Multivariable Calculus

(Lecture-8)

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Differentiation

of

(Scalar Valued Function of Vector Variable)

(Scalar Field)

$$F : S \subseteq \mathbb{R}^2 \to \mathbb{R}$$

Learning Outcome of this lecture

In this lecture, We learn for a scalar field $F : S \subseteq \mathbb{R}^2 \to \mathbb{R}$:

- Total Derivative of F at X_0
- F is differentiable at $X_0 \Rightarrow F$ is continuous at X_0

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Differential Calculus for $F : S \subseteq \mathbb{R}^2 \to \mathbb{R}$

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Wish List:

- F is differentiable at $X_0 \Rightarrow F$ is continuous at X_0.
- Sum, product and chain rules hold for $DF(X_0)$.

Differentiability of $f : (c, d) \subseteq \mathbb{R} \to \mathbb{R}$

f is differentiable at $a \in (c, d)$ if there exists $\alpha \in \mathbb{R}$ such that

$$\alpha = lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

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In other words, f is differentiable at a if there exists $\epsilon = \epsilon(h)$ and a constant α satisfying

$$f(a+h) - f(a) = h.\alpha + h.\epsilon$$

such that $\epsilon \to 0$ as $h \to 0$.

Differentiability of $F : S \subseteq \mathbb{R}^2 \to \mathbb{R}$

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Definition

A function $F : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ is differentiable at a point $(a, b) \in S$ if there exist $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ and $\epsilon_1 = \epsilon_1(h, k)$ and $\epsilon_2 = \epsilon_2(h, k)$ such that

$$f(a+h, b+k) - f(a, b) = h.\alpha_1 + k.\alpha_2 + h\epsilon_1 + k\epsilon_2$$

such that $\epsilon_1, \epsilon_2 \to 0$ as $(h, k) \to (0, 0)$.

Differentiability of $F : S \subseteq \mathbb{R}^2 \to \mathbb{R}$

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A function $F : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ is differentiable at a point $(a, b) \in S$ if there exist $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ and $\epsilon_1 = \epsilon_1(h, k)$ and $\epsilon_2 = \epsilon_2(h, k)$ such that

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such that $\epsilon_1, \epsilon_2 \to 0$ as $(h, k) \to (0, 0)$.

We call the pair (α_1, α_2) the total derivative of F at (a, b).

Differentiability of $F : S \subseteq \mathbb{R}^2 \to \mathbb{R}$

Fact: If (α_1, α_2) is the total derivative of F at (a, b), then letting $(h, k) \to (0, 0)$ along the x-axis and y-axis, we have $\alpha_1 = f_x(a, b)$ and $\alpha_2 = f_y(a, b)$, respectively.

Example-1

Show that the following function is NOT differentiable at $(0,0)$:

$$F(x,y)=\begin{cases} x\sin\frac{1}{x}+y\sin\frac{1}{y} & \text{if} \quad xy\neq 0\\ 0 & \text{if} \quad xy=0. \end{cases}$$

Solution:

$$|F(x,y)-F(0,0)|=|x\sin\frac{1}{x}+y\sin\frac{1}{y}-0|=\leq|x|+|y|\leq 2\sqrt{x^2+y^2}$$

Choosing $\delta=\frac{\epsilon}{2}$, implies that F is continuous at $(0,0)$.
We have

$$F_x(0,0)=\lim_{h\to 0}\frac{F(h,0)-F(0,0)}{h}=\lim_{h\to 0}\frac{0-0}{h}=0.$$

$$F_y(0,0)=\lim_{k\to 0}\frac{F(0,k)-F(0,0)}{k}=\lim_{k\to 0}\frac{0-0}{k}=0.$$

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Example-1

If F is differentiable at $(0,0)$, then $\alpha_1 = 0 = \alpha_2$, and there exist ϵ_1 and ϵ_2 such that

$$F(0+h, 0+k) - F(0,0) = h.0 + k.0 + h\epsilon_1 + k\epsilon_2$$

where $\epsilon_1, \epsilon_2 \to 0$ as $(h,k) \to (0,0)$.

$$\Rightarrow F(h,k) = h\epsilon_1 + k\epsilon_2$$

where $\epsilon_1, \epsilon_2 \to 0$ as $(h,k) \to (0,0)$.

Note: $(h,k) \to (0,0)$ means (h,k) can approach to $(0,0)$ from any direction.

Along $h = k$ path, we get

$$\lim_{h\to 0} \sin\frac{1}{h} \to 0, \text{ which is a contradiction.}$$

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Example-2

The function F defined by $F(x, y) = \sqrt{|xy|}$ is NOT differentiable at the origin.

Solution: If F is differentiable at $(0, 0)$, then $\alpha_1 = 0 = \alpha_2$, and there exist ϵ_1 and ϵ_2 such that

$$F(0 + h, 0 + k) - F(0, 0) = h.0 + k.0 + h\epsilon_1 + k\epsilon_2$$

$$F(h, k) = h\epsilon_1 + k\epsilon_2$$

where $\epsilon_1, \epsilon_2 \to 0$ as $(h, k) \to (0, 0)$. Along the path $h = k$, we get

$$F(h, h) = h(\epsilon_1 + \epsilon_2) \Rightarrow \frac{|h|}{h} = \epsilon_1 + \epsilon_2.$$

This implies that $\epsilon_1 + \epsilon_2 \nrightarrow 0$ as $h \to 0$ along the line $h = k$.

Another definition of differentiability of $F : S \subseteq \mathbb{R}^2 \to \mathbb{R}$

Let S be open in $\mathbb{R}^2$.

Definition

A function $F : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ is differentiable at a point $(a,b) \in S$ if

$$\lim_{(h,k)\to(0,0)} \frac{|F(a+h,b+k) - F(a,b) - F_x(a,b)h - F_y(a,b)k|}{\sqrt{h^2+k^2}} = 0.$$

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Example-3

Show that the following function is differentiable at $(0,0)$:

$$F(x,y)=\begin{cases}\frac{x^2y^2}{x^2+y^2} & \text{if} \quad (x,y)\neq(0,0)\\ 0 & \text{if} \quad (x,y)=(0,0).\end{cases}$$

Solution: Here $F_x(0,0)=0=F_y(0,0)$. By taking $h=r\cos\theta$ and $k=r\sin\theta$ we get

$$\frac{|F(0+h,0+k)-F(0,0)-F_x(0,0)h-F_y(0,0)k|}{\sqrt{h^2+k^2}}=\frac{r^4\cos^2\theta\sin^2\theta}{r^3}$$

$$=r\cos^2\theta\sin^2\theta \Rightarrow \lim_{r\to 0} r\cos^2\theta\sin^2\theta\to 0.$$

Hence, F is differentiable at $(0,0)$.

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Example-4

Show that the following function is NOT differentiable at $(0,0)$:

$$F(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0). \end{cases}$$

Solution: Here $F_x(0,0) = 0 = F_y(0,0)$. By taking $h = r\cos\theta$ and $k = r\sin\theta$ we get

$$\frac{|F(0+h,0+k) - F(0,0) - F_x(0,0)h - F_y(0,0)k|}{\sqrt{h^2+k^2}} = \frac{r^3\cos^2\theta\sin\theta}{r^3}$$

$$= \cos^2\theta\sin\theta \Rightarrow \lim_{r\to 0}\cos^2\theta\sin\theta \;\; \text{does not exist.}$$

Hence, F is not differentiable at $(0,0)$.

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