Lecture - 15 (ODE)

How to find particular integral :

$y = y_c(x) + y_p(x)$

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_n(x)\,y = F(x), \quad x\in[a,b] \qquad (1)$$

where $a_0(x) \neq 0$, $a_0(x), a_1(x), a_2(x), \cdots a_n(x)$ & $F(x)$ are continuous on $[a,b]$.

The corresponding Hom DE is

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_n(x)\,y = 0. \qquad (2)$$

As we already know that the solⁿ of (1) is given by

$$y = y_c(x) + y_p(x),$$

where $y_c(x)$ = solⁿ of corresponding Hom DE (2) (Complementary fⁿ)

$y_p(x)$ = particular solⁿ of nonhom DE (1). (particular integral).

We have already done how to calculate $y_c(x)$. (in the previous lecture) (solⁿ of corresponding Hom DE).

In this lecture, we will learn how to calculate $y_p(x)$ (particular integral)

(i.e particular solⁿ of non hom DE).

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0$$

A.E

$$a_0 m^2 + a_1 m + a_2 = 0$$

$$\Rightarrow m = m_1, m_2$$

$$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

How to calculate $y_p(x)$:

(i) Method of Undetermined Coefficients

(applicable only if DE is with constant coefficients)

(ii) Variation of Parameters.

Method of Undetermined Coefficients:

Consider the linear non hom DE

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \text{---} + a_n y = F(x) \quad \text{--- (1)}$$

$a_0 \neq 0$, $a_0, a_1, a_2, \text{---}\ a_n$ are constants.

The corresponding hom DE is

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \text{---} + a_n y = 0 \quad \text{--- (2)}$$

The auxiliary eqn or characteristic eqn is

$$p(m) = a_0 m^n + a_1 m^{n-1} + \text{---} + a_n = 0$$

Method of undetermined coefficients is applicable if $F(x)$ is of the following forms:

(i) $F(x) = k e^{\alpha x}$, $k \neq 0$, α constant

(ii) $F(x) = e^{\alpha x}(k_1 \cos\beta x + k_2 \sin\beta x)$

(iii) $F(x) = x^n$,

Case-I: If $F(x) = b e^{\alpha x}$,

Subcase-I If α is not a root of ch eqn, ($p(\alpha) \neq 0$), then $\boxed{y_p(x) = A e^{\alpha x}}$,

where A is an undetermined coefficient.

Since $y_p(x)$ is particular soln of (1).

$\Rightarrow$ by substituting the value of y_p in (1), you can determine A.

Subcase-II: If α is a root of ch eqn (A.E) of multiplicity r (α is repeated r times as a root of A.E), then

$$\boxed{y_p(x) = A x^r e^{\alpha x}.}$$

Determine the value of A by substitution in (1).

Example: find the particular soln of

$$y'' - 4y = 2e^{x}$$

S_1^n: Here $f(x) = 2e^x$. $(ke^{\alpha x})$

$(\alpha = 1)$

Then A.E is

$p(m) = m^2 - 4 = 0$

$\Rightarrow \quad m = \pm 2$;

$\boxed{y_c(x) = c_1 e^{2x} + c_2 e^{-2x}}$

$\Rightarrow \alpha = 1$ is not the root of A.E.

Thus $y_p(x) \equiv Ae^x$

Substituting the value of y_p in the given eqn, we get

$$y_p'' - 4y_p = 2e^x$$

$$\Rightarrow Ae^x - 4Ae^x = 2e^x$$

$$\Rightarrow (A - 4A)e^x = 2e^x$$

$$\Rightarrow A - 4A = 2 \Rightarrow 3A = \cancel{2}.$$

$$\Rightarrow A = \frac{-2}{3}$$

$$\Rightarrow \boxed{y_p(x) = \frac{-2}{3} e^x} \quad \text{Ans.}$$

$\Rightarrow$ The general soln is $y = y_c(x) + y_p(x)$

$\rightarrow \quad y = c_1 e^{2x} + c_2 e^{-2x} - \frac{2}{3} e^{x}$

Example: find the general solⁿ of

$$y''' - 3y'' + 3y' - y = \underline{2e^{x}}$$

$(ke^{\alpha x})$, $\underline{\alpha = 1}$

Solⁿ: Here $F(x) = 2e^{x}$,

A·E is

$$m^3 - 3m^2 + 3m - 1 = 0$$

$$\Rightarrow (m-1)^3 = 0$$

$$\Rightarrow m = 1, 1, 1$$

$y_c(x) = (c_1 + c_2 x + c_3 x^2) e^{x}$

$\Rightarrow \alpha = 1$ is a root of ch eqⁿ (A.E) of multiplicity 3.

$$\therefore y_p(x) = A x^{r} e^{\alpha x}$$

$$y_p(x) = A x^3 e^{\alpha x} = \underline{A} x^3 e^{x}.$$

$A = \frac{1}{3}$ (Calculate it yourself)

$$y_p(x) = \frac{1}{3} x^3 e^{x}.$$

$$y = y_c(x) + y_p(x)$$

$$\boxed{y = (c_1 + c_2 x + c_3 x^2) e^x + \frac{1}{3} x^3 e^x.}$$

Case-II: If $F(x) = e^{\alpha x}(b_1 \cos\beta x + b_2 \sin\beta x)$,

Subcase-I If $\alpha + i\beta$ is not a root of A.E.

