Multivariable Calculus

(Lecture-17)

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Example: Triple Integration

Evaluate $\iiint_D 2xdV$ where D is the region bounded by the planes $x = 0, y = 0, z = 0$ and $2x + 3y + z = 6$.

Solution: Note that $0 \leq z \leq 6 - 2x - 3y$ and $(x, y) \in R$, where R is the domain given by

$$R = \{(x, y) : 0 \leq y \leq 2 - \frac{2}{3}x, 0 \leq x \leq 3\}.$$

$$\begin{aligned}
\iiint_D 2xdV &= \int_{x=0}^{3} \int_{y=0}^{2-\frac{2}{3}x} \int_{z=0}^{6-2x-3y} 2xdzdydx \\
&= \int_{x=0}^{3} \int_{y=0}^{2-\frac{2}{3}x} 2x(6 - 2x - 3y)dydx \\
&= \int_{x=0}^{3} 2x \left[(6 - 2x)y - \frac{3}{2}y^2 \right]_{y=0}^{2-\frac{2}{3}x} dx = 9.
\end{aligned}$$

Example: Triple Integration

Find the volume of the solid in the first octant bounded by the coordinate planes, the plane $y + z = 2$, and the cylinder $x = 4 - y^2$.

Solution: Imagine the solid as the piece of parabolic cylinder over the region R of the xy-plane bounded by $x = 4 - y^2$ in the first quadrant. Over the solid z varies from 0 to $2 - y$ and domain for x, y is given by

$$R = \{(x, y) : 0 \leq y \leq \sqrt{4 - x}, 0 \leq x \leq 4.\}$$

Hence

$$Volume = \int_{x=0}^{4} \int_{y=0}^{\sqrt{4-x}} \int_{z=0}^{2-y} dzdydx = \int_{x=0}^{4} \int_{y=0}^{\sqrt{4-x}} (2 - y)dydx$$

$$= \int_{x=0}^{4} \left[2\sqrt{4 - x} - \frac{(4 - x)}{2} \right] dx = \frac{20}{3}.$$

Example

Evaluate: $\int_0^3 \int_{\sqrt{\frac{x}{3}}}^1 e^{y^3}\, dy\, dx$

Solution: Here $0 \le x \le 3$ and $\sqrt{\frac{x}{3}} \le y \le 1$. Using change of order of integration, then $0 \le y \le 1$ and $0 \le x \le 3y^2$. Now we have

$$\int_0^3 \int_{\sqrt{\frac{x}{3}}}^1 e^{y^3}\, dy\, dx = \int_0^1 \int_0^{3y^2} e^{y^3}\, dx\, dy$$

$$= \int_0^1 |x|_{x=0}^{3y^2} e^{y^3}\, dy$$

$$= \int_0^1 3y^2 e^{y^3}\, dy$$

$$= e - 1.$$

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Example

Find the volume of the solid whose base is the region in the $xy-$ plane that is bounded by the parabola $y = 4 - x^2$ and the line $y = 3x$, while the top of the solid is bounded by the plane $z = x + 4$.

Solution:

$$Volume = \iiint_D dzdydx = \int_{x=-4}^{1} \int_{y=3x}^{4-x^2} \int_{z=0}^{x+4} dzdydx$$

$$= \int_{x=-4}^{1} \int_{y=3x}^{4-x^2} (x+4)dydx$$

$$= \int_{x=-4}^{1} (x+4) \left| y \right|_{y=3x}^{4-x^2} dx$$

$$= \int_{x=-4}^{1} \left[(x+4)(4-x^2-3x) \right] dx$$

$$= \frac{625}{12}.$$

Examples

Find the volume of the portion of the cylinder $x^2 + y^2 = 1$ intercepted between the plane $z = 0$ and the paraboloid $x^2 + y^2 = 4 - z$.

Solution:

$$Volume = \iiint_D dzdydx = \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{z=0}^{4-x^2-y^2} dzdydx$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=0}^{4-r^2} dz\, rdr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (4 - r^2) rdrd\theta$$

$$= \int_{\theta=0}^{2\pi} \left| 2r^2 - \frac{r^4}{4} \right|_{r=0}^{1} d\theta$$

$$= \frac{7}{4} \int_{\theta=0}^{2\pi} d\theta = \frac{7\pi}{2}.$$