Matrices :-

Def:- A rectangular array of numbers is called a "matrix."

* The horizontal arrays of a matrix are called its "ROWS."
* The vertical arrays are called its "COLUMNS."

* A matrix A of order $m \times n$ can be represented in the following form:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

where a_{ij} is the entry at the intersection of the i^{th} row and j^{th} column.

* In a more concise manner, we denote $A = [a_{ij}]$.
* Some books also use $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ to represent a matrix.

* A matrix having only one column is called a "Column vector".
* A matrix having only one row is called a "ROW VECTOR"

Remark:- Whenever a vector is used, it should be understood from the context whether it is a row vector or a column vector.

Defn: 1) Equal Matrix :- Two matrices $A = [a_{ij}]$ & $B = [b_{ij}]$ having the "same order" $m \times n$ are equal if

$$\boxed{a_{ij} = b_{ij} \quad \text{for each } 1 \le i \le m,\ 1 \le j \le n.}$$

2) Zero Matrix :- A matrix in which each entry is zero, is called "zero Matrix", denoted by O.

Ex :- $O_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

3) Square Matrix :- A matrix having the number of rows equal to numbers of columns is called a "square Matrix"

4) Diagonal Matrix :- A square matrix is called diagonal if

$$\boxed{a_{ij} = 0 \quad \text{for } i \ne j.}$$

Remark :- Let $A = [a_{ij}]$, $1 \le i, j \le n$. Then the enteries "$a_{11}, a_{22}, \dots, a_{nn}$" are called the "diagonal entries". and form the principal diagonal of A.

5) Scalar matrix :- A diagonal matrix is called scalar matrix if $a_{11} = a_{22} = \dots = a_{nn}$ i.e all the diagonal entries are equal.

6) A Identity Matrix :- A square matrix $A = [a_{ij}]$ with

$$a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases}$$ is called the identity matrix;

denoted by I_n (i.e identity matrix of order $n \times n$).

7) A square matrix is called upper triangular matrix if $a_{ij} = 0$ for $i > j$

8) A square matrix is called Lower triangular matrix if $a_{ij} = 0$ for $i < j$.

9) A square matrix is called triangular if it is either a lower triangular or a upper triangular.

Examples:-

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 2 \end{bmatrix}_{3\times3}}_{\text{Square matrix}} \qquad \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times3}}_{\text{Zero matrix}}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}}_{\text{Identity Matrix}} \qquad \underbrace{\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}_{3\times3}}_{\text{Scalar Matrix}} \qquad \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 2 \end{bmatrix}_{3\times3}}_{\text{upper triangular matrix}}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 4 & 2 \end{bmatrix}_{3\times3}}_{\text{Lower triangular Matrix.}}$$

ALGEBRA OF MATRICES :-

(i) **Addition of Matrices** :→ Let $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$.

Then

$$A+B = [a_{ij} + b_{ij}]_{m\times n}.$$

(ii) **Multiplying a Scalar to a Matrix** :→ $A = [a_{ij}]_{m\times n}$.

Then for any $k \in \mathbb{R}$,

$$kA = [k\, a_{ij}]_{m\times n}.$$

Ex:- $A = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix}$.

Then

(i) $A+B = \begin{bmatrix} 2+1 & 4+0 & 6+2 \\ 1+1 & 2+0 & 6+0 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 8 \\ 2 & 2 & 6 \end{bmatrix}$

(ii) $5A = \begin{bmatrix} 5\times 2 & 5\times 4 & 5\times 6 \\ 5\times 1 & 5\times 2 & 5\times 6 \end{bmatrix} = \begin{bmatrix} 10 & 20 & 30 \\ 5 & 10 & 30 \end{bmatrix}$.

Properties :- For any given Matrix A, B, C & $\alpha, \beta \in \mathbb{R}$.

(i) $A+B = B+A$ ⟶ (Commutativity)

(ii) $(A+B)+C = A+(B+C)$ (Associativity)

(iii) $\alpha(\beta A) = (\alpha\beta)A$

(iv) $(\alpha+\beta)A = \alpha A + \beta A$

(v) $A + O = O + A = A$, where O is zero matrix.
& "O" is called the additive identity.

(vi) $A + B = O$, zero matrix. Then B is called additive inverse of A. & Moreover $\boxed{B = -A}$

Def^n: Matrix Multiplication :→ (Product of Matrices).

