Predicates and Quantifiers

Predicates & Quantifiers

- A generalization of propositions - *predicates*: propositions which contain variables
- Predicates become propositions once every variable is bound- by
  - assigning it a value from the *Universe of Discourse* U

    or

  - quantifying it

Predicates & Quantifiers (cont.)

- Examples:
  - Let U = Z, the integers = {. . . -2, -1, 0 , 1, 2, 3, . . .}
    - P(x): x > 0 is the predicate. It has no truth value until the variable x is bound.
  - Examples of propositions where x is assigned a value:
    - P(-3) is false,
    - P(0) is false,
    - P(3) is true.
  - The collection of integers for which P(x) is true are the positive integers.

Predicates & Quantifiers (cont.)

- P(y) V ¬P(0) is not a proposition. The variable y has not been bound. However, P(3) V ¬P(0) is a proposition which is true.
- Let R be the three-variable predicate R(x, y z):
  x + y = z
- Find the truth value of

R(2, -1, 5), R(3, 4, 7), R(x, 3, z)

Predicates & Quantifiers (cont.)

- Quantifiers
  - Universal

P(x) is true <u>for every x</u> in the universe of discourse.
Notation: *universal quantifier*

$$\forall x\ P(x)$$

'For all x, P(x)', 'For every x, P(x)'

The variable x is bound by the universal quantifier producing a proposition.

Predicates & Quantifiers (cont.)

- Example: U = {1, 2, 3}

$$\forall x\, P(x) \Leftrightarrow P(1) \wedge P(2) \wedge P(3)$$

Predicates & Quantifiers (cont.)

- Quantifiers (cont.)
  - Existential
    - P(x) is true <u>*for some x*</u> in the universe of discourse.

      Notation: *existential quantifier*

      $$\exists x \, P(x)$$

      ‘There is an x such that P(x),’ ‘For some x, P(x)’, ‘For at least one x, P(x)’, ‘I can find an x such that P(x).’

      Example: U={1,2,3}

      $$\exists x \, P(x) \Leftrightarrow P(1) \vee P(2) \vee P(3)$$

Predicates & Quantifiers (cont.)

REMEMBER!

A predicate is <u>not</u> a proposition until *all* variables have been bound either by quantification or assignment of a value!

Predicates & Quantifiers (cont.)

- Equivalences involving the negation operator

$$\neg(\forall x\ P(x)) \Leftrightarrow \exists x\ \neg P(x)$$

$$\neg(\exists x\ P(x)) \Leftrightarrow \forall x\ \neg P(x)$$

- Distributing a negation operator across a quantifier changes a universal to an existential and vice versa.

  - $\neg(\forall x\ P(x)) \Leftrightarrow \neg(P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n))$
    $\Leftrightarrow \neg P(x_1) \vee \neg P(x_2) \vee \ldots \vee \neg P(x_n)$
    $\Leftrightarrow \exists x\ \neg P(x)$

Predicates & Quantifiers (cont.)

- Multiple Quantifiers: read left to right . . .
  - Example: Let U = R, the real numbers,

    $P(x,y): xy = 0$

$$\forall x\ \forall y\ P(x, y)$$
$$\forall x\ \exists y\ P(x, y)$$
$$\exists x\ \forall y\ P(x, y)$$
$$\exists x\ \exists y\ P(x, y)$$

The only one that is false is the first one.

What's about the case when $P(x,y)$ is the predicate $x/y=1$?

Predicates & Quantifiers (cont.)

- Multiple Quantifiers: read left to right . . .
  - Example: Let U = {1,2,3}. Find an expression equivalent to $\forall x\, \exists y\, P(x, y)$ where the variables are bound by substitution instead:

Expand from inside out or outside in.

Outside in:

$$\exists y\, P(1, y) \wedge \exists y\, P(2, y) \wedge \exists y\, P(3, y)$$

$$\Leftrightarrow [P(1,1) \vee P(1,2) \vee P(1,3)] \wedge$$

$$[P(2,1) \vee P(2,2) \vee P(2,3)] \wedge$$

$$[P(3,1) \vee P(3,2) \vee P(3,3)]$$

Predicates & Quantifiers (cont.)

- Converting from English (Can be very difficult!)

“Every student in this class has studied calculus”
transformed into:

“For every student in this class, that student has studied calculus”

C(x): “x has studied calculus”
$\forall x\, C(x)$

This is one way of converting from English!