Linear Transformation :→

Def^n: Let V, W be vector spaces over $\mathbb{F}$. A map $T: V \to W$ is called a linear transformation (map) if

$$T(x+y) = T(x) + T(y) \quad \forall\ x, y \in V.$$

$$T(\alpha x) = \alpha T(x) \quad \forall\ \alpha \in \mathbb{F},\ x \in V.$$

Or Equivalently, $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ $\forall\ \alpha, \beta \in \mathbb{F}$, $x, y \in V$.

Example ①: Define $T: \mathbb{R} \to \mathbb{R}^2$ as $T(x) = (x, 2x)$.

Then T is a linear Transformation.

Sol^n: Let $x, y \in \mathbb{R}$, $\alpha \in \mathbb{R}$, Then we have

$$T(x+y) = (x+y,\ 2(x+y)) = (x, 2x) + (y, 2y) = T(x) + T(y)$$

$$T(\alpha x) = (\alpha x,\ 2\alpha x) = \alpha(x, 2x) = \alpha T(x).$$

2) Define $T: \mathbb{R}^2 \to \mathbb{R}^3$ as $T(x,y) = T(x+y,\ |x|,\ y)$

Then T is "NOT" a linear Transformation.

$\because$ $T(X+Y) \neq T(X) + T(Y)$, where $X, Y \in \mathbb{R}^2$. ($\because |x+y| \leq |x| + |y|$)

Also $T(\alpha X) \neq T(\alpha X)$ where $X \in \mathbb{R}^2$, $\alpha \in \mathbb{R}$., $X = (x, y)$

For this, take $\alpha = -1$, Then $T(\alpha(x, y)) = T((\alpha x, \alpha y)) = (\alpha x + \alpha y, |\alpha x|, \alpha y)$

i.e $T(-1(x, y)) = (-x-y,\ |x|,\ y)$

$\alpha T(x) = \alpha(x+y,\ |x|,\ y)$ $\Rightarrow -T(x) = -1(x+y,\ |x|,\ y)$
$= (-x-y,\ -|x|,\ y)$

Thus, $T(-(x,y)) \neq -1\,T(x)$

Ex 3:- Define $T: \mathbb{R}^n \longrightarrow \mathbb{R}$ as

$T(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} a x_i$, $a \in \mathbb{R}$. Then T is a linear map

Sol^n: Let $X = (x_1, x_2, \dots, x_n)$, $Y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$, $\alpha, \beta \in \mathbb{R}$. Then

$$\therefore T(\alpha X + \beta Y) = T((\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \dots, \alpha x_n + \beta y_n))$$

$$= \sum_{i=1}^{n} a(\alpha x_i + \beta y_i)$$

$$= \sum_{i=1}^{n} a \alpha x_i + \sum_{i=1}^{n} \beta a y_i$$

$$= \alpha \sum_{i=1}^{n} a x_i + \beta \sum_{i=1}^{n} a y_i$$

$$= \alpha T(x) + \beta T(y).$$

(ii) For any i, $1 \le i \le n$, $T_i: \mathbb{R}^n \longrightarrow \mathbb{R}$ defined as $T_i(X) = x_i$, where $(X = (x_1, x_2, \dots, x_n)$.

Then T is a linear map.

as
$$T_i(\alpha X + \beta Y) = T_i((\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \dots, \alpha x_n + \beta y_n))$$

$$= \alpha x_i + \beta y_i$$

$$= \alpha T_i(X) + \beta T_i(Y).$$

(iii) For any fixed vector $a = (a_1, a_2, \dots, a_n)$, $T: \mathbb{R}^n \longrightarrow \mathbb{R}$ defined as

$$T((x_1, x_2, \dots x_n)) = \sum_{i=1}^{n} a_i x_i$$ is a linear map.

$$T(\alpha X + \beta Y) = T((\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \dots, \alpha x_n + \beta y_n))$$

$$= \sum_{i=1}^{n} a_i(\alpha x_i + \beta y_i) = \sum_{i=1}^{n} a_i \alpha x_i + \sum_{i=1}^{n} \beta a_i y_i$$

$$= \alpha \sum_{i=1}^{n} a_i x_i + \beta \sum_{i=1}^{n} a_i y_i = \alpha T(x) + \beta T(y).$$

Ex: Define $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ by $T((x,y)) = (x+y,\ 2x-y,\ x+3y)$ is a linear map.

