Mathematics

For Data Science I

WEEK 5 - WEEK 8

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Week 8

Mathematics for Data Science 1
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Lecture – 27
Solution of Quadratic Equation Using Factorization

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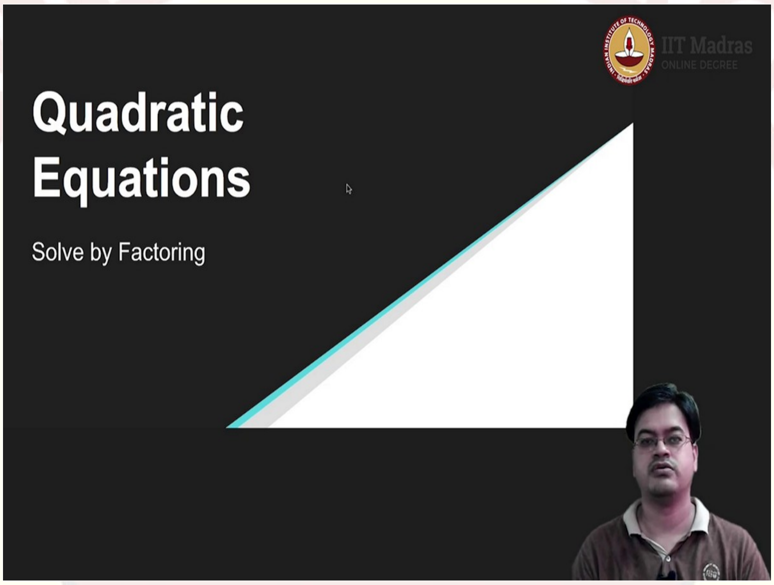

So, in the last video, we have seen how to find the roots of a Quadratic Equation by graphing the associated quadratic function.

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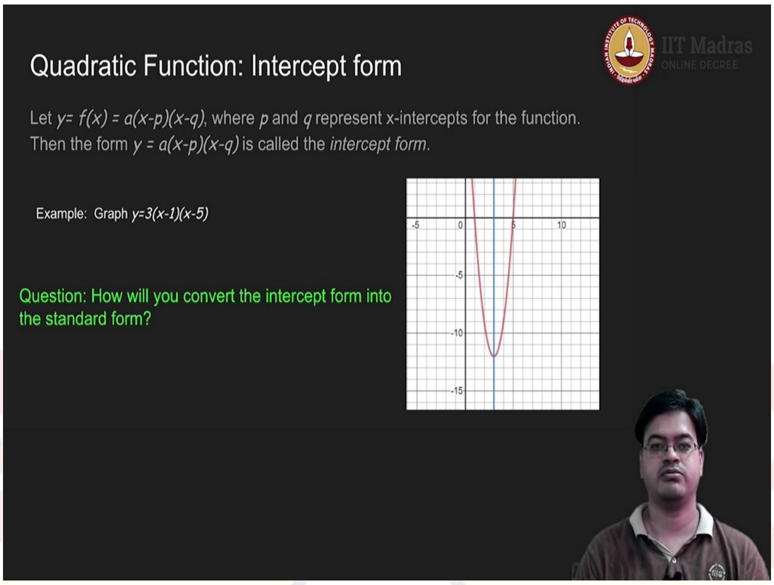

In this video, we will see how to find the solution of a quadratic equation by a well known method called factoring method. For that, we will define one new form of a quadratic function that is intercept form.

What is an intercept form? If $y=f(x)$ which is a quadratic function is written in this form , where and $(x-q)$ are called binomials ok. Where this p and q are nothing but x intercepts of the quadratic function.

So, essentially what you have done is, you have seen a graph of a function and you have located the two intercepts x intercepts of the function. Whenever the expression is possible in this form you are writing it. So, $y=a(x-p)(x-q)$, and this form is called the intercept form.

So, let us try to see one example of intercept form which is let us say you have been given this intercept form $y=3(x-1)(x-5)$. And you also know that these 1 and 5 are x intercepts of the quadratic function, that means, you have been given two values. Can you find the third value? The answer is yes.

So, at point 1, when , the value is 0; at point 5, the value is 0. Now, using the logic that I gave you in the previous video, you can actually see that there will be some axis of symmetry between this 1 and 5, because 1 and 5 both take value 0 right. So, the axis of symmetry will be nothing but the distance between these two points divided by 2. So, $\frac{1+5}{2}=3$. I am sorry it will not be a distance, it is just the sum of these two points divided by 2, average of these two points, that should be the correct terminology. So, it should be average of two points. So, $\frac{1+5}{2}=3$

Now, $x=3$ will be the axis of symmetry for this particular function if at all the this are the roots of the this are the x intercepts of the function right. So, now, I have given you $x=3$ is the axis of symmetry. So, just substitute the value 3 in this particular graph in this particular equation, and you will get 3 - 1 = 2, and 3 – 5 = -2, that means, $2\times-2=-4, -4\times3=-12$. So, you got three values. What are those three values? (1, 0), (5, 0), (3, -12).

So, based on this information, you can easily plot the graph which will look like this. As you can see this is the value -12 here, value -12 is here and 0, 0. So, you can easily plot this graph. You can connect the smooth curve using this.

Now, the main question is once given this kind of expression, how to convert this expression into a standard form? So, that is the question that I will post down. How will you convert the intercept form into the standard form? Just by multiplying the two binomials. So, for multiplying the two binomials, we have one rule which I will state in the next slide.

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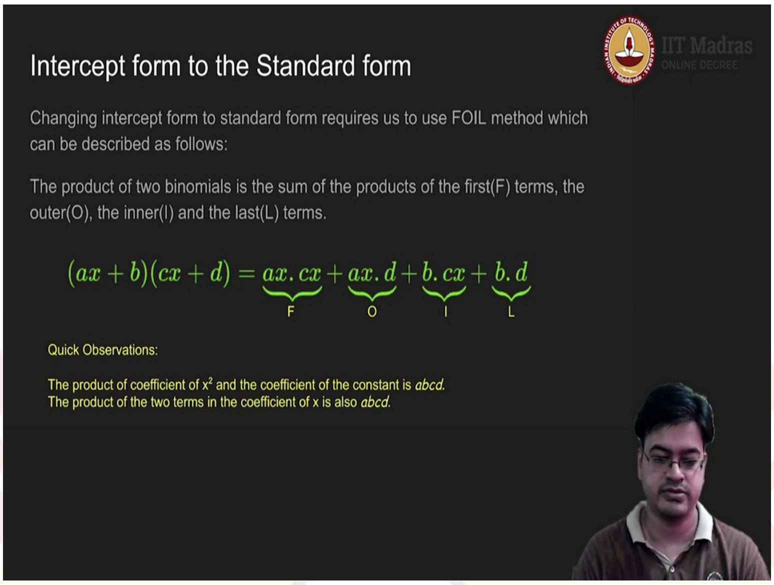

So, how the conversion from intercept form to standard form will happen using a method called FOIL method which is described below. So, the product of two binomials as I already mentioned that $(x-p)$ and $(x-q)$ these are the two binomials. So, the product of two binomials is sum of the product of first terms the outer, the inner and the last terms. So, let me make it more precise by demonstrating it.

So, let us consider this expression which is . Now, $(ax+b)(cx+d)$ what I will do is I will first take the first term of this expression, and the first term of this expression, and multiply them together that is the first term over here by the sum of the product of the first terms I mean this term.

Then I will take the inner term ok, then I will take the outer term, sorry, then I will take the outer term that is b is the outer term here, sorry, not b is not the outer term, b is the inner term. You have ax which is the outer term, and d which is the outer term. So, now you just multiply them together which gives me $ax.d$ which is the outer term product of the outer term.

Then you take the inner terms that is b and cx. So, $b.cx$, this is the inner term. And $b.d$ are the last terms. First term, outer term, inner term and last term that way we will multiply these things together. That means, I will get the first term as acx^2 ; second term is $(ad+bc)x$; and the third term as bd .

Now, if you look at, so basically the ac is the term which is the coefficient of x^2 , $ax+bc$ is the $ad+bc$ is the term which is the coefficient of x , and bd is the term which is the which is the constant term right.

So, now, a quick observation you can make is if you look at the product of this first term and the last term, what will you get $ac \times bd$, so it is $abcd$, the product is $abcd$ right. So, the product of the coefficient of x^2 and the coefficient of constant is $abcd$. In a similar manner, if you consider the coefficients of x , a d and b c are the coefficients of x, $ad+bc$, so if you take product of these two terms, again it will be $abcd$. This is a crucial observation which we will need while converting a standard form to intercept form and vice versa.

Just remember this the product of the coefficient of x^2 and the coefficient of constant is $abcd$. And the product of the two terms of the coefficients of x is $abcd$; both of them are $abcd$. So, this we will use to convert our expression into standard form, and convert our expression in intercept form in various ways. So, that observation is very crucial for us.

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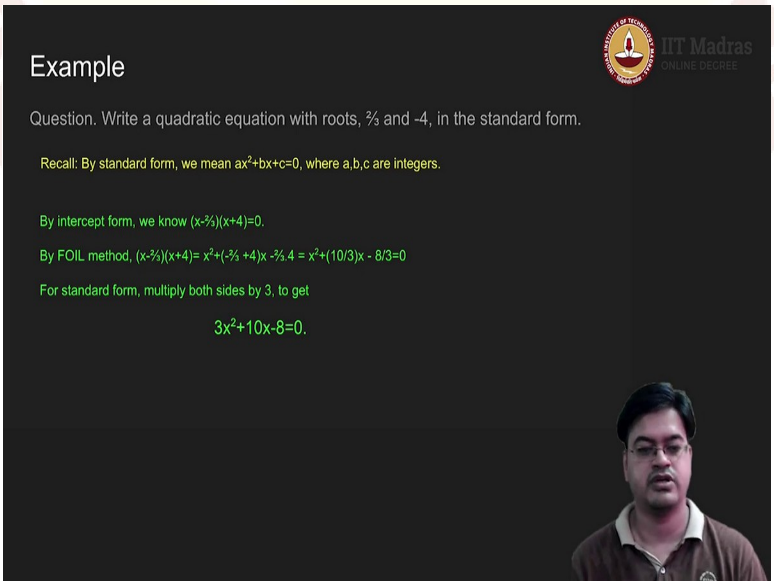

So, let us do take one example and see how we can apply our knowledge which we have gained in this particular video along with the previous videos to solve this problem. So, the question is the write a quadratic equation with roots 2, $\frac{2}{3}$ and -4 in the standard form ok. So, let us recollect what is a standard form. Standard form is of $ax^2+bx+c=0$; and a, b, c all are integers; and $a \neq 0$. This is the standard form; we have already seen that ok.

So, now, if I want to write this, we will use our knowledge about intercept form, and we can easily write this expression as $(x-\frac{2}{3})(x+4)=0$ because $\frac{2}{3}$ and -4 are the roots. Yes, but this equation is not in the standard form. So, now in the previous slide, we have seen that in order to convert this into a standard form, we will use a FOIL method. So, let us try to use a FOIL method. So, the what is a here? $ax+b$ that is a is 1, b is $\frac{-2}{3}$, c is 1, d is 4 ok.

Now, you use FOIL method that is first terms. So, first terms is 1×1 , so it will retain $1x^2$, Then $ad+bc$, so $(\frac{-2}{3})(1)$ and a is 1, 4 $4 . -\frac{2}{3}$ that is the product here $\frac{-2}{3}+4$ this is the term which have coefficient of x , and then $\frac{-2}{3}.4$ which is the term here. So, this is successful application of FOIL method.

Now, let us rewrite all these things that is you can sum this and write the sum that is x^2 , so $4 \times 3 = 12, 12-2=10$, so $x^2+\frac{10}{3}x-\frac{8}{3}$ right. Is this equation in the standard form? No, because for standard form a, b, c, all must be integers. So, what I will do is, I will multiply this equation with 3 on both sides. So, if I multiply on both sides with 3, then I get $3x^2+10x-8=0$ that is the solution to this question. So, the quadratic equation in standard form is $3x^2+10x-8=0$. So, we have solved.

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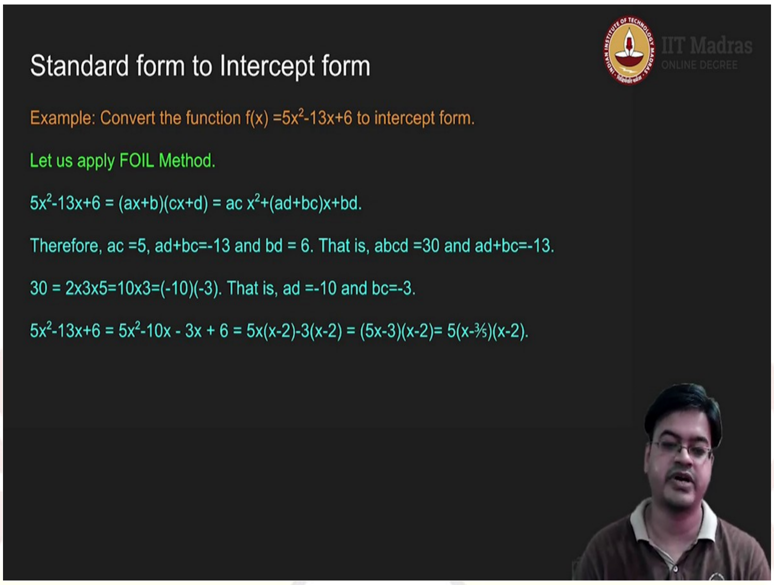

Let us now go further and try to see how I will convert a standard form into an intercept form. Again we will use FOIL method, but in a reverse manner. So, I want to convert a function quadratic function which is given to me $5x^2-13x+6$ to intercept form, that means, I want to write $a(x-p)(x-q)$. So, how will I convert this?

So, what I will do is, I will take this particular function $5x^2-13x+6$, and apply FOIL method to it. How to apply FOIL method to it? I will equate this to be equal to $(ax+b)(cx+d)$. Then based on FOIL method, I have this expression which is ac^2 sorry $acx^2+(ad+bc)x+bd$. Now, remember we have done some observations that is the product of this and this is $abcd$ right.

So, now, I can equate this equation with this equation. So, term containing x^2 will be equated with term containing x^2. So, I will get $ac=5$, $ad+bc=-13$ and $bd=6$. Then from this expression I can also derive an information that is $abcd$ that is the product of the first and the last term and the product of the terms contained in the sum is 30. So, $5\times6=30$; and $ad+bc=-13$.

Now, my job becomes crucial. My job is to guess what those two terms will be ad and bc right. So, that their product is 30, and if you sum over them, then it must be -13. For that

I will use the prime factorization theorem that was introduced in week-1. So, if you look at this expression 30, I will get prime factors as $2 \times 3 \times 5$.

Now, I want the product to be equal to 30, and I want the sum to be equal to -13. So, based on this, what I can derive is if at all this, this term has to be negative, I should have some negative factors over here and both of them should be negative factors. In particular if I combine 5 and 2, I will get 10 and 3, and 10 + 3 =13, but it is not giving me -13.

So, I will use a trick that multiplication of two negative numbers will become a positive number. So, it is $(-10)(-3)$ which will give me 30; at the same time, it will be the sum will be -13. So that means, my ad is -10; bc is -3. It does not matter, you can switch also. You can write bc as -3, and ad as -10 also, it does not matter.

So, now I will substitute these values into this expression, essentially I will rewrite this expression. So, I will write this expression as $5x^2 - 10x - 3x + 6$ ok. Then what I will do is I will look at the first two terms; first two terms, and I will take the greatest common factor from these two terms that is $5x$. So, I will take $5x$, whatever is remaining I will put in a bracket that is $x-2$.

Here also I will do take the greatest common factor out that is -3, so $-3(x-2)$. Now, you can see these $(x-2)$ s are same. So, essentially this expression will come if I have $(5x-3)(x-2)$. Now, is this in the intercept form? No, still it is not in the intercept form. What is the intercept form? It is $a(x-p)(x-q)$. So, I will just divide everything by 5 in this expression and take the 5 out. So, $5(x-\frac{3}{5})(x-2)$ this is the intercept form.

So, using FOIL method, I have converted this expression into an intercept form. An expression was given to me in standard form; I have converted it into intercept form. Let us see few more examples as this concept is quite intricate. You may need some practice, you solve as many problems as possible, but I will give you some demo cases, so that it will be easy to distinguish for you.

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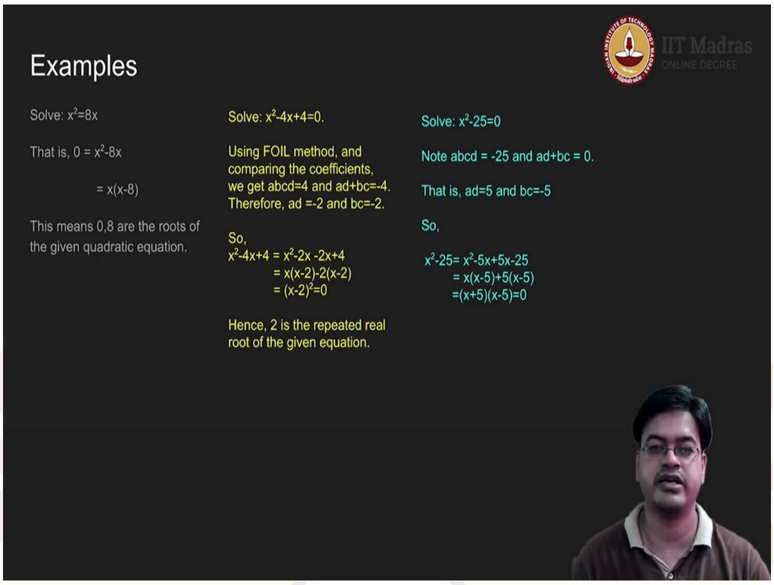

So, let us take let us say you have you have been asked to solve this equation; $x^2=8x$. Now, here you do not need, you do not really need a FOIL method. What you need is, just rearrange $x^2=8x$ and you just take out the greatest common factor which is x. So, x this will give you $x(x-8)$. So, if I want to solve this, I know $x-0$ and $x-8$ are the things. So, 0 and 8 are the roots of this given quadratic equation. Simple, this solves our problem for such a simple case, where the constant term is absent right.

Now, let us take another example, $x^2-4x+4=0$. Now, in this case, you will use FOIL method obviously but the essence of FOIL method reduces to that the coefficients of x are of the form $ad+bc$, and the product of this and this is $abcd$. So, the product $abcd$ is 4, and $ad+bc$ is -4, this is what it reduces to ok.

So, if abcd is 4, and ad + bc is -4, is there any other way out 4 can be factorized only in one way that is 2×2; 2×2. And ad + bc is -4, that means, both of them should be negative -2×-2. So, ad is -2, bc is -2. Substitute it in the master equation where you can write $x^2-2x-2x+4$. So, you have substituted it in master equation.

Now, you go ahead and take out the greatest common factors out, the first expression will have x out, the second expression will have 2 out. Then again these are product of binomials. So, it will be $(x-2)^2$ = 0 given in the expression. So, what is the root of this equation? 2; 2 is the real repeated root of this equation.

Let us go ahead and solve one more example. $x^2-25=0$. This is quite interesting 25, you can see is a perfect square 5, and I want to find the root of this equation. Again I will use FOIL method, abcd is -25, and ad + bc is 0 right. 5 is a perfect square. So, 5×5 is the factorization, but -25 is there. So, one will be, one 5 will be with a positive sign, another 5 will be with a negative sign. So, ad can be +5, and bc can be -5 or vice versa, it does not matter.

Substitute this substitute this knowledge into this expression. So, you take $x^2-25-5x+5x$, take out the greatest common factors that is x and 5 respectively, you will get this kind of expression. And then you just rewrite them as product of two numbers that is $(x+5)(x-5)=0$, and you have solved. Remember in all these expressions we have written this in intercept form.

So, all these expressions are written in intercept form. Once you write an expression in intercept form, it is very easy to find the roots of the equation, or in fact once you write in the intercept form you have already figured out the roots of the equation.

Mathematics for Data Science 1
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Solution of quadratic equation using Square method
Indian Institute of Technology, Madras
Lecture 28

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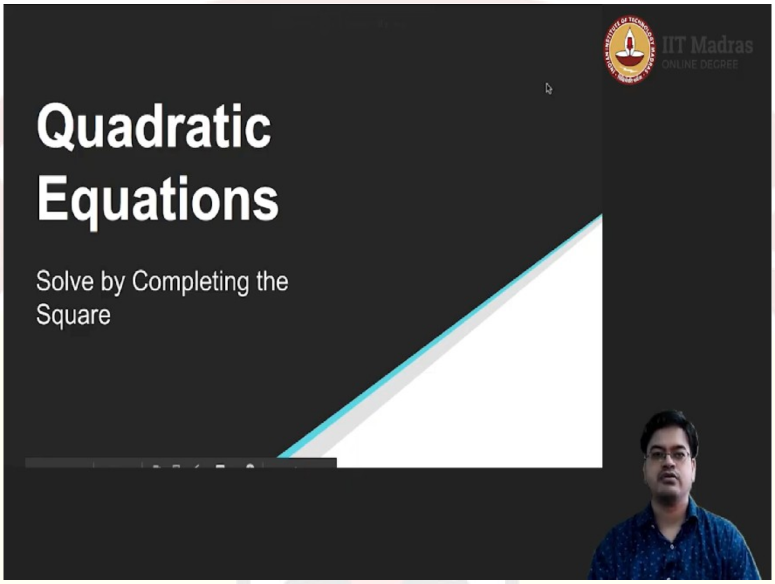

In this video we are going to learn one more interesting method called Solving quadratic equations or for finding the roots of quadratic equation. The method named completing the Square method also it has a very good connection with a very well-known or very popularly known as Quadratic formula. So, we will explore that connection towards the end of this video.

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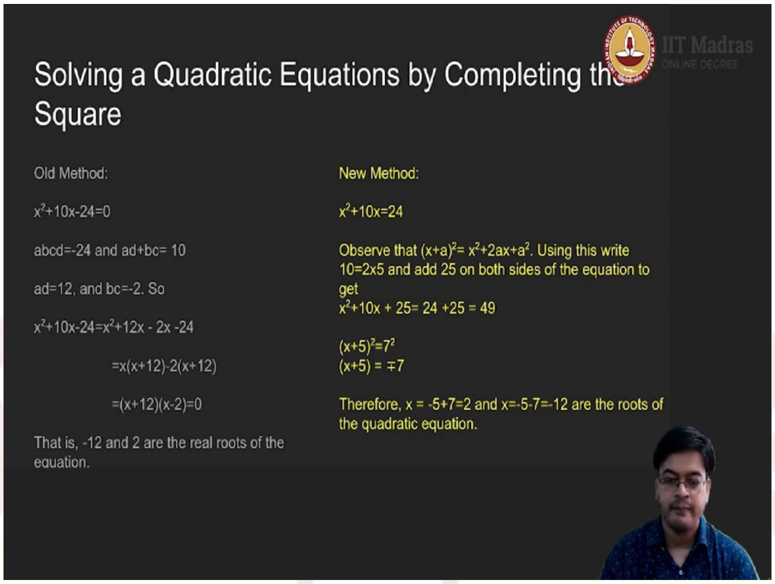

So, let us start I will demonstrate this method through some examples. So, let us first understand or revise what we done in the earlier stage or in the earlier video. We have used a method of Factoring that I have called as old method. So, let us take this example $x^2+10x-24=0$. If you use the method of factoring you need to identify what is this term 24 and 1. So, -24×1 so - 24 is the product of the leading coefficient and the constant term and $ad+bc=10$.

So, I have this setup which is $abcd=24$ and $ad+bc=10$. So, I will essentially use a prime factorization theorem and get the prime factors of -24 so that if you rearrange the prime factors in such a way that the sum should be equal to 10. One such rearrangement is 12 and -2. So, ad is 12 and bc is -2. So, I got this and then based on our factorization technique I will substitute this 12 and 2 for this coefficient if x and I will get this expression which is $x^2+12x-2x-24$.

Then I will use the greatest common factor technique that is I will take out x common, 2 common from the last 2 terms and I will get this expression. So, finally, I got $(x+12)(x-2)=0$ and therefore, I will get the roots of the equation are -12 and 2. Now, somebody came up and thought, that why should I bother what is this last term? It is just a constant right, so I will replace this constant with something and I will work on it.

So, from that particular though comes the new method, which is the method of completing the square. So, what that person did is just rewrite this expression into this form that is $x^2+10x=24$. Now, the next question that person asked is if I look at this x^2+10x, do I know something that will make this particular expression as a complete square. So, what do I mean by complete square let us understand?

So, complete square means $(x+a)^2$ if I want to write $(x+a)^2$ then what I need to do here is to add some number and subtract some number or to add this same number on both sides right. So, in this case if you look at this expression that is $(x+a)^2$ which is $x^2+2ax+a^2$. So, this a is the number that I am looking for. Now, in this case if I consider this expression and if I want to add a number which will typically be a^2 what that a^2 should be, Is the first question.

So, to answer that let us equate this 10 to this 2a, so $10=2a$ therefore $a=5$ is the answer. So, what will be a^2? a^2 will be 5^2 which is 25. So, now I got a number a^2 to add and subtract from both sides. So, what I will do is I will add the number 25 on both sides so once I add the number 25 on both sides for this expression. I get $x^2+10x+25=24+25$, it turns out here in this case that the number is 49 which is also a perfect square.

But it need not be the case all the time. So, now what I know here is this number this particular expression is nothing but $(x+5)^2$ from this formula. Formula that is given here and then what is other side is 7^2. So, I can rewrite this expression as $(x+5)^2=7^2$ wonderful.

So, I got something in terms of squares, now had it been only one square then in the situation was easy I would have equated to $x+5=\pm 7$. But there will be two situations because both the terms on the left-hand side and on the right-hand side are squares. Now, you just write $x+5=\pm 7$ and $-(x+5)=\pm 7$ that will give us four cases.

But if you look at these two expressions, they will eventually reduce to the same two expressions that is $x+5=\pm 7$. So, it is sufficing to consider only two equations $x+5=\pm 7$. Once I have considered this then I know the solution right, so it is just a matter of substituting the values and doing some little bit of algebra. So, you subtract 5 from both sides so $x = -5 + 7$ which will give me 2, and $x = -5 - 7$ which will give me -12.

These are the roots of the quadratic equation, these exactly match with the roots that we have got -12 and 2, and here 2 and -12. So, the solution set is same therefore now it is a personal choice which method to prefer but what is a choice that is available if you have some difficulty in factoring this. Let us say this is not 24 but some absurd number and you have some difficulty in factoring this finding prime factorization.

What you are doing here is you are not using this particular property when you are doing this example. When you are solving this example through this method you are not using this particular property so you can get rid of this property and you do not have to worry about. One note of caution is you cannot use this method when the number given here becomes negative in this side because square root of a negative number is not defined.

So, after adding this a and the number still remains negative you cannot use this method because according to this method there is no real solution whereas we do not know. So, this method had some limitations but it is quite powerful in solving the problems okay. We will come to how to overcome the limitations in a certain in the next few slides and we will see its beautiful connection with the quadratic formula.

Mathematics for Data Science 1
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Lecture – 29
Quadratic formula

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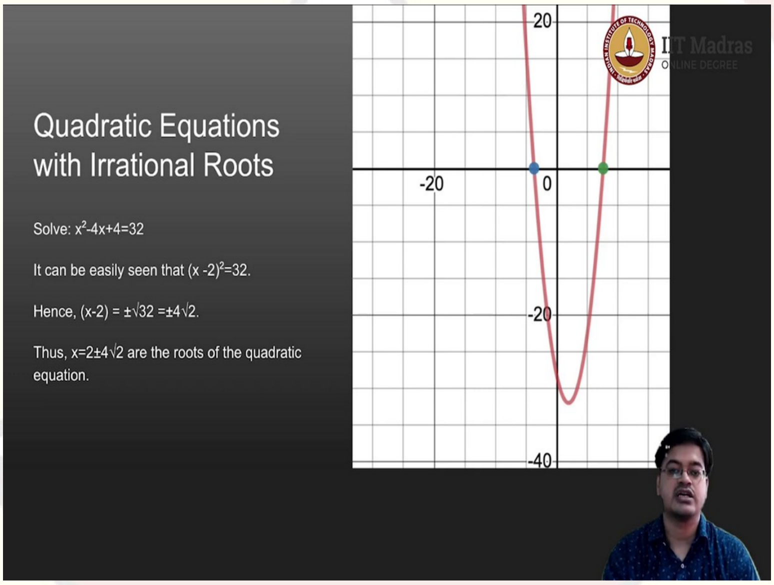

So, let us now go when the right-hand side is not a perfect square. In this case the right-hand side was a perfect square. So, when right hand side is not a perfect square, let us take this expression where; $x^2-4x+4=32$. So, I have already done the first step, I have added it. So, you can see the left-hand side is nothing, but $(x-2)^2$ which is equal to 32 and 32 is not the perfect square.

So, in such case what will happen? So, you will equate you will go in a similar manner $(x-2)^2=32$, you will take a positive square root of the left-hand side and $\pm\sqrt{32}$. $\sqrt{32}$ can be decomposed into 16×2. So, $\sqrt{16}$ is 4. So, it is $\pm 4\sqrt{2}$.

So, you will get 2 two roots; $2 \pm 4\sqrt{2}$ are the roots of the quadratic equation and they are irrational roots because, $\sqrt{2}$ is an irrational number ok. It is interesting to verify this result using a graph because, that will give us the clear cut understanding where does this $2+4\sqrt{2}$ are mapped. The two green dots over here represent the location of the roots ok. So, this is how you will solve a quadratic equation using the method of completing the squares.

Now let us explore the relation between quadratic equations method finding the roots using quadratic equations. Sorry; finding the roots of quadratic equations using completing the square method and its connection with quadratic formula.

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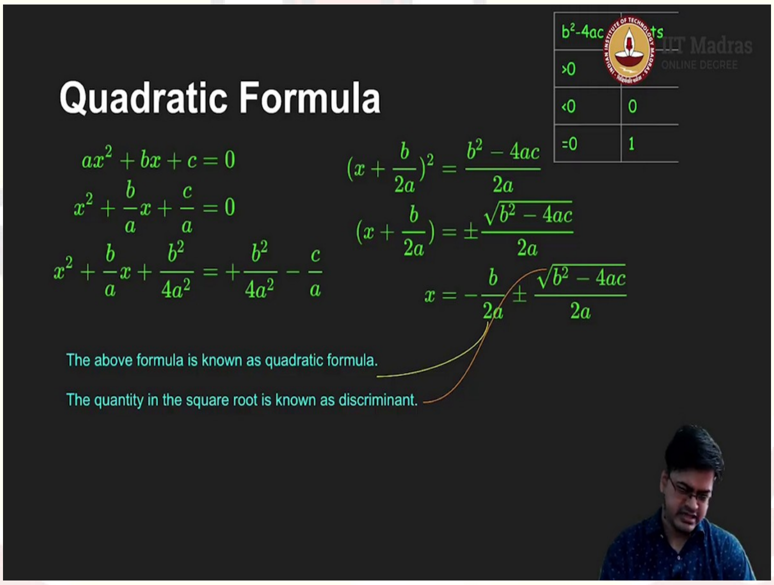

So, for this let us take a general quadratic function equated to 0 that is; $ax^2+bx+c=0$. I n is the second step is because $a \neq 0$. I can easily divide a I can easily divide by a; so that will give us the second step.

Now, as far as we can understand what we have here is, $\frac{c}{a}$ is the constant term. So, as per if we go by our method of completing the square, we will push this $\frac{c}{a}$ on the other side. So, it will take a negative sign that is what you are seeing here $\frac{-c}{a}$.

And, now, $\frac{b}{a}$ was the term $\frac{b}{a}$ was the term, if it is a complete square $\frac{b}{2a}\times 2$ will have come. So, I will get a term which is our a in the earlier expression that will be $\frac{b}{2a}$. So, $(\frac{b}{2a})^2$ will be $\frac{b^2}{4a^2}$, which is the term that I will add on both sides. So, I have added on both sides $\frac{b^2}{4a^2}$.

Now, look at this expression carefully, what is this expression? This is $(x+\frac{b}{2a})^2$ and this is some constant. So, I will use this that is; $(x+\frac{b}{2a})^2$ is equal to we can rearrange this term, $4a^2$ is the LCM. So, you just multiply by 4a over here you will get $4a^2$ there and divide by 4a so you will get $\frac{b^2-4ac}{2a}$. This is what you will get if you rearrange these terms hm. Sorry, it is wrong.

It is $\frac{b^2-4ac}{4a^2}$ it is. This is wrong. It should be $4a^2$ and then once you take the square root of this then you will get $(x+\frac{b}{2a})=\frac{\pm\sqrt{b^2-4ac}}{2a}$.

So, effectively using the same method of completing the square, the root of this equation will be $x=\frac{-b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$; as easy as this. So, this method is very powerful; this is what we have done using method of completing the squares and it gives us a general formula which is called quadratic formula. So, this formula is known as quadratic formula and the term over here in the square root is known as discriminant.

What is the quadratic formula? This complete expression on the right-hand side is a quadratic formula. And the term in the square root since b^2-4ac is called the discriminant. Why? Because it discriminates.

Let us see, if this $b^2-4ac>0$; that simply means we have two real roots. If this $b^2-4ac=0$; that means, we have only one repeated root. And if $b^2-4ac<0$, then we are actually taking square root of a negative number which will go to the complex domain. So, it has no real roots.

So, let us summarize this method or the summary of this method into a table.

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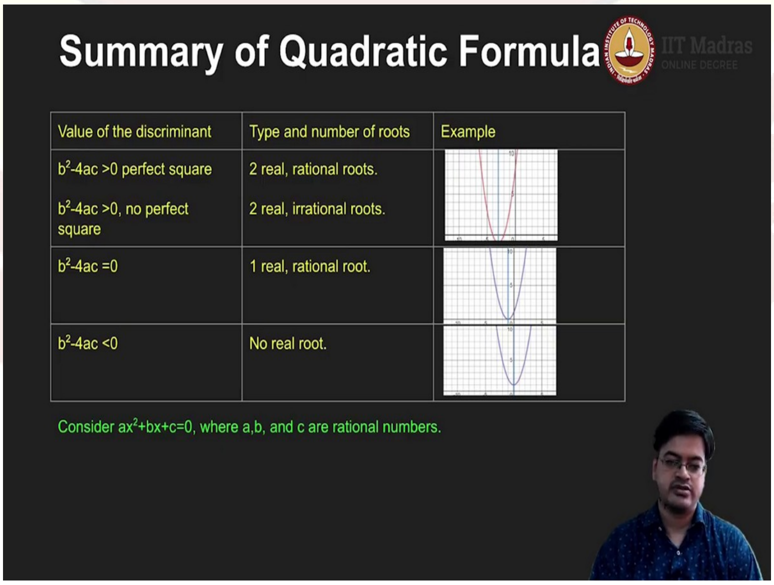

So, value of the discriminant suppose $b^2-4ac>0$ and the discriminant b^2-4ac is a perfect square; that means, I know the square root of it then we have two real rational roots. If $b^2-4ac>0$, but it is not a perfect square; then I have two real irrational roots.

We have already seen in week 1 that real number line is divided into rational numbers and irrational numbers. So, this is the splitting which will help. So, if $b^2-4ac>0$, but not a perfect square I will get irrational number. If b^2-4ac is a perfect square, I will get a rational number. If $b^2-4ac=0$, then I will get one real rational root. And if $b^2-4ac<0$, I do not have any root.

Let us demonstrate it through some graphs which we have seen which we have already seen. So, here is the example; where $b^2-4ac>0$. These are the two roots of this quadratic equation which are given here.

Let us say $b^2-4ac=0$, this is the root only one root it has and it is repeated. And if $b^2-4ac<0$, our example was x^2+1. So, in this case right it never touches the minimum; so you will get this particular expression ok. $b^2-4ac<0$, it never really touches the x axis. That is the verification that we have something like this ok.

So, let us go to the next slide which is actually yeah of course, these are the conditions that are required where a, b, c are rational numbers because, I am telling that this is this is will be a rational root. So, if it is not rational then what you need to do is; you suppress this, you do not need to say anything about this.

If they are rational numbers then whatever I am saying over here is true and whatever I am saying over here is true. If they are not rational numbers still you will have two real roots, but I cannot say whether they will be rational or irrational. And you will you may have one real root, but it can be irrational also. For example, $(x-\pi)^2$ or $b^2-4ac=0$ and then still you will get root as a π which is an irrational number.

So, that is so in order to distinguish between rational and irrational you need a condition that a, b, c are rational numbers. If you do not want to distinguish between rational and irrational numbers you do not need this condition. You can have a, b, c as real numbers; only condition that will prevail is $a \neq 0$.

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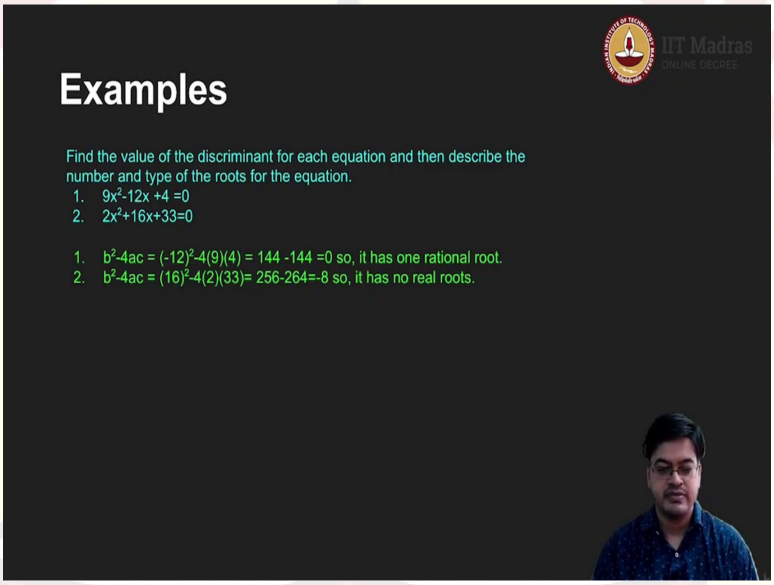

So, let us go ahead and see some examples where I will use the discriminant formula or quadratic formula to distinguish between the roots. So, the question itself says; find the value of discriminant for each equation and then describe the number and type of roots for the equation.

So, let us take the first example is $9x^2-12x+4=0$. So, I want to evaluate b^2-4ac. So, it is b is -12, a is a is 9 and c is 4. So, b^2-4ac I want to evaluate for this. In a similar manner let us take the next equation which $2x^2+16x+33=0$; where b will be 16, a will be 2 and c will be 33.

Let us evaluate b^2-4ac for the first equation that is; $(-12)^2-4\times9\times4$. So, 9 4s are 36, 36 4s are 144 and 12^2 is also 144. So, 144 - 144, if you refer to the previous table it has only one repeated rational root.

You go to the second example $b^2 - 4ac$; b is 16, a is 2 and c is 33. So, if you look at it 16^2 is 256, 33×8 is 264 yes. So, I got 256 - 264; that means, I got -8. Therefore, the $b^2 - 4ac < 0$, and hence it will have no real root ok. This is the summary of using the discriminant method.

The discriminant method or the quadratic formula actually gives you a number of ways to handle the problem. So, in short what we have seen today is, we can solve an equation given that I know the values of a, b and c using the quadratic formula. So, let us summarize what are all the methods that we have studied in this particular example ok.

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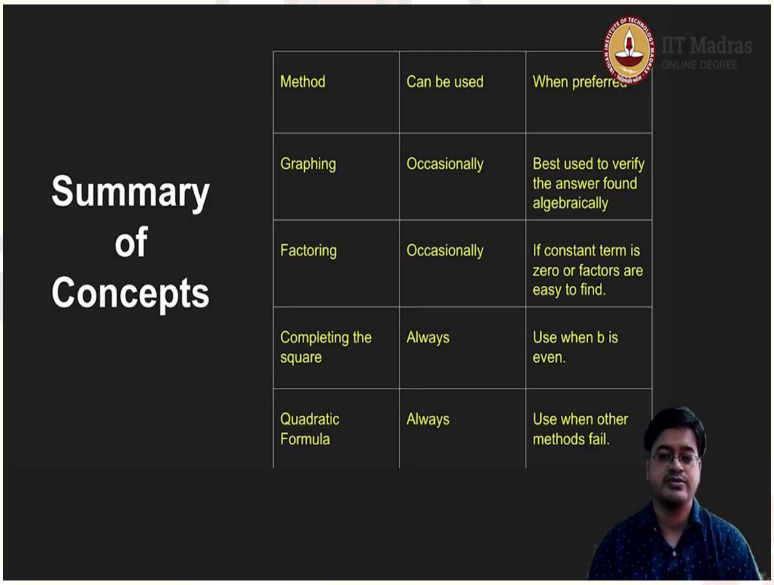

So, the summary of concepts is; let us say I have a method which is called graphing method. This is the method which we started with. When do we use this method? The graphing method actually unless your solutions are integers will not give you a good result.

But it is a best method to verify your results or verify the results that you have actually found algebraically. If there is any calculation mistake or something it will be revealed very easily. So, the graphing method is very helpful when you want to verify the result, but you can also use it to find roots of the equation occasionally.

Similarly, factoring method also suffers from the disadvantage that; it may not be helpful if the factors are not easily visible. For example, you may get the constant term to be equal to 26.2 in the quadratic equation. The constant term is 26.2 and then, you may have tough time in visualizing the factors.

So, in such cases factoring method need not be used, but it is very helpful if the constant term is 0 or the factors are easy to find you can actually guess the factors if the nice numbers like 49, 24 all these nice numbers are there then you can very well go with factorization method.

The method of completing the squares all the time it works; it is very easy when b is even. Otherwise, you will have problems if the right-hand side in the method of completing the square is negative, that is you will go to complex domain and then you may have some problems which we are not dealing with in this course. So, for us it can be always used when b is even ok.

Now last method is quadratic formula, which is derived from which is derived from method of completing the square. And this gives you all the time this is this will give the answer all the time it is always helpful for. So, for our purposes when we are studying these methods these two methods; completing the square and quadratic formula will always give the answer irrelevant of whether the coefficients are rational numbers, irrational numbers, or some absurd numbers ok.

Now, let us go to one more concept which is called axis of symmetry. I have not given any derivation about axis of symmetry yet.

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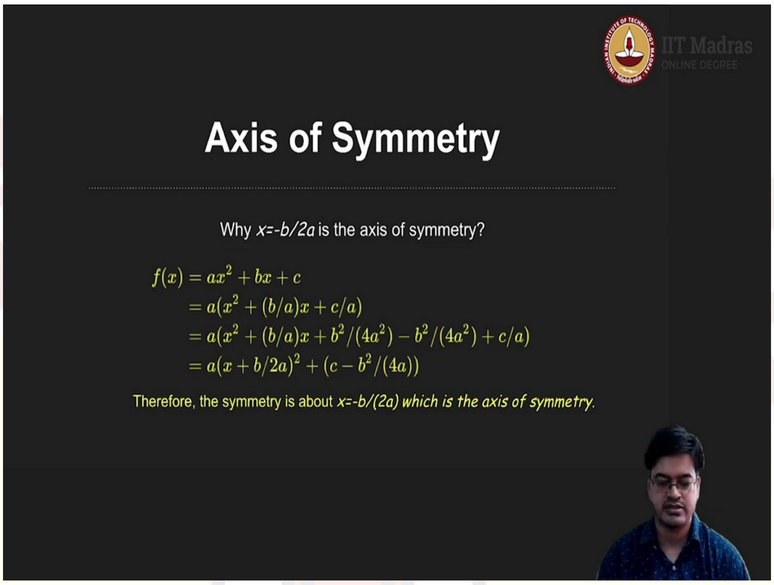

So, let us start with axis of symmetry. We already know while graphing the graphing the quadratic function it is very important to know the axis of symmetry. And we have boldly claimed at $x = \frac{-b}{2a}$ is the axis of symmetry. Now I will answer the question why $x = \frac{-b}{2a}$ is the axis of symmetry. This is an application of method of completing the square.

So, let us assume that I have been given a general quadratic function $f(x) = ax^2 + bx + c$. $a \neq 0$. So, I will pull out a common and therefore, my expression will become $a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$. Now, when I was completing the square I was throwing $\frac{c}{a}$ on the right hand side, but this time that provision is not there.

So, I will retain $\frac{c}{a}$ only thing is I will split the entire expression. So, when I split the entire expression, I will get a as it is and this expression as it is here $\frac{b}{2a}$ should come;

that means, $\frac{b^2}{4a^2}$ I will add and subtract $\frac{b^2}{4a^2}$ ok. And therefore, I will get this expression. Once I get this expression, I will recognize this term this entire term so these three terms together as $(x+\frac{b}{2a})^2+c-\frac{b^2}{4a}$ ok.

Now you look at this equation correctly. For example, this is a quadratic function written in a different form. What is this number? This is some constant ok. This is some constant and now you look at this number this is actually deciding the symmetry around x symmetry on x axis.

If you put $\frac{-b}{2a}$ as one vertical line; everything because $y=x^2$ it is symmetric about that y axis, everything will be symmetric about $x=\frac{-b}{2a}$. This expression defines the symmetry of the relation or the symmetry of the function because this is nothing, but just a constant on y axis.

Therefore, this x is equal x. So, basically you will write $(x+\frac{b}{2a})^2=0$. So, $x=\frac{-b}{2a}$ that vertical line is the axis of symmetry for this expression. So, the symmetry about $x=\frac{-b}{2a}$. Therefore, this is known as axis of symmetry that answers the quadratic equation axis of symmetry question. Why $x=\frac{-b}{2a}$ is the axis of symmetry?

This ends our topic on quadratic functions and quadratic equations.

Mathematics for Data Science 1
Week 05 - Tutorial 01

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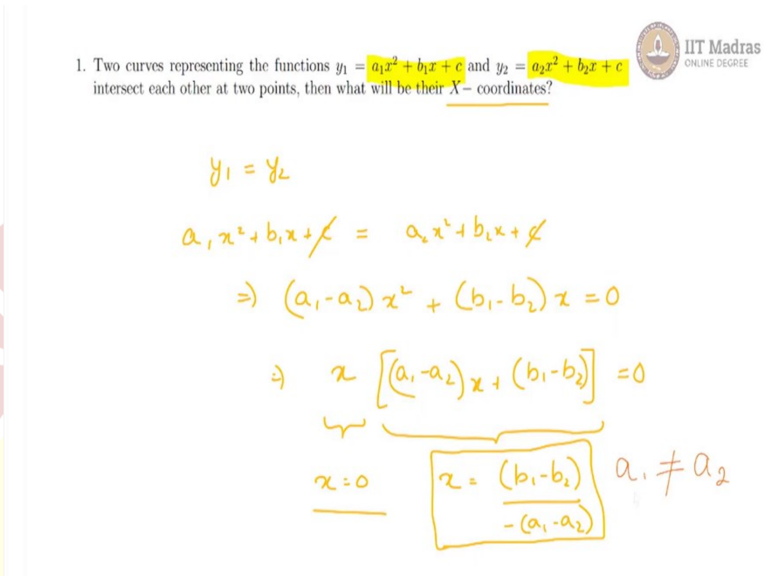

Hello Mathematics students, in this week's tutorial we will look at question related to quadratic functions. In our first question here we have two quadratic functions given to us and they intersect each other at two points and what will be there x coordinates. Clearly if they intersecting each other that means the x and y will be same. So, that is mean $y_1 = y_2$ and this is what we are trying to solve for.

So, $a_1x^2 + b_1x + c$ should be equal to $a_2x^2 + b_2x + c$ and the x is supposed to be same. So, anyway we can cancel off the c here. So this gives us $(a_1 - a_2)x^2 + (b_1 - b_2)x = 0$. This would imply this is us $x[(a_1 - a_2)x^1 + (b_1 - b_2)] = 0$. So, this corresponds to two different solutions. So, if we took this part to be 0 then $x = 0$ as one solution.

And the next will give us $x = \frac{b_1 - b_2}{-(a_1 - a_2)}$. So, this is a product of two terms and that product of two terms is 0 which means either of two terms has to be 0. So, one solution is x being 0 and the other one you get this as the solution. Now this is only a valid solution if a_1 and a_2 are not equal because a denominator cannot be 0. Therefore, $a_1 \neq a_2$ is a condition that needs to be satisfied.

So, these are the two x coordinates one is 0 and the other is $\frac{b_1 - b_2}{a_2 - a_1}$.

Mathematics for Data Science 1
Week 05 - Tutorial 02

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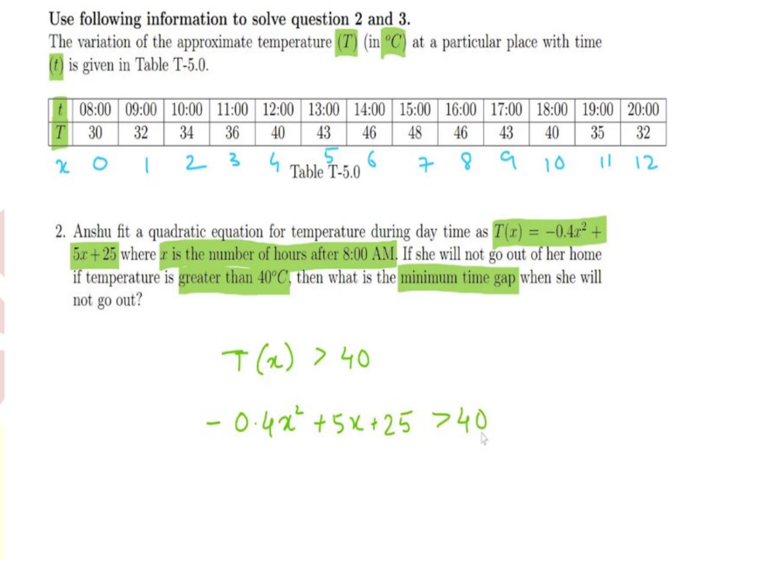

We are supposed to use this information this particular table to solve question 2 and 3. We will do a question 2 now. And this table will give us the variation of approximate temperature T. So, this is a temperature T in °C at a particular place with time small t, so this is the time. So, the time and the respective temperatures are given in this table.

And Anshu fit a quadratic equation for temperature during day time as $T(x) = -0.4x^2 + 5x + 25$. Where x is the number of hours after 8 am. So, x begins from 0 for $x = 0$. If we wrote additionally here this is 0, this is 1, this is 2, this is 3, this is 4, 5, 6, 7, 8, 9, 10, 11 and 12. So, we have x going from 0 to 12. If she will not, so if Anshu will not go out of her home the temperature is greater than 40 degrees.

So, greater than 40 degrees and Anshu will not go out of the home. Then what is the minimum time gap when she will not go out? Which means what is the time when the temperature is greater than 40. And this is on the basis of this particular quadratic equations. So, we are essentially trying to solve this as $T(x) > 40$. So, that means $-0.4x^2 + 5x + 25 > 40$.

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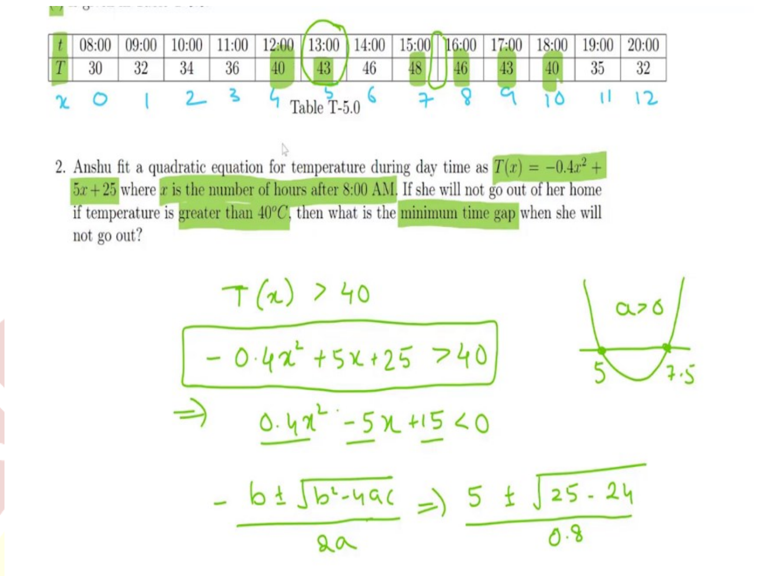

And that would indicate that $0.4x^2 - 5x + 15 < 0$. So, if we took all the LHS to the RHS this is what you will get and this is an upward facing parabola. So, the parabola will be like this and we are looking for the portion where you have the value, the y value to less than 0. So, that would be happen between the roots for this we find out the roots using the formula $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

And how did we know that this parabola is upward facing because a is greater than 0, a here is 0.4 b is -5 and c is 15. So, these roots will come out to be $5 \pm \sqrt{25 - 4 \times 0.4 \times 15}$ that is 16×0.4 that is 6 ×4 that is 24. So, divided by $2a$ is 0.8.

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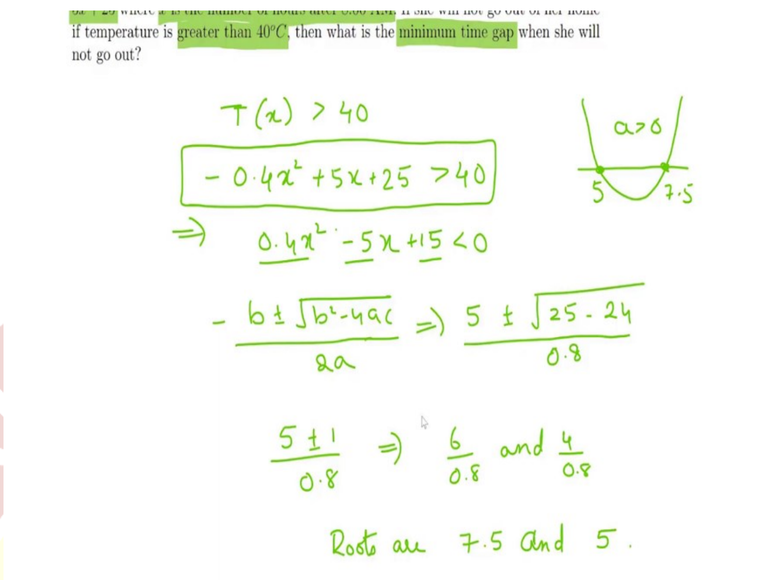

So, our roots are 5 plus or minus 1 divided 0.8 which is one is 6/ 0.8 and the other is 4/ 0.8. So, that gives us the roots as 7.5 and 5. So, these are the roots 5 and 7.5 and that means, this condition that is the temperature being greater than 40 is satisfied between 5 hours and 7.5 hours. That would be from here till somewhere in between here that is 15, 30. So, from 1 pm to 3:30 pm is the time suggested by the curve fit that Anshu has drawn but clearly this is wrong because it is already 43 here, and 48, here and 46, and 43, and 40, and 40 so it is a much larger duration where the temperature is greater than 40 degrees Celsius. So, this particular curve fit is pretty bad.

Mathematics for Data Science 1
Week 05 - Tutorial 03

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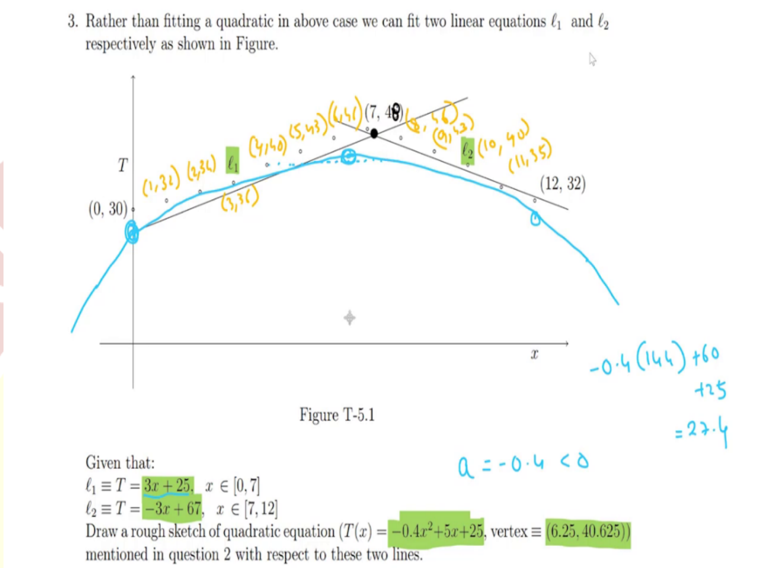

A third Question is related to the second question. So, in case you have not the second question please go back and see it. And here we are trying to say that instead of fitting a quadratic we can fit two linear equations l1 and l2. They have already provided us with the two equations which are this and this. So, $l_1: y = 3x + 25$ and $l_2: y = -3x + 67$ and the curves are already given.

Now the question is asking us to draw rough sketch of the quadratic equation that was fit and the vertex is also provided for us with respect to these two lines. So, I think it is useful if we can just mark out the points here. So, (0 , 30),(7 , 46) and (12 , 32) are already given. The remaining ones were, this is (1, 32), this is (2 , 34), this one is (3 , 36), this one is (44 , 40), this is (5 , 43), this is (6 , 46). So, this question has a problem here this should be (7, 48) that point.

And this is (8, 46) again and this is (9 , 43) and this point is (10 , 40) this is (11 , 35) and that may have (12 , 32). So, these are points and for us to do the rough sketch. The vertex is at 6.25 so the vertex should somewhere here and it is at 40.625. So, this is the horizontal (4, 40), then vertex is somewhere here. So, clearly our quadratic is below the points that we have been given. And this being the x^2 coefficient is -0.4 which is less than 0 so it is a down turned parabola.

And let us look at the two points that we know for sure 0 and $x = 0$ this parabola is going to give us 25 the quadratic equation going to give us 25. Which is definitely below so somewhere here it appearing to be intersecting with this line. So let us look at, we have $3x + 25$ so at $x = 0$ the quadratic equation and the l1 line meet. And $x = 12$, we have 0.4 into $-0.4 \times (144) + 60 + 25$ which give us about 27.4, I think.

So, that is below this somewhere here so our quadratic is going to look like this. Something like this so it is quite inaccurate for the given data. So, this is a very bad curve fit.

Mathematics for Data Science 1
Week 05 - Tutorial 04

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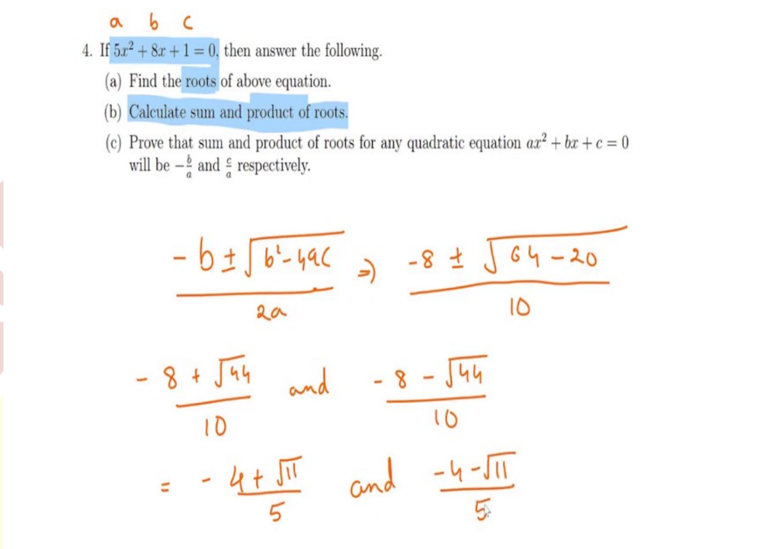

This is the pretty straight forward question, we are given a quadratic equation and we are asked to find the roots and also calculate the sum and product of roots. So, the roots we are going to get from the formula again which is $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. So, that gives us, here in this case a is 5, b is 8 and c is 1. So, we have $\frac{-8\pm\sqrt{64-4\times5}}{10}$.

So, we have $\frac{-8+\sqrt{44}}{10}$ and $\frac{-8-\sqrt{44}}{10}$. And if we simplify it taking 2 common out, you will get $\frac{-4+\sqrt{11}}{5}$ because the 4 comes out of the square root and becomes 2. And $\frac{-4-\sqrt{11}}{5}$.

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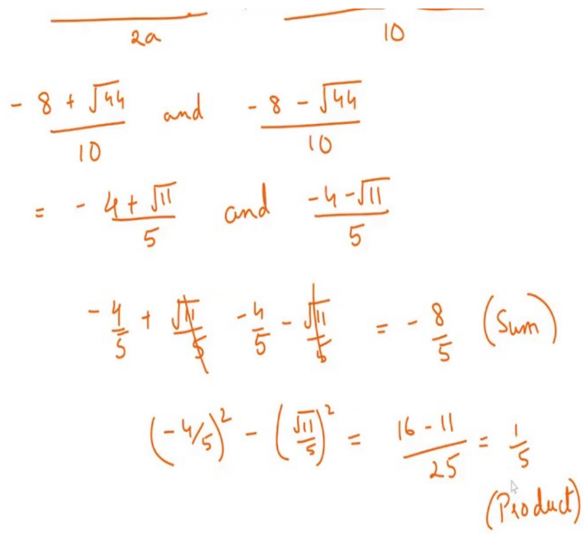

And the sum of these roots is if you just add them up you will get t $\frac{-4}{5} + \frac{\sqrt{11}}{5}$. So, these get canceled so you get $\frac{-8}{5}$ is the sum. And in terms of product you basically doing $(a + b) \times (a - b)$ so you will get $(\frac{-4}{5})^2 - (\frac{\sqrt{11}}{5})^2$. So, that gives us $\frac{16-11}{25} = \frac{1}{5}$. So, this is the product of the roots. $\frac{-8}{5}$ and $\frac{1}{5}$ are just sum and product of the roots respectively.

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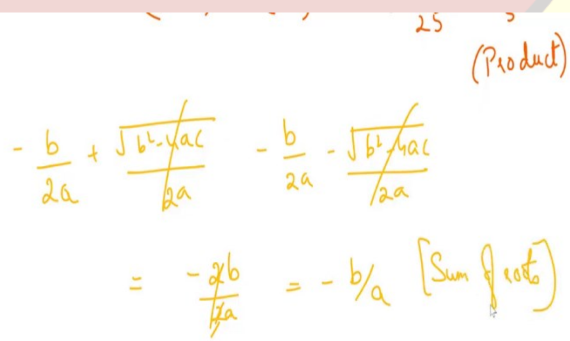

$$\left(\frac{-b}{2a}\right)^2 - \left(\frac{\sqrt{b^2-4ac}}{2a}\right)^2$$

$$= \frac{b^2}{4a^2} - \frac{(b^2-4ac)}{4a^2}$$

$$= \frac{\cancel{b^2}-\cancel{b^2}+\cancel{4}\cancel{a}c}{\cancel{4}a^{\cancel{2}}} = \frac{c}{a}$$

[Product]

The question is asking us to prove that the sum and product of roots for any quadratic equation will be this and this respectively. $\frac{-b}{a}$ and $\frac{c}{a}$ respectively. So, all we need to do for this is to sum $\frac{-b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}$. This we are summing with $\frac{-b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}$ a. So, clearly these two cancel off and you are left with $\frac{-2b}{2a}$, 2 and 2 cancel off and you have $\frac{-b}{a}$ is sum of roots.

And when we do the product again it is in the $(a+b)(a-b)$ form so we will get$(\frac{-b}{2a})^2 - (\frac{\sqrt{b^2-4ac}}{2a})^2$ which gives us $\frac{b^2}{4a^2}-\frac{(b^2-4ac)}{4a^2}$. So, that gives us $\frac{b^2-(b^2-4ac)}{4a^2}$, b^2-b^2 cancels off then we have 4 4 going away $a\ and\ a$ going away so you were left with $\frac{c}{a}$. So, this is the product of roots for a quadratic equation.

Mathematics for Data Science 1
Week 05 - Tutorial 05

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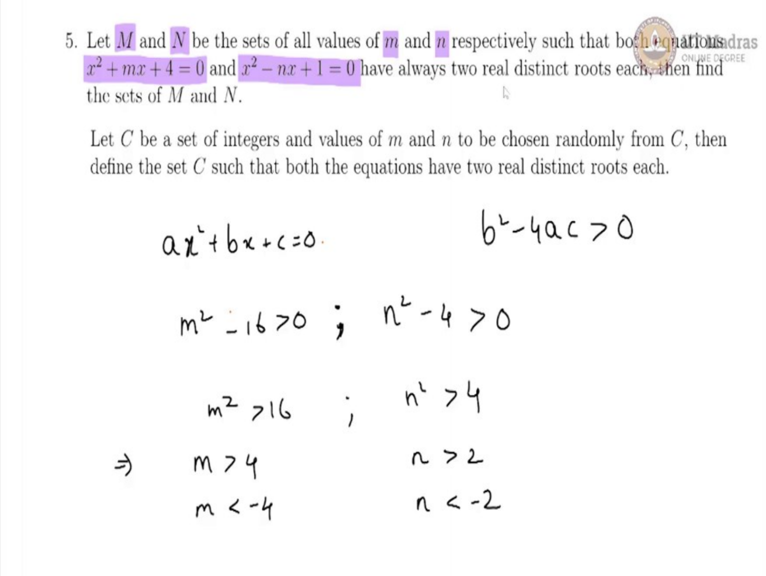

In this question we have 2 sets capital M and capital N which are sets of all values of small m and small n respectively such that these two equations have always two distinct real roots each, then find the sets M and N. Let us finish this part first. So, for a quadratic equation $ax^2 + bx + c = 0$ to have distinct real roots, the discriminant which is basically the value $b^2 - 4ac > 0$.

So, for this first equation that would be $m^2 - 16 > 0$ and simultaneously, we need for the second equation $n^2 - 4 > 0$. So, $m^2 > 16$ and $n^2 > 4$ and this would imply m is positive and greater than 4 or m is negative and lesser than -4. And here this would imply similarly n is positive and greater than 2 or n is negative and lesser than -2. So, these are all the possible values for which you will have two real distinct roots for these equations.

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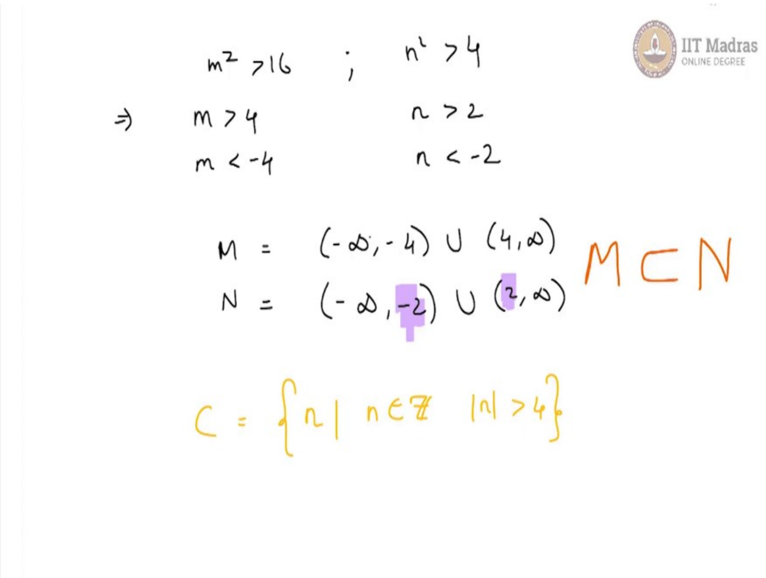

So, your set M would be the union of two intervals, one is a $(-\infty, -4) \cup (4, \infty)$. And set N is similarly $(-\infty, -2) \cup (2, \infty)$. Now the next part of the question, C is a set of integers and values of m and n are to be chosen randomly from C, then define the set C such that both equations have two distinct real roots each.

So, this is necessarily one single set we are taking and m and n should be chosen from that set. So, we clearly cannot have m being -2 or 2 or even -3 or 3. The set we are looking for is some sort of an intersection of capital M and capital N because both small m and small n should be drawn from this. And in this case that intersection will just be the set capital M because M is necessarily a subset of N.

However, C is also given out to be a set of integers, so it is not just the intersection of m and n, it is the set of integers which belong to the intersection and this case that intersection is only capital M where therefore, we have this set coming up as C.

Mathematics for Data Science 1
Week 05 - Tutorial 06

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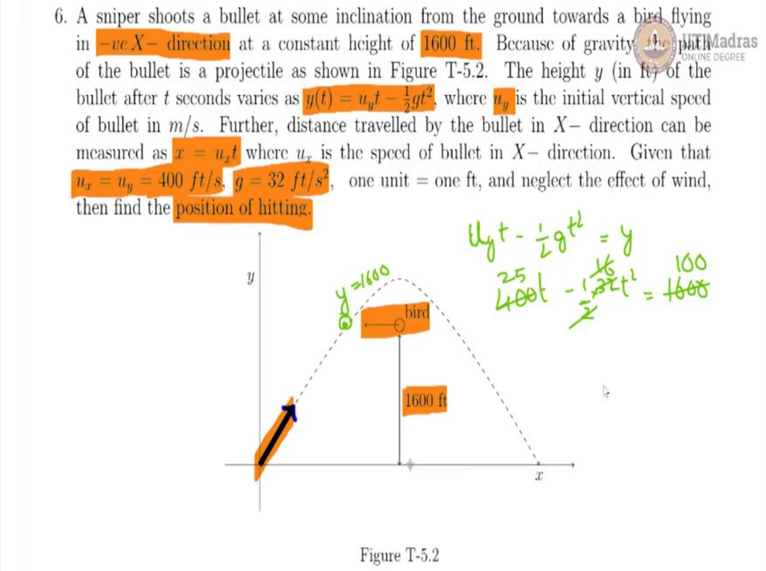

Figure T-5.2

In our sixth question, there is a sniper who shoots a bullet at some inclination from the ground towards a bird flying in the $-x$ direction, some bird is flying in the $-x$ direction at a constant height of 1600 feet. Because of gravity, the path of the bullet is projected as shown in this diagram. So, this is the bullet, it is going in this particular parabolic path and this is the bird which is going in the $-x$ direction at a constant height of 1600 feet.

Now, they have given the height y of the bullet at t seconds as this function, this is a quadratic function $y = u_y t - \frac{1}{2}gt^2$, where u_i is the initial vertical speed and that is also given here, it is equal to 400 feet per second and the value of g is also given here, 32 $feet/s^2$. And then, the distance travelled by the bullet in x direction is given by $x = u_x t$ and $u_x = u_y$ = 400 feet per second, neglecting the effect of the wind and everything, find the position of hitting?

Where will the bullet hit the bird and that would be here where y = 1600 for the bullet. So, let us use the y equation and the y equation is $u_y t - \frac{1}{2}gt^2$. So, $u_y t - \frac{1}{2}gt^2 = y$, so we know y is supposed to be 1600 and u_i is 400, so we get $400t - \frac{1}{2}g, g\ is\ 32t^2$. So, 2 ones and 2 16s, now you can cancel off 16 here with this is equal to and this becomes 100 and this becomes 25.

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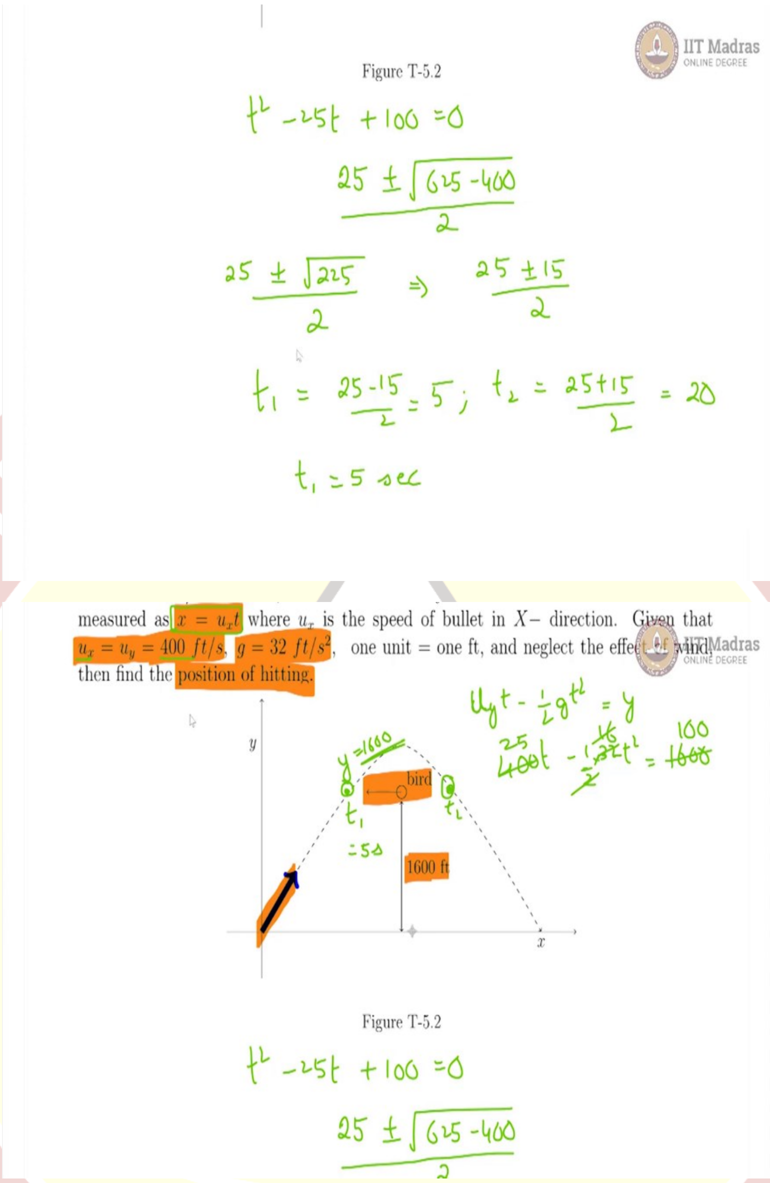

So, we get a quadratic equation which is $t^2 - 25t + 100 = 0$ and if we solve for the roots of this equation, we get the time when y is 1600 and we will get 2 times because y is 1600 twice on this path. So, we will get t_1 and t_2, we are looking for t_1 because that is where the bullet will hit the bird. So, your two roots are using the formula $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$, here you will get it as $\frac{25 \pm \sqrt{625-400}}{2}$.

So, that gives us $\frac{25 \pm \sqrt{225}}{2}$ which then given us $\frac{25 \pm 15}{2}$. So, we have one solution, $t_1 = \frac{25-15}{2}$ and $t_2 = \frac{25+15}{2}$. So, this is equal to 5 and this is equal to 20. Clearly, t_1 =5 seconds is where our bullet will hit the bird. This is t_1 =5 seconds. And we already know the y coordinate of this place so, for finding the position what is left is to find the x coordinate which we will get from $x = u_x t$ where, u_x is already given to be 400.

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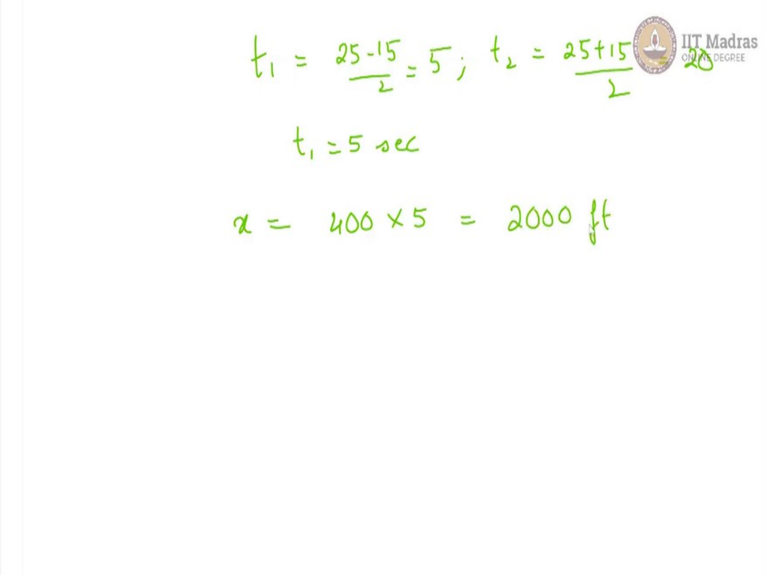

So, $x = 400 \times t_1$ which is 5 and that is equal to 2000 feet.

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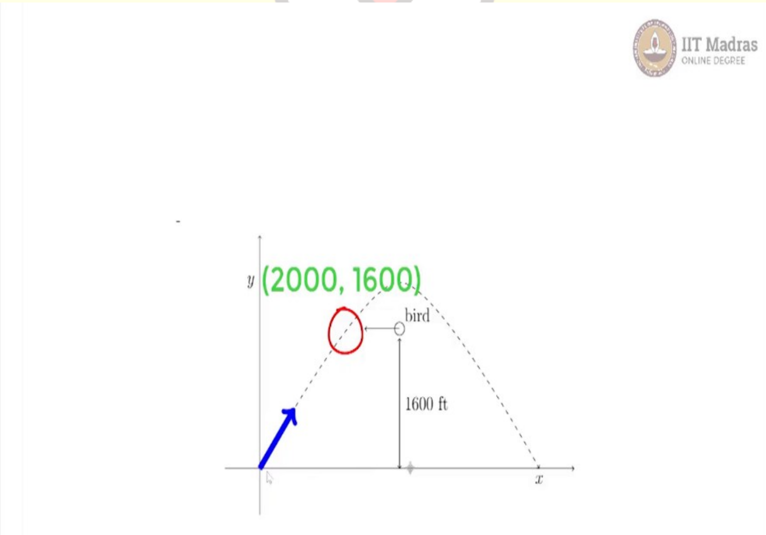

Figure T-5.2

Thus, the x coordinate for the point of hitting is 2000 and the y coordinate is 1600 feet. And this is the point where it hits.

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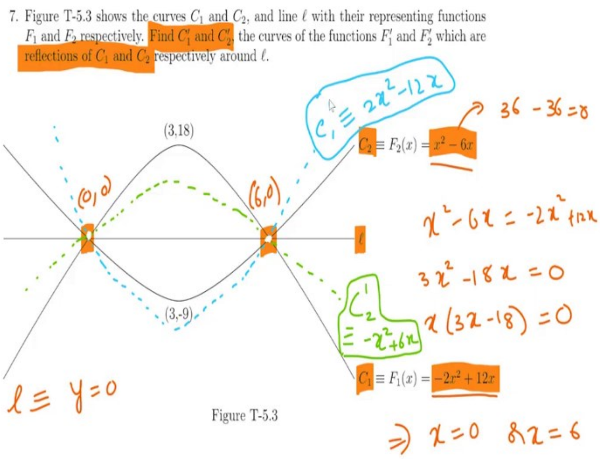

In this question there are these two curves C_1 and C_2 which are both quadratic curves and there is this line l which is passing through these two intersection point. So, line l is passing through the intersection points of these two parabolas. They are asking find C_1' and C_2', the curves of the functions F_1' and F_2' which are reflections of C_1 and C_2 respectively around l which means for C_1 the reflection would be something like this, about l it would be something like this and for C_2 the reflection would be something like this and these are what we are trying to find out, C_1' and C_2'.

So, this should be C_2' and this would be C_1'. For all of these, we have to first find the line l and that we can find when we solve for the equality of these two functions. So, we are taking $x^2 - 6x = -2x^2 + 12x$. And that gives us $3x^2 - 18x = 0$ and that further gives us $x\,(3x - 18) = 0$ that indicates $x = 0$ or $x = 6$.

So, this point has coordinate $x = 0$ and this point has coordinate $x = 6$. We need to find the y coordinates for these points now. For that we substitute $x = 0$ and we get in this equation or this equation I wrote this and we get $y = 0$. So, this point is essentially the origin. Whereas, for this point we substitute $x = 6$ and we get $36 - 36$ which is 0. So, this point would then be (6, 0).

So, essentially this is a horizontal line which is $y = 0, l$ is $y = 0$. So, now we are just looking for reflections about the x axis because $y = 0$ as the x axis. And that would give us directly the negative coefficients of the same things. So, C_1' would then be $2x^2 - 12x$ whereas, C_2' would now be $-x^2 + 6x$. Thank you.

