0	0	1	0
v_2	0	0	0	0	1
v_3	1	0	0	0	0
v_4	0	1	0	0	0

(c)

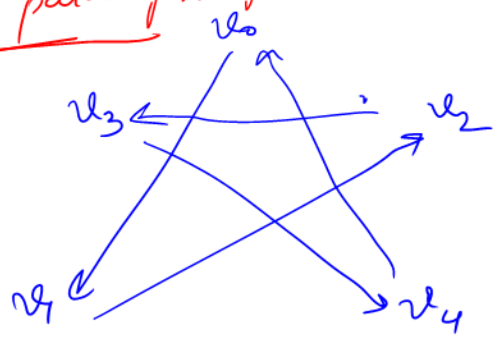
	v_0	v_1	v_2	v_3	v_4
v_0	0	1	0	0	1
v_1	1	0	1	0	0
v_2	0	1	0	1	0
v_3	0	0	1	0	1
v_4	1	0	0	1	0

(d)

	v_0	v_1	v_2	v_3	v_4
v_0	0	0	1	1	0
v_1	0	0	0	1	1
v_2	1	0	0	0	1
v_3	1	1	0	0	0
v_4	0	1	1	0	0

Second Method:

A^2 represents the path of length 2.



(1) vertices have path of length 2 from $v_0 \Rightarrow$

$v_0 \to v_1 \to v_2$

which means only v_2 is reachable in path of length 2 from v_0.

(2) $v_1 \Rightarrow$ $v_1 \to v_2 \to v_3$; $v_1 \to v_3$ } only v_3

(3) $v_2 \Rightarrow$ $v_2 \to v_3 \to v_4 \Rightarrow v_2 \to v_4$ } only v_4

(4) $v_3 \Rightarrow$ $v_3 \to v_4 \to v_0$ } only v_0.

(5) $v_4 \Rightarrow$ $v_4 \to v_0 \to v_1$ } only v_1

Therefore

$$A^2 = \begin{array}{c|ccccc} & v_0 & v_1 & v_2 & v_3 & v_4 \\ \hline v_0 & & & 1 & & \\ v_1 & & & & 1 & \\ v_2 & & & & & 1 \\ v_3 & 1 & & & & \\ v_4 & & 1 & & & \end{array}$$

2 MUTIPLE SELECT QUESTIONS:

4. Suppose $G = (V, E)$ is a directed graph, where $V = \{1, 2, 3, 4, 6, 7, 12\}$. There is an edge from a to b ($a \neq b$), that is, $(a, b) \in E$ if and only if $a|b$ (a divides b). [Ans: a,b,d]

   Which of the following can be a topological sorting of the graph G ?

   (a) $1, 2, 4, 3, 6, 12, 7$
   (b) $1, 7, 2, 4, 3, 6, 12$
   (c) $7, 1, 2, 4, 3, 6, 12$
   (d) $1, 2, 3, 4, 7, 6, 12$

If $V = \{1, 2, 3, 4, 6, 7, 12\}$

$\Rightarrow$ 1 devides all other elements therefore there will edges from 1 to all other elements.

$\Rightarrow$ 2 devides 4, 6, and 12 so there will be edges from 2 to 4, 6 and 12 respectively.

$\Rightarrow$ 3 devides 6 and 12

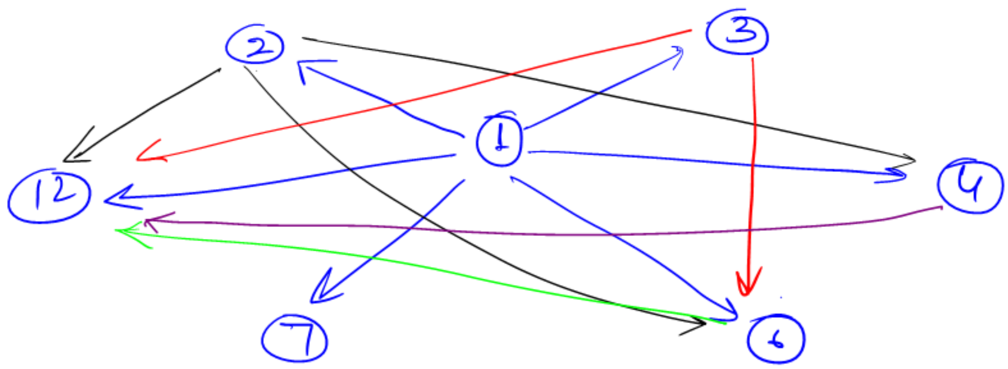
$\Rightarrow$ 4 devides only 12

$\Rightarrow$ 6 devides only 12

$\Rightarrow$ 7 and 12 will not devide any other elements

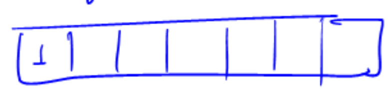
So no edges from 7 and 12.

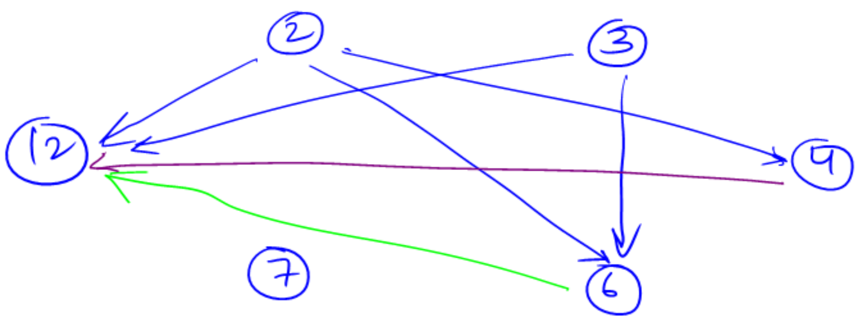
Therefore if represent as graph:



Now 1 has indegree zero and if we remove 1.

Topological order.



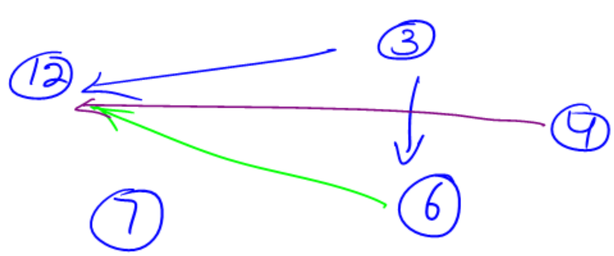


Now can remove any vertex among 7, 2, and 3. Therefore we will match with options. In option c the sequence starts with 7 which is not possible.