Sol^n: Let $X = (x_1, x_2)$, $Y = (y_1, y_2) \in \mathbb{R}^2$, $\alpha, \beta \in \mathbb{R}$.

$$T(\alpha X + \beta Y) = T((\alpha x_1 + \beta y_1,\ \alpha x_2 + \beta y_2))$$

$$= (\alpha x_1 + \beta y_1 + \alpha x_2 + \beta y_2,\ 2(\alpha x_1 + \beta y_1) - (\alpha x_2 + \beta y_2),\ \alpha x_1 + \beta y_1 + 3(\alpha x_2 + \beta$$

$$= (\alpha x_1 + \alpha x_2 + \beta y_1 + \beta y_2,\ 2\alpha x_1 - \alpha x_2 + 2\beta y_1 - \beta y_2,\ \alpha x_1 + 3\alpha x_2 + \beta y_1 + 3\beta y_2))$$

$$= \alpha(x_1 + x_2,\ 2x_1 - x_2,\ x_1 + 3x_2) + \beta(y_1 + y_2,\ 2y_1 - y_2,\ y_1 + 3y_2)$$

$$= \alpha T((x_1, x_2)) + \beta T((y_1, y_2))$$

$$= \alpha T(X) + \beta T(Y)$$

i.e. $T(\alpha X + \beta Y) = \alpha T(X) + \beta T(Y) \quad \forall\ \alpha, \beta \in \mathbb{R},\ X, Y \in \mathbb{R}^2$

4) Let A be a $m \times n$ real matrix. Define a map $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ by $T_A(X) = AX \quad \forall\ X^t = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$.

Then T_A is a linear Transformation.

As
$$T_A(\alpha X + \beta Y) = A(\alpha X + \beta Y)$$
$$= \alpha AX + \beta AY$$
$$= \alpha T_A(X) + \beta T_A(Y).$$

Excercise: $D: P_n(\mathbb{R}) \longrightarrow P_n(\mathbb{R})$ by

$$D(p(x)) = \frac{d}{dx} p(x)$$ (called Derivative map).

i.e $D(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1}$.

Then one can easily check that D is a linear Transformation (
for $p(x), q(x) \in P_n(\mathbb{R})$, $\alpha, \beta \in \mathbb{R}$, we have

As $D(\alpha p(x) + \beta q(x)) = \frac{d}{dx}(\alpha p(x) + \beta q(x))$

$$= \alpha \frac{d}{dx}(p(x)) + \beta \frac{d}{dx}(q(x))$$

$$= \alpha D(p(x)) + \beta D(q(x)).$$

Ex: $T: P_n(\mathbb{R}) \longrightarrow \mathbb{R}$ defined as.

$$T(p_n(x)) = \int_0^x p(t)\, dt.$$

$$T(a_0 + a_1 x + \cdots + a_n x^n) = a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \cdots + \frac{a_n}{n+1} x^{n+1}.$$

Then T is a linear map, known as integration map.

$\because T(\alpha p(x) + \beta q(x)) = \int_0^x (\alpha p(t) + \beta q(t))\, dt = \alpha \int_0^x p(t)\, dt + \beta \int_0^x q(t)\, dt$

$$= \alpha T(p(x)) + \beta T(q(x))$$

Ex. $T: M_{n\times n}(\mathbb{R}) \longrightarrow \mathbb{R}$ as

$$T(A) = \text{trace}(A)$$

Then T is a linear map.

Sol^n: $T(\alpha A + \beta B) = \text{trace}(\alpha A + \beta B)$

$$= \alpha\, \text{trace}(A) + \beta\, \text{trace}(B)$$

$$= \alpha T(A) + \beta T(B).$$

Result: Let $T: V \longrightarrow W$ be a Linear Transformation. Suppose that 0_V is the zero vector in V and 0_W is the zero vector in W,

Then $\boxed{T(0_V) = 0_W.}$

Pf:- Since $0_V = 0_V + 0_V$, we have

$$T(0_V) = T(0_V + 0_V) = T(0_V) + T(0_V)$$

$$\Rightarrow T(0_V) = 0_W \quad \text{as} \quad T(0_V) \in W.$$

Def^n: Zero Transformation: $\rightarrow$ Let V be a vector space. and let $T: V \longrightarrow W$ be defined as $T(v) = 0 \quad \forall\, v \in V$.

