MATHEMATICS FOR DATA SCIENCE - I

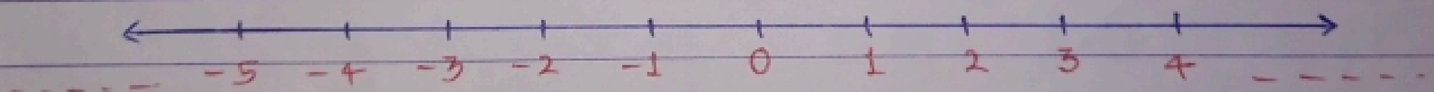
* Natural numbers

→ Numbers keep a count of object
→ 1, 2, 3, 4 - - - - .
→ 0 to represent no objects at all.
→ Natural numbers: N = {0, 1, 2, - - - - }
→ Sometimes N_0 to emphasize 0 is included

→ Addition, subtraction, multiplication, division

* Integers

→ 5-6 is not a natural number
→ Extend the naturals with negative numbers.
→ -1, -2, -3 - - - - .
→ Integers: Z = { - - - - -, -3, -2, -1, 0, 1, 2, 3 - - - }
→ Number line. —



* Multiplication and exponentiation

→ 7×4 — make 4 groups of 7
→ $m \times n = \underbrace{m + m + m - - - - - - + m}_{n \text{ times}}$

→ Notation = $m \times n$, $m \cdot n$, mn
→ $m \times m = m^2$
→ $m \times m \times m = m^3$

Date:
Page:

→ $m^K = \underbrace{m \times m \times m \cdots \times m}_{K \text{ times}}$ — m to the power K

→ Multiplication is repeated addition
→ Exponentiation is repeated multiplication

* Division

→ Division is repeated substraction

* Factors
→ a divides b if b mod a = 0
  * $a|b$
→ b is a multiple of a
→ $4|20$, $7|63$, $32|1024$, ...
→ $4 \nmid 19$, $9 \nmid 100$, ----
  a is a factor of b if $a|b$

→ Factors occur in pairs — factors of 12 are
→ {1, 12}, {2, 6}, {3, 4}
→ .... Unless the numbers ~~are~~ a perfect square —
→ factors of 36 —
→ {1, 36}, {2, 18}, {3, 12}, {4, 9}, {6}

* Prime numbers
→ p is prime if it has only two factors {1, p}
  * 1 is not a prime — only one factor

→ Prime numbers are 2, 3, 5, 7, 11, 13 ------.
* Sieve of Eratosthenes — remove multiples of p

→ Every numbers can be decomposed into prime factors

* $12 = 2 \cdot 2 \cdot 3 = 2^2 \cdot 3$
* $126 = 2 \cdot 3 \cdot 3 \cdot 7 = 2 \cdot 3^2 \cdot 7$

→ The decomposition is unique — prime factorization

* Rational numbers

→ Can not represent $19 \div 5$ as an integer

→ Fraction : $3\frac{4}{5}$

→ Rational number : $\frac{p}{q}$, p and q are integers

→ Numerator p, denominator q

→ Use $\mathbb{Q}$ to denote rational numbers

→ The same number can be written in many ways

→ $\frac{3}{5} = \frac{6}{10} = \frac{30}{50} = \ldots\ldots$

→ Useful to add, subtract, compare rationals.

→ $\frac{3}{5} + \frac{3}{4} = \frac{12}{20} + \frac{15}{20} = \frac{27}{20}$

→ $\frac{3}{5} < \frac{3}{4}$ because $\frac{12}{20} < \frac{15}{20}$

* Reduced form

→ Representation is not unique

→ $\frac{3}{5} = \frac{6}{10} = \frac{30}{50} = \ldots\ldots$

→ Reduced form: p/q.

→ where p, q have no common factors.

→ Reduced form of $\frac{18}{60}$ is $\frac{3}{10}$

* Greatest Common Divisor: gcd $(18, 60) = 6$
→ Recall prime factorization
→ $18 = 2 \cdot 3 \cdot 3$, $60 = 2 \cdot 2 \cdot 3 \cdot 5$
→ Common ~~p~~rime factors are 2, 3
→ Can find gcd (m, n) more efficiently.

⇒ Density
→ For each integers we have a next integer and a
→ previous integer
→ For m, next is $m+1$, previous is $m-1$
Next :- No integer between m and $m+1$
Previous :- No integers between m and $m-1$

⇒ Not possible for rationals
→ Between any two rationals we can find another one
→ Suppose $\frac{m}{n} < \frac{p}{q}$

→ Their average $\left(\frac{m}{n} + \frac{p}{q}\right)/2$ lies between them.

→ Rationals are dense, integers are discrete

Beyond rationals
→ Rational numbers are dense
→ Between any two rationals we can find another one

Is every point on the number line a rational number?

→ For an integer m, its square is $m^2 = m \cdot m$
→ Square root m, $\sqrt{m}$, is r such that $r \cdot r = m$
→ Perfect squares — 1, 4, 9, 16, 25 - - - - , 256 - - - - - -
→ Square roots — 1, 2, 3, 4, 5 - - - - - 16 - - - -

* What about integers that are not perfect squares?

→ $\sqrt{2}$ cannot be written as $\frac{p}{q}$

→ Yet we can draw a line of length $\sqrt{2}$
→ $\sqrt{2}$ is irrational
* Real numbers: R — all rational and irrational numbers

→ Like rationals, real numbers are dense
→ if $r < r'$, then $\left(\frac{r + r'}{2}\right)$ lies between r and r'

→ Some well known irrational numbers
$\pi = 3.1415927 - - - - -$
$e = 2.7182818 - - - - -$

* Can we stop with real numbers?
→ what about $\sqrt{-1}$
→ For any real number r, r^2 must be positive —
→ law of signs for multiplication.

* $\sqrt{-1}$ is complex number

* Fortunately we don't need to worry about them!

Date:
Page:

* Sets

→ A set is a collection of items
- → Days of the week —
  {Sun, Mon, Tue, Wed, Thu, Fri, Sat}
- → Factors of 24: {1, 2, 3, 4, 6, 8, 12, 24}
- → Primes below 15: {2, 3, 5, 7, 11, 13}

→ Sets may be infinite
- → Different types of numbers: N, Z, Q, R

→ No requirement that members of a set have uniform type.
- → Set of objects in a painting
  ~~Spot of~~

⇒ Order, duplicates, cardinality

→ Sets are unordered
- → {Kohli, Dhoni, Raina}
- → {Raina, Kohli, Dhoni}

→ Duplicates don't matter (unfortunately?)
- → {Kohli, Dhoni, Pujara, Kohli}

Cardinality :- number of items in a set
→ For finite sets, count the items
- → {1, 2, 3, 4, 6, 8, 12, 24} has cardinality 8
- → May not be obvious that a set is finite

* What about infinite sets?

→ Is Q biggar than Z?
→ Is R biggar than Q?
- Separate discussion

Date: ____
Page: ____

*** Describing sets, membership**

→ finite sets can be listed out explicitly.

→ {Kohli, Pujara, Dhoni}

→ $\{1, 2, 3, 4, 5, 6, 7, 8\}$

→ Infinite sets cannot be listed out

→ $N = \{0, 1, 2, 3 \ldots\}$ is not formal notation

→ Not every collection of items is a set

→ Collection of all sets is not a set

→ Russell's Paradox: Seperate discussion

→ Items in a set are called elements

→ Membership: $x \in X$, x is an element of X

→ $5 \in Z$, $\sqrt{2} \in Q$

*** Subsets**

→ X is a subset of Y

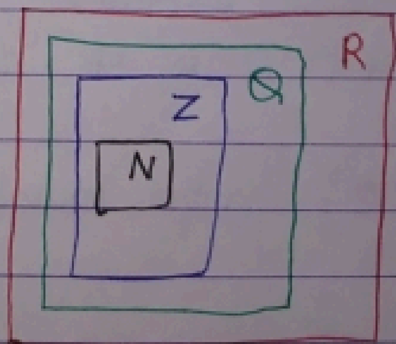
→ Every element of X is also an element of Y

→ Notation:- $X \subseteq Y$

→ Examples—

→ {Kohli, Pujara} ⊆ {Kohli, Dhoni, Pujara}

→ Primes $\subseteq N$, $N \subseteq Z$, $Z \subseteq Q$, $Q \subseteq R$



→ Every set is a subset of itself: $X \subseteq X$

→ $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$

→ Proper Subset:- $X \subseteq Y$ but $X \neq Y$

→ Notation: $X \subset Y$, $X \subseteq Y$

→ $N \subseteq Z$, $Z \subseteq Q$, $Q \subseteq R$

Date :
Page :

* The empty set and the powerset

→ The empty set has no element — Φ

→ $\Phi \subseteq X$ for every set X

→ Every element of Φ is also in X

→ A set can contain other sets

* Powerset — Set of subsets of a set.

→ $X = \{a, b\}$

→ Powerset is $\{\Phi, \{a\}, \{b\}, \{a, b\}\}$

→ Set with n element has 2^n subsets

→ $X = \{x_1, x_2 ---- x_n\}$

→ In a subset, either include or exclude each x_i

→ 2 choices per element, $\underbrace{2 \cdot 2 \cdot 2 \cdots 2}_{n \text{ times}} = 2^n$ subsets

* Subsets and binary numbers

→ $X = \{x_1, x_2, ---- x_n\}$

→ n bit binary numbers

→ 3 bits: 000, 001, 010, 011, 100, 101, 110, 111

→ Digit i represents whether x_i is included in a subset

→ $X = \{a, b, c, d\}$

→ 0101 is $\{b, d\}$

→ 0000 is Φ, 1111 is X

→ 2^n n bit numbers

* Constructing subsets

* Set comprehension

→ The subset of even integers

→ $\{x \mid x \in Z, x \bmod 2 = 0\}$

→ Begin with an existing set, Z

→ Apply a condition to each element in that set

→ $x \in Z$ such that $x \bmod 2 = 0$

→ Collect all the elements that match the condition

* Examples:-

→ The set of perfect squares

→ $\{m \mid m \in N, \sqrt{m} \in N\}$

→ The set of rationals in reduced form

→ $\{p/q \mid p, q \in Z, \gcd(p, q) = 1\}$

* Intervals

→ Integers from -6 to $+6$

→ $\{z \mid z, z \in Z, -6 \leq z \leq 6\}$

→ Real numbers between 0 and 1

→ Closed interval $[0, 1]$ — include endpoints

→ $\{r \mid r \in R, 0 \leq r \leq 1\}$

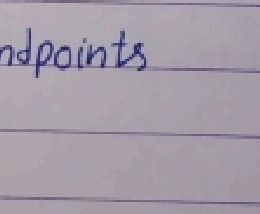
→ Open interval $(0, 1)$ — exclude endpoints

→ $\{r \mid r \in R, 0 < r < 1\}$

→ Left open $(0, 1]$

→ $\{r \mid r \in R, 0 < r \leq 1\}$

* Union



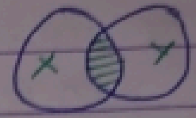
⇒ Union — Combine X and Y, $X \cup Y$

$\{a, b, c\} \cup \{c, d, e\} = \{a, b, c, d, e\}$

* Intersection, complement

~~Union — combine X and Y, X ∪ Y~~

⇒ Intersection — element common to X and Y, $X \cap Y$



→ $\{a, b, c, d\} \cap \{a, d, e, f\} = \{a, d\}$

⇒ Set difference — element in X that are not in Y, X / Y or $X - Y$

→ $\{a, b, c, d\} \setminus \{a, d, e, f\} = \{b, c\}$

⇒ Complement — elements not in X, $\overline{X}$ or X^c

→ Define complement relative to larger set, universe

→ Complement of prime numbers in N are composite numbers

* Sets Examples:—

→ Squares of the even integers

→ $\{x^2 \mid x \in \mathbb{Z}, x \bmod 2 = 0\}$ $\{0, 4, 16, 36, 64, 100 \ldots\}$

