Lecture 4 - ODE (Exact Differential Equations)

Total Differential of function of two variables:

If $F(x,y)$ is a f^n of two variables having continuous partial derivatives, then total differential is defined as

$$\boxed{dF = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy}$$

If $dF = 0 \Rightarrow \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 0$

$$\Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\,\frac{dy}{dx} = 0$$

$dF = 0 \Rightarrow F(x,y) = \text{constant}$

$F(x,y) = C$

If $F(x,y) = C \Rightarrow dF = 0$

$$\Rightarrow \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 0$$

$\Rightarrow$ we can generate a first order ODE correspond to a one parameter family of curves.

Exact Differential: An expression of the form

$$M(x,y)\,dx + N(x,y)\,dy$$

is called an exact differential if $\exists$ a

f^n $F(x,y)$ s.t

$$\boxed{M(x,y)\,dx + N(x,y)\,dy = dF(x,y)}$$

Example: $y\,dx + x\,dy = d(xy)$

Exact Differential eq^n: If $M\,dx + N\,dy$ is an exact differential, then the eq^n

$$\underbrace{M\,dx + N\,dy}_{d(F)} = 0$$

is ~~ci~~ called an exact DE.

Example: (i) $y\,dx + x\,dy = 0$ is an exact DE. $d(xy)$

(ii) $x^2y^3\,dx + x^3y^2\,dy = 0$ is an exact DE

$$x^2y^3\,dx + x^3y^2\,dy = d\left(\frac{x^3y^3}{3}\right)$$

(iii) $$\boxed{\frac{y\,dx - x\,dy}{y^2} = 0}$$ because $\frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right)$

exact DE

$$d\left(\frac{x}{y}\right) = \frac{y\,dx \quad x\,dy}{y^2}$$

(iv) $y\,dx - x\,dy = 0$ → not Exact DE

Necessary & Sufficient condition to check exactness:

Consider the eqⁿ

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \text{--- (1)}$$

Then ~~(1) will be~~ the necessary & sufficient condition for the exactness of (1) is

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Example $y^2\,dx + 2xy\,dy = 0$ — (2)

Comparing the (2) with $M(x,y)\,dx + N(x,y)\,dy = 0$

$M(x,y) = y^2$ and $N(x,y) = 2xy$

$\frac{\partial M}{\partial y} = 2y$ $\qquad$ $\frac{\partial N}{\partial x} = 2y$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}} \rightarrow$$

$\Rightarrow$ (2) is an exact DE.

Example: $(2x\sin y + y^3 e^x)\,dx + (x^2\cos y + 3y^2 e^x)\,dy = 0$ — (1)

Here $M = 2x\sin y + y^3 e^x$

$N = x^2\cos y + 3y^2 e^x$

$\frac{\partial M}{\partial y} = 2x\cos y + 3y^2 e^x \;\Big|\; \frac{\partial N}{\partial x} = 2x\cos y + 3y^2 e^x$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\Rightarrow$ (1) is an exact DE.

Solution of an Exact DE:

Let us take

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is an exact DE.

$$\boxed{M(x,y)\,dx + N(x,y)\,dy} = dF(x,y)$$

$$\Rightarrow dF(x,y) = 0$$

$$\Rightarrow F(x,y) = C \rightarrow \text{Sol}^n \text{ of Exact DE.}$$

$$M(x,y)\,dx + N(x,y)\,dy = df(x,y)$$

$$\Rightarrow M(x,y)\,dx + N(x,y)\,dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$\Rightarrow \boxed{M(x,y) = \frac{\partial f}{\partial x} \quad \Big| \quad N(x,y) = \frac{\partial f}{\partial y}}$$

The sol^n of an exact DE is

$$F(x,y) = C,$$

where $\frac{\partial f}{\partial x} = M(x,y) \;\Big|\; \frac{\partial f}{\partial y} = N(x,y)$

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How to find solⁿ of an Exact DE :

The sol^n of an exact DE can be calculated by finding $F(x,y)$.

Step-1 :
$$\frac{\partial F}{\partial x} = M(x,y)$$

Integrate on both the sides w.r.t x.

$$F(x,y) = \int M(x,y)\,dx + \phi(y)$$

where $\phi(y)$ is a constant of integration

Step-2 : Determine the f^n $\phi(y)$ using the condition $\frac{\partial F}{\partial y} = N(x,y)$

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(\int M(x,y)\,dx\right) + \frac{d\phi(y)}{dy}$$

Example : Solve the eq^n
$$(3x^2+4xy)\,dx + (2x^2+2y)\,dy = 0 \quad (1)$$

Check the exactness : Comparing the given eq^n $M\,dx + N\,dy = 0$

$$M = 3x^2+4xy, \quad N = 2x^2+2y$$

$$\frac{\partial M}{\partial y} = 4x \qquad \frac{\partial N}{\partial x} = 4x$$

$\rightarrow$ (1) is an exact DE

Sol^n of (1): We need to find a f^n $f(x,y)$

s.t.

$$\frac{\partial f}{\partial x} = M(x,y)$$

$$\frac{\partial f}{\partial y} = N(x,y)$$

$$\frac{\partial f}{\partial x} = M(x,y)$$

Integrate on both the sides w.r.t 'x',

$$F(x,y) = \int M(x,y)\,dx + \phi(y)$$

$$= \int (3x^2+4xy)\,dx + \phi(y)$$

$$= \frac{3x^3}{3} + \frac{4x^2}{2}y + \phi(y)$$

$$\Rightarrow \boxed{F(x,y) = x^3 + 2x^2y + \phi(y)} \rightarrow y^2 + C_1$$

$$\frac{\partial f}{\partial y} = 2x^2 + \phi'(y)$$

$$\frac{\partial f}{\partial y} = N(x,y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = 2x^2 + 2y$$

$$\Rightarrow \cancel{2x^2} + \phi'(y) = \cancel{2x^2} + 2y$$

$$\Rightarrow \phi'(y) = 2y$$

$$\Rightarrow \phi(y) = y^2 + C_1$$

$$F(x,y) = x^3 + 2x^2y + y^2 + C_1$$

Sol^n of (1) is

$$F(x,y) = C$$

$$\Rightarrow x^3 + 2x^2y + y^2 + C_1 = C$$

$$\Rightarrow \boxed{x^3 + 2x^2y + y^2 = C - C_1 = C_0}$$