Solutions - Tutorial Sheet 8

Problem-1: Find all real valued C^1 solutions $y(x)$ of the differential equation

$$xy'(x) + y(x) = x, \quad x \in (-1,1).$$

Solution: Given DE is

$$x\,y'(x) + y(x) = x$$

$$\Rightarrow \quad y'(x) = \frac{x-y}{x}$$

$$\Rightarrow \quad y' = \frac{x-y}{x} = 1 - \frac{y}{x}$$

Put $y = vx$

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Thus, we have

$$v + x\frac{dv}{dx} = 1 - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = 1 - v - v = 1 - 2v$$

$$\Rightarrow \quad \frac{dv}{1-2v} = \frac{dx}{x}$$

$$\Rightarrow \quad \frac{\log(1-2v)}{-2} = \log x + \log c$$

$$\Rightarrow \quad -\frac{1}{2}\log\left(1 - 2\frac{y}{x}\right) = \log cx \qquad \left[\because v = \frac{y}{x}\right]$$

$$\Rightarrow \quad \log\left(1 - 2\frac{y}{x}\right) = -2\log cx$$

$$\Rightarrow \quad \log\left(\frac{x-2y}{x}\right) = \log(cx)^{-2}$$

$$\Rightarrow \quad \frac{x-2y}{x} = \frac{1}{(cx)^2} \quad \Rightarrow \quad c^2x^2\left(\frac{x-2y}{x}\right) = 1$$

$$\Rightarrow \quad c^2x(x-2y) = 1$$

Thus, we have

$$x(x-2y) = \frac{1}{c^2} = c_1$$

$$\Rightarrow \quad x(x-2y) = c_1 \quad \Rightarrow y = \frac{x}{2} - \frac{c_1}{2x},$$

The c^1 solution $y(x)$ is obtained by putting $c_1 = 0$.

i.e $\boxed{y(x) = \frac{x}{2}}$ Ans

Alternative: Given DE is

$$xy' + y = x$$

$$\Rightarrow \quad y' + \frac{y}{x} = 1$$

Here, $I.F = e^{\int \frac{1}{x} dx} = e^{\log x} = x$

Thus the solution is

$$y \times I.F = \int (1) \times I.F \, dx + c$$

$$\Rightarrow y \cdot x = \int x \, dx + C$$

$$\Rightarrow xy = \frac{x^2}{2} + c$$

$$\Rightarrow y = \frac{x}{2} + \frac{c}{x}$$

The c^1 solution can be obtained by putting $c_1 = 0$.

i.e $\boxed{y(x) = \frac{x}{2}}$

Problem-2: Under what conditions, the following DE's are exact?

(a) $[h(x)+g(y)]\,dx + [f(x)+h(y)]\,dy = 0$

(b) $(x^3+xy^2)\,dx + (ax^2y+bxy^2)\,dy = 0$

(c) $\left(\frac{1}{x^2}+\frac{1}{y^2}\right)dx + \left(\frac{cx+1}{y^3}\right)dy = 0.$

Solution: (a) $[h(x)+g(y)]\,dx + [f(x)+h(y)]\,dy = 0$

Comparing the given DE with $M(x,y)\,dx + N(x,y)\,dy = 0$, we get

$$M = h(x)+g(y), \qquad N = f(x)+h(y)$$

$$\frac{\partial M}{\partial y} = g'(y), \qquad \frac{\partial N}{\partial x} = f'(x)$$

Condition of exactness is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$\Rightarrow \boxed{g'(y) = f'(x)} \quad \text{Ans.}$$

(b) $(x^3+xy^2)\,dx + (ax^2y+bxy^2)\,dy = 0$

Comparing the given DE with $M(x,y)\,dx + N(x,y)\,dy = 0$, we get

$$M = x^3+xy^2, \qquad N = ax^2y+bxy^2$$

$$\frac{\partial M}{\partial y} = 2xy, \qquad \frac{\partial N}{\partial x} = 2axy + by^2$$

Condition of exactness is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$\Rightarrow 2xy = 2axy + by^2$$

Comparing the coefficients of xy and y^2, we get

$$2a = 2 \quad \text{and} \quad b = 0$$

$$\Rightarrow \boxed{a = 1, \quad b = 0} \quad \text{Ans.}$$

(C) $\left(\frac{1}{x^2}+\frac{1}{y^2}\right)dx+\left(\frac{cx+1}{y^3}\right)dy=0$ (4)

Comparing the above equation with $M(x,y)\,dx+N(x,y)\,dy=0$, we get

$$M(x,y)=\frac{1}{x^2}+\frac{1}{y^2} \quad \text{and} \quad N(x,y)=\frac{cx+1}{y^3}$$

$$\frac{\partial M}{\partial y}=\frac{-2}{y^3} \quad \text{and} \quad \frac{\partial N}{\partial x}=\frac{c}{y^3}$$

Condition of exactness is $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

$$\Rightarrow \frac{-2}{y^3}=\frac{c}{y^3}$$

$$\Rightarrow \boxed{c=-2}$$ Ans.

