SOLUTION:-

Question. 1

(a) $$A = \begin{bmatrix} 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Step 1:-
Use $R_1 \longrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & -2 & 1 & 0 \end{bmatrix} \rightarrow \text{is the Row echelon form.}$$

. Since there are three non-zero rows. $\therefore \rho(A) = 3$

Row Reduced :- Step 2:- $R_3 \longrightarrow -\frac{1}{2} R_3$.

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1/2 & 0 \end{bmatrix}$$

$R_1 \longrightarrow R_1 - R_3$

$R_2 \longrightarrow R_2 - R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 3/2 & 1 \\ 0 & 1 & 0 & 1/2 & 1 \\ 0 & 0 & 1 & -1/2 & 0 \end{bmatrix}$$

(b)

(b) $A = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 2 & 1 \\ 0 & 4 & 0 & 1 & 1 \end{bmatrix}$

<u>Step 1</u>:- Use $R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 3 & 2 & 1 \\ 0 & 4 & 0 & 1 & 1 \end{bmatrix}$$

<u>Step 2</u>:- $R_3 \to R_3 - R_2$
$R_4 \to R_4 - 4R_2$

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 4 & 1 & 1 \\ 0 & 0 & 4 & -3 & 1 \end{bmatrix}$$

<u>Step 3</u>:- $R_4 \to R_4 - R_3$

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 4 & 1 & 1 \\ 0 & 0 & 0 & -4 & 0 \end{bmatrix}$$

This is the row echelon form. $\rho(A) = 4$

~~Step 4~~:- <u>Row Reduced form</u>

<u>Step 4</u> :- Use $R_3 \to \frac{1}{4} R_3$ in step 3,
$R_4 \to -\frac{1}{4} R_4$

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

<u>Step 5</u> — $R_1 \to R_1 - 2R_3$
$R_2 \to R_2 + R_3$

$$\begin{bmatrix} 1 & 0 & 0 & -1/2 & 1/2 \\ 0 & 1 & 0 & 5/4 & 1/4 \\ 0 & 0 & 1 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

<u>Step 6</u>:- $R_1 \to R_1 + \frac{1}{2} R_4$
$R_2 \to R_2 - \frac{5}{4} R_4$
$R_3 \to R_3 - \frac{1}{4} R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 1/2 & 1/2 \\ 0 & 1 & 0 & 0 & 1/4 \\ 0 & 0 & 1 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$ Ans

(C) $\begin{bmatrix} 0 & 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$

Step 1:- $R_3 \to R_3 - R_1$ $\sim \begin{bmatrix} 0 & 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & -3 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$

Step 2:- $R_3 \to R_3 + 2R_2$ $\sim \begin{bmatrix} 0 & 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$

Step 3:- $R_4 \to R_4 + R_3$ $\sim \begin{bmatrix} 0 & 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ is the Row echelon form.

$\rho(A) = 4.$

Step 4:- For Row Reduced Echelon form, we need to proceed further,

$R_3 \to -R_3$ $\sim \begin{bmatrix} 0 & 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

Step 5:- $R_1 \to R_1 - 3R_2$ $\sim \begin{bmatrix} 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

Step 6 :-

$R_1 \to R_1 + R_3$
$R_2 \to R_2 - R_3$

$$\sim \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

step 7 :-

$R_1 \to R_1 - R_4$
$R_2 \to R_2 - R_4$

$$\sim \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \text{ Ans}$$

Ques 2 :-

$$ax + by + cz = b_1$$
$$dy + ez = b_2$$
$$fz = b_3$$

No solⁿ : $f = 0$, $b_3 \neq 0$, $a, b, c, d, e, b_1, b_2 \in \mathbb{R}$ $\cancel{d \neq 0, a \neq 0}$

Unique solⁿ : $f \neq 0$, $a, b, c, d, e, b_1, b_2, b_3 \in \mathbb{R}$ $d, a \neq 0$

Infinitely Many solⁿ : $f = 0$, $b_3 = 0$, $a, b, c, d, e, b_1, b_2 \in \mathbb{R}$.

(3) (a) $x+2y+z=1$

$3x+7y+6z=5$

$-2x-y+7z=4$

$\underline{\text{Sol}^n}$: Augumented Matrix of given system is given by

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 3 & 7 & 6 & 5 \\ -2 & -1 & 7 & 4 \end{array}\right]$$

$\underline{\text{Step 1}}$: $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 + 2R_1$ $\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 3 & 9 & 6 \end{array}\right]$

$R_3 \to R_3 - 3R_2$, $\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

z is the free variable. $\therefore$ $x_3 = t$, $t \in \mathbb{R}$

x, y. are leading (pivot) element.