→ Generate Elements drawn from existing set

→ Filter Select elements that satisfy a constraint

→ Transform Modify selected elements

* More filters

→ Rationals in reduced form

→ $\{p/q \mid p/q \in \mathbb{Q}, \gcd(p, q) = 1\}$

→ Reals in interval $[-1, 2)$

→ $\{r \mid r \in \mathbb{R}, -1 \leq r < 2\}$

\# Cubes of first 5 natural numbers:-

$Y = \{n^3 \mid n \in \{0, 1, 2, 3, 4\}\}$

\# Cubes of first 500 natural numbers:-

$Y = \{n^3 \mid n \in \{0, 1, 2, 3 \ldots 498, 499\}\}$

Use set comprehension to define first 500 natural numbers.
$X = \{n \mid n \in N, n < 500\}$

Now, a more readable version
$X = \{n \mid n \in N, n < 500\}$
$Y = \{n^3 \mid n \in X\}$

Perfect squares :—

Integers whose square root is also an integer
$\{z \mid z \in Z, \sqrt{z} \in Z\}$

All squares are positive, so this is the same as
$\{n \mid n \in N, \sqrt{n} \in N\}$

Alternatively, generate all the perfect squares
$\{n^2 \mid n \in N\}$

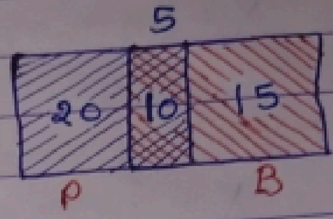
Extend the definition to rationals
$\frac{9}{16} = \left(\frac{3}{4}\right)^2$ is a square $\frac{1}{2} \neq \left(\frac{p}{q}\right)^2$ for any p, q is not
$\{q \mid q \in Q, \sqrt{q} \in Q\}$, or $\{q^2 \mid q \in Q\}$

* Choose the generator as required

* Counting problems

÷ In a class 30 students took Physics, 25 took Biology and 10 took both, and 5 took neither. How many students are there in the class?

→ Draw sets for Physics (P) and Biology (B)



→ 10 students are in $P \cap B$

→ This leaves 20 students in $P \setminus B$

→ Took Physics, but did not take Biology

→ Likewise 15 students in $B \setminus P$

→ Took Biology, but did not Physics

→ 5 students in $\overline{P \cup B}$

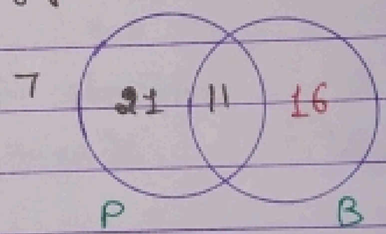
→ In the class, but took neither Physics nor Biology

Class strength: $5 + 20 + 10 + 15 = 50$

In a class of 55 students, 32 students took Physics, 11 took both Physics and Biology, and 7 took neither. How many students took Biology but not Physics?

→ $7 + 21 + 11 + x = 55$

→ $x = 55 - 39 = 16$



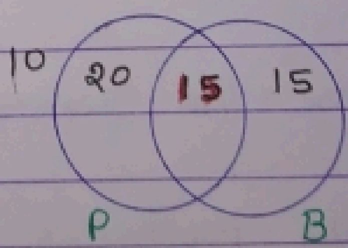
In a class of 60 students, 35 students took Physics, 30 took Biology and 10 took neither. How many took both Physics and Biology?

→ $|Y|$: Cardinality of Y (numbers of elements)

→ $|P| + |B| = 35 + 30 = 65$

→ $|P \cup B| = 60 - 10 = 50$

→ So $65 - 50 = 15$ must have taken both



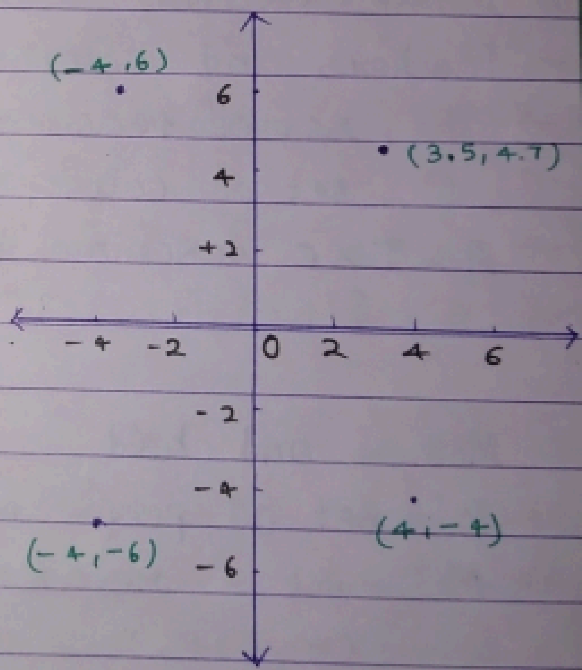
Note :- Venn diagrams can be useful to work out problems involving sets.

* Relations :

→ A set is a collection of items
→ We can combine sets to form new ones
→ $X \cup Y$, $X \cap Y$, $X \setminus Y$
→ $\bar{X}$ with respect to Y

→ Define subsets using set comprehension
→ Odd integers
→ $\{ z \mid z \in Z, z \bmod 2 = 1 \}$
→ Rationals not in reduced form
→ $\{ p/q \mid p, q \in Z, \gcd(p, q) > 1 \}$
→ Reals in $[3, 17)$
→ $\{ r \mid r \in R, 3 \le r < 17 \}$

Cartesian Product



→ $A \times B = \{ (a, b) \mid a \in A, b \in B \}$
  → Pair up elements from A and B
  → $A = \{0, 1\}$, $B = \{2, 3\}$
  → $A \times B = \{ (0,2), (0,3), (1,2), (1,3) \}$

→ In a pair, the order is important
→ $(0,1) \neq (1,0)$

→ For sets of numbers, visualize product as two dimensional space
→ $N \times N$
→ $R \times R$

Binary Relations

→ Select some pairs from the Cartesian product

→ Combine Cartesian product with set comprehension.

→ $\{(m,n) \mid (m,n) \in N \times N,\ n = m+1\}$

→ $\{(0,1),(1,2),(2,3) \ldots\ldots (17,18)\}$

→ Pairs (d,n) where d is a factor of n

→ $\{(d,n) \mid (d,n) \in N \times N,\ d \mid n\}$

→ $\{(1,1) \ldots\ldots (2,82) \ldots\ldots (14,56) \ldots\}$

* Binary relation $R \subseteq A \times B$
* Notation:- $(a,b) \in R$, aRb

* More Relation —

→ Teachers and courses

→ T, set of teachers in a college

→ C, set of courses being offered

→ $A \subseteq T \times C$ describes the allocation of teachers to courses

→ $A = \{(t,c) \mid (t,c) \in T \times C,\ t \text{ teaches } c\}$

→ Mother and child

→ P, set of people in a country

→ $M \subseteq P \times P$ relates mothers to children

→ $M = \{(m,c) \mid (m,c) \in P \times P,\ m \text{ is the mother of } c\}$

* Points at distance 5 from $(0,0)$

→ Distance from $(0,0)$ to (a,b) is $\sqrt{a^2+b^2}$

→ $\{(a,b) \mid (a,b) \in R \times R,\ \sqrt{a^2+b^2} = 5\}$

→ $\{(0,5),(5,0),(3,4),(-3,-4) \ldots\ldots\}$

→ A circle with center at $(0,0)$

* Rationals in reduced form

→ A subset of Q

→ $\{p/q \mid (p,q) \in Z \times Z, \gcd(p,q) = 1\}$

→ .... but also a relation on $Z \times Z$

→ $\{(p,q) \mid (p,q) \in Z \times Z, \gcd(p,q) = 1\}$

* Beyond binary relations

→ Cartesian products of more than two sets ...

→ Pythagorean triples

→ Square of the hypotenuse is the sum of the squares on the opposite sides..

→ $\{(a,b,c) \mid (a,b,c) \in N \times N \times N, a,b,c > 0, a^2 + b\}$

* Corner of squares

→ A corner is a point $(x,y) \in R \times R$

→ $((x_1,y_1),(x_2,y_2),(x_3,y_3),(x_4,y_4))$ are related if they are four corner square

* For instance:-

→ $((0,0),(0,2),(2,2),(2,0))$

→ $((0.5,0),(0,0.5),(0.5,1),(1,0.5))$

⇒ $Sq \subset R^2 \times R^2 \times R^2 \times R^2$

* Back to binary relations

⇒ Identity relation: $I \subseteq A \times A$

→ $I = \{(a,b) \mid (a,b) \in A \times A, a = b\}$

→ $I = \{(a,a) \mid (a,a) \in A \times A\}$

→ $I = \{(a,a) \mid a \in A\}$

⇒ Reflexive relations: $R \subseteq A \times A$, $I \subseteq R$

→ $\{(a,b) \mid (a,b) \in N \times N, a,b > 0, a \mid b\}$

→ $a \mid a$ for all $a > 0$

* Symmetric relations:

→ $(a,b) \in R$ if and only if $(b,a) \in R$

→ $\{(a,b) \mid (a,b) \in N \times N, \gcd(a,b) = 1\}$

→ $\{(a,b) \mid (a,b) \in N \times N, |a-b| = 2\}$

* Transitive relations:

→ If $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$

→ $\{(a,b) \mid (a,b) \in N \times N, a|b\}$

→ If $a|b$ and $b|c$ then $a|c$

→ $\{(a,b) \mid (a,b) \in R \times R, a < b\}$

→ If $a<b$ and $b<c$ then $a<c$

* Antisymmetric relations:

→ If $(a,b) \in R$ and $a \neq b$, then $(b,a) \notin R$

→ $\{(a,b) \mid (a,b) \in R \times R, a < b\}$

→ If $a<b$ then $b \nless a$

→ $M \subseteq P \times P$ relates mothers to children

→ If $(p,c) \in M$ then $(c,p) \notin M$

Equivalence relations

→ Reflexive, symmetric and transitive

→ Same remainder modulo 5

→ $7 \bmod 5 = 2$, $22 \bmod 5 = 2$

→ If $a \bmod 5 = b \bmod 5$ then $(b-a)$ is a multiple of 5

→ $Z \bmod 5 = \{(a,b) \mid a,b \in Z, (b-a) \bmod 5 = 0\}$

→ Divides integers into 5 group based on remainder when divided by 5

→ An equivalence relation partitions a set

→ Groups of equivalent elements are called equivalence classes

* Functions
⇒ A rule to map inputs to outputs
⇒ Convert x to x^2
→ The rule : $x \mapsto x^2$
→ Give it a name : $sq(x) = x^2$
→ Input is a parameter

→ Need to specify the input and ~~outputs~~ output sets
→ Domain : Input set
domain (sq) = $\mathbb{R}$

→ Codomain : Output ~~sets~~ set of possible values
Codomain (sq) = $\mathbb{R}$

→ Range : Actual values that the output can take
range(sq) = $\mathbb{R}_{\geq 0} = \{r \mid r \in \mathbb{R}, r \geq 0\}$

$f : X \to Y$, domain of f is X, codomain is Y

* Functions and relations
→ Associate a relation R_f with each function f
→ $R_{sq} = \{(x, y) \mid x, y \in \mathbb{R}, y = x^2\}$
→ Additional notation : $y = x^2$
→ $R_f \subset$ domain $(f) \times$ range (f)

→ Properties of R_f
→ Defined on the entire domain
→ For each $x \in$ domain(f), there is a pair $(x, y) \in R_f$
⇒ Single - valued
→ For each $x \in$ domain (f), there is exactly one $y \in$ codomain (f) such that $(x, y) \in R_f$

Date :
Page :

⇒ Drawing f as a graph is plotting R_f

B

* Lines

→ $f(x) = 3.5x + 5.7$

→ 3.5 is the slope

→ 5.7 is intercept where the line crosses the y axis, where $x = 0$

Changing the slope and intercept produce different lines

→ $f(x) = 3.5x - 1.2$

→ $f(x) = 2x + 5.7$

→ $f(x) = -4.5x + 2.5$

In all the cases

→ Domain = $\mathbb{R}$

→ Codomain = Range = $\mathbb{R}$

* More functions

→ $x \mapsto \sqrt{x}$

* Is this a function?

