Ordinary Differential Equations(EMAT102L)
(Lecture-13 and 14)

Department of Mathematics
Bennett University, India

Outline of the Lecture

We will learn

- Higher Order Differential Equations
- Results Related to Higher Order Differential Equations
- Homogeneous Linear Differential Equation with constant coefficients

The general form of an n-th order linear differential equation is

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F(x) \tag{1}$$

where the coefficients $a_i(x)$; $i = 0, 1, \cdots n$ and $F(x)$ are continuous and $a_0(x) \neq 0$ for every $x \in I$.

The above equation is said to be **homogeneous** if $F(x) = 0$ and **nonhomogeneous** if $F(x) \neq 0$.

Consider the initial value problem (IVP) for an nth order linear nonhomogeneous ODE

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F(x),$$

with the initial conditions $y(x_0) = c_0,\ y'(x_0) = c_1, \cdots y^{n-1}(x_0) = c_n$.

Existence and Uniqueness Theorem for an nth order linear nonhomogeneous IVP

If $a_0(x)$, $a_1(x)$, $a_2(x)$, $\cdots$ $a_n(x)$ and $F(x)$ are continuous functions on an interval I where $a_0(x) \neq 0$ and $x_0 \in I$, then the above initial value problem has a **unique solution** $y(x)$ in the interval I.

Note: This is the sufficient condition only.

Consider the initial value problem (IVP) for an nth order Homogeneous linear ODE

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0,$$

with the initial conditions $y(x_0) = 0,\ y'(x_0) = 0, \cdots y^{n-1}(x_0) = 0$

Existence and Uniqueness Theorem for nth Order Homogeneous Linear IVP

If $a_0(x), a_1(x), a_2(x), \cdots\ a_n(x)$ are continuous functions on an interval I where $a_0(x) \neq 0$ and $x_0 \in I$, then the above initial value problem has a **unique solution** $y(x) = 0$ for all x in the interval I.

Note: This is the sufficient condition only.

In the following theorem, we observe that the sum of two or more solutions of a homogeneous linear DE is also a solution.

Theorem

Superposition principle-Homogeneous equations: *Let $y_1, y_2, \cdots y_n$ be solutions of the n-th order homogeneous DE*

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \tag{2}$$

where the coefficients $a_i(x)$; $i = 0, 1, \cdots n$ are continuous and $a_0(x) \neq 0$ for every $x \in I$. Then the linear combination

$$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

where the $c_i; i = 1, 2, \cdots n$ are arbitrary constants, is also a solution to (2) on the same interval.

Definition

A set of functions $f_1(x), f_2(x) \cdots f_n(x)$ are said to be **linearly dependent** on an interval I if there exists constants $c_1, c_2, \cdots c_n$, not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for every $x \in I$. If the set of functions is not linearly dependent on the interval, it is said to be linearly independent.

In other words, a set of functions $f_1(x), f_2(x) \cdots f_n(x)$ is **linearly independent** on an interval if the only constants for which

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \;\; \forall \;\; x \Rightarrow c_1 = c_2 = \cdots = c_n = 0$$

Example

- The functions $f_1(x) = \cos^2 x, f_2(x) = \sin^2 x, f_3(x) = \sec^2 x, f_4(x) = tan^2 x$ are linearly dependent on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Example

- The functions $f_1(x) = \cos^2 x, f_2(x) = \sin^2 x, f_3(x) = \sec^2 x, f_4(x) = tan^2 x$ are linearly dependent on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
  Since
  $$c_1 \cos^2 x + c_2 \sin^2 x + c_3 \sec^2 x + c_4 \tan^2 x = 0$$
  for $c_1 = c_2 = c_4 = 1, c_3 = -1$.
  We used here $\sin^2 x + \cos^2 x = 1$ and $1 + \tan^2 x = \sec^2 x$.

Definition

Suppose each of the functions $f_1(x), f_2(x) \cdots f_n(x)$ possesses at least $n-1$ derivatives. The determinant

$$W(f_1, f_2, \cdots f_n) = det \begin{pmatrix} f_1 & f_2 & f_3 & . & . & . & f_n \\ f_1' & f_2' & f_3' & . & . & . & f_n' \\ f_1'' & f_2'' & f_3'' & . & . & . & f_n'' \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ f_1^{(n-1)} & f_2^{(n-1)} & f_3^{(n-1)} & . & . & . & f_n^{(n-1)} \end{pmatrix}$$

where the prime denote derivatives, is called the **Wronskian** of the functions $f_1, \cdots f_n$.

