Lecture - 9th (ODE)

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0 \quad \text{——— (1)}$$

When this initial value problem is going to have a solution.

To answer this, we have Picard's Existence Theorem.

Picard's Existence Theorem:

(i) Let $f(x,y)$ be continuous in the rectangle $R: |x-x_0| \le a,\ |y-y_0| \le b$.

(ii) $f(x,y)$ is bounded in R.
ie $|f(x,y)| \le M$.

Then $\exists$ a soln of (1) in $|x-x_0| \le h$,
where $h = \min(a, \frac{b}{M})$.

Example:

$$\left.\begin{aligned} \frac{dy}{dx} &= x^2+y^2 \\ y(0) &= 0 \end{aligned}\right\} \text{— (1)} \qquad R: |x| \le 1,\ |y| \le 2.$$

$f(x,y) = x^2+y^2$,
$x_0 = 0,\ y_0 = 0$,
$a = 1,\ b = 2$.

$$\left[\begin{aligned} &\frac{dy}{dx} = f(x,y), \\ &y(x_0) = y_0 \\ &R: |x-x_0| \le a \\ &\quad\ |y-y_0| \le b \end{aligned}\right.$$

(i) 'f(x, y) being a poly. f^n is cts in R.

(ii) $|f(x,y)| = |x^2+y^2|$

$\leq |x|^2+|y|^2 = (1)^2+(2)^2$

$= 5$

$\Rightarrow \boxed{M=5}$

Using Picard's Existence Theorem, $\exists$ a soln of (1) in

$|x-x_0| \leq h$, where $h = \min\left(a, \frac{b}{M}\right)$

$\Rightarrow$ $|x| \leq h$, where $h = \min\left(1, \frac{2}{5}\right)$

$= \frac{2}{5}$

$\Rightarrow$ $\boxed{|x| \leq \frac{2}{5}}$

Picard's Existence & Uniqueness Theorem:

$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \qquad R: |x-x_0| \leq a$
$|y-y_0| \leq b$

(1)

(i) Let f(x, y) be continuous in R
(ii) f(x,y) is bounded in R.
] existence of soln.

(iii) f(x,y) satisfies Lipschitz condition w.r.t y in R.

i.e $|f(x,y_1) - f(x,y_2)| \le K|y_1 - y_2|$ $\left[\left|\frac{\partial f}{\partial y}\right| \le K\right]$

Then $\exists$ unique solⁿ of (1) in
$|x - x_0| \le h$, where $h = \min\left(a, \frac{b}{M}\right)$

Example: $\frac{dy}{dx} = x^2 + e^{-y^2}$, $y(0) = 0$,
$R : |x| \le \frac{1}{2}$, $|y| \le 1$.

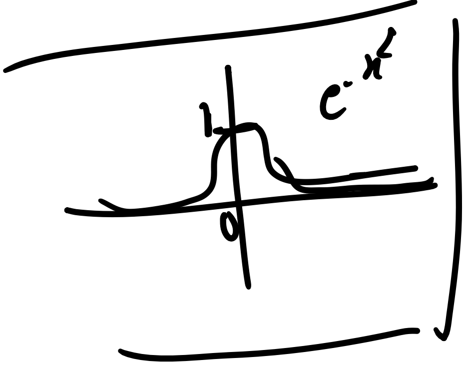
Comparing the given eqⁿ with
$\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$
$R : |x - x_0| \le a$, $|y - y_0| \le b$

$f(x,y) = x^2 + e^{-y^2}$, $x_0 = 0$, $y_0 = 0$,
$a = \frac{1}{2}$, $b = 1$.

(i) $f(x,y)$ is continuous in R as the sum of two cts fⁿs is cts in R.