→ $5^2 = (-5)^2 = 25$

→ $\sqrt{25}$ gives two options

→ By convention take positive square root

⇒ What is the domain?

→ Depends on codomain

→ Negative numbers do not have real square roots

→ If codomain is $\mathbb{R}$, domain is $\mathbb{R} \geq 0$

→ If codomain is the set $\mathbb{C}$ of complex numbers, domain is $\mathbb{R}$.

* Types of Functions :—

* **Injective**: Different inputs produces different outputs — one-to-one
  - → If $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$
  - → $f(x) = 3x+5$ is injective
  - → $f(x) = 7x^2$ is not: for any a, $f(a) = f(-a)$

* **Surjective**: Range is the codomain — onto
  - → For every $y \in$ codomain(f), there is an $x \in$ domain(f) such that $f(x) = y$
  - → $f(x) = -7x+10$ is surjective
  - → $f(x) = 5x^2+3$ is not surjective for codomain R
  - → $f(x) = 7\sqrt{x}$ is not surjective for codomain R

* **Bijective**: 1-1 correspondaence between domain and codomain
  - → Every $x \in$ domain(f) maps to a distinct $y \in$ Codomain(f)
  - → Every $y \in$ codomain(f) has a unique pre-image $x \in$ domain(f) such that $y = f(x)$

Theorem :—

→ The function is bijective if and only if it is injective and ~~sub~~ subjective.

→ From the definition, if a function is bijective it is injective and surjective.

→ Suppose a function f is injective and surjective :—

→ Injectivity guarntees that f satisfies the first condition of bijection.

→ Surjectivity says every $y \in$ Codomain(f) has a pre-image. Injectivity guarntees this pre-image is unique.

* Bijections and cardinality

→ For finite sets we can count the items

* What if we have two large sacks filled with marbles?

→ Do we need to count the marbles in each sack?

→ Pull out marbles in pairs, one from each sack

→ Do both sacks become empty simultaneously?

→ Bijection between the marbles in the sacks.

→ For infinite sets

→ Number of lines is the same as $R \times R$

→ Every line $y = mx + c$ is determined uniquely by (m, c) and vice versa

→ For every ~~pairs~~ pair of points (x_1, y_1) and (x_2, y_2), there is a unique line passing through both points.

→ Number of lines is same as cardinality of $R \times R$

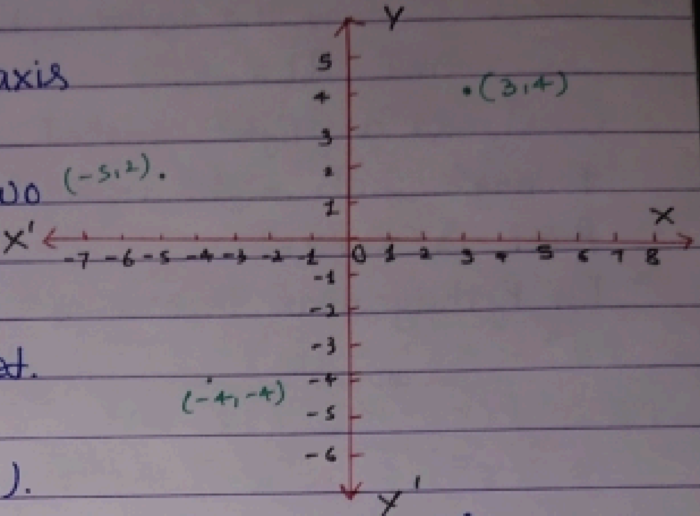
→ Does this show that $(R \times R) \times (R \times R) = R^2 \times R^2$ has the same cardinality as $R \times R$?

→ The correspondence is not a bijection — many pairs of points describe the same line

→ Be careful to establish that a function is a bijection.

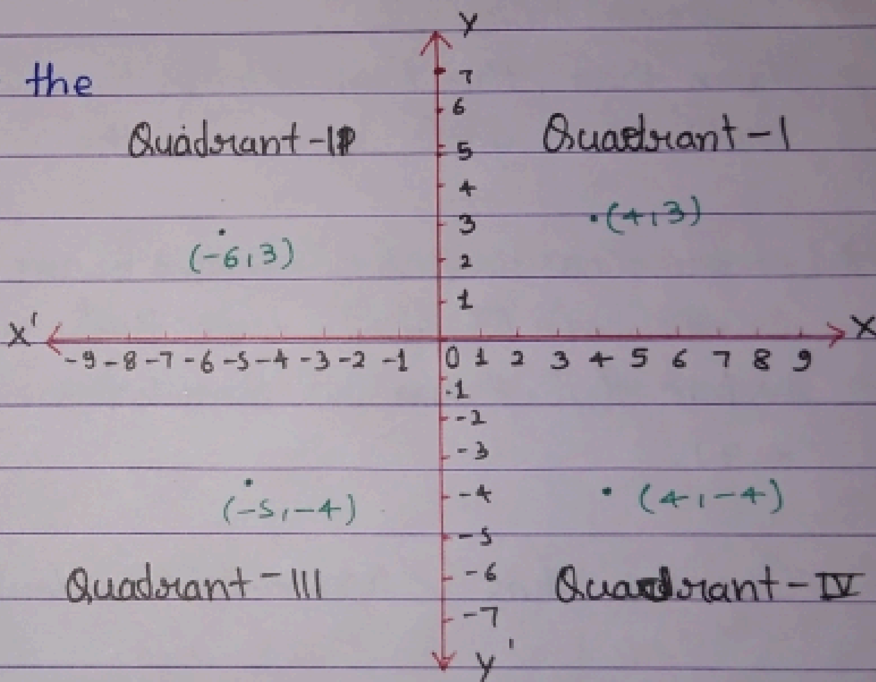
* Elements of Coordinate Geometry - Axes, Points and Lines

- → The horizontal line is called x-axis
- → The vertical line is called y-axis
- → The point of intersection of these two lines is called origin.
- → Any point on the coordinate plane can be represented by an ordered pair (x, y).
- → For example, $P = (3,4)$, $Q = (-5,2)$.



Coordinate Plane

- → The coordinate axes split the coordinate plane into four quardrants and two axes.
- → Quardrants I : $(+,+)$
- → Quardrants II : $(-,+)$
- → Quardrants III : $(-,-)$
- → Quardrants IV : $(+,-)$
- → x-axis : $(\pm, 0)$
- → y-axis : $(0, \pm)$
- → Origin : $(0,0)$



Quadrants in the Coordinate system

Distance of a Point from Origin :-

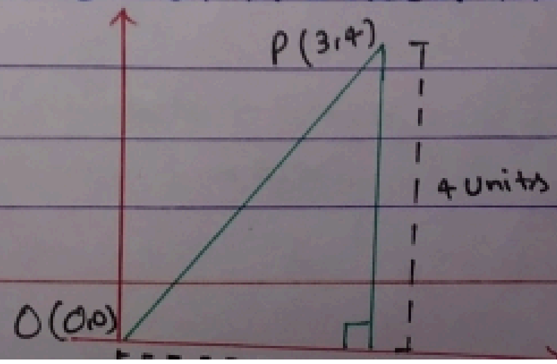
Goal :- To find the distance of Point $P(3,4)$ from the origin.

Drop a perpenidicular on x-axis which intersects the x-axis at $(3,0)$

By Pythgorean Theorem,

$$OP^2 = OQ^2 + QP^2$$

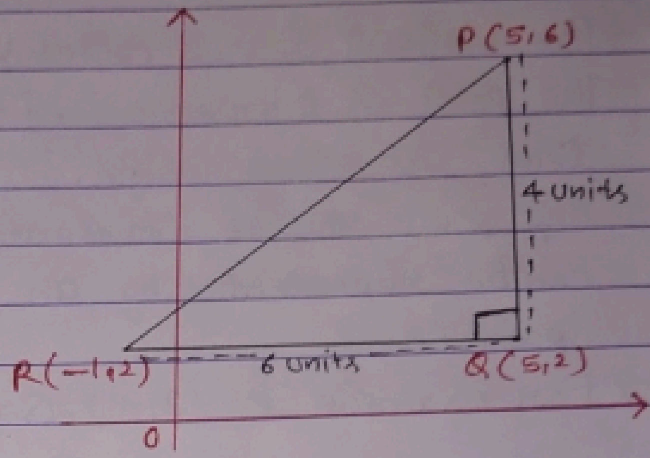
Hence, $\sqrt{OQ^2 + QP^2} = \sqrt{3^2 + 4^2} = 5$



Distance Between Any Two Points

* Goal: To find the distance between any two Points $P(x_1, y_1)$ and $R(x_2, y_2)$

→ Construct a right-angled triangle with right angle at Point $Q(x_1, y_2)$



→ By Pythagorean Theorem,

$$PR^2 = QR^2 + PQ^2$$

$$PR = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

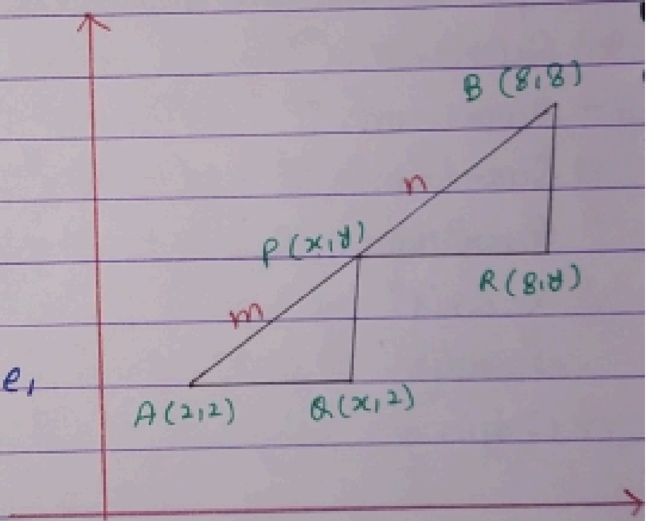
$$= \sqrt{6^2+4^2} = \sqrt{52} = 2\sqrt{13}$$

Section Formula

* Given that, the Point P cuts the line segment AB in the m:n ratio. Our goal is to find the coordinates of P.

→ Let the Coordinates of A and B are (x_1, y_1) and (x_2, y_2) respectively.

→ Assume that P has the coordinates (x, y).



→ Observe that $\triangle AQP \sim \triangle PRB$. Hence,

$$\Rightarrow \frac{m}{n} = \frac{AP}{PB} = \frac{AQ}{PR} = \frac{PQ}{BR}$$

$$\Rightarrow \frac{m}{n} = \frac{x-x_1}{x_2-x} = \frac{y-y_1}{y_2-y}$$

$$\Rightarrow x = \frac{mx_2+nx_1}{m+n}, \quad y = \frac{my_2+ny_1}{m+n}$$

Area of a Triangle Using coordinates

Goal: To find area of $\triangle ABC$ with known coordinates.

