Solutions - Tutorial Sheet 7

Problem-1 Determine the order and degree of the following differential equations. Also state that whether they are linear or nonlinear:

(a) $\frac{dy}{dx} + 8x\left(\frac{dy}{dx}\right)^2 = x^2 y$.

Solⁿ: Order = 1, Degree = 2, Nonlinear

(b) $xy = \sqrt{x}\left(\frac{dy}{dx}\right) + k/\left(\frac{dy}{dx}\right)$

Solⁿ:

$$xy = \sqrt{x}\left(\frac{dy}{dx}\right) + \frac{k}{\left(\frac{dy}{dx}\right)}$$

$$\Rightarrow xy\frac{dy}{dx} = \sqrt{x}\left(\frac{dy}{dx}\right)^2 + k$$

Order = 1, Degree = 2, Nonlinear

(c) $\frac{dy}{dx} + x^2 y = x e^x$.

Solⁿ: Order = 1, Degree = 1, Linear

(d) $\frac{d^7x}{dt^7} + \left(\frac{d^5x}{dt^5}\right)\left(\frac{d^3x}{dt^3}\right) + x = t$

Solⁿ: Order = 7, Degree = 1, Non-Linear

(e) $\left(\frac{dr}{ds}\right)^4 = \sqrt{\left(\frac{d^2r}{ds^2}\right) + 1}$

Solⁿ: Order = 2, Degree = 1, Non-Linear

$$\left[\left(\frac{dr}{ds}\right)^4\right]^2 = \frac{d^2r}{ds^2} + 1 \qquad \text{(Squaring on both the sides)}$$

$$\Rightarrow \left(\frac{dr}{ds}\right)^8 = \frac{d^2r}{ds^2} + 1$$

Problem-2 Show that $y = a\cos(mx+b)$ is a solution of the DE

$$\frac{d^2y}{dx^2} + m^2y = 0.$$

Solution:

Given $y = a\cos(mx+b)$ ——— (1)

$$\Rightarrow \frac{dy}{dx} = -a\sin(mx+b)\cdot m = -am\sin(mx+b)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -am\cdot\cos(mx+b)\cdot(m)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -am^2\cos(mx+b) = -m^2y \quad (\text{Using } (1))$$

$$\Rightarrow \frac{d^2y}{dx^2} + m^2y = 0$$

$\Rightarrow y = a\cos(mx+b)$ is a solution of the given DE.

Problem-3: For each of the following families of curves, find a DE (of least order) for which each member of the family is a solution.

(a) $\{y = c_1e^x + c_2e^{-3x} : c_1, c_2 \in \mathbb{R}\}$ (b) $\{y = x\sin(x+c) : c \in \mathbb{R}\}$

Solution:

Given $y = c_1e^x + c_2e^{-3x}$, $c_1, c_2 \in \mathbb{R}$

$$\frac{dy}{dx} = c_1e^x - 3c_2e^{-3x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = c_1e^x + 9c_2e^{-3x}$$

Thus $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = 12c_2e^{-3x} = -3(-4c_2e^{-3x})$

$$= -3\left(\frac{dy}{dx} - y\right)$$

$$\left[\because \frac{dy}{dx} - y = -4c_2e^{-3x}\right]$$

$$\Rightarrow \boxed{\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 0}$$

Which is the required DE corresponding to the given family of curves.

(b) Given $y = x \sin(x+c)$ —— (1)

$\Rightarrow \frac{dy}{dx} = x\cos(x+c) + \sin(x+c)$ —— (2)

from (1), $\frac{y}{x} = \sin(x+c)$

$\Rightarrow \frac{y^2}{x^2} = \sin^2(x+c)$

$\Rightarrow \cos^2(x+c) = 1 - \sin^2(x+c) = 1 - \frac{y^2}{x^2}$

From (2), we get

$$\frac{dy}{dx} = x \cdot \sqrt{1 - \frac{y^2}{x^2}} + \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} = x \cdot \frac{\sqrt{x^2-y^2}}{x} + \frac{y}{x} = \sqrt{x^2-y^2} + \frac{y}{x}$$

$$\Rightarrow x\frac{dy}{dx} = x\sqrt{x^2-y^2} + y$$

$$\Rightarrow x\frac{dy}{dx} - y = x\sqrt{x^2-y^2}$$

$$\Rightarrow \left(x\frac{dy}{dx} - y\right)^2 = x^2(x^2-y^2)$$

Which is the required DE corresponding to the given family of curves.

