Week 4: Expected value

Subsection 1

Introduction

Summarizing data

- How to provide summary information about a large set of data?
  - Minimum value, maximum value, average value, median value
- Average value of a set of data is very useful in practice.
  - Average marks in a class exam
  - Run rate, batting average in cricket (and similar numbers in many sports)
- What does the average represent?
  - One number to represent a large set of numbers
  - Useful in comparisons and in so many scenarios
- Probability theory: expected value of a random variable is an important theoretical construct that represents the average value
  - It is used to express various types of summary information

Example: Casino math

Place bets on the sum seen when two die are rolled.

- Bets and returns
  - Under 7: get money back
  - Over 7: get money back
  - Equal to 7: get 4 times the money back

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Suppose you bet 1 unit of money.

Bet	Gain	Prob
Under 7	0	5/12
	-1	7/12
Over 7	0	5/12
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Equal 7	$+3$	1/6
	-1	5/6

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	-1	$5/6$

How did the casino decide on above returns? How to bet?

Expected value of a discrete random variable

Definition (Expected value)

Suppose X is a discrete random variable with range T_X and PMF f_X. The expected value of X, denoted $E[X]$, is defined as

$$E[X] = \sum_{t \in T_X} t \, f_X(t), = \sum_{t \in T_X} t \, P(X=t)$$

assuming above sum exists.

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- Other names: mean of X, ~~average value~~ of X
- $E[X]$ may or may not belong to the range of X
- $E[X]$ has the same units as X

Examples: Easy summation given PMF

1. $X \sim \text{Bernoulli}(p)$

$$X \in \{\overset{1-p}{0}, \overset{p}{1}\}$$

$$E[X] = 0(1-p) + 1(p) = p = P(X=1)$$

Examples: Easy summation given PMF

1. $X \sim \text{Bernoulli}(p)$

$$E[X] = 0(1-p) + 1(p) = p$$

$1/6 \quad 1/6 \quad 1/6 \quad 1/6 \quad 1/6 \quad 1/6$

2. $X \sim \text{Uniform}\{1, 2, 3, 4, 5, 6\}$

$$E[X] = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = 3.5$$

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3. Value of a lottery ticket (in Rs.) is $\{\overset{1/1000}{200}, \overset{27/1000}{20}, \overset{972/1000}{0}\}$.

$$E[X] = 200\cdot\frac{1}{1000} + 20\cdot\frac{27}{1000} + 0\cdot\frac{972}{1000} = \text{Rs. } 0.56$$

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4. Change in temperature (in °C) is $\{\overset{1/5}{-2}, \overset{1/5}{-1}, \overset{1/2}{0}, \overset{1/10}{1}\}$.

$$E[X] = -2 \cdot \frac{1}{5} - 1 \cdot \frac{1}{5} + 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{10} = -0.5 \text{ °C}$$

Example: Uniform$\{a, a+1, \ldots, b\}$

$\frac{1}{b-a+1}$

$X \sim \text{Uniform}\{a, a+1, \ldots, b\}$

$$E[X] = a \cdot \frac{1}{b-a+1} + (a+1) \cdot \frac{1}{b-a+1} + \cdots + b \cdot \frac{1}{b-a+1}$$

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How to simplify sum?

Identity: $a + (a+1) + \cdots + b = (b-a+1)\,\frac{a+b}{2}$

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Identity: $a + (a+1) + \cdots + b = (b-a+1)\,\frac{a+b}{2}$

$$E[X] = \frac{a+b}{2}$$

Examples: More involved summation

$(1-p)^{k-1}p$

1. $X \sim \text{Geometric}(p)$ $\quad X \in \{1, 2, \ldots, k, \ldots\}$

$$E[X] = \sum_{t=1}^{\infty} t(1-p)^{t-1}p$$

$$= p + 2(1-p)p + 3(1-p)^2 p + \cdots$$

Examples: More involved summation

1. $X \sim \text{Geometric}(p)$

$$E[X] = \sum_{t=1}^{\infty} t(1-p)^{t-1}p$$

2. $X \sim \text{Poisson}(\lambda)$

$$E[X] = \sum_{t=0}^{\infty} t\ \overbrace{e^{-\lambda}\frac{\lambda^t}{t!}}^{P(X=t)}$$

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$$E[X] = \sum_{t=0}^{\infty} t\ e^{-\lambda}\frac{\lambda^t}{t!}$$

3. $X \sim \text{Binomial}(n, p)$

$$E[X] = \sum_{t=0}^{n} t\ \overbrace{\binom{n}{t} p^t(1-p)^{n-t}}^{P(X=t)}$$

$$nC_t \leftrightarrow \binom{n}{t} = \frac{n!}{t!(n-t)!}$$

How to simplify sum?

1. Difference Equation (DE): $a_{t+1} - r\,a_t = \underbrace{b_t}_{\text{another sequence}}$ $(r \neq 1)$

$$\sum_{t=1}^{n} a_t = \frac{a_1 - r\,a_n}{1-r} + \frac{1}{1-r}\sum_{t=1}^{n-1} b_t$$

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2. Geometric Progression (GP): $a_{t+1} - r\,a_t = 0$ $(r \neq 1)$

$$\sum_{t=1}^{n} a_t = \frac{a_1 - ra_n}{1-r} \quad \overset{|r|<1}{\underset{n\to\infty}{\Longrightarrow}} \quad \frac{a_1}{1-r}$$

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3. Exponential Function:

$$\sum_{t=0}^{\infty} e^{-\lambda}\frac{\lambda^t}{t!} = 1 \iff e^{\lambda} = \sum_{t=0}^{\infty}\frac{\lambda^t}{t!}$$

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$$\sum_{t=0}^{\infty} e^{-\lambda}\frac{\lambda^t}{t!} = 1$$