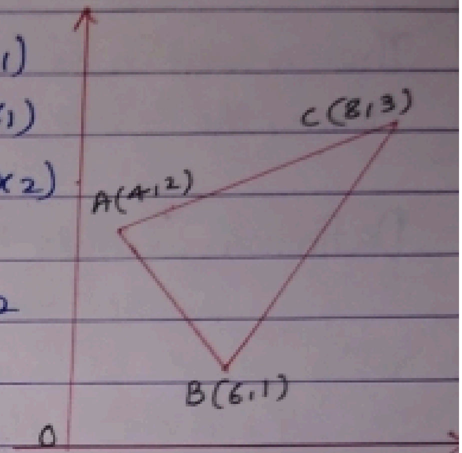
Let the coordinates of the vertics be $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$

$A(\triangle ABC) = A(ADFC) - A(ADEB) - A(BEFC)$

$A(ADFC) = \frac{1}{2}(AD+CF) \times DF = \frac{1}{2}(y_1+y_3)(x_3-x_1)$

$A(ADEB) = \frac{1}{2}(AD+EB) \times DE = \frac{1}{2}(y_1+y_2)(x_2-x_1)$

$A(BEFC) = \frac{1}{2}(BE+CF) \times EF = \frac{1}{2}(y_2+y_3)(x_3-x_2)$



$\triangle ABC = \frac{1}{2}(2+3)\times 4 - \frac{1}{2}(2+1)\times 2 - \frac{1}{2}(1+3)\times 2$

$= 10-3-4 = 3$ square units.

$\triangle ABC = \frac{1}{2}\left| x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2) \right|$

Slope of a line

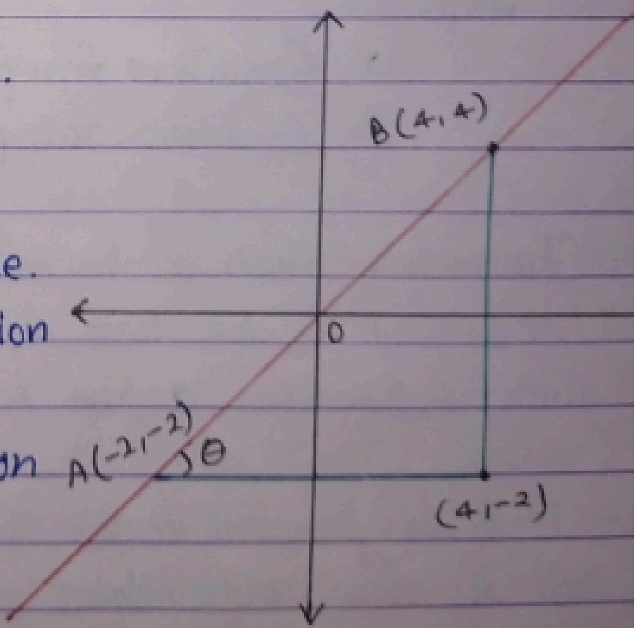
Goal:- To find the slope of a line, given on a coordinate plane

→ Identify two points on the line, say, $A(x_1, y_1)$ and $B(x_2, y_2)$

→ Construct a right angled triangle with a right angle at the Point $M(x_2, y_1)$.

→ Define

$$\Rightarrow \quad m = \frac{MB}{AM} = \frac{y_1-y_2}{x_1-x_2} = \tan\theta.$$

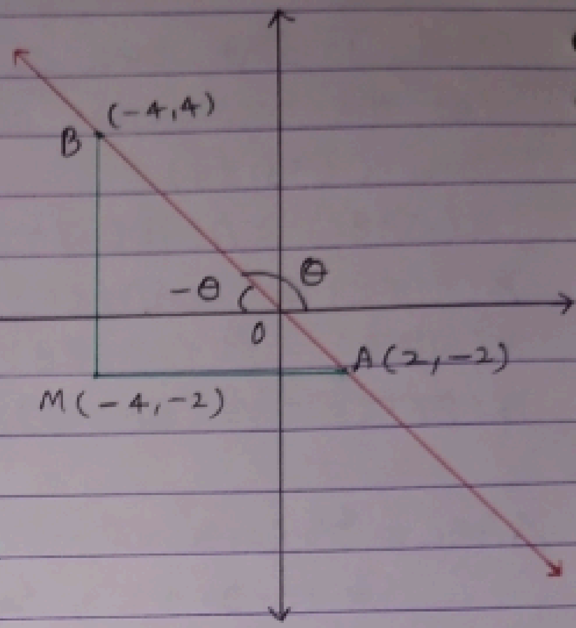


→ The m is called slope of a line.

→ θ is called the ~~intersecti~~ inclination of the line with positive X-axis, measured in anticlockwise direction

→ $0° \leq \theta \leq 180°$

Slope of line (Continued)



Observe that the lines parallel to X-axis have inclination of 0°. Hence the slope $m = \tan 0 = 0$.

The inclination of a vertical line is 90°. Hence, the slope m is undefined.

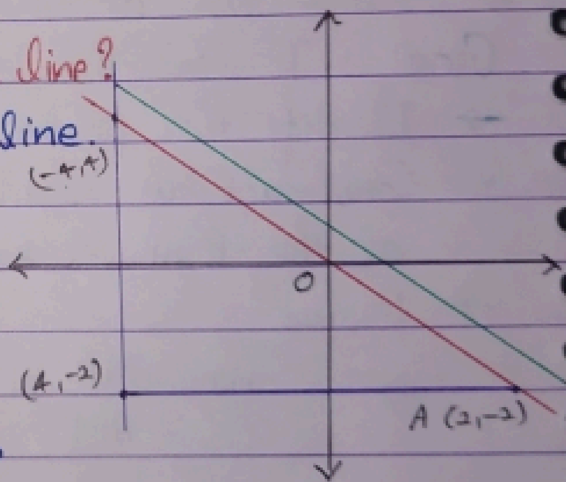
Definition: If θ is the inclination of a line l, then $\tan\theta$ is called the slope or gradient of line l.

If $\theta \neq 90^\circ$, then $m = \tan\theta$

$$m = \tan(180^\circ - \theta) = -\tan\theta = \frac{y_1 - y_2}{x_1 - x_2}.$$

Can slope of a line uniquely determine a line?

No, it can not uniquely determine the line.



How is the slope useful?

→ To explore:

⇒ Conditions for parallel lines

⇒ Conditions for perpendicular lines

Characterization of Parallel Lines via slope

→ Let l_1 and l_2 be two non-vertical lines with slopes m_1 and m_2 with inclinations α and β respectively.

→ If l_1 is parallel to l_2, then $\alpha = \beta$

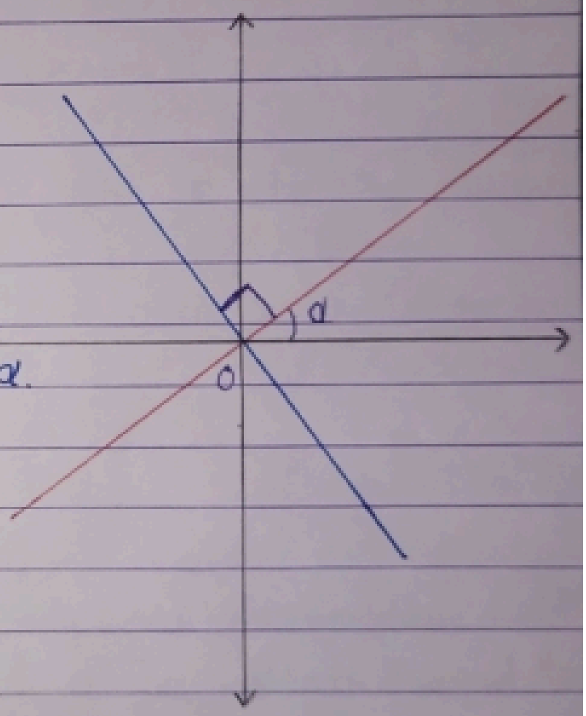
→ It is clear that $\tan\alpha = \tan\beta$.

→ Hence, $m_1 = m_2$

→ Assume $m_1 = m_2$. Then $\tan\alpha = \tan\beta$.
→ Since, $0° \le \alpha, \beta \le 180°$, $\alpha = \beta$.
→ Therefore, l_1 is parallel to l_2

Two non-vertical lines l_1 and l_2 are parallel if and only if their slopes are equal.

* Characterization of Perpendicular Lines via slope:
Let l_1 and l_2 be two non-vertical lines with slopes m_1 and m_2 with inclinations α and β respectively.



→ If l_1 is perpendicular to l_2, then $90 + \alpha = \beta$.
→ Now, $\tan\beta = \tan(90+\alpha) = -\cot\alpha = -1/\tan\alpha$.
→ Hence, $m_2 = -1/m_1$ or $m_1 m_2 = -1$.

→ Assume $m_1 m_2 = -1$. Then $\tan\alpha\tan\beta = -1$.
$\tan\alpha = -\cot\beta = \tan(90+\beta)$ or $\tan(90-\beta)$.
→ Hence, α and β differ by $90°$ which proves l_1 is perpendicular to l_2

* Two non-vertical lines l_1 and l_2 are perpendicular if and only if $m_1 m_2 = -1$

→ Relation of Angles between the two lines and their slopes
Let l_1 and l_2 be two non-vertical lines with slopes m_1 and m_2 with inclinations α_1 and α_2 respectively.

Date :
Page :

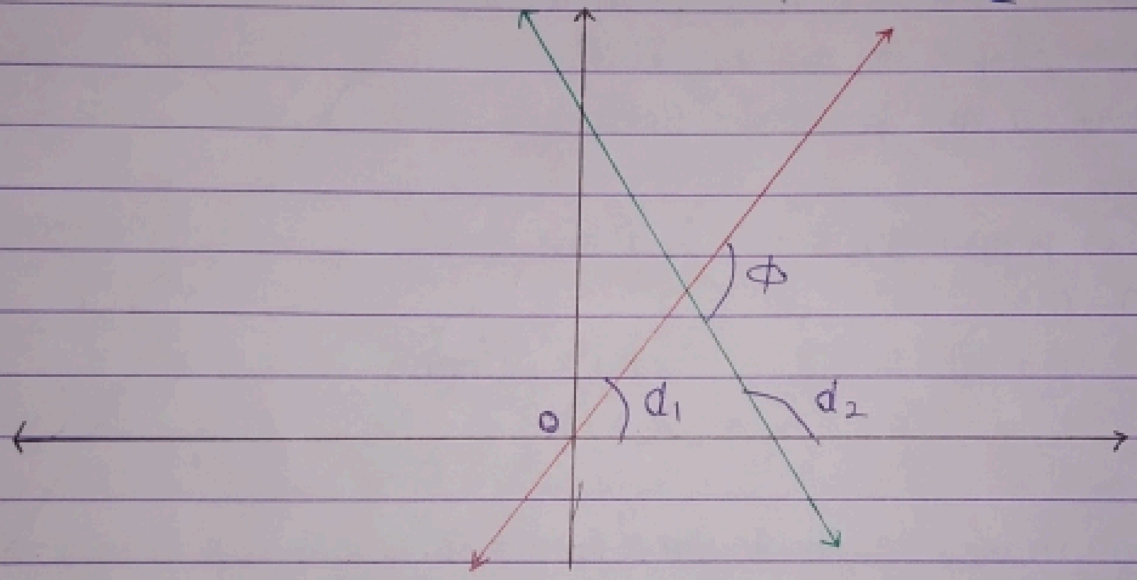
→ Suppose l_1 and l_2 intersects and let ϕ and θ be the ~~adjent~~ adjacent angles formed by l_1 and l_2

→ Now $\theta = a_2 - a_1$, for $a_1, a_2 \neq 90°$

→ Then

$$\Rightarrow \tan\theta = \tan(a_2 - a_1) = \frac{\tan\alpha_2 - \tan\alpha_1}{1 + \tan\alpha_1 \cdot \tan\alpha_2} = \frac{m_2 - m_1}{1 + m_1 m_2}, \; m_1 m_2 \neq -1$$

$$\Rightarrow \tan\phi = \tan(180 - \theta) = -\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$



Representation of a line

⇒ How to represent a line uniquely?

