Laplace Transforms

35 Marks for (ODE)

(7 Marks for Laplace transform)

15 Marks → Linear Algebra

System of DE's

$$\left.\begin{aligned} \frac{dx}{dt} &= 2x - 3y \quad \text{—— (1)} \\ \frac{dy}{dt} &= y - 2x \quad \text{—— (2)} \end{aligned}\right\} \longrightarrow \begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

Take Laplace transform on both the sides of (1),

$$L[x'(t)] = L[2x - 3y]$$

$$\Rightarrow \quad s\,L[x(t)] - x(0) = 2L[x] - 3L[y]$$

$$\left[\because L[f'(t)] = sF(s) - f(0)\right]$$

$$\Rightarrow (s-2)\, L[x(t)] \quad 8 = -3\, L[y(t)]$$

$$\Rightarrow (s-2)\, L[x(t)] + 3\, L[y(t)] = 8 \qquad \text{——— (3)}$$

Taking Laplace transform on both sides of (2), we get

$$L[y'(t)] = L[y(t)] - 2\, L[x(t)]$$

$$\Rightarrow s\, L[y(t)] - y(0) = L[y(t)] - 2\, L[x(t)]$$

$$\Rightarrow (s-1)\, L[y(t)] + 2\, L[x(t)] = 3$$

$$\Rightarrow 2\, L[x(t)] + (s-1)\, L[y(t)] = 3 \qquad \text{——— (4)}$$

$$\left.\begin{aligned} (s-2)\, L[x(t)] + 3L[y(t)] &= 8 \\ 2\, L[x(t)] + (s-1)\, L[y(t)] &= 3 \end{aligned}\right\}\begin{aligned} &\text{— (3)} \\ &\text{— (4)} \end{aligned}$$

$$L[x(t)] = \frac{5}{s+1} + \frac{3}{s+4},$$

$$L[y(t)] = \frac{5}{s+1} - \frac{2}{s-4}$$

$$L[x(t)] = \frac{5}{s+1} + \frac{3}{s+4}$$

$$\Rightarrow \quad x(t) = L^{-1}\left[\frac{5}{s+1} + \frac{3}{s+4}\right]$$

$$\Rightarrow \boxed{x(t) = 5e^{-t} + 3e^{-4t}}$$

$$\boxed{L^{-1}\left[\frac{1}{s-a}\right] = e^{at}}$$

$$y(t) = L^{-1}\left[\frac{5}{s+1} - \frac{2}{s-4}\right]$$

$$\boxed{y(t) = 5e^{-t} - 2e^{4t}}$$

Solve

$$\left.\begin{aligned} \frac{d^2x}{dt^2} - x + 5\frac{dy}{dt} &= t \quad &-(1) \\ \frac{d^2y}{dt^2} - 4y - 2\frac{dx}{dt} &= -2 \quad &-(2) \end{aligned}\right.$$

$- \; x(0) = 0, \quad x'(0) = 0,$

$y(0) = 1, \quad y'(0) = 0$

$$\boxed{\begin{aligned} &L[x''(t)] \\ &= s^2 L[x(t)] - s\,x(0) \\ &\quad - x'(0) \end{aligned}}$$

$$\boxed{L[t] = \frac{1}{s^2}}$$

$$L[x''(t)] - L[x(t)] + 5\,L[y'(t)] = L[t]$$

$$\Rightarrow s^2 L[x(t)] - s\,x(0) - x'(0) - L[x(t)] + 5\big(s\,L[y(t)] - y(0)\big) = \frac{1}{s^2}$$

$$\Rightarrow (s^2-1)\,L[x(t)] + 5\,s\,L[y(t)] = \frac{1}{s^2} + 5 \qquad (3)$$

from (2)

$$L[y''(t)] - 4\,L[y(t)] - 2\,L[x'(t)] = L[-2]$$

$$\Rightarrow s^2 L[y(t)] - s\,y(0) - y'(0) - 4\,L[y(t)] - 2\,(s\,L[x(t)] - x(0)) = \frac{-2}{s}$$

$$\Rightarrow (s^2-4)\,L[y(t)] - s - 2s\,L[x(t)] = \frac{-2}{s}$$

$$\Rightarrow -2s\,L[x(t)] + (s^2-4)\,L[y(t)] = \frac{-2}{s} + s \quad (4)$$

Solving (3) & (4),

$$L[y(t)] = \frac{1}{s} - \frac{2}{3}\cdot\frac{s}{s^2+4} + \frac{2}{3}\cdot\frac{s}{s^2+1}$$

$$\Rightarrow y(t) = L^{-1}\left(\frac{1}{s} - \frac{2}{3}\cdot\frac{s}{s^2+4} + \frac{2}{3}\cdot\frac{s}{s^2+1}\right)$$

