VECTOR SPACES :→

Definition :- A vector space over $\mathbb{F}$, denoted $V(\mathbb{F})$, is a non-empty set, satisfying the following properties :

(i) **Vector addition :-** To every pair $u, v \in V$, there corresponds a unique element $u+v$ in V such that
(i.e V is closed under addition).

a) **Commutative :-** $u+v = v+u$.

b) **Associative law :-** $(u+v)+w = u+(v+w)$.

c) **Additive identity :-** There is a unique element 0 in V such that $u+0 = u \quad \forall\ u \in V$.

d) **Additive inverse :-** For every $u \in V$, there is a unique element $-u \in V$ such that $u+(-u) = 0$

(ii) **Scalar addition :-** For each $u \in V$ & $\alpha \in \mathbb{F}$, there corresponds a unique element $\alpha u \in V$ such that
(i.e V is closed under scalar multiplication)

a) $\alpha(\beta u) = (\alpha\beta) u \quad \forall\ \alpha, \beta \in \mathbb{F},\ u \in V$

b) $1 \cdot u = u \quad \forall\ u \in V,\ 1 \in \mathbb{F}$

(iii) **Distributive laws :→**
For any $\alpha, \beta \in \mathbb{F}$ and $u, v \in V$, the following distributive law hold :-

a) $\alpha(u+v) = \alpha u + \alpha v$

b) $(\alpha+\beta) u = \alpha u + \beta u$.

Remark:- 1) The elements of $\mathbb{F}$ are called scalars and the elements of V are called Vectors.

2) If $\mathbb{F} = \mathbb{R}$ then the vector space V is called a Real Vector space.

3) If $\mathbb{F} = \mathbb{C}$ then the vector space V is called a complex vector space.

Result:→ Let V be a vector space over $\mathbb{F}$. Then

(i) $u + v = u \Rightarrow v = 0$

(ii) $\alpha u = 0$ iff either $\alpha = 0$ or u is zero vector

(iii) $(-1)u = -u$ for every $u \in V$.

Proof:- (i) $u + v = u \Leftrightarrow -u + (u+v) = -u + u$

$\Leftrightarrow (-u+u) + v = 0$ (Associative law & $u - u = 0$)

$\Leftrightarrow 0 + v = 0$ ($\because 0 + v = v$.)

$\Leftrightarrow v = 0$

(ii) Suppose $\alpha u = 0$.

If $\alpha = 0$. Then we are done.

If $\alpha \neq 0$ then $\frac{1}{\alpha}$ exist. and

$$0 = \frac{1}{\alpha} \cdot 0 = \frac{1}{\alpha}(\alpha u) = \left(\frac{\alpha . 1}{\alpha}\right) u = 1 \cdot u = u \Rightarrow \underline{u = 0}$$

(iii) $0 = 0u = (1 + (-1))u = u + (-1)u$

$\Rightarrow (-1)u = -u$.

Examples :

1) The set $V = \mathbb{R}$ of real numbers, with the usual addition and scalar multiplication forms a vector space over $\mathbb{R}$.

2) $V = \mathbb{R}^2 := \{(x_1, x_2) : x_1, x_2 \in \mathbb{R}\}$.

Then for $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}$, $\alpha \in \mathbb{R}$, define

$$x + y = (x_1 + y_1, x_2 + y_2)$$

$$\alpha x = (\alpha x_1, \alpha x_2).$$

Then V is a vector space over $\mathbb{R}$.

3) Let $V = \mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}, 1 \le i \le n\}$ be the set of n-tuples of real numbers.

For $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$, $\alpha \in \mathbb{R}$.

$x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$. (Component wise addition or coordinate wise ")

$\alpha x = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$. (component or coordinate wise scalar multiplication)

Then $\mathbb{R}^n$ is a vector space over $\mathbb{R}$.

Remark : $\mathbb{R}^n$ is not a vector space over $\mathbb{C}$, as it is not closed under scalar multiplication.

4) $V = \mathbb{R}^+$ (Set of all positive numbers), $\mathbb{F} = \mathbb{R}$.

Then V is NOT A VECTOR SPACE under usual addition and scalar multiplication.

$\because \alpha = -1 \in \mathbb{F}$ & $v = 1 \in \mathbb{R}^+$ Then $\alpha v = -1 \notin \mathbb{R}^+$.

But if we define $u + v = u.v$ $\forall\, u, v \in V = \mathbb{R}^+$

$\alpha v = v^{\alpha}$ $\forall\, \alpha \in \mathbb{F} = \mathbb{R}$, $v \in V = \mathbb{R}^+$

Then $V = \mathbb{R}^+$ is a vector space over $\mathbb{R}$ with 1 as the additive identity.

5) For fix a positive integer n, and let $M_n(\mathbb{R})$ denote the set of all $n \times n$ matrices with real entries.

Then $M_n(\mathbb{R})$ is a vector space with vector addition and scalar multiplication defined by

$$A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$$
$$\alpha A = \alpha [a_{ij}] = [\alpha a_{ij}].$$

6) Fix a positive integer n. Consider the set $P_n(\mathbb{R})$, of all polynomial of degree $\leq n$ with coefficients from $\mathbb{R}$ in the determinate x. Algebraically,

$$V := P_n(\mathbb{R}) = \{ a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n : a_i \in \mathbb{R}, 1 \leq i \leq n \}.$$

Then $P_n(\mathbb{R})$ is vector space with the addition and scalar multiplication defined as

For $f(x), g(x) \in P_n(\mathbb{R})$ i.e. $f(x) = a_0 + a_1 x + \cdots + a_n x^n$,
$g(x) = b_0 + b_1 x + \cdots + b_n x^n$

$$f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1) x + \cdots + (a_n + b_n) x^n.$$

$$\alpha f(x) = \alpha a_0 + \alpha a_1 x + \cdots + \alpha a_n x^n \quad \text{for } \alpha \in \mathbb{R}.$$

7) Let $C([a,b]) = \{ f : [a,b] \to \mathbb{R} : f \text{ is cts} \}$
= Set of all real valued continuous function defined on $[a,b]$.

Then for $f, g \in C[a,b]$ & $\alpha \in \mathbb{R}$, define

$(f+g)(x) = f(x) + g(x)$ (pointwise addition)

$(\alpha f)(x) = \alpha f(x)$ $\forall x \in [a,b]$. (pointwise scalar multiplication)

Then $C([a,b])$ is a ~~vector~~ real vector space.