Theorem

Let $y_1, y_2, \cdots y_n$ be n solutions of the homogeneous linear n-th order DE

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \tag{3}$$

where the coefficients $a_i(x)$; $i = 0, 1, \cdots n$ are continuous and $a_0(x) \neq 0$ for every $x \in I$. Then the set of solutions $\{y_1, y_2 \cdots y_n\}$ is ***linearly independent*** *on I if and only if*

$$W(y_1, y_2, \cdots y_n) \neq 0$$

for every x in the interval I.

Theorem

The Wronskian $W(y_1, y_2, \cdots y_n)$ of n solutions $y_1, y_2 \cdots y_n$ of (3) is either identically zero or never zero on the interval.

Recall the homogeneous linear n-th order DE

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \tag{4}$$

where the coefficients $a_i(x)$; $i = 0, 1, \cdots n$ are continuous and $a_0(x) \neq 0$ for every $x \in I$.

Fundamental Set of Solutions

Any set $\{y_1, y_2, \cdots y_n\}$ of n linearly independent solutions of the homogeneous linear n-th order DE (4) on an interval I is said to be a **fundamental set of solutions** on the interval.

Theorem

There exists a fundamental set of solutions for the homogeneous linear n-th order DE (4) on an interval I.

Theorem

Let $\{y_1, y_2, \cdots y_n\}$ be a fundamental set of solutions for the homogeneous linear n-th order DE (4) on an interval I. Then the general solution of the equation (4) on the interval I is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where $c_i, i = 1, 2, \cdots n$ are arbitrary constants.

Example

The functions $y_1(x) = e^x$, $y_2(x) = e^{2x}$, and $y_3(x) = e^{3x}$ satisfy the DE

$$y''' - 6y'' + 11y' - 6y = 0.$$

Example

The functions $y_1(x) = e^x$, $y_2(x) = e^{2x}$, and $y_3(x) = e^{3x}$ satisfy the DE

$$y''' - 6y'' + 11y' - 6y = 0.$$

Since $W(e^x, e^{2x}, e^{3x}) = 2e^{6x} \neq 0$ for every real x.
Therefore the functions y_1, y_2, y_3 form a fundamental set of solutions on $(-\infty, \infty)$.
Thus the general solution is

$$y(x) = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$$

Theorem

Consider the nonhomogeneous 2^{nd} order linear ODE

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F(x): \quad a < x < b. \qquad (NH)$$

where $a_i(x);\ i = 0, 1, 2, \cdots n$ are continuous function on (a, b) and $a_0(x) \neq 0$ for any $x \in (a, b)$. Let

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \qquad (H)$$

be the corresponding homogeneous equation.
If $y_c(x)$ is a solution of (H) and $y_p(x)$ is a solution of (NH) then

$$y(x) = y_c(x) + y_p(x)$$

is a solution of (NH).

Proof:

since $y_c(x)$ is a solution of (H) we get

$$a_0(x)\frac{d^n y_c}{dx^n} + a_1(x)\frac{d^{n-1} y_c}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy_c}{dx} + a_n(x)y_c = 0 \tag{5}$$

Proof:

since $y_c(x)$ is a solution of (H) we get

$$a_0(x)\frac{d^n y_c}{dx^n} + a_1(x)\frac{d^{n-1} y_c}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy_c}{dx} + a_n(x)y_c = 0 \tag{5}$$

and since $y_p(x)$ is a solution of (NH) we get

$$a_0(x)\frac{d^n y_p}{dx^n} + a_1(x)\frac{d^{n-1} y_p}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy_p}{dx} + a_n(x)y_p = F(x) \tag{6}$$

Proof:

since $y_c(x)$ is a solution of (H) we get

$$a_0(x)\frac{d^n y_c}{dx^n} + a_1(x)\frac{d^{n-1} y_c}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy_c}{dx} + a_n(x)y_c = 0 \tag{5}$$

and since $y_p(x)$ is a solution of (NH) we get

$$a_0(x)\frac{d^n y_p}{dx^n} + a_1(x)\frac{d^{n-1} y_p}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy_p}{dx} + a_n(x)y_p = F(x) \tag{6}$$

Adding equations (5) and (6), we get

$$a_0(x)\frac{d^n (y_c + y_p)}{dx^n} + a_1(x)\frac{d^{n-1} (y_c + y_p)}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{d(y_c + y_p)}{dx} + a_n(x)(y_c + y_p) = F(x)$$

which implies that the function $y_c + y_p$ is also a solution of (NH).

