Laplace Transforms(EMAT102L)
(Lecture-18 and 19)

Department of Mathematics
Bennett University, India

We will learn

- Inverse Laplace Transforms
- Application of Laplace Transforms in DE

Inverse Laplace Transform

If $\mathcal{L}\{f(t)\} = F(s)$, then $f(t)$ is said to be the **inverse Laplace transform** of $F(s)$. We then write

$$\mathcal{L}^{-1}\{F(s)\} = f(t).$$

Inverse Laplace Transform of Elementary Functions

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Linearity

The inverse Laplace transform is linear, i.e.,

$$\mathcal{L}^{-1}\{a_1F_1(s) \pm a_2F_2(s)\} = a_1\mathcal{L}^{-1}\{F_1(s)\} + a_2\mathcal{L}^{-1}\{F_2(s)\}$$

Properties of Inverse Laplace Transforms

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Theorem

If $\mathcal{L}^{-1}\{F(s)\} = f(t)$, then $\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$.

Example

Find $\mathcal{L}^{-1}\left\{\dfrac{s-2}{s^2+4s+13}\right\}$.

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Solution:
We can write

$$\frac{s-2}{s^2+4s+13}=\frac{s-2}{(s+2)^2+3^2}=\frac{s+2-4}{(s+2)^2+3^2}=\frac{s+2}{(s+2)^2+3^2}-\frac{4}{(s+2)^2+3^2}.$$

$$\mathcal{L}^{-1}\left\{\frac{s-2}{s^2+4s+13}\right\}=e^{-2t}\cos 3t-e^{-2t}\frac{4}{3}\sin 3t=\frac{e^{-2t}}{3}[3\cos 3t-4\sin 3t].$$

Example

Find $\mathcal{L}^{-1}\left\{\dfrac{s^2}{(s+3)^3}\right\}$.

Example

Find $\mathcal{L}^{-1}\left\{\dfrac{s^2}{(s+3)^3}\right\}$.

Solution: Using the partial fractions, we have

$$\frac{s^2}{(s+3)^3} = \frac{1}{s+3} - \frac{6}{(s+3)^2} + \frac{9}{(s+3)^3}.$$

Therefore,

$$\mathcal{L}^{-1}\left\{\frac{s^2}{(s+3)^3}\right\} = e^{-3t} - 6te^{-3t} + \frac{9}{2}t^2e^{-3t}.$$

Convolution of Functions

Let $f(t)$ and $g(t)$ be two smooth functions. The convolution $f * g$ is a function defined by

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

Note: $f * g = g * f$.

Theorem

If $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$, then

$$\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\left\{ \int_0^t f(\tau)g(t-\tau)d\tau \right\} = \mathcal{L}\{f(t)\}.\mathcal{L}\{g(t)\}.$$

$$\Rightarrow \mathcal{L}\{f(t) * g(t)\} = F(s).G(s).$$

or

$$\Rightarrow \mathcal{L}^{-1}\{F(s).G(s)\} = f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

Example based on Convolution Theorem

Example

Find $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+1)(s-2)}\right\}$.

Example

Find $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+1)(s-2)}\right\}$.

Solution: Let $F(s) = \dfrac{1}{s-2}$ and $G(s) = \dfrac{s}{s^2+1}$.

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = e^{2t} \text{ and } \mathcal{L}^{-1}\{G(s)\} = g(t) = \cos t.$$

Using the convolution theorem, we get

$$\begin{aligned}
\mathcal{L}^{-1}\{F(s).G(s)\} &= f(t) * g(t) \\
\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)(s-2)}\right\} &= e^{2t} * \cos t \\
&= \int_0^t e^{2\tau}\cos(t-\tau)d\tau \\
&= \frac{2}{5}e^{2t} + \frac{1}{5}(\sin t - 2\cos t).
\end{aligned}$$

ODEs with constant coefficients

Example (First Order ODE)

Solve the differential equation

$$\frac{dx}{dt} + 3x = 0, x(0) = 1.$$

ODEs with constant coefficients

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Solve the differential equation

$$\frac{dx}{dt} + 3x = 0, x(0) = 1.$$

Solution: By taking Laplace transform on both sides of the equation,

$$\begin{aligned}
&\mathcal{L}\left\{\frac{dx}{dt}\right\} + \mathcal{L}\{3x\} = 0 \\
&\Rightarrow s\mathcal{L}\{x\} - x(0) + 3\mathcal{L}\{x\} = 0 \\
&\Rightarrow (s+3)\mathcal{L}\{x\} = 1 \\
&\Rightarrow \mathcal{L}\{x\} = \frac{1}{s+3}
\end{aligned}$$

