Q.1 Let A be a 5×5 matrix with real entries having eigenvalues $3+i$, 2, 5 such that trace of A is 7. Find others eigen values of A.

Sol^n Let λ_1 and λ_2 be others eigen values.

$\lambda_1 = 3-i$ [$\because$ Complex eigen values always come in conjugate pair]

$tr(A) = 7$

$3+i+2+5+3-i+\lambda_2 = 7$

$\Rightarrow \lambda_2 = \cancel{8} -6$

Q.2 Is zero matrix of order 2×2 diagonalizable?

Sol^n Let $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

for eigen values, $\begin{vmatrix} -\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 = 0$

$\lambda = 0, 0$

So. A.M corresponding to $\lambda = 0$ is 2.

For eigen vectors $\rightarrow$ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$0\cdot x_1 + 0\cdot x_2 = 0$

Let $x_1 = s$, $x_2 = t$

$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\therefore$ eigen space corresponding to $\lambda = 0$ is $\text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

$\therefore$ G.M $= 2$

$\because$ A.M $=$ G.M $= 2$

$\therefore$ A is diagonalizable.

Q.3 Find the eigen values and eigen vectors of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Also calculate the eigen values of A^{-2}.

Sol^n

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 1 = 0$$

$$\Rightarrow \lambda = \pm i$$

Corresponding to $\lambda = i$, eigen vector is $\begin{bmatrix} i \\ 1 \end{bmatrix}$

Corresponding to $\lambda = -i$, eigen vector is $\begin{bmatrix} -i \\ 1 \end{bmatrix}$

for eigen values of A^{-2} i.e. $(A^{-1})^2$

eigen values of A^{-1} are $\frac{1}{i}, \frac{-1}{i}$

eigen values of $(A^{-1})^2$ are $\left(\frac{1}{i}\right)^2, \left(-\frac{1}{i}\right)^2$

$\therefore$ eigen values of A^{-2} are $-1, -1$.

Q.4 Consider $\mathbb{R}^3$ with standard inner product. Let $S = \{(1,1,1), (2,-1,0), (1,0,1)\}$. Then find an orthonormal set T with $L(S) = L(T)$

Sol^n Let $u_1 = (1,1,1)$, $u_2 = (2,-1,0)$, $u_3 = (1,0,1)$

$\therefore$ $S = \{u_1, u_2, u_3\}$

By Gram Schmidt orthogonalization process, we can find orthogonal set $\{v_1, v_2, v_3\}$ to S.

$$v_1 = u_1 = (1,1,1)$$

$$\langle u_2, v_1 \rangle = (2,-1,0).(1,1,1) = 2-1 = 1$$

$$\langle v_1, v_1 \rangle = (1,1,1).(1,1,1) = 3$$

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle v_1}{\langle v_1, v_1 \rangle}$$

$$= (2,-1,0) - \frac{(1,1,1)}{3}$$

$$= \left(\frac{5}{3}, \frac{-4}{3}, -\frac{1}{3}\right)$$

$\langle u_3, v_1 \rangle = (1,0,1)\cdot(1,1,1) = 1+1 = 2$

$$\langle u_3, v_2 \rangle = (1,0,1)\cdot\left(\frac{5}{3}, -\frac{4}{3}, -\frac{1}{3}\right)$$
$$= \frac{5}{3} - \frac{1}{3}$$
$$= \frac{4}{3}$$

$$\langle v_2, v_2 \rangle = \left(\frac{5}{3}, -\frac{4}{3}, -\frac{1}{3}\right)\cdot\left(\frac{5}{3}, -\frac{4}{3}, -\frac{1}{3}\right)$$
$$= \frac{25}{9} + \frac{16}{9} + \frac{1}{9}$$
$$= \frac{42}{9}$$
$$= \frac{14}{3}$$

$$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle v_1}{\langle v_1, v_1 \rangle} - \frac{\langle u_3, v_2 \rangle v_2}{\langle v_2, v_2 \rangle}$$
$$= (1,0,1) - \frac{2(1,1,1)}{3} - \frac{\frac{4}{\cancel{3}}\left(\frac{5}{3}, -\frac{4}{3}, -\frac{1}{3}\right)}{\frac{14}{\cancel{3}}}$$
$$= (1,0,1) - \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right) - \frac{2}{7}\left(\frac{5}{3}, -\frac{4}{3}, -\frac{1}{3}\right)$$
$$= \left(-\frac{3}{21}, -\frac{6}{21}, \frac{9}{21}\right)$$
$$= \left(-\frac{1}{7}, -\frac{2}{7}, \frac{3}{7}\right)$$

orthonormal set $T = \left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right\}$

$$= \left\{ \frac{1}{\sqrt{3}}(1,1,1), \sqrt{\frac{3}{14}}\left(\frac{5}{3}, -\frac{4}{3}, -\frac{1}{3}\right), \sqrt{\frac{7}{2}}\left(-\frac{1}{7}, -\frac{2}{7}, \frac{3}{7}\right) \right\}$$

Q. 5 Find a non zero vector w that is orthogonal to $u = (1, 2, 1)$ and $v = (1, 0, 1)$ in $\mathbb{R}^3$.

Sol^n $\because$ w is orthogonal to u and v

$\therefore$ $\langle w, u \rangle = 0$ and $\langle w, v \rangle = 0$

Let $w = (w_1, w_2, w_3)$

$(w_1, w_2, w_3) \cdot (1, 2, 1) = 0$

$w_1 + 2w_2 + w_3 = 0$ —— (1)

$(w_1, w_2, w_3) \cdot (1, 0, 1) = 0$

$w_1 + w_3 = 0$ —— (2).

In eqn (1) and (2), w_3 is free variable

Let $w_3 = t$

on solving, we have

$(w_1, w_2, w_3) = (-t, 0, t)$

$w = t(-1, 0, 1)$, $t \in \mathbb{R}$.

Q. 6 If $\langle u, v \rangle = 2 + i$. Then calculate $\langle (1+i)u, (1+i)v \rangle$

Sol^n

$$\langle (1+i)u, (1+i)v \rangle = (1+i)\overline{(1+i)}\langle u, v \rangle$$
$$= (1+i)(1-i)(2+i)$$
$$= (1-i^2)(2+i)$$
$$= 2(2+i)$$
$$= 4 + 2i$$