Department of Mathematics, Bennett University
Engineering Calculus (EMAT101L)
Practice Problem Sheet 1

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1. Find the infimum and supremum of the set $S = \{\frac{m}{|m|+n} : n \in \mathbb{N}, m \in \mathbb{Z}\}$.

2. Let $\{a_n\}$ be a sequence of real numbers such that each of the subsequences $\{a_{2n}\}$, $\{a_{2n-1}\}$ and $\{a_{3n}\}$ converges. Show that $\{a_n\}$ is convergent.

3. If $b > 0$, then show that $\lim\limits_{n\to\infty} \sqrt[n]{b} = 1$.

4. For $a \in [0, 3]$, let $x_1 = a$ and $x_{n+1} = \dfrac{1}{4}(x_n^2 + 3)$ for all $n \geq 2$. Examine the convergence of the sequence $\{x_n\}$ for different values of a. Also, find $\lim x_n$ whenever it exists.

5. Given $a, b \in \mathbb{R}$, let $x_1 = a, x_2 = b$ and $x_n = \dfrac{1}{2}(x_{n-1} + x_{n-2})$ for $n \geq 3$. Show that $\{x_n\}$ is a Cauchy sequence and show that $\lim\limits_{n\to\infty} x_n = \dfrac{1}{3}(a + 2b)$.

6. Let $\{a_n\}$ be a sequence of real numbers. Define the sequence $\{s_n\}$ by $s_n = \dfrac{1}{n}\sum\limits_{i=1}^{n} a_i$. If $\{a_n\}$ converges to a, then show that the sequence $\{s_n\}$ also converges to a. But converse is not true.

7. Show that the sequence defined below is bounded monotonically increasing sequence and converges to $\sqrt{3}$
$$a_1 = 1, a_{n+1} = \left(\frac{3 + a_n^2}{2}\right)^{1/2}.$$

8. Check if the following sequences are Cauchy sequences or not

   (a) $a_n = \sum\limits_{k=1}^{n} \frac{1}{k!}$

   (b) $a_1 = 1, a_{n+1} = \left(1 + \frac{(-1)^n}{2^n}\right) a_n, n \in \mathbb{N}$.

Solutions for Practice Problem Sheet 1

1. S is bounded above by 1 and bounded below by -1, $\sup = 1$ and $\inf = -1$.

2. Let $a_{2n} \to a, a_{2n-1} \to b$ and $a_{3n} \to c$. Clearly, $\{a_{6n}\}$ is a subsequence of $\{a_{2n}\}$ and $\{a_{3n}\}$. Hence, $a_{6n} \to a$ and $a_{6n} \to c$. This implies $a = c$.
   Again, $\{a_{3(2n-1)}\}$ is a subsequence of $\{a_{2n-1}\}$ and $\{a_{3n}\}$. Hence, $a_{3(2n-1)} \to c$ and $a_{3(2n-1)} \to b$. This implies $b = c$. Hence, $a = b = c$. Now show that if the subsequences $\{a_{2n}\}$ and $\{a_{2n-1}\}$ both converge to the same limit, then $\{a_n\}$ is also converges to the same limit.

3. First assume that $b > 1$. Let $a_n = b^{\frac{1}{n}} - 1$. As $b > 1$, $a_n > 0$ for all $n \in \mathbb{N}$. Further,
   $$b = (1 + a_n)^n \geq 1 + na_n.$$
   Then $0 \leq a_n \leq \dfrac{b-1}{n}$. Thus $a_n \to 0$, i.e., $b^{\frac{1}{n}} \to 1$ as $n \to \infty$.
   Now if $b < 1$, then take $c = \frac{1}{b}$ and it is easy to show the result.