Mathematics for Data Science 1
Week 05 - Additional Lecture

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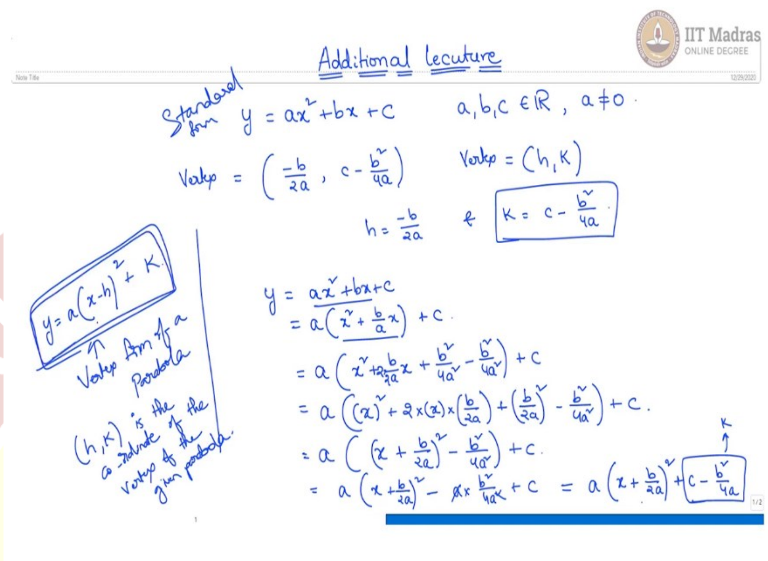

Hello, everyone, today we will discuss a small topic related to the forms of parabola. In other words we are going to see the relation between the standard form of a parabola and the vertex form of the parabola, we already know the standard form of a parabola which is given by $y = ax^2 + bx + c$ where these a, b, c belong to real and $a \neq 0$. From this standard form we will try to derive the vertex form of a parabola.

Now, from this equation we know that the coordinate of the vertex of the parabola is vertex will be x coordinate will be $\frac{-b}{2a}$ and y coordinate will be $c - \frac{b^2}{4a}$, this we already see in the previous lecture. Now, let us denote this coordinate of the vertex as (h, k). So, our h will be nothing but $\frac{-b}{2a}$ and k will be $c - \frac{b^2}{4a}$ .We have obtained the required data, now let us start the deriving.

We have $y = ax^2 + bx + c$, I will take a common from these two terms, I will get $a\left(x^2 + \frac{b}{a}x\right) + c$, also I will add and subtract $\frac{b^2}{4a^2}$ to this term, I will get $a(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2})$+c.

Now, also I will multiply with 2 and divide by 2 here, now I will rewrite this $a(x^2 + 2 \times x \times \frac{b}{2a} + (\frac{b}{2a})^2 - \frac{b^2}{4a^2})$+c. So, if we observe this $x^2 + 2 \times x \times \frac{b}{2a} + (\frac{b}{2a})^2$, this is in the form of $p^2 + 2pq + q^2$, we can write this as $(p + q)^2$.

So, writing like that, we get $(x + \frac{b}{2a})^2 - \frac{b^2}{(2a)^2} + c$. Now, if I multiply a we get $a[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}]$+c, a and this cancelled, finally I will obtain this will be equal to $a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$.

If we observe $c - \frac{b^2}{4a}$ is k here and $\frac{b}{2a}$ will be $-h$, so if we substituted h and k in this equation we get $y = a(x-h)^2 + k$, this is the vertex form, vertex form of the form of a parabola, where this $(h\,, k)$ is the coordinate of the vertex of the given parabola, this is the standard form and from this standard form we derived the vertex form.

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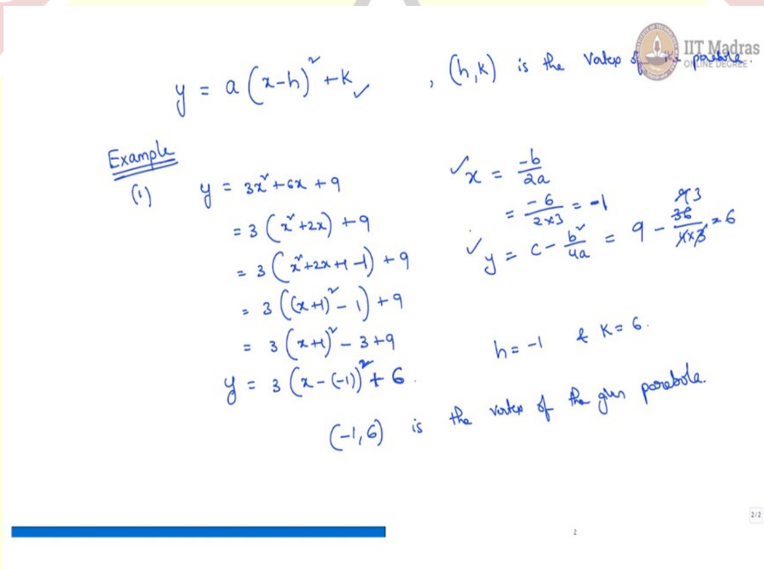

So, we have got the vertex form of the parabola which will be like this $a(x-h)^2 + k$ where $(h\,, k)$ is the vertex of the parabola. Now, let us see one example to understand this vertex form clearly. So, suppose we have an equation of a parabola given like this $y = 3x^2 + 6x + 9$ now we try to write in vertex form, so I will take 3 common from the first two terms, I will get $x^2 + 2x + 9$, so in order to make this a perfect square I will add 1 and subtract 1.

So, $3(x^2 + 2x + 1 - 1) + 9$, so 3 times this can be written as $3((x+1)^2 - 1) + 9$, which gives us $(3(x-(-1))^2 + 6)$. So, we have got the equation $y = (3(x-(-1))^2 + 6)$. So, if we equate it with this vertex form we get $h = -1$ and $k = 6$. So, our vertex will be at point $(-1\,, 6)$ is the vertex of the given parabola.

So, we will just cross verify it, we know that if we have a standard form we can calculate the x coordinate of the vertex, so x coordinate of this vertex will be $x = \frac{-b}{2a}$, so here b is 6 and a is 3, so if I substitute that-6 by 2×3 which I will get $x = -1$ and we know the y coordinate as $c - \frac{b^2}{4a}$ here we have c is $9 - b$ is 6 so b^2 is 36 / $4a$ is 3 so 4 9's 9 3's, so I will get 6. So, my vertex point will be at $(-1, 6)$, if we solve y, solve through standard form also.

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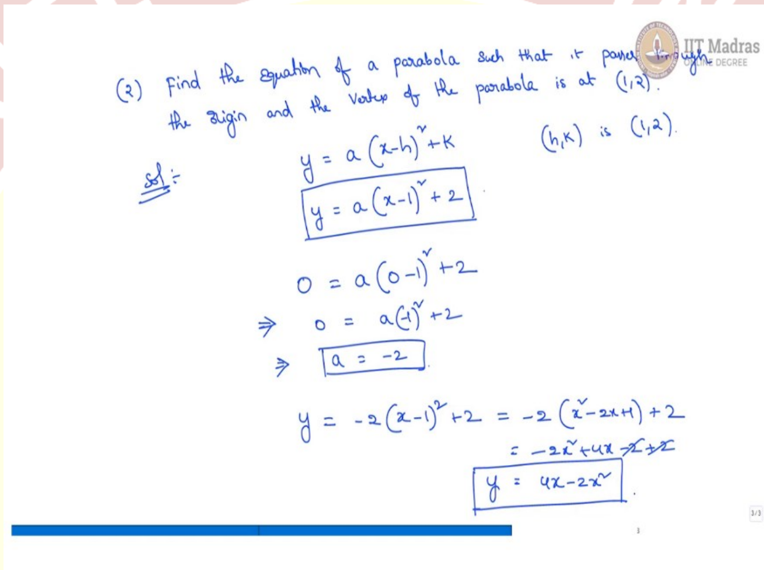

Now, let us see one more example, find the equation of a parabola such that it passes through the origin and the vertex of the parabola is at $(1, 2)$. So, as we know the vertex form given by y is equal to a times x minus h whole square plus k, here we have given that (h, k) is nothing but (1 , 2).

So, if we substitute that our equation will be simplified to $a(x - 1)^2 + 2$, also it is given that this equation passes through the origin that means 0 , 0 should satisfy this equation. So, if we substitute it we get the value of a, so $0 = a(0 - 1)^2 + 2$, this implies $0 = a(-1)^2 + 2$, which implies again $a = -2$.

So, our final equation of the parabola will be equal to $y = -2(x - 1)^2 + 2$, if you open that $y = -2x^2 + 4x - 2 + 2$, so this will be get cancelled and $4x - 2x^2$, so $y = 4x - 2x^2$ is the equation of the parabola that passes through the origin and the vertex of this parabola will be at $(1, 2)$. Thank you.

Mathematics for Data Science 1

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Lecture – 30
Polynomials

Let us introduce Polynomials. So, today we are going to see how the polynomials look like, how they behave. So, let us start with polynomials. Let us go ahead and see what expressions do we call as polynomial.

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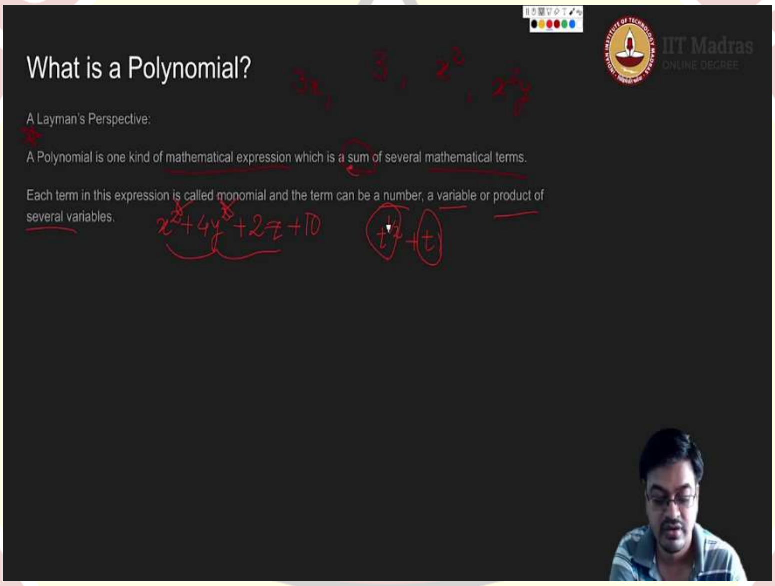

So, for that let us take the first point where we will take a Layman's perspective and, we will try to understand what a Layman will think of a polynomial. In order to do that let us first take a Layman's perspective and see what Layman will think. So, for a Layman, a polynomial it is some kind of mathematical expression which is a sum of several mathematical terms.

Then we asked Layman what do you mean by mathematical terms? The answer is each term in this expression, each term in this expression, that is mathematical term in this expression can be a number, a variable, or a product of several variables.

So, according to Layman each term this mathematical term can include a number, a variable, or product of several variables. These are the things that are allowed. So, basically then, let us take one example for this. And let us see if I have $3x$ this is a number and some variable. So, I have a constant 3, I have some number like x^2, I have some term like x^2y, all this contribute to something called polynomial ok.

Now, take a more significant number that is say $x^2 + 4y^2 + 2z + 10$; will this contribute to be a polynomial? Yes, because it is sum of a number which is 10 here, a variable. There are many variables 1, 2, 3, there are three variables, and product of several variables; in particular here we have x^2 and here we have y^2. So, this also qualifies to be a polynomial.

Then according to this, suppose what I will write is here, let us say some expression of the form $t^{\frac{1}{2}} + t$; is this expression a polynomial? Layman will say yeah, it can be a polynomial because, $t^{\frac{1}{2}}$, if you square this number you will get this, correct. So, this is one variable this is another variable and therefore, we are actually having a polynomial.

So, then we went and asked mathematician, what is a mathematician's perspective of a polynomial?

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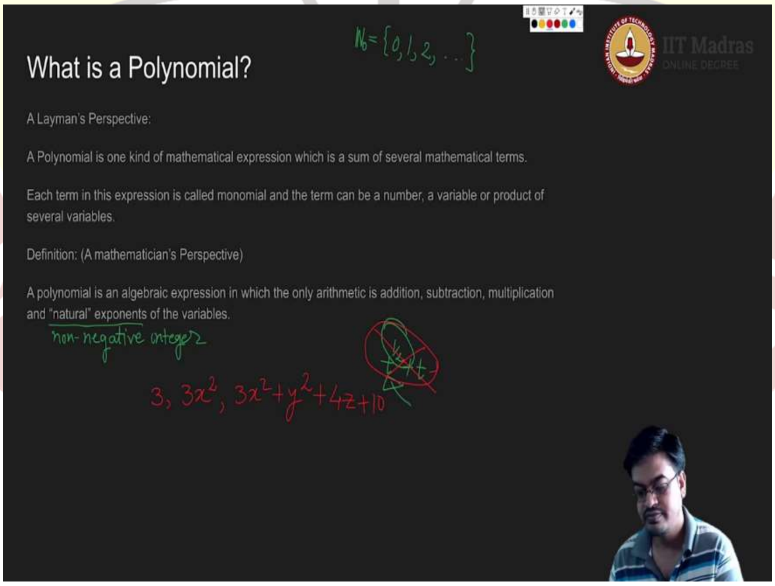

So, that we will define as a definition. So, a mathematician said a polynomial is nothing, but an algebraic expression in which only arithmetic is addition, subtraction, multiplication and this is interesting, he mentions it as natural exponents of the variables. Natural, by natural I mean the way we defined a set of natural numbers in our first week I mean natural means 0, 1, 2 and so on.

So, this is my set of natural numbers which include 0, if you want to emphasize it you can put it as N_0. Otherwise, you can call this set as set of whole numbers or set of non-negative integers. So, the definition can be twisted like this will have non-negative integer exponents. If you do not want any ambiguity, we can say that non-negative integer exponents of the variables.

So, then we if we go back to that earlier expression which is $t^{\frac{1}{2}} + t$, the all other expressions will qualify to be a variable, but this expression will not qualify to be a variable, why? Because this $t^{\frac{1}{2}}$ by definition is not a natural number, it is a rational number. We will come to it later, but this cannot qualify definitely so this cannot qualify as a polynomial if we go by this definition.

We have already seen many examples. Let us re-iterate them; one example was constant 3, another was $3x^2$, another one was $3x^2 + y^2 + 4z + 10$, all these are qualified to be polynomials. But this expression does not qualify to be a polynomial. So, this we will consider later as well and we will give our rational reason why it is not a polynomial.

So, we now we know what is polynomial. Now, it is time to see why is the name polynomial? Why do we call this as polynomial? That is what we will see now. So, let us go ahead and see something about the nomenclature. Why do we call them polynomials?

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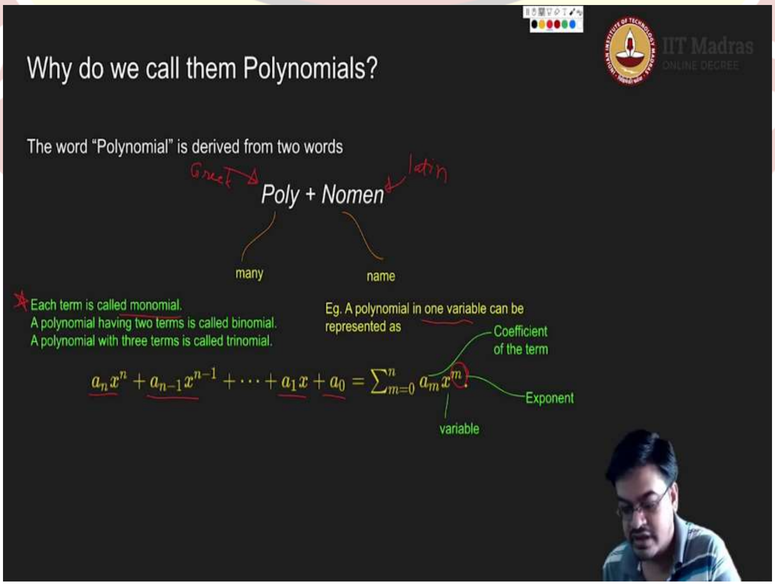

So, polynomials essentially is derived from two words one word is poly, second word is nomen. This is a Greek word this word and this has roots in Latin. The word poly essentially means many and the word nomen essentially means names or a terms. So, in our case it turns out to be terms. So, an expression having many terms is called polynomial ok.

Now, each term of the because it has many terms each term of the polynomial will be called as monomial, each term of a polynomial will be called as monomial. Then, if the polynomial has only two terms then you will call it as binomial. If the polynomial has only three terms then you will call it as trinomial.

So far, if you can label them you can label them, but in general we will treat them as polynomials. And remember that we will include this also; a monomial is also a polynomial for us. We will not distinguish between monomial and polynomial. Of course, monomial enjoy some different set of properties, but we will keep them with polynomials.

So, let us take one example. For example, a polynomial in one variable can be represented as a_nx^n this is the highest term, $a_{n-1}x^{n-1}$, a_1x^1, a_0 right. I am assuming that this a_n's not equal to 0. Otherwise the if they are 0, then the polynomial may extend to infinity. I do not want that. So, I am assuming that these $a_n{}'$s are not 0.

Now, if you can rewrite this using the notation of summation in this manner and therefore, this a_m will have a specific name and it will be called as coefficient of the term. Because this is a polynomial in one variable, x is the variable we are interested in x is the variable, and this m is the exponent of the variable.

Now, remember in order that this term to be a polynomial, this m should always be a natural exponent; by natural I mean the one that is non-negative integer. If it is not nonnegative integer then I cannot classify this as a polynomial ok. Let us go to the next step and see some examples of polynomials and try to identify whether the given expressions are polynomials or not.

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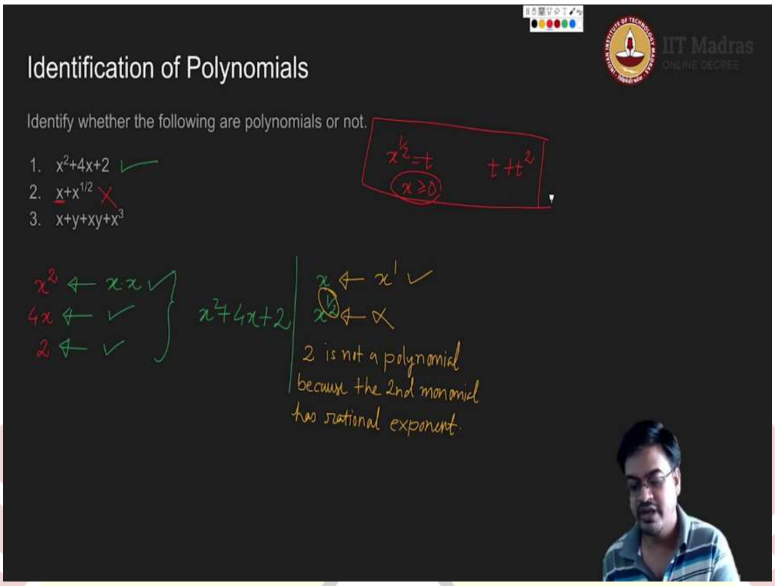

So, here is the question about identification of polynomials. Identify whether the followings are polynomials or not. The 1st one; $x^2 + 4x + 2$, what can you say about this? So, the first let us take term by term x^2, $4x$ and 2. So, the all these are monomials involved in this polynomial.

So, when I take x^2 it is nothing, but variable x multiplied by x. So, it is a product of two variables. So, this is ok. When I take $4x$ it is a number and a variable. So, this is also I do not have any and finally, this is just a number. So, together and the expression given it is sum of these that is; $x^2 + 4x + 2$, expression given it is sum of this. Therefore, this is a valid polynomial form.

Let us go ahead ok. Again, the same expression has come $x + x^{\frac{1}{2}}$. So, now, you look at the terms that are involved x and $x^{\frac{1}{2}}$. Now, if you look at the terms involved x and $x^{\frac{1}{2}}$, then this it is simply a variable raised to 1, x^1, right. So, I do not have any problem, this is a valid term because there is no issue with this; $x^{\frac{1}{2}}$ this term is not a valid term because, it has rational exponent. So, this is not correct.

So, this second expression 2 is not a polynomial, why? We need to justify we need to write a reason because, the 2nd monomial has rational exponent. This is an interesting observation. So, this does not qualify to be, to be a polynomial. So, I can erase this. This is not a polynomial.

Now, some people may say that what is a big deal? I can put $x^{\frac{1}{2}}$, let's say t and you can rewrite this expression as $t + t^2$, but remember when I am putting x raised to half as t, I am putting an explicit assumption on this x that is; this x should be greater than or equal to 0. So, I cannot define this polynomial on the entire real line.

So, we will refrain from doing such assumptions and therefore, it would not be a polynomial. Let us go to the next example, this example ok. So, let me erase the previous ones so that I will have some space.

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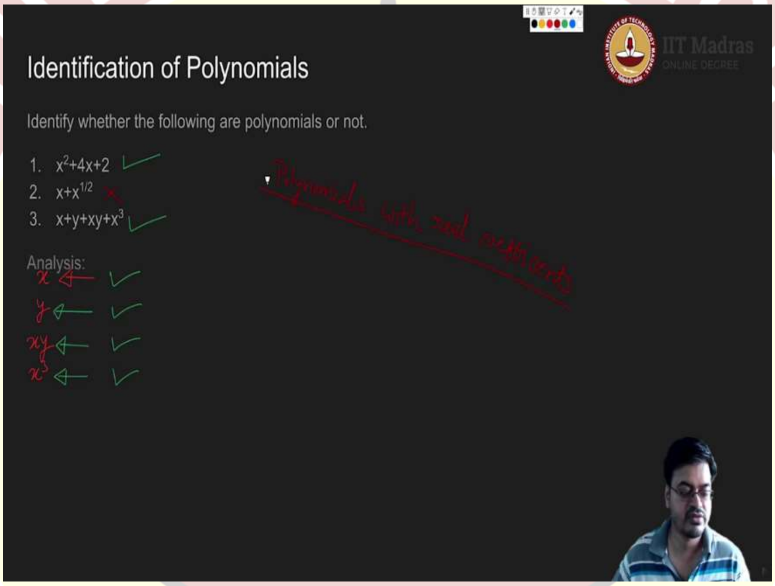

We will write all the terms as usual $x,\ y,\ xy,\ and\ x^3$. We will analyse the each monomial one by one, is this valid? Let me change the colour. Is this valid? Yes, it is valid because it is just a variable y? Yes, it is valid just a variable; product of several variables; product exponent natural exponent of single variable? Yes, it is valid.

So, according to me this and this qualify as a polynomial. And this do not qualify as a polynomial. So, we have identified what are the polynomials and how they look like. So, our identification part is complete. In particular we are dealing with polynomials having real coefficients because, all the numbers that I am giving you are real numbers.

So, just remember this fact we are only handling polynomials with real coefficients. If you go to the further branches of mathematics you may have polynomials with simply integer coefficients, you may have polynomials with complex coefficients, we are not dealing with them. So, this is how we will identify whether a polynomial whether a given expression is a polynomial or not.

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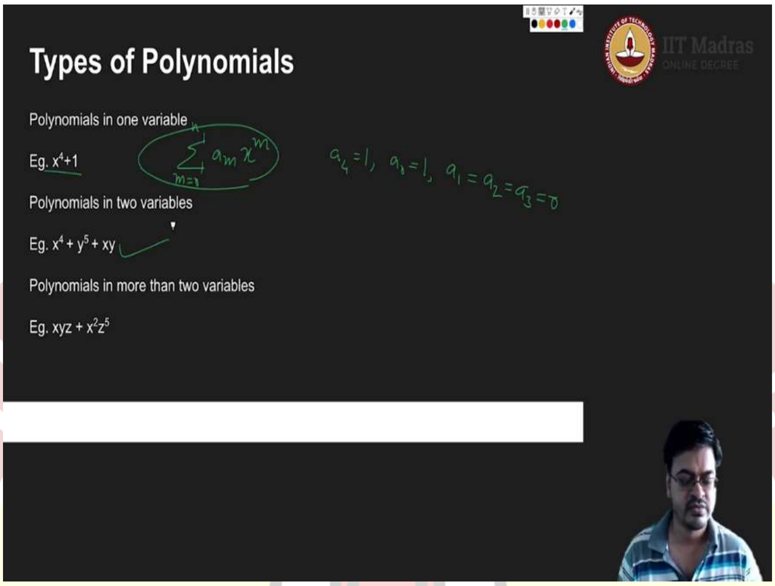

Let us go ahead and try to describe what are the types of polynomials, that we can encounter. We have already seen them; we are just enlisting them for the sake of completeness. So, there can be polynomials in one variable which will typically look like $\sum_{m=0}^{n} a_m x^m$. So, that let me rewrite it.

So, that was our expression; summation over $\sum_{m=0}^{n} a_m x^m$. So, this particular thing falls into that category. What is what will be here in this particular case $a_4 = 1$, $a_0 = 1$ and all others like a_1, a_2 and a_3 , all of them are 0. So, this is how we will describe the polynomial. So, this is a polynomial in one variable.

You can encounter a polynomial in two variables. For example, we have already seen some examples this is a polynomial in two variables and you can have similar expression, but now, you will have $a_m b_m$ and a_{mn} or something of that sort, to indicate the powers of these exponents. So, we will not indulge into a mathematical representation of this, but you can have polynomials in two variables.

In a similar manner you can have polynomials in three variables or more than two variables. So, here is an example of a polynomial in more than two variables. And these are the types of polynomials that you may encounter with real coefficients. So, this summarizes the topic of representation of polynomials.

Now, let us go ahead and see some further properties of these polynomials.

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Lecture – 31
Degree of Polynomials

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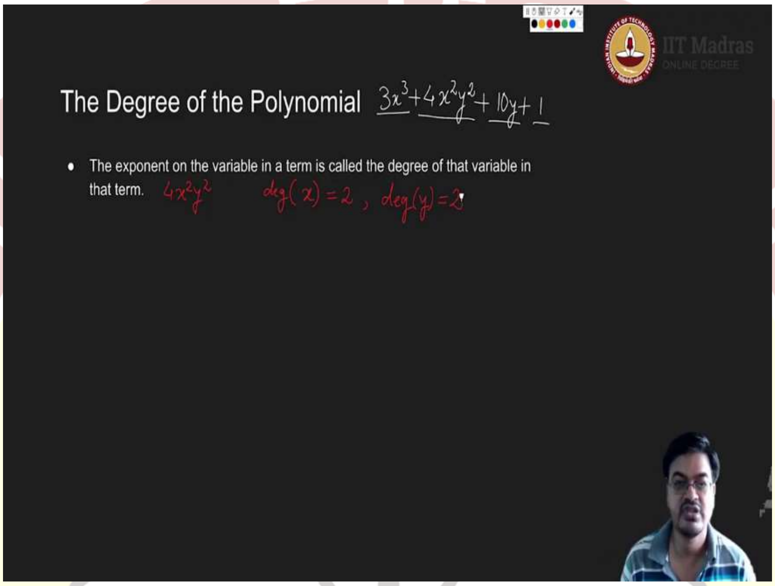

So, in particular, if I want to tell something about a Polynomial, an important property is a degree of the polynomial. So, what is the degree of the polynomial? For demonstration purposes, let me take one example. Let us say my example is $3x^3 + 4x^2y^2 + 10y + 1$, this is my example.

Then, I say this is the example. So, if I want to decide the degree of the polynomial, we have already seen each term itself is a polynomial. So, $3x^2$ is one; $4x^2y^2$ is one; $10y$ is one and this 1 is one. So, I want to identify the degree of each term as well.

In each term, there are many variables. For example, you would look at this term, if you look at this term then there are 2 variables. So, I want to have a complete understanding. So, in order to define the degree of a polynomial, I will start with defining the degree of the variable. So, the exponent on the variable in a term, the exponent on the variable in a term is called degree of that variable in that term. So, for demonstration purposes, let us take the expression $4x^2y^2$.

So, in this particular expression or in this particular monomial, how many variables are involved? One variable is x, second variable is y. So, what I am saying is now in this term, the degree of x; the degree of x let me abbreviate it as degree; degree of x is 2 and the degree of y, variable y is also 2, ok. So, this is how I will describe the degree of the variable.

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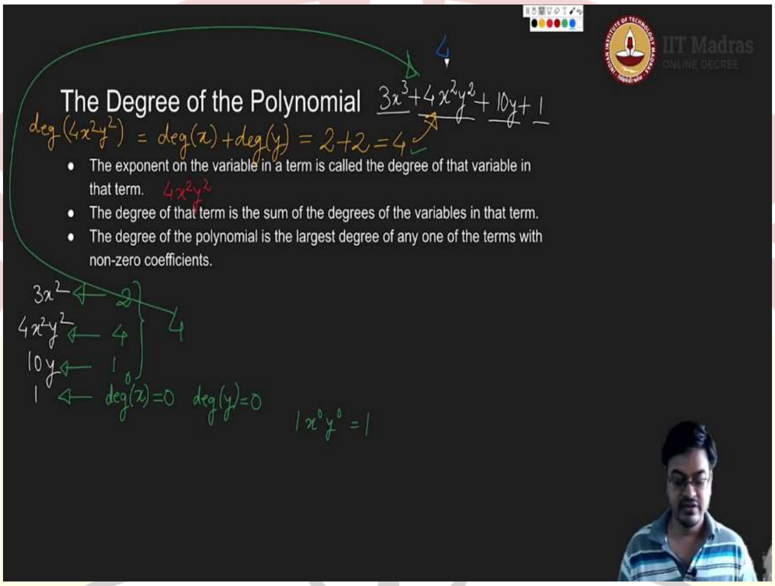

Now, let us take this term as a term and say what is the degree of this term, right. So, let me erase this particular portion which is actually blocking our view ok. So, the degree of that term, this term, we have already seen the degree of x is something and degree of y is something.

Degree of x was 2 and degree of y was 2, the degree of the term is the sum of the degrees of those variables in the term. That means if I look at this expression which is $4x^2y^2$, then and I ask for the degree of this term, then it is essentially degree of x plus degree of y that is 2 + 2 which is equal to 4.

So, degree of this term the second term in this expression is 4, fine. Now, we will answer the question, what is the degree of a polynomial? So, the degree of the polynomial is the largest degree of among these all the terms of any one of the terms with nonzero coefficients or terms will exist only when there are nonzero coefficients.

So, let us try to see how we can solve this problem. So, we will try to list all the degrees. So, if I take the first term that is $3x^2$, second term is $4x^2y^2$, then the next term is $10y$ and the last term is 1 which is the constant ok.

So, now, we will talk in terms of degrees. So, what is the degree of this particular term? It has only one variable x which has which is raised to the second power. So, the degree of this term is actually 2. What is the degree of this term? We have already seen here, the degree of this term is 4.

What is the degree of this term? The degree of this term is again y, means y^1. So, the exponent is 1, Interesting. Now, what is the degree of this term? Now, remember this is an expression in two variables; x and y. So, what then, I will ask a question what is the degree of x and what is the degree of y?

Now, you can also see that $1x^0y^0 = 1$. So, degree of x is naturally equal to 0 and degree of y is also equal to 0, right. Therefore, I can write the degree of this expression is 0 ok. Now, which one is the largest among these four? 0, 1, 2, 4? 4 is the largest. So, the degree of this particular polynomial is 4. So, this degree is actually 4, then write it here 4. So, this is a polynomial of degree 4 ok.

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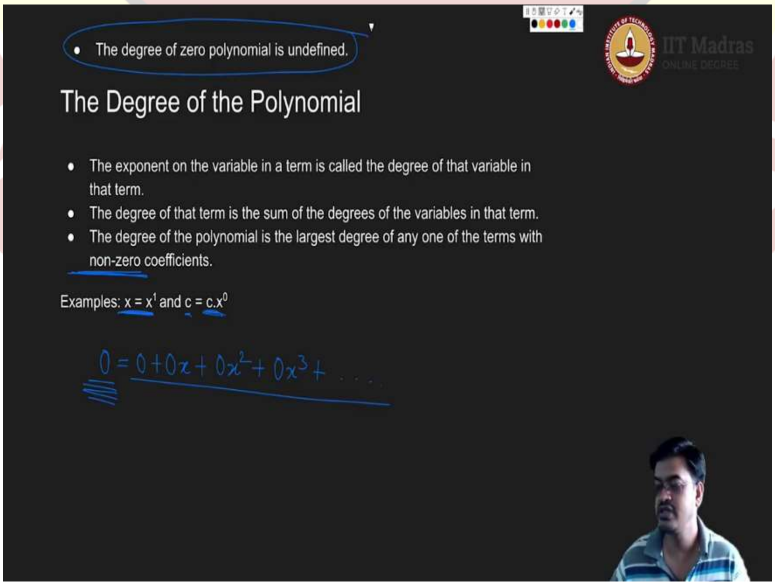

So, in this contest in while finding the degree of this particular polynomial, we have seen two things. What are those two things? If the coefficient is if the variable is x, then this is

x^1, if it is a constant, then $c \times x^0$. In our case because the polynomial was having 2 variables, it is cx^0y^0.

Interesting question comes when we try to see polynomials at some other things. Let us say if I want to describe 0, when c is nonzero, it is ok, but if $c = 0$, then what? Then, you can see $0 = 0 + 0x + 0x^2 + 0x^3 \ldots$, the matter is complicated further and so on right.

It will continue. So, if the point given, then this number is 0, then we will call this as 0 polynomial and we cannot define the degree of this polynomial because for a degree, we need a nonzero coefficient, just remember this in mind.

Therefore, the degree of 0 polynomial is always undefined this is an interesting fact which will be used when we use the division algorithm. The degree of 0 polynomial is undefined. So, you can use it in a more interesting manner that is what I can say. So, the degree; this is what? Degree of 0 polynomial is undefined.

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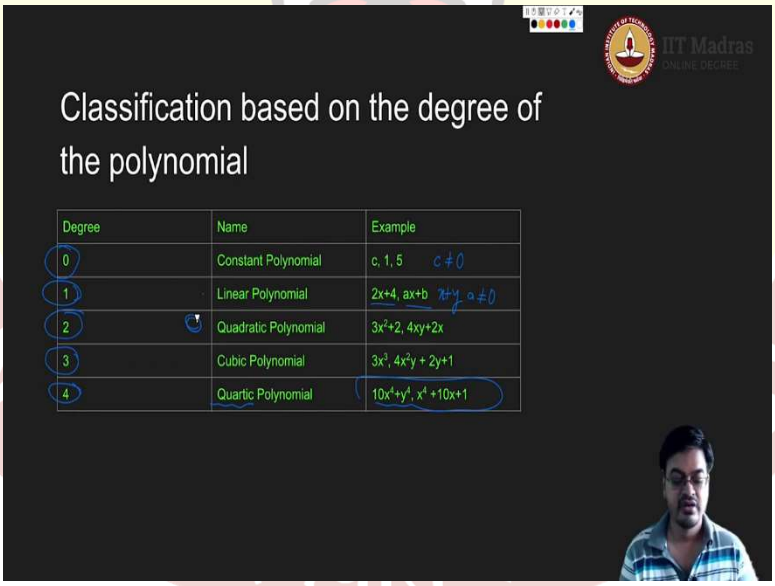

So, we have understood what is the degree of the polynomial. So, in particular, based on the degrees, now we have introduced one classification. So, based on the degrees, how the polynomials can be classified. So, if the polynomial has degree 0, then it is constant and this constant can never be equal to 0.

This is an important assumption. Then, if the polynomial is of degree 1, linear polynomial, then you will have a polynomial in this form. When I write this, then I should write $a \neq 0$, if I have a quadratic polynomial, if the polynomial ok.

So, here these are the polynomials in one variable, then I am considering a linear polynomials. If I am considering a polynomial of the form $x + y$, this is still a linear polynomial; but it is a polynomial in 2 variables. So, you can also encounter such polynomials in linear, but the crucial fact is degree is 1.

Second one is a quadratic polynomial which is of this form and here you can have polynomial in two variables, three variables or whatever way you want. Then, you will get a cubic polynomial which will have all terms containing degree 3, the highest term, highest monomial will have degree 3.

So, this is these are the examples of degree 3 polynomials. Similarly, degree 4 polynomials are called quartic polynomials and they will be given in this form and similarly, degree 5 polynomials are called quintic or quantic polynomials which will be represented with degree 5 polynomials, right.

So, and in general, you have a general term which is called polynomial. So, to be if you want to be specific, you can use this classification and say it is a quadratic polynomial, then you are giving more information about it.

This is what today's lecture meant to be. So, we have introduced the topic of polynomials.

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Lecture - 32
Algebra of polynomials: Addition & Subtraction

In this video, we will start with polynomials and we will try to do some Algebra with Polynomials. Or in other words you can say we will try to understand some operations on polynomials like Addition and Subtraction. So, let us move on.

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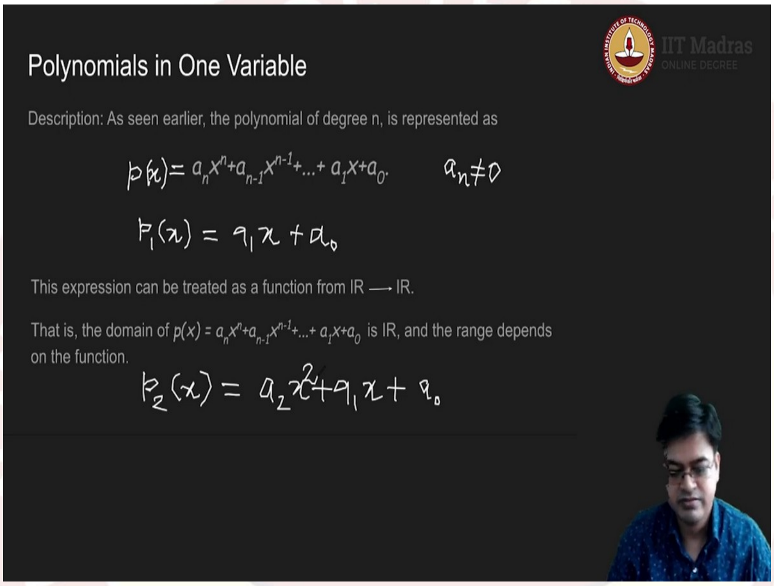

In order to simplify our calculations, we will only focus on polynomials in one variable; whereas all the operations that we are discussing can be done on polynomials with multiple variables. In order to pinpoint the thing, we will recollect how polynomials in one variable look like.

So, a polynomial of degree n in one variable can be represented in this form; $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$. So, you can actually correlate this with, let us say this is a monomial with 0 degree, this is a monomial with 1 degree and so on, if you go on this way, this is the monomial with n th degree. In order that this polynomial to qualify as a polynomial with n th degree, we need something, we need one condition; that condition is actually I need this to be a polynomial of the term of degree n to be non-zero.

So, that forces me to write $a_n \neq 0$, this is a condition that require that is required for writing a polynomial of nth degree. Remember here the argument is only one that is the variable is only one x. So, I can also assign this as something called $p(x)$, and now you can as well treat this $p(x)$ as a function of one variable which is interesting.

So, if you assign this as a function of one variable, the next question is; how is this function, what is the domain and co-domain and range of this function? So, the function runs from $\mathbb{R}$ to $\mathbb{R}$. So, it is a function from real line to real line; whereas the range typically depends on function.

For example, if I take a function like let us say $p_1(x)$ is one function, which is $a_1 x + a_0$ and if I take this function, then it is a linear function; we have already seen this function, this is an equation of a real line, equation of a line. And if a 1 is not equal to 0, then this function actually represents a real line. If $a_1 \neq 0$, it also represents a real line, but it is some constant.

So, it is a horizontally real line. So, now, the range of this function for $a_1 \neq 0$ is entire real line. Whereas if you look at some other function, let us say $p_2(x) = a_2 x^2 + a_1 x + a_0$. Now, this particular function represents a parabola, which we have seen in our topic on quadratic functions.

And you know depending on the sign of a_2, the parabola can open upward or downward; if it opens upwards, the range is the minimum value and any point beyond that; if it opens downwards, the range is the maximum value and any point below it. So, depending on the choice of the function, the ranges may differ. We will deal with polynomials as function when we will study the graphing of polynomials.

Right now, we are interested in algebraic properties of this polynomial. So, we will focus ourselves on the algebra of the polynomials that is addition, subtraction, multiplication, division.

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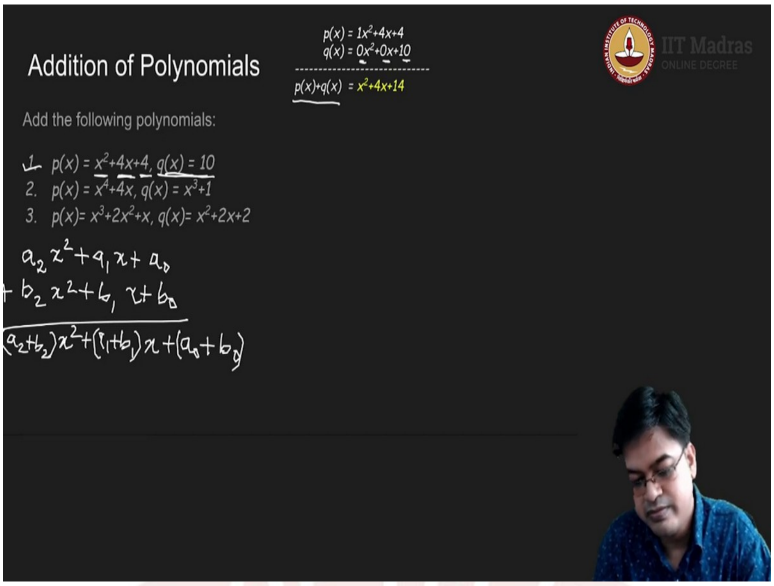

So, let us move ahead and try to understand addition of polynomials. We have done this in some sense, for example, currently the polynomial that we have written that $a_n x^n + a_{n-1} x^{n-1} + \ldots$, is also addition of some kind; but it is addition of monomials. So, let us try to see, if I have been given to two polynomials, how will I add them?

To help us in understanding and developing general theory for addition of polynomials, we will consider these three examples. Remember, the first example is actually one polynomial added with another monomial; second one both are two polynomials, but there are no clashing terms, like they the exponents are different for both the polynomials, you can check and the third one has few clashing terms.

So, we will demonstrate the addition of polynomials through these three things and we will formalize this into a theory. So, let us start with the first expression, $p(x) = x^2 + 4x + 4$. So, we are starting with this, this particular expression. So, $p(x)$ is x^2. So, if I am writing x^2, then it essentially means I am multiplying this with 1, the coefficient is 1; if I am starting with $4x$, then I do not have to do anything and this is 4.

So, in this case in our standard form for a quadratic polynomial, what is a standard form for a quadratic polynomial? $a_2 x^2 + a_1 x + a_0$ this is our standard form. So, in this particular thing, you can identify $a_2 = 1, a_1 = 4$ and $a_0 = 4$.

In a similar manner, I will look at this particular expression which is $q(x)$. Now, you notice the fact that $q(x)$ is just a constant polynomial, $q(x)$ do not have any terms which are related to square or related to a linear term. And, I want to add this polynomial to a given expression.

So, how will I add? So, let us bring in the terms related to square term and related to linear term; if I bring in those terms, the associated coefficients will be 0 right, the associated coefficients will be 0. So, I can write this term as $0x^2+0x+10$.

Now, because of this, let us write it in a generalized setting; $b_2 x^2+b_1 x+b_0$, right. So, now, I am trying to add these two polynomials. So, what is our recipe? We will consider the terms with like powers that is like exponents, ok. So, let me try to add the things.

So, if I consider this, this particular expression that is given here. So, I have $1x^2$, 1 minute. So, let me bring in my mouse pointer here. So, I have $1x^2+0x^2$. So, 1 + 0, I will get again singleton x^2; then 4 + 0 that will give me $4x$, 4 +10 will give me 14. So, essentially I can see that this expression should have a formulation which is of the form $x^2+4x+14$.

So, if I now try to do it in a more general settings, then how will I compare with this general setting. Let us see it here. So, I want to add these two. So, just add. So, in a similar manner, if I add these two; what I am getting is $(a_2+b_2)x^2+(a_1+b_1)x+a_0+b_0$, just to remember this format. So, what I am writing here is essentially.

Another point that to note with this example is; the first one was a polynomial of degree 2, the second expression $q(x)$ was a polynomial of degree 0. Now, the resultant expression that is $p(x)+q(x)$, what is the degree of this polynomial? It is a polynomial of degree 2. So, it is the maximum of degree of the first polynomial and degree of the second polynomial. So, we have roughly understood the settings that we need maximum of 1 and 2, let me write the findings in a different way.

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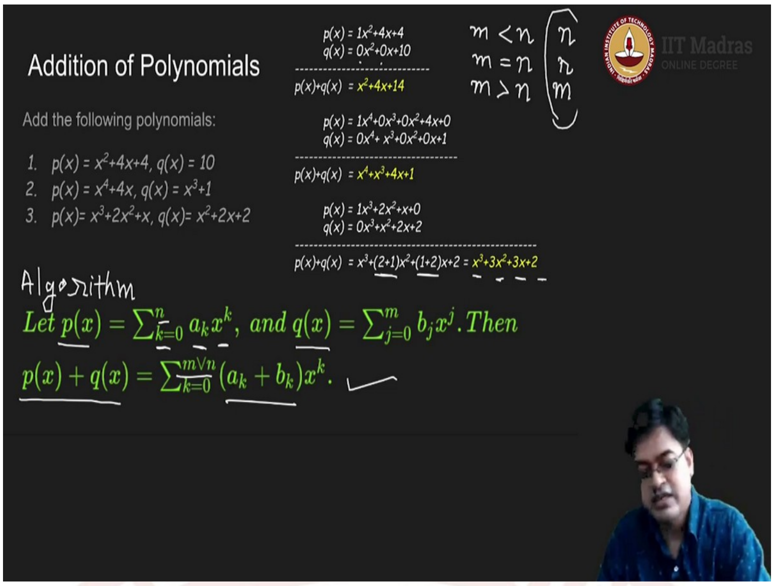

So, if I have a polynomial of degree m and let us say if I have a polynomial of degree m and degree n, these are the two polynomials; and if they satisfy a relation that m is less than n, then the resultant polynomial will have a degree n. If I have a polynomial where degree m is equal to n ok, then the resultant polynomial will still have a degree n.

And if I have a case where m is greater than n, then the resultant what will be the; so if we switch this $q(x)$ to $p(x)$ and $p(x)$ to $q(x)$, this case will happen and the resultant polynomial will have degree m. So, just remember this in mind; that means it is always maximum of m and n, if the resultant polynomial is having polynomials underlying polynomials with different degrees. So, with this understanding, let us attack the second problem.

So, the second problem has $p(x)$ which is x which is a polynomial of degree 4. So, I have written all other terms which were not there in the polynomial by multiplying with 0. In a similar manner, I have written the second polynomial $q(x)$ which is a polynomial of degree 3; but we want the maximum degree to survive right or is essentially in this expression the maximum degree will survive, therefore I have this kind of expression.

So, the resultant polynomial we know from our discussion will be a polynomial of degree 4, and therefore I need to consider all the terms that correspond to each of the degrees. So, what is that term corresponding to degree 4? In the first expression that is

$p(x)$ is 1, the coefficient is 1 and the term corresponding to degree 4 in the second polynomial that is $q(x)$ is degree 0.

So, I will get 1 + 0, which is 1; $1x^4$. In a similar manner you can see, for x^3 it is $1x^3$; x^2 there is no survivor both are 0, so $0x^2, 4+0x$ and 0 + 1 times 1. So, it is 1. So, the resultant that you are interested in is x^4+x^3+4x+1. Again I will reiterate, this time it will be; if you consider a generalized polynomial, it will be $a_4x^4+a_3x^3+a_2x^2+a_1x+a_0$.

And in a similar manner $q(x)$ will $b_4x^4+b_3x^3+b_2x^2+b_1x+b_0$. And if you sum over them, what we have written in yellow is essentially sum of $a_4+b_4=1, a_3+b_3=1, a_1+b_1=4, a_0+b_0=1$; a_2 and b_2 will sum to 0, because it does not have any non-zero coefficient. So, this is how we will handle the third, this is how we have handled the second problem.

And the term containing the highest degree survive, therefore the degree of polynomial is 4, ok. So, let us go back to the third problem, let us come ahead and solve the third problem; $p(x)$ and $q(x)$, again similar setting highest degree is degree 3. So, the term corresponding to degree 3 will survive. So, the polynomial with lower degree, I will bring it to degree 3. So, essentially, I will multiply with coefficient 0 for a degree 3 term.

Again by same logic, I will add the two terms and therefore, I will get the corresponding answer. So, there were cross terms, like the term corresponding to x^2 was crossing; for example, both polynomials had terms corresponding to x^2.

So, you can see the difference here, we are just adding 2 + 1, 1 + 2. So, all these things are happening and together we are writing the result x^3+3x^2+3x+2. So, from this we can derive, if you are clear with these three examples then we can derive a general formula; otherwise pause and look at each of the terms, you will be able to understand the general formula in a bit better manner if you pause and review these additions.

So, let us come to the general formula, you must have paused and understood the additions. So, if I have a polynomials of the form $\sum_{k=0}^{n} a_k x^k$. And if you assume that $a_n \neq 0$, then this is a polynomial of degree n.

In a similar manner you have taken a second polynomial $q(x)=\sum_{j=0}^{m} b_j x^j$. Now it does not matter whether m is greater than n or m is less than n, this particular thing will give you the answer, ok. What is the resultant? So, if I want to add these two polynomial functions $p(x)$ and $q(x)$, then $p(x)+q(x)$ will essentially show this kind of representation.

So, what we are actually saying is, choose which one is the maximum m or n ok? Whichever is the maximum? Take that maximum. So, it is maximum of m and n ; sum will run from k is equal to 0 to n. Let us say for argument purposes this is the highest degree that is n is the highest degree, then match the degree of the highest degree for other coefficients, for example, for j equal to $m+1$ to n put all $b_j's$ to be equal to 0.

If you do so, this is what we have done here. So, in this case the first degree was 2 and the second degree of was 0. So, in this case we matched the degree and substituted all the coefficients here to be equal to 0. So, you do a similar thing over here in general and then just add the coefficients $(a_k+b_k)x^k$, and this should give you the final answer. So, this is in fact an algorithm for adding the polynomials, so this is algorithm for adding the polynomials.

What is the, what are the steps in the algorithm? First identify degrees of both polynomials, choose the polynomial with highest degree that will be the degree of the resultant polynomial. Take the polynomial of least degree that is step 2, take the polynomial of least degree, add all the coefficients which are of the degree higher than the polynomial and multiply them with coefficients 0.

Once you do that you are ready to do the addition, add the two using this formulation. So, this is how you can program, you can actually program into a computer for addition of polynomials. Now, let us try to understand this with subtraction. What is the difference between subtraction and addition? Both are essentially same, but in subtraction you are multiplying the second polynomial by -1.

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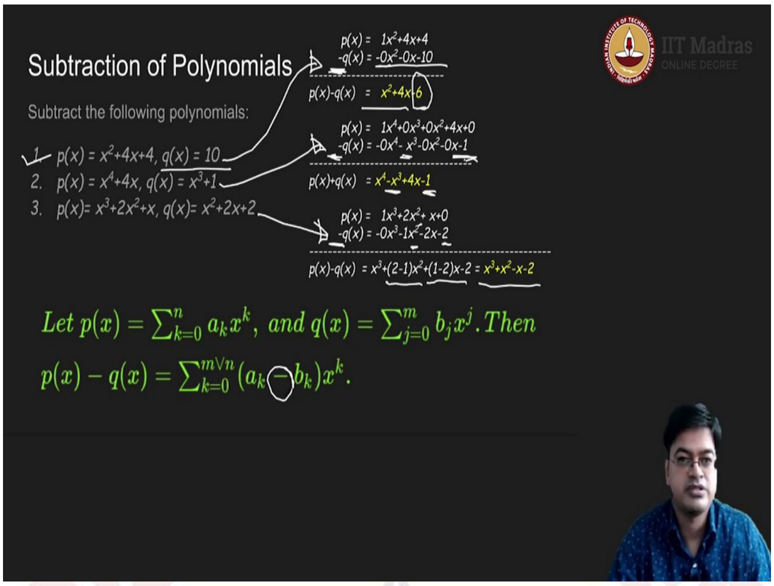

We have already seen how subtraction will happen. So, here is a quick overview of these examples. So, this is, these are the polynomials, same polynomials now we are subtracting. And what we are doing by subtracting? I mean we have multiplied with -1, just look at here all these terms.

The procedure is exactly the same, it is just that first we have to multiply by -1 and put the polynomial appropriately. So, let us start with first example, but it will be a quick run, because $p(x)$ is this. So, there is no change, but I want to subtract $p(x)$ from, I want to subtract $q(x)$ from $p(x)$. So, this polynomial will be multiplied with -1 that is what is done here, so $-q(x)$.

So, correspondingly all coefficients are negated, just look at these terms; all coefficients are negated and therefore. Because there were no cross terms, so you will not find any difference in the first two terms; but the significant difference is there in the third term which is actually -6. In a similar manner, take the second question and you are multiplying $q(x)$ with -1.

So, $-0x^4 - x^3 - 0x^2 - 0x - 1$, right. Again because there were no cross terms, there were no additions. So, this also will have a minimal effect where the second term, these terms will be with the negative sign, right. So, this -1 was there. So, this will have a negative sign.

So, there is a minimal impact. The third example that we have taken will face a major impact; for example, $p(x)-q(x)$. Now, you take this $-q(x)$ here, if you take that $-q(x)$ here, then $p(x)-q(x)$; the first term will be as it is x^3, because it is coming from this point here there was a clash of x^2. So, 2 in when we added, it was 2 + 1. So, everything became 3; here it is $(2-1)x^2$. In a similar manner $(1-2)x$ and then the final term was -2.

So, essentially you got the expression in this form, which is x^3+x^2-x-2; but the key principles remain the same, except for multiplying with -1. Multiplying with -1, because that was a polynomial of degree 0 will not change the degree of the polynomial. So, once it is not changing the degree of the polynomial, all the rules which were possible for addition remain intact.

For example, you have to choose the degree which is maximum of the polynomial; there is no change in the degree except for the multiplication of a minus sign. So, that multiplication of minus sign is absorbed here. So, now, in the new rule it will be $p(x)-q(x)$ will be k is equal to 0 to maximum of m and n there is a remained intact; and earlier when we were adding it was $a(k)+b(k),$ now it is $a(k)-b(k),$ ok.

So, I hope you have understood addition and subtraction of the polynomials, both are essentially same and the resultant what we are getting is again a polynomial. In the next video, we will take a closer look at multiplication of polynomials.

Thank you.

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Lecture - 33
Algebra of polynomials: Multiplication

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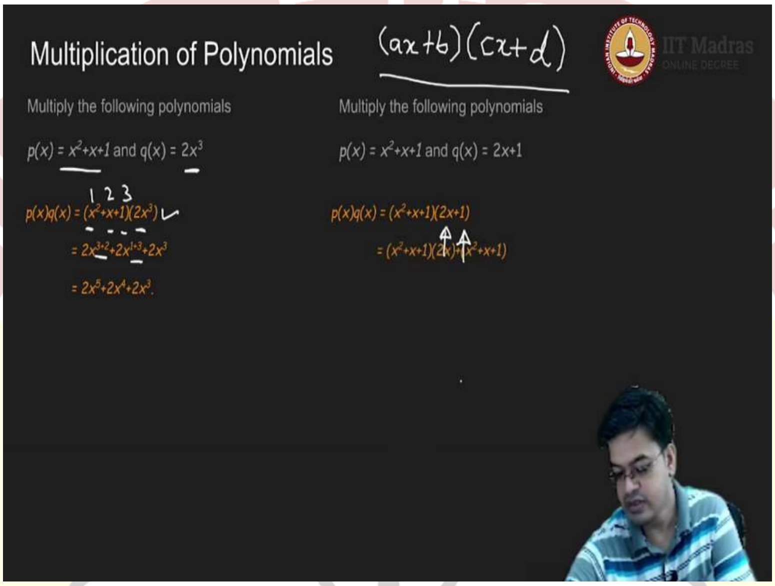

In this video, we will learn how to multiply two polynomials. Let us start with basics of multiplication of polynomials. We already know how to multiply two binomials. For example, if you have been given two binomials of the form $ax + b + cx + d$, then you know how to multiply these two binomials that is we will use the foil method. However, in this context, we want to generalize the settings for multiplication of polynomials of arbitrary degree.

So, let us see, let us start with some simple monomials with through examples. So, here is a polynomial given to you $p(x) = x^2 + x + 1$ and $q\ x = 2x^3$. The question is do I know how to multiply these two polynomials? Remember this one is called monomial, it has only one term. So, a standard rule of multiplication will mean we have seen this in our quadratic functions that I will consider the product in this manner.

Once I consider the product in this manner, what we will do is we will try to multiply each term of this $2x^3$ with each term of this polynomial. So, there are three terms. And for each term this $2x^3$will be multiplied. So, if I do that the law of exponents will apply.

For example, $x^3 \times x^2$ will mean x^{2+3}. So, once I apply we apply the law of exponents and add the exponents, obviously, 2 was a constant coefficient of x^3 which will be multiplied throughout the expression. And therefore, the resultant is this which we can simplify as $2\,x^5 + 2\,x^4 + 2\,x^3$. This is how we will multiply a monomial.

Now, as you can see the this polynomial has three terms 1, 2 and 3. So, it is not a binomial; it is a trinomial. So, my foil method will not work here. So, foil method will work only for these kind of expressions which are binomials. So, let us go ahead and try to consider a similar expression that is a quadratic expression and another binomial, and try to see how can I extend the basis of foil method right.

So, here is a binomial $2\,x + 1$. And here is a general polynomial quadratic polynomial which is $x^2 + x + 1$ same. Now, what will you do? So, naturally you will consider $p\,(x) \times q\,(x)$ which will be written in this form. Now, if I want to extend the basis whatever I did for monomial, that means, I need to convert this into two monomials.

So, what are those two monomials? One monomial is $2x$; another monomial is 1. So, if I treat them separately that is if I write them in this manner, let me erase this, that is I have written them in this manner.

Then what can I do about it, that means, now this turned out to be a same expression instead of x^3, here it is x that is all is the difference right. So, whatever I did here, I can do it here. And the last term is actually multiplied with 1 which it suppressed because multiplication with 1 will not change anything. So, I do not have to worry about the last term.

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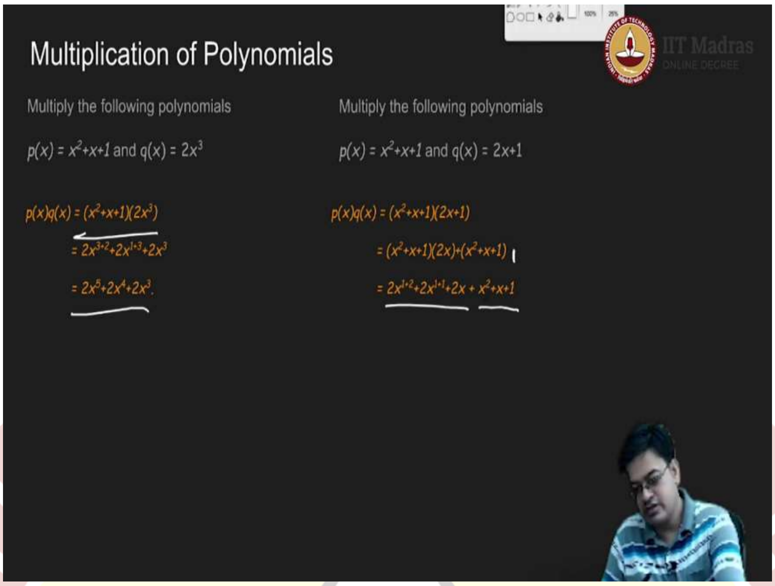

Now, I will multiply this $2x$ with all the terms in for of $p(x) = x^2 + x + 1$ which is similar to this particular thing. So, I will get $2\,x^{1+2} + 2x^{1+1} + 2x + x^2 + x + 1$. Now, the job is very simple.

You can treat this as one polynomial, and this one as a second polynomial, and then we have to add. How we add polynomials? We will add polynomials by matching the exponents, matching the exponents of x. So, if I want to add these two polynomials, what will I do, I will simply match the exponents and I will add them which is given here.

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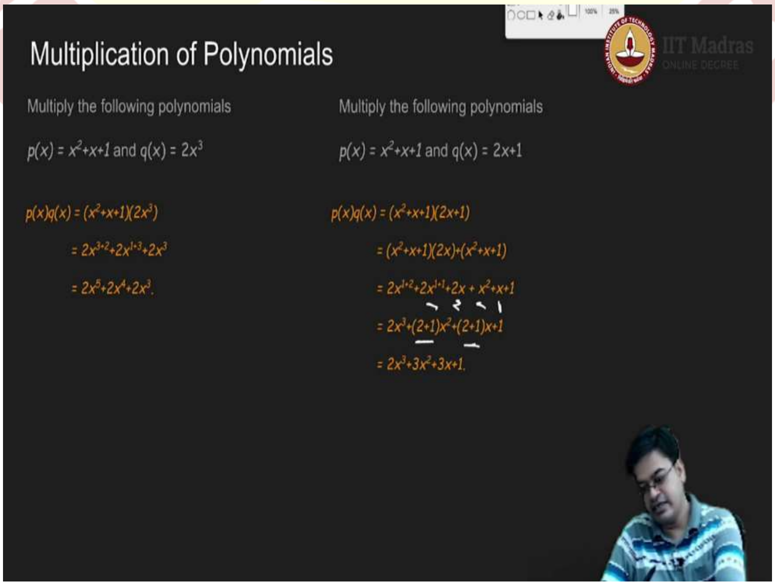

So, in this case 2 x^3, there is no competing term for x^3. So, it remains 2; x^2 comes here and here, therefore, I added the two which gives me 2+1, in a similar manner the terms containing x are these two. So, I have added these two, so 2+ 1 $x + 1$ which is similar to what we have seen in the last video of addition of polynomials. And therefore, we get the answer to be equal to $2x^3 + 3x^2 + 3x + 1$.

So, effectively what we have done is we know how to multiply the terms term by term. And finally, if at all I want to seek an extension of a foil method, it will be a term by term multiplication of polynomials, that means, you take the polynomial of least degree and multiply it with the polynomial of highest degree term by term, add those term match the powers and then write your answer. So, this is one prototype that we can follow for finding multiplication of polynomials or result of the multiplication of polynomials.

Now, the next question is can I generalize this method or can I answer it programmatically, that means, can I give a simple formula for what the coefficient of one part x^m will be? For example, in this case can I give a general formula what will be the coefficient of $3x^2$ provided I know polynomials $p(x)$ and $q(x)$. So, to answer that, let us go ahead and try to find a general formulation of this form of this formula.

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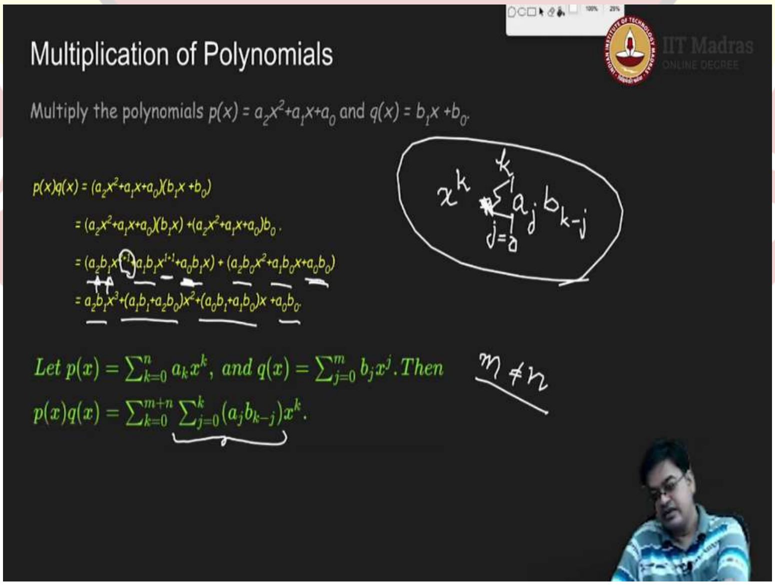

Let us go ahead. And if you are asked given one quadratic polynomial and one linear polynomial, you are asked to compute $p(x) \times q(x)$, how will you go about this? This is what our task is now. So naturally I will write $p(x) \times q(x)$, and then I will convert each of them into monomials that is one monomial will be b_1x, and second monomial will be b_0.

In this case, what will happen is we will simply multiply them as a separate term by term multiplication. So, in earlier case our b_0 was 1 when we studied one example. But here we are considering a general expression, and none of the expressions are 0 that is what we are assuming none of the coefficients at a_2, a_1, a_0, $b_1$and b_0 none of them are 0.

For example, if you consider $b_0 = 0$, then this term itself will vanish the second term itself will vanish; you will not have the second term. So, we are assuming that all terms remain in the loop ok. So, now it simple, the job is multiplying these two polynomials, and you will get some answers that is ok, but now our main worry is to find a pattern in these answers ok.

So, now, when I multiplied this, if you look at this particular expression that is $(a_2b_1x^{2+1} + a_1b_1x^{1+1} + a_0b_1x^1) + (a_2b_0x^2 + a_1b_0x^1 + a_0b_0)$. Here you take a pause and examine the terms. For example, this term contains the coefficient of x^3, this is $2 + 1$.

So, x^3. So, in that case, what is happening here is if you look at the suffixes of the coefficients this is a_2, this is b_1, so together they will sum to 3. In a similar manner, you look at this term which contains x^2. And you look at the suffixes of the coefficients that is a_1b_1, together they will sum to the exponent that is a 1+1=2. So, this should be a coefficient of x^2.

Then if this logic is correct, what should be the coefficient of a constant? The coefficient of the constant that is x^0. So, the coefficient of the constant must be a_0b_0. In a similar manner you can ask the question what is a coefficient of x? If you asked that question, you will naturally get the answer you collect all the in all the coefficients such that their suffixes will sum to 1 that is $a_1b_0 + b_1a_0$. So, is there anything called b_1a_0? Yes, it is here.

So, this what we have actually done is we have figured out a pattern; that means, if I want to find the coefficient of x^k, then better the sum should be some a_jb_{k-j}, so that they both will sum, they both will sum to it is not equal to the this is I am saying x raise to coefficient of x^k will be equal to of the will be of the form $a_j + b_{k-j}$. So, with this understanding, let us go further and try to rewrite this sum ok.

So, once I have rewritten this sum, my analogy is further amplified. For example, if you look at the coefficient of x^2, yes, it was it is $a_1 b_1$ and $a_2 b_0$ which is the coefficient of x^2, so that also means this means if I can sum over this j from 0 to what point to a point where I want the sum the exponent is raised to k, then I will get all possible combinations where sum is actually k.

In a similar manner, you can pause this video and verify whether you are getting the same expression for x^1 and all others right. So, with this understanding, I am ready to generalize this demonstration or this theory for a polynomial of an arbitrary order.

Let us consider polynomials of degree n and m, and try to find the general answer for them, and that answer will be in this form. So, if you are given a polynomial of degree $n, p(x)$, and if you are given another polynomial of degree $m, q(x)$, let us say $m \neq n$.

Even if $m = n$ it does not matter, but for our purposes let us take $m \neq n$, then what will be the coefficient of each of the x^k's? The coefficient is actually given here, $\sum_{j=0}^{k} a_j b_{k-j}$ this is what we have figured out in this expression is the coefficient of x^k.

Then the question is how far the degree will go? The degree will go till $m + n$ $m \neq n$; if $m = n$ then the degree will go to $2n$ that is ok. So, $k = 0$ to $m + n$, and each of the coefficient of x^k will be $\sum_{j=0}^{k} a_j b_{k-j}$. Now, let us demonstrate this idea with one example. Let us go ahead and see one example of this idea.

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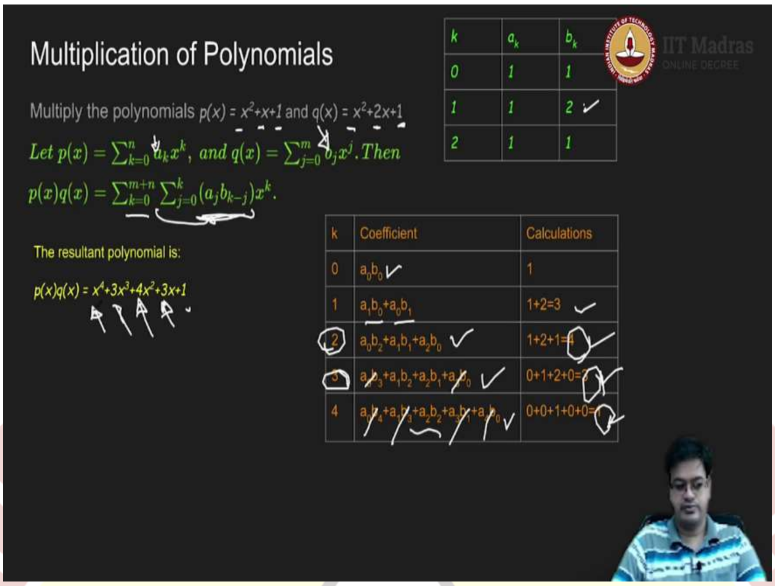

So, now, you have been given two polynomials two quadratic polynomials and you are asked to compute the multiplication of these two polynomials. One way is very simple you will go with term by term multiplication, and it simply means you have to multiply the terms of second polynomial with the first polynomial in a term by term fashion, or you can actually use the formula that I have given you in the previous slide. So, you can pause this video, and try to compute by yourself or you can go along with me.

So, let us recall that formula again that is $p(x)$ is equal to sum a, so my polynomial is a polynomial of degree n, and $q\ (x)$ is a polynomial of degree m. In this case, in this particular example, the polynomial the first polynomial is of degree 2 as well as the second polynomial is of degree 2.

So, in order to find the product of these two polynomials, what do we need to find is we simply need to find the coefficients of x^k. So, let us first identify what are a_k's and what are b_k's, j is a dummy index. So, it does not matter.

So, let us first identify what are a_k's and b_k's. So, a_0 as you can see is 1, b_0 is 1, a 1 is 1 again, b_1 is 2, correct, this is correct, and then a_2 and b_2 both are 1. So, I have enlisted all the coefficients of this particular expression, $p\ (x)$ and expressions $p\ (x)$ and $q\ (x)$. Now, we need to use this formula, then this formula which gives me the sum. So, let us use this formula and figure out.

Remember, all the coefficients that are not listed here. For example, what will be a_4, if at all, I will write a_4, what will be a_4 in this expression? It will be 0. What will be a_3 in this expression? It will be 0. So, all the coefficients that are not listed here are 0s. Keep this in mind and try to answer the question.

So, now, computation of coefficient; it is very easy. So, let us start with 0th degree term that is constant term. So, here $k = 0$. So, the summation will actually go from $j = 0$ to 0, that means, it will have only one term which is $a_0 b_0$.

What is $a_0 b_0$? Look here 1 into 1, so it will give you 1 ok. Let us go for a degree 1 term. So, j is equal to 0 to 1, j is equal to 0 to 1, so it will have, $a_0 b_1 + a_1 b_0$ these two terms are there. So, let us compute them through this table a_1 is 1, b_0 is 1, so this will retain 1. a_0 is 1; b_1 is 2, so it will give you 2. So, together it is 1+ 2 = 3.

Let us go for a second order term that is the monomial with degree 2. So, in this case, j will run from 0 to 2. So, I will have $a_0 b_2$, $a_1 b_1$, $a_3 b_0$, $a_1 b_1$, $a_2 b_0$, $a_0 b_2$, $a_1 b_1$, this is correct. Just go ahead and compute these terms, a_0 is 1, b_2 is 1, so you will get 1, $a_1$is 1, b_1 is 2, so you will get 2. And $a_2 b_0$ that is a_2 is 1, b_0 is 1, so you will get another 1. So, you will get the sum to be 4.

Let us go for a third term x^3 term, and just simply substitute this. So, we need to find all possible combinations. So, if it is a degree 3 term and we start with a_0, it will be $a_0 b_3$, $a_1 b_2$, $a_2 b_1$, $a_3 b_0$, these are the terms. And then you simply compute them.

Remember here now we came up with b_3. What is b_3? b_3 is not listed here, that means, b b_3 must be 0. In a similar manner here a_3 must be 0 correct. So, these 2 terms are chopped off right away they are 0. So, let us focus on the other 2 terms the first term you can easily verify because b_2 is 1, and a_1 is 1. And $a_2 b_1$, b_1 is 2, a_2 is 1, so it will be 2. So, 1+ 2=3; this is correct.

Now, the final term is a degree 4 term, correct. If you do a term wise multiplication, what you will come up with is because the degree 4 will be contributed by the highest order terms.

So, you will simply multiply $x^2 \times x^2$, and you will get only 1 term. But in this formulation what we are doing here is we are taking all possible terms of degree 4. So, even though they are 0, we will first list them, and we will put them as 0s.

So, now, when we consider degree 4 term, I will get a_0b_4, a_1b_3, a_2b_2, a_2b_2, a_3b_1 and a_4b_0. So, all these terms are here. And most of the terms will obviously, be 0 only 1 term is a contributor.

For example, a_0b_4 is 0, a_4b_4 will be 0, a_1b_3 is 0, a_3b_1 is 0. Why? Because b_4, b_3, a_3, a_4 all are 0 only term that will contribute is a_2b_2 which will be 1×1, so 1. So, this gives us a clear cut answer, and this is a systematic way to multiply two polynomials.

Therefore, the resultant polynomial $p(x) \times q(x)$ simply write the terms from this table, so this is a coefficient of x^0 is 1, so the constant term 1 is here coefficient of x^1 is 3, so $3x$ is here. So, in a similar manner x^2 coefficient of x^2 is 4. So, you will get $4x^2$ here ok; so $3x^3$ correct.

So, this is also done. And then x^4 has only 1 term as 1, so x^4. Therefore, you got the resultant polynomial to be equal to this. Now, remember one side note the multiplication of two polynomials will always fetch you a polynomial again ok. Next operation is division which we will see in the next video, but the division of two polynomials will not always lead to a polynomial. We will see that in the next video.

Bye for now.

Thank you.

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Lecture – 34
Algebra of polynomials: Division

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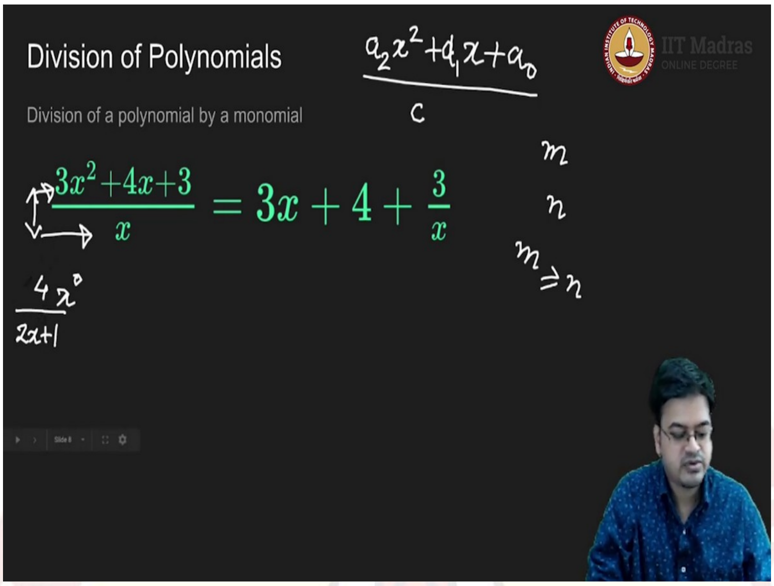

In this video, let us have look at Division of polynomials. What is a division of polynomial? We have already familiar with division of polynomials, but we have not done in a rigorous manner and we do not know all possible cases that can occur while considering division of polynomials, that is why it is important to look at division of polynomials.

We know some cases like for example, if I have been given a polynomial say $a_2x^2+a_1x+a_0$ and if I am told that if this polynomial is divided by a constant say c. Then, I know what is the resultant polynomial. It will be $\frac{a_2x^2}{c}+\frac{a_1x}{c}+\frac{a_0}{c}$ that will be the polynomial. So, this case, we are already familiar with.

Now, let us go to one more level of extension. Suppose, this polynomial is divided by a monomial; that means, we are considering a division of a polynomial by a monomial.

Monomial means, the polynomial that contains only one term, only one variable term. So, in that case, let us take this example. So, $\frac{3x^2+4x+3}{x}$, ok.

So, notice few factors here. In this case, when I am considering a division of two polynomials, the numerator and the denominator; the denominator should always have a degree smaller than the degree of the numerator. If it is not the case, let us say the numerator has degree m and the denominator has degree n, then what I am saying is the degree of the numerator m should always be greater than or equal to n. If it is not the case, then the division is not possible ok.

For example, let us consider one case, where I am considering a constant polynomial let us say 4 and I am dividing it by some polynomial which is $2x+1$. Here, I cannot divide this; I cannot divide by this polynomial because there is no corresponding x term. Here it is x^0.

So, I cannot divide this polynomial because the degree of the polynomial plays a crucial role. So, in this case, the division is not possible, I have to keep this function as it is. Let us keep this point in our mind and consider division of polynomials. So, now, I am dividing a polynomial with a monomial, how will you handle this?

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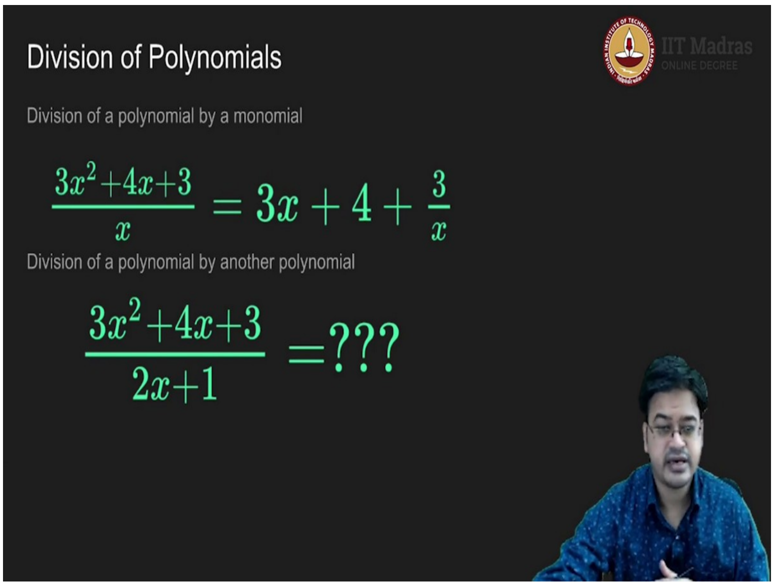

So, a monomial is simply x here. So, what I will do is I will split this with this addition sign, I will split each of them in separate terms. So, I will consider the term $\frac{3x^2}{x}$ that will give me a term, when I consider this term it will give me a term $3x$.

When I consider $\frac{4x}{x}$, I will get a term 4.

Now, as I mentioned earlier when I consider the term $\frac{3}{x}$, the degree of 3 is a constant because 3 is a constant polynomial the degree is 0. So, I cannot divide this polynomial. So, this will automatically influence this decision that it will remain as it is, that is $\frac{3}{x}$. So, these are some key things while dividing polynomial by a monomial.

Now, the key idea is I want to divide a polynomial with another polynomial. Let me erase this first. So, now, I want to divide a polynomial with another polynomial. So, how will I go about this?

That is I want to find something of this sort. Let us address this question in a video ok. So, apparently, I do not have any practical way to divide this right now; but from whatever theory I learnt about quadratic functions, can I derive something? That is what the question is. So, we will try to figure out some more methods in this video.

