<u>Result</u> : converges $\Rightarrow$ bounded.

$\{(-1)^n\}$ $\{n\}_{n=1}^{\infty}$

bounded + (monotone) $\Rightarrow$ converges

<u>monotone</u> : $a_n \le a_{n+1} \quad \forall n \in \mathbb{N}$

$a_n \ge a_{n+1} \quad \forall n \in \mathbb{N}$

$a_n = 1 - \frac{1}{n}$ $\quad a_{n+1} - a_n$

$= 1 - \frac{1}{n+1} - 1 + \frac{1}{n}$

$= \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} > 0$

$\Rightarrow \{a_n\}$ is increasing

$a_n = \frac{1}{n}$ $\quad \{(-1)^n\}$ $\left\{\frac{(-1)^n}{n}\right\}$ ✓

<u>Result</u> : ① increasing & bounded above $\Rightarrow$ convergent., $\sup\{a_n\}$

② decreasing + bounded below $\Rightarrow$ conv, $\inf\{a_n\}$

<u>Ex</u> : $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$

$a_{n+1} - a_n > 0$

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \quad \frac{1}{n+(n-2)} + \frac{1}{n+(n-1)} + \frac{1}{n+n}$$

$$a_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}$$

$a_{n+1} - a_n$

$$= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$$

$$= \frac{1}{2n+1} + \frac{1}{2}\frac{1}{(n+1)} - \frac{1}{n+1}$$

$$= \frac{1}{2n+1} - \frac{1}{2}\frac{1}{n+1} > 0$$

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$$

$$< \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n} = 1$$

$$a_n < 1 \quad \forall n \in \mathbb{N}$$

$$a_n \to 1$$

<u>subsequence</u>: $\{a_n\}$ $\{n_1, n_2, \ldots\}$

$\{a_{n_1}, a_{n_2}, a_{n_3}, \ldots\}$ $\quad n_1 < n_2 < n_3 \ldots$

$\{a_{n_k}\}$ of $\{a_n\}$

$$a_n = \frac{1}{n}, \quad n_k = k^2, \ k^3$$

$$\{a_{n_k}\} = \left\{\frac{1}{k^2}\right\}$$

Result: $\{a_n\} \to L$, every subseque of $\{a_n\} \to L$

$\{1, 0, 1, 0, \dots\}$

$\hookrightarrow \{0, 0, \dots\} \to 0$

$\{1, 1, \dots\} \to 1$

① a_n s.t $a_{2n} \to l$, $a_{2n-1} \to l$.

$\Rightarrow \{a_n\} \to l$.

Bolzano - Weierstrass $\{(-1)^n\}$

$\{1, 0, 1, 0, \dots\}$, $\{(-1)^n\}$

Theorem:- $\{a_n\}$ $|a_{n+1} - a| \le r|a_n - a|$

$0 < r < 1$, $a \in \mathbb{R}$. $\forall n \in \mathbb{N}$.

$\Rightarrow a_n \to a$

proof:- $|a_{n+1} - a| \le r|a_n - a|$

$|a_n - a| \le r|a_{n-1} - a|$

$$0 < |a_{n+1} - a| \leq r\,|a_n - a|$$

$$\leq r \cdot r\,|a_{n-1} - a|$$

$$= r^2\,|a_{n-1} - a|$$

$$\leq r^3\,|a_{n-2} - a|$$

$$\leq \dots \leq r^n\,|a_1 - a|$$

$$\downarrow$$

$$0$$

$|a_{n+1} - a| \to 0$ as $n \to \infty$

$\Rightarrow a_n \to a$

Q: $\{a_n\}$ $|a_{n+1} - a| < |a_n - a|$, $a \in \mathbb{R}$

$a_n \to a$??

EX:- $a_n = \dfrac{n+1}{n} \to 1$

$a_{n+1} = \dfrac{n+2}{n+1}$, $a = 0$

$a_{n+1} < a_n$

$a_{n+1} - a_n < 0$

$$\frac{n+2}{n+1} < \frac{n+1}{n}$$

$$\left|\frac{n+2}{n+1} - 0\right| < \left|\frac{n+1}{n} - 0\right|$$

Cauchy Sequence:

$\{a_n\}$ $\epsilon > 0, \exists N \in \mathbb{N}$ s.t.

$|a_n - a_m| < \epsilon \quad \forall n, m \geq N$

$a_1 \; a_2 \ldots \ldots \boxed{a_N}, a_{N+1}, a_{N+2}, \ldots$

EX 1: $a_n = \{\frac{1}{n}\}$ cauchy sequence.

$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m}$ — step 1

choose $N \in \mathbb{N}$ s.t. $\frac{1}{N} < \frac{\epsilon}{2}$ — 2

$n, m \geq N$, $\frac{1}{n} \leq \frac{1}{N} < \frac{\epsilon}{2}$

$\frac{1}{m} \leq \frac{1}{N} < \frac{\epsilon}{2}$ — 3

$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, $\forall n, m \geq N$

EX 2 $a_n = \frac{n}{n+1}$

$|a_n - a_m| = \left|\frac{n}{n+1} - \frac{m}{m+1}\right|$

$= \left|\frac{n-m}{(n+1)(m+1)}\right|$

$\leq \cancel{\frac{n}{(n+1)(m+1)}} \cancel{+} \cancel{\frac{m}{(n+1)(m+1)}}$

$$\leq \frac{n}{(n+1)(m+1)} + \frac{m}{(n+1)(m+1)}$$

$|a-b| \leq |a|+|b|$

$$\boxed{\begin{array}{l} n < n+1 \\ m < m+1 \end{array}}$$

$$< \frac{1}{m+1} + \frac{1}{n+1} < \frac{1}{m} + \frac{1}{n}$$

$$\therefore |a_n - a_m| \leq \frac{1}{m} + \frac{1}{n}$$

choose $N \in \mathbb{N}$ s.t $\frac{1}{N} < \frac{\epsilon}{2}$

<u>Ex 3</u> $a_n = \frac{n+1}{n}$

<u>Ex 4</u> : $a_n = (-1)^n$

<u>Result</u> cauchy sequence $\Longleftrightarrow$ convergent

<u>contractive sequence</u>

$\{a_n\}$ $|a_{n+2} - a_{n+1}| \leq \alpha |a_{n+1} - a_n|$

$0 < \alpha < 1$ $\forall n \in \mathbb{N}$

Ex: $a_1 = 1$, $a_{n+1} = 1 + \frac{1}{a_n}$

$a_n \geq 1 \quad \forall n \in \mathbb{N}$