Improper Integrals

Comparison test :-

Suppose $0 \le f(x) \le g(x)$
$\forall\, x \geq a$

Then ① $\int_a^\infty g(x)\,dx$ conv
$\Rightarrow \int_a^\infty f(x)\,dx$ conv.

② $\int_a^\infty f(x)\,dx$ div $\Rightarrow \int_a^\infty g(x)\,dx$ div.

Ex 1 :- $\int_1^\infty \frac{dx}{x^2(1+e^x)}$

$$\frac{1}{x^2(1+e^x)} < \frac{1}{x^2} \quad \forall\, x \geq 1$$

$\int_1^\infty \frac{1}{x^2}\,dx$ Conv. $\Rightarrow \int_1^\infty \frac{dx}{x^2(1+e^x)}$ Conv.

Ex 2: $\int_1^\infty \frac{x^3}{x+1}\,dx$

$$\frac{x^3}{x+1} > \frac{x^2}{2} \quad \forall\, x \geq 1$$

$x > 1$
$2x > 1+x \Rightarrow \frac{1}{1+x} > \frac{1}{2x}$
$\Rightarrow \frac{x^3}{1+x} > \frac{x^3}{2x} = \frac{x^2}{2}$

$\int_1^\infty \frac{x^2}{2}\,dx$ div.
$\Rightarrow \int_1^\infty \frac{x^3}{x+1}\,dx$ div.

Ex :- $\int_1^\infty \frac{1}{1+\sqrt{x}}\,dx$

$$\frac{1}{1+\sqrt{x}} \geq \frac{1}{2\sqrt{x}} \quad \forall\, x \geq 1$$

$\int_1^\infty \frac{1}{2\sqrt{x}}\,dx$ div
$\Rightarrow \int_1^\infty \frac{1}{1+\sqrt{x}}\,dx$ div.

EX:- $\int_1^\infty \frac{\sqrt{x}}{1+x^5}\,dx$, conv or div??

Limit comparison test:-

Let f, g define & positive $\forall x \geq a$ and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$. Then

① if $L > 0$, then $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ both conv or div together.

② $L = 0$ and $\int_a^\infty g(x)\,dx$ conv
$\Rightarrow \int_a^\infty f(x)\,dx$ conv.

③ $L = \infty$ and $\int_a^\infty g(x)\,dx$ div $\Rightarrow \int_a^\infty f(x)\,dx$ div.

EX1: $\int_1^\infty \frac{dx}{\sqrt{x+1}}$. $f(x) = \frac{1}{\sqrt{x+1}}$

$g(x) = \frac{1}{\sqrt{x}}$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1 > 0.$$

$\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ is div $\Rightarrow \int_1^\infty \frac{dx}{\sqrt{x+1}}$ div.

EX2: $\int_1^\infty \frac{dx}{1+x^2}$ $f(x) = \frac{1}{1+x^2}$

$g(x) = \frac{1}{x^2}$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1 > 0$$

$\int_1^\infty \frac{1}{x^2}\,dx$ conv $\Rightarrow \int_1^\infty \frac{dx}{1+x^2}$ conv.

Improper integral of 2nd kind

Let f defined on $[a,c)$ and $f \in R[a, c-\epsilon]$ $\forall \epsilon > 0$. Then $\int_a^c f(x)\,dx = \lim_{\epsilon\to 0}\int_a^{c-\epsilon} f(x)\,dx$

if limit exists & finite $\Rightarrow$ Imp. int conv.

otherwise Imp. int. div.

Ex:- $\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{\epsilon \to 0} \int_\epsilon^1 \frac{dx}{\sqrt{x}} = \lim_{\epsilon \to 0} 2(1-\sqrt{\epsilon})$
$= 2$

Ex:- $\int_0^1 \frac{1}{x^p}\, dx = \lim_{\epsilon \to 0} \int_\epsilon^1 \frac{1}{x^p}\, dx$

$= \lim_{\epsilon \to 0} \left[\frac{x^{-p+1}}{1-p} \right]_\epsilon^1$

$= \lim_{\epsilon \to 0} \left(\frac{1}{1-p} - \frac{\epsilon^{-p+1}}{1-p} \right)$

$= \begin{cases} \frac{1}{1-p} & \text{if } p<1 \\ \text{div} & \text{if } p \geq 1 \end{cases}$

$\boxed{\int_1^\infty \frac{1}{x^p}\, dx \quad \text{conv } p > 1, \quad \text{div } p \leq 1}$

Result:- suppose f is discontinuous at $a_1, a_2, \ldots, a_n$ in $[a, b]$. Then
$\int_a^b f(x)\,dx = \int_a^{a_1} f + \int_{a_1}^{a_2} f + \cdots + \int_{a_n}^b f$.
If all int. on Right side conv
$\Rightarrow \int_a^b f$ conv.
otherwise, $\int_a^b f$ div.

