# Solutions - Mid Term Examination (EMAT 101L)

**Solution-1:** $S$ is bounded above by $\frac{1}{4}$ and bounded below by $-1$.

Therefore $\sup. S = \frac{1}{4}$ and $\inf. S = -1$.

**Solution-2 (a):** Let $f$ be the function defined by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

then $f$ is discontinuous at every point of $\mathbb{R}$.

**Solution-2(b):** Take $x_n = \frac{1}{n+1}$, $y_n = \frac{1}{n}$,

then $|x_n - y_n| = \left|\frac{1}{n+1} - \frac{1}{n}\right| = \frac{1}{n(n+1)} \to 0$ as $n \to \infty$.

But $|f(x_n) - f(y_n)| = |n+1-n| = 1$.

Therefore $|f(x_n) - f(y_n)| \to 1 \neq 0$ as $n \to \infty$.

Thus $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1)$.

**Solution-3:**

$$\sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}$$

We have $\frac{1}{\sqrt{n^2+1}} \geq \frac{1}{\sqrt{n^2+k}}$ $\forall$ $k = 1, 2, \ldots, n$.

$$\therefore \quad \frac{n}{\sqrt{n^2+1}} \geq \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} \qquad \text{———— (1)}$$

Also, $\frac{1}{\sqrt{n^2+n}} \leq \frac{1}{\sqrt{n^2+k}}$ $\forall$ $k = 1, 2, \ldots, n$

$$\Rightarrow \quad \frac{n}{\sqrt{n^2+n}} \leq \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} \qquad \text{———— (2)}$$

From (1) and (2), we get

$$\frac{n}{\sqrt{n^2+n}} \leq \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} \leq \frac{n}{\sqrt{n^2+1}}$$

Using Sandwich theorem, we get

$$\boxed{\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} = 1}$$

Solution-4 : Given $\displaystyle\sum_{n=0}^{\infty} \frac{(x+2)^{3n}}{5^n}$.

Let $a_{3n} = \dfrac{1}{5^n}$

Now, $\displaystyle\lim_{n\to\infty} |a_{3n}|^{\frac{1}{3n}} = \lim_{n\to\infty} \left(\frac{1}{5^n}\right)^{\frac{1}{3n}} = \frac{1}{5^{1/3}}$.

Thus radius of convergence $R = 5^{1/3}$.

Given series converges for $|x+2| < 5^{1/3}$.

$$\Rightarrow \quad -5^{1/3} < x+2 < 5^{1/3}$$

$$\Rightarrow \quad -2-5^{1/3} < x < -2+5^{1/3}.$$

At $x = -2-5^{1/3}$, $\displaystyle\sum_{n=0}^{\infty} \frac{(-5^{1/3})^{3n}}{5^n} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{5^n}{5^n}$

$$= \sum_{n=0}^{\infty} (-1)^n, \quad \text{diverges}.$$

At $x = -2+5^{1/3}$, $\displaystyle\sum_{n=0}^{\infty} \frac{(5^{1/3})^{3n}}{5^n} = \sum_{n=0}^{\infty} 1$, diverges.

Therefore, the domain of convergence is $(-2-5^{1/3}, -2+5^{1/3})$.

## Solution-5:

$$f(x) = \begin{cases} x^2 \ln \frac{1}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

For $x > 0$, $f'(x) = 2x \ln \frac{1}{x} - x$ and $\lim_{x \to 0^+} f'(x) = 0$.

For $x < 0$, $f'(x) = 2x \ln \frac{1}{|x|} - x$ and $\lim_{x \to 0^-} f'(x) = 0$.

As $f'(0) = \lim_{h \to 0} h \ln\left(\frac{1}{|h|}\right) = \begin{cases} 0, & h > 0 \\ 0, & h < 0 \end{cases}$

$\therefore f'(0) = 0$.

$\therefore \lim_{x \to 0} f'(x) = 0 = f'(0)$.

Thus $f'$ is continuous at $0$.

## Solution-6 (a):

$$\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2},$$

Take $a_n = \dfrac{\sqrt[n]{n}}{n^2}$ and $b_n = \dfrac{1}{n^2}$

Then $\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} n^{1/n} = 1$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Therefore, by limit comparison test $\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2}$ converges.

**Solution-6(b):** $\sum_{n=1}^{\infty} \frac{n!}{10^n}$,

Take $a_n = \frac{n!}{10^n}$, then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)!}{10^{n+1}} \cdot \frac{10^n}{n!}$

$= \lim_{n\to\infty} \frac{n+1}{10} = \infty > 1.$

Hence, $\sum_{n=1}^{\infty} \frac{n!}{10^n}$ diverges.