$y + 3z = 2 \Rightarrow y = 2-3t$

$x+2y+z=1 \Rightarrow x = 1 - t - 4 + 6t = -3+5t$

i.e $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3+5t \\ 2-3t \\ t \end{bmatrix} = \underbrace{\begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}}_{\text{General form of sol}^n}$.

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Solution using Gauss Jordan Method:

Augmented matrix is

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 3 & 7 & 6 & 5 \\ -2 & 1 & 7 & 4 \end{array}\right]$$

$R_2 \to R_2 - 3R_1, \quad R_3 \to R_3 + 2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 3 & 9 & 6 \end{array}\right]$$

$R_1 \to R_1 - 2R_2, \quad R_3 \to R_3 - 3R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -5 & -3 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Here $\text{rank}[A] = 2 = \text{rank}[A|b] < 3 =$ number of variable

$\Rightarrow$ The given system has infinitely many solutions.

Here z is the free variable and x, y are leading elements.

Let $z = t, \quad t \in \mathbb{R}$

$y + 3z = 2 \quad \Rightarrow \quad y = 2 - 3t$

$x - 5z = -3 \quad \Rightarrow \quad x = -3 + 5t$

Thus, we have $x = -3+5t, \; y = 2-3t, \; z = t$,

ie $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3+5t \\ 2-3t \\ t \end{bmatrix}, = \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}.$

4. (a) Solve the system

$$2x_1 + x_2 + x_3 = 0$$
$$x_1 + x_3 = 0$$
$$2x_2 + x_3 = 0$$

over $\mathbb{Z}_3$.

Sol^n:- We have

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

$R_2 \to 2R_2 - R_1$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\approx \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} 2x_1 + x_2 + x_3 = 0 \\ 2x_2 + x_3 = 0 \end{matrix}$$

$R(A) = 2$

Taking $x_3 = s$

$2x_2 = -s$ $\quad (\because -1 = 2 \text{ in } \mathbb{Z}_3)$

$2x_2 = 2s \Rightarrow x_2 = s$

$2x_1 = -x_2 - x_3$

$\Rightarrow 2x_1 = -2s$

$\Rightarrow x_1 = -s$

$\Rightarrow x_1 = 2s$ $\quad (\because -1 = 2 \text{ in } \mathbb{Z}_3)$

4)(6)

$$2x_1 + x_2 + x_3 = 2$$
$$x_2 + x_3 = 1 \qquad \text{over } \mathbb{Z}_7.$$
$$x_3 = 4$$

$\underline{\text{Sol}^n}$:

$x_3 = 4$

$x_2 = 1 - x_3 = 1 - 4 = -3 \doteq 4$ in $\mathbb{Z}_7$.

$2x_1 + x_2 + x_3 = 2 \Rightarrow 2x_1 = 2 - x_2 - x_3$

$= 2 - 4 - 4$

$= 2 - 8$

$= -6$

$= 1.$

$\Rightarrow 2x_1 = 1$

$\Rightarrow x_1 = 1 \times 2^{-1} = 4$

$$\boxed{x_1 = x_2 = x_3 = 4 \text{ in } \mathbb{Z}_7}$$

5 Find the values of a, b, c such that the graph of the polynomial $p(x) = ax^2+bx+c$ passes through the points $(1, 2)$, $(-1, 6)$ and $(2, 3)$.

Solution:

$$p(1)=2 \Rightarrow a+b+c=2$$

$$p(-1)=6 \Rightarrow a-b+c=6$$

$$p(2)=3 \Rightarrow 4a+2b+c=3$$

Thus we have a system of equations

$$a+b+c=2$$
$$a-b+c=6$$
$$4a+2b+c=3$$

Augmented Matrix is

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 6 \\ 4 & 2 & 1 & 3 \end{array}\right]$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 4R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 0 & 4 \\ 0 & -2 & -3 & -5 \end{array}\right]$$

Applying $R_2 \to -\frac{1}{2} R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & +2 \\ 0 & 1 & 0 & -2 \\ 0 & -2 & -3 & -5 \end{array}\right]$$

Applying $R_3 \to R_3 + 2R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & -3 & -9 \end{array}\right]$$

Thus, we have

$$a+b+c = 2$$
$$b = -2$$
$$-3c = -9 \Rightarrow c=3$$

$$\Rightarrow \boxed{a=1, \ b=-2, \ c=3}$$ Ans.

