Ordinary Differential Equations(EMAT102L) (Lecture-1)

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Texts/References

1. S. L. Ross, Differential Equations, John Wiley & Sons, Inc., 2004.
2. E.A.Coddington, An introduction to Ordinary Differential Equations, Prentice Hall India, 1995.
3. E.Kreyszig, Advanced Engineering Mathematics, John Wiley & Son Inc.,2011.

Outline of the Lecture

We will learn

- Differential Equation
- Examples
- Classification of Differential Equations
  - Type: ODE/PDE
  - Order
  - Linear/Nonlinear
- Solution of a DE
  - General, Particular and Singular Solution

An equation containing derivatives is called a **differential equation**.

Differential equations arise naturally in the study of various physical phenomena and problems in Science and Engineering.

Newton's Equation of Motion

For example the motion of a particle of mass m under the influence of a force can be represented by the Newton's equation of motion

$$m\frac{d^2x}{dt^2} = F \tag{1}$$

where

- $x(t)$: displacement of the particle
- $\dfrac{d^2x}{dt^2}$: acceleration of the particle.
- m : mass of the object.
- F : net force exerted on the object.

(Here the time t is the independent variable and x denotes the dependent variable (the solution) of the equation.)

The equation (1) is a differential equation, which is the mathematical model of a falling object. This is an example to show how differential equations arise in nature.

How to solve this equation?

To solve this, we need to find a function $x(t)$ which satisfies this equation.

We will learn this soon in this course!

As an another example, consider the population growth of a specie. If

- $y(t)$ represents the population of a specie at time t.
- b and d denote the birth and death rate respectively of the specie.

then by assuming that the rate of change of the population y at time t is proportional to the population at t, i.e,

$$\frac{dy}{dt} \propto y(t),$$

we get the mathematical model

Population Growth Model

$$\frac{dy}{dt} = (b-d)y(t) \tag{2}$$

The equations (1) and (2) are simple examples of ordinary differential equations.

Differential Equations:Formal definition

Definition

An equation involving derivatives of one or more dependent variables w.r.t. one or more independent variables is called a **differential equation**.

Examples

1. $\dfrac{d^2y}{dx^2} + xy\left(\dfrac{dy}{dx}\right)^2 = 0.$
2. $\dfrac{d^4x}{dt^4} + 5\dfrac{d^2x}{dt^2} + 3x = \sin t$
3. $\dfrac{\partial v}{\partial s} + \dfrac{\partial v}{\partial t} = v$
4. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$

A differential equation can be classified based on its type as Ordinary Differential Equation (ODE) or Partial Differential Equation (PDE).

Ordinary Differential Equation

A differential equation involving derivatives of one or more dependent variables w.r.t. a single independent variable is called an **ordinary differential equation(ODE)**.

Examples

(i) $\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 = 0.$ (ii) $\frac{d^4x}{dt^4} + 5\frac{d^2x}{dt^2} + 3x = \sin t.$

Partial Differential Equation

A differential equation involving partial derivatives of one or more dependent variables w.r.t. more than one independent variables is called **partial differential equation(PDE)**.

Examples

(i) $\dfrac{\partial v}{\partial s} + \dfrac{\partial v}{\partial t} = v$

(ii) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$

In this course we will deal with ODE's only.

Definition

The **order** of a differential equation(ODE or PDE) is the order of the highest derivative appearing in the equation.

Degree of the differential equation

The **degree** of a differential equation is the power of the highest order derivative involved in the differential equation(after the differential equation has been made free from radicals and fractions as far as derivatives are concerned).

Examples

1. $x\left(\frac{dy}{dx}\right)^2 + y^2 = 1$. (Order=1, degree=2).
2. $\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 = 0$. (Order=2, degree=1)
3. $1 + \left(\frac{d^2y}{dx^2}\right)^{1/2} = \frac{dy}{dx}$. After rationalizing, we get $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx} - 1\right)^2$. This equation has order 2 and degree 1. Note that the degree is not $1/2$.
4. $y\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^3 + y^2 = e^x$. (Order=2, degree=1).
5. $\frac{d^ny}{dx^n} + x\frac{d^{n-1}y}{dx^{n-1}} + x^2\frac{d^{n-2}y}{dx^{n-2}} + \cdots x^n y = \sin x$. (Order=n, degree=1).
6. $\frac{d^4x}{dt^4} + 5\frac{d^2x}{dt^2} + 3x = \sin t$. (Order=4, degree=1)
7. $(y')^2 + y = 0$. (Order=1, degree=2)

General Form of an ODE

The general form of an ordinary differential equation is

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots \frac{d^ny}{dx^n}\right) = 0$$

or

$$f(x, y, y', y'', \cdots, y^{(n)}) = 0$$

where f is any function, x denotes the independent variable, y denotes the dependent variable.