$$y(t) = 1 - \frac{2}{3}\cdot\cos 2t + \frac{2}{3}\cos t$$

$$L^{-1}\left(\frac{1}{s}\right) = 1$$

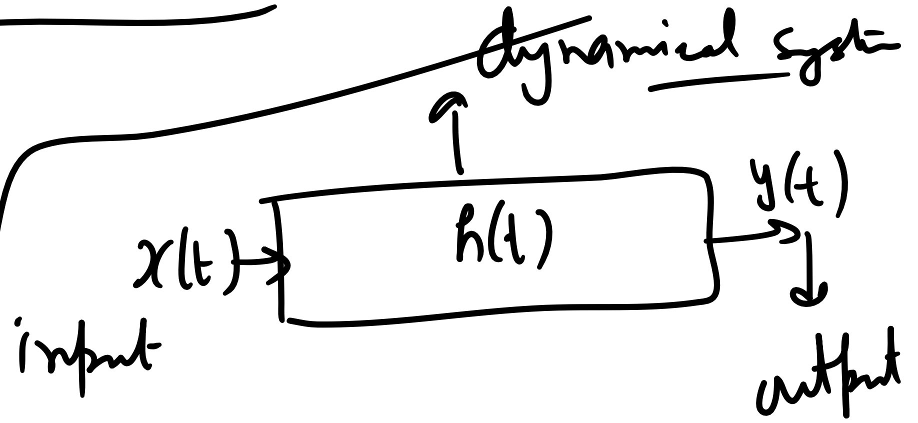
$$L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$$

$$L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$$

$$(s^2-1)L[x] + 5sL[y] = \frac{1}{s^2}+5$$

$$-2s\,L[x] + (s^2-4)L[y] = \frac{-2+s^2}{s}$$

Convolution of two functions:



$$y(t) = x(t) * h(t)$$

Convolution of f & g is defined by.

$$f * g = \int_0^t f(\tau)\cdot g(t-\tau)\,d\tau.$$

$$L[f * g] = L[f(t)]\cdot L[g(t)]$$

$$\Rightarrow L\left[\int_0^t f(\tau)\cdot g(t-\tau)\,d\tau\right] = F(s)\cdot G(s)$$

$$\Rightarrow \boxed{L^{-1}\left(F(s)\cdot G(s)\right) = \int_0^t f(\tau)\cdot g(t-\tau)\,d\tau}$$

Sol. Find $L^{-1}\left(\dfrac{1}{s(s^2+1)}\right)$

$$L^{-1}\left(F(s)\cdot G(s)\right), \quad F(s) = \frac{1}{s^2+1},$$

$$G(s) = \frac{1}{s}.$$

$$F(s) = \frac{1}{s^2+1}, \quad G(s) = \frac{1}{s}$$

$$\Rightarrow L^{-1}(F(s)) = L^{-1}\left(\frac{1}{s^2+1}\right) = \sin t = f(t)$$

$$L^{-1}(G(s)) = L^{-1}\left(\frac{1}{s}\right) = 1 = g(t)$$

$$L^{-1}(F(s) \cdot G(s)) = f(t) * g(t)$$

$$= \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$$= \int_0^t \sin\tau \cdot (1)\, d\tau$$

$$= -\left|\cos\tau\right|_0^t = -\cos t + 1$$

$$\Rightarrow \boxed{L^{-1}\left(\frac{1}{s(s^2+1)}\right) = 1 - \cos t.}$$

$L^{-1}\left(\dfrac{1}{s(s^2+1)}\right)$

$$\boxed{\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}}$$

$$= \frac{1}{s} - \frac{s}{s^2+1}$$

$$\Rightarrow L^{-1}\left(\frac{1}{s(s^2+1)}\right) = L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{s}{s^2+1}\right)$$

$$= 1 - \cos t$$

$$\int e^{ax}\cos(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos(bx+c) + b\sin(bx+c)\right]$$

$$\int e^{ax}\sin(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin(bx+c) - b\cos(bx+c)\right]$$

$L[1] = \frac{1}{s},\ s>0$	$L^{-1}\left(\frac{1}{s}\right) = 1$
$L[t] = \frac{1}{s^2}$	$L^{-1}\left(\frac{1}{s^2}\right) = t$
$L[t^n] = \frac{n!}{s^{n+1}}$	$L^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{n!}$
$L[\cos at] = \frac{s}{s^2+a^2}$	$L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$

$$L[\sin at] = \frac{a}{s^2+a^2} \qquad L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$$

$$L[e^{at}] = \frac{1}{s-a} \qquad L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$$

Properties

① If $L[f(t)] = F(s)$, then

$$L[e^{at} f(t)] = F(s-a)$$

② If $L[f(t)] = F(s)$, then

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}(F(s))$$

③ If $L[f(t)] = F(s)$, then

$$L[f'(t)] = sF(s) - f(0)$$

$$L[f''(t)] = s^2F(s) - sf(0) - f'(0)$$

$$\boxed{\begin{array}{l}\text{If } L^{-1}[F(s)] = f(t), \text{ then} \\ L^{-1}[F(s-a)] = e^{at} f(t)\end{array}}$$