Ques 1: Find the Laplace transform of the following functions.

(a) t^2+at+b

Solution: (a) Given function is

$$f(t) = t^2 + at + b.$$

$$\Rightarrow L[f(t)] = L[t^2] + a\,L[t] + b\cdot L[1]$$

$$= \frac{2!}{s^3} + a\cdot\frac{1}{s^2} + b\cdot\frac{1}{s}$$

$\left[\because L[t^n] = \frac{n!}{s^{n+1}}\right.$ if n is an integer and $\left.L[1] = \frac{1}{s}\right]$

$$= \frac{2}{s^3} + \frac{a}{s^2} + \frac{b}{s}$$

$$\Rightarrow \boxed{L[f(t)] = \frac{2}{s^3} + \frac{a}{s^2} + \frac{b}{s}}$$ Ans

(b) $3\sin 5t - 2\cos 3t$

Solution: Given function is

$$f(t) = 3\sin 5t - 2\cos 3t$$

$$\Rightarrow L[f(t)] = 3\,L[\sin 5t] - 2\,L[\cos 3t]$$

$$= 3\cdot\frac{5}{s^2+(5)^2} - 2\cdot\frac{s}{s^2+(3)^2}$$

$\left[\because L[\sin at] = \frac{a}{s^2+a^2},\ s>0\right.$

$\left.L[\cos at] = \frac{s}{s^2+a^2},\ s>0\right]$

$$\Rightarrow L[f(t)] = \frac{15}{s^2+25} - \frac{2s}{s^2+9}$$

$$\Rightarrow \boxed{L[3\sin 5t - 2\cos 3t] = \frac{15}{s^2+25} - \frac{2s}{s^2+9}}$$ Ans

(c) te^{5t}

Solution:

Given function is

$$f(t) = t\,e^{5t}.$$

Since $L[t] = \frac{1}{s^2}$ $\left[\because L[t^n] = \frac{n!}{s^{n+1}},\ s>0 \text{ if } n \text{ is an integer}\right]$

$$\Rightarrow \boxed{L[te^{5t}] = \frac{1}{(s-5)^2}} \quad \text{Ans.}$$

$$\boxed{\because \text{If } L[f(t)] = F(s), \text{ then } L[e^{at} f(t)] = F(s-a).}$$

Alternative way:

Given function is

$$f(t) = te^{5t}.$$

Since $L[e^{5t}] = \frac{1}{s-5}$ $\left[\because L[e^{at}] = \frac{1}{s-a},\ s>a\right]$

$$\Rightarrow L[te^{5t}] = (-1)^1 \frac{d}{ds}\left(\frac{1}{s-5}\right) = -\frac{d}{ds}\left(\frac{1}{s-5}\right)$$

$$\left[\because \text{If } L[f(t)] = F(s), \text{ then } L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}(F(s))\right]$$

$$\Rightarrow L[te^{5t}] = (-)(-1)(s-5)^{-2} = \frac{1}{(s-5)^2}$$

$$\Rightarrow \boxed{L[te^{5t}] = \frac{1}{(s-5)^2}} \quad \text{Ans}$$

(d) $t^2 e^{-at} \sin bt$.

Solution: Given function is

$$f(t) = t^2 e^{-at} \sin bt.$$

Since $L[\sin bt] = \dfrac{b}{s^2+b^2}$ $\quad \left[\because L[\sin at] = \dfrac{a}{s^2+a^2},\ s>0\right]$

$$\Rightarrow L[e^{-at}\sin bt] = \frac{b}{(s+a)^2+b^2}$$

[If $L[f(t)] = F(s)$, then
$L[e^{at} f(t)] = F(s-a)$]

Thus $L[t^2 e^{-at} \sin bt] = (-1)^2 \dfrac{d^2}{ds^2}\left(\dfrac{b}{(s+a)^2+b^2}\right)$