Now we can check along option a –

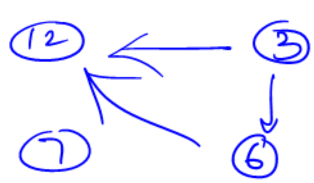
1 2 4 3 6 12 7

removing 2



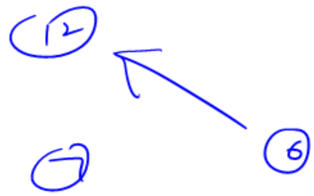
we can remove either 3 or 4 or 7

removing 4



we can remove either 7 or 3

Removing 3



we can remove either 6 or 7

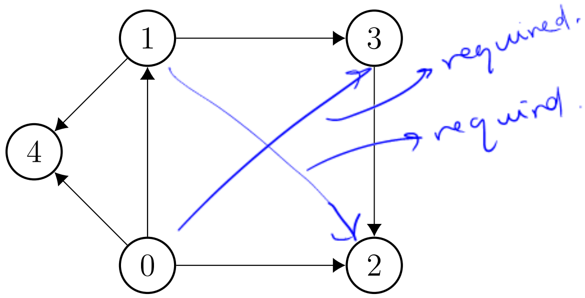
Removing 6

Then order 1 2 4 3 6 7 12 is possible.

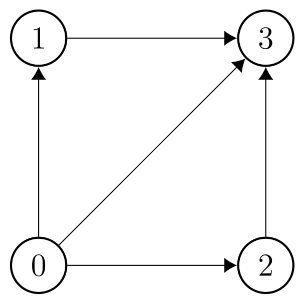
We can also check for option b and d similarily.

5. Which of the following graphs represent its own transitive closure? [Ans: b,c]

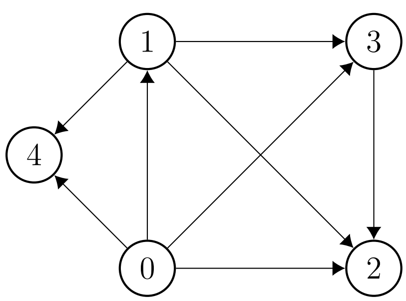
(a)



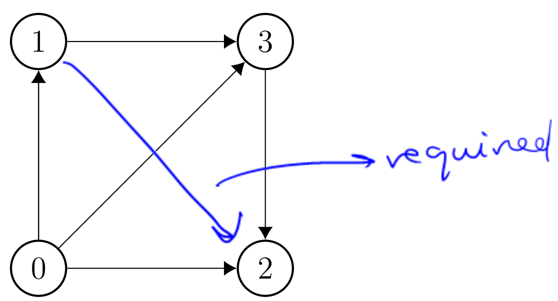
(b)



(c)



(d)



option ⓐ (1, 3) and (3, 2) are in R.
Therefor (1, 2) should also be in R which is not there.

option ⓑ

(0, 1) and (1, 3) is in R and therefore (0, 3) should be in R which there.

(0, 2) and (2, 3) $\in$ R and (0, 3) also in R.

There fore option ⓑ is correct.

option ⓒ:
The required edges in option ⓐ to make transitive are presented in option ⓒ.

6. Which of the followings options are correct? [Ans: b,d]

(a) If G is a graph with n vertices then length of a path in G is bounded by $n-2$.

(b) If G is a directed graph, then the sum of the in-degrees of all the vertices is equal to the sum of out-degrees of all the vertices.

(c) $A = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$ can represent the adjacency matrix of an undirected graph G.

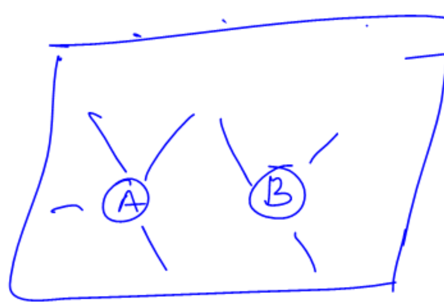
(d) If G is an undirected graph with exactly two vertices of odd degree, then those two vertices are connected in G.

option (a) $\Rightarrow$ bound is $(n-1)$ so incorrect.

option (b) every edge contributes to one outdegree and one indegree in a directed graph. There fore correct.

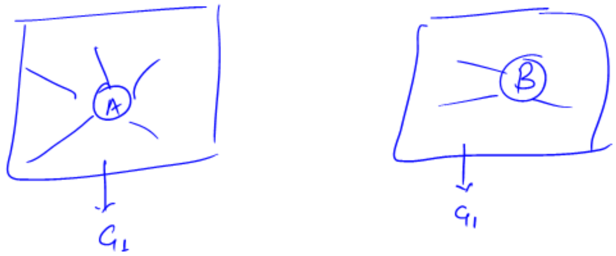
option (c) For undirected graph A_{ij} should be equal to A_{ji}. Here $A_{23} \neq A_{32}$.

Option (d) Let a graph G which have only 2 odd degree vertex.



A connected graph G.

Now if we think that G is not connected then we will get minimum two separate graph let us say G_1 and G_2 and each graph will have one vertex with odd degree.



Now G_1 is also a graph and it should have the properties of graph like sum of degrees of all vertices should be even.

But if see in G_1 the sum of all vertices will be odd which is not possible. Therefore the graph G should be connected.

7. Which of the following options are correct? [Ans: b,c]

(a) If $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, then $A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(b) If $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, then $A^n = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ for all $n > 2$.

can be calculated

(c) If A is a 3×3 matrix and $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $AI = IA = A$.

(d) If A is a 3×3 matrix and $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $AI \neq A$.

Soln: let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow AI = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = A.$

Now $IA = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = A$ Therefore $AI = IA = A$.

Therefore option (c) is correct and option (D) is wrong.

