SET Theory

Sets

- Common Universal Sets

  - R = reals
  - N = natural numbers = {0,1, 2, 3, . . . }, the *counting* numbers
  - Z = all integers = {. . , -3, -2, -1, 0, 1, 2, 3, 4, . .}
  - Z^+ is the set of positive integers

- Notation:

  x is a member of S or x is an element of S:

  $$x \in S.$$

  x is not an element of S:

  $$x \notin S.$$

Sets

- A set is a collection or group of objects or *elements* or *members*. (Cantor 1895)
  - A set is said to contain its elements.
  - There must be an underlying universal set U, either specifically stated or understood.

Sets

- Notation:
  - list the elements between braces:

    $$S = \{a, b, c, d\}=\{b, c, a, d, d\}$$

    (Note: listing an object more than once does not change the set. Ordering means nothing.)
  - specification by predicates:

    $$S= \{x|\ P(x)\},$$

    S contains all the elements from U which make the predicate P true.
  - brace notation with ellipses:

    $$S = \{ . . . , -3, -2, -1\},$$

    the negative integers.

Sets

Subsets

- Definition: The set A is a *subset* of the set B, denoted $A \subseteq B$, iff

$$\forall x\,[x \in A \rightarrow x \in B]$$

- Definition: The *void* set, the *null* set, the *empty* set, denoted $\varnothing$, is the set with no members.

Note: the assertion $x \in \varnothing$ is <u>always</u> false. Hence

$$\forall x\,[x \in \varnothing \rightarrow x \in B]$$

is always true(vacuously). Therefore, $\varnothing$ is a subset of every set.

Note: A set B is always a subset of itself.

Sets

- Definition: If $A \subseteq B$ but $A \neq B$ the we say A is a *proper* subset of B, denoted $A \subset B$ (in some texts).
- Definition: The set of all subset of a set A, denoted P(A), is called the *power set* of A.
- Example: If $A = \{a, b\}$ then

  $P(A) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$

Sets

- Definition: The number of (distinct) elements in A, denoted |A|, is called the *cardinality* of A.

  If the cardinality is a natural number (in N), then the set is called *finite*, else *infinite*.

- Example: A = {a, b},

  |{a, b}| = 2,

  |P({a, b})| = 4.

  A is finite and so is P(A).

  Useful Fact: |A|=n implies |P(A)| = 2^n

Sets

- N is infinite since |N| is not a natural number. It is called a *transfinite cardinal number*.
- Note: Sets can be both members and subsets of other sets.
- Example:

  A = {∅,{∅}}.

  A has two elements and hence four subsets:

  ∅, {∅}, {{∅}}. {∅,{∅}}

  Note that ∅ is both a member of A and a subset of A!
- Russell's paradox: Let S be the set of all sets which are not members of themselves. Is S a member of itself?
- Another paradox: Henry is a barber who shaves all people who do not shave themselves. Does Henry shave himself?

Sets

- Definition: The *Cartesian product* of A with B, denoted

  A x B, is the set of <u>ordered pairs</u> {<a, b> | a ∈ A ∧ b ∈ B}

  Notation:

$$\underset{i=1}{\overset{n}{\times}} A_i = \{ < a_1, a_2, \ldots, a_n > a_i \in A_i \}$$

  Note: The Cartesian product of anything with ⊘ is ⊘. (why?)

  - Example:

    A = {a,b}, B = {1, 2, 3}

    AxB = {<a, 1>, <a, 2>, <a, 3>, <b, 1>, <b, 2>, <b, 3>}

    What is BxA? AxBxA?

  - If |A| = m and |B| = n, what is |AxB|?

Set Operations

- Propositional calculus and set theory are both instances of an algebraic system called a

Boolean Algebra.

The operators in set theory are defined in terms of the corresponding operator in propositional calculus

As always there must be a universe U. All sets are assumed to be subsets of U

Set Operations

- Definition:

  Two sets A and B are equal, denoted $A = B$, iff

  $$\forall x\, [x \in A \leftrightarrow x \in B].$$

  - Note: By a previous logical equivalence we have

$$A = B \text{ iff } \forall x\, [(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$$

or

$$A = B \text{ iff } A \subseteq B \text{ and } B \subseteq A$$

Set Operations

- Definitions:
  - The *union* of A and B, denoted A U B, is the set $\{x \mid x \in A \vee x \in B\}$
  - The *intersection* of A and B, denoted A ∩ B, is the set

    $$\{x \mid x \in A \wedge x \in B\}$$

    Note: If the intersection is void, A and B are said to be *disjoint*.
  - The *complement* of A, denoted $\overline{A}$, is the set $\{x \mid \neg(x \in A)\}$