Problem - 3: Examine the following differential equations for exactness. Solve them by finding appropriate integrating factors if necessary: (5)

(a) $(\sin x \cdot \tan y + 1)\,dx - \cos x \cdot \sec^2 y\,dy = 0$.

(b) $e^x dx + (e^x \cot y + 2y \operatorname{cosec} y)\,dy = 0$.

(c) $(3xy + y^2)\,dx + (x^2 + xy)\,dy = 0$.

(d) $y\,dx + (2x - ye^y)\,dy = 0$

Solution: (a) $(\sin x \cdot \tan y + 1)\,dx - \cos x \cdot \sec^2 y\,dy = 0$ ——— (1)

Comparing the given DE with $M(x,y)\,dx + N(x,y)\,dy = 0$, we get

$M(x,y) = \sin x \cdot \tan y + 1$, $\quad N(x,y) = -\cos x \cdot \sec^2 y$

$\frac{\partial M}{\partial y} = \sin x \cdot \sec^2 y$, $\quad \frac{\partial N}{\partial x} = \sin x \cdot \sec^2 y$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\Rightarrow$ The given DE is exact.

Now, we need to find function $F(x,y)$ such that

$\frac{\partial F}{\partial x} = M(x,y)$ and $\frac{\partial F}{\partial y} = N(x,y)$ ——— (2)

$\Rightarrow \frac{\partial F}{\partial x} = \sin x \cdot \tan y + 1$

$\Rightarrow F(x,y) = \int (\sin x \cdot \tan y + 1)\,dx + \phi(y)$

$\Rightarrow F(x,y) = -\cos x \cdot \tan y + x + \phi(y)$, ——— (3)

where $\phi(y)$ is a constant of integration.

Differentiating the above equation w.r.t. 'y', we get

$\frac{\partial F}{\partial y} = -\cos x \cdot \sec^2 y + \phi'(y)$

$\Rightarrow N(x,y) = -\cos x \cdot \sec^2 y + \phi'(y)$ (Using (2))

$$\Rightarrow -\cos x \cdot \sec^2 y = -\cos x \cdot \sec^2 y + \phi'(y)$$

$$\Rightarrow \phi'(y) = 0$$

$$\Rightarrow \phi(y) = \text{Constant} = C_1$$

Substituting the above value of $\phi(y)$ in ③, we get

$$F(x, y) = -\cos x \cdot \tan y + x + C_1$$

The general solution of the given DE ① is given by

$$F(x, y) = C$$

$$\Rightarrow -\cos x \cdot \tan y + x + C_1 = C$$

$$\Rightarrow -\cos x \cdot \tan y + x = C - C_1 = C_0$$

$$\Rightarrow \boxed{-\cos x \cdot \tan y + x = C_0}$$

Ans.

3(b) $e^x \, dx + (e^x \cot y + 2y \operatorname{cosec} y) \, dy = 0$. —— ①

Solution: Comparing the given equation with

$$M(x, y) \, dx + N(x, y) \, dy = 0,$$

we get

$$M(x, y) = e^x, \quad N(x, y) = e^x \cot y + 2y \operatorname{cosec} y$$

$$\Rightarrow \frac{\partial M}{\partial y} = 0, \quad \frac{\partial N}{\partial x} = e^x \cot y$$

$$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\Rightarrow$ the given equation is not exact.

We need to find integrating factor:

Consider

$$\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M} = \frac{e^x \cot y - 0}{e^x} = \cot y = f(y)$$

which is a function of y only.

Thus integrating factor (I.F.) $= e^{\int f(y)\,dy} = e^{\int \cot y\,dy}$

$$= e^{\log(\sin y)} = \sin y$$

Multiplying the given equation (1) by $\sin y$, we get

$$e^x \sin y\,dx + \left(e^x \cot y + 2y \operatorname{cosec} y\right)\sin y\,dy = 0 \quad \text{--- (2)}$$

Now, $M = e^x \sin y$, $\quad N = \left(e^x \cot y + 2y \operatorname{cosec} y\right)\sin y$

$\Rightarrow \frac{\partial M}{\partial y} = e^x \cos y$, $\quad \frac{\partial N}{\partial x} = e^x \cot y \cdot \sin y = e^x \frac{\cos y}{\sin y} \cdot \sin y$

$\Rightarrow \frac{\partial M}{\partial y} = e^x \cos y$ and $\frac{\partial N}{\partial x} = e^x \cos y$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\Rightarrow$ (2) is an exact DE.

