Ex:- $f : [0,2] \to \mathbb{R}$,

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x^2, & 1 < x \le 2. \end{cases}$$

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{2 - x^2 - 1}{x - 1} = -2$$

$$\ne$$

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{x - 1}{x - 1} = 1$$

$\therefore$ f is not diff. at $x = 1$

$$f'(x) = \begin{cases} 1, & 0 \le x < 1 \\ -2x, & 1 < x \le 2 \end{cases}$$

$$f' : [0,1) \cup (1,2] \to \mathbb{R}.$$

- $(f \cdot g)'(c) = f'(c) \cdot g(c) + g'(c) \cdot f(c)$

Chain Rule:— If f is diff. at c

g " " at $f(c)$

$h(x) = g \circ f(x) = g(f(x))$

- h is diff at c, $h'(c) = g'(f(c)) \cdot f'(c)$.

Ex:- $$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x \sin \frac{1}{x}}{x} = \lim_{x \to 0} \sin \frac{1}{x}$$

does not exist.

So, f is not diff. at $x = 0$.

Ex:- $$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \ne 0 \\ 0, & x = 0. \end{cases}$$

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin \frac{1}{x}}{x} = \lim_{x \to 0} x \sin \frac{1}{x} = 0$$

f is diff at $x = 0$.

$x \neq 0$, $f'(x) = 2x \sin\frac{1}{x} - \cos\frac{1}{x}$

$$f'(x) = \begin{cases} 2x \sin\frac{1}{x} - \cos\frac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} 2x\sin\frac{1}{x} - \cos\frac{1}{x}$$

$$= 0 - \lim_{x\to 0} \cos\frac{1}{x}$$ does not exist.

i.e, f' is not conts at $x=0$.

Ex:- $f(x) = \begin{cases} x^3 \sin\frac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$ find f' check continuity of f'.

Local extremum

① $x=c$ is point of local maximum of $f(x)$, if $\exists\ \delta > 0$ s.t $|x-c| < \delta$
$\Rightarrow f(c) \geq f(x)$.

② $x=c$ is point of local minimum of $f(x)$, if $\exists\ \delta > 0$ s.t $|x-c| < \delta$
$\Rightarrow f(c) \leq f(x)$.

Result :- If f is diff. on (a,b) and $c \in (a,b)$ is a point of local max or min of f. Then $f'(c) = 0$

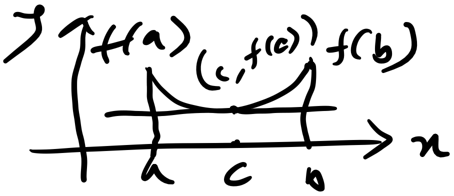
Remark:- $f(x) = x$, $f: \mathbb{R} \to \mathbb{R}$
$[0,1]$, $c = 0$ or 1 then above Result not true.

max $f = 1$, $f'(x) = 1\ \forall x \in [0,1]$

Rolle's Theorem :- Let $f: [a,b] \to \mathbb{R}$.

① f is conti on $[a,b]$.
② f is diff. on (a,b).
③ $f(a) = f(b)$. Then $\exists\ c \in (a,b)$ s.t $f'(c) = 0$

Ex:- $x^{13}+7x^3-5=0$ has exactly one (real) root.



$f(x) = x^{13}+7x^3-5$

$f(0) = -5 < 0$

$f(1) = 3 > 0$

by IVT, $\exists\ c \in (0,1)$ s.t $f(c)=0$,

suppose, $x_1, x_2 > 0$ roots of $f(x)$.

$[x_1, x_2]$ $f(x_1)=f(x_2)=0$, by Rolle's

$\exists\ y \in (x_1, x_2)$ s.t $f'(y)=0$

$f'(x) = 13x^{12}+21x^2 = x^2 \boxed{(13x^{10}+21)} \neq 0$

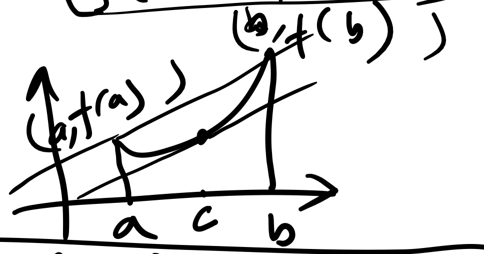
can not have positive real root

contradiction.

$\sqrt{-1} = i$

Mean value theorem:

$\boxed{f(a) \neq f(b)}$



$f:[a,b] \to \mathbb{R}$.

① f is conti. on $[a,b]$

② f is diff on (a,b)

Then $\exists\ c \in (a,b)$ s.t $\boxed{\dfrac{f(b)-f(a)}{b-a} = f'(c)}$

Ex:- $|\cos x - \cos y| \le |x-y| \quad \forall\ x,y \in \mathbb{R}$.

$f(x) = \cos x$, let $x, y \in \mathbb{R}$.

$[x,y] \to \dfrac{f(y)-f(x)}{y-x} = f'(c), \ c \in (x,y)$.

$\left|\dfrac{\cos y - \cos x}{y-x}\right| = |-\sin c| \le 1$

$\Rightarrow |\cos y - \cos x| \le |y-x|$

$\Rightarrow |\cos x - \cos y| \le |x-y|$

Ex:- $|\sin x - \sin y| \le |x-y| \quad \forall\ x,y \in \mathbb{R}$.

Ex:- If f is diff on (a,b), $f'=0 \Rightarrow f$ is const.