4. Binomial formula: $\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n$

$a = p$, $b = 1-p \Rightarrow \overset{RHS=1}{\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}} = 1$

Examples: More involved summation

1. $X \sim \text{Geometric}(p)$ (DE: $a_1 = p$, $r = 1 - p$, $b_t = r^t p$)

$E[X] = 1/p$

$$E[X] = \sum_{t=1}^{\infty} \underbrace{t(1-p)^{t-1}p}_{a_t} = 1/p$$

$a_{t+1} = (t+1)(1-p)^t p$

$a_{t+1} - (1-p)a_t = (1-p)^t p$

Examples: More involved summation

1. $X \sim$ Geometric(p) (DE: $a_1 = p$, $r = 1-p$, $b_t = r^t p$)

$$E[X] = \sum_{t=1}^{\infty} t(1-p)^{t-1} p = 1/p$$

2. $X \sim$ Poisson(λ): $E[X] = \sum_{t=0}^{\infty} t \ e^{-\lambda} \frac{\lambda^t}{t!}$

$E[X]=\lambda$

$$E[X] = e^{-\lambda}\left(0 \cdot + \cancel{1} \cdot \frac{\lambda}{\cancel{1}!} + \cancel{2} \cdot \frac{\lambda^2}{\cancel{2}!_{1!}} + \cancel{3} \cdot \frac{\lambda^3}{\cancel{3}!_{2!}} + \cdots\right)$$

$$= \lambda e^{-\lambda}\left(\underbrace{1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \cdots}_{e^{\lambda}}\right) = \lambda$$

Examples: More involved summation

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$E[X] = \lambda$

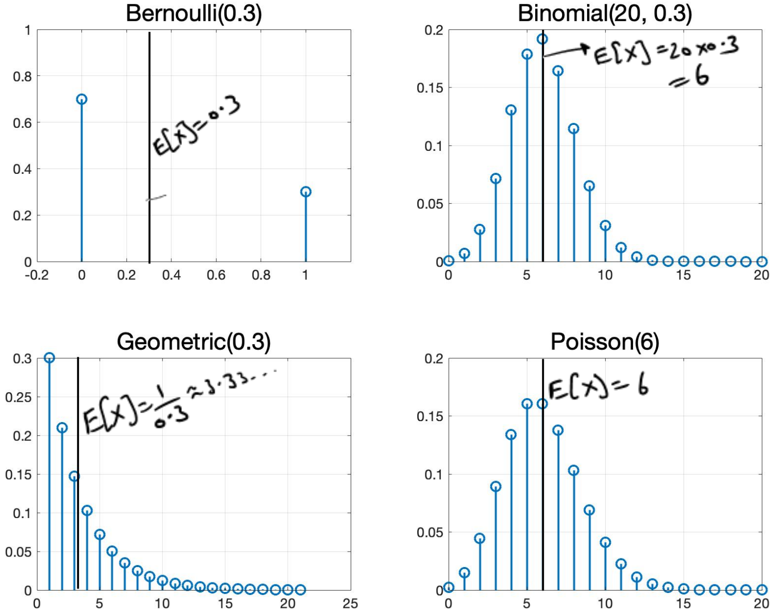
$$E[X] = e^{-\lambda}\left(0 \cdot +1 \cdot \frac{\lambda}{1!} + 2 \cdot \frac{\lambda^2}{2!} + 3 \cdot \frac{\lambda^3}{3!} + \cdots\right)$$
$$= \lambda e^{-\lambda}\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \cdots\right) = \lambda$$

3. $X \sim$ Binomial(n, p): $E[X] = \sum_{t=0}^{n} t \binom{n}{t} p^t (1-p)^{n-t}$

$E[X] = np$

$$E[X] = \sum_{t=1}^{n} \frac{\cancel{t}\; n!}{\underset{(t-1)!}{\cancel{t!}}(n-t)!} p^t (1-p)^{n-t} = np \sum_{t=1}^{n} \frac{(n-1)!}{(t-1)!(n-t)!} p^{t-1}(1-p)^{n-t}$$
$$= np\underline{(p + (1-p))^{n-1}} = np$$

PMF and expected value



Subsection 2

Properties of expected value

Example: Casino math

Bet	Gain	Prob
Under 7	0	5/12
	−1	7/12
Over 7	0	5/12
	−1	7/12
Equal 7	+3	1/6
	−1	5/6

Suppose you bet 1 unit of money with probability p_1 each on "Under 7" or "Over 7" and probability $1 - 2p_1$ on "Equal to 7". What is the expected gain?

$$\overbrace{p_1\cdot\frac{5}{12}\times 0 + p_1\cdot\frac{7}{12}\times(-1)}^{\text{Under 7}} + \overbrace{p_1\cdot\frac{5}{12}\times 0 + p_1\cdot\frac{7}{12}\times(-1)}^{\text{Over 7}} + (1-2p_1)\cdot\frac{1}{6}\cdot 3 + (1-2p_1)\cdot\frac{5}{6}(-1)$$

$$= -\frac{7p_1}{6} - (1-2p_1)\cdot\frac{2}{6} = \frac{-7p_1-2+4p_1}{6} = \frac{-(2+3p_1)}{6}$$

Negative

Constant random variable and positive random variable

$$E[X] = \sum_{t \in T_X} t\, P(X = t)$$

Constant random variable and positive random variable

$$E[X] = \sum_{t \in T_X} t\, P(X = t)$$

1. Consider a constant c as a random variable X with $P(X = c) = 1$.

$$E[c] = c \cdot 1 = c$$

Constant random variable and positive random variable

$$E[X] = \sum_{t \in T_X} t\, P(X = t)$$

1. Consider a constant c as a random variable X with $P(X = c) = 1$.

$$E[c] = c \cdot 1 = c$$

2. Suppose X takes only non-negative values, i.e. $P(X \geq 0) = 1$. Then,

$$E[X] \geq 0$$

Expected value of a function of random variables

Theorem (Expected value of a function)

