# IIT Madras

ONLINE DEGREE

**Mathematics for Data Science 1**
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**Week - 01**
**Lecture – 11**
**Degrees of infinity**

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So, when we looked at the sets of numbers, we said that we have various kinds of infinite sets – the natural numbers, integers, reals, the rationals, some of them are discrete, some of them are dense. And the question that we asked was whether they all have the same size, or there are more of one than another?

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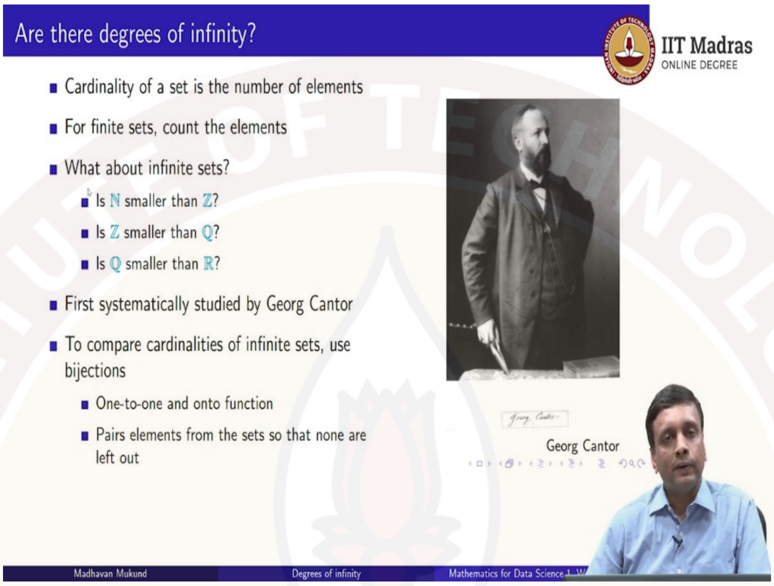


So, the question that we want to ask is, are there degrees of infinity? So, we know that for a set the cardinality denotes a number of elements, and if it is a finite set we just have to count these elements. So, for a finite set, there is no problem about cardinality which is the count the number of elements and we are done.

We get a natural number which is the cardinality of the set. Now, the question is what do we do for infinite sets. So, let us look at the natural numbers for instance. So, in which we move from the natural number to the integers, we added negative numbers. So, clearly we have added an infinite set of numbers we roughly doubled the set. So, is the set of natural numbers are same as the integer number in size or not?

Similarly, when we move from the integers to the rationals, we move from a discrete set where we had a next and previous element to a dense set where between any two element there is an another element. So, this suggests that there should be more rational than reals rationals than integers, but is that true or not?

And finally, when we move from rationals to real numbers we added a whole bunch of irrational numbers which cannot be expressed in the form $\frac{p}{q}$. So, clearly the real numbers have a large number of new things which are not in the rationals. So, again is the set of reals larger than the set of rationals or not? So, this study of the cardinality of infinite sets was actually undertaken by Georg Cantor in the 1870s. And as we have seen when we studied functions the correct way to compare the cardinality of infinite sets is to use a bijection.

So, what is the bijection? The bijection is one-to-one and an onto function. In other words, it allows us to map one set to another set in such a way that two elements are always map to two different elements and everything on the other side is map 2 from something here that is the onto part. So, it is one-to-one no onto elements map to the same one, and it is onto no element on the right hand side is missed out.

So, intuitively what this allows us do through this function this bijection is to pair up the elements from the one side with the elements from the another side. So, I take an element on the left hand side, through the bijection I pair up it with an element of right hand side. And because it is one-to-one and onto, this pair this paring actually exhaustibly covers all the elements in both sides or nothing is left out.

So, we have paired up everything and therefore, the two sides have the same cardinality. So, this is the technique that we will investigate in order to resolve these questions about the cardinalities of the infinite sets of numbers that we have discussed above.

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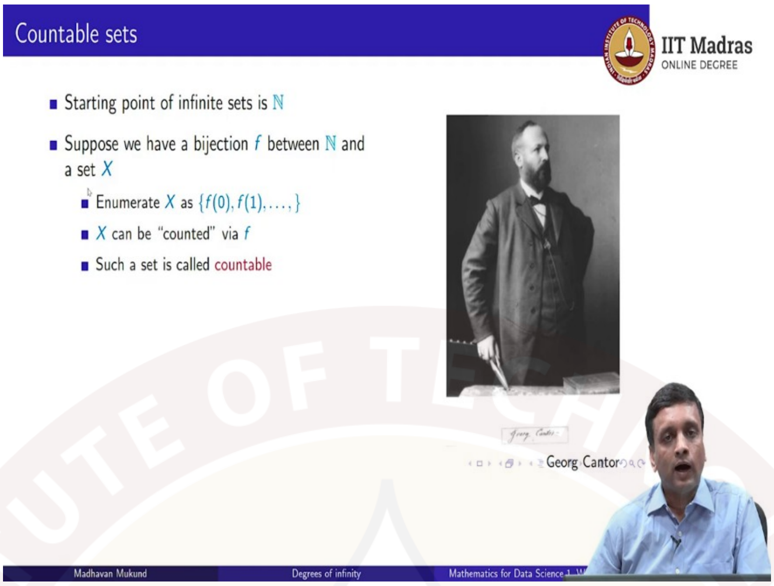


So, our starting point is the set of natural number, because this is the first infinite set that we have to begin with. When we start counting we realized that there is no largest number because we can always add 1. And so if we take all the finite numbers that we can used to count, we get an infinite set called the natural numbers.

