Improper Integrals
(Lecture 23 & 24)

Engineering Calculus

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- Let the function $f(x)$ defined on unbounded interval $[a, \infty)$ or $(-\infty, b]$ and $f \in \mathcal{R}[a, b]$ for all $b > a$.
- The function is not defined at some points on the interval $[a, b]$.
- **Improper integral of first kind:** Suppose f is a bounded function defined on $[a, \infty)$ or $(-\infty, b]$ and $f \in \mathcal{R}[a, b]$ for all $b > a$.

Definition

The improper integral of f on $[a, \infty)$ is defined as

$$\int_a^\infty f(x)dx := \lim_{b \to \infty} \int_a^b f(x)dx,$$

and the improper integral of f on $(-\infty, b]$ is defined as

$$\int_{-\infty}^b f(x)dx := \lim_{a \to -\infty} \int_a^b f(x)dx.$$

If the limit exists and is finite, we say that the improper integral converges. If the limit goes to infinity or does not exist, then we say that the improper integral diverges.

Examples

1. $\displaystyle\int_1^\infty \frac{1}{x^2}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}dx = \lim_{b\to\infty} 1 - \frac{1}{b} = 1.$
2. $\displaystyle\int_0^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty}\int_0^b \frac{dx}{1+x^2} = \lim_{b\to\infty} \tan^{-1} x \,|_0^b = \frac{\pi}{2}.$
3. $\displaystyle\int_1^\infty \frac{1}{x^p}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^p}dx = \lim_{b\to\infty} \frac{x^{-p+1}}{1-p}\,|_1^b = \frac{b^{-p+1}}{1-p} - \frac{1}{1-p} = \frac{1}{p-1}$ if $p > 1$.

   Thus $\displaystyle\int_1^\infty \frac{1}{x^p}dx$ converges if $p > 1$ and diverges if $p \leq 1$.

Comparison test

Suppose $0 \leq f(x) \leq g(x)$ for all $x \geq a$, then

(a) $\displaystyle\int_a^\infty f(x)dx$ converges if $\displaystyle\int_a^\infty g(x)dx$ converges.

(b) $\displaystyle\int_a^\infty g(x)dx$ diverges if $\displaystyle\int_a^\infty f(x)dx$ diverges.

Proof: Define $F(x) = \int_a^x f(t)dt$ and $G(x) = \int_a^x g(t)dt$. Then by properties of Riemann integral, $0 \le F(x) \le G(x)$ and we are given that $\lim_{x\to\infty} G(x)$ exists. So $G(x)$ is bounded. F is monotonically increasing and bounded above. Therefore, $\lim_{x\to\infty} F(x)$ exists.

Examples

1. $\int_1^\infty \frac{dx}{x^2(1+e^x)}$. Note that $\frac{1}{x^2(1+e^x)} < \frac{1}{x^2}$ and $\int_1^\infty \frac{dx}{x^2}$ converges.
2. $\int_1^\infty \frac{x^3}{x+1}dx$. Note that $\frac{x^3}{x+1} > \frac{x^2}{2}$ on $[1,\infty)$ and $\int_1^\infty x^2 dx$ diverges.
3. $\int_1^\infty \frac{1}{1+\sqrt{x}}$. Note that $\frac{1}{1+\sqrt{x}} \ge \frac{1}{2\sqrt{x}}$ on $[1,\infty)$ and $\int_1^\infty \frac{1}{\sqrt{x}}dx$ diverges. Therefore $\int_1^\infty \frac{1}{1+\sqrt{x}}$ diverges.
4. $\int_1^\infty \frac{\sqrt{x}}{1+x^5}$. Note that $\frac{\sqrt{x}}{1+x^5} \le \frac{1}{x^{3/2}}$ and $\int_1^\infty \frac{dx}{x^{3/2}}$ converges.

Limit comparison test

Let $f(x), g(x)$ are defined and positive for all $x \geq a$ and $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = L$.

(a) If $L \in (0, \infty)$, then the improper integrals $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ are either both convergent or both divergent. i.e., $\int_a^\infty f(x)dx$ converges $\iff \int_a^\infty g(x)$ converges.

