Department of Mathematics
Bennett University
EMAT101L: July-December, 2018
Tutorial Sheet-2 (Multivariable Calculus)

1) Examine

- Continuity of f at $(0,0)$
- Existence of partial derivatives f_x and f_y at $(0,0)$
- Existence of the directional derivatives $D_u f$ at $(0,0)$ along each unit vector u
- Differentiability of f at $(0,0)$

for each of the following functions:

(a)

$$f(x,y) = \begin{cases} \frac{x}{y} & if \quad y \neq 0, \\ 0 & if \quad y = 0. \end{cases}$$

Hint:

- Limit does not exist along $y = mx$ path so f is not continuous at $(0,0)$.
- $f_x(0,0) = f_y(0,0) = 0$.
- Take unit vector $u = (a,b) \neq (0,0)$. Then the directional derivatives

$$D_u f(0,0) = \lim_{t \to 0} \frac{f(at,bt)}{t} = \lim_{t \to 0} \frac{a}{bt}$$

do not exist.

- f is not differentiable at $(0,0)$ as direction derivatives do not exist for $u = (a,b) \neq (0,0)$.

(b)

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & if \quad (x,y) \neq (0,0), \\ 0 & if \quad (x,y) = (0,0). \end{cases}$$

Hint:

- Limit does not exist along $y = mx$ path so f is not continuous at $(0,0)$.
- $f_x(0,0) = f_y(0,0) = 0$.
- Take unit vector $u = (a,b) \neq (0,0)$, then the directional derivatives

$$D_u f(0,0) = \lim_{t \to 0} \frac{f(at,bt)}{t} = \lim_{t \to 0} \frac{ab}{t(a^2+b^2)}$$

do not exist.

- f is not differentiable at $(0,0)$ as direction derivatives do not exist for $u = (a,b) \neq (0,0)$.

(c)

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2} & if \quad (x,y) \neq (0,0), \\ 0 & if \quad (x,y) = (0,0). \end{cases}$$

Hint:

- f is continuous at $(0,0)$. Use definition with $\delta = \epsilon$.
- $f_x(0,0) = f_y(0,0) = 0$.
- Take unit vector $u = (a,b) \neq (0,0)$, then the directional derivatives exist. Indeed,
$$D_u f(0,0) = \lim_{t \to 0} \frac{f(at,bt)}{t} = \frac{ab^2}{a^2+b^2}.$$
- f is not differentiable at $(0,0)$. Prove by contradiction, follow the class notes.

(d)

$$f(x,y) = \begin{cases} \frac{y}{|y|}\sqrt{x^2+y^2} & if \quad y \neq 0, \\ 0 & if \quad y = 0. \end{cases}$$

Hint:

- f is continuous at $(0,0)$. Use definition with $\delta = \epsilon$.
- $f_x(0,0) = 0$ and $f_y(0,0) = 1$.
- Take unit vector $u = (a,b), a \neq 0, b \neq 0$, then Directional derivatives
$$D_u f(0,0) = \frac{b}{|b|}.$$
- f is not differentiable at $(0,0)$. Prove by contradiction, follow the class notes.

2) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\frac{1}{x^2+y^2} & if \quad (x,y) \neq (0,0), \\ 0 & if \quad (x,y) = (0,0). \end{cases}$$

(i) Show that f is continuous. (ii) Find f_x and f_y at every $(x,y) \in \mathbb{R}^2$. (iii) Show that the partial derivatives of f are not bounded in any disc (howsoever small) around $(0,0)$. (iv) Examine the differentiability at every point $(x,y) \in \mathbb{R}^2$.

Hint:

- f is continuous at $(0,0)$. Use definition with $\delta = \sqrt{\epsilon}$.
- By definition: $f_x(0,0) = 0 = f_y(0,0)$. For $(x,y) \neq (0,0)$, we have
$$f_x(x,y) = 2x(\sin\frac{1}{x^2+y^2} - \frac{1}{x^2+y^2}\cos\frac{1}{x^2+y^2}).$$
$$f_y(x,y) = 2y(\sin\frac{1}{x^2+y^2} - \frac{1}{x^2+y^2}\cos\frac{1}{x^2+y^2}).$$

- Clearly, because of the second term in f_x and f_y, they are unbounded in every neighborhood of $(0, 0)$.
- (Sufficient condition for differentiability: If partial derivatives exist and continuous at a point, then f is differentiable at that point). f is differentiable everywhere. Indeed, for $(x_0, y_0) \neq (0, 0)$ use the sufficient condition and for $(x_0, y_0) = (0, 0)$ use the definition (follow the class notes).

