Question paper Solution :-

① Is the given matrix in the row reduced echelon form? If not, find its row reduced echelon form. $\begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 3 \end{bmatrix}$

Solution :- Ans - No, The given matrix is not is RREF.

Let $A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 3 \end{bmatrix}$

$R_1 \leftrightarrow R_2$, we obtain $\sim \begin{bmatrix} 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

$R_1 \to R_1 - R_2 \quad \sim \begin{bmatrix} 0 & 1 & 0 & +1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

Thus $RREF(A) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

$Rank(A) = 2.$

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2) Under what condition on "a" The following system has (a) Unique sol^n
(b) No sol^n.
(c) Infinitely many sol.

The given system is
$$x_1 + 2x_2 + 3x_3 = 2$$
$$x_1 + ax_2 + x_3 = 6$$
$$4x_3 = 8$$

Consider the Augmented Matrix
$$[A|b] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 1 & a & 1 & 6 \\ 0 & 0 & 4 & 8 \end{array}\right]$$

$$R_2 \to R_2 - R_1, \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & a-2 & -2 & 4 \\ 0 & 0 & 4 & 8 \end{array}\right] \quad \text{—— (1)}$$

—— [1]

If $a-2 \neq 0$, then
$$\text{rank}(A|b) = 3 = \text{rank}(A) = \text{number of variables}.$$

$\Rightarrow$ the given system of linear equations has unique solution.

Optional

Now, we have
$$4x_3 = 8 \Rightarrow x_3 = 2$$
$$(a-2)x_2 - 2x_3 = 4 \Rightarrow (a-2)x_2 - 4 = 4 \Rightarrow x_2 = \frac{8}{a-2}$$
$$x_1 + 2x_2 + 3x_3 = 2 \Rightarrow x_1 + 2\left(\frac{8}{a-2}\right) + 3(2) = 2$$
$$\Rightarrow x_1 + \frac{16}{a-2} = -4$$
$$\Rightarrow x_1 = -4 - \frac{16}{a-2} = \frac{-4(a-2)-16}{a-2}$$

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Optional

$$\Rightarrow x_1 = \frac{-4a+8-16}{a-2} = \frac{-4a-8}{a-2}$$

Thus, we have

$$x_1 = \frac{-4a-8}{a-2}, \quad x_2 = \frac{8}{a-2}, \quad x_3 = 2$$

If $a-2=0$, then we have from ①,

$$\begin{bmatrix} 1 & 2 & 3 & : & 2 \\ 0 & 0 & -2 & : & 4 \\ 0 & 0 & 4 & : & 8 \end{bmatrix}$$

$$R_3 \to R_3 + 2R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & : & 2 \\ 0 & 0 & -2 & : & 4 \\ 0 & 0 & 0 & : & 16 \end{bmatrix}$$

Here $\text{rank}(A) = 2$

but $\text{rank}(A|b) = 3$

$\Rightarrow \quad \text{rank}(A) \neq \text{rank}(A|b)$.

Thus the given system of linear equations has no solution.

There is no value of 'a' for which the given system of linear equations has infinitely many solutions.

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(3) Let $\mathbb{P}_4(\mathbb{R}) = \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 : a_0, a_1, a_2, a_3, a_4 \in \mathbb{R}\}$

$W = \{p(x) \in \mathbb{P}_4(\mathbb{R}) : p(1) = p(-1) = 0\}$

$= \{p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 :$ $p(1) = a_0 + a_1 + a_2 + a_3 + a_4 = 0$ — ①

$p(-1) = a_0 - a_1 + a_2 - a_3 + a_4 = 0\}$ — ②

& $a_i \in \mathbb{R}$, $0 \le i \le 4$

$= \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 :$ $a_0 + a_2 + a_4 = 0$ (by adding ① & ②)

$a_1 + a_3 = 0$ (by subtracting ① & ②).

& $a_i \in \mathbb{R} : 0 \le i \le 4\}$.

$= \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 : a_0 = -(a_2 + a_4),\ a_3 = -a_1\}$

$= \{-(a_2 + a_4) + a_1 x + a_2 x^2 - a_1 x^3 + a_4 x^4 : a_i \in \mathbb{R},\ i = 1, 2, 4\}$

$= \{a_1(x - x^3) + a_2(x^2 - 1) + a_4(x^4 - 1) : a_1, a_2, a_4 \in \mathbb{R}\}$

$= \text{span}\{\underbrace{x - x^3,\ x^2 - 1,\ x^4 - 1}_{S}\}$

Span (S) is always a subspace of $\mathbb{P}_4(\mathbb{R})$.
Infact, it is a smallest subspace containg S.

To show:- $\{x - x^3, x^2 - 1, x^4 - 1\}$ is linearly independent.

$\alpha(x - x^3) + \beta(x^2 - 1) + \gamma(x^4 - 1) = 0$

$\Rightarrow (-\beta - \gamma) + \alpha x + \beta x^2 - \alpha x^3 + \gamma x^4 = 0$

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comparing the like power of x, we obtain

$\alpha = 0$, $\beta = 0$, $\gamma = 0$

Thus, the set S is linearly independent.

So, S forms a basis for W.

$\dim W = 3$.