Suppose $X_1, \ldots, X_n$ have joint PMF $f_{X_1 \cdots X_n}$ with range of X_i denoted T_{X_i}. Let $g : T_{X_1} \times \cdots \times T_{X_n} \to \mathbb{R}$ be a function, and let $Y = g(X_1, \ldots, X_n)$ have range T_Y and PMF f_Y. Then,

$$E[g(X_1, \ldots, X_n)] = \overbrace{\sum_{t \in T_Y} t\, f_Y(t)}^{E[Y]} = \sum_{t_i \in T_{X_i}} g(t_1, \ldots, t_n) f_{X_1 \cdots X_n}(t_1, \ldots, t_n).$$

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- We have seen how to find f_Y, PMF of a function of multiple random variables
- The above theorem says: To find $E[Y]$, you do not always need f_Y. The joint PMF of $X_1, \ldots, X_n$ can be used directly.
- Simple sounding property with far-reaching consequences!

Examples for illustration

1. $X \sim \{\overset{1/5}{-2}, \overset{1/5}{-1}, \overset{1/5}{0}, \overset{1/5}{1}, \overset{1/5}{2}\}$, $g(X) = X^2 \sim \{\overset{1/5}{0}, \overset{2/5}{1}, \overset{2/5}{4}\}$

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$$E[g(X)] = 0 \cdot \frac{1}{5} + 1 \cdot \frac{2}{5} + 4 \cdot \frac{2}{5} = 2$$

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$$E[g(X)] = 0 \cdot \frac{1}{5} + 1 \cdot \frac{2}{5} + 4 \cdot \frac{2}{5} = 2$$

$$E[g(X)] = (-2)^2 \cdot \frac{1}{5} + (-1)^2 \cdot \frac{1}{5} + (0)^2 \cdot \frac{1}{5} + (1)^2 \cdot \frac{1}{5} + (2)^2 \cdot \frac{1}{5} = 2$$

$g(x)$ $\quad$ $P(X=-2)$ $\quad$ $\sum_t t^2 P(X=t)$

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2. $(X, Y) \sim \text{Uniform}\{\overset{1/6}{(0,0)}, \overset{1/6}{(1,0)}, \overset{1/6}{(0,1)}, \overset{1/6}{(1,1)}, \overset{1/6}{(-1,1)}, \overset{1/6}{(1,-1)}\}$

$$g(X, Y) = X^2 + XY + Y^2 \sim \{\overset{1/6}{0}, \overset{4/6}{1}, \overset{1/6}{3}\}$$

x	y	$g(x,y)$	Pr
0	0	0	1/6
1	0	1	1/6
0	1	1	1/6
1	1	3	1/6
-1	1	1	1/6
1	-1	1	1/6

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2. $(X, Y) \sim \text{Uniform}\{(0,0), (1,0), (0,1), (1,1), (-1,1), (1,-1)\}$

$$g(X, Y) = X^2 + XY + Y^2 \sim \{\overset{1/6}{0}, \overset{4/6}{1}, \overset{1/6}{3}\}$$

$$E[g(X, Y)] = 0 \cdot \frac{1}{6} + 1 \cdot \frac{4}{6} + 3 \cdot \frac{1}{6} = 7/6$$

Examples for illustration

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$$E[g(X, Y)] = 0 \cdot \tfrac{1}{6} + 1 \cdot \tfrac{4}{6} + 3 \cdot \tfrac{1}{6} = 7/6$$

$g(1,-1)$ $P(X=1, Y=-1)$

$$E[g(X, Y)] = 0 \cdot \tfrac{1}{6} + (1) \cdot \tfrac{1}{6} + (1) \cdot \tfrac{1}{6} + (3) \cdot \tfrac{1}{6} + (1) \cdot \tfrac{1}{6} + (1) \cdot \tfrac{1}{6} = 7/6$$

$g(0,0)$ $P(X=0, Y=0)$

Linearity of expected value

1. $E[cX] = cE[X]$ for a random variable X and a constant c

Proof:

$$E[cX] = \sum_{t \in T_X} ct\, f_X(t) = c \sum_{t \in T_X} t\, f_X(t) = cE[X]$$

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2. $E[X+Y] = E[X] + E[Y]$ for any two random variables X, Y

Proof:

$$\begin{aligned} E[X+Y] &= \sum_{t_1 \in T_X, t_2 \in T_Y} (t_1 + t_2) f_{XY}(t_1, t_2) \\ &= \sum_{t_1 \in T_X, t_2 \in T_Y} t_1\, f_{XY}(t_1, t_2) + \sum_{t_1 \in T_X, t_2 \in T_Y} t_2\, f_{XY}(t_1, t_2) \\ &= E[X] + E[Y] \end{aligned}$$

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$$E[aX + bY] = aE[X] + bE[Y]$$

One of the most useful properties of expected value!

Illustrative examples

1. $(X, Y) \sim \text{Uniform}\{(0,0),(1,0),(0,1),(1,1),(-1,1),(1,-1)\}$

$$g(X,Y) = X^2 + XY + Y^2 \sim \{\overset{1/6}{0}, \overset{4/6}{1}, \overset{1/6}{3}\}$$

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$$E[g(X, Y)] = 7/6$$

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$E[g(X, Y)] = E[X^2 + XY + Y^2] = E[X^2] + E[XY] + E[Y^2] = 7/6$

$(E[X^2] = 4/6,\ E[XY] = -1/6,\ E[Y^2] = 4/6)$

Illustrative examples

1. $(X, Y) \sim \text{Uniform}\{(0,0),(1,0),(0,1),(1,1),(-1,1),(1,-1)\}$

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Note: Expected value of the form $E[g(X) + h(Y)]$ can be computed with marginal PMFs, and it does not depend on the joint PMF.

