Taylor's Theorem, Taylor Series and Power Series
(Lecture 17 & 18)

Engineering Calculus

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- If a function f has an *nth* derivative at a point a, then we can construct an *nth* degree polynomial P_n such that $P_n(a) = f(a)$ and $P_n^{(k)}(a) = f^{(k)}(a)$ for $k = 0, 1, 2, \cdots, n$.. The polynomial is

$$P_n(x) := \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

This polynomial P_n is called the *nth* **Taylor polynomial for** f **at** a. Then $f(x) = P_n(x) + R_n(x)$ in a neighbourhood of a, where R_n is the reminder. Also, we expect $R_n(x) \to 0$ as $x \to a$.

Theorem

Let $f : I \to \mathbb{R}$ be such that f and its derivatives of order m are continuous on I and $f^{(m+1)}(x)$ exists in a neighbourhood of $x = a \in I$. Then for any x in I there exists a point $c \in (a, x)$ (or $c \in (x, a)$) such that

$$f(x) = f(a) + f'(a)(x-a) + ... + f^{(m)}(a)\frac{(x-a)^m}{m!} + R_m(x),$$

where $R_m(x) = \dfrac{f^{(m+1)}(c)}{(m+1)!}(x-a)^{m+1}$.

Proof: Define the functions F and g as

$$F(y) = f(x) - f(y) - f'(y)(x-y) - .... - \frac{f^{(m)}(y)}{m!}(x-y)^m,$$

$$g(y) = F(y) - \left(\frac{x-y}{x-a}\right)^{m+1} F(a).$$

Then $g(a) = 0$. Also $g(x) = F(x) = f(x) - f(x) = 0$. Therefore, by Rolle's theorem, there exists some $c \in (a, x)$ such that

$$g'(c) = 0 = F'(c) + \frac{(m+1)(x-c)^m}{(x-a)^{m+1}} F(a).$$

On the other hand, from the definition of F,

$$F'(c) = -\frac{f^{(m+1)}(c)}{m!}(x-c)^m.$$

Hence $F(a) = \dfrac{(x-a)^{m+1}}{(m+1)!} f^{(m+1)}(c)$ and the result follows.

Examples

(i) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!}e^c, c \in (0,x)$ or $(x,0)$ depending on the sign of x.

(ii) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + ... + \frac{x^n}{n!}\sin(c + \frac{n\pi}{2}), c \in (0,x)$ or $(x,0)$.

(iii) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ... + \frac{x^n}{n!}\cos(c + \frac{n\pi}{2}), c \in (0,x)$ or $(x,0)$.

Problem

Find the order n of Taylor Polynomial P_n, about $x = 0$ to approximate e^x in $(-1, 1)$ so that the error is not more than 0.005

Solution: We know that $p_n(x) = 1 + x + ... + \frac{x^n}{n!}$. The maximum error in [-1,1] is

$$|R_n(x)| \leq \frac{1}{(n+1)!} \max_{[-1,1]} |x|^{n+1} e^x \leq \frac{e}{(n+1)!}.$$

So n is such that $\frac{e}{(n+1)!} \leq 0.005$ or $n \geq 5$.

Problem

Find the interval of validity when we approximate $\cos x$ with 2nd order polynomial with error tolerance 10^{-4}.

Solution: Taylor polynomial of degree 2 for $\cos x$ is $1 - \frac{x^2}{2}$. So the remainder is $(\sin c)\frac{x^3}{3!}$. Since $|\sin c| \leq 1$, the error will be at most 10^{-4} if $|\frac{x^3}{3!}| \leq 10^{-4}$. Solving this gives $|x| < 0.084$.

Taylor's series

Suppose f is infinitely differentiable at a and if the remainder term in the Taylor's formula, $R_n(x) \to 0$ as $n \to \infty$. Then we write

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

This series is called Taylor series of $f(x)$ about the point a.

- Suppose there exists $C = C(x) > 0$, independent of n, such that $|f^{(n)}(x)| \leq C(x)$. Then $|R_n(x)| \to 0$ if $\lim_{n\to\infty} \frac{|x-a|^{n+1}}{(n+1)!} = 0$. Using **Ratio test**, one can show that $\lim_{n\to\infty} \frac{|x-a|^{n+1}}{(n+1)!} = 0$.

Remark

If $a = 0$, the formula obtained in Taylor's theorem is known as *Maclaurin's formula* and the corresponding series that one obtains is known as *Maclaurin's series.*

Examples

(i) $f(x) = e^x$.

In this case $R_n(x) = \dfrac{x^{n+1}}{(n+1)!} f^{(n+1)}(c) = \dfrac{x^{n+1}}{(n+1)!} e^c = \dfrac{x^{n+1}}{(n+1)!} e^{\theta x}$, for some $\theta \in (0, 1)$.

Therefore for any given x fixed, $\lim\limits_{n\to\infty} |R_n(x)| = \lim\limits_{n\to\infty} \left(\dfrac{x^{n+1}}{(n+1)!} \right) e^{\theta x} = 0$.