(b) If $L = 0$, then $\int_a^\infty f(x)dx$ converges if $\int_a^\infty g(x)dx$ converges. i.e, $\int_a^\infty g(x)dx$ converges $\implies \int_a^\infty f(x)dx$ converges.

(c) If $L = \infty$, then $\int_a^\infty f(x)dx$ diverges if $\int_a^\infty g(x)dx$ diverges. i.e., $\int_a^\infty g(x)dx$ diverges $\implies \int_a^\infty f(x)dx$ diverges.

Examples

(1) $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x+1}}$. Take $f(x) = \frac{1}{\sqrt{x+1}}$ and $g(x) = \frac{1}{\sqrt{x}}$. Then $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 1$ and $\displaystyle\int_1^\infty g(x)dx$ diverges. So by above theorem, $\displaystyle\int_1^\infty f(x)dx$ diverges.

(2) $\displaystyle\int_1^\infty \frac{dx}{1+x^2}$. Take $f(x) = \frac{1}{1+x^2}$ and $g(x) = \frac{1}{x^2}$. Then $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 1$ and $\displaystyle\int_1^\infty g(x)dx$ converges. So by above theorem, $\displaystyle\int_1^\infty f(x)dx$ converges.

(3) $\displaystyle\int_0^\infty \frac{x}{\cosh x}dx$. Let $f(x) = \dfrac{x}{\cosh x} = \dfrac{2xe^x}{e^{2x}+1} \sim xe^{-x}$. So choose $g(x) = xe^{-x}$. Then $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 2$ and $\displaystyle\int_0^\infty g(x)$ converges.

Improper integrals of second kind

Let $f(x)$ be defined on $[a, c)$ and $f \in \mathcal{R}[a, c-\epsilon]$ for all $\epsilon > 0$. Then we define

$$\int_a^c f(x)dx = \lim_{\epsilon\to 0}\int_a^{c-\epsilon} f(x)dx.$$

Then $\int_a^b f(x)dx$ is said to converge if the limit exists and is finite. Otherwise, we say improper integral $\int_a^b f(x)dx$ diverges.

- Suppose $a_1, a_2, ...a_n$ are finitely many discontinuities of $f(x)$ in $[a, b]$. Then

$$\int_a^b f(x)dx = \int_a^{a_1} f(x)dx + \int_{a_1}^{a_2} f(x)dx + \int_{a_2}^{a_3} f(x)dx + \cdots + \int_{a_n}^{b} f(x)dx$$

If all the improper integrals on the right hand side converge, then we say the improper integral of f over $[a, b]$ converges. Otherwise, we say it diverges.

Examples

1. $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{\epsilon\to 0}\int_\epsilon^1 \frac{dx}{\sqrt{x}} = \lim_{\epsilon\to 0} 2(1-\sqrt{\epsilon}) = 2.$
2. $\displaystyle\int_0^1 \frac{1}{x^p}dx = \lim_{\epsilon\to 0}\int_\epsilon^1 \frac{1}{x^p}dx = \lim_{\epsilon\to 1}\frac{x^{-p+1}}{1-p}\Big|_\epsilon^1 = \frac{1}{1-p} - \frac{\epsilon^{-p+1}}{1-p} = \frac{1}{1-p}$ if $p<1$.

   Thus $\displaystyle\int_0^1 \frac{1}{x^p}dx$ converges if $p<1$ and diverges if $p \geq 1$.

Comparison test

Suppose $0 \leq g(x) \leq f(x)$ for all $x \in [a,c)$ and are discontinuous at c.

(a) If $\displaystyle\int_a^c f(x)dx$ converges then $\displaystyle\int_a^c g(x)dx$ converges.

(b) If $\displaystyle\int_a^c g(x)dx$ diverges then $\displaystyle\int_a^c f(x)dx$ diverges.

Limit comparison test

Suppose $f(x), g(x) > 0$ be continuous in $[a, c)$ and $\lim_{x \to c} \frac{f(x)}{g(x)} = L$. Then

1. If $L \in (0, \infty)$. Then $\displaystyle\int_a^c f(x)dx$ and $\displaystyle\int_a^c g(x)dx$ both converge or diverge together.
2. If $L = 0$ and $\displaystyle\int_a^c g(x)dx$ converges then $\displaystyle\int_a^c f(x)dx$ converges.
3. If $L = \infty$ and $\displaystyle\int_a^c g(x)dx$ diverges then $\displaystyle\int_a^c f(x)dx$ diverges.