Problem: Throw a fair die twice

What is the expected value of the sum of the two numbers seen?

X: first number $\quad$ Y: second number

$$E[X+Y] = E[X] + E[Y]$$
$$= 3.5 + 3.5 = 7$$

Problem: Expected value of Binomial(n, p)

Suppose $Y \sim \text{Binomial}(n, p)$.

$$E[Y] = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} = np$$

- Such summations may be difficult to simplify

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- Alternative method using linearity of expected value

Let $X_1, \ldots, X_n$ be i.i.d. Bernoulli(p). Then, $E[X_i] = p$ and

$$Y = X_1 + \cdots + X_n \sim \mathrm{Binomial}(n, p)$$

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So, $E[Y] = E[X_1] + \cdots + E[X_n] = np$

Zero-mean random variable

A random variable X with $E[X] = 0$ is said to be a zero-mean random variable.

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Translation of a random variable

$X + c$, where c is a constant, is a "translated" version of X.

- Range of $X + c$ is $\{t + c : t \in T_X\}$, which is a translated version of T_X, the range of X
- $P(X + c = t + c) = P(X = t)$ and the PMF is "translated" as well

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- $P(X + c = t + c) = P(X = t)$ and the PMF is "translated" as well

Translation by expected value

Important: $E[X]$ is a constant.

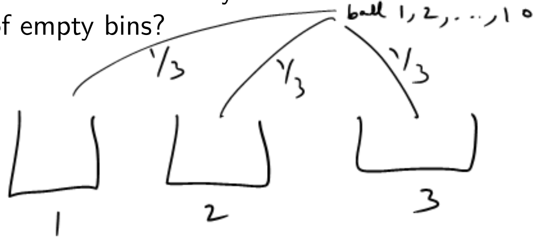
$Y = X - E[X]$ is a translated version of X and $E[Y] = 0$.

So, $X - E[X]$ is a zero-mean random variable.

"centering"

Problem: Balls on bins

Suppose 10 balls are thrown uniformly at random into 3 bins. What is the expected number of empty bins?



$$X_i = \begin{cases} 1, & \text{if bin } i \text{ is empty} \\ 0, & \text{otherwise} \end{cases} \qquad i = 1, 2, 3$$

$$P(X_i = 1) = \frac{2^{10}}{3^{10}} = \left(\frac{2}{3}\right)^{10}, \quad P(X_i = 0) = 1 - \left(\frac{2}{3}\right)^{10}$$

$$Y = \#\text{ empty bins} = X_1 + X_2 + X_3$$

"Finding f_Y is a little hard"

$$E[Y] = E[X_1] + E[X_2] + E[X_3] = 3 \cdot \left(\frac{2}{3}\right)^{10}$$

Subsection 3

Variance

Motivation

How good is expected value in describing a random variable?

- Let $X \sim \{\overset{1}{10}\}$, $Y \sim \{\overset{1/2}{9}, \overset{1/2}{11}\}$, $Z \sim \{\overset{1/2}{0}, \overset{1/2}{20}\}$.

$$E[X] = E[Y] = E[Z] = 10$$

However, the three random variables are quite different!

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However, the three random variables are quite different!

Center and spread

- Expected value: represents "center" of a random variable
- We need some indicator of "spread" about the expected value

Variance and standard deviation

Definition (Variance and standard deviation)

The variance of a random variable X, denoted $\mathrm{Var}(X)$, is defined as

$$\mathrm{Var}(X) = E[\underbrace{(X - \overbrace{E[X]}^{\text{some constant}})^2}_{\text{function of } X}].$$

The standard deviation of X, denoted $\mathrm{SD}(X)$, is defined as

$$\mathrm{SD}(X) = +\sqrt{\mathrm{Var}(X)}.$$

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$$\mathrm{SD}(X) = +\sqrt{\mathrm{Var}(X)}.$$

- Variance is expected value of the random variable $(X - E[X])^2$.
  - $\mathrm{Var}(X) = \sum_{t \in T_X} (t - E[X])^2 P(X = t)$
  - Variance is non-negative, and standard deviation is well-defined.
  - Units of $\mathrm{SD}(X)$ are same as units of X.
- Intuitively, the more the "spread" in T_X, the more will be $\mathrm{Var}(X)$

Example

$X \sim \{\overset{1}{10}\}$, $Y \sim \{\overset{1/2}{9}, \overset{1/2}{11}\}$, $Z \sim \{\overset{1/2}{0}, \overset{1/2}{20}\}$

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Variances and standard deviations are different!

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$$E[X] = E[Y] = E[Z] = 10$$

Variances and standard deviations are different!

$$\mathrm{Var}(X) = (\overset{t \in T_X}{10} - \overset{E[X]}{10})^2 \cdot 1 = 0,\ \mathrm{SD}(X) = 0$$

Example

$$X \sim \{\overset{1}{10}\},\ Y \sim \{\overset{1/2}{9}, \overset{1/2}{11}\},\ Z \sim \{\overset{1/2}{0}, \overset{1/2}{20}\}$$

$$E[X] = E[Y] = E[Z] = 10$$

Variances and standard deviations are different!

$$\mathrm{Var}(X) = (10 - 10)^2 \cdot 1 = 0,\ \mathrm{SD}(X) = 0$$

$t \in T_Y$ $E[Y]$ $P(Y=t)$

$$\mathrm{Var}(Y) = (9 - 10)^2 \cdot \frac{1}{2} + (11 - 10)^2 \cdot \frac{1}{2} = 1,\ \mathrm{SD}(X) = 1$$

Example

$X \sim \{\overset{1}{10}\},\ Y \sim \{\overset{1/2}{9}, \overset{1/2}{11}\},\ Z \sim \{\overset{1/2}{0}, \overset{1/2}{20}\}$