General solution of (NH):

From the last Theorem, we can conclude the following.
If $y_1, y_2, \cdots y_n$ are n linearly independent solutions of (H) and $y_p(x)$ is a solution of (NH). Then the general solution of (NH) can be expressed as

$$y(x) = \sum_{i=1}^{n} c_i \, y_i(x) + y_p(x) = y_c(x) + y_p(x)$$

where the first term $y_c(x)$ is called the **complementary function** and $y_p(x)$ is called the **particular solution** or **particular integral** of (NH).

Example: If $y = x$ is the solution of the nonhomogeneous equation

$$\frac{d^2y}{dx^2} + y = x.$$

and $y = \sin x$ is a solution of the corresponding homogeneous equation

$$\frac{d^2y}{dx^2} + y = 0.$$

Then by the previous Theorem, the sum

$$\sin x + x$$

is the solution of the given nonhomogeneous equation.

Solution of *n*th order homogeneous linear equation with constant coefficients

Consider the *n*th order homogeneous linear equation

$$a_0\frac{d^n y}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = 0 \qquad (7)$$

where $a_0 \neq 0$, a_1,$a_2 \cdots\ a_n$ are real constants.

Solution of *n*th order homogeneous linear equation with constant coefficients

Consider the *n*th order homogeneous linear equation

$$a_0\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_{n-1}\frac{dy}{dx}+a_ny=0 \tag{7}$$

where $a_0 \neq 0$, a_1,a_2 $\cdots$ a_n are real constants.

- Now, we will try to find the general solution of the above equation .
- We need to find a function which can be the solution of the above equation?

Solution of nth order homogeneous linear equation with constant coefficients

Consider the nth order homogeneous linear equation

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0 \tag{7}$$

where $a_0 \neq 0$, a_1,a_2 $\cdots$ a_n are real constants.

- Now, we will try to find the general solution of the above equation .
- We need to find a function which can be the solution of the above equation?
- For this, we need a function such that its derivatives are constant multiples of itself.
- Do we know any function f having this property

$$\frac{d^k f(x)}{dx^k} = cf(x) \ \forall \ x$$

- **Answer:** Yes, exponential function e^{mx}, where m is a constant such that

$$\frac{d^k}{dx^k}(e^{mx}) = m^k e^{mx}$$

- Thus we seek solution of (7) of the form $y = e^{mx}$, where the constant m will be choosen such that e^{mx} does satisfy the equation.

- **Answer:** Yes, exponential function e^{mx}, where m is a constant such that

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- Thus we seek solution of (7) of the form $y = e^{mx}$, where the constant m will be choosen such that e^{mx} does satisfy the equation.
- Assume that $y = e^{mx}$ is a solution of equation (7) for certain m, we have

$$y' = me^{mx}, \frac{d^2y}{dx^2} = m^2e^{mx}, \cdots \frac{d^ny}{dx^n} = m^ne^{mx}.$$

- Substituting in (7), we get

$$a_0m^ne^{mx} + a_1m^{n-1}e^{mx} + \cdots + a_{n-1}me^{mx} + a_ne^{mx} = 0$$
$$(a_0m^n + a_1m^{n-1} + \cdots + a_{n-1}m + a_n)e^{mx} = 0$$

- Since $e^{mx} \neq 0$, we obtain the polynomial equation

$$a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n = 0. \tag{8}$$

- This equation is called the **auxiliary equation** or **characteristic equation** of the given differential equation (7).
- We note that if $y = e^{mx}$ is a solution of (7), then the constant m should satisfy the equation (8).
- Hence to solve (7), we write the auxiliary equation (8) and solve it for m.
- Since equation (8) is a polynomial of degree n. Therefore it has n roots(real or complex).
- Thus three cases arise according as the roots of the auxiliary equation (8).
  - (i) The roots are real and distinct.
  - (ii) The roots are real and repeated.
  - (iii) The roots are complex.

Case I: The roots are real and distinct

Suppose the roots of the auxiliary equation (8) are n distinct real numbers say

$$m_1, m_2, \cdots m_n.$$

Then

$$e^{m_1 x}, e^{m_2 x}, \cdots e^{m_n x}$$

are distinct n solutions of (7).
Also, these n solutions are linearly independent.($\because W(e^{m_1 x}, e^{m_2 x}, \cdots e^{m_n x}) \neq 0$)
Thus the general solution of (7) can be written as

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}.$$

where c_1, c_2, $\cdots$ c_n are arbitrary constants.

Example

Consider the differential equation $4y'' - 20y' + 24y = 0$.