- Let $f \in \mathcal{R}[a, b]$ for all $b > a$. Then $\int_a^\infty f(x)dx$ converges absolutely if $\int_a^\infty |f(x)|dx$ converges.

Theorem

If the integral $\int_a^\infty |f(x)|dx$ converges, then the integral $\int_a^\infty f(x)dx$ converges.

Proof: Note that $0 \le f(x) + |f(x)| \le 2|f(x)|$. So the improper integral $\int_a^\infty f(x) + |f(x)|dx$ converges by comparison theorem above. Also $\int_a^\infty |f(x)|$ converges. Therefore

$$\int_a^\infty f(x)dx = \int_a^\infty (f(x) + |f(x)|)dx - \int_a^\infty |f(x)|dx$$

also converges.

- The converse of the above theorem is not true. For example, $\displaystyle\int_{\pi}^{\infty} \frac{\sin x}{x} dx$.

  This integral does not converge absolutely.

$$
\begin{aligned}
\int_{\pi}^{\infty} \frac{|\sin x|}{x} dx &= \sum_{n=1}^{\infty} \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{x} dx \\
&\geq \sum_{n=1}^{\infty} \frac{1}{(n+1)\pi} \int_{n\pi}^{(n+1)\pi} |\sin x| \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+1)\pi} \int_{0}^{\pi} \sin x = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n+1}.
\end{aligned}
$$

On the other hand, by integration by parts,

$$
\begin{aligned}
\lim_{b\to\infty} \int_{\pi}^{b} \frac{\sin x}{x} dx &= \lim_{b\to\infty} \int_{\pi}^{b} \frac{\sin x}{x} dx \\
&= \lim_{b\to\infty} \left(\frac{-\cos b}{b} + \frac{1}{\pi} - \int_{\pi}^{b} \frac{\cos x}{x^2} dx \right) \\
&= \frac{1}{\pi} - \int_{\pi}^{\infty} \frac{\cos x}{x^2} dx,
\end{aligned}
$$

The limits on the right exist by comparison test.

Examples

1. $\displaystyle\int_1^\infty \frac{\sin x}{x^3}dx$ converges.

   Note that $\left|\dfrac{\sin x}{x^3}\right| \leq \left|\dfrac{1}{x^3}\right|$ and $\displaystyle\int_1^\infty \frac{dx}{x^3}$ converges.

2. $\displaystyle\int_0^\infty \frac{e^{-x^2}\sin x}{\log(1+x)}$ converges.

   We have $\displaystyle\int_0^\infty \frac{e^{-x^2}\sin x}{\log(1+x)}dx = \int_0^{10} \frac{e^{-x^2}\sin x}{\log(1+x)}dx + \int_{10}^\infty \frac{e^{-x^2}\sin x}{\log(1+x)}dx.$

   Note that $\displaystyle\lim_{x\to 0} \frac{e^{-x^2}\sin x}{\log(1+x)} = 1$. Therefore the integral is proper at $x = 0$. For $x > 10$

$$|f(x)| \leq \frac{e^{-x^2}}{\log(1+x)} < e^{-x^2} \leq e^{-x}.$$

   Hence $\displaystyle\int_{10}^\infty \frac{e^{-x^2}\sin x}{\log(1+x)}$ converges by comparison test. Thus $\displaystyle\int_0^\infty \frac{e^{-x^2}\sin x}{\log(1+x)}$ converges.

Example 1

Show that $\int_1^\infty \frac{\sin x}{x^p}$ and $\int_1^\infty \frac{\cos x}{x^p}dx$ converges for all $p > 0$.