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Variances and standard deviations are different!

$$\mathrm{Var}(X) = (10 - 10)^2 \cdot 1 = 0,\ \mathrm{SD}(X) = 0$$

$$\mathrm{Var}(Y) = (9 - 10)^2 \cdot \frac{1}{2} + (11 - 10)^2 \cdot \frac{1}{2} = 1,\ \mathrm{SD}(X) = 1$$

$$\mathrm{Var}(Z) = (0 - 10)^2 \cdot \frac{1}{2} + (20 - 10)^2 \cdot \frac{1}{2} = 100,\ \mathrm{SD}(X) = 10$$

Example: Throw a die

$X \sim \text{Uniform}\{1, 2, 3, 4, 5, 6\}$

$$E[X] = 3.5$$

Example: Throw a die

$X \sim \text{Uniform}\{1,2,3,4,5,6\}$

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$$\begin{aligned}
\text{Var}(X) &= (1-3.5)^2 \cdot \frac{1}{6} + (2-3.5)^2 \cdot \frac{1}{6} + (3-3.5)^2 \cdot \frac{1}{6} \\
&\quad + (4-3.5)^2 \cdot \frac{1}{6} + (5-3.5)^2 \cdot \frac{1}{6} + (6-3.5)^2 \cdot \frac{1}{6} \\
&= \frac{35}{12} = 2.916\ldots \\
\text{SD}(X) &= 1.7078\ldots
\end{aligned}$$

$\sqrt{\frac{35}{12}}$

Example: Throw a die

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\end{aligned}$$

Given PMF of X with a small range, variance and standard deviation can be readily computed.

Properties: Scaling and translation

Let X be a random variable. Let a be a constant real number.

1. $\mathrm{Var}(aX) = a^2\mathrm{Var}(X)$
2. $\mathrm{SD}(aX) = |a|\mathrm{SD}(X)$
3. $\mathrm{Var}(X + a) = \mathrm{Var}(X)$
4. $\mathrm{SD}(X + a) = \mathrm{SD}(X)$

Pf. $E\left[(X+a-\underbrace{E[X+a]})^2\right]$

$= E\left[(X+\cancel{a}-E[X]-\cancel{a})^2\right]$

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3. $\mathrm{Var}(X + a) = \mathrm{Var}(X)$
4. $\mathrm{SD}(X + a) = \mathrm{SD}(X)$

- Contrast with similar properties for $E[X]$

Alternative formula for variance

Theorem

The variance of a random variable X is given by

$$Var(X) = E[X^2] - E[X]^2.$$

$$E[X^2] = \sum_{t \in T_X} t^2 P(X=t)$$

$$E[X]^2 = \left(\sum_{t \in T_X} t P(X=t) \right)^2$$

Alternative formula for variance

Theorem

The variance of a random variable X is given by

$$Var(X) = E[X^2] - E[X]^2.$$

Proof:

- $\text{Var}(X) = E[(X - E[X])^2] = E[X^2 - 2E[X]X + E[X]^2]$
- By linearity of expected value,

$$\text{Var}(X) = E[X^2] - 2E[E[X]X] + E[E[X]^2]$$

- Since $E[X]$ is a constant,

$$\text{Var}(X) = E[X^2] - 2E[X]E[X] + E[X]^2 = E[X^2] - E[X]^2$$

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$E[X]$, $E[X^2]$: first and second moment, $\mathrm{Var}(X)$: second central moment

Sum and product of independent random variables

For any two random variables X and Y (independent or dependent),

$$E[X + Y] = E[X] + E[Y].$$

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Suppose X and Y are **independent** random variables. Then, more is true.

1. $E[XY] = E[X]E[Y]$

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1. $E[XY] = E[X]E[Y]$
2. $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

Proof

$$E[XY] = \sum_{\substack{t_1 \in T_X,\\ t_2 \in T_Y}} t_1 t_2 \underbrace{P(X=t_1, Y=t_2)}_{X \,\&\, Y \text{ indep: }} = P(X=t_1)P(Y=t_2)$$

$$= \sum_{t_1 \in T_X} \left(\sum_{t_2 \in T_Y} t_1 t_2 P(X=t_1) P(Y=t_2) \right)$$

over t_2

factoring IS crucial

$$= \sum_{t_1 \in T_X} \left(t_1 P(X=t_1)\right) \underbrace{\left(\sum_{t_2 \in T_Y} t_2 P(Y=t_2) \right)}_{E[Y]}$$

$$= E[Y] \underbrace{\left(\sum_{t_1 \in T_X} t_1 P(X=t_1) \right)}_{E[X]}$$

$$= E[X]E[Y] \longrightarrow$$ Not always true if X & Y are dependent

Sum and product of independent random variables

For any two random variables X and Y (independent or dependent),

$$E[X+Y] = E[X] + E[Y].$$

Suppose X and Y are **independent** random variables. Then, more is true.

1. $E[XY] = E[X]E[Y]$
2. $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$ Pf: Exercise.

Example: Sum of two dice (independent)

What is the variance of the sum of two dice?

