# Multivariable Calculus

(Lecture-10)

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Multiple Integration

of

(Scalar Valued Function of Vector Variable)

(Scalar Field)

$$F : R \subseteq \mathbb{R}^n \rightarrow \mathbb{R},\ n = 2, 3$$

# Learning Outcome of this lecture

In the next few lectures, we learn about the Riemann-Darboux integration of a bounded scalar valued function $f : \mathcal{R} \subseteq \mathbb{R}^n \to \mathbb{R}$ over simple and bounded region $\mathcal{R}$.

- Double Integrals
  - Double Integral of $f : \mathcal{R} \subseteq \mathbb{R}^2 \to \mathbb{R}$ where $\mathcal{R}$ is a rectangular region in $\mathbb{R}^2$
  - Iterated Integrals of $f$ and Fubinis Theorem

# Recall: Riemann integration of $f : [a, b] \to \mathbb{R}$

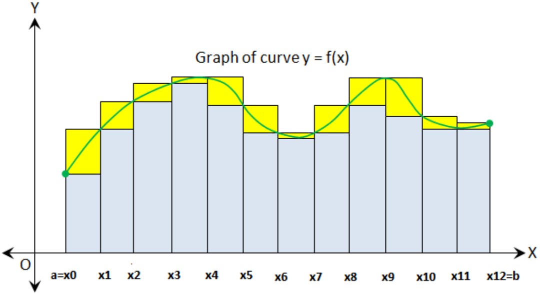
Partition of $[a, b]$: $\quad a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$

For a partition $P = [x_0, x_1, \ldots, x_n]$ of $[a, b]$, let

$$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\} \quad m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$$

Lower and Upper sums:

$$L(\mathcal{P}, f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \quad L(\mathcal{P}, f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$



# Continuation of previous slide

$$\text{Lower sum}: \quad L(\mathcal{P},f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$

$$\text{Upper sum}: \quad U(\mathcal{P},f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

$$\text{Lower integral}: \quad \int_{\bar{a}}^{b} f = \sup_{\mathcal{P}} L(\mathcal{P},f)$$

$$\text{Upper integral}: \quad \int_{a}^{\bar{b}} f = \inf_{\mathcal{P}} U(\mathcal{P},f)$$

Riemann integral: If Lower integral = Upper integral, then $f$ is Riemann integrable on $[a, b]$ and the common value is the Riemann integral of $f$ on $[a, b]$, denoted by $\int_a^b f$.

Note: Integration gives area bounded by the graph of the function and $x$-axis.

Riemann-integral of $f : \mathcal{R} \subset \mathbb{R}^2 \to \mathbb{R}$, where $\mathcal{R}$ is a rectangular region in $\mathbb{R}^2$.

# Partition or Mesh or Grid of a Rectangular Region

Let $\mathcal{R}$ be the bounded rectangular region given by

$\mathcal{R} = \{(x, y) \subset \mathbb{R}^2 \,:\, a \leq x \leq b \text{ and } c \leq y \leq d\}$ where $a < b \,\&\, c < d$.

Consider a partition $P_x$ of $[a, b]$ given by

$$P_x : \; a = x_0 < x_1 < x_2 < \cdots < x_{m-1} < x_m = b.$$

Consider a partition $P_y$ of $[c, d]$ given by

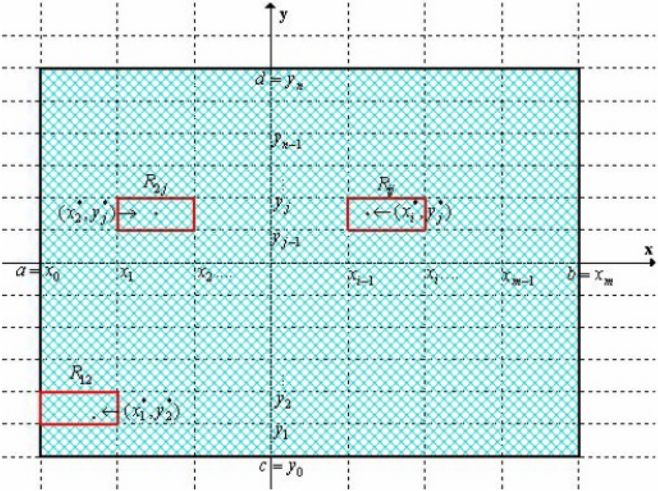
$$P_y : \; c = y_0 < y_1 < y_2 < \cdots < y_{n-1} < y_n = d.$$

