Series
(Lecture 9 & 10)

Engineering Calculus

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Definition

Let $\{a_n\}$ be a sequence of real numbers.

(a) An expression of the form

$$a_1 + a_2 + \ldots + a_n + \ldots$$

is called an infinite series.

(b) The number a_n is called as the n^{th} term of the series.

(c) The sequence $\{s_n\}$, defined by $s_n = \sum_{k=1}^{n} a_k$, is called the sequence of partial sums of the series.

(d) If the sequence of partial sums converges to a limit L, we say that the series converges and its sum is L.

(e) If the sequence of partial sums does not converge, we say that the series diverges.

Example

If $0 < x < 1$, then $\sum_{n=0}^{\infty} x^n$ converges to $\dfrac{1}{1-x}$.

Solution: Let us consider the sequence of partial sums $\{s_n\}$, where $s_n = \sum_{k=0}^{n-1} x^k$. Here

$$s_n = \sum_{k=0}^{n-1} x^k = \frac{1-x^n}{1-x} = \frac{1}{1-x} - \frac{x^n}{1-x}, \; n \in \mathbb{N}.$$

As, $0 < x < 1$, $x^n \to 0$ as $n \to \infty$. Hence $s_n \to \dfrac{1}{1-x}$. Thus the given series converges to $\dfrac{1}{1-x}$.

Example

The series $\sum_{n=1}^{\infty} \log(\frac{n+1}{n})$ diverges.

Hint: $S_n = \log(n+1) \to \infty$ as $n \to \infty$.

Example (Telescopic series)

Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges to 1.

Solution: Consider the sequence of partial sums $\{s_n\}$. Then

$$s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{n+1} \to 1.$$

Thus the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges and it converges to 1.

Theorem on Telescopic series

Suppose $\{a_n\}$ is a sequence of non-negative real numbers such that $a_n \to L$. Then the series $\sum (a_n - a_{n+1})$ converges to $a_1 - L$.

Proof: Let P_n be the sequence of partial sum of the series $b_n = a_n - a_{n+1}$. Then $P_n = a_1 - a_n \to a_1 - L$ as $n \to \infty$.

Lemma

(a) If $\sum_{n=1}^{\infty} a_n$ converges to L and $\sum_{n=1}^{\infty} b_n$ converges to M, then $\sum_{n=1}^{\infty}(a_n + b_n)$ converges to $L + M$.

(b) If $\sum_{n=1}^{\infty} a_n$ converges to L and if $c \in \mathbb{R}$, then the series $\sum_{n=1}^{\infty} ca_n$ converges to cL.

Theorem (Necessary condition for convergence)

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Proof: Suppose $\sum_{n=1}^{\infty} a_n = L$. Then the sequence of partial sums $\{s_n\}$ also converges to L. Now

$$a_n = s_n - s_{n-1} \to L - L = 0 \text{ as } n \to \infty.$$

- It is possible that $a_n \to 0$ and $\sum_{n=1}^{\infty} a_n$ diverges.

  Example: $\sum_{n=1}^{\infty} \log(\frac{n+1}{n})$ diverges, however $\log(\frac{n+1}{n}) \to 0$.

Convergence

- If $\lim_{n\to\infty} a_n \neq 0$. Then $\sum_{n=1}^{\infty} a_n$ diverges.

Example: If $x > 1$, then the series $\sum_{n=0}^{\infty} x^n$ diverges.

Solution: Assume that the series $\sum_{n=0}^{\infty} x^n$ converges. Then $x^n \to 0$. But as $x > 1$, $x^n \geq 1$ for all $n \in \mathbb{N}$ and hence $\lim_{n\to\infty} x^n \geq 1$, which is a contradiction. Hence the series $\sum_{n=1}^{\infty} x^n$ diverges.

Theorem (Necessary and sufficient condition for convergence)

Suppose $a_n \geq 0$ for all n. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\{s_n\}$ is bounded above.

Proof: Note that under the hypothesis $\{s_n\}$ is an increasing sequence.