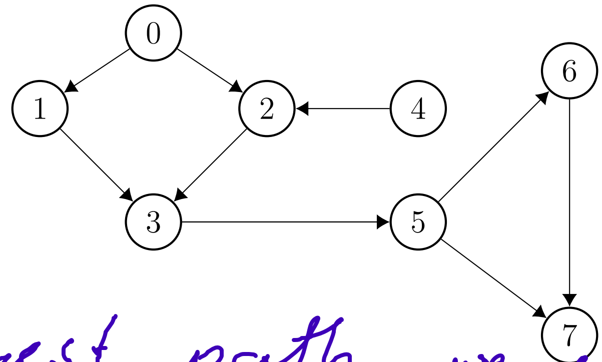
3 NUMERICAL ANSWER TYPE:

8. Let G be an undirected graph with 8 vertices and all vertices have degree 4. How many edges are there in the graph G? [Ans: 16]

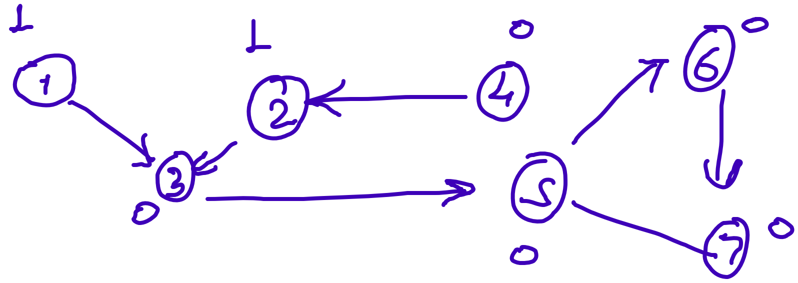
If there are 8 vertices and each vertex has 4 degree then the sum of all degrees of all vertices = 8 X 4 = 32

As we know the number of edges is half of the sum of degrees of all vertices, therefore 16 edges.

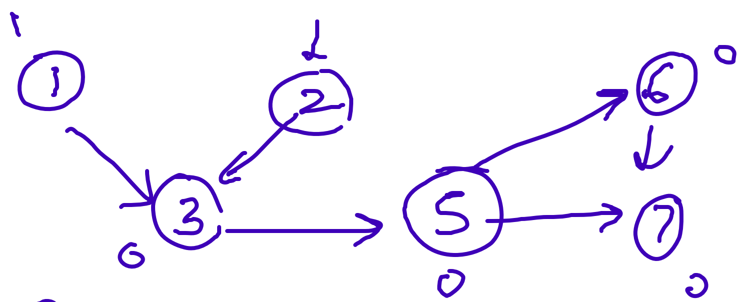
9. The longest path of the below DAG contains x edges. Find x [Answer: 5]



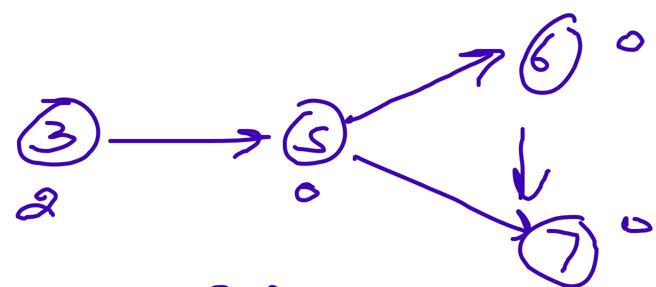
So for longest path we will first remove 0.



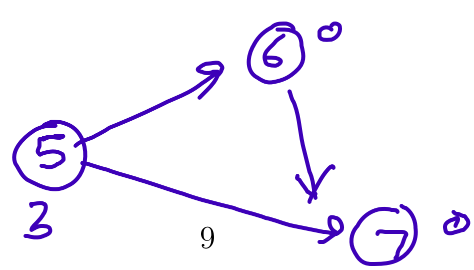
remove 4



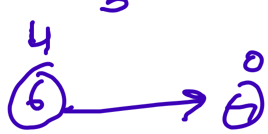
remove 1 and 2



remove 3



remove 5



, remove 6 → L.P = 5

Ans.

Mathematics for Data Science - 1
Practice Assignment Solution
Week 12

1 MULTIPLE CHOICE QUESTIONS:

1. Consider the following weighted graph in Figure PA-12.1.

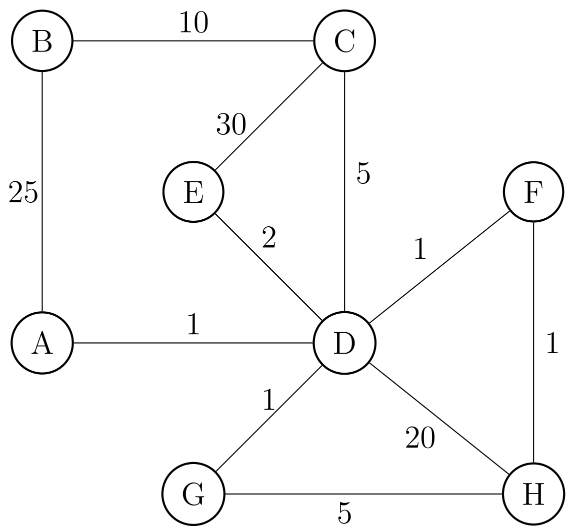


Figure PA-12.1

The shortest weighted path from H to B is [Ans: d]

(a) $H \to G \to D \to A \to B$

(b) $H \to D \to C \to B$

(c) $H \to F \to D \to E \to C \to B$

(d) $H \to F \to D \to C \to B$

Answer

As the edge weights are positive we can use Dijkstra's algorithm

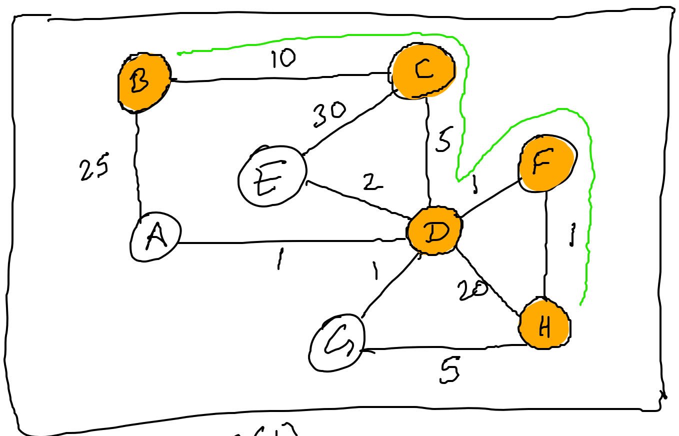


Figure (1)

Clearly from Figure (1) the shortest path to reach vertex B from vertex H is $\boxed{H \to F \to D \to C \to B}$ as in this path the total weight is minimum which is 17.