Then $\mathcal{P} = (P_x, P_y)$ partitions $\mathcal{R}$ into $mn$ subrectangles or cells as follows. Set for $1 \leq i \leq m$ and $1 \leq j \leq n$,

$$R_{ij} = \{(x, y) \in \mathcal{R} \,:\, x_{i-1} \leq x \leq x_i \text{ and } y_{j-1} \leq y \leq y_j\}.$$

$$\mathcal{P} = \{R_{ij} \,:\, 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}.$$

Then $\mathcal{P}$ is a partition or mesh or grid of the rectangular region $\mathcal{R}$.



y
x
$d = y_n$
$y_{n-1}$
$R_{2j}$
$R_{ij}$
$(x_2^*, y_j^*)$
$y_j$
$(x_i^*, y_j^*)$
$y_{j-1}$
$a = x_0$
$x_1$
$x_2 \cdots$
$x_{i-1}$
$x_i \cdots$
$x_{m-1}$
$b = x_m$
$R_{12}$
$y_2$
$(x_1^*, y_2^*)$
$y_1$
$c = y_0$

# Mesh Size or Grid Size or Norm of a Partition

Let $\mathcal{R}$ be the bounded rectangular region given by

$$\mathcal{R} = \{(x, y) \subset \mathbb{R}^2 \,:\, a \le x \le b \text{ and } c \le y \le d\}.$$

## Definition

If $\mathcal{P} = \{R_{ij} \,:\, 1 \le i \le m \text{ and } 1 \le j \le n\}$ is a mesh or grid of the rectangular region $\mathcal{R}$ then

- the diameter of the set $R_{ij}$ given by

$$d(R_{ij}) = |R_{ij}| = \mathrm{diam}(R_{ij}) = \sup\{|X - Y| \,:\, X \in R_{ij} \text{ and } Y \in R_{ij}\},$$

- the mesh size or grid size or norm of $\mathcal{P}$ is defined by

$$\|P\| = \sup\{\mathrm{diam}(R_{ij}) \,:\, 1 \le i \le m \text{ and } 1 \le j \le n\}.$$

- Let $\mathcal{R}$ be the bounded rectangular region given by

$$\mathcal{R} = \{(x, y) \subset \mathbb{R}^2 \; : \; a \leq x \leq b \; \text{ and } c \leq y \leq d\}.$$

- Let $f : \mathcal{R} \subset \mathbb{R}^2 \to \mathbb{R}$ be a bounded function on $\mathcal{R}$.
- Let $\mathcal{P} = \{R_{ij} \; : \; 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$ be a mesh of the rectangular region $\mathcal{R}$.
- On each subrectangle/ cell $R_{ij}$, $1 \leq i \leq m$, $1 \leq j \leq n$, set

$$\begin{aligned}
M_{ij} &= \sup\{f(x, y) \; : \; (x, y) \in R_{ij}\} \\
m_{ij} &= \inf\{f(x, y) \; : \; (x, y) \in R_{ij}\} \\
\Delta A_{ij} = \text{Area of } R_{ij} &= (x_i - x_{i-1})(y_j - y_{j-1}) = \Delta x_i \Delta y_j,
\end{aligned}$$

where $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_j = y_j - y_{j-1}$.

# Upper Sum, Lower Sum, Upper Integral, Lower Integral

Upper Riemann sum : $$U(\mathcal{P}, f) = \sum_{i=1}^{m} \sum_{j=1}^{n} M_{ij} \, \Delta A_{ij}$$

Lower Riemann sum : $$L(\mathcal{P}, f) = \sum_{i=1}^{m} \sum_{j=1}^{n} m_{ij} \, \Delta A_{ij}$$

Upper integral of $f$ over $\mathcal{R}$ : $$\overline{\iint_{\mathcal{R}}} f = \inf_{\mathcal{P} \in \Pi} U(\mathcal{P}, f)$$

Lower integral of $f$ over $\mathcal{R}$ : $$\underline{\iint_{\mathcal{R}}} f = \sup_{\mathcal{P} \in \Pi} L(\mathcal{P}, f)$$

where $\Pi$ denotes the collection of all meshes $\mathcal{P}$ of the rectangular region $\mathcal{R}$.

