Multivariable Calculus

(Lecture-4)

Department of Mathematics
Bennett University
India

24^{th} October, 2018

Learning Outcome of the Lecture

We learn

- Limits of Functions $F : S \subseteq \mathbb{R}^n \to \mathbb{R}$
- Limit and Iterated Limits
- Continuity of Functions $F : S \subseteq \mathbb{R}^2 \to \mathbb{R}$
- Properties of continuous functions

BENNETT UNIVERSITY
A TIMES GROUP INITIATIVE

Limits of Functions $F : S \subseteq \mathbb{R}^n \to \mathbb{R}$

We can similarly define limits of a function $F : S \subseteq \mathbb{R}^n \to \mathbb{R}$.

Limits of Functions $F : S \subseteq \mathbb{R}^n \to \mathbb{R}$

We can similarly define limits of a function $F : S \subseteq \mathbb{R}^n \to \mathbb{R}$.

Definition

Let $F : S \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function. Let $A = (x_1, x_2, \ldots, x_n)$ be a limit point of S.

Limits of Functions $F : S \subseteq \mathbb{R}^n \to \mathbb{R}$

We can similarly define limits of a function $F : S \subseteq \mathbb{R}^n \to \mathbb{R}$.

Definition

Let $F : S \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function. Let $A = (x_1, x_2, \ldots, x_n)$ be a limit point of S. A point $L \in \mathbb{R}$ is said to be the limit of F at A if

Limits of Functions $F : S \subseteq \mathbb{R}^n \to \mathbb{R}$

We can similarly define limits of a function $F : S \subseteq \mathbb{R}^n \to \mathbb{R}$.

Definition

Let $F : S \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function. Let $A = (x_1, x_2, \ldots, x_n)$ be a limit point of S. A point $L \in \mathbb{R}$ is said to be the limit of F at A if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$X \in S \quad \text{and} \quad 0 < \|X - A\| < \delta \quad \Rightarrow \quad |F(X) - L| < \epsilon.$$

Limits of Functions $F : S \subseteq \mathbb{R}^n \to \mathbb{R}$

We can similarly define limits of a function $F : S \subseteq \mathbb{R}^n \to \mathbb{R}$.

Definition

Let $F : S \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function. Let $A = (x_1, x_2, \ldots, x_n)$ be a limit point of S. A point $L \in \mathbb{R}$ is said to be the limit of F at A if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$X \in S \quad \text{and} \quad 0 < \|X - A\| < \delta \quad \Rightarrow \quad |F(X) - L| < \epsilon.$$

In this case we write

$$\lim_{X \to A} F(X) = L,$$

and say that $\lim_{X \to A} F(X)$ exists and is equal to L.

Limits of Functions $F : S \subseteq \mathbb{R}^n \to \mathbb{R}$

We can similarly define limits of a function $F : S \subseteq \mathbb{R}^n \to \mathbb{R}$.

Definition

Let $F : S \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function. Let $A = (x_1, x_2, \ldots, x_n)$ be a limit point of S. A point $L \in \mathbb{R}$ is said to be the limit of F at A if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$X \in S \quad \text{and} \quad 0 < \|X - A\| < \delta \quad \Rightarrow \quad |F(X) - L| < \epsilon.$$

In this case we write

$$\lim_{X \to A} F(X) = L,$$

and say that $\lim_{X \to A} F(X)$ exists and is equal to L.

In other words, we say that $F(X)$ approaches L as X approaches A or $F(X)$ has the limit L as X tends to A.

Important Theorem on Limits of Functions

Theorem

Let $F : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a function. Then, $F(X)$ can be written as

$$F(X) = (f_1(X), f_2(X), \ldots, f_m(X)) \quad \text{for } X \in S,$$

where $f_i (1 \leq i \leq m)$ are component functions of F.

