Tutorial sheet 3

① Find all the subspaces of $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$.

Solution: subspace of $\mathbb{R}$ — $\{0\}$, $\mathbb{R}$.

$\mathbb{R}^2$ — $\{0\}$, line passes through the origin, $\mathbb{R}^2$

$\mathbb{R}^3$ — $\{0\}$, $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$.

$\{0\}$, line passes through the origin,

plane plasses through the origin,

$\mathbb{R}^3$.

2) Is $\mathbb{R}^2$ a vector space over $\mathbb{Z}_2$.

Ans — $\mathbb{R}^2$ is not closed under scalar multiplication, as $(\alpha+\beta)u = 0 \quad \forall u \in \mathbb{R}^2$

$\alpha = \beta = 1$ Then $\alpha+\beta \doteq 0$ &

$\alpha u + \beta u = u + u = \underset{1}{2u} \in \mathbb{R}^2$ (Addition in vector not in scalar)

3) $V = \mathbb{R}$. $x+y = x-y$, $\alpha x = -\alpha x$.

which vector space axioms are not satisfied here.

$\underline{\text{Sol}^n}$: V is closed under addition, as $x, y \in \mathbb{R}$

$\Rightarrow x+y = x-y \in \mathbb{R}$.

Commutative — $\forall x, y \in \mathbb{R}$, $x+y = y+x$

$x+y = x-y$

$y+x = y-x = -(x-y) = -(x+y)$

Thus, commutative law does not hold.

Associative law:- $(x+y)+z \neq x+(y+z) \quad \forall\, x, y, z \in \mathbb{R}$

$$(x+y)+z = (x-y)+z = x-y-z.$$

$$x+(y+z) = x+(y-z) = x-y+z$$

Identity:- $x+0 = 0+x = x$.

Now $\quad x+0 = x-0 = x$

$$0+x = 0-x = -x.$$

No identity element exist.

Inverse :- does not exist.

Scalar Multiplication:- $\quad \alpha x = -\alpha x.$

Scalar Multiplication as $-\alpha x \in \mathbb{R}$.

V is closed under scalar multiplication

$\rightarrow$ $(\alpha\beta)x \neq \alpha(\beta x)$ As $(\alpha\beta)x = -\alpha\beta x$, $\quad \alpha(\beta x) = -\alpha(\beta x) = +\alpha\beta x$

$\rightarrow$ $1.x = -x \neq x.$

$\rightarrow$ $(\alpha+\beta)x \neq \alpha x + \beta x \quad \forall\, \alpha, \beta \in \mathbb{F}, \; x \in V.$

Now, $(\alpha+\beta)x = -(\alpha+\beta)x$
$= -\alpha x - \beta x$

$$\alpha.x + \beta x = \alpha x - \beta x = -\alpha x + \beta x$$

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\# $\alpha(x+y) \neq \alpha x + \alpha y \quad \forall\ \alpha \in \mathbb{F},\ x, y \in V.$

$$\alpha(x+y) = \alpha(x-y) = -\alpha(x-y) = -\alpha x + \alpha y$$

$$\alpha x + \alpha y = \alpha x - \alpha y = -\alpha x + \alpha y$$

(4) $V = \mathbb{R}^+$ = set of all positive real numbers.

(a) Show that V is not a vector space over $\mathbb{R}$ w.r.t usual addition & scalar multiplication

Sol^n: V is closed under addition, as sum of two positive real no's is again a +ve real no.

but V is not closed under scalar multiplication as $\alpha = -1$, $x \in \mathbb{R}^+$ Then $\alpha x = -x$ is a negative no. which does not belong to V.

(b) For $\alpha \in \mathbb{R}$, $u, v \in \mathbb{R}^+$. Define $u+v = uv$, $\alpha u = u^\alpha$.

Then V is a vector space over $\mathbb{R}$.

Sol^n: For $u, v \in \mathbb{R}^+ \Rightarrow u+v = uv \in \mathbb{R}^+$.

for $u, v, w \in \mathbb{R}^+$, we have

1) $u+v = uv = vu = v+u$ (Commutative property hold)

2) $(u+v)+w = (uv)+w = uvw$ (Associative property ahold)
$u+(v+w) = u(v+w) = uvw$

3) $1+u = 1\cdot u = u$

$u+1 = u\cdot 1 = u$

i.e $1+u = u+1 = u$

i.e 1 is the additive identity.

