Multivariable Calculus

(Lecture-7)

Department of Mathematics
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Differentiation

of

Scalar Valued Function of Vector Variable

(Scalar Field)

$$F : S \subseteq \mathbb{R}^2 \to \mathbb{R}$$

Learning Outcome of the lecture

In this lecture, We learn Differentiation of $F : S \subseteq \mathbb{R}^2 \to \mathbb{R}$, where S is an open set of $\mathbb{R}^2$.

- Directional Derivatives
- Directional Derivatives versus Continuity

Directional Derivatives of $F : S \subseteq \mathbb{R}^2 \to \mathbb{R}$

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Definition

The directional derivative of F at the point $X_0 = (x_0, y_0)$ in the direction of u is defined by

$$(D_u F)(X_0) := \lim_{t \to 0} \frac{F(X_0 + tu) - F(X_0)}{t}$$

provided the limit exists.

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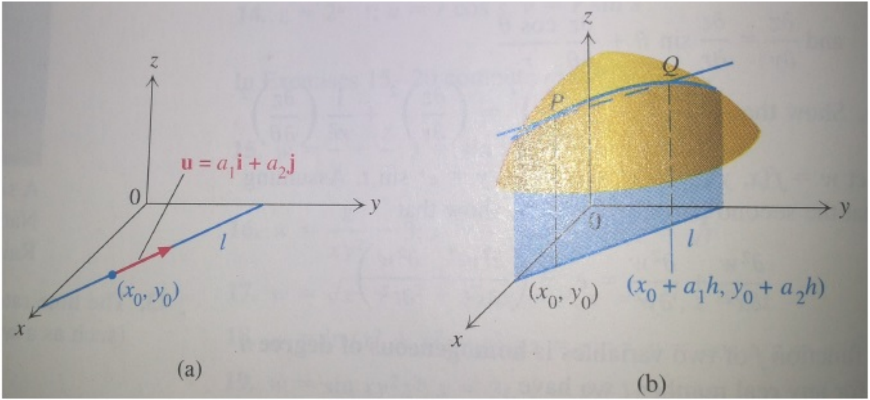
$(D_u F)(X_0) =$ Rate of change of F at X_0 in the direction of u.

Note that $\frac{\partial F}{\partial x_i}(X_0) = (D_{e_i} F)(X_0)$.

Picture explaining of $(D_uF)(X_0)$ where $F : S \subseteq \mathbb{R}^2 \to \mathbb{R}$



Example-1

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Question: Find the directional derivative of F at the point $X_0 = (1, 2)$ in the direction of unit vector $u = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$.

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Solution:

$$
\begin{aligned}
(D_u F)(X_0) &= \lim_{t \to 0} \frac{F(X_0 + tu) - F(X_0)}{t} \\
&= \lim_{t \to 0} \frac{F(1 + t\frac{\sqrt{3}}{2}, 2 + \frac{t}{2}) - F(1, 2)}{t} \\
&= \lim_{t \to 0} \frac{\left(1 + t\frac{\sqrt{3}}{2}\right)\left(2 + \frac{t}{2}\right) - 2}{t} \\
&= \lim_{t \to 0} \frac{1}{2} + \sqrt{3} + t\frac{\sqrt{3}}{4} \\
&= \frac{1}{2} + \sqrt{3}
\end{aligned}
$$

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Example-2: $D_u F(0)$ exists for all u , but F is not continuous at 0

Let $F(x,y) = \frac{xy^2}{x^2+y^4}$ for $(x,y) \neq (0,0)$ and $F(0,0) = 0$. Let $u = (a,b)$ be an unit vector in $\mathbb{R}^2$.

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Question: Compute $(D_uF)(0,0)$. Also examine the continuity of F at $(0,0)$.

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$$(D_uF)(0,0) = \lim_{t \to 0} \frac{F((0,0) + t(a,b)) - F(0,0)}{t}$$

$$= \lim_{t \to 0} \frac{F(at,bt) - F(0,0)}{t}$$

$$(D_uF)(0,0) = \begin{cases} \frac{b^2}{a}, & \text{if} \quad a \neq 0 \\ 0, & \text{if} \quad a = 0. \end{cases}$$

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The function F is not continuous at 0 (Hint: Path $x = ky^2$ where $k > 0$).

Example-3: Partial Derivatives Exist, but Directional Derivative along other directions do not exist

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Example-3: Partial Derivatives Exist, but Directional Derivative along other directions do not exist

Let $F(x,y) = \frac{xy}{x^2+y^2}$ for $(x,y) \neq (0,0)$ and $F(0,0) = 0$.

Question: Show that $D_1F(0,0)$ and $D_2F(0,0)$ exist and equal to 0.

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Question: Show that $D_1F(0,0)$ and $D_2F(0,0)$ exist and equal to 0. Let $u = (a,b)$ be an unit vector in $\mathbb{R}^2$ with $a \neq 0$ and $b \neq 0$. Show that $D_uF(0,0)$ does not exist.

$$D_1F(0,0) = \frac{\partial F}{\partial x}(0,0) = \lim_{h\to 0} \frac{F(h,0) - F(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

$$D_2F(0,0) = \frac{\partial F}{\partial y}(0,0) = \lim_{k\to 0} \frac{F(0,k) - F(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0.$$

$$D_uF(0,0) = \lim_{t\to 0} \frac{F((0,0)+t(a,b)) - F(0,0)}{t} = \lim_{t\to 0} \frac{F(at,bt)}{t}$$

$$= \lim_{t\to 0} \frac{ab}{t(a^2+b^2)} \text{ does not exist.}$$