① $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}$.

Solⁿ: Characterstic polynomial = $\det(A-\lambda I)$

$$\det(A-\lambda I) = \begin{vmatrix} 4-\lambda & 1 & -1 \\ 2 & 5-\lambda & -2 \\ 1 & 1 & 2-\lambda \end{vmatrix}$$

Char. Poly. = $\lambda^3 - 11t^2 + 39t - 45$

Ch. equation $\rightarrow$ $\lambda^3 - 11t^2 + 39t - 45 = 0$

Eigenvalues — Roots of ch. equation.

Now, $\lambda^3 - 11\lambda^2 + 39\lambda - 45 = 0$

$\Rightarrow (\lambda-3)(\lambda^2 - 8\lambda + 15) = 0$

$\Rightarrow (\lambda-3)(\lambda-5)(\lambda-3) = 0$

$\Rightarrow \lambda = 3, 3, 5.$

Algebraic multiplicity of $\lambda = 3$ $= 2$.

Algebraic multiplicity of $\lambda = 5$ $= 1$.

E_λ = Eigenvector corresponding to $\lambda = 3$ = $\{X : (A-\lambda I)X = 0\}$

$$E_3 = \left\{ X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : (A-3I)X = 0 \right\}$$

$$A - 3I = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 1 & 1 & -1 \end{bmatrix}. \qquad \text{R.EF of } (A-3I) = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A-3I)X \Rightarrow \quad x + y - z = 0.$$

Here y, z are free variable. $y = t$, $z = s$

$$x = -y + z = -t + s$$

$$\therefore \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -t+s \\ t \\ s \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

G.M = $\dim(E_\lambda) = 2$., As.

$$E_\lambda = \left\{ t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} : t, s \in \mathbb{R} \right\}$$

$$= \operatorname{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

G.M of $\lambda = 3$ = 2 = A.M of eigenvalue 3.

$$E_5 = \left\{ X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : (A - 5I)X = 0 \right\}$$

$$A - 5I = \begin{bmatrix} -1 & 1 & -1 \\ 2 & 0 & -2 \\ 1 & 1 & -3 \end{bmatrix}$$

R.EF of $(A-5I) :-$

$R_2 \to R_2 + 2R_1$

$R_3 \to R_3 + R_1$

$$\begin{bmatrix} -1 & 1 & -1 \\ 0 & 2 & -4 \\ 0 & 2 & -4 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} -1 & 1 & -1 \\ 0 & 2 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

Here z is free variable. $z = t$, $t \in \mathbb{R}$.

From. $(A-5I)X = 0 \implies -x + y - z = 0$

$\cancel{y} - 2z = 0 \implies y = 2z = 2t$

$$x = z = t$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Thus, $E_5 = \left\{ t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \ t \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$

$\dim E_5 = \underline{1}$.

G.M of E.V 5 = 1 = A.M of eigenvalue 5.

Thus A is diagonalizable.

$$P = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

& $$D = P^{-1} A P = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

$$f(A) = P\, f(D)\, P^{-1}$$

$$= P \begin{bmatrix} f(3) & 0 & 0 \\ 0 & f(3) & 0 \\ 0 & 0 & f(5) \end{bmatrix} P^{-1}.$$

Example :-
(i)

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 3 \\ -1 & 0 & 3 \end{bmatrix}.$$

$\underline{\text{Sol}^n}$: First, we have $tI - A = \begin{bmatrix} t-1 & 0 & 1 \\ -1 & t-1 & 3 \\ -1 & 0 & t-3 \end{bmatrix}$

So,

$$\text{Ch. poly } p(t) = \det(tI - A)$$

$$= (t-1)\begin{vmatrix} t-1 & 3 \\ 0 & t-3 \end{vmatrix} + (-1)\begin{vmatrix} 1 & t-1 \\ 1 & 0 \end{vmatrix}$$

$$= (t-1)^2(t-3) + (t-1)$$

$$= (t-1)\left[(t-1)(t-3) + 1\right]$$

$$= (t-1)(t-2)^2$$

Thus, the eigenvalues are $1, 2, 2$.

\# For $\lambda = 1$, we want to find the null space of $(A - I)$

i.e. $A - I = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 1 & 0 & -3 \end{bmatrix}$.

