Eigen Values, Eigen Vectors & Diagonalization:

Motivation: Let A be a real symmetric matrix. Consider the following problem:

Maximize (Minimize) $x^t A x$ s.t $x \in \mathbb{R}^n$ & $x^t x = 1$

To solve this, consider the Lagrangian.

$$L(x,\lambda) = x^t A x - \lambda(x^t x - 1)$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j - \lambda\left(\sum_{i=1}^{n} x_i^2 - 1\right)$$

Partially differentiating $L(x,\lambda)$ w.r.t x_i, $1 \le i \le n$, we get

$$0 = \frac{\partial L}{\partial x_1} = 2a_{11}x_1 + 2a_{12}x_2 + \cdots + 2a_{1n}x_n - 2\lambda x_1$$

$$0 = \frac{\partial L}{\partial x_2} = 2a_{21}x_1 + 2a_{22}x_2 + \cdots + 2a_{2n}x_n - 2\lambda x_2$$

$$\vdots$$

$$0 = \frac{\partial L}{\partial x_n} = 2a_{n1}x_1 + 2a_{n2}x_2 + \cdots + 2a_{nn}x_n - 2\lambda x_n$$

we get $\boxed{AX = \lambda X}$

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Consider a system of n. ordinary differential eq^n of the form

$$\frac{dy(t)}{dt} = Ay \quad \text{—(1)}, \quad t \geq 0,$$ where A is a real $n \times n$ matrix, y is a column vector.

To obtain a sol^n, let us assume that $y(t) = c\,e^{\lambda t}$ —(2) is a sol^n of (1). Then, we get

$$\lambda e^{\lambda t} c = A e^{\lambda t} c \iff (A - \lambda I)c = 0. \quad \text{—(3)}$$

So, (2) is a sol^n of (1) iff λ & c satisfy (3).

i.e given a matrix of order n, we are lead to find a pair (λ, c) s.t $c \neq 0$, & (3) has a sol^n.

— Suppose A be a matrix of order n. In general, we ask the question :-

For what values of $\lambda \in \mathbb{F}$, there exists a non-zero vector $X \in \mathbb{F}^n$ s.t $\boxed{AX = \lambda X.}$ —(A)

Here $\mathbb{F}^n$ standfor either vector space $\mathbb{R}^n(\mathbb{R})$ or $\mathbb{C}^n(\mathbb{R})$

eq^n (A) equivalent to $\boxed{(A - \lambda I)X = 0}$. —(B)

We know that (B) has a non-zero sol^n if rank $(A - \lambda I) < n$ or equivalently, $\det(A - \lambda I) = 0$.

• We observe that $\det(A-\lambda I)$ ~~is~~ a polynomial in λ of degree n.

We are therefore lead to the following definition:—

<u>Definition</u>: Let A be a matrix of order n. The polynomial $\underline{\det(A-\lambda I)}$ is called the characteristic poly. of A and is denoted by $p(\lambda)$.

The eqn $\boxed{p(\lambda)=0}$ is called the characteristic eqn of A.

If $\lambda \in \mathbb{F}$ is a root of the characteristic eqn $p(\lambda)=0$, then λ is called characteristic root or ch. value of A. or eigenvalue.

<u>Result</u>: Let $A=[a_{ij}]$; $a_{ij} \in \mathbb{F}$ for $1 \le i,j \le n$. Suppose $\lambda=\lambda_0 \in \mathbb{F}$ is a root of the ch. eqn. Then $\exists$ a non-zero $V \in \mathbb{F}^n$ s.t $AV=\lambda_0 V$.

<u>Pf</u>: λ_0 is a root of ch. eqn. $\therefore \det(A-\lambda_0 I)=0$

$\Rightarrow$ $A-\lambda_0 I$ is singular.

$\therefore$ The linear homogeneous system has a non zero soln.

#. We note that each scalar multiple kv of an eigenvector belonging to λ is also such an eigenvector.

$A(kV) = kAV = \lambda(kV)$.

