SYSTEMS OF LINEAR EQUATIONS:

Def^n: A linear equation in unknowns $x_1, x_2, \dots, x_n$ is an equation that can be put in the standard form

$$\boxed{a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b,} \quad \text{—— (A)}$$

where $a_1, a_2, \dots, a_n$ and b are constants.

The constants "a_K" is called the "co-efficient of x_K" and "b" is called the "constant term of the equation".

Def^n: If the constant term in equation (A) is zero. Then it is called "Homogeneous" equation.

i.e $a_1 x_1 + a_2 x_2 + \dots + a_n x_n = 0$.

if $b \neq 0$ Then the linear equation $a_1 x_1 + \dots + a_n x_n = b$ is called "Non-homogeneous".

Def^n: A solution of the linear equation (A) is a list of values for the unknowns or equivalently a vector u in K^n, say

$x_1 = k_1, \quad x_2 = k_2, \quad \dots, x_n = k_n$ or $u = (k_1, k_2, \dots, k_n)$.

Consider some examples of Linear system :-

(i) Suppose $a, b \in \mathbb{R}$. Consider the system $ax = b$.

a) if $a \neq 0$ then the system has a UNIQUE SOLUTION $x = b/a$.

b) If $a = 0$ and

(i) $b \neq 0$, then the system has NO SOLUTION.

(ii) $b = 0$, then the system has INFINITE NO. OF SOLUTIONS, namely all $x \in \mathbb{R}$.

(b) Now, we consider a system with 2 equations ~~and~~ in 2 unknowns.

Consider the equation $ax + by = c$. if either a or $b \neq 0$ then this is the linear equation a line in $\mathbb{R}^2$.

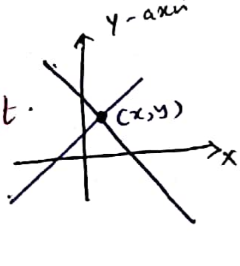
Thus for the system

$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2,$$

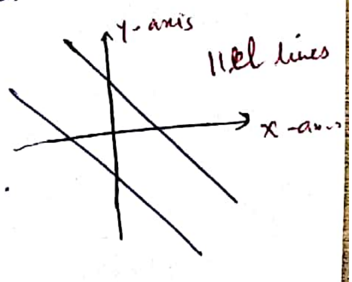
the set of solutions is given by the points of intersection of the two lines.

Then three cases arises :-

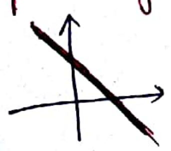
(a) Unique Solution :- If they intersect at one pt.



(b) No solution — If two lines are ||el lines



(c) Infinitely Many Soln :- If both the lines coincide.



(3) Consider 3 equations in 3 unknowns.

A linear equation $ax + by + cz = d$ represent a plane in $\mathbb{R}^3$ provided $(a, b, c) \neq (0, 0, 0)$.

Unique Sol^n:- Theree planes intersect at a point.

Infinitely Many Sol^n:- Three planes intersect on a line.

No. Solution:→ Three. parallel lines as intersection of two planes taken two at a time.

Definition :- Linear System :-

A Linear system of m equations in n-unknown $x_1, x_2, \dots, x_n$ is a set of equations of the form

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m. \end{aligned}\right\} \quad (A)$$

where $a_{ij} \in \mathbb{R}$, $1 \le i \le m$, $1 \le j \le n$, $b_i \in \mathbb{R}$.

- The Linear system (A) is called HOMOGENEOUS if $b_1 = b_2 = 0 = \dots = b_m$.

and Non-Homogeneous otherwise. i.e $b_i \neq 0$ for some i.

We rewrite the above equations in the form $Ax = b$,

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \& \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

The matrix "A" is called the COEFFICIENT MATRIX.

and the block matrix

$[A \quad b]$ is called the AUGMENTED MATRIX OF the linear system (A).

Remark :- 1) Observe that the ith row of the augmented matrix $[A \ b]$ represent the ith equation and the jth column of the co-efficient matrix A corresponds to coefficients of the jth variable x_j.

i.e. for $1 \le i \le m$, $1 \le j < n$, The entry a_{ij} of the co-efficient matrix A corresponds to the ith equation and jth variable x_j.

2) For a system of linear equation $AX = b$, The system $AX = 0$ is called the Associated Homogeneous system.

Def^n:- A solution of Linear system $AX = b$ is a column vector Y with entries $y_1, y_2, \cdots y_n$ such that the linear system (A) is satisfied by subtituting y_i in place of x_i.

i.e if $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ or $[y_1, y_2, \ldots y_n]^t$. Then $AY = b$ holds.

Note :- A zero n-tuple $X = 0$ is always a solution of the system $AX = 0$ and is called TRIVIAL SOL.

and a non-zero n-tuple X, if it satisfies $AX = 0$ is called a "NON-TRIVIAL SOLUTION."

Example :-

$x - y - z = 2, \quad 3x - 3y + 2z = 16, \quad 2x - y + z = 9.$

SOL :- The given system of linear equation can be represent as

$x - y - z = 2 \quad — (1)$

$3x - 3y + 2z = 16 \quad — (2)$

$2x - y + z = 9 \quad — (3)$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 3 & -3 & 2 & 16 \\ 2 & -1 & 1 & 9 \end{array}\right]$$

Applying $\boxed{(2) - 3 \times (1)}$ & $\boxed{(3) - 2 \times (1)}$

$R_2 \rightarrow R_2 - 3R_1$ $\quad R_3 \rightarrow R_3 - 2R_1$

i.e Multiplying equ (1) by 3 and subtract it from eqn (2)

subtract 2 times of eqn (1) from (2)

$x - y - z = 2$

$5z = 10$

$y + 3z = 5$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 0 & 5 & 10 \\ 0 & 1 & 3 & 5 \end{array}\right].$$

Interchanging the 2nd & 3rd eqn or Interchange $R_2 \rightarrow R_3$

$x - y - z = 2$

$y + 3z = 5$

$5z = 10$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 5 & 10 \end{array}\right]$$

By backward subtitution, we obtain $z = 2, \; y = -1, \; x = 3$ is a solution of the given system of equations