Find X s.t $(A - I)X = 0$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Row echelon of $A - I = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

Free variable is x_2. $(A - I)X = 0 \Rightarrow x_1 = 0,\ x_3 = 0,\ x_2 = t$

Thus, sol^n is $\begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

2

Example :-
(i)

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 3 \\ -1 & 0 & 3 \end{bmatrix}.$$

Sol^n :- First, we have $tI - A = \begin{bmatrix} t-1 & 0 & 1 \\ -1 & t-1 & 3 \\ -1 & 0 & t-3 \end{bmatrix}$

So,

Ch. poly $p(t) = \det(tI - A)$

$$= (t-1)\begin{vmatrix} t-1 & 3 \\ 0 & t-3 \end{vmatrix} + (-1)\begin{vmatrix} 1 & t-1 \\ 1 & 0 \end{vmatrix}$$

$$= (t-1)^2(t-3) + (t-1)$$

$$= (t-1)\left[(t-1)(t-3) + 1\right]$$

$$= (t-1)(t-2)^2$$

Thus, the eigenvalues are $1, 2, 2$.

For $\lambda = 1$, we want to find the null space of $(A - I)$.

i.e $A - I = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 1 & 0 & -3 \end{bmatrix}.$

Find X s.t $(A - I)X = 0$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Row echelon of $A - I = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

Free variable is x_2. $(A - I)X = 0 \Rightarrow x_1 = 0, x_3 = 0, x_2 = t$

Thus, sol^n is $\begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Thus, eigenspace corresponding to eigenvalue 1

is $= \left\{ \alpha \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} : \alpha \in \mathbb{R} \right\}$

$\dim E_1 = 1$.

For $\lambda = 2$, we want to find the eigenspace.

i.e. Find null space of $A - 2I$. $= \{ X : (A-2I) X = 0 \}$,

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$A - 2I = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & -3 \\ 1 & 0 & -1 \end{bmatrix}$$

Row echelon form, $R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

free variable is x_3. Take $x_3 = t$. Then $(A-2I) X = 0$

$\Rightarrow$ $x_2 - 2x_3 = 0 \Rightarrow x_2 = 2x_3 = 2t$

$x_1 - x_3 = 0 \Rightarrow x_1 = x_3 = t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Thus, 2- eigenspace is 1-dimensional & spanned by $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

Algebraic Multiplicity — corresponding to "1" is 1.
"2" is 2.

Geometric Multiplicity corresponding to "1" is 1
" " "2" is 1.

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Since Algebraic Multiplicity $\neq$ Geometric Multiplicity.

Therefore, A is not diagonalizable. So, we can not find P.

(ii) Calculate A^{-1}.

As we know by Caley Hamilton Theorem, Every square matrix satisfy its characterstic eqn.

$$\therefore \quad (A-I)(A-2I)^2 = 0$$

$$\Rightarrow (A-I)(A^2+4I-4A) = 0$$

$$\Rightarrow A^3 + 4A - 4A^2 - A^2 - 4I + 4A = 0$$

$$\Rightarrow A^3 - 5A^2 + 8A - 4I = 0$$

Multiplying by A^{-1}, we obtain

$$A^2 - 5A + 8I - 4A^{-1} = 0$$

$$\Rightarrow A^{-1} = \frac{1}{4}\left[A^2 - 5A + 8I\right]. \text{ Ans}$$

Example:
② (ii)

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Sol^n:

$$tI - A = \begin{bmatrix} t & 0 & 0 \\ -1 & t & 1 \\ 0 & -1 & t \end{bmatrix}.$$

So, characteristic polynomial: $p(t) = \det(tI-A)$

$$= t \begin{vmatrix} t & 1 \\ -1 & t \end{vmatrix} = t(t^2+1)$$

So, ch. eq^n is $t(t^2+1) = 0$

Roots of ch. eq^n (eigenvalues) $= 0, \pm i$.

Eigen vector corresponding to eigenvalue λ="0".

i.e we want to find null space of $(\lambda I - A) = -A$.

$$-A = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

Row echelon form of $A \rightarrow$ $R_1 \leftrightarrow R_2$ $\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$

$R_3 \leftrightarrow R_2$ $\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$R_1 \rightarrow -R_1$
$R_2 \rightarrow -R_2$ $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$,

$x_3 =$ free variable.