3) Find the directional derivative of $f(x, y) = y^3 - 2x^2 + 3$ at the point $(1, 2)$ in the direction of $u = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Also, find the directional derivative of $f(x, y) = log(x^2 + y^2)$ at $(1, -3)$ in the direction of $v = (2, -3)$.
Hint: By using definition:

$$D_u f(1, 2) = 6\sqrt{3} - 2 \qquad \text{and} \quad D_v f(1, -3) = \frac{11}{5\sqrt{13}}.$$

4) Find the directional derivative of $f(x, y) = x^2 - 3xy$ along the parabola $y = x^2 - x + 2$ (That is, in the parametric form $x(t) = t$ and $y(t) = t^2 - t + 2$) at the point $(1, 2)$. (Note: When a direction is given in terms of a curve, then one must take the direction as the (unit) tangent vector to the curve at that point).
Hint: Here $r(t) = (x(t), y(t)) = (t, t^2 - t + 2) \Rightarrow r'(t) = (1, 2t - 1)$. By using definition:

$$D_u f(1, 2) = -\frac{7}{\sqrt{2}}.$$

5) A golf ball leaves the ground at a $30°$ angle at a speed of $90ft/sec$. Will it clear the top of a $30ft$ tree $135ft$ away? (All launch angles are assumed to be measured from the horizontal. All projectiles are assumed to be fired from the origin over horizontal ground, unless stated otherwise).
Hint: Position of the ball at time t at angle θ with speed v_0 in xy-plane is given by

$$x(t) = (v_0 \cos\theta)t \quad \text{and} \qquad y(t) = (v_0 \sin\theta)t - \frac{gt^2}{2}.$$

When $x = 135ft$, calculate time t, i.e., $135 = 90\cos(30°)t \Rightarrow t \approx 1.732$, thus at time $t \approx 1.732$, ball is at $135ft$ away. Now at this position calculate y (height from the ground),i.e., $y \approx 29.94$, which is less than 30. So the ball will not clear the top of the tree.

6) An object in a space has initial position $\overrightarrow{R_0} = x_0\hat{\imath} + y_0\hat{\jmath} + z_0\hat{k}$ and initial velocity $\overrightarrow{V}_0$ and undergoes a constant acceleration $-g\hat{k}$. Show that the position of the object at any time t is given by $\overrightarrow{R(t)} = -\frac{gt^2}{2}\hat{k} + t\overrightarrow{V_0} + \overrightarrow{R_0}$. (Note that an elementary application of this problem is the motion of an object that remains near some point P on the earth's surface and moves only under the influence of the earth's gravity).
Hint: We know,

$$\overrightarrow{a(t)} = -g\hat{k} = \frac{dV}{dt} = \frac{d^2R}{dt^2}$$

so by successive integration yields the position of the object at time t by using initial conditions $V(0) = V_0$ and $R(0) = R_0$,

$$\overrightarrow{R(t)} = (x(t), y(t), z(t)) = -\frac{gt^2}{2}\hat{k} + t\overrightarrow{V_0} + \overrightarrow{R_0}$$

7) Consider a coordinate system so that the xy-plane represents the ground and a player is standing at origin. A ball is hit 4 feet above the ground at $100 ft/sec$ and at an angle of $\frac{\pi}{6}$ with respect to the ground by the player. After the hit, the ball travels in the yz-plane only (under the influence of the earth's gravity $g \approx 32 ft/sec$). How long does it take for the ball to hit the ground?
Hint: Given: Initial position $r_0 = 4\hat{k}$, Initial speed $||v_0|| = 100$ and initial angel $\theta = \frac{\pi}{6}$. In yz-plane, unit vector along the ray $\theta = \frac{\pi}{6}$, $\cos(\frac{\pi}{6})\hat{j} + \sin(\frac{\pi}{6})\hat{k}$. Therefore initial velocity

$$v_0 = 100\left(\cos(\frac{\pi}{6})\hat{j} + \sin(\frac{\pi}{6})\hat{k}\right) = 50\left(\sqrt{3}\hat{j} + \hat{k}\right).$$

From the previous exercise we know that position of the ball is given by $\overrightarrow{R(t)} = -\frac{gt^2}{2}\hat{k} + t\overrightarrow{V_0} + \overrightarrow{R_0}$. Thus,

$$\overrightarrow{R(t)} = -\frac{gt^2}{2}\hat{k} + 50\left(\sqrt{3}\hat{j} + \hat{k}\right)t + 4\hat{k}.$$

$$\overrightarrow{R(t)} = 50\sqrt{3}t\hat{j} + (4 + 50t - \frac{gt^2}{2})\hat{k}$$

When the ball hits the ground, the $\hat{k}$ component of $R(t)$ must be 0. This happens at the time $t > 0$ such that $4 + 50t - \frac{gt^2}{2} = 0$, taking $g = 32$, we get $t = 3.2$. So the ball hits the ground after approximately 3.2 seconds.