Now, supposing we find a bijection between the set of natural numbers and some other set X, does not matter what this set is, but supposing there is a bijection. We can pair of the natural numbers with the elements of X. This means that we can actually effectively enumerate the elements of X, we can take the number paired with 0, f (0) and call that the beginning of X, then f (1) is an X element, f (2) and so on.

And because we are doing this kind of enumerating X, we can count X in a way via f and so we call any such set countable. So, countable set is one which can be bijectively paired up with the set of natural numbers. So, when we are looking at other sets, we will first check whether they are countable or if not we have to argue that they cannot be counted.

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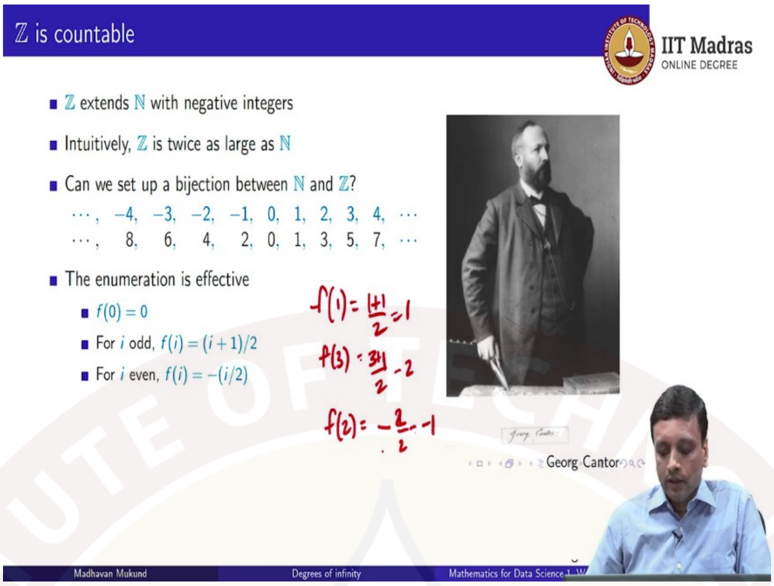


So, let us begin with set of integers and show that it is countable. So, why should be, why should it not be countable, or why should it be a surprise if it is countable? Well, because $Z$ extends $N$ with negative integers right. So, for every, if you do not count 0 in the calculation, for every positive natural number there is a corresponding negative integer in $Z$.

So, $Z$ is referring twice as big as $N$; for +1 you have -1; +2 you have -2 and so on right. So, it seems contradictory that you can double the set, and still have the set of the same size that you started with. So, the question now is for $Z$ to be countable, can we set up a bijection between the natural numbers and $Z$?

So, let us look at $Z$ as we do on the number line. So, it starts from some $-\infty$ and then it comes to -4, -3, -2, -1, 0, 1, 2, 3, 4 and continues. So, we start our enumeration at 0. So, we enumerate 0, the 0 of $Z$ as the 0th element, then we map 1 to +1, map 2 to -1. What do we do next? Well, we map 3 to +2, and 4 to -2.

So, we keep zigzagging to the right hand to the left, we count $Z$ by starting with the center moving right one, moving left one, moving right one, moving left one. So, in this way we could now enumerate the number +3 as 5, -3 as 6, +4 as 7, -4 as 8. So, in this way we can actually enumerate $Z$ effectively. So, f (0) is 0 as we saw. If i is odd for example, 1 then f (i) is $\frac{i+1}{2}$.

So, f(1) for instance is (1 + 1)/ 2 = 1; f(3) = (3 + 1) / 2 = 2 and so on. So, if f is odd, I have $\frac{i+1}{2}$. And if it is even like 2, then I take - $\frac{i}{2}$. So, I take -2/2 which is -1. If it is 4, I get -4/2 which is -2 right. So, we have actually given an effective way of assigning a position in some sense or count to every number in $Z$, and this shows that the set of integers is actually countable.

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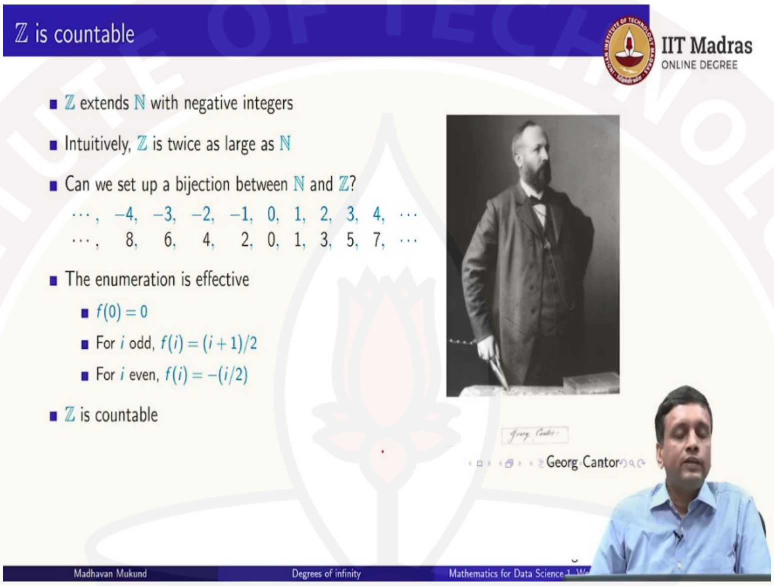


