Ordinary Differential Equations(EMAT102L)
(Lecture-10 and 11)

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Outline of the Lecture

We will learn

- Second Order Linear Differential Equation
- Solution of Second Order DE
- Linearly Dependent/Independent Functions
- Wronskian
- Abel's Formula

Second Order Linear ODE

The general form of a second order differential equation is

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = F(x), x \in I$$

Here I is an interval contained in R and the functions $a_0(x)$, $a_1(x)$, $a_2(x)$ and F are real valued continuous functions defined on I and $a_0(x) \neq 0$.

The above equation is called **homogeneous** if $F(x) = 0$ for all x otherwise it is called **nonhomogeneous**.

Examples

$y'' - y = 0$ **(Linear, Homogeneous)**

$y'' + y' + y = \sin x$ **(Linear, Nonhomogeneous)**

$y'' + 3xy' + x^3y = e^x$ **(Linear, Nonhomogeneous)**

Solution of Second Order ODE

Consider the second order ODE

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = F(x), x \in I \quad (1)$$

A function y defined on an interval I is called a solution of the second order ODE if

- y is twice differentiable.
- y satisfies equation (1).

Examples

1. e^x, e^{-x} are solutions of $y'' - y = 0$.
2. $\sin x$ and $\cos x$ are solutions of $y'' + y = 0$.

Second Order Initial Value Problem

Consider the initial value problem (IVP) for a second order linear ODE

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = F(x),\ \ y(x_0) = c_0,\ \ y'(x_0) = c_1$$

Existence and Uniqueness Theorem for Second Order IVP

If $a_0(x)$, $a_1(x)$, $a_2(x)$ and $F(x)$ are continuous functions on an interval I where $a_0(x) \neq 0$ and $x_0 \in I$, then the above initial value problem has a **unique solution** $y(x)$ in the interval I.

Note: This is the sufficient condition only.

Consider the initial value problem (IVP) for a second order Homogeneous linear ODE

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0,\ \ y(x_0) = 0,\ \ y'(x_0) = 0$$

Existence and Uniqueness Theorem for Second Order IVP

If $a_0(x)$, $a_1(x)$ and $a_2(x)$ are continuous functions on an interval I where $a_0(x) \neq 0$ and $x_0 \in I$, then the above initial value problem has a **unique solution** $y(x) = 0$ for all x in the interval I.

Note: This is the sufficient condition only.

Superposition Principle

Consider the second order Homogeneous linear ODE

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \tag{2}$$

If y_1, y_2 are two solutions of the linear second order homogeneous differential equation (2), then

$$c_1y_1 + c_2y_2, c_1, c_2 \in \mathbb{R}$$

is also a solution of the above equation. That is, any linear combination of solutions of the homogeneous linear differential equation (2) is also a solution of (2).

Linearly Dependent Functions

The functions $f(x)$ and $g(x)$ are said to be **linearly dependent** on an interval I if there exist constants a, b**, not all zero**, such that

$$af(x) + bg(x) = 0$$

for every $x \in I$.

Linearly Independent Functions

The functions $f(x)$ and $g(x)$ are said to be **linearly independent** on an interval I if there exist constants a, b such that

$$af(x) + bg(x) = 0 \;\forall\; x \in I \Rightarrow a = b = 0$$

for every $x \in I$.

Examples

1. The functions x and $2x$ are linearly dependent on the interval $0 \leq x \leq 1$. For there exist constants c_1 and c_2, not both zero, such that

$$c_1 x + c_2(2x) = 0$$

   for all x on the interval $0 \leq x \leq 1$. For example, let $c_1 = 2$, $c_2 = -1$.

2. The functions $f_1(x) = \sin 2x$ and $f_2(x) = \sin x \cos x$

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   for all x on the interval $0 \leq x \leq 1$. For example, let $c_1 = 2$, $c_2 = -1$.
2. The functions $f_1(x) = \sin 2x$ and $f_2(x) = \sin x \cos x$ are linearly dependent on the interval $(-\infty, \infty)$ because $f_1(x)$ is a constant multiple of $f_2(x)$.
3. The functions x and x^2 are linearly independent on $0 \leq x \leq 1$.
   Since $c_1 x + c_2 x^2 = 0$ for all x on $0 \leq x \leq 1$ implies that both $c_1 = 0$ and $c_2 = 0$.
4. The functions $f_1(x) = x$ and $f_2(x) = |x|$ are linearly independent on $(-\infty, \infty)$.
   Neither of the functions is a constant multiple of the other on $(-\infty, \infty)$ but linearly dependent on $(0, \infty)$ and $(-\infty, 0)$.

Definition

The **Wronskian** of two differentiable functions $f(x)$ and $g(x)$ is defined by

$$W(f,g) = W(f,g)(x) = det\begin{pmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{pmatrix} = f(x)g'(x) - f'(x)g(x)$$

Some Results on Wronskian

Theorem

Let y_1, y_2 be two solutions of the homogeneous linear Second order DE

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \tag{1}$$

*on an interval I. Then the set of solutions $\{y_1, y_2\}$ is **linearly independent** on I if and only if*

$$W(y_1, y_2) \neq 0$$

for every x in the interval I.