The auxiliary equation is

$$4m^2 - 20m + 24 = 0.$$

$$\Rightarrow m_1 = 2, m_2 = 3.$$

That is, the roots are real and distinct.
$\therefore$ The general solution is

$$y(x) = c_1 e^{2x} + c_2 e^{3x}$$

where c_1 and c_2 are arbitrary constants.
Verify that e^{2x} and e^{3x} are linearly independent.
$\because$ their Wronskian is $\neq 0$

Case II- If the roots are real and repeated.

We will study this case by considering a simple example.
Consider the differential equation

$$y'' - 6y' + 9y = 0.$$

The auxiliary equation is

$$m^2 - 6m + 9 = 0$$

$$\Rightarrow (m-3)^2 = 0.$$

The roots of this equation are

$$m_1 = 3, m_2 = 3$$

which are real but not distinct.
Corresponding to the root m_1, we have the solution e^{3x}, and corresponding to m_2, we have the same solution e^{3x}.

Case II- If the roots are real and repeated(cont.).

- We can write the combination $c_1e^{3x} + c_2e^{3x} = (c_1 + c_2)e^{3x} = C_3e^{3x}$, which is involving only one arbitrary constant.

Case II- If the roots are real and repeated(cont.).

- We can write the combination $c_1 e^{3x} + c_2 e^{3x} = (c_1 + c_2)e^{3x} = C_3 e^{3x}$, which is involving only one arbitrary constant.
- So
$$y = C_3 e^{3x}$$
is not the general solution of the given differential equation.
- We need to find another linearly independent solution.
- But how shall we proceed to do so?

Case II- If the roots are real and repeated(cont.).

- We can write the combination $c_1e^{3x} + c_2e^{3x} = (c_1 + c_2)e^{3x} = C_3e^{3x}$, which is involving only one arbitrary constant.
- So
$$y = C_3e^{3x}$$
is not the general solution of the given differential equation.
- We need to find another linearly independent solution.
- But how shall we proceed to do so?
- Using the method of reduction of order, we find that the another linearly independent solution is
$$xe^{3x}.$$
- Thus the general solution of the given equation is
$$y = c_1e^{3x} + c_2xe^{3x}$$
$$y = (c_1 + c_2x)e^{3x}$$

Theorem

Consider the n th- order homogeneous linear differential equation

$$a_0\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_n y=0$$

with constant coefficients.

1. If the auxiliary equation $a_0m^n+a_1m^{n-1}+\cdots+a_{n-1}m+a_n=0$ has the real root m occurring k times, then the part of the general solution of the given equation corresponding to this k fold repeated root is

$$y=(c_1+c_2x+c_2x^2+\cdots+c_kx^{k-1})e^{mx}.$$

Case II- If the roots are real and repeated

Theorem

Consider the n th- order homogeneous linear differential equation

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_ny = 0$$

with constant coefficients.

1. If the auxiliary equation $a_0m^n + a_1m^{n-1} + \cdots + a_{n-1}m + a_n = 0$ has the real root m occurring k times, then the part of the general solution of the given equation corresponding to this k fold repeated root is

$$y = (c_1 + c_2x + c_2x^2 + \cdots + c_kx^{k-1})e^{mx}.$$

2. If, further, the remaining roots of the auxiliary equation are the distinct real numbers $m_{k+1}, \cdots m_n$, the the general solution of the given equation is

$$y = (c_1 + c_2x + c_3x^2 + \cdots + c_kx^{k-1})e^{mx} + c_{k+1}e^{m_{k+1}x} + \cdots + c_ne^{m_nx}.$$

Theorem

Consider the n th- order homogeneous linear differential equation

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1. If the auxiliary equation $a_0m^n+a_1m^{n-1}+\cdots+a_{n-1}m+a_n=0$ has the real root m occurring k times, then the part of the general solution of the given equation corresponding to this k fold repeated root is

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2. If, further, the remaining roots of the auxiliary equation are the distinct real numbers $m_{k+1},\cdots m_n$, the the general solution of the given equation is

$$y=(c_1+c_2x+c_3x^2+\cdots+c_kx^{k-1})e^{mx}+c_{k+1}e^{m_{k+1}x}+\cdots+c_ne^{m_nx}.$$