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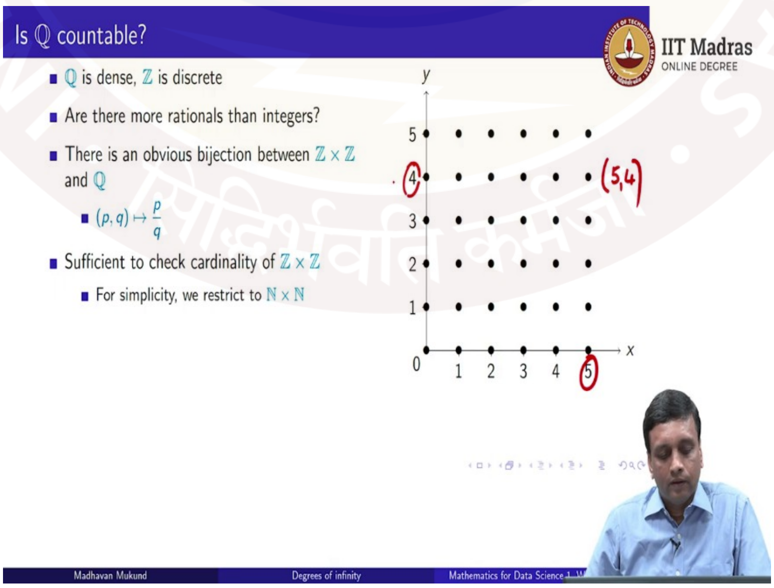

Now, what about the rationals? One reason why we might suspect that the rationals are not countable is because the rationals we saw a dense between any two rational numbers there is an another rational number.

Whether the integers and the rational numbers are discrete, you can always find a next number; and in the case of integers you can always find a previous number. For natural numbers 0 has no previous number, every other number has a previous and a next. So, given that rationals are dense and the integers are discrete, the question is are there more rationals than there are integers?

Now, there is an obvious bijection between pairs of integers and rationals because that is what a rational is, rational is a pair of integers p upon q. So, I can take a pair (p , q) and $Z$ cross $Z$ and directly connect it in an bijective way to the fraction $\frac{p}{q}$. So, every pair gives a unique rational number, every rational number gives me a unique pair.

There is no surprise here, there are no we are not talking about reduce forms of for example, we have different numbers like $\frac{1}{10}$, then we have $\frac{2}{20}$, and $\frac{3}{30}$, these are all different rational numbers they may represent the same value, but they represent different pairs. So, this is a clear bijection between $Z$ cross $Z$ and $Q$. So, $Z$ cross $Z$ has the same size of $Q$.

So, if we are looking at the cardinality of $Q$, we can also look at the cardinality of $Z$ cross $Z$. Because if we can measure the cardinality the size of $Z$ cross $Z$, then through this bijection, $Q$ must have the same size, there is no need to separately measure the size of $Q$.

So, instead of $Z$ cross $Z$ just to make the picture easier to see, we will actually do $N$ cross $N$, and then I will show you how to extend it to $Z$ cross $Z$. So, here is a picture of $N$ cross $N$. So, remember that we think of $N$ cross $N$ in a two-dimensional grid and at each point (i, j) I have a dot representing the pair (i, j). So, for instance this pair, this pair is (5 , 4), because it comes from the 5 and the 4 over here right. So, every dot in this pair in this grid is a pair in $N$ cross $N$.

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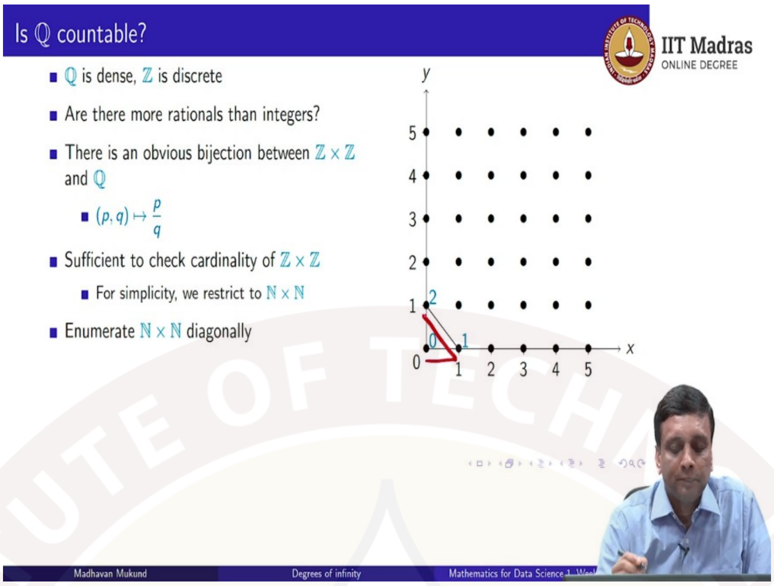


Now, I am going to enumerate this in a particular way. So, here is a one enumeration. So, you start with the 0th element as the element at the bottom left corner what is normally called the origin. Then you enumerate the first diagonal right, so you go from here and then you go right and then you go up. So, you enumerate in this way then you continue.

