Laplace Transforms(EMAT102L)
(Lecture-17)

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Learning Outcome of the Lecture

We will learn

- Laplace Transforms
- Properties of Laplace Transforms

- The Laplace transform is an integral transform since the transformation process involves an integral and It is named after **Pierre-Simon Laplace**.
- It takes a function of a real variable t (time) to a function of a complex variable s (frequency).
- **Advantage of Laplace Transform:**
  - Laplace transform directly gives the solution of differential equations with given boundary values without the necessity of first finding the general solution and then evaluating from it the arbitrary constants.
  - Moreover it reduces the problem of solving differential equation to an algebraic equation.

Functions of exponential order

A function $f(t)$ is said to be of **exponential order** α, if there exist constant $M > 0$ such that

$$|f(t)| \leq Me^{\alpha t}, t \geq 0.$$

Geometrically this condition implies that the graph of $f(t)$, $t \geq 0$ does not grow faster than the graph of the exponential function $g(t) = Me^{\alpha t}, \alpha > 0$.

Example

- Since
$$|t| \leq e^{t}, \ |e^{-2t}| \leq e^{t}, \ |\cos t| \leq e^{t}, t \geq 0.$$
So, these functions are of exponential order.
- For a function of the form $f(t) = e^{ct^2}, c > 0$, it is not possible to determine α and M such that $e^{ct^2} \leq Me^{\alpha t}$.
So, it is not of exponential order.

Laplace Transform of a function

Let $f(t)$ be a function defined for all positive values of t. Then, the **Laplace transform** of $f(t)$, denoted by $F(s) = \mathcal{L}\{f(t)\}$, is defined by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt,$$

provided this integral exists. Here s is a positive real number or a complex number.

Sufficient Condition for the existence of Laplace Transforms

Let $f : [0, \infty) \to \mathbb{R}$ be

- piecewise continuous (that is, a function which is continuous except at a finite number of points in its domain) on any finite interval $[0, b]$.
- of exponential order α.

Example

Consider the function f defined by

$$f(t) = 1, \text{ for } t > 0.$$

Then

$$\begin{aligned}
\mathcal{L}\{1\} &= \int_0^{\infty} e^{-st}.1dt = \lim_{R\to\infty} \int_0^{R} e^{-st}.1dt \\
&= \lim_{R\to\infty} \left[\frac{-e^{-st}}{s}\right]_0^R \\
&= \lim_{R\to\infty} \left[\frac{1}{s} - \frac{e^{-sR}}{s}\right] \\
&= \frac{1}{s}
\end{aligned}$$

for all $s > 0$. Thus

$$\mathcal{L}\{1\} = \frac{1}{s}, s > 0.$$

Example

Consider the function f defined by

$$f(t) = t, \text{ for } t > 0.$$

Then

$$\begin{aligned}
\mathcal{L}\{t\} &= \int_0^\infty e^{-st}.t\,dt = \lim_{R\to\infty}\int_0^R e^{-st}.tdt \\
&= \lim_{R\to\infty}\left[\frac{-e^{-st}}{s^2}(st+1)\right]_0^R \\
&= \lim_{R\to\infty}\left[\frac{1}{s^2} - \frac{e^{-sR}}{s}(sR+1)\right] \\
&= \frac{1}{s^2}
\end{aligned}$$

for all $s > 0$. Thus

$$\mathcal{L}\{t\} = \frac{1}{s^2}, s > 0.$$

Example

Consider the function f defined by

$$f(t) = e^{at}, \text{ for } t > 0.$$

Then

$$\begin{aligned}
\mathcal{L}\{e^{at}\} &= \int_0^\infty e^{-st}.e^{at}dt = \lim_{R\to\infty}\int_0^R e^{(a-s)t}dt \\
&= \lim_{R\to\infty}\left[\frac{e^{(a-s)t}}{a-s}\right]_0^R \\
&= \lim_{R\to\infty}\left[\frac{e^{(a-s)R}}{a-s} - \frac{1}{a-s}\right] \\
&= \frac{-1}{a-s} = \frac{1}{s-a}
\end{aligned}$$

for all $s > a$. Thus

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, s > a.$$

Example

Consider the function f defined by

$$f(t) = \sin bt, \text{ for } t > 0.$$

Then

$$\begin{aligned}
\mathcal{L}\{\sin bt\} &= \int_0^\infty e^{-st}.\sin bt dt = \lim_{R\to\infty}\left[-\frac{e^{-st}}{s^2+b^2}(s\sin bt + b\cos bt)\right]_0^R \\
&= \lim_{R\to\infty}\left[\frac{b}{s^2+b^2} - \frac{e^{-sR}}{s^2+b^2}(s\sin bR + b\cos bR)\right] \\
&= \frac{b}{s^2+b^2}
\end{aligned}$$