2. Suppose Dijkstra's algorithm is run on the graph below (Figure PA-12.2), starting at node A. In what order do the shortest distances to the other vertices get finalized?

[**Ans: a**]

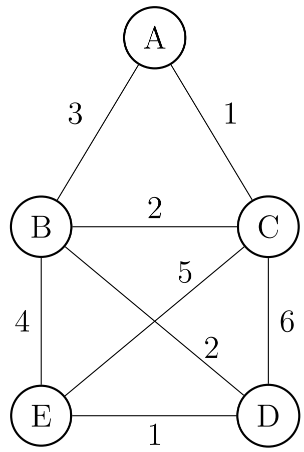


Figure PA-12.2

(a) A, C, B, D, E

(b) A, C, B, E, D

(c) A, C, D, B, E

(d) A, C, D, E, B

Answer

Using Dijkstra's algorithm the shortest distances to reach other vertices are shown in Figure(2)

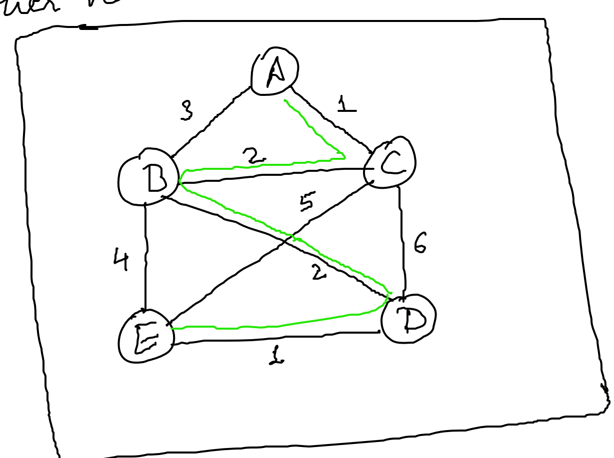


Figure (2)

Notice that from source vertex A to vertex C the distance is shortest as compared with the distance from vertex A to vertex B.

The same logic is applied for every other vertices and we get the shortest distance from vertex A to other vertices as follows $\boxed{A \to C \to B \to D \to E}$

3. If we perform Floyd-Warshall algorithm for the graph shown below (Figure PA-12.3),

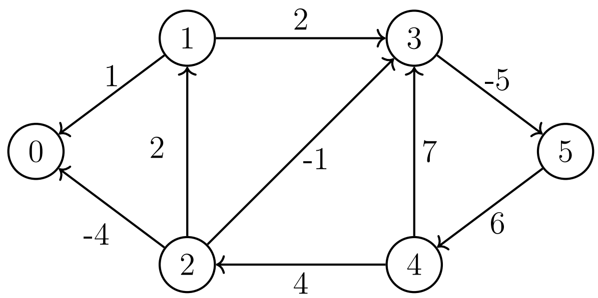


Figure PA-12.3

then which of the following matrices represents SP^4? [Ans: c]

(a)

SP^4	0	1	2	3	4	5
0	∞	∞	∞	∞	∞	∞
1	1	∞	∞	2	∞	∞
2	-4	2	∞	-1	∞	∞
3	∞	∞	$-\infty$	∞	∞	-5
4	∞	∞	4	7	∞	∞
5	∞	∞	∞	∞	6	∞

(b)

SP^4	0	1	2	3	4	5
0	∞	∞	∞	∞	∞	∞
1	1	∞	∞	2	∞	∞
2	-4	2	∞	-1	∞	∞
3	∞	∞	∞	∞	∞	-5
4	0	6	4	3	∞	∞
5	∞	∞	∞	∞	6	∞

(c)

SP^4	0	1	2	3	4	5
0	∞	∞	∞	∞	∞	∞
1	1	∞	∞	2	∞	-3
2	-4	2	∞	-1	∞	-6
3	∞	∞	∞	∞	∞	-5
4	0	6	4	3	∞	-2
5	∞	∞	∞	∞	6	∞

(d)

SP^4	0	1	2	3	4	5
0	∞	∞	∞	∞	∞	∞
1	1	∞	∞	2	∞	-3
2	∞	2	∞	-1	∞	-6
3	∞	∞	∞	∞	∞	-5
4	0	6	4	3	∞	∞
5	∞	∞	∞	∞	6	∞

Answer:

- To find: SP^4 adjacency matrix
- Approach: Find SP^0, SP^1, SP^2, SP^3 and then find SP^4

- Floyd-Warshall Algorithm

Let $SP^k[i,j]$ be the length of the shortest path from i to j via vertices $\{0, 1, \dots k-1\}$.

- Note:- $SP^0[i,j] = W[i,j]$; where $W[i,j]$ is weight of an edge from i to j.

For SP^1 find shortest path via vertex $\{0\}$

For SP^2 " " " vertices $\{0,1\}$

For SP^3 " " " $\{0,1,2\}$

SP^0	0	1	2	3	4	5
0	∞	∞	∞	∞	∞	∞
1	1	∞	∞	2	∞	∞
2	-4	2	∞	-1	∞	∞
3	∞	∞	∞	∞	∞	-5
4	∞	∞	4	7	∞	∞
5	∞	∞	∞	∞	6	∞

SP^1	0	1	2	3	4	5
0	∞	∞	∞	∞	∞	∞
1	1	∞	∞	2	∞	∞
2	-4	2	∞	-1	∞	∞
3	∞	∞	∞	∞	∞	-5
4	∞	∞	4	7	∞	∞
5	∞	∞	∞	∞	6	∞

SP^2	0	1	2	3	4	5
0	∞	∞	∞	∞	∞	∞
1	1	∞	∞	2	∞	∞
2	-4	2	∞	-1	∞	∞
3	∞	∞	∞	∞	∞	-5
4	∞	∞	4	7	∞	∞
5	∞	∞	∞	∞	6	∞

SP^3	0	1	2	3	4	5
0	∞	∞	∞	∞	∞	∞
1	1	∞	∞	2	∞	∞
2	-4	2	∞	-1	∞	∞
3	∞	∞	∞	∞	∞	-5
4	0	6	4	3	∞	∞
5	∞	∞	∞	∞	6	∞

SP^4	0	1	2	3	4	5
0	∞	∞	∞	∞	∞	∞
1	1	∞	∞	2	∞	-3
2	-4	2	∞	-1	∞	-6
3	∞	∞	∞	∞	∞	-5
4	0	6	4	3	∞	-2
5	∞	∞	∞	∞	6	∞

2 MULTIPLE SELECT QUESTIONS:

Using the graph below answer the following questions [Question 4 and 5]

