**Department of Mathematics, Bennett University**
**Engineering Calculus (EMAT101L)**
**Practice Problem Sheet 2**

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1. If the terms of the convergent series $\sum_{n=1}^{\infty} a_n$ are positive and forms a non-increasing sequence, then prove that $\lim_{n\to\infty} 2^n a_{2^n} = 0$.

2. Determine which of the following series converges/diverges:
$(a) \sum_{n=1}^{\infty} \frac{(\log n)^2}{n^{3/2}}, \quad (b) \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}, \quad (c) \sum_{n=1}^{\infty} \frac{1-n}{n2^n}.$

3. Determine which of the following series converges/diverges:
$(a) \sum_{n=1}^{\infty} \left(\frac{n-2}{n}\right)^n, \quad (b) \sum_{n=1}^{\infty} \frac{(\log n)^n}{n^n}.$

4. Find the value of $x$ for which the following series converges:
$(a) \sum_{n=0}^{\infty} \frac{x^{2n}}{a^n}, a \neq 0, \quad (b) \sum_{n=0}^{\infty} \frac{x^n}{n!n^n}.$

5. Test the convergence of the infinite series:
$(a) \sum_{n=2}^{\infty} \frac{1}{n\ln(n^3)}, \quad (b) \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sin\left(\frac{1}{n}\right).$

6. Test the convergence of the series: $(a) \sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}, \quad (b) \sum_{n=2}^{\infty} \frac{1}{(\log n)^x},\ x \in \mathbb{R}.$

## Solutions for Practice Problem Sheet 2

1. Let $t_n = a_1 + 2a_2 + ... + 2^n a_{2^n}$. Then $\{t_n\}$ converges. This implies $\sum_{n=1}^{\infty} 2^n a_{2^n}$ is convergent. Hence $\lim_{n\to\infty} 2^n a_{2^n} = 0$.

2. (a) Take $a_n = \dfrac{(\log n)^2}{n^{3/2}}$ and $b_n = \dfrac{1}{n^\alpha}$ where $1 < \alpha < \frac{3}{2}$. By limit comparison test series converges. (one can also use Cauchy condensation test i.e find the behaviour of the series $\sum 2^n a_{2^n}$.)

   (b) Take $b_n = \dfrac{1}{n^{3/2}}$. By limit comparison test series converges.

   (c) Take $b_n = \dfrac{1}{2^n}$. By limit comparison test series converges.

3. (a) Take $a_n = \left(\dfrac{n-2}{n}\right)^n$. Then $\lim_{n\to\infty} a_n = e^{-2} \neq 0$. Hence series diverges.

   (b) Take $a_n = \left(\dfrac{\log n}{n}\right)^n$. Then $\limsup_{n\to\infty} |a_n|^{1/n} = 0 < 1$. Hence series converges.

4. (a) Apply root test, $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \frac{x^2}{a}$ and series converges if $|x|^2 < a$.

   (b) $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = 0$. So every value of $x$, series converges.

5. (a) Diverges, use Cauchy condensation test, here, $2^n a_{2^n} = \frac{1}{3n\log 2}$.

   (b) Converges, $|a_n| \leq \frac{1}{n^{3/2}}$.

6. (a) Note that $\displaystyle\sum_{k=1}^{\infty} 2^k \frac{1}{2^k(\log 2^k)^p} = \sum_{k=1}^{\infty} \frac{1}{(k\log 2)^p} = \frac{1}{(\log 2)^p}\sum_{k=1}^{\infty}\frac{1}{k^p}$. Hence the series converges for $p > 1$ and diverges for $p \leq 1$.

   (b) Note that $\displaystyle\sum_{k=1}^{\infty} 2^k \frac{1}{(\log 2^k)^x} = \frac{1}{(\log 2)^x}\sum_{k=1}^{\infty}\frac{2^k}{k^x}$. Now let $a_k = \dfrac{2^k}{k^x}$. Then as $\displaystyle\lim_{k\to\infty}\frac{a_{k+1}}{a_k} = 2 > 1$, so $\displaystyle\sum_{k=1}^{\infty}\frac{2^k}{k^x}$ does not converge.