# Tutorial - 3 Solutions

(1) $T(n) = T(n-1) + n$.

We show that $T(n) = O(n^2)$ is a solution of

$T(n) = T(n-1) + n$ using substitution method.

We use mathematical induction.

Assume that ~~this~~ $T(n) = O(n^2)$ satisfies the equation $T(n) = T(n-1) + n$ for all $m < n$.

(i.e) $T(m) = O(m^2)$ ~~for all~~ is a solution of $T(m) = T(m-1) + m$ $\forall\, m < n$

Consider $T(n) = T(n-1) + n$

Since $n-1 < n$, we chose $m = n-1$ and use the above assumption

$$
\begin{aligned}
T(n) &= T(n-1) + n \\
&\le c(n-1)^2 + n \\
&\le cn^2 + n + c \\
&\le (c+2)\,n^2
\end{aligned}
$$

$\therefore T(n) = O(n^2)$

(2) Assume $O(m^{\log_3 4})$ be the solution of

$$T(m) = 4T(m/3) + m \text{ for all } m < n.$$

Consider $T(n) = 4T(n/3) + n$

Note that $n/3 < n$, hence above assumption is also true for $m = n/3 < n$.

$$\begin{aligned}
T(n) &= 4T(n/3) + n \\
&\le 4 \cdot c \cdot (n/3)^{\log_3 4} + n \\
&\le 4 \cdot c \cdot \frac{n^{\log_3 4}}{3^{\log_3 4}} + n \\
&\le c \cdot n^{\log_3 4} + n \\
&\le 2cn^{\log_3 4}
\end{aligned}$$

$$\therefore T(n) = O(n^{\log_3 4})$$

(3) Show that the solution $T(n) = T(n/3) + T(2n/3) + O(n)$ is $O(n \log n)$, using substitution method.

Assume that the statement is true for all $m < n$.

(i.e) $O(m \log m)$ is a solution of

$$T(m) = T(m/3) + T(2m/3) + O(m)$$

for all $m < n$

Consider $T(n) = T(n/3) + T(2n/3) + O(n)$

Since $n/3,\ 2n/3 < n$,

we use above assumption

$$T(n) \leq c_1 \cdot \frac{n}{3} \log \frac{n}{3} + c_2 \cdot \frac{2n}{3} \cdot \log \frac{2n}{3}$$

$$\leq \frac{c_1}{3} \cdot n \log n + c_2 \cdot \frac{2n}{3} \cdot \log 2n$$

$$\leq \frac{c_1}{3} \cdot n \log n + c_2 \cdot \frac{2n}{3} \cdot 2 \log n$$

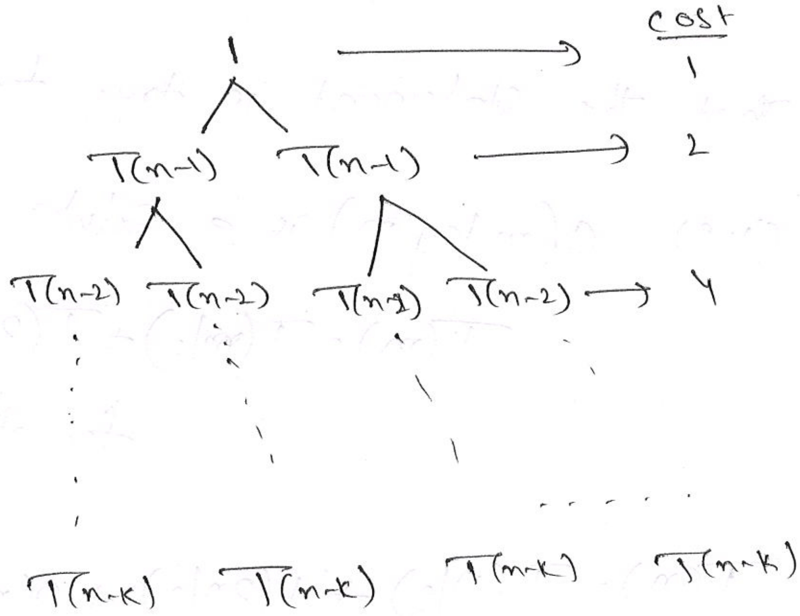
$$\leq n \log n \left( \frac{c_1}{3} + \frac{4c_2}{3} \right)$$