⇒ Given a point, how to decite wheather the point lies on a line?

→ In other words, for a given line l, we should have a definite expression that describes the line in terms of coordinates plane.

→ If the coordinates a given point P, satisfy the expression for the line l, then the point lies on the line L.

Date : ______
Page : ______

Horizontal and Vertical Lines:

Horizontal Lines: A line is horizontal line only if it is parallel to x-axis

→ To locate such a line, we need to specify the value it takes on Y-axis.

→ That is, the expression for such a line is of the form $y=a$

→ Then all points that lie on this line are ~~one~~ of the form (x,a).

Vertical Lines: A line is a vertical line only if it is parallel to Y-axis.

→ To locate such a line, we need to specify the value it takes on X-axis.

→ That is, the expression for such a line is of the form $x=b$.

→ Then, all points that lie on this line are of the form (b,y)

* Equation of a Line : Point-slope Form

→ For a non-vertical line l, with slope m and a fixed point $P(x_0,y_0)$ on the line, can we find the equation (algebraic representation) of the line?

→ Let $Q(x,y)$ be an arbitrary point on line l. Then, the slope of the line is given by

$$m = \frac{y-y_0}{x-x_0}$$

$$(y-y_0) = m(x-x_0) \quad \text{(Point-slope form)}$$

→ Any point $P(x,y)$ is on line l, if and only if the coordinates of P satisfy the above equation.

Date :
Page :

Q- Find the equation of a line through the points $P(5,6)$ with slope -2.

→ Let $Q(x,y)$ be an arbitrary point on this line. Then, using Point-slope form, we get.

$$-2 = \frac{y-6}{x-5}$$

$$(y-6) = 2(\cancel{5}-x) \text{ or } y = 16-2x$$

* Equation of a line : Two-Point Form

→ Let the line l pass through the points $P(x_1,y_1)$ and $Q(x_2,y_2)$.

→ Assume that $R(x,y)$ is an arbitrary point on the line l.

→ Then, the points P, Q, and R are collinear.

→ Hence, Slope of PR = Slope of PQ, Therefore. $\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$

$$(y-y_1) = \frac{y_2-y_1}{x_2-x_1}(x-x_1) \text{ (Two-Point Form)}$$

→ Any point $R(x,y)$ is on line l, if and only if, the coordinates of R satisfy the above equation.

Q- Find the equation of a line passing through $(5,10)$ & $(-4,-2)$

→ Let (x,y) be an arbitrary point on this line. Then by two-point form, we get

$$(y-10) = \frac{-2-10}{-4-5}(x-5)$$

$$3y = 4x+10.$$

Date:
Page:

Equation of Line: Slope-Intercept Form.

→ Let a line l with slope m cut Y-axis at c. Then c is called the y-intercept of the line l.

→ That is, the point $(0, c)$ lies on the line l.

→ Therefore, by Point-slope form, we get ~~y = m(x-d)~~.

$y - c = mx$, or $y = mx + c$

Let a line l with slope m cut x-axis at d. Then d is called the x-intercept of the line l.

That is, the point $(d, 0)$ lies on the line l.

Therefore, by Point-slope form, we get $y = m(x-d)$.

Q- Find the equation of a line with slope $1/2$ and y-intercept $-3/2$.

The equation of line is $y = \frac{1}{2}x - \frac{3}{2}$

Q. Find the equation of a line with slope $1/2$ and x-intercept 4.

The equation of the line is $y = \frac{1}{2}(x-4)$ or $2y - x + 4 = 0$.

Equation of a Line: Intercept Form

Suppose a line makes x-intercept at a and y-intercept at b. Then the two points on the line are $(a, 0)$ and $(0, b)$.

Using two-points form,

$(y - 0) = \frac{b-0}{0-a}(x-a)$ or $\frac{x}{a} + \frac{y}{b} = 1$

Q- Find the equation of a line having x-intercept at -3 and y-intercept at 3.

$\frac{x}{-3} + \frac{y}{3} = 1$ or $y = x + 3$.

* General Equation of a line :-

Different forms of Equation of Line	Representation	General Form $Ax+By+C=0$
Slope-Point Form	$(y-y_0)=m(x-x_0)$	$m=-\frac{A}{B}, y_0-mx_0=-\frac{C}{B}$
Slope-Intercept Form	$y=mx+c$ or $y=m(x-d)$	$m=-\frac{A}{B}, c=-\frac{C}{B}$ or $d=-\frac{C}{A}$
Two-Point Form	$(y-y_1)=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$	$\frac{y_2-y_1}{x_2-x_1}=-\frac{A}{B}, y_1+\frac{A}{B}x_1=-\frac{C}{B}$
Intercept Form	$\frac{x}{a}+\frac{y}{b}=1$	$a=-\frac{C}{A}, b=-\frac{C}{B}$

→ Any equation of the form $Ax+By+C=0$, where $A, B \neq 0$ simultaneously, is called general linear equation or general equation of a line.

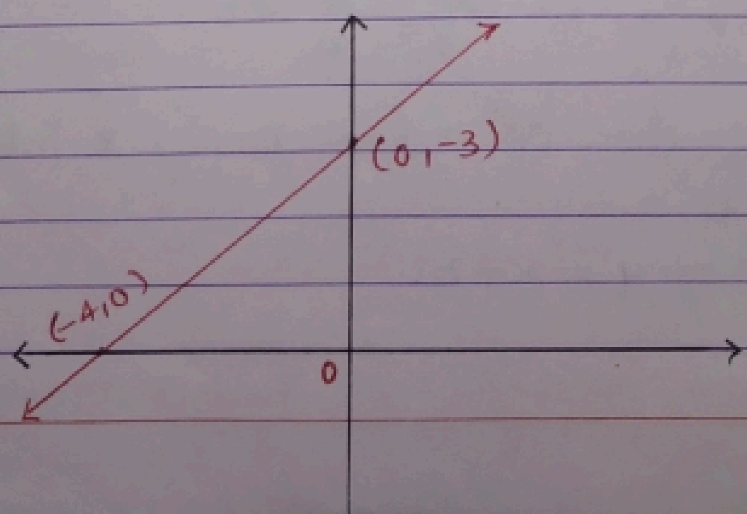
Q→ The equation of a line is $3x-4y+12=0$. Find the slope, x-intercept and y-intercept of the line.

Identify $A=3, B=-4$ and $C=12$

Using Intercept form, $a=-C/A=-4$ and $b=-C/B=3$.

Using Slope-intercept form, $y=\frac{3}{4}x+3$. Hence, $m=3/4$

Slope $=3/4$, x-intercept $=-4$ and y-intercept $=3$.



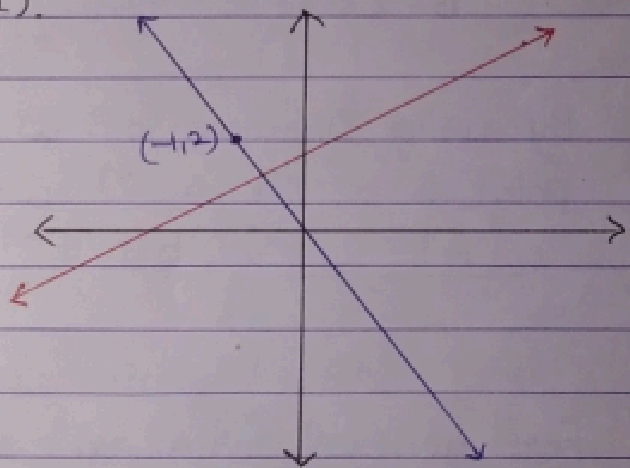
Q- Show that the two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, $b_1b_2 \neq 0$ are
- parallel if $a_1b_2 = a_2b_1$, and
- perpendicular if $a_1a_2 + b_1b_2 = 0$.

Two non-vertical lines l_1 and l_2 are parallel if and only if their slopes are equal.
Two non-vertical lines l_1 and l_2 are perpendicular if and only if $m_1m_2 = -1$.

Using slope-intercept form,
$m_1 = -\frac{a_1}{b_1}$ and $m_2 = \frac{-a_2}{b_2}$

If ~~there~~ these lines are parallel, then $a_1b_2 = a_2b_1$.
If the lines are perpendicular then, $a_1a_2 + b_1b_2 = 0$.

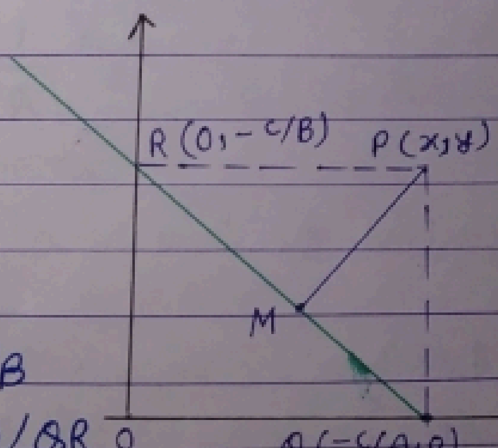
Q- Find the equation of a line perpendicular to the line $x-2y+3=0$ and passing through the point $(-1,2)$.
The slope of the given line is $m_1 = 1/2$.
The slope of a line perpendicular to the given line is $m_2 = -1/m_1 = -2$



To find the equation of the line passing through the point $(-1,2)$ and slope -2.
$(y-2) = -2(x+1)$ or $y = -2x$

* Distance of a Point from a Line

Goal:- To find the distance of the point $P(x_1, y_1)$ from the line l having equation $Ax + By + C = 0$
For $A, B \neq 0$, Using Intercept form,
x- intercept $= -C/A$ and y- intercept $= -C/B$
$A(\Delta PQR) = \frac{1}{2} QR \times PM$. Hence, $PM = 2A(\Delta PQR)/QR$



$$A(\Delta PQR) = \frac{1}{2}\left|x_1\left(\frac{-C}{B}\right) - \frac{C}{A}\left(y_1 + \frac{C}{B}\right)\right| = \frac{1}{2}\frac{|C|}{|AB|}|Ax_1 + By_1 + C|$$

Date :
Page :

$$QR = \sqrt{\frac{C^2}{A^2} + \frac{C^2}{B^2}} = \frac{|C|}{|AB|}\sqrt{A^2+B^2}.$$

$$PM = \frac{2A(\triangle PQR)}{QR} = \frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}.$$

* Distance between two Parallel Lines

Let l_1 and l_2 be two parallel lines with slopes m.

l_1: $y = mx + c_1$. Comparing with general form, we get x-intercept at $(-c_1/m)$.

l_2: $y = mx + c_2$ Comparing with general form, we get $A = -m$, $B = 1$ and $C = -c_2$.