$x_2 = 0$

$x_1 - x_3 = 0 \Rightarrow x_1 = x_3$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

So, the 0- eigenspace is 1- dimensional & spanned by $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

For $\lambda = i$, we want to find the nullspace of $(iI - A)$.

$$iI - A = \begin{bmatrix} i & 0 & 0 \\ -1 & i & 1 \\ 0 & -1 & i \end{bmatrix}$$

Row echelon form of $iI - A$, $R_2 \leftrightarrow R_1$ $\begin{bmatrix} -1 & i & 1 \\ i & 0 & 0 \\ 0 & -1 & i \end{bmatrix}$

$R_1 \to -R_1$ $\begin{bmatrix} 1 & -i & -1 \\ i & 0 & 0 \\ 0 & -1 & i \end{bmatrix}$

$R_2 \to R_2 - iR_1$ $\begin{bmatrix} 1 & -i & -1 \\ 0 & -1 & i \\ 0 & -1 & i \end{bmatrix}$

$R_3 \to R_3 - R_1$ $\begin{bmatrix} 1 & i & -1 \\ 0 & -1 & i \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & i & -1 \\ 0 & -1 & +i \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Thus, $x_3 = t$, $-x_2 + ix_3 = 0 \Rightarrow x_2 = +ix_3$, $x_1 + ix_2 + x_3 = 0$

$x_1 = -ix_2 + x_3 = 0$

Thus
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ it \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$$

Hence E_i = Eigenspace corresponding to $i = \left\{ t \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$

#. $\boxed{\dim E_i = 1.}$

For $\lambda = -i$, we want to find x s.t. $(A + iI)x = 0$.

$$A + iI = \begin{bmatrix} -i & 0 & 0 \\ -1 & -i & 1 \\ 0 & -1 & -i \end{bmatrix}$$

Row echelon form of $A + iI$ $\underset{R_2 \leftrightarrow R_1}{=}$
$$\begin{bmatrix} -1 & -i & 1 \\ -i & 0 & 0 \\ 0 & -1 & -i \end{bmatrix}$$

$R_2 \to iR_2 + R_1$
$$\begin{bmatrix} -1 & -i & 1 \\ 0 & -i & 1 \\ 0 & -1 & -i \end{bmatrix}$$

$R_3 \to R_3 + iR_2$
$$\begin{bmatrix} -1 & -i & 1 \\ 0 & -i & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

x_3 = free variable. $x_3 = t$

Then
$$\begin{bmatrix} -1 & -i & 1 \\ 0 & -i & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -x_1 - ix_2 + x_3 = 0$, $-ix_2 + x_3 = 0$

$x_2 = -ix_3$

$\Rightarrow x_1 = 0$

Thus, eigenvector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$.

$\dim(E_{-i}) = 1.$

Since $A \cdot M = G \cdot M = 1$ for every eigenvalue.

Hence, A is diagonalizable

(b) By calyley Hamilton Thm, $A(A^2+I)=0$ $\quad$ ($\because$ ch. eq^n $t(t^2+1)=0$)

$$\Rightarrow A^3 + A = 0$$

Since 0 is an eigenvalue of. A. Therefore $\det A = 0 \Rightarrow A$ is not invertible. & hence, we can not find the inverse.

$$P = \begin{array}{c} \begin{array}{ccc} \lambda=0 & \lambda=i & \lambda=-i \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & -i \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

. One can easily verify that

$$P^{-1} A P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix} = D.$$

\# $f(A) = P\, f(D)\, P^{-1}$.

$$= P \begin{bmatrix} f(\lambda_1) & 0 & 0 \\ 0 & f(\lambda_2) & 0 \\ 0 & 0 & f(\lambda_3) \end{bmatrix} P^{-1}$$

(iii)

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

\# $\det(\lambda I - A) = (\lambda-1)^2(\lambda-2)$

So, eigenvalues are $\lambda = 1, 1, 2$.