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# Riemann Integral / Double Integral of $f$

Let $f : \mathcal{R} \subset \mathbb{R}^2 \to \mathbb{R}$ be a bounded function on $\mathcal{R}$.

**Definition**

The function $f$ is said to be (Riemann) integrable or simply integrable over $\mathcal{R}$ if

$$\overline{\iint_{\mathcal{R}}} f = \underline{\iint_{\mathcal{R}}} f$$

In this case, the common value is called the (Riemann) integral of $f$ over $\mathcal{R}$ and is denoted by

$$\iint_{\mathcal{R}} f \quad \text{or} \quad \iint_{\mathcal{R}} f(x,y)\,dA \quad \text{or} \quad \iint_{\mathcal{R}} f(x,y)\,dxdy.$$

Since $f$ is a real valued function of two variables, it is also called the double integral of $f$ over the region $\mathcal{R}$.

Note: Double integral gives the volume of the solid bounded by the surface $z = f(x,y)$ and $xy$-plane.

# An Example of Integrable Function

Let $\mathcal{R} = \{(x,y) \subset \mathbb{R}^2 \ : \ a \leq x \leq b \ \text{ and } c \leq y \leq d\}$. Let $f(x,y) = k$ for $(x,y) \in \mathcal{R}$ where $k$ is a real constant. Then for any mesh $\mathcal{P}$ of $\mathcal{R}$

$$\begin{aligned} \text{Upper (Riemann) Sum} &= U(\mathcal{P}, f) = k(b-a)(d-c) \\ \text{Lower (Riemann) Sum} &= L(\mathcal{P}, f) = k(b-a)(d-c) \end{aligned}$$

Therefore

$$\begin{aligned} \text{Upper integral of } f &= \overline{\iint_{\mathcal{R}}} f = \inf_{\mathcal{P}}(U(\mathcal{P}, f)) = k(b-a)(d-c) \\ \text{Lower integral of } f &= \underline{\iint_{\mathcal{R}}} f = \sup_{\mathcal{P}}(L(\mathcal{P}, f)) = k(b-a)(d-c) \end{aligned}$$

So

$$\iint_{\mathcal{R}} f(x,y)\,dA = k(b-a)(d-c).$$

# An Example of Non-Integrable Function

Let $\mathcal{R} = \{(x, y) \subset \mathbb{R}^2 \,:\, a \leq x \leq b \text{ and } c \leq y \leq d\}$. Let $f(x, y) = 1$ for $(x, y) \in \mathcal{R}$ with $x \in \mathbb{Q} \& y \in \mathbb{Q}$, and $f(x, y) = 0$ otherwise. Then for any mesh $\mathcal{P}$ of $\mathcal{R}$

$$\begin{aligned} \text{Upper (Riemann) Sum} &= U(\mathcal{P}, f) = (b-a)(d-c) \\ \text{Lower (Riemann) Sum} &= L(\mathcal{P}, f) = 0 \end{aligned}$$

Therefore

$$\begin{aligned} \text{Upper integral of } f &= \overline{\iint_{\mathcal{R}}} f = \inf_{\mathcal{P}}(U(\mathcal{P}, f)) = (b-a)(d-c) \\ \text{Lower integral of } f &= \underline{\iint_{\mathcal{R}}} f = \sup_{\mathcal{P}}(L(\mathcal{P}, f)) = 0 \end{aligned}$$

So

$$\iint_{\mathcal{R}} f(x, y)\, dA \quad \text{does not exist.}$$

# Iterated Integrals of $f$

Let $f : \mathcal{R} \subset \mathbb{R}^2 \to \mathbb{R}$ be a bounded function on

$$\mathcal{R} = \{(x, y) \subset \mathbb{R}^2 \ : \ a \leq x \leq b \ \text{ and } c \leq y \leq d\}.$$