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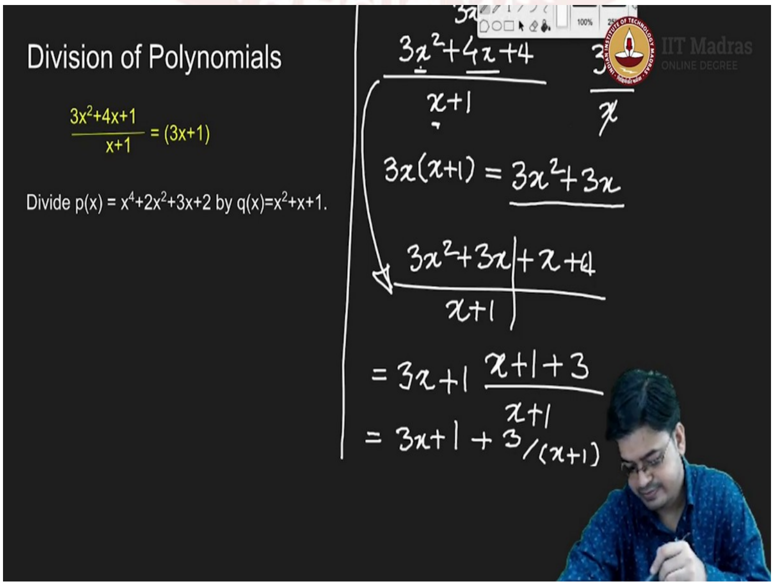

So, let us continue and take a question, this is the numerator that is given to me divided by another polynomial which is $x+1$ and I want to figure out what this will be equal to? Let me take this polynomial over here and try to figure out what this polynomial will be equal to. So, now, I have $3x^2+4x+1$ and it is divided by $x+1$. Now, if I want to divide the numerator by the denominator, what I should see is ok, the denominator has the highest degree which is x. The numerator has highest degree which is x^2 and now, how will I be able to get rid of the denominator for some at least for some terms?

So, in that quest, what I will see is I will simply take the first term over here and the first term over here and I will see like monomial, I will see what is $\frac{3x^2}{x}$. This I can do very easily because both are monomials. So, x vanishes with this square and I will I am left with $3x$. So, next thing that I will do is I will consider $3x(x+1)$. So, this actually gives me the answer $3x^2+3x$. Now, I will try to figure out this term in the expression that is given in the numerator.

So, if I want to figure out the expression that is given in the numerator, I can easily split this $4x$ as $3x+x$. If I can do so, that means, I can take this term and based on this logic, I can actually write this as $\frac{3x^2+3x+x+1}{x+1}$.

Now, I can intelligently split the term over here and I can divide this and I can take this as a separate term and divide this. So, now, you can readily see the answer will be here, $3x(x+1)$ that will get cancel off with $x+1$ and over here, it will be 1. So, the answer is $3x+1$. Therefore, such a division is possible, ok.

Let us verify whether the answer is $3x+1$; yes. So, I have demonstrated you how to divide a polynomial using simple method by the method of factorization that we have already used. Now, this is because $x+1$ was the factor of $3x^2+4x+1$.

What if $x+1$ is not a factor of $3x^2+4x+1$, what would have happened? Let us use this example to understand our findings. So, let me take a eraser and let me write that this instead of 1, let me put ok, let all other things remain constant, what makes it a factor; that $x+1$.

So, I will simply what I will simply do is I will simply change the term to 4 ok. Now, $x+1$ is no longer a factor, still I will continue with the same method, I will take this $\frac{3x}{x}$ 3x by x. So, I can consider this x^2+3x, only difference is this will be $x+4$. In that case, what happens is $3x^2+3x$. So, that will give me $3x$ into let me rewrite this as $\frac{1(x+1+3)}{x+1}$. So, that again gives me an edge that is this is nothing but $3x+\frac{1(x+1)+3}{x+1}$ getting cancelled.

So, this will remain as $\frac{3}{x+1}$. So, this is how even if it is not a factor, I can divide the polynomial. Now, as we have started by giving some for addition, multiplication, subtraction, we have given some algorithms. So, now, we need to identify such algorithm for division of polynomials. To do that, let us first solve this complicated problem in this simple manner and try to derive an algorithm and try to derive an algorithm for by solving this problem.

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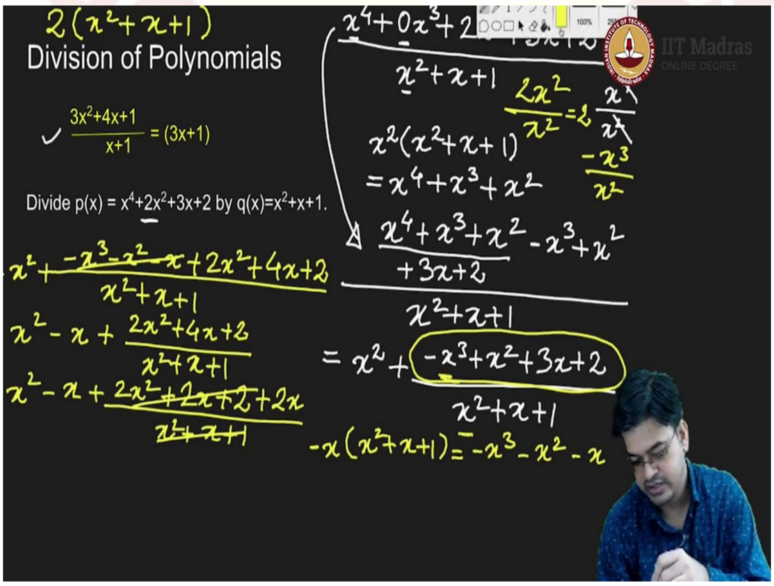

So, the problem is, I want to divide the terms divide a polynomial $p(x)=x^4+2x^2+3x+2$ by $q(x)=x^2+x+1$. So, the $p(x)$ is a polynomial of degree 4; $q(x)$ is a polynomial of degree 2. So, how will I go about this? So, again, I will apply a strain strategy that is I will start by writing x^4 plus remember here, it directly goes to $2x^2$. So, but I want the term containing x^3 also to be present $0x^3+2x^2+3x+2$ and this term is divided by x^2+x+1.

So, remember our first step in the last example was first you take the first term over here and take the first term over here. Then, take a consider a division of these monomials. This will give me x^2. So, what I will do now is, I will consider $x^2(x^2+x+1)$. This will give me actually $x^4+x^3+x^2$.

Now, as per our earlier strategy while solving this problem, we have adopted a strategy that I will add these terms over here. So, if I add these terms over here, then I will subtract appropriate terms over here. So, in this case, x^4 is already there, x^3 was not there and here, x^3 is there.

So, I need to subtract that x^3 from this expression and then, I need to subtract from $2x^2$, I will split this into two. So, let us rewrite this expression, that is the numerator of this expression, x^4 is already there. In order to cancel the denominator, I need x^3 over here.

So, I need to add x^3+x^2 ok; but this x^3 was not present here, it was $0x^3$. So, naturally the next step will be to eliminate x^3 from here. So, that it will retain a legacy of this term. So, if I have eliminated x^3, then it is $2x^2$ of which $1x^2$, I have taken out, so this will be another x^2+3x+2 as it is.

So, let me write that term as it is $3x+2$ and now, if I divide this term by x^2+x+1, then what I will get here is take these first three terms and keep the remaining term as it is ok. So, if I do that, then what will happen is this term x^2 will come out as common plus now, what happens?

This term vanishes, this term vanishes, this term because x^2, I can take out common; from these three terms, I can take out x^2 common that is what I have written and it cancels with the denominator. So, whatever is remaining are the remaining term that is $-x^3+x^2+3x+2$ and this thing is divided by x^2+x+1.

Now, is our division over? No, because the numerator over here has a higher degree than the denominator. Therefore, our division is not over. So, again, I will follow a similar step, I will simply change the color so that I will have a better view ok. So, let us change the color and have a better view of this.

So, let me write it here from this; from this step, I can go here and say ok. So, this is in fact, equal to x^2 plus now you look at this term $-x^3$ and x^2. So, you divide $\frac{-x^3}{x^2}$ which will give you $-x$. So, in this case, you will multiply $-x(x^2+x+1)$.

So, if you multiply $-x(x^2+x+1)$, what you will get over here is $-x^3-x^2-x$, this is what you will get. So, you write this term as it is, that is $-x^3-x^2-x$.

Now, from this term, you adjust the terms. So, $-x^3$ is already there, so I do not have to compensate for this term. But there is a plus x^2, there is a plus x^2 and here there is a $-x^2$. So, that will give me plus $2x^2$ because I am compensating for this extra $-x^2$ added in this term, then there is a $-x$ and over here it is plus $3x$.

So, I have to add one x for this $-x$. So, that will give me plus $4x$ plus and there is no competition for a constant term upon x^2+x+1. Now, you can take out x common and this will cancel off, this term will cancel off with this term by taking x common. So, it is x^2-x is in common plus what you are left with here is $2x^2+4x+2$ upon x^2+x+1 ok. Again, you will apply a similar procedure that is you will actually divide $\frac{2x^2}{x^2}$. So, you will get 2. So, essentially what you; so, when you do that, when you divide $\frac{2x^2}{x^2}$, you will get 2.

So, when you will multiply this number by 2, let me write it here that is $2(x^2+x+1)$ ok. So, in this case, what you will get is $(2x^2+2x+2)$ of which $2x^2$ is already there. So, I will continue over here itself x^2-x+2x^2 is already there, $2x^2$. So, let it be $2x^2+2x$, over here there is plus $4x$. So, I can split $2x$ over here plus $2x$

plus 2. So, that will again come plus 2 as it is here. So, what is remaining now is $2x$ upon x^2+x+1.

So, now if you look at this term, what you will get is you can take out 2 common and this will cancel off with this denominator and therefore, the final expression, I am running short of space. So, let me erase some terms over here. Let me erase some terms over here so that I will get some space.

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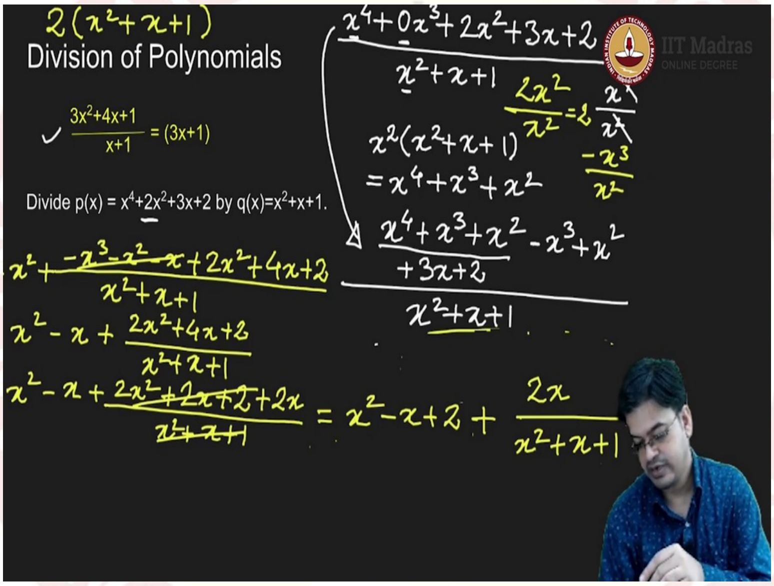

So, you can rewrite this as to be equal to $x^2-x+2+\frac{2x}{x^2+x+1}$. This will be the final answer to this division ok. So, this is how we can actually do a division of two polynomials ok. Let us remove this and see whether to verify whether we have got the final answer to be correct or not. So, I have removed it, you must have noted the answer.

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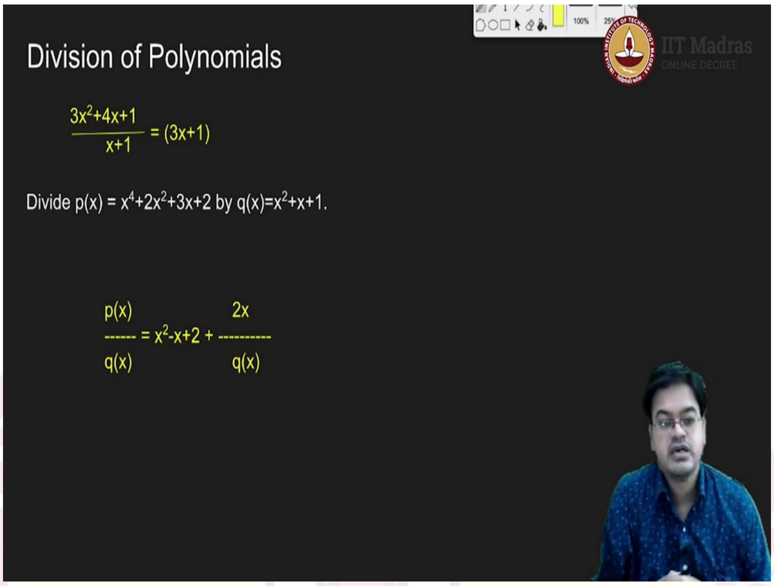

And the final answer that we have got here is $x^2 - x + 2 + \frac{2x}{q(x)}; q(x) = x^2 + x + 1$. Yes, so I have got the correct answer. So, here while doing this, we have derived one algorithm which we will emphasize in the next slide.

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Lecture – 35
Division Algorithm

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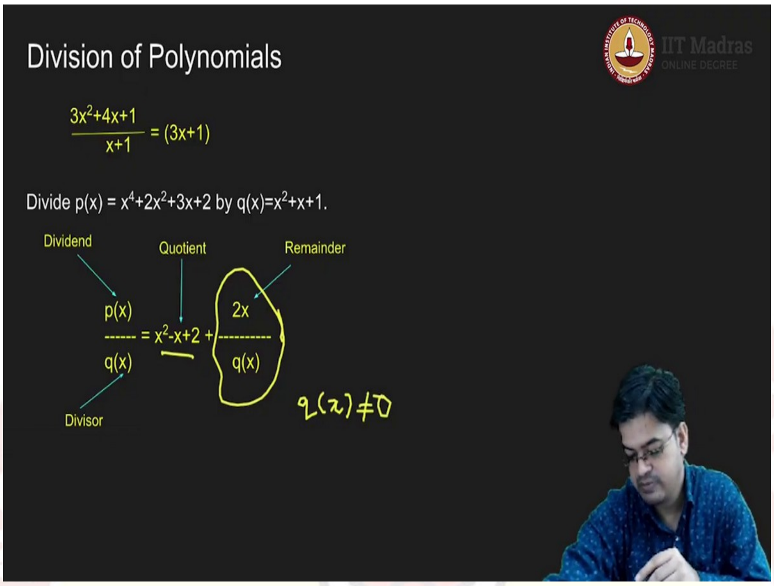

So, let us go to the next slide and emphasize the algorithm that we have derived just now. In order to understand the algorithm, you need some terminology. For example, this $p(x)$ is called the divident; the $q(x)$ is called the divisor. The term that you get over here, here is called the term that you get over here is called the quotient. And the $2x$ that you have got is called the remainder.

Remember you will declare something as a reminder only when the degree of the denominator is higher than the degree of the numerator, this is the strategy that we will follow.

So, now, you are very clear about the terminology, the numerator is the divident, the denominator is the divisor, the term the polynomial term that you get after dividing is called the quotient, and the rational and the remainder is something that where the degree of the numerator is smaller than the degree of the denominator.

This is also called a rational function. If you look at polynomial as a function, then division of two polynomials is a rational function, only condition that we are enforcing is $q(x)$ cannot be equal to 0, this is the condition is which is always in place. Let me eliminate this and let us go and study the algorithm.

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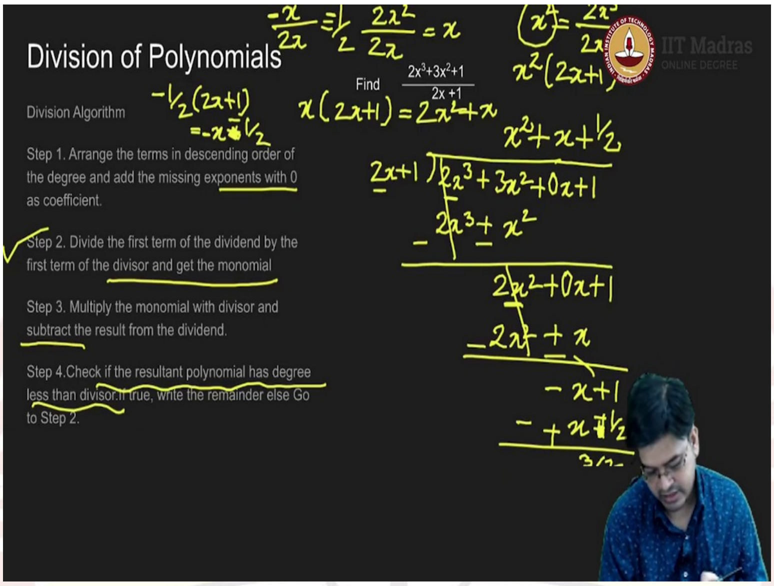

So, for division of polynomials, we will use the following division algorithm which we have derived just now, where in the first step what we will do is we will arrange the terms in the descending order of the degree, and add the missing exponent with 0 as a coefficient.

Then after adding the missing 0, 0 as a coefficient after adding the missing exponents, next what we will do is, we will take the first leading terms or the leading monomials, and we will divide the dividents monomial, the leading monomial of the divident and the leading monomial of the divisor together. And we will get some number which is which we will call as quotient, temporary quotient and that quotient we will multiply with our divident.

Once we multiply with our divident, what we will actually do is we will subtract that from the original expression for the polynomial that is our numerator. Whatever is remaining, we will treat that as the next divident. Once we treat that divident, then we will check if the degree of that new divident is higher than the degree of the denominator or divisor. If yes, then we will continue with the procedure; if no, we will terminate the procedure; this is how we will give the division algorithm.

Let us understand this division algorithm by using one example. So, here is an example. This is the numerator $2x^3+3x^2+1$ divided by $2x+1$, and I want to find the answer to this question. Let us figure out how to find the answer.

So, in the earlier quest, what I did is I have used the standard numerator denominator. Now, there is a popular method for division of the polynomial which is called long division, which works in a similar manner and the same division algorithm works, but you will have a better handle over the terms.

So, in this long division, what you will do is you will put a parenthesis over here, and you will put $2x+1$ outside the parenthesis, and you will put this term that is $2x^3+3x^2$. Now, remember the first step plus $0x+1$ ok. So, this is how we will write. Now, according to our standard terminology, what we will do is we will take the leading terms $2x$ and $2x^{3.}$ So, somewhere in the rough you do that. What is $2x^3$ divided by $2x$? This will give you $x^{2.}$.

So, you write x^2 over here, multiply x^2 with $2x+1$. Once you multiply x^2 with $2x+1$, write that term over here, $2x^3+x^2$. Now, according to our algorithm divide the first term of the dividend by the first term of the divisor and get the monomial that monomial is x^2 over here. Next step, multiply the monomial with the divisor and subtract the result from the divident. So, this is the result from the divident, result from multiplication, and you are subtracting it from the divident.

So, this will cancel off. So, this will give me 0 and $3x^2-x^2$ will give me $2x^2+0x+1$. This is the result ok. So, now, this result, I will check whether the degree of this result this polynomial that I have obtained is greater or smaller than this ok, that is what we will do. Check if the resultant polynomial has a degree less than the divisor that is not true.

So, we will go to step 2. What is the step 2? Which is this, divide the first term, first term of this divident with this that is you will divide $\frac{2x^2}{2x}$. So, what you will get here is

x . So, you will simply add x over here. And then you will multiply that x with $2x+1$. Once you do that, you will get $2x^2+x$. So, you write here $2x^2+x$.

Then what is the next step? You subtract it from the result, so minus, minus $2x^2$ vanishes, this gives me $-x+1$, ok. So, $-x+1$, again I will go to the same step because this degree is same, it is not less than the degree of the denominator.

So, I will again follow the same procedure; $\frac{x}{2x}$ which will give me $\frac{1}{2}$. So, naturally I will add $\frac{1}{2}$ over here. And once I add $\frac{1}{2}$ over here, when I multiply $\frac{1}{2}$ with $2x+1$, what I will get here is $x+\frac{1}{2}$. So, I will write that $x+\frac{1}{2}$.

But remember over here the thing was $-x$. So, I should what I should have done is I should have multiplied -1 to the x that means, $\frac{-x}{2}$. So, the answer is $\frac{-1}{2}$, and you will multiply $\frac{-1}{2}$ over here, so $-x$ this will not be plus this will be $\frac{-1}{2}$, so $-x-\frac{1}{2}$ which will be given a negative sign. So, this will be $x+\frac{1}{2}$. So, I will get the answer to be equal to $\frac{3}{2}$ ok. So, the answer is $\frac{3}{2}$.

So, what is what will be the resultant answer? This should not be plus, 1 minute, let me make it very clear. This cannot be plus; this should be minus, because I have to multiply with $\frac{-1}{2}$. And here it is the remainder is $\frac{3}{2}$. So, what I got here is $x^2+x-\frac{1}{2}$ and the as a quotient, and the remainder is $\frac{3}{2}$.

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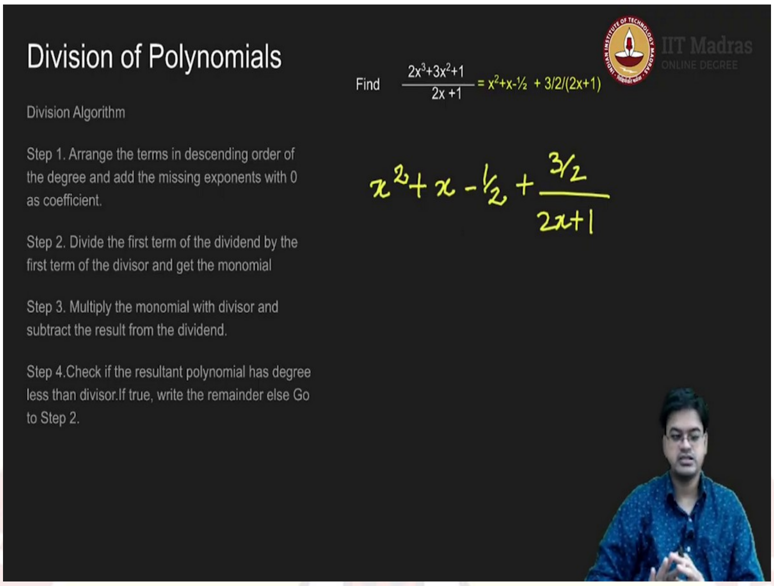

So, let me rewrite it again that is I got $x^2+x-\frac{1}{2}+\frac{\frac{3}{2}}{2x+1}$, this is what I got. Let me verify this result. And we have demonstrated the algorithm, yes, $x^2+x-\frac{1}{2}+\frac{\frac{3}{2}}{2x+1}$, this is how we will consider division of polynomials in general.

Mathematics for Data Science 1
Week 06 - Tutorial 01

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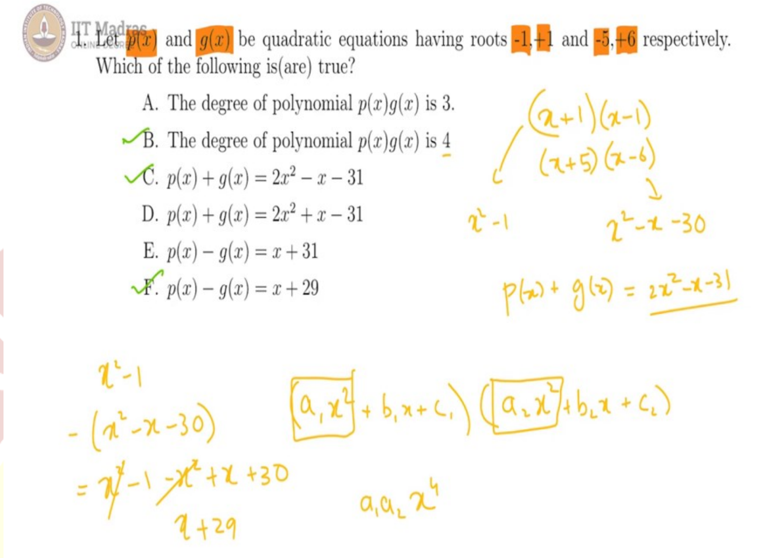

Hello, mathematics students. In this week's tutorials, we will look at some questions based on polynomials and the algebra of polynomials. In this question, we have two quadratic equations, which are $p(x)$ and $g(x)$, presumably equal to 0, and they have the roots, $-1+\ 1, -5+5$ respectively. Then the degree of the polynomial $p(x)\ \times g(x)$ is three, it is not because you have two quadratic equations, and you are multiplying them.

So, the x^2 terms will have to necessarily multiply, so $(a_1x^2+b_1x+c_1)\times(a_2x^2+b_2x+c_2)$, when you multiply these, this term, and this term will have to be multiplied and you are going to get $(a_1a_2x^4)$, so the degree has to be 4, which is this. So, B is correct. And then we have the sum, is equal to, so we need to find the respective quadratic equations now for this, so this would be $(x+1)\ \times(x-1)$, the other would be $(x+5)\ \times(x-6)$.

So, this gives us this is, x^2-1. And this is essentially x^2-x-30. So, when we add these two, we get $p(x)+g(x)=2x^2-x-31$. So, C is correct, and that would imply D is wrong. And now we are looking at the difference $p(x)-g(x)$ and that would give us $x^2-1-(x^2-x-30)$ =x square minus 1 minus of x square minus x minus 30, which is x^2-1-x^2+x+30. So, x^2-x^2 cancel off and you have $x+29$. So, E is wrong and F would be correct.

Mathematics for Data Science 1
Week 06 – Tutorial 02

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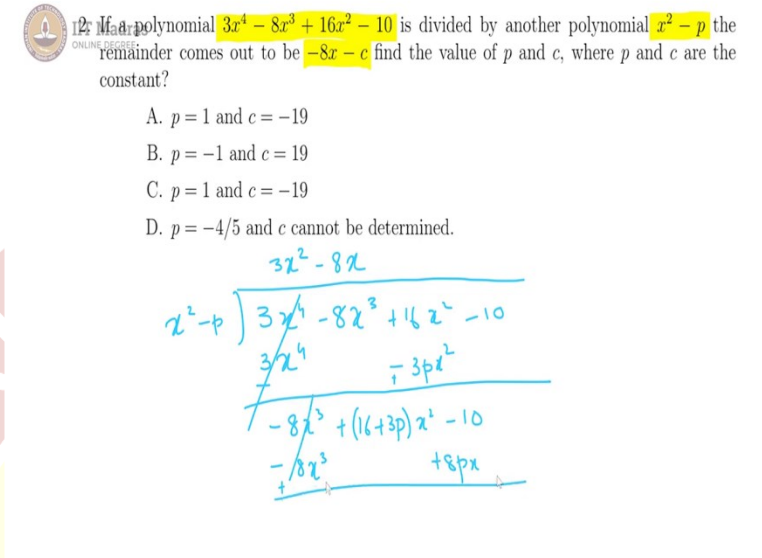

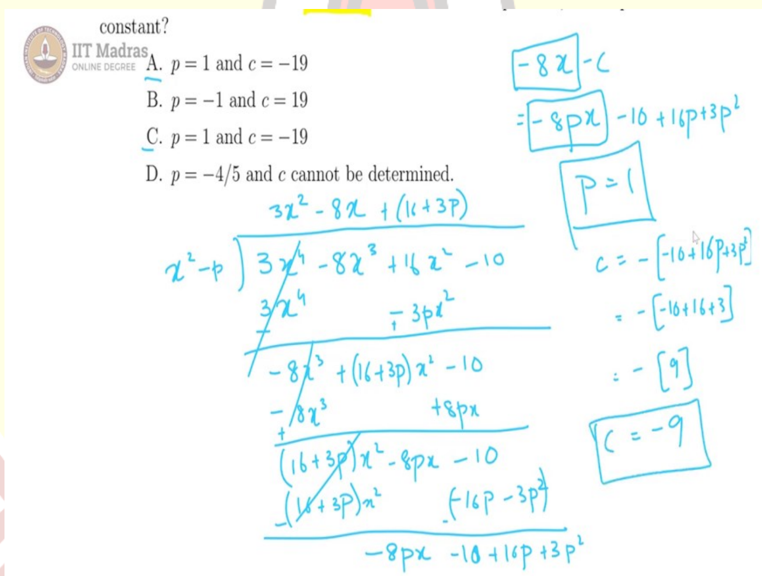

In question number two, there is this polynomial, $3x^4 - 8x^3 + 16x^2 - 10$ and is divided by another polynomial $x^2 - p$, then the remainder comes out to be $-8x - c$. They are saying find the value of p and c. So, let us do the division, then, we have $3x^4 - 8x^3 + 16x^2 - 10$ and here we have $x^2 - p$.

So, this gives us $3x^2$ to start with and so this will be $3x^4 - 3px^2$, so we should write it there, $-3px^2$. And this goes off, and we get $-8x^3$ + this becomes +. So, $16 + 3px^2 - 10$. So, now we have $-8x$ coming up here, which gives us $-8x^3 + 8x$, so +$8px$ and this goes off again.

So, we have $16+3px^2 - 8px - 10$. So, we again multiply by 16 plus 3p here, and that gives us $16+3px^2$. And there is no x term, we get minus $16p - 3p^2$, then this of course cancelled again. So, we are left with $-8px - 10 + 16p - 3p^2$, because this is being subtracted.

So, they are saying this remainder is$-8\ x -\ c$. And that is equal to $-8px - 10 + 16p - 3p^2$. So, the x terms have to be the same here, which gives $p = 1$. And then c would be the negative of $-10 + 16p + 3p^2$, which is equal to the negative of $-10 + 16 + 3$. So, that is the negative of 9, and so we get -9.

And that will indicate that none of the options are correct. So, this probably was supposed to be -9. We observed that option A and option C are in fact the same thing. So, one of this was probably supposed to be -9. Anyway, so our answer is that $p = 1$, and $c = 9$.

Mathematics for Data Science 1
Week 06 - Tutorial 03

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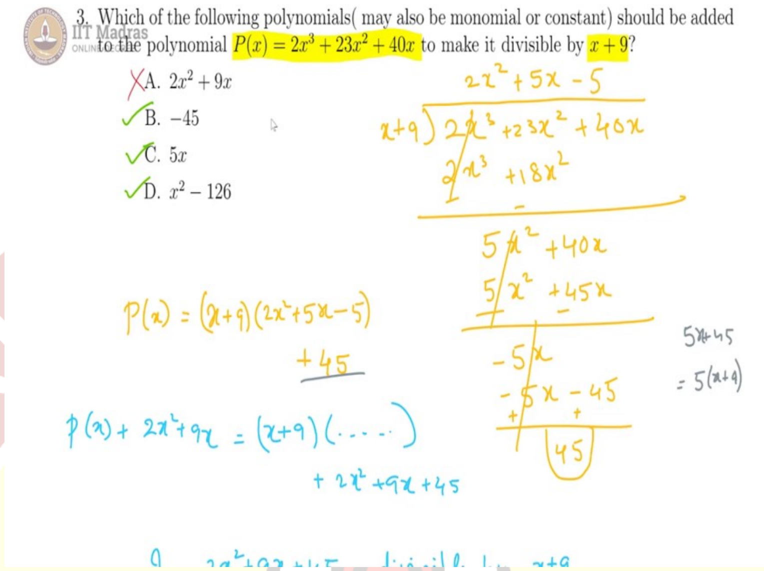

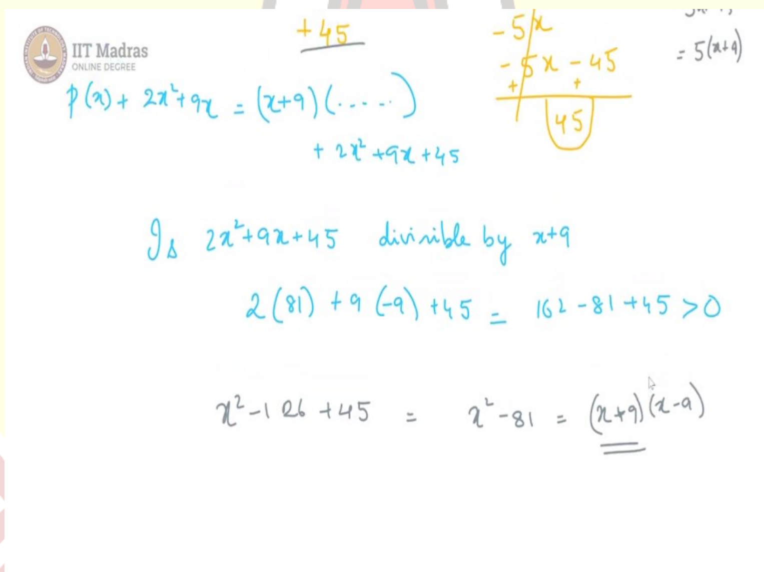

Now, we have this problem, which of the following polynomials should be added to the polynomial $p\ (x)$ to make it divisible by $x +\ 9$. So, we need to recognize that it is not necessary that there is only one polynomial that you add, because since it is only divisibility, we can add a number of polynomials to $p(x)$ and make it divisible by $x +\ 9$. So, we have to check for each of these cases.

So let us see, or what we can additionally do is, we can look at the remainder that we get by dividing $p(x)$ with this and then see what to do with that remainder. So, if we did the division, now, we have $2x^3 + 23x^2 + 40x$ and we are dividing it with $x + 9$. So, start with $2x^2$, so we get $2x^3 + 18x^2$. So, this cancels off, this gives us $5x^2 + 40x$.

So, we do $+5x$ additionally, then we get $5x^2 + 45x$, so negative and negative so we are left with $-5x$ and then that gives us a -5 additionally here, so we have $-5x - 45$, therefore these two go off and we are left with 45 as our remainder. So, $p(x)$ is essentially $(x + 9)$ into the quotient +45. So, if we subtracted 45 from $p(\ x)$, we will get divisibility by $(x + 9)$.

So, B is necessarily correct. Let us look at what happens if we added A, if we added A, $p(x) + 2x^2 + 9x$ is some multiple of some product of $(x + 9)$, and some quadratic plus $2x^2 + 9x + 45$. So, unless $2x^2 + 9x + 45$ is divisible by $(x + 9)$, $p(x)$ would not be divisible by $(x + 9)$.

So, what we should really be checking is $2x^2 + 9x + 45$. Is it divisible by $(x + 9)$? And the direct way to check it is to substitute $x = -9$, so you will get $2 \times 81 + 9 \times -9 + 45 = 162 - 81 + 45$, which is greater than 0, it is not equal to 0. So, no, A does not give us divisibility by $(x + 9)$.

What happens if we added $5x$, we get $5x + 45$. So, we have this 45 remainder, so we are getting $5x + 45$, which is equal to $5(x + 9)$, which is directly divisible by $(x + 9)$. So, this is correct too, c is also correct. And what happens if we added $x^2 - 126$, then we would get $x^2 - 126$+45 as the additional part upside from $(x + 9)$ into that quadratic, so this is equal to $x^2 - 81$, which is equal to $(x + \ 9\ )(x - 9)$. So, $(x + \ 9\ )$ is dividing this particular polynomial. So, we can add $x^2 - 26x - 126$ also, and get divisibility by $(x + \ 9\ )$.

Mathematics for Data Science 1
Week 06 - Tutorial 04

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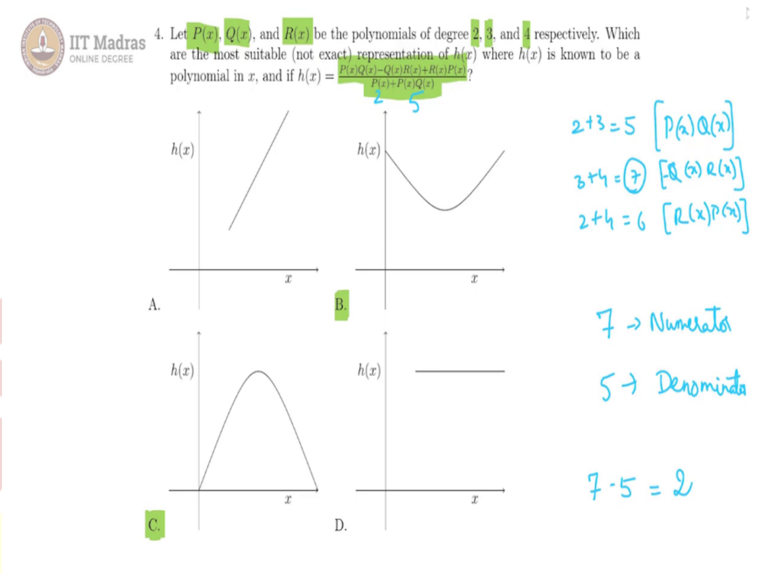

In this question we have 3 polynomials, $P(x), Q(x)$ and $R(x)$ and their degrees are given to be 2, 3 and 4 respectively. Which are the most suitable, although not necessarily exact representation of $h(x)$ where $h(x)$ is a polynomial in x and it is given as $\frac{P(x)\times Q(x)-Q(x)\times R(x)+R(x)\times P(x)}{P(x)+P(x)Q(x)}$. So, what we need to do here is to identify the degree of the numerator and the denominator.

Numerator degree $P(x) \times Q(x)$ will give $2 + 3 = 5$ that would be the degree of $P(x) \times Q(x)$, the degrees will add up and when we look at $-Q(x) \times R(x)$, then again the degrees will add up which will give us $3 + 4 = 7$, so this is from $-Q(x) \times R(x)$ and then $R(x) \times P(x)$ gives $2 + 4 = 6$. This is $R(x) \times P(x)$ degree.

And in the denominator $P(x)$ anyway has degree of 2 and $P(x) \times Q(x)$ we have seen has degree of 5. So, since we are adding all these polynomials together, the degree of the entire numerator is the maximum which is 7. So, we have 7 as a degree of the numerator and 5 as the degree of the denominator. Since it is a division, the powers will have to subtract, so degree of $h(x) = 7 - 5 = 2$. So, $h(x)$ is a quadratic and that would indicate B and C are possibly the curves because these look like quadratic curves. A and D are definitely straight lines.

Mathematics for Data Science 1
Week 06 - Tutorial 05

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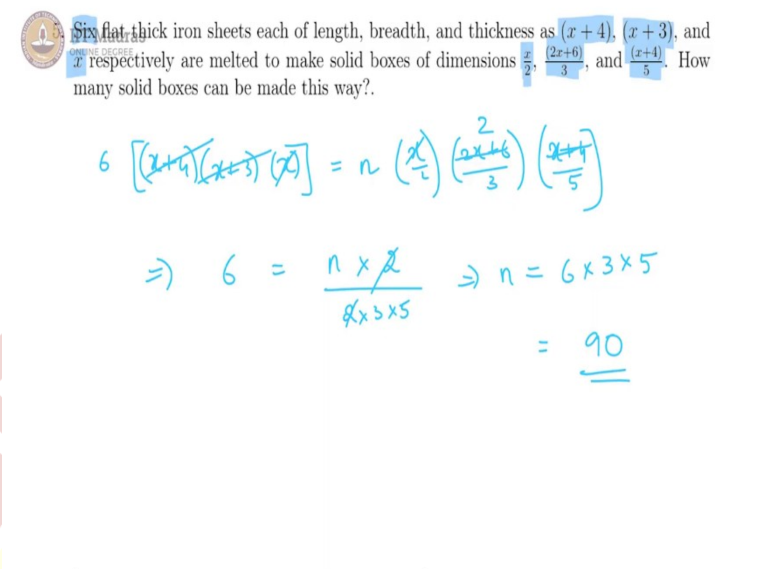

There are 6 flat, 6 of them, thick iron sheets each of length, breath and thickness $x + 4, x + 3$ and x respectively and they are melted to make solid boxes of dimensions $\frac{x}{2}, \frac{2x+6}{3}, \frac{x+4}{5}$. How many solid boxes can be made this way? So, basically the volume will have to be equal. So, first we find the volume of our 6 sheets put together that would be $6\left[(x+4) \times (x+3) \times x\right]$ and this would be equal to the volume of the solid boxes.

So, let us say there are n solid boxes and then the volume of each is $\frac{x}{2}, \frac{2x+6}{3}$ and $\frac{x+4}{5}$. So, now this x and this x cancels and this $x + 4$ and this numerator here cancels and $2x + 6$ is $(x + 3) \times 2$ so, this is one time and this is 2 times. So, what we get is $6 = \frac{n \times 2}{2 \times 3 \times 5}$. So, 2 and 2 also cancels. This implies $n = 6 \times 3 \times 5$ and that is 90. So, you get 90 boxes overall.

Mathematics for Data Science 1
Week 06 - Tutorial 06

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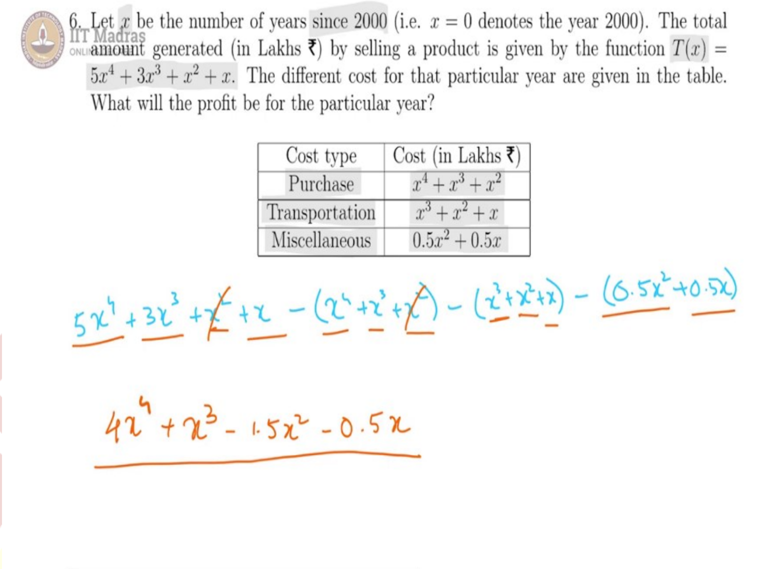

In this question, let x be the number of years since the year 2000, so $x = 0$ denotes the year 2000. And the total amount generated in lakhs by selling a product is given by $T(x)$. So, this is a polynomial which has the variable as a number of years since 2000, and the different cost of the particular years are given here. So, purchase cost is this polynomial, transportation cost is this polynomial, miscellaneous cost is this polynomial.

So, we now have to find out the profit for that year. So, that would just be $T(x)$ minus all these cost. So, it is $5x^4 + 3x^3 + x^2 + x - (x^4 + x^3 + x^2) - (x^3 + x^2 + x) - (0.5x^2 + 0.5x)$. So, this would be the total profit and for that we now have to look at the each x power term.

So, x^4, there are 2 terms, $5x^4$ and $-x^4$. So, we get $4x^4$and x^3 there are 3 terms, $3x^3$, and $-x^3$ and $-x^3$ here. So, we get x^3 and x^2x square terms there are 4, there is this x^2 and then there is this $-x^2$ and another $-x^2$ and minus $-0.5x^2$.

So, that will give us minus 1.5 x square because this and this cancels off and then we get $-1.5x^2$. And lastly the x term there is x and $-x$ which cancels off and $-0.5\ x$. So, $-0.5\ x$. So, this would be the total profit for that year.

Mathematics for Data Science 1
Week 06 - Tutorial 07

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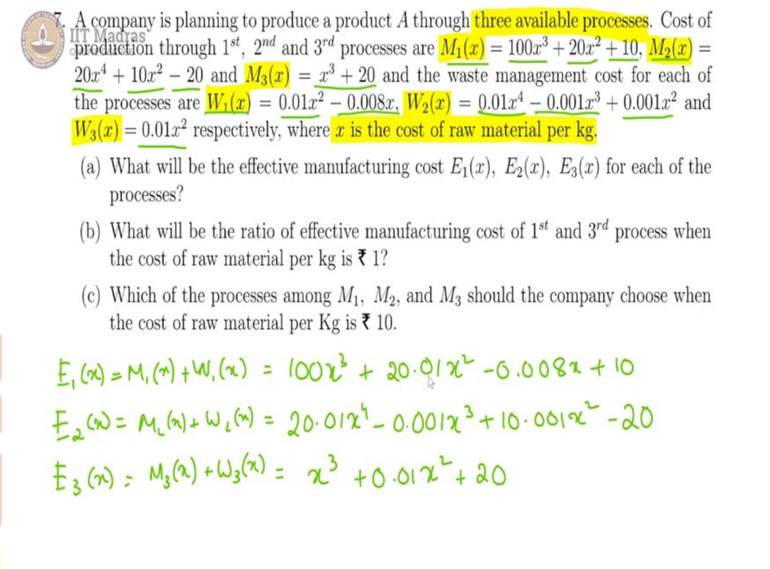

In this question, we have a company which is producing a product A through 3 processes and the cost of production are given as $M_1(x)$, $M_2(x)$ and $M_3(x)$. These are the 3 cost of production. They have given us polynomials of x. So, what is x? x is the cost of raw material per kilo. And now they are also giving us the waste management cost as $W_1(x)$, $W_2(x)$ and $W_3(x)$. What will be the effective manufacturing cost?

So, effective manufacturing cost simply has to be the sum of these. So, $E_1(x) = M_1(x) + W_1(x)$, so that is going to be so, M_1 is here, W_1 is here. So, M_1 would be $100x^3$ and there is no x^3 term in W_1, so we first write down $100x^3$, then there is an x^2 term here, $20x^2$, there is also an x^2 term here, 0.01. So, their sum will give us $20.01x^2$ and then there is no x term in M_1. There is an x term here, so you get $-0.008x$. So, this is also done. And then lastly we have the constant term which is $+10$. So, this is the effective manufacturing cost for process 1.

Likewise, process 2 would be $M_2(x) + W_2(x)$. So, here this is $M_2(x)$, this starts with an x^4 and W_2 also has an x^4 term. So, we have $20.01x^4$ plus there is no x^3 term in M_2, there is an x^3 term here, so there is no, this is not plus, we write $0.001x^3$, then the x^2 term there is a $10x^2$ here and a $0.01x^2$ here. So, $+ 10.001x^2$ and lastly there is a constant term and no constant term over there. So, this is -20. So, this is the E_2.

And then E_3 is going to be $M_3(x) + W_3(x)$ and in M_3 there is just 2 terms, the x^3 terms and a constant. W_3, there is only one term which is the x^2 term. So, we just write all of them together,

$x^3 + 0.01x^2 + 20$. So, this is the effective manufacturing cost for the third process and the three of them are given here.

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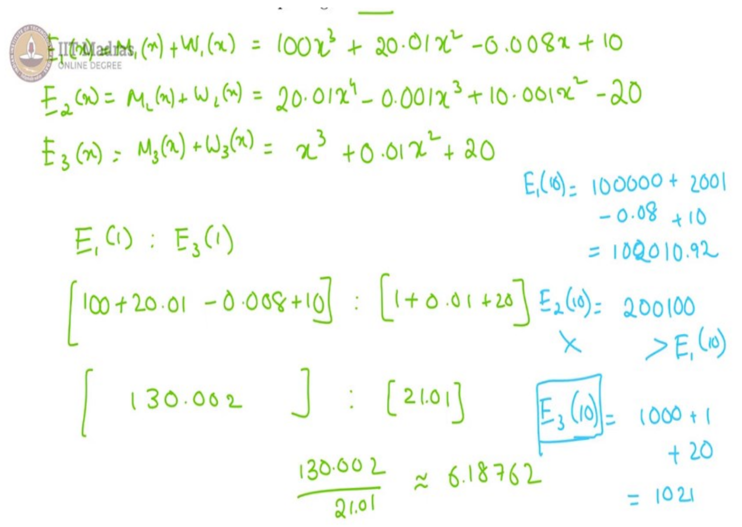

Now, what is the ratio of effective manufacturing cost of first and third processes when the cost of raw material per kg is rupees 1? So, basically we are looking for the ratio $E_1(1)$: $E_3(1)$ which is then we just substitute 1 in the E_1 term, so we get $[100 + 20.01 - 0.008 + 10]$: $[1 + 0.01 + 20]$ is to, so we get 130.002 is to 21.01. So, we have to put this down as a number 130.002 divided by 21.01 is roughly 6.18762.

Then we have the third question, third part of this question which says, which asks which of the processes M_1, M_2 and M_3 should the company chose when the cost of raw material per kg is 10? So, the company should chose the cheapest process. So, we have to find out $E_1(10)$, $E_2(10)$ and $E_3(10)$. And then if we looked at that $E_1(10)$ =$100x^3$ is 1 lakh plus $20.01x^2$ is $2001 - 0.08 + 10$. This is then 102010.92.

Moving on then $E_2(10)$ is 200100. So, $x^4 = 0^4$ so, this is what we get and this is 200100, so $x^4 = 10^4$, so this is what we get and this is 200100. We see that the remaining smaller terms, x^3, x^2 and constant term have small coefficients as well, 0.001 and 10.001. So, they will not really impact the value very much. So, we know that this is already larger than $E_1(10)$, so we do not consider it.

Let us look at $E_3(10)$ which is then $1000 + 1 + 20$, so this is just 1021 rupees. So, $E_3(10)$ is the least which is why the company should chose the third process as their process when x is equal to 10 rupees per kilo.

Mathematics for Data Science 1
Week 06 - Tutorial 08

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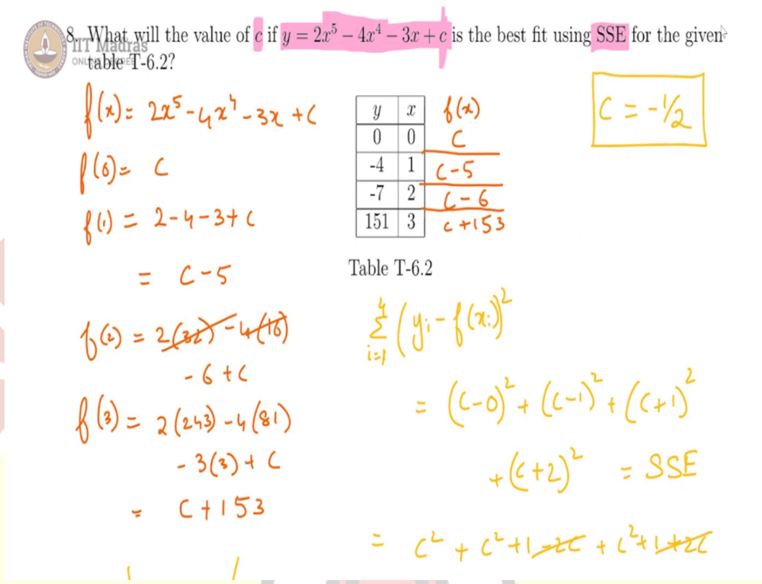

Our last question we are looking at the best fit for some data. So, this is the fit we have obtained a fifth degree polynomial for this data, these 4 points and they are asking what is the value of c, c is the constant term here. What is the value of c if this curve has to be the best fit using sum squared error? So, let us assume this curve is $f(x) = 2x^5 - 4x^4 - 3x + c$. So, we are going to have to also put up the $f(x)$ value, so $f(0)$ is then c because everything else is power of x, so $f(0)$=c, and then we have to look at $f(1)$ which is $2 - 4 - 3 + c$ that is equal to $c - 5$.

So, here this is $c - 5$ and then $f(2)$ is $2 \times 32 - 4 \times 16 - 6 + c$, now 2×32 is 64, 4×16 is 64, so these two cancel off, so you get 6 $c - 6$. And lastly, we have $f(3)$ which is $2 \times 243 - 4 \times 81 - 3 \times 3 + c$ so that gives us $c + 153$, so here it will be $c + 153$. So, for finding SSE we are going to have to do $f(x - y)$ or $(y - f(x))^2$.

So, $(y_i - f(x_i))^2$ and we are going to sum it from $i = 1$ to 4 and that gives us $(c - 0)^2 + (c - 1)^2 + (c - 1)^2 + (c + 1)^2 + (c + 2)^2$ So, this is the sum squared error.

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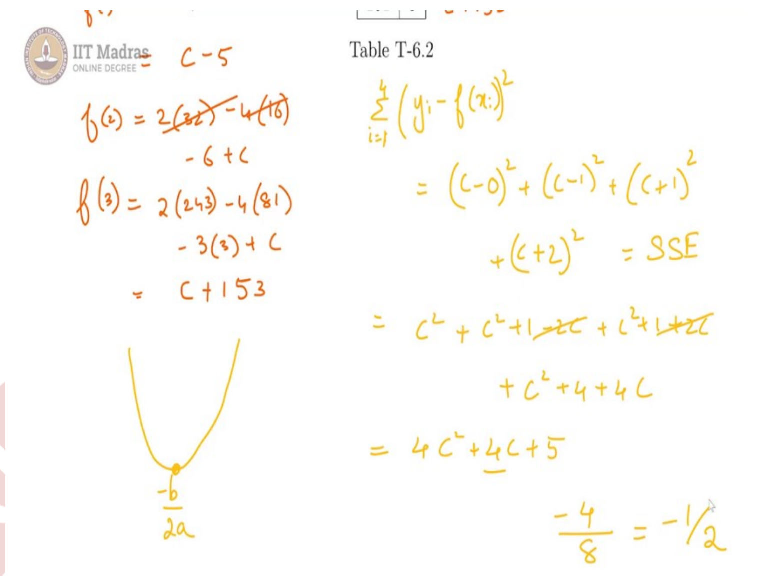

And we get $c^2 + c^2 + 1 - 2c + c^2 + 1 + 2c + c^2 + 4 + 4c$, so this $-\,2\,c$ and this $+\,2\,c$ cancels off and we arrive at $4c^2 + 4c + 5$, this is our sum squared error it is a quadratic in c and for minimum and this is also an upward facing quadratic because the coefficient of $c^2 > 0$, so it will be a parabola like this and the minimum occurs at this point which is the vertex of the parabola and that we know is $\frac{-b}{2a}$, here $-\,b = -4$ and $a = \ 4$ so $2a = 8$ so you get $\frac{-1}{2}$, so for $c = \frac{-1}{2}$, we get the minimum sum squared error. Thank you.

Mathematics for Data Science 1

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Lecture – 36
Graphs of Polynomials: Identification and Characterization

Hello friends, in this video, we will take up our next mission of about understanding the polynomials. This mission is given a graph of a function, whether we can identify the given function is a polynomial or not. If you have been given a polynomial equation, how will you put it on a graph paper?

So, the mission is twofold. First, If you have been given a graph of a function, you will identify whether this function is a polynomial or not. If yes, the, we will answer the second question that is can I derive the algebraic equation of this polynomial? The second part of the mission is we want to identify how the graph looks like if I have the equation of the polynomial. So, let us begin our journey about understanding the Graphs of the Polynomials.

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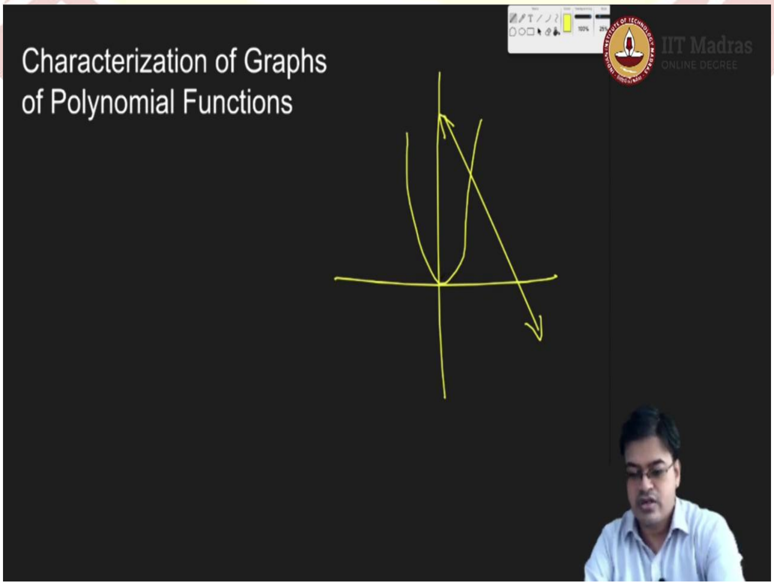

So, first of all let us recollect from our earlier experience that is linear functions and quadratic functions. If I am as linear functions and quadratic functions themselves are graphs of the functions. So, when I am plotting these two functions or when I am putting them on the graph paper, what happens? There you will never feel any abrupt jerk while drawing these functions. If you are trying to draw, you, for example, if you are trying to draw a line, then what you will do is you will simply draw a line, and then on graph paper. And there would not be any jerk for drawing the line.

In a similar manner if you are asked to plot a quadratic curve, you will find a axis of symmetry, and around the axis of symmetry you will do something like this, this is let us say this is the graph right, that means, the curve that you are trying to draw is has always been smooth. So, that one feature we can record in our mind. And say that the, if I have been given a polynomial function, the polynomial function must be smooth that means, I should be able to join the points effortlessly without having any jerk.

If there is any corner or edge in the graph then it better not be a polynomial function. Another thing is you can draw these graphs without lifting your pen; you can draw these graphs without lifting your pen that means, these graphs always are continuous.

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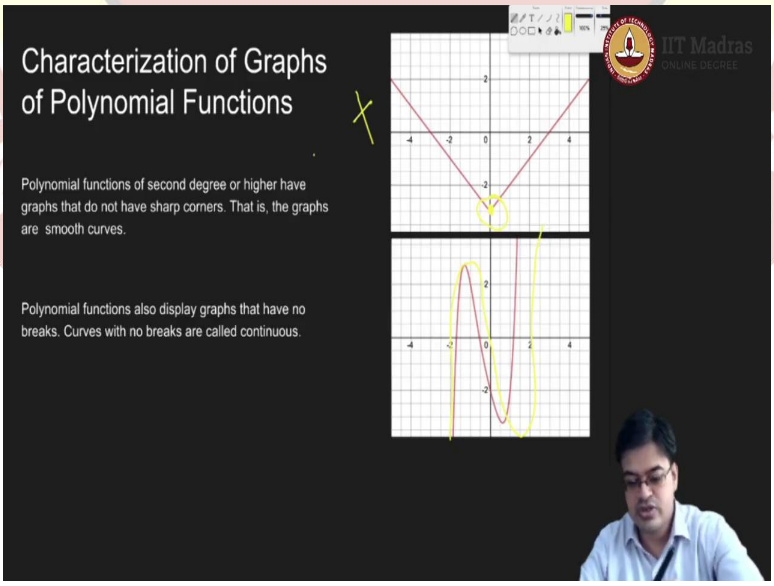

So, let us try to list these properties; first if you have polynomial of second degree or higher even a linear degree, the graphs do not have sharp corners that is the graphs are always smooth curves, this is a first feature that we will notice if I have been presented with graph of a function.

Second thing is polynomial functions always display graphs that have no breaks that is what I meant; the graph that I am drawing is always going to be continuous curve. Or you can say in better words, it is curves with no breaks are called continuous, and therefore the function itself will be continuous.

So, let us identify through two graphs. Let us have this graph. Now, is this a polynomial function? Does it satisfy the first criteria? That is it should be a smooth curve, is it a smooth curve? Yes, it is a smooth curve. But if you look at this point, then it had some sharp corner over here, this corner is very sharp. And therefore, I cannot qualify this as polynomial function; this is not a polynomial function.

Let us have a look at the next graph which is this. Now, here I can use my free hand skill to draw a curve and I can actually find out how I can draw better curve. For example, if I start drawing this curve, then I can easily pass through this. So, you will all the transitions are very smooth, because the transition is very smooth I can easily identify this to be a polynomial function.

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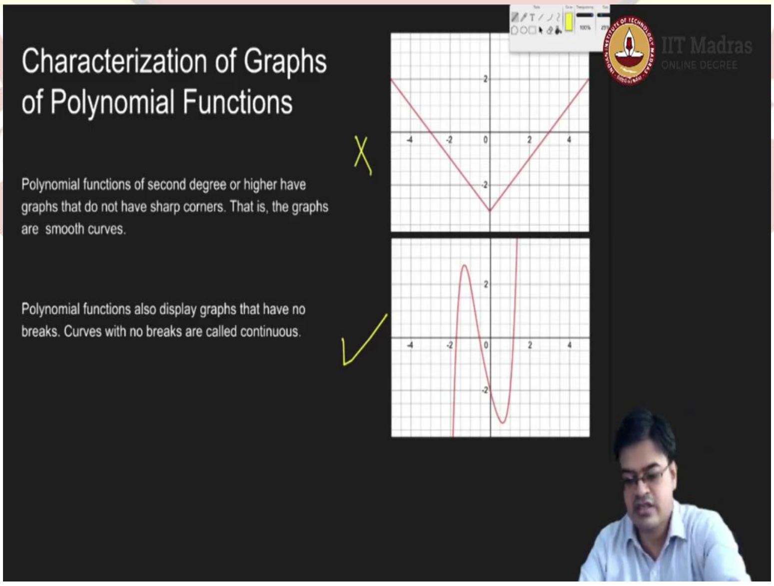

Therefore, this qualifies to be a polynomial function, whereas this does not qualified to be a polynomial function. Let us take a quick look at some other graphs and see whether they will qualify as polynomial function or not.

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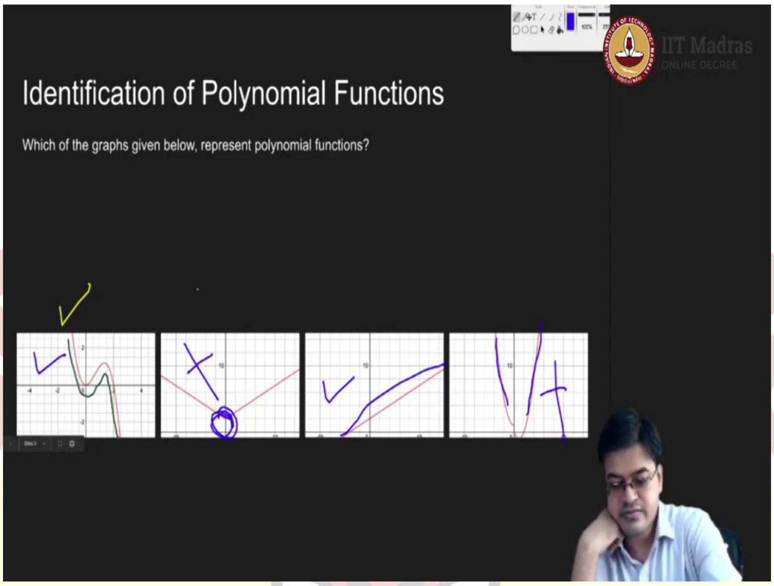

So, let us go ahead. And pose a question, which of the graphs given below represent polynomial functions? So, one by one I will unfold the graph, and we will argue for whether they are polynomial functions or not. This is the first graph. As I mentioned earlier, this also qualifies to be a polynomial function.

For example, if I want to draw a curve across this, I can easily draw a curve without lifting my pen and therefore, it qualifies to be continuous, and it has no breaks in between. So, therefore, it is continuous and it does not have any sharp edges, and the graph seems to be free hand. Therefore, it qualifies to be a polynomial function. So, my answer to this question is yes, this is a polynomial function ok.

Let us go ahead with the next graph ok. Now, what about this graph? Of course, we have argued for say similar graph in the earlier page that this graph does not seem to be a very neat graph, and it has a corner over here. This is the corner point of the graph. And therefore, this disqualifies to be a polynomial function; this is not a polynomial function. Again let me reiterate this was a valid polynomial function.

Next graph, let us look, let us try to see the next graph. This graph more or less seems to be a graph of a line, because it is a graph of a line. You can see this is also smooth. The transition is very smooth. So, again this will qualify as a polynomial function at least as far as the graph is visible. This is a graph of a line and it is a linear polynomial. So, it qualifies to be a polynomial function.

Let us go to the next graph ok. So, this graph is actually smooth. I can draw a curve over here, but at this point let me erase this graph; actually, you do it here. At this point, at this juncture, there is some problem. What is the problem? Over here, if I am drawing a curve over here, then I have to lift my pen come to a point 0 and then start drawing it.

So, this defeats the criteria that the graph should be continuous. Though the curves are very smooth, but at this point, this in this juncture, the problem is you cannot have a drawing without lifting your pen. Therefore, this will disqualify to be a polynomial function. This is not a polynomial function.

So, we have identified what is a polynomial function and what is not a polynomial function. Generally, whenever you have several ups and downs in the functions, we will estimate them or we will guess them to be a polynomial function if there is no corner as given in this graph, second graph to be precise.

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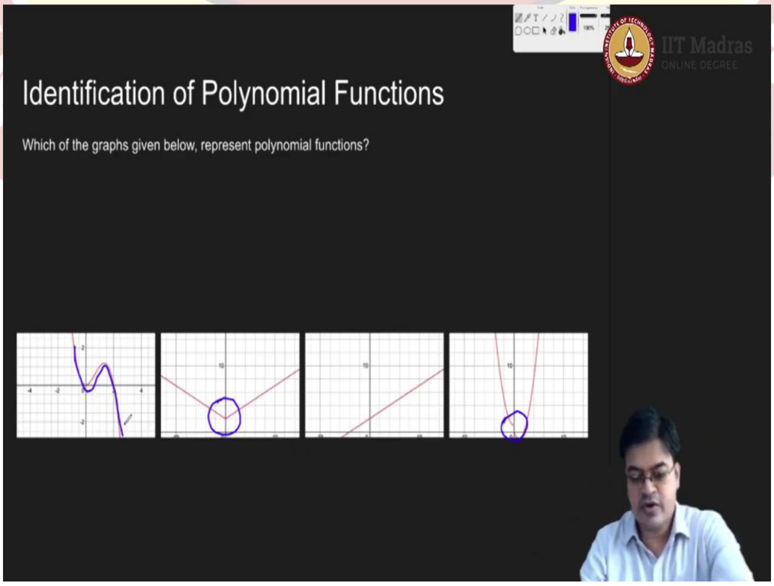

And if there is no there should not be any corner of this kind and there should not be any discontinuity of this kind, ok. Then we can easily safely say that the given function is a polynomial function ok.

And if you are looking at the ups and downs, these up and down of a function, those are the typical features of polynomial functions. With this knowledge, we are ready to handle polynomial functions because now, if you have been given a function on a graph paper you can identify whether the given function is a polynomial function or not. The next important criteria for about polynomial functions like quadratic equations is our identity our ability to identify zeros; zeros of the function.

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Lecture – 37
Zeroes of Polynomial Functions

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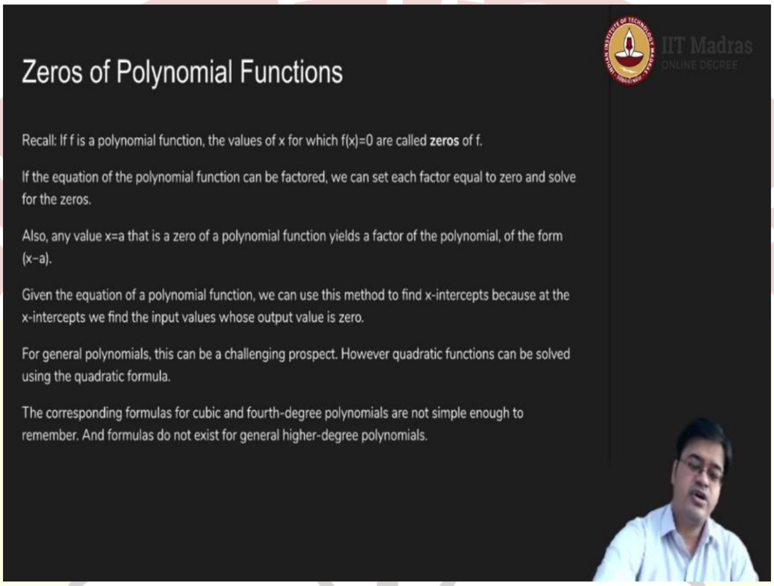

So, let us focus on Zeros of Polynomial Functions. So, for clarity, let us recall what is zero of a polynomial function. If f is a polynomial function, then the values of x for which $f(x) = 0$ is called zero of f. A value x of for which $f(x) = 0$ is called zero of f.

Now, when we studied quadratic functions, we had several methods of identifying the zeros of the quadratic functions. For example, we actually tried to graph the quadratic function because we knew some techniques, we actually plotted set of ordered pairs on a graph paper and join the curve smoothly, then we identified it is crucial to identify axis of symmetry and around axis of symmetry you can plot and wherever it intersects x axis, we will call that as a zero of a function. This is how we identified quadratic zeros of quadratic functions.

Another way that we used which will be helpful here is; factoring the quadratic function into factors given a quadratic function identify the factors and write the polynomial into intercept form. If you are able to do that, then you have again identified zeros of the polynomial because when you said that quadratic function to be equal to zero and if it is in a factored form, all the coefficients corresponding to that factor will be all the numbers corresponding to that factor will be zeros of the polynomial function.

So, now, we will focus on the factoring component of polynomial functions. So, if the equation of the polynomial function can be factored, then we can set that each factor to be equal to 0 and solve for zeros; this is an important step. But it as we have seen in quadratic functions, this is not always possible.

In such case, if you put some random values, if you throw in some random values in the function and you get something like $x = a$, you will get the value to be 0 that is also helpful.

Then, you can guess that is a factor and you can use the previous video to divide the polynomial by $(x - a)$ which will give you the remainder term and that remainder, you can actually figure out whether you can; you can consider factoring for that remainder or not all these things are possible or the other factor it is not remainder sorry it is the other factor. So, these are some possible ways.

Up to quadratic equations, we had some easy ways out easy way out like given the equation of a quadratic function, we can use this method to find x intercept because x for x intercepts, we get zeros that is what I explained earlier also. So, you can find x intercepts and you will easily get this. You can use the similar technique of finding x intercepts for a general polynomial function also, but it is very difficult to plot a polynomial function ok.

Given a graph of a polynomial function and you have identified based on our previous criteria, you have identified that this is a polynomial function, you can guess what are the zeros of the polynomial function that way this statement helps. But, if you go for higher order polynomials that is general polynomials, this can become messy, it can be really challenging.

Quadratic equations can be easily solved using quadratic formula we have a solution for quadratic equations. But, the cubic and four-degree polynomials have some formulae which you may study in your tutorials, but they are not easy enough to remember. And, for higher degree polynomials, you do not have any idea of how to approach finding zeros of the polynomial functions, you have to go by trial and error method and whatever knowledge you have about square, quadratic, linear and cubic polynomials.

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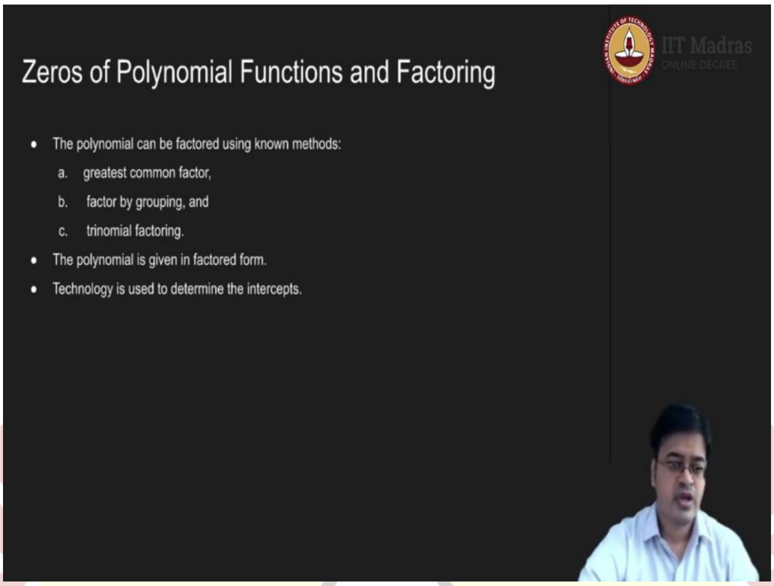

So, let us summarize what we have; what we have discussed just now. If I want to identify zeros of polynomial functions, the factoring technique is a crucial technique. So, what you can do is you can look at the polynomial and if you look at the polynomial, there is one easy way out that if you can identify the greatest common factor that is the greatest monomial that can be taken out common you can use that technique.

Once, if there is no such technique, if once that is available, the polynomial is more or less manageable, then you can use the technique of factor by grouping. So, you can create groups in that and see whether a anything is coming out common that is another technique. Another thing is you can instead of handling groups, you can decide to handle three terms at a time so, that is a trinomial factoring. This will be helpful when you have very high degree polynomial. So, these are the common methods for factoring the polynomials.

Once you can factor the polynomials each of them can be equated to 0 by writing a polynomial in a factored form. And then finally, if you are not very sure, then you can use some graphical tools which are available these days on computer or on the net one such tool is; Desmos which we are using in our presentations.

So, you can use those tools to determine the intercepts. In these tools basically, you will give of equation of a function and it will be graphed they will give the they will project the graph of a function right. So, this is our zeros of the polynomials and factoring play a crucial role.

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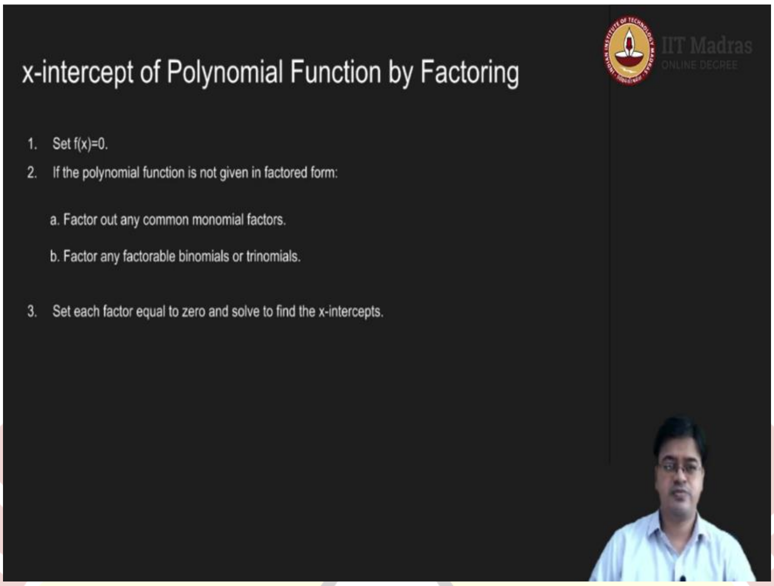

To understand this, let us see how to find x-intercept of a polynomial function by factoring. So, what we have discussed just now is we have set the equation that is $f(x) = 0$ in order to facilitate factoring $f(x) = 0$, then if the polynomial is given in factor form; factored form then equate each of them to be equal to 0 which we have seen for quadratic case also.

If it is not given in factor factored form, first in that you will look for is you take out some common monomial that is available in all the terms if that is that is there and you have taken out or if that is not there still you can go to the second step that is whatever at the rest of the terms you can factor them into factorable binomials or trinomials, you look for try to look for combinations which we have done successfully for quadratic equations well doing the factoring. So, you can do a similar thing over here.

And then finally, set each factor equal to zero that will give you the x-intercept. This is the; this is the strategy that we will follow for finding x-intercept of polynomial function by the method of factoring.

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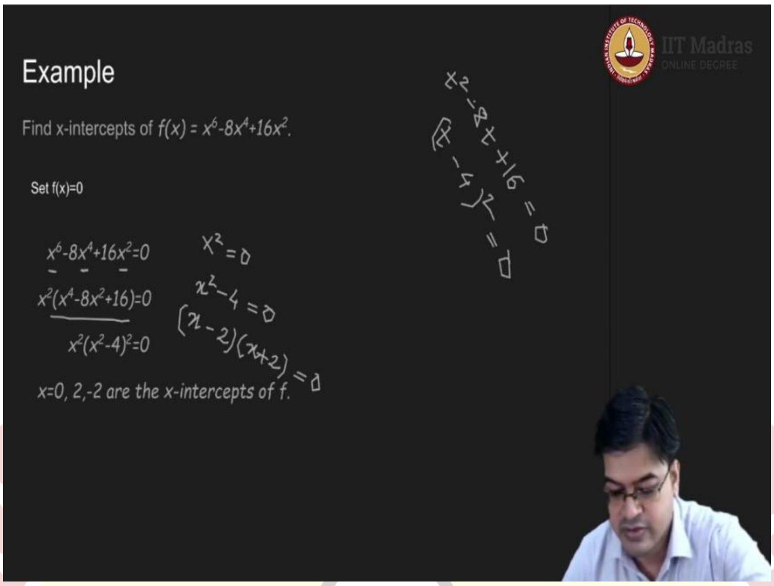

So, let us look at this example where we will follow the steps of the algorithm. So, the a question says; find x-intercepts of a function $x^6 - 8x^4 + 16x^2$. So, as per our algorithm or as per the steps given in the previous slide, I will set $f(x) = 0$ that is; $x^6 - 8x^4 + 16x^2 = 0$.

Now, you look at greatest common factor, a monomial that is common in all these terms that is x^2. So, what I will do is I will separate out this x^2, I have taken out this x^2 and now, you look at the other factor that is $x^4 - 8x^2 + 16$.

Now, this factor can be related to our quadratic equation of the form $t^2 - 8t + 16$. Can I factor this quadratic equation because there is no term corresponding to x^1 and there is no there are no odd terms essentially. So, I can use this and I can leverage the skill of quadratic equations to solve this equation and from quadratic equation point of view, I know this is $(t-4)^2 = 0$. So, instead of t here, it is x^2. So, that will give me $x^2 \times (x^2 - 4)^2 = 0$.

Now everything is looks in the form of x^2. So, what are the values of x? What are the feasible values of x? Those will be the x-intercepts. So, you can put x^2 is so, this will give me $x^2 = 0$ or $x^2 - 4 = 0$. So, $x^2 - 4$ can further be factored into $(x-2)(x+2) = 0$. And with this understanding, I can write $x = 0, 2, -2$ are the intercepts of f x-intercepts of f ok.

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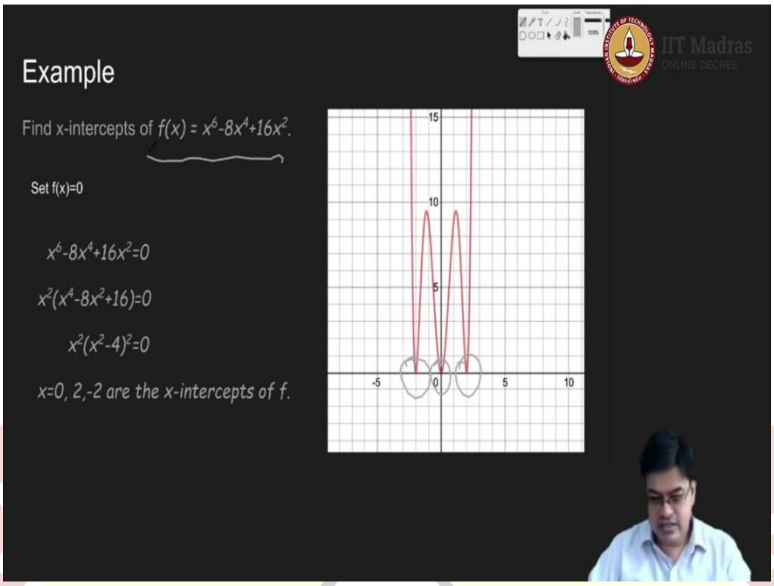

Now, as per the last step in the algorithm, you want to verify this result. How will you verify this result? Using the technology; so using Desmos, I have drawn this graph and you can verify that $x = -2$ which is here, $x = 0$ which is here and $x = 2$ which is here are all x-intercepts of a polynomial function given by these $f(x)$ ok. So, this is how we will identify x-intercepts.

Let us understand this strategy by looking at one more example.

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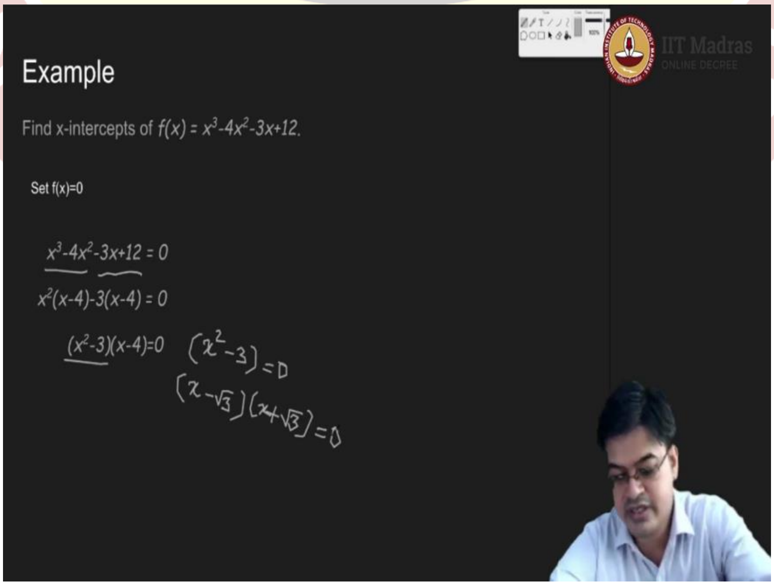

So, now here, we have been asked to find x-intercept of a polynomial function which is a cubic polynomial function $x^3 - 4x^2 - 3x + 12$ fine. So, as per our set up, this first step is set $f(x) = 0$. So, you have set $f(x) = 0$ that essentially gives me $x^3 - 4x^2 - 3x + 12 = 0$.

