Ordinary Differential Equations(EMAT102L) (Lecture-4)

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Outline of the Lecture

We will learn

- Exact Differential Equation
- Solution of Exact Differential Equation

Definition

Differential of a function of 2 variables: If $F(x, y)$ is a function of two variables with continuous first order partial derivatives in a region R of the xy-plane, then its differential dF is

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy.$$

In the special case when $F(x, y) = c$, where c is a constant, we have

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$$

So given a one-parameter family of functions $F(x, y) = c$, we can generate a first order ODE by computing the differential on both sides of the equation $F(x, y) = c$.

Definition

A differential expression

$$M(x,y)dx + N(x,y)dy \tag{1}$$

is called an **exact differential** in a region R of the xy-plane if it there exists a function F of two variables such that this expression equals the total differential $dF(x,y)$ for all $(x,y) \in R$. That means, expression (1) is an **exact differential** in R if there exists a function F such that $\dfrac{\partial F}{\partial x} = M(x,y)$ and $\dfrac{\partial F}{\partial y} = N(x,y)$ for all $(x,y) \in R$.

Exact Differential Equation

If $M(x,y)dx + N(x,y)dy$ is an exact differential, then the differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

is called an **exact differential equation**.

Examples

1. $x^2y^3dx + x^3y^2dy = 0$ is an exact differential equation since $x^2y^3dx + x^3y^2dy = d\left(\frac{x^3y^3}{3}\right)$.
2. $ydx + xdy = 0$ is an exact differential equation since $ydx + xdy = d(xy)$.
3. $\frac{ydx - xdy}{y^2} = 0$ is an exact differential equation since $\frac{ydx - xdy}{y^2} = d\left(\frac{x}{y}\right)$

Theorem

Consider the differential equation

$$M(x,y)dx + N(x,y)dy = 0 \tag{2}$$

Let $M(x,y)$ *and* $N(x,y)$ *be continuous and have continuous first order partial derivatives for all points* (x,y) *in a rectangular domain R. Then the necessary and sufficient condition for (2) to be an exact differential equation is*

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Examples

Example 1.

Consider the equation $y^2dx + 2xydy = 0$.
Here $M = y^2$ and $N = 2xy$. So,

$$\frac{\partial M}{\partial y} = 2y = \frac{\partial N}{\partial x}.$$

$\Rightarrow$ the given equation is an exact equation.

Example 2.

Consider the equation $ydx + 2xdy = 0$.
Here $M = y$ and $N = 2x$. So,

$$\frac{\partial M}{\partial y} = 1 \quad and \quad \frac{\partial N}{\partial x} = 2.$$

$\Rightarrow$ the given equation is **not** an exact equation.

Example 3.

Consider the equation $(2x\sin y + y^3e^x)dx + (x^2\cos y + 3y^2e^x)dy = 0$
Here $M = 2x\sin y + y^3e^x$ and $N = (x^2\cos y + 3y^2e^x)$. So,

$$\frac{\partial M}{\partial y} = 2x\cos y + 3y^2e^x = \frac{\partial N}{\partial x}.$$

$\Rightarrow$ the given equation is an exact equation.

Solution of an exact differential equation

Let us assume that the differential equation

$$M(x, y)dx + N(x, y)dy = 0 \tag{3}$$

is exact in rectangular domain R. Then a one parameter family of solutions of this differential equation is given by

$$F(x, y) = c$$

where F is a function such that $\dfrac{\partial F}{\partial x}(x, y) = M(x, y)$ and $\dfrac{\partial F}{\partial y}(x, y) = N(x, y)$ for all $(x, y) \in R$ and c is an arbitrary constant.

How to find solution for an exact differential equation?

For a given exact DE, $M(x,y)dx + N(x,y)dy = 0$, the function $F(x,y)$ can be found either by inspection or by the following procedure:

- **Step 1.** Integrate $\dfrac{\partial F}{\partial x} = M(x,y)$ with respect to x to obtain

$$F(x,y) = \int M(x,y)dx + \phi(y),$$

  where $\phi(y)$ is a constant of integration.

- **Step 2.** To determine the function $\phi(y)$, differentiate the above equation with respect to y, to obtain

$$\frac{\partial F}{\partial y}(x,y) = \frac{\partial}{\partial y}\left(\int M(x,y)dx\right) + \frac{d\phi(y)}{dy}.$$

- **Step 3.** Use the condition

$$\frac{\partial F}{\partial y}(x,y) = N(x,y) = \frac{\partial}{\partial y}\left(\int M(x,y)dx\right) + \frac{d\phi(y)}{dy}.$$

  Determine $\phi(y)$ and hence the function $F(x,y)$.

Example

Solve the equation

$$(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0$$

Solution:

To check whether the equation is exact or not:

Comparing with $Mdx + Ndy = 0$, we get

$$M = (3x^2 + 4xy) \; and \; N = (2x^2 + 2y)$$

$$\frac{\partial M}{\partial y} = 4x = \frac{\partial N}{\partial x}$$

So, the given DE is exact.

Solution of exact differential equation: We need to find $F(x, y)$ such that

$$\frac{\partial F}{\partial x} = M(x, y) = (3x^2 + 4xy)$$

$$\frac{\partial F}{\partial y} = N(x, y) = (2x^2 + 2y)$$

Example(cont.)

Step 1. Integrate $\frac{\partial F}{\partial x} = M(x, y)$ with respect to x.

$$F(x, y) = \int M(x, y)dx + \phi(y)$$

$$F(x, y) = \int (3x^2 + 4xy)dx + \phi(y)$$

$$\Rightarrow F(x, y) = x^3 + 2x^2y + \phi(y).$$

Step 2. Find the unknown function $\phi(y)$ using the condition $\frac{\partial F}{\partial y} = N(x, y)$.

$$\frac{\partial F}{\partial y} = 2x^2 + \frac{d\phi(y)}{dy} = 2x^2 + 2y$$

$$\frac{d\phi(y)}{dy} = 2y \Rightarrow \phi(y) = y^2 + c_0$$

where c_0 is an arbitrary constant.

$$So, F(x, y) = x^3 + 2x^2y + y^2 + c_0.$$

Step 3. Hence a one parameter family of solutions is $F(x,y) = c_1$ or

$$x^3 + 2x^2y + y^2 + c_0 = c_1$$

Combining the constant c_1 and c_0, we get

$$x^3 + 2x^2y + y^2 = c$$

where $c = c_1 - c_0$ is an arbitrary constant.

Thank You