$$E_1 = \{x : (A-I)x = 0\}.$$

$$= \left\{ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1 \text{ \& } x_3 \text{ are free valuable}, \; x_2 = 0 \right\}$$

$$= \begin{bmatrix} x_1 \\ 0 \\ x_3 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\dim E_1 = 2$.

$$E_2 = \{x : (A-2I)x = 0\} = \left\{ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : \begin{array}{l} x_1 = -x_2 \Rightarrow x_1 = -t/2 \\ 2x_2 = x_3 \Rightarrow x_2 = t/2 \\ x_3 = \text{free variable} = t \end{array} \right\}$$

$$= \left\{ t \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

$\dim E_2 = 1.$

$$P = \begin{array}{ccc} \lambda=1 & \lambda=1 & \lambda=2 \end{array}$$
$$P = \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 0 & 1/2 \\ 0 & 1 & 1 \end{bmatrix}, \qquad P^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D.$$

A.M corresponding to eigenvalue 1 = 2 $\Rightarrow$ A.M = G.M
G.M ———— = 2.

A.M corresponding to eigenvale 2 = 1 $\Rightarrow$ AM = G.M
G.M " " " 2 = 1.

P is diagonalizable.

Calculate A^{-1}, $(\lambda-1)^2(\lambda-2) = 0 \Rightarrow (\lambda^2+1-2\lambda)(\lambda-2) = 0$
$\Rightarrow \lambda^3+\lambda-2\lambda^2-2\lambda^2-2+4\lambda=0$
$\Rightarrow \lambda^3-4\lambda^2+5\lambda-2=0$

$$\Rightarrow (A-I)^2(A-2I) = 0$$
$$\Rightarrow A^3-4A^2+5A-2I = 0$$
$$\Rightarrow A^{-1} = \frac{1}{2}[A^2-4A+5I) \quad (\text{by multiplying } A^{-1}).$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Characteristic polynomial for A = $(\lambda-1)^3$

eigenvalues of A are $\lambda = 1, 1, 1$

A.M corresponding to 1 is 3.

Eigenspace corresponding to $\lambda = 1 = E_1 = \left\{ X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : (A-I)X = 0 \right\}$

$$= \left\{ X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : \begin{array}{l} x_3 = 0 \\ x_2 = 0 \\ x_1 = t \text{ (free variable)} \end{array} \right.$$

$$= \left\{ t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\}.$$

$\dim E_1 = 1$

G.M corresponding to $\lambda = 1$ = 1

A.M $\neq$ G.M

$\Rightarrow$ A is not diagonalizable so, we can not find P.

By Cayley Hamilton thm, $(A-I)^3 = 0$

$$\Rightarrow A^3 - I - 3A^2 + 3A = 0$$

Multiplying by A^{-1}, we obtain,

$$A^2 - A^{-1} - 3A + 3I = 0$$

$$\Rightarrow \boxed{A^{-1} = A^2 - 3A + 3I}.$$

$f(t) = t^3 - 5t - 4$

$f(A) = A^3 - 5A - 4I.$

(we know that

$A^3 - I - 3A^2 + 3A = 0$

$\boxed{A^3 = 3A^2 - 3A + I}$

$= 3A^2 - 8A - 3I$

Thus $f(A) = 3A^2 - 8A - 3I.$

$$= 3\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} - 8\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= 3\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 8 & 8 & 8 \\ 0 & 8 & 8 \\ 0 & 0 & 8 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 6 & 9 \\ 0 & 3 & 6 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 11 & 8 & 8 \\ 0 & 11 & 8 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & -2 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & -8 \end{bmatrix}$$

3) All the eigenvalues of $n \times n$ Hermitian/symmetric matrix A are real.

Solution :- Let λ be an eigenvalue of Hermitian matrix corresponding to eigenvector x.

i.e $Ax = \lambda x$ & $A = A^*$

Then $x^* A = x^* A^* = (Ax)^* = (\lambda x)^* = \bar{\lambda} x^*$.

Hence, $\lambda x^* x = = x^*(\lambda x) = x^*(Ax) = (\bar{\lambda} x^*) x = \bar{\lambda} x^* x$

$\Rightarrow (\lambda - \bar{\lambda}) x^* x = 0$.

But x is an eigen vector & hence $x \neq 0$ and so that the real no. $\|x\|^2 = x^* x$ is non zero as well.