Then T is called zero Transformation.

Also, T is a linear map. as
$$\begin{aligned} T(\alpha v + \beta w) &= 0 \\ &= \alpha 0 + \beta 0 \\ &= \alpha T(v) + \beta T(w) \end{aligned}$$

Def^n: Identity Transformation:-

Let $I : V \longrightarrow V$ defined as $I(v) = v \quad \forall\, v \in V$.

Then I is a linear Transformation. such a linear transformation is called the identity transformation.

Rank- Nullity Theorem:-

Let V, W be finite dimensional vector space over the same field $\mathbb{F}$. $T: V \to W$ be a linear Transformation.

We define

1) $R(T) = \{ T(x) : x \in V \}$ (Range space)

2) $N(T)$ or $Ker(T)$ $= \{ x \in V : T(x) = 0 \}$ (Null space or Kernal)

Then $R(T)$ is a subspace of W.

$N(T)$ is a subspace of V.

Moreover, $\dim(R(T)) \leq \dim W$

$\dim(N(T)) \leq \dim V.$

3) T is one-one $\iff$ $N(T) = \{0\}$ is the zero subspace of V

4) $\dim(R(T)) = \dim V \iff$ T is one-one

$\iff N(T) = \{0\}.$

Rank - Nullity Theorem:- Let $T: V \to W$ be a linear transformation and V be a finite dimensional vector space Then.

$$\boxed{\dim(V) = \dim(R(T)) + \dim(N(T))}$$

Corollary:- $T: V \to V$ be a linear transformation on a finite dim. vector space V. Then

T is one-one $\iff$ T is on-to $\iff$ T is invertible.

Example: $T: \mathbb{R}^3 \to \mathbb{R}^4$ as

$$T(x, y, z) = (x - y + z,\ y - z,\ x,\ 2x - 5y + 5z)$$

Find the range, null space and $\dim(R(T))$ & $\dim(N(T))$.

Sol^n: $N(T) = \{(x,y,z) : T(x,y,z) = (0,0,0,0)\}$

$$= \{(x,y,z) : (x-y+z,\ y-z,\ x,\ 2x-5y+5z) = (0,0,0,0)\}$$

$$= \left\{(x,y,z) : \begin{array}{c} x - y + z = 0 \\ y - z = 0 \\ x = 0 \\ 2x - 5y + 5z = 0 \end{array}\right\}$$

$$= \{(x,y,z) : x = 0,\ y - z = 0\}$$

$$= \{(0,y,y) : y \in \mathbb{R}\}$$

$$= \{y(0,1,1)\}$$

$$= \text{span}\{(0,1,1)\}$$

Thus, $N(T)$ is subspace of $\mathbb{R}^3$ spanned by $(0,1,1)$.

$\dim N(T) = 1$. , Basis of $N(T) = \{(0,1,1)\}$

From Rank-Nulity Thm, $\dim(R(T)) + \dim(N(T)) = \dim \mathbb{R}^3$

$$\Rightarrow \dim(R(T)) = 3 - 1 = 2.$$

$$R(T) = \{ T(x,y,z) \in \mathbb{R}^4 : (x,y,z) \in \mathbb{R}^3 \}$$

$$= \{ (x-y+z,\ y-z,\ x,\ 2x-5y+5z) : x,y,z \in \mathbb{R} \}$$

$$= \{ x(1,0,1,2) + y(-1,1,0,-5) + z(1,-1,0,5) \}$$

$$= \text{span}\{ \underset{v_1}{(1,0,1,2)},\ \underset{v_2}{(-1,1,0,-5)},\ \underset{v_3}{(1,-1,0,5)} \}$$

But $\{v_1, v_2, v_3\}$ are l. dependent as $v_2 = -v_3$

We need to extract linearly independent subset of $\{v_1, v_2, v_3\}$

i.e $\{v_1, v_2\}$ is l. i. subset of $\{v_1, v_2, v_3\}$

also $\{v_1, v_2\}$ span $R(T)$.