Question no - 6 .

Solution ÷ Let x_i denote the amount of F_i in mixture.

According to the problem, we have

$$x_1 + x_2 + x_3 + x_4 = 14$$
$$x_1 + 3x_2 + 2x_3 + x_4 = 29$$
$$4x_1 + x_3 + x_4 = 23.$$

The augmented matrix of given eqn is

$$[A|b] = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 14 \\ 1 & 3 & 2 & 1 & 29 \\ 4 & 0 & 1 & 1 & 23 \end{array}\right].$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - 4R_1$
$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 14 \\ 0 & 2 & 1 & 0 & 15 \\ 0 & -4 & -3 & -3 & -33 \end{array}\right]$$

$R_3 \to R_3 + 2R_2$
$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 14 \\ 0 & 2 & 1 & 0 & 15 \\ 0 & 0 & -1 & -3 & -3 \end{array}\right]$$

Thus, The general soln is

$x_4 = t$, $x_3 = 3 - 3t$, $x_2 = \dfrac{15 - 3 + 3t}{2} = 6 + \dfrac{3t}{2}$

$$x_1 = 14 - 6 + \frac{3t}{2} - (3 - 3t) - t = 5 + 7t/2.$$

As, x_3 must remain non-negative, $t \leq 1$.

consequently, at most 7.5 mg of F_2 may be used.

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Solution II:

II(a)

Here

$$[A|b] = \left[\begin{array}{cc|c} 2 & 3 & 3 \\ 4 & 5 & 5 \end{array}\right]$$

$$\begin{array}{c} (1) \\ (2)-2(1) \end{array} \left[\begin{array}{cc|c} 2 & 3 & 3 \\ 0 & -1 & -1 \end{array}\right]$$

Here number of unknowns = rank$[A]$ = rank$[A|b]$ = 2.

$\Rightarrow$ The system has a unique solution.

II(b)

Here $[A|b] =$

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 3 \\ 2 & 1 & -1 & 1 \\ 6 & 5 & 2 & 5 \end{array}\right]$$

$$\begin{array}{c} (1) \\ (2)-(1) \\ (3)-3(1) \end{array} \left[\begin{array}{ccc|c} 2 & 3 & 4 & 3 \\ 0 & -2 & -5 & -2 \\ 0 & -4 & -10 & -4 \end{array}\right]$$

$$\begin{array}{c} (1) \\ (2) \\ (3)-2(2) \end{array} \left[\begin{array}{ccc|c} 2 & 3 & 4 & 3 \\ 0 & -2 & -5 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Here number of unknowns = 3

and rank$[A]$ = rank$[A|b]$ = 2.

$\therefore$ The system has a one parameter family of solutions.

7(c) Here

$$[A|b] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 2 & 2 & -1 & 1 \\ 6 & 4 & 1 & 4 \end{array}\right]$$

$$\begin{array}{r} (1) \\ (2)-(1) \\ (3)-3(1) \end{array} \left[\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 1 & -2 & -2 \end{array}\right]$$

$$\begin{array}{r} (1) \\ (2) \\ (3)-(2) \end{array} \left[\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

Here $\text{rank}[A] = 2 < \text{rank}[A|b] = 3$.

$\therefore$ The system has no solution.

7(d) Here $[A|b] =$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 1 & 2 & 1 & -1 \\ -1 & 3 & 2 & 1 & 1 & 2 \\ 2 & 0 & 5 & 7 & 4 & -1 \\ -1 & 5 & 5 & 4 & 3 & 3 \end{array}\right]$$

$$\begin{array}{r} (1) \\ (2)+(1) \\ (3)-2(1) \\ (4)+(1) \end{array} \left[\begin{array}{ccccc|c} 1 & -1 & 1 & 2 & 1 & -1 \\ 0 & 2 & 3 & 3 & 2 & 1 \\ 0 & 2 & 3 & 3 & 2 & 1 \\ 0 & 4 & 6 & 6 & 4 & 2 \end{array}\right]$$

$$\begin{array}{r} (1) \\ (2) \\ (3)-(2) \\ (4)-2(2) \end{array} \left[\begin{array}{ccccc|c} 1 & -1 & 1 & 2 & 1 & -1 \\ 0 & 2 & 3 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Here number of unknowns $= 5 > \text{rank}[A] = \text{rank}[A|b] = 2$.

The system has a three parameter family of solutions.