Problem-4: Find the solution of the IVP

$$y\,dy = x\,dx, \quad y(0)=\beta, \quad \beta \in \mathbb{R}.$$

Solution: Given $y\,dy = x\,dx$

Integrating on both sides, we get

$$\frac{y^2}{2} = \frac{x^2}{2} + c_1$$

$$\Rightarrow y^2 = x^2 + c, \quad c = 2c_1$$

Since $y(0)=\beta \Rightarrow \beta^2 = 0 + c$

$$\Rightarrow c = \beta^2$$

Thus, we have, $y^2 = x^2 + \beta^2$

$$\Rightarrow y = \pm\sqrt{\beta^2+x^2}$$

If $\beta = 0$, then we have $y = \pm x$ are the solutions of the given IVP.

If $\beta \neq 0$, then

for $\beta > 0$, $y = \sqrt{\beta^2+x^2}$ is the solution.

($y = -\sqrt{\beta^2+x^2}$ is not the solution in this case as $y(0) \neq \beta$)

for $\beta < 0$, $y = -\sqrt{\beta^2+x^2}$ is the solution.

($y = \sqrt{\beta^2+x^2}$ is not the solution in this case)

Problem-5: Consider the equation $y'(x) = ky(x)$, $0 < x < \infty$, where c is a real constant. Then

(a) Show that if ϕ is any solution and $\psi(x) = \phi(x) e^{-kx}$, then $\psi(x)$ is a constant.

(b) If $k < 0$, then show that every solution tends to zero as $x \to \infty$.

Solution: Given equation is

$$y'(x) = k\, y(x) \quad \text{———} \; (1)$$

$$\Rightarrow \frac{y'}{y} = k$$

Integrating, we get

$$\log y = kx + c_1$$

$$\Rightarrow \quad y = c\, e^{kx}, \quad \text{where } c = e^{c_1}$$

(a) If ϕ is any solution of (1),

$$\Rightarrow \phi(x) = c\, e^{kx}$$

Given, $\psi(x) = \phi(x)\, e^{-kx} = c e^{kx} \cdot e^{-kx} = c$

$$\Rightarrow \psi(x) = c$$

$\Rightarrow$ $\psi(x)$ is a constant.

(b) Since $y(x) = c e^{kx}$,

If $k < 0$, then $y(x) \to 0$ as $x \to \infty$

Problem-6: What can you say about the solution of the DE (6)

$$\left|\frac{d^2y}{dx^2}\right| + \left|\frac{dy}{dx}\right| + y^2 + 2 = 0 \ ?$$

Solution:

Suppose $\phi(x)$ is a solution of the given DE on some interval I. Then

$$|\phi''(x)| + |\phi'(x)| + (\phi(x))^2 + 2 = 0, \ \forall x \in I.$$

But, $|\phi''(x)|, |\phi'(x)| \geq 0$ and $(\phi(x))^2 \geq 0$

Thus,

$$|\phi''(x)| + |\phi'(x)| + (\phi(x))^2 + 2 \geq 2, \ \forall x \in I,$$

which leads to a contradiction.

Problem-7 Consider the DE $\frac{dy}{dx} = y^4+6$.

(a) Show that there exist no constants solutions of the above DE.

(b) Is it possible for the solution curve to have any relative extrema?

Solution: (a) Given DE is

$$\frac{dy}{dx} = y^4+6$$

$$\Rightarrow \frac{dy}{dx} > 0 \quad \forall \quad x.$$

Therefore $y = y(x)$ (solution curve) should be strictly increasing.

That is why there exist no constant solutions of the given DE.

(b) As mentioned in the part (a), $\frac{dy}{dx} > 0$.

$\Rightarrow$ Solution curve $y(x)$ must be strictly increasing.

$\Rightarrow \frac{dy}{dx}$ can never be equal to zero.