Definition

If every term in a differential equation $f(x, y, y', y'', \cdots, y^{(n)}) = 0$ is linear in $y, y', y'', \cdots, y^{(n)}$.
i.e, If a differential equation is of the form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = g(x) \tag{3}$$

where $a_0(x), a_1(x) \cdots a_n(x)$ and $g(x)$ are continuous functions of x and $a_0(x) \neq 0$ for any x.
Then equation (3) is called the n^{th} order **linear differential equation**. If it is not linear, then the differential equation is called **nonlinear ordinary differential equation**.

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Note that in a linear differential equation $f(x, y, y', y'', \cdots, y^{(n)}) = 0$,

- *the dependent variable y and its various derivatives occur in the first degree only. For instance, expressions like* y^2, xy^2, $\left(\frac{dy}{dx}\right)^2$ *should not appear in the differential equation.*
- *There should not be products of dependent variable and any of its derivatives. For instance, expressions like* $y\frac{dy}{dx}$ *should not appear in the differential equation.*
- *No transcendental functions of the dependent variable and its derivatives should occur in the differential equation. For instance, expressions like* $\sin y$, e^y, $\log y$ *should not appear in the equation.*

Examples

- $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 6y = 0.$ (Linear ODE)
- $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 6y^2 = 0.$ (Nonlinear ODE)
- $\dfrac{d^4y}{dx^4} + x^2\dfrac{d^3y}{dx^3} + x^3\dfrac{dy}{dx} = xe^x.$ (linear ODE)
- $\dfrac{d^2y}{dx^2} + 5y\left(\dfrac{dy}{dx}\right)^3 + 6y = 0.$ (Nonlinear ODE)
- $y\dfrac{d^2y}{dx^2} + 5\left(\dfrac{dy}{dx}\right) + 6y = 0.$ (Nonlinear ODE)

Definition

A function $y = f(x)$ is called a solution of a differential equation $f(x, y, y', y'', \cdots, y^{(n)}) = 0$ on any interval I if

- $y = f(x)$ is differentiable (as many times as the order of the equation) on I.
- y satisfies the differential equation for all $x \in I$.

The curve (the graph) of $y = f(x)$ is called a **solution curve**.

Example

Show that $y = ce^{-2x}$ is a solution of $y' + 2y = 0$ on $\mathbb{R}$ for a constant $c \in \mathbb{R}$.

Solution: By direct differentiation, we have

$$y' = -2ce^{-2x} = -2y$$

.

$$\Rightarrow y' + 2y = 0.$$

Example

Show that for any constant $a \in \mathbb{R}$, $y = \dfrac{a}{1-x}$ is a solution of $(1-x)y' - y = 0$ on $(-\infty, 1)$ or on $(1, \infty)$.

Solution:

$$\frac{dy}{dx} = \frac{a}{(1-x)^2}$$

$$\Rightarrow (1-x)y' - y = 0$$

$\Rightarrow y = \dfrac{a}{1-x}$ is a solution of $(1-x)y' - y = 0$ on $(-\infty, 1)$ or on $(1, \infty)$.

Note that y is not a solution on any interval containing 1.

Example

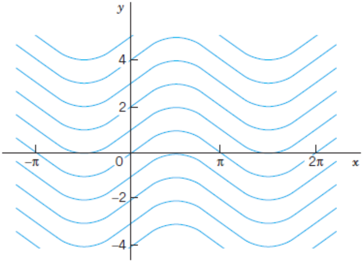
Consider the ODE

$$\frac{dy}{dx} = \cos x$$

The above ODE can be solved directly by integration on both sides. We obtain

$$y = \int \cos x + c = \sin x + c$$

where c is an arbitrary constant. This is a **family of solutions**. Each value of c gives one of these curves. The following figure shows some of them, for $c = -3, -2, -1, 0, 1, 2, 3, 4$.



To start with, let us try to understand a structure of a first order differential equation of the form

$$f(x, y, y') = 0 \tag{4}$$

One parameter family of solutions is given by $g(x, y, c) = 0$.

This one parameter family of solutions is called general solution of given ODE.

General Solution

Solution containing an arbitrary constant is called a general Solution of ODE.

For example, $y = x^2 + c$ is a general solution of the ODE $\dfrac{dy}{dx} = 2x$.

Particular Solution

Solution corresponding to a particular value of constant is called a particular solution of ODE.

For example, $y = x^2$ is a particular solution of the ODE $\dfrac{dy}{dx} = 2x$. This solution is obtained from the general solution $y = x^2 + c$ by assigning the value 0 to the constant c.

Singular Solution

If a solution to an ODE cannot be obtained from a general solution, then it is called a **singular solution**.

Example

$$\frac{dy}{dx} = (y-3)^2 \tag{5}$$

Then

$$\int \frac{dy}{(y-3)^2} = \int dx$$

$$\Rightarrow y - 3 = -\frac{1}{x+c}$$

.

$$y = 3 - \frac{1}{x+c} \tag{6}$$

where c is an arbitrary constant. We note that $y = 3$ is also a solution of (5). But no value of c in (6) gives $y = 3$.

Thus the solution $y = 3$ is a singular solution.


Thank You