The following integrals

$$\int_{x=a}^{x=b} \left( \int_{y=c}^{y=d} f(x, y) dy \right) dx \quad \text{and} \quad \int_{y=c}^{y=d} \left( \int_{x=a}^{x=b} f(x, y) dx \right) dy$$

are called iterated integrals of $f$ over $\mathcal{R}$.

$$\text{Double integral of } f \text{ over } \mathcal{R} = \iint_{\mathcal{R}} f(x, y) dA.$$

# Fubini's Theorem for rectangular regions

Let $f : \mathcal{R} \subset \mathbb{R}^2 \to \mathbb{R}$ be a bounded function on

$$\mathcal{R} = \{(x, y) \subset \mathbb{R}^2 \; : \; a \leq x \leq b \;\text{ and } c \leq y \leq d\}.$$

**Theorem**

If $f$ is continuous in $\mathcal{R}$ then the double integral of $f$ over $\mathcal{R}$ exist and both the iterated integrals of $f$ also exist, and

$$\begin{aligned} \iint_{\mathcal{R}} f(x,y)dA &= \int_{x=a}^{x=b} \left( \int_{y=c}^{y=d} f(x,y)dy \right) dx \\ &= \int_{y=c}^{y=d} \left( \int_{x=a}^{x=b} f(x,y)dx \right) dy. \end{aligned}$$

The above theorem gives a method to evaluate the double integral of $f$ through evaluation of iterated integrals of $f$.

# Another Version of Fubini's Theorem

**Theorem**

Let $f : \mathcal{R} \to \mathbb{R}$ be integrable over $\mathcal{R} = \{(x, y) \subset \mathbb{R}^2 \ : \ a \leq x \leq b \ \text{and} \ c \leq y \leq d\}$.

- For each $y \in [c, d]$, define the function $F_y(x) = f(x, y)$ for $x \in [a, b]$. If $F_y : [a, b] \to \mathbb{R}$ is integrable over $[a, b]$ for each $y$ then
$$\iint_{\mathcal{R}} f(x, y)dA = \int_{y=c}^{y=d} \left( \int_{x=a}^{x=b} f(x, y)dx \right) dy.$$
- For each $x \in [a, b]$, define the function $F_x(y) = f(x, y)$ for $y \in [c, d]$. If $F_x : [c, d] \to \mathbb{R}$ is integrable over $[c, d]$ for each $x$ then
$$\iint_{\mathcal{R}} f(x, y)dA = \int_{x=a}^{x=b} \left( \int_{y=c}^{y=d} f(x, y)dy \right) dx.$$

# Example: Evaluating the Iterated Integrals

Let $f(x,y) = x^2y$ for $(x,y) \in \mathcal{R} = [-1,2] \times [0,2]$. Compute $\iint_{\mathcal{R}} f(x,y)dA$ by means of computing its iterated integrals.

$$\begin{aligned}
\int_{y=0}^{y=2} \left( \int_{x=-1}^{x=2} x^2y\,dx \right) dy &= \int_{y=0}^{y=2} y \left( \int_{x=-1}^{x=2} x^2 dx \right) dy = \\
\int_{y=0}^{y=2} y \left( \left[ \frac{x^3}{3} \right]_{x=-1}^{2} \right) dy &= \int_{y=0}^{y=2} 3y dy = \left[ \frac{3y^2}{2} \right]_{y=0}^{2} = 6
\end{aligned}$$

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$$\begin{aligned}
\int_{x=-1}^{x=2} \left( \int_{y=0}^{y=2} x^2y\,dy \right) dx &= \int_{x=-1}^{x=2} x^2 \left( \int_{y=0}^{y=2} y dy \right) dx = \\
\int_{x=-1}^{x=2} x^2 \left( \left[ \frac{y^2}{2} \right]_{y=0}^{2} \right) dx &= \int_{x=-1}^{x=2} 2x^2 dx = \left[ \frac{2x^3}{3} \right]_{x=-1}^{2} = 6
\end{aligned}$$