4) Inverse, $u \in \mathbb{R}^+$ Then $\frac{1}{u}$ is the inverse of u.

$u + \frac{1}{u} = u\cdot\frac{1}{u} = 1$

$\frac{1}{u} + u = \frac{1}{u}\cdot u = 1.$

Scalar Multiplication:- $\alpha u = u^{\alpha}$ $\forall$ $u \in \mathbb{R}^+, \alpha \in \mathbb{R}$

(V) It is closed under scalar multiplication.

$(\alpha\beta)u = u^{\alpha\beta}$

$\alpha(\beta u) = \alpha u^{\beta} = u^{\beta\alpha} = u^{\alpha\beta}$

$\boxed{\text{i.e } (\alpha\beta)u = \alpha(\beta u)}$

$1\cdot u = u^{1} = u.$ $\Rightarrow$ $\boxed{1\cdot u = u.}$

Distributive law:-

$(\alpha+\beta)u = u^{\alpha+\beta} = u^{\alpha}\cdot u^{\beta} = u^{\alpha} + u^{\beta} = \alpha u + \beta u =$

$\alpha(u+v) = \alpha(uv) = (uv)^{\alpha} = u^{\alpha}v^{\alpha} =$

$\alpha u + \alpha v = u^{\alpha} + v^{\alpha} = u^{\alpha}v^{\alpha}$

Thus $\alpha(u+v) = \alpha u + \alpha v$

Thus all the properties of vector space are satisfied.

⑤ (a) $Sl_n = \{A \in M_n(\mathbb{C}) : \text{trace}(A) = 0\}$ is a subspace of $M_n(\mathbb{C})$.

Solⁿ: $Sl_n \neq \phi$ $\because A = $ Zero matrix $\in Sl_n$ $\text{trace}(A) = 0$.

let $A \in Sl_n$, $B \in Sl_n$. $\Rightarrow \text{trace}(A) = 0$, $\text{trace}(B) = 0$

To show:- $A + B \in Sl_n$

$$\text{trace}(A+B) = tr(A) + tr(B) = 0 + 0 = 0. \text{ (a real no)}.$$

To show:- $\alpha \in \mathbb{C}$, $A \in Sl_n$ Then $\alpha A \in Sl_n$.

$$\text{trace}(\alpha A) = \alpha \, \text{trace}(A) = \alpha \cdot 0 = 0.$$

Thus Sl_n is a subspace of $M_n(\mathbb{C})$.

(b) $W = Sym_n = \{A \in M_n(\mathbb{C}) : A = A^\theta = \bar{A}^t\}$.

Solⁿ: W is not a subspace of $M_n(\mathbb{C})$

As, $A, B \in M_n(\mathbb{C}) \Rightarrow A = A^\theta$, $B = B^\theta$

$$(A+B)^\theta = A^\theta + B^\theta = A + B \Rightarrow A + B \in W$$

But $\alpha = i \in \mathbb{C}$ Then $(iA)^\theta = -iA^\theta = -iA \notin W$

i.e $\alpha A \notin W$.

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(c) $W = Skew_n = \{A \in M_n(\mathbb{C}) : A^\theta = -A\}$.

Sol^n: Let $A, B \in W \Rightarrow A^\theta = -A, \quad B^\theta = -B$.

To show: $A+B \in W$ i.e. $(A+B)^\theta = -(A+B)$.

Consider, $(A+B)^\theta = A^\theta + B^\theta = -A-B = -(A+B)$

i.e. $A+B \in W$.

- Scalar multiplication does not hold in W, as

let $\alpha = i \in \mathbb{C}$, $A \in W$ Then

$$(\alpha A)^\theta = (iA)^\theta = -i A^\theta = -i(-A) = iA = \alpha A$$

i.e. $\alpha A \notin W$.

(d) Is the set of all invertible matrices subspace of $M_n(\mathbb{R})$.

Sol^n: NO,

As $A = $ Identity. $\therefore$ A is invertible, det $A = 1$

$B = -$ Indentity B is invertible as det $B = -1$

$A+B = $ Zero matrix.

det $(A+B) = 0$. $\Rightarrow$ $A+B$ is not invertible.

$\therefore$ Set of all invertible matrices does not form a subspace of $M_n(\mathbb{R})$.