It follows that the solution curve cannot have any relative extrema at any point on it.

8 The population of a certain country is known to increase at a rate proportional to the number of people presently living in the country. If after 2 years, the population has doubled, and after three years, the population is 20,000, estimate the number of people initially living in the country.

Sol: Let $N(t)$ denotes the number of people initially living in the country at time t and let N_0 denotes the number of people initially living in the country.

Using the given condition,

$$\frac{dN}{dt} \propto N$$

$$\Rightarrow \frac{dN}{dt} = kN$$

$$\Rightarrow \frac{dN}{N} = kt$$

$$\Rightarrow \log N = kt + \log C_1$$

$$\Rightarrow \log \frac{N}{C_1} = kt \Rightarrow \frac{N}{C_1} = e^{kt}$$

$$\Rightarrow N(t) = C_1 e^{kt}.$$

At $t=0$, $N = N_0$

$$\Rightarrow N_0 = C_1$$

Thus,

$$N(t) = N_0 e^{kt}.$$

At $t=2$, $N = 2N_0$

$$\Rightarrow 2N_0 = N_0 e^{k(2)}$$

$$\Rightarrow 2 = e^{k(2)} \Rightarrow 2k = \log 2 \Rightarrow k = \frac{1}{2}\log 2 = 0.347$$

Thus, we have

$$N(t) = N_0 \, e^{0.347t}$$

At $t=3$, $N = 20,000$.

$$\Rightarrow 20,000 = N_0 \, e^{0.347(3)} = N_0 \cdot (2.832)$$

$$\Rightarrow \boxed{N_0 = 7062}$$ Ans.

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Problem-9: Solve the following initial value problems:

(a) $\frac{dy}{dx} = (1+y^2)\tan x$, $y(0) = \sqrt{3}$

(b) $\frac{dy}{d\theta} = y \sin\theta$, $y(\pi) = -3$

Solution: (a) $\frac{dy}{dx} = (1+y^2)\tan x$ ——— (1)

$\Rightarrow \frac{dy}{1+y^2} = \tan x \, dx$

Integrating on both the sides, we get

$$\int \frac{dy}{1+y^2} = \int \tan x \, dx + c$$

$\Rightarrow \tan^{-1} y = -\log|\cos x| + c$ ——— (2)

Since $y(0) = \sqrt{3}$

$\Rightarrow \tan^{-1}\sqrt{3} = -\log|\cos 0| + c = c$

$\Rightarrow c = \tan^{-1}\sqrt{3}$

Substituting the value of c in (2), we get

$$\boxed{\tan^{-1} y = -\log|\cos x| + \tan^{-1}\sqrt{3}}$$ Ans.

(b) $\frac{dy}{d\theta} = y \sin\theta$, $y(\pi) = -3$

$\Rightarrow \frac{dy}{y} = \sin\theta \, d\theta$

Integrating, we get $\log|y| = -\cos\theta + \log c_1$

$\Rightarrow \log \frac{y}{c_1} = -\cos\theta \Rightarrow \frac{y}{c_1} = e^{-\cos\theta}$

$$\Rightarrow \quad y = c_1 e^{-\cos\theta}$$

Since $\quad y(\pi) = -3 \quad \Rightarrow \quad -3 = c_1 e^{-\cos\pi} = c_1 e^{-(-1)} = c_1 e$

$$\Rightarrow \quad c_1 = \frac{-3}{e}$$

Thus, we have $\boxed{y = \frac{-3}{e} e^{-\cos\theta}}$ Ans

Problem-10: Solve the following ODEs:

(a) $(x^3+3xy^2)\,dx + (y^3+3x^2y)\,dy = 0$.

