Laplace Transforms (time domain $\rightarrow$ frequency domain)

Let $f(t)$ be a function defined for all $t \geq 0$, then Laplace transform of $f(t)$ is defined by

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt, \quad s > 0$$
$$= F(s)$$

provided the integral exists.

Laplace Transform of Some Elementary functions:

(i) $L[1] = \frac{1}{s}, \quad s > 0$

$$\boxed{L[1] = \int_0^\infty e^{-st}(1)\, dt}$$

(ii) $L\{t\} = \frac{1}{s^2}, \quad s > 0$

(iii) $L\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$, n is a positive integer

$\boxed{\Gamma(n+1) = n! \text{ if } n \text{ is an integer}}$

or $\frac{\Gamma(n+1)}{s^{n+1}}$, n (not an integer)

(iv) $L\{e^{at}\} = \frac{1}{s-a}, \quad s > a$

(v) $L\{\sin at\} = \frac{a}{s^2 + a^2}$

(vi) $L\{\cos at\} = \dfrac{s}{s^2+a^2}$.

Linearity Property.

$$L\{af(t) + bg(t)\} = aL\{f(t)\} + bL\{g(t)\}$$

Example $L\{\sin^2 at\}$

$$= L\left\{\frac{1 - \cos 2at}{2}\right\} = L\left\{\frac{1}{2}\right\} - \frac{1}{2} \times L\{\cos 2at\}$$

$$= \frac{1}{2} L\{1\} - \frac{1}{2} L\{\cos 2at\}$$

$$= \frac{1}{2} \cdot \frac{1}{s} - \frac{1}{2} \frac{s}{s^2+(2a)^2}$$

$$\boxed{\begin{aligned} L\{1\} &= \frac{1}{s} \\ L\{\cos at\} &= \frac{s}{s^2+a^2} \end{aligned}}$$

$$\Rightarrow L\{\sin^2 at\} = \frac{1}{2s} - \frac{s}{2(s^2+4a^2)}$$

Find $L\{\cosh at\}$ and $L\{\sinh at\}$

$$L\{\cosh at\} = L\left\{\frac{e^{at}+e^{-at}}{2}\right\}$$

$$\boxed{\begin{aligned}\cosh t &= \frac{e^{t}+e^{-t}}{2}\\ \sinh t &= \frac{e^{t}-e^{-t}}{2}\end{aligned}}$$

$$= \frac{1}{2}L\{e^{at}\} + \frac{1}{2}L\{e^{-at}\}$$

$$\Rightarrow L\{\cosh at\} = \frac{1}{2}\cdot\frac{1}{s-a} + \frac{1}{2}\cdot\frac{1}{s+a}$$

$$\left(L\{e^{at}\} = \frac{1}{s-a}\right)$$

$$= \frac{s}{s^2-a^2}$$

$$L\{\sinh at\} = \frac{a}{s^2-a^2}.$$

First Shifting Theorem

If $L\{f(t)\} = F(s)$, then

$$L\{e^{at} f(t)\} = F(s-a)$$

Proof:

$$L\{e^{at} f(t)\} = \int_0^\infty e^{-st} \cdot e^{at} f(t)\, dt$$

$$\boxed{L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s)}$$

$$= \int_0^\infty e^{-(s-a)} f(t)\, dt$$

$$\boxed{L\{e^{at} f(t)\} = F(s-a)}$$

Example: find $L\{e^{at} \cdot t\}$

Since $L\{t\} = \frac{1!}{s^{1+1}}$ $\quad \boxed{L\{t^n\} = \frac{n!}{s^{n+1}}}$

$\Rightarrow L\{t\} = \frac{1}{s^2}$

$$L\{e^{at}\cdot t\} = \frac{1}{s^2}\Big|_{s\to s-a} = \frac{1}{(s-a)^2}$$

$$\Rightarrow \boxed{L\{e^{at}\cdot t\} = \frac{1}{(s-a)^2}}$$

Example ~~$L\{t^2 e$~~ find $L[e^{at}\cos bt]$.

Since $L[\cos bt] = \dfrac{s}{s^2+b^2}$

$$L[e^{at}\cos bt] = \frac{s}{s^2+b^2}\Big|_{s\to s-a}$$

$$= \frac{s-a}{(s-a)^2+b^2}$$

Property :- III

If $L\{f(t)\} = F(s)$, then

$$L\{t\, f(t)\} = -\frac{d}{ds}(F(s))$$

$$L[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2}(F(s))$$

In general,

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}(F(s))$$

Example: find $L[t^2 \sin bt]$?

Since $L[\sin bt] = \frac{b}{s^2+b^2} = F(s)$

$$\Rightarrow L[t^2 \sin bt] = (-1)^2 \frac{d^2}{ds^2}\left(\frac{b}{s^2+b^2}\right)$$

$$= \frac{d^2}{ds^2}\left(\frac{b}{s^2+b^2}\right)$$

$$= \frac{6bs^2-2b^3}{(s^2+b^2)^3}$$

Example: find $L\{t\, e^{-4t} \sin 3t]$?

$$L[\sin 3t] = \frac{3}{s^2+9}$$

$$\Rightarrow L[e^{-4t} \sin 3t] = \frac{3}{(s+4)^2+9}$$

$$L[te^{-4t}\sin 3t] = (-1)\frac{d}{ds}\left(\frac{3}{(s+4)^2+9}\right)$$

$$= -3(-1)\left((s+4)^2+9\right)^{-2}\cdot 2(s+4)$$

$$\Rightarrow L[te^{-4t}\sin 3t] = \frac{6(s+4)}{\left((s+4)^2+9\right)^2}$$

Property - IV

If $f(t)$ is a differentiable function of t, then

$$L[f'(t)] = sF(s) - f(0),$$

where $F(s) = L[f(t)]$.

