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Lecture - 7 (ODE)

Orthogonal Trajectories

$$\underbrace{\frac{dy}{dx} + P(x)\, y = Q(x) \cdot y^n}$$ —— ①

$$\Rightarrow \frac{1}{y^n}\frac{dy}{dx} + \frac{P(x)}{y^n} \cdot y = Q(x)$$

$$\Rightarrow \frac{1}{y^n}\frac{dy}{dx} + P(x) \cdot y^{1-n} = Q(x)$$ —— ②

Let $\boxed{y^{1-n} = z}$

$$\Rightarrow (1-n)\, y^{-n} \frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow \frac{1}{y^n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dz}{dx}$$

$\Rightarrow$ ② becomes,

$$\frac{1}{(1-n)}\frac{dz}{dx} + P(x)\, z = Q(x)$$

$$\Rightarrow \frac{dz}{dx} + \underbrace{(1-n)\, P(x)}\, z = Q(x)\,(1-n)$$

Which is a linear DE.

\# $\frac{dy}{dx} + y = x y^2$ —— ①

Dividing the eqn ① with y^2, we get

$$\frac{}{y^2}\frac{dy}{dx} + \frac{y}{y^2} = x$$

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y} = x \quad \text{——} \quad (2)$$

$$\text{Let } \frac{1}{y} = z \Rightarrow \frac{-1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$$

$\Rightarrow$ (2) becomes,

$$-\frac{dz}{dx} + z = x \Rightarrow \underline{\frac{dz}{dx} - z = -x}$$

$$I.F = e^{\int(-1)\,dx} = e^{-x}$$

$$z\cdot e^{-x} = \int (e^{-x})\cdot(-x)\,dx + C$$

$$\Rightarrow z\cdot e^{-x} = -\int \underset{I}{x}\,\underset{II}{e^{-x}}\,dx + C$$

$$= -\left[x\frac{e^{-x}}{-1} - \int \frac{e^{-x}}{-1}\,dx\right] + C$$

$$= -\left[-xe^{-x} + \frac{e^{-x}}{-1}\right] + C$$

$$\Rightarrow z\cdot e^{-x} = xe^{-x} + e^{-x} + C$$

$$\Rightarrow z = (x+1) + Ce^{x}$$

$$\Rightarrow \boxed{\frac{1}{y} = (x+1) + Ce^{x}}$$

Orthogonal Trajectories

Suppose you are given a family of curves

$$F(x, y, c) = 0 \quad \text{--- (1)}$$

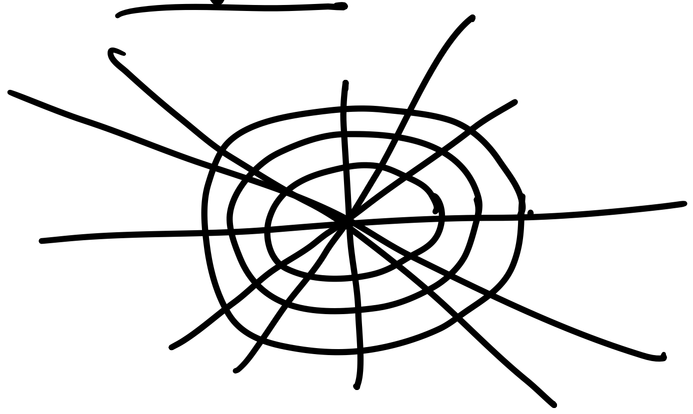
and we wish to find another family of curves

$$G(x, y, c) = 0$$

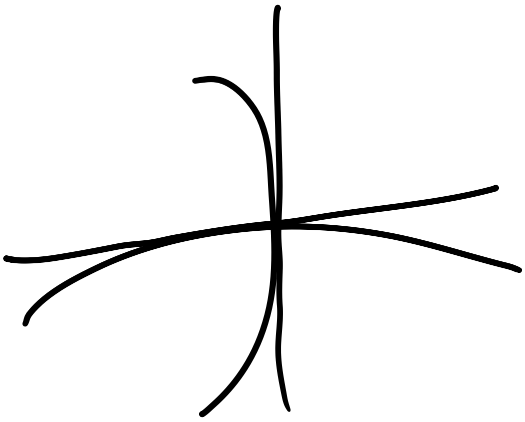
which intersects every member of (1) orthogonally (1).