6 Let $C([-1,1])$ be the set of all real valued continuous functions on the interval $[-1,1]$. Let

$$W_1 = \{ f \in C([-1,1]) : f(\tfrac{1}{2}) = 0 \}$$

and $$W_2 = \{ f \in C([-1,1]) : f(\tfrac{1}{4}) = 5 \}$$

Are W_1, W_2 subspaces of $C([-1,1])$.

Solution: Here $W_1 = \{ f \in C([-1,1]) : f(\tfrac{1}{2}) = 0 \}$.

(~~Since zero~~)

(i) 0 (zero function) $\in W_1$ as $0(\tfrac{1}{2}) = 0$ (Here $0 \to$ zero function)

(ii) Let $f, g \in W_1$ and $\alpha, \beta \in F$,

then
$$(\alpha f + \beta g)(\tfrac{1}{2}) = \alpha f(\tfrac{1}{2}) + \beta g(\tfrac{1}{2})$$
$$= \alpha \cdot 0 + \beta \cdot 0 \qquad \left[\because f, g \in W_1 \Rightarrow f(\tfrac{1}{2}) = 0 \text{ \& } g(\tfrac{1}{2}) = 0 \right]$$
$$= 0$$
$$\Rightarrow (\alpha f + \beta g)(\tfrac{1}{2}) = 0$$
$$\Rightarrow \alpha f + \beta g \in W_1$$
$\Rightarrow$ W_1 is a subspace of $C[-1,1]$.

Consider $W_2 = \{ f \in C([-1,1]) : f(\tfrac{1}{4}) = 5 \}$.

Since $0(\tfrac{1}{4}) = 0 \neq 5$

$\Rightarrow$ 0 (zero function) $\notin W_2$

$\Rightarrow$ W_2 is not a subspace of $C([-1,1])$.

7) To show :- $U \cap W$ is a subspace of V if U & W are subspace of V.

Solⁿ: $S = U \cap W = \{u : u \in U \text{ \& } u \in W\}$.

To show: S is closed under addition and scalar multiplication.

Let $u, v \in S \Rightarrow u \in U,\ u \in W$
$\qquad\qquad\qquad v \in U,\ v \in W$

i.e $u, v \in U$ and U is a subspace of V.

$\boxed{\therefore\ u+v \in U \text{ and } \alpha u \in U \ \forall\ \alpha \in \mathbb{F};\ u \in U.}$ ①

Illy, $u, v \in W$ and W is a subspace of V

$\boxed{\therefore\ u+v \in W \ \&\ \alpha u \in W \ \forall\ \alpha \in \mathbb{F},\ u \in W}$ ②

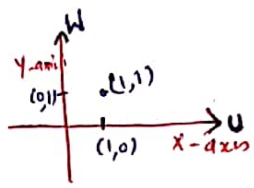
Thus. ① & ② $\Rightarrow u+v \in U \cap W,\ \alpha u \in W \cap U$.

(ii) $U \cup W$ may (need) not be a subspace.

Let $U = \{(x,0) : x \in \mathbb{R}\}$ be a subspace of $\mathbb{R}^2$

Also $W = \{(0,y) : x \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$.

$S = U \cup W = \{(x,0) : x \in \mathbb{R}\} \cup \{(0,y) : y \in \mathbb{R}\}$
is not a subspace.



As, $u_1 = (1,0) \in S$, $u_2 = (0,1) \in S \Rightarrow u_1 + u_2 = (1,1) \notin S$.

(ii) $U \cup W$ is a subspace of W if either $U \subseteq W$ or $W \subseteq U$.

Pf:- $S = U \cup W$.

Let $u, v \in S = U \cup W$

$\Rightarrow$ either $u \in W$ or $u \in U$, Ily either $v \in U$ or $v \in W$.

Thus, we have the following cases. and suppose $U \subseteq W$

(i) $u \in U$, $v \in U$ $\Rightarrow u, v \in U$.

(ii) $u \in U$, $v \in W$ $\Rightarrow u, v \in W$ as $u \in U \subset W$

(iii) $u \in W$, $v \in U$ $\Rightarrow u, v \in W$ as $v \in U \subset W$

(iv) $u \in W$, $v \in W$ $\Rightarrow u, v \in W$

Since U & W are subspace of V. $\therefore$ $u+v \in U$ or W

and $U, W \subseteq U \cup W$

Thus $u+v \in U \cup W$

Ily $\alpha u \in U \cup W$

Thus $U \cup W$ is a subspace of V.

