Find $L^{-1}\left\{\frac{s^2}{(s+3)^3}\right\}$

$$= L^{-1}\left\{\frac{1}{s+3} - \frac{6}{(s+3)^2} + \frac{9}{(s+3)^3}\right\}$$

$$= e^{-3t} - 6te^{-3t} + \frac{9}{2}t^2e^{-3t}$$

$$\boxed{\frac{s^2}{(s+3)^3} = \frac{A}{s+3} + \frac{B}{(s+3)^2} + \frac{C}{(s+3)^3}}$$

Application of Laplace transform in solving ODE

$$\frac{dx}{dt} + 3x = 0, \quad x(0) = 1.$$

Taking Laplace transform on both sides,

$$L[x'(t)] + 3L[x(t)] = L[0]$$

$$\Rightarrow sL[x(t)] - x(0) + 3L[x(t)] = 0$$

$$\boxed{L[f'(t)] = sF(s) - f(0), \text{ where } F(s) = L[f(t)]}$$

$$\Rightarrow \quad (s+3)\,L[x(t)] - 1 = 0$$

$$\Rightarrow \quad L[x(t)] = \frac{1}{s+3}$$

Take inverse Laplace transform on both the sides,

$$x(t) = L^{-1}\left(\frac{1}{s+3}\right) = e^{-3t}$$

$$\boxed{x(t) = e^{-3t}}$$

Solve $y' - 3y = 4e^{5t}$, $y(0) = 6$.

Take Laplace transform on both the sides

$$s\,L[y(t)] - \underline{y(0)} - 3L[y(t)] = 4\,L[e^{5t}]$$

$$\Rightarrow \quad s\,L[y(t)] - 6 - 3L[y(t)] = 4\left(\frac{1}{s-5}\right)$$

$$\Rightarrow (s-3)\,L[y(t)] = \frac{4}{s-5} + 6$$

$$\Rightarrow L[y(t)] = \frac{4}{(s-5)(s-3)} + \frac{6}{s-3}$$

$$\Rightarrow y(t) = L^{-1}\left[\frac{4}{(s-5)(s-3)}\right] + 6L^{-1}\left[\frac{1}{s-3}\right]$$

$$= L^{-1}\left[\frac{2}{s-5} - \frac{2}{s-3}\right] + 6L^{-1}\left[\frac{1}{s-3}\right]$$

$$\Rightarrow \boxed{y(t) = 2e^{5t} - 2e^{3t} + 6e^{3t}}$$

$\frac{d^2x}{dt^2} + x = t, \quad x(0)=1, \; x'(0) = -2$

Take Laplace transform on both the sides,

$$\underline{L[x''(t)]} + L[x(t)] = L[t]$$

$$\Rightarrow s^2 L[x(t)] - s\,x(0) - x'(0) + L[x(t)] = \frac{1}{s^2}$$

$$\boxed{L[f''(t)] = s^2F(s) - sf(0) - f'(0)}$$

$$(s^2+1)\,L[x(t)] - s + 2 = \frac{1}{s^2}$$

$$\Rightarrow (s^2+1)\,L[x(t)] = \frac{1}{s^2} + s - 2$$

$$\Rightarrow \quad L[x(t)] = \frac{1}{s^2(s^2+1)} + \frac{s}{s^2+1} - \frac{2}{s^2+1}$$

$$\Rightarrow x(t) = L^{-1}\left[\underline{\frac{1}{s^2(s^2+1)}} + \frac{s}{s^2+1} - \frac{2}{s^2+1}\right]$$

$$= L^{-1}\left[\frac{1}{s^2} - \frac{1}{s^2+1} + \frac{s}{s^2+1} - \frac{2}{s^2+1}\right]$$

$$= L^{-1}\left[\frac{1}{s^2} - \frac{3}{s^2+1} + \frac{s}{s^2+1}\right]$$

$$= L^{-1}\left(\frac{1}{s^2}\right) - 3L^{-1}\left[\frac{1}{s^2+1}\right] + L^{-1}\left[\frac{s}{s^2+1}\right]$$

$$\boxed{y(t) = t - 3\sin t + \cos t}$$