Ordinary Differential Equations(EMAT102L)
(Lecture-16)

Department of Mathematics
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Outline of the Lecture

We will learn

- How to find Particular Integral
  - Method of undetermined coefficients(already done)
  - Method of Variation of Parameters

Method of variation of parameters

- Method of Undetermined Coefficients is a restricted method which cannot apply to find particular integral for such type of equations like

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Consider the second order non-homogeneous linear equation with variable coefficients

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = F(x), \tag{1}$$

where $a_0(x) \neq 0$ and $a_0(x), a_1(x), a_2(x), F(x)$ are continuous in $[a, b]$.

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- Suppose that y_1 and y_2 are linearly independent solutions of the corresponding homogeneous equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0.$$

Then the complementary function of the given equation is

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x).$$

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Then the complementary function of the given equation is
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- In the **method of variation of parameters**, we replace the arbitrary constants c_1 and c_2 in the complementary function by respective function $A(x)$ and $B(x)$ which will be determined so that the resulting function
$$y_p(x) = A(x)y_1(x) + B(x)y_2(x) \tag{2}$$
is the particular integral of equation (1).

Method of variation of parameters(cont.)

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$$y_p'(x) = A'y_1 + Ay_1' + B'y_2 + By_2' = (A'y_1 + B'y_2) + (Ay_1' + By_2') \tag{3}$$

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- If we differentiate this equation again, then these equations would contain the second derivative A'' and B'' of the unknown functions. In order to avoid the second order derivatives, we impose the condition

$$A'y_1 + B'y_2 = 0.$$

With this condition, (3) reduces to

$$y_p'(x) = Ay_1' + By_2',$$

we obtain

$$y_p''(x) = Ay_1'' + A'y_1' + By_2'' + B'y_2'$$

- Since $y_p(x)$ satisfies the given equation. Therefore, substituting the expressions for $y_p(x),\ y_p'(x)$ and $y_p''(x)$ in equation (1), we obtain

$$
\begin{aligned}
& a_0(x)[Ay_1'' + A'y_1' + By_2'' + B'y_2'] + a_1(x)[Ay_1' + By_2'] + a_2(x)[Ay_1 + By_2] = F(x) \\
& \Rightarrow a_0(x)[A'y_1' + B'y_2'] + A[a_0(x)y_1'' + a_1(x)y_1' + a_2(x)y_1] \\
& \qquad\qquad + B[a_0(x)y_2'' + a_1(x)y_2' + a_2(x)y_2] = F(x)
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& \qquad + B[a_0(x)y_2'' + a_1(x)y_2' + a_2(x)y_2] = F(x)
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- Since, $y_1(x)$ and $y_2(x)$ are the solutions of the corresponding homogeneous equation, we obtain

$$a_0(x)[A'y_1' + B'y_2'] = F(x), \ or \ A'y_1' + B'y_2' = \frac{F(x)}{a_0(x)}.$$

- Thus we have two imposed conditions.

$$\begin{aligned} A'y_1 + B'y_2 &= 0 \\ A'y_1' + B'y_2' &= \frac{F(x)}{a_0(x)} \end{aligned}$$

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- Since y_1 and y_2 are linearly independent solutions of the corresponding homogeneous equation. Therefore the determinant of coefficients of this system is

$$\text{Wronskian } W(y_1, y_2) = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1y_2' - y_2y_1' \neq 0$$

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Hence the system has a unique solution. Solving the system of equations, we get

$$A' = -\frac{F(x)y_2}{a_0(x)(y_1y_2' - y_2y_1')}, \quad B' = \frac{F(x)y_1}{a_0(x)(y_1y_2' - y_2y_1')}$$

- Therefore

$$A' = -\frac{F(x)y_2}{a_0(x)W(y_1,y_2)}, \quad B' = \frac{F(x)y_1}{a_0(x)W(y_1,y_2)}$$

Integrating, we obtain

$$A(x) = -\int \frac{F(x)y_2}{a_0(x)W(y_1,y_2)}dx, \quad B(x) = \int \frac{F(x)y_1}{a_0(x)W(y_1,y_2)}dx \tag{4}$$

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- Thus we have particular integral y_p of the given equation is defined by

$$y_p(x) = A(x)y_1(x) + B(x)y_2(x).$$

where $A(x)$ and $B(x)$ are given by (4).