Then, the second a step if you have any common monomial, there is no common monomial because the last term is a constant term so, you cannot figure out a common monomial. Then is there any pattern? Can you look at two-two terms each binomials or trinomials because there are four terms, it is better to look at binomial terms.

So, if you look at the first two terms, you can see that you can throw out x^2 as a common thing, if you throw out x^2 as a common thing, then you will be stayed with $(x - 4)$ as a term as a one factor. And if you look at these two terms, then again if you take out 3 common - 3 common, then you will get $(x - 4)$. So, using the technique of binomial, binomials in this case, I am able to see this kind of factoring possible.

Good, that essentially means I can rewrite this expression as $(x^2 - 3)(x - 4) = 0$. Then, I want to solve this $(x^2 - 3)$ that is all is remaining which is a quadratic equation. So, you can easily solve using quadratic formula or a factoring, but here in this case, I know the factors so, that will be $(x - \sqrt{3})$ and $(x + \sqrt{3}) = 0$.

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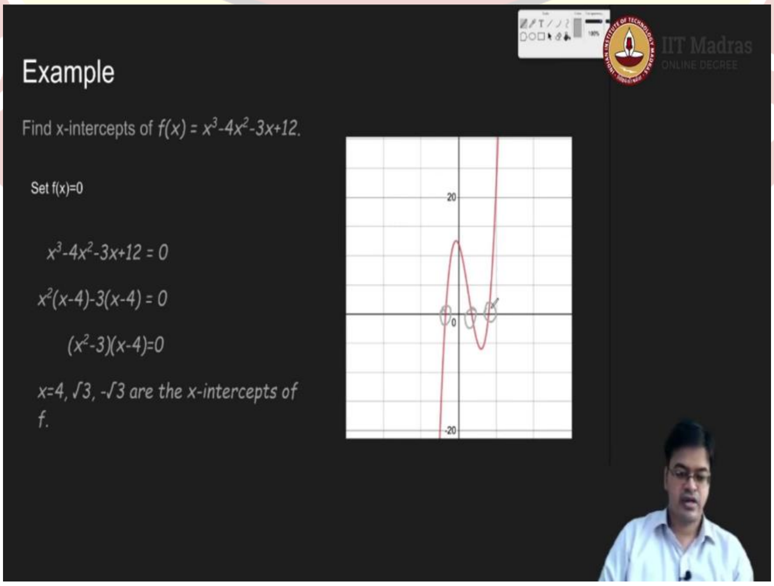

And therefore; therefore, the solution of this quadratic equation is well known that is $x = 4, +\sqrt{3}, -\sqrt{3}$ are the x-intercepts of the function. The final step is I want to verify using some technology or a graphing tool. This is the graph of a function.

So, in this case, you can easily verify there are three roots: first root this one which is a occurs, it occurs at , $-\sqrt{3}$, this one this is , $\sqrt{3}$, this one is 4. So, these are the four these are the three roots of a cubic polynomial. Roots or x-intercepts or a zeros of a cubic polynomial ok.

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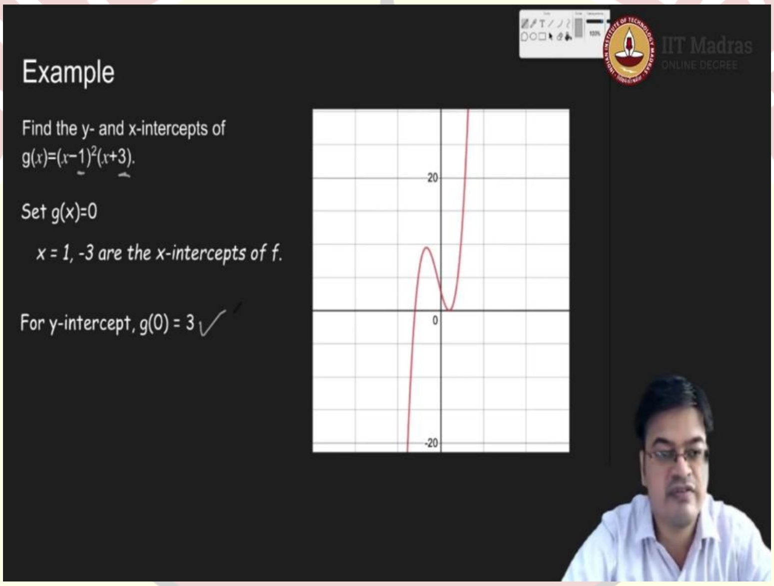

So, let us go ahead and see; let me remove this blocks. Another example where I want, I am interested in finding x-intercepts as well as y-intercepts of a polynomial function which is given in a factored form.

So, the polynomial is given in factored form. So, visually you will be able to guess the roots. So, as a standard set up, we will set $g(x) = 0$. Once you said $g(x) = 0$, it is very clear that $x = 1$ and $x = -3$ are the x-intercepts of f.

What about y-intercept? What is the y-intercept at all? So, y-intercept is where x is given to be 0. So, simply substitute $x = 0$ in the expression of $g(x)$, you will get $g(0)$, which is $0 \times (-1)$; the whole square that is 1+0+3 that is 3 so, 1×3 is 3 so, your $g(0) = 3$.

So, this is how you will figure out x-intercepts and y-intercepts of the function and this is the graph of that function. Using technology, I have verified.

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Lecture – 38
Graphs of Polynomials: Multiplicities

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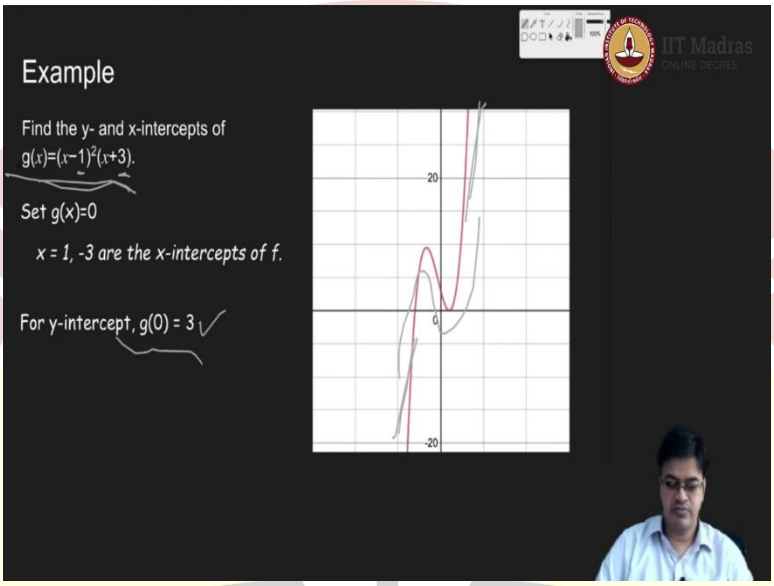

So, now, so far, we have mastered two skills; given a graph of a polynomial function, I know whether the given function is I can identify whether the given function is a polynomial function or not, ok. The second thing that we have seen is from algebraic expression of polynomial function whether it is in factored form or non-factored form, I have some set of rules or algorithm which will help me to identify, the roots of the polynomial or the zeros of the polynomial.

So, with this knowledge, can I explicitly write a polynomial function, or do I need to know something more about it? That is what the question that is troubling us. For example, the knowledge about x-intercepts in this case, and the knowledge of y-intercept, is this helping us to understand how the polynomial will look like?

For example, how will I decide the polynomial is going down from here, polynomial is going up from here, and it will stay going up forever, or when will this kind of shape come, the curve when will the rise and fall will happen, I do not know anything about this right now. What I know is simply the function should be smooth, yes, this is a smooth function.

But, from this graph can I write this equation? Seems to be difficult right now, but we will get the handle over it in due course of time.

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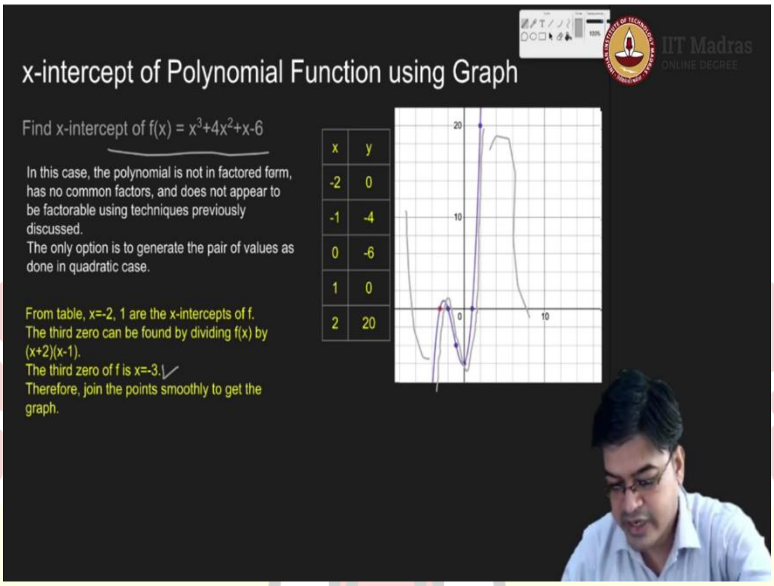

So, let us now look at the x-intercepts or identifying the x-intercepts using graph. So, you have been given a polynomial function. You have used a technology to identify the graph, but still you are not convinced.

And, you want to try it by your hand. So, how will you do it? That is the question. So, if I want to find the x-intercept, the given polynomial is a cubic polynomial, fine. So, this polynomial is not given in a factored form; I cannot find greatest common factor that is not possible.

Then, if I want to find something which is like binomial thing that is $(x + 4)$, but the rest the other term is $(x - 6)$. So, I am not actually getting these two tricks done. So, there is no way in which, I can factor this polynomial. So, one crude way, if you do not know how to go about, is to plot the pair of values as we have done in quadratic case.

So, simply find out what are the function values at some points. So, these are some standard points, I have drawn them symmetrically 0, 1, 2, -1, and - 2. When I considered these two points, because the function is very nice, I accidentally came across two zeroes that are - 2 and 1, good.

So, -2 and 1 are the x-intercepts of f which is clear from the table. Now, can I use this knowledge to find the third zero? The answer is yes. And, we know the long division. So, what you do is you consider $(x+2)$ as one factor and $(x-1)$ as another factor. You multiply $(x+2)(x-1)$ and treat that as a divisor, and take $f(x)$ as a dividend, and do the long division.

If you do the long division; you will get, you may pause the video and try for yourself; otherwise you will get the third 0 to be equal to $x = -3$; that is $x + 3$ is another factor. And, this is a cubic polynomial, so it cannot have any other factor, can I have at most 3 roots. So, you got this $(x+3)$, $(x+2)$, and $(x-1)$ as the factored form of this equation. Then, it is easy to plot the equation along with this table of values.

So, what you will do is, you will simply put up the points, you will simply put up the point. So, over here I know something and I know I have figured out the third root to be equal to $x = -3$; therefore, I can put that point as well.

So, this is $x = -3$ and, how to draw a line passing through; so, the next step is joining the line. So, you can draw join a smooth line passing through these points, then at this point it will turn up; at this point it will turn up, but to connect to this point, it has to go down, and then I do not have any idea. So, right now I can draw only up to this, right.

We will analyze further and see the cubic polynomial cannot turn more than two times, so that we will that we will come later. But, right now I can draw it this way. So, let us see what our graphical tool gives us. Yes, where the previous image and this image are slightly perturb but they are exactly matching. So, this is how the behaviour of the function will be.

So, now we have slightly better edge over drawing the graph of a function, when we have been given a equation of this form, but still I do not know why this is not turning up or why this is not coming down, I need to understand these things in a better way by using some analytical tools. For a moment, we got the correct graph.

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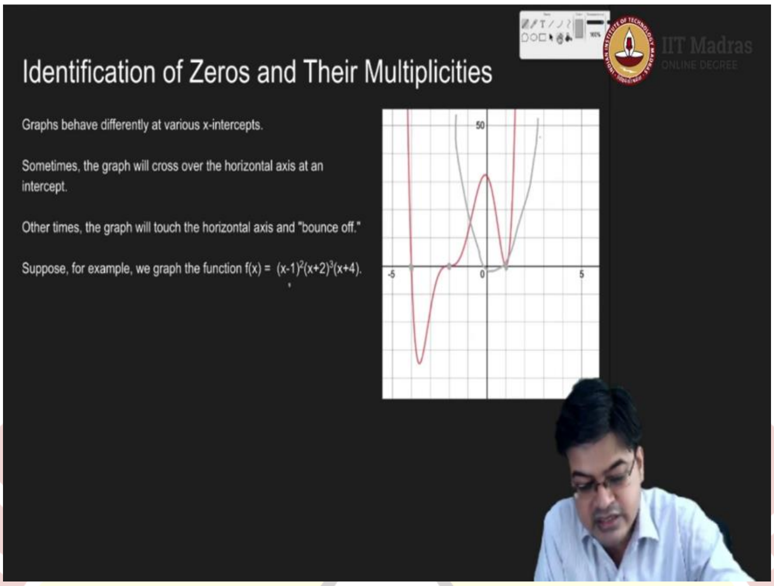

So, let us move ahead and try to see the behaviour of the graphs around the intercepts. For that, it is important to know the multiplicities of factor. So, in particular, graphs behave differently for at various x-intercepts.

We can go back to the previous case, where everything is of linear order that is $x + 3$, $x + 2$ and $x + 1$, the graph was behaving like this. If you go further back; why would this graph behave like this? Over here, when $x = -3$ is a factor, the graph the graph was actually like a straight line and over here it turned around.

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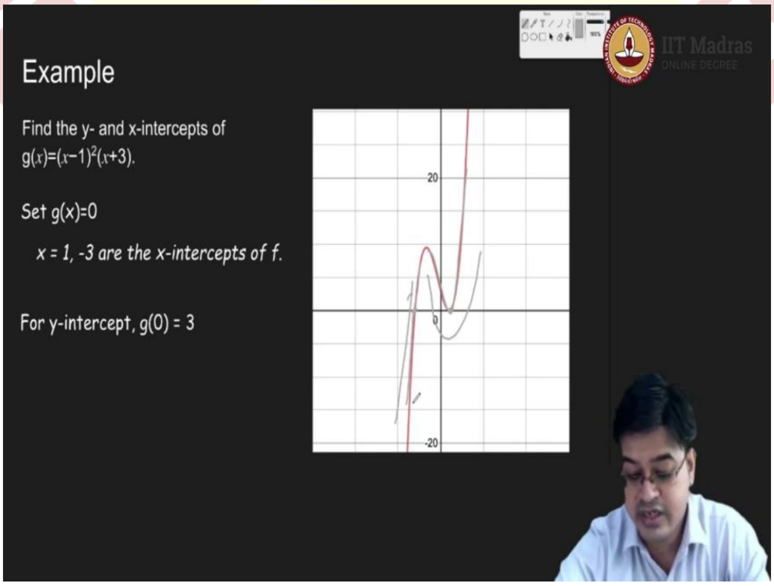

So, why should it turn around the factor? So, for example, $x + 3$ is one factor that is -3, and here $x - 1$ is one factor. But, for this factor it turned around; and for this factor it crossed, it cross the x-axis. So, why is this happening? So, I need to have a deep understanding of this. For that, we will discuss the next that is what we will discuss in the next slide. So, is it related to the function being appearing the factor appearing multiple times? That is what we will try to see.

So, as mentioned as shown in the earlier slide, that the graph can cross over the horizontal axis or it may bounce off; that means, it will touch and go up that is tangential to that axis. So, why is this happening at x-intercept? That, so in that case let for that making the understanding clear we will write a polynomial in a factored form which is $(x-1)^2(x+2)^3(x+4)$, right. And, let us draw that polynomial using technology or graphing tool, ok.

So, now some crucial things, let us identify the factors first; $x - 1$ that is $x = 1$, this is the factor that we are talking about. Then, $x = -2$, this is the factor that we are talking about and $x + 4$ this is the factor that we are talking about -4.

Now, at these points, what is happening, what exactly is happening at these points? So, when I consider the factor $(x-1)^2$, because it is quadratic and if I recollect the graph of a quadratic function, it behaved some it is not to the scale, but it behaved something like this, right. It will never cross x-axis.

So, a similar feature is visible over here, when I consider the graph of this function. So, if I consider a graph of this function, because $x - 1$ is coming twice; it is square $(x-1)^2$ is there; so, what I am getting is the behaviour is of the quadratic nature, ok.

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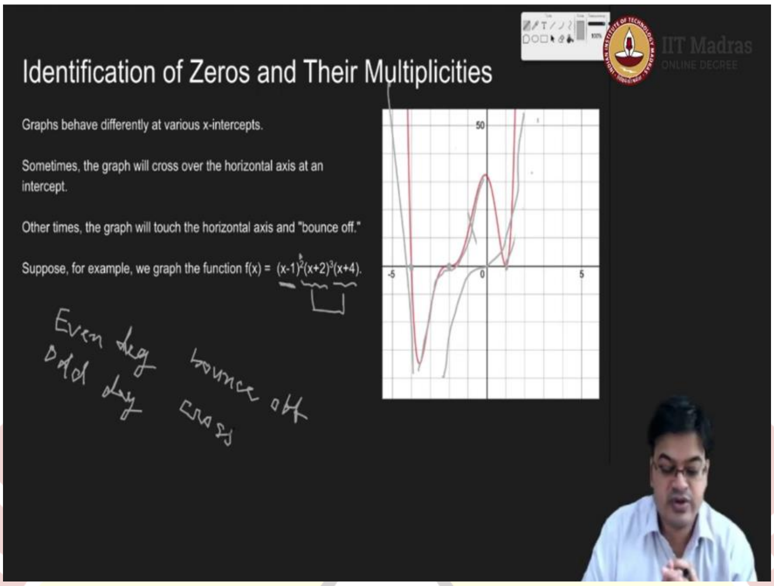

Now, let us look at that $(x+2)^3$. What is a graph of $(x)^3$? A graph of $(x)^3$ is somewhat like this; it crosses x-axis y is equal to x cube, it crosses x-axis. So, now that behaviour is evident when I consider that instead of x, I consider $(x+2)^3$ that behaviour is evident over here. It actually cuts and crosses x-axis.

And, if you look at the third factor that is $x+\ 4$ which is $x=-4$, it is behaving like a straight line that is also. So, what is happening here? I have two things; one and this one. So, in these both cases we have odd degree polynomials and, the odd degree polynomials as we know actually cross x-axis. And, in this case I have an even degree polynomial which is actually bouncing off the x-axis, this is a typical feature.

So, if I have even degree what we are saying is, if the polynomial is a, the factor is of even degree, then it will bounce off; that means, it will not cross x-axis but, if the polynomial as odd degree, then it will actually cross x-axis. So, these are the two typical features that we will employ while for plotting the functions which are of polynomial nature once we understand the factors.

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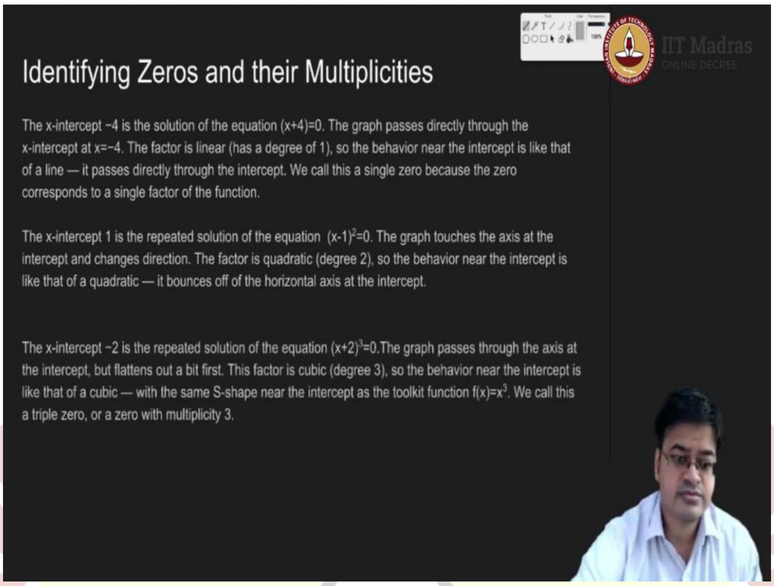

So, in the next slide, I have given a general description of these factors, you can go through these slides later, but it is essentially the same that I have said just now, ok.

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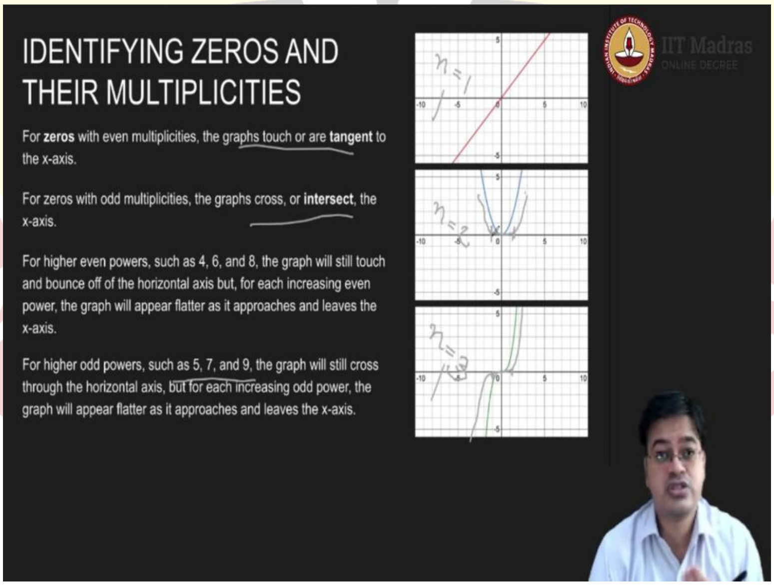

So, now for identifying zeros and their multiplicity. What do I mean by multiplicity? How often that factor is appearing. In the previous case the factor 1 was appearing twice, factor minus 2 was appearing thrice, and other factor was appearing only ones. So, if I want to identify the zeros and their multiplicities, I should look at the shapes of the curves. For example, if you look at the first graph, here the degree of the polynomial $n = 1$, here $n = 2$, $y = x^2$ this is, and here $n = 3$.

As mentioned earlier, it is more; it is more evident now, that when I have odd degrees, the curve actually passes through x-axis, when I have even degrees we can draw $y = x^4$, but this will be slightly broad and it will cut the x-axis, it will be somewhat like this. Let us not get into that. But, for odd degrees you will get something of this form, or even degrees you will get the bouncing off pattern and for odd degrees you will get a pattern which is actually crossing, one minute.

Let me reiterate this that; this is very as this is very important. If you have an odd degree polynomial, then you are almost sure you are sure to cross x-axis. If you have even degree polynomial, you will never cross x-axis at that point, you will simply bounce off from x-axis, ok. So, that gives us some more clarity. So, if the zeros of the polynomial or the factor has even multiplicities, the graph will touch or is tangent to x-axis or zeros with odd multiplicities, the graphs cross or intersect the x-axis.

Now, if you look at the even powers which are 4, 6 and 8, how will you guess, what is the strength of the power? So, in that case the graph will still touch x-axis it will bounce off, but which with each increasing even power it will appear to be flatter and flatter while approaching the zero and leaving from the zero. For example, the base will broaden; in this case, it will be something like this if it is x^4; x^6 further flattening.

If in a similar manner when you have odd powers like 5, 7 and then the graph will appear to be more flat over here, and while leaving also it will leave slowly and then it will decay at very fast rate. So, this is the typical feature from the bulge at these intervals you can actually guess the multiplicity of a polynomial. That is the importance of this slide.

So, now we have added one more weapon in our arsenal that is we will identify the multiplicities of the zeros. First we identify zeros. So, at the step zero is we identified given a function where whether it is a polynomial function or not. Then, we identified the x-intercept of the function that is zeros of the functions or roots of the functions.

After identifying roots of the functions, roots how many times repeated that is what, we have identified here in this by using the graphical tools. This is quite powerful. And, you can use it more often to understand the polynomial function. When you will actually solve some problems on identifying the polynomial functions, you will get a better hold of it.

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Lecture – 39
Graphs of Polynomials: Behavior at X-intercepts

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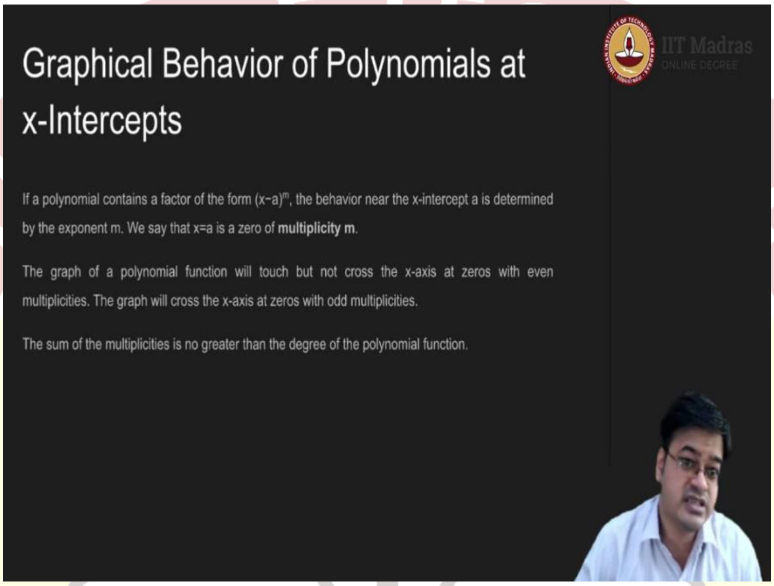

So, let us go ahead and look at the Graphical Behavior of Polynomials at X-Intercepts. So, in particular if the given polynomial has a factor of the form $(x-a)^m$, this m is called the multiplicity of the polynomial. And you will say $x = a$, is a zero of a polynomial f with multiplicity m. This is to fix the terminology.

Now, the graph of a polynomial function will touch, but not cross x axis at zeros with even multiplicities and the graph will cross x axis at zeros with odd multiplicity. We have iterated it enough number of times.

Also, one important thing is the degree of the polynomial cannot exceed the sum of the multiplicities, or the sum of the multiplicities is always less than or equal to the degree of the polynomial function, this is quite common sense right. If it exceeds, if it actually exceeds the polynomial degree then it is a polynomial of higher degree.

So, then now why it is not equal to, can be one question. The sum of multiplicities you will say will always be equal to the degree of the polynomial function. For that we need to understand that all the roots all the zeros of the polynomials or the only way we are identifying zeros of the polynomials is by identifying the x intercepts.

So, all x intercepts are real roots of the polynomials, but as in the quadratic case we have seen that some of the polynomials, some of the quadratic equations do not have real roots. In such cases the x intercepts are not visible.

I will demonstrate it further through some examples.

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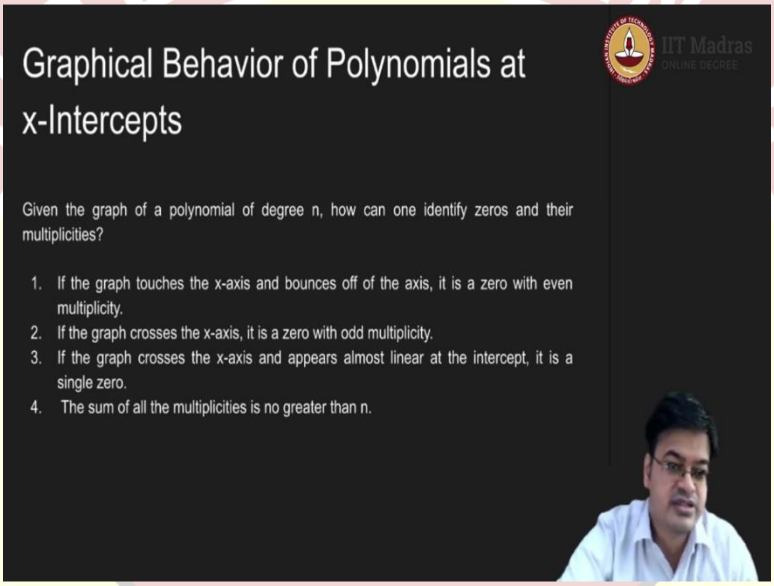

So, let us try to see. So, given a polynomial of degree n, that is a graph of a polynomial of a degree n. We want to identify zeros and their multiplicities; this is our goal. So, you are you have been told that this polynomial is of degree n and this is the graph of the polynomial, how, what will you do about it?

So, in that you will look at all the coordinates, where the graph touches x-axis, you take them. If the graph touches x-axis and bounces off the x-axis then, it is a zero with even multiplicity. If the graph actually crosses x axis, it is a zero with odd multiplicity. And finally, when you will conclude you have to take care that because the polynomial is of degree n, the sum of the multiplicities should never exceed the actual degree of the polynomial.

Another thing if the graph crosses x axis and appears almost linear at the intercept, then it is a single order; that means, it appeared only once; it is a linear function. And finally, that is what I explained the sum of the multiplicities is no greater than this fine.

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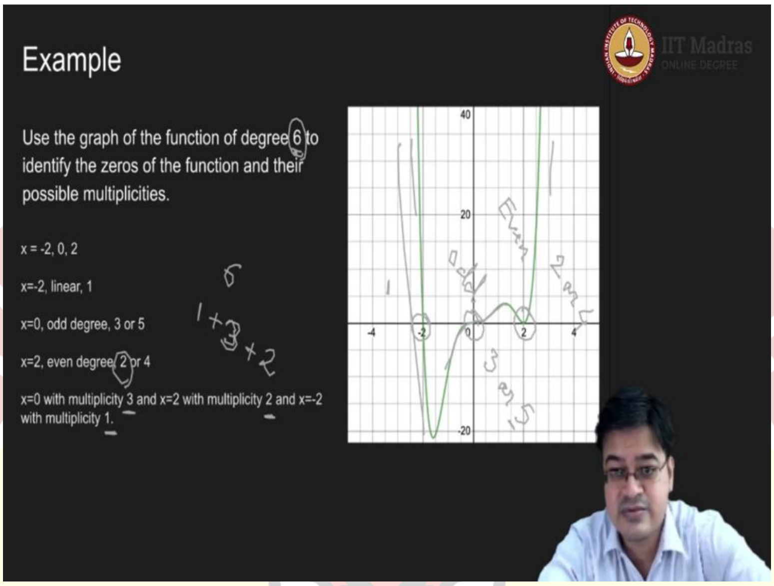

So, let us go ahead and see some examples and let us see whether we can apply these principles in action. So, use the graph of a function of degree 6. So, it is a degree 6 polynomial, which is given to you and this is the graph, wonderful.

So, now you can easily see $-2, 0$ and 2. So, x intercepts are $-2, 0$ and 2 fine. Then if you, let us start from left; so, at x is equal to minus 2, how is the behavior? It is more like a straight line, it is more like a straight line. So, at minus 2, I feel the behavior is linear or it is a onetime event.

At 0, what is happening is; it is having this S shape, somewhat twisted S shape. And that is indicative of odd degree that is indicative of odd degree and degree can be 3 or 5 or 7 or 9 I do not know right, but I have been given that the polynomial is of order 6. So, at most it can have a degree 5 right, the multiplicity 5.

Now, look at this particular junction, it actually bounces off the x axis. So, this is a typical trait of even degree polynomial. So, what can be the degree? It can be 2 or 4 right; so it can be 2 or 4 ok. Now, we need to collate this information. So, x intercept $x = -2$ is linear, there is no doubt about it. $x = 0$, you have odd degree 3 or 5 is the degree and $x = 2$ it is even degree.

Now, together sum of the degree should be equal to 6, of which this 1 is fixed. So, now, I can assign 3 or 5. If I assign 5 then $1 + 5 = 6$ and if $1 + 5 = 6$ that essentially means; there is no root of the form $x = 2$ that is not possible. So, I have to assign 3 over here. Once I assign 3 over here, then I do not have any other choice, but to choose 2 over here.

So, therefore, I have identified the multiplicities of the factors like $x = -2$ will have a multiplicity of 1, $x = 2$ has multiplicity of 2 and $x = 0$ has multiplicity of 3. So, this is how we will identify the factors, and identify the zeros of the functions, and identify their multiplicities. This is much better for drawing the graph of a function.

Still, I have not answered a question that why this should go to infinity and why this also should go to infinity; those things are not clear, but we will come to them later. Right now, we have much better understanding about zeros of the polynomial functions and their multiplicities and how we can use them to understand the function.

Now, another interesting thing that you can ask yourself is ok, I have seen this function and I know the multiplicities of zeros and everything. Can I use this knowledge to actually tell what is the equation of the function or not. We can try our hand on it, but we may not be able to because $x = 0$ is with multiplicities 3.

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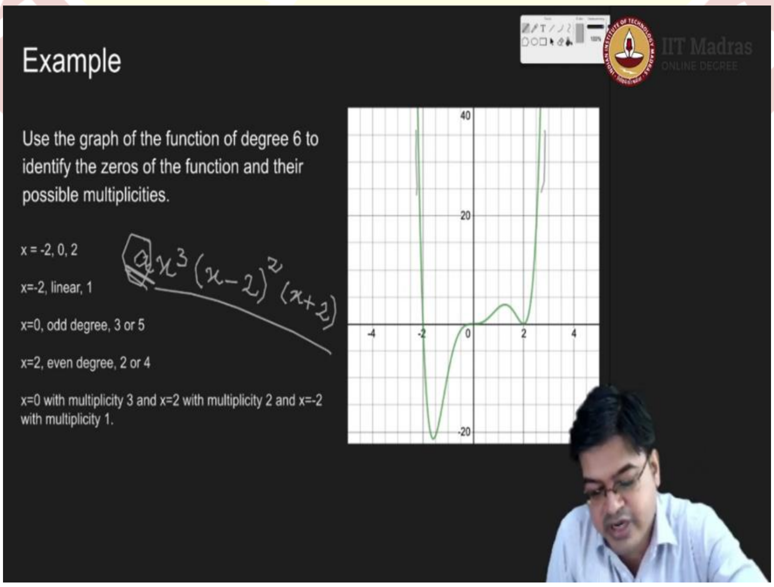

That gives me x^3. $x = 2$ with multiplicity 2 that gives me $(x - 2)$ the whole square and $x = -2$ with multiplicity 1 which will give me $(x + 2)$ ok.

So, now I can say that this is the polynomial function that is graphed here, but how will I verify? So, for that I need something which is non-zero. So, you can choose some point and check whether this is there, but there is a catch over here.

It need not match the values that are given here. So, what we will do is though the factors are correct the polynomial is of degree 6 here the degree is 6, we will put some unknown a over here and we will determine this a by putting the actual values.

We will come to it later when we have better understanding about the behavior of this kind, but right now you keep this in this point in mind; that we do not know this a. So, we cannot actually give the exact equation of the polynomial with reference to these points.

Though we know the form of the polynomial, but we do not have accuracy up to the exact matching on the coordinate plane with numbers. Just remember this point and this juncture and let us go ahead and do some other problems.

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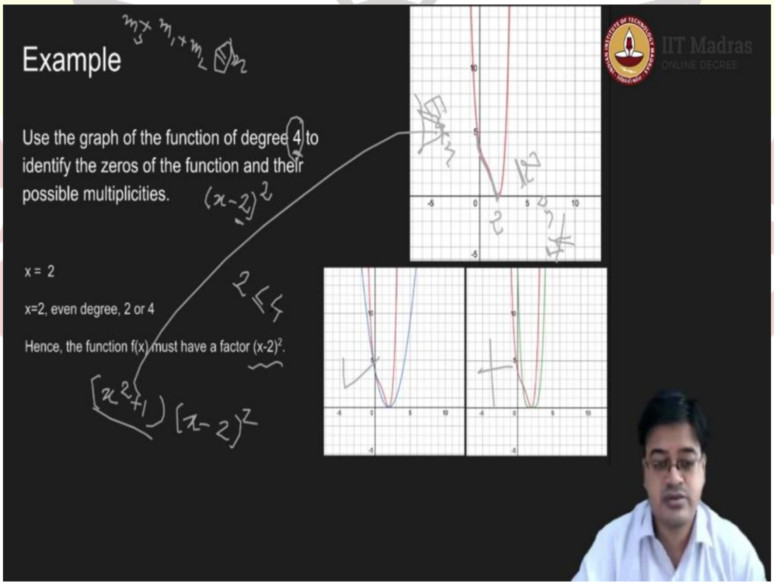

So, in the next problem is of similar nature. I have a polynomial of degree 4 and I have been projected with a graph of this polynomial function.

Now, this is an interesting example because, you have a polynomial of degree 4. In earlier case we had a polynomial of degree 6 and all 6 roots were actually visible. In this case I have a polynomial of degree 4, but there is only one root that is visible or one 0 that is visible.

So, what is that number? It is 2 ok. And based on our understanding of the algorithms what we know is the graph actually bounces off; the graph actually bounces off the x axis which is 2. So, this is a typical trait of an even degree polynomial, or even degree even multiplicity. So, in this case I can say it can have a multiplicity of 2 or 4. I cannot exceed beyond 4 because the given polynomial has degree 4 only ok.

So, now because the given polynomial has degree 4, it is safe to assume that this polynomial is of degree 4 right. But if this polynomial is of degree 4 you can see there is a perturbation of the shape over here. It is not the shape of a polynomial of degree 4, for example; y=x^4 will not be in this form. So, I can rule out the degree 4 constraint.

Therefore, I do not have any other choice, but to say that the polynomial is of degree 2, the multiplicity of this particular factor 0 is 2; that means, I have a factor of the form $(x-2)^2$ the whole square that is all I can say in this case. Yeah, $x = 2$, is of even degree 2 or 4 and hence the function based on the reasoning that I have given it must have a factor of $(x-2)^4$.

Let us understand this graphically as well. This is a function, the blue line over here, is actually $(x-2)^2$. As you can see it passes very closely to through the function and the other line is $(x-2)^4$, other graph the green line is $(x-2)^4$.

Now, if you look at graph of $(x-2)^4$ closely there is no possibility of changing the shape. You can scale with that unknown a, but you cannot change the shape of the function. So, this graph is actually ruled out. So, graphically we have understood why we are ruling out and this graph is not ruled out. So, this is somewhat familiar. So, it will have some factor of this form.

Now, one exercise for you is this graph actually though it is a polynomial of degree 4, the way we have constructed is we have multiplied (x^2+1) with $(x-2)^2$.

And if you look at this particular factor x square plus 1 because you, now what you can do is you can actually consider this $(x-2)^2$ as one factor. And you can see whether this x^2 you will get the same graph by multiplying this. The beauty of this example is that this (x^2+1) has no real roots.

Therefore, the degree though the degree of the polynomial is 4, I was not able to find the two missing roots; those are not in the real domain. They are in the complex domain. So, those things are there that is why the question that the sum of the multiplicities will always be less than or equal to the degree of the polynomial.

So, let us say there are three factors m_1, m_2, m_3 are the multiplicities. Then $m_1+m_2+m_3 \leq n$; that answers this question. So, in this case I was actually able to find for these real roots only 2. So, 2 is less than or equal to 4. So, that is why the; this is the example, why I cannot say that the sum of the multiplicities is always equal to the degree of the polynomial.

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Lecture – 46
Graphs of Polynomials: End behavior

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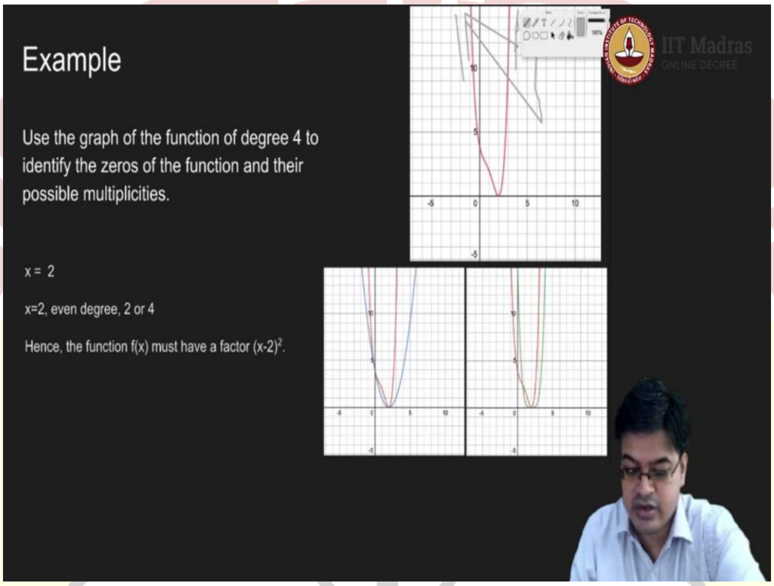

So, now we have understood how multiplicities affect the polynomial and how we are able to find the multiplicities of the polynomial functions with some factors, correct? Still we do not have an answer to a question that what why what is deciding this behavior that this function will go to infinity, this function will go up as usual, how this behavior is decided, we do not have any answer for that.

Let us try to understand that through end through what is called end behavior of the polynomials. So, the next slide is actually the end behavior of the polynomials ok. So, let us go to the next slide, it is end behavior of the polynomials.

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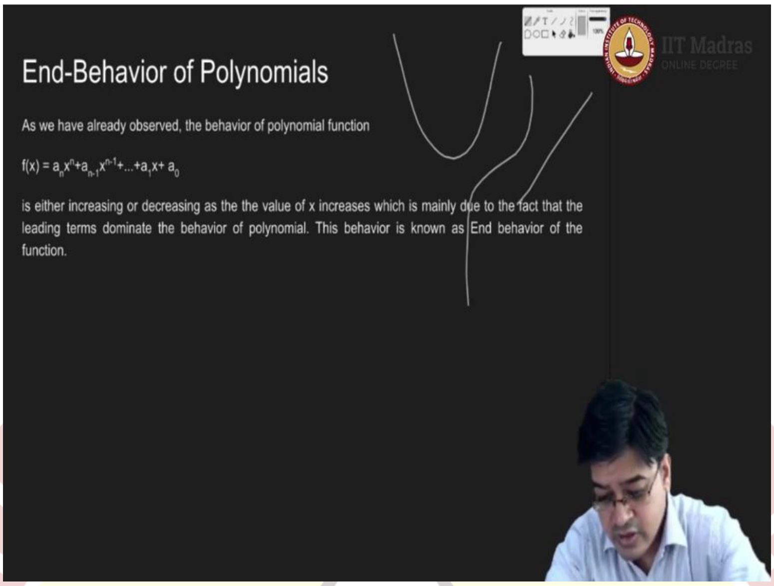

So, what is an end behavior of the polynomial? In order to understand end behavior, let us define an end behavior properly based on our understanding of quadratic equations. So, when we studied quadratic functions, we looked at the term of the form $a_2x^2 + a_1x + a_0$ right and then, we talked about a_2x^2 whether $a_2 > 0$ or $a_2 < 0$, then we decided the behavior of the function.

If $a_2 > 0$, then we said yes, if $a_2 > 0$, the function will take its minimum and therefore, it will go from both sides to infinity. If $a_2 < 0$, then the function will take its maximum and from both sides, it will go down and it will be unbounded. Now, from the graphs that you have seen in the earlier lectures as well as in this lecture, it is very clear; it is very clear that these functions, polynomial functions are either increasing or decreasing based on the way they wish right.

So, for example, it can be like this also. So, or it can go like this also or it can be a straight line as well, if it is linear or it can move like this. All these are polynomial functions. So, now, we want to have a better understanding. So, what is the behavior of a function after it has passed through all the roots is the question right, that is the term that was troubling us a lot.

So, for quadratic equations, we have decided that it is basically based on a 2 because quadratic equation the highest degree is 2. So, a 2 it is. So, now, in a similar manner, if I want to consider a polynomial function $f\,(x)$=$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$. Then, the behavior should be decided by this term.

Why should I make this claim? Because if you look at x^n, what we are looking for is as the value of x increases or as the value of x decreases. Now, it is not in that zone, where it is passing through many roots. So, it has passed through all its possible roots and now, after that how the function will behave? There is no determining factor right.

So, in such case, the only determining factor is the term $a_n x^n$; why? Because for large values of x this term x^n will dominate all other terms corresponding to x; x^n raised to n will dominate x^{n-1}, and so on that is when x is becoming large. When x is becoming small that is x is tending to $-\infty$, the term x will be the small x^n will be the smallest possible term or if we that n is of even degree, still it will be the largest possible term.

In any case, the behavior of $a_n x^n$ will play a dominant role in identifying the behavior beyond roots of the polynomials or beyond zeros of the polynomials. This behavior we will call as end behavior of a function and for polynomial functions, it is determined by the leading terms that is $a_n x^n$.

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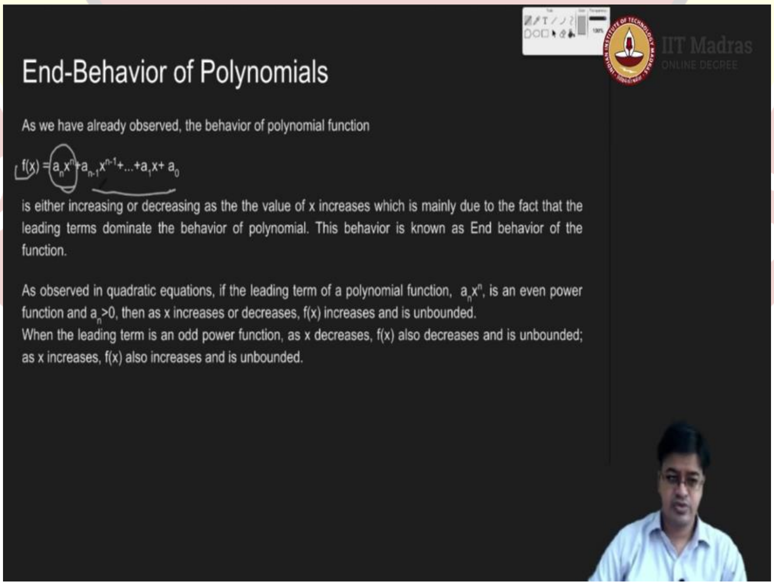

If this $a_n > 0$, and x^n that n is a even power exponent is even, then as x increases or decreases, it is very similar to quadratic. As x increases or decreases, $f(x)$ will always go to infinity. If $a_n < 0$ n is an even exponent, then whether x increases or decreases, $f\,(x)$

will go to $-\infty$. It will go on decreasing. Good. Then, what if $a_n x^n$ that n is the exponent which is of odd power or exponent is odd. What happens?

If $a_n > 0$, then as the function increases, $f(x)$ also increases. If $a_n > 0$ and it is of odd power as x increases, $f(x)$ also increases; as x decreases, $f(x)$ also decreases and both are going to infinity; one is going to ∞, another one is going to $-\infty$. They both are unbounded. Similar thing can be argued for $a_n < 0$. So, in order to improve our understanding, I have tabulated this zone; 1 minute, let me remove this part ok.

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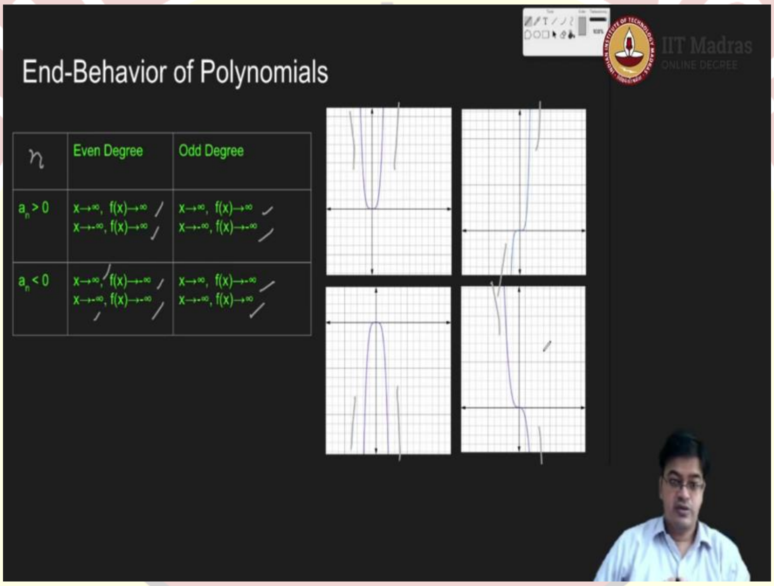

So, this is the better understanding. So, now, you look at the leading term $a_n x^n$ So, this is referring to n, n is of even degree, n is of odd degree. So, if n is of even degree and $a_n > 0$, x tending to ∞, x becoming larger and larger, $f(x)$ will become ∞; $f(x)$ will also increase. $a_n > 0$, x tending to $-\infty$; that means, x is becoming smaller and smaller and smaller; but because the polynomial is of even degree, it will again go to ∞.

In a similar manner, if $a_n > 0$ and the polynomial the leading exponent is of odd degree, then as x tends to ∞, $f(x)$ tends to ∞. You can imagine a function of the form x^3. Similarly, if $a_n > 0$, x tends to $-\infty$, $f(x)$ will also go to $-\infty$because $f(x)$ will also keep decreasing.

Remember polynomials of odd degree crossover x axis, if you link that point to this, then naturally it is very easy to and visualize the behavior of the polynomials. I will demonstrate these two graphs again, once again to reiterate the point. If $a_n < 0$, now $a_n < 0$; that means, x becoming larger, $f(x)$ the term, the leading term of $f(x)$ will be negative more and more negative.

So, $f(x)$ will tend to $-\infty$; but if x is becoming smaller and smaller, the exponent is of even degree, still $f(x)$ will again go to $-\infty$ because $a_n < 0$. Come back to odd degree, here the exact replica of what we have done for odd degree when $a_n > 0$ will happen.

So, in this case when $a_n > 0$, x tending to ∞, we will make bring this $f(x)$ to go to ∞; but in this case, it will bring it to $-\infty$ and similar case is true for the other part that is x tending to $-\infty$, $f(x)$ will tend to ∞. Let us visualize it through graphs.

Let us take this first block even degree $a_n > 0$. Imagine a function of the form x^2 or x^4 as x tends to ∞; both of them are going up. Just remember this figure that will clear this understanding.

Let us go to odd degree with $a_n > 0$, as x tends to ∞, here this is going up. This is going down right. Just imagine a figure of x^3 for the convenience. When $a_n > 0$, just imagine a figure of $-x^2$ or $-x^4$, both of them should naturally go down. That is what is written here as well. In a similar manner, just consider $-x^3$.

So, whatever was going down, will go up and whatever was going up, will go down that is what I meant when I said this. So, now we have much better hold over end behavior of polynomials. Now, you can look at the graph of a polynomial function and you by looking at the end behavior, you can say whether the polynomial, the leading term of the polynomial is of odd degree or even degree.

That is one more understanding, one more level of understanding that we have achieved through understanding this end behavior. But that is not over. We further need better understanding of the functions.

Mathematics for Data Science 1

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Lecture – 41
Graphs and Polynomials: Turning points

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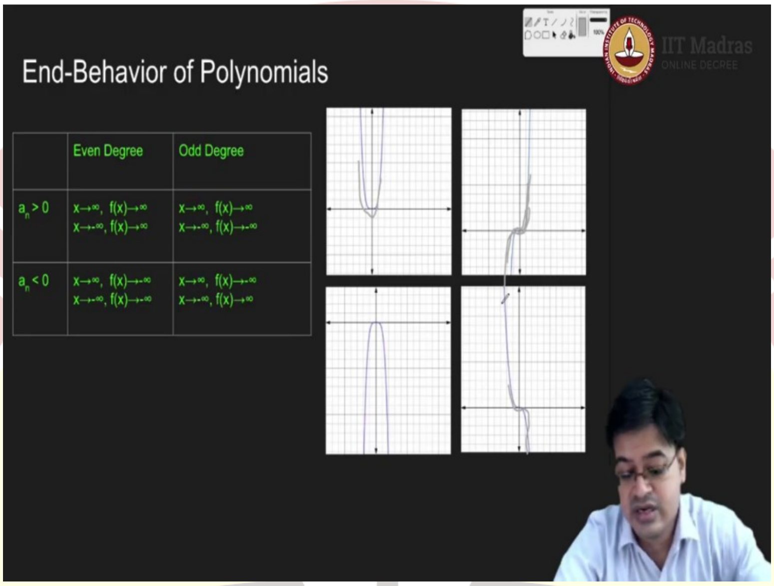

Now, we have understood how the graphs are behaving at x intercepts or zeros of the polynomial functions. How the graphs are; what is the end behavior of the graphs as x tends to ∞ or $-\infty$. Further, we need understanding that this turning. So, this is that this is one turning; though this looks as one turning, but these are actually two terms, here the function actually started.

This was, this function was increasing and then, it flattened out and then again, it turned to decreasing, right. So, these are the two twists. So, these are called turning points. This is one turning point, this is another turning point. So, now, in order to graph a function, you also need to understand, what is the meaning of these turning points? And, how many turning points it can have; a function can have.

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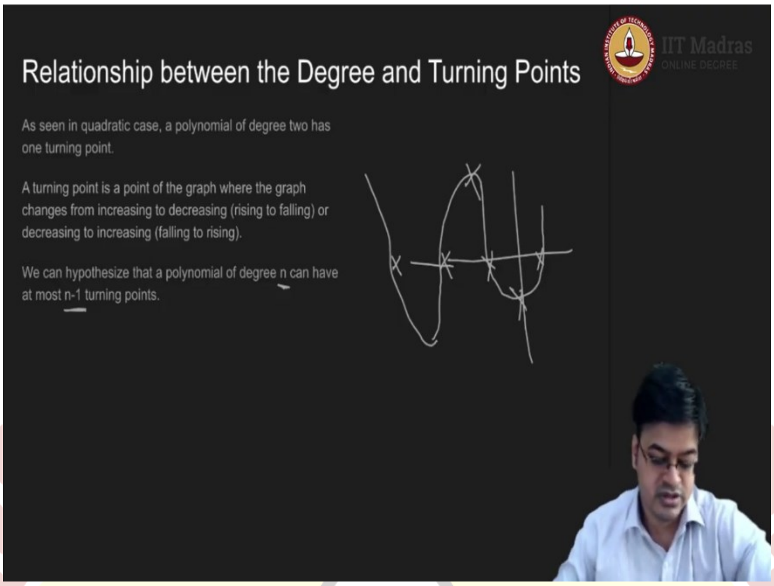

To understand that let us go to the understanding of relationship between the degree and turning points. When we had a quadratic function, when we were studying a quadratic function, our turning point was only one right that is the axis of symmetry that we have drawn. So, that was there was only one turning point. If you look at the function x^3, it actually turns here and here. So, it goes up and down. So, what can I say about the turning points, that is why I need to precisely define what is a turning point.

So, a definition of turning point can be read as a turning point is a point on the graph, where the graph changes its behavior from increasing to decreasing. In other words, rising to falling or decreasing to increasing in other words, falling to rising. So, what are the locations at which these turning points are available? These turning points bear a special relation with the theory of calculus; you need to identify these turning points.

We need to study the theory of calculus which you will do in math 2 of this particular course program. But for a moment, assume that we may not be able to accurately understand the turning points. But, we can have a rough estimate of turning points. So, in particular, if a polynomial is of degree n, then we can hypothesize that, ok. So, a polynomial of degree n, let us go with this understood logic.

A polynomial of degree n can have at most n roots or n zeros. This is very clear. So, in order to achieve these n zeros at most n zeros, suppose the polynomial has all n zeros which are real, then to achieve these zeros, every time the function crosses through one zero to come to another zero, it has to take a turn. So, how many such turns the function can take?

Let us say, we are very familiar with quadratic function. So, for example, if there are two real roots of the quadratic function which are here, then I am drawing this here. Then, axis of symmetry, then here; this is how the quadratic function will behave. Suppose, the function is cubic and now, this the other root is here, then how will the function behave? It has to again come back, take a turn and pass through this point, right. So, what we are doing is if it is cubic polynomial, it has all the roots, all possible roots on the real line, then it is turning here and it is turning here.

So, cubic polynomial will return 2 times. If it is a 4 degree polynomial and another root is here, then again it has to turn and it has to pass through this root. So, it has to turn 3 times.

So, in general, a polynomial of degree n can have at most $n - 1$ turning points. Is that safe to hypothesize? You can actually see the proof of this, when you will study maths 2. But right now for our understanding, what we are saying is if the polynomial is of degree n, then it will have at most $n - 1$ turning points.

This gives us some added advantage for drawing the graphs, though we may not be precisely able to locate the turning point; but we know that there will be a turning point. So, if I understand this, then I want to use this knowledge in drawing the graph of polynomial functions. So, how will I use this knowledge for drawing the graph of polynomial functions? That is what we will see now. Let me clear this, hide it and go to the next level.

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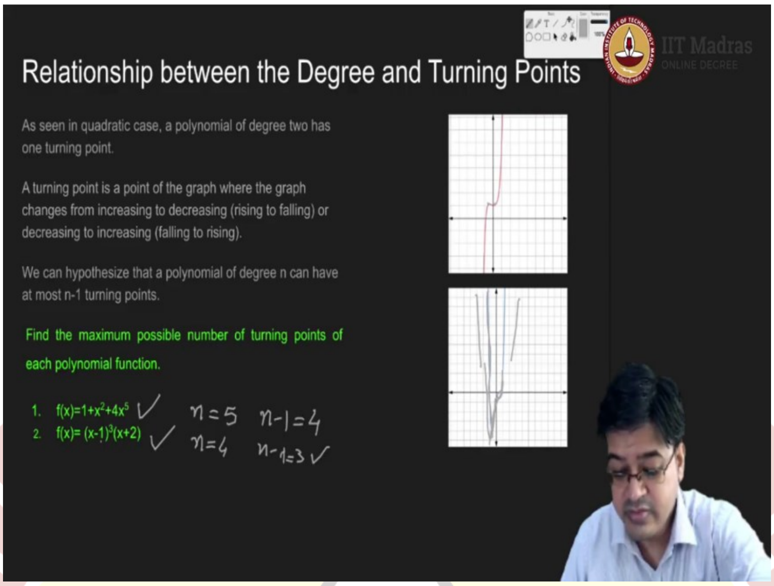

So, in order to understand how to graph, first let us look at some functions and see whether we can identify how many turning points these functions have. So, first function is $1 + x^2 + 4x^5$. So, what are the maximum possible turning points? Maximum possible turning points are 4.

Will the function actually turn 4 times? No, this is just a check whether verification test, whether we have drawn the graph correctly or not. If you have drawn the graphs which has more than 4 turning points; so, in this case, it is a polynomial of degree 5.

So, it will have $n -\ 1$ which is equal to 4 turning points, at most 4 turning points; it is not saying, it will have 4. So, if you have drawn a graph of this function and you have got 5 turning points, then definitely the graph of a function is wrong, ok. So, let us look at the graph of this function.

You can see in the graph, it is visible that there is one turning point here and another turning point here. So, it is actually showing only 2 turning points, but the graph can have at most 4. So, $2\ \leq\ 4$. Therefore, the graph must be correct, ok.

Let us go to the next this polynomial function is $(x-1)^3(x+2)$. So, this is a polynomial of degree 4 ok; this is a polynomial of degree 4 and therefore, it can have at most 3 turning points. So, $n =\ 4$ and $(n -\ 1) =\ 3$. So, it can have at most 3 turning points. You can see, so let us look at second example that we have displayed in the last slide

$f(x) = (x-1)^3(x+2)$ as you can see from the graph, this particular function, is a polynomial of degree 4 and therefore from our hypothesis we know that it can have at most $(n-1)$ turning points which is 3. So it can have at most 3 turning points. Now this is a crucial thing. When I draw the graph of this function, I get this kind of graph. Let me zoom in a bit and if you look at this point which is located over here it is $x = 1$ actually shows , $(x-1)$ actually shows cubic behavior, but this point something strange happens, OK and then at this point there is another zero, so actually I can locate the two zeros. In order to compensate for these two zeros, I need to have at least one turning point here, so this is the turning point based on our definition of turning point, but this point which is located here is also a very special point. For us right now the solution to this problem is this particular function $f(x) = (x-1)^3(x+2)$ has one turning point. which is located at this point, right? Another thing that when I claim that there can be at most $(n-1)$ turning points, so another thing that can be easy to verify is whether actually those $(n-1)$ turns happen or not. For that let me go to another example, which is $f(x) = (x-2)(x-4)(x+3)$. What is a feature of this example? The factorization has all three distinct terms. When I considered $f(x) = (x-1)^3$(x+2) the one term is repeated thrice the root 1is repeated thrice if I consider a corresponding equation. So in this case all these are distinct so when I plot a graph of this function, $(x-2)$, 2 is here, $(x-4)$, 4 is here, $(x+3)$ so $(x = -3)$ will be the zero and therefore to attain all three zeros, you can see this is the first term that we have taken and this is the second term that we have taken so here $n = 3$ and actually there are $n-1$ which is equal to 2 turning points. So my hypothesis is correct which is verified by examples and using the theory of Math 2 or Mathematics For Data Science 2 you can verify this hypothesis holds true or you can prove the hypothesis theoretically. In particular, let us come back to this example where I am dealing with $(x-1)^3$ at this particular function something strange happens. If you can relate this to the slope of the function then you will come to know that the slope of the function becomes 0 at this particular point and that is where this point can also be classified as a turning point if we know some advanced theory in mathematics in particular if we know calculus in mathematics but right now we are not identifying this as a turning point. This is the limitation of the current state of knowledge that we have, so we are not identifying this point as turning point. We have identified only one turning point, but when you study the theory of calculus, you will find another definition of turning point and in that, this also will be identified as a turning point. Therefore unless you study calculus you can not be precise in graphing the functions. Thank you.

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Lecture – 42
Graphs of Polynomials: Graphing and Polynomial creation

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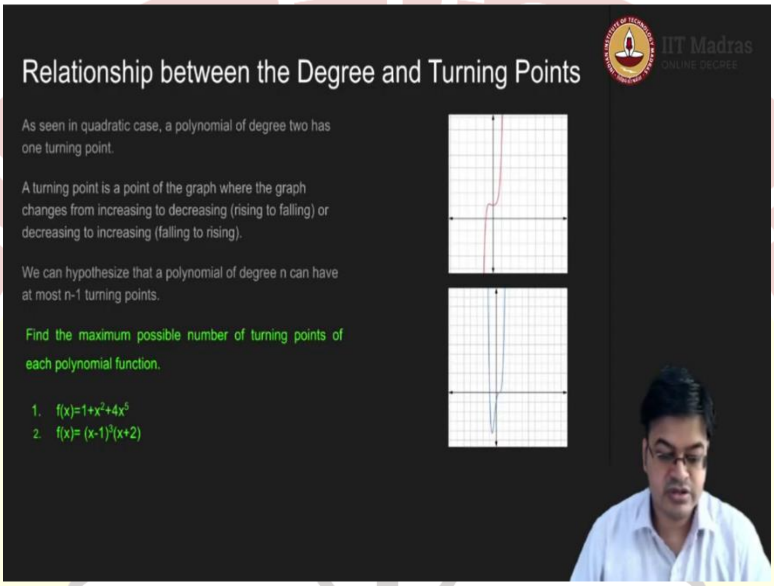

So, now I want to use all this knowledge to plot a Polynomial Function or graph a Polynomial Function.

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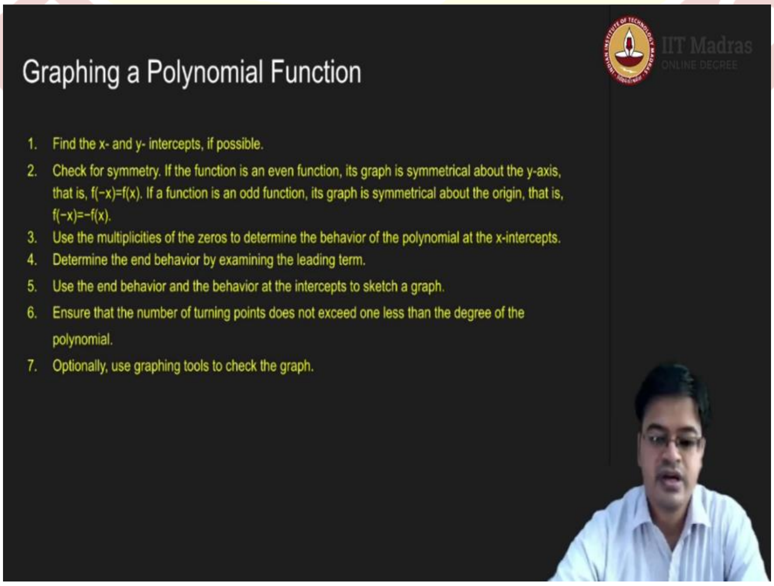

So, let us reiterate what are the things that we have seen. For graphing the polynomial function, one way is to find the tabular form and try to graph it as in a crude manner. More knowledgeable way is, you follow this algorithm that is find x intercept, y intercept if possible because it may happen that they do not have any real roots and you may not be able to get x-intercept, all the x-intercepts right.

Then for graphing it is helpful to check the symmetry that is; if $f(x)$ and $f(-x)$ are same if it is an even degree polynomial; that means, you have symmetry about y axis. If it is an odd function you can check whether they are symmetric about origin that is $f(-x) = -f(x)$. Typical case is the first symmetry is $y = x^2$, it is an even degree polynomial and it is symmetric. So, once you have drawn here for $-x$ you have to just keep the mirror image.

That is how it helps in graphing. In a similar manner a $y = x^3$ is a odd degree polynomial and $f(-x) = -f(x)$. Therefore, whatever you got about origin if you reflect about origin then you will be able to retain the same shape; you do not have to compute explicitly. This is the way this checking of symmetry helps.

Next identify the zeros; x intercepts we have already identified. So, you have identified the zeros. Then you identify their multiplicities. If you identify the multiplicities of the polynomials you know the behavior of the polynomials at x intercept. You just recollect; multiplicity the sum of the multiplicities of all zeros cannot exceed the degree of the polynomial that you have to keep in mind. After identifying the multiplicity you know the behavior at the zeros of the polynomial function.

Now, you want to know the behavior beyond zeros of the polynomial function that is; the end behavior. So, end behavior you can use the leading term and you can identify the behavior. Remember the table that we have shown for identifying the end behavior.

And finally, you use the end behavior the behavior at intercepts to sketch the graph. Turning points - the number of turning points can be identified we may not be able to locate exactly where the turning point is. For that, you need the tools of calculus to identify the exact location of a turning point.

And when you identify those when you roughly estimate the turning points; kindly ensure that the number of turning points do not exceed one less than the degree of the polynomial.

So, if the degree of the polynomial is n, the number of turning point should not exceed n -1 ok. And finally, you can use technology to sketch the graph. So, use graphing tools like Desmos or some other tools for graphing the function ok.

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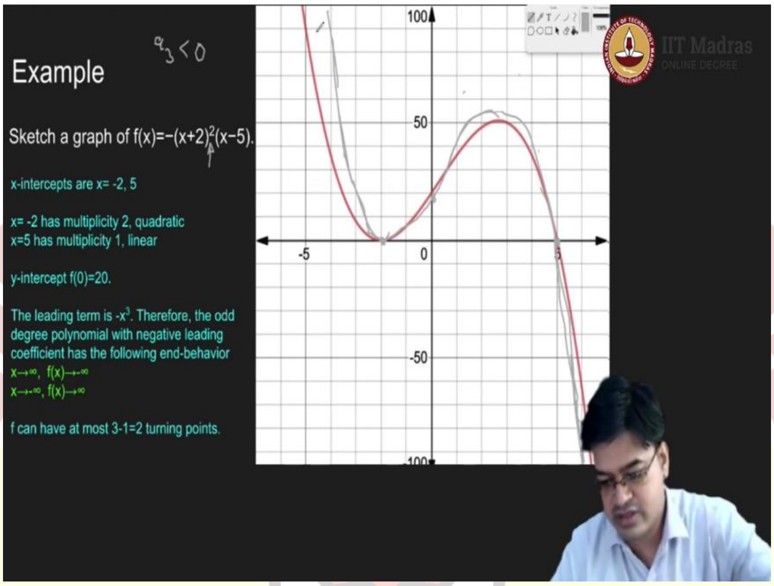

So, let us see this in action. So, here is an example I want to sketch a graph of this polynomial function; $-(x+2)^2(x-5)$. Obviously, I have figured out oh it is a $-(x+2)^2$. So, the first thing that we; so, I want to graph this function. So, the first thing that I want to find is the x intercept, because it is given in factored form it is no brainer; $x = -2$ which is this point $x = -2$ and then $(x - 5)$. So, $x = 5$ which is this point.

These two are the zeros of the polynomial functions; $x = -2$ has multiplicity 2 and it is an even degree polynomial. So, over here the behavior of function I am trying to sketch it will come from here, it will go from here. So, I know the behavior of the function is of this form, it will just pass through the axis.

And $x = -5$ I do not know the exact values, but roughly it will be more of the linear form and it will pass through the point -5, then it will come down over here and then it will pass through this point. So, up to this I am ok. Now, you look at this polynomial if you look at this polynomial, then the polynomial will be a cubic polynomial; it will have a negative term.

So, essentially $a_3 < 0$ ok. So, the end behavior of this polynomial because $a_3 < 0$ as $x \to \infty$ this function will tend to infinity. Yes, and as $x \to -\infty$ the function will naturally go up like this.

So, this is the vague understanding of the behavior. If I want to get more precise on what values this is roughly the shape of the function. If I want to get more precise on what values the function takes, I can consider the y intercept as well that is I will put $y;\ x = 0$. So, it will be 2^2 4 $2^2 4$ yes, and into -5 that will give me $-20 + 20$. So, this intercept that I have drawn is wrong. It should be somewhere here + 20. So, let me erase this and redraw the function again.

Let us take the eraser. So, it may not go this high as well. So, over here the font behavior of the function ok; so, let me again go back to the marker. And the function may cut here itself pass through this point and join this point. Yes, so, this bulge will not be there because this function is linear over here. It may be of this form. So, let me again erase this part yeah.

So, let us see. So, we have identified the end behavior ok, final check that number of turning points. The function is cubic, so it can have at most two turning points, there are only two turning points: one is here, one is here fine. So, let us see whether whatever we have said is correct or not. So, let me hide this first.

So, x intercept is -2 and 5, no problem. $x = -2$ has multiplicity 2. So, the quadratic behavior should be plotted there. Yes, $x = 5$ has multiplicity 1. So, a linear behavior is plotted here assume this is a line. So, linear behavior is a plot must be plotted here. And, then $f(y)$ intercept $f(0)$ is 20 which we corrected we were not correct in the initial stages and the leading term is $-x^3$.

So, therefore, the odd degree polynomial with negative leading coefficient has the following end behavior; as $x \to \infty$ $f(x) \to -\infty$, this is the behavior that we have plotted; $x \to \pm\infty$ $f(x) \to \infty$, this is also correct. And f can have at most $3 - 1 = 2$ turning points, this is the behavior right.

So, now, I was roughly ok in drawing the graph of a function. This is because I do not exactly know the behavior of the turning points. So, I will be roughly ok in drawing the graph of a function, but not exactly. If you want to be more precise you can actually tabulate the values around some critical points and then you can figure out. This is when the formula is given to you.

Now, the question can be asked that what if the formula is not given to you, but you have been given only a function. And from the graph you need to identify the polynomial.

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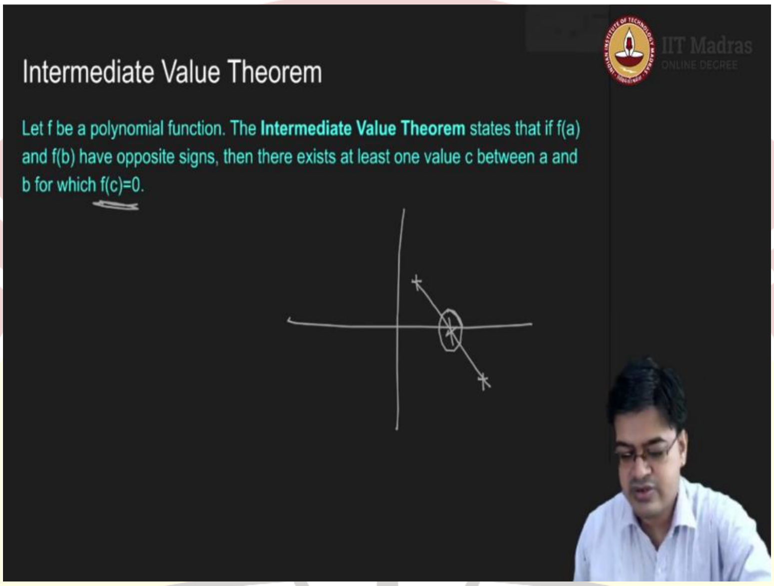

In such cases one theorem which will help you a lot, I will not use this theorem in a rigorous manner. But it will help you a lot, is intermediate value theorem because we are dealing with continuous functions. This intermediate value theorem is valid for all continuous functions.

What this theorem says is, a polynomial function is a continuous function. So, let f be a polynomial function, then the intermediate value theorem states that; if $f(a)\ and\ f(b)$ have opposite signs; that means. So, let us say $f(a) > 0$, and $f(b) < 0$ and $a > b$, then there exist at least one c between a and b such that $f(c) = 0$; that is essentially the meaning.

For example, I have this coordinate plane my value of $f(a)$ is here, and $f(b)$ is here, and the function that is given to me is a continuous function right. So, finally, it has to pass through the x axis to reach the value here right. So, in such cases we will say that; this is the 0 of the polynomial that is what we are calling as c, $f(c)$.

So, you using this you when you are actually having trouble in finding the zeros of the function, you can actually evaluate two values any two values of opposite signs. And if you evaluate any two values of opposite signs, then you know that there is some root some 0 in between that will improve that you will gain a confidence by doing these things.

So, this is an important theorem in mathematics, intermediate value theorem. You can use this to find the roots of the polynomial when you are having difficulty in identifying the roots of the polynomial. So, you simply put $f(a)$ and $f(b)$ if they have opposite sign, then there is at least one root in between; that is the meaning. You can use this theorem to your advantage.

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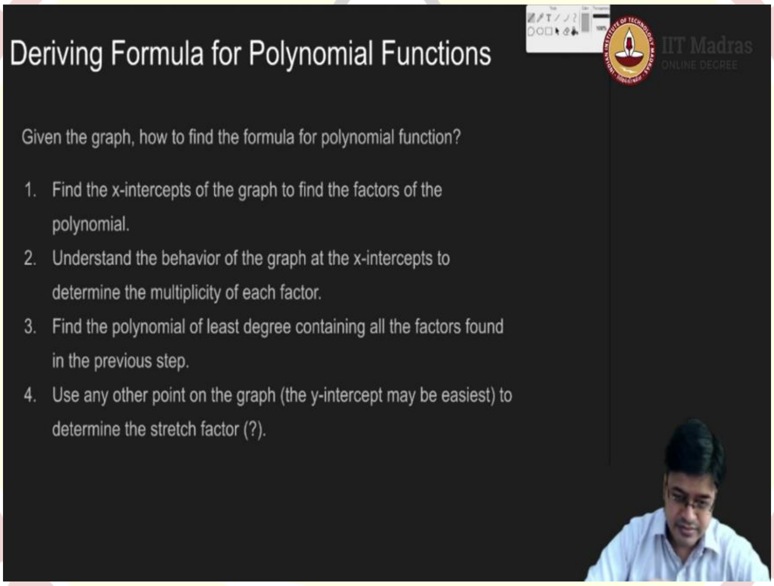

So, using this theorem we can actually derive a formula for polynomial function. You use this theorem to identify the zeros; rest of the methodology is similar. So, how to derive a formula for polynomial functions? So, given a graph of a polynomial, how to find with in coordinate axis? You have all the numbers attached to it, then the question can be asked as to how to find the polynomial function the algebraic expression of a polynomial function?

So, in that case our modus operandi is similar to what we have done. Find the x-intercepts from the graph. Find the factors of the polynomial, this we already know. Understand the behavior of x intercepts around x intercepts to get more understanding of the x intercepts that is zeros of the polynomial about their multiplicities.

So, you will find multiplicity of each factor. Once you have gained understanding identify the end behavior that also you have to do. Next, after doing that you find the least degree polynomial containing these factors. What are the factors? Those are x intercepts that you have figured out. You have also seen the end behavior, so the least degree polynomial which will give you that particular function behavior.

Once the least degree polynomial is figured out you use any point on the graph that is why the coordinate axis is important, the numbers are important. You use any point on the graph, in particular y intercept is the easiest and in that case you can determine the stretch factor.

The stretch factor over here is the unknown a that I have told you while figuring out the factors in one of the examples. So, that is the stretch factor. It will be more clear when we will solve the examples ok. So, this is our recipe for attacking the problem of deriving the formula given a graph.

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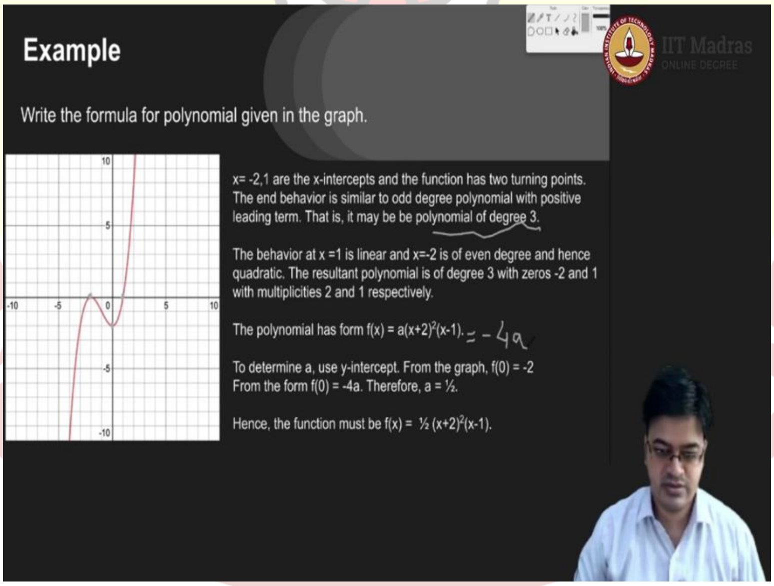

So, let us try to apply this recipe to one example. So, write the formula of a polynomial given in the graph, the graph is here ok. So, I will go around and try to find the x intercepts of this graph. So, one x intercept is here which I think is $x = 1$ and other x intercept is -2. So, -2 and 1 are the x intercepts; y intercept over here is-2, $0-2$ is the y intercept. So, we have identified x and y intercepts.

The graph actually seem to have two turning points. So, the least degree if it has two turning points the least degree polynomial will be because $n - 1 = 2$. So, the least degree polynomial should be cube degree 3 polynomial right ok. And since it is crossing over this end from end behavior also you will have some understanding that it is yeah, it should be an odd degree polynomial. So, therefore, the polynomial may be of degree 3, correct.

It should be an odd degree polynomial; it has only two turning points. So, the least degree of the polynomial is 3. Now what you will do next? Next I want to identify the multiplicities, that is $x = 1$ it the function more or less seems to be linear and at $x = -2$, the function more or less seems to be quadratic.

So, it is very easy in this case because, $x = -2$ is a even degree behavior, x is equal to because it is bouncing off. So, it is a even degree behavior and $x = 1$ is linear behavior and the polynomial is of degree 3 or more, but odd degree. So, the first instance is you guess the function to be of the form $(x + 2)^2(x - 1)$. So, now, I have not yet used the information that the intercept the y intercept is happening at -2, correct.

So, that information I have to use now because that is the function value that I have. These are the based-on factors we are basically equating to 0 right. So, the a may be missed out. So, where the non-zero value comes you should be able to figure out. You can you are free to choose any value, but for me it is better to choose y intercept.

