Multivariable Calculus (Lecture-4)

Department of Mathematics Bennett University India

24th October, 2018





Learning Outcome of the Lecture

We learn

- Limits of Functions $F: S \subseteq \mathbb{R}^n \to \mathbb{R}$
- Limit and Iterated Limits
- Continuity of Functions $F: S \subseteq \mathbb{R}^2 \to \mathbb{R}$
- Properties of continuous functions

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In other words, we say that F(X) approaches L as X approaches A or F(X) has the limit L as X tends to A.



Important Theorem on Limits of Functions

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Let $F: S \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a function. Then, F(X) can be written as

$$F(X) = (f_1(X), f_2(X), \dots, f_m(X))$$
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Let X_0 be a limit point of S and let $L = (A_1, A_2, \dots, A_m)$ be a point in \mathbb{R}^m . Then,

$$\lim_{X \to X_0} F(X) = L$$

if and only if

$$\lim_{X \to X_0} f_i(X) = A_i \quad \text{for } 1 \le i \le m.$$





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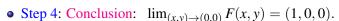
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• Step 3: Third Component function f_3

$$\lim_{(x,y)\to(0,0)} f_3(x,y) = \lim_{(x,y)\to(0,0)} 5 = 5$$

• Step 4: Conclusion: $\lim_{(x,y)\to(0,0)} F(x,y)$ does NOT exist.





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$$\lim_{(x,y)\to(0,0)} \frac{x^2 y^2}{x^2 + y^2} = \lim_{r\to 0^+} \frac{r^4 \cos^2 \theta \sin^2 \theta}{r^2}$$
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Sequences Criteria for Limits of Functions

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For every sequence $\{X_k\}$ in S (with $X_k \neq X_0$ for all k) converging to X_0 the sequence $\{F(X_k)\}$ converges to L.



Iterated Limits of Scalar Valued Functions

Let $f: S \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a real valued function. Let (x_0, y_0) be a limit point of S. The limits

$$\lim_{x \to x_0} \lim_{y \to y_0} f(x, y), \quad \text{if it exists}$$

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Question: What is the relation between the existence of these three limits? - Analyze.



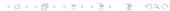
Examples

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$$f(x,y) = \frac{x^2}{x^2 + y^2}$$
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Still $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.



Note on Iterated Limits and (Double) Limit

Important Note:

- Existence of the (double) limit does **NOT** guarantee existence of iterated limits.
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- If (double) limit and iterated limits exist then they are all equal.

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NOTE: Limit and Continuity are closely related. Justify?





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<u>While</u> in limit at X_0 is L, then we have for every $\epsilon > 0$ there exists $\delta > 0$ such that

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Let $F: S \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a function. Then, F(X) can be written as

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