Department of Mathematics, Bennett University
Engineering Calculus (EMAT101L)
Solutions for Tutorial Sheet 1

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1. (a) $S = \{0, 2\}$ is bounded, max = sup = 2 and min = inf = 0.

   (b) S is bounded above by $\frac{1}{2^2}$ and bounded below by -1. max = sup = $\frac{1}{2^2}$ and min = inf = -1.

   (c) $S = \{0, \pm\frac{\sqrt{3}}{2}\}$ is bounded, max = sup = $\frac{\sqrt{3}}{2}$ and min = inf = $-\frac{\sqrt{3}}{2}$.

   (d) S is bounded above by $\frac{1}{2}$ and bounded below by 0, sup = $\frac{1}{2}$ and inf = 0.

2. Suppose $\alpha = \sup S$. Then it is a least upper bound of S. So, we have $s \leq \alpha$ for all $s \in S$. Next, suppose $\epsilon > 0$. If there is no $s \in S$ such that $\alpha - \epsilon < s$, then we have $s \leq \alpha - \epsilon < \alpha$ for all $s \in S$. This implies that $\alpha - \epsilon$ is an upper bound. This contradicts the fact that α is the least upper bound.

3. Suppose this is not true then $x < y$ implies $y - x > 0$. Let $\epsilon = \frac{1}{2}(y - x) > 0$. By the given hypothesis, $x + \frac{1}{2}(y - x) > y$ which implies $x > y$ a contradiction.

4. For any $\epsilon > 0$, we have $x < y + \epsilon$, $y - \epsilon < x$. Then by above problem we have $x \leq y$ and $y \leq x$.

5. Clearly $0 \in (-\frac{1}{n}, \frac{1}{n})$ for each $n \in \mathbb{N}$. Then $0 \in \cap_{n\in\mathbb{N}}(-\frac{1}{n}, \frac{1}{n})$. Let $0 \neq x \in \cap_{n\in\mathbb{N}}(-\frac{1}{n}, \frac{1}{n})$. Then $|x| > 0$ and by Archimedean property we get a contradiction.

6. If $r \in \mathbb{Q}$, then take $x_n = r$ for all n. Otherwise, for every n choose a rational number x_n such that $r < x_n < r + \frac{1}{n}$.

7. (a) Let $\epsilon > 0$. Then $|\frac{2n}{2+n} - 2| = \frac{4}{2+n} < \frac{4}{n}$. Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\epsilon}{4}$. Then for all $n \geq N$, $|\frac{2n}{2+n} - 2| < \frac{4}{n} < \epsilon$ implies $\lim\limits_{n\to\infty} \frac{2n}{2+n} = 2$.

   (b) Let $\epsilon > 0$. Then $|\frac{5}{1+n^2} - 0| < \frac{5}{n^2}$. Choose $N \in \mathbb{N}$ such that $\frac{1}{N^2} < \frac{\epsilon}{5}$. Then for all $n \geq N$, $|\frac{5}{1+n^2} - 0| < \frac{5}{n^2} < \epsilon$ implies $\lim\limits_{n\to\infty} \frac{5}{1+n^2} = 0$.

   (c) Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $\frac{1}{N^p} < \epsilon$ then for all $n \geq N$, $\frac{1}{n^p} < \epsilon$ implies $\lim\limits_{n\to\infty} \frac{1}{n^p} = 0$.

   (d) Using the sum of n terms of geometric series, one can check that $a_n = 1 - \frac{1}{10^n}$. Thus $|a_n - 1| = \frac{1}{10^n}$ so that $|a_n - 1| < \epsilon$ if and only if $10^n > \frac{1}{\epsilon}$. Let $N \in \mathbb{N}$ be such that $10^N > \frac{1}{\epsilon}$. Then for every $\epsilon > 0$, we have $|a_n - 1| < \epsilon$ for all $n \geq N$. Therefore $a_n \to 1$ as $n \to \infty$.

8. We have $a_{n+1} = \frac{1}{2}(a - a_n) = \frac{1}{2}(a - \frac{1}{2}(a - a_{n-1}))$. Continuing this way, we get

$$a_{n+1} = a\left(\frac{1}{2} - \frac{1}{4} + \frac{1}{8} + \cdots + \frac{(-1)^{n-1}}{2^n}\right) + \frac{(-1)^n}{2^n}a_1.$$

Thus

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a\left(\frac{1}{2} - \frac{1}{4} + \frac{1}{8} + \cdots + \frac{(-1)^{n-1}}{2^n}\right) + \lim_{n\to\infty} \frac{(-1)^n}{2^n} a_1 = \frac{a}{2} . \frac{1}{1+\frac{1}{2}} = \frac{a}{3}.$$

9. (a) True. Note that $\sum_{i=1}^{n} \frac{1}{\sqrt{n^2+i}} \leq 1$. Also $\sum_{i=1}^{n} \frac{1}{\sqrt{n^2+n}} \leq \sum_{i=1}^{n} \frac{1}{\sqrt{n^2+i}}$. Now use sandwich theorem.

   (b) True. We have $\frac{1}{n^2} \leq a_n \leq \frac{n+1}{n^2}$. By sandwich theorem, $\lim_{n\to\infty} a_n = 0$.

   (c) False. $\lim_{n\to\infty} \left(\frac{a^{n+1}+b^{n+1}}{a^n+b^n}\right) = b$, since $\frac{a}{b} < 1$ and $\left(\frac{a}{b}\right)^n \to 0$ as $n \to \infty$.

10. Let $|a_n| \leq M$ for all n and $b_n \to 0$. So given $\epsilon > 0$, there exists N such that $|b_n| < \frac{\epsilon}{M}$ for all $n \geq N$. Therefore, $|a_n b_n| \leq M|b_n| \leq \epsilon$ for all $n \geq N$.

    Take $a_n = (-1)^n$ and $b_n = 1$ for all n. Then $\{a_n\}$ is bounded and $b_n \to 1 (\neq 0)$ but $\{a_n b_n\}$ does not converge.