[If $L[f(t)] = F(s)$, then
$L[t^n f(t)] = (-1)^n \dfrac{d^n}{ds^n}(F(s))$]

$$\Rightarrow L[t^2 e^{-at}\sin bt] = \frac{d^2}{ds^2}\left(\frac{b}{(s+a)^2+b^2}\right)$$

$$= \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{b}{(s+a)^2+b^2}\right)\right]$$

$$= \frac{d}{ds}\left[\frac{-2b(s+a)}{((s+a)^2+b^2)^2}\right]$$

$$\Rightarrow \boxed{L[t^2 e^{-at}\sin bt] = \frac{2b\left((s+a)^2-b^2\right)}{(s+a)^2+b^2}}$$

Ans

(e) $t e^{2t} \sin 4t$.

Solution: Given function is

$$f(t) = t e^{2t} \sin 4t.$$

Since $L[\sin 4t] = \dfrac{4}{s^2+16}$ $\qquad \left[\because L[\sin at] = \dfrac{a}{s^2+a^2}\right]$

$$\Rightarrow L[e^{2t} \sin 4t] = \frac{4}{(s-2)^2+16}$$

$$\left[\because \text{If } L[f(t)] = F(s), \text{ then } L[e^{at} f(t)] = F(s-a)\right]$$

Thus $L[t e^{2t} \sin 4t] = (-1) \dfrac{d}{ds}\left(\dfrac{4}{(s-2)^2+16}\right)$

$$\left[\text{If } L[f(t)] = F(s), \text{ then } L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}(F(s))\right]$$

$$\Rightarrow L[t e^{2t} \sin 4t] = -\frac{d}{ds}\left(\frac{4}{s^2-4s+4+16}\right)$$

$$= \frac{-d}{ds}\left(\frac{4}{s^2-4s+20}\right)$$

$$= \frac{-d}{ds}\left(\frac{4}{s^2-4s+20}\right)$$

$$= (-4)(-1)(s^2-4s+20)^{-2} \cdot (2s-4)$$

$$\Rightarrow L[t e^{2t} \sin 4t] = \frac{4(2s-4)}{(s^2-4s+20)^2} = \frac{8(s-2)}{(s^2-4s+20)^2}$$

$$\Rightarrow \boxed{L[t e^{2t} \sin 4t] = \frac{8(s-2)}{(s^2-4s+20)^2}} \quad \text{Ans}$$

(f) $\cos(\omega t+\theta)$.

Solution: Here $f(t) = \cos(\omega t+\theta)$

$$\Rightarrow f(t) = \cos\omega t\cdot\cos\theta - \sin\omega t\cdot\sin\theta$$

Thus

$$L[f(t)] = L[\cos\omega t\cdot\cos\theta] - L[\sin\omega t\cdot\sin\theta]$$

$$= \cos\theta\cdot L[\cos\omega t] - \sin\theta\cdot L[\sin\omega t]$$

$$= \cos\theta\left(\frac{s}{s^2+\omega^2}\right) - \sin\theta\cdot\left(\frac{\omega}{s^2+\omega^2}\right)$$

$\left[\because L[\cos at] = \frac{s}{s^2+a^2} \text{ and } L[\sin at] = \frac{a}{s^2+a^2}\right]$

$$\Rightarrow \boxed{L[f(t)] = \frac{1}{s^2+\omega^2}\left[s\cos\theta - \omega\sin\theta\right]}$$ Ans.

(g) $t^n\cdot e^{at}$

Solution: Here $f(t) = t^n\cdot e^{at}$

Since $L[t^n] = \frac{n!}{s^{n+1}}$, $s>0$

$$\Rightarrow L[t^n\cdot e^{at}] = \frac{n!}{(s-a)^{n+1}}$$

$$\left[\because \text{If } L[f(t)] = F(s), \text{ then } L[e^{at}\cdot f(t)] = F(s-a)\right]$$

Thus $$\boxed{L[t^n\cdot e^{at}] = \frac{n!}{(s-a)^{n+1}}}$$ Ans

Problem-2 Find a function $f(t)$ such that $F(s) = \frac{4}{(s-1)^3}$.

