Ordinary Differential Equations(EMAT102L) (Lecture-7)

Department of Mathematics
Bennett University, India

We will learn

- Orthogonal Trajectories

One among the many applications of differential equation is to find a curve that intersects the given family of curves at right angles.

In other words, given a family F of curves, we wish to find a curve(or curves) G which intersect orthogonally with any member of F (whenever they intersect).

Such a family of curves G is called **orthogonal trajectories** of the family F.

Note that at the common point of intersection, the product of the slope of their tangents is -1.

In case, the families F and G are identical, then we say that the given family of curves is self orthogonal.

Definition

Let

$$F(x, y, c) = 0 \tag{1}$$

be a given one-parameter family of curves in the xy plane. A curve that intersects the curves of the family (1) at right angles is called an **orthogonal trajectory** of the given family.

Procedure for Finding the Orthogonal Trajectories of a Given Family of Curves

Step 1. From the equation

$$F(x,y,c)=0$$

of the given family of curves, find the differential equation

$$\frac{dy}{dx}=f(x,y)$$

of this family.

Step 2. In the differential equation $\frac{dy}{dx}=f(x,y)$ so found in Step 1, replace $\frac{dy}{dx}$ by $\frac{-1}{\frac{dy}{dx}}$. This gives the differential equation

$$\begin{aligned}\frac{-1}{\frac{dy}{dx}} &= f(x,y)\\ \Rightarrow \frac{dy}{dx} &= -\frac{1}{f(x,y)}\end{aligned}$$

of the orthogonal trajectories.

Step 3. Obtain a one-parameter family

$$G(x,y,c)=0$$

of solutions of the above differential equation, we get the desired family of orthogonal trajectories.

Definition

Suppose

$$\frac{dy}{dx} = f(x,y)$$

represents the DE of the family of curves. Then, the slope of any orthogonal trajectory is given by $\dfrac{dy}{dx} = -\dfrac{1}{f(x,y)}$ which is a DE of the orthogonal trajectories.

Example

Consider the family of circles $x^2 + y^2 = c^2$.

Solution: Step 1. We first find the differential equation of the given family

$$x^2 + y^2 = c^2$$

Differentiate w.r.t. x, we obtain

$$x + y\frac{dy}{dx} = 0$$
$$\Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$

Step 2. We now find the differential equation of the orthogonal trajectories by replacing $\frac{dy}{dx}$ by $\frac{-1}{\frac{dy}{dx}}$ in the above equation, obtaining

$$\frac{-1}{\frac{dy}{dx}} = \frac{-x}{y}$$
$$\Rightarrow \frac{dy}{dx} = \frac{y}{x}$$

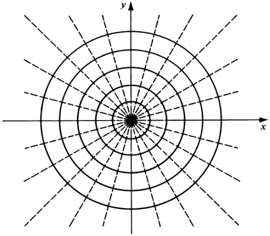
Step 3. Solving the above differential equation by separating variables, we get

$$\frac{dy}{y} = \frac{dx}{x}$$

Integrating, we obtain

$$y = cx$$

as the equation of the orthogonal trajectories of the family of circles.



Example

Find the orthogonal trajectories of the family of parabolas $y = cx^2$.

Solution:
Step 1. We first find the differential equation of the given family

$$y = cx^2 \tag{2}$$

Differentiating, we obtain

$$\frac{dy}{dx} = 2cx \tag{3}$$

Eliminating the parameter c between Equations (2) and (3), we obtain the differential equation of the family (2) in the form

$$\frac{dy}{dx} = \frac{2y}{x} \tag{4}$$

Step 2. We now find the differential equation of the orthogonal trajectories by replacing $\dfrac{dy}{dx}$ in (4) by its negative reciprocal, obtaining

$$\frac{dy}{dx} = -\frac{x}{2y} \tag{5}$$

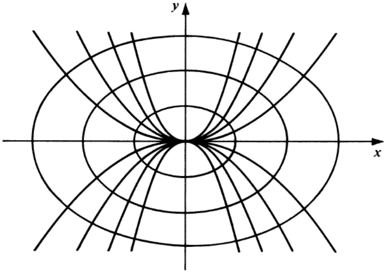
Step 3. Solving the differential equation (5) by separating variables, we get

$$2ydy = -xdx$$

Integrating, we obtain the one-parameter family of solutions of (5) in the form

$$x^2 + 2y^2 = k^2$$

where k is an arbitrary constant. This is the family of orthogonal trajectories of (2). It is clearly a family of ellipses with center at the origin and major axes along the x-axes. Some members of the original family of parabolas and some of the orthogonal trajectories (the ellipses) are shown in the following figure.


y
x

Example 1

Find the orthogonal trajectories of the family of curves

$$y^2 = x + c$$

Example 2

Show that the following family of curves

$$y^2 = 4a(x + a), a \in \mathbb{R}$$

is self orthogonal.

Thank You