Such a family of curves G is called the orthogonal trajectories of the family of curves (1).

Example: $x^2+y^2=a^2 \longrightarrow$ family of straight lines $y = cx$



family of straight lines $y=cx$ is the family of orthogonal trajectories to the family of circles.



Procedure to find Orthogonal Trajectories to the given family of curves:

given family of curves

$$F(x, y, c) = 0 \quad \text{—— (1)}$$

↓

Obtain the DE

$$\left(\frac{dy}{dx}\right)_1 = f(x, y)$$

family of orthogonal trajectories

$$G(x, y, c) = 0 \quad \text{—— (2)}$$

↓

$$\left(\frac{dy}{dx}\right)_2 = g(x, y) = \frac{-1}{\left(\frac{dy}{dx}\right)_1}$$

Since (1) & (2) are orthogonal.

$\Rightarrow$ product of slope of their tangents $= -1$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_1 \cdot \left(\frac{dy}{dx}\right)_2 = -1$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_2 = \frac{-1}{\left(\frac{dy}{dx}\right)_1}$$

Steps to follow:

Let $F(x, y, c) = 0$ — (1) be a one parameter family of curves.

Step - 1

Obtain the DE corresponding to (1).

$$\frac{dy}{dx} = f(x, y) \quad \text{——— (2)}$$

Step - 2:

Replace $\frac{dy}{dx} = \frac{-1}{\frac{dy}{dx}}$ in (2),

we get

$$\frac{-1}{\frac{dy}{dx}} = f(x, y)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-1}{f(x, y)} \quad \text{——— (3)}$$

which is the DE of orthogonal trajectories.

Step - III:

Obtain one parameter family of curves by solving (3),

$G(x, y, c) = 0$.

which is the family of orthogonal trajectories of the given family of curves $F(x, y, c) = 0$.

Example: find the orthogonal trajectories of the family of circles $x^2+y^2=a^2$.

Sol^n Given family of curves is

$$x^2+y^2=a^2 \qquad \text{——— (1)}$$

Step-I: Differentiating (1) w.r.t. x, we get

$$2x+2y\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{x}{y}$$

Step-II: Replace $\frac{dy}{dx}$ by $\frac{-1}{\frac{dy}{dx}}$, we get

$$\frac{-1}{\frac{dy}{dx}}=-\frac{x}{y} \Rightarrow \frac{dy}{dx}=\frac{y}{x}$$

Step-III: $\frac{dy}{dx}=\frac{y}{x} \Rightarrow \frac{dy}{y}=\frac{dx}{x}$

$$\Rightarrow \log y=\log x+\log c$$

$$\Rightarrow \boxed{y=cx}$$

Which is a family of straight line.

Example: find the orthogonal trajectories to the family of parabolas $y = cx^2$.

Solⁿ: Given family is

$$y = cx^2 \quad \text{--- (1)}$$

Step-I: Obtain the DE corresponding to (1),

Differentiating (1) w.r.t. x, we get

$$\frac{dy}{dx} = 2cx$$

$$\Rightarrow \quad c = \frac{1}{2x}\frac{dy}{dx}$$

$\Rightarrow$ (1) becomes,

$$y = \frac{1}{2x}\frac{dy}{dx} \cdot x^2$$

$$\Rightarrow \quad y = \frac{x}{2}\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2y}{x}.$$

Step-2: Replace $\frac{dy}{dx}$ by $\frac{-1}{\frac{dy}{dx}}$, we get

$$\frac{-1}{\frac{dy}{dx}} = \frac{2y}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-x}{2y}$$

Step-3:

$$\frac{dy}{dx} = \frac{-x}{2y}$$

$$\Rightarrow \quad 2y\,dy = -x\,dx + C$$

$$\Rightarrow \quad y^2 = -\frac{x^2}{2} + C$$

$$\Rightarrow \quad \boxed{2y^2 + x^2 = c^2}$$

Which is a family of ellipses.