So, y intercept is -2, we have already seen that, but if you put what is y intercept? It is $f\ (0)$. So, if you put this in the function form the value of 0 in the function form over here you will get actually this is to be equal to $-4a$. So, if you are getting this to be equal to $-4a$, then $-4a$ must be equal to -2; that means, $a = \frac{1}{2}$ great.

So, if $a = \frac{1}{2}$you substitute this value into the function. So, $f(x) = \frac{1}{2}(x + 2)^2(x - 1)$, fantastic. So, you have you got an algebraic expression. Now, to match this algebraic expression, you use the technology that is graphing tool to plot the function and you can verify the result for yourself that yes, this is the function that we have actually plotted ok.

So, this is the complete understanding of two-step mission that is; given an algebraic expression how to graph the polynomial function. Given the graph of a polynomial function, how to write an algebraic expression of a polynomial function. This ends our topic on polynomial functions.

Thank you.

Mathematics for Data Science 1
Week 07 - Tutorial 01

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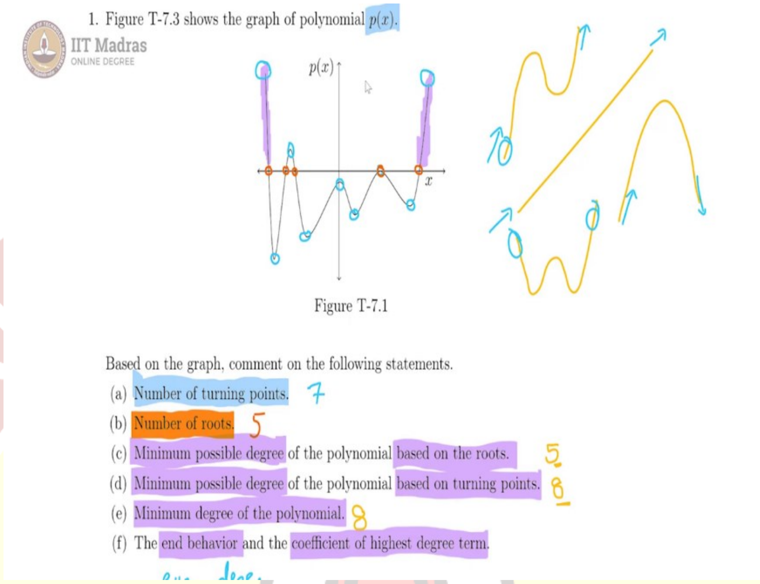

Hello mathematics students. In this tutorial we are going to look at questions based on graphs of polynomials. So, in this question there is this polynomial $p(x)$ whose graph is given here and we are supposed to comment on the following statements, the number of turning points, so that is easy so there is a turn here, 1, 2, 3, 4, 5, 6 and 7. So, there are 7 turning points. And then we are asked the number of roots, so roots would be where the polynomial touches or cuts the x axis so that is 1, 2, 3 and 4 and 5, so there are 5 distinct roots.

Now, what is the minimum possible degree? Minimum possible degree of this polynomial based on the number of roots. So, the minimum possible degree would be the same as the number of roots so if there are n roots to a polynomial then it should have a degree of at least n, so 5 is the minimum possible degree of this polynomial based on the roots. But now they are asking what is the minimum possible degree based on the turning points.

So, here we see this thing a straight line has no turning points, a quadratic equation has 1 turning point and a cubic would have 2 turning points at most likewise a quartic that is a fourth degree polynomial would have 3 turning points at most. So, if you have n turning points, then the minimum possible degree of the polynomial would be $n + 1$. So, here that is 8.

Now, what would be the minimum degree of the polynomial given all the information we have? Then you know that it has to be at least 8 the five which is on the basis of the roots is a lesser number than it and we know already that it has to be at least 8, so the minimum degree of the polynomial should be the greater of these two which is 8 because 6 and 7 and 5 are not allowed on the basis of turning points.

And then we are being asked what is the end behavior and the coefficient of the highest degree term. So, the end behavior shows that the polynomial is coming from ∞ and going to ∞ which means the degree of the polynomial is definitely even. So, we can say that it is an even degree polynomial.

So, as you can see we have just drawn these basic raw curves for the linear and quadratic and cubic and quartic polynomial. So, linear which is an odd degree polynomial it comes from -∞ and it goes to +∞ whereas quadratic a parabola here it is coming from -∞ and it is going to -∞ .

So, when it is even degree you see that the ends of the curves are in the same directions. Similarly, for quartic here this is coming from ∞ and going to ∞, whereas for a cubic this is coming from -∞ and going to +∞. So, here this is coming from ∞ and going to infinity.

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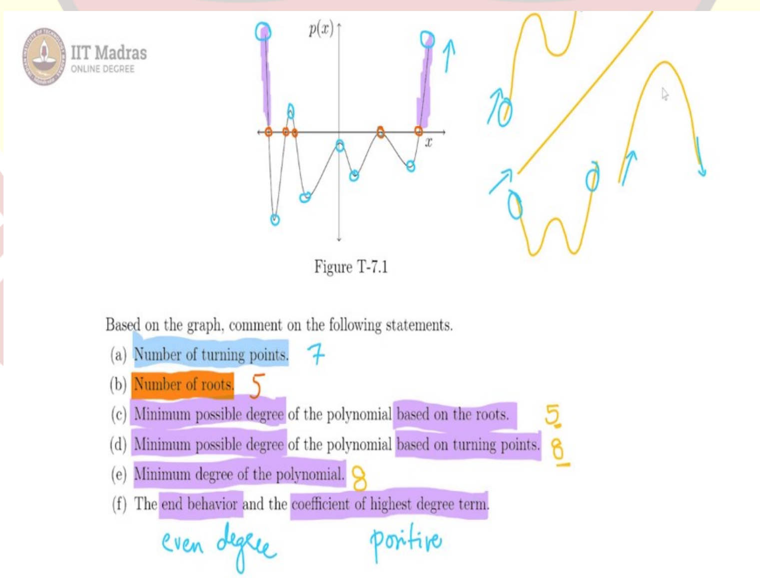

Therefore, this is a even degree polynomial and the coefficient of the highest degree term. So, the coefficient of the highest degree term determines whether the behavior of the polynomial as x increases whether it is going to $+\infty$ or $-\infty$, if the coefficient of the highest degree term is positive, it goes to $+\infty$. So, if this is going to $+\infty$ so this has to be positive coefficient for the highest degree term.

Mathematics for Data Science 1
Week 07 Tutorial 02

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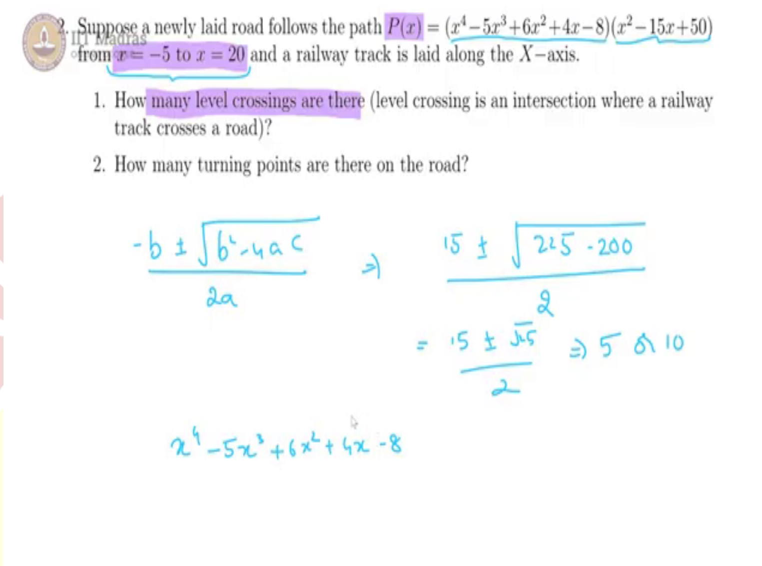

Now second question there is newly laid road which follows the path of this polynomial about some coordinate system, from $x = -5$ to $x = 20$. And railway track is laid along the x axis. So how many level crossings are there? So what we are interested in is; how many times does the x axis cut this polynomial? And for that we have to find the roots of this polynomial because roots give when the polynomial is touching or cutting the x axis.

Now this is of quartic forth degree polynomial multiplied with the quadratic polynomial, so the degree is 6, so at best we could have 6 roots but let us find out what these roots are. The easy way to start is to first find the roots of the quadratic, so that would be using $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. We get $\frac{-15\pm\sqrt{225-200}}{2}$, which is $\frac{-15\pm\sqrt{25}}{2}$ that is essentially 5 or 10.

So you get $\frac{10}{2}$ or $\frac{20}{2}$ so 5 or 10 those are the two roots and they are both within the given range. Anyway now we look at the other part, the quartic part. So here we have $x^4 - 5x^3 + 6x^2 + 4x - 8$. In this situations what is typically suggested is that we do a little bit of trial an error, we try out with the basic small integers and we see if we can find any roots at all.

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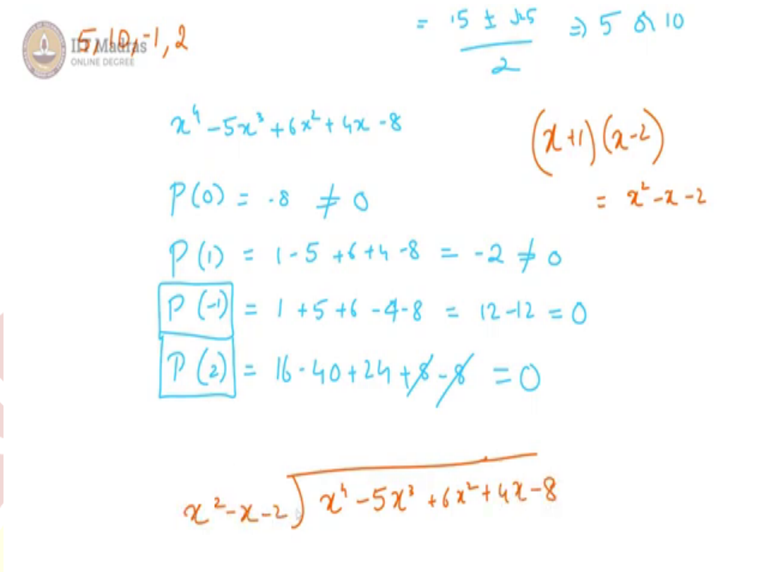

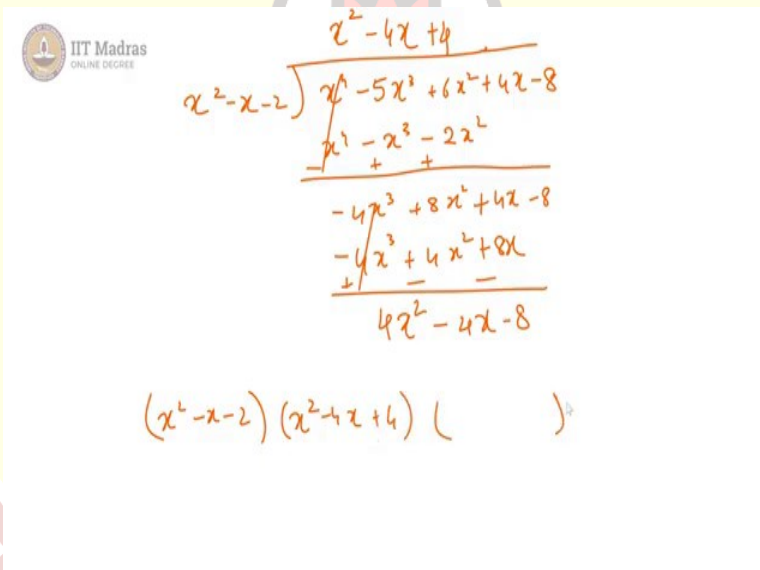

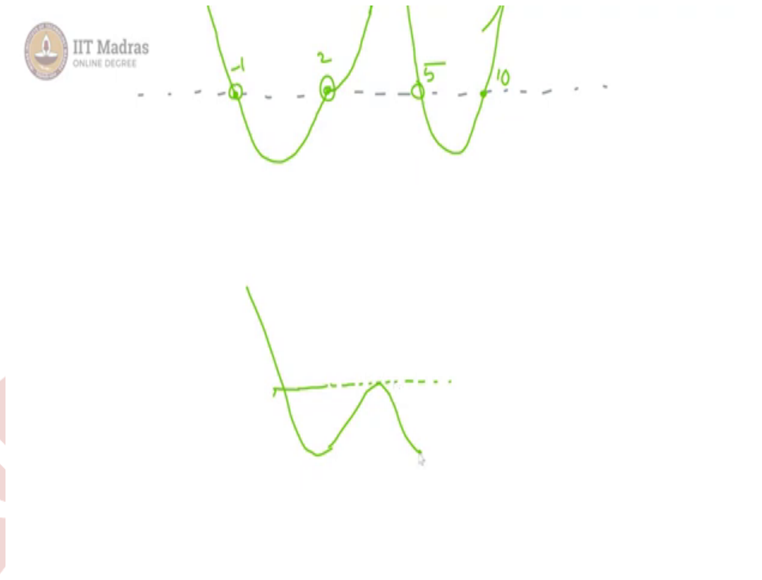

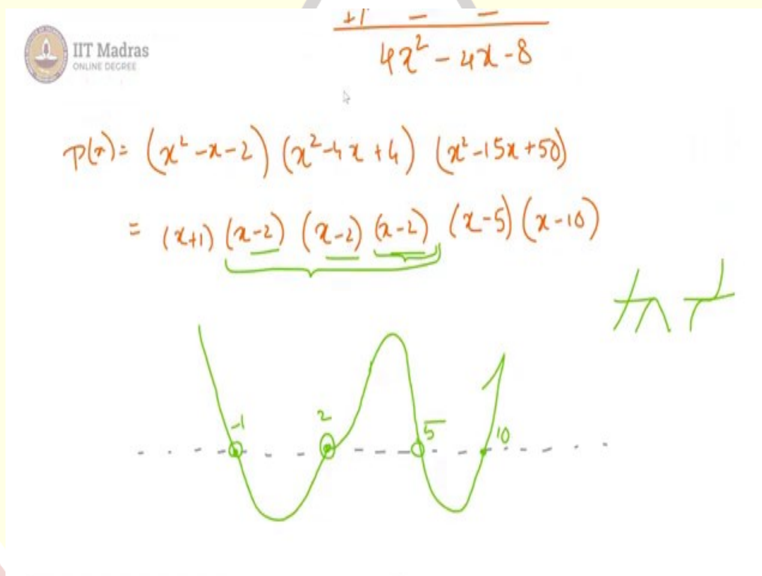

So let us start with $p(0)$, $p(0)$ is -8 which is clearly $\neq 0$, so 0 is not a root then we have $p\ (1)$ which is $1 - 5 + 6 + 4 - 8 = -2$ which is again $\neq 0$, so not a root. Then we try $p(-1)$ and we get $1 + 5 + 6 - 4 - 8$, so this is equal to $12 - 12 = 0$. So yes $p(-1)$ gives you 0 which means we have another root that is - 1.

So let us note down our roots that we have found here, roots we have found so far are 5, 10 and - 1. Now going back to our trial and error let us try $p\ (2)$ and $p\ (2)$ gives us $16 - 8 \times 5 = 40 + 6 \times 4 = 24 + 8 - 8$. So we get $16 + 24 = 40, 40 - 40 = 0$, so this is 0. So we have another root that we have found. So we now have two roots for our quartic and those two roots give us another quadratic which is $(x + 1)(x - 2)$ that is $x^2 - x - 2$.

So if we divide our quartic with quadratic we will get the other quadratic within it. So here we have $x^4 - 5x^3 + 6x^2 + 4x - 8$ and we divide it with $x^2 - x - 2$ so here go x^2 so $x^4 - x^3 + (m-2)x^2$ -,+ and + cancel this of you get $-4x^3 + 8x^2 + 4x - 8$.

And then we do $-4x$ times this, $-4x^3 + 4x^2 + 8x$, so + - and - cancel this and here we have $4x^2 - 4x - 8$. And that is just 4 times this, so + 4. So our quartic, so is basically x$(x^2 - x - 2)(x^2 - 4x + 4)$ and this gives the quartic and additionally we have to also multiply for our p of x we have to multiply the other quadratic which is $x^2 - 15x + 50$, this one.

So this is p of x totally, an if we further separate it out into all its roots we get this one as we know is x + 1 into x - 2 and this is if you notice x - 2 to the whole square, so $(x-2)(x-2)$ and then here this we have found the roots already which is $x - 5$ into what was the other root; the other root was 10, $x - 10$.

So these are our roots and the coefficient of x power 6 will be positive clearly. So therefore this is an even degree polynomial and thus if we have to sketch the graph it look something like this, it comes from infinity and what is the least lowest root here, the lowest root is - 1. So at - 1 if we draw this as the x axis at - 1 you have one root it crosses the x axis and then it goes around and it comes to 2.

But 2 is a triple root, so what happens with a root if it is a single root it crosses the x axis but if it is a double root it will just touch the x axis and come back but since it is a triple root it actually crosses the x axis. So here we do have a crossing and then afterwards at 5 and 10 we will have, so this will be for two this will be for - 1, this will be for 5 and this will be for 10. This is just a rough plotting of the graph.

The question was how many times does it intersect the x axis, so we have to draw this basic sketch and we find that the intersection are 4. If x - 2 was not a triple root, if it were a double root or a quadruple root like if it is there are 2 times or 4 times then the graph would be very different. It would be x - 1 would still be the same but at 2 you would not actually see a intersection, you will just see a touching. It would not be a cut.

So therefore we have to check how many times the $\sqrt{2}$ occurs. Since it is an odd number of times we can say it is actually cutting x axis and that gives us a number of level crossings is 4. And how many turning points are there? Now we can look at our graph and quickly tell; 1, 2 and 3; 3 turning points.

Mathematics for Data Science 1
Week 07 - Tutorial 03

In this question Saraswati bought an 8 gram gold chain for rupees 40000, so we can presume that 1 gram is 5000 rupees on first November. And after 10 months that is August 2021, she sold the chain and bought a new 10 gram gold chain by paying an additional 10000 rupees. Suppose, the rate of the gold per gram is denoted by $G(t)$ and it is a function of time $G(t)$ is given to be this cubic polynomial here and we are taking t is to be 0 at the time when Saraswati bought her first gold chain. So, t is a number of months since her buying her first gold chain.

Now, what is the when $G(t)$ is a polynomial of the rate for both used and new good. So, all gold has the same rate as what we are considering and what is the rate of gold per gram when she sold her first chain. So, after 10 months at $t = 10$ is what we are really looking for. So, that means we are looking for $G(10)$ and that gives us $0.07 \times 1000 - 1.4 \times 100 + 7 \times 10 + 5$ and this is $70 - 140 + 70 + 5$. So, that is actually 5. So, the rate is back to 5000 per gram. So, it is again rupees 5000 per gram.

Now, if she had sold the first gold chain after 6 months how much extra would she have paid for buying the 10 grams gold chain? So, after 6 months we have to find the price, the rate, so that would be $G(6)$ and that is 0.07×6^3 is $216 - 1.4 \times 36 + 7 \times 6 + 5$. And then we get this is $15.12 - 1.4 \times 36 = 50.4 + 42 + 5 = 42 + 5 = 47 + 15.12 = 62.12$ that is $62.12 - 50.4 = 11.72$ that would give us then the rate is 11720 rupee per gram. And Saraswati is selling 8 grams at this price that would mean she basically has to pay for the additional 2 gram and that would be 2×11720 which is equal to rupees 23440, this is how much she pays extra for her 10 gram gold chain.

Mathematics for Data Science 1
Week 07- Tutorial 04

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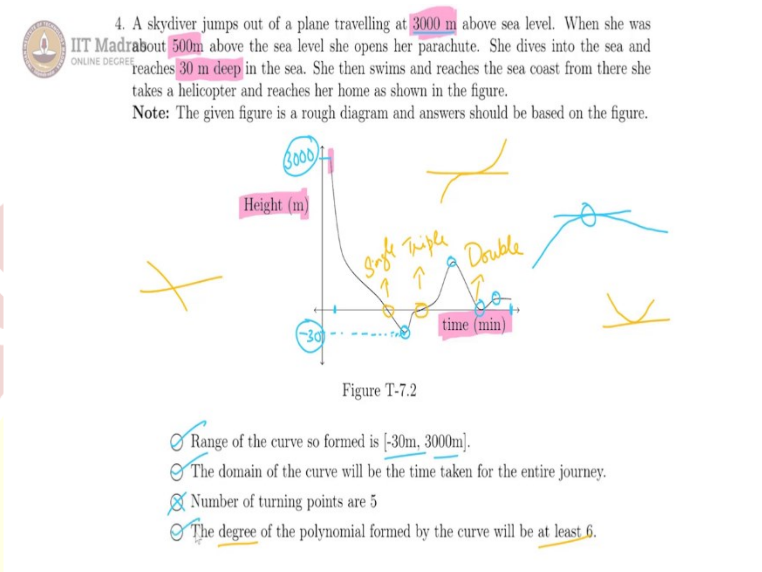

A skydiver jumps out of a plane travelling at 3000 meter above sea level. And when she was about 500 meter above the sea level, she opens a parachute. And then she dives into the sea and reaches 30-meter-deep into the sea. Then she swims and reaches a sea coast from there she takes a helicopter and reaches her home as shown in the figure.

So this figure is between the height and time there is no x coordinate in this. This is only about the y coordinate taking sea level to be 0 and the time. So initially she is way above sea level, so this point is going to be our 3000 because as where she is jumping from and then she is dropping quite quickly and then she slowly dropping after she opens a parachute. And she reaches under the sea so here it is negative till she goes to the point where it is - 30 meter below sea level.

And from there she swims out and she takes a helicopter and she goes. So we are supposed to see which of these option are correct and the range of the curve so formed is - 30 to 3000 which is true - 30 to 3000 is her total y coordinate range. And the domain of the curve will be the time taken for the entire journey that is true so your curve starts here when she jumps till the point she reaches home.

So this is correct this is also correct. Number of turning points are 5 so I see only 1, and 2, 3 turning points. So this is not maybe this is a turning point I am not able to say because it appears to go a little like this and then bend down. So probably this is a turning point. But either way there are not 5 they are less than 5 so this is wrong. And then the degree of the polynomial formed by the curve will be at least 6.

Now we have to look at the roots here. So let us take this root this is a single root it just cuts the $x$axis like this, whereas this is a more, if the x axis like this, it is kind of touching it this way and that only happens if your root is a triple root at least. So it cannot happen for single root and it does not happen for any even powered root because for root which occurs even number of times you would not cut the x axis.

So this has to be at least a triple root. So this is a single root the first one is the single root this one we are assuming it is a triple root because they are asking for minimum degree at least so we are looking for what it what the number of roots is in the minimum. And here there is a root which occurs an even number of times because it is touching the x axis and turning around it is not actually crossing the x axis.

So this is at least a double root so we have one +3 +2. So we have at least 6 roots therefore the degree also has to be at least 6. So this is also correct.

Mathematics for Data Science 1
Week 07- Tutorial 05

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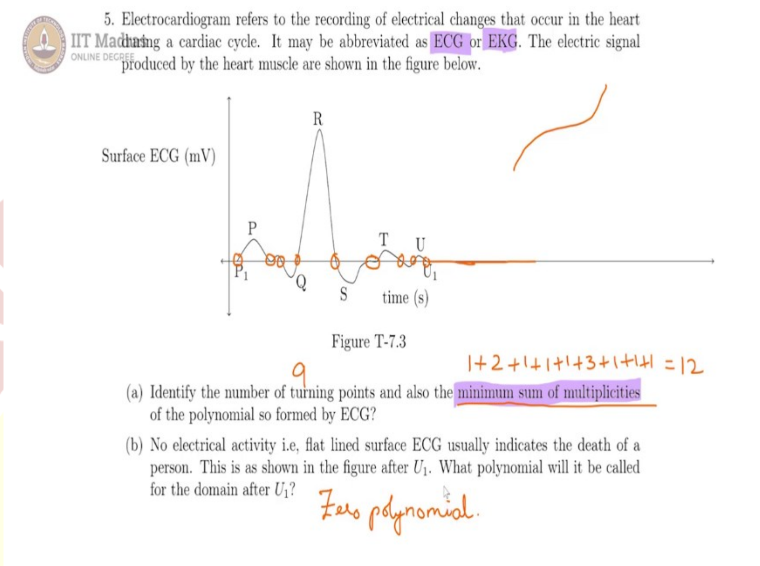

In this question, we are looking at an Electrocardiogram which is often called ECG or EKG. This is a recording of electrical changes that occur in your heart during a cardiac cycle. So here we have some ECG shown to us as a polynomial. And they are asking identify the number of turning points. Ok? So that will be 1, this is 2, this is 3, 4, 5, 6, and here this is not a turning point, it is flattening out like this and rising. Therefore it is not a turning point. We already have 1,2,3,4,5,6 so this is 7,8,9. That means we have 9 turning points. And then they are asking for the minimum sum of multiplicities for that we look at the roots and so this one is directly cutting through this root is directly cutting through the axis.

So the multiplicity of this is 1 and here this it is touching and coming back so it has to have an even multiplicity. So the minimum is 2 and then here again this and this are both 1 each $+1$, this 1 also should be 1. And here we see this flat lined situation which occurs when you have an odd multiplicity but not 1. So the minimum there would be 3 and then this is a 1 this is a 1 and this also has to be 1.

So plus $1 + 1 + 1$ which gives us all put together 3, 4, 5, 6, 9, 10, 11, 12, 12. So the minimum sum of multiplicities is 12. No electric activity that is flat lined surface ECG usually indicates the death of a person. This is as shown in the figure after U_1 so after U_1 we presume that it is a, it is basically along the x axis what polynomial will it be called for the domain after U_1. Clearly a 0 polynomial. It is simply y is equal to a constant. So it has no degree and it is a 0 polynomial.

Mathematics for Data Science 1
Week 07- Tutorial 06

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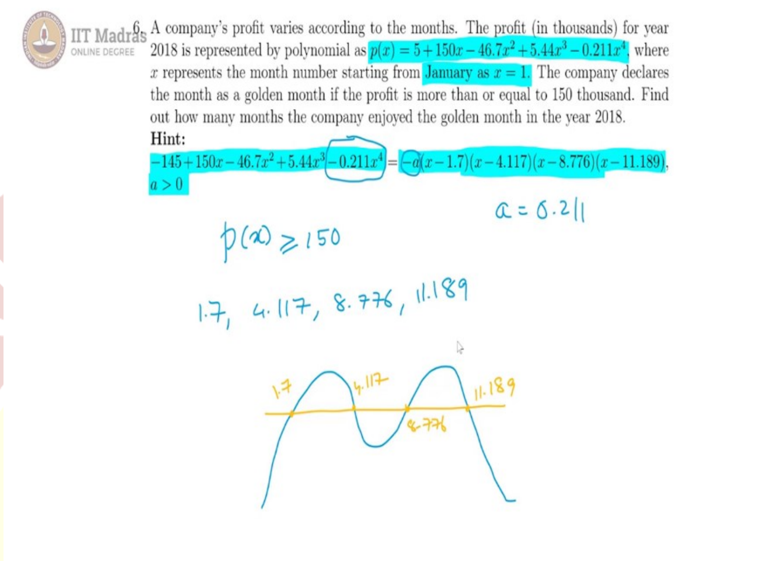

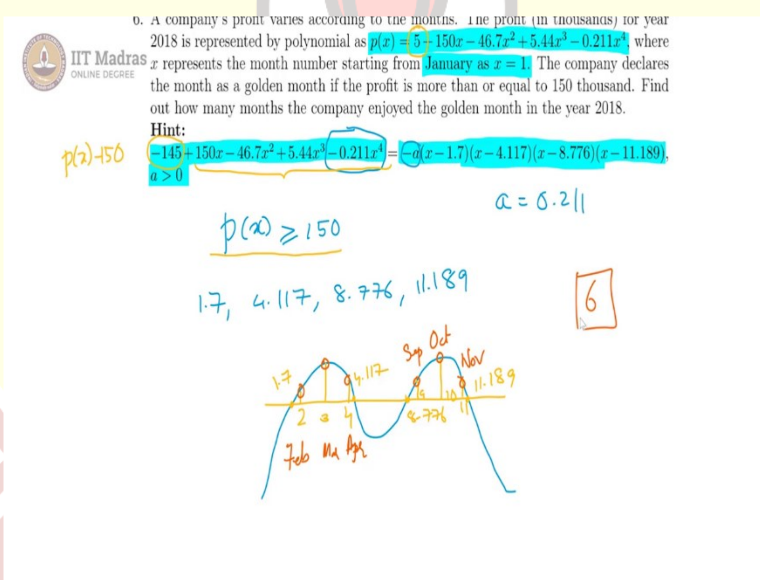

So here we have a company's profit varies according to the months. So they are going to have a profit versus time along in the profit in thousands for year 2018 is represented by this polynomial $p(x)$ is equal to this quartic polynomial fourth power polynomial where x represents the month number starting from January as $x = 1$. So January is x=1. The company declares the month as a golden month if the profit is ≥150 thousand.

So golden month is when $p(x) \geq 150$. Find out how many months the company enjoyed the golden month in the year 2018. So how many times does this happen? Alright, and there is a hint also given to us where this cortex is apparently equal to. So they have basically given us the roots of the polynomial and a does not matter, $a > 0$.

And so we can also tell what a is a has to be minus 0.211 because that is the coefficient of x^4 here and minus a will be the coefficient of x^4 and the RHS therefore a has to be equal to 0.211, $-a = -0.211$. Therefore, $a = 0.211$. Anyway so now given that we already have the roots, the roots are essentially 1.7, 4.117, 8.776, and 11.189. So these are the roots and the coefficient of the highest power of x is negative.

It also even so our curve is going to be something like this. Where the x axis cutting it here and that would mean this point is 1.7, this is 4.117, this is 8.776, and this is 11.189. So all of this is given to us but what we are supposed to find is to related to $p(x) \geq 150$. So given that $p(x)$ is all this stuff plus 5 and here we have the same terms of $x - 145$, we can see that this particular polynomial given to us is simply $p(x) - 150$.

So whenever this polynomial that we have drawn here is greater than it. We have a golden month so that would be month 2, 3, and even 4, then 5, 6, 7, and 8 do not come in and here we would have month 9, and 10, and also 11. So as you can see this, this, this 3 and again 3 here. So it is overall 6 months during which we can also tell which ones is Feb, March, and April, and here this is September, October, and November.

So 6 months are the golden months in that year for this company.

Mathematics for Data Science -1
Week 07 - Tutorial 07

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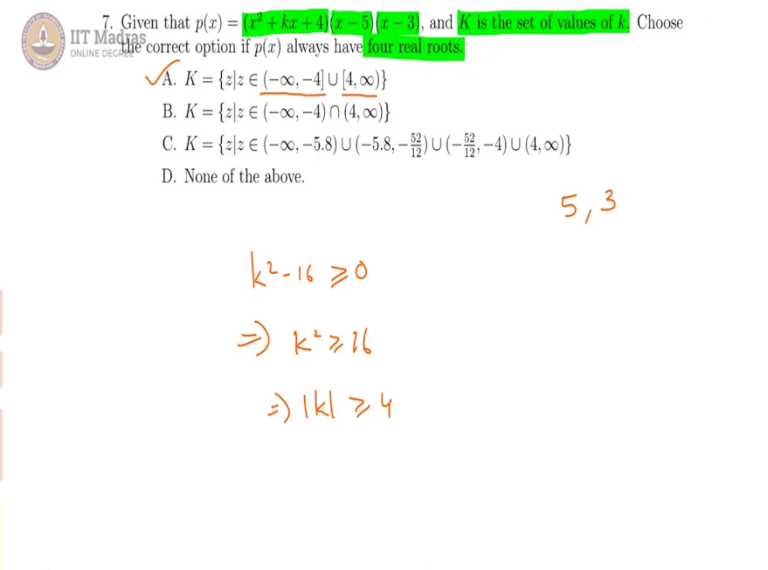

In this question, we are given a polynomial $p(x)$ which is a product of a quadratic with a monomial and another monomial. And the quadratic has some variable k in it, capital K is the set of values of this small k, choose a correct option if $p(x)$ always has 4 real roots but they need not be distinct and we already know that 5 and 3 are roots because of these two monomials. So, what is remaining is that our quadratic equation also should have roots.

And for that the discriminant which is $k^2 - 16$ should be $\geq$0. That would indicate $k^2 \geq 16$, thus k, the magnitude of $k \geq 4$. If $k \geq 4$ you get a repeated root you get the same root twice, so what corresponds which option corresponds to this is a because you go from $-\infty$ to -4 and then 4 to $+\infty$ and their union and 4 and $-$ 4 are with closed intervals therefore, they are included.

Mathematics for Data Science -1
Weel 07-Tutorial 08

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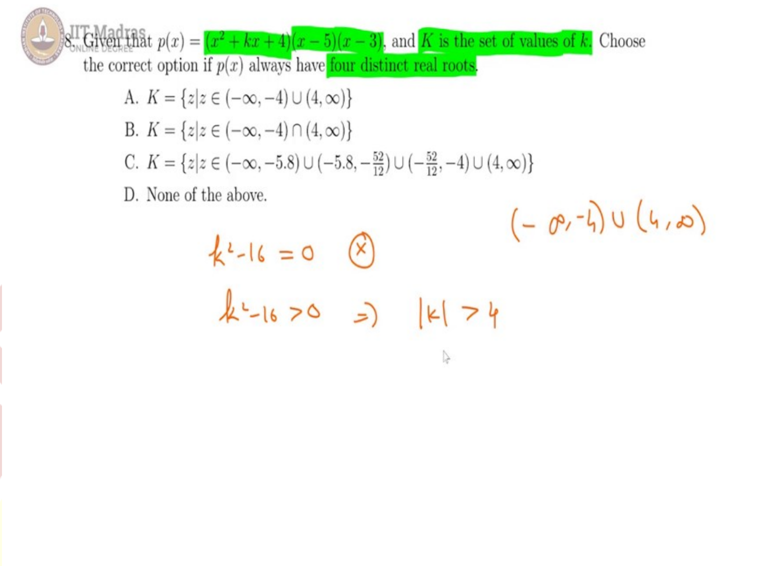

Question 8 is very closely related to question 7, we again have the same, quadratic into monomial into monomial is the same polynomial and again the same set is given to us. Now we have to see the correct option for $p(x)$ to have four distinct real roots, that means a root should not be equal to each other and that is a catch.

So, we have already seen that $k^2 - 16$ the discriminant being equal to 0 will give us equal roots. So, this case is not done, this time the discriminant has to be greater than 0, so that would indicate $|k| > 4$, so you will have $(-\infty, -4) \cup (4, \infty)$ for the quadratic condition. The other condition here is that the roots for the quadratic should not be equal to 5 or 3.

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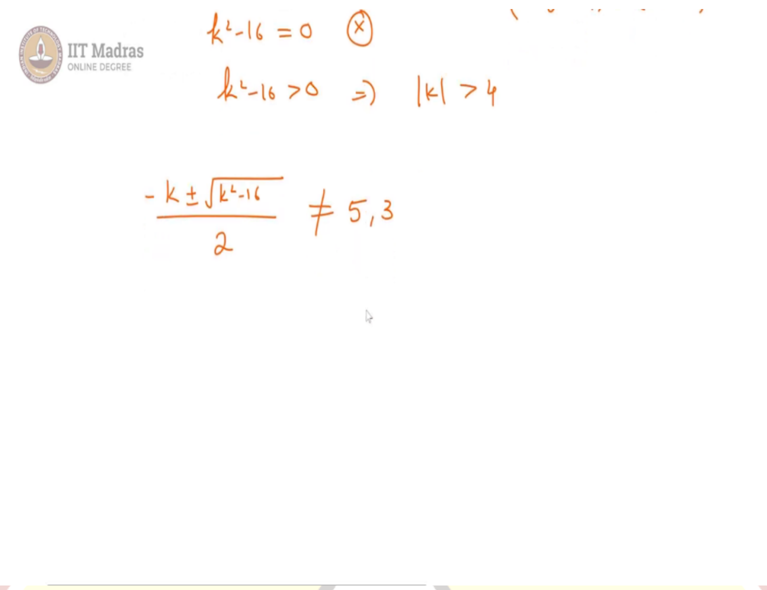

So, these routes which are $\frac{-k\pm\sqrt{k^2-16}}{2}$ this should not be equal to 5 or 3.

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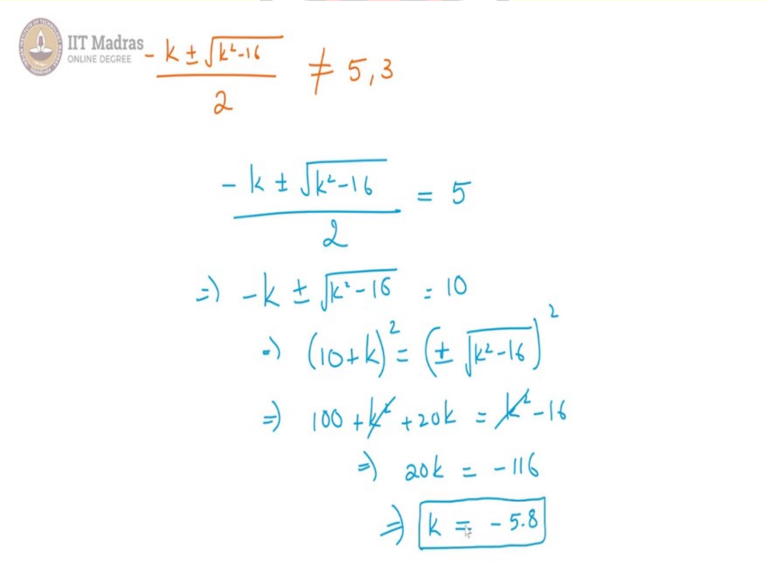

So, for finding that condition let us start with $\frac{-k\pm\sqrt{k^2-16}}{2} = 5$ let us start with this and you get $-k \pm \sqrt{k^2-16} = 10$ and that would mean $10 + k = \pm\sqrt{k^2-16}1$.

Now if you square this we do not need to worry about the plus or minus, so let us square it and we will reach $100 + k^2 - 16$, k^2 and k^2 goes away. So we get $20k = -116$ and that would imply $k = -5.8$. So when $k = -5.8$ the root of the quadratic part will be equal to 5.

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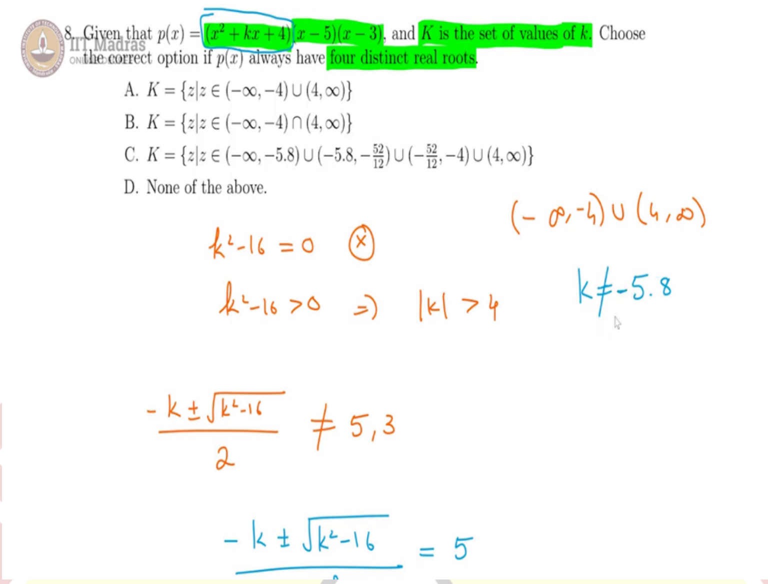

So the root of this part will be equal to 5 and that is not allowed, so we should somehow eliminate 5.8 from this set, -5.8 from this set.

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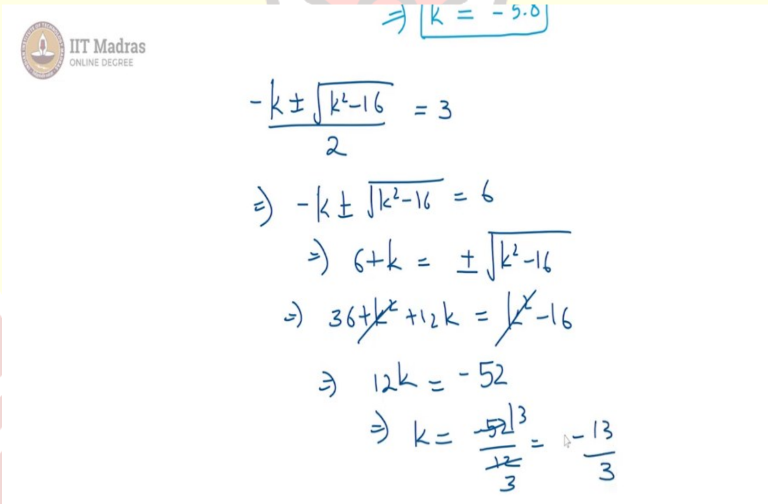

And further let us check for three case where $\frac{-k \pm \sqrt{k^2-16}}{2} \neq 3$ so we first check when is it equal to 3 and that gives us $-k \pm \sqrt{k^2 - 16} = 6$ that gives us $\pm\sqrt{k^2 - 16} = 6 + k$ and that further gives us $36 + k^2 + 12k = k^2 - 16$. So k^2 and k^2 canceled off and that gives us $12k = -52$ this implies $k = \frac{-52}{12}$ which is essentially for 3 and 4 13, so $\frac{-13}{3}$.

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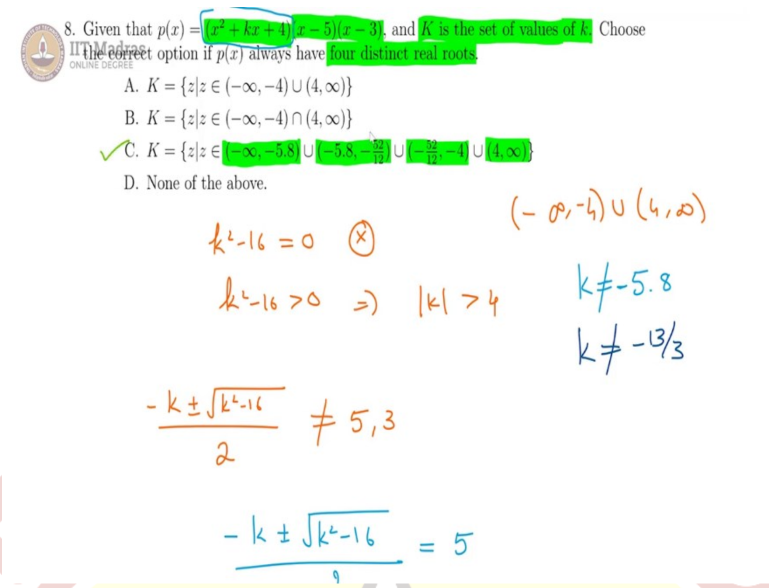

So, k should not be also be equal to $\frac{-13}{3}$, so which of these options does that is here we see option c is goes from $(-\infty, -5.8) \cup (-5.8, \frac{-13}{3})$ and keeping it open interval we are basically exploding -5.8 and similarly the open interval on the $\frac{-13}{3}$ side on in this and this is essentially excluding $\frac{-13}{3}$ and lastly we are doing the union with $4, \infty$. So, this is correct, we are excluding all values from -4 and 4 and also excluding -5.8 and also excluding $\frac{-13}{3}$.

Mathematics for Data Science -1
Week 07-Tutorial 09

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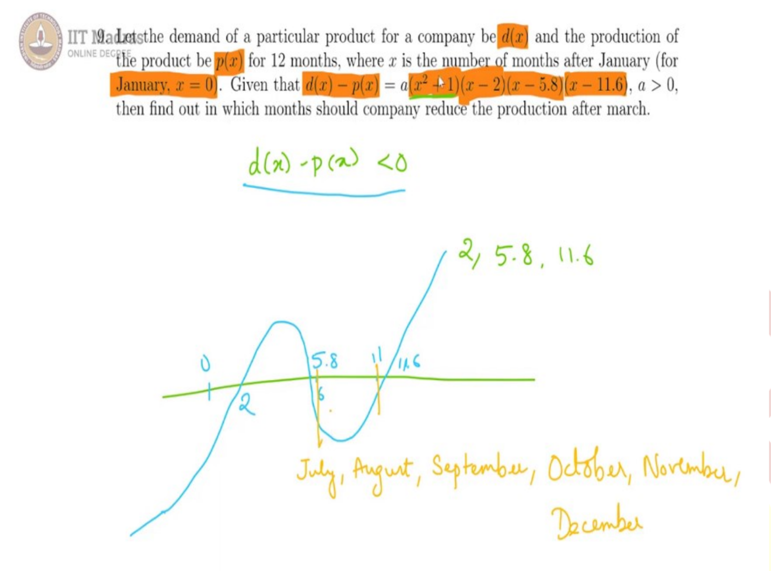

For our last question, there is a company and they are making a particular product and the demand of the particular product is us $d(x)$, the production of the same product is $p(x)$ for 12 months, where x is the number of months after January and for January we are taking x is equal to 0. And then they have given us $d(x) - p(x)$, as a polynomial and this is essentially a quadratic multiplied by a monomial by another monomial and another monomial.

So, we have a fifth degree polynomial here $d(x) - p(x)$, then find out which months should company reduce production after March. So, reduced production would mean $p(x)$ is greater, that means $d(x) - p(x) < 0$ and we are interested in those situations where this curve is less than 0 and so we just try to graph this curve and $x^2 + 1$ has no real roots.

So, the only roots here are 2 and 5.8 and 11.6, so our curve is going to look something like and since a is positive, then since the coefficient of the highest power is positive we will have this how polynomial go to ∞ as x goes to ∞ and then we have a situation like this because this odd polynomial it goes to -∞ here and there are only three roots, there is 2 and 5.8 and 11.6.

So, we are looking for when is it negative and that we have to do only in the 0 to 12 range because we are only looking for months of 1 year, actually even 12 is not correct because we are starting from 0 we are only going till 11. So, if the root is 11.6 then 11 is somewhere here and this is a 5.8, 6 is somewhere here and so these are the months where you get negative. Now six is not June it is actually July because Jan is taken to be 0.

So, we have July and then August is 7, then September would be 8 and October is 9, November is 10 and December is 11 and all these months you have the function being, the polynomial being lesser than 0. So, these are the months where they should reduce production and they are mentioning after March, if we looked at it before March, then you would have also have to consider January and 0 and 1 which is January and February but since they are specifically saying after March this is the set of months where the company should reduce production, thank you.

Mathematics for Data Science 1
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Lecture – 8.1
One-to-One Function: Definition & Tests

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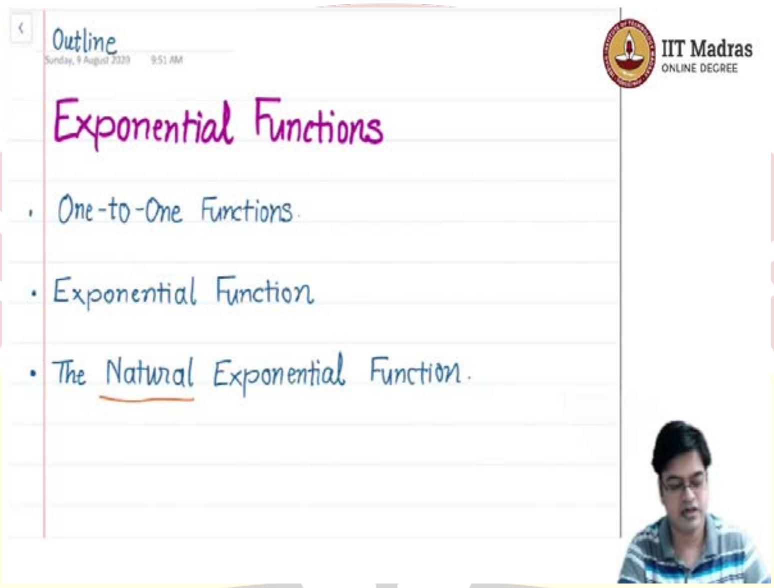

Hello students. Today, we are going to start a new unit called exponential functions and logarithmic functions. In that our goal today is to understand exponential functions. For studying exponential functions, a concept of a One-to-One Function is very much relevant.

So, we will first study that concept. Then, we will go to a definition of exponential function and properties of exponential functions as we studied for polynomial and straight line case. Then, we will define one interesting function which is called the natural exponential function, and I will justify why it is called natural and what are the properties that it shares with the mathematics world.

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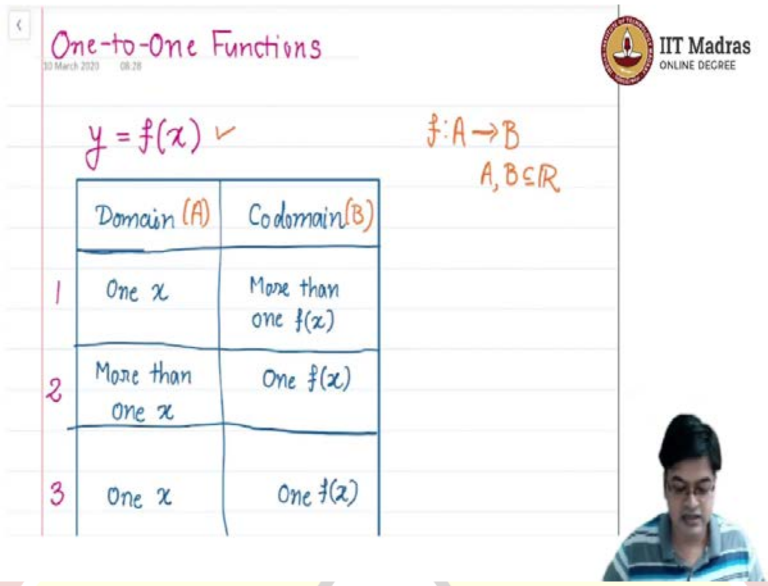

So, let us start and I think it is time to start and let us start. So, first let us go to a concept of one-to-one function. In order to define the concept of one-to-one function, let us quickly recall, what is a concept of a function? So, whenever I talk about function, I will talk about $y = f(x)$. This f when I say I need some domain let us say A and I need some co-domain B.

In general function can be defined on any two sets, but for us let us consider to simplify our understanding. So, that we will understand in terms of coordinate plane A and B to be subsets of real line, ok. So, now my domain according to this definition is A, and co-domain is B. So, now, what is the function? Function is a relation between one set to the other set. So, it is a mapping that assigns values from one set to the values of other set.

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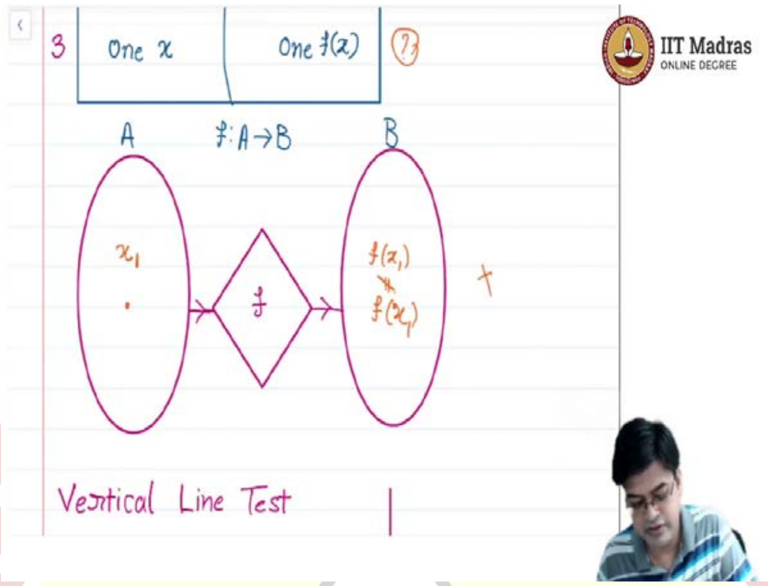

Let us describe this function as something like this, ok. So, let us see this is the set A which we will call as domain, and this is the; this is the set B that we will call as co-domain. This is A this is B. What does the function do? If we feed some value of x from this set, it will process the value and spit out or give us $f(x)$.

It is like a popcorn machine; you are feeding in the corn and getting out the popcorn. So, this is how the function works. Now, let us see, what are the cases that can happen? Suppose, for same value we are getting different outputs that, is the case, when you consider one x more than one $f(x)$, ok.

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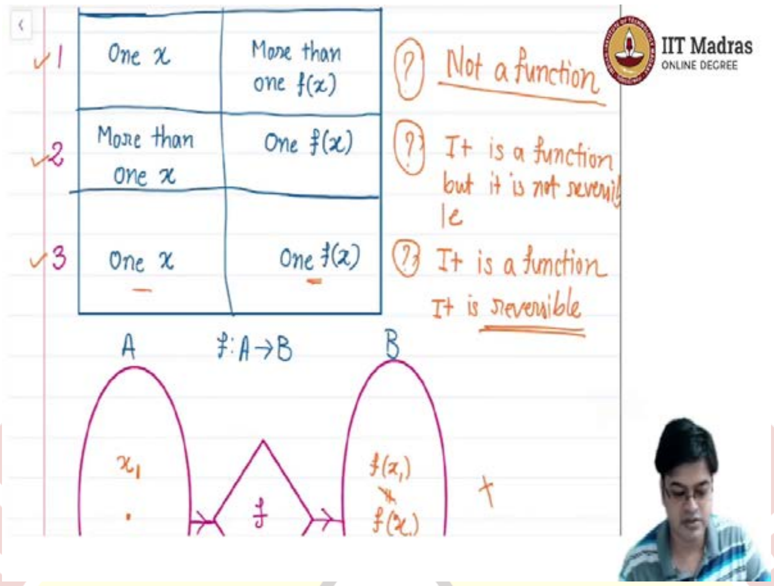

So, now, is this a function is a question mark. We will soon answer the question. Then, it can so happen that you have fed two different values and you are getting the same output, ok. Is this a function? We will answer the question. And, for every single output that you produce, there is a unique output that is produced by f, that is a one $f(x)$ only.

So, let us analyze these three again is this a function this. So, let us analyze all these three things together. So, let us start with the first case that is, one x more than one $f(x)$. The, what do I mean by one x more than one $f(x)$. Suppose, I have put x_1. If I put x_1 as a value if ones gave me $f(x_1)$, other time it gave me $f(x_1)$ which are not equal.

Is this a function? Is this a well-behaved function? It is not. So, I will say no, no this is not a function. Therefore, the first thing the first case I will say is not a function. Because, we are dealing with coordinate plane, let us understand this function with the coordinate plane.

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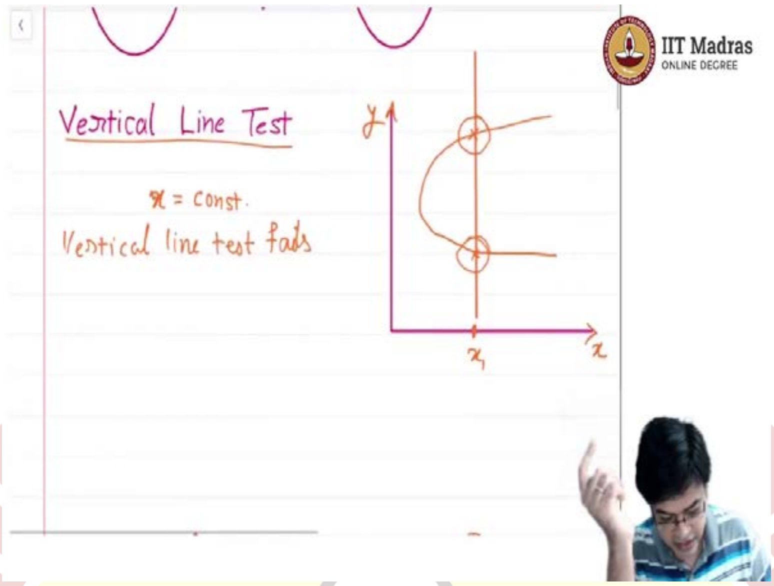

So, what happens here is suppose, I have been I am taking the value one value x_1 on x −axis. Let us say this is x −axis, let us say this is y −axis. And now, I am taking one value of x_1. So, one I once I fed in I got something which is $f(x_1)$ here. And, other time I fed in I got something which is $f(x_1)$ here, ok. Now, what is this? This is actually while if at all some curve I have to draw, it can be like this.

And, what happens here is for same value of x you are getting two different values. Then, your function is not well behaved, because I do not know which value will come when I fed in x_1. So, in that case how will you identify whether something is a function or not? Sometimes, if you have a graph of a function the way it is given; it is very easy to see; what is not a function.

For example, if you take a line which is $x =$ constant. If you take your line which is $x =$ constant and if I draw that line vertically. For example, let us say here this is the line $x =$ constant; $x = x_1$ in fact I have drawn. So, if I draw this line vertically, then I can see that there are two points at which, this line intersects the graph of a function.

When such a thing happens, we say a vertical line test that is, this is the vertical line right it is a parallel to y −axis. Therefore, this line passes through two points; that means, there is something fishy about the function and we will use this as a vertical line test. And say that, because vertical line test fails, this particular function is not a function sorry, vertical line test fails and hence, this cannot be a function.

Can you imagine such a function where can you imagine such a relation where this is not a function. For example, ok. Let us not imagine right now. Let us come to the next case and generalize it to the other set up. So, now let us see here. So, first thing is not a function that is very clear. Now, let us take the second case which is more than one x and one $f(x)$, ok.

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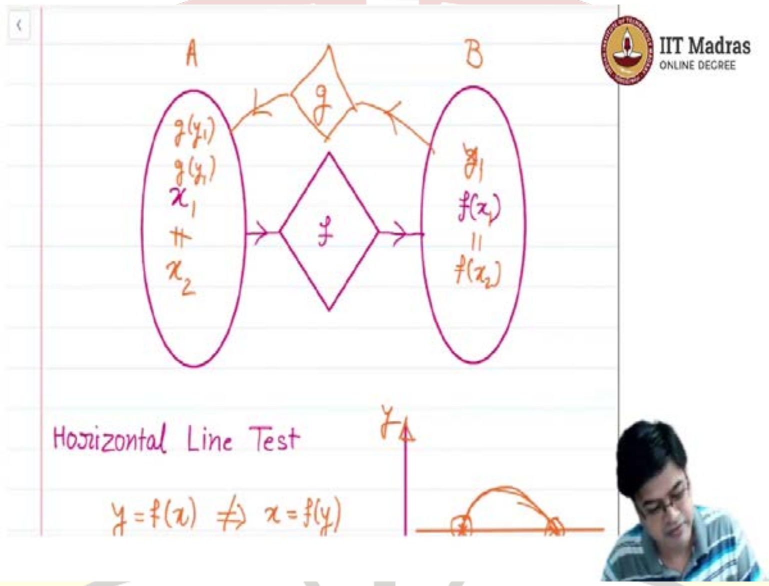

So, let us understand it on a paper. So, they are more than one x, x_1 and x_2, both are not equal, but somehow, they give the output which is equal, ok. In such case, what happens? So, as usual general this is domain set A, this is a co-domain B; f is processing unit and I have processed I have given I have fed in two different values of exercise, but the output produced is same.

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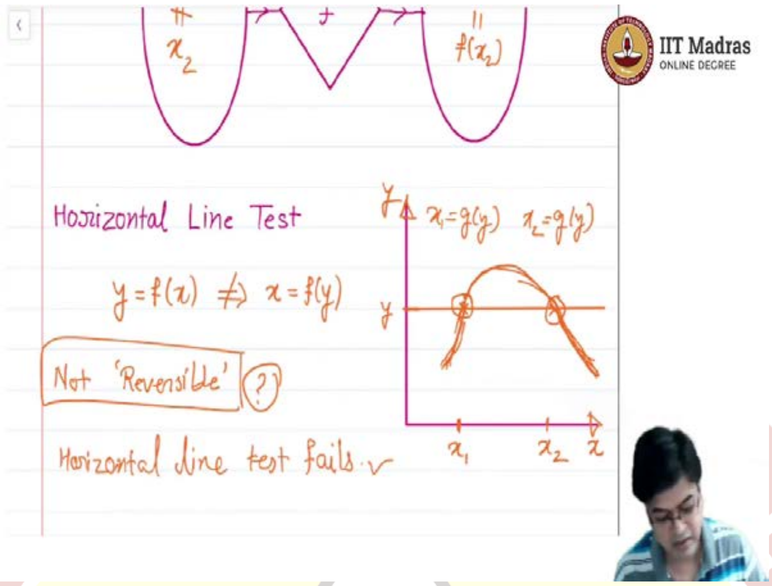

So, is this a function or not? The answer is this is a function. And, in this case let us see, how it happens? So, let us have some understanding about x_1 and x_2. So, naturally $x_1 \neq x_2$ and I got $f(x_1)$ which is this and I got $f(x_2)$which is this, ok. Both are same.

And, then how will the function look like? I can join a curve like this, where the function actually passes through these two points and I have a curve like this. So, if this is the curve then these two points are same. And, do you call this as a function? Yes, we call this as a function. Based on our understanding of the function lecture we call this as a function. In this case, something interesting has happened. Let me analyze it in a more thorough manner.

For example, when I considered the first case let me go to the first case. I have drawn a vertical line and I said that, because of this vertical line I can say this is not a function; I can say this is not a function. Now, the similar graph has appeared over here, but now if I draw a horizontal line I have a horizontal line, which passes through two points and I am saying it is a function, correct?

So, if I rotate this graph by say 90 degrees and flip it over then, what I am getting is a graph similar to this function. So, this actually helps me in understanding that, if I want to write y as a function of x, I am able to write it. But, if I want to write x as a function of y that is simply just flip this by rotate this by 90 degrees and flip the y axis. That will give you the exact understanding of the picture. And, from this to this I cannot go, ok. And therefore, the horizontal line if I draw it passes through more than one points.

So, what will happen is; suppose, if I take; if I take x_1 here, if I take a point in this domain y_1. And, if I try to setup another processing unit let us say g, and if I feed that $y_1$into g, what I get if I use this f, I will not get something which is similar that is I will get something called $g(y_1)$ once. And, if I feed in again y_1, I will get something else as $g(y_1)$. That is what is happening.

For example, here if you locate this point, this point it is say y. Then, once you feed y into g then, $g(y)$ you will get as x_1 and if you feed it other time $g(y)$, you will get as x_2. Something interesting happens. What I am trying to do is, I am trying to reverse the function and I an easily say that this function is not reversible.

If this is a interesting point which will help us in gaining more understanding of exponential functions. So, if this function is not reversible and a horizontal line test actually detects whether a function is reversible or not; so, this horizontal line test fails; horizontal line test actually fails in this case.

So, what happens here? Here, when you applied vertical line test that is case 1, it was not at all a function. Here it is a function, but our conclusion is it is not reversible. Let us look at this 3rd case, where only one x is there and one $f(x)$ is there. So, here is x, $f(x)$.

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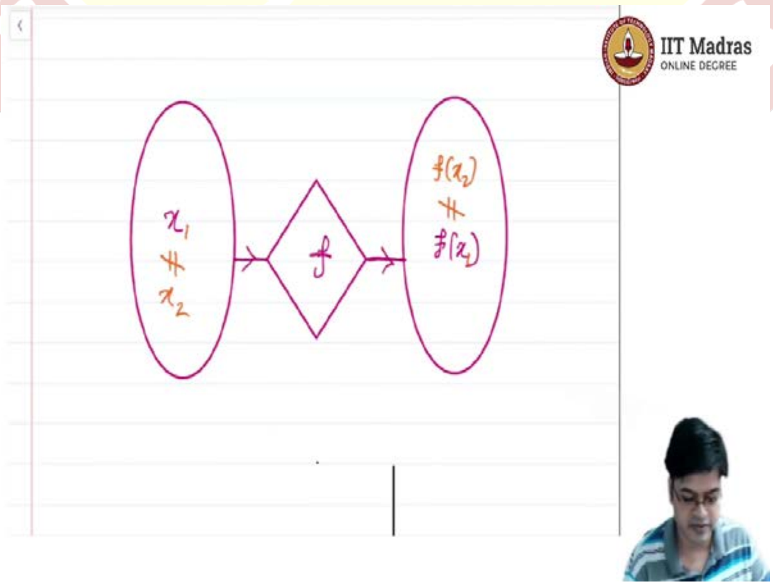

So, if I substitute x_1 and x_2 as two different values, then I will get through this processing unit, I will get $f(x_1)$and I will get $f(x_2)$ and both of them will not be equal to each other for $x_1 \neq x_2$. That is the only way it can happen right; one x, one x to one $f(x)$. So, if I take different $f(x)$ I will get different values, ok.

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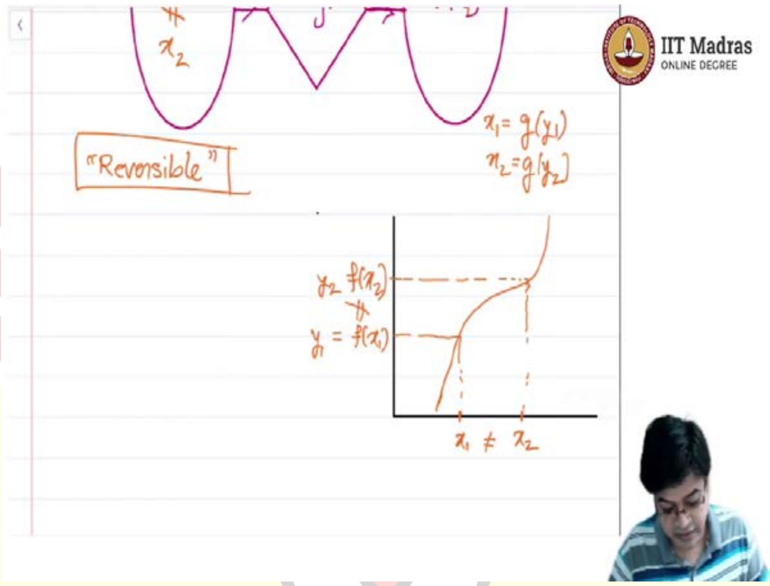

So, what will be a typical behavior of such functions? Let us try to figure out. Let us say this is the; this is one function. So, is this function one-to-one? The answer is yes, because if I take x_1 x_2 here this is $f(x_1)$, this is $f(x_2)$. So, for every one, if $x_1 \neq x_2$, $f(x_1)$ is not going to be equal to $f(x_2)$. Such function is called one-to-one function. And is it a function? It is a properly defined function, yes.

So, now, I can summarize using this as, this is a function. In earlier case, we actually characterized whether it is reversible or not. What can you say about this new function that you have defined? This function; so, now because for $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$. So, if I start this as y_1 and if I start with this as y_2, I can easily retrace back x_1 x_2 that is g there will be some function g such that, $x_1 = g(y_1)$ and $x_2 = g(y_2)$.

So, this function in some sense is actually reversible. We will come to this point in more detail in the next section. Right now, remember that the function that is one-to-one is reversible and it is reversible. We need to be more precise and I will give you a word for reversible function in the later lectures. But right now, it is an important observation that there are three cases. These three cases actually deal with a function.

The first one is not a function; first one is not a function, why? Because we had dealing with the coordinate plane. The vertical line test fails when then you pass a vertical line that vertical line will pass through two points or more than two points, then it is not a function.

So, then we said the second case. The second case was more than one x and one $f(x)$. In this case, what happened is we were able to find a function properly, but we were not able to revert the function. So, we said the function is not reversible. And, on what basis we have said? We have said on the basis of horizontal test, ok. And, we have related this with our first case that is, if the function is not reversible then the other part that is g is not also is not a function as well.

Third case where we have one x and one $f(x)$ we showed that it is a function and it is reversible also. This brings us to a question that, one-to-one functions how easy are they to identify; how easy are they to identify? So, let us properly define a one-to-one function. One in the domain, one in the co-domain and there is a clear cut association that is given by two.

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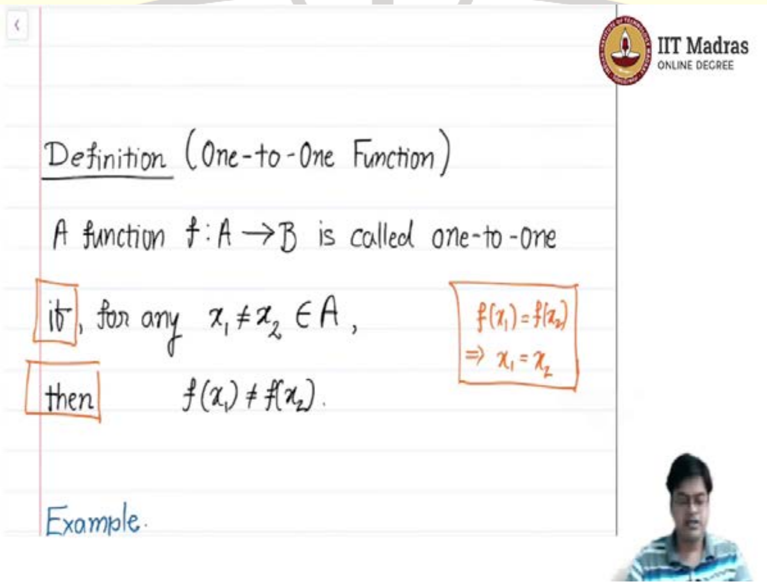

So, our definition; give a function f is said to be one-to-one if, for every $x_1 \neq x_2$ which belong to the domain of a function, $f(x_1) \neq f(x_2)$, ok. The other interpretation is, if $f(x_1) = f(x_2)$, this should imply $x_1 = x_2$ this is the other interpretation of definition, but we will use this as a definition. But, sometimes it may be difficult to prove this thing in that case, you can prove the one written in the orange.

Mathematics for Data Science 1
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Lecture – 8.2
One-to-one Function: Examples & Theorems

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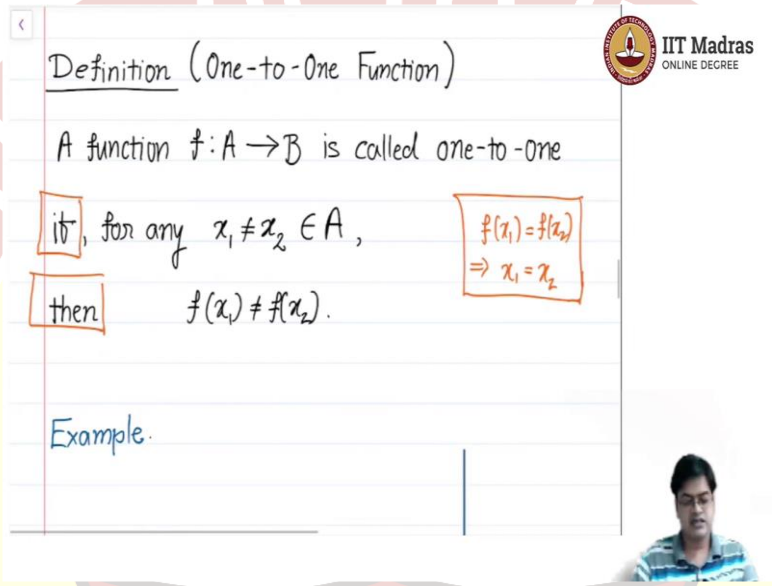

So, let discuss some examples of a functions that are one to one and not one to one. So, for this let us first take $f(x) = |x|$; is this function one to one?

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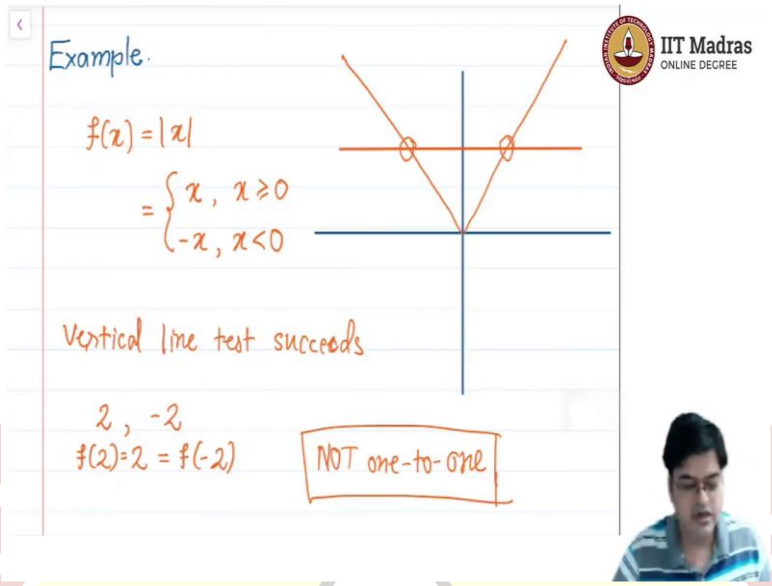

Try let us try to answer this question. So, let me write this function properly. So, if $f(x) = x$ for $x \geq 0$ and $-x$ for $x < 0$. So, it is actually a straight line on a passing through the origin like this 40 at a 45 degree angle and the $-x$ is this line ok. So, it is a V shape 90 degrees V; so, is this function one to one? First of all let us let us not take the argument, first of all is this a function $(x) = |x|$?

Pass a vertical line, take a vertical line and pass it through this; is there at if there is any point where two points more than one points pass through this function pass through that line then it is not a function. So, vertical line test is successful therefore, it is a function. Vertical line test says that it is a function succeeds and we know it is a function ok. Now, the question is the function one to one? Right. So, you pass a horizontal line. So, let me pass one horizontal line somewhere, let us take this horizontal line. Now, is the function one to one?

For $x_1 \neq x_2$, I got the same $f(x)$ ok. So, how will I prove it is not one to one? Let us take a value which is say 2 and -2; these are the two values, $f(2) = 2 = f(-2)$. Therefore, this is not going to be a one to one function. So, it is not one to one function. So, our conclusion is it is not one to one function. Then do we know functions that are one to one?

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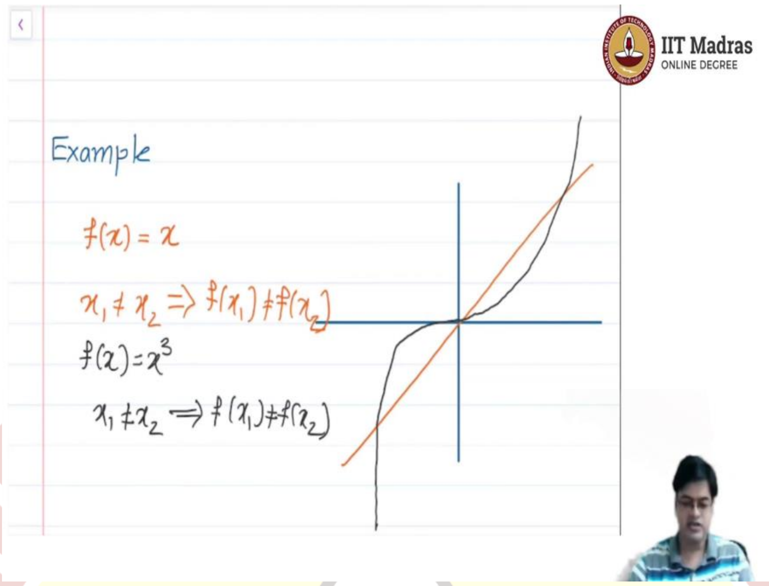

So, since we have taken $f(x) = |x|$, let us take a function $f(x) = x$; is this function one to one? It is a straight line passing through the origin, is this function one to one?

Let us take horizontal, first let us check whether this is a function, take a horizontal line, pass it through this pass it horizontally, the line parallel to x −axis. So, just drag x −axis up and down; do you see any point touching more than one point? No. So, it is a valid function; then, sorry yeah you have to pass the vertical line first ok.

Start with $f(x) = x$, take a vertical line which is y −axis, slide it to the left, slide it to the right. Do you see any where it has more than one points? No. So, it is a valid function. Then take a horizontal line, pass it from the top to bottom; see if you are getting any two points together for on that line; no. Therefore, this function is actually one to one because $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$ which is more or less expected right.

Because $f(x) = x$ therefore, $x_1 \neq x_2$ will give $f(x_1) \neq f(x_2)$. So, what about it is an exercise then what about if you take a cubic functions? So, cubic function will pass like this sorry, it is not a correct diagram of a cubic function. So, cubic function let us change the color as well, cubic function will have something like this, symmetry will be retained and then this will go down.

So, if this function, now check whether this function is one to one or not. Again the exercise is very similar, pass a let the x − axis go up and down, see if you are finding any two points together. So, let us say this function is $f(x) = x^3$ and now you can easily make out that for $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$. So, again through horizontal line test, I have detected that the function is one to one.

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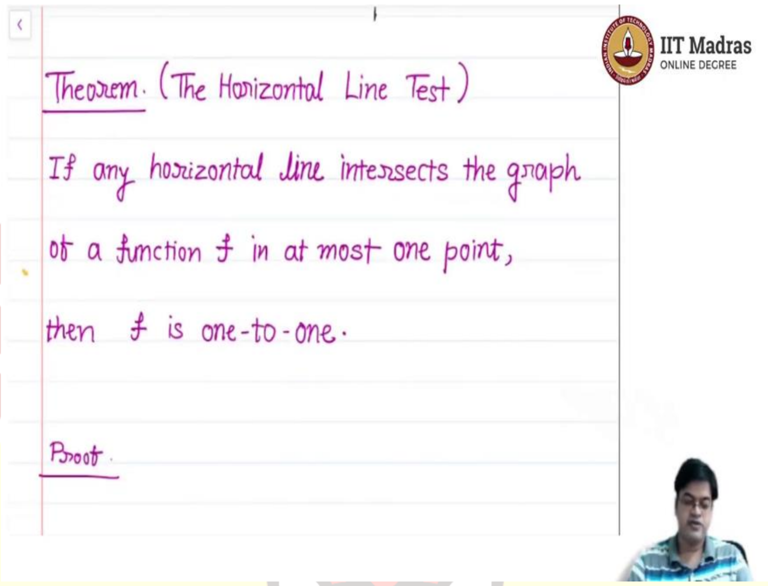

So, let us write this particular test as a theorem. If any horizontal line intersects the graph of a function in at most one point, then the function is one to one ok. So, then what we will show here, if you want the proof of this what we will show here is if the function is not one to one then it will intersect some horizontal line will intersect the graph of a function in more than one point ok. So, that is very easy to prove. So, I will prove it graphically.

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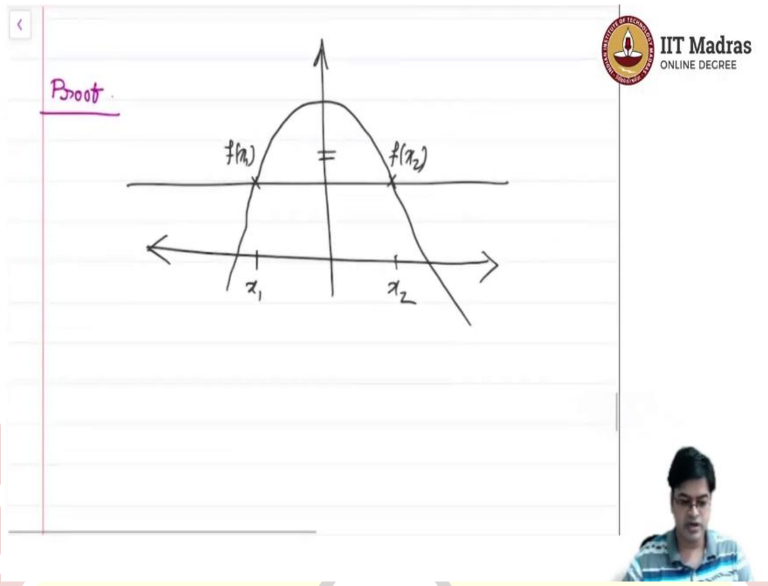

So, if the function is not one to one, let us say this is x −axis, this is y −axis. If the function is not one to one, I can take this point and call this as x_1 and I can take this point as call this as x_2. This is how I can make function not one to one and then pass a curve passing through these two points and pass the horizontal line over here which we have done several times now by now.