Theorem

The Wronskian $W(y_1, y_2)$ of two solutions y_1, y_2 of (1) is either identically zero or never zero on the interval.

Example

Show that the solutions $\sin x$ and $\cos x$ of $y'' + y = 0$ are linearly independent.

Solution: Here $W(\sin x, \cos x)$=

$$\det \begin{pmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{pmatrix} = -\sin^2 x - \cos^2 x = -1 \neq 0$$

for all real x. Thus $W(\sin x, \cos x) \neq 0$ for all real x.
So, we conclude that $\sin x$ and $\cos x$ are linearly independent solutions of the given differential equation on every real interval.

Result 1.

If y_1 and y_2 have a common zero at point x_0 in the interval $[a, b]$, then y_1 and y_2 are linearly dependent.

Solution: Since y_1 and y_2 have common zero at $x_0 \in [a, b]$,

$$\Rightarrow y_1(x_0) = y_2(x_0) = 0$$

So,

$$W(y_1, y_2)(x_0) = \det \begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix} = 0$$

$$\Rightarrow W(y_1, y_2)(x_0) = 0 \text{ for some point } x_0 \in [a, b].$$

$\Rightarrow y_1$ and y_2 are linearly dependent.
Note: Here y_1 and y_2 are the solutions of the same differential equation.

Result 2.

If y_1 and y_2 have a relative maxima or minima at some common point $x_0 \in [a, b]$, then y_1 and y_2 are linearly dependent.

Solution: Since y_1 and y_2 have a relative maxima or minima at some common point $x_0 \in [a, b]$,

$$\Rightarrow y_1'(x_0) = y_2'(x_0) = 0$$

So,

$$W(y_1, y_2)(x_0) = \det \begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix} = 0$$

$$\Rightarrow W(y_1, y_2)(x_0) = 0 \text{ for some point } x_0 \in [a, b].$$

$\Rightarrow y_1$ and y_2 are linearly dependent.

Note: Here y_1 and y_2 are the solutions of the same differential equation.

Definition

If $\{y_1, y_2\}$ are two linearly independent solutions of the homogeneous linear second order DE

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \tag{3}$$

where $a_0(x) \neq 0$, $a_i(x), i = 1, 2$ are continuous functions on an interval I, then the set $\{y_1, y_2\}$ is said to be the **fundamental set of solutions** on the interval I.

Theorem

There exists a fundamental set of solutions(Linearly independent solutions) for the homogeneous linear second order DE (3) on an interval I.

Theorem

Let $\{y_1, y_2\}$ be a fundamental set of solutions for the homogeneous linear second order DE (3) on an interval I. Then the general solution of the equation (3) on the interval I is

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where c_1, c_2 are arbitrary constants.

Note:

- For a homogeneous linear second order ODE, if we know two linearly independent solutions, then every solution can be obtained with the linear combination of these two linearly independent solutions.
- That is, if y_1, y_2 are two linearly independent solutions of the homogeneous linear second order DE, then the general solution $y(x)$ can be written as the linear combination of these solutions. i.e,

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

  where c_1 and c_2 are arbitrary constants.

Example

1. The functions $y_1(x) = e^{3x}$ and $y_2(x) = e^{-3x}$ are both solutions of the homogeneous linear equation $y'' - 9y = 0$ on the interval $(-\infty, \infty)$.

Example

1. The functions $y_1(x) = e^{3x}$ and $y_2(x) = e^{-3x}$ are both solutions of the homogeneous linear equation $y'' - 9y = 0$ on the interval $(-\infty, \infty)$.
    - Here Wronskian $W(e^{3x}, e^{-3x}) = -6 \neq 0$ for every $x \in (-\infty, \infty)$.
    - So the solutions y_1, y_2 are linearly independent on $(-\infty, \infty)$.
    - Hence we can conclude that $\{y_1, y_2\}$ is a fundamental set of solutions.
    - Therefore $y(x) = c_1 e^{3x} + c_2 e^{-3x}$ is the general solution of the equation on $(-\infty, \infty)$.

Abel's Theorem

If y_1 and y_2 are solutions of the DE

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$$

where $a_0(x) \neq 0$, $a_i(x), i = 1, 2$ are continuous functions on an open interval I, then the Wronskian $W(y_1, y_2)(x)$ is given by

$$W(y_1, y_2)(x) = c\exp\left[-\int \frac{a_1(x)}{a_0(x)}dx\right],$$

where c is a certain constant that depends on y_1 and y_2, but not on x.
Further, $W(y_1, y_2)(x)$ is either zero for all $x \in I$ (if $c = 0$) or else is never zero in I (if $c \neq 0$).

Problem

Let y_1 and y_2 be two linearly independent solutions of

$$y'' + (\sin x)y = 0 \text{ in } [0, 1]$$

Let $g(x) = W(y_1, y_2)$, then show that $g'(x) = 0$.

Solution: Here $a_0(x) = 1, a_1(x) = 0, a_2(x) = \sin x$.
Therefore, by Abel's formula,

$$g(x) = W(y_1, y_2) = c \exp\left[-\int \frac{a_1(x)}{a_0(x)} dx\right] = ce^0 = c.$$

$$\Rightarrow g'(x) = 0.$$


Thank You