Tutorial Sheet 5
Inner Product Spaces

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1. Find an inner product in $\mathbb{R}^2$ such that the following condition hold:

$$\|(1,2)\| = \|(2,-1)\| = 1 \quad \text{and} \quad \langle (1,2),(2,-1)\rangle = 0$$

2. Let x, $y \in \mathbb{R}^n$. Then prove the following:

   (a) $\langle x, y\rangle = 0$ if and only if $\|x-y\|^2 = \|x\|^2 + \|y\|^2$, (This is called Pythagoras Theorem).

   (b) $\|x\| = \|y\| \iff \langle x+y, x-y\rangle = 0$, ($x$ and y form adjacent sides of a rhombus as the diagonals $x+y$ and $x-y$ are orthogonal).

   (c) $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$, (This is called the Parallelogram Law).

   (d) $4\langle x, y\rangle = \|x+y\|^2 - \|x-y\|^2$ (This is called the polarisation identity).

   (e) If x, $y \in \mathbb{C}^n(\mathbb{C})$, then $4\langle x, y\rangle = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$ (This is called the polarisation identity).

3. (a) Let $\{u_1 = (1,1,1,1), u_2 = (1,0,1,0), u_3 = (0,1,0,1)\}$ be a linearly independent set in $\mathbb{R}^4(\mathbb{R})$. Find an orthonormal set $\{v_1, v_2, v_3\}$ such that $L(u_1, u_2, u_3) = L(v_1, v_2, v_3)$.

   (b) Find an orthonormal basis for $\mathcal{P}_2(\mathbb{R})$, where the inner product is given by $\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)$.

4. Let V be an inner product space. Let W be a non-empty set. Then

$$W^\perp = \{v \in V : \langle v, w\rangle = 0 \text{ for all } w \in W\}.$$

   (a) If $W = \{(x,y,z) \in \mathbb{R}^3 : x + y - z = 0\}$. Then find $W^\perp$ with respect to the standard inner product. Also find basis and dimension of W, $W^\perp$

   (b) Let V be the vector space of all $n \times n$ real matrices. Then V is a real inner product space with the inner product given by $\langle A, B\rangle = tr(AB^t)$. If W is the subspace given by $W = \{A \in V : A^t = A\}$, determine the dimension of $W^\perp$.