Multivariable Calculus

(Lecture-2)

Department of Mathematics
Bennett University
India

17^{th} October, 2018

Learning Outcome of the Lecture

We learn

- First Topic: Visualizing functions
  - $F : I \subseteq \mathbb{R} \to \mathbb{R}^n$ where $n = 2, 3$ - Curves
  - $f : S \subseteq \mathbb{R}^2 \to \mathbb{R}$: Contour Lines, Level Curves, Contour Plots
  - $f : S \subseteq \mathbb{R}^3 \to \mathbb{R}$: Level Surfaces

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- Second Topic: Coordinates Systems
  - $\mathbb{R}^2$: Cartesian Coordinates, Polar Coordinates
  - $\mathbb{R}^3$: Cartesian Coordinates, Cylindrical Coordinates, Spherical Coordinates

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  - $\mathbb{R}^2$: Cartesian Coordinates, Polar Coordinates
  - $\mathbb{R}^3$: Cartesian Coordinates, Cylindrical Coordinates, Spherical Coordinates
- Third Topic: Topology of Sets in $\mathbb{R}^n$:
  - Open Set, Closed Set, Bounded Set, Compact Set, Connected Set, Convex Set

First Topic

Visualizing Functions

Curves as Vector Valued Functions of Real Variable

Definition

Plane Curve:

A curve in the $2D$-plane $(\mathbb{R}^2)$ is defined as a continuous function $f : I \subseteq \mathbb{R} \to \mathbb{R}^2$ where I is an interval in $\mathbb{R}$.

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- Parabola in $\mathbb{R}^2$: $f(t) = (t, t^2)$ for $t \in \mathbb{R}$.
- Straight Line Segment joining the point A and B in $\mathbb{R}^n$: $f(t) = tB + (1-t)A$ for $t \in [0, 1]$.

Curves in $2D$ or $3D$

Curves in $\mathbb{R}^2$ can be visualized by drawing the curve on the two-dimensional coordinates system ($2D$ Plane).

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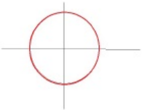

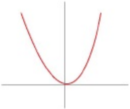

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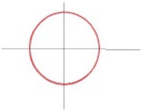

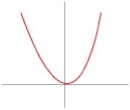

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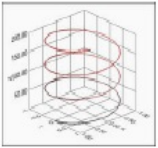

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- A collection of level curves is called a contour plot.

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Let $z = f(x, y)$ be a real valued function.

Definition

Contour Lines:

The contour line is the curve in which the surface $z = f(x, y)$ is cut by the plane $z = c$. We also call the contour line as the trace of the graph of f in the plane $z = c$.

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Contour Plot or Contour Map:
A set/collection of level curves for $z = f(x, y)$ is called a contour plot or contour map of f.

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Contour Line $(z = c)$ = Collection of points (x, y, c) such that $f(x, y) = c$.

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The idea behind Contour Plot is to provide three dimensional information in a two-dimensional setting.

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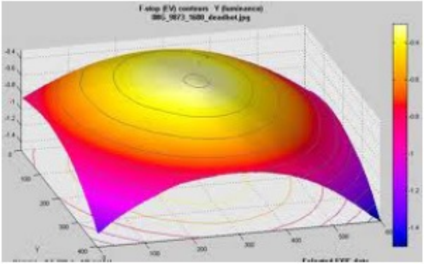
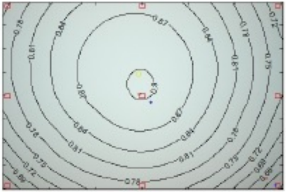
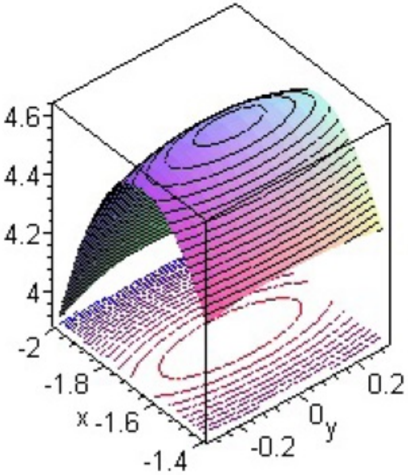
Picture Example: Contour Lines, Level Curves, Contour Plot

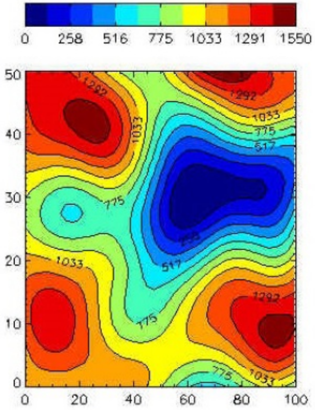




Picture Example: Contour Lines and Level Curves



Picture Example: Some Contour Plot



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Example: Contour Lines, Level Curves, Contour Plot

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Find the contour line and level curve of $f(x,y)$ for the value $c = 0$?