$\therefore$ $\{v_1, v_2\}$ is basis for $R(T)$.

More over, $R(T) = \{ \alpha v_1 + \beta v_2 : \alpha, \beta \in \mathbb{R} \}$

$$= \{ \alpha(1,0,1,2) + \beta(-1,1,0,-5) : \alpha, \beta \in \mathbb{R} \}$$

$$= \{ (\alpha-\beta,\ \beta,\ \alpha,\ 2\alpha-5\beta) : \alpha, \beta \in \mathbb{R} \}$$

$$= \left\{ (x,y,z,w) \in \mathbb{R}^4 : \begin{array}{l} x+y-z=0 \\ 5y-2z+w=0 \end{array} \right\}.$$

Ex:- $T: M_{2\times 2}(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$ defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b)x^2 + (b+c)x$$

To show:- T is linear and determine null(T), Range(T).

Sol^{n}: **To show:-** T is linear.

$$T(\alpha A + \beta B) = T\left(\alpha\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \beta\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$$

$$= T\left(\begin{bmatrix} \alpha a_1 + \beta a_2 & \alpha b_1 + \beta b_2 \\ \alpha c_1 + \beta c_2 & \alpha d_1 + \beta d_2 \end{bmatrix}\right)$$

$$= (\alpha a_1 + \beta a_2 + \alpha b_1 + \beta b_2)x^2 + (\alpha b_1 + \beta b_2 + \alpha c_1 + \beta c_2)x$$

$$= [\alpha(a_1+b_1) + \beta(a_2+b_2)]x^2 + [\alpha(b_1+c_1) + \beta(b_2+c_2)]x$$

$$= \alpha[(a_1+b_1)x^2 + (b_1+c_1)x] + \beta[(a_2+b_2) + (b_2+c_2)]x$$

$$= \alpha\, T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + \beta\, T\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$$

Thus $= \alpha T(A) + \beta T(B)$

T is a linear map.

$$\mathrm{Ker}(T) = \mathrm{Null}(T) = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : T(A) = 0_{\text{zero poly.}} \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : (a+b)x^2 + (b+c)x = 0 \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{matrix} a+b=0 \\ b+c=0 \end{matrix} \Big\rangle \Rightarrow \begin{matrix} a=c \\ b=-c \end{matrix} \right\}$$

$$= \left\{ \begin{bmatrix} c & -c \\ c & d \end{bmatrix} \right\}$$

$$= \left\{ c \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} : c, d \in \mathbb{R} \right\}$$

$$= \text{span} \left\{ \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{v_2} \right\}$$

$\{v_1, v_2\}$ is linearly independent.

$\therefore$ $\{v_1, v_2\}$ forms a bases for $\text{Ker}(T)$.

$\dim(\text{Ker}(T)) = 2$.

By Rank-nullity Thm, $\dim(R(T)) + \dim(N(T)) = \dim(M_{2\times 2}(\mathbb{R}))$

$$\Rightarrow \dim(R(T)) = 4 - 2 = 2.$$

$$R(T) = \left\{ T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) \right\}$$

$$= \left\{ (a+b)x^2 + (b+c)x : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) \right\}$$

$$= \left\{ ax^2 + b(x+x^2) + cx : a, b, c \in \mathbb{R} \right\}$$

$$= \text{span} \{ \underbrace{x, x^2}_{\to \text{ l.i}} \} \qquad (\because b(x+x^2) = b.x + b.x^2)$$

~~dim~~ Thus $\{x, x^2\}$ form a basis for $R(T)$.

Ex :- Define $T: M_{2\times 2}(\mathbb{R}) \longrightarrow M_{2\times 2}(\mathbb{R})$ defined by

$$T(A) = A + A^t.$$

Then show that 1) T is a linear map.

2) Determine $R(T)$, $N(T)$, $\dim(R(T))$, $\dim(N(T))$.