(ii) $f(x) = \sin x$.

Here $|R_n(x)| \leq \dfrac{|x|^{2n+1}}{(2n+1)!} \left| \sin(c + \dfrac{n\pi}{2}) \right|$. Now use the fact that $|\sin x| \leq 1$ and follow as in (i).

Maxima and Minima: Derivative test

- A point $x = a$ is called critical point of the function $f(x)$ if $f'(a) = 0$.
- A point $x = a$ is a local maxima if $f'(a) = 0$ and $f''(a) < 0$.
- Suppose $f(x)$ is continuously differentiable in an interval around $x = a$ and let $x = a$ be a critical point of f. Then $f'(a) = 0$. By Taylor's theorem around $x = a$, there exists $c \in (a, x)$ (or $c \in (x, a)$) such that

$$f(x) - f(a) = \frac{f''(c)}{2}(x-a)^2.$$

  If $f''(a) < 0$. Then $f''(c) < 0$ in $|x - a| < \delta$ as $f''(x)$ is continuous at $x = a$. Hence $f(x) < f(a)$ in $|x - a| < \delta$, which implies that $x = a$ is a local maximum.
- Similarly, a point $x = a$ is a local minima if $f'(a) = 0,\ f''(a) > 0$.
- Also the above observations show that if $f'(a) = 0, f''(a) = 0$ and $f^{(3)}(a) \neq 0$, then the sign of $f(x) - f(a)$ depends on $(x-a)^3$. i.e., it has no constant sign in any interval containing a. Such point is called point of inflection or saddle point.
- If $f'(a) = f''(a) = f^{(3)}(a) = 0$, then we again have $x = a$ is a local minima if $f^{(4)}(a) > 0$ and is a local maxima if $f^{(4)}(a) < 0$.

Theorem

Let f be a real valued function that is differentiable $2n$ times and $f^{(2n)}$ is continuous at $x = a$. Then

(a) If $f^{(k)}(a) = 0$ for $k = 1, 2, \cdots, 2n-1$ and $f^{(2n)}(a) > 0$ then a is a point of local minimum of $f(x)$.

(b) If $f^{(k)}(a) = 0$ for $k = 1, 2, \cdots, 2n-1$ and $f^{(2n)}(a) < 0$ then a is a point local maximum of $f(x)$.

(c) If $f^{(k)} = 0$ for $k = 1, 2, \cdots, 2n-2$ and $f^{(2n-1)}(a) \neq 0$, then a is point of inflection. i.e., f has neither local maxima nor local minima at $x = a$.

• Suppose $f(x)$, $g(x)$ are differentiable n times, and $f^{(n)}, g^{(n)}$ are continuous at a and $f^{(k)}(a) = g^{(k)}(a) = 0$ for $k = 0, 1, 2, \cdots, n-1$. Also if $g^{(n)}(a) \neq 0$. Then by Taylor's theorem,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f^{(n)}(x)}{g^{(n)}(x)} = \frac{f^{(n)}(a)}{g^{(n)}(a)}$$

Similarly, we can derive a formula for limits as x approaches infinity by taking $x = \frac{1}{y}$.

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{y \to 0} \frac{f(1/y)}{g(1/y)} = \lim_{y \to 0} \frac{(-1/y^2)f'(1/y)}{(-1/y^2)g'(1/y)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$$

- Given a sequence of real numbers $\{a_n\}_{n=0}^{\infty}$, the series $\sum_{n=0}^{\infty} a_n(x-c)^n$ is called **power series** with center c. The series converges for $x = c$.
- Power series is a function of x provided it converges for x. If a power series converges, then the domain of convergence is either a bounded interval or the whole of $\mathbb{R}$.
- The translation $x' = x - c$ reduces a power series around c to a power series around 0.
- Consider the series for $c = 0$, i.e., the power series around 0 of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \cdots + a_n x^n + \cdots . \tag{1}$$

Even though the functions appearing in (1) are defined over all of $\mathbb{R}$, it is not to be expected that the series (1) will converge for all x in $\mathbb{R}$. For example, the series

$$\sum_{n=0}^{\infty} n! x^n, \quad \sum_{n=0}^{\infty} x^n, \quad \sum_{n=0}^{\infty} x^n/n!,$$

converge for x in the sets

$$\{0\}, \quad \{x \in \mathbb{R} : |x| < 1\}, \quad \mathbb{R}, \quad \text{respectively.}$$

Theorem

If $\sum a_n x^n$ converges at $x = r$, then $\sum a_n x^n$ converges for $|x| < |r|$.

Proof: We can find $C > 0$ such that $|a_n r^n| \le C$ for all n. Then

$$|a_n x^n| \le |a_n r^n| |\frac{x}{r}|^n \le C |\frac{x}{r}|^n.$$

Conclusion follows from comparison theorem.

Theorem

If $\sum a_n x^n$ diverges at $x = r$, then $\sum a_n x^n$ diverges for $|x| > |r|$.