Thus, $\lambda = \bar{\lambda}$

i.e λ is real no.

In case of symmetric matrix. Just replace x^* with x^t.

OR $*$ with t (transpose).

Proof follows similar as above.

Detail :- $AX = \lambda X$ & $A = A^t$

$\overline{AX} = \bar{\lambda} \bar{X}$ (Taking conjugate)

$\Rightarrow \bar{A} \bar{X} = \bar{\lambda} \bar{X}$

$\Rightarrow A \bar{X} = \bar{\lambda} \bar{X}$ ($\because$ A is sym. with real entries)

Taking transpose both sides,

$\bar{X}^t A^t = \lambda \bar{X}^t$

Post multiplying by X, & $A^t = A$

$\bar{X}^t A X = \lambda \bar{X}^t X$

$\Rightarrow \bar{X}^t \lambda X = \bar{\lambda} \bar{X}^t X$ ($\because AX = \lambda X$)

$\Rightarrow (\lambda - \bar{\lambda}) \bar{X}^t X = 0 \Rightarrow \bar{\lambda} = \lambda$

4) The 3x3 matrix A has $(1,0,1)^t$ & $(1,1,0)^t$ as eigenvector, both the eigenvalue 4, and its trace is 2.
Find the determinant of A. and ch. poly. of A.

Solⁿ:- The matrix A is 3×3 matrix so it has 3 eigen values.

Eigenvector corresponding to eigenvalues 4, contain the vector $(1,0,1)^t$, $(1,1,0)^t$.

Therefore dim of eigenspace at least 2.

It means that eigenvalue 4 has multiplicity at least 2.

Let the other eigenvalue $= \lambda$.

Then trace $= 2 \Rightarrow$ sum of eigenvalues $= 2$

$$\Rightarrow \lambda + 4 + 4 = 2$$

$$\Rightarrow \lambda = -6$$

Hence, determinant A = product of eigenvalue

$$= -6 \times 4 \times 4 = -96.$$

ch. poly. of A $= (\lambda-4)^2(\lambda+6)$

$$= (\lambda^2+16-8\lambda)(\lambda+6)$$

$$= \lambda^3 + 16\lambda - 8\lambda^2 + 6\lambda^2 + 96 - 48\lambda$$

$$= \lambda^3 - 2\lambda^2 - 32\lambda + 96$$

(5). The matrix A has ch. poly. $t(t+1)(t-2)$. what is the characteristic poly. of A^3.

Sol^n:- The eigenvalues of A are $0, -1, 2$.

$\therefore$ The eigen values of A^3 are $0, -1, 8$.

Thus, the ch. poly of A^3 is $(\lambda-0)(\lambda+1)(\lambda-8)$

$$= (\lambda^2+\lambda)(\lambda-8)$$
$$= \lambda^3+\lambda^2-8\lambda^2-8\lambda$$
$$= \lambda^3-7\lambda^2-8\lambda.$$

(6) (a) The eigen values of A & A^t are same.

Sol^n: $\det(A-\lambda I) = \det((A-\lambda I)^t)$ $\quad (\because \det A = \det A^t)$

$$= \det(A^t - \lambda I)$$

Since the ch. poly. of A & A^t are same implies same ch. eq^n. But eigen vectors/ch. roots may not be same.

Exa:- $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $A^t = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. eigenvalues of A & A^t are $0, 0$. But ch. vector of $A = x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $A^t = x_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(b). If $\lambda \neq 0$ is an eigenvalue of A then $1/\lambda$ is an eigenvalue of A^{-1}. But eigen vectors are same.

Sol^n. Ch. eq^n is $\det(A-\lambda I) = 0$

$$\Rightarrow \det(A - \lambda A^{-1}A) = 0$$
$$\Rightarrow \det(\lambda\lambda^{-1}A - \lambda A^{-1}A) = 0$$
$$\Rightarrow \det(\lambda(\lambda^{-1}I - A^{-1})A) = 0 \Rightarrow \lambda^n \det(\lambda^{-1}I - A^{-1}) \det A = 0 \Rightarrow \boxed{\det(\lambda^{-1}I - A^{-1}) = 0}$$

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(c) If λ is an eigenvalue of A then λ^k is an eigenvalue of A^k for any natural no. k.