Solution: Since, for $x > 0$, we have

$$\left|\frac{\sin x}{x^p}\right| \leq \frac{1}{x^p} \text{ and } \left|\frac{\cos x}{x^p}\right| \leq \frac{1}{x^p},$$

by comparison test, $\int_1^\infty \frac{\sin x}{x^p}$ and $\int_1^\infty \frac{\cos x}{x^p}$ converges for all $p > 1$. We show that $\int_1^\infty \frac{\sin x}{x^p}$ converges for all $p > 0$.

$$\int_1^t \frac{\sin x}{x^p}dx = -\frac{\cos x}{x^p}\Big|_1^t - p\int_1^t \frac{\cos x}{x^{p+1}}dx = \cos 1 - \frac{\cos t}{t^p} - p\int_1^t \frac{\cos x}{x^{p+1}}dx.$$

By above $\int_1^\infty \frac{\cos x}{x^{p+1}}$ converges for all $p > 0$. Also $\lim_{t\to\infty} \frac{\cos t}{t^p} = 0$. Hence $\int_1^\infty \frac{\sin x}{x^p}$ converges for all $p > 0$. Similarly we can show that $\int_1^\infty \frac{\cos x}{x^p}dx$ converges for all $p > 0$.

Example 2

Show that $\int_0^1 \frac{\sin x}{x^p}$ converges for all $p < 2$ and $\int_0^1 \frac{\cos x}{x^p} dx$ converges for all $p < 1$.

Solution: For $p \leq 0$, $\int_0^1 \frac{\sin x}{x^p}$ and $\int_0^1 \frac{\cos x}{x^p}$ exist as Riemann integrals. So let $p > 0$. Since, for $x > 0$, we have

$$\left|\frac{\sin x}{x^p}\right| = \left|\frac{\sin x}{x}\right| \frac{1}{x^{p-1}} \leq \frac{1}{x^{p-1}} \text{ and } \left|\frac{\cos x}{x^p}\right| \leq \frac{1}{x^p},$$

by comparison test, $\int_0^1 \frac{\sin x}{x^p}$ converges for $p < 2$ and $\int_0^1 \frac{\cos x}{x^p}$ converges for all $p < 1$.

Remark

From Examples 1 and 2, we have

$$\int_0^\infty \frac{\sin x}{x^p} dx \text{ converges for } 0 < p < 2.$$

and

$$\int_0^\infty \frac{\cos x}{x^p} dx \text{ converges for } 0 < p < 1.$$

- Sometimes improper integrals may be transformed into proper integrals:

  For example, consider the improper integral $I = \int_1^3 \frac{dx}{\sqrt{x}\sqrt{3-x}}$. Taking the transformation $y = \frac{1}{3-x}$, we get $I = \int_{1/2}^{\infty} \frac{dy}{y\sqrt{3y-1}}$. This is an improper integral of first kind. Instead, if we choose the transformation $3-x=u^2$ then $I = \int_0^{\sqrt{2}} \frac{2udu}{u\sqrt{3-u^2}}$, which is a proper integral.

Remark

Note that the **symmetric** limit could be convergent but the limit may not exist.
For example,

$$\int_{-1}^{1} \frac{dx}{x^3} = \int_{-1}^{0} \frac{dx}{x^3} + \int_{0}^{1} \frac{dx}{x^3} = \lim_{\epsilon_1 \to 0} \int_{-1}^{-\epsilon_1} \frac{dx}{x^3} + \lim_{\epsilon_2 \to 0} \int_{\epsilon_2}^{1} \frac{dx}{x^3} = \frac{1}{2} \lim_{\epsilon_1, \epsilon_2 \to 0} \left(\frac{1}{\epsilon_1^2} - \frac{1}{\epsilon_2^2} \right).$$

If we take $\epsilon_1 = \epsilon_2$, then the limit exits and is equal to 0. But if we take $\epsilon_1 = \frac{1}{(n+1)^2}, \epsilon_2 = \frac{1}{n^2}$, then $\epsilon_1, \epsilon_2 \to 0$ as $n \to \infty$ and the above limit does not exist. So through different sequences, we are getting different limits. Therefore the integral diverges.

Cauchy principal value

- Consider the improper integral $I = \int_0^\infty \sin x \, dx$. Then we have $I = \lim_{a\to\infty}(1 - \cos a)$, which does not exist. Similarly, $\int_{-\infty}^0 \sin x \, dx$ does not exist. But

$$\lim_{c\to\infty}\int_{-c}^{c} \sin x dx$$

exists and is equal to 0. Though the improper integral does not exist, this symmetric limit exists. This is called **Cauchy principal value** of improper integral.