$$\therefore T(n) = O(n \log n)$$

4

# Recursion tree method

$T(n) = 2T(n-1) + 1$




The ~~tree~~ leaves in the tree are $T(1)$

$\Rightarrow n-k=1$

$k = n-1$

$= O(n)$

$\therefore$ height of the tree $= (n-1)$

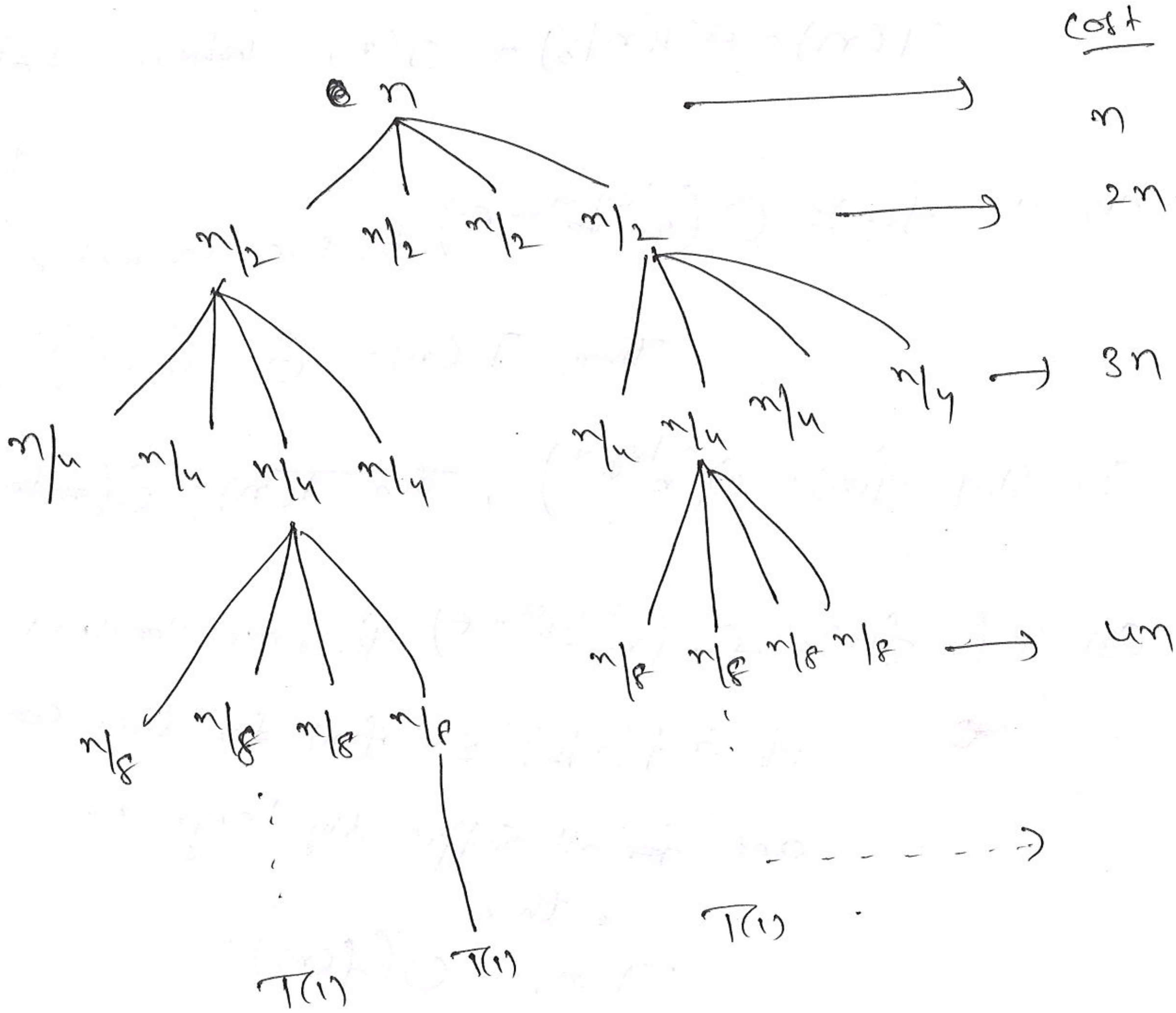
cost of any level $i$ = cost at $i-1 \times 2$

$\therefore$ total cost $= \sum_{i=1}^{n-2}$ cost at level $i$ + the number of leaves at level $(n-1)$

$= \sum_{i=1}^{n-2} 2^i + 2^{n-1}$

$= O(2^n)$

5 $T(n) = 4T(n/2) + cn.$ 5




∴ The height of the tree = $O(\log n)$

~~The no. of~~ The no. of nodes at level $i = 2 *$ no. of nodes at level $i-1$.