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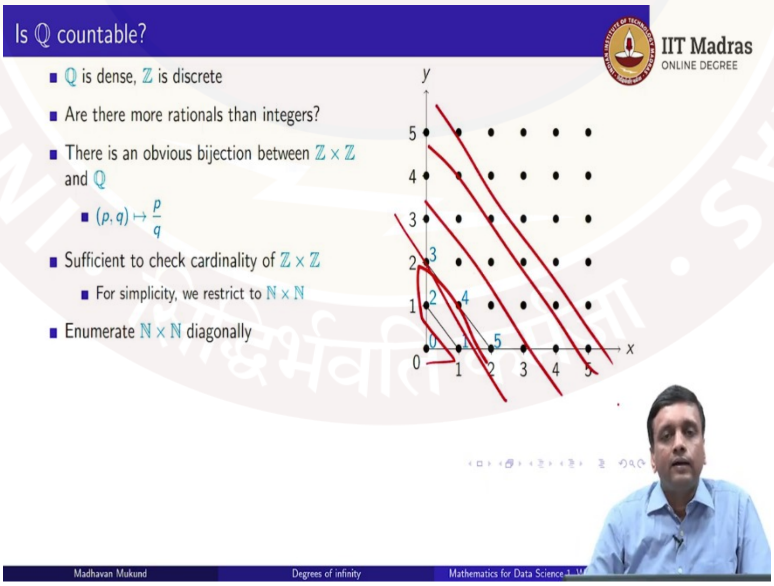


So, you started from here went up, then you up there, and come back down again right. So, you can slice this thing like this right. So, you can slice this gird like this, and enumerate it diagonal by diagonal.

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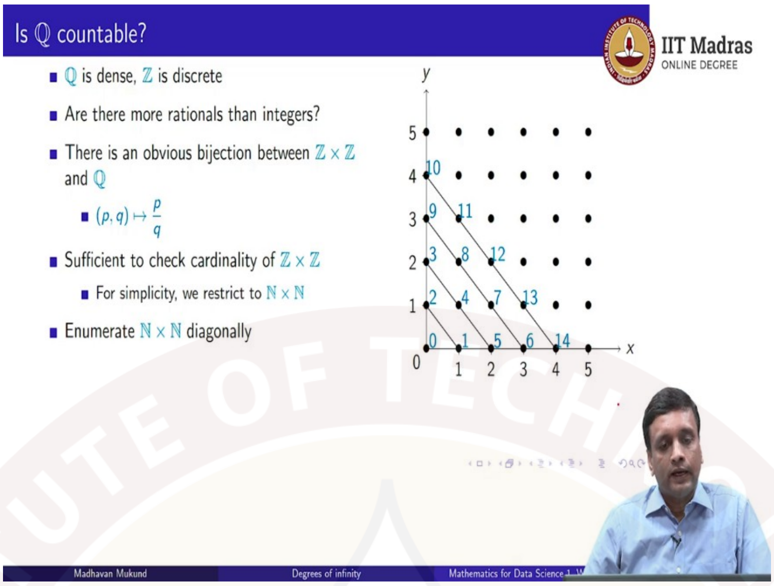


So, this gives us an effective enumeration of $N$ cross $N$.

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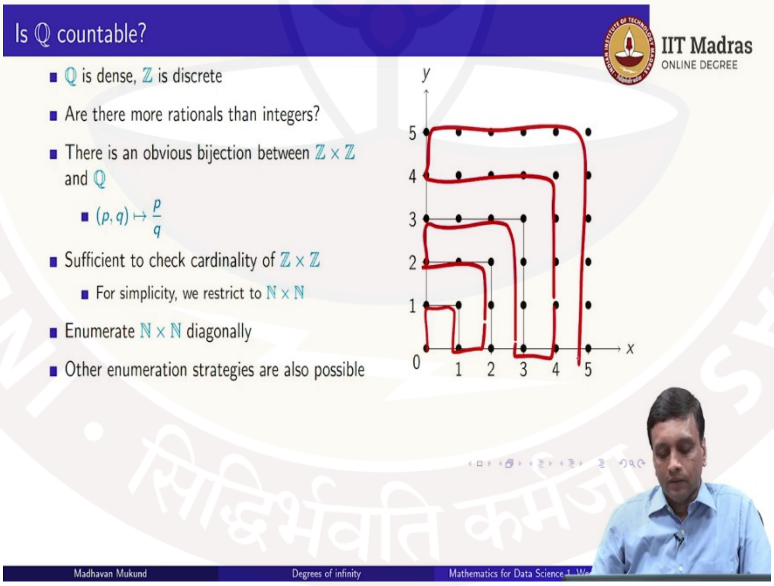


But we can also enumerate in different ways. For instance, we can enumerate in these larger and larger squares. So, we can start here, then finish this, then do this, then do this, then do this and so on right. So, long if we do not miss out any point in the grid we are done.

