Ordinary Differential Equations(EMAT102L)
(Lecture-6)

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Outline of the Lecture

We will learn

- Linear Equation
- Bernoulli's Equation(Reducible to Linear Equation)

Recall that a **first order linear ODE** is one in which the dependent variable and its first order derivative occur in the first degree only. That is, a first order linear ODE has the form

$$a_0(x)\frac{dy}{dx} + a_1(x)y = g(x) \tag{1}$$

where $a_0(x) \neq 0$ and $a_0(x), a_1(x), g(x)$ are continuous in an interval I.

Definition

A first order linear ODE (of the above form (1)) is called **homogeneous** if $g(x) = 0$ and **non-homogeneous** otherwise.

Definition

By dividing both sides of equation (1) by the leading coefficient $a_0(x)$, we obtain a more useful form of the above first order linear ODE, called the **standard form**, given by

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{2}$$

where $P(x) = \dfrac{a_1(x)}{a_0(x)}$, $Q(x) = \dfrac{a_2(x)}{a_0(x)}$.
Equation (2) is called the **standard form** of a first order linear ODE.

Note that a linear ODE can be converted into an exact ODE by using integrating factor

$$\mu = e^{\int P(x)dx}$$

How to solve a Linear ODE

Theorem

The linear differential equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

has an integrating factor of the form

$$\mu(x) = e^{\int P(x)dx}$$

A one-parameter family of solutions of this equation is

$$y \times I.F. = \int Q(x) \times I.F.dx + c$$

or

$$\left[ye^{\int P(x)dx}\right] = \int Q(x)e^{\int P(x)dx}dx + c$$

or

$$y = e^{-\int P(x)dx}\left(\int Q(x)e^{\int P(x)dx}dx + c\right)$$

A first order linear differential equation in the dependent variable x and independent variable y is of the form

$$\frac{dx}{dy} + P(y)x = Q(y)$$

Then it has an integrating factor of the form

$$\mu(y) = e^{\int P(y)dy}$$

A one-parameter family of solutions of this equation is

$$x \times I.F. = \int Q(y) \times I.F. dy + c$$

or

$$\left[xe^{\int P(y)dy}\right] = \int Q(y)e^{\int P(y)dy}dy + c$$

or

$$x = e^{-\int P(y)dy}\left(\int Q(y)e^{\int P(y)dy}dy + c\right)$$

Example

Solve $x\dfrac{dy}{dx} - 4y = x^6 e^x$.

The standard form of this ODE is

$$\frac{dy}{dx} + \left(\frac{-4}{x}\right) y = x^5 e^x.$$

On comparing with $\dfrac{dy}{dx} + P(x)y = Q(x)$, we get

$$P(x) = \frac{-4}{x} \text{ and } Q(x) = x^5 e^x.$$

Integrating Factor (I.F) $= e^{\int P(x)dx} = e^{\int \frac{-4}{x}dx} = \dfrac{1}{x^4}$

Example(cont.)

Solution of the given ODE is given by

$$y \times I.F. = \int Q(x) \times I.F. dx + c$$

$$\Rightarrow y \times \frac{1}{x^4} = \int x^5 e^x . \frac{1}{x^4} dx + c \Rightarrow \frac{y}{x^4} = \int x e^x dx + c$$

$$\Rightarrow \frac{y}{x^4} = x e^x - e^x + c \Rightarrow y = x^5 e^x - x^4 e^x + c x^4$$

Example

Consider the differential equation

$$y^2 dx + (3xy - 1)dy = 0$$

Solution: Solving for $\frac{dy}{dx}$, we get

$$\frac{dy}{dx} = \frac{y^2}{1 - 3xy}$$

which is not linear in y.
Writing the above equation in the form

$$\frac{dx}{dy} = \frac{1 - 3xy}{y^2}$$

or

$$\frac{dx}{dy} + \frac{3}{y}x = \frac{1}{y^2}$$

Now the above equation is of the form

$$\frac{dx}{dy} + P(y)x = Q(y)$$

Which is linear in x.

Example(cont.)

Thus $I.F = e^{\int P(y)dy} = e^{\int \frac{3}{y}dy} = y^3$.
So, the solution of the above ODE is

$$x \times I.F. = \int Q(y) \times I.F.dy + c$$

$$\Rightarrow x \times y^3 = \int \frac{1}{y^2}.y^3 dy + c$$

$$\Rightarrow x.y^3 = \frac{y^2}{2} + c$$

$$\Rightarrow x = \frac{1}{2y} + \frac{c}{y^3}$$

Problems for Practice

Example 1.

Solve $\dfrac{dy}{dx} + \left(\dfrac{2x+1}{x}\right) y = e^{-2x}$.

Example 2.

Solve the differential equation

$$\frac{dy}{dx} + \frac{2}{x}y = 5x^2.$$

Example 3.

Solve the differential equation

$$y' \cos x - y \sin x = \sec^2 x.$$

Example 4.

Solve the differential equation

$$(x + 2y^3)dy - ydx = 0.$$

A differential equation of the form

$$\frac{dy}{dx} + P(x)y = Qy^n \tag{3}$$

where n is any real number, is called **Bernoulli's differential equation** named after the **Swiss mathematician James Bernoulli(1654-1705)**.

Note that when $n = 0$ or 1, Bernoulli's DE is a linear DE.
Method of Solution: Multiply by y^{-n} throughout the DE (4) to get

$$\frac{1}{y^n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x) \tag{4}$$

Use the substitution $z = y^{1-n}$. Then $\dfrac{dz}{dx} = (1-n)\dfrac{1}{y^n}\dfrac{dy}{dx}$.
Substituting in equation (4), we get $\dfrac{1}{1-n}\dfrac{dz}{dx} + P(x)z = Q(x)$, which is a linear DE.

Example

Solve the Bernoulli's DE $\dfrac{dy}{dx} + y = xy^3$.

Solution: Multiplying the above equation throughout by y^3, we get

$$\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{y^2} = x$$

Putting $z = \dfrac{1}{y^2}$, we get $\dfrac{dz}{dx} - 2z = -2x$, which is a linear DE.
Integrating Factor (I.F.) = $e^{-\int 2dx} = e^{-2x}$.
Therefore the solution is

$$z.e^{-2x} = \left[-2\int xe^{-2x}dx + c\right]$$

$$z = e^{2x}\left[-2\int xe^{-2x}dx + c\right] = x + \frac{1}{2} + ce^{2x}.$$

Putting back $z = \dfrac{1}{y^2}$ in this, we get the final solution

$$\frac{1}{y^2} = x + \frac{1}{2} + ce^{2x}.$$

Problems for practice

Example

Solve the Bernoulli equation $y' + xy - 2xy^2 = 0$

Example

Solve the Bernoulli equation $x^3y' = x^2y - y^4 \cos x, y(0) = 1$


Thank You