BENNETT UNIVERSITY
A TIMES GROUP INITIATIVE

Important Theorem on Limits of Functions

Theorem

Let $F : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a function. Then, $F(X)$ can be written as

$$F(X) = (f_1(X), f_2(X), \ldots, f_m(X)) \quad \text{for } X \in S,$$

where $f_i (1 \leq i \leq m)$ are component functions of F.
Let X_0 be a limit point of S and let $L = (A_1, A_2, \ldots, A_m)$ be a point in $\mathbb{R}^m$. Then,

$$\lim_{X \to X_0} F(X) = L$$

if and only if

$$\lim_{X \to X_0} f_i(X) = A_i \quad \text{for } 1 \leq i \leq m.$$

BENNETT UNIVERSITY
A TIMES GROUP INITIATIVE

Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be the function defined by

$$F(x,y) = \left(\frac{\sin x}{x}, \frac{x^2 y}{x^2 + y^2}, 5y \right)$$

for all (x,y) with $x \neq 0$.Examine the existence of $\lim_{(x,y)\to(0,0)} F(x,y)$.

Let $F:\mathbb{R}^2 \to \mathbb{R}^3$ be the function defined by

$$F(x,y) = \left(\frac{\sin x}{x}, \frac{x^2 y}{x^2+y^2}, 5y\right)$$

for all (x,y) with $x \neq 0$.Examine the existence of $\lim_{(x,y)\to(0,0)} F(x,y)$.

- Step 1: First Component function f_1

BENNETT UNIVERSITY
A TIMES GROUP INITIATIVE

Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be the function defined by

$$F(x, y) = \left(\frac{\sin x}{x}, \frac{x^2 y}{x^2 + y^2}, 5y \right)$$

for all (x, y) with $x \neq 0$.Examine the existence of $\lim_{(x,y)\to(0,0)} F(x, y)$.

- Step 1: First Component function f_1

$$\lim_{(x,y)\to(0,0)} f_1(x, y) = \lim_{(x,y)\to(0,0)} \frac{\sin x}{x} = 1$$

Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be the function defined by

$$F(x, y) = \left(\frac{\sin x}{x}, \frac{x^2 y}{x^2 + y^2}, 5y \right)$$

for all (x, y) with $x \neq 0$.Examine the existence of $\lim_{(x,y)\to(0,0)} F(x, y)$.

- Step 1: First Component function f_1

$$\lim_{(x,y)\to(0,0)} f_1(x, y) = \lim_{(x,y)\to(0,0)} \frac{\sin x}{x} = 1$$

- Step 2: Second Component function f_2

Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be the function defined by

$$F(x, y) = \left(\frac{\sin x}{x}, \frac{x^2 y}{x^2 + y^2}, 5y \right)$$

for all (x, y) with $x \neq 0$.Examine the existence of $\lim_{(x,y)\to(0,0)} F(x, y)$.

- Step 1: First Component function f_1

$$\lim_{(x,y)\to(0,0)} f_1(x, y) = \lim_{(x,y)\to(0,0)} \frac{\sin x}{x} = 1$$

- Step 2: Second Component function f_2

$$\lim_{(x,y)\to(0,0)} f_2(x, y) = \lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2 + y^2} = 0$$

Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be the function defined by

$$F(x,y) = \left(\frac{\sin x}{x}, \frac{x^2 y}{x^2+y^2}, 5y \right)$$

for all (x,y) with $x \neq 0$.Examine the existence of $\lim_{(x,y)\to(0,0)} F(x,y)$.

- Step 1: First Component function f_1

$$\lim_{(x,y)\to(0,0)} f_1(x,y) = \lim_{(x,y)\to(0,0)} \frac{\sin x}{x} = 1$$

- Step 2: Second Component function f_2

$$\lim_{(x,y)\to(0,0)} f_2(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2} = 0$$

- Step 3: Third Component function f_3

BENNETT UNIVERSITY
A TIMES GROUP INITIATIVE

Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be the function defined by

$$F(x,y) = \left(\frac{\sin x}{x}, \frac{x^2 y}{x^2+y^2}, 5y \right)$$

for all (x,y) with $x \neq 0$.Examine the existence of $\lim_{(x,y)\to(0,0)} F(x,y)$.