Let $A = [a_{ij}]_{m\times n}$. $B = [b_{ij}]_{n\times r}$.

Then the product $AB = [c_{ij}]_{m\times r}$, with

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

Remark:- The product AB is defined iff
The no. of column of A = The no. of rows of B.

Ex. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \end{bmatrix}_{2\times 3}$, $B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 1 & 0 & 4 \end{bmatrix}_{3\times 2}$.

Then $[AB]_{2\times 3} = \begin{bmatrix} 4 & 2 & 19 \\ 3 & 4 & 18 \end{bmatrix}$.

Note that the product AB is defined but BA is not defined.

However, For square matrices A & B of same order,
both the product AB & BA are defined.

Properties: (i) $A(BC) = (AB)C$ (associative)

(ii) for any $k \in \mathbb{R}$, $k(AB) = (kA)B = A(kB)$.

(iii) $A(B+C) = AB + AC$. (multiplication distributes over addition)

(4) $AI_n = I_nA = A$, where I is identity matrix of order n & A is $n\times n$ matrix.

(v) $AB \neq BA$, in general.
i.e multiplication is not commutative.

(vi) $AI_n = I_n A = I$, ~~For~~ A is an $n \times n$ matrix.

Then I_n is called "multiplicative" identity.

(vii) If $\exists$ a matrix B s.t

$$AB = BA = I_n.$$

Then B is called multiplicative inverse of A.

(viii) Cancellation Law "DOES NOT" Holds in Matrices

i.e $\boxed{AB = AC \not\Rightarrow B = C.}$

Example:- $A =$ zero Matrix, $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $C = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

Then $AB = AC = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ but $B \neq C$.

$\boxed{AB = 0 \not\Rightarrow A = 0 \text{ or } B = 0}$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Transpose of a matrix :-

The transpose A^t of $A = [a_{ij}]_{n\times m}$ is defined as $A^t = [b_{ij}]_{m\times n}$

where

$$b_{ij} = a_{ji} \quad \forall\ i, j.$$

Ex :-

$$A = \begin{bmatrix} 2 & 3 & 6 \\ 4 & 5 & 7 \end{bmatrix}, \quad A^t = \begin{bmatrix} 2 & 4 \\ 3 & 5 \\ 6 & 7 \end{bmatrix}$$

Symmetric Matrix :- A matrix "A" over $\mathbb{R}$ is called symmetric

if $\boxed{A^t = A}$.

Skew symmetric :- if $\boxed{A^t = -A}$

Orthogonal matrix :- if $A A^t = A^t A = I$.

Nilpotent Matrix :- if $\exists$ an integer k s.t $A^k = 0$.

$$A^k := \underbrace{A\ A\ A\ A\ \cdots\ A}_{k \text{ times}}.$$

The least positive integer "k" for which $A^k = 0$ is called "ORDER OF NILPOTENCY."

Idempotent Matrix :- if $A^2 = A$.

Normal Matrix :- $A A^t = A^t A$

Trace of a matrix $A_{n\times n}$:- Sum of diagonal entries

i.e If $A = [a_{ij}]_{n\times n}$. Then $\text{Trace}(A) = tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$.

Examples :-

SYMETETRY MATRIX :- $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 4 & 6 \\ 2 & 6 & 3 \end{bmatrix}$

$\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$.

SKEW - SYMMETRIC MATRIX :- $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$

ORTHOGONAL MATRIX :- $A = \begin{bmatrix} 1/\sqrt{3} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{bmatrix}$, $\begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix}$

IDEMPOTENT MATRIX :- $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

NILPOTENT MATRIX :- $A = \begin{array}{c} \begin{matrix} a_{11} & a_{12} \end{matrix} \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array}$, $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \Rightarrow A^2 = 0$,

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}_{3\times 3} \Longrightarrow, \boxed{A^3 = 0}$$

Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 5 & 9 \\ 0 & 1 & 7 \end{bmatrix}$. Then Trace of $A = 1+5+7$
$= 13.$

Remarks :- It is easy to verify that

For any square matrix A, $P = \frac{A + A^t}{2}$ is symmetric and $Q = \frac{1}{2}(A - A^t)$ is skew-symmetric.

Also, $A = P + Q$

Let A and B be symmetric Matrix. Then AB is symmetric iff $AB = BA$.