Confusion in terminologies

Because of the close association of contour lines with level curves there is no firm agreement about which word to use for which kind of curve. <u>The convention in our course is that</u>

Level curves lie in the domain of f
(in the plane $z = 0$)

while

Contour lines lie on the surface defined by f
(in the appropriate plane $z = c$).

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That is, for any c, the set of points (x, y, z) for which $f(x, y, z) = c$ (form a surface and) is called a level surface of the function.

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- $c < 0$: There is no level surface in this case.

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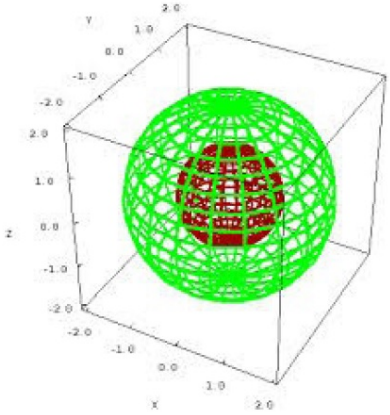
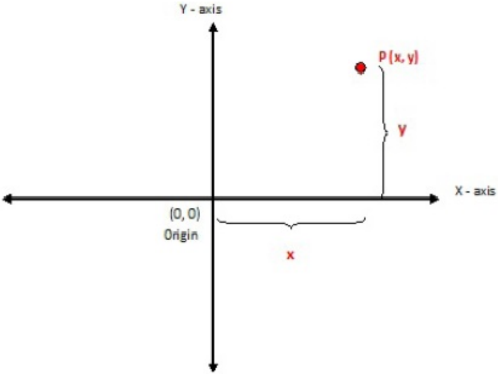
Example: Level Surfaces



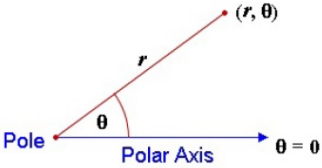
Coordinate Systems

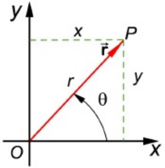
$\mathbb{R}^2$: Cartesian Coordinates/ Rectangular Coordinates

Any point P in the plane (in $2D$) can be assigned coordinates in the rectangular (or cartesian) coordinates system as (x, y).



$\mathbb{R}^2$: Polar Coordinates





- For each nonzero point $P = (x, y) \neq= (0, 0)$ the polar coordinates (r, θ) of P are given by the equations

$$x = r\cos\theta,\ y = r\sin\theta, \quad \text{or} \quad x^2 + y^2 = r^2, \quad \frac{y}{x} = \tan\theta.$$

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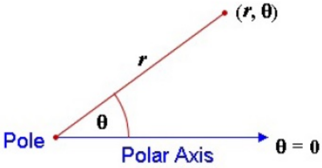

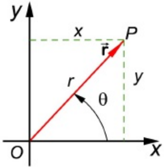

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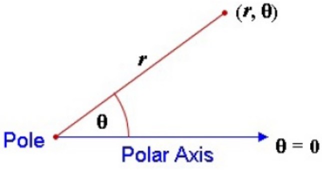
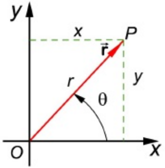
$\mathbb{R}^2$: Polar Coordinates





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- The points (r, θ) and $(r, \theta + 2n\pi)$ where n is any integer denote the same (geometrical) point.
- When P is origin, then $r = 0$ but θ is undetermined, so nothing can be done to remove the ambiguity (in defining ?) there.

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$\mathbb{R}^3$: Cartesian Coordinates/ Rectangular Coordinates

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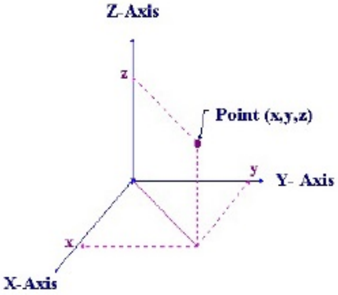
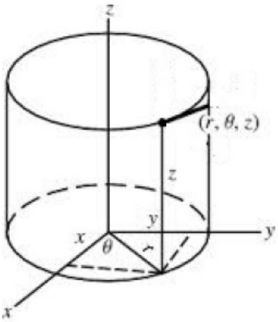
Select a point O to be the origin of the coordinate system, and choose three mutually perpendicular straight lines at O to be x, y, and z coordinate axes.