Prob. 8

① $ax + z = 2$

$ax + ay + 4z = 4$

$ay + bz = b$

Sol^n: Consider the augmented matrix

$$\left[\begin{array}{ccc|c} a & 0 & 1 & 2 \\ a & a & 4 & 4 \\ a & a & b & b \end{array}\right]$$

$R_2 \to R_2 - R_1$
$$\sim \left[\begin{array}{ccc|c} a & 0 & 1 & 2 \\ 0 & a & 3 & 2 \\ 0 & a & b & 1 \end{array}\right]$$

$R_3 \to R_3 - R_2$
$$\sim \left[\begin{array}{ccc|c} a & 0 & 1 & 2 \\ 0 & a & 3 & 2 \\ 0 & 0 & b-3 & 1 \end{array}\right].$$

No sol^n: $\rho(A) \neq \rho(A|b)$.

$b - 3 = 0 \Rightarrow \boxed{b = 3}$, $a \in \mathbb{R}$.

Also, if $a = 0$, $b - 3 \neq 0$
$\rho(A) = 1$
$\rho(A|b) = 2$ $\Rightarrow$ No sol^n.

Unique sol^n: $\rho(A) = \rho(A|b) = 3$.

$b - 3 \neq 0$, $a \neq 0$.

i.e $b \neq 3$, $a \neq 0$.

Sol^n are –
$(b-3)z = 1 \Rightarrow z = \frac{1}{b-3}$, $b \neq 3$
$ay + 3z = 2 \Rightarrow y = \frac{2-3z}{a}$, $a \neq 0$
$ax + z = 2$.

Prob :- 3

(2) Consider the system

$$x + ay = 4 \quad , \quad ax + 9y = b.$$

(a) Find the value of "a" for which the system have a unique solution.

(b) Find those pairs of value (a, b) for which the system has more than one solution.

(c) No sol^n.

Sol^n: a) Consider $x + ay = 4 \Rightarrow x = 4 - ay$.

Subtituting the value of x in $ax + 9y = b$, we obtain

$$4a - a^2 y + 9y = b$$

$$\Rightarrow y(9 - a^2) = -4a + b$$

$$\text{i.e } \boxed{(9 - a^2)y = b - 4a} \quad \text{—— (A)}$$

Thus, the system has unique sol^n iff the coefficient of y in (A) is not zero.

$$\text{i.e } 9 - a^2 \neq 0 \Rightarrow \boxed{a \neq \pm 3}$$

(b) The system has more than one sol^n if both sides of (A) are zero.

$$\text{i.e } 9 - a^2 = 0 \quad , \quad b - 4a = 0$$

$$\Rightarrow a = \pm 3 \quad , \quad \boxed{b = 4a}$$

$$a = 3, \Rightarrow b = 12 \quad , \quad a = -3 \Rightarrow b = -12$$

Thus, $(3, 12)$, $(-3, -12)$ are the pairs for which sys. has more than one sol^n.

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(c) No - sol^n :-

$9 - a^2 = 0$, $b - 4a \neq 0$

$\Rightarrow a = \pm 3$, $b \neq 4a$

i.e for the pairs of (a, b) . $(3, 12)$, $(-3, -12)$, The system has no sol^n.

OR Using the Rank - Method -

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(3) Prob 8

$$x_1 + x_2 + 2x_3 = b$$
$$x_2 + x_3 + 2x_4 = 0$$
$$x_1 + x_2 + 3x_3 + 3x_4 = 0$$
$$2x_2 + 5x_3 + ax_4 = 3.$$

Sol^n: Consider the Augmented Matrix of the system

$$\left[\begin{array}{cccc:c} 1 & 1 & 2 & 0 & b \\ 0 & 1 & 1 & 2 & 0 \\ 1 & 1 & 3 & 3 & 0 \\ 0 & 2 & 5 & a & 3 \end{array}\right]$$

$R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{cccc:c} 1 & 1 & 2 & 0 & b \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 & -b \\ 0 & 2 & 5 & a & 3 \end{array}\right]$$

$R_4 \to R_4 - 2R_2$

$$\left[\begin{array}{cccc:c} 1 & 1 & 2 & 0 & b \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 & -b \\ 0 & 0 & 3 & a-4 & 3. \end{array}\right]$$

$R_4 \to R_4 - 3R_3$

$$\left[\begin{array}{cccc:c} 1 & 1 & 2 & 0 & b \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 & -b \\ 0 & 0 & 0 & a-13 & 3+3b \end{array}\right]$$

$$(a-13)x_4 = 3+3b$$

No sol^n: $\rho(A) \neq \rho(A|b)$

if $a-13 = 0$, $3+3b \neq 0$.