Solution: $f(t) = L^{-1}[F(s)] = L^{-1}\left[\frac{4}{(s-1)^3}\right]$

Since $L^{-1}\left[\frac{1}{s^3}\right] = \frac{t^2}{2!}$ $\left[\because \text{ If } L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!}, \; n \text{ is an integer}\right]$

$$\Rightarrow L^{-1}\left[\frac{1}{(s-1)^3}\right] = e^t \cdot \frac{t^2}{2!}$$

$$\left[\because \text{ If } L^{-1}[F(s)] = f(t), \text{ then } L^{-1}[F(s-a)] = e^{at} f(t)\right]$$

$$\Rightarrow L^{-1}\left[\frac{4}{(s-1)^3}\right] = 4e^t \cdot \frac{t^2}{2!} = 2t^2 e^t.$$

$$\Rightarrow \boxed{f(t) = L^{-1}[F(s)] = 2t^2 e^t.}$$

Ques 3 (a) Find the inverse Laplace transform of the following functions.

(a) $\frac{1}{s(s+1)}$

Solution: Given function is

$$F(s) = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1} \quad \text{(Using partial fractions)}$$

$$\Rightarrow L^{-1}[F(s)] = L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s+1}\right)$$

$$= 1 - e^{-t} \quad \left[\because L^{-1}\left(\frac{1}{s-a}\right) = e^{at} \text{ and } L^{-1}\left(\frac{1}{s}\right) = 1\right]$$

$$\Rightarrow \boxed{L^{-1}[F(s)] = 1 - e^{-t}}$$

3(b) $\dfrac{s-5}{s^2-10s+61}$

Solution: Given $F(s) = \dfrac{s-5}{s^2-10s+61} = \dfrac{s-5}{s^2-10s+25-25+61}$

$$= \frac{s-5}{(s-5)^2+36} = \frac{s-5}{(s-5)^2+(6)^2}$$

$$\Rightarrow L^{-1}[F(s)] = L^{-1}\left[\frac{s-5}{(s-5)^2+(6)^2}\right]$$

Since $L^{-1}\left[\dfrac{s}{s^2+6^2}\right] = \cos 6t$ $\quad \left[\because L^{-1}\left[\dfrac{s}{s^2+a^2}\right] = \cos at\right]$

$$\Rightarrow L^{-1}\left[\frac{s-5}{(s-5)^2+(6)^2}\right] = e^{5t}\cos 6t$$

$\left[\because L^{-1}[F(s)] = f(t), \text{ then } L^{-1}[F(s-a)] = e^{at} f(t)\right]$

$$\Rightarrow \boxed{L^{-1}[F(s)] = L^{-1}\left[\frac{s-5}{(s-5)^2+(6)^2}\right] = e^{5t}\cos 6t}$$ Ans.

3(c) $\dfrac{(s+1)(s+3)}{s(s+2)(s+8)}$

Solution: Given $F(s) = \dfrac{(s+1)(s+3)}{s(s+2)(s+8)} = \dfrac{A}{s} + \dfrac{B}{s+2} + \dfrac{C}{s+8}$

$$\Rightarrow F(s) = \frac{(s+1)(s+3)}{s(s+2)(s+8)} = \frac{3}{16s} + \frac{1}{12(s+2)} + \frac{35}{48(s+8)}$$

