Reduced Row Echelon form:

A matrix "A" is said to be in reduced row echelon form if it satisfies the following properties:-

1) "A" is in row echelon form.
2) The leading entry in each nonzero row is "1".
3) Each column containing a leading element 1 has zero everywhere else.

Example:- The following matrices are in reduced row echelon form

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Result:- 1) Every matrix has unique reduced row echelon form.

2) Reduced row echelon form of invertible matrix is an Identity mat

Example:- Reduce the following matrices in reduced row echelon fo

1) $$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2 \sim \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$R_1 \to R_1 - R_2 \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_3} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{\text{Ans.}}$$

2) $A = \begin{bmatrix} 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{bmatrix}$

$R_2 \rightarrow R_2 - R_1$ $\begin{bmatrix} 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{bmatrix}$

$R_4 \rightarrow R_4 - 2R_2$ $\begin{bmatrix} 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$

$R_1 \rightarrow R_1 + R_3$
$R_2 \rightarrow R_2 - R_3$
$R_4 \rightarrow R_4 + 2R_3$ $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ Ans.

3) $A = \begin{bmatrix} 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$

$R_1 \rightarrow \frac{1}{2} R_1$ $\sim \begin{bmatrix} 1 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$

$R_3 \leftrightarrow R_2$ $\begin{bmatrix} 1 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ Ans.

Gauss - Jordon Elimination Method :

There are following steps to solve a system of equations $Ax = b$ using Gauss Jordon Method.

1) Write the augmented matrix $[A|b]$.

2) Use the elementary row operations to convert $[A|b]$ into row reduced Echelon form.

3) Use back subtitution method to solve the equivalent system which corresponds to the reduced row echelon form.

Ex :- Solve the following system using Gauss - Jordon elimination Method :

$$x + 2y + z = 1$$
$$y + 3z = 2$$
$$4z = 8$$

Sol^n : Step 1 : Consider the augmented matrix

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 4 & 8 \end{array}\right]$$

Step 2 :- Reduce $[A|b]$ into RREF using elementary row operatio

$$R_3 \to \frac{1}{4} R_3 \quad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$R_1 \to R_1 - R_3$$
$$R_2 \to R_2 - 3R_3 \quad \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$R_1 \to R_1 - 2R_2 \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

<u>Step 3</u> :- Using Back subtitution method,

$z = 2, \quad y = -4, \quad x = 7.$

Elementary Matrix :→

A square matrix E of order "n" is called an elementary matrix if it is obtained by applying exactly one elementary row operation to the identity matrix, I_n.

Remark :→ $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Then $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$

E_1, E_2, E_3 are elementary matrix as.

E_1 is obtained by $R_2 \leftrightarrow R_3$

E_2 is obtained by $R_3 \to k R_3$, $k \neq 0$.

E_3 is obtained by $R_2 \to R_2 + k R_1$

Ex :→ Let $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 0 & 3 & 4 \\ 3 & 4 & 5 & 6 \end{bmatrix}$. Then

$R_2 \leftrightarrow R_3$ $\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 0 & 3 & 4 \\ 3 & 4 & 5 & 6 \end{bmatrix}$.

$E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ (This is obtained by using $R_2 \leftrightarrow R_3$ in Identity matrix).

$$E_{23}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 0 & 3 & 4 \\ 3 & 4 & 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 & 0 \\ 3 & 4 & 5 & 6 \\ 2 & 0 & 3 & 6 \end{bmatrix}.$$

i.e, interchanging the two rows of the matrix A is same as multiplying on the left by the corresponding elementary matrix.

In other words, we see that the left multiplication of elementary matrixes to a matrix results in elementar[y] row operation.

Thus, we can see that if A & B are two $m \times n$ matrices. Then A & B are row equivalent iff $B = PA$, where P is product of elementary matrices.

If A is invertible. Then A is row equivalent to I_n.

Thus, $\underbrace{E_1 E_2 \cdots E_k}_{\text{Product of elementary matrices.}} A = I_n$.

Gauss Jordon Method For Computing Inverse :-

Let A be $n \times n$ matrix.

Step:- 1) Write down $[A \mid I_n]$.

2). Use elementary row operation to reduce $[A \mid I_n]$ into $[B \mid C]$.

If A is invertible. Then $[A \mid I_n]$ will be converted to $[I_n \mid C]$, where $C = A^{-1}$.

If A is not invertible. Then $[A \mid I_n]$ can never be reduced into $[I_n \mid C]$.

Example:- Compute A^{-1} Using Gauss Jordon method,

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Sol^n:

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_1 \to R_1 - R_3$

$R_2 \to R_2 - R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] = [I_3 \mid A^{-1}]$$

Ex: Compute A^{-1} using Gauss Jordon Method, where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Sol^n: Consider the matrix $\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$

$R_1 \leftrightarrow R_2$ $\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$ $\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & -3 & -1 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \end{array}\right]$

$R_3 \to -3R_3 + R_2$ $\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & -3 & -1 & 1 & -2 & 0 \\ 0 & 0 & -4 & 1 & 1 & -3 \end{array}\right].$

$R_1 \to 4R_1 + R_3$
$R_2 \to +4R_2 - R_3$ $\left[\begin{array}{ccc|ccc} 4 & 8 & 0 & 1 & 5 & -3 \\ 0 & -12 & 0 & 3 & -9 & 3 \\ 0 & 0 & -4 & 1 & 1 & -3 \end{array}\right]$

$R_1 \to 3R_1 + 2R_2$ $\left[\begin{array}{ccc|ccc} 12 & 0 & 0 & 9 & -3 & -3 \\ 0 & -12 & 0 & 3 & -9 & 3 \\ 0 & 0 & -4 & 1 & 1 & -3 \end{array}\right]$

$$R_1 \to \frac{1}{12}R_1 \quad R_2 \to \frac{-1}{12}R_2 \quad R_3 \to -\frac{1}{4}R_3 \qquad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & -1/4 & -1/4 \\ 0 & 1 & 0 & -1/4 & 3/4 & -1/4 \\ 0 & 0 & 1 & -1/4 & -1/4 & 3/4 \end{array}\right]$$

Thus

$$A^{-1} = \begin{bmatrix} 3/4 & -1/4 & -1/4 \\ -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & 3/4 \end{bmatrix}$$

Result: Let A be an $n \times n$ matrix. Then the following are equivalent :-

1) A is invertible.
2) $AX = b$ has a unique solution for every b in $\mathbb{R}^n$.
3) $AX = 0$ has only the trivial solution.
4) The reduced row echelon form of A is I_n.
5) A is product of elementary matrices.