for all $s > 0$. Thus

$$\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}, \; s > 0.$$

Example

Consider the function f defined by

$$f(t) = \cos bt, \text{ for } t > 0.$$

Then

$$\begin{aligned}
\mathcal{L}\{\cos bt\} &= \int_0^\infty e^{-st}.\cos btdt = \lim_{R\to\infty}\left[-\frac{e^{-st}}{s^2+b^2}(-s\cos bt + b\sin bt)\right]_0^R \\
&= \lim_{R\to\infty}\left[-\frac{e^{-sR}}{s^2+b^2}(-s\cos bR + b\cos bR) + \frac{s}{s^2+b^2}\right] \\
&= \frac{s}{s^2+b^2}
\end{aligned}$$

for all $s > 0$. Thus

$$\mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2}, s > 0.$$

1. $\mathcal{L}\{1\} = \frac{1}{s}, s > 0.$
2. $\mathcal{L}\{t\} = \frac{1}{s^2}, s > 0.$
3. $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, s > 0$, if n is a positive integer.
4. $\mathcal{L}\{t^\alpha\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}, s > 0$ if $\alpha + 1 > 0$, α not an integer.

Note: Gamma function $\Gamma(\alpha)$ is defined by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt, \alpha > 0.$$

We have $\mathcal{L}\{t^\alpha\} = \int_0^\infty t^\alpha e^{-st} dt = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$, substituting $st = w$.

Laplace transform of some elementary functions

1. $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}, s > a.$
2. $\mathcal{L}\{\sin at\} = \dfrac{a}{s^2+a^2}, s > 0.$
3. $\mathcal{L}\{\cos at\} = \dfrac{s}{s^2+a^2}, s > 0.$
4. $\mathcal{L}\{\sinh at\} = \dfrac{a}{s^2-a^2}, s > |a|.$
5. $\mathcal{L}\{\cosh at\} = \dfrac{s}{s^2-a^2}, s > |a|.$

Laplace Transforms of some elementary functions

	$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$
2	t	$1/s^2$
3	t^2	$2!/s^3$
4	t^n $(n = 0, 1, \cdots)$	$\dfrac{n!}{s^{n+1}}$
5	t^a (a positive)	$\dfrac{\Gamma(a+1)}{s^{a+1}}$
6	e^{at}	$\dfrac{1}{s-a}$

	$f(t)$	$\mathcal{L}(f)$
7	$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
8	$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
9	$\cosh at$	$\dfrac{s}{s^2-a^2}$
10	$\sinh at$	$\dfrac{a}{s^2-a^2}$
11	$e^{at}\cos \omega t$	$\dfrac{s-a}{(s-a)^2+\omega^2}$
12	$e^{at}\sin \omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$

Theorem (Linearity of Laplace Transform)

If $f(t)$ and $g(t)$ are two functions whose Laplace transforms exist, then

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

where a and b are arbitrary constants.

Proof:

$$\begin{aligned}
\mathcal{L}\{af(t) + bg(t)\} &= \int_0^\infty (af(t) + bg(t))e^{-st}dt \\
&= \int_0^\infty (af(t)e^{-st} + bg(t)e^{-st})dt \\
&= a\int_0^\infty f(t)e^{-st}dt + b\int_0^\infty g(t)e^{-st}dt \\
&= a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}
\end{aligned}$$

Example

Find the Laplace transform of

$$f(t) = \sin^2 at.$$

Solution:

Since $\sin^2 at = \dfrac{1-\cos 2at}{2}$, we have

$$\mathcal{L}\{\sin^2 at\} = \mathcal{L}\left\{\frac{1-\cos 2at}{2}\right\} = \mathcal{L}\left\{\frac{1}{2} - \frac{\cos 2at}{2}\right\}$$

$$= \frac{1}{2}\mathcal{L}\{1\} - \frac{1}{2}\mathcal{L}\{\cos 2at\}.$$

Since

$$\mathcal{L}\{1\} = \frac{1}{s},\ \mathcal{L}\{\cos 2at\} = \frac{s}{s^2+4a^2}.$$

Thus

$$\mathcal{L}\{\sin^2 at\} = \frac{1}{2}.\frac{1}{s} - \frac{1}{2}\frac{s}{s^2+4a^2} = \frac{2a^2}{s(s^2+4a^2)}.$$

Example

Find the Laplace transform of $\cosh at$ and $\sinh at$.