(10)

$$\Rightarrow L^{-1}[F(s)] = L^{-1}\left[\frac{(s+1)(s+3)}{s(s+2)(s+8)}\right]$$

$$= L^{-1}\left[\frac{3}{16s} + \frac{1}{12(s+2)} + \frac{35}{48(s+8)}\right]$$

$$= \frac{3}{16} L^{-1}\left[\frac{1}{s}\right] + \frac{1}{12} L^{-1}\left[\frac{1}{s+2}\right] + \frac{35}{48} L^{-1}\left[\frac{1}{s+8}\right]$$

$$\left[\because L^{-1}[a_1 F_1(s) + a_2 F_2(s)] = a_1 L^{-1}[F_1(s)] + a_2 L^{-1}[F_2(s)]\right]$$

$$= \frac{3}{16}\cdot 1 + \frac{1}{12} e^{-2t} + \frac{35}{48}\cdot e^{-8t} \qquad \left[\because L^{-1}\left[\frac{1}{s-a}\right] = e^{at} \text{ and } L^{-1}\left[\frac{1}{s}\right] = 1\right]$$

$$\Rightarrow f(t) = \frac{3}{16} + \frac{1}{12} e^{-2t} + \frac{35}{48} e^{-8t}.$$

Thus

$$\boxed{L^{-1}[F(s)] = f(t) = \frac{3}{16} + \frac{1}{12} e^{-2t} + \frac{35}{48} e^{-8t}}$$

Ans.

Ques 4: Solve the following initial value problems using Laplace transforms

(a) $y' + 4y = e^t$, $y(0) = 2$.

Solution: Given DE is

$$y' + 4y = e^t.$$

Taking Laplace transform on both the sides, we get

$$L[y'] + 4L[y] = L[e^t]$$

$$\Rightarrow s\,L[y(t)] - y(0) + 4\,L[y(t)] = \frac{1}{s-1}$$

$$\left[\begin{array}{l} \because \text{ If } L[f(t)] = F(s), \text{ then} \\ \qquad L[f'(t)] = s\,F(s) - f(0) \\ \text{and} \qquad L[e^{at}] = \dfrac{1}{s-a} \end{array}\right]$$

$$\Rightarrow (s+4)\,L[y(t)] - 2 = \frac{1}{s-1} \qquad [\because y(0) = 2]$$

$$\Rightarrow (s+4)\,L[y(t)] = \frac{1}{s-1} + 2 \quad \boxed{\cancel{= \frac{1 + 2s - 2}{s-1}} \;\cancel{\neq \frac{2s-1}{s-1}}}$$

$$\Rightarrow L[y(t)] = \frac{1}{(s-1)(s+4)} + \frac{2}{s+4}$$

$$\Rightarrow L[y(t)] = \frac{1}{5(s-1)} + \frac{1}{(-5)(s+4)} + \frac{2}{s+4}$$

[Use Partial fractions]

$$\Rightarrow L[y(t)] = \frac{1}{5(s-1)} + \frac{9}{5(s+4)}$$

$$\Rightarrow y(t) = \frac{1}{5}L^{-1}\left[\frac{1}{s-1}\right] + \frac{9}{5}L^{-1}\left[\frac{1}{s+4}\right] = \frac{1}{5}e^t + \frac{9}{5}e^{-4t}$$

(Taking inverse Laplace transform on both sides). Ans.

4(b) $y'' - 2y' - 3y = 10 \sinh 2t, \quad y(0) = 0, \quad y'(0) = 4.$

Solution: Given DE is

$$y'' - 2y' - 3y = 10 \sinh 2t$$

Taking Laplace transform on both sides, we get

$$L[y''] - 2L[y'] - 3L[y] = 10 L[\sinh 2t]$$

$$\Rightarrow s^2 L[y(t)] - s\,y(0) - y'(0) - 2[s L[y(t)] - y(0)] - 3L[y(t)] = 10\left(\frac{2}{s^2-4}\right) \qquad \left[\because L[\sinh at] = \frac{a}{s^2-a^2}\right]$$

$$\Rightarrow s^2 L[y(t)] - s(0) - 4 - 2s L[y(t)] + 2(0) - 3L[y(t)] = \frac{20}{s^2-4}$$

$$\Rightarrow (s^2-2s-3) L[y(t)] = 4 + \frac{20}{s^2-4} = \frac{4s^2-16+20}{s^2-4}$$

$$\Rightarrow (s^2-2s-3) L[y(t)] = \frac{4(s^2+1)}{(s^2-4)}$$

$$\Rightarrow L[y(t)] = \frac{4(s^2+1)}{(s^2-4)(s^2-2s-3)} = \frac{4(s^2+1)}{(s-2)(s+2)(s-3)(s+1)}$$

$$\Rightarrow L[y(t)] = \frac{-5}{3(s-2)} + \frac{1}{s+2} + \frac{2}{s-3} + \frac{2}{3(s+1)}$$