Observation: The linear system $AX = \lambda X$ has a sol^n $X = 0$ for every $\lambda \in \mathbb{F}$. So, we consider only those $X \in \mathbb{F}^n$ that are non-zero and are solutions of the linear system $AX = \lambda X$.

Definition: Eigenvalue & Eigen vector:

If the linear system $AX = \lambda X$ has a non-zero sol^n $X \in \mathbb{F}^n$ for some $\lambda \in \mathbb{F}$, then

1) $\lambda \in \mathbb{F}$ is called an eigenvalue of A.
2) $0 \neq X \in \mathbb{F}^n$ is called eigen vector corresponding to the eigenvalue λ of A and
3) the tuple (λ, X) is called a eigenpair.

#

Remark: To understand the difference b/w a ch. value and a eigenvalue, we give the following example:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

Ch. pol^n — $\det(A - \lambda I) = \lambda^2 + 1$. , Ch. polynomial is $\lambda^2 + 1 = 0$

Ch. roots , $\lambda = \pm i$

1. If $\mathbb{F} = \mathbb{C}$, i.e A is considered as a complex matrix. Then the roots of $p(\lambda) = 0$ in $\mathbb{C}$ are $\pm i$.

$\rightarrow \quad \det(A-\lambda I) = (-1)^n \det(\lambda I - A)$

Caley Hamilton Thm: Every matrix A is a root of its ch. eq^n

Remark: largest power of o (More than n)
3. Inverse

Suppose $A = [a_{ij}]_{n\times n}$ is a triangular matrix.

Then $(tI - A)$ is a triangular matrix with diagonal entries $t - a_{ii}$. Hence.

$$\text{Ch. pol} = \det(\lambda I - A) = (t - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn})$$

Observation: The roots of ch. poly are the diagonal elements of A.

or eigen values are diagonal elements of A.

Ex: $A = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$.

$$\text{ch. poly} = |tI - A| = \begin{vmatrix} \lambda-1 & -3 \\ -4 & \lambda-5 \end{vmatrix} = (\lambda-1)(\lambda-5) - 12$$
$$= \lambda^2 - 5\lambda - 7.$$

By Caley Hamilton: $A^2 - 5A - 7 = 0$.

$$A^2 - 5A - 7I = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} - 5\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

?. Find A^{-1}, Using Caley Hamilton Thm,

We have $A^2 - 5A - 7I = 0$.

premultiplying by A^{-1}, we obtain,

$$7A^{-1} = -5I + A$$

$$7A^{-1} = -\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 4 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{-1}{7}\begin{bmatrix} 5 & -3 \\ -4 & 1 \end{bmatrix}.$$

(3) $A^2 = 6A - 7I = \begin{bmatrix} 6 & 18 \\ 24 & 30 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -1 & 18 \\ 24 & 23 \end{bmatrix}$.

Ex:- $\begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & -2 & -1 \\ -2 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ (Find inverse using Caley Hamilton)

Similar matrix :- Let A & B are two $n \times n$ matrix. A is said to be similar to B if $\exists$ a non singular matrix B s.t $B = P^{-1}AP$.

Diagonalizable matrix :- Let A be an $n \times n$ matrix. The matrix A is said to be diagonalizable if there exist a non-singular matrix P such that $B = P^{-1}AP$ is a diagonal matrix.

Remark: 1) Suppose A and B are similar matrices.

Then A and B have same ch. poly.

Character poly of $B = \det(\lambda I - B)$

$$= \det(\lambda I - P^{-1}AP)$$

$$= \det(\lambda P^{-1}IP - P^{-1}AP)$$

$$= \det(P^{-1}(\lambda I - A)P)$$

$$= \det P^{-1} \cdot \det(\lambda I - A) \cdot \det P$$

$$= \det(\lambda I - A)$$

$$= \text{ch. poly. of } A.$$

Hence, eigen values are also same.

→ simplest form of characteristic poly. of degree 2 or 3.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

~~then~~

ch. poly $= \lambda^2 - (a+d)\lambda + ad - bc$

$= \lambda^2 - (\text{trace } A)\lambda + \det A$

→ $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ ch. poly → $\left(\lambda^3 - (\text{trace} A)\lambda^2 + (A_{11} + A_{22} + A_{33})\lambda - \det A\right)$

where A_{ii} denote the cofactor of a_{ii}.

