EPHY105L (Fall Semester 2018-2019)
Solutions to Problem Sheet 3

1. (a) Potential: $V = \frac{5}{r^2} cos\theta$

$$\vec{\boldsymbol{E}} = -\nabla V = -\left(\frac{\partial V}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\widehat{\boldsymbol{\theta}} + \frac{1}{r sin\,\theta}\frac{\partial V}{\partial \phi}\widehat{\boldsymbol{\phi}}\right)$$

$$= \frac{5}{r^3}\left(2cos\theta\,\hat{\boldsymbol{r}} + sin\theta\,\widehat{\boldsymbol{\theta}}\right) \qquad (2)$$

At r = 2, θ =π/2,

$$\vec{\boldsymbol{E}} = \frac{5}{8}\widehat{\boldsymbol{\theta}}$$

2. Inside the conductor we must have $\vec{\boldsymbol{E}} = 0$. Let the surface charge density be σ. Then the field due to the surface charge density within the conductor will be

$$\vec{\boldsymbol{E}} = -\frac{\sigma}{\epsilon_0}\hat{\boldsymbol{z}}. \qquad (1)$$

Thus since this field has to cancel the applied field, we must have

$$E_{\mathbf{0}} = \frac{\sigma}{\epsilon_0} \qquad (1)$$

or

$$\sigma = \epsilon_0 E_{\mathbf{0}}$$

3. For electrostatic fields $\nabla \times \vec{\boldsymbol{E}} = 0$. For the given vector field

$$\nabla \times \vec{\boldsymbol{E}} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & 3xz^2 & -2xz \end{vmatrix} \neq 0$$

Since the given vector field does not satisfy the required condition, the field cannot represent an electrostatic field.

4. $\vec{\boldsymbol{E}} = \frac{10^{-6}}{4\pi\epsilon_0}\frac{[(x-x_0)\hat{x}+(y-y_0)\hat{y}+(z-z_0)\hat{z}]}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{3/2}} = \frac{10^{-6}}{4\pi\epsilon_0}\frac{[(3-3)\hat{x}+(5-2)\hat{y}+(0-0)\hat{z}]}{[(3-3)^2+(5-2)^2+(0-0)^2]^{3/2}} = \frac{10^{-6}}{4\pi\epsilon_0}\frac{1}{9}\hat{y}$

5. (a) Let us consider a Gaussian sphere of radius r' inside the sphere. So, charge enclosed inside that sphere is

$Q_{encl} = \int \rho dV = 4\pi \int_0^{r'} r^2\,\rho_0\left(1 - \frac{4r}{3R}\right)dr = \left(\frac{4}{3}\right)\pi r'^3\rho_0\left(1 - \frac{r\prime}{R}\right).$

Hence using Gauss's law in integral form we obtain

$$E4\pi r'^2 = \left(\frac{4}{3\epsilon_0}\right)\pi r'^3\rho_0\left(1 - \frac{r'}{R}\right) \Rightarrow E = \frac{r'\rho_0}{3\epsilon_0}\left(1 - \frac{r'}{R}\right) \Rightarrow$$

$$\vec{\boldsymbol{E}} = \frac{r'\rho_0}{3\epsilon_0}\left(1 - \frac{r\prime}{R}\right)\hat{\boldsymbol{r}}.$$

However, the total charge enclosed by the sphere is

$$Q_{encl} = \int \rho dV = 4\pi \int_0^{R} r^2\,\rho_0\left(1 - \frac{4r}{3R}\right)dr = \left(\frac{4}{3}\right)\pi R^3\rho_0\left(1 - \frac{R}{R}\right) = 0.$$

Hence the electric field outside the sphere is zero.

(b) $\vec{\nabla}\cdot\vec{E} = 0$ and $\vec{\nabla}\times\vec{E} = 0$ outside the sphere.

Inside the sphere $\vec{\nabla}\times\vec{E} = 0$ is zero again.

However, inside the sphere

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} = \frac{\rho_0}{\epsilon_0}\left(1 - \frac{4r}{3R}\right)$$

Note that

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{r\rho_0}{3\epsilon_0}\left(1 - \frac{r}{R}\right)\right) = \frac{\rho_0}{\epsilon_0}\left(1 - \frac{4r}{3R}\right).$$

Which is just Gauss's law in differential form.

So

$$\left[\vec{\nabla} \cdot \vec{E}\right]_{r=\frac{4R}{5}} = \frac{\rho_0}{\epsilon_0}\left(1 - \frac{16}{15}\right) = -\frac{\rho_0}{15\epsilon_0}.$$

6. The electric field produced by the spherical charge distribution is as if the entire charge was concentrated at the center. Hence the problem reduces to a pair of charges +Q and -Q/4 separated by a distance d along the x-axis.

   It also follows from the problem that the zero of the electric field will exist on the x-axis. Since the charges are of opposite sign, the zero of the electric field will be at a value of x greater than d (to the right of the negative charge).

   Consider a point at a distance l from the negative charge and to the right of the charge. The electric field at this point will be zero if

$$\frac{Q}{4\pi\epsilon_0(d+l)^2} = \frac{Q/4}{4\pi\epsilon_0 l^2}$$

   Solving for l we get $l = d$.
7. Since the electric field within the conductor is zero, the inner surface will have a charge of + 1 μC and the outer surface would have a total charge of -1 μC distributed uniformly across the surface since the surface is spherical. Thus the charge density on the outer surface would be $10^{-6}/4\pi$ C/m^2
8. If the surface charge density on the metallic sphere is σ, then the electric field just outside the surface of the sphere is σ/ϵ_0. Since the breakdown electric field of air is 30 kV/cm or 3 x 10^6 V/m, in order that breakdown does not occur, $\sigma/\epsilon_0 < 3 \times 10^6$ V/m. Now the potential of the sphere is $V = \sigma R/\epsilon_0$. Thus the radius of the sphere must satisfy $R > \frac{90\times10^3}{3\times10^6} =$ 30 mm.