Given $\quad \frac{dy}{dx} = \frac{-(x^3+3xy^2)}{y^3+3x^2y}$. ——— (1)

Put $\quad y = vx$

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Using this, (1) becomes,

$$v + x\frac{dv}{dx} = \frac{-(1+3v^2)}{v^3+3v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\frac{(1+3v^2)}{v^3+3v} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\frac{1-3v^2-v^4-3v^2}{v^3+3v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{-(v^4+6v^2+1)}{v^3+3v}$$

$$\Rightarrow \quad \frac{-(v^3+3v)}{v^4+6v^2+1}\,dv = \frac{dx}{x}$$

$$\Rightarrow \quad -\frac{1}{4}\,\frac{(4v^3+12v)}{v^4+6v^2+1}\,dv = \frac{dx}{x}$$

Integrating on both sides, we get

$$-\frac{1}{4}\int\left(\frac{4v^3+12v}{v^4+6v^2+1}\right)dv = \int\frac{dx}{x} + \log c, \quad \text{(c is an arbitrary constant)}$$

$$\Rightarrow -\frac{1}{4}\log(v^4+6v^2+1) = \log x + \log c$$

$$\Rightarrow -\frac{1}{4}\log\left(\frac{y^4}{x^4}+6\frac{y^2}{x^2}+1\right) = \log cx \qquad \left[\because v = \frac{y}{x}\right]$$

$$\Rightarrow \log\left(\frac{y^4}{x^4}+6\frac{y^2}{x^2}+1\right)^{-1/4} = \log cx$$

$$\Rightarrow \left(\frac{y^4}{x^4}+\frac{6y^2}{x^2}+1\right)^{-1} = c^4x^4$$

$$\Rightarrow \left(\frac{y^4+6x^2y^2+x^4}{x^4}\right)^{-1} = c^4x^4$$

$$\Rightarrow \frac{x^4}{y^4+6x^2y^2+x^4} = c^4x^4$$

$$\Rightarrow x^4+y^4+6x^2y^2 = \frac{1}{c^4} = c_1 \qquad \left(\text{where } c_1 = \frac{1}{c^4}\right)$$

$$\Rightarrow \boxed{x^4+y^4+6x^2y^2 = c_1}$$

Ans.

(b) $\left(x\tan\frac{y}{x}+y\right)dx - x\,dy = 0$

Solution:

$$\frac{dy}{dx} = \frac{x\tan\left(\frac{y}{x}\right)+y}{x} \quad \text{—— (1)}$$

Put $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Using this (1) becomes,

$$v + x\frac{dv}{dx} = \frac{x\tan(v) + vx}{x} = v + \tan v$$

$$\Rightarrow \quad x\frac{dv}{dx} = v + \tan v - v = \tan v$$

$$\Rightarrow \quad \frac{dv}{\tan v} = \frac{dx}{x}$$

$$\Rightarrow \quad \frac{\cos v}{\sin v}\,dv = \frac{dx}{x}$$

$$\Rightarrow \quad \log \sin v = \log x + \log c$$

$$\Rightarrow \quad \sin v = cx$$

$$\Rightarrow \quad \sin\left(\frac{y}{x}\right) = cx, \qquad \left[\because v = \frac{y}{x}\right]$$

where c is an arbitrary constant.

(C) $$\frac{dy}{dx} = \frac{4x + 6y + 5}{3y + 2x + 4}$$

$$\left[\frac{dy}{dx} = \frac{2(2x+3y)+5}{2x+3y+4}\right] \quad (1)$$

Observe that this DE is of the form

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}, \quad \text{where } \frac{a_1}{a_2} = \frac{b_1}{b_2}.$$

Put $2x + 3y = z$ (2) $\Rightarrow 2 + 3\frac{dy}{dx} = \frac{dz}{dx}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{3}\left(\frac{dz}{dx} - 2\right) \qquad \text{——— } (3)$$

Using (2) and (3), (1) becomes

$$\frac{1}{3}\left(\frac{dz}{dx} - 2\right) = \frac{2z+5}{z+4}$$

$$\Rightarrow \quad \frac{dz}{dx} = \frac{3(2z+5)}{z+4} + 2 = \frac{8z+23}{z+4}$$

$$\Rightarrow \quad \frac{z+4}{8z+23}\,dz = dx$$

$$\Rightarrow \quad \left[\frac{1}{8} + \frac{9}{8(8z+23)}\right] dz = dx$$

$$\Rightarrow \quad \frac{z}{8} + \frac{9}{8}\left(\frac{\log(8z+23)}{8}\right) = x + c$$