Theorem

If a power series $\sum_{n=0}^{\infty} a_n x^n$ be neither nowhere convergent nor everywhere convergent, then there exists a positive real number R such that the series converges absolutely for all real x satisfying $|x| < R$ and diverges for all x satisfying $|x| > R$.

- The real number R in the above theorems is called the **radius of convergence** of the power series. The interval $(-R, R)$ is called the **interval of convergence** of the power series.

Theorem

Consider the power series $\sum_{n=0}^{\infty} a_n x^n$. Suppose $\beta = \limsup \sqrt[n]{|a_n|}$ and $R = \frac{1}{\beta}$ (we define $R = 0$ if $\beta = \infty$ and $R = \infty$ if $\beta = 0$). Then

1. $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < R$.
2. $\sum_{n=0}^{\infty} a_n x^n$ diverges for $|x| > R$.
3. No conclusion if $|x| = R$.

Theorem

Consider the power series $\sum_{n=0}^{\infty} a_n x^n$. Suppose $\beta = \limsup \left|\frac{a_{n+1}}{a_n}\right|$ and $R = \frac{1}{\beta}$. Then

1. $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < R$.
2. $\sum_{n=0}^{\infty} a_n x^n$ diverges for $|x| > R$.
3. No conclusion if $|x| = R$.

Examples

Find the interval of convergence of $(i)\ \sum \frac{x^n}{n}$, $(ii)\ \sum \frac{x^n}{n!}$, $(iii)\ \sum 2^{-n}x^{3n}$.

(i) $\beta = \limsup |\frac{a_{n+1}}{a_n}| = 1$, and we know that the series does not converge for $x = 1$. So the interval of convergence is $[-1, 1)$.

(ii) $\beta = \limsup |\frac{a_{n+1}}{a_n}| = 0$. Hence the series converges everywhere.

(iii) To see the subsequent non-zero terms, we write the series as $\sum 2^{-n}(x^3)^n = \sum 2^{-n}y^n$. For this series $\beta_y = \limsup \sqrt[n]{|a_n|} = 2^{-1}$. Therefore, $\beta_x = 2^{-1/3}$ and $R = 2^{1/3}$.

Theorem (Term by term differentiation and integration)

Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < R$. Then

1. $\sum_{n=0}^{\infty} n a_n x^{n-1}$ converges in $|x| < R$ and is equal to $f'(x)$.
2. $\sum_{n=0}^{\infty} \frac{1}{n+1} a_n x^{n+1}$ converges in $|x| < R$ and is equal to $\int f(x)dx$.

- A power series is infinitely differentiable with in its radius of convergence.

Results

(a) Let R be the radius of convergence of $\sum a_n x^n$ and let K be a closed and bounded interval contained in the interval of convergence $(-R, R)$. Then the power series converges uniformly on K.

(b) The limit of a power series is continuous on the interval of convergence. A power series can be integrated term-by-term over any closed and bounded interval contained in the interval of convergence.

(c) A power series can be differentiated term-by-term within the interval of convergence.

(d) Let $\sum\limits_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R(> 0)$ and $f(x)$ be the sum of the series on $(-R, R)$. Then $f^k(0) = k! a_k,\ k = 0, 1, 2, \cdots$.

(e) Every power series $\sum\limits_{n=0}^{\infty} a_n x^n$ with radius of convergence $R(> 0)$ is the Taylor's series about 0 of its sum function f.

Question

If a function f, having derivatives of all orders on some neighbourhood $N(0)$ of 0, be chosen first and the Taylor's series $\sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$ be constructed, does this power series will have f as its sum function on $N(0)$?

- The answer is **No**.

Example

Consider the function

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Then $f^n(0) = 0$ for $n = 0, 1, 2, \cdots$. The Taylor's series of f about 0 is $0 + 0 + 0 + \cdots$ and this converges to 0, and not to f, on $N(0)$.

Remark

If a series is convergent at an endpoint, then the differentiated series may or may not be convergent at this point. For example, the series $\sum_{n=1}^{\infty} x^n/n^2$ converges at both endpoints $x = 1, -1$. However, the differentiated series given by $\sum_{n=1}^{\infty} x^{n-1}/n$ converges at $x = -1$ but diverges at $x = 1$.

Theorem

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R(> 0)$. If the series converges at the end point R of the interval of convergence $(-R, R)$, then the series is uniformly convergent on the closed interval $[0, R]$.

Result

We write $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$ for $|x - c| < R$ if and only if the sequence $\{R_n(x)\}$ of remainders converges to 0 for each x in some interval $\{x : |x - c| < R\}$. In this case, the power series is the Taylor expansion of f at c.

Example

Find the Taylor series of $f(x) = \tan^{-1} x$ and a domain of its convergence.
Solution:

$$\tan^{-1} x = \int \frac{dx}{1+x^2} = \int 1 - x^2 + x^4 - \cdots$$
$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

Taking $x = 1$ we get

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \tan^{-1}(1) = \frac{\pi}{4}.$$

Though the function $\tan^{-1} x$ is defined on all of $\mathbb{R}$, we see that the power series converges on $(-1, 1)$. Also, the series converges at $x = 1, -1$.

Thank You