Department of Mathematics, Bennett University
Engineering Calculus (EMAT101L)
Solutions for Tutorial Sheet 2

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1. Since $\{a_n\}$ is bounded, let $|a_n| \le M$ for all n. Therefore, $|s_n| \le M$. Now, to prove $\{s_n\}$ is a monotone let $a_n \le a_{n+1}$ for all n. Then

$$\begin{aligned} s_{n+1} - s_n &= \frac{1}{n+1}\sum_{i=1}^{n} a_i + \frac{1}{n+1}a_{n+1} - \frac{1}{n}\sum_{i=1}^{n} a_i \\ &= \frac{1}{n+1}a_{n+1} - \frac{1}{n(n+1)}\sum_{i=1}^{n} a_i \\ &\ge \frac{1}{n+1}a_{n+1} - \frac{1}{n(n+1)} \cdot n a_{n+1} \\ &= 0. \end{aligned}$$

$\Rightarrow s_{n+1} \ge s_n$ for all n.

2. Use induction to show that the sequence $\{s_n\}$ is bounded below by $\frac{1}{2}$. Then note that if $s_n > \frac{1}{2}$, we have
$$s_{n+1} - s_n = -\frac{2}{3}s_n + \frac{1}{3} < 0.$$
Hence $\{s_n\}$ is nonincreasing.

3. $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) = \frac{1}{2}\left(\sqrt{a_n} - \frac{\sqrt{2}}{\sqrt{a_n}}\right)^2 + \sqrt{2} \Rightarrow a_{n+1} \ge \sqrt{2}, \ \forall\ n$, i.e. $\{a_n\}$ is bounded below. Also $2a_{n+1} - a_n = \frac{2}{a_n} \Rightarrow 2a_{n+1} - 2a_n = \frac{2}{a_n} - a_n = \frac{2-a_n^2}{a_n} \le 0$ $\Rightarrow a_{n+1} \le a_n$. Hence $\{a_n\}$ is nonincreasing. The limit of the sequence $\{a_n\}$ is $\sqrt{2}$.

4. (a) Note that the given sequence takes the value $\frac{\sqrt{3}}{2}$, $\frac{-\sqrt{3}}{2}$, 0 infinitely times. Therefore $\limsup a_n = \frac{\sqrt{3}}{2}$, $\liminf a_n = -\frac{\sqrt{3}}{2}$.
   (b) $\limsup a_n = +\infty$, $\liminf a_n = 0$.
   (c) $\limsup a_n = 0$, $\liminf a_n = 0$.

5. $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x| < 1$, then $\lim\limits_{n\to\infty} a_n = 0$.

6. Notice that $n^3 + n + 1 \le n^3 + n + i \le n^3 + n + n$ for all $1 \le i \le n$. This implies that $\frac{n^2}{n^3+2n} \le \frac{n^2}{n^3+n+i} \le \frac{n^2}{n^3+n+1}$ for all $1 \le i \le n$. Thus by taking the summation from $i = 1$ to n, we get
$$\frac{n.n^2}{n^3+2n} \le x_n \le \frac{n.n^2}{n^3+n+1}$$
and hence $x_n \to 1$.