By using Distance of a point from a line formula, where point is $(-c_1/m, 0)$, we get

$$\frac{|A(-c_1/m) + B(0) + C|}{\sqrt{A^2+B^2}} = \frac{|c_1-c_2|}{\sqrt{1+m^2}}.$$

For general form, $m = A/B$, $c_1 = -C_1/B$ and $c_2 = -C_2/B$, then

$$d = \frac{|C_1-C_2|}{\sqrt{A^2+B^2}}$$

Q → Find the distance of the point $(3,-5)$ from the line $3x-4y-26=0$.

$Ax+By+C=0$ implies $A=3, B=-4$ and $C=-26$.

Also $(x_1,y_1) = (3,-5)$. Then

$$d = \frac{|3(3)-4(-5)-26|}{\sqrt{3^2+(-4)^2}} = \frac{3}{5}.$$

Q> Find the distance between parallel lines $3x-4y+7=0$ and $3x+4y+5=0$.

Observe that $A=3$, $B=-4$ and $C_1=7$, $C_2=5$, Then

$$d=\frac{|7-5|}{\sqrt{3^2+(-4)^2}}=\frac{2}{5}.$$

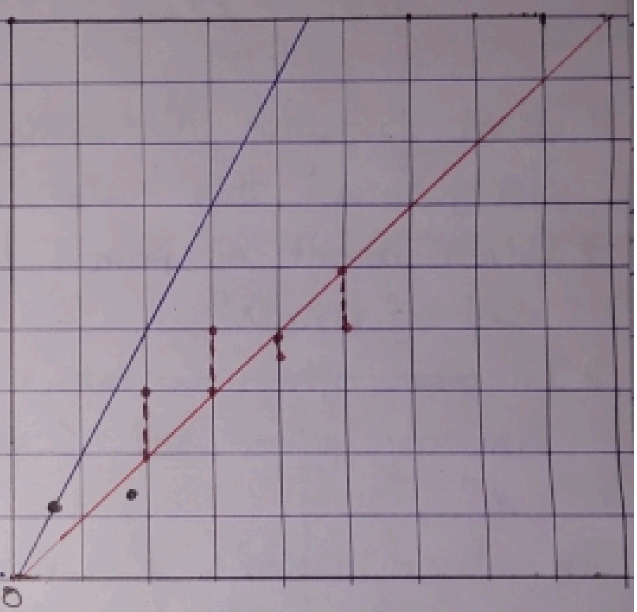
Examples: Real-World:—

It is known that $V=IR$.

That is, Voltage = current × Resistance

In our context, this represents a line passing through the origin.

That is $y=mx$, where y is the voltage, x is the current and m is the resistance.



You have been asked to perform an experiment to verify this phenomenon.

How to say mathematically which line is better?

Let the equation of the lines be $y=x$ and $y=2x$

x_i	y_i
1	2
5	4
7	8
8	9
9	8.7
10	9

From the set of observations, (x_i, y_i) $i=1,2,3,4,5,6$.

We can consider the square of the differences.

$\sum_{i=1}^{6}(y_i-x_i)^2$ and $\sum_{i=1}^{6}(y_i-2x_i)^2$

The first difference is 5.09 and the second difference is 328.49.

Therefore, the line is better than the second line.

Distance of a Set of Points from a line.

Apart from perpendicular distance, we can also talk about the distance which is parallel to y-axis.

Consider the set of points $\{(x_i, y_i) | i = 1, 2, \dots n\}$ and a line with equation $y = mx + c$.

Then the squared sum of the distance of set of points from the line is defined as –

$$SSE = \sum_{i=1}^{n} (y_i - mx_i - c)^2.$$

* Least Square Motivation :–

In general, this raises the following question.

→ Given a set of points, how to find the line that fits the given set of points?

→ In ~~over~~ other words, what is the equation of the best fit line for given set of points?

In other words, if I need to find the equation of the line $y = mx + c$, then the question can be reframed into two questions.

⇒ What is the value of m and c that best fits the given set of points

⇒ What is a meaning?

Best Fit : Given a set of n points, $\{(x_i, y_i) | i = 1, 2, \dots n\}$, define....

$$SSE = \sum_{i=1}^{n} (y_i - mx_i - c)^2.$$

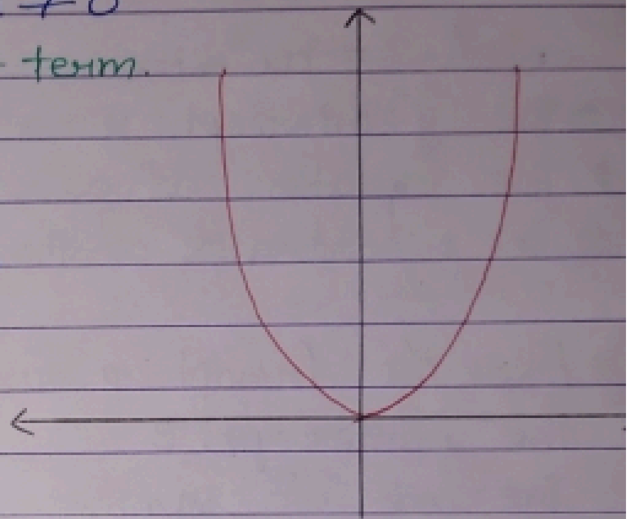
Find the value of m and c that mimimizes SSE.

Quadratic Function - Graphing :-

⇒ Quadratic Function (Definition)

→ A quadratic function is described by an equation of the form
$f(x) = ax^2 + bx + c$, where $a \neq 0$

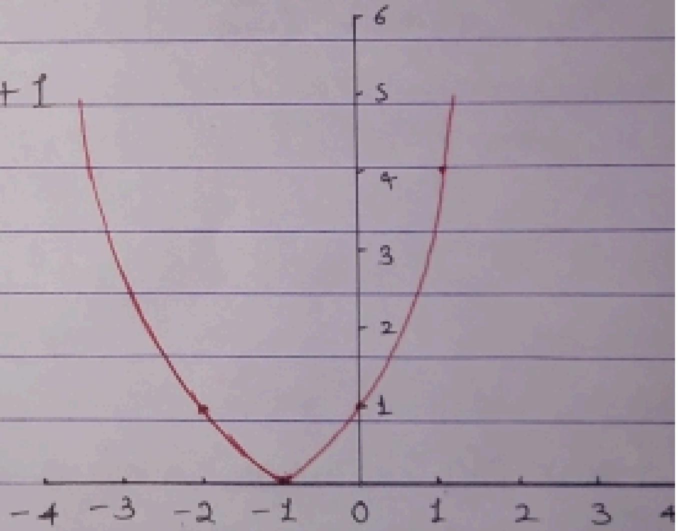
Quadratic term (ax^2), Linear term (bx), Constant term (c).



→ The graph of any quadratic function is called parabola.

→ To graph a quadratic function, plot the ordered pairs on the coordinate plane that satisfy the function.

Example ^Graph a function $f(x) = x^2 + 2x + 1$



→ Generate a table of ordered pairs satisfying the function.

→ Plot the points on the coordinate plane.

→ Connect a smooth curve joining the points.

→ Important Observations :-

x	y
-2	1
-1	0
0	1
1	4

→ All parabolas have an axis of symmetry. That is, if the graph paper containing the graph of parabolas is folded along the axis of symmetry the portion of parabola on either sides will exactly match each other.

Date :
Page :

→ The point at which the axis of symmetry intersects the parabola is called the vertex.

→ The y-intercept of a quadratic function is C.

Let $f(x) = ax^2 + bx + c$, where $a \neq 0$

→ The y-intercept: $y = a(0)^2 + b(0) + c = c$

→ The equation of axis of symmetry: $x = -b/(2a)$ (to be derived later)

→ The x-coordinates of the vertex: $-b/(2a)$.

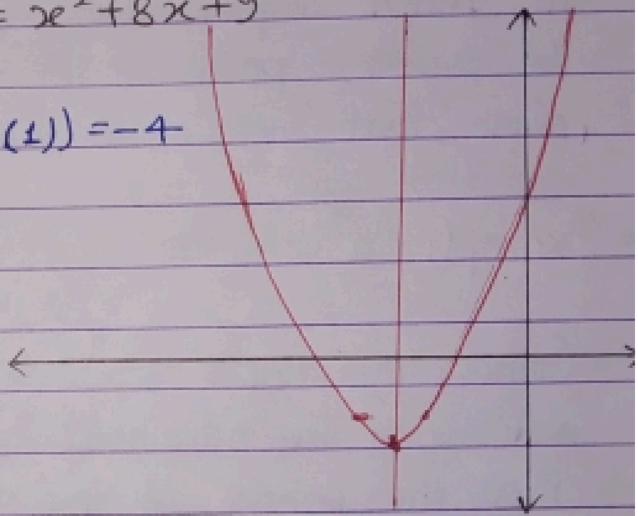
⇒ Example: Graph a function $f(x) = x^2 + 8x + 9$

The y-intercept: 9

The axis of symmetry: $x = -8/(2(1)) = -4$

The vertex: $(-4, -7)$

x	y
-3	-6
-4	-7
-5	-6



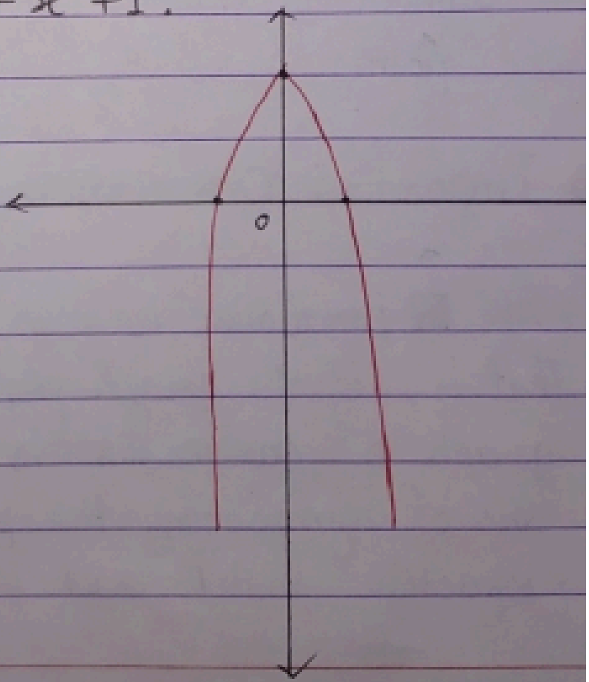
Example: Graph a function $f(x) = -x^2 + 1$.

The y-intercept : 1

The axis of symmetry: $x = 0$

The vertex: $(0, 1)$

x	y
-1	0
0	1
1	0



Date :
Page :

Maximum and Minimum Values

The y-coordinate of the vertex of a given quadratic function is the minimum or maximum values attained by the function.

* The graph of a quadratic function $f(x) = ax^2+bx+c$, where $a \neq 0$ is:

→ Opens up and has minimum value, if $a > 0$.
→ Opens down and has maximun value if $a < 0$.
→ The range of a quadratic function is:-

$R \cap \{f(x) \mid f(x) \geq f_{min}\}$ or $R \cap \{f(x) \mid f(x) \leq f_{max}\}$

Example:—

* Let $f(x) = x^2 - 6x + 9$.

(1)- Determine whether f has minimum or maximum value. If so, what is the value?

(2)- State the domain and the range of f.

→ Observe that $a=1$, $b=-6$ and $c=9$.
Since, $a>0$, the function opens up and has the minimum value

→ The minimum value is given by y-coordinate of the vertex.
→ The x-coordinate of the vertex is $-b/(2a) = 3$. Therefore, the minimum value is $f(3) = 0$.