Now, we need to find a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M(x,y), \qquad \frac{\partial F}{\partial y} = N(x,y)$$

$\Rightarrow \frac{\partial F}{\partial x} = e^x \sin y$ and $\frac{\partial F}{\partial y} = \left(e^x \cot y + 2y \operatorname{cosec} y\right)\sin y$ --- (3)

$$\Rightarrow F(x,y) = \int e^x \sin y\,dx + \phi(y)$$

where $\phi(y)$ is a constant of integration.

$$\Rightarrow \quad F(x,y) = e^x \sin y + \phi(y) \qquad \text{——— (4)}$$

Differentiate the above equation partially w.r.t. 'y', we get

$$\frac{\partial F}{\partial y} = e^x \cdot (+\cos y) + \phi'(y)$$

$$\Rightarrow \left(e^x \cot y + 2y \operatorname{cosec} y\right) \sin y = +e^x \cos y + \phi'(y) \qquad (\text{Using } (5))$$

$$\Rightarrow \quad e^x \cos y + 2y = +e^x \cos y + \phi'(y)$$

$$\Rightarrow \quad \phi'(y) = 2y$$

$$\Rightarrow \quad \phi(y) = y^2 + c_1$$

Substituting the value of $\phi(y)$ in (4), we get

$$F(x,y) = e^x \sin y + y^2 + c_1$$

The solution of the given DE is given by

$$F(x,y) = c$$

$$\Rightarrow e^x \sin y + y^2 + c_1 = c$$

$$\Rightarrow e^x \sin y + y^2 = c - c_1 = c_0$$

$$\Rightarrow \boxed{e^x \sin y + y^2 = c_0}$$

Ans.

3(c) $\quad (3xy + y^2)\,dx + (x^2 + xy)\,dy = 0.$ ——— (1)

Comparing the above DE with $M(x,y)\,dx + N(x,y)\,dy = 0$, we get

$$M(x,y) = 3xy + y^2, \quad N = x^2 + xy$$

$\Rightarrow \frac{\partial M}{\partial y} = 3x+2y, \qquad \frac{\partial N}{\partial x} = 2x+y$

$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\Rightarrow$ the given DE is not exact.

We need to find integrating factor.

Consider $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \dfrac{3x+2y-2x-y}{x^2+xy} = \dfrac{x+y}{x(x+y)} = \frac{1}{x} = f(x)$

which is a function of x only.

Thus $I.F = e^{\int f(x)\,dx} = e^{\int \frac{1}{x}\,dx} = e^{\log x} = x$.

Multiplying the given DE(1) by x, we get,

$(3xy+y^2)x\,dx + (x^2+xy)x\,dy = 0$

$\Rightarrow (3x^2y+xy^2)\,dx + (x^3+x^2y)\,dy = 0$. —— (2)

Now, $M = 3x^2y + xy^2, \qquad N = x^3+x^2y$

$\frac{\partial M}{\partial y} = 3x^2+2xy, \qquad \frac{\partial N}{\partial x} = 3x^2+2xy$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\Rightarrow$ (2) is an exact DE.

Alternative Method to find sol^n of exact DE

Thus the solution is given by

$\int\limits_{y\text{-constant}} M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = C$

$$\Rightarrow \int_{y=\text{Constant}} (3x^2y + xy^2)\,dx + \int 0\,dy = C$$

$$\Rightarrow \frac{3x^3y}{3} + \frac{x^2y^2}{2} = C$$

$$\Rightarrow \boxed{x^3y + \frac{x^2y^2}{2} = C}$$ Ans.

(d) $y\,dx + (2x - ye^y)\,dy = 0$. ——— (1)

Comparing the given DE with $M(x,y)\,dx + N(x,y)\,dy = 0$, we get

$$M(x,y) = y, \quad N(x,y) = 2x - ye^y$$

$$\Rightarrow \frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = 2$$

$$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\Rightarrow$ (1) is not an exact DE.

We need to find integrating factor.

Consider $$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{2-1}{y} = \frac{1}{y} = f(y),$$

which is a function of y only.

Thus I.F $= e^{\int f(y)\,dy} = e^{\int \frac{1}{y}\,dy} = e^{\log y} = y$.

Multiplying (1) by y, we get

$$y^2\,dx + (2xy - y^2e^y)\,dy = 0.$$

Now, $M = y^2$, $\quad N = 2xy - y^2 e^y$

$$\Rightarrow \quad \frac{\partial M}{\partial y} = 2y, \quad \frac{\partial N}{\partial x} = 2y$$

$$\Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\Rightarrow$ (2) is an exact DE.

The solution is given by

$$\int_{y=\text{Constant}} M\, dx + \int (\text{terms of } N \text{ not containing } x)\, dy = c$$

$$\Rightarrow \int_{y=\text{Constant}} (y^2)\, dx + \int (-y^2 e^y)\, dy = c$$

$$\Rightarrow \quad xy^2 - \left(y^2 e^y - \int 2y e^y\, dy\right) = c$$

$$\Rightarrow \quad xy^2 - y^2 e^y + 2(ye^y - e^y) = c$$

$$\Rightarrow \quad xy^2 - y^2 e^y + 2ye^y - 2e^y = c$$

$$\Rightarrow \quad \boxed{xy^2 - (y^2 - 2y + 2)e^y = c}$$ Ans.