$$\Rightarrow \quad \frac{2x+3y}{8} + \frac{9}{64}\log\left(8(2x+3y)+23\right) = x + c$$

$$\Rightarrow \quad \frac{2x+3y}{8} + \frac{9}{64}\log\left(16x+24y+23\right) = x + c$$

$$\Rightarrow \quad 2x+3y + \frac{9}{8}\log(16+24y+23) = 8x + 8c$$

$$\Rightarrow \quad -6x+3y + \frac{9}{8}\log(16x+24y+23) = 8c$$

$$\Rightarrow \quad \boxed{y - 2x + \frac{3}{8}\log(16x+24y+23) = \frac{8c}{3} = c',}$$

where c' is an arbitrary constant. Ans

(d) $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$

The above DE is of the form

$$\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}, \quad \text{where } \frac{a_1}{a_2} = \frac{1}{2} \neq \frac{b_1}{b_2} = \frac{2}{1}.$$

Use the substitution $x = X+h$ and $y = Y+k$, ——— (1)

where h and k are the constants to be determined.

Then we have $dx = dX$, $dy = dY$ and the given equation becomes

$$\frac{dY}{dX} = \frac{(X+2Y)+(h+2k-3)}{(2X+Y)+(2h+k-3)} \quad \text{——— (2)}$$

Choose h and k such that

$$h+2k-3=0 \quad \text{and} \quad 2h+k-3=0, \quad \text{——— (3)}$$

Solving these equations, we get $h=1$, $k=1$.

So from (1), we have $X = x-1$ and $Y = y-1$.

Using (3) in (2), we get

$$\frac{dY}{dX} = \frac{X+2Y}{2X+Y} \quad \text{——— (4)}$$

which is a homogeneous DE.

Take $Y = VX \Rightarrow \frac{dY}{dX} = V + X\frac{dV}{dX}$.

Thus (4) becomes,

$$V + X\frac{dV}{dX} = \frac{X+2VX}{2X+VX} = \frac{1+2V}{2+V}$$

$$\Rightarrow X\frac{dV}{dX} = \frac{1+2V}{2+V} - V = \frac{1+2V-2V-V^2}{2+V}$$

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$$\Rightarrow \quad X\frac{dV}{dX} = \frac{1-V^2}{2+V}$$

$$\Rightarrow \quad \frac{2+V}{1-V^2}\,dV = \frac{dX}{X}$$

$$\Rightarrow \quad \frac{2+V}{(1-V)(1+V)}\,dV = \frac{dX}{X}$$

$$\Rightarrow \quad \left[\frac{1}{2(1+V)} + \frac{3}{2(1-V)}\right] dV = \frac{dX}{X}$$

$$\Rightarrow \quad \frac{1}{2}\log(1+V) + \frac{3}{2}\frac{\log(1-V)}{-1} = \log X + \log C$$

$$\Rightarrow \quad \frac{1}{2}\left[\log(1+V) - \log(1-V)^3\right] = \log CX$$

$$\Rightarrow \quad \frac{1}{2}\log\frac{(1+V)}{(1-V)^3} = \log CX$$

$$\Rightarrow \quad \log\frac{(1+V)}{(1-V)^3} = \log C^2X^2$$

$$\Rightarrow \quad \frac{1+\frac{Y}{X}}{\left(1-\frac{Y}{X}\right)^3} = C^2X^2$$

$$\Rightarrow \quad \left(\frac{X+Y}{X}\right) \times \left(\frac{X^3}{(X-Y)^3}\right) = C^2X^2$$

$$\Rightarrow \quad \frac{X^2(X+Y)}{(X-Y)^3} = C^2X^2$$

$$\Rightarrow \quad (X+Y) = C^2(X-Y)^3 \quad \Rightarrow \quad C^2(X-Y)^3 = X+Y$$

$$\Rightarrow c^2\left[(x-1)-(y-1)\right]^3 = x-1+y-1$$

$$\left[\because X = x-1,\ Y = y-1\right]$$

$$\Rightarrow c^2(x-y)^3 = x+y-2$$

$$\Rightarrow c'(x-y)^3 = x+y-2,$$ where $c' = c^2$ is an arbitrary constant.