⇒ Domain = R and Range = $R \cap \{f(x) \mid f(x) \geq 0\}$

Q- A tour bus in Chenni serves 500 customers per day. The change is ₹40/- per person. The owner of the bus service estimate that the company would lose 10 passengers per day for each ₹4/- fare ~~hike~~ hike.
How much should the fare be in order to maximize the income of the company?

→ Let x denotes the number of ₹4/- fare hike. Then the price per passenger is $40+4x$, and the number of passengers is $(500-10x)$. Therefore, the income is
$$I(x) = (500-10x)(40+4x) = -40x^2 + 1600x + 20000.$$

→ In this case, $a=-40$, $b=1600$ and $c=20000$, and the maximum value attained will be $I(-b/2a) = I(20) = 36000$.

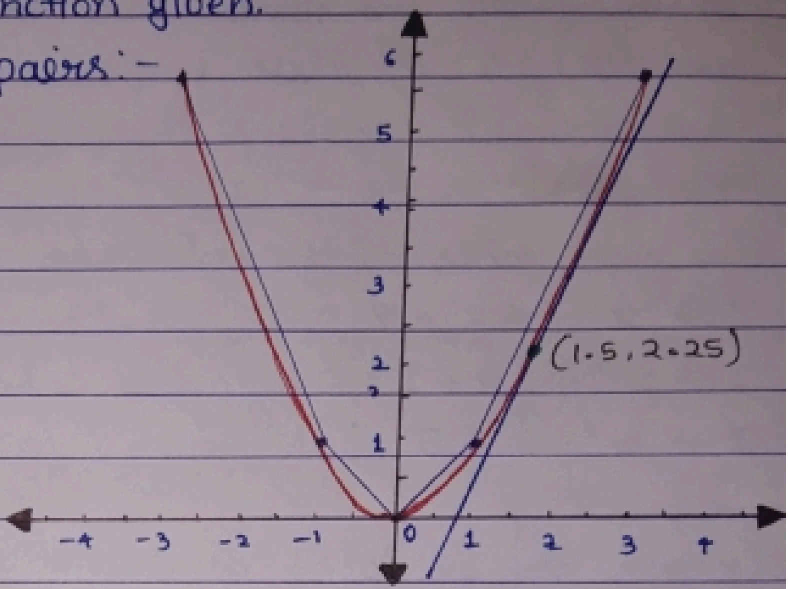
→ This means the company should make 20 fare-hikes of ₹4/- in order to maximize the income. That is the new fare $= 40 + 4 \times 20 =$ ₹120/-

* Slope of a quadratic function:-
➔ Given a quadratic function, $f(x) = ax^2+bx+c$, where $a \neq 0$, how to determine the slope of f?

Recall, for a linear function $y = g(x) = mx+c$, we have calculated the ratio of change in y and change in x and observed that it remains ~~that~~ constant and is m. We also showed that $m = \tan\theta$, where θ is the inclination with positive X-axis.

Let us use similar analogy for a quadratic function and define slope of a quadratic function.
We now discuss the concept using a simple example.

Let $y = x^2$ be a quadratic function given.
Let us tabulate the ordered pairs:-



x_i	y_i	$y_i - y_{i-1}$
—	—	—
-2	4	
-1	1	-3
0	0	-1
1	1	1
2	4	3

The slope of $f(x) = x^2$ is $2x$.

x_i	y_i	$y_i - y_{i-1}$	
-2	$4a-2b+c$		
-1	$a-b+c$	$-3a+b$	
0	c	$-a+b$	$2a$
1	$a+b+c$	$a+b$	$2a$
2	$4a+2b+c$	$3a+b$	$2a$

From the table, it is clear that the slope of $f = 2ax+b$

Also note that the slope denotes the rate of change of y with respect to x.

Hence, slope $= 0$ means the function has either maximum or minimum which happens when $2a+b=0$. That is, $x = -b/(2a)$.

* Quadratic Equations - Solve by Graphing.

→ Roots of Equation and Zeros of the Functions

→ The solutions to a quadratic equation are called roots of the equation.

→ One method of finding the roots of a quadratic equation is to find Zeros of a related quadratic function.

→ Observe that the zeros of ~~the~~ a function are x-intercepts of its graph and these are the solutions of related equation as $f(x) = 0$ at these points.

Examples:—

Find the roots of the following equations.

1 → $x^2 + 6x + 8 = 0$.

2 → $x^2 + 2x + 1 = 0$.

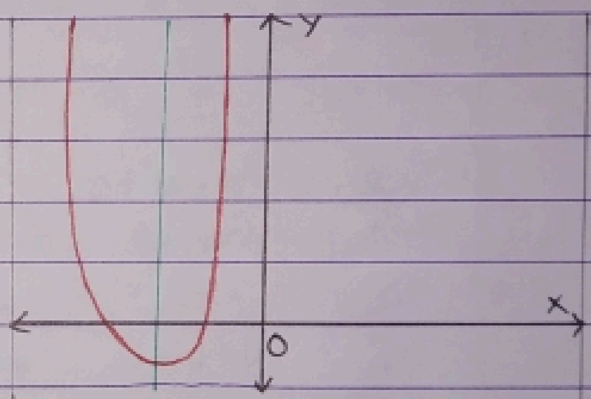
3 → $x^2 + 1 = 0$.

Graph the related quadratic functions using axis of symmetry and vertex.

→ Axis of symmetry: $x = -3$

→ The roots are $-4, -2$

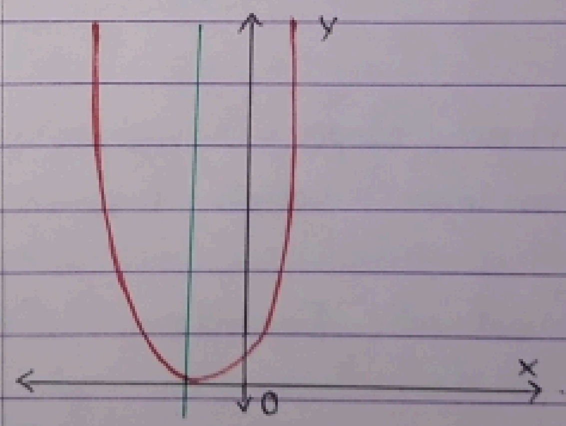
→ Two real roots



→ Axis of symmetry: $x = -1$

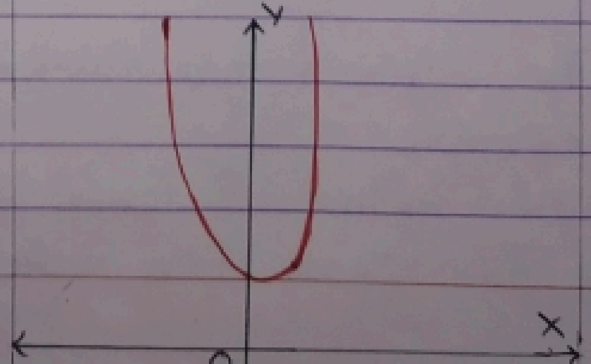
→ The roots are $-1, -1$

→ One real root.



→ Axis of symmetry: $x = 0$

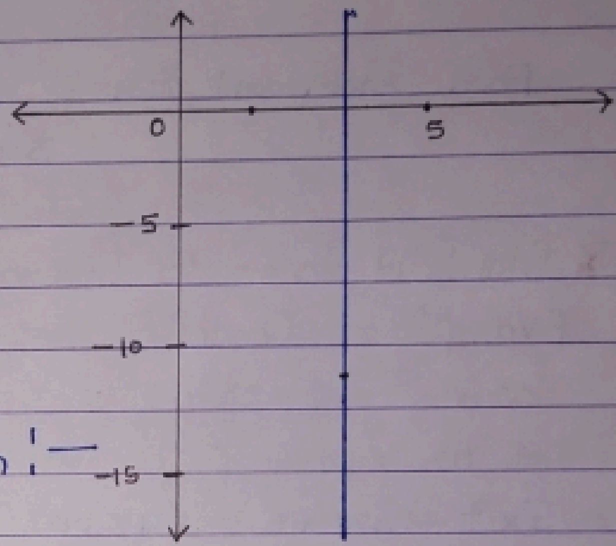
→ No real roots.



* Quadratic Equations :- Solve by Factoring

→ Quadratic Equations Intercept form

Let $y = f(x) = a(x-p)(x-q)$, where p and q represent x-intercepts for the function. Then the form $y = a(x-p)(x-q)$ is called the intercept form.



Example: Graph $y = 3(x-1)(x-5)$

How will you convert the intercept form into standard form?

* Intercept form to the Standard form :-

Changing intercept form to standard form requires us to use FOIL method. which can be described as follows:-

The product of two binomials is the sum of the products of the first (F) terms, the outer (O), the inner (I) and the last (L) terms.

$(ax+b)(cx+b) = ax \cdot cx + ax \cdot d + b \cdot cx + b \cdot d$

Quick Observations:

The product of coefficient of x^2 and the coefficient of the constant is abcd.

The product of the two terms in the coefficient of x is also abcd.

Example:-

Question:- Write a quadratic equation with roots, $2/3$ and -4, in the standard form.

Recall: By standard form, we mean $ax^2+bx+c=0$, where a,b,c are integers.

By intercept form, we known $(x-2/3)(x+4)=0$

By FOIL method. $(x-2/3)(x+4) = x^2+(-2/3+4)x - 2/3\cdot 4$

$$= x^2+(10/3)x - 8/3 = 0$$

For standard form, multiply both sides by 3, to get

$$3x^2+10x-8=0$$

* Standard form to Intercept form

Example:- Convert the function $f(x) = 5x^2-13x+6$ to intercept form.

Let us apply FOIL method.

$5x^2-13x+6 = (ax+b)(cx+d) = acx^2+(ad+bc)x+bd.$

Therefore, $ac=5$, $ad+bc=-13$ and $bd=6$. That is, $abcd=30$ and $ad+bc=-13$.

$30 = 2\times3\times5 = 10\times3 = (-10)(-3)$ That is, $ad=-10$ and $bc=-3$.

$5x^2-13x+6 = 5x^2-10x-3x+6 = 5x(x-2)-3(x-2) = (5x-3)(x-2)$

$$= 5(x-3/5)(x-2).$$

1- Solve: $x^2=8x$

That is, $0 = x^2-8x$

$$= x(x-8)$$

This means 0, 8 are the roots of the given quadratic equation.

2- Solve: $x^2-4x+4=0$.

Using FOIL method, and comparing the coefficients, we get $abcd=4$ and $ad+bc=-4$. Therefore, $ad=-2$ and $bc=-2$.

So, $x^2-4x+4 = x^2-2x-2x+4 = x(x-2)-2(x-2) = (x-2)^2=0$

Hence, 2 is the repeated real root of the given equation.

3- Solve : $x^2-25=0$

Note $abcd = 25$ and $ad+bc=0$

That is, $ad = 5$ and $bc = -5$

So.

$$x^2-25 = x^2-5x+5x-25 = x(x-5)+5(x-5)$$
$$=(x+5)(x-5)=0$$

Hence, -5, 5 are the roots of the given quadratic equation.

* Quadratic Equations :- Solve by Completing the Square

Old Method

$x^2+10x-24=0$

$abcd=-24$ and $ad+bc=10$

$ad=12$, and $bc=-2$. So

$$x^2+10x-24 = x^2+12x-2x-24$$
$$=x(x+12)-2(x+12)$$
$$=(x+12)(x-2)=0$$

That is, -12 and 2 are the real roots of the equation

New Method.