Sol^n : To show : T is a linear map.

Let $\alpha, \beta \in \mathbb{R}$, $A, B \in M_{2\times 2}(\mathbb{R})$. Then

$$\begin{aligned} T(\alpha A + \beta B) &= \alpha A + \beta B + (\alpha A + \beta B)^t \\ &= \alpha A + \beta B + \alpha A^t + \beta B^t \\ &= \alpha (A + A^t) + \beta (B + B^t) \\ &= \alpha T(A) + \beta T(B) \end{aligned}$$

Thus, T is a linear map.

$$\begin{aligned} \text{Ker } T &= \{ A \in M_{2\times 2} : T(A) = 0 \} \\ &= \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \\ &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \\ &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{matrix} a = 0 \\ d = 0 \end{matrix}, \ b = -c \right\} \\ &= \left\{ \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} : b \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\} \\ &= \left\{ b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} : b \in \mathbb{R} \right\} \end{aligned}$$

Ker T is a subspace of $M_{2\times 2}$ s.t

$\dim(\mathrm{Ker}(T)) = \mathrm{Null}(T) = 1$

and $\mathrm{Basis}(\mathrm{Null}(T)) = \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$

Also, Ker T = set of all skew symmetric matrices.

$\dim(\mathrm{Range}(T)) + \dim(N(T)) = \dim(M_{2\times 2}(\mathbb{R}))$

$\Rightarrow \dim(R(T)) = 4 - 1 = 3.$

$$R(T) = \{ T(A) : A \in M_{2\times 2}(\mathbb{R}) \}$$

$$= \{ A + A^t : A \in M_{2\times 2}(\mathbb{R}) \}$$

$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} 2a & b+c \\ c+b & 2d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\} \quad \text{— (1)}$$

$$= \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} : \begin{array}{l} x = 2a, \quad w = 2d \\ y = z = b + c. \end{array} \right\}$$

$$= \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} : \begin{array}{l} x, w \in \mathbb{R} \\ y = z. \end{array} \right\} \quad \text{— (2)}$$

= Set of all symmetric matrices.

Using (1)

$$= \left\{ 2a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (b+c) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 2d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right.$$

OR

from (2)

$$= \left\{ x \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + z \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$= \text{span} \left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{v_3} \right\}$$

Also, $\{v_1, v_2, v_3\}$ are l.i. so forms a basis for $R(T)$

$\dim(R(T)) = 3.$

Now, we prove a result which relates a linear transformation T with its value on a basis of the domain space.

Thm: Let $T: V \longrightarrow W$ be a linear Transformation.

Let $\{u_1, u_2, \ldots, u_n\}$ be an ordered basis of V.

Then the linear transformation T is a linear combination of the vectors $T(u_1), T(u_2), \ldots, T(u_n)$.

In other words, T is determined by $T(u_1), T(u_2), \ldots, T(u_n)$.

Proof: Let $v \in V$. and $\{u_1, u_2, \ldots, u_n\}$ be an ordered basis of V

$\therefore v = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$.

Since T is linear. so, we have

$$T(v) = T(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n)$$

$$= \alpha_1 T(u_1) + \alpha_2 T(u_2) + \cdots + \alpha_n T(u_n) \quad (\because T \text{ is linear})$$

Thus to find a linear map, it is sufficient to know the values of T at basis vectors.

Ex: (1) $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is a linear map such that

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\\1\end{bmatrix}$$

$$T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\\2\end{bmatrix}$$

$$T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\\1\end{bmatrix}$$

Find $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = ?$

Solⁿ: Note that $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$ form a basis for $\mathbb{R}^3$.

Let $\begin{bmatrix}x\\y\\z\end{bmatrix} \in \mathbb{R}^3$.