And therefore, $f(x_1)$ and $f(x_2)$ are same, they both are same. Therefore, the function is not one to one, that essentially proves the point that if a horizontal line intersects the graph of a function in at most one point then f is one to one good.

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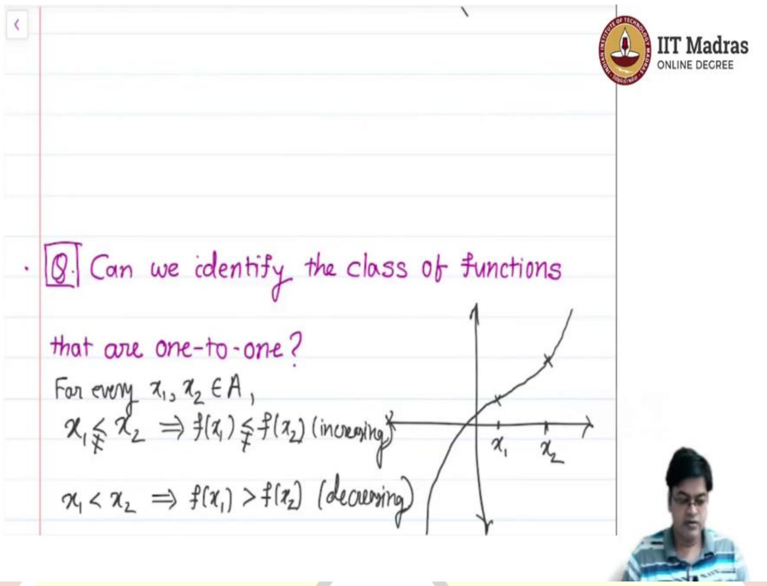

So, we are good to go now. Next thing that we will come is can we identify the class of functions that are one to one? So, what class of functions can you immediately think are one to one? For example, we have also seen some functions like if $x_1 \leq x_2$ then $f(x_1) \leq f(x_2)$ or let us not put this strict equality; let us put this way strictly increasing.

So, what does what do I mean by ok; let us can we question is can we identify the class of functions that are not one to one? So, I can; obviously, think of function of this form $x_1 < x_2$, $f(x_1) < f(x_2)$. Let me plot it and then the my imagination will work fine. So, this function is something like if x_1 is to the left of x_2 then $f(x_1)$ should always be less to the left of $f(x_2)$

Or, if you are plotting it on the y −axis then $f(x_1)$ below $f(x_2)$, this is the intuition and you can draw line joining these two points. Let it go ahead and this is true for every x_1, x_2 belonging to A this is true; then I am done.

But, this function have a name that is they are called increasing functions ok. In a similar manner, if I multiply this function with minus sign. Then I will get a function which is decreasing function and that can be written as $x_1 < x_2$ employs $f(x_1) > f(x_2)$ and this is called decreasing function.

Now, you look at any increasing function and apply your horizontal line test. What is the horizontal line test? Just now we have seen that if you take the horizontal line, roll it across the axis across $y -$ axis and there should not be more than one point intersecting that line at any given point in time ok. So, this increasing function and decreasing function will satisfy this phenomena.

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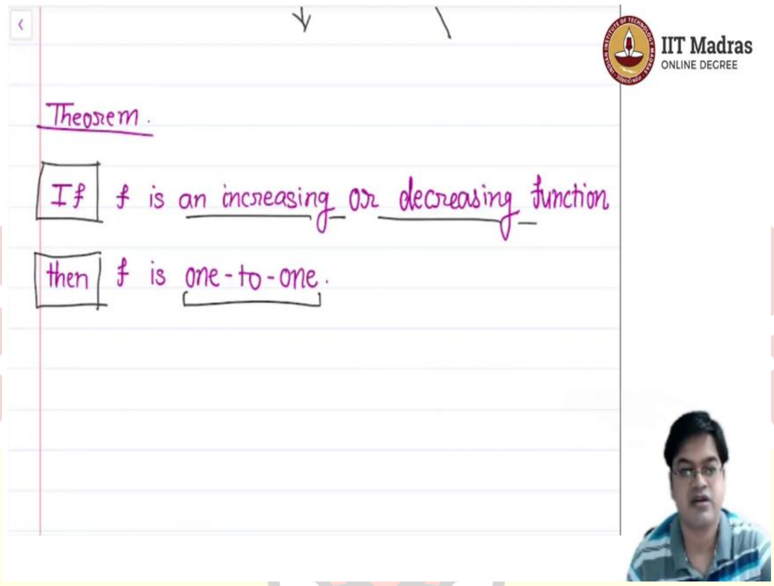

And therefore, we can easily write this as through horizontal line test that, if f is an increasing function or a decreasing function then f is one to one.

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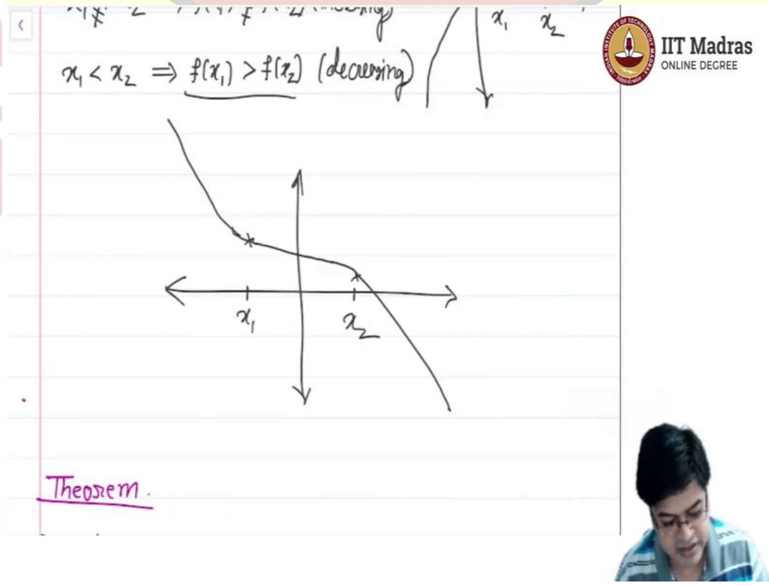

Let us see one decreasing function as well. What happens when the function is decreasing? As I go from left to right there is a $x_1$is here, x_2 is here. As I go from left to right, I get x_1 here and now according to the condition $f(x_1) > f(x_2)$. So, it will be somewhere here and I can have a curve passing through this point in this manner ok.

This is true for every x_1 and x_2 belonging to the domain. And therefore, using our line test, horizontal line test we can easily see that the function whether it is increasing or decreasing, they are one to one. This gives us a big class of functions that are one to one. So, one to one functions are not rare to find, one to one functions are abundant in nature and they are reversible as well.

With this insight, we will go to our next topic which is exponential functions.

Mathematics for Data Science 1
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Lecture – 8.3
Exponential Functions: Definitions

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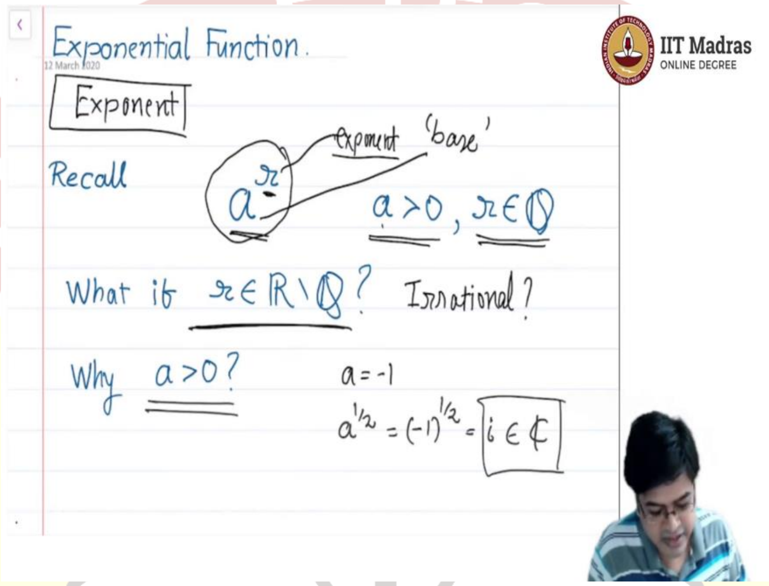

Welcome back. Next topic is Exponential Function. So, in this topic, what we will see is first we will identify with some known terminology that is exponent. We have already know seen this exponent. Where we allowed integer powers and then while defining the exponents, we allowed rationals also. So, when we defined exponents, they were of the form a^r and we always assume $a > 0$ and $r \in \mathbb{Q}$.

Now, I want to define an exponential function. So, as the name suggest, exponential has to do something with the exponent. So, what we are doing here when I am considering a function of this form, I am raising something, some number to the power of a where a will be popularly called as base and r is the exponent. So, this is base and this is exponent.

Now, if I want to define exponential functions on real line, then it is mandatory for me to define this a^r for $r \in \mathbb{R}\setminus\mathbb{Q}$. This is real line minus set of rational numbers that is I am talking about set of irrational numbers. So, I do not know as of now what is a definition of exponent form of set of irrational numbers ok.

The next question that we have seen is why is $a > 0$ that for which you know the answer. Let us say $a = -1$ now$a^{1/2} = (-1)^{1/2} = i$ which belongs to complex set of complex numbers, but I do not want to deal with complex number so, I am avoiding a to be greater than 0. In general, you can define a to be a negative number and then deal with complex numbers. We do not want to indulge into that conversation. So, I do not; I do not want to define $a < 0$.

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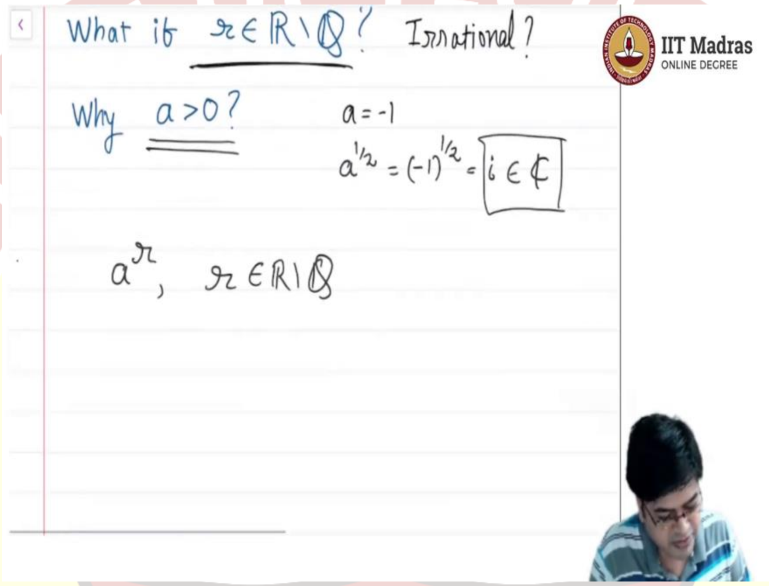

So, $a < 0$ is eliminated now the question is $a^r$and r belongs to irrational where $r \in \mathbb{R}\setminus\mathbb{Q}$ what will happen in this case? Or how will I define rational number?

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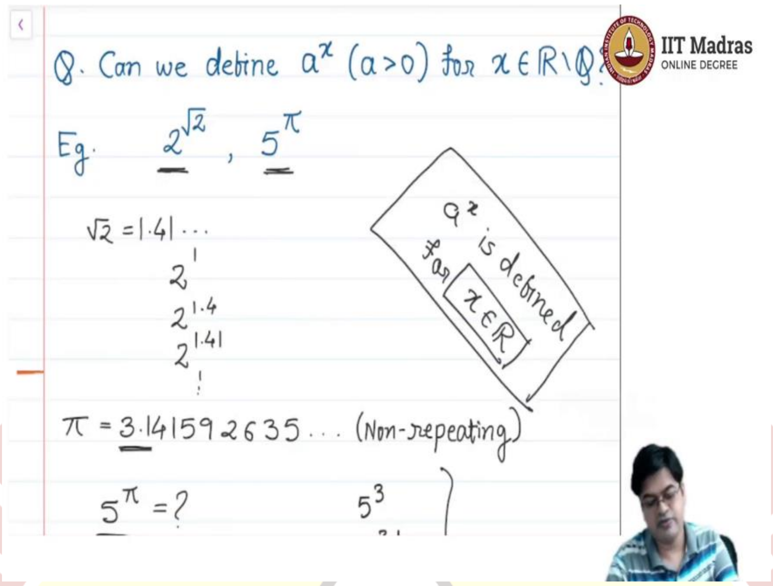

To be precise, let us ask a question that is can I define $2^{\sqrt{2}}$ or 5^{π} ok. So, in this case, there is no direct way to answer this question, but I will definitely have a strategy which is a calculus-based strategy to answer this question.

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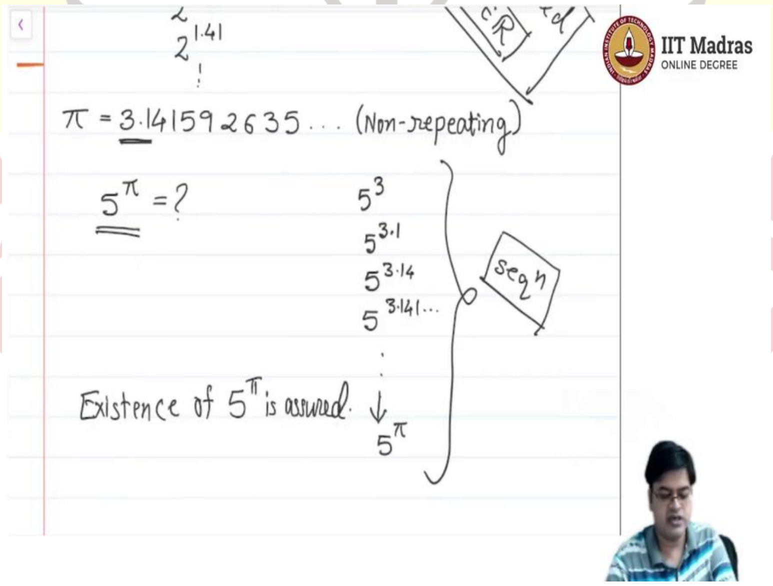

Let us consider this π and the value the numerical approximation of π is actually we all know π is an irrational number and 3.14592635… and this thing is non repeating it will continue till infinity right. So, now, what I need to understand is from what I know, can I define the number 5^{π}? So, anyway I cannot define it accurately right now.

So, based on my understanding, I am asking you a question that is 5^3? Right if so, then next question is $5^{3.1}$ defined? So, what I am doing here is this if yes, can I define $5^{3.14}$? Now, you remember all these a approximations are actually a rational approximations 3 is a rational number, 3.1 is 31 by 10 which is again a rational number, 3.14 is 314 by 100 which is again a rational number and I can go on like this there is 3.141 and so on.

So, if I continue this way, I will reach somewhere; I will reach somewhere and that somewhere I will call as 5^{π}. So, in principle, I can actually define a raised to irrational number. This you will study when you will study a topic of sequences which is outside the scope of this syllabus. So, we will assume that you have to trust me on this that 5^{π} is well defined.

In a similar manner, you can do an exercise for $2^{\sqrt{2}}$. So, $\sqrt{2} = 1.41$ ... and something. So, again you will go with 2^1 is defined $2^{1.4}$ is defined, $2^{1.41}$ is defined and so on and you will reach somewhere that is $2^{\sqrt{2}}$.

So, this way we are very clear that a^x is defined for $x \in \mathbb{R}$. This sets up the platform for defining an exponential function, this is very important a raised to x is well defined for $x \in r$.

This answer is given by convergence of sequences which is outside the scope of the syllabus, but we know that it exist for sure. So, I am guaranteeing the existence of 5^{π}; existence of 5^{π} is assured. In case you are interested, you can take a basic course in analysis or; in analysis or calculus where you will study these things ok.

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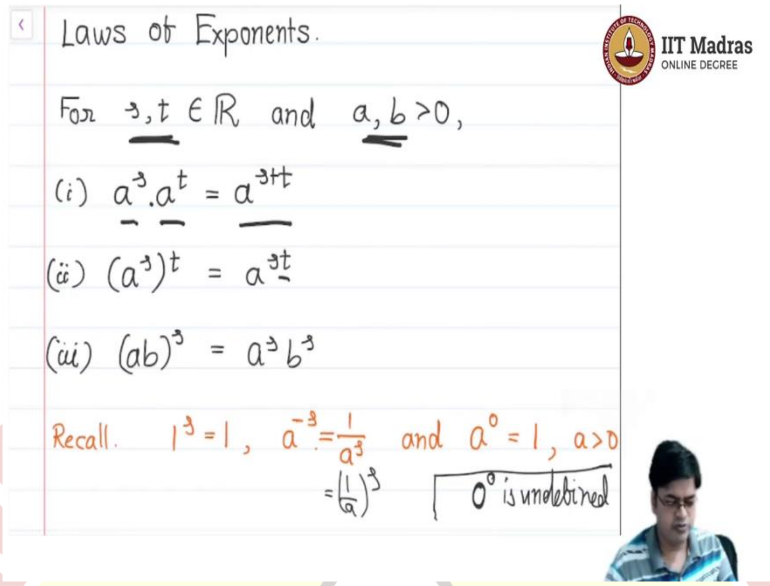

So, now let us go to let us recall all of these laws you already know simple laws of exponents. Earlier, we have we knew the laws of exponents for only rational numbers. Now, we are talking about the real numbers. So, $s, t \in \mathbb{R}$, $a, b > 0$, s and t will play a role of exponents, a and b will play a role of bases ok. So, then it is very easy to prove you might have proved. $a^s a^t = a^{s+t}$.

Remember here, product here is becoming addition here these are crucial points $(a^s)^t = a^{st}$. So, a raised to operation is becoming a product here. $(ab)^s = a^s b^s$ and then; obviously, you need to know that $1^s = 1$ for every $s \in \mathbb{R}$, $a^{-s} = \frac{1}{a^s} = \left(\frac{1}{a}\right)^s$, $a^0 = 1$. Remember where your $a > 0$ because 0^0 is undefined ok.

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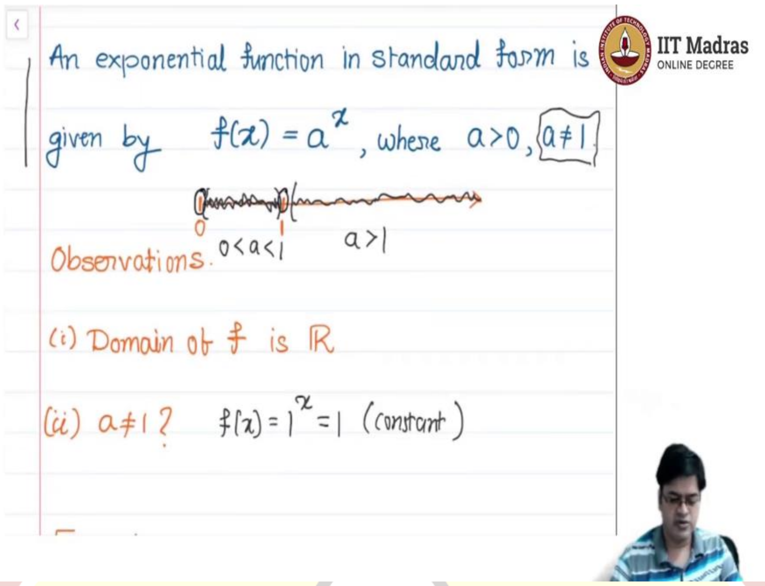

So, with this understanding, we have revised laws of exponents which will which we will use the left and right. So, you better remember all these laws and therefore, we are ready to set a framework of exponential function. So, here is our definition. An exponential function in the standard form is given by $f(x) = a^x$, where $a > 0, a \neq 1$. These are new condition that we have introduced.

We have seen why $a > 0$, but here they are saying $a \neq 1$. So, this needs further analysis, we will analyze it in due course. So, right now, if you look at the values of a, $a > 0$; that means, all these values are allowed and $a > 1$; that means, all these values are also allowed. Bearing the values 0 and 1 right.

So, the first from the definition, the first observation that you can figure out is because you have bared the value 0 and 1, the function $f(x) = a^x$ will have a domain which is entire real line. For every , $x \in \mathbb{R}$, we should be able to compute a^x ok.

Then, let us analyze this is then observation: why $a \neq 1$? Let us put $a = 1$. So, $f(x) = 1^x$, but from the laws of exponent what you know? $1^s = 1$.

Therefore, $1^x = 1$ in fact; it is nothing, but a constant function. I am not interested in handling a constant function right which nothing, but a horizontal line $y = 1$ is the graph of a function; I am not interested in this. So, let us not call this as exponential function that is what we are saying in the definition.

So, hence forth, we will never talk about $a = 1$, $a = 0$ or $a < 0$. So, if you have a real line, you will have an expression of this form where you are talking about this interval, open interval and this interval, which is an infinite interval. So, you have two characterizations which is $0 < a < 1$, $a > 1$ these are the two characterizations that you got over this thing.

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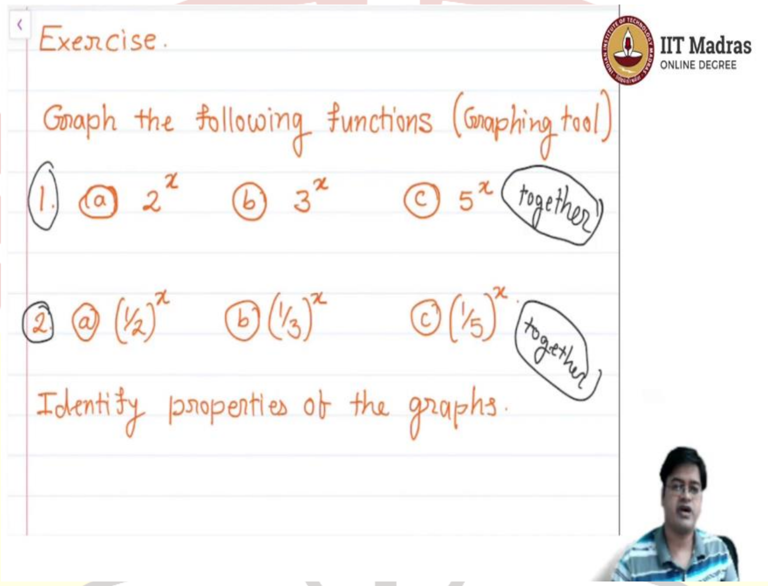

Now it will be interesting to use some graphical tool and see what are some functions of this kind look like. So, here is an exercise that I will give you. Use some graphing tool like Desmos and plot these functions together. For example, you plot the functions given in 1 using Desmos we just put $f(x)$ is equal to this, $f(x)$ is equal to this and this and plot all these three graphs together without any understanding about the behavior of the function you plot all three of them together.

Then, use the 2nd graph and put all these three things together. Identify the properties of the graph that is through which points they pass through is there any difference in the graphs of 1 and 2. So, identify all these properties like we did in polynomials and after doing that again return back to this video and we will see some of the functions that are given here by a graph and we will analyze those functions. So, right now you pause the video, come back in the next video.

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Lecture – 8.4
Exponential Functions: Graphing

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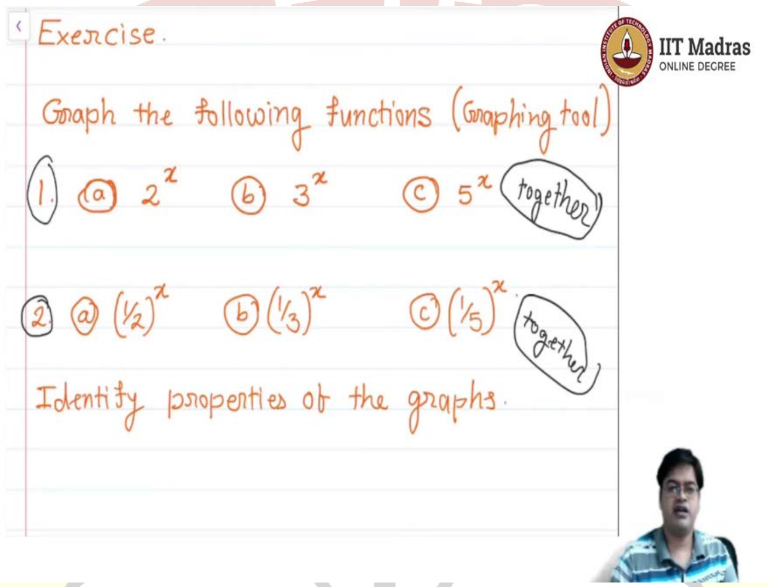

Welcome back. So, I hope you must have done your exercises and you must have developed some understanding about the exponential functions. Let us try to collect recollect that understanding through 2 examples given here.

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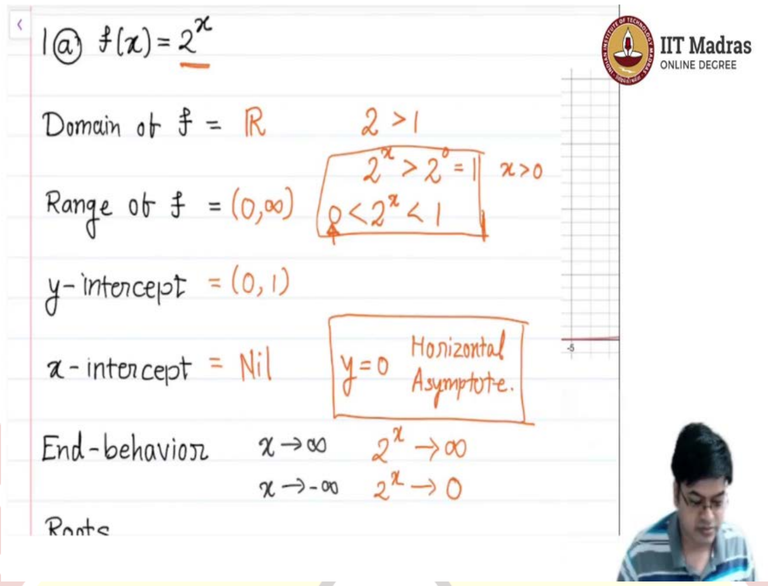

So, let us first take 1 a which is $f(x) = 2^x$. If you have used DESMOS, you must have got the figure of the function. But prior to receiving the figure of the function, let us see what should be the domain of a function.

We have already discussed in greater detail that the domain of a function can be a $\mathbb{R}$, entire real line. Now, if you look at this function which is 2^x, this $2 > 1$ and the $2^x > 2^0$ which is equal to 1, $2^x > 2^0$ whenever x is positive correct.

Now, because $x > 0$, then $2^x > 2^0$ ok. So, if $x < 0$, what will happen? 2^x, when $x < 0$ will always be less than 1. This is also possible. But when this 2 raised to; can this 2^x become negative? No. So, it is always greater than 0.

So, if you have this understanding, then you can easily write the function has a range which is $(0, \infty)$. So, there is a split from when you consider a point 1, there is something happening at point $(0, 1)$ right. What is $(0, 1)$? $(0, 1)$ actually is an y −intercept ok, something is happening at $(0, 1)$ because I have put 0 here for then it is I am getting 1.

So, $(0, 1)$ is also y −intercept and there is something happening which is going below 0. Is going below 1, your graph is going below 1 and therefore, this particular thing is going down, but it never goes below 0. This is an interesting fact because if you consider 2^x, it never goes below 0.

It cannot go to a negative number. Therefore, will it touch the $X -$ axis? It will not touch $X -$ axis. In fact, $x -$ intercept is nil ok, but it is approaching 0. So, the something that is approaching 0, so $x -$intercept is actually it will never touch it; but it will actually go along that line. So, this $y = 0$, it will touch at infinity ok. So, such a thing, we call as horizontal asymptote ok.

So, such a thing you call as horizontal asymptote. So, with this understanding, these are the things that I can make out directly without looking at the graph. So, let us now look at the graph ok, before going to that, let us see what happens to the end behavior. End behavior of a function as $x \to \infty$.

So, as 2^x, you consider a function 2^x as x increases, this also increases. In fact it increases at a rapid rate than x. So, this also should tend to infinity and as $x \to -\infty$, we have already figured out $y = 0$ is the horizontal asymptote. So, 2^x will actually go to 0 ok.

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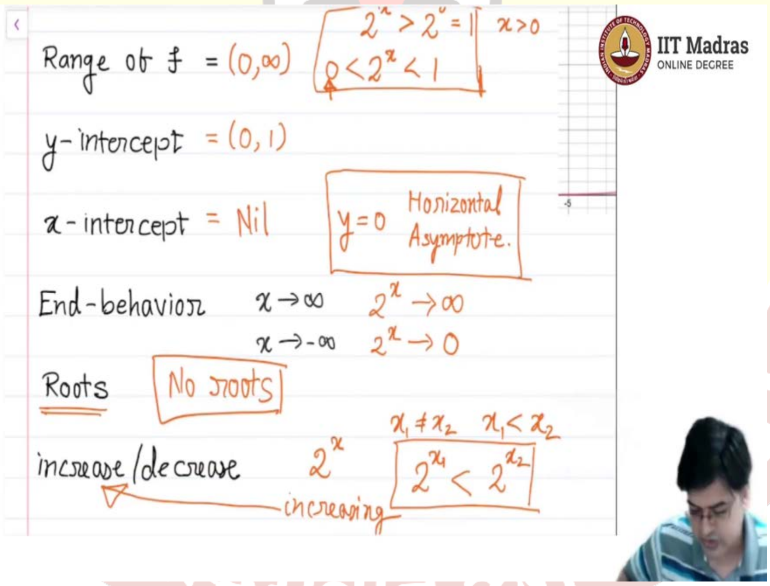

Then, the question that we used to quantify while considering the function, what are the roots of this function. So, do they have any roots? In fact, using graphical method, it is very clear that it never touches 0. So, there are no roots and the functions increase and decrease.

So, the domains of increase and decrease like polynomials, we studied domains of increase and decrease; but here, I think my claim is no need to identify the domains of increase and decrease. Why? Because you look at a function 2^x, let us take $x_1 \neq x_2$ or $x_1 < x_2$, without loss of generality, we can take this. Then, what can you say about 2^{x_1} and 2^{x_2}?

See $x_1 < x_2$, so naturally if it is raised to the power 2; 2^{x_1} and 2^{x_2}, this relation should hold. So, what I am saying is the function is actually an increasing function and increasing functions are 1 to 1. Therefore, I do not have any doubt that the increase and decrease, it is only increasing; throughout the real line, the function is only increasing.

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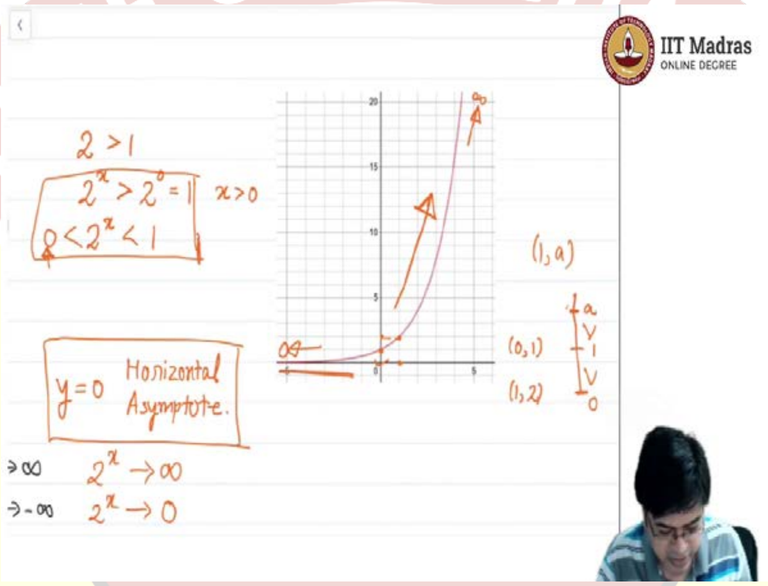

So, let us look at the graph of a function $f(x) = 2^x$. Let us identify the points. So, here you can identify a point right. So, this point we have seen as $y -$ intercept and that point was (0, 1) right. Then, the one in this case, let us look at this point which is 1 and where will it go? It will actually tell you 2.

So, the point is (1, 2), the second point ok. So, these 2 points are very special points, they tell you something. So, in particular, had it not been 2^x, but a^x, then that point would have been $(1, a)$ and if you mimic this graph over here y x is over here ok, this is a point 1, this is the point 0 and this is the point which is a.

So, that says $a > 1$; this relation is there, is greater than 0 yeah and therefore, the graph was a point which lies here, which is here right. As $x \to \infty$, this graph actually goes to infinity; as $x \to -\infty$, this graph goes to 0. These two points is these two point and this is an increasing function.

As you come from left to right, it increases. So, this is an increasing function, $y = 0$ is the horizontal asymptote, that is very clear ok. The range of a function is 0 to infinity, that is also very clear. The domain of a function is entire real line, $\mathbb{R}$.

So, we have got all the details necessary for finding this. Now, what it so special about 2^x, if I replace this 2 with 3, still I will have y −intercept to be 0, 1 because 3^0 is also 1 and I will again have domain of f to be equal to $\mathbb{R}$; range of f to be equal to 0 infinity; no x −intercept; $y = 0$ will be horizontal asymptote; $x \to \infty$, $a^x \to \infty$, $x \to -\infty$, $a^x \to 0$. There are no roots. The function is only increasing.

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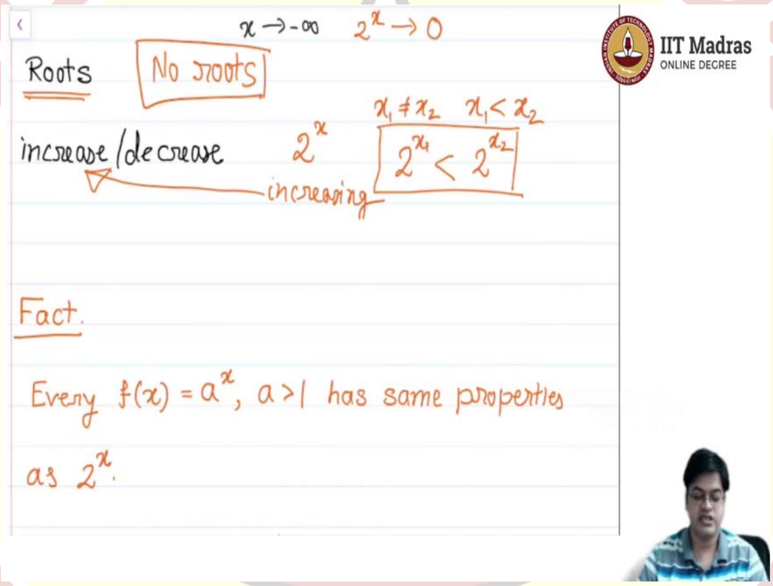

And therefore, I will state this as a fact that every $f(x) = a^x$, for $a > 1$ will have same properties as 2^x. So, I do not there is no need to draw different different values. The behavior is same only the values will change.

For example, in this case, where you have seen the graph of this (1, 2) is a point; (1, 2) is a point, suppose I consider 3^x, (1, 3) will be the point. So, only the values are changing; but the shape, the behavior, everything else that is listed here remains the same. Therefore, you do not have to draw a graph every time, only thing is you need to evaluate the values in general.

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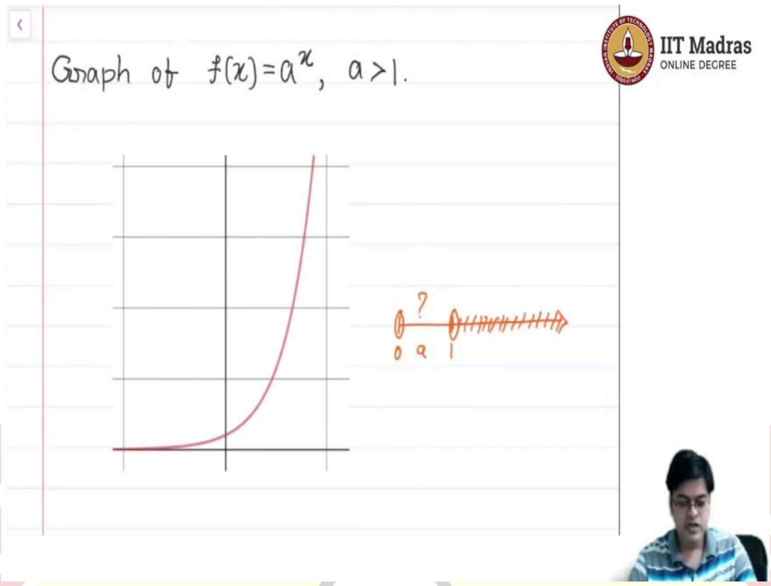

So, what is the graph of $f(x) = a^x$ in general? It is this way for $a > 1$. So, remember that line that we have drawn which is that the line for a, where we have eliminated these 2 points such as 0, this is 1, we have identified what is the case for $a > 1$. You have also identified the case, where $0 < a < 1$. So, let us go back and see what happens when $0 < a < 1$. So, if a lies here how is the behavior? So, you have already analyzed.

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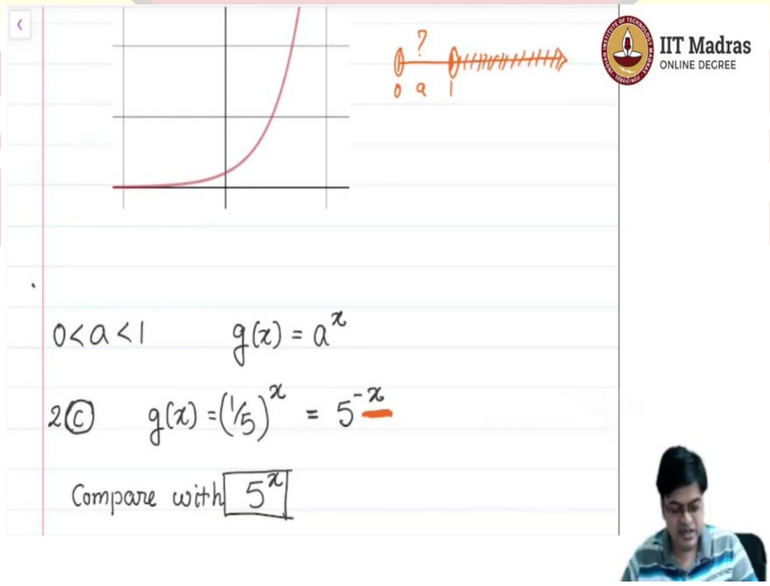

And let us take this function as $g(x)$ and take it to be $g(x) = a^x$ and this is $\left(\frac{1}{5}\right)^x$. Now, you do not really have to draw this graph, what you can do is ok. So, $g(x) = 5^{-x}$. So, here x is replaced by $-x$. So, what will be the change in the behavior?

So, when x is replaced by $-x$, you know its reflection across $Y -$ axis, you have solved many examples in the assignments. This $Y -$ axis, this is x; then when I put it as $-x$, it will be simply reflected along $Y -$ axis.

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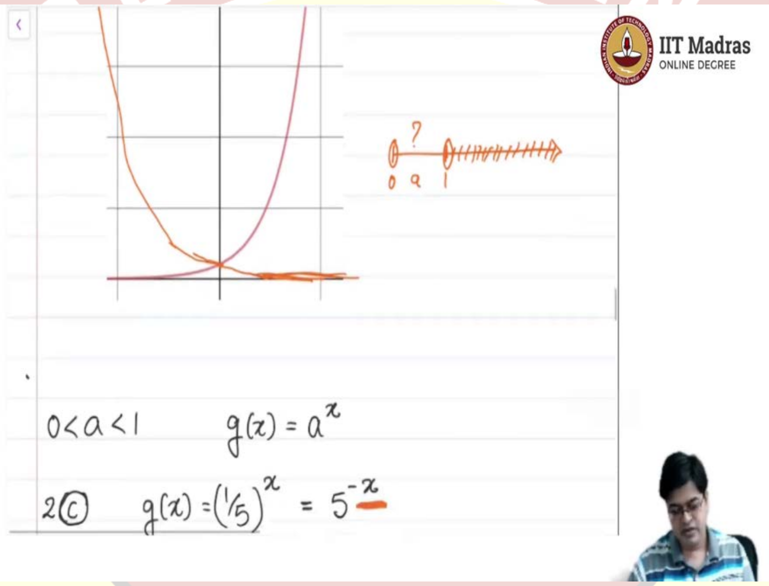

So, if you look at this graph and try to draw a graph of this function, then it should be something like coming from here going here, it should be something like this, it should actually look like a reflection along $Y -$ axis. So, let us try to show it as reflection ok. This will actually go very close, but never touch.

So, let me erase this ok. So, this is how it will look like. So, without actually thinking about anything else, you can simply draw a graph of $\left(\frac{1}{5}\right)^x$; but still let us try to do it in regular set up.

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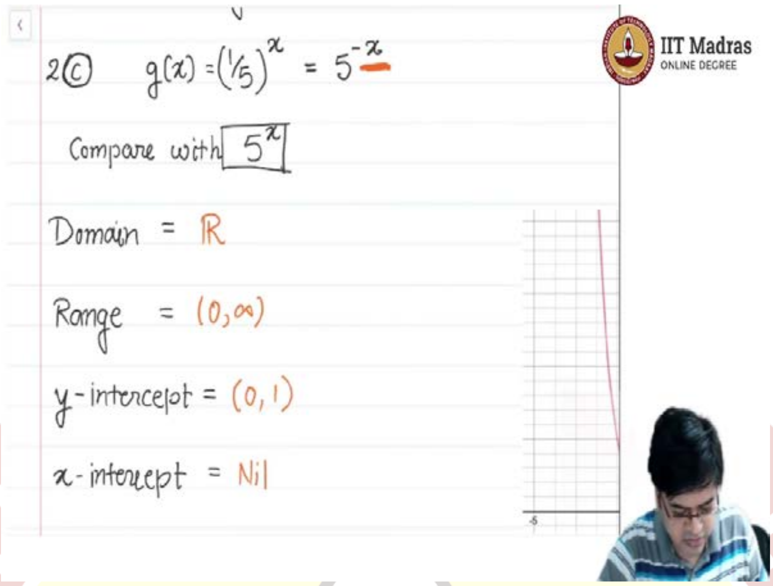

So, what will be the domain of this function? The domain of this function is very clear because we have used it several times, the domain of this function will be real line. Range, nothing changes; (0, ∞) because it is a reflection across $Y -$ axis. So, let us look at this function.

So, the domain will be ℝ; range will be (0, ∞). What will be the $y -$ intercept? Because it is a reflection, so $y -$ intercept would not change, so it will be 0, 1 only. $x -$ intercept will be nil, there would not be any $x -$ intercept.

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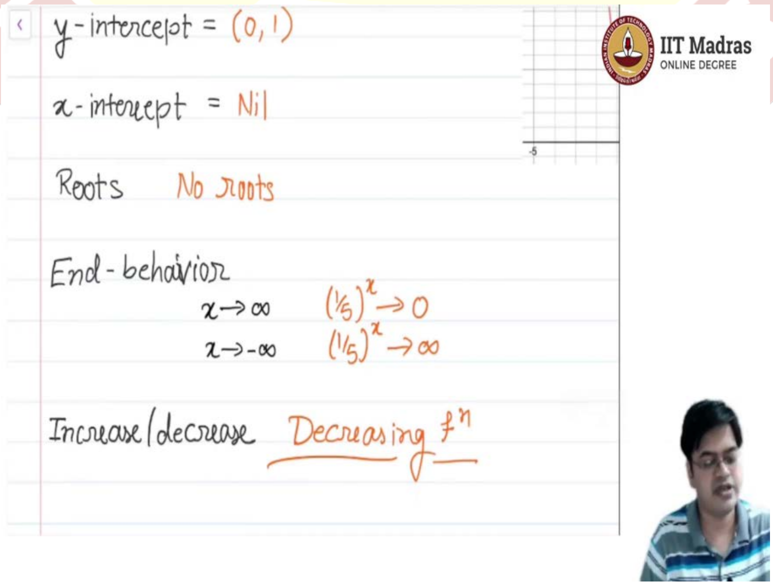

And therefore, no roots and what about the end behavior? End behavior is like $x \to \infty$, $x \to -\infty$. So, when $x \to \infty$, the end behavior will be because it is a reflection you see.

So, when $x \to \infty$ there, it was going to ∞. So, and $x \to -\infty$, function 5^x would have behaved, it will go to 0. So, that reflection will make this a^x or $\left(\frac{1}{5}\right)^x$ whatever is the function $\left(\frac{1}{5}\right)^x$, let me do it properly.

So, this will make $\left(\frac{1}{5}\right)^x$ to go to 0 and this function $\left(\frac{1}{5}\right)^x$ will go to infinity ok. Good. Then, because it is a reflection, the increasing thing will become decreasing. So, there is no intelligence here. So, this will be in fact a decreasing function wonderful. So, we have analyzed everything without taking much efforts. This is the beauty of once you understand the functions on graphical plane.

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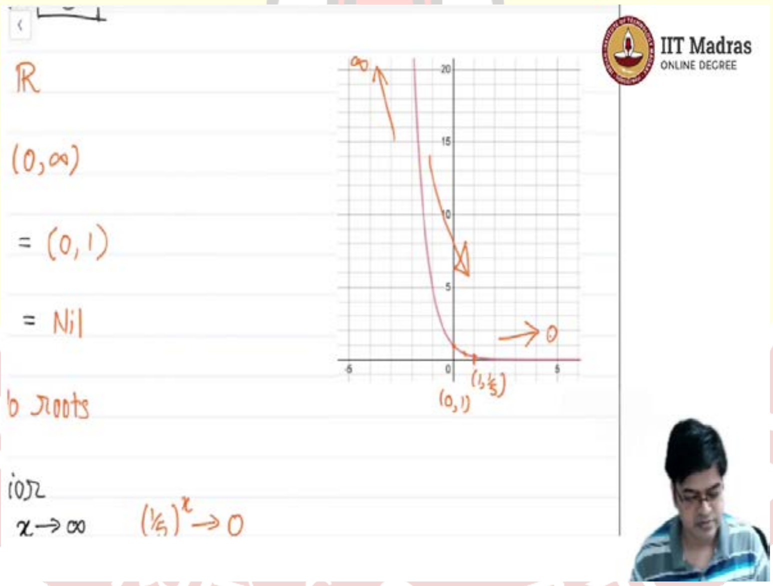

So, here is the graph of a function which is given to us $\left(\frac{1}{5}\right)^x$, you also might have plotted and naturally, the we will analyze whether it coincides with our thing. So, this is a point (0, 1), now it is $\frac{1}{5}$. So, your point will be somewhere here, sorry this is 5. So, the point 1 is here and this point is $\frac{1}{5}$.

So, $\left(1, \frac{1}{5}\right)$, this is done. Then, as $x \to \infty$, this function goes to 0. As $x \to -\infty$ that is this way, this function actually goes to ∞ and this function is decreasing. From left to right if you come, you are actually coming down. So, it is a decreasing function. So, this completely gives us an understanding of what the graph of a function will look like.

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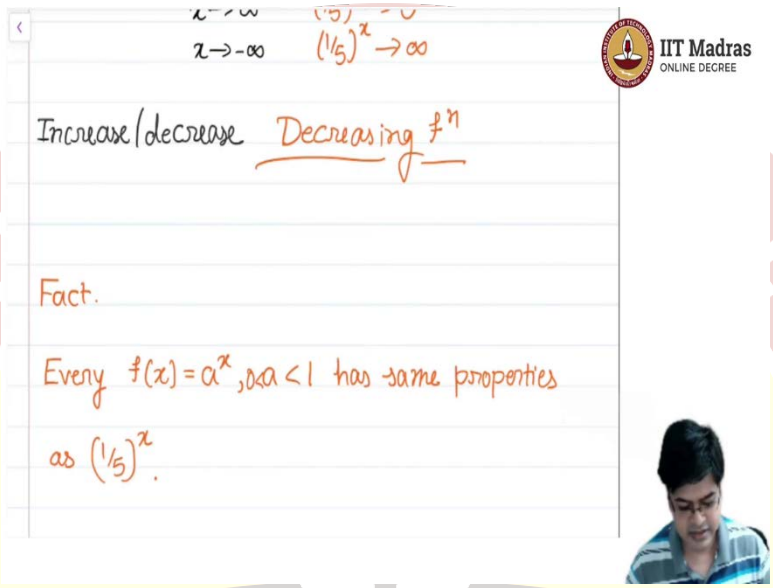

Also, the same fact is true that every $f(x) = a^x$, where $0 < a < 1$ has same properties as $\left(\frac{1}{5}\right)^x$. Therefore, it is a representative class. So, you do not have to worry about the because it is a representative class, you have to worry about all other functions. All other functions will have a similar behavior.

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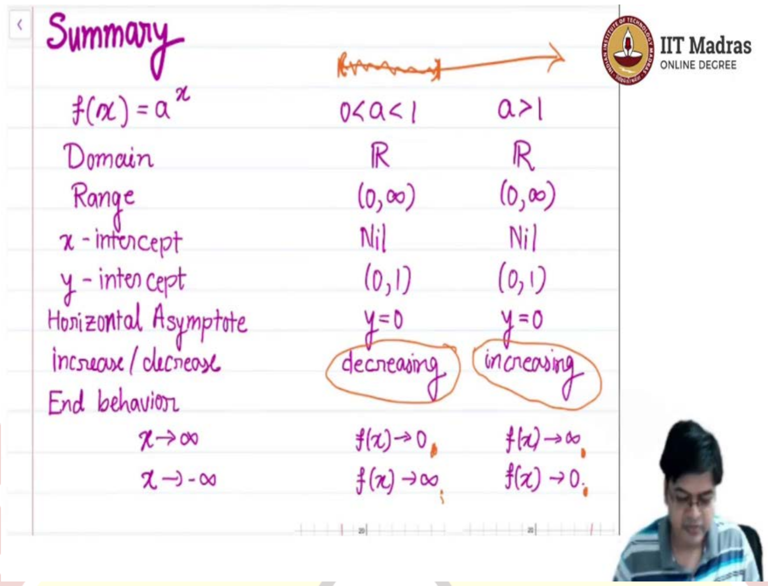

So, we have done a lot, let us summarize these things in a neat table which is this. So, this is the summary of the table. So, if I have been given a function $f(x) = a^x$, then to be more precise, let me draw a line here. This is a line; it does not look like a line, but assume that this is a line.

This is the point 1, then I am talking about $0 < a < 1$ that this zone. In this zone, the domain of a function is ℝ; range of a function is $(0, \infty)$. There are no $x -$ intercepts, no; $y -$intercept is 0, 1. Horizontal asymptote $y = 0$ is there. The function is decreasing. The end behavior as $x \to \infty$, $f(x) \to 0$; as $x \to -\infty$, $f(x) \to \infty$ correct.

Then, you look at the function which is $a > 1$, domain is real line, range is $(0, \infty)$, nil; (0, 1), $y -$ intercept is (0, 1). Horizontal asymptote is $y = 0$. The only distinguishing feature is the function is increasing here and a function is decreasing here and because it is increasing and decreasing, the end behavior changes that is because it is decreasing, it will decrease to 0 because it is bounded below by 0 and because this is increasing, it will increase to infinity, but here it will go to 0 ok.

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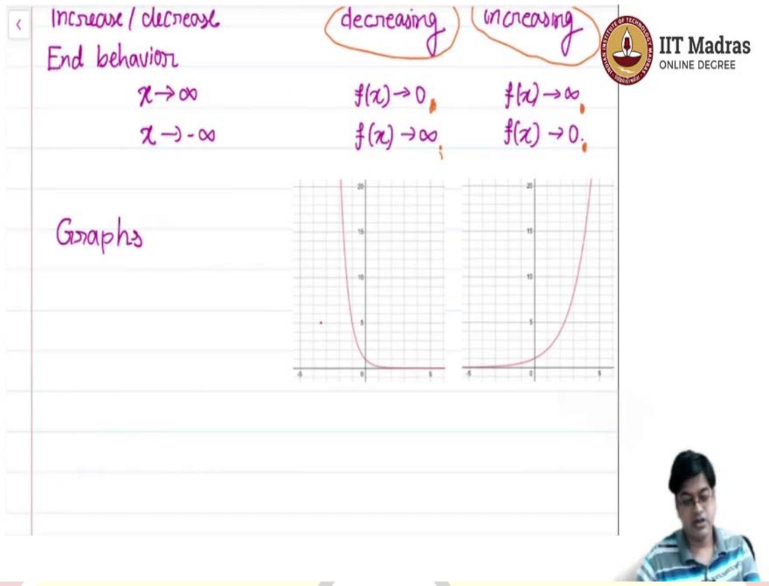

Then finally, you see the prototypes, just look at the graphs of these two functions ok. This ends our topic on exponential functions. Now, we will introduce something which is called natural exponential function in the next video.

Mathematics for Data Science 1
Professor Neelesh S Upadhye
Department of Mathematics
Indian Institute of Technology, Madras
Lecture 47
Natural Exponential Function

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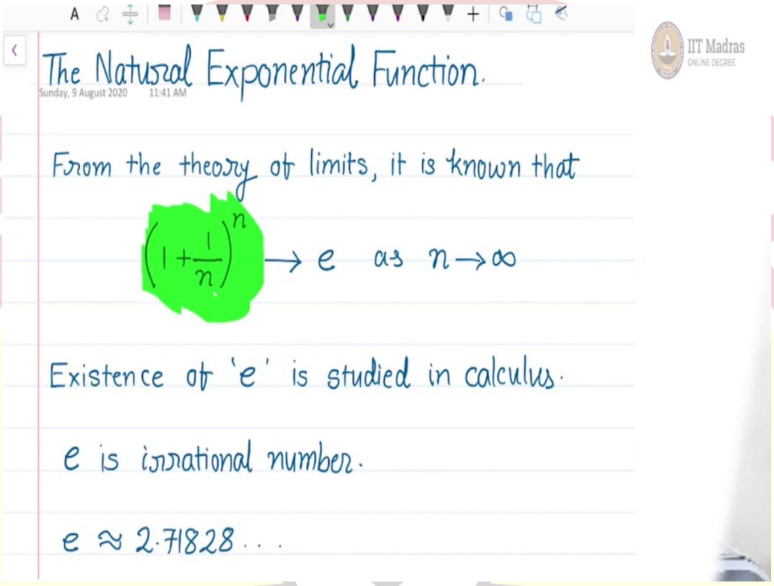

Hello friends, in this video we are going to talk about yet another important function among all exponential functions is natural exponential function. So, basically the theory of natural exponential function is derived from calculus. So, in order to understand how relation natural functional, natural exponential function arises, we need to study the theory of limits. In particular, this natural exponential function is dependent on something raised to the exponent that something is in irrational number that is called e.

And I will make sure by the end of this video you will understand why this number e is very important. In particular, when we talk about number e or ratio or a limit of some quantity is important which is shown here. So, from the theory of limits it is known that whenever you are talking about $(1+\frac{1}{n})^n$ this particular limit it actually converges to e.

So, now unless you understand the concepts of limits, you may not be able to have complete understanding of this concept, but still I will give you some intuition behind this number e. So, though, as I mentioned earlier, the existence of e is actually studied in the field of calculus. For that you may have to do the course which is maths 2, Math for Data Science 2 and you have to agree with me on certain facts without knowing them or you have to trust me that e is an irrational number and e is approximately equal to 2.71828 and so on. It is an irrational number. So, it will go to never ending decimal representation, it will continue on the right.

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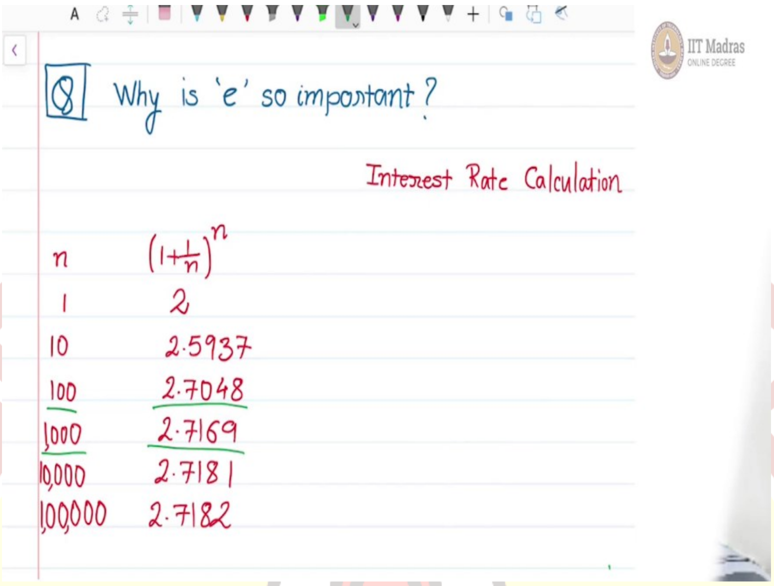

So, these are the facts about e. Now the question that we asked at the beginning of the video is why is 'e' so important? So, to answer that question, let us first look at the behaviour of this particular number as a limit. So, when I say that n goes without bounds, the number the this particular function $f(n) = (1+\frac{1}{n})^n$ converges to e. What do I mean? Let me put it in a proper formal way.

So, I have generated a table over here for our convenience and let us understand. So, when I substitute this n, the value of $n = 1$, this number $(1+\frac{1}{n})^n$ is simply 2. When I substitute $n = 10$, the number becomes 2.5937. So, does that mean this function will go without bounds? The answer is no that is why we get the convergence. And such type of questions are studied in Calculus.

So, when you substitute the further values of n that is, n=*100* you have substituted, you got 2.7048. When you substitute $n = 1000$, you will get 2.7169. Now, you can see that you approaching closer to the ideal value of e. And when you put n =*10000*, you get 7181 still because we are writing up to 4 decimal places, we are not really very close to it, but we will be, we are very close to it, but we are not at that point.

But when I put n is equal to 1 lakh, then I get a value of e which is 2.7182 and that is actually exact representation of this number e up to 4 decimal places, correct up to 4 decimal places.

Now, if you go on further and put higher and higher values e like you can put it to be a 1 million, 10 lakhs and then you will see you will get further improvement, but because we are focusing only on 4 digits after decimal, we this requirement is enough for us, this much calculation is enough for us. Another thing, another aspect in which this e becomes very crucial in accounts this interested calculations.

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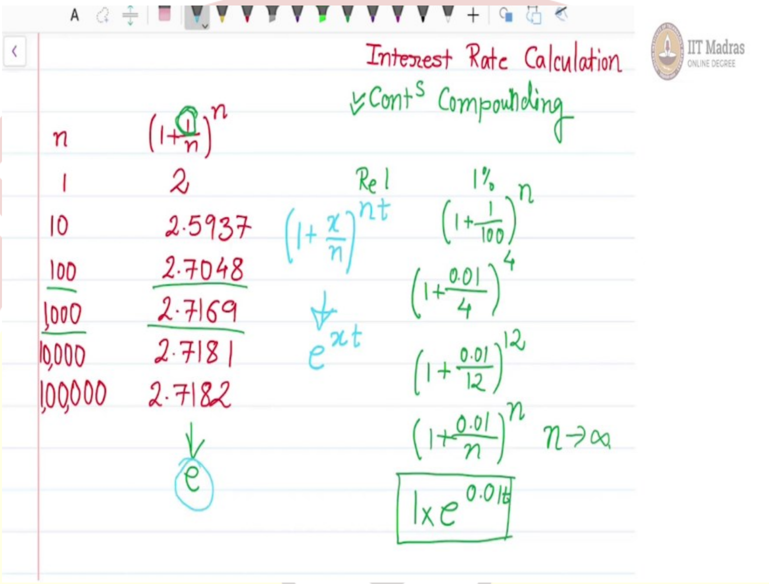

You must have heard the term by now of continuous compounding and continuous compounding actually means the taking ratios with respect to e or taking exponents with respect to e. So, let me demonstrate to you in this manner. Let us say you have invested rupee 1 in a bank and bank is offering 1 % interest rate and you have invested it for 1 year. So, in that case what is the answer? 1 plus 1 upon 100 raised to 1, this is the answer, 1.01 1 % you will get if you have invested rupee 1, you will get 1 paisa of interest.

So, now if you go on like this you will have something like $(1+\frac{1}{100})^1$ raised to so whatever number of years you have invested in raised to n. Now, when you actually look at the procedure of the bank, banks do not give you the interest which are given annually, but they credit the interest quarterly. So, in that case what you need to understand is the interest rate is actually given in a quarter, so it is computed on quarter and whatever interest you have accumulated, that interest will be taken into account for the next quarter.

So in that case, basically what bank is doing is bank is taking this interest rates which is 0.01 and it is actually dividing it into 4 quarters, 4 parts because they are giving you a quarterly interest and then you are actually getting this multiplied in this fashion. So here, if you look at the interest rate, instead of 1, I have 0.01 as the number and for 0.01 I got this number. So, if the bank decides, so I will revise the bank decides that I am revising, bank is revising the interest rate every month, then what will happen?

Then the same logic for single year, for single year remember $(1 + \frac{0.01}{12})^{12}$. So, if the bank starts revising the interest rate infinitely often, then we are actually talking about something like $(1 + \frac{0.01}{n})^n$ and this $n \to \infty$. In this case, according to our judgement, according to this, this number was 1 here, now it is 0.01 and this number converge to e. So, based on this understanding, if you apply the same logic and try to calculate this thing, then it will converge to e raised to 0.1.

This is an interesting revelation. That means, if you invest rupee 1, you just take that rupee $1 * e^{0.01}$ that will be the interest accumulated along with the original capital in are bank if the bank follows continuous compounding. This is how whenever you study finance, you calculate the interest rate. So, this is for the period of 1 year. Now if you add the period in terms of time, then it will be $e^{0.01t}$. So, this is how e becomes important. Let us now replace this 1 % by a generic number which is x.

So, what I am talking about now is $(1 + \frac{x}{n})^n$ and now from the discussion that we have done this will converge to e^x and when I add the time that is it is more than 1 year, then I have something like n_t and that is where I will get x_t. So, these are simple understanding why the number e is very important. e typically comes when you are considering a continuous compounding.

I hope I have made the relevance of the number e, irrational number e very clear and it is an irrational number and its exact value is given by this particular expression. It is not exact but it is approximate which is suitable for our purposes.

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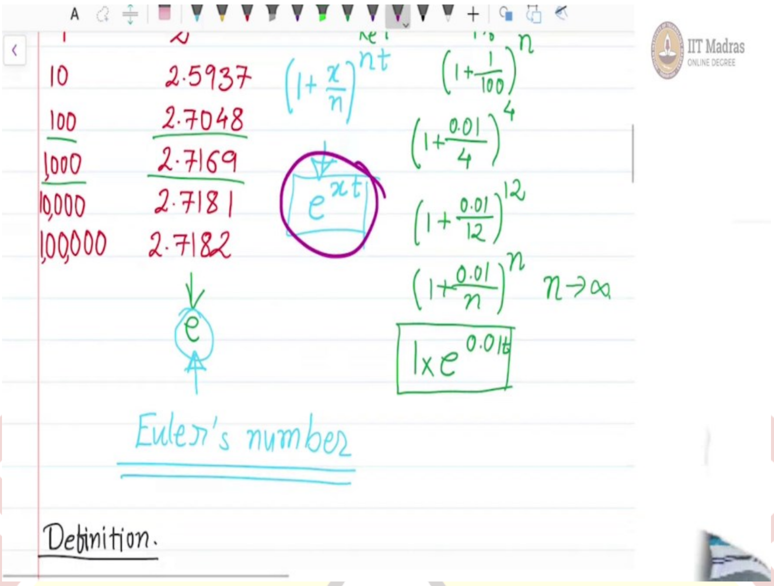

Now let us go further and understand what is a function that we have defined here and it is why it is called natural exponential function? So, this number e as I mentioned now naturally comes when you are considering continuous compounding. It also comes very naturally in the field of Differential Equations which is also relaying on our calculus. So, this number e has a special name when you consider differential equations as a area which is called Euler's Number.

So, you can Google and you can search the meaning of Euler's number and why it is relevant. So, that is how this e is called a natural exponential. So, now let us formally define the function that we have just now seen which is e raised to x_t as a natural exponential function.

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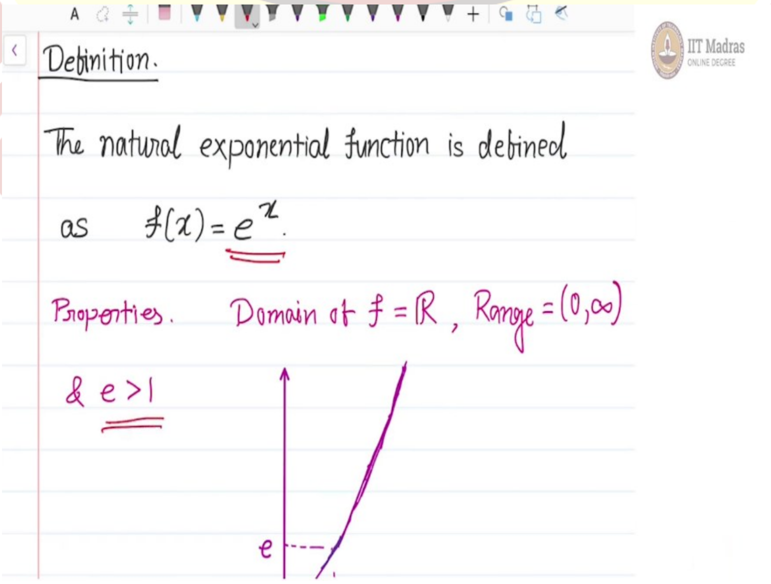

So, a natural exponential function is defined as $f(x) = e^x$. Then, you may ask a question, what are some interesting properties of this natural exponential function? Now, the properties will be very similar to the exponential function that we have studied, but it is special in some sense. We will see its specialness in a when we will study some special properties of this natural exponential function e^x. Let us list all the properties.

Domain of f, domain of this function will be set of real numbers and range of the function is positive real line that is 0 to ∞. As you have seen e, the value of e is 2.7182 so e is natural greater than 1.

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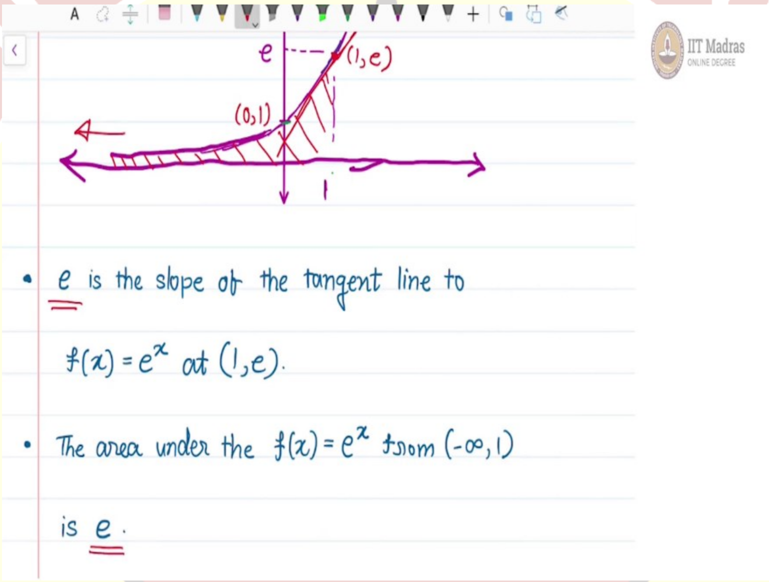

So, if you recollect whatever we studied for exponential functions, you will get the graph of exponential function in this manner where there are two typical points 0,1 is one typical point that it passes through and it will always pass through 1,e. As x tends to infinity, the function goes without bounds, as $x \to -\infty$, the function asymptotically goes to the x axis so $y = 0$ is the horizontal asymptote for this function, we have already seen that. For general exponential function same properties hold true.

Now what makes e special? And what is something special that is not true with general exponential function. So, in this case, if you look at the point (1,e) and if you draw a tangent to a line tangent to the curve, that is a line passing through this particular point, the slope of this line will be e, that is very special. So, e is the slope of the line that is tangent to the curve y is equal to e raised to x at (1,e). So, that is one thing.

Then if you look at the area that is covered under this curve from $-\infty$ to 1, that area is actually *e,* the irrational number *e.* This you will learn when you will study calculus in maths 2. So, that is very important.

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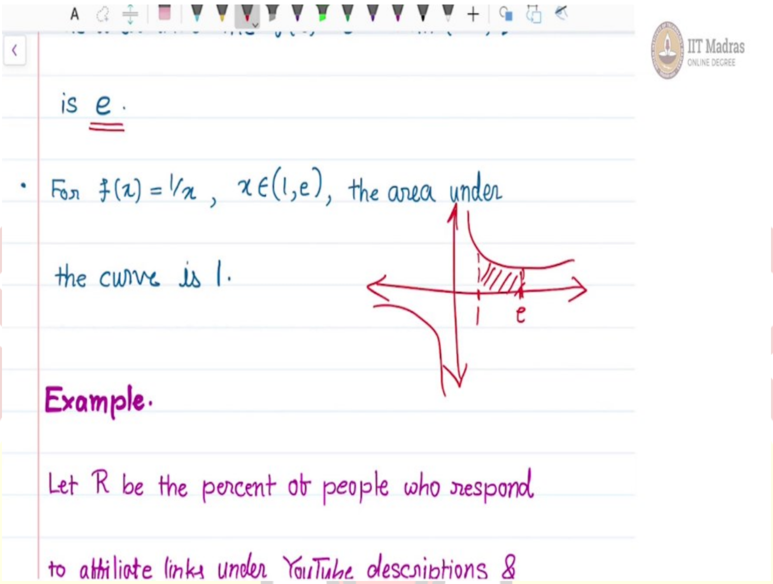

And the third thing that is very important which will not happen in general with other exponential function is if you draw a curve *f(x)* is *e*, if you draw a function $f(x) = \frac{1}{x}$. So, you may be familiar with that function, it will be something like this and something like this. And in this particular case, if you look at the area under the curve in the in the range 1 to *e*, this particular area, this area is a unit area for *f(x)* =*x* and remember this *e* is an irrational number so still it will be a unit area.

Why it is so? this is a matter of calculus to explore, but these are the things, these are some of the things that makes the function $f(x) = e^x$ special function. Let us now understand this function better by considering an example which will deal with our real life problems.

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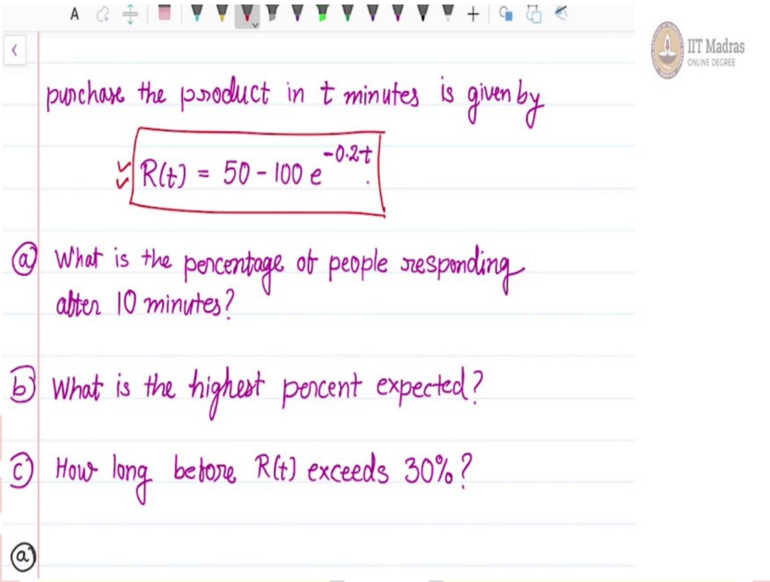

So, here is an example which says that let R be the percentage of people who respond to affiliate links under YouTube descriptions and the purchase and they purchase the product in t minutes and that particular purchasing thing is a function of time so it is given as the $R_t = 50 - 100e^{-0.2t}$. So, let me give you a brief understanding of the problem.

So, now when you watch some video on YouTube if you, the speaker in the YouTube says that there are some affiliate links below in the description. Now, if you click on that link and go to the affiliate site, then what you will do is, either you will purchase or you will not purchase. If you will purchase, the speaker or the channel owner will get some amount of commission.

Now, here the person who is actually giving the affiliate links is interested in finding the number of people who are responding in t minutes. So, he has devised a function which is available in YouTube statistics so based on the data available, he has derived a function, we are taking the function as it is. So, that function is $R_t = 50 - 100e^{-0.2t}$.

Now, he is interested in answering these questions. What percentage of people responding after 10 minutes? So, how many percentage of people responded after 10 minutes? Then, based on this function, what is the highest percentage expected? And the third question is how long before $R_t > 30$ %? The response rate being 30 % is also a good enough rate.

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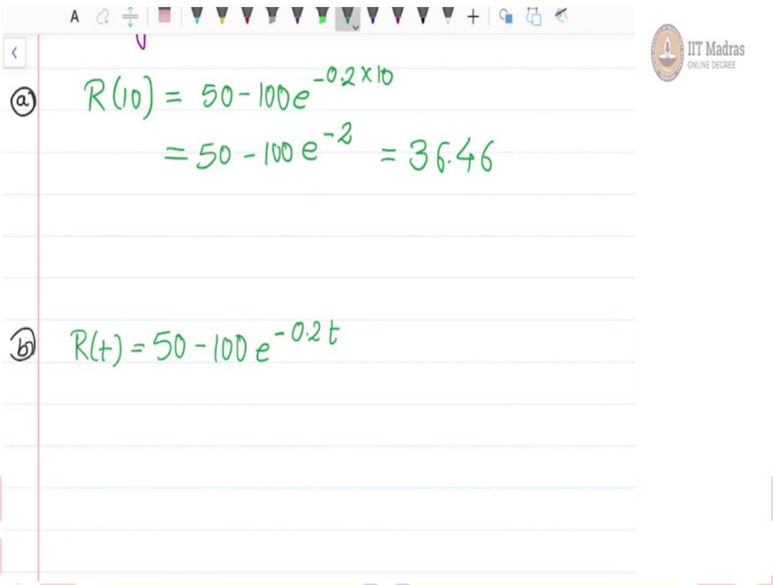

So, because you are just putting some affiliate links. So, let us try to understand what percentage or people will respond after 10 minutes? That means, I want to essentially evaluate the function as R_{10}. So, if I substitute this, it will be $50 - 100e^{-0.2\times10}$. that is simply if you rewrite this as 50 -100 times, this is 2/ 10 which can be simplified to e^{-2}. And then you can actually calculate the function e to by value of e^{-2} and you can put the value, that value is 36.46. So, this you can do it using calculator.

Now, let us look at the second question. What is the highest percentage expected? Now you have to think about this function which is $50 - 100e^{-0.2t}$. Now, you look at the function which is $e^{-0.2t}$ or e^{x}.

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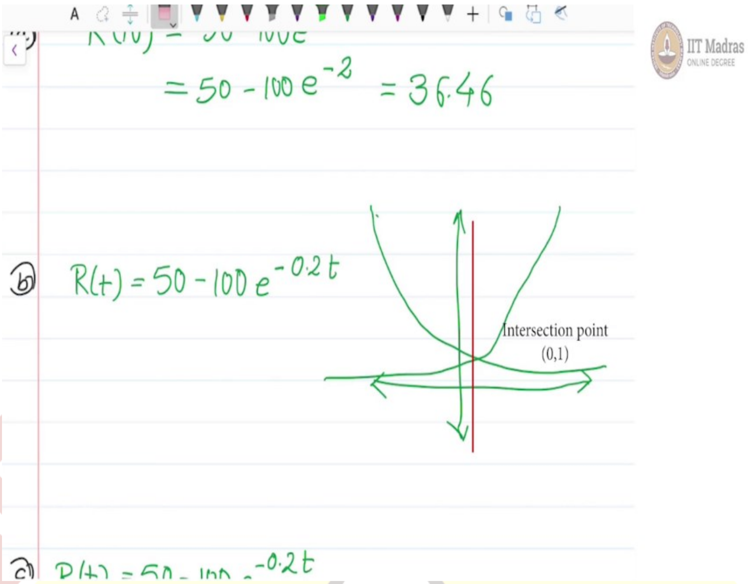

You already know the graph of the function which is e raised to x, $f(x) = e^x$. Now, how will the function look like when you are talking about $f(x) = e^{-x}$ has a graph of this form, roughly this form. Now, when you are talking about − of x, when you are talking about − of x, you are actually talking a reflection of this graph along this so that will give you some graph of this kind, it will never cross x axis but it will go this way.

So, now you have a understanding of how the graph of e^{-t} will look like. But here are some scaling versions, scaled versions like 100, this is 100 and this is 50. So, now this graph is actually multiplied with − 100, this graph is actually multiplied with -100, but multiplying with − 100 will again, what it will? It will actually keep the graph in a similar manner but it will actually because it is multiplied with − 100, it will shift in some sense like this.

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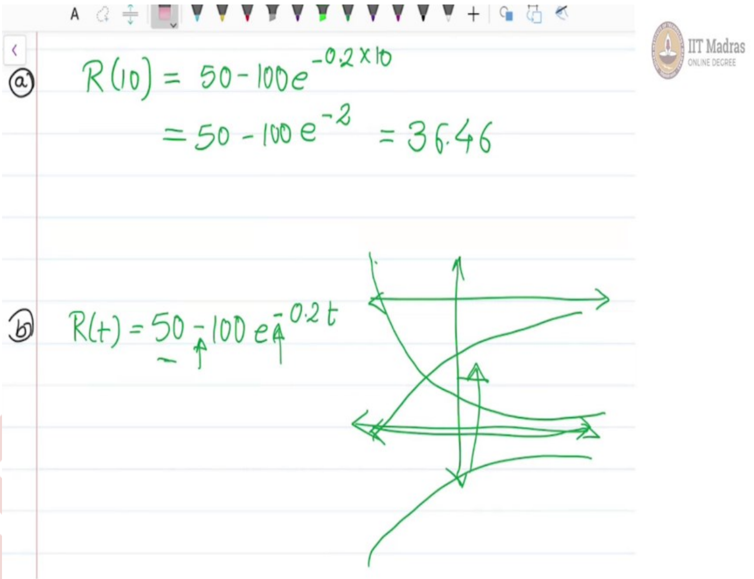

One minute let me chose and erase it. So, it will shift like, it will flip here and it will shift like this. And then, now when you are adding 50 to it, this graph will actually go up by 50 units. So, this way the changes will happen to the graph and finally graph will look something like this. You can actually check for yourself. So, basically first multiplying this − sign will have an effect of reflecting the graph along y axis, then multiplied with − 100 will reflect the graph along x axis and then adding 50 will shift the graph by 50 unit.

So, you have a fairly good understanding of the graph. Now, you just apply your knowledge that what is the highest percentage expected? So, in this case, if you understand this, the horizontal asymptote over here is actually shifted to 50 units because you are transferring to 50 and the graph actually let me clear up the image.