Problem 4: Suppose $M(x,y)\,dx + N(x,y)\,dy = 0$ has an integrating factor $\mu(x,y)$ such that $df = \mu M\,dx + \mu N\,dy$ is an exact differential. Show that the equation has an infinite number of integrating factors by demonstrating that the product $\mu G(f)$, where G is an arbitrary continuous function from $\mathbb{R}$ to $\mathbb{R}$, is also an integrating factor.

Solution: We have

$$\mu M\,dx + \mu N\,dy = df$$

Multiplying the above equation by $G(f)$ on both the sides, we get

$$\mu G(f)\,M\,dx + \mu G(f)\,N\,dy = G(f)\,df.$$

$$\Rightarrow \mu G(f)\,M\,dx + \mu G(f)\,N\,dy = d\left(\int G(f)\,df\right)$$

$$\Rightarrow \mu G(f)\,M\,dx + \mu G(f)\,N\,dy = dv,$$

where $v = \int G(f)\,df$.

Thus $\mu G(f)\,M\,dx + \mu G(f)\,N\,dy = dv$

is an exact differential, where $v = \int G(f)\,df$.

Hence $\mu G(f)$ is an integrating factor.

Problem-5 Solve the following linear/ reducible to linear ODEs: (13)

(a) $$(x+2y^3)\frac{dy}{dx} = y$$

Solution:
$$\frac{dx}{dy} = \frac{x+2y^3}{y}$$
$$\Rightarrow \frac{dx}{dy} - \frac{x}{y} = 2y^2$$

Which is of the form $\frac{dx}{dy} + P(y)x = Q(y)$.

Here $P(y) = -\frac{1}{y}$ and $Q(y) = 2y^2$

$$I.F. = e^{\int P(y)dy} = e^{\int -\frac{1}{y}dy} = e^{-\log y} = \frac{1}{y}$$

Thus the solution is
$$(IF)\cdot x = \int Q(y) \times I.F\, dy + C$$
$$\Rightarrow \frac{1}{y}\cdot x = \int 2y^2 \cdot \frac{1}{y}\, dy + C$$
$$\Rightarrow \boxed{\frac{x}{y} = y^2 + C}$$
Ans

(b) $$(1+y^2) + (x - e^{\tan^{-1}y})\frac{dy}{dx} = 0$$

Solution:
$$(1+y^2)\frac{dx}{dy} + x = e^{\tan^{-1}y}$$
$$\Rightarrow \frac{dx}{dy} + \left(\frac{1}{1+y^2}\right)x = \frac{e^{\tan^{-1}y}}{1+y^2}$$

Which is of the form $\frac{dx}{dy} + P(y)x = Q(y)$.

Here $P(y) = \frac{1}{1+y^2}$, $Q(y) = e^{\frac{\tan^{-1}y}{1+y^2}}$

$\therefore$ The solution is given by

$$x \cdot e^{\tan^{-1}y} = \int Q(y) \cdot e^{\tan^{-1}y}\, dy + c$$

$$\Rightarrow \quad x \cdot e^{\tan^{-1}y} = \int \frac{1}{1+y^2}\, dy + c$$

$$\Rightarrow \quad \boxed{x \cdot e^{\tan^{-1}y} = \tan^{-1}y + c} \quad \text{Ans.}$$

(C) $\quad x\frac{dy}{dx} + y = x^2y^2 \quad \left(\text{Bernoulli's DE} \quad \frac{dy}{dx} + P(x)\,y = Q(x)\cdot y^n\right)$

Solution: Dividing both sides by y^2, we get

$$\frac{x}{y^2}\frac{dy}{dx} + \frac{y}{y^2} = x^2$$

$$\Rightarrow \quad \frac{x}{y^2}\frac{dy}{dx} + \frac{1}{y} = x^2 \quad \text{——— (1)}$$

Put $\frac{1}{y} = z \quad \Rightarrow \quad \frac{-1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$

$$\Rightarrow \quad \frac{1}{y^2}\frac{dy}{dx} = -\frac{dz}{dx}$$

With this substitution, (1) becomes

$$-x\frac{dz}{dx} + z = x^2$$

$$\Rightarrow \quad \frac{dz}{dx} - \frac{z}{x} = -x$$

$$I.F = e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = \frac{1}{x}$$

Thus the solution is

$$z \times I.F = \int (-x) \times I.F \; dx + c$$

$$\Rightarrow z \cdot \frac{1}{x} = \int \frac{-x}{x} dx + c$$

$$\Rightarrow \frac{z}{x} = -x + c$$

$$\Rightarrow \frac{1}{yx} = -x + c \qquad \left[\because z = \frac{1}{y}\right]$$

$$\Rightarrow \boxed{y = \frac{1}{x(c-x)}}$$ Ans.