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3$

$S_1 = \text{trace } A$, $S_2 = A_{11} + A_{22} + A_{3}$, $S_3 = \det A$,

S_k is the sum of the principal minor of order k.

* Characterization of Diagonal Matrix :-

Let A be $n \times n$ matrix, which is similar to a digonal matrix iff A has n-linearly independent eigenvectors.

In this case, the diagonal elements of D are the corresponding eigenvalues and $D = P^{-1}AP$, where P is the matrix whose columns are the eigenvectors.

Remark : (1) "If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigenvalues of a matrix A with corresponding eigenvectors $x_1, x_2, \ldots x_k$. Then the set $\{x_1, x_2, \ldots, x_k\}$ is linearly independent".

Pf: This is done by induction.
As for $k=1$, obious.
Assume for $k=m$, $1 \leq m < k$
We prove the result for $k = m+1$. Consider,

$$c_1 x_1 + c_2 x_2 + \cdots + c_{m+1} x_{m+1} = 0 \quad \text{— (1)}$$

for $c_1, \ldots, c_{m+1}$ are unknown.

$$A(c_1 x_1 + c_2 x_2 + \cdots + c_{m+1} x_{m+1}) = A(0) = 0.$$

$$\Rightarrow c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 + \cdots + c_{m+1} \lambda_{m+1} x_{m+1} = 0 \quad \text{— (2)}$$

$-\lambda_1 \times$ (1) + (2), we obtain

$$c_2(\lambda_2 - \lambda_1)x_2 + c_3(\lambda_3 - \lambda_1)x_3 + \cdots + c_{m+1}(\lambda_{m+1} - \lambda_1)x_{m+1} = 0$$

These are m-vector, $\therefore$ By Induction hypothesis, $(\lambda_{m+i} - \lambda_1) c_{m+i} = 0$

But $\lambda_1 \neq \lambda_{m+i}$ $\therefore c_{m+i} = 0$

Hence $c_1 x_1 = 0 \Rightarrow c_1 = 0$

3) The eigenvectors corresponding to distinct eigenvalues of $n \times n$ matrix A are linearly independent.

Properties : Suppose the ~~product~~ characteristic poly. $p(\lambda)$ of an-n-square matrix A is a product of n- distinct factor say $p(\lambda) = (\lambda - a_1)(\lambda - a_2) \cdots (\lambda - a_n)$

Then A is similar to the diagonal matrix D with diagonal entries a_i.

#→ Let A be a square matrix over the complex field $\mathbb{C}$. Then A has at least one eigenvalue.

#. Suppose a matrix A can be diagonalizable. ~~Then~~

i.e $D = P^{-1} A P$

Then $A = P D P^{-1}$

$A^2 = P D P^{-1} P D P^{-1} = P^{-1} D^2 P$

$A^m = P D^m P^{-1} = P \, \mathrm{dia}\{d_1^m, d_2^m, \ldots, d_n^m) P^{-1}$.

2) More Generally, for any polynomial $f(t)$,

$$f(A) = f(PDP^{-1})$$
$$= P f(D) P^{-1}$$
$$= P \, \mathrm{dig}(f(d_1), f(d_2), \ldots, f(d_n)) P^{-1}.$$

3) Furthermore, if diagonal entries of D are non-negative,

$$B = P \, \mathrm{dia}(\sqrt{d_1}, \sqrt{d_2}, \ldots, \sqrt{d_n}) P^{-1}$$

Then B is a non-negative square root of A.

i.e $B^2 = A$ & the eigenvalues of B are non-negative.

⇒ If "λ" is an eigenvalue of the matrix A, then the algebraic multiplicity of "λ" is defined to be the multiplicity of λ as a root of ch. poly. of A.

Geometric multiplicity of λ is defined to be the dimension of eigen space.

① Problem Set :

1) $A = \mathrm{diag}(d_1, d_2, \dots d_n)$, $d_i \in \mathbb{R}$, $1 \le i \le n$.