4. If $\{x_n\}$ converges, then $\ell = \lim x_n$ satisfies $\ell^2 - 4\ell + 3 = 0$. Hence $\ell = 1$ or $\ell = 3$. We have $x_{n+1} - x_n = \frac{1}{4}(x_n^2 - x_{n-1}^2)$ for all $n > 1$. Also $x_2 - x_1 = \frac{1}{4}(a-1)(a-3)$.
   Case 1: If $a = 3$, then $x_n = 3$ for all $n \in \mathbb{N}$, and hence $\{x_n\}$ converges to 3.
   Case 2: If $1 < a < 3$, then $x_2 < x_1$ and we get $x_{n+1} < x_n$ for all $n \in \mathbb{N}$. Also in this case $x_n > 1$ for all $n \in \mathbb{N}$. (Because $x_{n+1} - 1 = \frac{1}{4}(x_n^2 - 1)$ for all $n \in \mathbb{N}$ and $x_1 > 1$.) Hence $\{x_n\}$ converges to 1. Note that $x_n \not\to 3$ as $\lim x_n = \inf\{x_n : n \in \mathbb{N}\} \leq x_1 = a < 3$.
   Case 3: If $0 \leq a \leq 1$, then $x_2 \geq x_1$ and we get $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$. Also in this case $x_n \leq 1$ for all $n \in \mathbb{N}$. Hence $\{x_n\}$ converges to 1.

5. We have $x_{n+2} - x_{n+1} = (-\frac{1}{2})(x_{n+1} - x_n)$ for all $n \in \mathbb{N}$, so that $|x_{n+2} - x_{n+1}| = \frac{1}{2}|x_{n+1} - x_n|$ for all $n \in \mathbb{N}$. It follows that $\{x_n\}$ is a Cauchy sequence in $\mathbb{R}$ and therefore $\{x_n\}$ converges.

   Again, $x_{n+1} - x_n = (-\frac{1}{2})(x_n - x_{n-1}) = \cdots = (-\frac{1}{2})^{n-1}(x_2 - x_1)$ for all $n \in \mathbb{N}$. This yields $x_n - x_1 = (x_n - x_{n-1}) + \ldots + (x_2 - x_1) = [(-\frac{1}{2})^{n-2} + \cdots + 1](x_2 - x_1) = \frac{2}{3}[1 - (-\frac{1}{2})^{n-1}](b - a)$. If $\ell = \lim x_n$, then $\ell - a = \frac{2}{3}(b - a)$ and so $\ell = \frac{1}{3}(a + 2b)$.

6. $|s_n - a| = \frac{1}{n}\sum\limits_{i=1}^{n}|a_i - a| = \frac{1}{n}\sum\limits_{i=1}^{N}|a_i - a| + \frac{1}{n}\sum\limits_{i=N+1}^{n}|a_i - a|$. Then proof follows by using the convergence of the sequence $\{a_n\}$.

   Let $a_n = (-1)^n$, then $\{a_n\}$ is divergent. Then $\frac{1}{n}\sum\limits_{i=1}^{n} a_i = 0$, if n is even and $\frac{1}{n}\sum\limits_{i=1}^{n} a_i = -\frac{1}{n}$, if n is odd. Now $\lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{i=1}^{n} a_i = 0$.

7. Using induction, one can show that $a_n \leq \sqrt{3}$. Using this one can show that

$$a_{n+1}^2 - a_n^2 = \frac{3}{2} - \frac{a_n^2}{2} \geq 0.$$

8. (a) Without loss of generality we can assume that $m > n$ hence
$\mid a_m - a_n \mid < \sum\limits_{k>n} \frac{1}{k!} < \sum\limits_{k>n} \frac{1}{2^k} < \frac{1}{2^{n-1}} < \epsilon \,, \forall n > N$. Hence the sequence is Cauchy.
(b) Using $AM - GM$ inequality,

$$a_n \leq \left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{2^2}\right)\cdots\left(1 + \frac{1}{2^n}\right) \leq \left(\frac{1}{n}\left(n + \sum_{i=1}^{n} \frac{1}{2^i}\right)\right)^n \leq \left(1 + \frac{1}{n}\right)^n < 3.$$

Therefore, $|a_{n+1} - a_n| < \frac{3}{2^n}$ and now using triangle inequality, for $m > n$

$$|a_m - a_n| \leq |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + \cdots + |a_{n+1} - a_n|.$$

Using the above estimate we can show that $\{a_n\}$ is Cauchy.