(d) $y^{1/2}\frac{dy}{dx} + y^{3/2} = 1, \quad y(0) = 4$

Solution: Let $y^{3/2} = z$

$$\Rightarrow \frac{3}{2} y^{1/2} \frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow y^{1/2} \frac{dy}{dx} = \frac{2}{3}\frac{dz}{dx}$$

Using this, the given equation becomes,

$$\frac{2}{3}\frac{dz}{dx} + z = 1$$

$$\Rightarrow \frac{dz}{dx} + \frac{3}{2} z = 1$$

$$IF = e^{\int \frac{3}{2} dx} = e^{\frac{3}{2}x}$$

$\therefore$ Solution is

$$z \cdot e^{\frac{3}{2}x} = \int (1) \cdot e^{\frac{3}{2}x} dx + c$$

$$\Rightarrow z \cdot e^{\frac{3}{2}x} = \frac{e^{\frac{3}{2}x}}{\frac{3}{2}} + c$$

$$\Rightarrow z \cdot e^{\frac{3}{2}x} = \frac{2}{3} e^{\frac{3}{2}x} + C$$

$$\Rightarrow y^{3/2} \cdot e^{\frac{3}{2}x} = \frac{2}{3} e^{\frac{3}{2}x} + C$$

$$\Rightarrow \boxed{y^{3/2} = \frac{2}{3} + Ce^{-3/2x}}$$

Ans.

Problem-6(a) Find the orthogonal trajectories to the family of curves $x^2+y^2=cx$.

Solution: Given family of curves is

$$x^2+y^2 = cx \quad \text{——— (1)}$$

Differentiating (1) w.r.t. 'x', we get

$$2x+2y\frac{dy}{dx} = c$$

Substituting the value of c in (1), we get

$$x^2+y^2 = \left(2x+2y\frac{dy}{dx}\right)x = 2x^2+2xy\frac{dy}{dx}$$

$$\Rightarrow \quad y^2-x^2 = 2xy\frac{dy}{dx} \quad \text{——— (2)}$$

Which is the differential equation corresponding to the given family of circles (1).

Replacing $\frac{dy}{dx}$ by $\left(\frac{-1}{\frac{dy}{dx}}\right)$, the differential equation of the required orthogonal trajectories is

$$y^2-x^2 = 2xy\left(\frac{-1}{\frac{dy}{dx}}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2xy}{x^2-y^2} \quad \text{——— (3)}$$

Which is a homogeneous DE.

Put $y=vx \Rightarrow \frac{dy}{dx} = v+x\frac{dv}{dx}$

$\therefore$ (3) gives, $\quad v+x\frac{dv}{dx} = \frac{2x(vx)}{x^2-v^2x^2} = \frac{2v}{1-v^2}$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v}{1-v^2} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v - v + v^3}{1-v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v(1+v^2)}{1-v^2}$$

$$\Rightarrow \frac{dx}{x} = \left(\frac{1-v^2}{v+v^3}\right) dv$$

$$\Rightarrow \frac{dx}{x} = \left(\frac{1}{v} - \frac{2v}{1+v^2}\right) dv$$

Integrating, we get

$$\log x = \log v - \log(1+v^2) + \log c$$

$$\Rightarrow x = \frac{cv}{1+v^2}$$

$$\Rightarrow x = \left(\frac{\frac{cy}{x}}{1+\frac{y^2}{x^2}}\right)$$

$$\Rightarrow x = \frac{c\frac{y}{x}\cdot x^2}{x^2+y^2} = \frac{cxy}{x^2+y^2}$$

$$\Rightarrow \boxed{x^2+y^2 = cy}$$, c being parameter.

Which are the required orthogonal trajectories of the given family of circles.

(b) Find the value of n such that the curves $x^n+y^n=c$ are orthogonal trajectories of the family $y=\frac{x}{1-c_1x}$. (19)

Solution: Given family of curves is

$$y=\frac{x}{1-c_1x}$$

$$\Rightarrow \frac{1}{x}-\frac{1}{y}=c_1 \qquad \text{(1)}$$

Differentiating (1), we get

$$-\frac{1}{x^2}+\frac{1}{y^2}\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{y^2}{x^2} \qquad \text{(2)}$$

Which is the DE corresponding to the given family of curves (1)

