Riemann Integral

Result :- $f : [a,b] \to \mathbb{R}$ bounded on $[a,b]$ and $\{P_n\}$ be seq^n of partition of $[a,b]$ s.t $\|P_n\| \to 0$ as $n \to \infty$. Then

① $\lim_{n\to\infty} U(P_n, f) = \overline{\int_a^b} f$.

② $\lim_{n\to\infty} L(P_n, f) = \underline{\int_a^b} f$.

EX :- $f(x) = x$ on $[0,1]$ Then find $\underline{\int_0^1} f$ and $\int_0^1 f$.

$$P_n = \left(0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right) \text{ of } [0,1]$$

$$\|P_n\| = \frac{1}{n} \to 0 \text{ as } n \to \infty$$

$$U(P_n, f) = M_1(x_1 - x_0) + M_2(x_2 - x_1) + \cdots + M_n(x_n - x_{n-1})$$

$M_1 = \sup_{x \in [0, \frac{1}{n}]} x = \frac{1}{n}$

$M_2 = \sup_{x \in [\frac{1}{n}, \frac{2}{n}]} x = \frac{2}{n}$

$M_n = \sup_{x \in [\frac{n-1}{n}, 1]} x = 1$

$$= \frac{1}{n} \cdot \frac{1}{n} + \frac{2}{n} \cdot \frac{1}{n} + \cdots + 1 \cdot \frac{1}{n}$$

$$= \frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n}{n^2}$$

$$= \frac{1}{n^2}[1 + 2 + \cdots + n]$$

$$= \frac{1}{n^2} \cdot \frac{n(n+1)}{2}$$

$\lim_{n\to\infty} U(P_n, f) = \frac{1}{2} = \overline{\int_0^1} f$

$L(P_n, f) = m_1(x_1 - x_0) + m_2(x_2 - x_1) + \cdots + m_n(x_n - x_{n-1})$

$m_1 = \inf_{x \in [0, \frac{1}{n}]} x = 0$

$m_2 = \inf_{x \in [\frac{1}{n}, \frac{2}{n}]} x = \frac{1}{n}$, $m_n = \inf_{x \in [\frac{n-1}{n}, 1]} x = \frac{n-1}{n}$

$$L(P_n, f) = 0 \cdot \frac{1}{n} + \frac{1}{n} \cdot \frac{1}{n} + \cdots + \frac{n-1}{n} \cdot \frac{1}{n}$$

$$= \frac{1}{n^2}[1 + 2 + \cdots + (n-1)]$$

$$= \frac{1}{n^2} \frac{n(n-1)}{2}$$

$$\lim_{n \to \infty} L(P_n, f) = \frac{1}{2} = \underline{\int_0^1} f$$

$$\therefore \underline{\int_0^1} f = \overline{\int_0^1} f = \frac{1}{2} = \int_0^1 f$$

<u>EX:-</u> $f(x) = x^2$ on $[0,1]$, find $\overline{\int_0^1} f$ & $\underline{\int_0^1} f$.

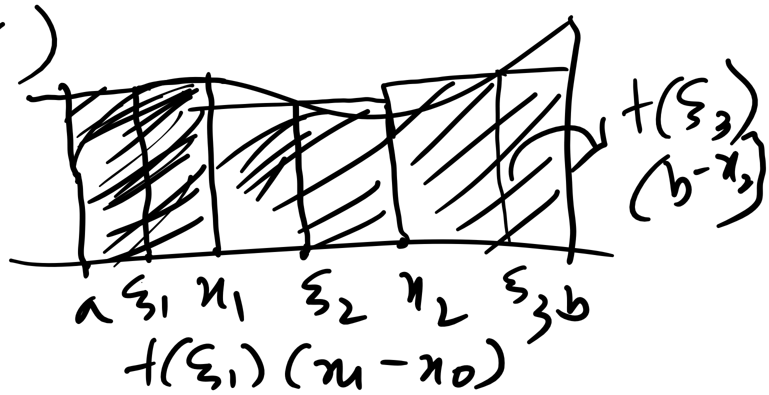
<u>Necessary & sufficient condition of Integrability:-</u>

① A bounded function $f : [a, b] \to \mathbb{R}$ is integrable if and only if for any $\epsilon > 0$ $\exists \{P_\epsilon\}$ s.t $U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon$.

② $f : [a, b] \to \mathbb{R}$ is integrable if and only if there exists seq^n of partition $\{P_n\}$ of $[a, b]$ s.t $U(P_n, f) - L(P_n, f) \to 0$ as $n \to \infty$.

