Ordinary Differential Equations(EMAT102L) (Lecture-2)

Department of Mathematics
Bennett University, India

Outline of the Lecture

We will learn

- Initial Value Problems
- Formation of differential equations
- Separable Equations

A first order ODE can be expressed as $F(x,y,y') = 0$ or $\dfrac{dy}{dx} = f(x,y)$.

Initial value problem (IVP)

A differential equation along with an initial condition is called an initial value problem (IVP), i.e.,

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0.$$

Geometrically, the IVP is to find an integral curve of the DE that passes through the point (x_0, y_0).

Example

Given an amount of a radioactive substance, say 0.5 gm, find the amount present at any later time.

Solution:

Physical Information. Experiments show that at each instant a radioactive substance decomposes and is thus decaying in time proportional to the amount of substance present.
We solve the given problem in three steps.

Step 1: Setting up a mathematical model of the physical process. Let us denote by $y(t)$ the amount of substance present at any time t. We know by the physical law, $\frac{dy}{dt}$ is proportional to $y(t)$. This gives the first-order ODE

$$\frac{dy}{dt} = -ky$$

where the constant k is positive and negative sign taken due to decay.
Initially at time $t = 0$, amount is 0.5gm, i.e., $y(0) = 0.5$gm.

Step 2: Mathematical solution. Now we have the mathematical model of the physical process is the initial value problem

$$\frac{dy}{dt} = -ky(t), y(0) = 0.5$$

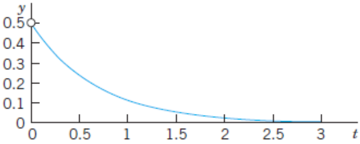
Solution of the above initial value problem is

$$y(t) = ce^{-kt}$$

Determine the value of c using the initial condition $y(0) = 0.5$, we get $c = 0.5$. Thus we have

$$y(t) = 0.5e^{-kt}.$$

Step 3: Interpretation of the solution The function $y(t) = 0.5e^{-kt}$ gives the amount of radioactive substance at time t. It starts from the correct initial amount and decreases with time because k is positive. The limit of y as $t \to \infty$ is zero.



What can we say about the solutions of the following IVPs?

- $\dfrac{dy}{dx} = \dfrac{2y}{x}, \; y(2) = 4, \; (y = x^2$, Unique Solution)
- $\dfrac{dy}{dx} = \dfrac{2y}{x}, \; y(0) = 4$ (No Solution)
- $\dfrac{dy}{dx} = \dfrac{2y}{x}, \; y(0) = 0$ (Infinitely Many Solutions)

Thus we observe that an initial value problem can have unique, infinitely many solutions or no solution.

Problem

Which of the following IVP's have unique solution, Infinitely many solution or unique solution?

- $\dfrac{dy}{dx} = 2x, y(0) = 0$
- $x\dfrac{dy}{dx} = y - 1, y(0) = 1$
- $\left|\dfrac{dy}{dx}\right| + |y| = 0, y(0) = 1$

Suppose we are given a family of curves containing n arbitrary constants. Then we can obtain an nth order differential equation whose solution is the given family as follows.

Working Rule to form the differential equation from the given equation containing 'n' arbitrary constants:

- **Step I.** Write the equation of the given family of curves.
- **Step II.** Differentiate the equation of step I, n times so as to get n additional equations containing the n arbitrary constants and derivatives.
- **Step III.** Eliminate n arbitrary constants from the equations obtained in step I and step II. Thus we obtain the required differential equation involving a derivative of nth order.

Example

Find the differential equation for a family of circles with center at $(1, 0)$ and arbitrary radius 'a'.

Solution

Equation of family of of circles with center at $(1, 0)$ and arbitrary radius 'a' is

$$(x-1)^2 + y^2 = a^2,$$

where a is a constant. Differentiating the above equation w.r.t. 'x', we get

$$\begin{aligned} 2(x-1) + 2y.\frac{dy}{dx} &= 0 \\ y.\frac{dy}{dx} + (x-1) &= 0 \end{aligned}$$

which is the required differential equation.

Example

Find the differential equation corresponding to the family of curves $(x-c)^2 + y^2 = 1$, where c is a constant.

Solution

Differentiating the given equation w.r.t. 'x', we get

$$(x-c) + yy' = 0 \Rightarrow x - c = -yy'$$

Substituting the value of $x - c$ in the given family of curves, we get

$$(yy')^2 + y^2 = 1.$$

Example

Find the differential equation of all circles of radius '2'.