Then $p(\lambda) = \prod_{i=1}^{n} (\lambda - d_i)$. is the ch. eq^n.

So, eigen pair are

$(d_1, e_1), (d_2, e_2) \dots, (d_n, e_n)$

$e_i = (0, 0, \dots, \underset{\substack{\downarrow \\ \text{ith place}}}{1}, \dots 0)$

② $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Characterstic eq^n $(1-\lambda)^2 = 0$.

eigen value — 1, 1 (repeated eigen value.

eigen vector — $x_1(1, 0)$

$(A - I_2)X = 0$

$\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\boxed{x_2 = 0}$

Two eigen values 1, 1 but only one eigenvector.
(Not diagonalizable)

③ $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\det(A - \lambda I_2) = (1-\lambda)^2$

ch. roots 1, 1.

Any two linearly independent vector of $\mathbb{R}^2$ are eigenvector.

4) $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

$\det(A - \lambda I) = (\lambda - 3)(\lambda + 1)$.

ch. eq^n has roots $3, -1$.

eigen ~~vector~~ pair — $(3, (1,1)^t)$, $(-1, (1,-1)^t)$ (Check.

In this case, we have Two distinct eigenvalue and the corresponding eigen vectors are also linearly indepedent (Diagonalizable).

5). $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $\det(A - \lambda I_2) = \lambda^2 - 2\lambda + 2$.

The ch. roots are $1+i$, $1-i$

1) Hence A has no real value over $\mathbb{R}$.

2) It has two roots over $\mathbb{C}$.

Correspondinp eigen vectors are $(i,1)$ & $(1,i)$

Eigen pair $(1+i, (i,1))$ & $(1-i, (1,i))$.

So, A has $(1, (1,i)^t)$ & $(-i, (i,1)^t)$ as eigen pair

$AX = iX$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} i x_1 \\ i x_2 \end{bmatrix}$$

$x_2 = i x_1$

$-x_1 = i x_2$

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ i x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$AX = -iX$

$x_2 = -i x_1$

$-x_1 = -i x_2$

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -i x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Thus $\left(i, \begin{pmatrix} 1 \\ i \end{pmatrix}\right)$, $\left(-i, \begin{pmatrix} 1 \\ -i \end{pmatrix}\right)$ are two eigen pair.

* Note: A & A^t have same ch. poly.

* If $\lambda \neq 0$ is an eigen value of A Then $\frac{1}{\lambda}$ is an eigen value of A^{-1}.

$$0 = \det(A - \lambda I) = \det(A - \lambda A^{-1} A)$$

$$= \det(\lambda \lambda^{-1} A - A A^{-1} A)$$

$$= \det(\lambda (\lambda^{-1} I - A^{-1}) A)$$

$$= \lambda^n \det(\lambda^{-1} I - A^{-1}) \det A$$

$$\Rightarrow \det(\lambda^{-1} I - A^{-1}) = 0.$$

Ex ÷ Let $A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$.

Let $v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Then

$$A v_1 = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = v_1$$

$$A v_2 = 4 v_2.$$

i.e $1, 4$ are eigen values corresponding to eigenvector v_1 & v_2.

Since v_1 & v_2 are l.i. & hence form a basis of $\mathbb{R}^2$.

Accordingly A is diagonalizable.

Furthermore, let $P = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$ (i.e whose columns are eigenvector).

$$P = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} \frac{1}{3} & -1/3 \\ 2/3 & 1/3 \end{bmatrix}.$$

A is similar to Diagonal matrix.

$$D = P^{-1} A P = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}.$$

Accordingly,

$$A^4 = P D^4 P^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 256 \end{bmatrix} \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 171 & 85 \\ 170 & 86 \end{bmatrix}.$$

Moreover, Suppose $f(t) = t^3 - 5t^2 + 3t + 6$.

$f(1) = 5$, $f(4) = 2$.

$$f(A) = P\, f(D)\, P^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -1/3 \\ 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}.$$

→ Positive Square root of A

$$B = P \sqrt{D}\, P^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 5/3 & 1/3 \\ 2/3 & 4/3 \end{bmatrix}.$$

$\rightarrow$ $A = \begin{bmatrix} 2 & 1 & -2 \\ 2 & 3 & -4 \\ 1 & 1 & -1 \end{bmatrix}$ (9.27)

Q:- 1) Find all eigen values of A.