Sol^n:- We have
$$AX = \lambda X.$$

Now, $A^k X = A^{k-1}(AX) = A^{k-1}(\lambda X) = \lambda A^{k-1} X = \lambda A^{k-2}(AX)$

$$= \lambda A^{k-2}(\lambda X)$$
$$= \lambda^2 A^{k-2} X$$
$$\vdots$$
$$= \lambda^k X.$$

Thus, λ^k is an eigenvalue of A^k.
But eigenvector of A & A^k are same.

(d) If A & B are $n \times n$ matrices with A non singular then BA^{-1} & $A^{-1}B$ have the same set of eigenvalues.

Solution:- $\det(\lambda I - BA^{-1})$
$$= \det(\lambda A A^{-1} - A A^{-1} B A^{-1})$$
$$= \det(A(\lambda I - A^{-1}B) A^{-1})$$
$$= \det(A) \det(\lambda I - A^{-1}B) \det A^{-1}$$

since A is invertible $\Rightarrow \det A \neq 0$, $\det A^{-1} \neq 0$

Thus $\det(\lambda I - BA^{-1}) = 0 \iff \det(\lambda I - A^{-1}B) = 0.$

$\Rightarrow BA^{-1}$ & $A^{-1}B$ have same set of eigenvalues.

(e) Similar matrices have same eigenvalue.

Suppose A and B are similar matrix Then ∃ a matrix P s.t

$$\boxed{P^{-1}AP = B}$$

Sol^n: Suppose λ is an eigen value of A corresponding to eigen-vector X

i.e $AX = \lambda X$.

$$\begin{aligned} B(P^{-1}X) &= P^{-1}AP(P^{-1}X) \\ &= P^{-1}A(PP^{-1})X \\ &= P^{-1}AIX \\ &= P^{-1}AX \\ &= P^{-1}\lambda X \\ &= \lambda P^{-1}X \end{aligned}$$

i.e $B(P^{-1}X) = \lambda P^{-1}X$

⇒ $P^{-1}X$ is an eigen vector of B corresponding to eigenvalue λ.

Thus, A & B have same eigenvalue.

Also if X is an eigenvector of A corresponding to λ

Then $P^{-1}X$ is an eigenvector of B corresponding to λ.

Relationship b/w Eigenvectors :-

(b) A & A^{-1} have same eigenvector.

Let λ be the eigenvalue corresponding to eigenvector X.

Then $AX = \lambda X$

multiplying by A^{-1}, we obtain

$$IX = \lambda A^{-1} X$$

$$\Rightarrow \frac{1}{\lambda} X = A^{-1} X$$

i.e $A^{-1} X = \frac{1}{\lambda} X$

$\Rightarrow$ X is an eigenvector of A corresponding to eigenvalue $1/\lambda$.

(c) A and A^k both have same eigenvector although eigenvalues are different.

(d) Suppose X is an eigenvector of BA^{-1} corresponding to eigenvalue λ.

Then $BA^{-1} X = \lambda X$.

Claim:- $A^{-1}X$ is an eigenvector of $A^{-1}B$ corresponding to eigenvalue λ.

$$(A^{-1}B)(A^{-1}X) = A^{-1} B A^{-1} X$$
$$= A^{-1}(\lambda X)$$
$$= \lambda A^{-1} X$$

i.e $A^{-1}B(A^{-1}X) = \lambda(A^{-1}X)$

7) Construct a non diagonal 2×2 matrix that is diagonalizable but not invertible.

Sol^n: Since the matrix is not invertible.

$\therefore$ 0 is an eigenvalue.

Let the other eigenvalue is 1.

Thus, the ch. poly. $\lambda^2 - \lambda$.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$|A - \lambda I| = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - (a+d)\lambda + ad - bc$$

$\Rightarrow$ $ad - bc = 0$

$a + d = 1$

Take $a = 1$, then $d = 0$

$\Rightarrow$ $bc = 0$ $\Rightarrow$ either $b = 0$ or $c = 0$.

or both $b = c = 0$.

Hence $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

with a double eigenvalue $\lambda = 0$.