- Step 1: First Component function f_1

$$\lim_{(x,y)\to(0,0)} f_1(x,y) = \lim_{(x,y)\to(0,0)} \frac{\sin x}{x} = 1$$

- Step 2: Second Component function f_2

$$\lim_{(x,y)\to(0,0)} f_2(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2} = 0$$

- Step 3: Third Component function f_3

$$\lim_{(x,y)\to(0,0)} f_3(x,y) = \lim_{(x,y)\to(0,0)} 5y = 0$$

BENNETT UNIVERSITY
A TIMES GROUP INITIATIVE

Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be the function defined by

$$F(x,y) = \left(\frac{\sin x}{x}, \frac{x^2 y}{x^2+y^2}, 5y \right)$$

for all (x,y) with $x \neq 0$.Examine the existence of $\lim_{(x,y)\to(0,0)} F(x,y)$.

- Step 1: First Component function f_1

$$\lim_{(x,y)\to(0,0)} f_1(x,y) = \lim_{(x,y)\to(0,0)} \frac{\sin x}{x} = 1$$

- Step 2: Second Component function f_2

$$\lim_{(x,y)\to(0,0)} f_2(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2} = 0$$

- Step 3: Third Component function f_3

$$\lim_{(x,y)\to(0,0)} f_3(x,y) = \lim_{(x,y)\to(0,0)} 5y = 0$$

- Step 4: Conclusion: $\lim_{(x,y)\to(0,0)} F(x,y) = (1,0,0)$.

BENNETT UNIVERSITY
A TIMES GROUP INITIATIVE

Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be the function defined by

$$F(x, y) = \left(\frac{\sin x}{x}, \frac{xy}{x^2 + y^2}, 5 \right)$$

for all (x, y) with $x \neq 0$. Examine the existence of $\lim_{(x,y)\to(0,0)} F(x, y)$.

Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be the function defined by

$$F(x, y) = \left(\frac{\sin x}{x}, \frac{xy}{x^2 + y^2}, 5 \right)$$

for all (x, y) with $x \neq 0$. Examine the existence of $\lim_{(x,y)\to(0,0)} F(x, y)$.

- Step 1: First Component function f_1

$$\lim_{(x,y)\to(0,0)} f_1(x, y) = \lim_{(x,y)\to(0,0)} \frac{\sin x}{x} = 1$$

Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be the function defined by

$$F(x, y) = \left(\frac{\sin x}{x}, \frac{xy}{x^2 + y^2}, 5 \right)$$

for all (x, y) with $x \neq 0$. Examine the existence of $\lim_{(x,y)\to(0,0)} F(x, y)$.

- Step 1: First Component function f_1

$$\lim_{(x,y)\to(0,0)} f_1(x, y) = \lim_{(x,y)\to(0,0)} \frac{\sin x}{x} = 1$$

- Step 2: Second Component function f_2

$$\lim_{(x,y)\to(0,0)} f_2(x, y) = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2} \text{ does NOT exist.}$$

Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be the function defined by

$$F(x, y) = \left(\frac{\sin x}{x}, \frac{xy}{x^2 + y^2}, 5 \right)$$

for all (x, y) with $x \neq 0$. Examine the existence of $\lim_{(x,y)\to(0,0)} F(x, y)$.

- Step 1: First Component function f_1

$$\lim_{(x,y)\to(0,0)} f_1(x, y) = \lim_{(x,y)\to(0,0)} \frac{\sin x}{x} = 1$$

- Step 2: Second Component function f_2

$$\lim_{(x,y)\to(0,0)} f_2(x, y) = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2} \text{ does NOT exist.}$$

- Step 3: Third Component function f_3

$$\lim_{(x,y)\to(0,0)} f_3(x, y) = \lim_{(x,y)\to(0,0)} 5 = 5$$

Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be the function defined by

$$F(x,y) = \left(\frac{\sin x}{x}, \frac{xy}{x^2+y^2}, 5 \right)$$

for all (x, y) with $x \neq 0$. Examine the existence of $\lim_{(x,y)\to(0,0)} F(x,y)$.

- Step 1: First Component function f_1

$$\lim_{(x,y)\to(0,0)} f_1(x,y) = \lim_{(x,y)\to(0,0)} \frac{\sin x}{x} = 1$$

- Step 2: Second Component function f_2

$$\lim_{(x,y)\to(0,0)} f_2(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \text{ does NOT exist.}$$

- Step 3: Third Component function f_3

$$\lim_{(x,y)\to(0,0)} f_3(x,y) = \lim_{(x,y)\to(0,0)} 5 = 5$$

- Step 4: Conclusion: $\lim_{(x,y)\to(0,0)} F(x,y)$ does NOT exist.