Taking inverse Laplace transform on both sides, we get

$$x = e^{-3t}.$$

Example

Consider the problem

$$y' - 3y = 4e^{5t}, y(0) = 6.$$

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Consider the problem

$$y' - 3y = 4e^{5t}, y(0) = 6.$$

Solution: Given DE is

$$y' - 3y = 4e^{5t}.$$

Taking Laplace transform on both sides, we get

$$\mathcal{L}\{y'(t)\} - 3\mathcal{L}\{y(t)\} = 4\mathcal{L}\{e^{5t}\}.$$

$$s\mathcal{L}\{y(t)\} - y(0) - 3\mathcal{L}\{y(t)\} = \frac{4}{s-5}.$$

$$\Rightarrow (s-3)\mathcal{L}\{y(t)\} - 6 = \frac{4}{s-5}.$$

$$\Rightarrow \mathcal{L}\{y(t)\} = \frac{6}{s-3} + \frac{4}{(s-3)(s-5)} = \frac{6}{s-3} - \frac{2}{s-3} + \frac{2}{s-5}.$$

$$\Rightarrow \mathcal{L}\{y(t)\} = \frac{4}{s-3} + \frac{2}{s-5}.$$

Taking inverse Laplace transform on both sides, we get

$$y(t) = 4e^{3t} + 2e^{5t}.$$

ODEs with constant coefficients

Example (Second Order ODE)

Solve the following differential equation

$$\frac{d^2x}{dt^2} + x = t, x(0) = 1, \frac{dx}{dt}(0) = -2.$$

ODEs with constant coefficients

Example (Second Order ODE)

Solve the following differential equation

$$\frac{d^2x}{dt^2} + x = t, x(0) = 1, \frac{dx}{dt}(0) = -2.$$

Solution: By taking Laplace transform on both sides of the equation,

$$\mathcal{L}\left\{\frac{d^2x}{dt^2}\right\} + \mathcal{L}\{x\} = \mathcal{L}\{t\}$$

$$\Rightarrow s^2\mathcal{L}\{x\} - sx(0) - x'(0) + \mathcal{L}\{x\} = \frac{1}{s^2}$$

$$\Rightarrow (s^2 + 1)\mathcal{L}\{x\} = \frac{1}{s^2} + s - 2$$

$$\Rightarrow \mathcal{L}\{x\} = \frac{1}{s^2(s^2+1)} + \frac{s-2}{s^2+1} = \frac{1}{s^2} + \frac{s}{s^2+1} - \frac{3}{s^2+1}.$$

Taking inverse transform on both sides, we get

$$x(t) = t + \cos t - 3\sin t.$$

System of Differential Equations

Example (First Order)

$$\left.\begin{aligned} \frac{dx}{dt} &= 2x - 3y, \\ \frac{dy}{dt} &= y - 2x, \end{aligned}\right\} \quad x(0) = 8, y(0) = 3$$

System of Differential Equations

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Solution: Taking Laplace transform on both sides of the first equation,

$$(s-2)\mathcal{L}\{x\} + 3\mathcal{L}\{y\} = 8.$$

Similarly, taking Laplace transform on both sides of the second equation,

$$2\mathcal{L}\{x\} + (s-1)\mathcal{L}\{y\} = 3.$$

Solving the above equations, we get

$$\mathcal{L}\{x\} = \frac{5}{s+1} + \frac{3}{s-4}, \mathcal{L}\{y\} = \frac{5}{s+1} - \frac{2}{s-4}.$$

By taking the inverse Laplace transform, we have

$$x(t) = 5e^{-t} + 3e^{4t}, \quad y(t) = 5e^{-t} - 2e^{4t}.$$

System of Differential Equations

Example 5(Second Order)

$$\left.\begin{aligned} \frac{d^2x}{dt^2} - x + 5\frac{dy}{dt} &= t, \\ \frac{d^2y}{dt^2} - 4y - 2\frac{dx}{dt} &= -2 \end{aligned}\right\} \quad x(0) = 0, x'(0) = 0, y(0) = 1, y'(0) = 0$$

System of Differential Equations

Example 5(Second Order)

$$\left.\begin{aligned} \frac{d^2x}{dt^2} - x + 5\frac{dy}{dt} &= t, \\ \frac{d^2y}{dt^2} - 4y - 2\frac{dx}{dt} &= -2 \end{aligned}\right\} \quad x(0) = 0, x'(0) = 0, y(0) = 1, y'(0) = 0$$

Solution: Taking Laplace transform on both sides of the first equation,

$$\begin{aligned} (s^2 - 1)\mathcal{L}\{x\} + 5s\mathcal{L}\{y\} - 5 &= \frac{1}{s^2} \\ -2s\mathcal{L}\{x\} + (s^2 - 4)\mathcal{L}\{y\} - s &= \frac{-2}{s} \end{aligned}$$

Eliminating $\mathcal{L}\{x\}$ from the above equations, we get

$$\mathcal{L}\{y\} = \frac{1}{s} - \frac{2}{3}\frac{s}{s^2+4} + \frac{2}{3}\frac{s}{s^2+1}$$

Taking inverse Laplace transform on both sides, we get

$$y(t) = 1 - \frac{2}{3}\cos 2t + \frac{2}{3}\cos t$$

Substituting back into the second original equation, we get

$$x(t) = -t - \frac{5}{3}\sin t + \frac{4}{3}\sin 2t.$$

Formula	Name, Comments
$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$ $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Definition of Transform Inverse Transform
$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	Linearity
$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$ $\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)$	s-Shifting (First Shifting Theorem)
$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$ $\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$ $\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{(n-1)}f(0) - \cdots$ $\cdots - f^{(n-1)}(0)$	Differentiation of Function
$\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{1}{s}\mathcal{L}(f)$	Integration of Function


Thank You