Solution: The equation of all circles of radius '2' is given by

$$(x-h)^2+(y-k)^2=4, \tag{1}$$

where h and k are taken to be arbitrary constants.
Differentiating the above equation w.r.t.'x', we get

$$(x-h)+(y-k)y'=0 \tag{2}$$

Differentiating again w.r.t.'x', we get

$$1+(y')^2+(y-k)y''=0$$

or

$$y-k=-\frac{\{1+(y')^2\}}{y''} \tag{3}$$

Putting this value of $(y-k)$ in (2), we get

$$x-h=-(y-k)y'=\frac{(1+y'^2)}{y''}.y'. \tag{4}$$

Substituting the values of $x-h$ and $y-k$ from (3) and (4) in (1), we get

$$\frac{\{1+(y')^2\}^2(y')^2}{(y'')^2}+\frac{\{1+(y')^2\}^2}{(y'')^2}=4 \; or \; \{(1+(y')^2\}^3=4(y'')^2.$$

We will now discuss different methods of finding the solutions of first order ODEs. These methods are described as below:

- Separation of Variables(Separable Equations)
- Reducible to Separable Equation
- Homogeneous Equation(Reducible to Separable)
- Equations reducible to Homogeneous
- Exact Differential Equation
- Reducible to Exact Differential Equation(Integrating Factors)
- Linear Differential Equation
- Reducible to Linear Differential Equation(Bernoulli's Equation)

Definition

Separable Equation: A first order differential equation of the form

$$\frac{dy}{dx} = g(x)h(y)$$

is called **separable** or to have **separable variables**.

Such ODEs can be solved by direct integration: Write $\frac{dy}{dx} = g(x)h(y)$ as $\frac{dy}{h(y)} = g(x)dx$ and then integrate both sides, we get

$$\int \frac{dy}{h(y)} = \int g(x)dx + c$$

$\Rightarrow H(y) = G(x) + c$, where c is a constant of integration.

Example

Solve $y' = y(y-1)$.

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Solution.

$$
\begin{aligned}
\frac{dy}{y(y-1)} &= dx \\
\Rightarrow \int \frac{dy}{y(y-1)} &= \int dx \\
\Rightarrow \int \left(\frac{-1}{y} + \frac{1}{y-1} \right) dy &= x + c \\
\log\left(\frac{y-1}{y}\right) &= x + c \\
\frac{y-1}{y} &= e^{x+c} \\
y &= \frac{1}{1-e^{x+c}}
\end{aligned}
$$

where c is a constant of integration.

Example

Solve the differential equation $y\dfrac{dy}{dx} = (\cos^2 3x)(y+1), y(0) = 0.$

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Separating the variables and integrating, we get

$$\int \frac{y}{y+1}dy = \int \cos^2 3x dx + c.$$

$$\Rightarrow \int dy - \int \frac{dy}{y+1} = \int \left(\frac{\cos 6x + 1}{2}\right) dx + c.$$

$$\Rightarrow y - \log(y+1) = \frac{1}{2}\left(\frac{\sin 6x}{6} + x\right) + c$$

is the general solution.
The initial condition $y(0) = 0$ gives $c = 0$.
So, the required solution is $y - \log(y+1) = \dfrac{1}{2}\left(\dfrac{\sin 6x}{6} + x\right).$

Example

Solve

$$e^x \frac{dy}{dx} = e^{-y} + e^{-2x-y}$$

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This equation can be rewritten as $\dfrac{dy}{dx} = e^{-x}e^{-y} + e^{-3x-y}$, which is the same as

$$\frac{dy}{dx} = e^{-y}(e^{-x} + e^{-3x}).$$

This equation is now in separable variables form.

$$\frac{dy}{e^{-y}} = (e^{-x} + e^{-3x})dx$$

Integrating, we get the required solution as

$$e^y = -e^{-x} - \frac{e^{-3x}}{3} + c,$$

where c is a constant of integration.

Some Problems for Practice

1. $$\frac{dy}{dx} + \sqrt{\frac{(1+y^2)}{(1+x^2)}} = 0$$

   Answer: $\sinh^{-1} x + \sinh^{-1} y = c$, where c is a constant.

2. $$\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$$

   Answer: $e^y = \dfrac{x^3}{3} + e^x + c$

3. $$y - x\frac{dy}{dx} = a(y^2 + \frac{dy}{dx})$$

   Answer: $y = c(a+x)(1-ay)$

4. Find the equation of the curve passing through $(1,1)$, whose differential equation is

   $$(y - yx)dx + (x + xy)dy = 0$$

   Answer: $\log|xy| + (y - x) = 0$

Example

Consider the IVP

$$\frac{dy}{dx} = 3y^{2/3}; y(0) = 0$$

If $y \neq 0$, then

$$\frac{dy}{y^{2/3}} = 3dx \Rightarrow 3y^{1/3} = 3(x + c) \Rightarrow y = (x + c)^3$$

Use initial condition $y(0) = 0$, we get $c = 0$, i.e, $y = x^3$.
Observe that the constant solution $y = 0$ is lost while solving the IVP

$$\frac{dy}{dx} = 3y^{2/3}; y(0) = 0$$

by separable variables method.
Recall that such solutions are called Singular solutions of the given ODE.

Thank You