Multivariable Calculus

(Lecture-12 & 13)

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Multiple Integration

of

(Scalar Valued Function of Vector Variable)

(Scalar Field)

$$F : R \subseteq \mathbb{R}^n \to \mathbb{R}, \; n = 2, 3$$

(Continuation....)

Learning Outcome of this lecture

In the last lectures, we have learnt double integral over rectangular/non-rectangular region.

In this lecture, we learn double integrals over simple and bounded region $\mathcal{R}$ with the help of change of Variables.

- Change of order of integration
- Applications of Double Integrals
- Change of Variables
  - Polar coordinates
  - General Transformation

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Change of order of integration

Sometimes changing the order of integration makes computation much easier.
Example: Compute $I = \int_0^1 \left(\int_{\sqrt{x}}^1 \sqrt{1+y^3} dy \right) dx$?
Rewrite the integral as $I = \int_{x=0}^{x=1} \left(\int_{y=\sqrt{x}}^{y=1} \sqrt{1+y^3} dy \right) dx$
Draw the region of integration:
Note that the given integration is not easy to compute with respect to y as it involves the square root in y. Thus change the order of integration, that means first with respect to x and then y.
Carefully find the limits in x and change the order of limits in integration.
We get the following after changing the order of integration.

$$I = \int_{x=0}^{x=1} \left(\int_{y=\sqrt{x}}^{y=1} \sqrt{1+y^3} dy \right) dx = \int_{y=0}^{y=1} \left(\int_{x=0}^{x=y^2} \sqrt{1+y^3} dx \right) dy.$$

Common Applications of Double Integrals

- Area of a closed, bounded plane region $\mathcal{R}$ is given by

$$\text{Area} = \iint_{\mathcal{R}} dA$$

- $$\text{Average value of } f \text{ over } \mathcal{R} = \left(\frac{1}{\text{Area of } \mathcal{R}}\right) \iint_{\mathcal{R}} f \, dA$$

- If $f(x, y) \geq 0$ for all $(x, y) \in \mathcal{R}$ where $\mathcal{R}$ is a simple region in $\mathbb{R}^2$ then the volume V of the solid region bounded above by the surface $z = f(x, y)$ and below by the the region $\mathcal{R}$ in the xy-plane is given by

$$\text{Volume} = \iint_{\mathcal{R}} f \, dA$$

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Examples

Let $\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \ : \ 0 \leq x \leq 1 \text{ and } x^2 \leq y \leq x\}$.
Find the area of the region $\mathcal{R}$.

Answer:

$$\int_{x=0}^{1} \int_{y=x^2}^{x} dA = \frac{1}{6}.$$

Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.

Answer:

$$\int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - y) dA = 16\pi.$$

Change of Variables

in

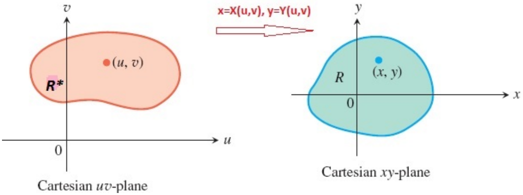
Double Integrals

Transforming Double Integrals from One System to Another System

Change of Variables $(x, y) \to (u, v)$

Suppose that a region $\mathcal{R}^*$ in the uv-plane transformed one-to-one into the region $\mathcal{R}$ in the xy-plane by equations

$$x = X(u, v) \quad \text{and} \quad y = Y(u, v).$$



Then $f(x, y)$ defined on $\mathcal{R}$ can be thought of as a function $f(X(u, v), Y(u, v))$ on $\mathcal{R}^*$.

Question: How is the integral of $f(x, y)$ over $\mathcal{R}$ related to the integral of $f(X(u, v), Y(u, v))$ over $\mathcal{R}^*$?

Continuation of previous slide

Question: How is the integral of $f(x,y)$ over $\mathcal{R}$ related to the integral of $f(X(u,v),Y(u,v))$ over $\mathcal{R}^*$?