(i.e $p(\alpha + i\beta) \neq 0$), then

$$y_p(x) = e^{\alpha x}(A \cos\beta x + B \sin\beta x),$$

where A and B are undetermined coefficients. (By substituting the value of y_p into the given eqn, we can calculate A & B)

Subcase-II: If $\alpha + i\beta$ is a root of ch eq^n of multiplicity r, then

$$y_p(x) = x^r e^{\alpha x}(A \cos\beta x + B \sin\beta x)$$

Example: $y'' + 2y' + 2y = 4e^x \sin x$

Here $F(x) = 4e^x \sin x$

$$\boxed{e^{\alpha x}(b_1 \cos\beta x + b_2 \sin\beta x)}$$

$\alpha = 1, \beta = 1$

$\boxed{\alpha + i\beta = 1 + i}$

A.E (ch eqⁿ) is

$$m^2 + 2m + 2 = 0$$

$$\Rightarrow \quad m = \frac{-2 \pm \sqrt{4-8}}{2}$$

$$\Rightarrow m = -1 \pm i$$

$$y_c(x) = e^{-x}(c_1 \cos x + c_2 \sin x)$$

$\Rightarrow$ $\alpha + i\beta = 1 + i$ is not the root of ch. eqⁿ.

$$\Rightarrow \quad y_p(x) = e^x(A\cos x + B\sin x),$$

where A and B are undetermined coefficients.

Since $y_p(x)$ is the particular solⁿ of given DE

$\Rightarrow$ $y_p(x)$ satisfies the given DE.

$$\left(A = -B = \frac{1}{2}\right).$$

$$y_p(x) = e^x\left(\frac{1}{2}\cos x - \frac{1}{2}\sin x\right)$$

$$= \frac{1}{2}e^x(\cos x - \sin x)$$

$$\underline{y = y_c(x) + y_p(x)}$$

Example:

$$y'' + y = \sin x$$

$e^{\alpha x}(b_1 \cos \beta x + b_2 \sin \beta x)$

$f(x) = \sin x$, $\alpha = 0$, $\beta = 1$.

$\alpha + i\beta = i$

A.E is

$$m^2 + 1 = 0$$

$$\Rightarrow m = \pm i$$

$\Rightarrow \alpha + i\beta = i$ is the root of ch eq^n of multiplicity 1.

$$\Rightarrow y_p(x) = x\, e^{0 \cdot x}(A \cos x + B \sin x)$$

$$\Rightarrow y_p(x) = x(A \cos x + B \sin x)$$

$A = -\frac{1}{2}$, $B = 0$

Case-III: If $F(x) = x^n$,

Subcase: If 0 is not the root of A.E,

then $y_p(x) = A_n x^n + A_{n-1} x^{n-1} + \cdots + A_1 x + A_0$

where $A_n, A_{n-1}, — A_1, A_0$ are undetermined coefficients.

Subcase-II: If 0 is root of ch eqn of multiplicity r, then

$$y_p(x) = x^r(A_n x^n + A_{n-1}x^{n-1} + — — + A_1 x + A_0)$$

Example:

$$y''' - y'' + y' - y = x^2$$

Here $f(x) = x^2$, $n = 2$.

(A.E) Ch eqn is

$$m^3 - m^2 + m - 1 = 0$$

$$\Rightarrow m^2(m-1) + 1(m-1) = 0$$

$$\Rightarrow (m-1)(m^2+1) = 0$$

$$\Rightarrow m = 1, \pm i. \quad y_c(x) = c_1 e^x + (c_2 \cos x + c_3 \sin x)$$

Since '0' is not the root of A.E.

$$\therefore y_p(x) = A_2 x^2 + A_1 x + A_0.$$

$$A_0 = 0, \ A_1 = -2, \ A_2 = -1$$

$$y_p(x) = -(x^2+2x)$$

$$y(x) = y_c(x) + y_p(x)$$

$$\boxed{y(x) = c_1 e^{x} + c_2 \cos x + c_3 \sin x - (x^2+2x)}$$

Example: $y'' + y = 2\sin x + \sin 2x$

$y'' + y = 2\sin x$ | $y'' + y = \sin 2x$