sequence

$$f : \mathbb{N} \to \mathbb{R}.$$

$$f(n) = n^2 \quad \{a_n\}_{n=1}^{\infty}$$

$$\{n\}_{n=1}^{\infty}, \ \{2\}_{n=1}^{\infty}$$

convergence

$$a_n \to L$$

for any $\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t

$$|a_n - L| < \epsilon \quad \forall n \geq N$$

$$\{a_1, a_2, \dots, a_N, \dots\}$$

$L-\epsilon \quad L \quad L+\epsilon$

Ex:

$$a_n = \frac{1}{n}, \quad \lim_{n \to \infty} \frac{1}{n} = 0$$

$$\left|\frac{1}{n} - 0\right| < \epsilon \quad \forall n \geq N$$

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \epsilon$$

choose $N \in \mathbb{N}$ s.t $\frac{1}{N} < \epsilon$

$n \geq N$, $\frac{1}{n} < \epsilon$, $\frac{1}{n} \leq \frac{1}{N} < \epsilon$

$\epsilon > 0$, $\exists N$ s.t $|a_n - 0| < \epsilon$ $\forall n \geq N$

$\phantom{\epsilon > 0, \exists N \text{ s.t } |}\overset{\shortparallel}{1/n}$

<u>Ex 2</u> $\quad a_n = \frac{1}{n^p} \longrightarrow 0$, $p > 0$

$$\left|\frac{1}{n^p} - 0\right| = \frac{1}{n^p}$$

choose $N \in \mathbb{N}$ s. that $\frac{1}{N^p} < \epsilon$

$n \geq N$, $\frac{1}{n^p} < \epsilon$

$\epsilon > 0$, $\exists N \in \mathbb{N}$, $\underset{\left|\frac{1}{n^p} - 0\right|}{\overset{\shortparallel}{\frac{1}{n^p}}} < \epsilon$, $\forall n \geq N$

<u>Ex 3</u> $\quad a_n = \frac{n}{n+1} \longrightarrow 1 = L$

$$|a_n - L| = \left|\frac{n}{n+1} - 1\right|$$

$$= \frac{1}{n+1} \cancel{< \frac{1}{n}}$$

Sandwich Theorem:

$\{a_n\}, \{b_n\}, \{c_n\}$, $\quad a_n \le b_n \le c_n \quad \forall n \in \mathbb{N}$

$\lim_{n\to\infty} a_n = l$, $\lim_{n\to\infty} c_n = l$

Then $\lim_{n\to\infty} b_n = l$

Ex1 $\quad \lim_{n\to\infty} \frac{\sin n}{n} = 0$

$$\frac{-1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$0 \qquad 0 \qquad 0$$

Ex2: $\quad \lim_{n\to\infty} \frac{(-1)^n}{2^n} = 0$

$$\frac{-1}{2^n} \le \frac{(-1)^n}{2^n} \le \frac{1}{2^n}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$0 \qquad 0 \qquad 0$$

$\underline{Ex3}$:— $\lim_{n \to \infty} \left(n^{1/n} - 1\right) = 0$

$a_n = n^{1/n} - 1$

$\lim_{n \to \infty} a_n = 0$

$a_n + 1 = n^{1/n}$

$\Rightarrow n = (a_n + 1)^n$

$= 1 + {}^nC_1 a_n + {}^nC_2 a_n^2 + \cdots + {}^nC_n a_n^n$

$= 1 + n a_n + \frac{n(n-1)}{2} a_n^2 + \cdots + a_n^n$

$\geq \frac{n(n-1)}{2} a_n^2$

$n \geq \frac{n(n-1)}{2} a_n^2$

$0 \leq a_n^2 \leq \frac{2\cancel{n}}{\cancel{n}(n-1)}$

$0 \leq a_n \leq \sqrt{\frac{2}{n-1}}$

$\hookrightarrow 0$

Result $\{a_n\}$ $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$

① $L<1$, $a_n \to 0$

② $L>1$, $a_n \to \infty$

Ex 1: $a_n = \frac{2^n}{n^4}$, $a_{n+1} = \frac{2^{n+1}}{(n+1)^4}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)^4}\cdot\frac{n^4}{2^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2 \cdot n^4}{(n+1)^4}\right|$$

$a_n \to \infty$

$$= \lim_{n\to\infty}\frac{2}{(1+\frac{1}{n})^4}$$

$$= 2 > 1$$

Ex 2 :- $\lim_{n\to\infty}\frac{n}{2^n} = 0$??

$a_n = \frac{n}{2^n}$, $a_{n+1} = \frac{n+1}{2^{n+1}}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2} < 1$$

EX3: $a_n = n \cdot y^{n-1}, \quad 0 < y < 1$

$$\lim_{n\to\infty} a_n = ??$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1.)\cdot y^n}{n \cdot y^{n-1}}\right|$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)\cdot |y|$$

$$= |y| = y < 1$$

⊛ $a_n \to \infty$, as $n \to \infty$

if for any $M > 0$, $\exists\, N \in \mathbb{N}$ s.t

$$|a_n| \geq M \quad \forall n \geq N$$

$$\{a_1, a_2, \dots, a_N, \dots\}$$

$$\begin{array}{cc} a_N & \dots \\ \text{\rotatebox{90}{$\leq$}} & \text{\rotatebox{90}{$\leq$}} \\ M & M \end{array}$$

$a_n \to -\infty$ as $n \to \infty$

if $M > 0$, $\exists\, N \in \mathbb{N}$ s.t $a_n < -M$

$$\forall\, n \geq N$$

$\{n\}$

Ex 1 $\quad a_n = \frac{n^2}{n+1} \to \infty$ as $n \to \infty$

$M > 0$, $\exists N \in \mathbb{N}$ s.t $a_n \geq M$

$\exists$ there exist $\quad \forall n \geq N$

$$a_n = \frac{n^2}{n+1} \geq \frac{n^2}{2n} = \frac{n}{2}$$

choose, $N \in \mathbb{N}$ s.t $\frac{N}{2} > M$

$n \geq N$, $\frac{n}{2} \geq \frac{N}{2} > M$

$\frac{n^2}{n+1} \geq \frac{n}{2} > M \quad \forall n \geq N$

$a_n \geq M \quad \forall n \geq N$

$\sqrt{n+1}$, $\sqrt{n}$