Ordinary Differential Equations(EMAT102L)
(Lecture-9)

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Outline of the Lecture

We will learn

- Picard's Existence and Uniqueness Theorem
- Picard's Iteration Method

Theorem

Let R be a rectangle and (x_0, y_0) be an interior point of R, let

- $f(x, y)$ be continuous at all points (x, y) in

$$R : |x - x_0| \leq a, |y - y_0| \leq b.$$

- Bounded in R, that is, $|f(x, y)| \leq M$ for all $(x, y) \in R$.
- f satisfies the Lipschitz condition with respect to y in R, that is, $|f(x, y_1) - f(x, y_2)| \leq K|y_1 - y_2|$ for all $(x, y_1), (x, y_2) \in R$.

Then, the initial value problem

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0$$

has a unique solution $y(x)$, defined for all x in the interval $|x - x_0| \leq h$, where

$$h = \min\left(a, \frac{b}{M}\right).$$

Example

Show that the solution of the following IVP is unique. Then find the interval of existence of the solution.

$$\frac{dy}{dx} = x^2 + e^{-y^2},\ y(0) = 0,\ R : |x| \leq \frac{1}{2}, |y| \leq 1$$

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Solution:

- Here $f(x, y) = x^2 + e^{-y^2}$ is continuous and consider

$$\begin{aligned} |f(x,y)| &= |x^2 + e^{-y^2}| \\ &\leq |x|^2 + \left|\frac{1}{e^{y^2}}\right| \\ &\leq \frac{1}{4} + 1 = \frac{5}{4} \end{aligned}$$

$$\text{Thus } M = \frac{5}{4}$$

Thus existence of the solution is guaranteed.

Example(cont.)

- Now, we check Lipschitz Condition.

$$\frac{\partial f}{\partial y} = e^{-y^2}(-2y) = \frac{-2y}{e^{y^2}}$$

$$\Rightarrow \left|\frac{\partial f}{\partial y}\right| = \left|\frac{-2y}{e^{y^2}}\right| = \left|\frac{2y}{e^{y^2}}\right| \leq K$$

So, $f(x,y)$ satisfies Lipschitz condition also.

- Thus, we have $f(x,y)$ satisfies all the three conditions. So, the given IVP has unique solution in $|x - x_0| \leq h$

$$\Rightarrow |x| \leq h, \text{ where } h = \min\left(\frac{1}{2}, \frac{1}{5/4}\right) = \min\left(\frac{1}{2}, \frac{4}{5}\right)$$

$$\Rightarrow h = \frac{1}{2}.$$

$$\text{Thus } |x| \leq \frac{1}{2}.$$

Example

Consider

$$\frac{dy}{dx} = x^2 + y^2, y(0) = 0$$

in the rectangle $R : |x - 0| \leq 3, |y - 0| \leq 5$.

Solution:

- Here $x_0 = 0, y_0 = 0, a = 3, b = 5$.
  $f(x, y) = x^2 + y^2$ is continuous and $|f(x, y)| = |x^2 + y^2| \leq |x|^2 + |y|^2 \leq 34$, thus existence of the solution is guaranteed.
- For uniqueness of the solution, we need to check Lipschitz condition.
  $|f(x, y_1) - f(x, y_2)| = |x^2 + y_1^2 - x^2 - y_2^2| = |y_1^2 - y_2^2| = |(y_1 - y_2)(y_1 + y_2)| \leq 10|y_1 - y_2|$,
  Thus uniqueness of solution is guaranteed.
- Interval of existence of unique solution is $|x - 0| \leq h$, where

$$h = \min\left(a, \frac{b}{M}\right) = \min\left(3, \frac{5}{34}\right) = \frac{5}{34}.$$

Objective

To solve

$$\frac{dy}{dx} = f(x,y), \;\; y(x_0) = y_0. \tag{1}$$

Procedure:

1. Integrate both side of (1) to obtain

$$y(x) - y(x_0) = \int_{x_0}^{x} f(s, y(s))ds$$

$$y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))ds \tag{2}$$

2. The initial approximation is $y_0(x) = y_0$. Solve (2) by iteration:

$$y_1(x) = y_0 + \int_{x_0}^{x} f(s, y_0(s))ds$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(s, y_1(s))ds$$

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$$y_n(x) = y_0 + \int^{x} f(s, y_{n-1}(s))ds$$

Then $y_0, y_1, \cdots\ y_n$ are called **Picard's Successive Approximations** to the IVP (1). Under the assumptions of existence-uniqueness theorem, the sequence of approximations converges to the solution $y(x)$ of (1). That is,

$$y(x) = \lim_{n\to\infty} y_n(x).$$

and is well defined on the interval $|x - x_0| \leq h = \min(a, \frac{b}{M})$, i.e. $\forall x \in [x_0 - h, x_0 + h]$

Example

Solve

$$y' = -y, y(0) = 1.$$

using Picard's Iteration Method.

Example

Solve
$$y' = -y, y(0) = 1.$$
using Picard's Iteration Method.

Solution:

1. Let $y_0(x) = y_0 = 1$, then the successive approximations are :

$$y_1(x) = y_0 + \int_{x_0}^{x} f(s, y_0(s))ds = 1 + \int_0^x (-1)ds = 1 - x.$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(s, y_1(s))ds = 1 - \int_0^x (1-s)ds = 1 - x + \frac{x^2}{2!}$$

.

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$$y_n(x) = 1 - x + \frac{x^2}{2!} + \cdots + \frac{(-1)^n}{n!}x^n.(By\ induction)$$

Solution

$$y(x) = \lim_{n\to\infty} y_n(x) = e^{-x}.$$

is the solution of the given IVP.

Problem for practice

Example

Solve

$$\frac{dy}{dx} = xy, y(0) = 1.$$

using Picard's iteration Method.


Thank You