Then $\begin{bmatrix}x\\y\\z\end{bmatrix} = \alpha\begin{bmatrix}1\\0\\0\end{bmatrix} + \beta\begin{bmatrix}1\\1\\0\end{bmatrix} + \gamma\begin{bmatrix}1\\1\\1\end{bmatrix}$

$\Rightarrow \gamma = z, \quad \beta = y - z, \quad \alpha = x - y$;

i.e. $\begin{bmatrix}x\\y\\z\end{bmatrix} = (x-y)\begin{bmatrix}1\\0\\0\end{bmatrix} + (y-z)\begin{bmatrix}1\\1\\0\end{bmatrix} + z\begin{bmatrix}1\\1\\1\end{bmatrix}$

Applying T on both sides,

$$T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = (x-y)\,T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) + (y-z)\,T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) + z\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)$$

$$= (x-y)\begin{bmatrix}1\\2\\1\end{bmatrix} + (y-z)\begin{bmatrix}0\\1\\2\end{bmatrix} + z\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}x-y+z\\2x-y-z\\x+y-z\end{bmatrix}$$ Ans.

(2)

$T : \mathbb{P}_2(\mathbb{R}) \longrightarrow \mathbb{R}$ as

$T(1) = 2$

$T(x) = 4$

$T(x^2) = 0$

Find $T(p(x))$?

Sol^n:

$$p(x) = a_0 + a_1 x + a_2 x^2$$

$$T(a_0 + a_1 x + a_2 x^2) = a_0 T(1) + a_1 T(x) + a_2 T(x^2)$$

$$= 2a_0 + 4a_1$$

i.e. $\boxed{T(a_0 + a_1 x + a_2 x^2) = 2a_0 + 4a_1}$

3) $T : M_{2\times2}(\mathbb{R}) \longrightarrow M_{2\times2}(\mathbb{R})$

$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Find $T\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right) = ?$

Sol^n:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = aT\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) + bT\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) + cT\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) + dT\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= a\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} + b\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Inverse Linear Transformation: $T: V \to W$ be a linear transformation. If the map T is one-one and on-to. Then the map $T^{-1}: W \to V$ defined by

$$T^{-1}(w) = v \quad \text{whenever } T(v) = w$$

is called the inverse of the linear transformation T.

Ex: Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ as

$$T((x,y)) = (x+y,\ x-y).$$

Find $T^{-1}: \mathbb{R}^2 \to \mathbb{R}^2$ as $T^{-1}(x) = T^{-1}(x,y) = \left(\frac{x+y}{2}, \frac{x-y}{2}\right)$.

Sol^n: One can easily see that $\ker T = \{(0,0)\}$. As

$= \{(x,y) : T((x,y)) = (0,0)\}$

$= \{(x,y) : (x+y, x-y) = (0,0)\}$

$= \{(x,y) : \begin{array}{l} x+y=0 \\ x-y=0 \end{array}\}$

$= \{(x,y) : x=0, y=0\}$

$= (0,0)$.

$\Rightarrow T$ is one-one.

$\Rightarrow T^{-1}$ exist.

$$T^{-1}(v) = w \iff v = T(w)$$

$$T(v_1, v_2) = T(w_1, w_2)$$

$$\Rightarrow (v_1, v_2) = (w_1 + w_2, w_1 - w_2)$$

$$\Rightarrow \begin{array}{l} w_1 + w_2 = v_1 \\ w_1 - w_2 = v_2 \end{array}$$

$$\Rightarrow w_1 = \frac{v_1+v_2}{2}, \quad w_2 = \frac{v_1 - v_2}{2}$$

i.e. $$T((v_1, v_2)) = \left(\frac{v_1+v_2}{2}, \frac{v_1-v_2}{2}\right).$$

2) $T: \mathbb{R}^{n+1} \longrightarrow \mathbb{P}_n(\mathbb{R})$ defined by

$$T((a_1, a_2, \ldots, a_n, a_{n+1})) = a_1 + a_2 x + \cdots + a_n x^n + a_{n+} x^{n+1}.$$

Then T is a linear map and T is one-one.

Therefore T^{-1} exist.

& one can find that $T^{-1}: P_n(\mathbb{R}) \to \mathbb{R}^{n+1}$ as

$$T^{-1}(a_1 + a_2 x + a_2 x^2 + \cdots + a_{n+1} x^{n+1}) = (a_1, a_2, \ldots, a_n, a_{n+1})$$

One can also, verify that $(T \circ T^{-1}) = (T^{-1} \circ T) = I$.