$a = 13$, $b \neq -1$

Unique sol^n: $\rho(A) = \rho(A|b) = 4$

if $a-13 \neq 0$, $b \in \mathbb{R}$

$a \neq 13$, $b \in \mathbb{R}$.

Sol^n: $\boxed{x_4 = 3+3b/a-13}$

~~got~~ $x_3 + 3x_4 = -b \Rightarrow \boxed{x_3 = -b-3x_4}$

$x_2 + x_3 + 2x_4 = 0 \Rightarrow x_2 = -x_3 - 2x_4$
$= b + 3x_4 - 2x_4$
$\Rightarrow \boxed{x_2 = b + x_4}$

$x_1 + x_2 + 2x_3 = b \Rightarrow x_1 = b - x_2 - 2x_3$
$= \cancel{b} - \cancel{b} - x_4 + 2b + 6x_4$
$\Rightarrow \boxed{x_1 = 2b + 5x_4}$

Infinitely many sol^n:

$a-13 = 0$, $3+3b = 0$
i.e. $a = 13$, $b = -1$.

Take $x_4 = t$.

Then $x_3 = 1-3t$, $x_2 + x_3 + 2x_4 = 0 \Rightarrow x_2 = -1-t$, $x_1 + x_2 + 2x_3 = -1 \Rightarrow x_1 = -2+7t$

Q Prove the following

(a) Show that if x_1 and x_2 are any two solutions of a linear system of equations $Ax=b$, then x_1-x_2 is a solution of the associated homogeneous system $Ax=0$.

(b) Show that if x_1 is any solution of the original system $Ax=b$, every solution is of the form x_1+x_2, where x_2 is a solution of the associated homogeneous system.

(c) Show that if x_1 and x_2 are both solutions of a given homogeneous system of equations $Ax=0$ and if c_1 and c_2 are numbers, then $c_1x_1+c_2x_2$ is also a solution.

(d) Show that if $x_1, x_2, \ldots, x_k$ are all solutions of a given homogeneous system of equations and if $c_1, c_2, \ldots, c_k$ are numbers, then $c_1x_1+c_2x_2+\ldots+c_kx_k$ is also a solution.

Solution: (a) Given that x_1 is a solution of $Ax=b$

$$\Rightarrow Ax_1 = b \quad \text{—— (1)}$$

and x_2 is also a solution of $Ax=b$

$$\Rightarrow Ax_2 = b \quad \text{—— (2)}$$

Consider $A(x_1-x_2) = Ax_1 - Ax_2 = b-b = 0$

$\Rightarrow$ x_1-x_2 is a solution of homogeneous system $Ax=0$.

(b) Since x_1 is a solution of $Ax=b$

$$\Rightarrow Ax_1 = b \quad \text{—— (3)}$$

and x_2 is a solution of the associated homogeneous system $Ax=0$

$$\Rightarrow Ax_2 = 0 \quad \text{—— (4)}$$

Consider $A(x_1+x_2) = Ax_1 + Ax_2$

$= b+0$

$\Rightarrow A(x_1+x_2) = b$

$\Rightarrow$ x_1+x_2 is a solution of $Ax=b$.

(c) Since x_1 is a solution of $Ax=0$

$\Rightarrow Ax_1 = 0$ ——— (5)

and x_2 is a solution of $Ax=0$.

$\Rightarrow Ax_2 = 0$ ——— (6)

Consider

$A(c_1x_1+c_2x_2) = c_1Ax_1 + c_2Ax_2$

$= c_1 \cdot 0 + c_2 \cdot 0$

$= 0$

$\Rightarrow A(c_1x_1+c_2x_2) = 0$

$\Rightarrow$ $c_1x_1+c_2x_2$ is a solution of $Ax=0$.

(d) Since $x_1, x_2, \ldots, x_k$ are all solutions of $Ax=0$

$\Rightarrow Ax_1=0,\ Ax_2=0, \ldots, Ax_k=0$

Consider $A(c_1x_1+c_2x_2+\cdots+c_kx_k) = c_1Ax_1+c_2Ax_2+\cdots+c_kAx_k$

$= c_1 \cdot 0 + c_2 \cdot 0 + \cdots + c_k \cdot 0$

$= 0$

$\Rightarrow A(c_1x_1+c_2x_2+\cdots+c_kx_k) = 0$

$\Rightarrow$ $c_1x_1+c_2x_2+\cdots+c_kx_k$ is a solution of $Ax=0$.