Solution:

Since $\cosh at = \dfrac{e^{at} + e^{-at}}{2}$ and $\sinh at = \dfrac{e^{at} - e^{-at}}{2}$, so we have using the linearity property,

$$\begin{aligned} \mathcal{L}\{\cosh at\} &= \frac{1}{2}(\mathcal{L}\{e^{at}\} + \mathcal{L}\{e^{-at}\}) \\ &= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{s}{s^2 - a^2}. \end{aligned}$$

$$\begin{aligned} \mathcal{L}\{\sinh at\} &= \frac{1}{2}(\mathcal{L}\{e^{at}\} - \mathcal{L}\{e^{-at}\}) \\ &= \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) = \frac{a}{s^2 - a^2}. \end{aligned}$$

Theorem (First Shifting Theorem)

If $\mathcal{L}\{f(t)\} = F(s)$, *then*

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

Proof:

$$\begin{aligned}
\mathcal{L}\{e^{at}f(t)\} &= \int_0^\infty e^{-st}\{e^{at}f(t)\}dt \\
&= \int_0^\infty e^{-(s-a)t}f(t)dt \\
&= F(s-a)
\end{aligned}$$

Example

Find the Laplace transform of

$$e^{at}\cos bt.$$

Solution: Let $f(t) = \cos bt$. Then $\mathcal{L}\{e^{at}\cos bt\} = F(s-a)$, where

$$F(s) = \mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2}.$$

Thus using the first shifting theorem, we get

$$\mathcal{L}\{e^{at}\cos bt\} = \left.\frac{s}{s^2+b^2}\right|_{s\to(s-a)} = \frac{s-a}{(s-a)^2+b^2}$$

Example

Find the Laplace transform of $e^{at}t$.

Solution: Let $f(t) = t$, Then $\{e^{at}.t\} = F(s-a)$, where

$$F(s) = \mathcal{L}\{t\} = \frac{1}{s^2}.$$

Thus using the first shifting theorem, we get

$$\mathcal{L}\{e^{at}t\} = \frac{1}{(s-a)^2}.$$

Theorem

If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s)$ and in general

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

Example

Find the Laplace transform of

$$t^2 \sin bt.$$

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Let $f(t) = \sin bt$, Then $\{t^2 \sin bt\} = (-1)^2 \dfrac{d^2}{ds^2}[F(s)]$, where

$$F(s) = \mathcal{L}\{\sin bt\} = \frac{b}{s^2 + b^2}.$$

From this, we have
$$\frac{d}{ds}(F(s)) = \frac{-2bs}{(s^2 + b^2)^2}.$$

and
$$\frac{d^2}{ds^2}(F(s)) = \frac{6bs^2 - 2b^3}{(s^2 + b^2)^3}.$$

Thus, we have
$$\mathcal{L}\{t^2 \sin bt\} = \frac{6bs^2 - 2b^3}{(s^2 + b^2)^3}.$$

Example

Find the Laplace transform of

$$te^{-4t}\sin 3t.$$

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Solution:

$$\begin{aligned}\mathcal{L}\{te^{-4t}\sin 3t\} &= \mathcal{L}\{t\sin 3t\}\Big|_{s\to s+4} = -\frac{d}{ds}\mathcal{L}\{\sin 3t\}\Big|_{s\to s+4}\\ &= \cdots = \frac{6(s+4)}{((s+4)^2+9)^2}.\end{aligned}$$

Derivative property of Laplace transform

Theorem

If $f(t)$ is a differentiable function of t and $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

Proof.

$$\begin{aligned}
\mathcal{L}\{f'(t)\} &= \int_0^\infty e^{-st} f'(t)\,dt \\
&= \left[|e^{-st} f(t)|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt \right] \\
&= -f(0) + sF(s).
\end{aligned}$$

Derivative property of Laplace transform

Theorem

If $f(t)$ is twice differentiable function of t and $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

Proof:

$$\begin{aligned}
\mathcal{L}\{f''(t)\} &= -f'(0) + s\mathcal{L}\{f'(t)\} \\
&= -f'(0) + s[-f(0) + sF(s)] \\
&= -f'(0) - sf(0) + s^2F(s).
\end{aligned}$$

The previous results can be generalized:

Theorem

If $f(t)$ is an n times differentiable function of t and $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\{f^n(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

Exercises

Prove the following:

1. $\mathcal{L}\{t^2 e^{at}\} = \dfrac{2}{(s-a)^3}$.
2. $\mathcal{L}\{t \sin at\} = \dfrac{2as}{(s^2+a^2)^2}$.
3. $\mathcal{L}\left\{\sin 2t \sin 3t\right\} = \dfrac{12s}{(s^2+1)(s^2+25)}$.
4. $\mathcal{L}\left\{\dfrac{e^{bt}-e^{at}}{b-a}\right\} = \dfrac{1}{(s-a)(s-b)}$.


Thank You