<u>Riemann Sum:-</u> $S(P, f) = \sum_{i=1}^{n} \underbrace{f(\xi_i)}_{\xi_i \in [x_{i-1}, x_i]} (x_i - x_{i-1})$

$L(P,f) \leq S(P,f) \leq U(P,f)$



$m = \inf_{x \in [a,b]} f(x)$

$M = \sup_{x \in [a,b]} f(x)$

$$m(b-a) \leq L(P,f) \leq S(P,f) \leq U(P,f) \leq M(b-a)$$

<u>Darboux Theorem</u>: $f:[a,b] \to \mathbb{R}$ be Riemann integrable. Then for any $\epsilon > 0$ $\exists$ $\delta > 0$ s.t for any partition P with $\|P\| < \delta \Rightarrow \left| S(P,f) - \int_a^b f \right| < \epsilon$.

<u>Remark</u>:- If we have $\{P_n\}$ s.t $\|P_n\| \to 0$ as $n \to \infty$. Then $U(P_n,f) - L(P_n,f) \not\to 0$ as $n \to \infty$ Then f is not integrable.

<u>Result</u>:- ① If f is bounded and monotone on $[a,b]$. then $f \in R[a,b]$.

② If f is continuous on $[a,b]$. Then $f \in R[a,b]$

③ If f is bounded function which has finitely many discontinuity. Then $f \in R[a,b]$.

<u>Ex</u>:- $f(x) = \begin{cases} -1, & -2 \leq x < 0 \\ 0, & x = 0 \\ 1 & 0 < x \leq 2 \end{cases}$

f has discontinuity at $x = 0$. $f \in R[0,2]$.

(4). f is bounded, continuous on $[a,b]$ except on a infinite set $S \subset [a,b]$ s.t number of limit points of S is finite. Then $f \in R[a,b]$.

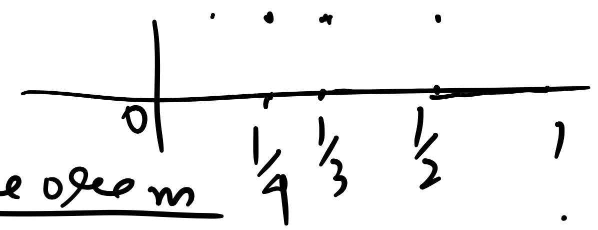
$f:[0,1] \to \mathbb{R}$.

EX:- $f(x) = \begin{cases} 1 & x = \frac{1}{n}, n \in \mathbb{N}, n \geq 2 \\ 0 & , x \neq \frac{1}{n} \end{cases}$

f has discontinuity at $x = \frac{1}{n}$

$S = \{ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \}$

limit point of $S = \{0\}$



Mean Value Theorem

Result:- Let f is continuous on $[a,b]$, then $\exists\ \xi \in [a,b]$ s.t

$$\int_a^b f(x)\,dx = f(\xi)(b-a).$$

Fundamental theorem:- Let f is continuous on $[a,b]$ and let $\phi(x) = \int_a^x f(t)\,dt$. then ϕ is differentiable and

$$\phi'(x) = f(x).$$

$\boxed{\text{If } \phi'(x) = f(x), \text{ we call } \phi \text{ is antiderivative of } f.}$

EX:- $f(x) = \begin{cases} 1 & , 0 \leq x \leq 1 \\ x & , 1 \leq x \leq 2. \end{cases}$ Verify that ϕ defined by $\phi(x)=\int_0^x f(t)\,dt$, $x\in[0,2]$ is diff. on $[0,2]$ and $\phi'(x)=f(x)$, $x\in[0,2]$.

Sol^n:- f is continuous on $[0,2]$

$\therefore f$ is integrable on $[0,2]$

i.e, $\int_0^x f(t)\,dt$, exist $\forall\, x\in[0,2]$

$$\phi(x)=\int_0^x f(t)\,dt.$$

For $0\le x\le 1$, $\phi(x)=\int_0^x 1.dt = x$.

For, $1\le x\le 2$, $\phi(x)=\int_0^x f(t)\,dt$

$$=\int_0^1 1\,dt+\int_1^x t.dt = 1+\left[\frac{t^2}{2}\right]_1^x$$

$$=\frac{1}{2}+\frac{x^2}{2}$$

$$\phi(x)=\begin{cases} x, & 0\le x\le 1\\ \frac{1}{2}(1+x^2), & 1<x\le 2.\end{cases}$$

$\phi'(1)=??\ (1)$

$$\phi'(x)=\begin{cases} 1, & 0\le x\le 1\\ x, & 1<x\le 2\end{cases}$$

$\phi'(x)=f(x)$, $x\in[0,2]$.