    Note: Alternative notation is A^c, and $\{x|x \notin A\}$.
  - The *difference* of A and B, or the *complement* of B *relative* to A, denoted A - B, is the set $A \cap \overline{B}$

    Note: The (absolute) complement of A is U - A.
  - The *symmetric difference* of A and B, denoted $A \oplus B$, is the set

    $$(A - B) \cup (B - A)$$

Set Operations

- **Examples:**
  $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

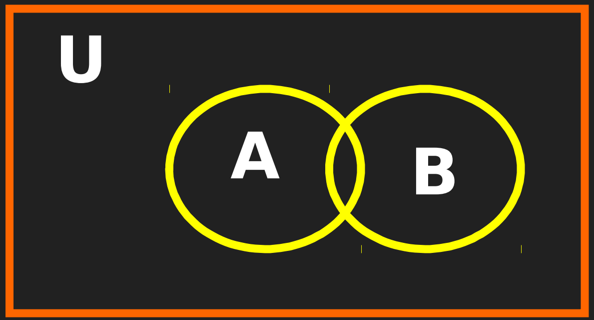
  $A = \{1, 2, 3, 4, 5\}$,
  $B = \{4, 5, 6, 7, 8\}$. Then

  - $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$
  - $A \cap B = \{4, 5\}$
  - $\overline{A} = \{0, 6, 7, 8, 9, 10\}$
  - $\overline{B} = \{0, 1, 2, 3, 9, 10\}$
  - $A - B = \{1, 2, 3\}$
  - $B - A = \{6, 7, 8\}$
  - $A \oplus B = \{1, 2, 3, 6, 7, 8\}$

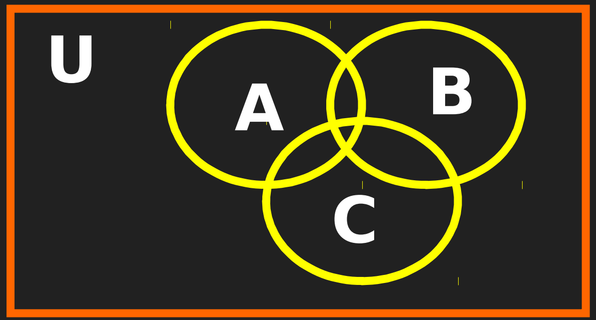
Set Operations

Venn Diagrams

- A useful geometric visualization tool (for 3 or less sets)
- The Universe U is the rectangular box
- Each set is represented by a circle and its interior
- All possible combinations of the sets must be represented



For 2 sets



For 3 sets

- Shade the appropriate region to represent the given set operation.

Set Operations

- Set Identities
  - Set identities correspond to the logical equivalences.
  - Example:

The complement of the union is the intersection of the complements:

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

Proof: To show:

$$\forall x\, [x \in \overline{A \cup B} \leftrightarrow x \in \overline{A} \cap \overline{B}]$$

To show two sets are equal we show for all x that x is a member of one set if and only if it is a member of the other.

- We say

'Let x be arbitrary.'

Then we can treat the predicates as propositions:

Assertion	Reason
$x \in \overline{A \cup B} \Leftrightarrow x \notin [A \cup B]$	Def. of complement
$x \notin A \cup B \Leftrightarrow \neg[x \in A \cup B]$	Def. of $\notin$
$\Leftrightarrow \neg[x \in A \vee x \in B]$	Def. of union
$\Leftrightarrow \neg x \in A \wedge \neg x \in B$	DeMorgan's Laws
$\Leftrightarrow x \notin A \wedge x \notin B$	Def. of $\notin$
$\Leftrightarrow x \in \overline{A} \wedge x \in \overline{B}$	Def. of complement
$\Leftrightarrow x \in \overline{A} \cap \overline{B}$	Def. of intersection

Set Operations

- Note: As an alternative which might be easier in some cases, use the identity

$$A = B \Leftrightarrow [A \subseteq B \text{ and } B \subseteq A]$$

- Example:

  Show $A \cap (B - A) = \emptyset$

  The void set is a subset of every set. Hence,

$$A \cap (B - A) \supseteq \emptyset$$

  Therefore, it suffices to show

$$A \cap (B - A) \subseteq \emptyset \qquad \text{or} \qquad \forall x\, [x \in A \cap (B - A) \rightarrow x \in \emptyset]$$

  So as before we say 'let x be arbitrary'.