Ordinary Differential Equations(EMAT102L)
(Lecture-8)

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Outline of the Lecture

We will learn

- Lipschitz Condition
- Picard's Existence Theorem
- Examples

Recall that an initial value problem can be described as

$$\frac{dy}{dx} = f(x,y), y(x_0) = y_0$$

An initial value problem can have unique, infinitely many solutions or no solution.

- $\frac{dy}{dx} = \frac{2y}{x}, y(2) = 4$, (Unique Solution, $y = x^2$)
- $\frac{dy}{dx} = \frac{2y}{x}, y(0) = 4$ (No Solution)
- $\frac{dy}{dx} = \frac{2y}{x}, y(0) = 0$ (Infinitely Many Solutions)

Thus there arise two following fundamental questions.

Existence of a Solution

Under what conditions an initial value problem

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0$$

has atleast one solution.

Uniqueness of a Solution

Under what conditions an initial value problem can have a **unique** solution.

The answer to the above questions is **Picard's Existence Theorem** and **Picard's Existence and Uniqueness Theorem**. But before discussing about these theorems, we need some definitions.

Bounded Function

Let f be a real function defined on R, where R is the domain of the xy-plane. The function f is said to be bounded in R if there exists a positive real number M such that

$$|f(x,y)| \leq M \ \forall \ (x,y) \in R$$

Definition

Let f be defined on R, where R is the domain of the xy- plane. The function f is said to satisfy Lipschitz Condition (with respect to y) in R if there exists a constant $K > 0$ such that

$$|f(x, y_1) - f(x, y_2)| \leq K|y_1 - y_2|$$

for every $(x, y_1), (x, y_2) \in R$. The smallest such constant K is called the **Lipschitz constant**. We say f is Lipschitz continuous in R with respect to y.

Example

The function $f(x) = x^2$ is Lipschitz continuous in $[-1, 4]$.
Consider

$$\begin{aligned} |f(x_1) - f(x_2)| &= |x_1^2 - x_2^2| = |x_1 + x_2||x_1 - x_2| \\ &\leq (|x_1| + |x_2|)|x_1 - x_2| \\ &\leq 8|x_1 - x_2| \end{aligned}$$

Here Lipschitz constant is 8.

Lipschitz Continuity $\Rightarrow$ continuity.

But Continuity $\nRightarrow$ Lipschitz continuity.

Counter Example

Consider the function $f(x, y) = \sqrt{y}$

Here f is continuous for all y. But f doesn't satisfy Lipschitz condition in any region that includes $y = 0$ as for $y_1 = 0$, we have

$$\frac{|f(x, y_1) - f(x, y_2)|}{|y_1 - y_2|} = \frac{\sqrt{y_2}}{y_2} = \frac{1}{\sqrt{y_2}}$$

which can be made as large as possible.

Sufficient condition for Lipschitz condition

Result

If $\dfrac{\partial f}{\partial y}$ exists and is bounded for all $(x, y) \in R$, then f satisfies Lipschitz condition w.r.t. y in R, where the Lipschitz constant is given by

$$K = \sup_{(x,y)\in R} \left| \frac{\partial f}{\partial y} \right|.$$

Example

Show that $f(x, y) = 1 + y^2$ satisfies Lipschitz condition in rectangle R defined by $R : |x| \leq 1, |y| \leq 2$.

Solution. Here we have

$$\frac{\partial f}{\partial y} = 2y$$

which is bounded in R. So, the Lipschitz constant is

$$K = \sup_{(x,y)\in R} \left| \frac{\partial f}{\partial y} \right| = \sup_{(x,y)\in R} |2y| = 4$$

Note: Boundedness of $\frac{\partial f}{\partial y}$ is sufficient condition but not necessary for Lipschitz condition.

Counter Example

Consider the function $f(x,y) = x^3|y|$, where R is the rectangle defined by $|x| \leq 1$, $|y| \leq 2$.

- f satisfies

$$|f(x,y_1) - f(x,y_2)| = |x^3|y_1| - x^3|y_2|| \leq |x|^3|y_1 - y_2| \leq |y_1 - y_2|$$

  for all $(x,y_1), (x,y_2) \in R$.
- Therefore f satisfies Lipschitz condition (with respect to y) in R.

But the partial derivative $\frac{\partial f}{\partial y}$ does not exist in R.

Existence Theorem

Theorem

Let R be a rectangle and (x_0, y_0) be an interior point of R. Let

- $f(x, y)$ be continuous at all points (x, y) in

$$R : |x - x_0| \leq a, |y - y_0| \leq b.$$

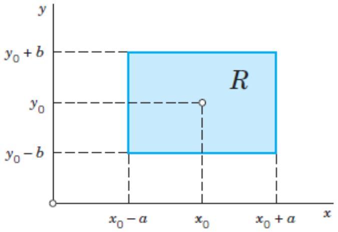
- Bounded in R, that is, $|f(x, y)| \leq M$ for all $(x, y) \in R$.

Then, the initial Value Problem

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0$$

has at least one solution $y(x)$ defined for all x in the interval $|x - x_0| \leq h$, where

$$h = \min\left(a, \frac{b}{M}\right).$$



Rectangle R in the existence and uniqueness theorems

Existence Theorem-Example

Example

Check whether the solution of the following IVP

$$\frac{dy}{dx} = 2x^2 + 3y^2,\ \ y(0) = 1,\ \ R : |x| \leq 1, |y - 1| \leq 1$$

exists or not and if it exists, then find the interval.

Solution:

- Here $f(x, y) = 2x^2 + 3y^2$. Consider

$$\begin{aligned} |f(x, y)| &= |2x^2 + 3y^2| \\ &\leq 2|x|^2 + 3|y|^2 \\ &= 2.1 + 3.4 = 14 \end{aligned}$$

$$\Rightarrow M = 14.$$

- Since $f(x, y)$ is a polynomial.
  $\Rightarrow f(x, y)$ is continuous. Thus both the conditions of Existence Theorem are satisfied.

- So, by Existence Theorem, the solution exists and it exists in

$$|x - x_0| \leq h \Rightarrow |x - 0| \leq h \Rightarrow |x| \leq h$$

where

$$h = min(1, 1/14) \Rightarrow h = \frac{1}{14}$$

$$\Rightarrow |x| \leq \frac{1}{14}.$$

Thank You