For example: The y and z axes are in the plane, and the x-axis is perpendicular to this plane and points toward the reader.



$\mathbb{R}^3$: Cylindrical Coordinates

A cylindrical coordinate system consists of polar coordinates (r, θ) in a plane together with a third coordinate z measured along an axis perpendicular to the $r\theta$-plane which is the xy-plane. This means that the z-coordinate in the cylindrical coordinate system is the same as the z-coordinate in the cartesian system.



Relation between Cylindrical and Rectangular Coordinates

Cylindrical and Rectangular coordinates are related by the following equations:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

where $r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$.

Example

Problem: Describe the set of point $P(r, \theta, z)$ whose cylindrical coordinates satisfy:
(a) $r =$ constant, (b) $\theta =$ constant, (c) $z =$ constant.

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- The equation $\theta = \theta_0$ describes the plane that contains the z-axis and makes an angle θ_0 with the positive x-axis.

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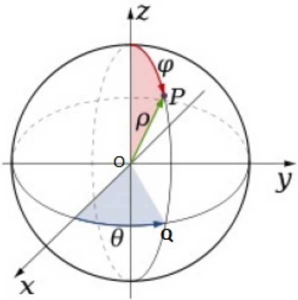
- The equation $\theta = \theta_0$ describes the plane that contains the z-axis and makes an angle θ_0 with the positive x-axis.
- The equation $z = k$ describes a plane perpendicular to the z-axis.

$\mathbb{R}^3$: Spherical Coordinates

Spherical coordinates are useful when there is a center of symmetry that we can take as the origin. The spherical coordinates (ρ, ϕ, θ) of a given point P are shown in the following Figure



Here $\rho = \sqrt{x^2 + y^2 + z^2}$.

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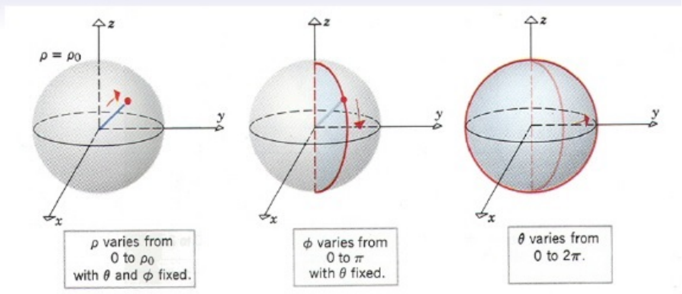
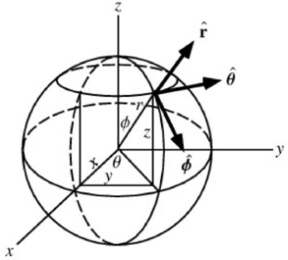
Note: Incidentally, some books give spherical coordinates in the order (ρ, θ, ϕ) with the ϕ and θ reversed, and you should watch out for this when you read elsewhere.



z
y
x
$\rho = \rho_0$
ρ varies from
0 to ρ_0
with θ and ϕ fixed.
ϕ varies from
0 to π
with θ fixed.
θ varies from
0 to 2π.



z
$\hat{\mathbf{r}}$
$\hat{\theta}$
$\hat{\phi}$
r
ϕ
z
y
x
θ
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y

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- The equation $\theta = \theta_0$ describes a half-plane, containing the z-axis whose trace in the xy-plane is the ray that makes an angle θ_0 with the positive x-axis.

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Topology of Sets in the Euclidean Space $\mathbb{R}^n$

The Euclidean Space $\mathbb{R}^n$

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \,:\, x_i \in \mathbb{R} \text{ for } 1 \leq i \leq n\}.$$

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It is also denoted by $B_r(A)$.
It is also called an (open) neighborhood of A.
it is also denoted by $N(A, r)$ or $N(A)$ or $N_r(A)$.

Interior Point and Limit Point

Let S be a subset of $\mathbb{R}^n$.

Definition

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A point X_0 is said to be a limit point(or accumulation point or cluster point) of S if every neighborhood $N(X_0)$ of X_0 contains at least one point from S which is not equal to X_0.

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The set of all limit points of S is called the derived set of S and is denoted by S' or $Der(S)$.

Boundary Point and Exterior Point

Let S be a subset of $\mathbb{R}^n$. The complement of the set S in $\mathbb{R}^n$ is defined as $S^c = \{X \in \mathbb{R}^n, : \; X \notin S\} = \mathbb{R}^n \setminus S$.

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Basically a closed set is a set together with its boundary points. It can be characterized as: A set S is closed if and only if it contains all its limit points.

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The above definition of compact set is true only in the Euclidean spaces.

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Note: Every convex set is connected, but converse is not true.