X: first die $\quad Y$: second die

X & Y: independent

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$$

$$= 2\sqrt{\frac{35}{12}} = \sqrt{\frac{35}{3}}$$

Variance of common distributions

Distribution	Expected value	Variance
Bernoulli(p)	p	$p(1-p)$
Binomial(n,p)	np	$np(1-p)$
Geometric(p)	$1/p$	$(1-p)/p^2$
Poisson(λ)	λ	λ
Uniform$\{1,\ldots,n\}$	$(n+1)/2$	$(n^2-1)/12$

① $X \sim \left\{ \overset{1-p}{0}, \overset{p}{1} \right\}$ $E[X] = p$ $Var(X) = p - p^2$

$E[X^2] = 1 \cdot p = p$ $= p(1-p)$

② $X \sim \text{Binomial}(n,p)$ $E[X] = np$

$X = X_1 + \cdots + X_n$ $Var(X) = Var(X_1) + \cdots + Var(X_n)$

$X_i \sim \underline{\underline{iid}}$ Bernoulli(p) $= np(1-p)$ (more involved)

$$E[X^2] = \sum_{k=0}^{n} k^2 \binom{n}{k} p^k (1-p)^{n-k} = \underbrace{np(1-p)}_{Var(X)} + \underbrace{(np)^2}_{E[X]^2}$$

Working

Standardised random variables

Definition

A random variable X is said to be standardised if

$$E[X] = 0, \mathrm{Var}(X) = 1.$$

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Theorem

Let X be a random variable. Then,

$$Y = \frac{X - E[X]}{SD(X)}$$

is a standardised random variable.

$$E[Y] = \frac{1}{SD(X)}\left(E[X] - E[X]\right) = 0 \qquad \mathrm{Var}(Y) = \mathrm{Var}\Bigg(\frac{X}{\underbrace{SD(X)}_{\text{constant}}}\Bigg) = \frac{\mathrm{Var}(X)}{SD(X)^2} = 1$$

→ PMF of Y is "very similar" to PMF of X.

Existence of expected value and variance

- There are random variables s.t. $E[X]$ goes to ∞ (or $-\infty$)
  - $X \sim \{\overset{1/2}{1}, \overset{1/4}{2}, \overset{1/8}{4}, \ldots, \overset{1/2^n}{2^{n-1}}, \ldots\}$
  - $E[X^2]$ will go to ∞ in this case and $\mathrm{Var}(X)$ is ill-defined
- There are random variables s.t. $E[X]$ is not well-defined
  - $X \sim \{\overset{1/2}{1}, \overset{1/4}{-2}, \overset{1/8}{4}, \ldots, \overset{1/2^n}{(-2)^{n-1}}, \ldots\}$
  - $E[X^2]$ will go to ∞ in this case and $\mathrm{Var}(X)$ is ill-defined
- There are random variables s.t. $E[X]$ is finite, but $E[X^2]$ goes to ∞
  - $\mathrm{Var}(X)$ goes to ∞ too $\quad X \in \{1, 2, 3, \ldots\} \quad P(X=k) \propto \frac{1}{k^3}$
- In this course, we will generally consider well-behaved random variables with finite mean and variance

Subsection 4

Covariance and correlation

Motivation

Consider the following two joint PMFs for two random variables X and Y.

f_{XY}	$X=0$	$X=1$	f_Y
$Y=0$	$1/\cancel{2}4$	$1/\cancel{2}4$	$1/2$
$Y=1$	$1/\cancel{2}4$	$1/\cancel{2}4$	$1/2$
f_X	$1/2$	$1/2$	

f_{XY}	0	1	f_Y
0	0	$1/2$	$1/2$
1	$1/2$	0	$1/2$
f_X	$1/2$	$1/2$	

Motivation

Consider the following two joint PMFs for two random variables X and Y.

f_{XY}	0	1
0	$1/\cancel{2}4$	$1/\cancel{2}4$
1	$1/\cancel{2}4$	$1/\cancel{2}4$

f_{XY}	0	1
0	0	1/2
1	1/2	0

- Same marginal PMFs in both cases. So, mean and variance are the same for X and Y.
- However, the two cases are very different
  - X and Y are independent in one case
  - Value of X determines the value of Y in the other

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f_{XY}	0	1
0	1/2	1/2
1	1/2	1/2

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  - Value of X determines the value of Y in the other

How to summarize relationship between random variables?

Covariance

Definition (Covariance)

Suppose X and Y are random variables on the same probability space. The covariance of X and Y, denoted $\mathrm{Cov}(X, Y)$, is defined as

$$\mathrm{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])].$$

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- $\mathrm{Cov}(X, Y)$ is positive
  - $(X - E[X])(Y - E[Y])$ tends to be positive
  - When X is above/below its average, Y tends to be correspondingly above/below its average

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  - When X is above/below its average, Y tends to be correspondingly above/below its average
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  - $(X - E[X])(Y - E[Y])$ tends to be positive
  - When X is above/below its average, Y tends to be correspondingly above/below its average
- $\mathrm{Cov}(X, Y)$ is negative
  - $(X - E[X])(Y - E[Y])$ tends to be negative
  - When X is above/below its average, Y tends to be correspondingly below/above its average
- $\mathrm{Cov}(X, Y) = 0$
  - X and Y are said to be "uncorrelated"

Examples: positive and negative covariance

1. X: height of a person, Y: weight of a person
    - A higher value of X tends to result in a higher value for Y
    - We expect $\text{Cov}(X, Y)$ to be positive

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2. X: Rainfall during monsoon, Y: Farmer debt
   - A higher value of X tends to result in a lower value for Y
   - We expect $\text{Cov}(X, Y)$ to be negative

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2. X: Rainfall during monsoon, Y: Farmer debt
   - A higher value of X tends to result in a lower value for Y
   - We expect $\text{Cov}(X, Y)$ to be negative
3. X: Runs in an IPL over, Y: Wickets in the same over
   - We expect $\text{Cov}(X, Y)$ to be negative

Example: Computing covariance

	$X=-1$	$X=0$	$X=1$	f_Y
$Y=-1$	$1/15$	$2/15$	$2/15$	$1/3$
$Y=0$	$2/15$	$1/15$	$2/15$	$1/3$
$Y=1$	$2/15$	$2/15$	$1/15$	$1/3$
f_X	$1/3$	$1/3$	$1/3$	

$E[Y]=0$

$E[X]=0$

$$\mathrm{Cov}(X,Y)=E\big[(X-E[X])(Y-E[Y])\big] = (-1)(-1)\cdot\frac{1}{15}+(-1)(1)\cdot\frac{2}{15}$$

$$+(-1)(1)\cdot\frac{2}{15}+(1)(1)\cdot\frac{1}{15}$$

$$=\frac{-2}{15}$$

Properties

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4. Covariance is a "linear" quantity
   1. $\text{Cov}(X, aY + bZ) = a\text{Cov}(X, Y) + b\text{Cov}(X, Z)$
   2. $\text{Cov}(aX + bY, Z) = a\text{Cov}(X, Z) + b\text{Cov}(Y, Z)$

Covariance and independence

1. If X and Y are independent, X and Y are uncorrelated, i.e. $\text{Cov}(X, Y) = 0$
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   - Proof: $\mathrm{Cov}(X, Y) = E[XY] - E[X]E[Y] = 0$
2. If X and Y are uncorrelated, they may be dependent.