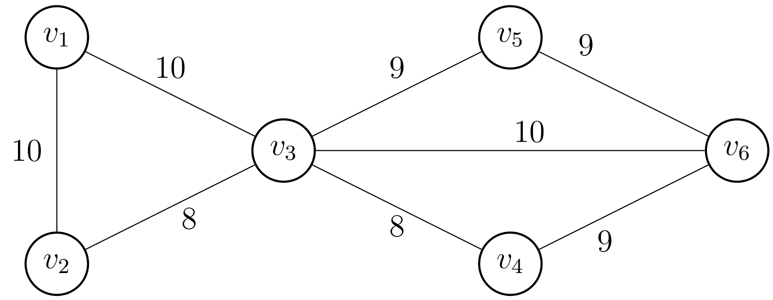
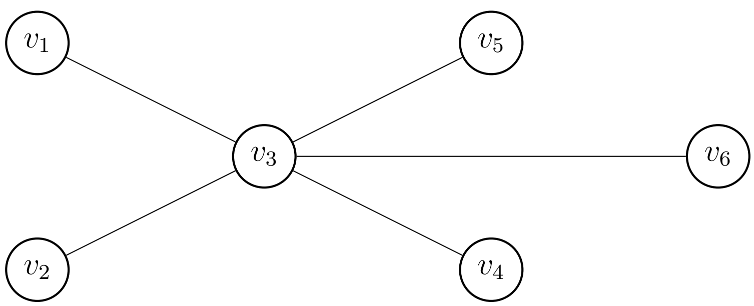


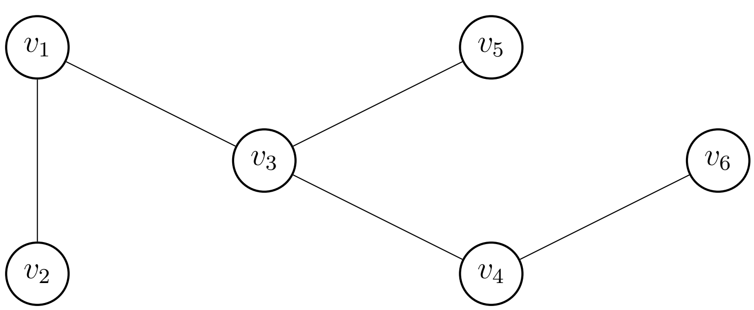
Figure PA-12.4

4. Which of the following could be the minimum cost spanning tree computed by running Prim's algorithm on the graph in Figure PA-12.4? [Ans: c,d]

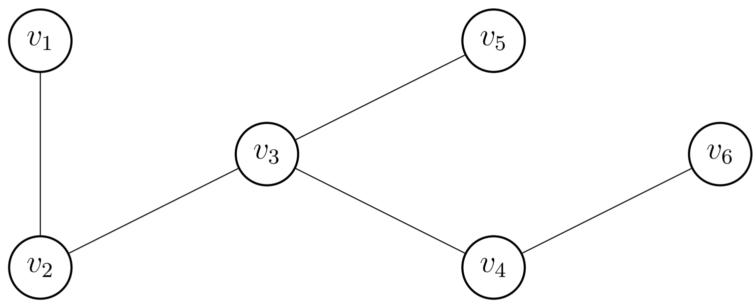
   (a)



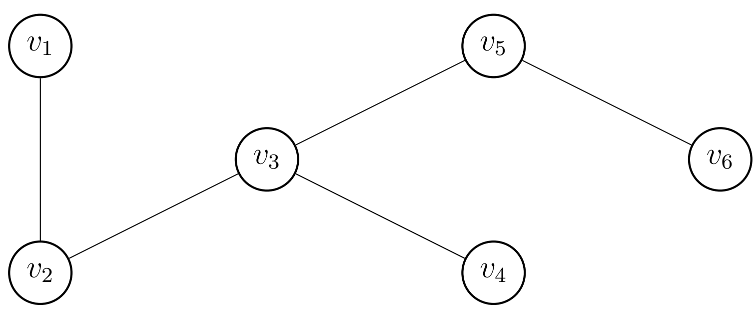
   (b)



(c)



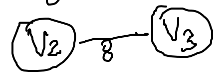
(d)



Answer:

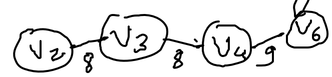
Prim's algorithm:

Step1: Select the edge with minimum cost (edge weight)

For Eg: 

Step2: Now check for all the edges adjacent to (V_1, V_2). Select the one which has the lowest weight and include it in the tree. For eg:  (Make sure no cycles are formed)

Step3: Repeat step 2 by adding one more edge {adjacent to (V_2, V_3) or (V_3, V_4)} which has the lowest weight edge

Eg:  or 

Step4:- Repeat the above step to cover all the edges to obtain MCST.

Note:- In the given graph (Figure PA-12.4) there are many edges which have same weights and thus we can have multiple MCST.