$\Rightarrow$ Taking inverse Laplace transform on both sides, we get

$$y(t) = \frac{-5}{3} L^{-1}\left(\frac{1}{s-2}\right) + L^{-1}\left(\frac{1}{s+2}\right) + 2L^{-1}\left(\frac{1}{s-3}\right) + \frac{2}{3} L^{-1}\left(\frac{1}{s+1}\right)$$

$$\Rightarrow \boxed{y(t) = \frac{-5}{3} e^{2t} + e^{-2t} + 2e^{3t} + \frac{2}{3} e^{-t}.}$$

Which is the solution of the given IVP.

Ques 5: Solve the following system of differential equations using Laplace transforms.

(a)
$$\left.\begin{aligned} y_1' + y_2 &= 2\cos x \\ y_1 + y_2' &= 0 \end{aligned}\right\}$$

$$y_1(0) = 0, \quad y_2(0) = 1.$$

Solution:

Given system of DE is

$$y_1' + y_2 = 2\cos x \qquad \text{———} (1)$$

$$y_1 + y_2' = 0 \qquad \text{———} (2)$$

Taking the Laplace transform of both sides of the above differential equations, we get

$$\left.\begin{aligned} L[y_1'] + L[y_2] &= L[2\cos x] \\ L[y_1] + L[y_2'] &= L[0] \end{aligned}\right\}$$

$$\Rightarrow \quad s\,L[y_1(x)] - y_1(0) + L[y_2(x)] = 2 \cdot \frac{s}{s^2+1}$$

$$L[y_1(x)] + s\,L[y_2(x)] - y_2(0) = 0$$

$$\left[\because \text{If } L[f(t)] = F(s), \text{ then } L[f'(t)] = s\,F(s) - f(0) \right]$$

$$\Rightarrow \quad s\,L[y_1(x)] - 0 + L[y_2(x)] = \frac{2s}{s^2+1}$$

$$\Rightarrow \quad s\,L[y_1(x)] + L[y_2(x)] = \frac{2s}{s^2+1}$$

and
$$L[y_1(x)] + s\,L[y_2(x)] = 1 \qquad [\because y_2(0) = 1]$$

Thus we have two equations

$$\left.\begin{aligned} s\,L[y_1(x)] + L[y_2(x)] &= \frac{2s}{s^2+1} \\ \text{and} \quad L[y_1(x)] + s\,L[y_2(x)] &= 1 \end{aligned}\right\}$$

Denote $L[y_1(x)] = Y_1(s)$ and $L[y_2(x)] = Y_2(s)$, then the above system of equations becomes

$$\left.\begin{aligned} s\,Y_1(s) + Y_2(s) &= \frac{2s}{s^2+1} \\ Y_1(s) + s\,Y_2(s) &= 1 \end{aligned}\right\}$$

Solving the above system of equations, we get

$$Y_1(s) = \frac{1}{s^2+1} \quad \text{and} \quad Y_2(s) = \frac{s}{s^2+1}$$

$$\Rightarrow \quad L[y_1(x)] = \frac{1}{s^2+1} \quad \text{and} \quad L[y_2(x)] = \frac{s}{s^2+1}$$

$$\Rightarrow \quad y_1(x) = L^{-1}\left[\frac{1}{s^2+1}\right] = \sin x \quad \text{and} \quad y_2(x) = L^{-1}\left[\frac{s}{s^2+1}\right] = \cos$$