∴ the ~~cost~~ # of nodes at level $-\log n = n$.

∴ cost $= \sum_{i=1}^{\log n - 1} (i+1)n$ + # of nodes at the last level

$= \sum_{i=1}^{\log n - 1} (i+1)n + n$

$= O(n \log n)$

## (6) Master Theorem (MT)

$T(n) = aT(n/b) + f(n)$ where ~~a, b ≥ 1 and~~ $a \geq 1$ & $b > 1$ are constants.

(1) If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$.

Then $T(n) = \Theta(n^{\log_b a})$

(2) If $f(n) = \Theta(n^{\log_b a})$, Then $T(n) = \Theta(n^{\log_b a} \log n)$

(3) If $f(n) = \Omega(n^{\log_b a + \epsilon})$, for some constant $\epsilon > 0$, and

if $a f(n/b) \leq c \cdot f(n)$ for some constant $c < 1$

and ~~for~~ all sufficiently large $n$,

then

$T(n) = \Theta(f(n))$.

Q:

(a) $T(n) = 4T(n/2) + n^2$

$a = 4,\ b = 2$

$\therefore f(n) = \Theta(n^{\log_2 4}) \Rightarrow$ case (2) for M.T

$\Rightarrow T(n) = \Theta(n^2 \log n)$

(b) $T(n) = 2T(n/4) + \sqrt{n}$

$a = 2,\ b = 4 \Rightarrow \log_b a = \log_4 2$

case (2) of M.T

$\therefore T(n) = \Theta(\sqrt{n} \log n)$.

(c) $T(n) = T(n/2) + 2^n$

$a=1,\ b=2 \quad \log_2 1 = 0.$

case (3) of M.T

$\Rightarrow T(n) = \Theta(2^n)$

(d) $T(n) = 2T(n/2) + n\log n$

$a=2,\ b=2 \quad \log_2 2 = 1$

$f(n) = n\log n$

it seems that it ~~falls~~ falls in case (3) of M.T

but ~~the~~ ~~$\log a$~~ $\dfrac{f(n)}{n^{\log_b a}} = \dfrac{n\log n}{n} = \log n$

is non-polynomial

So, MT cannot be applied

(e) $T(n) = 3T(n/2) + n$

$a=3,\ b=2 \quad \log_2 3 > 1$

case (1) of M.T

$T(n) = \cancel{to}\ \Theta(n^{\log_2 3})$

(f) $T(n) = 3T(n/3) + n$

$a=3,\ b=3,\ \log_3 3 = 1$

Case (2) of M.T

$T(n) = \Theta(n\log n).$

(g) $T(n) = 0.5\,T(n/2) + 1/n$

$a = 0.5 < 1$

Master Theorem Cannot apply

(h) $T(n) = 2T(n/4) + n^2$

$a=2,\ b=4 \quad \log_b a = \log_4 2 = 0.5$

Case(3) of M.T, $\dfrac{f(n)}{n^{\log_b a}} = \dfrac{n^2}{n^{\log_4 2}}$ is polynomial

$\Rightarrow$ ~~T(n) =~~

$\Rightarrow T(n) = \Theta(n^2)$

(i) $T(n) = 3T(n/4) + n\log n$

$a=3,\ b=4 \quad \log_4 3 < 1$ (Seems case(3)

$\therefore$ ~~it fe~~ ~~cas~~ &

$$\frac{f(n)}{n^{\log_b a}} = \frac{n\log n}{n^{\log_4 3}} = n^{0.21}\log n$$

is polynomial

$\Rightarrow$ By case(3) of M.T

$T(n) = \Theta(n\log n)$

(j) $T(n) = T(7n/10) + n$

$a=1, \quad b = 10/7$

$\log_b a = \log_{10/7} 1 = 0.$

Case (3) of M.T (check $\frac{f(n)}{n^{\log_{10/7} 1}} = f(n) = n$ is polynomial

$\Rightarrow T(n) = \Theta(n)$

(k). $T(n) = 2^n T(n/2) + n^n$

$a = 2^n$ not a constant

$\Rightarrow$ Master theorem cannot apply

(l). $T(n) = \sqrt{2}\, T(n/2) + \log n$

$a = \sqrt{2}, \quad b = 2$

$\log_b a = \log_2 \sqrt{2} = 1/2$

Case (1) of M.T

$\Rightarrow T(n) = \Theta(n^{\log_b a}) = \Theta(n^{1/2}).$