Example

Find the general solution of

$$y''' - y'' - y' + y = 0$$

The auxiliary equation is

$$m^3 - m^2 - m + 1 = 0$$

The roots of the auxiliary equation are $1, 1, -1$.
The general solution is

$$y = (c_1 + c_2x)e^x + c_3e^{-x}.$$

Example

If the roots of the auxiliary equation are $2, 2, 2, -1$. Then, the general solution of corresponding DE is

$$y = (c_1 + c_2 x + c_3 x^2)e^{2x} + c_4 e^{-x}.$$

Case III: If the roots are Conjugate Complex

- Suppose that the auxiliary equation has the complex root $a + ib$ (a, b real, $i^2 = -1$, $b \neq 0$) which is nonrepeated.
- Then since the coefficients are real, the conjugate complex number $a - ib$ is also a nonrepeated root.
- Therefore the corresponding part of the general solution is

$$k_1 e^{(a+ib)x} + k_2 e^{(a-ib)x},$$

  where c_1 and c_2 are arbitrary constants.
- It is desirable to replace these by two real independent solutions.

Case III: If the roots are Complex(cont.)

For this, consider

$$\begin{aligned} k_1 e^{(a+ib)x} + k_2 e^{(a-ib)x} &= k_1 e^{ax} e^{ibx} + k_2 e^{ax} e^{-ibx} \\ &= e^{ax}[k_1 e^{ibx} + k_2 e^{-ibx}] \\ &= e^{ax}[k_1(\cos bx + i\sin bx) + k_2(\cos bx - i\sin bx)] \\ &\quad (\text{ Using Euler's Formula } e^{i\theta} = \cos\theta + i\sin\theta.) \\ &= e^{ax}[(k_1+k_2)\cos bx + i(k_1-k_2)\sin bx] \\ &= e^{ax}[c_1\cos bx + c_2\sin bx] \end{aligned}$$

where $c_1 = k_1 + k_2$, $c_2 = i(k_1 - k_2)$ are two new arbitrary constants.
Thus the general solution corresponding to the nonrepeated conjugate complex roots $a \pm ib$ is

$$e^{ax}[c_1\cos bx + c_2\sin bx].$$

Theorem

Consider the n th- order homogeneous linear differential equation

$$a_0\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_ny=0$$

with constant coefficients.

1. If the auxiliary equation $a_0m^n+a_1m^{n-1}+\cdots+a_{n-1}m+a_n=0$. has the conjugate complex roots $a+ib$ and $a-ib$, neither repeated, then the corresponding part of the general solution of the given differential equation is

$$y=e^{ax}[c_1\cos bx+c_2\sin bx].$$

2. If, however, $a+ib$ and $a-ib$ are each k-fold roots of the auxiliary equation, then the general solution of the given equation is

$$y=e^{ax}[(c_1+c_2x+c_3x^2+\cdots+c_kx^{k-1})\cos bx+(c_{k+1}+c_{k+2}x+c_{k+3}x^2+\cdots+c_{2k}x^{k-1})\sin bx].$$

Example

Find the general solution of

$$y'' + y = 0$$

The auxiliary equation is

$$m^2 + 1 = 0$$

$$\Rightarrow m = \pm i$$

Here the roots of the auxiliary equation are conjugate complex numbers $a \pm ib$, where $a = 0$, $b = 1$.

The general solution is

$$y = e^{0x}(c_1 \cos x + c_2 \sin x).$$

$$y = (c_1 \cos x + c_2 \sin x)$$

Example

Find the general solution of

$$y'' - 6y' + 25y = 0$$

The auxiliary equation is

$$m^2 - 6m + 25 = 0$$

$$\Rightarrow m = 3 \pm 4i$$

Here the roots of the auxiliary equation are conjugate complex numbers $a \pm ib$, where $a = 3$, $b = 4$.
The general solution is

$$y = e^{3x}(c_1 \cos 4x + c_2 \sin 4x).$$

Example

If the roots of the auxiliary equation are $1+2i, 1-2i, 1+2i, 1-2i$, then the general solution of corresponding DE is

$$y = e^{x}[(c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x].$$

Example

Solve the initial value problem

$$y'' - 6y' + 25y = 0, y(0) = -3, y'(0) = -1$$

The auxiliary equation is

$$m^2 - 6m + 25 = 0$$

$$\Rightarrow m = 3 \pm 4i$$

Here the roots of the auxiliary equation are conjugate complex numbers $a \pm ib$, where $a = 3$, $b = 4$.

The general solution is

$$y = e^{3x}(c_1 \cos 4x + c_2 \sin 4x).$$

Since

$$y(0) = -3 \Rightarrow c_1 = -3.$$

$$y'(0) = -1 \Rightarrow c_2 = 2.$$

Thus the solution is

$$y = e^{3x}(2 \cos 4x - 3 \sin 4x).$$


Thank You