Definition

The Cauchy principal value of improper integral of first kind is defined as

$$\text{CPV} \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx.$$

For the improper integral of second kind, with $c \in (a,b)$ as point of discontinuity of $f(x)$ as

$$\text{CPV} \int_a^b f(x)dx = \lim_{\delta\to 0}\int_a^{c-\delta} f(x)dx + \lim_{\delta\to 0}\int_{c+\delta}^{b} f(x)dx.$$

Examples

First kind: Consider $\int_{-\infty}^{\infty} x^{2n+1}dx$ for $n = 1, 2, 3, \ldots$. Then $\lim_{a\to\infty} \int_{-a}^{a} x^{2n+1} = 0$. But the improper integrals $\int_{0}^{\infty} x^{2n+1}$ and $\int_{-\infty}^{0} x^{2n+1}$ does not converge.

Second kind: Consider $\int_{-1}^{1} x^{-(2n+1)}dx$, for $n = 1, 2, 3, \ldots$. Then evaluate

$$\int_{-1}^{-\epsilon} x^{-(2n+1)} + \int_{\epsilon}^{1} x^{-(2n+1)},$$

to see that the the limit is 0.

Gamma function

The *Gamma function* defined as improper integral for $p > 0$,

$$\Gamma(p) = \int_{0}^{\infty} x^{p-1}e^{-x}dx.$$

This integral is improper of second kind in the neighbourhood of 0 as x^{p-1} goes to infinity as $x \to 0$ (when $p < 1$). Since the domain of integration is $(0, \infty)$, the integral is improper of first kind.

Convergence

To prove the convergence, we divide the integral into

$$\Gamma(p) = \int_0^1 x^{p-1}e^{-x}dx + \int_1^\infty x^{p-1}e^{-x}dx = I_1 + I_2.$$

To see the convergence of I_1, we take $f(x) = x^{p-1}e^{-x}$ and $g(x) = x^{p-1}$. Then

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = 1 \quad \text{and} \quad \int_0^1 x^{p-1}dx \quad \text{converges.}$$

To see the convergence of I_2, we take $f(x) = x^{p-1}e^{-x}$ and $g(x) = \frac{1}{x^2}$. Then

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty} x^{2+p-1}e^{-x} = 0 \quad \text{and} \quad \int_1^\infty \frac{1}{x^2}dx \quad \text{converges.}$$

Hence by limit comparison test, the integral converges.

Beta function

The *Beta function* defined as improper integral for $p > 0, q > 0$,

$$\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx.$$

If $p > 1$ and $q > 1$, then the integral is definite integral. If $p < 1$ or $q < 1$, this integral is improper of second kind at 0 or 1.

Convergence

To prove the convergence, we divide the integral as

$$\int_0^1 x^{p-1}(1-x)^{q-1}dx = \int_0^{1/2} x^{p-1}(1-x)^{q-1}dx + \int_{1/2}^1 x^{p-1}(1-x)^{q-1}dx = I_1 + I_2.$$

To see the convergence of I_1, we take $f(x) = x^{p-1}(1-x)^{q-1}$ and $g(x) = x^{p-1}$. Then

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}(1-x)^{q-1} = 1 \quad \text{and} \quad \int_0^{1/2} x^{p-1}dx \quad \text{converges.}$$

Similarly, for convergence of I_2, we take $f(x) = x^{p-1}(1-x)^{q-1}$ and $g(x) = (1-x)^{q-1}$.

