Lecture - 11th (ODE)

Second Order Differential Equations

30 minutes Syllabus for Main quiz test

→ ODE (12 lectures of ODE & 2 tut. sheets)

15 minutes Syllabus for Tut. quiz test

→ Linear transformation

Second Order ODE:

Linear Hom. ~~DE~~

$$a_0(x) \frac{d^2y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x)\, y = 0 \quad (1)$$

The set of solⁿs of second order Hom. DE eqⁿ forms a vector space over $\mathbb{R}$.

Superposition Principle: If y_1, y_2 are solⁿs of (1), then $c_1 y_1 + c_2 y_2$ is also a solⁿ of (1).

$y(x) = c_1 y_1 + c_2 y_2$, where y_1, y_2 are L.I solⁿs of (1).

basis → $\{y_1, y_2\}$ → fundamental set of solⁿs.

Linearly Independent | Linearly Dependent f^{n}s:

$\{f, g\}$ is called L-I

if $\alpha f + \beta g = 0$

$\Rightarrow \alpha = 0, \ \beta = 0$.

Wronskian: Wronskian of two differentiable f^{n}s y_1 & y_2 is defined as

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

If $W(y_1, y_2) \neq 0$, then y_1 & y_2 are L.I.

(provided ~~y_1 & y_2 are the sol^{n}s of~~ the same DE)

If $W(y_1, y_2) = 0$, then y_1 & y_2 are L.D.

[Wronskian is either identically zero or never zero

Example: Show that $\sin x$ & $\cos x$ are L.I sol^{n}s of

$$y'' + y = 0.$$

Sol^n:

$$y_1 = \sin x, \quad y_2 = \cos x$$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$$= -\sin^2 x - \cos^2 x$$

$$= -1 \neq 0$$

$$\Rightarrow W(y_1, y_2) \neq 0$$

$\Rightarrow$ y_1 & y_2 are L.I sol^{n}s of

$$y'' + y = 0.$$

The general sol^n is $y(x) = c_1 y_1 + c_2 y_2$

$$\boxed{y(x) = c_1 \sin x + c_2 \cos x}.$$

Example: If y_1 & y_2 ~~not~~ have common zero at any point $x_0 \in [a, b]$, then

y_1 and y_2 are ______ ? (provided y_1 & y_2 are the sol^{n}s of same DE)

$$y_1(x_0) = 0 = y_2(x_0).$$

$$W(y_1, y_2)(x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = 0$$

$\Rightarrow W(y_1, y_2)(x_0) = 0$ at $x_0 \in [a, b]$.

$\Rightarrow y_1$ and y_2 are L.D.

Example: If y_1 and y_2 have relative extrema at a pt $x_0 \in [a, b]$, then y_1 and y_2 are ______ ?

$$y_1'(x_0) = 0 = y_2'(x_0)$$

$$W(y_1, y_2)(x_0) = 0$$

$\Rightarrow y_1$ and y_2 are L.D.

Abel's formula:

Suppose y_1 and y_2 are the sol^{n}s of the DE

$$a_0(x)\, y'' + a_1(x)\, y' + a_2(x) y = 0, \quad x \in [a,b].$$

where $a_0(x) \neq 0$ and $a_0(x), a_1(x), a_2(x)$ are cts in $[a,b]$. Then

$$\boxed{W(y_1, y_2)(x) = C\, e^{-\int \frac{a_1(x)}{a_0(x)}\,dx}}$$

where C is a constant.

Example: Let y_1 and y_2 are sol^{n}s of

$$y'' + (\sin x)\, y = 0,$$

Let $g(x) = W(y_1, y_2)$, then prove that $g'(x) = 0$.

Here $a_0(x) = 1$, $a_1(x) = 0$, $a_2(x) = \sin x$

$$W(y_1, y_2)(x) = c\, e^{-\int \frac{a_1(x)}{a_0(x)}\,dx}$$

$$= c\, e^{-\int \frac{0}{1}\,dx} = C$$

$$g(x) = W(y_1, y_2)(x) = C$$

$\rightarrow g'(x) = 0.$

Existence and Uniqueness Theorem for IVP for Second Order Linear ODE:

$x \in (a,b)$

$x_0 \in [a,b]$

(1) where

$$\begin{cases} a_0(x) \dfrac{d^2y}{dx^2} + a_1(x) \dfrac{dy}{dx} + a_2(x) y = f(x) \\ y(x_0) = c_0, \quad y'(x_0) = c_1 \end{cases}$$

If $a_0(x) \neq 0$ and $a_0(x), a_1(x), a_2(x), f(x)$ are cts in $[a,b]$, Then (1) has unique solⁿ in $[a,b]$.

$$\begin{cases} (x-3) y'' + x y' + 4y = \sin x \\ y(-2) = 4, \quad y'(-2) = 3, \end{cases}$$

$(-\infty, 4)$ (cannot say anything)

Here $a_0(x) = x - 3 = 0$ at $x = 3$

$3 \in (-\infty, 4)$

$(-\infty, 3).$

(1)

$$\begin{cases} a_0(x) \dfrac{d^2y}{dx^2} + a_1(x) \dfrac{dy}{dx} + a_2(x) y = 0, \quad x \in (a,b) \\ y(x_0) = 0, \quad y'(x_0) = 0 \end{cases}$$

If $a_0(x) \neq 0$, $a_0(x), a_1(x), a_2(x)$ are ct in $[a,b]$, then (I) has unique sol^n $y(x) = 0$ in $[a,b]$.