Tutorial 2 Solutions

(1) Q:

(a) $2^{n+1} = O(2^n)$. $\Rightarrow$ ~~Not~~ Yes, it's true

$2^{n+1} = 2 \cdot 2^n$

$\Rightarrow$ Let $f(n) = 2^{n+1}$, $g(n) = 2^n$,

$\therefore$ $f(n) \leq c \cdot g(n)$ $\forall n \geq 1$. where $c = 2$.

(b) $2^{2n} \neq O(2^n)$

$2^{2n} = 2^n \cdot 2^n$, Let $f(n) = 2^{2n}$
$g(n) = 2^n$

$f(n) \leq c \cdot g(n)$ $\forall n \geq 1$ where $c = 2^n$

but c is not a fixed constant

as n is changing, c is also changing

& we cannot find a fixed c,

which satisfies $f(n) \leq c \cdot g(n)$

for all values of n.

(2) $f_1(n) = 10^n, \quad f_2(n) = n^{1/3},$

$f_3(n) = n^n, \quad f_4(n) = \log_2 n$

$f_5(n) = 2^{\sqrt{\log_2 n}}$

Result 1: $f_1(n) \le f_3(n)$. (i.e) $10^n \le c.n^n \;\forall\; n \geq 10.$

$\Rightarrow 10^n = O(n^n).$

Result 2: It is easy to see $f_4(n) = O(f_2(n))$

(i.e) $\log_2 n = O(n^{1/3})$

Since $\log_2 n = O(n^{\epsilon})$ for all $\epsilon > 0$.

Further, $f_2(n) = o(f_1(n))$. (i.e) $n^{1/3} = O(10^n)$.

$\boxed{f_4, f_2, f_1, f_3.}$

~~Result 3~~:

Now, we place f_5 at the right position.

Consider f_2 and f_5

$$\left.\begin{array}{l} f_2(n) = n^{1/3} \\ f_5(n) = 2^{\sqrt{\log_2 n}} \end{array}\right\} \Rightarrow \begin{array}{l} \log f_2 = \frac{1}{3}\log n \\ \log f_5 = \sqrt{\log n} \end{array}$$

$\Rightarrow \log f_5 = O(\log f_2) \Rightarrow f_5 = O(f_2)$

So, there are two possible orders

(1) f_4, f_5, f_2, f_1, f_3.

(2) f_5, f_4, f_2, f_1, f_3.

But, it is clear that $f_4 = O(f_5) \Rightarrow$ the correct order is $f_4, f_5, f_2, f_1,$ and f_3.

(3) $f_1(n) = n^{0.999999} \log n$

$f_2(n) = 1000000000 n$

$f_3(n) = (1.000001)^n$

$f_4(n) = n^2$

$f_5(n) = n^2 \log n.$

$\Rightarrow$ Result(1): $f_4(n) = O(f_5(n))$

and $f_2(n) = O(f_4(n))$

Further, $f_5(n) = O(f_3(n))$ since $f_3(n)$ is an exponential function.

$\rightarrow f_2.\ f_4\ \ f_5\ \ f_3$

Now we place f_1 at the right position,

Recall, that $\log n = O(n^t)$ for all $t > 0$

$\Rightarrow \log n \leq c.\, n^t$ for a small $t > 0$

$\Rightarrow$ We can chouse, a small t such that

$\log n\, n^{0.999999} \leq c.\, n^{0.999999 + t} \leq c.n,$

$\rightarrow f_1(n) \leq c.\, f_2(n).$

$\rightarrow f_1(n) = O(f_2(n)$

$\Rightarrow$ The order of the function in growth is

$f_1, \leq f_2 \leq f_4 \leq f_5 \leq f_3.$

(4)

(u) $f_1(n) = 2^{\sqrt{\log n}}$

$f_2(n) = n^{4/3}$

$f_3(n) = n(\log n)^3$

$f_4(n) = n^{(\log n)}$

$f_5(n) = 2^{2^n}$

$f_6(n) = 2^{n^2}$

Result: $f_6(n) = O(f_5(n))$ (i.e)

$2^{n^2} \leq O(2^{2^n})$.

Ily $f_1(n) \leq O(f_5(n))$

(i.e) $2^{\sqrt{\log n}} = O(2^{2^n})$

and $f_1(n) = O(f_4(n))$.

Consider f_2 & f_3

$$\left.\begin{aligned} f_2(n) &= n^{4/3} \\ f_3(n) &= n(\log n)^3 \end{aligned}\right] \Rightarrow \begin{aligned} f_2(n)/n &= n^{1/3} \\ f_3(n)/n &= (\log n)^3 \end{aligned}$$

From, the result of $\log n = O(n^t)$ for all $t > 0$

we can conclude that

$f_3 = O(f_2)$.

Ily $f_4(n) = O(f_3(n))$. & f_4

Further, $f_2(n) = O(f_6(n))$ ~~and~~

$\therefore$

~~$f_6 = f_5$~~

$\therefore f_1 \leq f_4 \leq f_3 \leq f_2 \leq f_6 \leq f_5$

(5)

Let $f(n)$ and $g(n)$ be two functions such that

$$f(n) = O(g(n))$$ ~~for all~~

(i.e) $\exists$ two constants, $c\ (>0)$ and n_0 such that

$$f(n) \leq c \cdot g(n) \quad \forall \ n \geq n_0.$$

$$\Rightarrow \quad g(n) \geq \frac{1}{c} \cdot f(n) \quad \forall \ n \geq n_0$$

$$\Rightarrow \quad g(n) = \Omega(f(n)).$$