Example:

Dependent, but uncorrelated

	$X=-1$	$X=0$	$X=1$	f_Y
$Y=0$	$1/8$	$1/4$	$1/8$	$1/2$
$Y=1$	$1/4$	$0 \neq \frac{1}{2}\cdot\frac{1}{4}$	$1/4$	$1/2$
f_X	$3/8$	$1/4$	$3/8$	

$E[Y] = 1/2$

$E[X] = 0$

$E[XY] = 0$

$E[XY] = E[X]E[Y]$

Correlation coefficient

Definition (Correlation coefficient)

The correlation coefficient or simply correlation of two random variables X and Y, denoted $\rho(X, Y)$, is defined as

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- Result: $-\text{SD}(X)\text{SD}(Y) \leq \text{Cov}(X, Y) \leq \text{SD}(X)\text{SD}(Y)$
  - Simplify $E\left[\left(\frac{X-E[X]}{\text{SD}(X)} + \frac{Y-E[Y]}{\text{SD}(Y)}\right)^2\right] \geq 0$ to get lower bound
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- Using the above, $-1 \leq \rho(X, Y) \leq 1$

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   - Y is a linear function of X.
4. $|\rho(X, Y)|$ is close to one
   - X and Y are strongly correlated
   - Increase in X is likely to match up with an increase in Y

Problem

Consider the following joint PMF, where $-1/4 \leq x \leq 1/4$.

	$X = 0$	$X = 1$	f_Y
$Y = 0$	$1/4 - x$	$1/4 + x$	$1/2$
$Y = 1$	$1/4 + x$	$1/4 - x$	$1/2$
f_X	$1/2$	$1/2$	

$E[Y] = 1/2$
$E[Y^2] = 1/2$

$E[XY] = \frac{1}{4} - x$

$E[X] = 1/2, \; E[X^2] = 1/2$

$\mathrm{Cov}(X,Y) = \frac{1}{4} - x - \frac{1}{2} \cdot \frac{1}{2} = -x$

$\mathrm{Var}(X) = \frac{1}{4}, \; \mathrm{Var}(Y) = \frac{1}{4}$

$\rho(X,Y) = \frac{-x}{\frac{1}{2} \cdot \frac{1}{2}} = -4x$

Problem

Consider the following joint PMF, where $-1/4 \le x \le 1/4$.

	$X = c$	$X = c+1$	f_Y
$Y = d$	$1/4 - x$	$1/4 + x$	$1/2$
$Y = d+1$	$1/4 + x$	$1/4 - x$	$1/2$
f_X	$1/2$	$1/2$	

$E[Y] = d + 1/2$

$E[X] = c + 1/2$

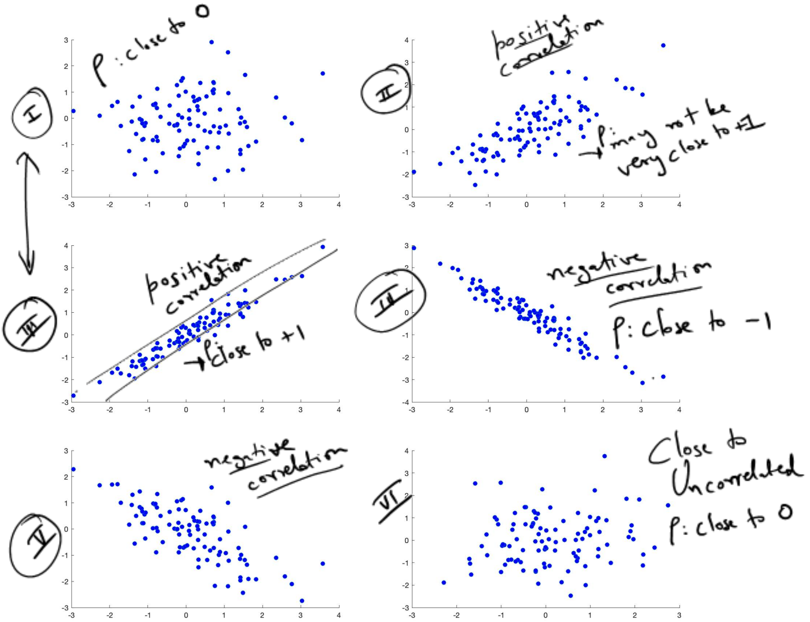
$$E[XY] = cd\left(\tfrac{1}{4} - x\right) + c(d+1)\left(\tfrac{1}{4} + x\right) + (c+1)d\left(\tfrac{1}{4} + x\right) + (c+1)(d+1)\left(\tfrac{1}{4} - x\right)$$

$$= cd + \frac{c}{2} + \frac{d}{2} + \frac{1}{4} - x$$

$\mathrm{Cov}(X,Y) = -x$

$\rho(X,Y) = -4x$

Scatter plots and correlation



Subsection 5

Bounds on probabilities using mean and variance

Notation for mean and variance

- If there is no confusion about the random variable X,
  - μ will denote the mean $E[X]$
  - σ^2 will denote the variance $\mathrm{Var}(X)$
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- If there are multiple random variables under discussion,
  - μ_X will denote the mean $E[X]$
  - σ_X^2 will denote the variance $\mathrm{Var}(X)$
  - σ_X will denote the standard deviation $\mathrm{SD}(X)$

What does mean say about the distribution?