BENNETT UNIVERSITY
A TIMES GROUP INITIATIVE

Converting to other coordinates system and then finding limit

Some times converting from cartesian coordinates system to other coordinates system ($\mathbb{R}^2$: Polar; $\mathbb{R}^3$: Cylindrical, Spherical) will make easier to compute the limits.

Converting to other coordinates system and then finding limit

Some times converting from cartesian coordinates system to other coordinates system ($\mathbb{R}^2$: Polar; $\mathbb{R}^3$: Cylindrical, Spherical) will make easier to compute the limits.

Compute $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}$ converting it into polar coordinates.

Converting to other coordinates system and then finding limit

Some times converting from cartesian coordinates system to other coordinates system ($\mathbb{R}^2$: Polar; $\mathbb{R}^3$: Cylindrical, Spherical) will make easier to compute the limits.

Compute $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}$ converting it into polar coordinates.

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = \lim_{r\to 0^+} \frac{r^4\cos^2\theta\sin^2\theta}{r^2}$$

$$= \lim_{r\to 0^+} r^2\cos^2\theta\sin^2\theta \leq \lim_{r\to 0^+} r^2 = 0$$

Therefore,

BENNETT UNIVERSITY
A TIMES GROUP INITIATIVE

Converting to other coordinates system and then finding limit

Some times converting from cartesian coordinates system to other coordinates system ($\mathbb{R}^2$: Polar; $\mathbb{R}^3$: Cylindrical, Spherical) will make easier to compute the limits.

Compute $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}$ converting it into polar coordinates.

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = \lim_{r\to 0^+} \frac{r^4\cos^2\theta\sin^2\theta}{r^2}$$

$$= \lim_{r\to 0^+} r^2\cos^2\theta\sin^2\theta \leq \lim_{r\to 0^+} r^2 = 0$$

Therefore,

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = 0.$$

BENNETT UNIVERSITY
A TIMES GROUP INITIATIVE

Sequences Criteria for Limits of Functions

Theorem

Let $F : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a function. Let X_0 be a limit point of S and let L be a point in $\mathbb{R}^m$. Then,

BENNETT UNIVERSITY
A TIMES GROUP INITIATIVE

Sequences Criteria for Limits of Functions

Theorem

Let $F : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a function. Let X_0 be a limit point of S and let L be a point in $\mathbb{R}^m$. Then,

$$\lim_{X \to X_0} F(X) = L$$

if and only if

Sequences Criteria for Limits of Functions

Theorem

Let $F : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a function. Let X_0 be a limit point of S and let L be a point in $\mathbb{R}^m$. Then,

$$\lim_{X \to X_0} F(X) = L$$

if and only if

For every sequence $\{X_k\}$ in S (with $X_k \neq X_0$ for all k) converging to X_0 the sequence $\{F(X_k)\}$ converges to L.

Iterated Limits of Scalar Valued Functions

Let $f : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a real valued function. Let (x_0, y_0) be a limit point of S. The limits

$$\lim_{x \to x_0} \lim_{y \to y_0} f(x, y), \quad \text{if it exists}$$

$$\lim_{y \to y_0} \lim_{x \to x_0} f(x, y), \quad \text{if it exists}$$

are called the iterated limits of f at (x_0, y_0).

Iterated Limits of Scalar Valued Functions

Let $f : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a real valued function. Let (x_0, y_0) be a limit point of S. The limits

$$\lim_{x \to x_0} \lim_{y \to y_0} f(x, y), \quad \text{if it exists}$$

$$\lim_{y \to y_0} \lim_{x \to x_0} f(x, y), \quad \text{if it exists}$$

are called the iterated limits of f at (x_0, y_0). Then, the limit

$$\lim_{(x,y) \to (0,0)} f(x, y), \quad \text{if it exists}$$

is called the (double) limit of f at (x_0, y_0).

