Determinants (Part 1)

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Determinant of a 1×1 matrix :
If $A = [a]$, a 1×1 matrix then $det(A) = a$

Determinant of a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

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Example

$$A = \begin{bmatrix} 2 & 3 \\ 6 & 10 \end{bmatrix}$$

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$$A = \begin{bmatrix} 5 & 2/3 \\ 6 & 3/7 \end{bmatrix}$$

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$$A = \begin{bmatrix} 5 & 2/3 \\ 6 & 3/7 \end{bmatrix} \qquad det(A) = 15/7 - 4 = -13/7$$

Determinant of a 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

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We will obtain the determinant by expanding with respect to the 1st row :

$$\begin{bmatrix} \boxed{a_{11}} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

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$$det(A) = a_{11} \times det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \times det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \times det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

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$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$\begin{bmatrix} a_{11} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} & a_{12} & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{bmatrix} \quad \begin{bmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{33} & \end{bmatrix}$$

$$\begin{aligned} det(A) &= a_{11} \times det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \times det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \times det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \end{aligned}$$

Examples

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 8 & 7 \\ 5 & 6 & 9 \end{bmatrix}$$

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Determinant of the Identity matrix

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad det(I_2) = 1 - 0 = 1$$

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$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$det(I_3) = 1 \times det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 0 \times det\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + 0 \times det\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= 1 \times det(I_2)$$

$$= 1 \times 1 = 1.$$

Determinant of a product of matrices

$$\text{Let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

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$$\begin{aligned} det(AB) &= (ae + bg)(cf + dh) - (af + bh)(ce + dg) \\ &= aecf + bgcf + aedh + bgdh - afce - bhce - afdg - bhdg \end{aligned}$$

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$$\begin{aligned}
det(AB) &= (ae+bg)(cf+dh) - (af+bh)(ce+dg) \\
&= aecf + bgcf + aedh + bgdh - afce - bhce - afdg - bhdg \\
&= bgcf + aedh - bhce - afdg \\
&= bcfg + adeh - bceh - adfg \\
&= (ad-bc)(eh-fg) \\
&= det(A)det(B).
\end{aligned}$$

It can be checked that for 3×3 matrices this equality holds.

Determinant of the inverse of a matrix

$$A A^{-1} = I = A^{-1} A$$

$$\det(A A^{-1}) = \det(I) = 1$$

$$\det(A)\det(A^{-1})$$

$$\Rightarrow \det(A^{-1}) = \frac{1}{\det(A)} = \det(A)^{-1}.$$

Properties : Switching two rows.

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Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Define $\tilde{A} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$.

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$$\text{Let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \qquad \text{Define } \tilde{A} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}.$$

$$det(\tilde{A}) = cb - da = -(ad - bc) = -det(A).$$

Switching two columns

$\tilde{A} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$

$\det(\tilde{A}) = bc - ad$

$= -(ad - bc) = -\det(A)$

This is also true for 3×3 matrices.

Properties : Adding multiples of a row to another row.

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Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Define $\tilde{A} = \begin{bmatrix} a+tc & b+td \\ c & d \end{bmatrix}$.

$[a \quad b] + t[c \quad d] = [a+tc \quad b+td]$

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Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Define $\tilde{A} = \begin{bmatrix} a+tc & b+td \\ c & d \end{bmatrix}$.

$$\begin{aligned} det(\tilde{A}) = \ (a+tc)d - (b+td)c &= ad + \cancel{tcd} - bc - \cancel{tdc} \\ &= ad - bc = det(A). \end{aligned}$$

$$\tilde{A} = \begin{bmatrix} a+tb & b \\ c+td & d \end{bmatrix}$$

$$\begin{aligned} det(\tilde{A}) &= (a+tb)d - b(c+td) \\ &= ad + \cancel{tbd} - bc - \cancel{btd} \\ &= ad - bc = det(A) \end{aligned}$$

Check this for 3x3 matrices.

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Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Define $\tilde{A} = \begin{bmatrix} a & tb \\ c & td \end{bmatrix}$.

$$det(\tilde{A}) = atd - tbc = t(ad - bc) = tdet(A).$$

$\tilde{\tilde{A}} = \begin{bmatrix} ta & tb \\ c & d \end{bmatrix}$ $det(\tilde{\tilde{A}}) = tad - tbc$
$= t(ad - bc) = t \, det(A)$.

Same thing for 3×3 matrices.

Thank you