$$\sum_{n=0}^{\infty} a_n (x-c)^n, \quad \lim_{n\to\infty} |a_n|^{1/n} = \beta$$

$$R = \frac{1}{\beta}.$$

This series conv. $|x-c| < R$.

$$\Rightarrow -R < x-c < R$$

$$\Rightarrow c-R < x < c+R.$$

Interval of conv $(c-R, c+R)$

at $x = c-R$, $\sum_{n=0}^{\infty} a_n (x-c)^n$ check it conv. or div.

$x = c+R$

$[c-R, c+R] \to$ Interval of conv.

Ex:- $\sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{2n}$, $a_n = \frac{(-1)^n}{2n}$

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|\frac{(-1)^n}{2n}\right|^{1/n} = \lim_{n\to\infty} \frac{1}{2^{1/n} \cdot n^{1/n}} = 1$$

$R = 1$

series conv for $|x-2| < 1$

$$\Rightarrow -1 < x-2 < 1$$

$$\Rightarrow 1 < x < 3$$

$x = 1$, $\sum_{n=1}^{\infty} \frac{(-1)^n (1-2)^n}{2n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{2n} = \sum_{n=1}^{\infty} \frac{1}{2n} \to$ div.

$x = 3$, $\sum_{n=1}^{\infty} \frac{(-1)^n (3-2)^n}{2n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n} \to$ conv.

$(1, 3] \to$ interval of conv.

Ex:- $\sum_{n=0}^{\infty} \frac{(x+2)^{3n}}{25^n}$, $a_{3n} = \frac{1}{25^n}$

$$\lim_{n\to\infty} |a_{3n}|^{1/3n} = \lim_{n\to\infty} \left(\frac{1}{25^n}\right)^{1/3n} = \frac{1}{5^{2/3}}$$

$R = 5^{2/3}$

series conv. $|x+2| < 5^{2/3}$

$\Rightarrow -5^{2/3} < x+2 < 5^{2/3}$

$\Rightarrow -2-5^{2/3} < x < -2+5^{2/3}$

at $x = -2-5^{2/3}$, $\sum_{n=0}^{\infty} \frac{(-5^{2/3})^{3n}}{5^{2n}} = \sum_{n=0}^{\infty} (-1)^{3n} \frac{5^{2n}}{5^{2n}}$

$= \sum_{n=0}^{\infty} (-1)^n \rightarrow$ div.

$x = -2+5^{2/3}$, $\sum_{n=0}^{\infty} \frac{(5^{2/3})^{3n}}{5^{2n}} = \sum_{n=0}^{\infty} 1 \rightarrow$ div.

Interval of conv. $(-2-5^{2/3}, -2+5^{2/3})$

Taylor's formula

Ex:- Find the error while app. $\sqrt{1+x}$ with $1+\frac{(x-0)}{2}$ in $|x|<0.01$

$f(x) = \sqrt{1+x}$, $P_1(x) = 1+\frac{x}{2}$

$R_1(x) = \frac{x^2}{2!} f''(c)$.

$f'(x) = \frac{1}{2}\frac{1}{(1+x)^{1/2}}$

$= \frac{x^2}{2}\left(-\frac{1}{4}\right)\cdot\frac{1}{(1+c)^{3/2}}$

$f''(x) = \left(-\frac{1}{4}\right)\frac{1}{(1+x)^{3/2}}$

$|R_1(x)| = \left|\frac{x^2}{2}\left(-\frac{1}{4}\right)\frac{1}{(1+c)^{3/2}}\right|$ $\quad 1+0.0001 > 1$

$< \frac{x^2}{2\cdot 4\cdot (1+c)^{3/2}}$ $\quad \frac{1}{1+c} < 1$

$< \frac{(0.01)^2}{8}$.

Ex:- Find the domain of validity when $\sin x$ is app. by $x - \frac{x^3}{6}$ with error of mag. no greater than 5×10^{-4}.