BENNETT UNIVERSITY
A TIMES GROUP INITIATIVE

Iterated Limits of Scalar Valued Functions

Let $f : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a real valued function. Let (x_0, y_0) be a limit point of S. The limits

$$\lim_{x \to x_0} \lim_{y \to y_0} f(x, y), \quad \text{if it exists}$$

$$\lim_{y \to y_0} \lim_{x \to x_0} f(x, y), \quad \text{if it exists}$$

are called the iterated limits of f at (x_0, y_0). Then, the limit

$$\lim_{(x,y) \to (0,0)} f(x, y), \quad \text{if it exists}$$

is called the (double) limit of f at (x_0, y_0).

Question: What is the relation between the existence of these three limits? - Analyze.

Examples

- Let $f(x,y) = \frac{x^2}{x^2+y^2}$ for $(x,y) \neq (0,0)$.

$$\begin{aligned}
\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) &= \lim_{x\to 0} 1 = 1 \\
\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) &= \lim_{y\to 0} 0 = 0
\end{aligned}$$

Examples

- Let $f(x,y) = \frac{x^2}{x^2+y^2}$ for $(x,y) \neq (0,0)$.

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = \lim_{x\to 0} 1 = 1$$
$$\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) = \lim_{y\to 0} 0 = 0$$

$\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Examples

- Let $f(x,y) = \frac{x^2}{x^2+y^2}$ for $(x,y) \neq (0,0)$.

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = \lim_{x\to 0} 1 = 1$$

$$\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) = \lim_{y\to 0} 0 = 0$$

$\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

- Let $f(x,y) = \frac{x^2y^2}{x^2y^2+(x-y)^2}$ if $x^2y^2 + (x-y)^2 \neq 0$.

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = \lim_{x\to 0} 0 = 0$$

$$\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) = \lim_{y\to 0} 0 = 0$$

Examples

- Let $f(x,y) = \frac{x^2}{x^2+y^2}$ for $(x,y) \neq (0,0)$.

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = \lim_{x\to 0} 1 = 1$$

$$\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) = \lim_{y\to 0} 0 = 0$$

$\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

- Let $f(x,y) = \frac{x^2y^2}{x^2y^2+(x-y)^2}$ if $x^2y^2 + (x-y)^2 \neq 0$.

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = \lim_{x\to 0} 0 = 0$$

$$\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) = \lim_{y\to 0} 0 = 0$$

Still $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Note on Iterated Limits and (Double) Limit

Important Note:

- Existence of the (double) limit does NOT guarantee existence of iterated limits.
- Existence of the iterated limits do NOT guarantee existence of (double) limit.
- If (double) limit and iterated limits exist then they are all equal.

Continuity of function $F : S \subseteq \mathbb{R}^2 \to \mathbb{R}$

Definition

Let $F : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a function. Let $X_0 \in S$. We say that F is continuous at the point X_0 if

Continuity of function $F : S \subseteq \mathbb{R}^2 \to \mathbb{R}$

Definition

Let $F : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a function. Let $X_0 \in S$. We say that F is continuous at the point X_0 if for each $\epsilon > 0$, there exists a real number $\delta > 0$ such that

Continuity of function $F : S \subseteq \mathbb{R}^2 \to \mathbb{R}$

Definition

Let $F : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a function. Let $X_0 \in S$. We say that F is continuous at the point X_0 if for each $\epsilon > 0$, there exists a real number $\delta > 0$ such that

$$X \in S, \quad \|X - X_0\| < \delta \quad \Rightarrow \quad |F(X) - F(X_0)| < \epsilon.$$

BENNETT UNIVERSITY
A TIMES GROUP INITIATIVE

Continuity of function $F : S \subseteq \mathbb{R}^2 \to \mathbb{R}$

Definition

Let $F : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a function. Let $X_0 \in S$. We say that F is continuous at the point X_0 if for each $\epsilon > 0$, there exists a real number $\delta > 0$ such that

$$X \in S, \quad \|X - X_0\| < \delta \quad \Rightarrow \quad |F(X) - F(X_0)| < \epsilon.$$

If F is continuous at each point of S then we say that F is continuous in the set S.