The diagonal element of a skew-sym. matrix are zero.

$tr(A+B) = tr(A) + tr(B)$

$trace(AB) = trace(BA)$

$trace(kA) = k\, trace(A)$ for any scalar k.

$(A^t)^n = (A^n)^t$, n is any +ve integer.

$(A^t)^t = A$.

$(A+B)^t = A^t + B^t$.

$(AB)^t = B^t A^t$

AA^t is a symmetric matrix.

Similarly, For any matrix A, A^tA is also a symmetric matrix.

Product of two symmetric matrix need not be sym.

Ex:- $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$A^t = A$, $B^t = B$ But $(AB)^t \neq AB$

Matrices Over Complex Numbers :→

Let $A = [a_{ij}]_{m \times n}$ over $\mathbb{C}$. Then,

* Conjugate of A, denoted by $\bar{A}$ is the matrix $B = [b_{ij}]$ with $b_{ij} = \overline{a_{ij}}$.

Ex :- $A = \begin{bmatrix} -1 & 4+3i & i \\ 0 & 1 & i-2 \end{bmatrix}$. Then $\bar{A} = \begin{bmatrix} -1 & 4-3i & -i \\ 0 & 1 & -i-2 \end{bmatrix}$

* Conjugate Transpose of A, denoted by A^*, is the matrix $B = [b_{ij}]$ with $b_{ij} = \overline{a_{ji}}$

Ex :- $A = \begin{bmatrix} -1 & 4+3i & i \\ 0 & 1 & i-2 \end{bmatrix}$. Then $A^* = \begin{bmatrix} -1 & 0 \\ 4-3i & 1 \\ -i & -i-2 \end{bmatrix}$

* A square matrix "A" over $\mathbb{C}$ is called
  (i) Hermitian if $\boxed{A^* = A}$
  (ii) Skew Hermitian if $\boxed{A^* = -A}$
  (iii) Unitary if $\boxed{A^* A = A A^* = I}$
  (iv) Normal if $\boxed{A^* A = A^* A}$

Remark :- If $A = [a_{ij}]$ with $a_{ij} \in \mathbb{R}$ (real no's). Then

$$\boxed{A^* = A^t}$$

Examples :-

Hermitian :- $A = \underbrace{\begin{bmatrix} a \in \mathbb{R} & c \\ -\bar{c} & b \in \mathbb{R} \end{bmatrix}}_{\text{General form}}$. $A = \begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix}$ Then $A^* = \begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix}$

Skew Hermitian :— $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$. $A^* = \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix} = -A$

Unitary :— $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$. Then $AA^* = A^*A = I$

The diagonal entries of a skew Hermitian matrices are either zero or pure imaginary.

For any matrix A, $P = \frac{A + A^*}{2}$ is Hermitian

$Q = \frac{A - A^*}{2}$ is skew Hermitian.

Also, $\boxed{A = P + Q}$

i.e A can be decompose as the sum of a Hermitian Matrices and a skew Hermitian matrices.

$(A+B)^* = A^* + B^*$.

$(A^*)^* = A$.

For a positive integer $n > 1$, let

$\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$.

Remark :- $\mathbb{Z}_n$ can be identify with the set of remainders of integers division by "n".

For any integer x,

denote by $[x]$ — remainder after division by "n".

Then $[x] \in \mathbb{Z}_n$ for any $x \in \mathbb{Z}$ = set of integers.

Ex :- Let $n = 6$. Then $[10] = 4$

$[-1] = 5$

Addition on $\mathbb{Z}_n$:- Let $a, b \in \mathbb{Z}_n$ Then $a+b = [x+y]$, where $a = [x]$, $b = [y]$.

Ex → Take $n = 3$, $a = 1$, $b = 2$

$a+b = 3 \bmod 3 = 0$

$[a+b]_3 = 0$.

Product on $\mathbb{Z}_n$:- For $a, b \in \mathbb{Z}_n$, let x & y be integer s.t $[x] = a$, $[y] = b$.

Then $a \cdot b = [xy]$.

Excercise:-

1) Let $A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}_{1\times n}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}_{n\times 1}$

Then calculate AB & BA.

(ii) Let n be a positive integer. Compute A^n for the following matrices –

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Can you guess a formula for A^n. prove it by induction.

(iii)