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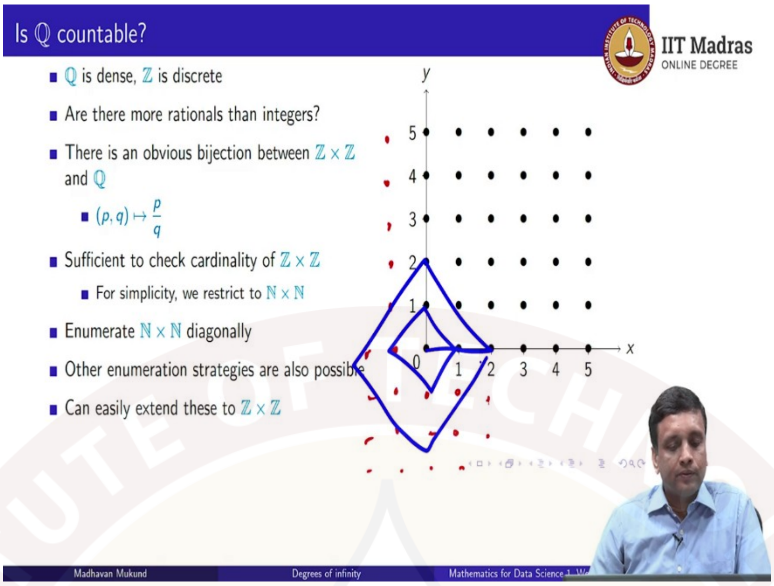


So, this shows us that $N$ cross $N$ is something that we can enumerate. Now, how would we do it for $Z$ cross $Z$? Well, it is very simple. If I had $Z$ cross $Z$, I would also have points on this side, and I would also have points below right. So, I would have points to the left and below 0 because I would had negative numbers.

So, now, if I wanted to enumerate $Z$ X $Z$, I would start here, then I would do this, and I would complete this diamond, then I would go here, and then go here, and then complete this diamond and so on right. So, instead of doing just the diagonal, I would extend the diagonal around to form a diamond, and in this way I would start from the center and spiral out so that I enumerate all the numbers in $Z$ X $Z$. So, $N$ X $N$ can be enumerated as we saw, and this can be easily extended to $Z$ X $Z$.

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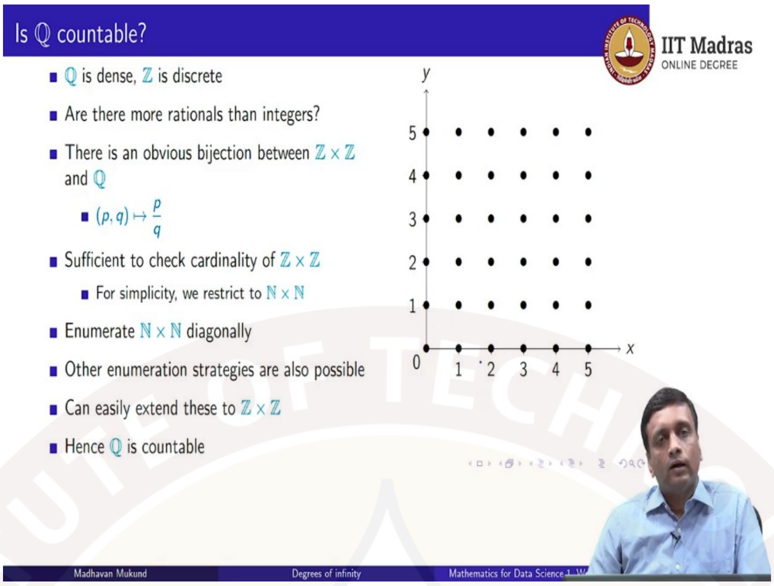


So, therefore, the set of rational numbers though it is dense and then it looks superficially to be much larger than the set of integers, actually both the integers and the rational numbers have the same number of elements which is quite surprising, but it is true.

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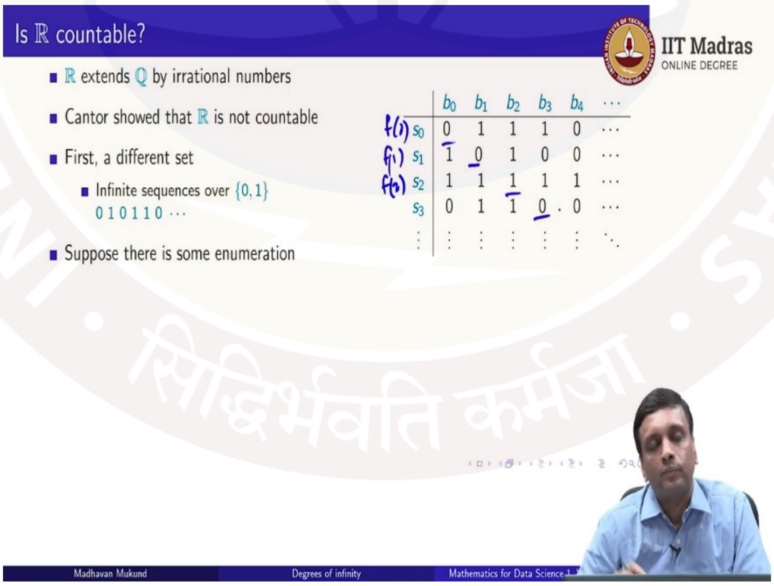


So, for all the infinite sets we have seen are countable right. Of course, the natural numbers are countable by definition, and then we saw integer are also countable, and the rational are also countable. So, what about the real numbers? So, how did we get to the real numbers? We took the rationals and then we added all these irrational numbers like √2, π, e and so on. So, Cantor showed that R actually is not countable. So, let us see how this proof works.

