Quiz - 2.

(1) Is $W = \{z \in \mathbb{C} : z = \bar{z}\}$ a subspace of $\mathbb{C}$ over $\mathbb{C}$? what can you say about W over $\mathbb{R}$.

Solution :- let $z_1, z_2 \in W$. Then $z_1, z_2 \in \mathbb{C}$ & $\bar{z}_1 = z_1$
$\bar{z}_2 = z_2$

To show: $z_1 + z_2 \in W$.

As $z_1, z_2 \in \mathbb{C} \Rightarrow z_1 + z_2 \in \mathbb{C}$.

$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2 = z_1 + z_2 \Rightarrow \overline{z_1 + z_2} = z_1 + z_2$.

$\Rightarrow z_1 + z_2 \in W$

To show:- W is closed under scalar multiplication over $\mathbb{R}$.

Let $\alpha \in \mathbb{R}$ & $z \in W \Rightarrow z \in \mathbb{C}$, $\bar{z} = z$.

Then $\alpha z \in \mathbb{C}$ & $\overline{\alpha z} = \bar{\alpha}\bar{z} = \alpha z$ $(\because \alpha \in \mathbb{R} \Rightarrow \bar{\alpha} = \alpha)$

Thus, W is a vector space over $\mathbb{R}$.

But if $\alpha = i \in \mathbb{C}$ Then $\overline{iz} = \bar{i}\bar{z} = -iz \neq iz$.

$\therefore$ W is not a vector space over $\mathbb{R}$.

(2) $\mathbb{C}^2(\mathbb{R}) = \{(a+ib, c+id) : a, b, c, d \in \mathbb{R}\}$

$= \{a(1,0) + b(i,0) + c(0,1) + d(0,i) : a, b, c, d \in \mathbb{R}\}$

$= \text{span}\{\underbrace{(1,0)}_{v_1}, \underbrace{(i,0)}_{v_2}, \underbrace{(0,1)}_{v_3}, \underbrace{(0,i)}_{v_4}\}$

It is easy to see that $S = \{v_1, v_2, v_3, v_4\}$ is l.i. Therefore

ther set $S = \{v_1, v_2, v_3, v_4\}$ form a basis of $\mathbb{C}^2(\mathbb{R})$

$\dim(\mathbb{C}^2(\mathbb{R})) = 4$.

$$\mathbb{C}^2(\mathbb{C}) = \{(a+ib,\ c+id)\}$$

$$= \{(a+ib)(1,0) + (c+id)(0,1)\}$$

$$= \text{span}\{\underbrace{(1,0)}_{v_1},\ \underbrace{(0,1)}_{v_2}\}$$

The vectors $\{v_1, v_2\}$ are l.i. therefore form a basis.

$\dim \mathbb{C}^2(\mathbb{C}) = 2$.

(3) Let $W = \{(x,y,z,w) \in \mathbb{R}^4 : x+y=0,\ w+z=0\}$. Find $\dim W$ and basis. Extend the basis of W to $\mathbb{R}^4$.

Solution :-

$$W = \{(x,y,z,w) \in \mathbb{R}^4 : x+y=0,\ w+z=0\}$$

$$= \{(x,y,z,w) \in \mathbb{R}^4 : x=-y,\ w=-z\}$$

$$= \{(-y,y,+z,-z) \in \mathbb{R}^4 : y,z \in \mathbb{R}\}$$

$$= \{y(-1,1,0,0) + z(0,0,1,-1)\}$$

$$= \text{span}\{\underbrace{(-1,1,0,0)}_{v_1},\ \underbrace{(0,0,1,-1)}_{v_2}\}$$

W is a subspace of $\mathbb{R}^4$ as span (W) is the smallest subspace of $\mathbb{R}^4$ containing W.

Also $\{v_1, v_2\}$ are linearly independent. Therefore, form a basis of W.

$\dim W = 2$.

As, we know that $\dim \mathbb{R}^4 = 4$.

So, we need to add two more vectors v_3, v_4.

$$\begin{array}{c} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \to v_3 \\ \to v_4 \end{array}$$

Thus $\{v_1, v_2, v_3, v_4\}$ form a basis for $\mathbb{R}^4$.