Department of Mathematics, Bennett University
Engineering Calculus (EMAT101L)
Practice Problem Sheet 3

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1. Show that the following limits are correct by using the $\epsilon - \delta$ definition of limit:

   $(a)\ \lim_{x\to 0} x^3 \cos\left(\frac{1}{x}\right) = 0, \quad (b)\ \lim_{x\to a} x^{1/3} = a^{1/3} \text{ in } \mathbb{R}^+, \quad (c)\ \lim_{x\to 2}(2x^2+10x+4) = 32,$

   $(d)\ \lim_{x\to c} x^2 = c^2, \quad (e)\ \lim_{x\to c}\sqrt{x} = \sqrt{c},\ x \geq 0, \quad (f)\ \lim_{x\to a}\frac{1}{x} = \frac{1}{a}, \text{ where } a \in \mathbb{R}\setminus\{0\}.$

2. Show that a polynomial of odd degree has at least one real root.

3. Show that the equation $(1-x)\cos x = \sin x$ has at least one solution in $(0,1)$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be such that for every $x, y \in \mathbb{R}$, $|f(x) - f(y)| \leq \alpha|x-y|$, $\alpha > 0$. Show that f is continuous.

5. Let $f : (-1,1) \to \mathbb{R}$ be a continuous function such that in every neighbourhood of 0, there exists a point x such that $f(x) = 0$. Then show that $f(0) = 0$.

6. Determine the points and nature of discontinuity of the following functions:

   $(a)\ \frac{x\tan x}{x^2+1}, \quad (b)\ f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q}. \end{cases}$

Solutions for Practice Problem Sheet 3

1. (a) Choose $\delta = \epsilon^{\frac{1}{3}}$.
   (b) Use $(a^3 - b^3) = (a-b)(a^2 + b^2 + ab)$ and choose $\delta = \frac{\epsilon}{a^{2/3}}$.
   (c) Choose $\delta > 0$ such that $(7 + \delta)\delta = \epsilon$.
   (d) Choose $\delta > 0$ such that $(c + \delta)\delta = \epsilon$.
   (e) Choose $\delta = \dfrac{\epsilon}{\sqrt{c}}$.
   (f) Choose $\delta = \min\left\{\dfrac{|a|}{2}, \dfrac{\epsilon|a|^2}{2}\right\}$.

2. Let $p(x) = a_n x^n + \cdots + a_1 x + a_0$, $a_n \neq 0$ and n is odd.

   Then $p(x) = x^n\left(a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n}\right)$. If $a_n > 0$ then $p(x) \to \infty$ as $x \to \infty$ and $p(x) \to -\infty$ as $x \to -\infty$. Thus by IVT, there exist x_0 such that $p(x_0) = 0$. Similar argument for $a_n < 0$.

3. $f(0) = 1$, $f(1) = -\sin 1 < 0$. Now use IVT.

4. Choose $\delta = \dfrac{\epsilon}{\alpha}$.

5. For every n, there exists $x_n \in (-1/n, 1/n)$ such that $f(x_n) = 0$, Since f is continuous at 0 and $x_n \to 0$, we have $f(x_n) \to f(0)$. Therefore, $f(0) = 0$.

6. (a) $\frac{x \tan x}{x^2+1} = \frac{x \sin x}{\cos x (x^2+1)}$. The points of infinite discontinuities are $(2n+1)\frac{\pi}{2}$, $n \in \mathbb{Z}$.
   (b) This function is discontinuous everywhere and has discontinuity of second kind at all points. To prove this first take $q \in \mathbb{Q}$. Then consider the sequence $\{q + \frac{1}{n}\} \in \mathbb{Q}$ which converges to q. Therefore $\lim\limits_{n\to\infty} f(q + \frac{1}{n}) = 1$. Now we can also take a sequence $\{q + \frac{e}{n}\}$ of irrational numbers which converges to q and $f(q + \frac{e}{n}) = 0$. So the $\lim\limits_{x \to q^+} f(x)$ does not exist. By taking sequences $\{q - \frac{1}{n}\}$ and $\{q - \frac{e}{n}\}$ one can see that $\lim\limits_{x\to q^-} f(x)$ does not exist.

   Now for $a \notin \mathbb{Q}$, we use the fact that: there exists sequence $\{q_n\} \in \mathbb{Q}$ such that $q_n \to a$ as $n \to \infty$. So one can take the sequences $\{q_n\}$ and $\{a \pm \frac{1}{n}\}$ to see that the left and right limits does not exist.