Continuity of function $F : S \subseteq \mathbb{R}^2 \to \mathbb{R}$

Definition

Let $F : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a function. Let $X_0 \in S$. We say that F is continuous at the point X_0 if for each $\epsilon > 0$, there exists a real number $\delta > 0$ such that

$$X \in S, \quad \|X - X_0\| < \delta \quad \Rightarrow \quad |F(X) - F(X_0)| < \epsilon.$$

If F is continuous at each point of S then we say that F is continuous in the set S.

NOTE: Limit and Continuity are closely related. Justify ?

Comparison between limit and continuity

- In continuity function must be defined at the point X_0, that is, X_0 must be an element of the domain S,

Comparison between limit and continuity

- In continuity function must be defined at the point X_0, that is, X_0 must be an element of the domain S, while in limit X_0 is a limit point of set S and it need not belong to the set S.

BENNETT UNIVERSITY
A TIMES GROUP INITIATIVE

Comparison between limit and continuity

- In continuity function must be defined at the point X_0, that is, X_0 must be an element of the domain S, while in limit X_0 is a limit point of set S and it need not belong to the set S.
- In continuity limit of the function at X_0 must be equal to $F(X_0)$ (that is, the function value at X_0).

Comparison between limit and continuity

- In continuity function must be defined at the point X_0, that is, X_0 must be an element of the domain S, while in limit X_0 is a limit point of set S and it need not belong to the set S.
- In continuity limit of the function at X_0 must be equal to $F(X_0)$ (that is, the function value at X_0). While in limit it need not be.

BENNETT UNIVERSITY
A TIMES GROUP INITIATIVE

Comparison between limit and continuity

- In continuity function must be defined at the point X_0, that is, X_0 must be an element of the domain S, while in limit X_0 is a limit point of set S and it need not belong to the set S.
- In continuity limit of the function at X_0 must be equal to $F(X_0)$ (that is, the function value at X_0). While in limit it need not be.
- In continuity at X_0, we have for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$X \in S \text{ and } \|X - X_0\| < \delta \quad \Rightarrow \quad |F(X) - F(X_0)| < \epsilon.$$

BENNETT UNIVERSITY
A TIMES GROUP INITIATIVE

Comparison between limit and continuity

- In continuity function must be defined at the point X_0, that is, X_0 must be an element of the domain S, while in limit X_0 is a limit point of set S and it need not belong to the set S.
- In continuity limit of the function at X_0 must be equal to $F(X_0)$ (that is, the function value at X_0). While in limit it need not be.
- In continuity at X_0, we have for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$X \in S \text{ and } \|X - X_0\| < \delta \quad \Rightarrow \quad |F(X) - F(X_0)| < \epsilon.$$

While in limit at X_0 is L, then we have for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$X \in S \text{ and } 0 < \|X - X_0\| < \delta \quad \Rightarrow \quad |F(X) - L| < \epsilon.$$

We can similarly define continuity of a function $F : S \subseteq \mathbb{R}^n \to \mathbb{R}$.

Definition

Continuity of function $F : S \subseteq \mathbb{R}^n \to \mathbb{R}$

We can similarly define continuity of a function $F : S \subseteq \mathbb{R}^n \to \mathbb{R}$.

Definition

Let $F : S \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function. Let $X_0 \in S$. We say that F is continuous at the point X_0 if

Continuity of function $F : S \subseteq \mathbb{R}^n \to \mathbb{R}$

We can similarly define continuity of a function $F : S \subseteq \mathbb{R}^n \to \mathbb{R}$.

Definition

Let $F : S \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function. Let $X_0 \in S$. We say that F is continuous at the point X_0 if for each $\epsilon > 0$, there exists a real number $\delta > 0$ such that

Continuity of function $F : S \subseteq \mathbb{R}^n \to \mathbb{R}$

We can similarly define continuity of a function $F : S \subseteq \mathbb{R}^n \to \mathbb{R}$.

Definition

Let $F : S \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function. Let $X_0 \in S$. We say that F is continuous at the point X_0 if for each $\epsilon > 0$, there exists a real number $\delta > 0$ such that

$$X \in S, \quad \|X - X_0\| < \delta \quad \Rightarrow \quad |F(X) - F(X_0)| < \epsilon.$$

BENNETT UNIVERSITY
A TIMES GROUP INITIATIVE

Continuity of function $F : S \subseteq \mathbb{R}^n \to \mathbb{R}$

We can similarly define continuity of a function $F : S \subseteq \mathbb{R}^n \to \mathbb{R}$.