So, actually he did not, he did have a separate proof that R is not countable, but later on he made another proof which is easier to present which starts with the different set. So, instead of looking at $R$, we will look at something which looks quite different. We will look at infinite sequences over 0, 1. So, an infinite sequence of a 0, 1 is just something like you just keep writing down 0 or 1 infinitely many times without stopping right.

So, what Cantor argued is that this set is not something that you can count. So, supposing you can enumerate the infinite sequences over 0, 1, then on the right to see some enumeration; we are not looking at a particular enumeration in some particular order. We are just saying is there any enumeration at all, so that I can write down the 0-th sequence. So, this is the 0th sequences, this is f (0) in some sense, this is f (1), this f (2) and so on.

So, I have just written f (0) as $s_0$, and f (1) as $s_1$, and so on. And each sequence has positions which I have written b for bits because these are binary digits 0 or 1. So, each sequence has an infinite sequence of bits which characterize what it is, and no 2 rows are the same they are all different infinite sequences of 0s and 1s right. So, hypothetically this table is an enumeration of such sequences. So, if this is a enumeration of all such sequences, can be derive a contradiction? So, this is how Cantor derived a contradiction.

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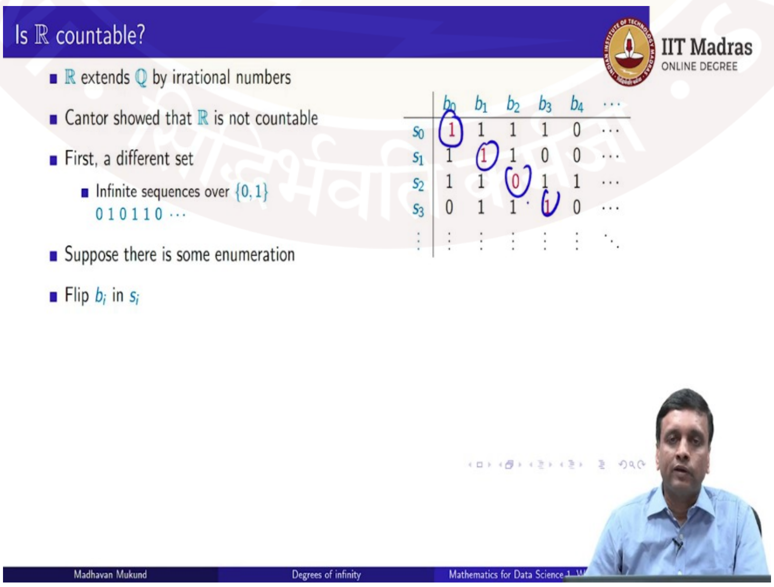

So, he said let us take each row and reverse the bit. And which bit to be reversed? Well, if we are in the ith row, then we reverse the ith bit. So, in the first row which is s 0, we reverse $b_0$ , in the second row. So, if you want go back, so this was 0, so we are here at 0 0 1 0. So, after flipping, it becomes 1 1 0 1 right. So, what we are doing is in 0th row, we are flipping $b_0$; in row $s_1$ we are flipping $b_1$; in $s_2$ we are flipping $b_2$, and so on.

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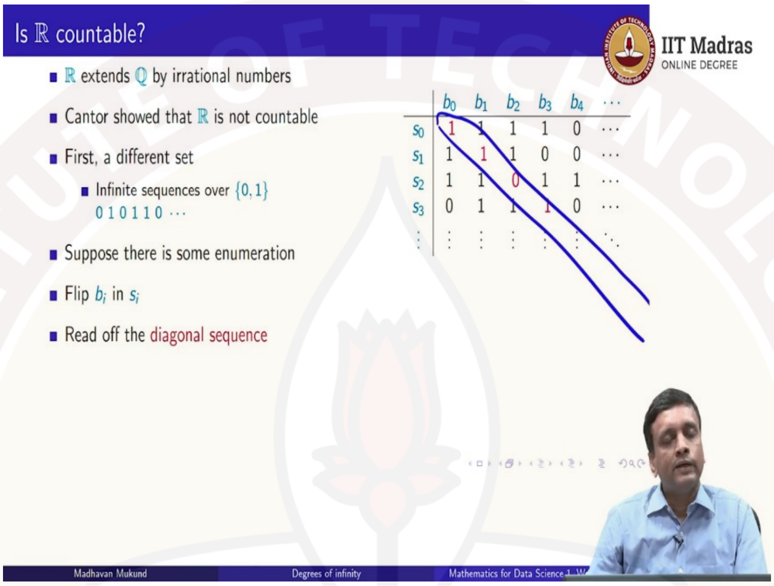


So, now this gives us a new sequence which we can read off diagonally right. The sequence consists of the red numbers which we have got by flipping the number at the i-th position in the i-th sequence. What can be say about this sequence? Well, first of all it is an infinite 0, 1 sequence.

