Ordinary Differential Equations(EMAT102L)
(Integrating Factors)

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Outline of the Lecture

We will learn

- Some Rules for finding integrating factors
- Examples

Some Rules for finding an Integrating Factor

Consider the DE $\quad M(x,y)dx + N(x,y)dy = 0 \qquad (1)$

Rule

If $M(x,y)dx + N(x,y)dy = 0$ is a homogeneous DE with $Mx + Ny \neq 0$, then

$$\frac{1}{Mx + Ny}$$

is an integrating factor for (1).

Example

Solve the equation

$$(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$$

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Solution: Comparing the given equation with $Mdx + Ndy = 0$, we have

$$M = (x^2y - 2xy^2) \text{ and } N = -(x^3 - 3x^2y)$$

$$\Rightarrow M_y = x^2 - 4xy \text{ and } N_x = -3x^2 + 6xy$$

So, the given equation is not exact.
We observe that M and N are homogeneous functions of same degree in x and y, so integrating factor is

$$I.F. = \frac{1}{Mx + Ny} = \frac{1}{x^3y - 2x^2y^2 - x^3y + 3x^2y^2} = \frac{1}{x^2y^2}$$

Multiplying the given ODE by I.F., we get

$$\left(\frac{1}{y}-\frac{2}{x}\right)dx-\left(\frac{x}{y^2}-\frac{3}{y}\right)dy=0$$

Now, for this equation

$$M_y=N_x=-\frac{1}{y^2}.$$

which is an exact DE.
Hence the solution is

$$\frac{x}{y}-2\log x+3\log y=c$$

Consider the DE $M(x,y)dx + N(x,y)dy = 0 \quad (1)$

Rule

If $M(x,y) = f_1(xy)y$ and $N(x,y) = f_2(xy)x$ and $Mx - Ny \neq 0$, where f_1 and f_2 are functions of the product xy, then

$$\frac{1}{Mx - Ny}$$

is an integrating factor for (1).

Example

Solve the equation

$$(xy^2 + 2x^2y^3)dx + (x^2y - x^3y^2)dy = 0$$

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Solution: Comparing the given equation with $Mdx + Ndy = 0$, we have

$$M = (xy^2 + 2x^2y^3) \text{ and } N = (x^2y - x^3y^2)$$

$$\Rightarrow M_y = 2xy + 6x^2y^2 \text{ and } N_x = 2xy - 3x^2y^2$$

So, the given equation is not exact.
We write the given equation in the form

$$(xy + 2x^2y^2)ydx + (xy - x^2y^2)xdy = 0$$

So, the integrating factor is given by

$$I.F. = \frac{1}{Mx - Ny} = \frac{1}{x^2y^2 + 2x^3y^3 - x^2y^2 + x^3y^3} = \frac{1}{3x^3y^3}$$

Multiplying the given ODE by I.F., we get

$$\left(\frac{1}{3x^2y} + \frac{2}{3x}\right) dx - \left(\frac{1}{3xy^2} - \frac{1}{3y}\right) dy = 0$$

Now, for this equation $M_y = N_x = -\dfrac{1}{3x^2y^2}$.
which is an exact DE. Hence the solution is

$$\frac{-1}{xy} + \log\frac{x^2}{y} = c$$

Rules for finding an integrating factor

Consider the DE $M(x,y)dx + N(x,y)dy = 0$ (1)

(General Rule)

If the functions $M(x,y)$ and $N(x,y)$ are polynomials in x, y, then

$$x^{\alpha}y^{\beta}$$

works as an I.F. for some appropriate values of α and β.

Example

Solve the equation

$$(4y^2 + 3xy)dx - (3xy + 2x^2)dy = 0$$

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Solution: Comparing the given equation with $Mdx + Ndy = 0$, we have

$$M = (4y^2 + 3xy) \text{ and } N = -(3xy + 2x^2)$$

$$\Rightarrow M_y = 8y + 3x \text{ and } N_x = -3y - 4x$$

So, the given equation is not exact.
We observe that M and N are polynomials in x and y.
Thus we suppose that

$$I.F. = x^{\alpha} y^{\beta}$$

for some $\alpha, \beta \in \mathbb{R}$. Now, We try to find α and β.
Multiplying the terms $M(x, y)$ and $N(x, y)$ by $x^{\alpha}.y^{\beta}$, we get

$$M(x, y) = x^{\alpha} y^{\beta}(4y^2 + 3xy) \text{ and } N(x, y) = -x^{\alpha} y^{\beta}(3xy + 2x^2)$$

Example(cont.)

Now for exactness,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\text{i.e, } 4(\beta+2)x^{\alpha}y^{1+\beta} + 3(1+\beta)x^{1+\alpha}y^{\beta} = -3(1+\alpha)x^{\alpha}y^{1+\beta} - 2(2+\alpha)x^{1+\alpha}y^{\beta}$$

Thus

$$4(\beta+2) = -3 - 3\alpha \Rightarrow 3\alpha + 4\beta = -11$$
$$3(1+\beta) = -2(2+\alpha) \Rightarrow 2\alpha + 3\beta = -7$$

Solving the above equations, we get

$$\alpha = -5, \beta = 1$$

So integrating factor is

$$I.F. = \frac{y}{x^5}$$

Multiplying the given ODE by I.F., we get

$$\left(\frac{4y^3 + 3xy^2}{x^5}\right) dx - \left(\frac{3xy^2 + 2x^2y}{x^5}\right) dy = 0$$

which is an exact DE. Hence the solution is

$$y^2(y+x) = cx^4$$

Rules to remember (for finding integrating factors)

Consider the DE $M(x,y)dx + N(x,y)dy = 0$ (1)

Rule 1

If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$(function of x-alone), then $e^{\int f(x)dx}$ is an integrating factor for (1).

Rule 2

If $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y)$(function of y-alone), then $e^{\int f(y)dy}$ is an integrating factor for (1).

Rule 3

If $M(x,y)dx + N(x,y)dy = 0$ is a homogeneous DE with $Mx + Ny \neq 0$, then $\dfrac{1}{Mx + Ny}$ is an integrating factor for (1).

Rule 4

If $M(x,y) = f_1(xy)y$ and $N(x,y) = f_2(xy)x$ and $Mx - Ny \neq 0$, where f_1 and f_2 are functions of the product xy, then $\dfrac{1}{Mx - Ny}$ is an integrating factor for (1).

Rule 5(General Rule)

If the functions $M(x,y)$ and $N(x,y)$ are polynomials in x, y, then $x^\alpha y^\beta$ works as an I.F. for some appropriate values of α and β.

Problem 1.

Solve the differential equation

$$y(2xy + e^x)dx - e^x dy = 0$$

Problem 2.

Solve the differential equation

$$(2x^2y^2 + y)dx = (x^3y - 3x)dy$$


Thank You