$x^2+10x=24$

Observe that $(x+a)^2 = x^2+2ax+a^2$

Using this write $10=2\times5$ and add ~~Do~~ 25 on both sides of the equation to get

$$x^2+10x+25 = 24+25 = 0$$
$$(x+5)^2=7^2, \quad (x+5) = \mp 7$$

Therefore, $x=-5+7=2$ and $x=-7-5 = -12$ are the roots of the quadratic equation.

* Quadratic Equations with irrational Roots:—

Solve : $x^2-4x+4=32$

It can easily seen that $(x-2)^2 = 32$.

Hence. $(x-2) = \pm\sqrt{32} = \pm 4\sqrt{2}$

Thus, $x = 2\pm4\sqrt{2}$ are the roots of the quadratic equations.

$$ax^2+bx+c=0$$

$$x^2+\frac{b}{a}x+\frac{c}{a}=0$$

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=+\frac{b^2}{4a^2}-\frac{c}{a}$$

b^2-4ac	roots
>0	2
<0	0
$=0$	1

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{2a}$$

$$\left(x+\frac{b}{2a}\right)=\frac{\pm\sqrt{b^2-4ac}}{2a}$$

$$x \pm \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$$

$$\boxed{x=\frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}}$$

The quantity in the square roots is known as discriminant.

The above formula is known as quadratic formula.

* Summary of Quadratic Formula :-

Value of the discriminant	Types and Number of roots
$b^2-4ac>0$ perfect square	2 real, rational roots
$b^2-4ac>0$, no perfect Sq.	2 real, irrational roots
$b^2-4ac=0$	1 real, rational roots
$b^2-4ac<0$	No real root.

Examples:—

⇒ Find the value of the discriminant for each equation and then describe the number and type of the roots for the equation.

1- $9x^2-12x+4=0$

2- $2x^2+16x+33=0$

1→ $b^2-4ac=(-12)^2-4(9)(4)=144-144=0$. So, it has one rational root.

2→ $b^2-4ac=(16)^2-4(2)(33)=256-264=-8$ So, it has no real roots.

Axis of Symmetry:—

Why $x=-b/2a$ is the axis of symmetry?

$$f(x)=ax^2+bx+c$$
$$=a(x^2+(b/a)x+c/a)$$
$$=a(x^2+(b/a)x+b^2/(4a^2))-b^2/(4a)+c/a)$$
$$=a(x+b/2a)^2+(c-b^2/(4a))$$

Therefore, the symmetry is about $x=-b/(2a)$ which is the axis of symmetry.

→ Summary of Concepts:—

Method	Can be used	When preferred
Graphing	Occasionally	Best used to verify the answer found algebraically.
Factoring	Occasionally	If constant term is Zero or factors are easy to find.
Completing the square	Always	Use when b is even.

—: Polynomials :—

*** What is a Polynomial?**

A Leyman's Perspective :-

A Polynomial is one kind of mathematical expression which is a sum of several mathematical terms.
Each term in this expression is called monomial and the term can be a number, a variable or product of several variables.

Definition :(A mathematician's Perspective) :-

A Polynomial is an algebraic expression in which the only arithmetic is addition, substraction, multiplication and "natural" exponents of the variables.

*** Why do we call them Polynomials?**

The word 'Polynomial' is derived from two words

Poly + Nomen

many   name

→ Each term is called monomial.
→ A Polynomial having two terms is called binomial.
→ A Polynomial having three terms is called trinomial.

Ex:- A Polynomial in one variable can be represented as —

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = \sum_{m=0}^{n} a_m x^m$$

Exponent

Variable

Coefficient of the term

* Identification of Polynomials:—

1. x^2+4x+2 — Polynomial
2. $x+x^{1/2}$ — not Polynomial
3. $x+y+xy+x^3$ — Polynomial

* Types of Polynomials:—

Polynomials in one variable
→ x^4+1

Polynomials in two variables
→ x^4+y^5+xy

Polynomials in more than two variables.
→ $xyz+x^2z^5$

* The Degree of Polynomial:—

→ The degree of zero polynomial is undefined.
→ The exponent on the variable in a term is called the degree of that variable in that term.
→ The degree of that term is the sum of the degree of the variables in that term.
→ The degree of the polynomial is the largest degree of any one of the terms with non-zero coefficients.

Examples:- $x = x^1$

$c = c \cdot x^0$

* Classification based on the degree of the polynomial.

Degree	Name	Example
0	Constant Polynomial	$c, 1, 5$
1	Linear Polynomial	$2x+4$, $ax+b$
2	Quadratic Polynomial	$3x^2+2$, $4xy+2x$
3	Cubic Polynomial	$3x^3$, $4x^2y+2y+1$
4	Quartic Polynomial	$10x^4+y^4$, $x^4+10x+1$

* Algebra with Polynomial :—

⇒ Polynomials in one variable :—

Description :- As seen earlier, the polynomial of degree n, is represented as -

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0.$$

This expression can be treated as a function from $\mathbb{R} \to \mathbb{R}$. That is, the domain of $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ is $\mathbb{R}$, and the range depends on the function.

⇒ Addition of Polynomials :-

Add the following polynomials :-

1. $p(x) = x^2 + 4x + 4$, $g(x) = 10$
2. $p(x) = x^4 + 4x$, $g(x) = x^3 + 1$
3. $p(x) = x^3 + 2x^2 + x$, $g(x) = x^2 + 2x + 2$

S.1

$$p(x) = 1x^2 + 4x + 4$$
$$q(x) = 0x^2 + 0x + 10$$
$$p(x) + q(x) = x^2 + 4x + 14$$

S.2 :- $p(x) = 1x^4 + 0x^3 + 0x^2 + 4x + 0$

$g(x) = 0x^4 + x^3 + 0\cdot x^2 + 0\cdot x + 1$

$p(x) + g(x) = x^4 + x^3 + 4x + 1$

S.3 :- $p(x) = x^3 + 2x^2 + x + 0$

$g(x) = 0x^3 + x^2 + 2x + 2.$

$p(x) + g(x) = x^3 + (2+1)x^2 + (1+2)x + 2 = x^3 + 3x^2 + 3x + 2$

Let $p(x) = \sum_{k=0}^{n} a_k x^k$, and $g(x) = \sum_{j=0}^{m} b_j x^j$. Then

$$p(x) + q(x) = \sum_{k=0}^{m \vee n} (a_k + b_k)\, x^k.$$

* Substraction of Polynomials :-

Substract the following polynomials:

1. $p(x) = x^2 + 4x + 4$, $g(x) = 10$
2. $p(x) = x^4 + 4x$, $g(x) = x^3 + 1$
3. $p(x) = x^3 + 2x^2 + x$, $g(x) = x^2 + 2x + 2$

S.1 :- $p(x) = 1x^2 + 4x + 4$

$-g(x) = -0x^2 - 0x - 10$

$p(x) - g(x) = x^2 + 4x - 6$

S.2 $p(x) = 1x^4 + 0\cdot x^3 + 0\cdot x^2 + 4x + 0$

$-q(x) = -0x^4 - x^3 - 0\cdot x^2 - 0x - 1$

$p(x) - q(x) = x^4 - x^3 + 4x - 1$

S.3 :- $p(x) = 1x^3 + 2x^2 + x + 0$

$-q(x) = -0x^3 + 1x^2 - 2x - 2$

$p(x) - q(x) = x^3 + (2-1)x^2 + (1-2)x - 2 = x^3 + x^2 - x - 2$

Let $p(x) = \sum_{k=0}^{n} a_k x^k$, and $g(x) = \sum_{j=0}^{m} b_j x^j$. Then.

$$p(x) - q(x) = \sum_{k=0}^{m \vee n} (a_k - b_k)\, x^k.$$

Date:
Page:

* Multiplication of Polynomials:—

→ Multiply the following polynomials.

→ $p(x) = x^2 + x + 1$ and $q(x) = 2x^3$

$$\begin{aligned} p(x)q(x) &= (x^2+x+1)(2x^3) \\ &= 2x^{3+2} + 2x^{1+3} + 2x^3 \\ &= 2x^5 + 2x^4 + 2x^3. \end{aligned}$$

→ $p(x) = x^2 + x + 1$ and $q(x) = 2x + 1$.

$$\begin{aligned} p(x)q(x) &= (x^2+x+1)(2x+1) \\ &= 2x^{1+2} + 2x^{1+1} + 2x + x^2 + x + 1 \\ &= 2x^3 + 2x^2 + 2x + x^2 + x + 1 \\ &= 2x^3 + 3x^2 + 3x + 1 \end{aligned}$$

Multiply the polynomials $p(x) = a_2x^2 + a_1x + a_0$ and $q(x) = b_1x + b_0$.

$$\begin{aligned} p(x)q(x) &= (a_2x^2 + a_1x + a_0)(b_1x + b_0) \\ &= (a_2x^2 + a_1x + a_0)(b_1x) + (a_2x^2 + a_1x + a_0)b_0. \\ &= a_2b_1x^3 + (a_1b_1 + a_2b_0)x^2 + (a_0b_1 + a_1b_0)x + a_0b_0. \end{aligned}$$

Let $p(x) = \sum_{k=0}^{n} a_k x^k$, and $q(x) = \sum_{j=0}^{m} b_j x^j$. Then

$$p(x)q(x) = \sum_{k=0}^{m+n} \sum_{j=0}^{k} (a_j b_{k-j}) x^k.$$

Multiply the polynomials $p(x) = x^2 + x + 1$ and $q(x) = x^2 + 2x + 1$

K	a_K	b_K
0	1	1
1	1	2
2	1	1

The resultant polynomial is:

$$p(x)q(x) = x^4 + 3x^3 + 4x^2 + 3x + 1$$

K	Coefficient	Calculations
0	a_0b_0	1
1	$a_1b_0 + a_0b_1$	$1+2=3$
2	$a_0b_2+a_1b_1+a_2b_0$	$1+2+1=4$
3	$a_0b_3+a_1b_2+a_2b_1+a_3b_0$	$0+1+2+0=3$
4	$a_0b_4+a_1b_3+a_2b_2+a_3b_1+a_4b_0$	$0+0+1+0+0=1$

* Division of Polynomials:—

⇒ Division of a polynomial by a monomial:—

$$\frac{3x^2+4x+3}{x} = 3x+4+\frac{3}{x}$$

⇒ Division of a polynomial by another polynomial:—

$$\frac{3x^2+4x+1}{x+1} = (3x+1)$$

Divide $p(x) = x^4+2x^2+3x+2$ by $q(x) = x^2+x+1$.

Dividend Quotient Remainder

$$\frac{p(x)}{q(x)} = x^2-x+2+\frac{2x}{q(x)}$$

Divisor

• Division Algorithm :—

Step-1: Arrange the terms in descending order of the degree and add the missing exponents with 0 as coefficient.

Step-2: Divide the first term of the dividend by the first term of divisor and get the monomial.

Date :
Page :

Step-3: Multiply the monomial with divisor and substract the result from the dividend.

Step-4: Check if the resultant polynomial has degree less than divisor. If true, write the remainder else go to step 2.

Find $2x^3+3x^2+1 / 2x+1$

$$
\begin{array}{r|l}
 & x^2 + x - 1/2 \\
2x+1 & 2x^3 + 3x^2 + 0\cdot x + 1 \\
 & \underline{\pm 2x^3 \pm x^2} \\
 & \quad 2x^2 + 0\cdot x \\
 & \quad \underline{\pm 2x^2 \pm x} \\
 & \qquad -x + 1 \\
 & \qquad \underline{\mp x \mp \tfrac{1}{2}} \\
 & \qquad\quad 3/2
\end{array}
$$

$$\therefore \quad \frac{2x^3+3x^2+1}{2x+1} = x^2+x-\frac{1}{2}+\frac{3/2}{(2x+1)}$$