Department of Mathematics
Bennett University
EMAT101L: July-December, 2018
Tutorial Sheet-3 (Multivariable Calculus)

1) Evaluate the following iterated integrals:
(i)$\int_0^{\ln 2}\int_1^{\ln 5} e^{2x+y}\,dy\,dx$

(ii) $\iint_{\mathcal{R}} \frac{\sqrt{x}}{y^2}\,dA$ $\qquad\mathcal{R}:\ 0\le x\le 4$ and $1\le y\le 2$

(iii)$\iint_{\mathcal{R}} y\,\sin(x+y)\,dA$ $\qquad\mathcal{R}:\ [-\pi,0]\times[0,\pi]$

(iv)$\iint_{\mathcal{R}} \frac{y}{1+x^2y^2}\,dA$ $\qquad\mathcal{R}:\ [0,1]\times[0,1]$

Answers:$(i)\ \frac{3}{2}(5-e)$
(ii) $\iint_{\mathcal{R}} \frac{\sqrt{x}}{y^2}\,dA = \int_{x=0}^{4}\int_{y=0}^{1}\frac{\sqrt{x}}{y^2}dydx = \int_{x=0}^{4}|-\frac{\sqrt{x}}{y}|_{y=1}^{2}dx = \int_{x=0}^{4}\frac{1}{2}x^{1/3}dx = |\frac{1}{3}x^{3/2}|_0^4 = \frac{8}{3}.$
(iii) $\iint_{\mathcal{R}} y\,\sin(x+y)\,dA = \int_{x=-\pi}^{0}\int_{y=0}^{\pi} y\,\sin(x+y)\,dydx$
$= \int_{x=-\pi}^{0}|-y\cos(x+y)+\sin(x+y)|_{y=0}^{\pi}dx$
$= \int_{x=-\pi}^{0}[\sin(x+\pi)-\pi\cos(x+\pi)-\sin x]dx = |-\cos(x+\pi)-\pi\sin(x+\pi)+\cos x|_{-\pi}^{0} = 4.$
(iv) $\iint_{\mathcal{R}} \frac{y}{1+x^2y^2}\,dA = \int_{y=0}^{1}\int_{x=0}^{1}\frac{y}{xy^2+1}dxdy$
$= \int_{y=0}^{1}|\tan^{-1}xy|_{x=0}^{1}dy = \int_{y=0}^{1}\tan^{-1}ydy = |y\tan^{-1}y-\frac{1}{2}\ln|1+y^2||_0^1 = \frac{\pi}{4}-\frac{1}{2}\ln 2.$

2) Write an iterated integral for $\iint_{\mathcal{R}} dA$ over the following region $\mathcal{R}$ using both vertically and horizontally simple regions:
(i) Bounded by $x=0$, $y=1$ and $y=\tan x$.
(ii)Bounded by $x=0$, $y=0$, $y=1$ and $y=\ln x$.
Answer: (i) $\iint_{\mathcal{R}} dA = \int_{x=0}^{\frac{\pi}{4}}\int_{y=\tan x}^{1} dydx = \int_{y=0}^{1}\int_{x=0}^{\tan^{-1}y} dxdy.$
(ii) $\iint_{\mathcal{R}} dA = \int_{x=0}^{1}\int_{y=0}^{1} dydx + \int_{x=1}^{e}\int_{y=\ln x}^{1} dydx = \int_{y=0}^{1}\int_{x=0}^{e^y} dxdy.$

3) Use the given transformations to transform the integrals and evaluate them:

(a) $u=3x+2y, v=x+4y$ and $I=\displaystyle\iint_R (3x^2+14xy+8y^2)\,dA$ where R is the region in the first quadrant bounded by the lines $y+\frac{3}{2}x=1, y+\frac{3}{2}x=3, y+\frac{1}{4}x=0$, and $y+\frac{1}{4}x=1$.

(b) $u=x+2y, v=x-y$ and $I=\displaystyle\int_0^{2/3}\int_y^{2-2y}(x+2y)e^{(y-x)}dA$

(c) $u=xy,\ v=x^2-y^2$ and $I=\displaystyle\iint_R (x^2+y^2)dA$, where R is the region bounded by $xy=1, xy=2, x^2-y^2=1$ and $x^2-y^2=2$.

Solution:

(a) Here we get $x = \frac{2u-v}{5}$ and $y = \frac{3v-u}{10}$. The Jacobian of the transformation is

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2/5 & -1/5 \\ -1/10 & 3/10 \end{vmatrix} = 1/10$$

The image is again a rectangle with sides $u = 2, u = 6, v = 0$ and $v = 4$. Hence

$$\iint_R (3x^2 + 14xy + 8y^2)\, dA = \int_{u=2}^{6} \left(\int_0^4 |J| uv dv \right) du = \frac{64}{5}$$

(b) The Jacobian $J = -\frac{1}{3}$ and $|J| = 1/3$. The given domain is the triangle bounded by $y = x, y = 0$ and $x + 2y = 2$. The image of this triangle under the transformation is again a triangle bounded by $v = 0, v = u$ and $u = 2$. Hence

$$\int_0^{2/3} \int_y^{2-2y} (x+2y) e^{(y-x)} dA = \frac{1}{3} \int_{u=0}^{2} \left(\int_0^u u e^{-v} \frac{1}{3} dv \right) du = \frac{1}{3}(3e^{-2} + 1)$$

(c) The domain of integration is the domain bounded by the hyperbolas $xy = 1, xy = 2, x^2 - y^2 = 1$ and $x^2 - y^2 = 2$. Under the given transformations this hyperbolas are transformed into the lines $u = 1, u = 2, v = 1$ and $v = 2$. The Jacobian J may be calculated using the relation $JJ^{-1} = 1 \Rightarrow J = \frac{1}{J^{-1}}$. Here,

$$J^{-1} = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = -2(x^2 + y^2).$$

Therefore

$$\iint_R (x^2 + y^2) dA = \int_{u=1}^{2} \left(\int_{v=1}^{2} (x^2 + y^2) \cdot \frac{1}{2(x^2 + y^2)} dv \right) du = \int_{u=1}^{2} \int_{v=1}^{2} \frac{1}{2} dv du = \frac{1}{2}.$$

4) Find the area of the following:

(a) The region lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$ in the first quadrant.

(b) The region common to the interiors of the cardioids $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$.

Solution:

(a) Area of the region R in the polar coordinates is given by the formula $A = \iint_R r dr\, d\theta$. By drawing a ray from origin, it is easy to see that the ray intersects the domain R at $r = 1$(closest from origin) and $r = 1 + \cos\theta$(farthest from origin). Hence the area is

$$A = \int_{\theta=0}^{\pi/2} \left(\int_1^{1+\cos\theta} r dr \right) d\theta = \frac{8 + \pi}{8}.$$

(b) The domain is symmetric with respect to x axis and y axis. So it is enough to evaluate the area in the first quadrant. Here as above we can see that the region is bounded by $r = 0$ and $r = 1 - \cos\theta$. Hence

$$A = 4 \int_0^{\pi/2} \left(\int_0^{1-\cos\theta} r dr \right) d\theta$$