Answers

Solution 1: $\int_{-\infty}^{\infty} e^{-t^2}\,dt$

$$= 2\int_0^{\infty} e^{-t^2}\,dt$$

Put $t^2 = x \Rightarrow 2t\,dt = dx$

$$= 2\int_0^{\infty} e^{-x}\cdot\frac{dx}{2\sqrt{x}} = \int_0^{\infty} e^{-x}\,x^{-1/2}\,dx$$

$$= \int_0^{\infty} e^{-x}\cdot x^{\frac{1}{2}-1}\,dx$$

$$= \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Thus $\boxed{\int_{-\infty}^{\infty} e^{-t^2}\,dt = \sqrt{\pi}}$

Solution 2: Let $f(x) = \cos x$, $a = \frac{\pi}{2}$.

The Taylor's expansion of $f(x)$ about the point $a = \frac{\pi}{2}$ is

$$f(x) = f\left(\frac{\pi}{2}\right) + f'\left(\frac{\pi}{2}\right)\cdot\left(x-\frac{\pi}{2}\right) + f''\left(\frac{\pi}{2}\right)\cdot\frac{\left(x-\frac{\pi}{2}\right)^2}{2!} + \cdots$$

We have $f\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$

$f'(x) = -\sin x,\quad f'\left(\frac{\pi}{2}\right) = -1$

$f''(x) = -\cos x,\quad f''\left(\frac{\pi}{2}\right) = 0$

$f'''(x) = \sin x,\quad f'''\left(\frac{\pi}{2}\right) = 1$

$f^{(iv)}(x) = \cos x,\quad f^{(iv)}\left(\frac{\pi}{2}\right) = 0$

$f^{(v)}(x) = -\sin x,$

$f^{(v)}\left(\frac{\pi}{2}\right) = -1$

Therefore, first three non-zero terms of the Taylor's expansion of $f(x)$ about $\frac{\pi}{2}$ is

$$f(x) = -\left(x-\frac{\pi}{2}\right) + \frac{\left(x-\frac{\pi}{2}\right)^3}{3!} - \frac{\left(x-\frac{\pi}{2}\right)^5}{5!}$$

Solution 3: Since f is Riemann integrable on $[0,1]$, f is bounded on $[0,1]$. So there exists $M>0$ such that

$$|f(x)| \leq M \quad \forall \; x \in [0,1].$$

Now,
$$\left|\int_0^1 x^n f(x)\,dx\right| \leq \int_0^1 |x^n f(x)|\,dx$$
$$\leq M \int_0^1 x^n\,dx = \frac{M}{n+1} \to 0 \text{ as } n \to \infty.$$

Hence, $$\lim_{n\to\infty} \int_0^1 x^n f(x)\,dx = 0.$$

Solution 4: Consider the sequence of partitions $\{P_n\}$, where
$$P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \text{—}, \frac{n-1}{n}, 1\right\},$$
then $$\|P_n\| = \frac{1}{n} \to 0 \text{ as } n \to \infty.$$

$$U(P_n,f) = M_1(x_1-x_0) + M_2(x_2-x_1) + \text{——} + M_n(x_n - x_{n-1})$$
$$= 2\cdot\frac{1}{n} + 2\cdot\frac{1}{n} + \text{——} + 2\cdot\frac{1}{n}$$
$$= 2$$

$$L(P_n, f) = m_1(x_1-x_0) + m_2(x_2-x_1) + \text{——} + m_n(x_n-x_{n-1})$$
$$= 1\cdot\frac{1}{n} + 1\cdot\frac{1}{n} + \text{——} + 1\cdot\frac{1}{n}$$
$$= 1$$

So, $\lim_{n\to\infty} \left(U(P_n,f) - L(P_n,f) \right) = 1$.

Hence f is not Riemann integrable.

Solution 5 (a) $\displaystyle\int_1^{\infty} \frac{x+1}{x^{3/2}}\, dx = \int_1^{\infty} \frac{1}{\sqrt{x}}\, dx + \int_1^{\infty} \frac{1}{x^{3/2}}\, dx$

and $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{x}}\, dx$ diverges.

Hence the given integral diverges.

(b) Let $f(x) = \dfrac{\sin(x^2)}{\sqrt{x}}$ and $g(x) = \dfrac{1}{\sqrt{x}}$.

Then $\displaystyle\lim_{x\to 0} \frac{f(x)}{g(x)} = 0$.

Also, $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$ is convergent.

Hence $\displaystyle\int_0^1 \frac{\sin(x^2)}{\sqrt{x}}\, dx$ is convergent.