Power series

$\{a_n\}_{n=0}^{\infty}$, $\sum_{n=0}^{\infty} a_n (x-c)^n$ power series with center c.

$c=0$, $\underline{\sum_{n=0}^{\infty} a_n x^n}$ power series at $c=0$

$\sum_{n=0}^{\infty} x^n \longrightarrow \{x \in \mathbb{R} : |x| < 1\}$ ✓

$\sum_{n=0}^{\infty} n!\, x^n \longrightarrow \{0\}$ ✓ $\Rightarrow$ nowhere convergent series.

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\mathbb{R}$ ✓, everywhere conv. series.

Result: ① $\sum_{n=0}^{\infty} a_n x^n$ conv. at $\underline{x=b}$,
conv, $\underline{|x| < |b|}$

② $\sum_{n=0}^{\infty} a_n x^n$ div. at $\underline{x=b}$.
$\Rightarrow$ div $|x| > |b|$

③ If $\sum_{n=0}^{\infty} a_n x^n$ neither nowhere conv
nor everywhere conv.
Then $\underline{\exists\, R > 0}$ s.t series conv. absolutely $\forall\ \underline{|x| < R}$ and div $|x| > R$.
$R \to$ Radius of conv.
$\underline{(-R, R)}$ interval of convergence

$(-\infty, \infty)$

$R > 0$
$c-R \quad c \quad c+R$

Method 1 $\sum_{n=0}^{\infty} a_n x^n$, $\boxed{\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = B}$

① $\sum a_n x^n$ conv $|x| < R \to (-R, R)$ $\quad R = \frac{1}{B}$
② $\sum a_n x^n$ div $|x| > R$
③ no conclusion $|x| = R$.

• method 2:

$\lim_{n\to\infty} |a_n|^{1/n} = \beta$, $R = \frac{1}{\beta}$.

① $\sum a_n x^n$ conv, $|x| < R$.

② div, $|x| > R$

③ no conclusion $|x| = R$.

Ex:- $\sum_{n=0}^{\infty} n!\, x^n$, $a_n = n!$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty}(n+1)$

$= \infty$

$R = \frac{1}{\infty} = 0$ $\xrightarrow{0<|x|<0} \{0\}$

Ex:- $\sum_{n=0}^{\infty} x^n$, $\boxed{a_n = 1}$, $R = 1$, $(-1, 1)$

$\to \sum_{n=0}^{\infty} a_n x^n$ $\lim \left|\frac{a_{n+1}}{a_n}\right|$ $|x| < 1$ $\Rightarrow -1 < x < 1$

$= 1$

$(-1, 1)$

Ex:- $\sum_{n=1}^{\infty} \frac{x^n}{n}$

$a_n = \frac{1}{n}$, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$

$(-1, 1)$

$x = -1$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \to$ conv.

$x = 1$, $\sum_{n=1}^{\infty} \frac{1}{n}$ div.

$\boxed{[-1, 1)}$

Ex:- $\sum_{n=0}^{\infty} 2^{-n} x^{3n}$, $a_{3n} = 2^{-n}$. $\sum_{n=0}^{\infty} a_n x^n$

$x^3 = y$

$\sum_{n=0}^{\infty} 2^{-n} y^n$

$a_n = 2^{-n}$

$|y| <$

$\lim_{n\to\infty} |a_{3n}|^{\frac{1}{3n}} = \lim_{n\to\infty} 2^{-\frac{n}{3n}}$

$= 2^{-\frac{1}{3}}$.

$\boxed{R = 2^{\frac{1}{3}}}$

$\sum_{n=0}^{\infty} 2^{-n} x^{3n} = \sum_{n=0}^{\infty} 2^{-n} y^n$, $y = x^3$

$= \sum_{n=0}^{\infty} a_n y^n$, $a_n = 2^{-n}$

$\lim_{n\to\infty} |a_n|^{1/n} = 2^{-1}$, $R = 2$

$|y| < 2$

$|x| < 2^{1/3}$

$\lim_{n\to\infty} |a_n|^{1/n}$ does not exist

or, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ " " "

Then $\limsup_{n\to\infty} |a_n|^{1/n} = \beta$. $R = \frac{1}{\beta}$.

$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \beta$, $R = \frac{1}{\beta}$.

Ex:- $\frac{1}{3} - x + \frac{x^2}{3^2} - x^3 + \frac{x^4}{3^4} - x^5 + \cdots =$

$a_0 = \frac{1}{3}$, $a_1 = -1$, $a_2 = \frac{1}{3^2}$, $a_3 = -1$,

$a_4 = \frac{1}{3^4}, \ldots$

$\lim |a_n|^{1/n} = 1$

$\lim |a_n|^{1/n} = \frac{1}{3}$

$\limsup_{n\to\infty} |a_n|^{1/n} = 1$. , $R = 1$.

$(-1, 1)$

Ex:- $x + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \cdots$

$R = ?$

$= a_0 + a_1 x + a_2 x^2 + \ldots \quad (-R, R)$

Theorem: $f(x) = \sum_{n=0}^{\infty} a_n x^n$ conv for $|x| < R$.

① $\sum_{n=0}^{\infty} n a_n x^{n-1}$ conv for $|x| < R$, and equal to $f'(x)$.

② $\sum_{n=0}^{\infty} \frac{a_n \cdot x^{n+1}}{n+1}$ conv in $|x| < R$ and equal to $\int f(x)\,dx$

Ex:- $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ con $[-1, 1]$ $\qquad \sum_{n=0}^{\infty} a_n x^n \quad R \quad (-R, R)$

$\sum_{n=1}^{\infty} \frac{x^{n-1}}{n}$ conv $(-1, 1)$

conv $x = -1$ $\qquad \sum_{n=0}^{\infty} n a_n x^{n-1} \quad R \quad (-R, R)$

div $x = 1$

Taylor's Theorem

If f has nth derivative at $x = a$

f by nth degree poly. P_n s.t.

$P_n(a) = f(a)$, $P_n^{(k)}(a) = f^{(k)}(a)$, $k = 0, 1, \ldots, n$

$$P_n(x) = \sum_{k=0}^{n} \frac{f^k(a)}{k!}(x-a)^k.$$

nth Taylor poly. for f at a.

$$f(x) = P_n(x) + \underbrace{R_n(x)}_{\text{Reminder or error}}$$