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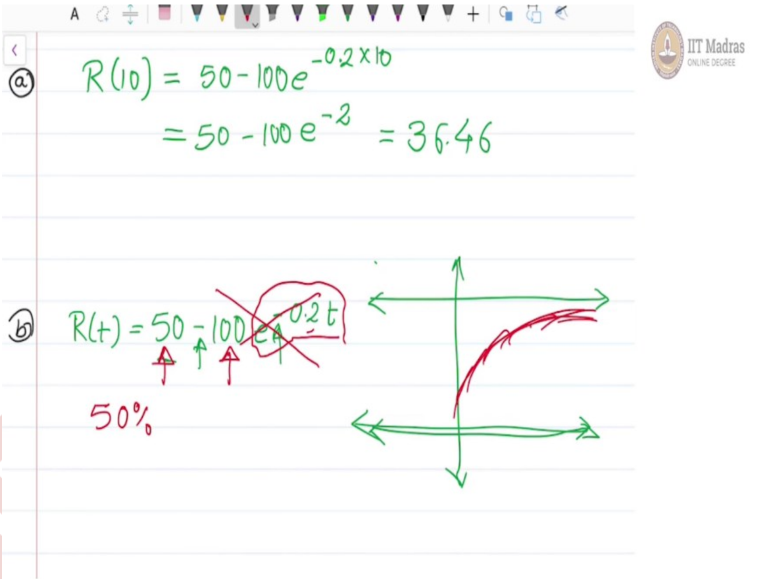

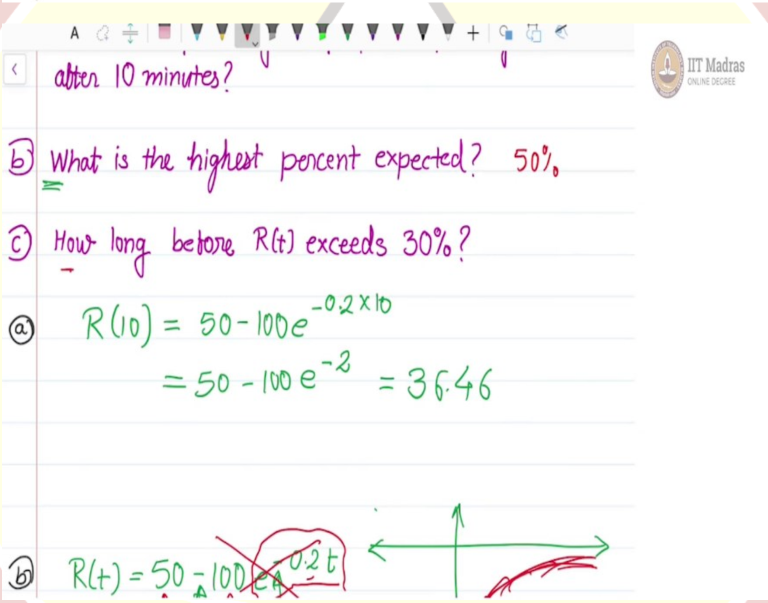

You can use your graphing tool also and verify that this is the graph. So the graph will look somewhat like this and it will asymptotically reach to 50 units. So, the highest percentage that is expected will be 50 %. It will not exceed 50 % based on the graphical analysis. Let us analyse this graph, instead of graphically analysing, let us look at this function e raised to − 0.2t, this function in itself will never exceed 1 and as x tends to infinity, this function will actually tend to 0.

That means, whether I am multiplying by 100 or I am multiplying by 10000, as $t \to \infty$, this function has to go to 0. So, this entire thing will go to 0 and therefore, 50 is the maximum that I can achieve. Therefore, my question b is answered as 50 %. Now let us look at this particular thing, how long it takes before R_t exceeds 30 %?

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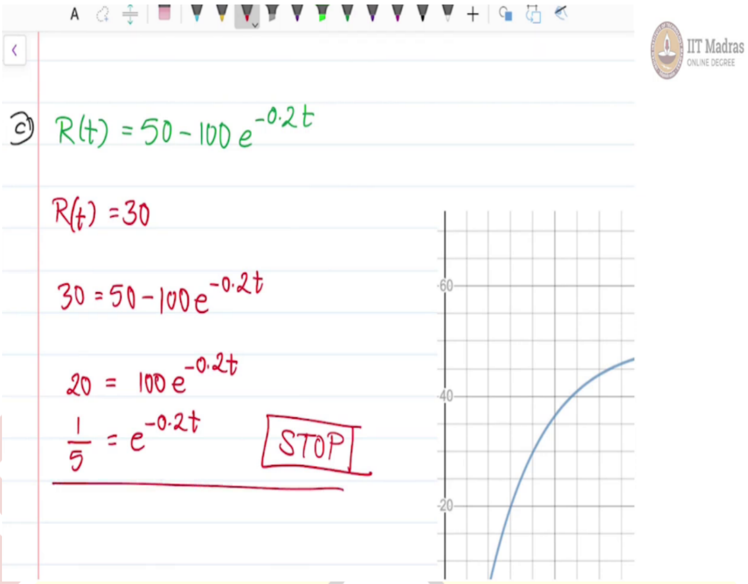

So, in this case you just look at the graph of the function, so as I have already described this will, this is how the graph of the function will look like. Now, there are two ways in which you can solve. Let us try to see whether we can go ahead formally and solve this. So, essentially I have been given that how long till R_t >30 %? That means R_t= 30, find are value of t such that $R_t = 30$?

So, 30 = 50 − 100$e^{-0.2t}$. So, 30 − 50 will give me something like − 20 = − 100$e^{-0.2t}$. Now, this is, these − signs will cancel themselves off so, this is this and then you simply rewrite this expression $\frac{20}{100}$is nothing but $\frac{1}{5} = e^{-0.2t}$. Now, I have to stop here because right now I do not have any ways to see what t will be when $\frac{1}{5} = e^{-0.2t}$. No analytical way is possible.

Then, what I will do is, so analytically I am stopping here and if I somehow I am able to figure out how to find t is equal to something, then I can answer this question. But let us now try to compute this graphically.

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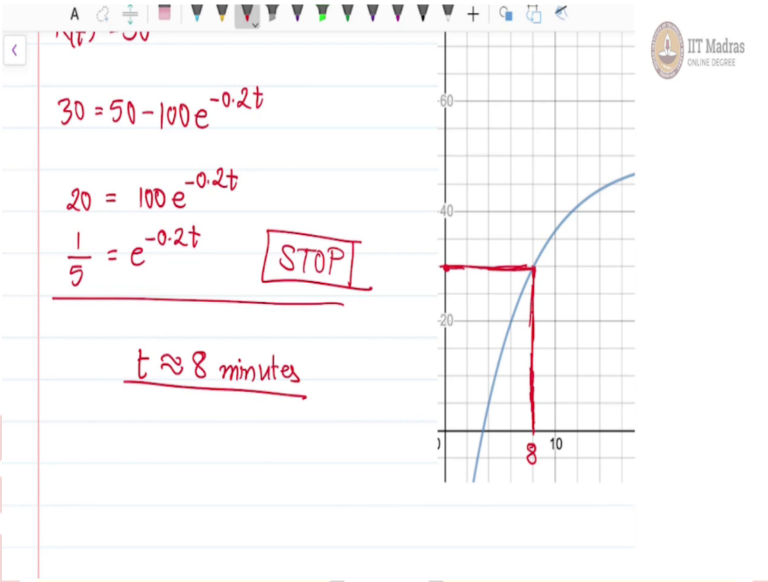

So, in this case, $R_t = 30$ is this point. So, now if you go along this line and then you map this onto x axis, remember x axis is nothing but the value of t. So, in this case, now you look at the mesh, this roughly turns out to be 8 that means, t will be approximately equal to 8 minutes. So, this is how we can without even solving the expression for this, graphically solve the expression for exponential functions. So, this is one live demo of that. That is all for now. So, we will meet in the next video where we will actually try to understand how to solve this particular problem. Thank you.

Mathematics for Data Science 1
Professor Neelesh S Upadhye
Department of Mathematics
Indian Institute of Technology, Madras
Lecture 48
Composite Functions

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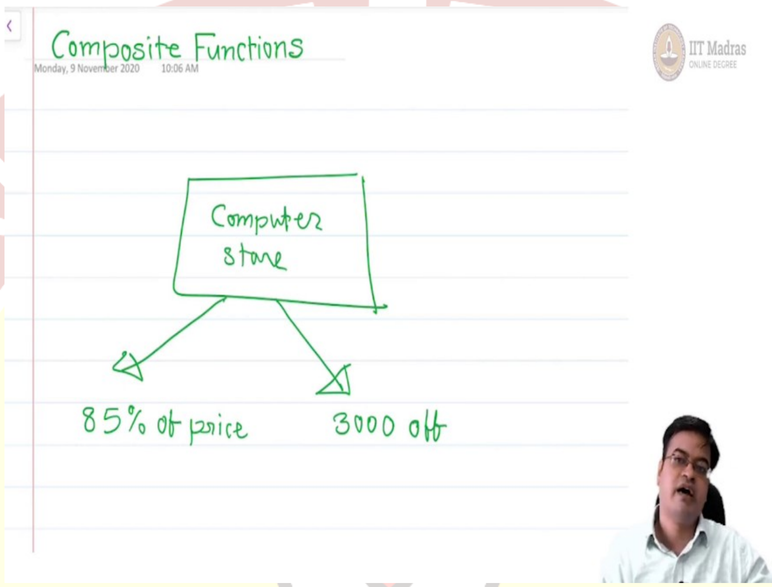

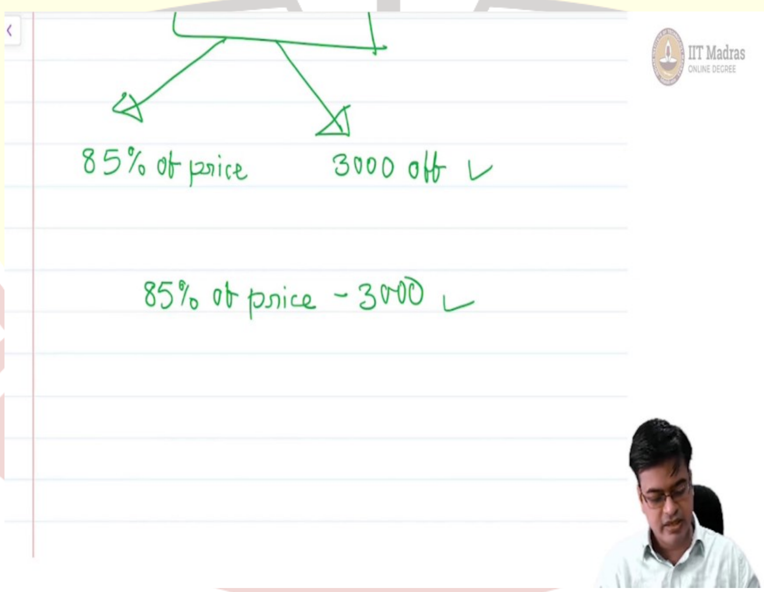

Hello students, today we are going to learn the concept of composite functions, what do you mean by a composite function? So, let me motivate this with an example. For example, it is known that you are a very good bargainer and your friend wants to buy a computer. So, your friend takes you to a computer store, so this is a computer store, in this computer store there are two offers available.

So, something is on sale, all items are on sale and there are two offers available, one offer is you will get 85% of the price, whatever you buy you will get the product at 85% of the price. And the other offer is you will get flat 3000 off on the MRP, the maximum retail price you will get 3000 off. So, these are the two offers that are available.

Obviously because you are a good bargainer, you bargain with a salesperson and you strike a deal that is the computer that you want to buy will be given to you at 85% of the price and of the amount, once the 85% of the price is decided further 3000 will be given you as a discount. So, there is a discount of rupees 3000 as well as you are getting 85% of the price.

Now, this kind of thing when we write mathematically can be considered as composite functions, you are in fact using these kind of tricks in a day-to-day life. So, let us see what happens when we put this mathematically and how composite functions arise. So, let us say the first draft that is 85% of the price. So, can I represent this as a function?

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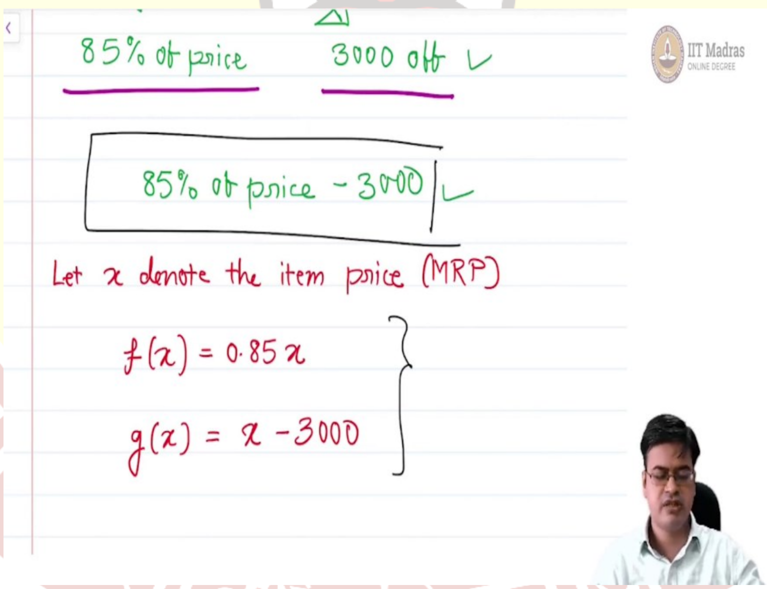

So, for cleanliness let us write let $x$denote the item price. So, let $x$denote the item price which is the MRP, you can write maximum retail price and on that you are getting 15% discount that is 85% of the price you are getting. So, I can write this particular offer as $f(x)$ which is nothing but 0.85 times x.

Now, the other offer that is on in this particular computer store is this. So, I can write this as g of $x$to be equal to if $x$is the MRP I will subtract 3000 rupees from x, so these are the two offers that are available. Now, what we did is we want best of both the offers. Now, when a store is offering these two offers it is safe to assume that you may not have any item that is less than 3000 rupees, you may not have any item on sale which is less than 3000 rupees, so your $x$will always be greater than 3000.

Another thing that you can assume that because the store is offering you this kind of thing, that is 85% of the price -3000, the store has already taken care of that they do not have to pay back any money, that means after giving the 85% of the price, the price should be greater than 3000, so all these conditions are assumed implicitly, which we will deal with them in later when we will formulate a problem.

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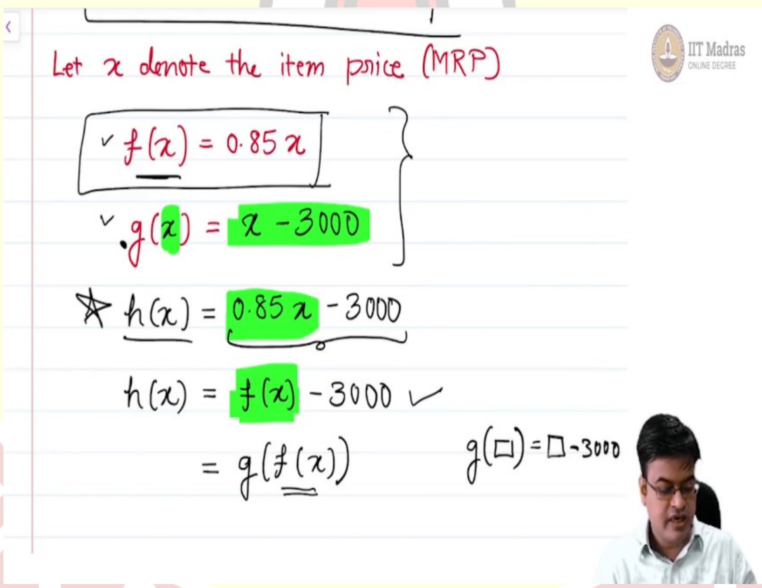

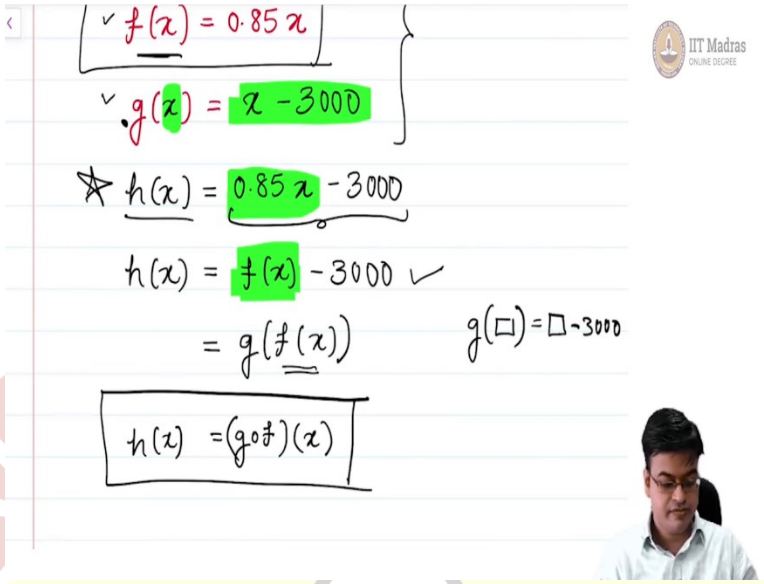

So, now the offer that you got if I want to write this offer mathematically I can write this as some function $h(x)$ which is equal to it is 85% of the price -3000. So, now when we are dealing with functions in mathematics it is good to see if I have some correspondence of the function h with these functions f and g, this is the question that we are trying to answer when we are studying composite functions.

So, let us first see what is being done over here, that is if I use this f then it is 0.85 x, so if I want to do something like this then I can write this as $h(x)$ is equal to $f(x) - 3000$ is that a safe assumption to do? Yes, of course because $f(x)$ is 0.85 x so what I am essentially doing is, I am for this particular term I am substituting $f(x)$, so it is a perfectly valid guess, fine.

Now, if you treat this f, if you treat this $f(x)$ as one argument like $x$then what you are actually doing, you are actually saying it is $x$minus 3000 that means instead of this x had it been $f(x)$ you would have written $f(x) - 3000$. So, I will use that knowledge and I will try to do, I will try to rewrite this as, this is g times f(x). Is this acceptable? Let us redo the math.

For example, what is g times $f(x)$? So, if you look at g of, f of, $g(x)$, so whatever is x you will write that $x - 3000$ or whatever, let me put it this way if g had some box inside it then I will write that box -3000. So, in particular, in that box right now $f(x)$ is written, so I will substitute it as $f(x)$ minus 3000, done?

And what is *f(x)?* Now, *f(x)* as you know is nothing but 0.85× x . Therefore, I can rewrite this function as *g(f(x))*. In mathematics you will rewrite this as *g(f(x))*, so my *h(x)* can also be written in terms of g and f in this fashion. So, this is the motivation for composition of two functions. So, in particular what we have seen is a practical example, we motivated it through a practical example of a computer store which is offering two kinds of sales, one is 85% of the price, another one is flat 3000 off on the MRP. So, after doing this you can easily guess that how will, how will I evaluate this function, how will I evaluate this function, that is what we have to see.

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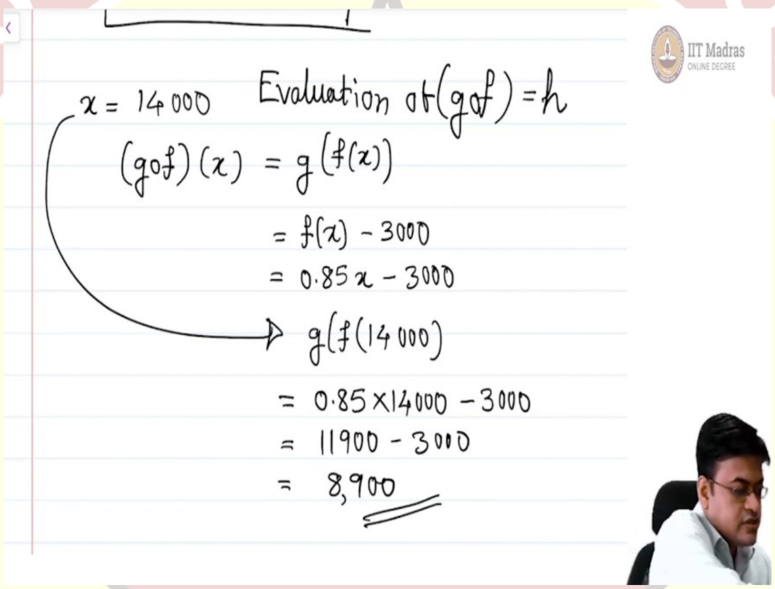

So, in particular let us say your $x$in this particular function is say you can take it to be 14000 let us say, 14000 is your $x$and you are asked to calculate *g o f(x).* So, how will you calculate? It is very simple, you will first insert g(f(x)). So, what is f(x)? f of x is nothing but point okay, let us follow the same notion the way we followed, so in particular in this case this is what will happen, this is going to be equal to $f(x) - 3000$.

What is $f(x)$? *f(x)* is going to be $0.85 \times (x - 3000)$, so I will substitute the value 14000 over here which will give me, so since my x is 14000 I will plug this value in, so I am calculating g of f of 14000. What will be g of f of 14000? Again you have to do a similar calculation which will give me 0.85 multiplied with 14000 -3000 so this I think comes out to be 11900 just check if I am calculating it correctly -3000 which will give me 8900, 3000, 900 as it is 11 -3, 8, yes, so the final answer is 8900, this is what, this is actually, what I have just now shown is evaluation of a composite function which is *g o f,* what is, which is actually *h*, there is nothing special in this, it is just a nomenclature that we are using.

But this kind of composition helps you in understanding lot of things. So, let me formally define what is the composition of a function and how we are going to handle them mathematically. Because composition of a function as you must have seen is again a function. So, natural questions about domain, range will arise and we will try to answer them as and when they come.

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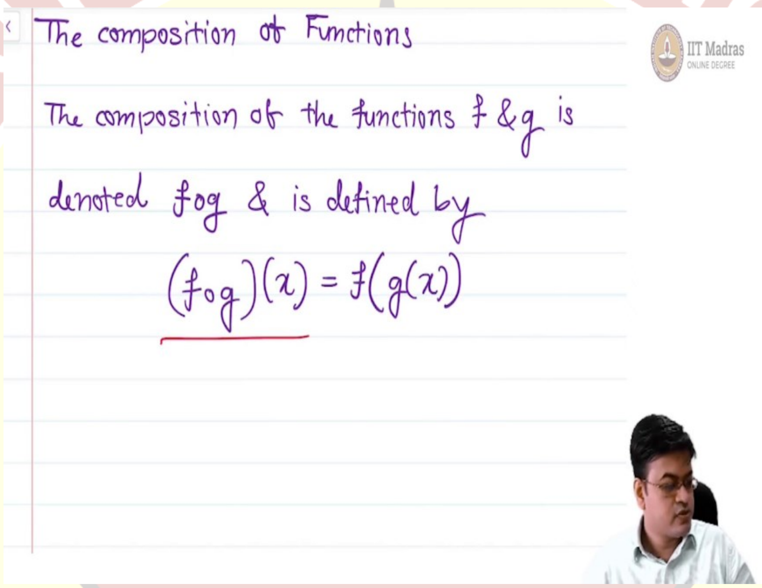

So, let me formally define the composition of functions. What is the composition of function? So, in particular we can write as the composition of functions f and g composition of the functions, there are two, at least two functions you need, functions f and g or we can write the composition of the function *f* with g that is also a valid terminology is denoted by, I have already defined this notation *f o g* and is defined by *f o g,* this is one function of *x*, so you can write this as *f(g(x))*.

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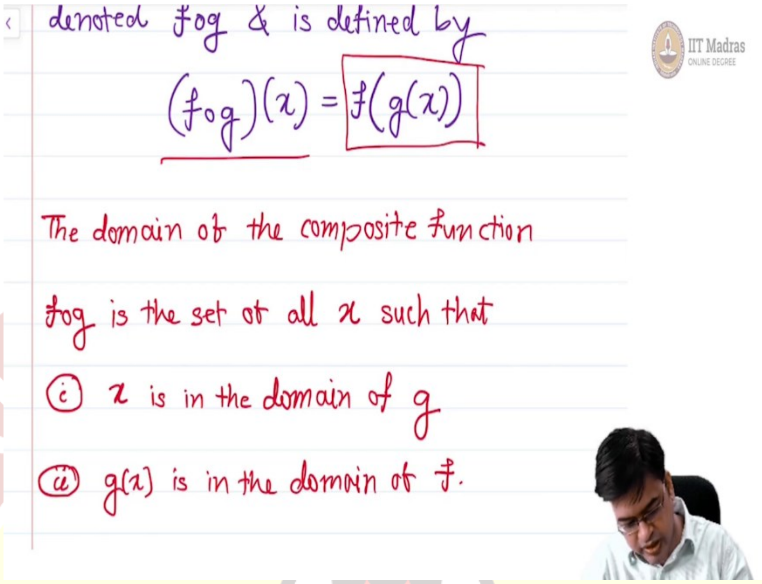

So, naturally the next question is what should be the domain of this function, so that we will answer as the domain of the composite function *f o g*, let me write it here, *f o g* is actually the set of, is the set of all *x*such that the two conditions we require and they are pretty evident, as we go further we will realize how these two conditions are evident.

So, the first condition is *x* is in the domain of g and second condition is it will be about *x*so if g *x*is something that you are figuring out. Now, that *g(x)* should be in the domain of *f, g(x)* is in the domain of *f*. So, now why these two conditions are required that is what we need to figure out. For that you need to focus on this particular component *f (g(x))*. Let us use this particular component and try to answer the question.

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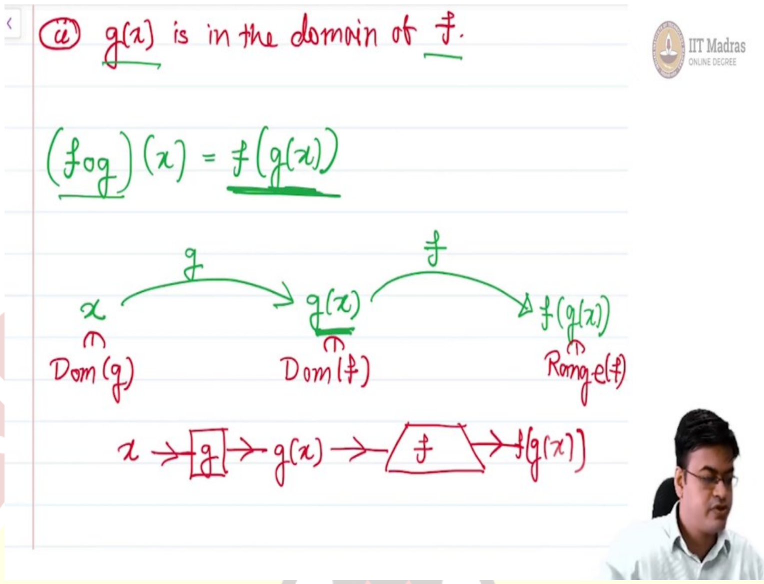

So, I have, if when I talk about *f o g(x),* what I am talking about is *f(g(x)).* Now, let us look at the first condition. If I want something to be in the domain of *f o g* that means it should be well defined, so when I input the value it should give me the output, if there is some ambiguity then it is not a properly defined function. So, let us say why this condition x is in the domain of g.

What if x is not in the domain of g? g of x is not defined, because g is defined only over domain of g, so g of $x$is not defined and therefore you need this condition that $x$should be in the domain of g. Now, when I am using this composite function, I am applying f to the value that is obtained by applying g, so it is g of $x$that is playing the part.

So, now if this g of x that is the value of $x$which is in the domain of g if that particular value g of $x$is not in the domain of f then again this f(g(x)) is not defined. Therefore, I need g of $x$also to be in the domain of f. So, in particular you can visualize it this way. So, if I have $x$then there is a map which maps everything that map is g and that maps it to a value called g of x.

Now, this *g(x)* should be in the domain of f because I will take this value to a function which is f(g(x)). So, this is another value and what is the application? *f* is the application, we are applying the function f to the value *g(x),* if this $g(\ x)$ is not, does not belong to domain of f then our function is not defined.

So, you can actually remember this diagram by using this particular, this belongs to, what it belongs to? It belongs to domain of g, this particular thing actually belongs to domain of f, this is my abbreviation for domain and this is nothing but the range of f, so this will be in the range of f but it can be smaller than the range of f because $g(x)$ may not cover the entire domain of f.

So, it can be smaller but this will belong to range of f or if you want to visualize it in a better manner there is something which is box, you feed an input to this box x, g is this box and it will throw out $g(x)$, so when you feed x, this will spit out $g(x)$. Now, for $g(x)$ to be fed into another let us say this is a trapezium and that trapezium is a function f, this is a machine, function machine that is $g(x)$.

If this $g(x)$ is not in the domain of f, then this machine is unable to produce the output, so I need this g(x) to be in the domain of f in order to get the output which is $f(g(x))$. So, this is how you can always remember how to compute the composition of two functions and what are the necessary steps that are required to compute this composition.

Mathematical for Data Science 1
Professor Neelesh S Upadhye
Department of Mathematics
Indian Institute of Technology, Madras
Lecture 49
Composition Functions: Examples

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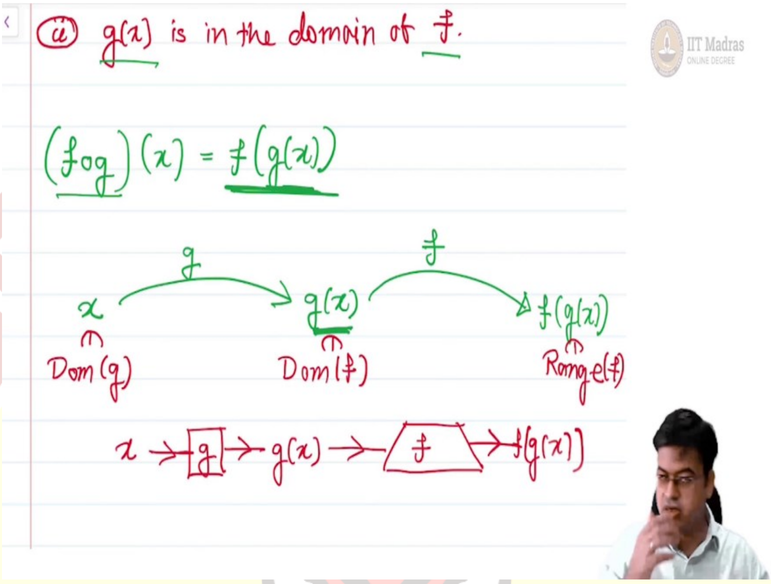

So, we have understood the theory, roughly the theory behind the function, composition, composite functions or composition of two functions. So, it's time to get some practice.

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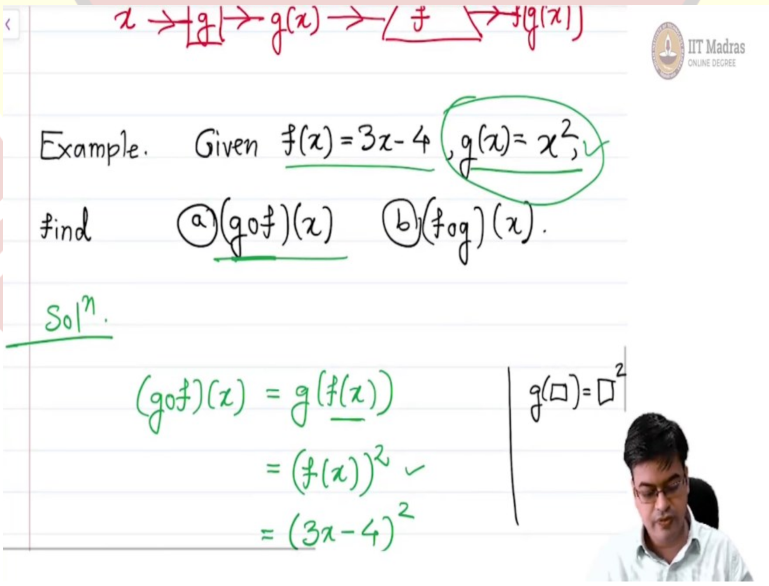

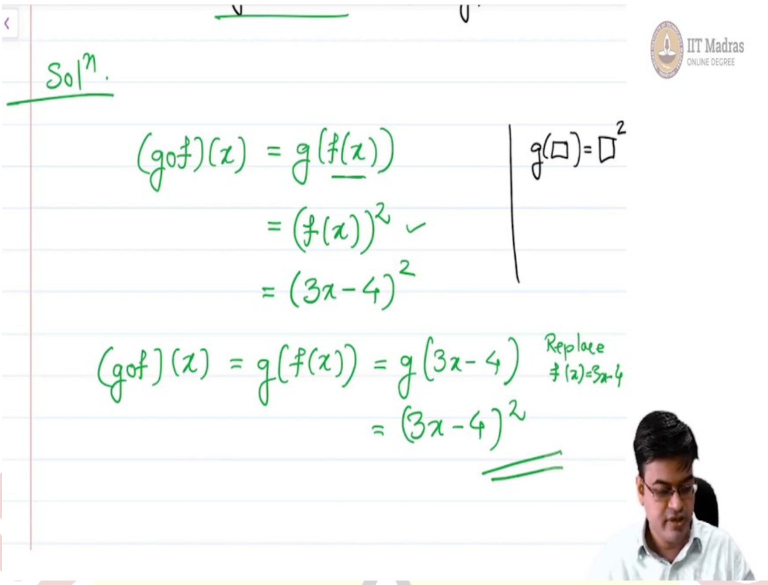

So, let me start with an example. And in that example, let us take you have been given two functions f(x)=3x-4, and g(x), which is equal to let us say x^2 these are the two functions that are given, then you are asked to find two things one is *g o f(x),* and the other one is obviously *f o g(x),* how to find this? Let us start, let us start with a solution.

So, what can be the solution let us take this function. So, let me write it properly, it is *g o f(x).* So, as per our theory, we have to write this as *g(f(x)).* So, *g(f(x)),* you can treat this as, what is *f(x)* now? $f(x) = 3x - 4$, and g(x) is x square. So, naturally *g(f(x)),* so, you go to this function, you treat this g as g. So, let me write it here, you treat this *g* as a *g* of a box, and *g(x)* is nothing but this box squared. So, in particular, if I want to write something about this function, this box right now has an argument which is *f(x).*

So, I will simply write this as f(x) squared, that is all. Now, the entire process is simplified. So, now, you do not have to worry about what g is, now it simply $f(x)^2$ what is the *f(x)* fit that when you in and you will get $(3x - 4)^2$. Another way to handle this is you can simply write *g o f(x)* as *g(f(x))* fit in the value of *f(x)* that is $g(3x - 4)$ and what is $g(3x - 4)$ as per our question, it is x^2. So, $3g(3x - 4)$ will be $(3x - 4)^2$.So, anyway whichever way is convenient to you, you proceed and you will get this answer correct.

So, what I have done here is I have replaced *f(x)* in this particular case, I have replaced $f(x) = 3x - 4$ in this particular case I have written *f(x)* and replaced what is *g(x).* So, both ways you can go now.

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$= (3x-4)^2$

$$(f \circ g)(x) = f(g(x))$$
$$= 3g(x) - 4$$
$$= 3x^2 - 4.$$

$f(\Delta) = 3\Delta - 4$

Sol^n.

$$(g \circ f)(x) = g(f(x))$$
$$= (f(x))^2$$
$$= (3x-4)^2$$

$g(\square) = \square^2$

$$(g \circ f)(x) = g(f(x)) = g(3x-4)$$
$$= (3x-4)^2$$

Replace $f(x) = 3x-4$

$x \to \boxed{g} \to g(x) \to \boxed{f} \to f(g(x))$

Example. Given $f(x) = 3x - 4$, $g(x) = x^2$,

find (a) $(g \circ f)(x)$ (b) $(f \circ g)(x)$.

Sol^n.

$$(g \circ f)(x) = g(f(x))$$
$$= (f(x))^2$$
$$= (3x-4)^2$$

$g(\square) = \square^2$

Let us go to the second problem that is a *f o g(x)* and *f o g(x)* is again can be written as *f of g(x).* Clear, there is no question, then there are two ways let us go it the first way, what is *f(g(x))?* So, what is *f(x)* here? $f(x) = 3x - 4$ here. So, I will write this as to be equal to $3g(x) - 4$.

So again, let me be very clear about this there should not be any confusion in this. So, what is $f(\Delta)$? Δ is an argument. So, this Δtriangle will be $3\Delta - 4$. So, now this triangle is replaced with *g(x)*, that is all. Therefore, your answer is *3x-3g(x)- 4*. But what is *g(x)?* Again, go back to the question *g(x)* is x^2 So, substituted here that means it will be $3x^2 - 4$ and this is the final answer for you in terms of *f o g(x).* So, we are seen how to write the compositions in both ways *g o f* and *f o g.*

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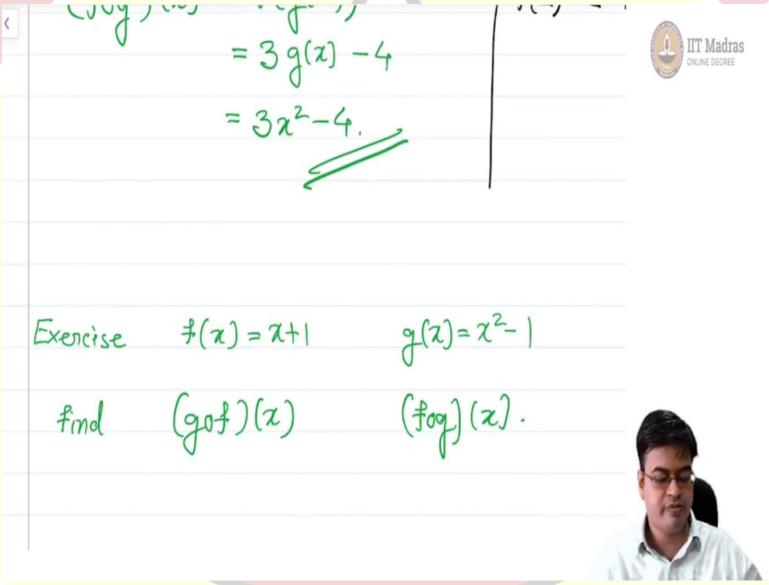

So, here is a quick exercise for you pause the video, do the exercise and get back the get the answer. So, $f(x) = x + 1$ and $g(x)$=$x^2 - 1$. Then simply find *g o f(x)* and *f o g(x)*. This is an exercise you stop and get the answer. It will be a good practice to revise the concepts.

Mathematics for Data Science 1
Professor Neelesh S Upadhye
Department of Mathematics
Indian Institute of Technology, Madras
Lecture 50
Composite Functions: Domain

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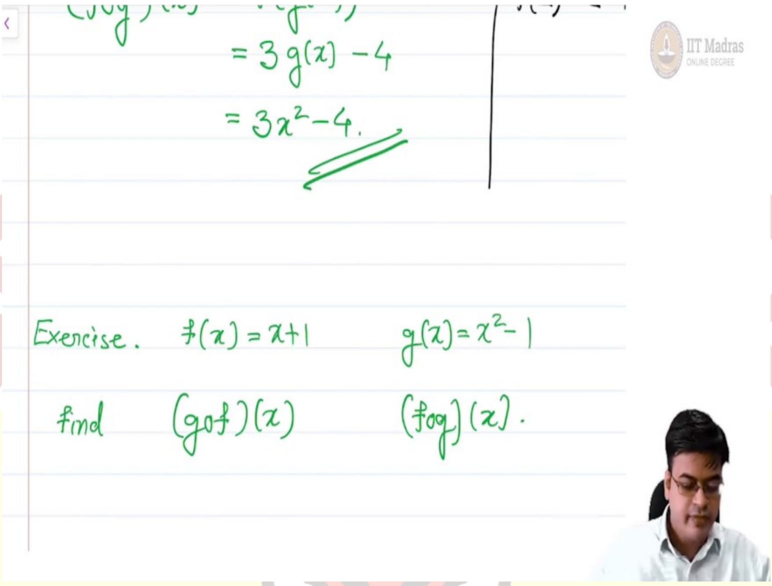

Let us now go further and talk about how to determine the domain of composite functions. So, this will be any important question that is determination of domain of a composite function.

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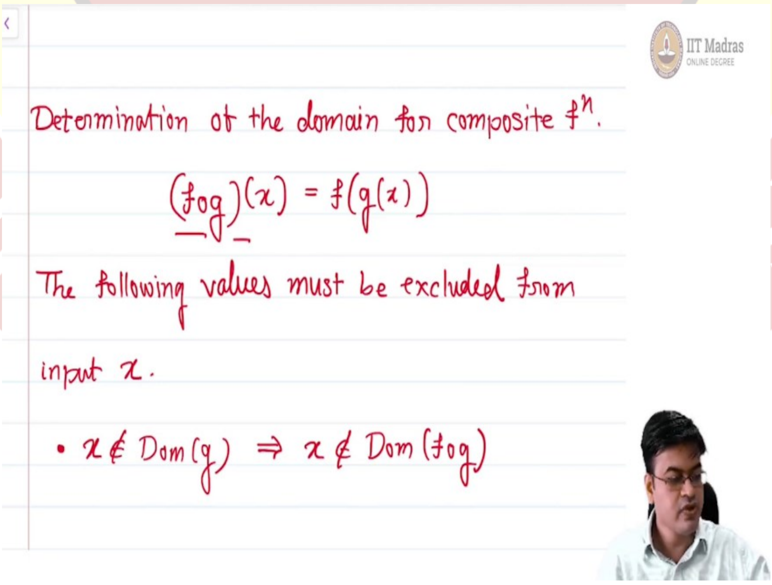

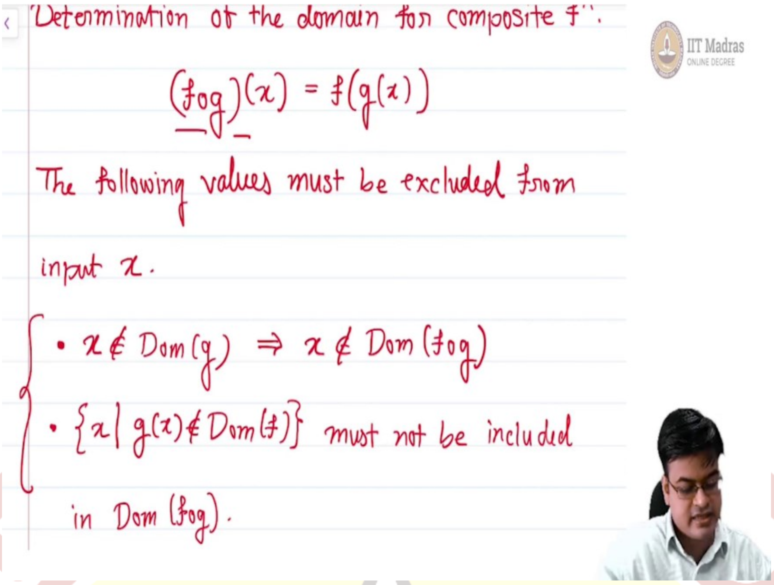

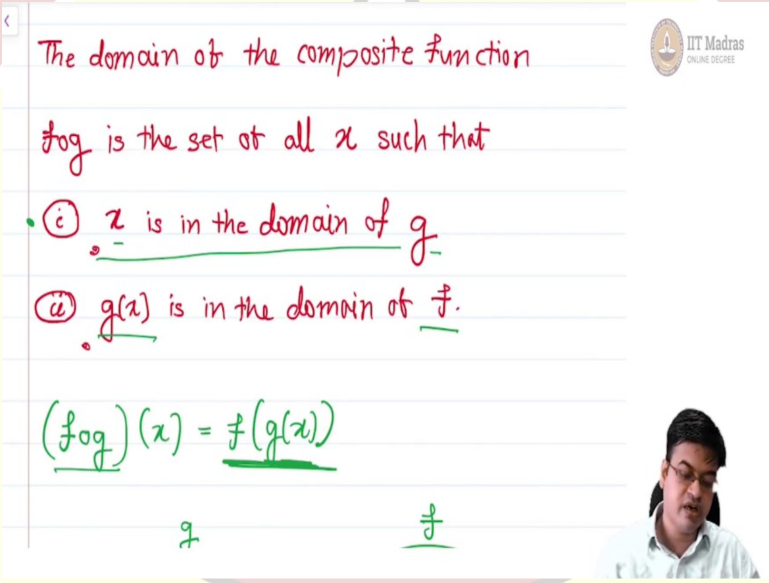

Determination of the domain let us say for domain for composite function, how will you determine this? So, I have let us say *f o g(x) = f(g(x)),* we are talking about all functions that are real value. So, in order to determine the domain there must be some rules that you should follow I will list the rules and that essentially says the following rules, the following rules must be followed and therefore, the following values must be excluded from input values of x.

So, this is again in concordance with what we have seen earlier that if you remember we have seen some conditions right, where *x* should be in the domain of *g* and *g(x)* should be in the domain of *f*. So, again what we are discussing now is in concordance with that, but here we were seeing what are the possible values.

Now, what we are seeing is what are the possible exclusions, that means, what value should be excluded from the input values. So, there are basically two rules the first rule which corresponds to the first rule of this that x should be in the domain of g that means, x if x is not in the domain of g then I cannot include it then x cannot be in the domain of the function $f\ o\ g$.

So, I am talking about $f\ o\ g$, when you talk about $g\ o\ f$, you will talk about the $x$belonging to domain of f implies. So, x does not belong to the domain of f implies x does not belong to domain of $g\ o\ f$. So, just remember the function the order in which they are taken it matters and in the similar manner, when I talked about $g(x)$ belonging to the domain of f. So, the set of all x's such that g $x$does not belong to the domain of f.

So, this is the set that you need to be careful about this set must not be included in domain of our function $f\ o\ g$ that is a composite function otherwise, we will have some ambiguity. So, in order to eliminate the ambiguity, we need to follow these two rules strictly very strictly. So, let me demonstrate how these rules can fail and then we will I will demonstrate it through an example and let me take that example as that is write it here.

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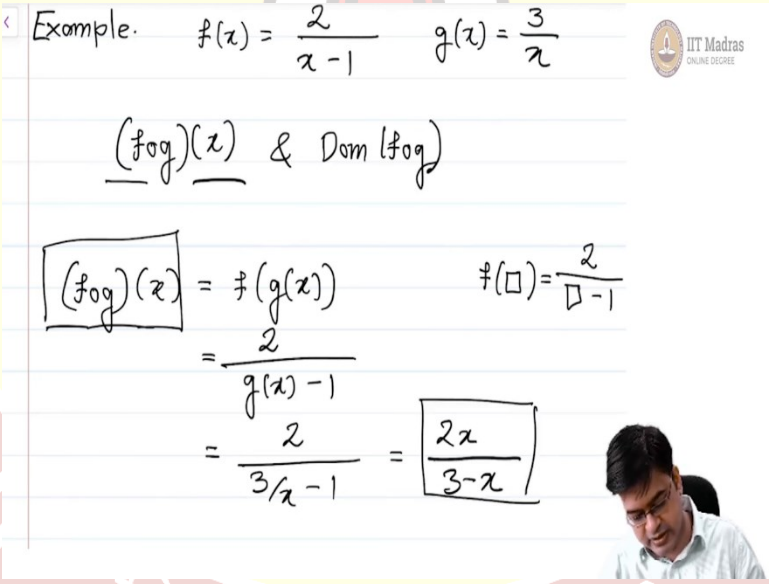

So example, so I have been given a function $f(x) = \frac{2}{x-1}$ and another function that is given to me is $g(x)= \frac{3}{x}$ and you want to find $f\ o\ g(x)$ and you also need to find a domain of this function $f\ o\ g$. What is the domain? Domain if you recollect from your week 1 it is nothing but the set of allowed values for which the function is well defined whatever input values you are fitting into the function, this function should be well defined this is the domain This is the notion of domain.

So, let us first see what is *f o g(x)?* And let us see if it gives you some hints about what can happen, correct? So, what is *f o g(x)?* Simply apply our definition it is *f(g(x)),* fine no confusion in this, then again you use that $f(\boxdot) = \frac{2}{\boxdot - 1}$. So, that gives me $\frac{2}{g(x)-1}$.

Now, what is *g(x),* it is $\frac{3}{x}$. So, substitute what is *g(x)?* So, it will be $\frac{3}{x-1}$, simplify this assume $x$is not equal to 0 and simplify this you will get $\frac{2x}{3-x}$. So, this is my *f o g(x).* Now, the question the second question that is asked is, so I have given answer what is a *f o g(x).*

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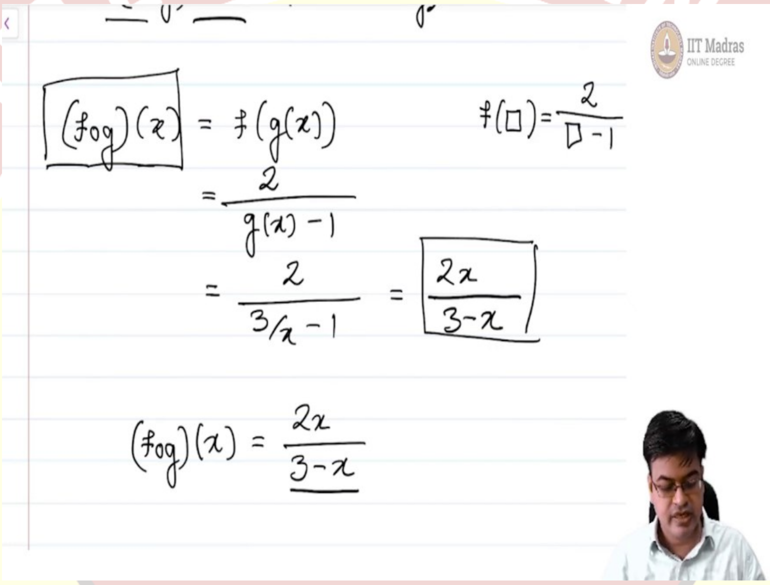

So, my $fog(x) = \frac{2x}{3-x}$. Now, if you look at this function, if you look at this function, you can simply see that at *x=3* this function is not defined, because the denominator is becoming 0. So, 6 / 0 is undefined. So, this function is not defined at *x=3*. So, the domain of this function must exclude 3 that is very well known.

But let us now see because of composition if I am eliminating any points, so here you look at this function which is *f(x).* And you look at this function which is *g(x)* and I am calculating *f o g(x).* So, if x does not belong to domain of g, then that function that particular value of x should not belong to domain of *f o g* that is the first rule that we have to implement.

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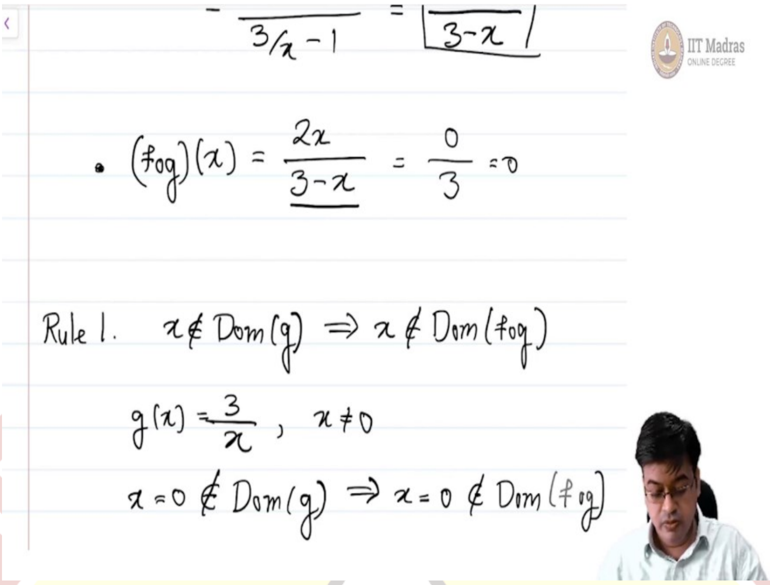

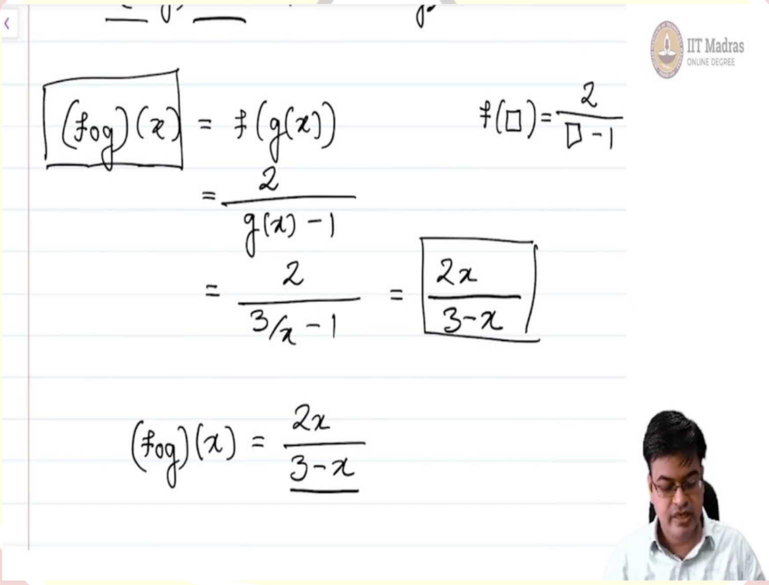

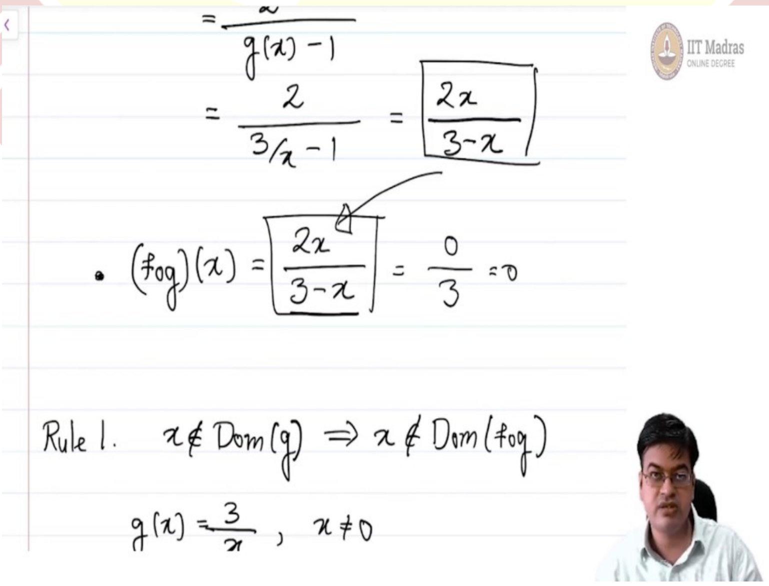

So, rule 1, what is the rule 1? If x does not belong to the domain of g that must imply x does not belong to the domain of f o g. So, what is that point? Let us look at what is *g(x)?* $g(x) = \frac{3}{x}$. So, this function is well defined only when $x \neq 0$. So, $x \neq 0$ not equal to 0. So, $x = 0$ cannot belong to domain of g. So, $x = 0$ do not belong to domain of g. So, naturally I will enforce that $x$equal to 0 should not belong to domain of $f\,o\,g$.

So, now, you may come up with some argument that when you look at this function, when you look at this function, if I substitute $x = 0$ if I substitute x=0, I am getting 0/3. Then this function is well defined because the answer is 0. That is what your argument will be. But no, why? I will tell you because when we when, when we were while we were coming to this particular form, what we were doing actually is we were multiplying a numerator and denominator by x or we are taking assuming $x$not equal to 0.

We are taking this x on the numerator on the numerator side and multiplying by $x$and that is where we have reached this point. If we had not assumed $x \neq 0$, then we would not have reached this point. Therefore, $x \neq 0$ is a valid condition still even when you cannot see anything visible over here, because I am composing the 2 functions where $x \neq 0$ is outside the domain.

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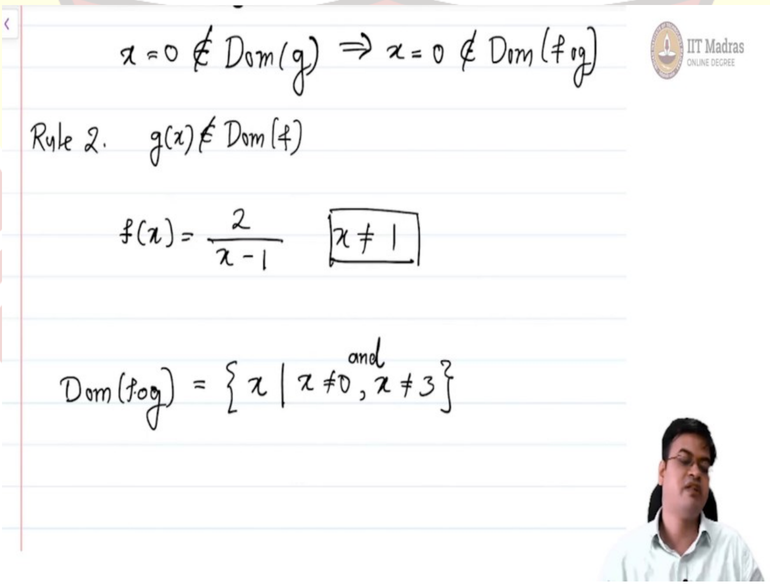

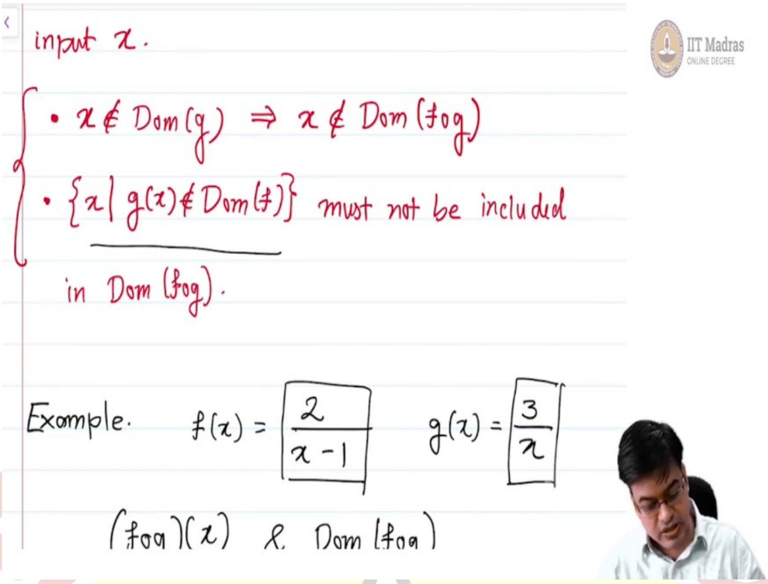

So, let us come to the next rule 2 that rule 2 was if *g(x)* does not belong to the domain of *f* then I am having a problem. So, that rule we have figured out like x says that *g(x)* does not belong to the domain of *f* must be excluded. So, let us look at our function *f* what is our function f it is 2 upon $x$minus 1 in this case x=*1* I have a function where the denominator is 0.

So, x is equal. So, let me write for the sake of completeness $f(x) = \frac{2}{x-1}$ this is well defined when $x \neq 1$. So, so this also this point $x \neq 1$ should also be eliminated from the domain of *f o g*. So, what should be the domain of a *f o g*? All other points the function *f* and *g* are well defined. So, domain of *f o g* must be set of all *x*'s belonging to real line such that $x \neq 0$ and $x \neq 3$, this comma means and or if you want me to be precise, I will write.

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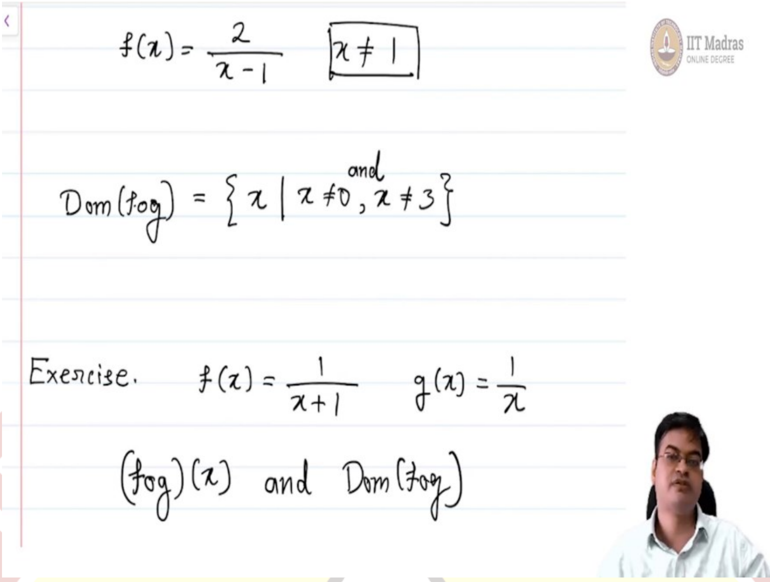

Another quick exercise that you can do in order to verify whether you have understood the concept of composition of function and the domain is you have been given 2 functions $f(x) = \frac{1}{x+1}$ and $g(x) = \frac{1}{x}$ and you are asked to find $f \, o \, g(x)$ and the domain of $f \, o \, g$. So, you can quickly solve this problem and check whether you have understood what we are supposed to understand. Thank you.

Mathematics for Data Science 1
Professor. Neelesh S Upadhye
Department of Mathematics
Indian Institute of Technology, Madras
Lecture No. 51
Inverse Functions

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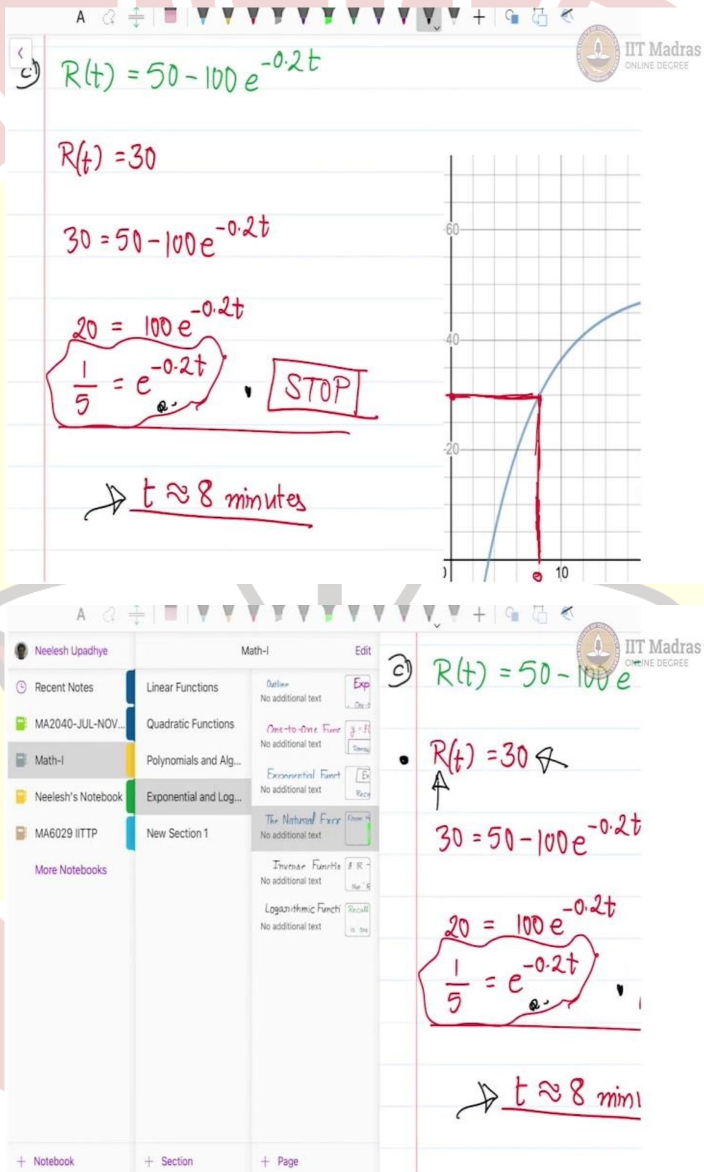

Hello Students, in the last video we have stumbled upon one concept, where we could not proceed. Then we came to…Let us go to the last videos last slide. So, here if you look at this particular concept we actually stopped while computing. And why we stop while computing is because we did not have enough information on, how to write t is equal to something given this equation.

So when, what we did we found escape out by plotting the lines across x and y $axis$, horizontal and vertical lines and figured out that the answer is 8 minutes. And that is how we concluded this is 8 minutes. Now when we start such a thing analytically that is R_t is given to be 30. What is the value of t? We want to answer such questions then we need to look at the function R and we need to understand whether this function is reversible or not.

Which is the case, in this case, in this particular function because we were able to map it uniquely. So, what are the important trades of this function R_t? R_t was a one to one function and it was increasing function. Hence, it was one to one. Therefore, we were able to find a reversal of the value 30 to the value of t which is 8.

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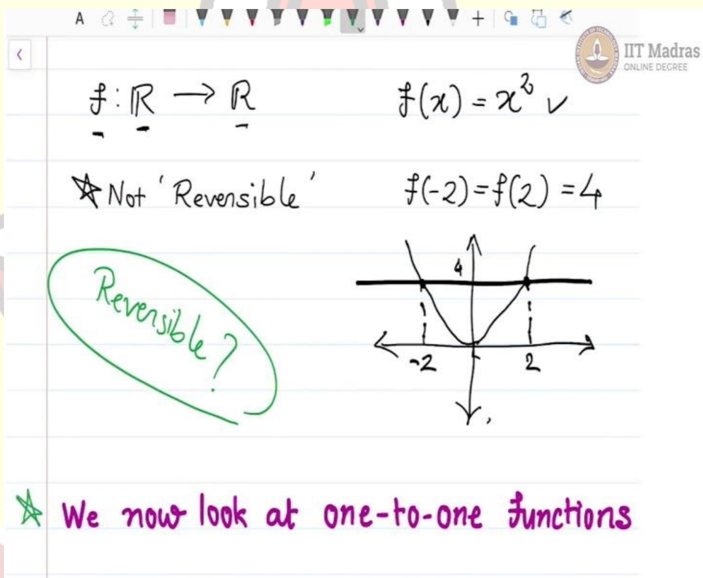

So, in order to find such reversible functions, we need to understand the theory which we will discuss now is the theory of inverse functions. So, when I talk about inverse functions, I am talking about functions from domain which is real line co domain which is real line. So, a function is define from real line to real line, then the immediate question that comes to our mind, are all functions reversible? And the immediate answer is a very well-known function that we have seen is, $f(x) = x^2$.

Now this function is not reversible because it feels to pass the horizontal line test, if you remember. So, $y = x^2$, if I try to plot, it will be something like this. Very close to something like this. And when I pass a horizontal line through this it passes through 2 points. And let these points be 2 and -2 . And that essentially means this, when I feed in the value 2, it will give you four. And when I feed in the value - 2, it will give you the answer to be equal to 4.

Now if this function is reversible, when I feed the value to 4 it can spit out the two values 2 and -2. So, it is not uniquely spitting out the value. Therefore, this function is termed as not reversible function. Such functions we cannot study the inverse properties or the properties of inverse functions. However, if you restrict the domain of this function instead of real line to only positive half of the real line, then you will get one to one correspondence between the values on x $axis$ and y $axis$ and then you can talk about inverse of these functions, when it is defined from 0 to ∞.

Now let us look, then the question is, this function is not reversible then which functions are reversible? That is a question that we can ask now, in order to answer this question, we need to study some class of functions. So, in last few videos we have already seen that one to one functions are nice functions. Any function that is either increasing or decreasing is one to one and therefore we can look at one to one functions for the class of reversible functions. So, here is our answer that we will start looking at the class of one to one functions.

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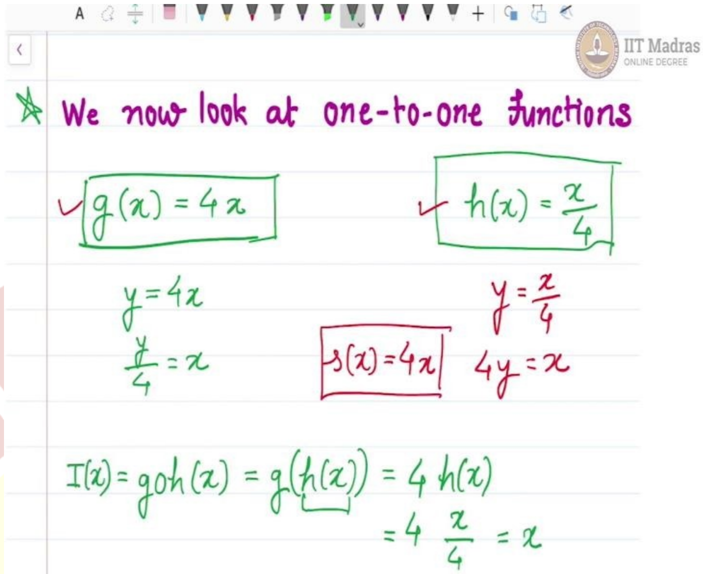

Let us look at a simple function a linear function g(x) is equal to $4x$. Is this function reversible or not? So in order to answer this question, let us look at $g(x) = 4x$. So, you can put $y = 4x$. If you look at $y = \ 4x$ from our basic understanding of linear equations or rather than linear equations an equation of a straight line. This is a straight line passing through origin having slope 4. So, if I want to find a point x on this a$x$is then I will simply transform this as $\frac{y}{4} = x$ and this transformation is unique. Therefore, I can write some function let us say *r(x)* as $\frac{x}{4}$. And this function will actually be giving be the inverse of this.

So, let us take this function, if this function $h(x) = \frac{x}{4}$. So, I do not need to write *r(x)*= $\frac{y}{4}$. $h(x) = \frac{x}{4}$. Now if I start with this function and I want to get value of x, what should I do? I will write, so I will write $y = \frac{x}{4}$ and in that case I will get $4y = x$. And therefore, I will get another function which is say $4x = s(x)$. So, essentially what we have seen is this g(x) and h(x) have something in common. So, let us recollect the notion of composition of two functions, and try to answer this question.

For e$x$ample, if I consider the function goh(x). Now this function is again a function and it will simply operate like g of h(x). And once you start with g of h(x), what you will do is, you will treat h(x) as an argument of g and put the values of h(x) inside. So, let us try to understand this, so it is like g(*h(x))* is actually, what is *g(x)?* 4 of x. So, it will be 4 $\times$ *h(x)*. Now what is $h(x)$? $h(x)$ is nothing but $\frac{x}{4}$ So, 4 times x by 4 which will give me x. So, what this function is, this function actually gives me identity function. $goh(x) = x$ and a similar manner

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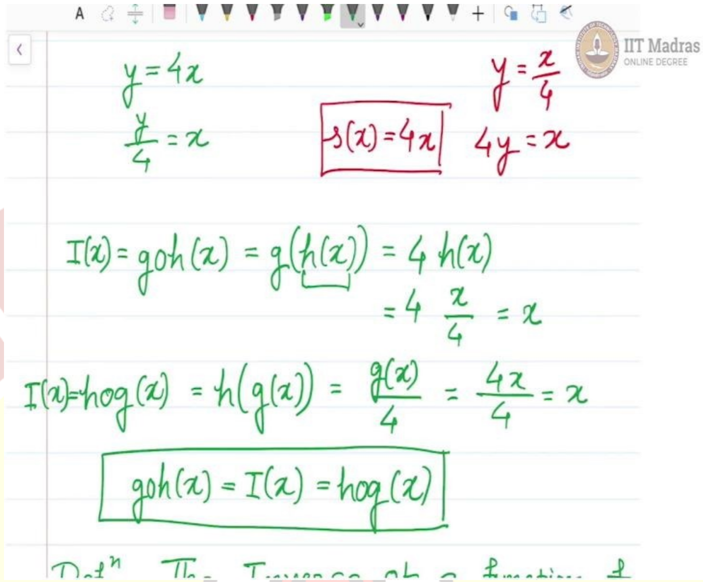

I can start thinking about *hog(x)*. now in this case, if you recollect the notion of composition of functions studied in week one, *h(g(x))*. So, if h of g(x) I will simply see what is *h(x)*. $\frac{g(x)}{4}$ and therefore what is g(x)? It is $4x$ therefore I will get $4x$/ 4 which is actually equal to x. And therefore this is also equal to identity function of x. So, now to summarize what I got is *G o h(x)*= $I(x) = hog(x)$. Now this becomes our definition of inverse function. And let us define it formally as the definition of inverse function.

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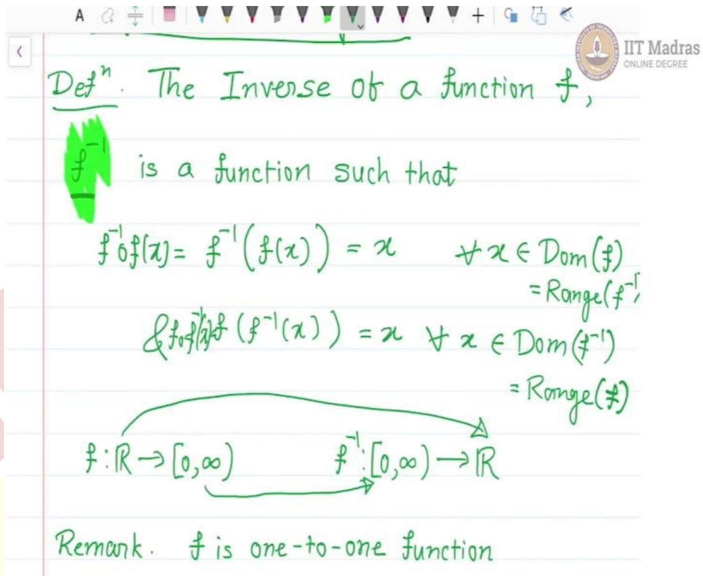

So, here is a definition of inverse function. The inverse function inverse of a function f, we denote it by .$f^{-1}$is actually a function this is our notation f^{-1} is actually a function such that.$f^{-1}f(x)$ or I can rewrite this as $f^{-1}f(x) = x$. Now here is a typical thing that comes for all x belonging to domain of f which is equal to range of f^{-1}. And $f(f^{-1}(x))$ or you can write this as $f(f^{-1}(x))$ being equal to $f(f^{-1}(x)) = x$ for all x belonging to domain of f^{-1} and range of f^{-1}

So, right now when I did this particular calculation I have assume that everything goes from real line to real line there was no such event. Because this function is define from real line to real line. And this function is also define from real line to real line. So, there was no consideration for domain and ranges. But sometimes it may so happen that your original function maybe define, let us say f is define from ℝ to[0,∞). If such a definition is there, then you need to worry about the domain of a function and the range of a function. Because here the domain of f is ℝ and range of f is 0 to ∞.

So, if I talk about f^{-1} of this, then naturally I cannot go over entire real line. I have to go over 0 ∞ and then I have to come to ℝ. So, this is how it will be define and therefore, the domain of f will become the range of f will become the domain of f and the domain of f will become the range of f^{-1}. This is the typical factor that you need to always remember. Now let us go ahead and improve our understanding about one to one functions.

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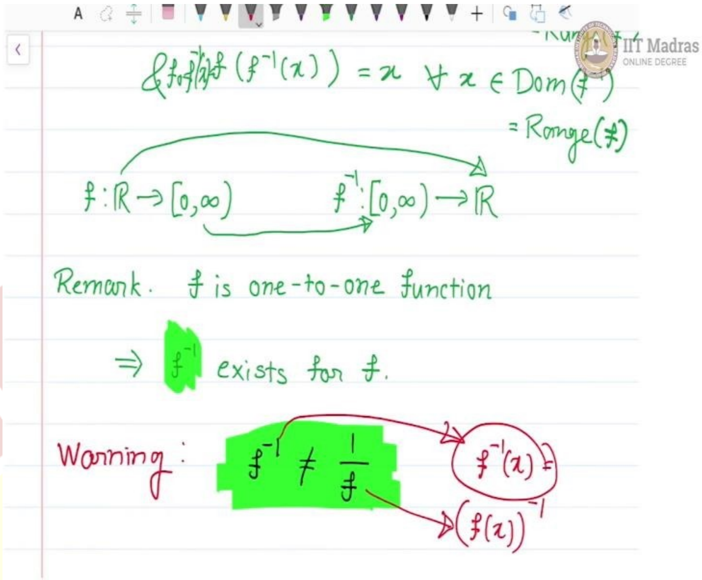

So, if the given function is one to one function then f^{-1}always exist for *f*. Now the notion may confuse you. So, let me give you one precise warning that the notion f^{-1}does not mean $\frac{1}{f}$. This is very important. Because you may quite often confuse f^{-1} with $\frac{1}{f}$. So, whenever we want to discuss in this course or in Mathematics, whenever we talk about $f^{-1}(x)$ it is simply means it an inverse function.

So, this is an inverse function and whenever you want to talk about the $\frac{1}{f}$. Then you should talk about $f(x) - f^{-1}$. So, this $f(x) - f^{-1} = \frac{1}{x}$ and this f^{-1} is actually has a meaning $f^{-1}(x)$ with this you always remember. Now if f is one to one function f^{-1} always exist for *f*. This you have to trust me. I cannot prove it right now with the current tools, so f^{-1} always exists.

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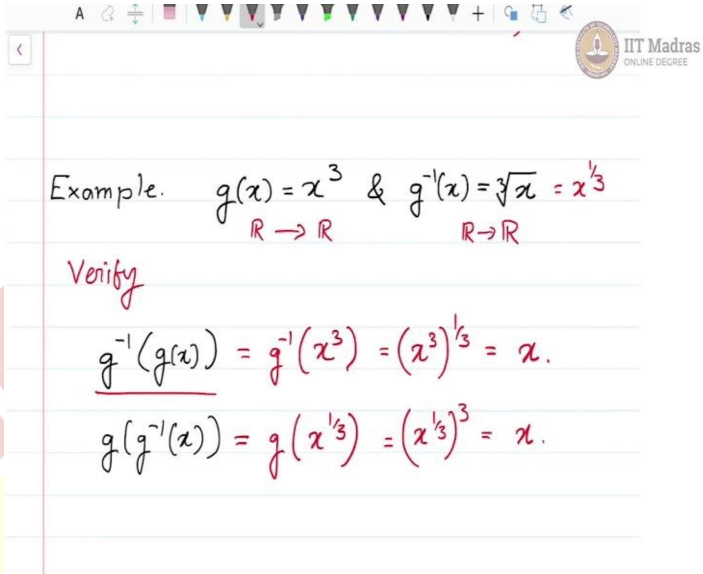

Let us take one example g(x) is equal to x cube and g^{-1} of x is $\sqrt[3]{x}$. This you can write as x raise to 1 by 3 as well. So that, this is simple to verify. So, now you want to verify that the given functions are actually inverses of each other. So, in this case let us first identify the domains, it is a real line to and range is real line. So, naturally for inverse also it's real line to real line. Now question about it. So, let us talk about g^{-1} $g(x)$.

Now if you recollect the definition of inverse function then naturally the inverse function is a function such that all this combinations, all this combinations should produce x $f^{-1}(f(x))$ or $f(f^{-1}(x))$ f. So, let us talk about $g^{-1}g(x)$). So, let us keep g^{-1}intact and put what is $g(x)$ which is x^3. Now this you substitute the function $g^{-1}$of x as $x^{1/3}$ then this becomes $(x^3)^{1/3}$. Then multiplication of indices a^{mn} applicable, so it will x. So, one way it is true.

Now the second way also you have to check. So, what you will do now, is you just write g within the box you write $g^{-1}(x)$ here, x raise to 1 by 3. And then simply put the function g so x raise to 1 by 3 the whole thing raise to 3 which again a raise to m n. So, this will also give you x domain and ranges we have already seen. So, whatever the conditions of that therefore g and $g^{-1}$are inverses of each other. So, $g^{-1}$is inverse of g .

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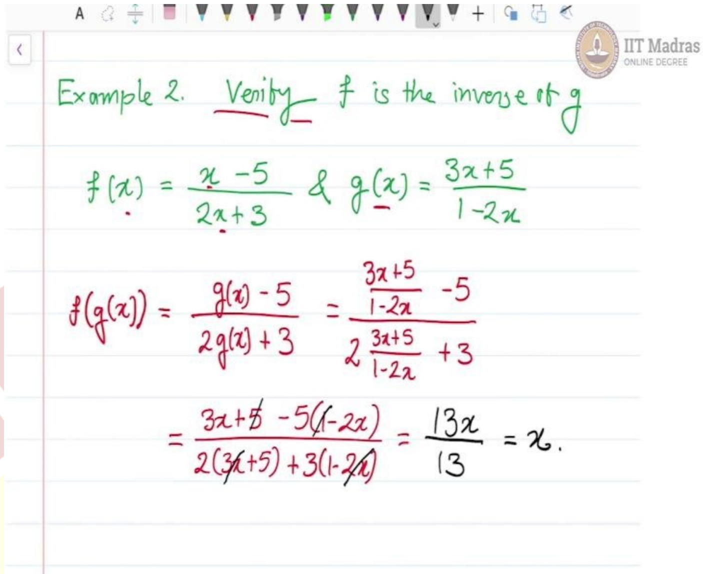

Let us take this example where we are suppose to verify, whether f and g are inverses of each other. So, let us try to verify, you can check the domain and co ranges of these functions. I will simply start with $f(g(x))$. So, if I start with $f\ (g(x))$ as per our notion what we will do? We will simply keep this $g\ x$ in place wherever f has an argument x. So, we will take this and we will put $g\ x$ wherever x is written there.