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Q. Let U and W be two subspaces of a vector space V. Define $U+W = \{u+w : u \in U, w \in W\}$. Show that $U+W$ is a subspace of V. Also show that $L(U \cup W) = U+W$.

Solution: Since U and W are subspaces of V.

$\Rightarrow \quad 0 \in U$

$\Rightarrow \quad 0 \in U \cup W$

$\Rightarrow \quad U \cup W \neq \phi$.

Let x, y be two elements of $U+W$.

Then by definition of $U+W$,

$$x = u_1 + w_1, \quad u_1 \in U, \quad w_1 \in W.$$
$$y = u_2 + w_2, \quad u_2 \in U, \quad w_2 \in W.$$

Let $\alpha, \beta \in F \Rightarrow \alpha x + \beta y = \alpha(u_1 + w_1) + \beta(u_2 + w_2)$

$$= (\alpha u_1 + \beta u_2) + (\alpha w_1 + \beta w_2).$$

Since U and W are subspaces of V.

$\therefore \alpha u_1 + \beta u_2 \in U$ and $\alpha w_1 + \beta w_2 \in W$.

Thus $\alpha x + \beta y \in U+W$.

Hence $U+W$ is a subspace of V.

Now, to show that $L(U \cup W) = U+W$.

Since $U \subseteq U+W$ and $W \subseteq U+W$

$\Rightarrow \quad U \cup W \subseteq U+W$

Since $L(U \cup W)$ is the smallest subspace of V, containing $U \cup W$ and $U+W$ is a subspace containing $U \cup W$, therefore

$$L(U \cup W) \subseteq U+W \qquad \text{——— (1)}$$

Let $x = u + w$ be any element of $U+W$, where $u \in U$, $w \in W$.

$\therefore$ $u, w \in U \cup W$

Now, $u + w = 1 \cdot u + 1 \cdot w$

$\Rightarrow$ $x = u + w$ is a linear combination of the elements $u, w \in U \cup W$.

$\Rightarrow$ $x \in L(U \cup W)$

$\therefore$ $U + W \subseteq L(U \cup W)$ ——— (2)

Thus from (1) & (2), we get

$$\boxed{U + W = L(U \cup W)}$$

9) Is $(4,5,5)$ a linear combination of $(1,2,3), (1,1,4)$ & $(3,3,2)$

Soln:

$(4,5,5) = \alpha(1,2,3) + \beta(1,1,4) + \gamma(3,3,2)$, we obtain

$$\alpha + \beta + 3\gamma = 4$$
$$2\alpha + \beta + 3\gamma = 5$$
$$3\alpha + 4\beta + 2\gamma = 5$$

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 2 & 1 & 3 & 5 \\ 3 & 4 & 2 & 5 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$
$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & -1 & -3 & -3 \\ 0 & 1 & -7 & -7 \end{array}\right]$$

$R_3 \to R_3 + R_2$
$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & -1 & -3 & -3 \\ 0 & 0 & -10 & -10 \end{array}\right]$$

Thus, $\gamma = +1$, $\beta + 3\gamma = 3 \Rightarrow \beta = 0$, $\alpha = 1$

Thus, $\alpha = 1$, $\beta = 0$ and $\gamma = 1$

$\Rightarrow (4,5,5) = 1(1,2,3) + 0(1,1,4) + 1(3,3,2)$ Ans.

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(10) Find the linear span of $S = \{(1,1,1), (2,1,3)\}$ over $\mathbb{R}$.

Solⁿ: $\text{Span}(S) = \{\alpha(1,1,1) + \beta(2,1,3) : \alpha, \beta \in \mathbb{R}\}$

$= \{(\alpha + 2\beta, \alpha + \beta, \alpha + 3\beta : \alpha, \beta \in \mathbb{R}\}$

$= \{(x, y, z) :$ $x = \alpha + 2\beta$, $y = \alpha + \beta$, $z = \alpha + 3\beta.\}$

$\Rightarrow \quad \alpha = 2y - x$ and $\beta = x - y$

Thus $\text{span}(S) = \{(2y-x)(1,1,1) + (x-y)(2,1,3) : x, y, z \in \mathbb{R}\}$.