Comparison test:- $0 \leq f(x) \leq g(x)$
$\forall\, x \in [a, c)$
and f, g are disconti at c. Then
① $\int_a^c g(x)\,dx$ conv $\Rightarrow \int_a^c f(x)\,dx$ conv.
② $\int_a^c f(x)\,dx$ div $\Rightarrow \int_a^c g(x)\,dx$ div.

Limit comparison test -
If $f, g > 0$ & f, g conti on $[a, c)$. and
$\lim_{x \to c} \frac{f(x)}{g(x)} = L$. Then.
① $L > 0$, $\int_a^c f$, $\int_a^c g$ both conv or div together.

② $L = 0$ and $\int_a^c g$ conv $\Rightarrow \int_a^c f$ conv.

③ $L = \infty$ and $\int_a^c g$ div $\Rightarrow \int_a^c f$ div.

Defn:- $f \in R[a,b]$ $\forall b > a$. Then $\int_a^\infty f(x)\,dx$ conv. absolutely if $\int_a^\infty |f(x)|\,dx$ conv.

Result: If $\int_a^\infty |f(x)|\,dx$ conv. Then $\int_a^\infty f(x)\,dx$ conv.

EX:- Show that $\int_1^\infty \frac{\sin x}{x^3}\,dx$ absolutely conv.

$$\left|\frac{\sin x}{x^3}\right| < \frac{1}{x^3} \quad \text{and } \int_1^\infty \frac{1}{x^3}\,dx \text{ conv.}$$

$$\Rightarrow \int_1^\infty \left|\frac{\sin x}{x^3}\right| dx \text{ conv.}$$

$$\Rightarrow \int_1^\infty \frac{\sin x}{x^3}\,dx \text{ absolutely conv.}$$

EX 2: Show that $\int_1^\infty \frac{\sin x}{x^p}\,dx$ conv $\forall p > 0$.

$$\int_1^\infty \frac{\sin x}{x^p}\,dx = \lim_{b\to\infty} \int_1^b \frac{\sin x}{x^p}\,dx$$

$$\int_1^b \frac{\sin x}{x^p}\,dx = \left[-\frac{\cos x}{x^p}\right]_1^b - p\int_1^b \frac{\cos x}{x^{p+1}}\,dx$$

$$= \cos 1 - \frac{\cos b}{b^p} - p\int_1^b \frac{\cos x}{x^{p+1}}\,dx$$

$\lim_{b\to\infty} \frac{\cos b}{b^p} = 0$, $\boxed{\int_1^\infty \frac{\cos x}{x^{p+1}}\,dx \text{ conv for } p>0}$

$\int_1^\infty \frac{\sin x}{x^p}\,dx$
$= \cos 1 - 0 - \lim_{b\to\infty} p\int_1^b \frac{\cos x\,dx}{x^{p+1}}$
conv for $p > 0$

$\boxed{\left|\frac{\cos x}{x^{p+1}}\right| < \frac{1}{x^{p+1}} \text{ and } \int_1^\infty \frac{1}{x^{p+1}}\,dx \text{ conv } p+1>1}$

Ex:- $\int_1^\infty \frac{\cos x}{x^p}\,dx$ conv $\forall\, p>0$.

Ex:- $\int_0^1 \frac{\sin x}{x^p}\,dx$ conv $\forall\, p<2$

$\int_0^1 \frac{\cos x}{x^p}\,dx$ conv $\forall\, p<1$

Remark: $\int_0^\infty \frac{\sin x}{x^p}\,dx$ conv $0<p<2$

$\int_0^\infty \frac{\cos x}{x^p}\,dx$ conv. $0<p<1$