Thus we have

$$\boxed{\left.\begin{aligned} y_1(x) &= \sin x \\ \text{and} \quad y_2(x) &= \cos x \end{aligned}\right\}}$$

Ans.

is the solution of the given system of differential equ

(b) $$\left.\begin{array}{l} x'' - 6x + 3y = 8e^t \\ y' - 2x - y = 4e^t \end{array}\right\}, \quad \begin{array}{l} x(0) = -1 \\ y(0) = 10 \end{array}$$ (25)

Solution:

Given system of DE is

$$\left.\begin{array}{l} x' - 6x + 3y = 8e^t \\ y' - 2x - y = 4e^t \end{array}\right\}$$

Taking the Laplace transform of both sides of the above differential equations, we get

$$L[x'] - 6L[x] + 3L[y] = 8L[e^t]$$
$$L[y'] - 2L[x] - L[y] = 4L[e^t]$$

$$\Rightarrow \left.\begin{array}{l} sL[x(t)] - x(0) - 6L[x(t)] + 3L[y(t)] = \dfrac{8}{s-1} \\ sL[y(t)] - y(0) - 2L[x(t)] - L[y(t)] = \dfrac{4}{s-1} \end{array}\right\}$$

$$\left[\because \text{ If } L[f(t)] = F(s), \text{ then } L[f'(t)] = sF(s) - f(0)\right]$$

$$\Rightarrow (s-6)L[x(t)] - (-1) + 3L[y(t)] = \frac{8}{s-1} \qquad [\because x(0) = -1]$$

$$\Rightarrow (s-6)L[x(t)] + 3L[y(t)] = \frac{8}{s-1} - 1 = \frac{8-s+1}{s-1} = \frac{9-s}{s-1}$$

and

$$(s-1)L[y(t)] - 2L[x(t)] = \frac{4}{s-1} \qquad [\because y(0) = 0]$$

$$\Rightarrow -2L[x(t)] + (s-1)L[y(t)] = \frac{4}{s-1}$$

Thus we have two equations

$$(s-6)\,L[x(t)] + 3\,L[y(t)] = \frac{9-s}{s-1}$$

and $$-2\,L[x(t)] + (s-1)\,L[y(t)] = \frac{4}{s-1}$$

Denote $L[x(t)] = X(s)$ and $L[y(t)] = Y(s)$, then the above system of equations becomes

$$(s-6)X(s) + 3\,Y(s) = \frac{9-s}{s-1}$$

and $$-2\,X(s) + (s-1)\,Y(s) = \frac{4}{s-1}$$

Solving the above system of equations, we get

$$X(s) = \frac{7-s}{(s-4)(s-1)} \quad \text{and} \quad Y(s) = \frac{2}{(s-1)(s-4)}$$

$$\Rightarrow X(s) = \frac{1}{s-4} - \frac{2}{s-1} \quad \text{and} \quad Y(s) = \frac{-2}{3(s-1)} + \frac{2}{3(s-4)}$$

$$\Rightarrow L[x(t)] = \frac{1}{s-4} - \frac{2}{s-1} \quad \text{and} \quad L[y(t)] = \frac{-2}{3(s-1)} + \frac{2}{3(s-4)}$$

$$\Rightarrow x(t) = L^{-1}\left[\frac{1}{s-4}\right] - 2\,L^{-1}\left[\frac{1}{s-1}\right] \quad \text{and} \quad y(t) = -\frac{2}{3}L^{-1}\left[\frac{1}{s-1}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{s-4}\right]$$

$$\Rightarrow x(t) = e^{4t} - 2e^{t} \quad \text{and} \quad y(t) = -\frac{2}{3}e^{t} + \frac{2}{3}e^{4t}$$

Thus we have

$$x(t) = e^{4t} - 2e^{t}$$

and $$y(t) = -\frac{2}{3}e^{t} + \frac{2}{3}e^{4t}$$

is the required solution of the given system of DE's.