Replacing $\frac{dy}{dx}$ by $\frac{1}{-\frac{dy}{dx}}$ in (2) ~~to~~ obtain the DE of orthogonal trajectories, we get

$$\frac{1}{-\frac{dy}{dx}}=\frac{y^2}{x^2}$$

$$\Rightarrow -\frac{x^2}{y^2}=\frac{dy}{dx}$$

$$\Rightarrow y^2\,dy=-x^2\,dx$$

Integrating, we get $\frac{y^3}{3}=\frac{-x^3}{3}+c_1$

$$\Rightarrow x^3+y^3=c, \qquad \boxed{c=3c_1}$$

$$\Rightarrow x^3+y^3=c,$$

Which is the orthogonal trajectories of the given family of curves. $\boxed{\text{Thus } n=3}$ Ans

6(c) Show that the family of parabolas $y^2 = 2cx + c^2$ is self-orthogonal: (20)

Solution: Given $y^2 = 2cx + c^2$ ——— (1)

Differentiating (1), we get

$$2y \frac{dy}{dx} = 2c$$

$$\Rightarrow \quad c = y \frac{dy}{dx} \qquad \text{——— (2)}$$

Eliminating c from (1) and (2), we get

$$y^2 = 2\left(y \frac{dy}{dx}\right)x + \left(y \frac{dy}{dx}\right)^2$$

$$\Rightarrow \quad y^2 = 2xy \frac{dy}{dx} + y^2 \left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \quad y = 2x \frac{dy}{dx} + y \left(\frac{dy}{dx}\right)^2 \qquad \text{——— (3)}$$

Which is the DE of (1).

Replacing $\frac{dy}{dx}$ by $\left(\frac{1}{-\frac{dy}{dx}}\right)$, the DE of orthogonal trajectories is

$$y = 2x \left(\frac{-1}{\frac{dy}{dx}}\right) + y \left(\frac{-1}{\frac{dy}{dx}}\right)^2$$

$$\Rightarrow \quad y \left(\frac{dy}{dx}\right)^2 = -2x \frac{dy}{dx} + y$$

$$\Rightarrow \quad y = 2x \frac{dy}{dx} + y \left(\frac{dy}{dx}\right)^2 \qquad \text{——— (4)}$$

Which is the same as the DE (3) of the given system (1). Hence the system of parabolas (1) is self orthogonal.

Problem-7: Suppose $P(x)$ is continuous on some closed interval I and a is a number in I. What can be said about the existence and uniqueness of solution of initial value problem $y' + P(x)y = 0$, $y(a) = 0$ (without solving)?

Solution: The given initial value problem is

$$y' + P(x)\, y = 0, \quad y(a) = 0$$

$$\Rightarrow \; y' = -P(x)\, y$$

$$\Rightarrow \; y' = f(x, y)$$

where $f(x, y) = -P(x)\, y$.

Since $P(x)$ is continuous on I.

$\Rightarrow$ $f(x,y) = -P(x)\cdot y$ is continuous on $I \times \mathbb{R}$.

~~There~~ Moreover $\dfrac{\partial f}{\partial y} = -P(x)$,

which is also continuous on $I \times \mathbb{R}$.

Therefore $f(x, y)$ satisfies Lipschitz condition on $I \times \mathbb{R}$.

[$\because$ $\left|\dfrac{\partial f}{\partial y}\right| = |-P(x)| \leq K$

as every continuous function on a closed interval is bounded.]

Therefore, by existence and uniqueness theorem, the given IVP has a unique solution on some subinterval of I containing a.

Problem-8 : Verify whether the following functions satisfy Lipschitz condition or not on the given sets R.

(i) $f(x,y) = x^3 \sin y$ on $R: |x| \le 2, -\infty < y < \infty$.

Solution: Here $\frac{\partial f}{\partial y} = x^3 \cos y$

$$\Rightarrow \left|\frac{\partial f}{\partial y}\right| = |x^3 \cos y| = |x|^3 . |\cos y|$$

$$\le (2)^3 . (1) \qquad \left[\because |\cos y| \le 1 \text{ and } |x| \le 2\right]$$

$$\le 8 \quad \text{in } R$$

$$\Rightarrow \left|\frac{\partial f}{\partial y}\right| \le 8 = K$$

Thus partial derivative of f w.r.t y exists and is bounded in R.

$\Rightarrow$ $f(x,y)$ satisfies Lipschitz condition on the domain R.

Alternatie :

Consider

$$|f(x,y_1) - f(x,y_2)| = |x^3 \cos y_1 - x^3 \cos y_2|$$

$$= |x|^3 \, |\cos y_1 - \cos y_2|$$

$$= |x|^3 . \left|-2 \sin\left(\frac{y_1+y_2}{2}\right) . \sin\left(\frac{y_1-y_2}{2}\right)\right|$$

$$\left[\because \cos C - \cos D = -2 \sin\left(\frac{C+D}{2}\right) . \sin\left(\frac{C-D}{2}\right)\right]$$

$$\le (2)^3 . (2)\,(1)\, \frac{|y_1-y_2|}{2} \qquad \left[\because \begin{array}{l} |\sin x| \le 1 \\ |\sin x| \le |x| \end{array}\right]$$

$$\le 8\,|y_1-y_2|$$

$$\Rightarrow |f(x,y_1)-f(x,y_2)| \le 8\,|y_1-y_2|$$

$\Rightarrow$ $f(x,y)$ satisfies Lipschitz Condition w.r.t. y in R.