Suppose the average marks in a course is $50/100$.

- What fraction of students will have marks ≥ 50?

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  - It cannot be 1. Why?

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  - It cannot be 1. Why?
  - Can it be 0.9? No.

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- What fraction of students will have marks ≥ 50?
  - It cannot be 0. Why?
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  - It cannot be 1. Why?
  - Can it be 0.9?

Average says something about the distribution of marks!

Standard units in statistics

- Consider a random variable X with mean μ and variance σ^2.
- In an experiment, X may take a value that is close to μ or away from μ.
- $X - \mu$: measures the distance of X from the mean μ.
  - Could be positive or negative

$\frac{X-\mu}{\sigma}$

- Standard units: The number of standard deviations that a realization of a random variable is away from the mean.
  - We expect $X - \mu$ to fall between $-c\sigma$ and $c\sigma$ for a small value of c
  - In other words, we expect X to fall between $\mu - c\sigma$ and $\mu + c\sigma$

Examples

- Throw a pair of die, $X =$ sum of the two numbers
  - $\mu = 7$, $\sigma \approx 2.42$
  - $P(|X - \mu| \leq \sigma) = P(4.58 \leq X \leq 9.42) = P(X \in \{5, 6, 7, 8, 9\} = 2/3$
    - ⋆ So, $P(|X - \mu| > \sigma) = 1/3$
  - $P(|X - \mu| > 2\sigma) = P(X \in \{2, 12\}) = 2/36 \approx 0.056$

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    - ★ So, $P(|X - \mu| > \sigma) = 1/3$
  - $P(|X - \mu| > 2\sigma) = P(X \in \{2, 12\}) = 2/36 \approx 0.056$
- $X \sim \text{Uniform}\{1, 2, \ldots, 100\}$
  - $\mu = 50.5$, $\sigma \approx 28.9$
  - $P(|X - \mu| > \sigma) = 58/100$
  - $P(|X - \mu| > 2\sigma) = 0$

Markov's inequality

Theorem (Markov's inequality)

Let X be a discrete random variable taking non-negative values with a finite mean μ. Then,

$$P(X \geq c) \leq \frac{\mu}{c}.$$

$c = 100\mu$

$P(X \geqslant 100\mu) \leq \frac{1}{100} = 0.01$

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Proof:

- $\mu = \sum_{t \in T_X} t\, P(X = t) = \sum_{t < c} t\, P(X = t) + \sum_{t \geq c} t\, P(X = t)$

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Proof:

- $\mu = \sum_{t \in T_X} t\, P(X = t) = \sum_{t<c} t\, P(X = t) + \sum_{t \geq c} t\, P(X = t)$
- Since the first sum is non-negative,

$$\mu \geq \sum_{t \geq c} t\, P(X = t) \geq \sum_{t \geq c} c\, P(X = t) = c \sum_{t \geq c} P(X = t) = cP(X \geq c)$$

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Theorem (Chebyshev's inequality)

Let X be a discrete random variable with a finite mean μ and a finite variance σ^2. Then,

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}.$$

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Proof:

- Apply Markov's inequality to $(X - \mu)^2$.

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Proof:

- Apply Markov's inequality to $(X-\mu)^2$.

Other forms:

Non-negative Markov

- $P(|X-\mu| \geq c) \leq \frac{\sigma^2}{c^2}$, $P(\underline{(X-\mu)^2} \geq \underline{k^2\sigma^2}) \leq \frac{1}{k^2}$
- $P(\underline{\mu-k\sigma} < \underline{X} < \underline{\mu+k\sigma}) \geq 1-\frac{1}{k^2}$ $\longleftarrow$ complement of $|X-\mu| \geq k\sigma$
- $P(X \geq \mu+k\sigma) + P(X \leq \mu-k\sigma) \leq \frac{1}{k^2}$ $|X-\mu| \geq k\sigma \rightarrow X \geq \mu+k\sigma$ (or) $X \leq \mu-k\sigma$
  - $P(X \geq \mu+k\sigma) \leq \frac{1}{k^2}$, $P(X \leq \mu-k\sigma) \leq \frac{1}{k^2}$

Compare actual and Chebyshev

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$$P(|X-\mu| \geq 2\sigma) \leq \frac{1}{4}$$

1. $X \sim \text{Binomial}(10, 1/2)$, $\mu = 5$, $\sigma = \sqrt{2.5} \approx 1.58$

$$P(|X-5| \geq 2\sigma) = P(X \in \{0,1,9,10\}) \approx 0.021 \leq 1/4$$

Compare actual and Chebyshev

Let X be a random variable with finite mean μ and variance σ^2.

$$P(|X - \mu| \geq 2\sigma) \leq \frac{1}{4}$$

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$$P(|X - 5| \geq 2\sigma) = P(X \in \{0, 1, 9, 10\}) \approx 0.021$$

2. $X \sim \text{Geometric}(1/4)$, $\mu = 4$, $\sigma \approx 3.46$

$$P(|X - 4| \geq 2\sigma) = P(X \in \{11, 12, \ldots\}) \approx 0.056 < 1/4$$

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- These are some of the most useful measures of expected centre and spread in practice.
  - Suppose the ~~average~~ number of accidents decreases by 10000 per day across the country. Is that a "significant" decrease?
  - If standard deviation of the number of accidents is known, we can find how high 10000 is in terms of standard deviations to answer above question.

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- There are a lot of important theoretical implications as well. We will see these later on.