Definition

Let $F : S \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function. Let $X_0 \in S$. We say that F is continuous at the point X_0 if for each $\epsilon > 0$, there exists a real number $\delta > 0$ such that

$$X \in S, \quad \|X - X_0\| < \delta \quad \Rightarrow \quad |F(X) - F(X_0)| < \epsilon.$$

If F is continuous at each point of S then we say that F is continuous in the set S.

Continuity of function $F : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$

We can similarly define continuity of a function $F : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$.

Continuity of function $F : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$

We can similarly define continuity of a function $F : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$.

Theorem

Let $F : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a function. Then, $F(X)$ can be written as

$$F(X) = (f_1(X), f_2(X), \ldots, f_m(X)) \quad \text{for } X \in S,$$

where $f_i (1 \leq i \leq m)$ are component functions of F.

Continuity of function $F : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$

We can similarly define continuity of a function $F : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$.

Theorem

Let $F : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a function. Then, $F(X)$ can be written as

$$F(X) = (f_1(X), f_2(X), \ldots, f_m(X)) \quad \text{for } X \in S,$$

where $f_i (1 \leq i \leq m)$ are component functions of F.
Let $X_0 \in S$. Then F is continuous at X_0, i.e.,

Continuity of function $F : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$

We can similarly define continuity of a function $F : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$.

Theorem

Let $F : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a function. Then, $F(X)$ can be written as

$$F(X) = (f_1(X), f_2(X), \ldots, f_m(X)) \quad \text{for } X \in S,$$

where $f_i (1 \leq i \leq m)$ are component functions of F.
Let $X_0 \in S$. Then F is continuous at X_0, i.e.,

$$\lim_{X \to X_0} F(X) = F(X_0)$$

Continuity of function $F : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$

We can similarly define continuity of a function $F : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$.

Theorem

Let $F : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a function. Then, $F(X)$ can be written as

$$F(X) = (f_1(X), f_2(X), \ldots, f_m(X)) \quad \text{for } X \in S,$$

where $f_i (1 \leq i \leq m)$ are component functions of F.
Let $X_0 \in S$. Then F is continuous at X_0, i.e.,

$$\lim_{X \to X_0} F(X) = F(X_0)$$

if and only if

Continuity of function $F : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$

We can similarly define continuity of a function $F : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$.

Theorem

Let $F : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a function. Then, $F(X)$ can be written as

$$F(X) = (f_1(X), f_2(X), \ldots, f_m(X)) \quad \text{for } X \in S,$$

where $f_i (1 \leq i \leq m)$ are component functions of F.
Let $X_0 \in S$. Then F is continuous at X_0, i.e.,

$$\lim_{X \to X_0} F(X) = F(X_0)$$

if and only if

$f_i(X)$ is continuous at X_0 for each i, i.e.,

Continuity of function $F : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$

We can similarly define continuity of a function $F : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$.

Theorem

Let $F : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a function. Then, $F(X)$ can be written as

$$F(X) = (f_1(X), f_2(X), \ldots, f_m(X)) \quad \text{for } X \in S,$$

where $f_i (1 \leq i \leq m)$ are component functions of F.
Let $X_0 \in S$. Then F is continuous at X_0, i.e.,

$$\lim_{X \to X_0} F(X) = F(X_0)$$

if and only if

$f_i(X)$ is continuous at X_0 for each i, i.e.,

$$\lim_{X \to X_0} f_i(X) = f_i(X_0) \quad \text{for } 1 \leq i \leq m.$$

Properties of continuous function

Results for continuity of the function similar to limit of the function.

Properties of continuous function

Results for continuity of the function similar to limit of the function.

- Sequential criteria for continuity at $X_0 \in S$.

Properties of continuous function

Results for continuity of the function similar to limit of the function.

- Sequential criteria for continuity at $X_0 \in S$.
- Algebra of continuous functions.

Properties of continuous function

Results for continuity of the function similar to limit of the function.

- Sequential criteria for continuity at $X_0 \in S$.
- Algebra of continuous functions.
- F is continuous if and only if all component functions of F are continuous.