Multivariable Calculus

(Lecture-5)

Department of Mathematics
Bennett University
India

26^{th} October, 2018

Learning Outcome of this lecture

We learn

- Vector Valued Function of Real Variable $F : S \subseteq \mathbb{R} \to \mathbb{R}^n$.
- Differentiation of $F : (a, b) \to \mathbb{R}^n$
- Integration of $F : [a, b] \to \mathbb{R}^n$
- Application of Differentiation and Integration of vector valued functions of real variable.

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Vector Valued Functions of Real Variable

(Vector Functions)

$$F : S \subseteq \mathbb{R} \to \mathbb{R}^n$$

(In particular, $n = 2$ and $n = 3$)

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Vector Valued ($\mathbb{R}^2$) Functions of Real Variable

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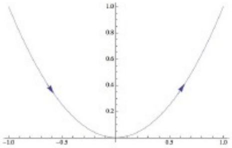

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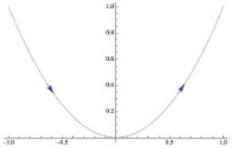

Remark: We can depict a vector valued function $F : \mathbb{R} \to \mathbb{R}^2$, by drawing only its range in the $2D$-plane. If we think $F(t)$ as a point in the xy-plane, then as t increases, $F(t)$ traces out a curve C in the plane, the arrow on the curve indicating the direction in which the curve is traced out as t increases.

Parametric Equations of the Curves (Parametric Curves)

Definition

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$$x = x(t) = f_1(t) \quad \text{and} \quad y = y(t) = f_2(t), \quad t \in I.$$

- The set of equations ($x = x(t), y = y(t)$ for $t \in I$ where $x(t)$ and $y(t)$ are continuous functions on I) is called a parametric equation of the curve Γ.

A curve may have different parametrization

Example:

$$x(t) = t \quad \text{and} \quad y(t) = t^2, \quad t \in \mathbb{R}.$$

is a parametric equation of the parabola $y = x^2$.

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For each real constant c,

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Vector Valued ($\mathbb{R}^3$) Functions of Real Variable

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- If $F(t)$ is continuous in an interval I, then it traces out a curve in $\mathbb{R}^3$ as t varies over I.
- We can rewrite it in parametric equations form as

$$x = f_1(t),\ y = f_2(t),\ z = f_3(t) \quad \text{for } t \in I$$

The parametric equations

$$x(t) = \cos t, \quad y(t) = \sin t, \quad z(t) = t \quad \text{for } t \in [0, 4\pi]$$

trace out a circular helix in $\mathbb{R}^3$.

Example of a Curve in $\mathbb{R}^3$

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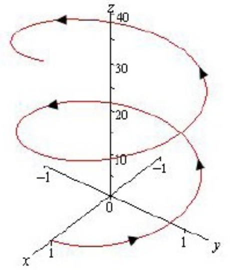

Differentiation

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Derivative of Vector Valued Function of One Variable

Definition

Let $F : S \subseteq \mathbb{R} \to \mathbb{R}^n$ where S is an open set in $\mathbb{R}$. Let $t_0 \in S$. The function F is said to be differentiable at the point t_0 if

$$\lim_{t \to t_0} \frac{F(t) - F(t_0)}{t - t_0} \quad \text{exists.}$$

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Note: An open set S in $\mathbb{R}$ will be an open interval or a union of finite/countable number of disjoint open intervals.

Example

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$$\lim_{t \to t_0} \frac{F(t) - F(t_0)}{t - t_0} = \lim_{t \to t_0} \frac{(t, 2, t^2) - (t_0, 2, t_0^2)}{t - t_0}$$

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&= \lim_{t \to t_0} \frac{\big((t - t_0, 2 - 2, t^2 - t_0^2)\big)}{t - t_0}
\end{aligned}$$

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\end{aligned}$$

$$F'(t_0) = (1, 0, 2t_0).$$

Relation Between Derivative of F and Derivatives of its Component Functions

Let $F : S \subseteq \mathbb{R} \to \mathbb{R}^n$ where S is an open set in $\mathbb{R}$. Then

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F is differentiable at t_0

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Each Component function $f_i,\ 1 \leq i \leq n$ is differentiable at t_0.

Further,

$$F'(t_0) = \left(f_1'(t_0), f_2'(t_0), \ldots, f_n'(t_0)\right).$$

Application of Theorem in Previous Slide

Let $F(t) = (t, 2, t^2)$ for $t \in R$. Find $F'(t)$.

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- Third Component Function of F: $f_3(t) = t^2$ for $t \in \mathbb{R}$.
  So $f_3'(t) = 2t$ for $t \in \mathbb{R}$.

