Tutorial Sheet 4
Vector Space, Subspace

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1. Find all the vector subspaces of $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$.

2. Is $\mathbb{R}^2$ a vector space over $\mathbb{Z}_2$.

3. Let $V = \mathbb{R}$. Define $x + y = x - y$ and $\alpha.x = -\alpha x$. Which vector space axioms are not satisfied here?

4. Let $V = \mathbb{R}^+ =$ set of all positive real numbers.

   (a) Show that V is not a vector space over $\mathbb{R}$ with respect to usual addition and scalar multiplication.

   (b) for $\alpha \in \mathbb{R}$, $u, v \in \mathbb{R}^+$, define $u + v = uv$ and $\alpha u = u^\alpha$. Then show that V is a vector space over $\mathbb{R}$.

5. Recall that $M_n(\mathbb{C})$ is the complex vector space of all $n \times n$ complex matrices. Which of the following subsets are subspaces of $M_n(\mathbb{C})$.

   (a) $sl_n = \{A \in M_n(\mathbb{C}) : trace(A) = 0\}$.

   (b) $Sym_n = \{A \in M_n(\mathbb{C}) : A = A^\theta\}$.

   (c) $Skew_n = \{A \in M_n(\mathbb{C}) : A + A^\theta = 0\}$.

   (d) Is set of all invertible matrices subspace of $M_n(\mathbb{R})$.

6. Let $C([-1, 1])$ be the set of all real valued continuous functions on the interval $[-1, 1]$. Let
$$W_1 = \left\{ f \in C([-1,1]) : f\left(\frac{1}{2}\right) = 0 \right\},$$
and
$$W_2 = \left\{ f \in C([-1,1]) : f\left(\frac{1}{4}\right) = 5 \right\}.$$
Are W_1, W_2 subspaces of $C([-1, 1])$?

7. Let U and W be two subspaces of a vector space V. Show that $U \cap W$ is a subspace of V. Also show that $U \cup W$ need not to be a subspace of V. When is $U \cup W$ a subspace of V?

8. Let U and W be two subspaces of a vector space V. Define $U + W = \{u + w : u \in U, w \in W\}$. Show that $U + W$ is a subspace of V. Also show that $L(U \cup W) = U + W$.

9. Is $(4, 5, 5)$ a linear combination of $(1, 2, 3)$, $(1, 1, 4)$ and $(3, 3, 2)$?
10. Find the linear span of $S = \{(1, 1, 1), (2, 1, 3)\}$ over $\mathbb{R}$.