2) Eigen vector of A.

3) Find a basis of each eigen space.

4) Is A diagonalizable.

Soln:- Ch. poly of A — $\lambda^3 - (\text{trace } A)\lambda^2 + (A_{11} + A_{22} + A_{33})\lambda + \det A$

trace of $A = 4$

$A_{11} = \begin{vmatrix} 3 & -4 \\ 1 & -1 \end{vmatrix} = -3+4 = 1$

$A_{22} = \begin{vmatrix} 2 & -2 \\ 1 & -1 \end{vmatrix} = -2+2 = 0$

$A_{33} = \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = 6-2 = 4$

$|A| = 2(1) - 1(2) - 2(-1) = 2.$

Ch. poly — $t^3 - 4t^2 + 5t - 2 = (t-1)^2(t-2)$

Eigen values, $\lambda = 1, 1, 2$

Eigen vector for $\lambda_1 = 1$

$(A - \lambda_1 I)x = 0 \Rightarrow \begin{pmatrix} 1 & 1 & -2 \\ 2 & 2 & -4 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$R_3 \to R_3 - R_1$, $R_2 \to R_2 - 2R_1$

$$\begin{pmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 + x_2 - 2x_3 = 0$$

$$x_1 = -x_2 + 2x_3$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_2 + x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

For $\underline{\underline{\lambda = 2}}$,

$$\begin{pmatrix} 0 & 1 & -2 \\ 2 & 1 & -4 \\ 1 & 1 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_2 \to R_2 - 2R_3$

$$\begin{pmatrix} 0 & 1 & -2 \\ 0 & -1 & 2 \\ 1 & 1 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_1 \to R_1 + R_2$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 2 \\ 1 & 1 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-x_2 + 2x_3 = 0$$

$$x_1 + x_2 - 3x_3 = 0$$

$$x_2 = 2x_3$$

$$x_1 = 3x_3 - 2x_3 = +x_3$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} +x_3 \\ 2x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Thus T is diagonalizable, because it has three independent eigenvectro.

$$S = \{u_1, u_2, u_3\}$$
$$= \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$$

$$D = \text{dia}(1, 1, 2)$$

Ex :

* Diagonalizing Real Symmetric Matrices :-

There are many real matrices A that are not diagonalizable. In fact, some real matrices may not have any (real) eigenvalues. However, if A is real symmetric matrix Then these problem dont exist.

Infact, 1) If A is real symmetric matrix. Then each root λ of its characteristic polynomial is real.

2) A is real symmetric matrix. Suppose u & v are eigenvectors of A belonging to distinct eigenvalues λ_1, λ_2. Then u & v are orthogonal.

i.e $\langle u, v \rangle = 0$.

1) & 2) gives the following fundamental result:-

Let A be real symmetric matrix. Then there exists an orthogonal matrix P such that $D = P^{-1}AP$ is diagonal.

<u>Remark:</u> The orthogonal matrix P is obtained by normalizing the basis of orthogonal eigenvectors of A. In such a case, We say A is "Orthogonally diagonalizable".

Ex :- Let $A = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$, a real symmetric matrix. Find an orthogonal matrix P s.t $P^{-1}AP$ is diagonal.

Solⁿ: Characteristic polynomial of A is $\det(A-\lambda I)$

$= \lambda^2 - 7\lambda + 6 = (\lambda-6)(\lambda-1)$

ch. eq^n : $(\lambda-6)\cdot(\lambda-1) = 0$

ch. roots $\Rightarrow \lambda = 6, \quad \lambda = 1$

(a) Eigen vector (ch. vector) corresponding to $\lambda = 6$,

$(A-6I)X = 0$

$\Rightarrow \begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \left.\begin{array}{l} -4x_1 - 2x_2 = 0 \\ -2x_1 - x_2 = 0 \end{array}\right\} \Rightarrow 2x_1 + x_2 = 0$

$\Rightarrow x_2 = -2x_1$

Eigen ~~space~~ vector $= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -2x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$

(b) Eigenvector corresponding to $\lambda = 1$:-

$(A-I)X = 0 \quad \Rightarrow \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 - 2x_2 = 0 \Rightarrow \boxed{x_1 = 2x_2}$

$$\text{Eigenvector} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $u_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $u_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

As, we see $u_1 \cdot u_2 = 2 - 2 = 0$

$\Rightarrow$ u_1 & u_2 are orthogonal.