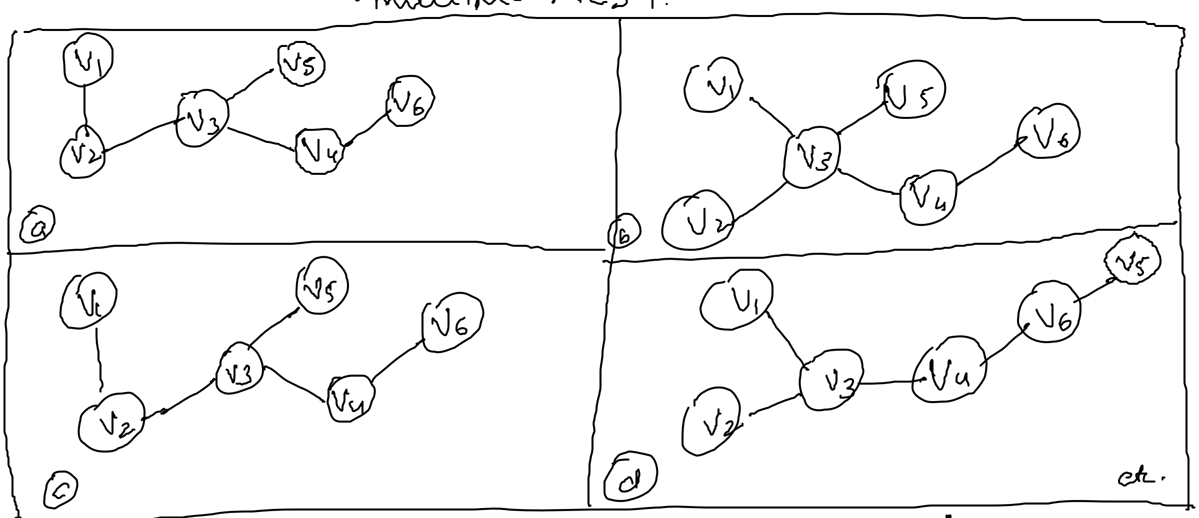
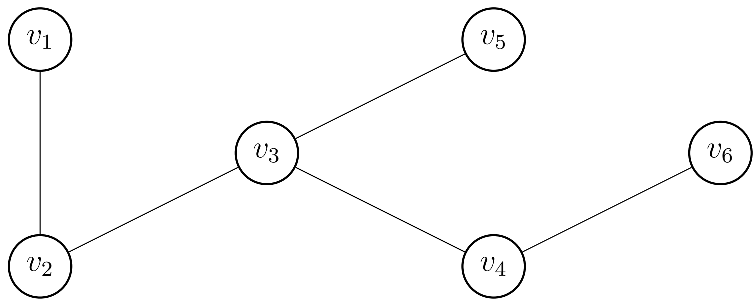


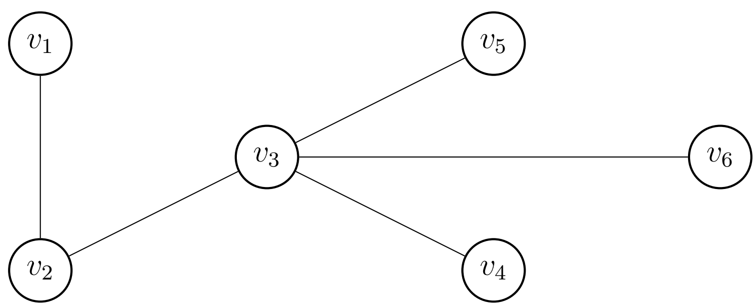
Figure (3): MCST using Prim's algorithm.

5. Which of the following could be the minimum cost spanning tree computed by running Kruskal's algorithm on the graph in Figure PA-12.4? [Ans: a,d]

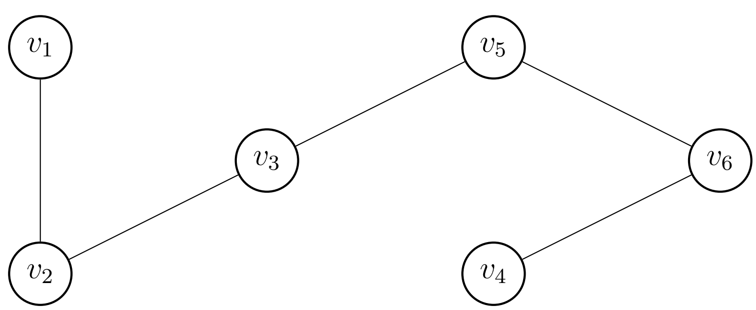
(a)



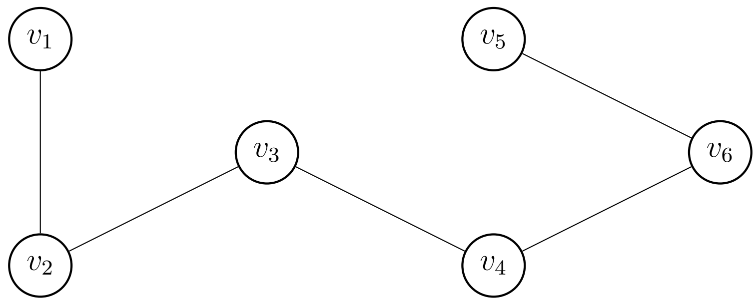
(b)



(c)



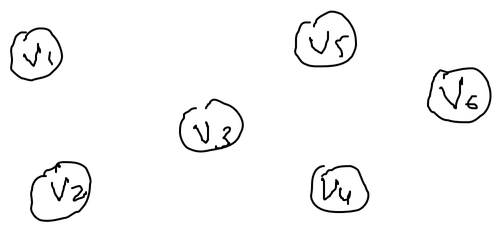
(d)



Answer:- Note:- In each step make sure cycle is not created.

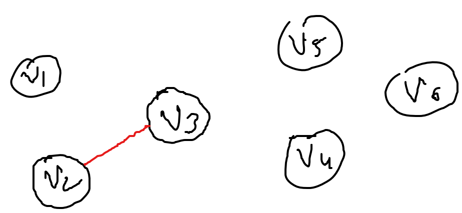
Step 1: For the graph given in (Figure PA-12.4) break them into n components (here $n=6$)

Eg:



Step 2: Connect 2 components in ascending order of cost.

Eg:



Note: We can connect either V_2—V_3 or V_3—V_4 as they have same edge weights. In above Eq. we are connecting vertices V_2 and vertex V_3

Step 3: Repeat step 2 until all the edges are connected.

see Figure (4).

