# Riemann Integral

**Fundamental Theorem** :- Let $f$ is continuous on $[a,b]$ and let $\phi(x)=\int_a^x f(t)\,dt$. Then $\phi$ is diff. and $\phi'(x)=f(x)$.

Remark :- continuity of $f$ is not a necessary condition for existance of anti-derivative of $f$.

Ex :- $f(x)=\begin{cases} 2x\sin\frac{1}{x}-\cos\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$

$f:[-1,1]\to\mathbb{R}$.

$f$ is not conti at $0$ $\qquad \phi:[-1,1]\to\mathbb{R}$.

$\phi(x)=\begin{cases} x^2\sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$

Then $\phi'(x)=f(x)\ \forall x\in[-1,1]$.

**2nd fundamental theorem** :

If $\phi$ is an anti-derivative of conti. function $f$. Then $\int_a^b f=\phi(b)-\phi(a)$

$\phi'(x)=f(x)\ .\ x\in[a,b]$.

Ex :- Let $f:[-2,2]\to\mathbb{R}$,

$f(x)=\begin{cases} 3x^2\cos\frac{\pi}{x^2}+2\pi\sin\frac{\pi}{x^2}, & x\neq 0 \\ 0, & x=0 \end{cases}$

show that $f$ is integrable on $[-2,2]$ and Evalute $\int_{-2}^{2} f$.

Sol$^n$ :- $f$ is bounded on $[-2,2]$ and $f$ is continuous except at $x=0$.

$\therefore$ $f$ is integrable on $[-2,2]$.

$\phi : [-2,2] \to \mathbb{R}$ s.t $\phi'(x) = f(x)$.

$$\phi(x) = \begin{cases} x^3 \cos\frac{\pi}{x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

$$\int_{-2}^{2} f(x)\,dx = \phi(2) - \phi(-2)$$
$$= 8\cos\frac{\pi}{4} + 8\cos\frac{\pi}{4} = 8\sqrt{2}.$$

<u>change of variable formula</u> :-

Let $u(t)$, $u'(t)$ continuous on $[a,b]$ and $f$ is continuous on $[u(a), u(b)]$

Then $\int_a^b f(u(x))\cdot u'(x)\,dx = \int_{u(a)}^{u(b)} f(y)\,dy$.

Ex:- Evalute $\int_0^1 x\sqrt{1+x^2}\,dx$. $\quad u(x) = y$

<u>soln</u>:- $u = 1+x^2$, $u' = 2x$.
$u(0) = 1$, $u(1) = 2$.

$$\int_0^1 x\sqrt{1+x^2}\,dx = \frac{1}{2}\int_1^2 \sqrt{u}\,du$$
$$= \frac{1}{3}\left(2^{3/2} - 1\right).$$

<u>Result (Darboux theorem)</u>.

$$\boxed{\begin{aligned} &S(P,f) = \sum_{k=1}^{n} f(c_k)\cdot(x_k - x_{k-1}) \\ &P = \{x_0, x_1, \ldots, x_n\} \\ &\qquad c_k \in [x_{k-1}, x_k]. \end{aligned}}$$

If $f : [a,b] \to \mathbb{R}$ be a Riemann integrable fun. Then for any $\{P_n\}$ of $[a,b]$ with

$\|P_n\| \to 0$ as $n \to \infty$. we have $\lim\limits_{n\to\infty} S(P_n,f) = \int_a^b f$.

$\odot$ $P_n = (x_0, x_1, \ldots, x_n)$ of $[a,b]$

$a = x_0 < x_1 < \cdots < x_n = b$.

$\boxed{x_i = x_0 + ih}$

choose, $x_1 - x_0 = x_2 - x_1 = \cdots = x_n - x_{n-1}$

$[a,b]$

$b = a + nh$

$\boxed{\frac{b-a}{n} = h}$

$\|P_n\| = \frac{b-a}{n}$ and $\lim\limits_{n\to\infty} \|P_n\| = 0$.

$$S(P_n,f) = \sum_{k=1}^{n} f(c_k) \cdot \frac{b-a}{n}$$

$$= \frac{b-a}{n} \sum_{k=1}^{n} f(c_k)$$

$$\int_a^b f = \lim_{n\to\infty} S(P_n,f) = \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} f(c_k)$$

. $[0,1]$ $\quad P_n = (0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1)$

$\|P_n\| = \frac{1}{n} \to 0$ as $n \to \infty$

$$S(P_n,f) = \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \cdot \frac{1}{n} = \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right).$$

$$\int_0^1 f = \lim_{n\to\infty} S(P_n,f) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right).$$

Evalute $\lim\limits_{n\to\infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}\right]$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \cdots + \frac{1}{1+\frac{n}{n}}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(\frac{1}{1+\frac{k}{n}}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right), \text{ where } f(x) = \frac{1}{1+x}.$$

$$= \int_0^1 f(x)\,dx = \int_0^1 \frac{1}{1+x}\,dx = \ln 2.$$

# Improper Integrals

① f is defined on unbounded interval $[a,\infty)$ or $(-\infty, b]$ and $f \in R[a,b]$ for all $b > a$.

② f is not defined at some point in $[a,b]$.

## Improper integral of 1st kind:-

If f is bounded on $[a,\infty)$ or $(-\infty, b]$ and $f \in R[a,b]$ $\forall$ $b>a$. Then

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$

If this limit exists and finite, then we say improper integral converges. and if limit goes to $\infty$ or does not exist we say Imp. int. diverges.

Ex:- $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx$

$\displaystyle = \lim_{b\to\infty}\left(1-\frac{1}{b}\right) = 1.$

Ex:- $\displaystyle\int_0^{\infty} \frac{dx}{1+x^2} = \lim_{b\to\infty}\int_0^b \frac{dx}{1+x^2} = \frac{\pi}{2}$

Ex:- $\displaystyle\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^p}\,dx.$

$\displaystyle = \lim_{b\to\infty}\left[\frac{x^{1-p}}{1-p}\right]_1^b$ if $p>1$

$\displaystyle = \lim_{b\to\infty}\left(\frac{b^{1-p}}{1-p} - \frac{1}{1-p}\right) = \frac{1}{p-1}$

$$\int_1^\infty \frac{1}{x^p}\,dx \quad \begin{array}{ll} \text{conv} & \text{if } p > 1 \\ \text{div} & \text{if } p \le 1. \end{array}$$