(1) $\Gamma(1) = \int_0^\infty e^{-x} dx = 1.$

(2) $\Gamma(\alpha+1) = \alpha\Gamma(\alpha).$
Integration by parts formula implies,

$$\Gamma(\alpha+1) = \int_0^\infty x^\alpha e^{-x} = -(x^\alpha e^{-x})|_0^\infty + \alpha \int_0^\infty x^{\alpha-1} e^{-x} dx = \alpha\Gamma(\alpha).$$

Therefore, $\Gamma(m+1) = m! \;\; \forall m \in \mathbb{N}.$

(3) $\Gamma(\frac{1}{2}) = \sqrt{\pi}.$

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = 4\int_0^\infty \int_0^\infty e^{-u^2} e^{-v^2} dudv, \quad \text{take } \; u = r\cos\theta, v = r\sin\theta,$$
$$= 4\int_0^\infty \int_0^{\pi/2} e^{-r^2} rdrd\theta = \pi.$$

(4) $\beta(m,n) = \beta(n,m).$
Substituting $t = 1-x$ in the definition of $\beta(m,n)$, we get

$$\beta(m,n) = \int_0^1 t^{n-1}(1-t)^{m-1} dt = \beta(n,m).$$

(5) $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta.$
Taking $x = \sin^2\theta$ in $\beta(m,n)$, we get

$$\beta(m,n) = \int_0^{\pi} \cos^{2m-2}\theta \sin^{2n-2}\theta \cos\theta \sin\theta d\theta = 2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta d\theta.$$

(6) $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

Problem

Evaluate $(i)\ \int_0^\infty x^{2/3} e^{-\sqrt{x}} dx, \quad (ii)\ \int_0^1 x^{\frac{3}{2}}(1-\sqrt{x})^{\frac{1}{2}} dx.$

Solution: (i) Take $t = \sqrt{x}$, then the given integral becomes

$$\int_0^\infty t^{4/3} e^{-t} 2t dt = 2\int_0^\infty e^{-t} t^{7/3} dt = 2\Gamma\left(\frac{10}{3}\right) = \frac{56}{27}\Gamma(1/3).$$

(ii) Again take $t = \sqrt{x}$, then the integral becomes

$$2\int_0^1 t^3 (1-t)^{1/2} t dt = 2\int t^4 (1-t)^{1/2} dt = 2\beta(5, 3/2) = 2\frac{\Gamma(5)\Gamma(3/2)}{\Gamma(13/2)} = \frac{512}{3465}.$$

- Consider an integral

$$I(\alpha) = \int_a^b f(x, \alpha)dx,$$

where the integrand is depend on the parameter α. At times we can differentiate under the integral sign to evaluate the integral. It is sometimes not possible and leads to wrong assertions. For example, we know that

$$I = \int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

It is easy to notice with change of variable formula, taking $tx = y$, that

$$I = I(t) = \int_0^\infty \frac{\sin(tx)}{x} dx = \frac{\pi}{2}.$$

Now differentiating this, taking derivative inside integral, we get

$$I'(t) = \int_0^\infty \cos(tx)dx = 0,$$

which doesn't make sense.

Theorem

Let

$$I(\alpha) = \int_a^b f(x, \alpha) dx.$$

(a) Suppose $f, \frac{d}{d\alpha} f(x, \alpha)$ are continuous functions for $x \in [a, b]$ and α in an interval containing α_0.

(b) Also $|f(x, \alpha)| \leq A(x), \ |\frac{d}{d\alpha} f(x, \alpha)| \leq B(x)$ such that A, B are integrable on $[a, b]$. If the domain in unbounded, then the improper integrals $\int_a^b A dx$, $\int_a^b B dx$ converge.

Then I is differentiable and

$$I'(\alpha) = \int_a^b \frac{d}{d\alpha} f(x, \alpha) dx.$$

Newton-Leibnitz formula

Let $h(x) = \int_{a(x)}^{b(x)} f(x, t) dt$. Then

$$h'(x) = \int_{a(x)}^{b(x)} \frac{df}{dx}(x, t) dt + f(x, b(x))\ b'(x) - f(x, a(x))\ a'(x).$$

Example

Evaluate $I(\alpha) = \int_0^\infty e^{-x} \frac{\sin \alpha x}{x} dx$.

By the above formula, $I'(\alpha) = \int_0^\infty e^{-x} \cos \alpha x \, dx = \frac{1}{1+\alpha^2}$. Therefore, $I(\alpha) = \tan^{-1} \alpha + C$.

Also $I(0) = \int_0^\infty e^{-x} \frac{\sin 0}{x} dx = 0$. Hence $C = 0$.

Thank You