How to find Particular Integral using Method of Variation of Parameters-Working Rule

Consider the second order non-homogeneous linear equation with variable coefficients

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- **Step 1.** Suppose that y_1 and y_2 are linearly independent solutions of the corresponding homogeneous equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0.$$

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  $$y_c(x) = c_1y_1(x) + c_2y_2(x).$$

- **Step 2.** Find Wronskian of functions y_1 and y_2.

  $$W = W[y_1, y_2] = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1y_2' - y_2y_1'.$$

- **Step 2.** Let $y_p = A(x)y_1 + B(x)y_2$, where

$$A(x) = -\int \frac{F(x)y_2}{a_0(x)W}dx, B(x) = \int \frac{F(x)y_1}{a_0(x)W}dx,$$

where $W = W(y_1, y_2)$.

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- **Step 3.** So, the particular integral is

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- **Step 4.** Thus, we have the general solution

$$y(x) = y_c + y_p = c_1y_1 + c_2y_2 + A(x)y_1 + B(x)y_2.$$

Example

Consider the differential equation

$$y'' + 4y' + 4y = e^{-2x} sin\, x.$$

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Solution:

- Here $a_0(x) = 1$, $a_1(x) = 4$, $a_2(x) = 4$, $F(x) = e^{-2x} \sin x$.
  The auxiliary equation of the corresponding homogeneous equation is

$$m^2 + 4m + 4 = 0.$$

  The characteristic roots are $m = -2, -2$.

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- The complementary function is given by
  $$y_c(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 e^{-2x} + c_2 x e^{-2x}$$
  where $y_1(x) = e^{-2x}$ and $y_2(x) = xe^{-2x}$ are two linearly independent solutions of the homogeneous equation.

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- The Wronskian of $y_1,\ y_2$ is given by

$$W(x) = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \begin{pmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & e^{-2x}(1-2x) \end{pmatrix} = e^{-4x}$$

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$$\begin{aligned} A(x) &= -\int \frac{F(x)y_2(x)}{a_0(x)W}dx = -\int x\sin x dx = x\cos x - \sin x \\ B(x) &= \int \frac{F(x)y_1(x)}{a_0(x)W}dx = \int \sin x dx = -\cos x \end{aligned}$$

- Thus the particular integral is

$$\begin{aligned} y_p(x) &= A(x)e^{-2x} + B(x)xe^{-2x} \\ &= (x\cos x - \sin x)e^{-2x} + (-\cos x)xe^{-2x} \\ &= -e^{-2x}\sin x. \end{aligned}$$

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- The general solution is

$$\begin{aligned} y(x) &= y_c(x) + y_p(x) \\ &= c_1e^{-2x} + c_2xe^{-2x} - e^{-2x}\sin x. \end{aligned}$$

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- The Wronskian of y_1, y_2 is given by

  $$W(x) = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} = 1$$

- By the method of the variation of parameters, we write the particular integral as

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Example(cont.)

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- **To find $A(x)$ and $B(x)$:**

$$\begin{aligned}
A(x) &= -\int \frac{F(x)y_2(x)}{a_0(x)W}dx = -\int \tan x \sin x dx \\
&= -\int \frac{\sin^2 x}{\cos x}dx = -\int \frac{1-\cos^2 x}{\cos x}dx \\
&= \sin x - \log(\sec x + \tan x). \\
B(x) &= \int \frac{F(x)y_1(x)}{a_0(x)W}dx = \int \tan x \cos x dx = -\cos x.
\end{aligned}$$

- Thus the particular integral is

$$\begin{aligned} y_p &= A(x)\cos x + B(x)\sin x = \sin x\cos x - \cos x\log(\sec x + \tan x) - \sin x\cos x \\ &= -\cos x\log(\sec x + \tan x). \end{aligned}$$

- The general solution is

$$y(x) = y_c + y_p = c_1\cos x + c_2\sin x + A(x)\cos x + B(x)\sin x$$
$$\Rightarrow y(x) = c_1\cos x + c_2\sin x - \cos x\log(\sec x + \tan x)$$

Thank You