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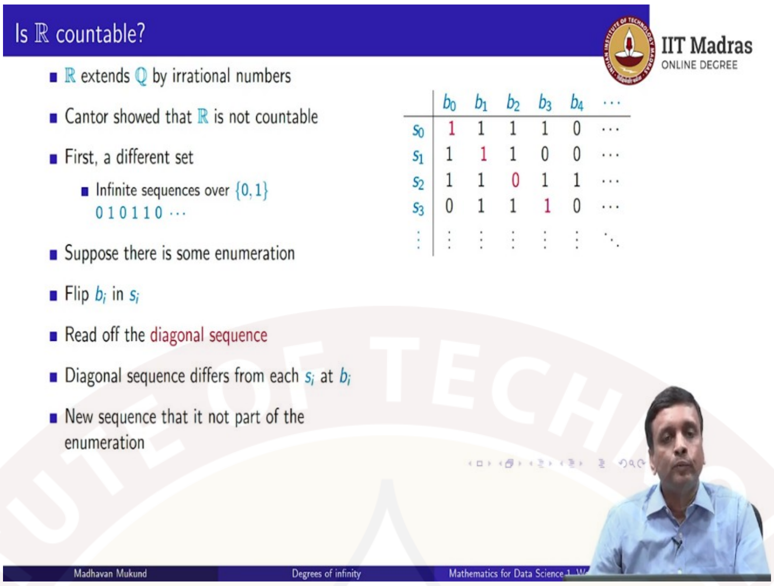


But this infinite 0, 1 sequence cannot be any of the rows in my table, because by construction if it is a row in my table it must be $s_j$ for some j, but at position j, $s_j$ has been flipped. So, this cannot be $s_j$ because if I had $s_j$ already in my table if the sequence is already in my table, the new sequence has the j-th bit flipped. So, diagonal sequence differs from each $s_i$ at $b_i$, and therefore, this new sequence that I have constructed cannot be part of the enumeration.

Now, it is important that we are shown this regardless of what the enumeration looks like, we have not made any assumption about the order in which we are enumerating. We have said no matter what sequence you have in mind in terms of enumeration, you would have to be able to write down the sequences one after the other table in a sequence of rows.

However you write it down, I will be able to construct this new diagonal sequence by taking the i-th bit in the i-th row and flipping it. So, however you enumerate it, I get a new sequence which is not part of your enumeration. Therefore, there is no possible way of enumerating 0, 1 sequences.

So, as we said this is not the question we asked, the question we asked is are the real numbers enumerable, are real numbers countable? And what we have actually argued is that 0, 1 sequences, infinite 0, 1 sequences are not countable. So, from here how do we get to the real number?

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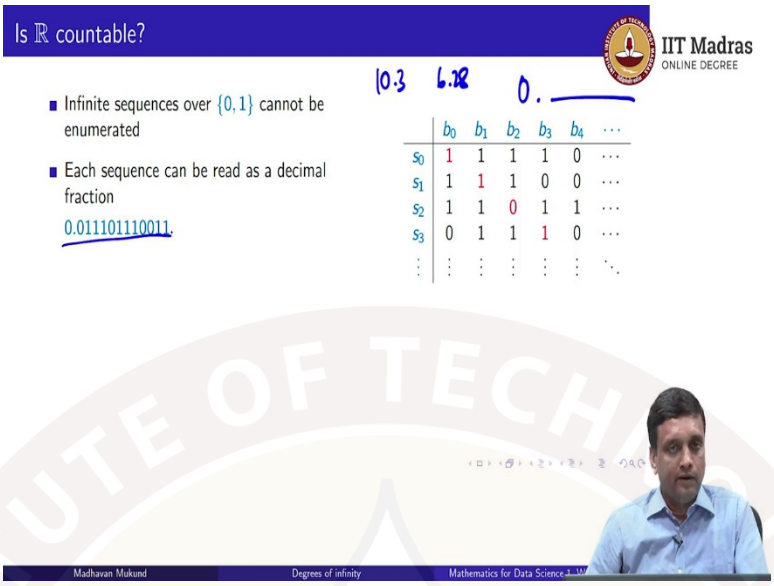


Well, it is one way to do this is to just think of these 0, 1 sequences as actually decimal fraction. Now, we know that we can write things like 10.3 and 6.28 and so on. So, now, we just restrict our self to writing in decimal fractions of the form 0 point something where everything on the right hand side of the decimal point is either a 0 or 1.

So, here is an example right. So, this is an example of a 0, 1 sequence represented as a decimal fraction. So, since each sequence is different, each such decimal fraction represents a different number.