(ii) $f(x,y) = y^{1/3}$ on $R: |x| \le 1, |y| \le 1$.

Consider

$$\frac{|f(x,y_1)-f(x,0)|}{|y_1-0|} = \frac{|y_1^{1/3}-0|}{|y_1|} = \frac{1}{y_1^{2/3}}, \quad |y_1| \le 1.$$

(for $y_1 > 0$ and $y_2 = 0$)

Since $\frac{1}{y_1^{2/3}}$ becomes unbounded as y_1 approaches zero.

$\Rightarrow$ $f(x,y)$ doesnot satisfy Lipschitz condition throughout any domain containing the line $y=0$, hence not in R.

(iii) $f(x,y) = x^2+y, \; |x| \le 1, \; |y| < \infty$.

Solution: Here $\frac{\partial f}{\partial y} = 1$

$$\Rightarrow \left|\frac{\partial f}{\partial y}\right| = |1| = 1$$

$\Rightarrow$ $\frac{\partial f}{\partial y}$ is bounded in $R: |x| \le 1, |y| < \infty$.

$\Rightarrow$ $f(x,y)$ satisfies Lipschitz condition w.r.t. 'y' in $R: |x| \le 1, |y| < \infty$.

Alternative:

$$|f(x,y_1)-f(x,y_2)| = |x^2+y_1-x^2-y_2|$$
$$= |y_1-y_2|$$

$$\Rightarrow |f(x,y_1)-f(x,y_2)| \le (1)\,|y_1-y_2|$$

Thus $f(x,y)$ satisfies Lipschitz condition w.r.t. 'y' in $R: |x| \le 1, |y| < \infty$.

Problem-9: Discuss the existence and uniqueness of solution for the following initial value problems (IVP) in the region $R: |x| \le 1, |y| \le 1$.

(a) $\frac{dy}{dx} = 3y^{2/3}, \quad y(0) = 0$.

Solution: Here $f(x, y) = 3y^{2/3}$,

(i) $f(x, y)$ is continuous on R.

(ii) $|f(x,y)| = |3y^{2/3}| \le 3 = M$ in $R: |x| \le 1, |y| \le 1$

Thus the given IVP has a solution in $|x| \le h$,

where $h = \min\left(a, \frac{b}{M}\right) = \min\left(1, \frac{1}{3}\right) = \frac{1}{3}$

$\Rightarrow |x| \le \frac{1}{3}$.

(By Picard's Existence Theorem)

But f doesnot satisfy Lipschitz condition in R, as for $y_1 > 0$ and $y_2 = 0$

$$\frac{|f(x, y_1) - f(x, 0)|}{|y_1 - 0|} = \left|\frac{3y_1^{2/3}}{|y_1|}\right| = \frac{3}{y_1^{1/3}},$$

which is unbounded in the neighbourhood of origin.

Thus uniqueness of the solutions for the given IVP is not guaranteed.

(Note that $y_1(x) = 0$ and $y_2(x) = x^3$ are two different solutions)

(b) $\frac{dy}{dx} = x^2+y^2, \quad y(0)=0.$ $\quad (R: |x| \le 1, |y| \le 1)$

Solution! Here $f(x,y) = x^2+y^2$; $x_0=0, y_0=0, a=1, b=1$

~~Here~~ (i) $f(x,y)$ is continuous in R.

(ii) $$|f(x,y)| = |x^2+y^2| \le |x|^2+|y|^2$$
$$\le 1+1 = 2 = M$$

$$\Rightarrow |f(x,y)| \le M (=2)$$

Thus $f(x,y)$ is bounded in R.

(iii) $$\frac{\partial f}{\partial y} = 2y$$

$$\Rightarrow \left|\frac{\partial f}{\partial y}\right| = |2y| \le 2(1) = 2 = K$$

$\Rightarrow$ the partial derivative of f w.r.t. 'y' is bounded.

$\Rightarrow$ $f(x,y)$ satisfies Lipschitz condition w.r.t. 'y' in R.

Therefore, using Picard's Existence and Uniqueness Theorem, the given IVP has unique solution in

$$|x| \le h, \text{ where } h = \min\left(a, \frac{b}{M}\right) = \min\left(1, \frac{1}{2}\right) = \frac{1}{2}$$

$$\Rightarrow |x| \le \frac{1}{2}.$$

(c) $\frac{dy}{dx} = \sin x \cdot \cos y + xy^2$, $y(0) = 0$ $R: |x| \le 1, |y| \le 1$.