So, let us do that exercise that is, $\frac{g(x)-5}{2g(x)+3} = \frac{\frac{3x+5}{1-2x}-5}{2\times\frac{3x+5}{1-2x}+3} = \frac{3x+5-5(1-2x)}{2(3x+5)+3(1-2x)} = \frac{13x}{13} = x.$ this is f o g(x). Now what is g(x)? g(x) is this so let us go ahead and substitute those values over, those functions in place of $g(x)$. So, it is $\frac{\frac{3x+5}{1-2x}-5}{2\times\frac{3x+5}{1-2x}+3}$. So, now it is a matter of your Algebra just simplify this. So, denominator both have $1-2x$ in common, so multiply the numerator by $1-2x$ and denominator by $1-2x$. So, that we will get rid of this. So, it will be $\frac{3x+5-5(1-2x)}{2(3x+5)+3(1-2x)}$.

So, what is a question, we have to verify that f is the inverse of g So, essentially I want to come up with a number with a function which is x f o g(x)= x, this is my end goal just remember this. Now you can simply (multi) simplify this $3x$ + 5 - 5 will get rid of this + 5. Let me change the color over here. So, this will get this one will get rid of this then this is - $2x$ - $10x$ and - $10x$ will become + $10x$. Because of this - sign and then 3. So, I will simply get here $13x$.

Now you look at the denominator which is 2 into $3x$ that is, $6x$ then look at the corresponding term here - $6x$. So, this x, terms corresponding to x will vanish and 3 and $2 \times 5 = 10$. So, I will give get the denominator to be equal to 13 and that will give me x as my answer. So, f o g(x) is verified. Does this complete, will this complete our verification of whether f is the inverse of g? No, because I want to check whether g is also the inverse of f. Then only the verification will be complete.

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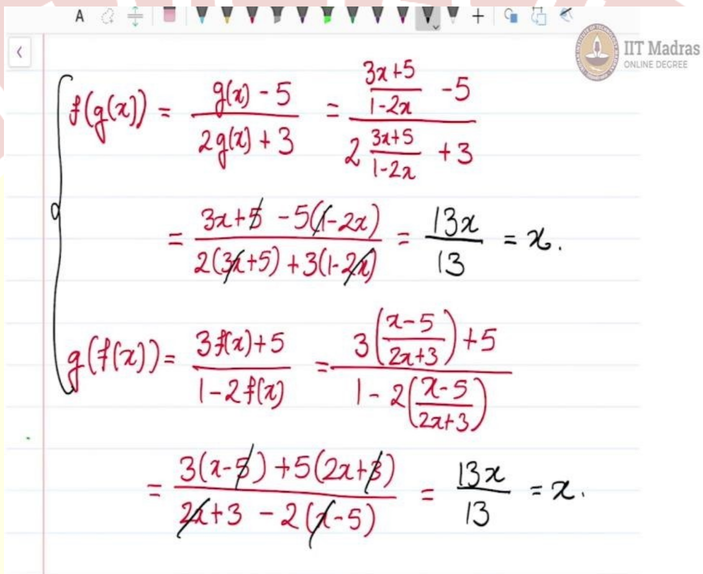

So, let us go ahead and do that, that is we will consider g f of x and that should give me x that is my end goal. So, now you look at what is a function g and put f x as it is everywhere. So, it is 3 times f x + 5 1 - 2 times f x. What is the next step take the functional form of f x and substitute it in the expression.

So, 3 into x - 5 upon $2x$ + 3 + 5 upon 1 - 2 times x - 5 upon $2x$ + 3 and then again the same logic applies multiply both sides by the $2x$ + 3 and then you will get 3 times x - 5 + 5 times $2x$ + 3 to be, upon 2 1 is there. So, $2x$ + 3 as it is - 2 into x - 5. Let us look at the simplified form let me change the color. So, $3x$ + $10x$ that will give me $13x$ here 3 into 5 - 15 + 5 into 3 + 15. So, this is taken care of vanished upon again the same logic applies $2x$ - $2x$ will vanish 2 into 5 will give me 10 and this 3 will give me 13.

Therefore, I got this domain and ranges you have already verified for yourself and therefore process is now complete because f of g(x) is x g o f of x is again x. So, we can verify that f is inverse of g as stated. So, this completes our discussion on inverse functions.

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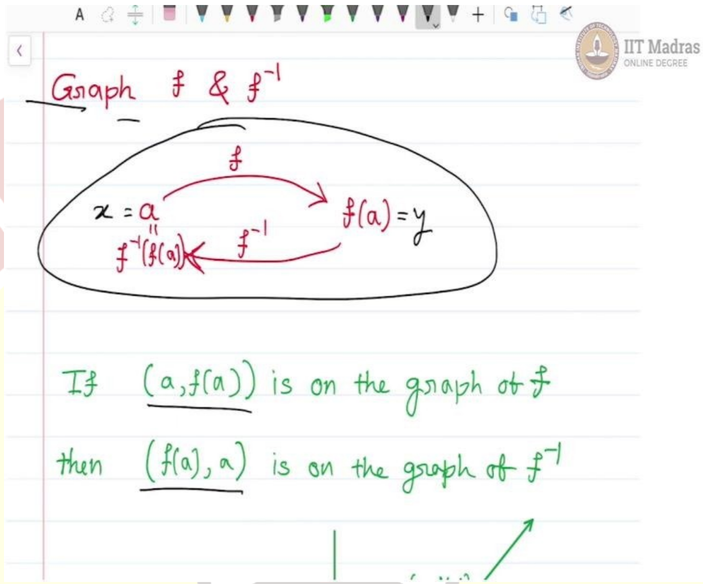

Now it is important to understand graphically what the inverse function is or how the graph of f and $f^{-1}$is changes. So, we already have a wage understanding of the graph of f and f^{-1}. Now let us look at it formally, so if I know something about f or the graph of f, then given a value of a. I am able to calculate f of a and a f of a is the payer which we call as graph of x, graph of f. You look at f^{-1}, what happens when you talk about f^{-1}, here f of a is actually on y a$x$is and a is on x a$x$is.

So, when you look at the inverse function the values on y a$x$is actually get convert into values of x a$x$is. And the values on x a$x$is will actually get converted into values on y a$x$is. So, this is the mapping that we have given. So, if you start with f of a which is y then you will talk about $f^{-1}(y)$of y and when you talk about $f^{-1}(y)$ you will actually get it to be equal to a. Because $y = f(a)$. and this is how the entire circle is complete. So, in particular if a and $f(a)$ is on the graph of f then f(a), a is the graph of f^{-1}. That is obvious.

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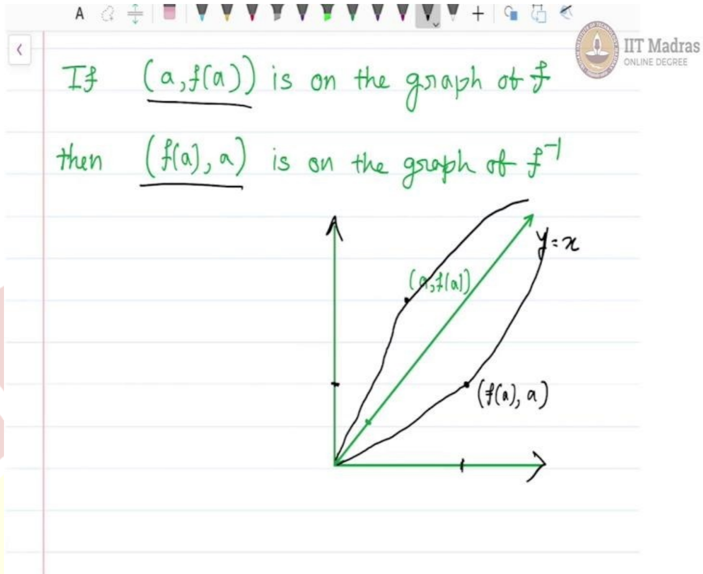

So, let us look at, let us imagine this is the graph. This is the graph of a straight line a, fa. So, you plot a line $y = x$ here. This is a line in $y = x$ and this is a point which is on the graph of $f(x)$. So, now you are saying that where will this point be, when I talk about f inverse. So, then we are also answering this question that wherever a was there it will be $f(a)$ now. And wherever $f(a)$ was there now there will be a.

So, in this case you just take this distance and you plot it, you just take the distance on the y axis and choose the distance on x axis here and take the distance on x axis for this point and put that distance over here. That means it will be somewhere here. And therefore, the point will be somewhere here and this point is actually $f(a)$. So, what we are actually doing when we are plotting the graph is actually we are reflecting our original function, in the original function is somewhat like this. Let us say, so it is somewhat like this.

Then what we are doing is we are actually reflecting it along y axis and it will be very similar function. Which will look like this. So, this is how the graph of inverse function will look like it is actually a reflection along y is equal to x or reflection along a function y $f(x)=x$.

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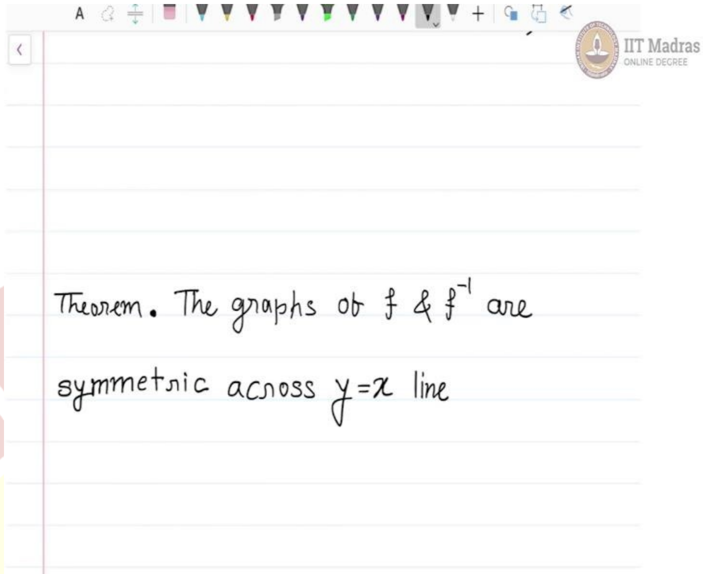

So, in particular the graph of f and f^{-1} are symmetric across the line y= x. This is what you have to remember all the time. If you want, you can prove the theorem but there is nothing it just a graphical prove that, if I want to compute this particular point and if I know that the inverse of this function e$x$ist, then you just take length on y and plot it across x and length of, length in x direction plot it across y direction. That is what I did a this is actually the prove of the theorem that the graph of f and f^{-1} are symmetric on y =x line. That completes our topic on inverse function. In the ne$x$t video we will deal with the inverse functions in a more restricted manner that is, inverse of this e$x$ponential functions.

Mathematics for Data Science 1
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Department of Mathematics
Indian Institute of Technology, Madras
Week 8 - Tutorial 1

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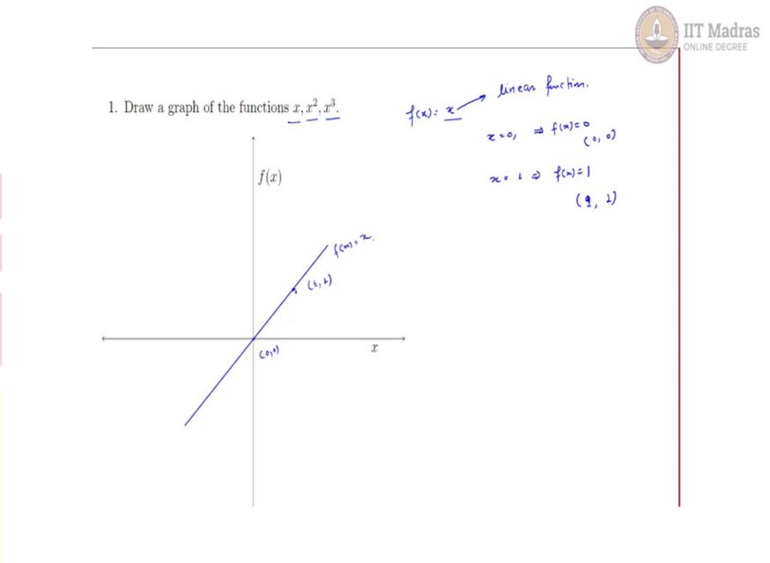

Welcome to week 8 tutorial questions. The first question is about to draw the graph of some functions those are x, x^2 and x^3, so first is x so if I write $f(x) = x$, what comes in my mind first is that this is a linear function. So, if this is a linear function, it will represent a line and to draw a line we need two points. So, let us take some randomly two points, if I take $x = 0$, what will I get? $f(x) = 0$, so my point is 0, 0 if I take $x = 1$, then I will get $f(x) = 1$ what the point is again this is 1, 1.

So, if I try to plot this line I will choose one point here this is (0, 0) and one point somewhere here, this is (1,1) and I will just draw the line, so it will go from here actually, so this is (1,1) and (0,0). So, this is the graph of $f(x) = x$.

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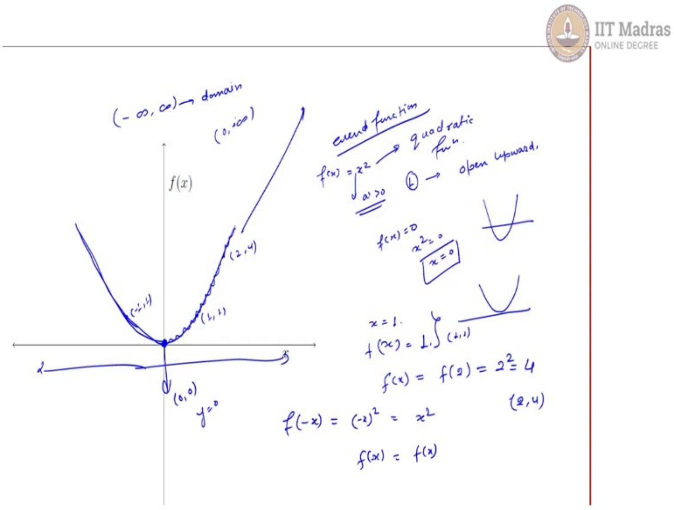

For the second, $f(x) = x^2$, what comes in my mind that is it is a quadratic function, so the quadratic function represent the parabola and parabola could be upward and downward. So, I can see that $a >\ 0$ which is actually 1, it represent that the parabola is open upward, what I need next that where it will touch or cross the *x*-axis and those are called the intercepts. So, I do fx equal to 0, then I will get $x^2 = 0$ which means $x = 0$ so my both roots are at origin because I am getting 1 root which is 0.

So, if I am getting one root it means it will never cross x axis, it will not be like it will be here only and as it is going from 0,0 we have only on option that this curve will look like some like this, what will be the points if I say from where it is passing through, so if I get $x = 1$, I will get $f(x) = 1$, 1 is curve equal to 1. If I get $f(x = 2)$, then it will be 2 ×2, 4.

So, here it is passing through 1,1 and here it is passing through 2,4, so this will be 1,1 and this will be 2,4, what will be this point at $x = -1$? As we know that this is a quadratic function and this is also an even function, even function why? Because if you put $f(x) - x$ equal to then it will be $(-x)^2$and it will give you x^2, so $f(-x) = f(x)$ that is why this is an even function and even function are the symmetric about y axis, even functions are symmetric about y axis.

So, if you plot this curve only then this part will be the mirror image around the y-axis only. So, this will be - 1,1, what will be the domain? You can see that this can go from here to here so - ∞ to ∞ is the domain and range we know that this vertex will decide the minimum and that is 0,0 which means y equal to 0 is the minimum and it can go here so and behaviour shows that at ∞ it will give ∞ only. So, the range will be 0, ∞.

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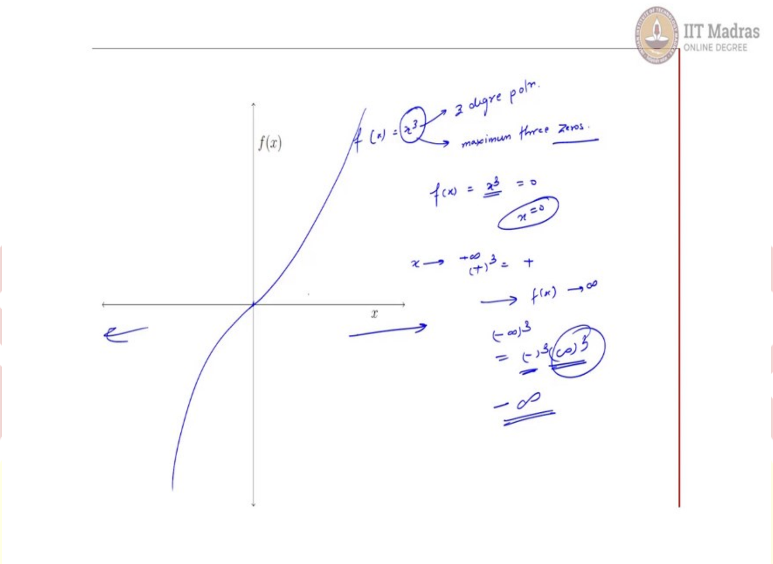

The third graph is of x^3, so if I take $f(x) = x^3$ what comes in my mind first that this is an 3 degree polynomial and if it is 3 degree polynomial it will have maximum 3 roots maximum 3 zeros, what does it mean by 3 zeros? It means this particular curve will cross x axis 3 times but let us see how much actually we are getting the zeros. So, if say $f(x)$ equal to sq I will get $x = 0$ only.

So, I am getting all three roots at 0,0 only because at $x = 0$, I will get 0 only. So, what does it really means that the curve will pass from here and it will never cross x axis again except 0. Now what else needs to see is the n behaviour, what will happen if x goes to positive ∞? Means what will happen if x goes like this? You can see this is the cube of x and if you take + the cube will give + which means at ∞the $f(x) =$∞.

So, n behaviour shows that when x goes to ∞fx will go also ∞. So, in positive side it will be positive ∞, so this will be like this. What will be happening in negative side of x? So, if I take - ∞this side, then after doing cube I will get - cube and ∞cube, so this is not actually we write just I am writing to understand, so this will give - and this will give some ∞ because ∞ is undefined so cube will also will be ∞but - so it will be downwards, so this is our x^3.

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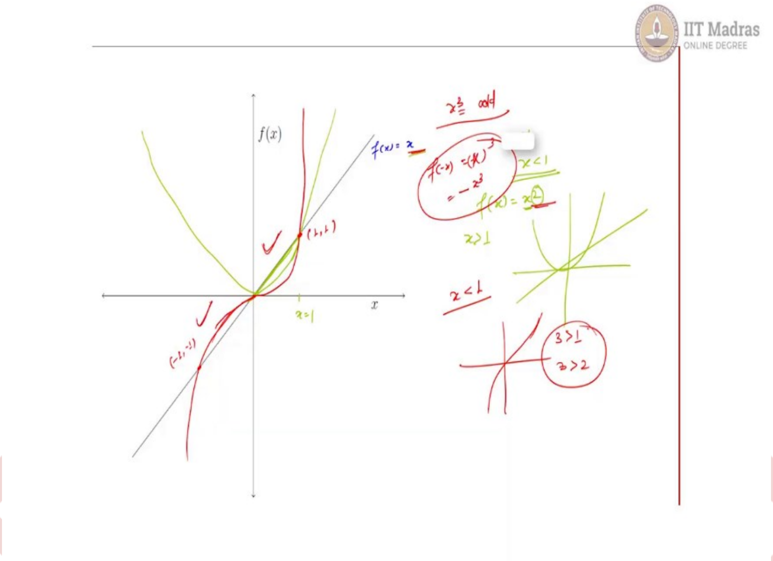

What if I try to draw all three graphs here at 1 coordinate plane, so this is actually $f(x)$=x we can see this is a linear, what will be fx of $f(x) = x^2$? So, $f(x) = x^2$ we saw that the graph was like this but how will it be. I mean will this part be above the line or below the line? So, we knew that if this x which is x>1, if x>1, the more power will give lower value means $at\ x =$ 1 before this the curve will give a lower value in comparison with x so in this part this will be like this.

And what happens in upper part? When $x > 0$ the more power will give the more value, so x^2x square has the higher degree than x then it will give higher value. And we know that this is a even function so this side it will be like this. What about $f(x) = x^3$? Again same thing we will get a graph like this but how because it has 3 degree which is greater than 1 and greater than 3 is greater than 2 also so in the domain less than 1 $x >$ 1, it will give the least value in comparison with x and x^2, so it will be this and when $x > 1$ it will be giving the maximum value in comparision with x and x^2 so this will be and similarly the same way here will happen here.

So, this is a point 1,1 where all are intersecting this is a point where - 1 and - 1 x and x^3 are intersecting, why this is showing the same here because this is an odd function, x^3 is an odd function, what does it mean by odd function, if I take $f\ (-x)$ - it will give $x - x^3$ and then - give - and x gives x. So, $-x^3$, if this is an odd function that will give me that the function is symmetric about origin so what does it mean by origin, the same behaviour will be happening in this and this coordinate.

Mathematics for Data Science 1
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Week 8 - Tutorial 2

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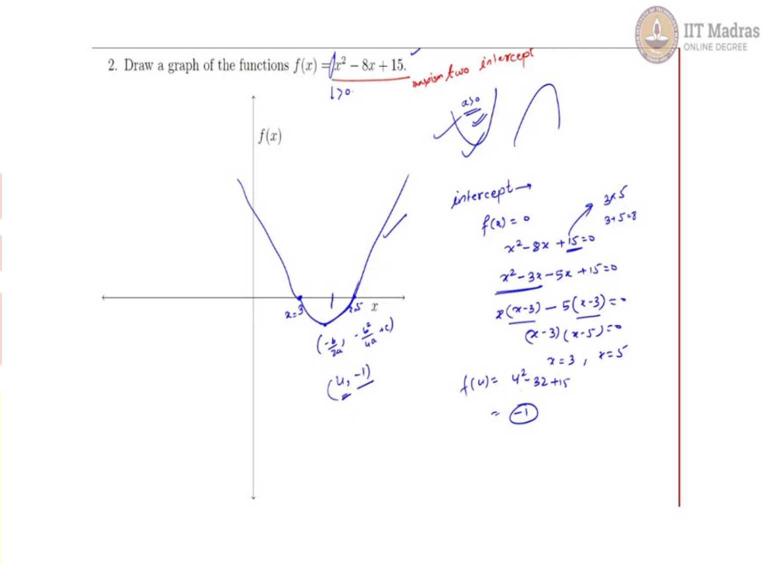

The second question is about drawing the graph of a quadratic function. So this is and quadratic function again as we did in question 1, it will give two, maximum two intercept and we know that the curve will look like this or this or it will happen when the x^2 the coefficient of $x^2 > 0$. So here, it is 1 so greater than 0, so this curve will be representing this function.

Now to find the intercept $f(x) = 0$, so $x^2 - 8x + 15 = 0$ if I do the prime factorization I will get 3× 5 and after summing of 3+ 5 I will get 8 so I will use a factorisation method to solve this. It will be $x^2 - 3x - 5x + 15 = 0$.

Now this will give me $x - 3$ when I take x common - 5 common x -3 =0 then x- 3 common it will give x-5=0 which means x=3 and x=5 are the intercepts or roots of this quadratic equation. If they are roots, it means the curve will cross x axis at these points, so if this is x =3 and this is x=5, here the curve will cross.

Now as we know that this is open upward parabola we will get one turning point if we are getting two intercepts. So, that turning point will be somewhere here, which means we have only one option that is this, so this is an rough diagram only, so this will be the two intercept and this will look like the parabola will look like this parabola. What will be this point? We can find this is the vertex and vertex we can find that using $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ formula. So, it will be - 8 / 2 and if I take - then it will be 4 which is between 3 and 5 it should be actually due to symmetricity.

Now what will be value at 4? $f(4) = 4^2 - 32 + 15$ which means $16 - 32 - 16 - 1$ this is correct actually, why this is a this should be negative we got negative, this should be between 3 and 5 we got 4.

Mathematics for Data Science 1
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Week 08- Tutorial 3

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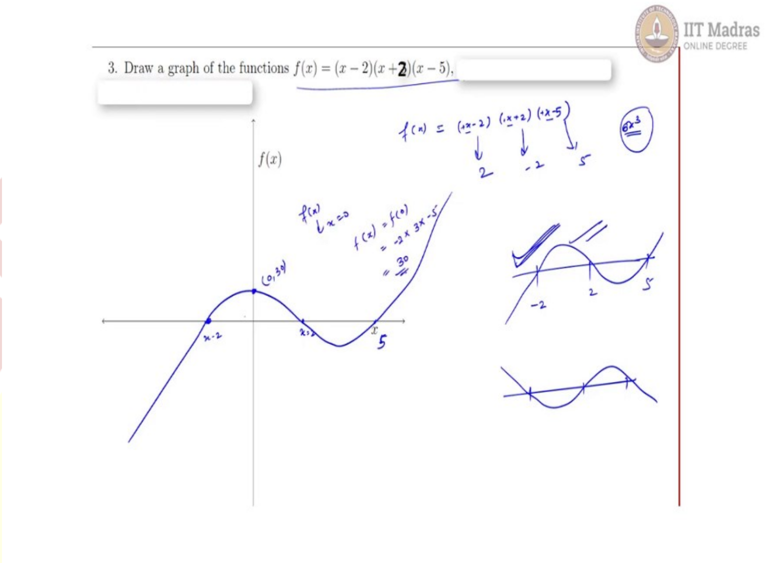

For our third question, we are getting a function, which is given in its intersect form. So, what is given actually, x -2 x +2 and x -5. So, this is actually 3 degree polynomial or function. And if it is given intercept form, it means we can clearly find the intercepts. So, this will give to this will be-2 and this will give 5, which means the curve will cross the x axis at these 3 points, what are those points, x =2, x =-2, x =5.

Now, the next thing is, is that how the curve will look like? If we are getting 3 intercepts, it means the curve will have 2 turning points, what does it mean? This is-2, this is 2 and this is 5, the curve will have 3 to 2 turning points, and those 2 turning points will be between 2 intercepts. So, this could be like this one case and the other case will be this and this which we, which I will take that will be dependent on the coefficient of the x^3.

So, coefficient of x^3 we will get x , x , x when we multiply these 3, we will get x^3. So, this is + this is + and this is +. So,+ and +. So, we will get positive coefficient of x^3 and positive coefficient of x^3 means, we will get this type of graph. Why? Because end behaviour shows if this is positive, the coefficient is positive we will get ∞ at x =∞ and-∞ at x =-∞. So, when this is pass then, we can understand that we have only one option to draw the graph and that will be this.

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Week 08 - Tutorial 04

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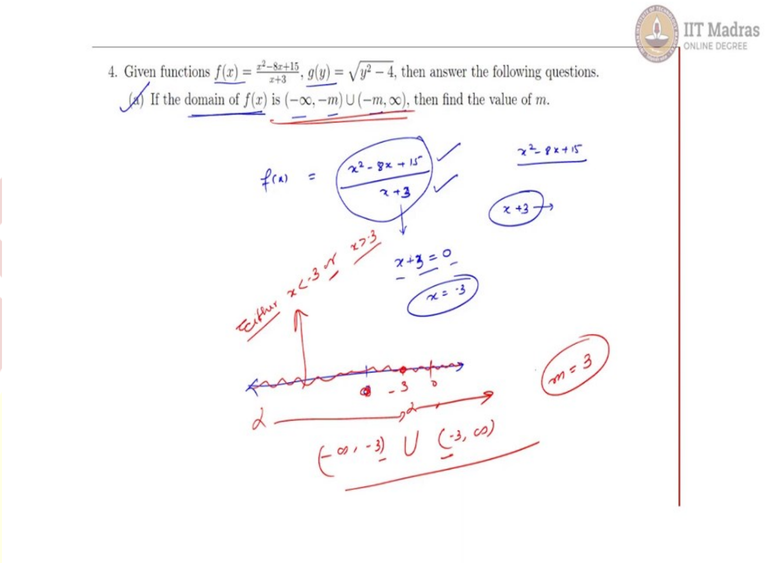

For our fourth question, we are given 2 functions a $f(x)$ and $g(y)$. And the question is asking the questions 2 questions are given a and b. And those are based on the given information. So, let us solve the first part a, it is saying that if the domain of $f(x)$ is (-∞,-m)∪(-m,∞), then find the value of m.

So, to find the value of m, first we will find the domain of $f(x)$. So, what $f(x)$ is given, $\frac{x^2-8\,x+15}{x+3}$, we will have real number the domain until unless this function has any problem at any specific x. So, we will see this function in 3 parts first part, numerator, then denominator and then the whole fraction.

So, our numerator is actually $x^2 - 8\,x + 15$. This is our quadratic function, and we know that there is no problem it is a quadratic function and quadratic function has the domain real numbers or real numbers are the domain of quadratic function, so no problem with the numerator, the denominator has x+3, is there any problem? No, if this is a function, then there is no problem with that, because it is a linear function and linear function has the domain from-∞ to ∞, where the problem is when we consider hole as a function, then we get that there could not be any function or any value which is 0 for denominator.

So, what does it mean we cannot allow a function when denominator is actually 0. So, when this is 0, $x = -3$, so the function is defined on real number except $x = -3$, what does it mean if I take the yellow line and this is 0, then our function is totally define and good till there, this will be 0 and this-3. So, till here and after here, so it is not defined at only this point, how can I write this?

So, if you consider either x is >-3 or x is <-3, the function is defined this resembles this statement, and when we use the statement either or then it shows ∪. So, this actually is-∞ to-3 and this part is actually-3 to ∞ and those will be open bracket because we are not including 3 and there those will be connected with sign ∪. And if you match these things with the given in the equation, will get that m = 3.

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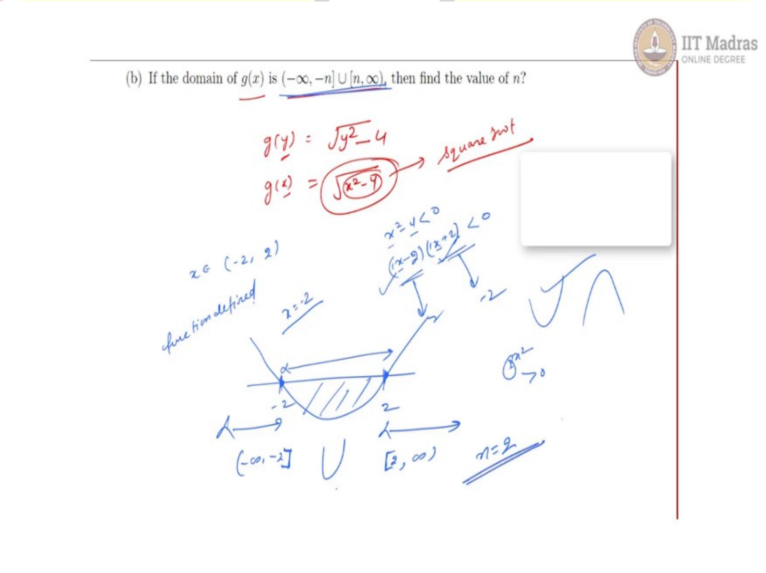

For our second part, we have $g(x)$ = this, and if you remember the question is given $g(y) = \sqrt{y^2 - 4}$. So, first we need to find what is $g(x)$. So, $g(x)$ means what we will just replace y with x, so it will be x squared-4 and this is under square function q square root function and we properly know that the square root function is not defined when the any value or any function gives the value > 0 in square root, it means this value should not be > 0.

So, will find when this value is > 0, x^2-4 > 0, what does it mean? This will be open like x-2 and x+2 > 0, this is a quadratic function and this has 2 zeros and 2 zeros are-2 n+2, this will give+2 and this will be-2. Now, you know that parabola either will look like this or will look like this, it will dependent on the coefficient of x, if you multiply x with x I will get x^2 only, there is 1 and 1. So, 1 and this is < 0, which means open upward will be the correct representation of this parabola and when it is open afford this will be the representation.

Now, you can see where the value of function is negative this part and what is this part this is this. So, if x value is from-2 to 2, then the function will provide negative value and we cannot get negative values, other thing that what will happen if $x = -2$, then the function is giving value 0 and if it is 0, is 0 acceptable in square root? Yes. So, we can take these 2 critical points here-2 and 2 which means the function is not defined when x belongs to-2 to 2 in open upward.

How can I write in another way that the function is defined for this reason and then this reason, what is this reason, this is-∞ to-2, as 2 is included open bracket here, as 2 is included I will close bracket as 2 is included. So, this is closed bracket and this is ∞ and how will I enter the join them using ∪. So, this is given here in the question, and if you see properly then n = 2 is our answer.

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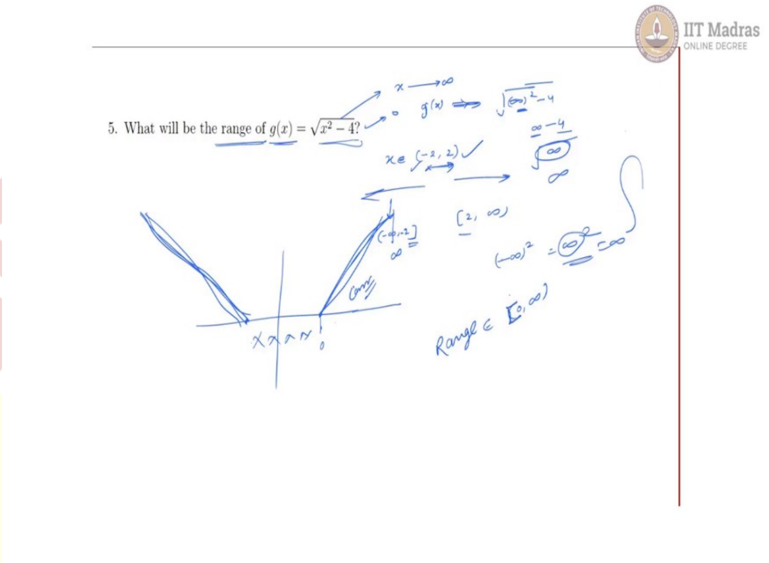

For our fifth question, we are supposed to find range of a function $g(x)$ which is a $x^2 - 4$. And you can understand that this function is actually not defined at x =, I mean in the domain of x = -2 to 2. At 2 they are defined because we get 0. So, for finding the range, first we need to find what the domain is. The domain will tell what will be the range.

So, we do not need to see in this region, the value of function in this region we do not need to see, we will look only where the x is this side or x is this side. So, we will see in -∞ to [2,∞), because here it is open so this will be closed, we discussed this in our earlier question fourth b you can see that.

Now, if we find that where are the function is defined only, I will try to draw a graph, a hypothetical graph which is rough diagram. So, at 2 this will give 0 and - 2, this will be - 2, this will give 0 what will happen if I take if I go towards and we have here. So, this is if I take x →∞, what will I get, $g(x)$ is = which is tending to actually ∞^2 - 4 and this will be the ∞^2 which will give again ∞ only there is nothing called ∞^2.

This is a very large number and the square of the large number it will be large that is all, and - 4 means what? ∞ - 4 this will again give ∞ and if you take square root then again this the ∞

only. So, in we have here so that the function gives the ∞ positive at positive ∞, which means it will be something, we cannot go in this reason, this reason we cannot go, so will stop here.

The same thing happened here because this was a quadratic function. If you take - ∞, then - ∞ square will also give ∞, square and we will get same thing here = ∞. So, this is actually symmetric, why? Because it was already quadratic. This is not a linear function do not see this as a line. This is a curve, it might be like this or this, we can verify with using Desmos.

So, I will recommend you to go to the Desmos and see that correct graph, but the rough diagram we can understand what will be the domain and range. So, for this domain we will get ranges here 0 and here ∞, which means the range is actually $[0,\infty)$. 0 will be involved or not. As we can get 0 at $x = 2$, then 0 will be involved.

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Week 08 – Tutorial 6

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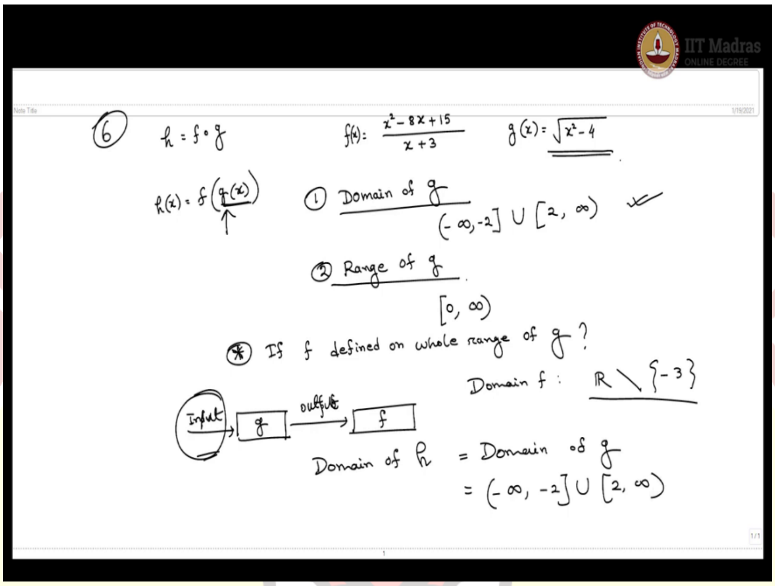

So, let us discuss the sixth question. So, in the sixth question we have to calculate the domain of the function h, where h is given as the composition function, the composite function fog, where $f(x) = \frac{x^2 - 8x + 15}{x+3}$. So, we can write it as $f(x)$ as it is a function of x and $g(x) = \sqrt{x^2 - 4}$. So, we have to calculate what is the domain of the function h?

So, $h(x) = f(g(x))$. So, how will we compute the domain of $h(x)$? Observe that the input value of h firstly come from the input value of g. So, let us begin with computing what is the domain of g x, I mean from where the g is taking its input. So, to begin with, calculate, so this is the first step, calculate the domain of g. So, you have seen earlier that domain of g is nothing but $(-\infty, -2] \cup [+2, +\infty)$. So, this is the domain of g.

So, from this set g is taking it input. Now, if we talk input from this set, then we will get some values as output from g and those output basically those are the, those are in the range of g and those output must be inside the domain of f, otherwise the f will not define. So, let us compute what is the range of g. So, you have seen that this is a square root function, so range of g is nothing but 0 to ∞. So, it is nothing but the non-negative real numbers.

Now, let us ask the question, is f defined on whole range of g? This is the question. If f is defined on the whole that means, if I am giving the input in the f here in f composite in the whole range of g, I mean whatever the outputs coming from g is accepted as the input of f, then the domain of the function is nothing but the domain of g, because domain of g means we are giving input, so let me write this is my g and this is my f, so we are giving input, getting some output and if all the output elements, I mean if all this set, if everything is in the output set is taken as the input of f, then these inputs, these set is nothing but the domain of h.

So, here, let us observe what is the domain of f, I mean what inputs f can take. So, domain of f is nothing but all the real numbers except -3, but observe that -3 is not in the range of g because range of g is nothing but non-negative real numbers. So, - 3 is not there. So, anything in the, which is in the range of g can be taken as input in the f. So, whatever I am giving input in the g, we are getting some output and all the outputs are taken as the input of f.

So, the domain of h here is nothing but domain of g. Remember, this is for this particular case. There may be some cases where this range of g, the whatever I am getting as the range of g cannot I mean all the elements may not be taken as an input of f. There can be some case, I will show an example there, but in this particular case, in this question number 6, we are seeing that domain of g is nothing but domain of h because the range, the range set is basically a subset of domain of f. So, let me write here. So, let me finish this at first. So, domain of g is nothing but this thing $(-\infty,-2] \cup [2, \infty)$. So, this is the domain of h.

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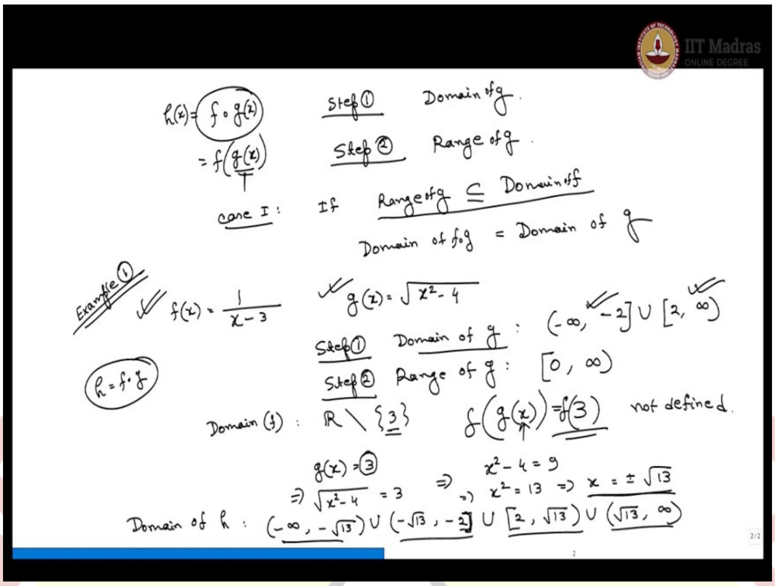

So, what we have seen here? We have seen that to find the domain of some composite function f composite, the step 1, we must calculate what is domain of g. If we calculate domain of g, then we have to calculate, so let me write this as $h(x) = fog\ (x)$. So, it is nothing but f of g x. So, these things are this g x, these are in range of g. So, step 2 must be calculating range of g.

Now, there some cases can arrive. So, if case 1, if range of g is subset of domain of f which we have seen in the question number 6, which we have seen in the example we have just discuss, if range of g is subset of domain of f, then see every value which are in the output of g can be taken as input in f, so no problem arises. So, domain of the composite function f composite is same as domain of g. So, this is domain of g, this is range of g. So, this is case 1.

Now, what other case can arrive? So, let us see another example before going to the case 2. So, I am taking my $f(x) = \frac{1}{x-3}$ and let g be the same, $\sqrt{x^2 - 4}$. So, what are the step, step 1 we have seen. Domain of g, domain of g is nothing but we have already seen this ∪. Step 2, range of g. So, range of g is nothing but 0 to ∞. Now, we come to the case 1, is this happen? Is range of g is a subset of range of domain of f?

So, let us calculate domain of f. Domain of f is set of all real number except 3 because when we put 3, this will become undefined, otherwise it is a defined everywhere. So, domain of f is real number, set of real number - this singleton set 3. So, you can see that this 3 is in range of g. So, we are getting some output from g which is not in the domain of f, so we cannot give that value in g for which this 3 is coming.

So, we cannot put that x as input for which g(x) is 3 because in that case this f3, this is not defined. So, we have to find those x in the domain of g for which g(x) is 3 and we will cut those elements out. So, what is g(x) is 3? g(x) is √over x square - 4 equal to 3, so if we solve this, we will get $x^2 - 4 = 9$, so $x^2 = 13$ which implies $x = \pm\sqrt{13}$. So, for these two values, $+\sqrt{13}$ and $-\sqrt{13}$, we are getting the output as 3 which is cannot, which cannot be given as input in f.

So, we have to eliminate this from the domain. So, in this case we have already seen domain of g is this, so we have to eliminate this + √13 from this side and - √13 from this side. So, + √13 from this side and - √13 from this side. So, eventually, our domain of h is - ∞ to $-\sqrt{13}$ and again ∪ - √13 because we are eliminating $-\sqrt{13}$. So, $-\sqrt{13}$ to - 2. This should be closed interval as it is there, 2 to + √13. See, + √13 is get 2, you can check.

So, this ∪ + √13 to ∞. So, this is the ∪ of these 4 intervals, this is the domain of h in this case. So, where h is fog. whereas f is this one and g is this one. So, let us write me as example 1. So, here you have observed that here case 1 is not satisfying so we are going to some other case. So, let me write it.

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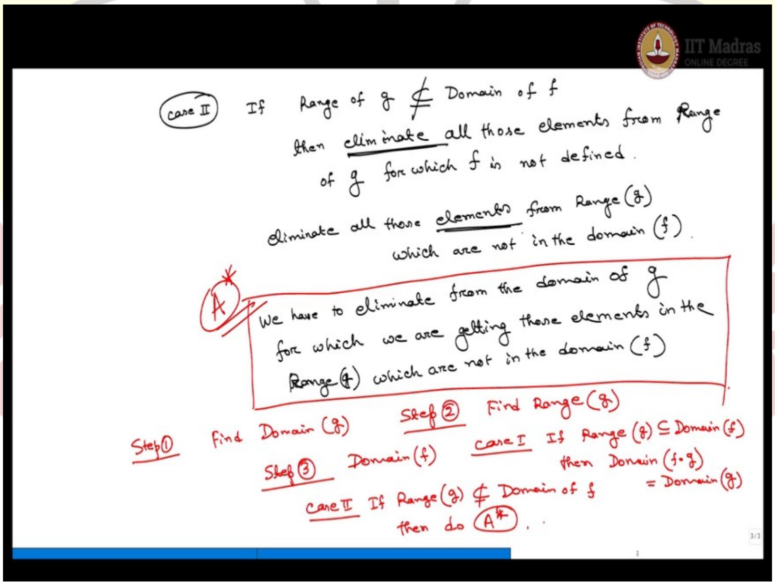

Case 2, if range of g is not a subset of domain of f, then eliminate all those element, all those elements from range of g for which f is not defined. What does it mean? It means eliminate all those element from range of g which is, which are not in the domain of h. What does it mean by elimination here? So, we have to eliminate from the domain of g, which element from the domain we have to eliminate? For which we are getting those element in the range of g which are not in the domain of f. So, this is my final conclusion.

So, let us recall the steps. Step 1, find domain if g. Step 2, find range of g. Now, after that two cases will arrive so, before going to the two cases, is better to calculate what is domain of f, what is domain of f. Now observe if, now case 1, if range of g is subset of domain of f, then it is easy, then domain of fog is same as domain of g. Case 2, if range of g is not subset of domain of f, then what we have to do? We have to do this thing. So, let me write it as A*, then do A*. So, this is the stepwise procedure to calculate the domain of f compo g.

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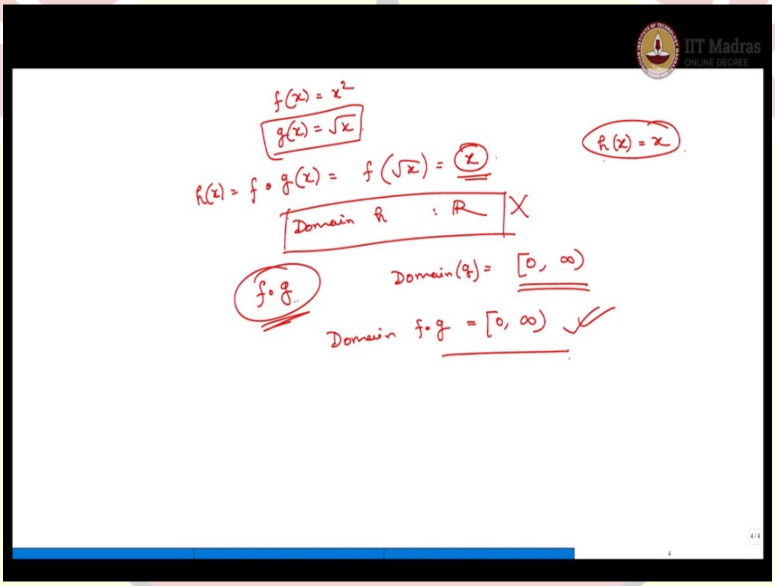

Now, you may ask was this the problem if we calculate the f composite beforehand and then solve this, find the domain? So, what I mean here let $f(x) = x^2$, $g(x) = \sqrt{x}$. So, what is fog here? So, $f(g(x))$ is $f(g(x))$ of is $\sqrt{x}$ so, $(\sqrt{x})^2$ whole square that means, x. So, you see that if we calculate this f composite beforehand you can see, so this is our $h(x)$.

So, as a function $h(x)$, the domain of $h(x)$ if I write only this thing, so what is the domain of this thing? This is whole real number, domain of h, let me write this, domain of h is whole real number. But when I am getting $h(x)$ as a composite function, as a composite function fog, then is g defined on whole $\mathbb{R}$, it is not the case. So, domain of g, what is domain of g? Domain of g is only non-negative real numbers because we cannot put negative real numbers under the square root, otherwise it will give us complex values.

So, domain of g for real valued function is $[0,\infty)$. Only the non-negative real numbers. So, this fog, the domain of fog must be a subset of this and if you calculate as we have stated the steps previously, then you will get domain of fog is nothing but domain of g which is 0 to ∞. But if you calculate the f compose beforehand, and then try to find what is the domain, then you will get domain of h is $\mathbb{R}$ which is not the case. So, this is the correct one. So, you have to follow the procedure which I have stated in this page. So, this is the correct procedure to find domain of fog. Thank you.

Mathematics for Data Science 1
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Week 08 – Tutorial 7

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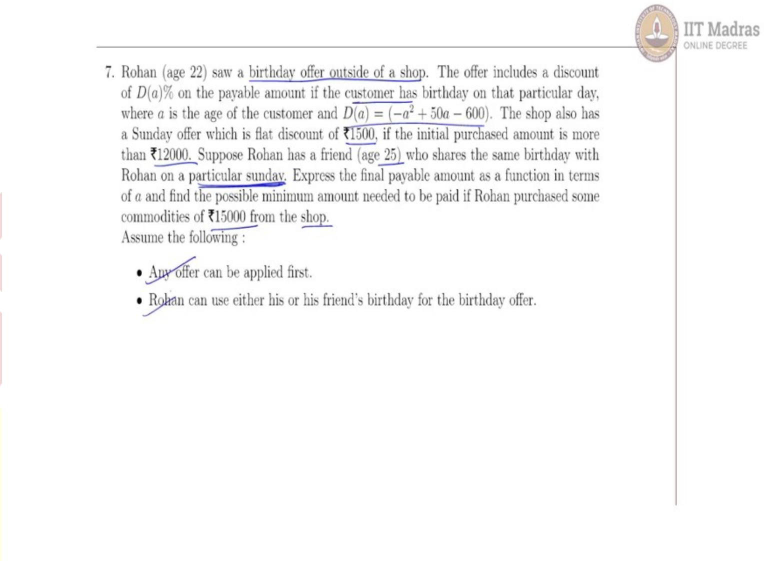

Our seventh question says that Rohan who has age 22 and he saw birthday offer outside a shop. The offer includes a discount of $D(a)$ percent on the payable amount if the customer has birthday on that particular day, where a is the age of the customer and the $D(a)$ is a function. And $D(a)$ is given like this. The shop also has a Sunday offer which is flat discount of 1500 if the initial purchase amount is more than 12000.

Suppose Rohan has a friend whose age is 25 who shares the same birthday with Rohan on a particular Sunday. So, Sunday and both the friends had the birthday on that Sunday also. Express the final payable amount as a function in terms of a and find the possible minimum amount needed to be paid if Rohan purchase some commodities of 15000 from the shop. It is also given that any offer can be applied first and Rohan can use his birthday or his friend birthday for the offer.

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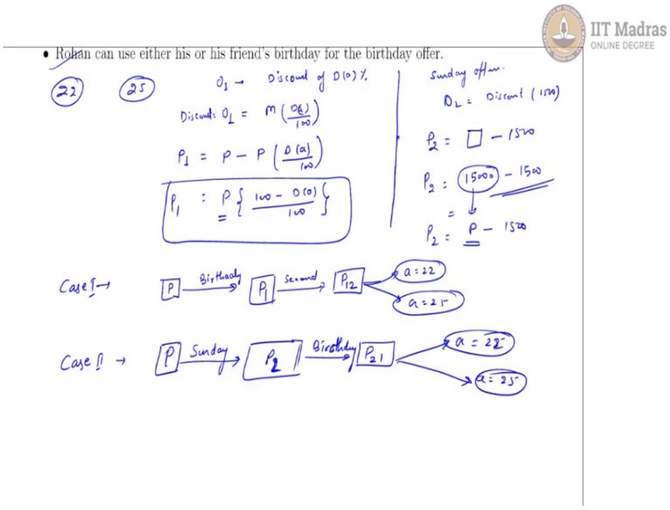

So, let us try to understand what the question is. So, question is about Rohan whose age is 22 and his friend whose age is 25. Rohan wants to use any of them the birthday. And the shop has offer, 2 offers. Offer 1 is the discount of $D(a)\%$. So, how will I apply offer 1 it means if the payable amount or purchase amount is m, then the $D(a)\%$ discount you will get and perceiving. So, this is how we write the discount when the percent discount is given. So, this is actually discount. So, offer 1 is a discount.

What is offer 2? Offer 2 is a Sunday offer which means if the purchase amount is less, more than 12000, so in that case the discount is 1500. So, offer 2 is the discount of 1500. So, this is the offer 2. The payable amount after applying first offer, first offer means individual first, only first offer if I apply that is, birthday offer, the payable amount will be the initially payable amount whatever the payable amount is our purchase amount-the discount and discount is what?

That p_m will be replaced by p because p is the payable amount so $\frac{D(a)}{100}$. And if you solve this you will get $p\{\frac{100-D(a)}{100}\}$. So, this is the offer 1. This is the payable amount if we apply the first offer. Now, what is the second offer? Second offer is same if the amount payable is this, then reduce 1500, this will be the payable amount p2. So, here the payable amount will be whatever the you purchase that is 15000 or I can say if I apply 15000 here, so -1500.

So, if I apply p_2 directly, so I will get this much amount if p is a payable amount rather than 1500, not in general case, so this will be 1500. Do not get confuse with this p and this p. So, these are two different cases. Now, the question, let us try to understand the question. There is a different scenarios. So, in case 1, this is the purchase amount which can I say p, then apply the first offer which is the birthday offer, I will get some p_1, then apply the second offer which is Sunday offer, I will get some p_{12}.

Then, there will be two case which $a=22$ or $a = 25$ should be used. So, for minimum payable amount this will be the case for case 1 when we go with this first. What is the case 2? p, the purchase amount-Sunday offer which is second offer, but I will write Sunday offer which means this will be p_1 and this will p_2 because we are getting the Sunday offer first, so p_2 and then this will be p_{21} if I apply the birthday offer, which is the first offer, birthday offer.

Then again I will have to decide what should I take, $a = 22$ his birthday or his friend birthday equal to 25. These are the two cases I need to think about. So, should we work on first case1?

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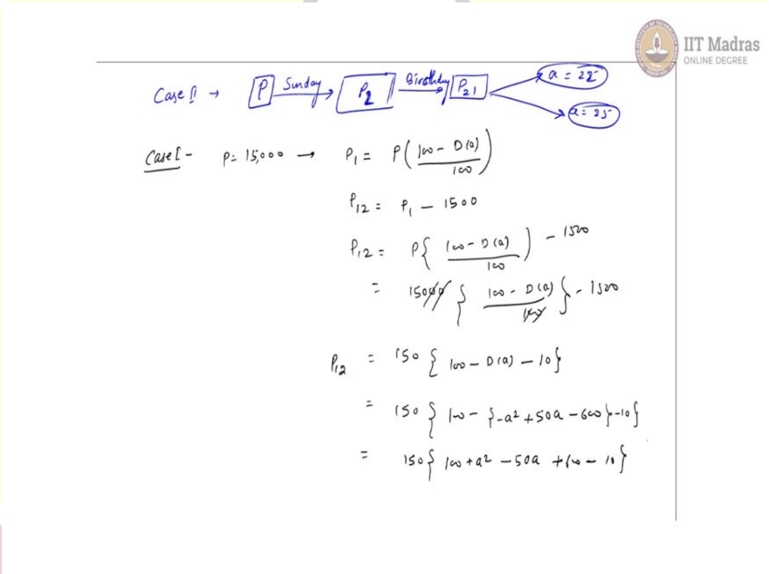

Case 1 is a p, so p is 15000. What is the birthday offer that discount of some percentage amount and that is p1 actually. So, $p_1 = p(\frac{100-D(a)}{100})$, we calculated this 100. So, this is p_1. What will be p_{12}? p_{12} is equal to reduce from here to here to get p_{12}. So, $p_1 - 1500 = p_{12}$. So, $p_{12} = p\left[\frac{100-D(a)}{100}\right] - 1500$So, what is p? $p = 15000\left[\frac{100-D(a)}{100}\right] - 1500$ Take 150 common, then you will get $100 - D(a) - 10$.

This is the final payable amount for the case when we apply first offer first and then second offer second. Now we need to choose what should be the a and what will be the minimum possible amount. For that purpose, I need to write a here and a $D(a)$ is$-a^2 + 50a - 600$-a. So, 150 100$-a^2 + 50a - 600 - 10$. So, 150, it will be $100 + a^2$-50 a + $600 - 10$.

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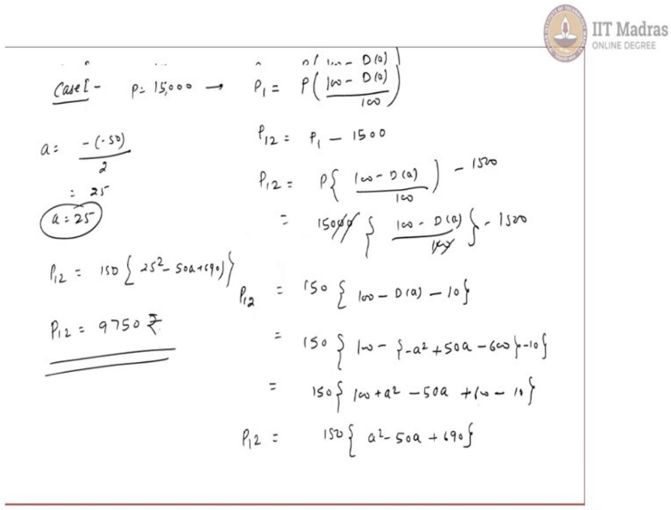

So, p_{12} is actually 150 600 700-10 so $a^2 - 50a$, 600 + 700-10 will be 690, + 690. Now you can understand this p_{12} is a quadratic function and possible minimum amount we need to find the vertex. As we did in week 4, so I will directly find the vertex. So, a will be for minimum possible amount, for paying minimum possible amount,-of which will be $-(-50)/2a$ means 2. It will be 25.

So, $a = 25$ means Rohan should use his friend's birthday for the purchasing so that he can get, he should, he will have to pay minimum amount. Now what will be p_{12} when $a = 25$ so put $a = 25$ and you get 150, $(25)^2 - 50a + 690$. So, after calculation, the payable amount for p_{12} will be 9750. What if we apply the second case? First Sunday offer, then the birthday offer.

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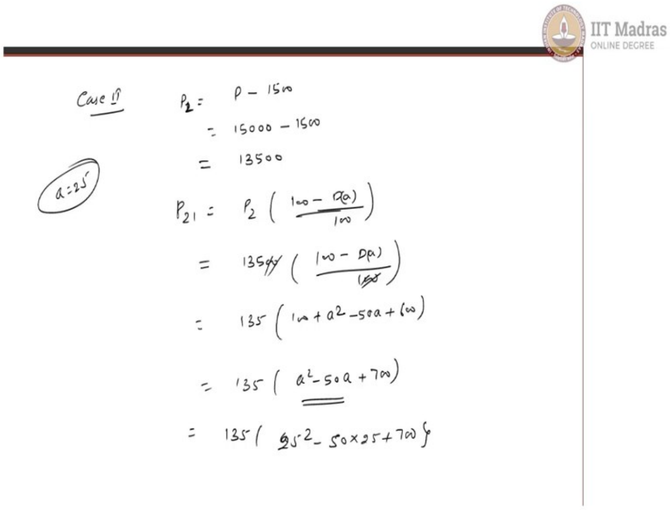

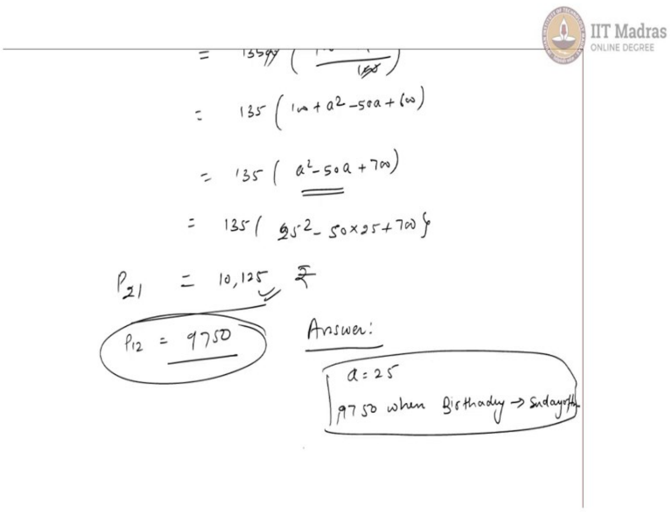

So, I will do it, in first as we have discussed first case, so for case 2, the p_1, p_2 will be $p-1500$. So, p is 15000-1500. It will be 13500. So, p_2 is 13500. Now, p_{21} will be applied on the p_2 and the discount will take $\frac{100-D(a)}{100}$ and that will be $\frac{13500-D(a)}{100}$. So, it is 135, 100-if I replace $D(a)$ with the function so it will be $a^2-50a+600$. So, I will get $135a^2-50a+600$.

Now, again we will follow the same procedure that minimal amount will be our $a=\frac{-b}{2}$ and again $a=25$ we are getting. So, you should use thus use his friend's birthday for the second case also. What will be the minimum payable amount in this case? $135a^2$ means $(25)^2 - 50 \times 25$+700. So, after solving this we will get p_{21}=10125.

So, obviously p_{12} which was around 9000, we solved here 9750 and p_{21} is 10125. So, clearly this is the least amount to be paid, so the answer for this question will be what? $a = 25$ which means Rohan should use his birthday, his friend's birthday and the payable amount will be 9750 when he uses the birthday offer first and then the Sunday offer. So, this is the answer.

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Mathematics for Data Science 1
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Week 08 - Tutorial 08

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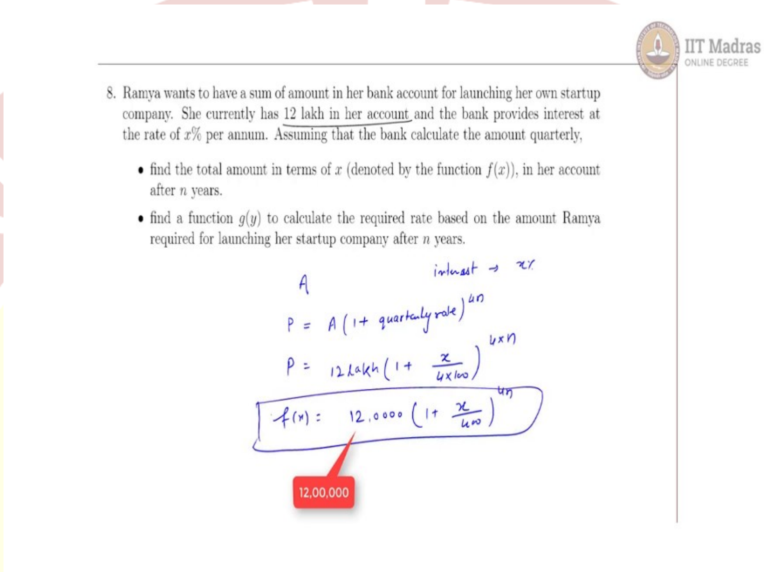

For our eighth question Ramya wants to have a sum of amount in her bank account for launching her own startup company. She currently has 12 lakhs in her account, so currently she has 12 lakhs in an account and then bank provides interest at the rate of $x\%$ per annum, assuming that the bank calculate the amount quarterly, quarterly means after 3 months, find the total amount in terms of x denoted by the function $f(x)$ in her account after n years.

So, if she has amount A in her account at this time and she gets interest at rate of $x\%$, assuming that the bank calculate the quarterly, so find the amount in the terms of x. So, total amount P in her account after time A is represented as the amount $A(1 + quarterly\,rate)$, rate is what, quarterly rate because we are getting spread by quarterly rate, quarterly then this will be this and how much years, n years. So, we know that the one year has 4 quarters, so n year will be having *4n* quarters, quarterly year.

So, A is 12 lakh I will say lakh, then $1 +$ what will be the quarterly? So, x is anum so I will divide 4 and then 100, $4 \times n$, so this is the amount in Ramya account after n year if we calculate the interest quarterly. Find this, so this is actually *x*, so it is represented by *f(x*), so *f(x)* 12 lakh means, so this will be the 12 lakh and $\left(1+\frac{x}{400}\right)4n$, this is the *f(x).*

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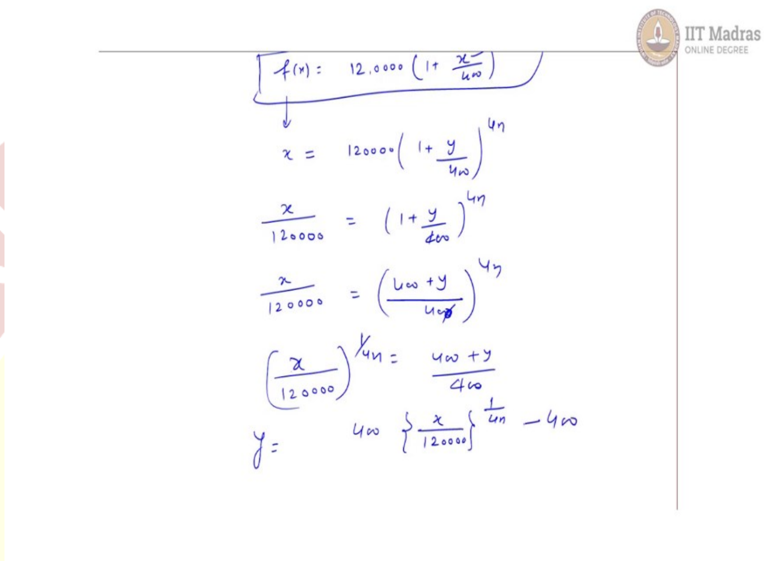

What is the second question? Find a function $g(y)$ to calculate the required rate based on the amount runway required for launching or startup after n years. So, this is a little thinking based question, what the question wants to say, find the function $g(y)$ to calculate the required rate, earlier we were given rate here and we were about to find the total sum, now the sum is given we need to find the rate, what we need to find the rate means, we need to find x and you know that when we need to find x we actually talk about the inverse function.

And how to calculate the inverse function you know, we just put $f(x)$, x and then 120000 1 plus we will replace x by some random variable y and then y so x divided by 1 you know this 4n will come here and this will be $\frac{4000+y}{400}$. So, y will be actually $400[(\frac{x}{120000})^{\frac{1}{4n}}] - 400$.

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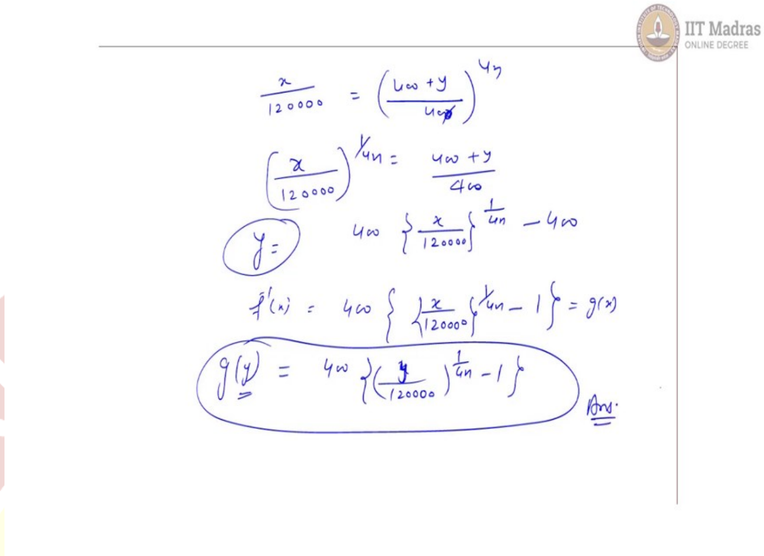

So, this is the inverse function of $x$and I can write it as f inverse and it is given as 400 you can take common $\frac{x}{12}$ like 1 by $4n - 1$ and the question is about to ask in $g(y)$. So, x represents we will just replace y with x so $\frac{400y}{12}$ $(4n - 1)$. So, do not get confused with this y and this one, both are different because here y is random variable and here we just gave that it could be $g(x)$ also, so we replace $g(x)$ with $g(y)$ here, so x is replaced by y. So, this is our the answer for the second question, thank you.

Mathematics for Data Science 1
Professor. Neelesh S Upadhye
Department of Mathematics
Indian Institute of Technology, Madras
Lecture No. 8.1
Additional lecture Inverse function

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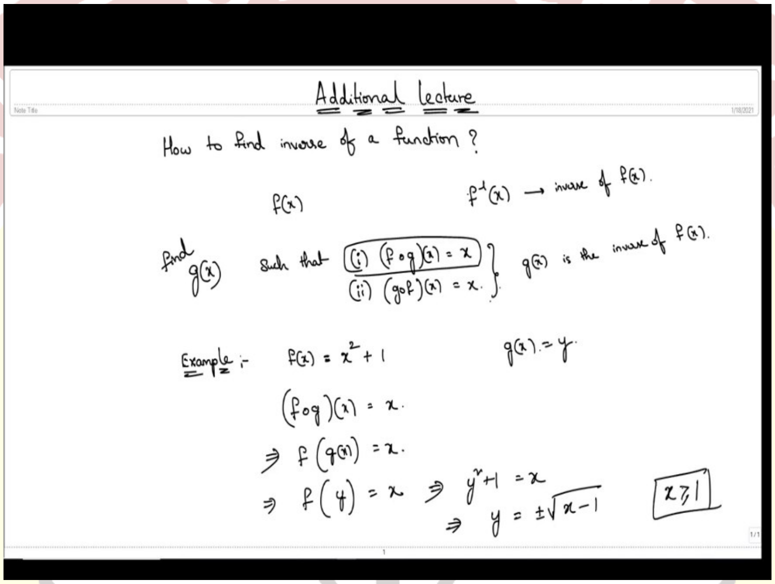

Hello everyone, today latest see how to find inverse of a function. Suppose $f(x)$ is a function, we try to find $f^{-1}(x)$ which is the inverse of $f(x)$, basically we try to find a function $g(x)$, we try to find $g(x)$ such that first $fog(x)$ will be x and also $gof(x)$ is x. So, when these two conditions are satisfied I can say, $g(x)$ is the inverse of $f(x)$. So, why do we need to check this two condition, why not only one is sufficient?

In order to find this let us take an example. Here is an example $f(x) = x^2 + 1$ we have to find the inverse of this function, let us say the inverse of this function is $g(x)$ let us assume. Now, we have this one condition, first condition is $fog(x) = x$, I will assume $g(x)$ =y. So, this implies $fog(x) = x$ and this imply f of if I substitute $g(x) = y$, $f(y) = x$ and I have $f(x) = x^2 + 1$.

So, this will imply $f(y) = y^2 + 1 = x$ if I take 1 to the right side and square root on both sides I will get finally $y = \pm\sqrt{x-1}$. So, in order to remove this complex ambiguities we take, we assume $x \geq 1$.

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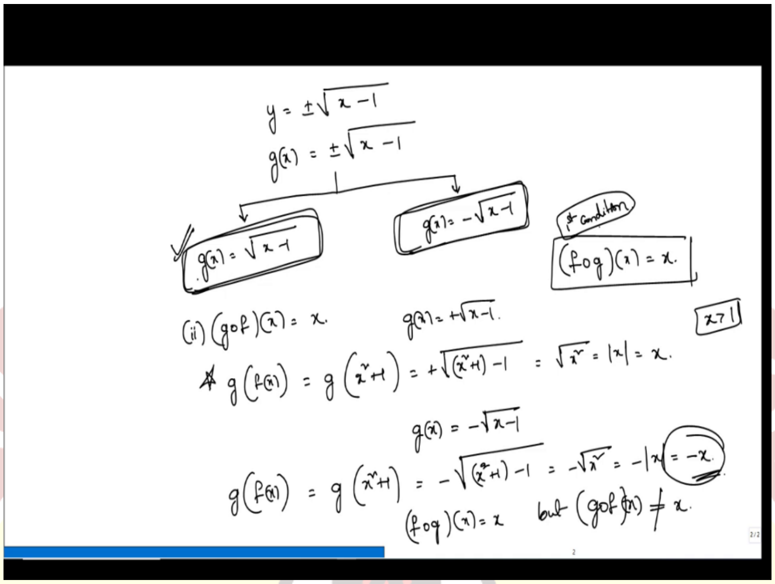

So, finally we have $y = \pm\sqrt{x-1}$ as we have assumed $y = g(x)$, so our finally $g(x)$ which is nothing but the inverse of the function $f(x) = \pm\sqrt{x-1}$,. So, I got two functions now, $g(x) = +\sqrt{x-1}$ and $g(x) = -\sqrt{x-1}$. So, we got these two functions that will satisfy the condition $fog(x) = x$ which is nothing but the first condition.

So, which one will be the inverse of $f(x)$? Whether $g(x) = \sqrt{x-1}$ or $g(x) = -\sqrt{x-1}$, that is why the second condition is also important which is our second condition is $gof(x) = x$. So, let us take two cases. Suppose my $gof(x) = -\sqrt{x-1}$. Now, $g\big(f(x)\big) = g$ of as I have $f(x) = x^2 + 1$that will be $g(x)^2 + 1 = \sqrt{x^2+1-1}$ which will be $\sqrt{x^2}$ which gives me $|x|$. As we have taken x>1, this will give me x. So, second condition is satisfied by this function $g(x)$.

Now, let us take the second case, where $g(x) = -\sqrt{x-1}$. Now, similarly I have $gof(x) = g(x)^2 + 1$ which is equals to $-\sqrt{x^2+1-1}$, which will be $-\sqrt{x^2}$, again the same thing $-|x| = -x$. So, if I take $g(x) = -\sqrt{x-1}$ my $fog(x)$will be x but my $gof(x) \neq x$. So, we can conclude that this function $g(x) = \sqrt{x-1}$ is the inverse of the function $f(x)$.

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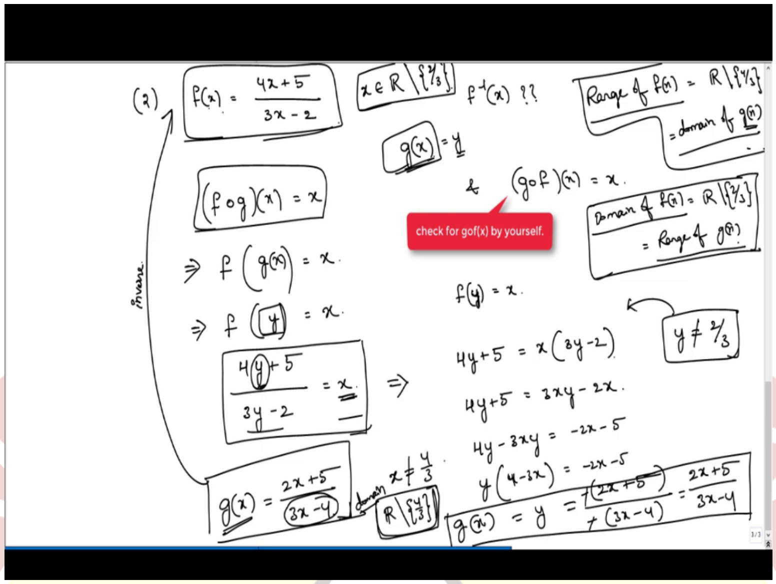

Now, let us take one more example $f(x) = \frac{(4x+5)}{(3x-2)}$. So, we have to find inverse of this function, so nothing but we have to find $f^{-1}(x)$, how do we find this? So, the same way let us assume there is a function $g(x)$ that is satisfying this condition $fog(x) = x$ and $gof(x) = x$.

Now, let us take this condition first, we have $f\big(g(x)\big) = x$. So, I will again assume $g(x) = y$, assume. So, this implies $f(y) = x$, so I have to find y such that my $f(y)$ giving me x. So, I have $f(x) = \frac{(4x+5)}{(3x-2)}$, so if I substitute now, I get $x = \frac{(4y+5)}{(3y-2)}$, so this is the condition I have and I have to find y in terms of x because y is a function of x, so y in terms of x and that y is nothing but $g(x)$ and that $g(x)$ is nothing but the inverse of this function.

So, if we solve this equation we get $4y + 5 = x(3y - 2)$. See if we are multiplying $3y - 2$ on both sides, that means we are assuming that $y \neq \frac{2}{3}$, so it is automatically comes in. Because the domain of this function f of x is x belongs to $\mathbb{R} - \frac{2}{3}$ so denominator should not be 0, so that is why the domain of this function will be $\mathbb{R} - \frac{2}{3}$.

So, now let us get back and solve this, we have $4y + 5 = 3xy - 2x$, so getting y terms on left side we get $4y - 3xy = -2x - 5$, so if I take $y(4 - 3x) = -(2x + 5)$, so ultimately we get $y = \frac{-(2x+5)}{-(3x-4)}$, so this get cancel and we get $\frac{2x+5}{3x-4}$ which will be the $g(x)$.

So, we got the function $g(x) = \frac{2x+5}{3x-4}$. So, this will be the inverse of the function $f(x)$, this is the inverse function. Now, if we see the domain of this $g(x)$, so if you see here the denominator if *3x -4* so it should not be equal to 0 that means x should not be equal to $\frac{4}{3}$, so the domain of $g(x)$will become set of real numbers $-\frac{4}{3}$, so this is the domain, this domain of $g(x)$ is nothing but the range of this function $f(x)$.

If we use this online graphing tools like the Desmos and put the function $f(x)$in that, we get an vertical asymptote at x is equals to $\frac{4}{3}$, that means $f(x)$ cannot take the value $\frac{4}{3}$. So, the range of this function $f(x)$, range of $f(x)$ will be set of $\mathbb{R} - \frac{4}{3}$ which will be the domain of $g(x)$ which is nothing but the inverse of the function $f(x)$. And we have one more thing this domain of $f(x) = \mathbb{R} - \frac{2}{3}$ which will be the range of $g(x)$.

So, finally this is how we find the inverse of any given function f of x and also the range of $f(x)$is equals to the domain of inverse of that function $f(x)$and the domain of that function $f(x)$ is equal to the range of the inverse of the function, of the function $f(x)$. Thank you.

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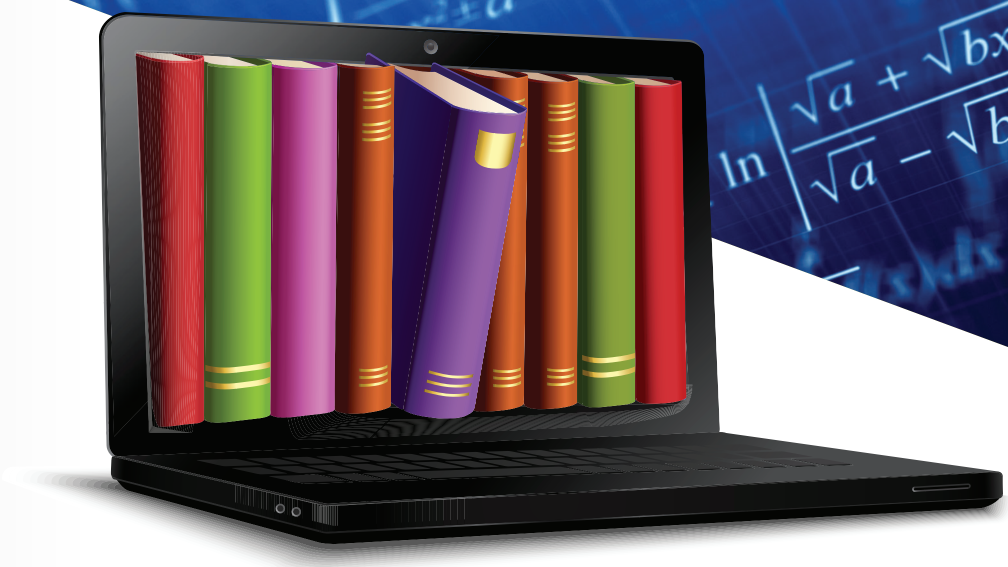

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