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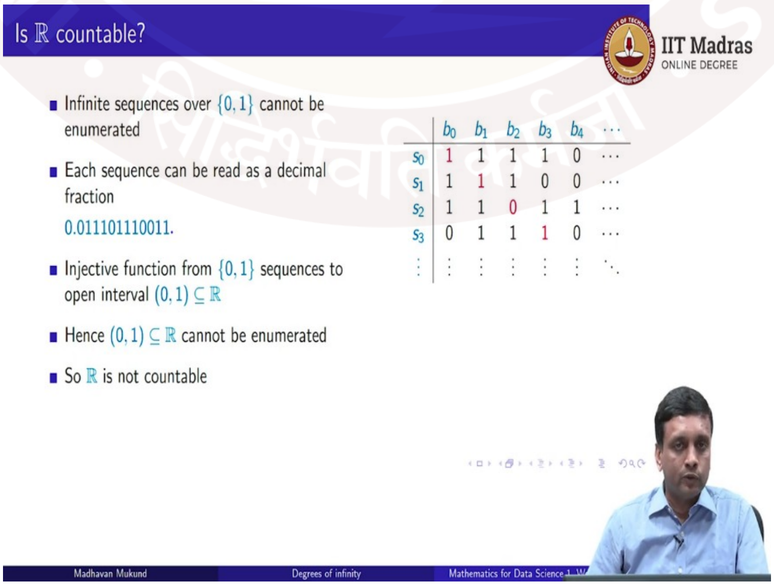

And these are all real numbers between 0 and 1, because they all have an integer part which is 0, and then we have something which is of course, we could have exactly 0 if we have all 0s ok. So, we definitely do not have, all we do not cannot get to 1, but we can think of these as numbers between 0 and 1.

So, each such sequence represents a different point in the interval 0 to 1. So, this is an injective function right. So, this is an injection that is a one-to-one function from infinite sequences 0, 1 to the interval (0, 1). Now, the interval (0, 1) is a very small fraction of the reals.

So, what this argument tells us is that in fact even this very small fraction of the reals is not countable because the set of underlying 0, 1 sequences not countable. So, if this even this small fraction of the reals is cannot be enumerated, then $R$ itself cannot be countable right. So, this is an indirect argument saying that not saying that $R$ itself is not countable directly, but saying that there is a small part of $R$, which is not countable. And since $R$ is much more than that, if the small part cannot be counted we have no hope of counting the whole thing.

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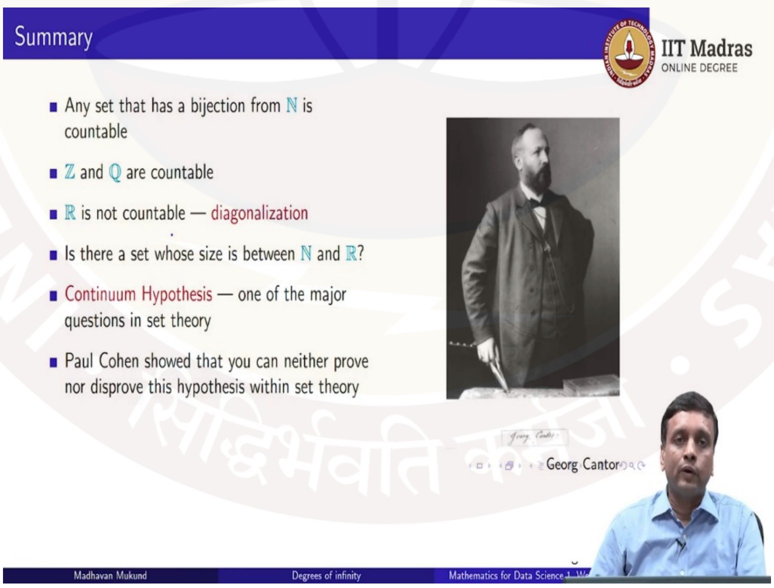


So, to summarize any set that has a bijection from $N$ is what we call a countable set. And we showed that the set of integers in the set of rationals are countable by describing a strategy to enumerate the sets. Now, this argument is due to Cantor which builds this diagonal sequence called diagonalization and has been used in many other proofs involving infinity after that. So, the proof of diagonalization by Cantor shows that the set of real numbers is not countable.

So, notice that the set of real numbers is not countable and the set of rationals is countable. What it does to the rationals to create the real numbers? We added the irrational numbers. So, actually the set of irrational numbers that we have added to the rationals must be itself uncountable, because we cannot take two countable sets and add them up and get an uncountable set. So, in other words, there are vastly more irrational numbers than there are rational numbers that is what it tells us.

Now, one question that we could ask is, is there anything in between? So, these are sets that we have been using intuitively. So, we have counted them. But can we construct something for instance which is not countable, but which is smaller than the reals right? So, is there such an infinite set?

Now, it turns out that this is a very non-trivial question. This question was actually posed when Cantor came up with this proof in the late 1800s, and it remained a very central opened question it was called the continuum hypothesis.

So, if you look at cardinal numbers in the finite sense, we have 1, 2, 3, 4, 5. So, we have a kind of small jumps between them, but we have a continuous sequence of numbers. Now, we seem to have this big jump between the real number the integers of the natural number and the real numbers, is there something in between or is it so, is there a continuum of these infinite numbers or these big jumps?

And this continuum hypothesis was a very important open question in set theory. And in the 1960s Paul Cohen actually showed that this is a question which cannot be proved or disproved. So, this is what is called independent. So, this is a fact which is independent of set theory using the axiom of the set theory, no way that you can either prove or disprove it.

So, both the fact that there is a such a set, and there is not such a set are consistent. So, these infinite sets lead to a lot of interesting questions, some of them are quite mindboggling, and they are quite counterintuitive. But if you are interested in these things, it is well-worth looking into them.