Solution: Here $f(x,y) = \sin x \cdot \cos y + xy^2$

(i) $f(x,y)$, is continuous in R.

(ii)
$$|f(x,y)| = |\sin x \cdot \cos y + xy^2|$$
$$\le |\sin x| \cdot |\cos y| + |x||y|^2$$
$$\le (1)(1) + (1)(1)^2 \qquad \left[\because |\sin x| \le 1, |\cos x| \le 1\right]$$
$$= 2$$

$$\Rightarrow |f(x,y)| \le 2 = M$$

Thus $f(x,y)$ is bounded in R.

(iii)
$$\frac{\partial f}{\partial y} = \sin x \cdot (-\sin y) + 2xy$$

$$\Rightarrow \left|\frac{\partial f}{\partial y}\right| = |-\sin x \cdot \sin y + 2xy| \le (1)(1) + 2(1)(1) = 3$$

$$\Rightarrow \left|\frac{\partial f}{\partial y}\right| \le 3 = K$$

Thus $f(x,y)$ satisfies Lipschitz condition w.r.t. 'y' in R.

Therefore, Using ~~Lipschitz~~ Picard's Existence and Uniqueness theorem, $\exists$ unique solution of the given IVP in

$$|x| \le h, \text{ where } h = \min\left(a, \frac{b}{M}\right) = \min\left(1, \frac{1}{2}\right) = \frac{1}{2}$$

$$\Rightarrow |x| \le \frac{1}{2}.$$

Problem-10: For the following initial value problems, compute the first three iterates using Picard's iteration method.

(i) $y' = x^2 + y^2 - 1, \quad y(0) = 1.$

Solution: Let $y_0(x) = 1$

$$y_1(x) = y_0 + \int_{x_0}^{x} f(s, y_0(s))\, ds$$

$$= 1 + \int_0^x f(s,1)\, ds \qquad \left[\because \begin{matrix} x_0 = 0 \\ y_0(x) = 1 \end{matrix}\right]$$

$$= 1 + \int_0^x (s^2 + 1^2 - 1)\, ds \qquad \left[\because f(x, y) = x^2 + y^2 - 1\right]$$

$$= 1 + \int_0^x s^2\, ds = 1 + \left|\frac{s^3}{3}\right|_0^x$$

$$\Rightarrow y_1(x) = 1 + \frac{x^3}{3}$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(s, y_1(s))\, ds$$

$$= 1 + \int_0^x f(s, y_1(s))\, ds = 1 + \int_0^x (s^2 + y_1^2 - 1)\, ds$$

$$= 1 + \int_0^x \left(s^2 + \left(1 + \frac{s^3}{3}\right)^2 - 1\right) ds$$

$$= 1 + \int_0^x \left(s^2 + 1 + \frac{s^6}{9} + \frac{2s^3}{3} - 1\right) ds$$

$$= 1 + \int_0^x \left(s^2 + \frac{s^6}{9} + \frac{2s^3}{3}\right) ds$$

$$= 1 + \left| \frac{s^3}{3} + \frac{s^7}{63} + \frac{2s^4}{12} \right|_0^x$$

$$= 1 + \frac{x^3}{3} + \frac{x^7}{63} + \frac{x^4}{6}$$

$$\Rightarrow y_2 = 1 + \frac{x^3}{3} + \frac{x^4}{6} + \frac{x^7}{63}$$

Thus the first three iterates are

$$\boxed{y_0(x) = 1, \quad y_1(x) = 1 + \frac{x^3}{3}, \quad y_2(x) = 1 + \frac{x^3}{3} + \frac{x^4}{6} + \frac{x^7}{63}}$$ Ans.

(ii) $y' = 1 + 2y^2, \quad y(0) = 0.$

Solution: Let $y_0(x) = 0$

$$y_1(x) = y_0 + \int_{x_0}^{x} f(s, y_0(s))\, ds$$

$$= 0 + \int_0^x f(s, 0)\, ds$$

$$= \int_0^x \left(1 + 2(0)^2\right) ds \qquad \left[\because f(x, y) = 1 + 2y^2\right]$$

$$= \int_0^x ds$$

$$= x$$

$$\Rightarrow y_1(x) = x$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(s, y_1(s))\, ds = 0 + \int_0^x f(s, y_1(s))\, ds$$

$$= \int_0^x f(s, y_1(s))\, ds = \int_0^x (1 + 2y_1^2)\, ds = \int_0^x (1 + 2s^2)\, ds$$

$$= \left| s + \frac{2s^3}{3} \right|_0^x$$

$$= x + 2\frac{x^3}{3}$$

$$\Rightarrow y_2(x) = x + \frac{2x^3}{3}$$

Thus the first three iterates are

$$y_0(x) = 0, \quad y_1(x) = x, \quad y_2(x) = x + \frac{2x^3}{3}.$$