(ii) $|f(x,y)| = |x^2 + e^{-y^2}|$
$\le |x|^2 + |e^{-y^2}|$
$\le \frac{1}{4} + 1 = \frac{5}{4}$

$\boxed{R: |x| \le \frac{1}{2}, |y| \le 1}$



$\Rightarrow |f(x,y)| \le \frac{5}{4} = M$

(iii) To check Lipschitz continuity:

$$f(x,y) = x^2 + e^{-y^2}$$

$$\frac{\partial f}{\partial y} = e^{-y^2}\cdot(-2y) = \frac{-2y}{e^{y^2}}$$

$$\left|\frac{\partial f}{\partial y}\right| = \left|\frac{-2y}{e^{y^2}}\right| \le 2(1)$$

$\boxed{|y| \le 1}$

$$\Rightarrow \left|\frac{\partial f}{\partial y}\right| \le 2 = K \text{ (Lipschitz constant)}$$

$\Rightarrow$ $f(x,y)$ satisfies Lipschitz condition w.r.t. y in R.

Thus all the conditions of Picard's Existence & Uniqueness theorem are satisfied.

$\Rightarrow$ $\exists$ unique solⁿ of the given IVP

in $|x - x_0| \le h$, $h = \min\left(a, \frac{b}{M}\right)$

$\Rightarrow |x| \le h$, $h = \min\left(\frac{1}{2}, \frac{1}{\frac{5}{4}}\right)$

$\Rightarrow |x| \le h, \quad h = \min\left(\frac{1}{2}, \frac{4}{5}\right) = \frac{1}{2}$

$\Rightarrow \boxed{|x| \le \frac{1}{2}}$.

<u>Example:</u>

$$\frac{dy}{dx} = y^2 + \cancel{\cos}(x+y), \quad y(0) = 0,$$

$R: |x| \le 1, |y| \le 2.$

$$f(x,y) = \underline{y^2 + \sin(x+y)}$$

(i) $f(x,y)$ is cts in R.

(ii) $|f(x,y)| = |y^2 + \sin(x+y)|$

$\le |y|^2 + |\sin(x+y)|$

$\le 4 + 1 = 5 = M$

$\Rightarrow |f(x,y)| \le M \ (=5)$

(iii) $|f(x,y_1) - f(x,y_2)|$

$= |y_1^2 + \sin(x+y_1) - y_2^2 - \sin(x+y_2)|$

$= |y_1^2 - y_2^2 + \sin(x+y_1) - \sin(x+y_2)|$

$\le |y_1^2 - y_2^2| + |\sin(x+y_1) - \sin(x+y_2)|$

$\le |y_1 - y_2|\,|y_1 + y_2| + \left|2\cos\frac{(x+y_1+x+y_2)}{2}\right.$

$\sin\left(\frac{x+y_1-x-y_2}{2}\right)$

$|x| \le 1,$
$|y| \le 2$

$|\sin x| \le |x|$
$|\cos x| \le 1$

$$\le |y_1-y_2|\,|y_1+y_2| + 2\left(\left|\cos\left(\frac{2x+y_1+y_2}{2}\right)\right| \cdot \sin\left(\frac{y_1-y_2}{2}\right)\right)$$

$$\le (|y_1|+|y_2|)\,|y_1-y_2|$$
$$+2(1)\cdot\left|\frac{y_1-y_2}{2}\right|$$

$$\le (2+2)\,|y_1-y_2| + |y_1-y_2|$$

$$\le |y_1-y_2|\,(4+1)$$

$$\le 5\,|y_1-y_2|$$

$$\Rightarrow \ |f(x,y_1)-f(x,y_2)| \le 5\,|y_1-y_2|$$

$\boxed{K=5}$

$$\left|\frac{\partial f}{\partial y}\right| = |2y+\cos(x+y)|$$

$$\le 2|y| + |\cos(x+y)|$$

$$\le 2(2)+(1) = 5 = K.$$

$\Rightarrow$ $f(x,y)$ satisfies Lipschitz condition w.r.t. y in R.

$\exists$ unique solⁿ of given IVP in

$$|x| \le h, \quad h = \min\left(a, \frac{b}{M}\right) = \min\left(1, \frac{2}{5}\right) = \frac{2}{5}$$

$$\Rightarrow |x| \le \frac{2}{5}.$$

Slve

$$\boxed{\frac{dy}{dx} = \underbrace{3y^{2/3}}_{f(x,y)}, \quad y(0) = 0, \quad |x| \le 1, \ |y| \le 1}$$

Check whether the solⁿ of the above IVP exists or not? (Also tell about uniqueness)