Example

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Solution: Consider the sequence of partial sums $\{s_n\}$, where $s_n = \sum_{k=1}^{n} \frac{1}{k}$. Now, let us examine the subsequence s_{2^n} of $\{s_n\}$. Here

$$\begin{aligned}
s_2 &= 1 + 1/2 = 3/2, \\
s_4 &= 1 + 1/2 + 1/3 + 1/4 > 3/2 + 1/4 + 1/4 = 2, \\
s_{2^n} &\geq 1 + \frac{1}{2} + 2.\frac{1}{4} + 4.\frac{1}{8} + \ldots + 2^{n-1}.\frac{1}{2^n} = 1 + \frac{n}{2}.
\end{aligned}$$

Thus the subsequence $\{s_{2^n}\}$ is not bounded above and as it is also increasing, it diverges. Hence the sequence diverges, i.e., the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Remark: Note that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=p}^{\infty} a_n$ converges for any $p \geq 1$.

Theorem (Comparison Test)

Let $\{a_n\}, \{b_n\}$ be sequences of positive reals such that $a_n \leq b_n$ for $n \geq k$ for some k. Then

1. If $\sum b_n$ converges then $\sum a_n$ converges.
2. If $\sum a_n$ diverges then $\sum b_n$ diverges.

Examples

1. The series $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ converges, because $\frac{1}{(n+1)^2} \leq \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.
2. The series $\sum_{n=1}^{\infty} \frac{1}{2n^2-n}$ converges, because $2n^2 - n \geq n^2$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.
3. The series $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$ diverges, because $\frac{1}{n+\sqrt{n}} \geq \frac{1}{2n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.
4. The series $\sum_{n=1}^{\infty} \frac{7}{7n-2}$ diverges, because $\frac{7}{7n-2} = \frac{1}{n-2/7} \geq \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.
5. The series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges, because $\frac{1}{n!} \leq \frac{1}{2^n}$ and $\sum_{n=0}^{\infty} \frac{1}{2^n}$ converges.

Theorem (Cauchy condensation test)

Let $\{a_n\}$ be an decreasing sequence of positive numbers. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges.

Examples

(1) Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, $p > 0$. Then we have $\sum_{n=1}^{\infty} 2^n \frac{1}{(2^n)^p} = \sum_{n=1}^{\infty} \frac{1}{(2^n)^{p-1}}$ which converges for $p > 1$ and diverges for $p \leq 1$.

(2) Consider the series $\sum_{n=2}^{\infty} \frac{1}{n \log n}$. Here $\sum_{n=2}^{\infty} 2^n \frac{1}{2^n \log 2^n} = \frac{1}{\log 2} \sum_{n=2}^{\infty} \frac{1}{n}$ which diverges. Hence the given series diverges.

Theorem (Limit comparison test)

Let $\{a_n\}$ and $\{b_n\}$ be two sequences of positive numbers. Then

(a) if $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or diverge together.

(b) if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(c) if $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Example

(1) Consider the series $\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$. Here $a_n = \frac{2n+1}{(n+1)^2}$. Let $b_n = \frac{1}{n}$. Then

$\frac{a_n}{b_n} = \frac{\left(\frac{2n+1}{(n+1)^2}\right)}{\frac{1}{n}} = \frac{2n^2+n}{n^2+2n+1} \to 2$ as $n \to \infty$. Further, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Thus by limit comparison test, the given series diverges.

Examples

(2) The series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ converges. Here $a_n = \frac{1}{2^n - 1}$. Let $b_n = \frac{1}{2^n}$. Then $\frac{a_n}{b_n} = \frac{2^n}{2^n - 1} \to 1$. Further, $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges and hence the given series converges.

(3) The series $\sum_{n=1}^{\infty} \frac{e^{-n}}{n^2}$ converges. Here $a_n = \frac{e^{-n}}{n^2}$ and $b_n = \frac{1}{n^2}$. Then $\frac{a_n}{b_n} = e^{-n} \to 0$ as $n \to \infty$. Further, $\sum \frac{1}{n^2}$ converges and hence the given series converges.

(4) The series $\sum_{n=1}^{\infty} \frac{1}{n} \log(1 + \frac{1}{n})$ converges. Take $b_n = \frac{1}{n^2}$. Then $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.