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By the Theorem mentioned in previous slide, we have

$$F'(t) = \big(f_1'(t), f_2'(t), f_3'(t)\big) = (1, 0, 2t) \quad \text{for } t \in \mathbb{R}.$$

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- Since $f_1(t) = |t|$ is not differentiable at $t = 0$, we conclude that $F(t)$ is not differentiable at $t = 0$.
- But F is differentiable in $\mathbb{R} \setminus \{0\}$.

$$F'(t) = (1, 0, 2t) \quad \text{for } t > 0$$

$$F'(t) = (-1, 0, 2t) \quad \text{for } t < 0$$

Example:

Let $G(t) = (1, 2, t^2)$ for $t \in \mathbb{Q}$ and $G(t) = (0, 2, t^2)$ for $t \in \mathbb{R} \setminus \mathbb{Q}$. Examine the differentiability of $G(t)$ on $\mathbb{R}$.

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(Reimann) Integration

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(In particular, $n = 2$ and $n = 3$)

Definition

Let $F : [a,b] \subseteq \mathbb{R} \to \mathbb{R}^n$ be a bounded function. Then

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Integration of Vector Valued Function of One Variable

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Further

$$\int_a^b F(t)dt = \left(\int_a^b f_1(t)dt, \int_a^b f_2(t)dt, \ldots, \int_a^b f_n(t)dt \right).$$

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$$\int_a^b F(t)dt = \left(\int_a^b f_1(t)dt, \int_a^b f_2(t)dt, \ldots, \int_a^b f_n(t)dt \right).$$

Note that the value of the integral $\int_a^b F(t)dt$ is an element in $\mathbb{R}^n$.

Example

Let $F(t) = \cos t\hat{i} + \hat{j} + 2t\hat{k}$ for $t \in [0, \frac{\pi}{2}]$. Compute $\int_a^b F(t)$.

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- Second Component Function of F: $f_2(t) = 1$ for $t \in [0, \frac{\pi}{2}]$.
  So $\int_0^{\frac{\pi}{2}} f_2(t)dt = \frac{\pi}{2}$.

Example

Let $F(t) = \cos t\hat{i} + \hat{j} + 2t\hat{k}$ for $t \in [0, \frac{\pi}{2}]$. Compute $\int_a^b F(t)$.

- First Component Function of F: $f_1(t) = \cos t$ for $t \in [0, \frac{\pi}{2}]$.
  So $\int_0^{\frac{\pi}{2}} f_1(t)dt = 1$.
- Second Component Function of F: $f_2(t) = 1$ for $t \in [0, \frac{\pi}{2}]$.
  So $\int_0^{\frac{\pi}{2}} f_2(t)dt = \frac{\pi}{2}$.
- Third Component Function of F: $f_3(t) = 2t$ for $t \in [0, \frac{\pi}{2}]$.
  So $\int_0^{\frac{\pi}{2}} f_3(t)dt = \frac{\pi^2}{4}$.

Example

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  So $\int_0^{\frac{\pi}{2}} f_2(t)dt = \frac{\pi}{2}$.
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  So $\int_0^{\frac{\pi}{2}} f_3(t)dt = \frac{\pi^2}{4}$.

By the Theorem mentioned in previous slide, we have

$$\int_0^{\frac{\pi}{2}} F(t)dt = \left(\int_0^{\frac{\pi}{2}} f_1(t)dt, \int_0^{\frac{\pi}{2}} f_2(t)dt, \int_0^{\frac{\pi}{2}} f_3(t)dt\right) = (1, \frac{\pi}{2}, \frac{\pi^2}{4}).$$

Application of Differentiation and Integration

of

Vector Valued Functions of Real Variable

(Vector Functions)

$$F : \subseteq \mathbb{R} \to \mathbb{R}^n$$

(In particular, $n = 2$ and $n = 3$)

Example

Suppose we do not know the path of a hang glider, but only its acceleration vector $\mathbf{a}(t) = -3\cos t\hat{i} - 3\sin t\hat{j} + 2\hat{k}$. We also know that initially (at time $t = 0$) the glider departed from the point $(3, 0, 0)$ with velocity $\mathbf{v}(0) = 3\hat{j}$. Find the glider's position as a function of t.

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Solution: Our goal is to find $\mathbf{r}(\mathbf{t})$ knowing:

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}(\mathbf{t})}{dt^2} = -3\cos t\hat{i} - 3\sin t\hat{j} + 2\hat{k}.$$

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Example

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The glider's position is given by

$$\mathbf{r}(\mathbf{t}) = 3\cos t\hat{i} + 3\sin t\hat{j} + t^2\hat{k}.$$