Lecture - 16

Method of Variation of Parameters

\# $y'' + 4y' + 4y = e^{-2x} \sin x$.

Here $a_0(x) = 1$, $a_1(x) = 4$, $a_2(x) = 4$,

$F(x) = e^{-2x} \sin x$

$$\boxed{a_0(x) y'' + a_1(x) y' + a_2(x) y = F(x)}$$

The A.E is

$m^2 + 4m + 4 = 0$

$\Rightarrow$ ~~$m = -2 \pm \sqrt{16}$~~

$(m+2)^2 = 0 \Rightarrow m = -2, -2$.

$$y_c(x) = (c_1 + c_2 x) e^{-2x}$$

$$y_c(x) = c_1 e^{-2x} + c_2 x e^{-2x}$$

$$= c_1 y_1 + c_2 y_2$$

$$\left.\begin{aligned} y_1 &= e^{-2x}, \\ y_2 &= x e^{-2x} \end{aligned}\right]$$

Let us assume

$$y_p(x) = A(x) y_1 + B(x) y_2,$$

where $A(x) = -\int \frac{F(x) y_2}{a_0(x) \cdot W(y_1, y_2)} dx$

$$B(x) = \int \frac{F(x)\ y_1}{a_0(x)\ W(y_1, y_2)}\, dx$$

$$W(y_1, y_2) = W\left(e^{-2x},\ x e^{-2x}\right)$$

$$= \begin{vmatrix} e^{-2x} & x e^{-2x} \\ -2e^{-2x} & -2x e^{-2x} + e^{-2x} \end{vmatrix}$$

$$W(y, y_2) = e^{-4x}$$

$$A(x) = -\int \frac{F(x)\ y_2}{a_0(x)\ W}\, dx$$

$$= -\int \frac{\cancel{e^{-2x}} \sin x \cdot x \cancel{e^{-2x}}}{(1)\ (\cancel{e^{-4x}})}\, dx$$

$$= -\int x \sin x\, dx$$

$$= -\left[x(-\cos x) - \int -\cos x\, dx \right]$$

$$A(x) = x \cos x - \sin x$$

$$B(x) = \int \frac{F(x)\, y_1}{a_0(x)\, W}\, dx$$

$$= \int \frac{\cancel{e^{-2x}} \sin x \cdot e^{-2x}}{(1)\,(\cancel{e^{-4x}})}\, dx$$

$$= -\cos x.$$

$$y_p(x) = A(x)\, y_1 + B(x)\, y_2$$

$$= (x\cos x - \sin x)\, e^{-2x} - \cos x\,(e^{-2x})x$$

$$= -\sin x\, e^{-2x}$$

$$y(x) = y_c(x) + y_p(x)$$

$$\boxed{y(x) = c_1 e^{-2x} + c_2 x e^{-2x} - \sin x\, e^{-2x}.}$$

<u>Example</u> $y'' + y = \tan x.$ $y = e^{mx}$

Here $a_0(x) = 1$, $a_1(x) = 0$, $a_2(x) = 1$,

$F(x) = \tan x$

A.E is $m^2 + 1 = 0 \Rightarrow m = \pm i$

$$\therefore y_c(x) = c_1 \underline{\cos x} + c_2 \underline{\sin x}$$
$$= c_1 y_1 + c_2 y_2$$

$$y_1 = \cos x, \quad y_2 = \sin x.$$

$$y_p(x) = A(x)\cos x + B(x)\sin x$$

$$A(x) = -\int \frac{F(x)\, y_2}{a_0(x)\, W}\, dx$$

$$B(x) = \int \frac{F(x)\, y_1}{a_0(x)\, W}\, dx$$

$$\boxed{\int \sec x\, dx = \log(\sec x + \tan x)}$$

$$y_p(x) = -\cos x \log(\sec x + \tan x)$$

$$y(x) = c_1 \cos x + c_2 \sin x - \cos x \log(\sec x + \tan x)$$

Laplace Transforms

Laplace transform is an integral transform

$$\underline{f(t)} \longrightarrow F(s)$$

$f(t)$ → time domain

$F(s)$ → Frequency domain

Signal Processing

Let $f(t)$ be a f^n defined for $t \in [0, \infty)$, then Laplace transform of $f(t)$ is defined as

$$\boxed{L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt}$$

$$= F(s), \quad \boxed{s>0}$$

(provided integral exists)

Sufficient conditions for the existence of Laplace Transforms

(i) If $f(t)$ is piecewise continuous

(ii) $f(t)$ is of exponential order α

$$|f(t)| \leq M e^{t\alpha}, \quad M>0$$

$|\cos t| \leq e^t$ $\qquad$ $|1| \leq e^t$

$e^{(t^2)}$, is not of exponential order

$\Rightarrow$ exponential order

① Find the Laplace transform of $f(t) = 1$, $t>0$.

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty e^{-st} \cdot (1)\, dt$$

$$= \lim_{R\to\infty} \int_0^R e^{-st}\, dt$$

$$= \lim_{R\to\infty} \left| \frac{e^{-st}}{-s} \right|_0^R$$

$$= \lim_{R\to\infty} \left[\frac{e^{-sR}}{-s} + \frac{1}{s} \right], \quad s>0$$

$$\Rightarrow \boxed{L[1] = \frac{1}{s}}, \quad \underline{s > 0}$$

(2) $f(t) = t$, find $L[t]$.

$$L[t] = \underline{\int_0^\infty e^{-st}\, t\, dt}$$

$$= \lim_{R\to\infty} \int_0^R t\, e^{-st}\, dt$$

$$= \lim_{R\to\infty} \left[\left| t\, \frac{e^{-st}}{-s} \right|_0^R - \int_0^R \frac{e^{-st}}{-s}\, dt \right]$$

$$= \lim_{R\to\infty} \left[\frac{R e^{-sR}}{-s} + 0 - \left. \frac{e^{-st}}{(-s)^2} \right|_0^R \right]$$

$(s>0)$ $$= \lim_{R\to\infty} \left[\frac{R e^{-sR}}{-s} - \left(\frac{e^{-sR}}{s^2} - \frac{1}{s^2} \right) \right]$$

$$= \frac{1}{s^2}$$

$$\Rightarrow \boxed{L[t] = \frac{1}{s^2}}$$

$L[1] = \frac{1}{s}$, $s>0$

$L[t] = \frac{1}{s^2}$, $s>0$

$L[t^\alpha] = \frac{\boxed{\alpha+1}}{s^{\alpha+1}}$, gamma fn

α not an integer

$L[t^n] = \frac{n!}{s^{n+1}}$, n is an integer

$L[e^{at}] = \frac{1}{s-a}$, $s>a$

$L[\sin bt] = \frac{b}{s^2+b^2}$

$L[\cos bt] = \frac{s}{s^2+b^2}$

$$L[\sinh bt] = \frac{b}{s^2-b^2}, \quad s^2 > b^2$$

$$L[\cosh bt] = \frac{s}{s^2-b^2}, \quad s^2 > b^2$$

$$\boxed{\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} \cdot e^{-t}\, dt}$$

$\alpha > 0$

$\Gamma(n+1) = n!$, if n is an integer