Normalizing u_1 & u_2, we obtain orthonormal vector.
$\|u_1\| = \sqrt{5}$, $\|u_2\| = \sqrt{5}$

$$\hat{u}_1 = \frac{u_1}{\|u_1\|} = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}, \quad \hat{u}_2 = \frac{u_2}{\|u_2\|} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{1}{\sqrt{5}} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}.$$

$$P^{-1}AP = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}.$$

As expected, $P^{-1}AP$ are the eigenvalues corresponding to the columns of P.

Ex :- let $A = \begin{bmatrix} 11 & -8 & 4 \\ -8 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix}$

Sol^n :- a) Ch. poly — $\lambda^3 - 6\lambda^2 - 135\lambda - 400$.

trace $A = 6$, $|A| = 400$, $A_{11} = 0$. $A_{22} = -60$ $A_{33} = -75$

$\sum A_{ii} = -135$

ch. roots, $(t+5)^2 (t-16) = 0$

$t = -5$ (Repeated twice), 16.

b) Orthogonal Basis :-

$\lambda = -5$

$$A + 5I = \begin{bmatrix} 16 & -8 & 4 \\ -8 & 4 & -2 \\ 4 & -2 & 1 \end{bmatrix}$$

$(A + 5I)X = 0 \Rightarrow 4x - 2y + z = 0$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cancel{-4x+2y} \quad x \\ y \\ -4x + 2y \end{bmatrix} = y \underbrace{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}_{v_1} + x \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$$

$v_2 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ s.t $0.a + b + 2c = 0$ $\quad 4a - 2b + c = 0$

$b + 2c = 0$

let $c = 4$. $b = -8$, $a = -5$

$$v_2 = \begin{bmatrix} -5 \\ -8 \\ 4 \end{bmatrix}$$

$\underline{\underline{\lambda = 16}}$

$$A - 16I = \begin{bmatrix} -5 & -8 & 4 \\ -8 & -17 & -2 \\ 4 & -2 & -20 \end{bmatrix}$$

$R_2 \to 5R_2 - 8R_1$, $R_3 \to 5R_3 + 4R_1$

$$\begin{bmatrix} -5 & -8 & 4 \\ 0 & -21 & -42 \\ 0 & -42 & 84 \end{bmatrix} \sim$$

$R_3 \to \frac{1}{2}R_3 - R_2$, $R_2 \to \frac{1}{21}R_2$

$$\begin{bmatrix} -5 & -8 & 4 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$-x_2 - 2x_3 = 0 \Rightarrow x_2 = -2x_3$

$-5x_1 - 8x_2 + 4x_3 = 0$

$-5x_1 + 16x_3 + 4x_3 = 0$

$x_1 = 4x_3$

$$v_3 = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

Then $\{v_1, v_2, v_3\}$ form a maximal set of non-zero orthogonal eigenvectors of B.

(c) Normalizing v_1, v_2, v_3 to obtain the orthonormal basis:

$\hat{v}_1 = \frac{v_1}{\sqrt{5}}$, $\hat{v}_2 = \frac{v_2}{\sqrt{105}}$, $v_3 = \frac{1}{\sqrt{21}}$

$$P = \begin{bmatrix} 0 & -5/\sqrt{105} & 4/\sqrt{21} \\ 1/\sqrt{5} & -8/\sqrt{105} & -2/\sqrt{21} \\ 2/\sqrt{5} & 4/\sqrt{105} & 1/\sqrt{21} \end{bmatrix}$$

$$D = P^{-1}AP = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 16 \end{bmatrix}$$