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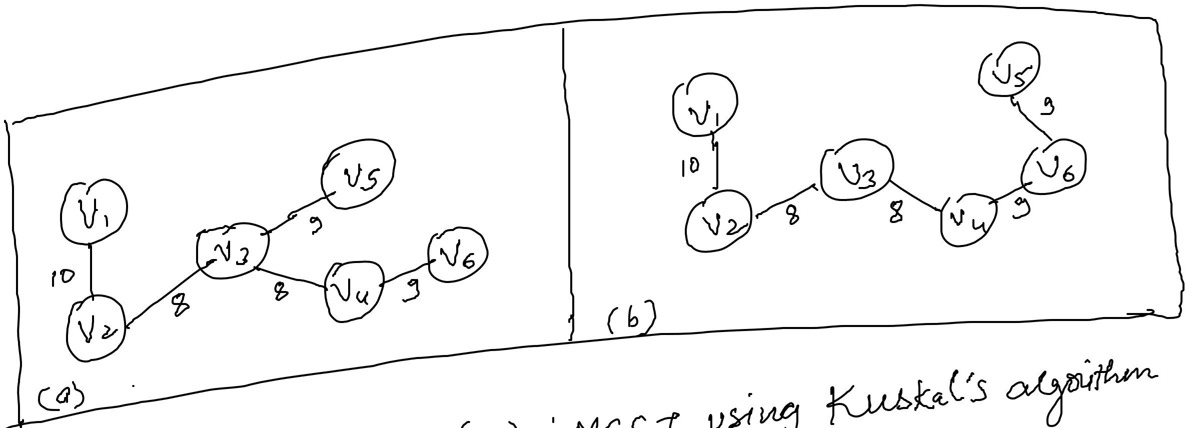

Figure(4) : MCST using Kuskal's algorithm

Note:- In the given graph (Figure PA-12.4) there are many edges which has same weights and thus we will have multiple MCST. Some of them are shown in Figure (4).

6. While using Bellman-Ford Algorithm for the graph shown below (Figure PA-12.5), let $D(v)$ be the shortest distance of vertex v from the source vertex 4 after 7 iterations.

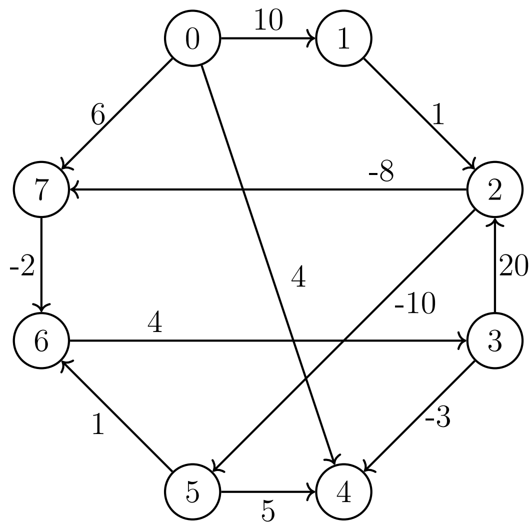

Figure PA-12.5

If the direction of edges in the graph are reversed, then which of the following is (are) CORRECT? [Ans: a,d]

(a) $D(0) = 2$

(b) $D(2) = 9$

(c) Bellman-Ford is not applicable for the new graph.

(d) $D(v)$ is negative for some vertex v.

Answer:-

Consider the new graph shown in Figure(10). after changing the directions of the edges of the graph given in Figure PA-12.5

Now, using Bellman-Ford algorithm we get adjacency matrix as shown in the $\boxed{\text{Table 1.}}$ Note, after 7 iterations we get the values of D(V) as shown in last column.

Clearly D(0) = 2 ; D(2) = -9 ; D(V) is negative for some vertex v.

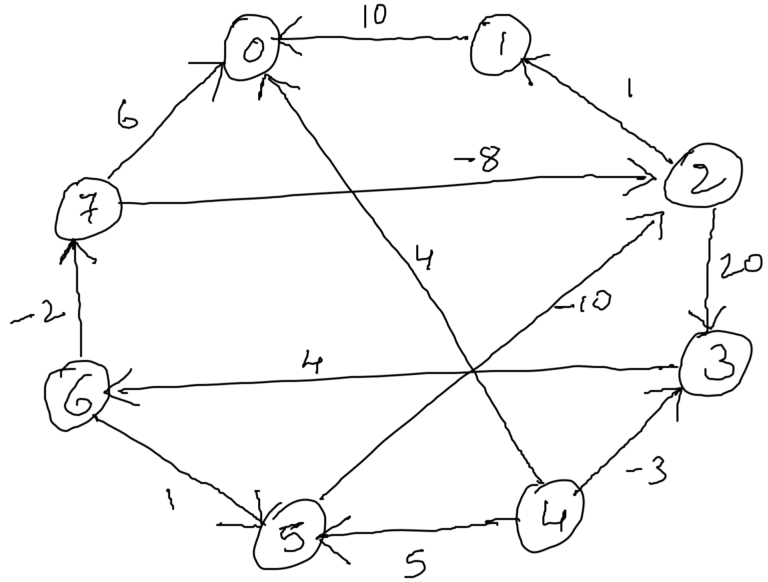

Figure (10)

v	D(v)							
0	∞	4	4	4	4	4	2	2
1	∞	∞	∞	-4	-4	-8	-8	-8
2	∞	∞	-5	-5	-9	-9	-9	-9
3	∞	-3	-3	-3	-3	-3	-3	-3
4	0	0	0	0	0	0	0	0
5	∞	5	5	2	2	2	2	2
6	∞	∞	1	1	1	1	1	1
7	∞	∞	∞	-1	-1	-1	-1	-1

Table 1

Bellman Ford is applicable for this new graph because there is no negative weight cycle.

7. Which of the following options are correct? [Ans:c, d]

(a) Let G be a weighted graph and in which the weights of all the edges are different. If we run a shortest path algorithm on G, then we will get a unique shortest path from the starting vertex to every other vertex.

(b) Suppose $G = (V, E)$ is a weighted graph, where $V = \{v_1, v_2, \ldots, v_n\}$. Let P be a shortest path from v_i to v_j $(i \neq j)$. If we increase the weight of each edge in the graph by one, then P will still be the shortest path from v_i to v_j.

(c) A graph G can have more than one spanning tree.