(5) The series $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$ converges. Take $b_n = \frac{1}{n^2}$. Then $\frac{a_n}{b_n} = 1$.

Definition (Absolute convergence)

(a) Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers. If $\sum_{n=1}^{\infty} |a_n|$ converges, we say that $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(b) If $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges, we say that $\sum_{n=1}^{\infty} a_n$ converges conditionally.

Examples

(1) The series $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n!}$ converges absolutely.

(2) The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges absolutely.

(3) The series $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n}$ converges conditionally.

(4) The series $\sum_{n=1}^{\infty} \dfrac{(-1)^{2n-1}}{2n-1}$ converges conditionally.

Theorem

If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ converges.

Tests for absolute convergence

Theorem (Comparison test)

Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers. Then, $\sum_{n=1}^{\infty} a_n$ converges absolutely if there is an absolutely convergent series $\sum_{n=1}^{\infty} c_n$ with $|a_n| \leq |c_n|$ for all $n \geq N, N \in \mathbb{N}$.

Theorem (Ratio test)

Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers. Let

$$a = \liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \text{ and } A = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

Then

(a) $\sum_{n=1}^{\infty} a_n$ converges absolutely if $A < 1$.

(b) $\sum_{n=1}^{\infty} a_n$ diverges if $a > 1$.

(c) the test fails if $a < 1 < A$.

Examples

(a) The series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges.

Here

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \frac{n!}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n \to e,$$

which is greater than 1. So $a = A = e > 1$. Thus the given series diverges.

(b) For every $x \in \mathbb{R}$, the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges.

Here

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} = \frac{x}{n+1} \to 0.$$

Therefore $a = A = 0 < 1$. Thus, for all $x \in \mathbb{R}$, the given series converges.

Theorem (Root test)

Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers. Let $A = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Then

(a) the series converges absolutely if $A < 1$;

(b) the series diverges if $A > 1$;

(c) the test fails if $A = 1$.

Examples

(1) Find the value of $x \in \mathbb{R}$ for which the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges or diverges.

Here $a_n = \frac{x^n}{n}$. Therefore, $\sqrt[n]{\left|\frac{x^n}{n}\right|} = \left|\frac{x}{\sqrt[n]{n}}\right| \to |x|$. Thus the series converges for $|x| < 1$ and diverges for $|x| > 1$.

(2) Find the value of $x \in \mathbb{R}$ for which the series $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$ converges.

Here $a_n = \frac{x^n}{n^n}$. Then, $\sqrt[n]{|a_n|} = \left|\frac{x}{n}\right| \to 0$. Thus the series converges for $x \in \mathbb{R}$.

(3) Check the convergence of the series $\sum a_n$, where $a_n = \begin{cases} \frac{n}{2^n}, & n \text{ is odd} \\ \frac{1}{2^n}, & n \text{ is even} \end{cases}$

Since $\limsup_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{2}$. Therefore the series converges.

(4) $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$ converges. Since $\limsup_{n\to\infty} \sqrt[n]{a_n} = 0$.

Alternating series

Definition

An alternating series is an infinite series whose terms alternate in sign. i.e. $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ is an alternating series.

Theorem (Leibniz's test)

Suppose $\{a_n\}$ is a sequence of positive numbers such that

(a) $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$ and

(b) $\lim_{n\to\infty} a_n = 0$,

then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.

Example

Consider the series $\sum_{n=1}^{\infty}(1 - 2^{1/n})(-1)^{n+1}$.

Here $a_n = 1 - 2^{1/n} \to 0$ as $n \to \infty$. Also $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. Hence the series converges.

Examples

(a) Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

Then a_n's of this series satisfies the hypothesis of the above theorem and hence the series converges.

(b) Consider the series $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{\log n}$.

Then $a_n = \frac{1}{\log n}$ satisfy the hypothesis of the above theorem and hence the series converges.

Result

(a) Grouping of terms of a convergent series does not change the convergence and the sum. However, a divergent series can become convergent after grouping of terms.

(b) Rearrangement of terms does not change the convergence and the sum of an absolutely convergent series.


Thank You