($f(t)$ is diff 2 times)

$$L[f''(t)] = s^2F(s) - sf(0) - f'(0).$$

$$L[f^{(n)}(t)] = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0).$$

(provided $f(t)$ is n times differentiable)

Proof

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \left| e^{-st} f(t) \right|_0^\infty - \int_0^\infty -s e^{-st} \cdot f(t)\,dt$$

$$= -f(0) + s\underbrace{\int_0^\infty e^{-st} f(t)\,dt}_{F(s)}$$

$$\Rightarrow L[f'(t)] = s F(s) - f(0)$$

$$\boxed{L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = F(s)}$$

Inverse Laplace Transforms

If $L[f(t)] = F(s) \rightarrow$ (Laplace transform of $f(t)$)

and $L^{-1}[F(s)] = f(t)$,

$f(t) \rightarrow$ Inverse Laplace transform of $F(s)$.

(i) $L[1] = \frac{1}{s} \Rightarrow L^{-1}\left[\frac{1}{s}\right] = 1$

(ii) $L[t] = \frac{1}{s^2} \Rightarrow L^{-1}\left[\frac{1}{s^2}\right] = t$

(iii) $L[t^n] = \frac{n!}{s^{n+1}} \Rightarrow L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!}$

(iv) $L[e^{at}] = \frac{1}{s-a} \Rightarrow L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

(v) $L[\sin at] = \frac{a}{s^2+a^2} \Rightarrow L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$

(vi) $L[\cos at] = \frac{s}{s^2+a^2} \Rightarrow L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

Example. Find ~~the~~ the inverse Laplace transform of $\frac{1}{s(s+1)}$

$$L^{-1}\left[\frac{1}{s(s+1)}\right]$$

$$L^{-1}\left[\frac{1}{s}-\frac{1}{s+1}\right]$$

$$= L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1}\right]$$

$$= 1 - e^{-t} \qquad \left[\because L^{-1}\left[\frac{1}{s}\right]=1,\quad L^{-1}\left[\frac{1}{s-a}\right]=e^{at}\right]$$

Example: $L^{-1}\left(\frac{s}{(s+1)(s+2)(s+3)}\right) = ?$

Property:

If $L[f(t)] = F(s) \Rightarrow L^{-1}[F(s)] = f(t)$

$\Rightarrow L[e^{at} f(t)] = F(s-a) \Rightarrow L^{-1}[F(s-a)]$

$= e^{at} f(t).$

$$\boxed{\text{If } L^{-1}[F(s)] = f(t), \text{ then } L^{-1}[F(s-a)] = e^{at}\cdot f(t)}$$

Example 1 find $L^{-1}\left[\frac{1}{(s-2)^2}\right]$

$$L^{-1}\left[\frac{1}{s^2}\right] = t$$

$$\boxed{L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!}}$$

$$\Rightarrow L^{-1}\left[\frac{1}{(s-2)^2}\right] = t \cdot e^{2t}$$

$$\boxed{\begin{array}{l} \text{If } L^{-1}[F(s)] = f(t) \\ \Rightarrow L^{-1}[F(s-a)] = e^{at} f(t) \end{array}}$$

find $L^{-1}\left(\frac{1}{(s-1)^2+4}\right)$

$$\Rightarrow L^{-1}\left(\frac{1}{s^2+4}\right) = \frac{1}{2}\sin 2t$$

$$\Rightarrow L^{-1}\left(\frac{1}{(s-1)^2+4}\right) = \frac{1}{2}e^{t}\sin 2t$$

$$L^{-1}\left(\frac{1}{s^2+4}\right) =$$

$$= \frac{1}{2}\sin 2t$$

$$\boxed{L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at}$$

find $L^{-1}\left(\frac{s+2}{(s+2)^2+3^2}\right)$

$$= e^{-2t}\cdot \cos 3t$$

$L^{-1}\left(\frac{s}{s^2+3^2}\right)$

$= \cos 3t$

$L(\cos 3t) = \frac{s}{s^2+9}$

find $L^{-1}\left(\frac{s-2}{s^2+4s+13}\right)$

$$L^{-1}\left(\frac{s-2}{s^2+4s+4-4+13}\right)$$

$$= L^{-1}\left(\frac{s-2}{(s+2)^2+9}\right)$$

$L(\sin 3t)$

$= \frac{3}{s^2+9}$

$L^{-1}\left(\frac{1}{s^2+9}\right)$

$= \frac{1}{3}\sin 3t$

$$= L^{-1}\left(\frac{s+2-2-2}{(s+2)^2+9}\right)$$

$$= L^{-1}\left(\frac{s+2}{(s+2)^2+9} - \frac{4}{(s+2)^2+9}\right) = e^{-2t}\cos 3t - \frac{4}{3}e^{-2t}\sin 3t$$