Answer: If X,Y and f have continuous partial derivatives and the "Jacobian" $J(u,v)$ is nonzero for all $(u,v) \in \mathcal{R}^*$, where

$$J = \frac{\partial(X,Y)}{\partial(u,v)} = \begin{bmatrix} \frac{\partial X}{\partial u} & \frac{\partial X}{\partial v} \\ \frac{\partial Y}{\partial u} & \frac{\partial Y}{\partial v} \end{bmatrix}.$$

Then the formula for transforming double integrals over the region $\mathcal{R}$ into double integrals over the region $\mathcal{R}^*$ can be written as

$$\iint_{\mathcal{R}} f(x,y)\,dxdy = \iint_{\mathcal{R}^*} f(X(u,v),Y(u,v)).|J(u,v)|\,du\,dv.$$

Example: Transforming into Double Integrals in Polar Coordinates

$$xy - \text{plane} \;\rightarrow\; r\theta - \text{plane}$$

In polar coordinates,

$$x = X(r,\theta) = r\cos\theta \quad y = Y(r,\theta) = r\sin\theta.$$

$$|J| = \left|\frac{\partial(X,Y)}{\partial(r,\theta)}\right| = \left|\begin{bmatrix} \frac{\partial X}{\partial r} & \frac{\partial X}{\partial \theta} \\ \frac{\partial Y}{\partial r} & \frac{\partial Y}{\partial \theta} \end{bmatrix}\right| = \left|\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}\right| = r.$$

Then

$$\iint_{\mathcal{R}} f(x,y)\,dxdy = \iint_{\mathcal{R}^*} f(r\cos\theta, r\sin\theta))\,r\,dr\,d\theta.$$

Example: Double Integral in Polar Coordinates

Find the area enclosed by the circle $x^2 + y^2 = a^2$ where $a > 0$ by converting into polar coordinates.

$$\begin{aligned}
\iint_{\mathcal{R}} f &= \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\, dx \\
&= \int_{r=0}^{a} \int_{\theta=0}^{2\pi} r\, dr\, d\theta \\
&= \int_{r=0}^{a} \left(\int_{\theta=0}^{2\pi} d\theta \right) r\, dr \\
&= \int_{r=0}^{a} 2\pi\, r\, dr \\
&= \pi a^2
\end{aligned}$$

Example: Change of Variable - General Transformation

Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, (a > 0, b > 0)$ by applying the simple transformation $x = au$ and $y = bv$.

Answer: Here $x = X(u, v) = au$ and $y = Y(u, v) = bv$.

Now, we need to calculate J using $X(u, v)$ and $Y(u, v)$:

$$J = \begin{vmatrix} \frac{\partial X}{\partial u} & \frac{\partial X}{\partial v} \\ \frac{\partial Y}{\partial u} & \frac{\partial Y}{\partial v} \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab.$$

Using the given transformation the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, (a > 0, b > 0)$ gets converted to unit circle $u^2 + v^2 = 1$. Thus,

$$Area = \iint_{\mathcal{R}} dx\, dy$$

$$= \iint_{\mathcal{R}^*} f(X(u, v), Y(u, v)).|J(u, v)|\, du\, dv.$$

$$= ab \iint_{\mathcal{R}^*} du\, dv = \pi ab.$$

Example

Using the transformation $u = 2x + 3y$ and $v = x - 3y$, find the value of the integral $\iint_{\mathcal{R}} \exp^{2x+3y} \cos(x - 3y)\, dx\, dy$, where $\mathcal{R}$ is the region bounded by the parallelogram with vertices $(0, 0)$, $(1, 1/3)$, $(4/3, 1/9)$, $(1/3, -2/9)$.

Answer: Under the given transformations, $u = 2x + 3y$ and $v = x - 3y$, i.e.,

$$x = X(u, v) = \frac{1}{3}(u + v) \quad \text{and} \quad y = Y(u, v) = \frac{1}{9}(u - 2v).$$

Now, we need to calculate J using $X(u, v)$ and $Y(u, v)$:

$$J = \begin{vmatrix} \frac{\partial X}{\partial u} & \frac{\partial X}{\partial v} \\ \frac{\partial Y}{\partial u} & \frac{\partial Y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{9} & -\frac{2}{9} \end{vmatrix} = -\frac{1}{9} \Rightarrow |J| = \frac{1}{9}.$$

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Example Cont...

$\mathcal{R}$ will be transformed into the rectangle $\mathcal{R}^*$ with vertices $(0,0)$, $(3,0)$, $(3,1)$ and $(0,1)$. Thus,

$$I = \iint_{\mathcal{R}} e^{2x+3y} \cos(x-3y)\, dx\, dy$$

$$= \iint_{\mathcal{R}^*} f(X(u,v), Y(u,v)).|J(u,v)|\, du\, dv.$$

$$= \int_{v=0}^{1} \int_{u=0}^{3} e^{u} \cos v du dv = \frac{1}{9} \sin 1(e^3 - 1).$$