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OR

(9) (a) Let

$$V = \left\{ \begin{bmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{bmatrix} ;\ a,b,c,d,e,f,g,h,i \in \mathbb{R} \right\}.$$

$$= \left\{ a \underbrace{\begin{bmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}}_{v_1} + b \underbrace{\begin{bmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}}_{v_2} + c \underbrace{\begin{bmatrix} 0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0 \end{bmatrix}}_{v_3} + d \underbrace{\begin{bmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{bmatrix}}_{v_4} \right.$$

$$+ e \underbrace{\begin{bmatrix} 0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}}_{v_5} + f \underbrace{\begin{bmatrix} 0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0 \end{bmatrix}}_{v_6} + g \underbrace{\begin{bmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0 \end{bmatrix}}_{v_7}$$

$$+ h \underbrace{\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0 \end{bmatrix}}_{v_8} + i \underbrace{\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0 \end{bmatrix}}_{v_9} + j \underbrace{\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1 \end{bmatrix}}_{v_{10}}.$$

$= \operatorname{span}\{v_1, v_2, \ldots, v_{10}\}$

$\dim V = 10.$

$B = \{v_1, v_2, \ldots, v_{10}\}$ forms a basis.

$\therefore \{v_1, v_2, \ldots v_{10}\}$ is l.i.

(b) $V - \{A \in M_{n\times n}(\mathbb{C}) : A$ is Heremitian$\}$ is not a subspace.

So, we can not find Basis & dim.

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(4) Justify your Answer :

(a) The set $S = \{(x,y,z) \in \mathbb{R}^3 : x \text{ is an irrational no}\}$ is a subspace of $\mathbb{R}^3$.

Solⁿ: Ans — False. because S is not closed under scalar multiplication.

Take $\alpha = \sqrt{2}$, $x = (\sqrt{2}, 2, 3)$ Then $\alpha x = (2, 2\sqrt{2}, 3\sqrt{2}) \notin S$

as 2 is not an irrational no.

(b) The maximum number of linearly independent vector in $\mathbb{R}^3$ are 3.

Solⁿ: True. Because $\dim \mathbb{R}^3 = 3$. and There are "3" elements in Basis and Basis contains maximum no. of linearly independent vectors.

(c) $\dim U = 3$, $\dim W = 5$ & $U + W = \mathbb{R}^8$. Then $U \cap W \neq \{0\}$.

Solⁿ: False.

We know that $\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$

$\Rightarrow \; 8 = 3 + 5 - \dim(U \cap W)$

$\Rightarrow \; \dim(U \cap W) = 0$

$\Rightarrow \; U \cap W = \{0\}$.

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(d) Is the map $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(x, y, z) = (|x|, y-z)$$

is linear.

Solution: False, as $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x,y,z) = (|x|, y-z)$ is not a linear transformation.

Let us take $X = (1, 2, 3)$ and $Y = (-1, 2, 3) \in \mathbb{R}^3$.

Then $T(1, 2, 3) = (|1|, -1) = (1, -1)$

and $T(-1, 2, 3) = (|-1|, -1) = (1, -1)$

$\Rightarrow$ $T(X) = (1, -1)$ and $T(Y) = (1, -1)$

$\Rightarrow$ $T(X) + T(Y) = (2, -2)$

Now, consider $X+Y = (0, 2, 3)$

$\Rightarrow$ $T(X+Y) = (0, -1) \neq T(X) + T(Y)$

Thus, we have

$T(X+Y) \neq T(X) + T(Y)$ for $X = (1, 2, 3)$
$Y = (-1, 2, 3) \in \mathbb{R}^3$.

$\Rightarrow$ $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (|x|, y-z)$ is not a linear transformation.

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5. $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear Transformation such that

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}3\\4\end{bmatrix}$$

Find $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)$.

Solⁿ: $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = T\left[x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix}\right)$

$$= xT\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + y\,T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) \quad (\because T \text{ is linear})$$

$$= x\begin{bmatrix}1\\2\end{bmatrix} + y\begin{bmatrix}3\\4\end{bmatrix}$$

$$= \begin{bmatrix}x+3y\\2x+4y\end{bmatrix} \quad \underline{\text{Ans}}.$$

6. Determine all the linear maps $T: \mathbb{R}^4 \to \mathbb{R}^5$, which is onto.

Solⁿ: Suppose There is a linear Transformation which is on-to.
By Rank-nullity Theorem.

$$\boxed{\dim(\text{Null}(T)) + \dim(\text{Range}(T)) = \dim(\mathbb{R}^4)} \quad (1)$$

Since T is on-to. $\therefore \dim(\text{Range}(T)) = 5$, $\dim(\mathbb{R}^4) = 4$

Thus from (1), we obtain

$$\dim(\text{Null}(T)) = 4-5 = -1$$, which is not possible.

$\therefore$ There is "no" linear transformation which is on-to.

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(7) Let A be $n \times n$ sqare matrix s.t det $A \neq 0$.

Then find the basis & dim of $W = \{ X \in \mathbb{R}^{n \times 1} : AX = 0 \}$.

Sol^n: We know that if det $A \neq 0$ Then $AX = 0$ has only trivial sol^n. i.e $X = \bar{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Then $W = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

dim $W = 0$ and There is no element in Basis.

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