(d) Suppose $G = (V, E)$ is a weighted graph and the weights of all the edges are positive. Let P be a shortest path from $a \in V$ to $b \in V$. If we double the weight of every edge in the graph G, then the shortest path remains same but the total weight of path changes.

Answer:-

option(a)

Let G be weighted graph as shown in Figure (7)

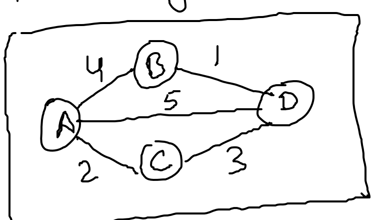

Figure (7)

Note: All the edge weights are different.

There are more than one shortest path to reach the vertex D from the starting vertex A. Thus, the statement as in option (a) is wrong.

Option (b)

Consider the graph shown in Figure (8)

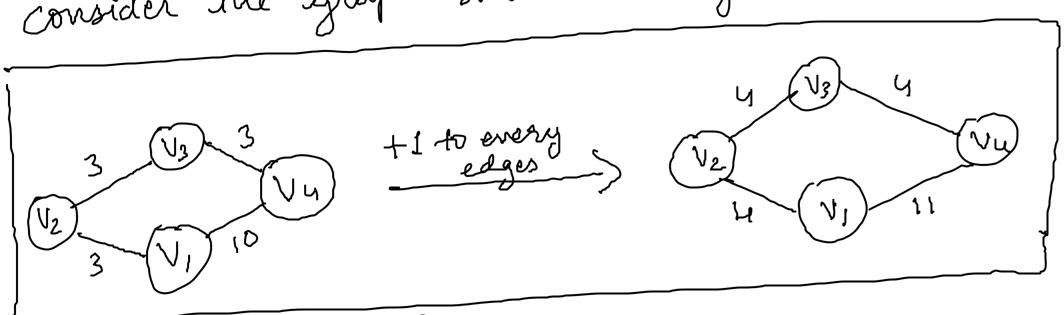

Figure (8)

Initially, the shortest path, to reach vertex V_4 from V_1 was $V_1 \to V_2 \to V_3 \to V_4$ (total cost is 9). If we increase the weights of each edge by one, (see Figure(8)) then the shortest path from V_1 to V_4 becomes $V_1 \to V_4$. So we are getting a new shortest path after the increament so option (b) is wrong.

option(c)

A simplest example could be a graph G, when all the edges have same weights.

option (d)

If we multiply all edge weights by 2, then the shortest path doesn't change. Because weights of all paths from a to b gets multiplied by 2. Here the number of edges in a path doesn't matter.

8. Which of the following options are correct? [Ans: a,d]

(a) Dijkstra's algorithm works for graphs having no negative weight edge.

(b) Floyd–Warshall algorithm works for graphs with negative weight cycles.

(c) Dijkstra's algorithm works on any graph without negative weight cycles.

(d) Shortest path problem is not applicable for a graph with a negative weight cycle.

Answer

(a) Dijkstra's algorithm work for graph with non-negative edges, is the basic condition for this algorithm. So option (a) is correct.

(b) Floyd-Warshall algorithm doesn't works for graphs with negative weight cycles. So, option (b) is wrong.

(c) This option is incorrect. For Dijkstra's algorithm to work, the edge weights must be non-negative. A graph can have negative edges even though there are no negative weight cycles.

For example: Figure (6)

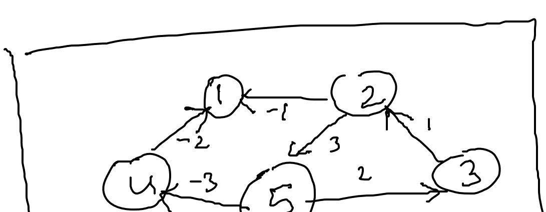

Figure (6)

vertices 2, 3 and 5 forms a non-negative cycle but there are edges b/w vertices 2 to 1, 4 to 1 and 5 to 4 with negative weight edges.

(d) Shortest path problem is applicable for non-negative weight cycle. So, option (d) is correct.

3 NUMERICAL ANSWER TYPE:

9. What is the weight of a minimum cost spanning tree of the graph given below (Figure PA-12.6)?
   [ans: 38]

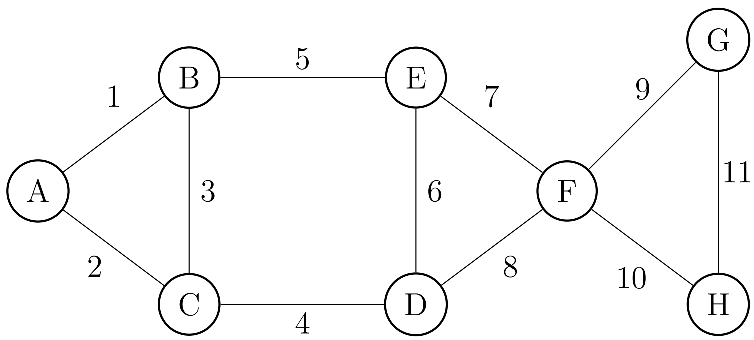

Figure PA-12.6

Answer:-

Using Prim's algorithm, MCST of graph shown in Figure PA-12.6 is (see Figure (5))

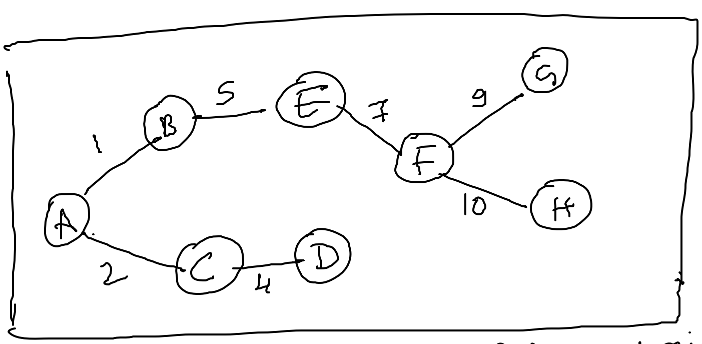

Figure (5): MCST using Prim's algorithm

Weight of MCST will be sum of weights of all the edges in MCST which is $1+2+4+5+7+9+10 = \underline{38}$

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