$f(x) = \sin x$, $P_3(x) = x - \frac{x^3}{6}$

$R_3(x) = \frac{x^4}{4!} f^{iv}(c)$

$R_3(x) = \frac{x^4}{4!} \sin c \quad , |\sin c| < 1$

$|R_3(x)| = \left|\frac{x^4}{4!} \sin c\right| < \left|\frac{x^4}{4!}\right| < 5 \times 10^{-4}$

$\Rightarrow |x| < \frac{3}{10}.$

$f(x) = P_n(x) + R_n(x) \rightarrow = \frac{f^{n+1}(c)}{n+1!}(x-a)^{n+1}$

$\hookrightarrow 2$

<u>Uni continuity</u>

<u>Defn</u> ① f is uni. conti on D. if $\epsilon > 0$
$\exists\ \delta > 0$ s.t $\forall x, y \in D$
$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon.$

<u>Defn</u> ② f is uni conti on $D \iff \{x_n\}, \{y_n\} \in D$
s.t $|x_n - y_n| \to 0$ than $|f(x_n) - f(y_n)| \to 0$
as $n \to \infty$
$f: D \to \mathbb{R}.$

③ f is uni conti on D. If $\{x_n\}$ is any cauchy seqn in $D \Rightarrow$
$\{f(x_n)\}$ is also cauchy seqn in $\mathbb{R}$.

<u>Result</u>

④ If f is conti on $[a,b] \Rightarrow f$ is Uni conti on $[a,b]$.

⑤ If f has discontinuity on $[a,b]$
$\downarrow$
(Removable discontinuity)
Then $\exists\ \tilde{f} \to$ extension function of f on $[a,b]$ s.t $\tilde{f}$ uni conti $[a,b]$

⑥ f is conti on (a,b).
$\tilde{f}$ on $[a,b] \Rightarrow \tilde{f}$ uni on $[a,b]$
$\Rightarrow \tilde{f} = f$ on (a,b)
$\Rightarrow f$ is Uni conti (a,b)

EX:- $f(x) = \frac{1}{x}$, $(0,1)$ → check f is uni conti or not.

$$\boxed{\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{x} = \infty}$$

$x_n = \frac{1}{n}$, $y_n = \frac{1}{n+1}$ → $|x_n - y_n| = \frac{1}{n(n+1)} \to 0$

$|f(x_n) - f(y_n)| = |n - n - 1| = 1 \not\to 0$

i.e f is not uni. conti on $(0,1)$

EX:- $f(x) = x^2$, $x \in \mathbb{R}$, check uni continuity.

$$\boxed{\lim_{x \to \infty} x^2 = \infty}$$

$x_n = n + \frac{1}{n}$, $y_n = n$, $|x_n - y_n| = \frac{1}{n} \to 0$

$|f(x_n) - f(y_n)| = 2 + \frac{1}{n^2} \to 2 \neq 0$.

$\therefore f$ is not uni conti.

EX:- $f(x) = e^{x^2} \sin x^2$, $x \in \underline{(0,1)}$

$$\tilde{f}(x) = \begin{cases} e^{x^2} \sin x^2, & x \in (0,1) \\ 0 & x = 0 \\ e \sin 1 & x = 1 \end{cases}$$

$\lim_{x \to c} f(x) = f(c)$

$\tilde{f} : [0,1] \to \mathbb{R}$ is conti

$\Rightarrow \tilde{f}$ is uni conti

$\tilde{f} = f$ on $(0,1)$

$\Rightarrow f$ is uni conti on $(0,1)$

$$f(x) = \begin{cases} x^3 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

check f' is conti or not

Step ① $\lim_{x \to 0} x^3 \sin \frac{1}{x} = 0 = f(0)$

i.e f is conti on $\mathbb{R}$

step ② $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$

$\therefore f'(0) = 0$

$f'(x) = \cancel{3x^2 \sin\frac{1}{x}} + x^3 \cos\frac{1}{x}\left(-\frac{1}{x^2}\right)$

$$f'(x) = \begin{cases} 3x^2 \sin\frac{1}{x} - x\cos\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

step 3 : $\lim_{x\to 0} 3x^2 \sin\frac{1}{x} - x\cos\frac{1}{x} = 0 = f'(0)$

f' is conti on $\mathbb{R}$.

$\boxed{\lim_{x\to 0} f''(x) \neq f''(0) \quad ??}$ → not. conti at $=0$

$$f(x) = \begin{cases} x\sin\frac{1}{x}, & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$\lim_{x\to 0} \frac{x\sin\frac{1}{x} - 0}{x - 0} = \lim_{x\to 0} \sin\frac{1}{x}$