On Bases :-

Theorem : 1. Let V be a vector space of dimension "n". Then

1) Any $n+1$ or more vectors in V are linearly dependent.

(ii) Any linearly independent set $S = \{v_1, v_2, \dots, v_n\}$ with "n" elements is a basis of V.

(iii) Any spanning set $T = \{v_1, v_2, \dots, v_n\}$ of V with "n" elements is a basis of V.

Theorem : 2 :- Suppose S spans a vector space V. Then

1) Any maximum number of linearly independent vectors in S form a basis of V.

2) Suppose one deletes from S every vector that is linear combination of preceding vectors in S. Then remaining the remaining vectors form a basis of V.

Theorem 3 : Let V be a vector space of finite dimension and let $S = \{v_1, v_2, \dots, v_r\}$ be a set of linearly independent vectors in V.

Then S is a part of a basis of V.

i.e S may be extended to a basis of V.

Dimension & subspace :-

Theorem :- Let W be a subspace of an n-dimensional vector space V. Then $\dim W \leq \dim V = n$.

In particular, if $\dim W = n$. Then $\boxed{W = V}$

Ex:- Let W be a subspace of the real vector space $\mathbb{R}^3$.

Then $\dim W$ can Only be $0, 1, 2, 3$.

The following cases apply :- (a) $\dim W = 0$ Then $W = \{0\}$, a pt.

(b) $\dim W = 1$, Then W is a line passes through the origin.

(c) $\dim W = 2$ Then W is a plane through the origin

(d) $\dim W = 3$, Then W is the entire space $\mathbb{R}^3$.

Ex :- $S = \{(1, 1, 0, 1\}, (0, 1, 1, 0), (0, 0, 0, 1)\}$.

Then Extend S as a basis of $\mathbb{R}^4$.

Sol^n: Check whether S is a linearly independent set or not.

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ which is in row echelon form.

$\rho(A) = 3$

$\therefore$ They are linearly independent vector.

As we know $\dim \mathbb{R}^4 = 4$. So, we need to add only one vector of $\mathbb{R}^4$.

Take (choose) an element s.t which has non-zero entry at 3rd place. $(0, 0, 1, 0)$

(i.e which can not be expressed as linear combination of all three vectors).

Thus, $\{(1,1,0,1), (0,1,1,0), (0,0,0,1), (0,0,1,0)\}$ are l.i & $\dim \mathbb{R}^4 = 4$ $\therefore$ Hence forms a basis for $\mathbb{R}^4$.

SUMS AND DIRECT SUMS :

Let U and W are subspace of V. Then $U+W$, defined as

$$U+W = \{ v : v = u+w \text{, where } u \in U, w \in W \}$$

is subspace of V.

Also, $U \cap W$ is a subspace of V.

But $U \cup W$ need not be a subspace. as

$U = \{(x,0) : x \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$.

$W = \{(0,y) : y \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$.

$U \cup W = \{(x,0) : x \in \mathbb{R}\} \cup \{(0,y) : y \in \mathbb{R}\}$ is not a subspace of $\mathbb{R}^2$

As $(1,0), (0,1) \in U \cup W$. but $(1,0)+(0,1) = (1,1) \notin U \cup W$.

$\Rightarrow$ $U \cup W$ is not closed under addition.

$U \cup W$ is a subspace of V if either $U \subseteq V$ or $W \subseteq V$

Theorem :→ Suppose U and W are finite-dimensional vector space of V. Then $U+W$ has finite dimension and

$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W).$$

Def^n: Direct sum : The vector space V is said to be the direct sum of its subspace U and W, denoted by $V = U \oplus W$, if every $v \in V$ can be written in one & only one way as

$$v = u + w \text{, where } u \in U, w \in W.$$

The following Theorem characterizes such a decomposition :-

Theorem : The vector space V is the direct sum of its subspace U and W iff (i) $V = U + W$ (ii) $U \cap W = \{0\}$.

Example : Let $V = \mathbb{R}^3$. Let $U =$ xy-plane, $W =$ yz-plane.
Then verify that $V \neq U \oplus W$.

Sol^n : $U =$ xy-plane $= \{(x, y, 0) : x, y \in \mathbb{R}\}$.

$W =$ yz-plane $= \{(0, y, z) : x, y \in \mathbb{R}\}$

U and W both are subspace of $\mathbb{R}^3$.

Let $(x, y, z) \in U \cap W$

i.e $(x, y, z) \in U \Rightarrow z = 0$

$(x, y, z) \in W \Rightarrow x = 0$

$\therefore (x, y, z) = (0, y, 0) \in U \cap W$

i.e $U \cap W = \{(0, y, 0) : y \in \mathbb{R}\} \neq \{0, 0, 0\}$.

Thus, $\mathbb{R}^3 \neq U \oplus W$.

Find (Calculate) the dim U, dim W, dim $U \cap W$, dim $(U+W)$

$$U = \{(x, y, 0) : x, y \in \mathbb{R}\}$$
$$= \{x(1,0,0) + y(0,1,0) : x, y \in \mathbb{R}\}$$
$$= \text{span}\{(1,0,0), (0,1,0)\}$$

Also, $\{(1,0,0), (0,1,0)\}$ are linearly independent as

$\alpha(1,0,0) + \beta(0,1,0) = (0,0,0)$

$\Rightarrow (\alpha, \beta, 0) = (0,0,0)$

$\Rightarrow \alpha = \beta = 0$.

$\therefore \dim U = 2$.

$$W = \{ (0, y, z) : y, z \in \mathbb{R} \}$$

$$= \{ y(0,1,0) + z(0,0,1) : y, z \in \mathbb{R} \}$$

$$= \text{span}\{ (0,1,0), (0,0,1) \}.$$

Also, it is easy to see that $\{ (0,1,0), (0,0,1) \}$ is linearly independent set. as $\alpha(0,1,0) + \beta(0,0,1) = (0,0,0)$
$\Rightarrow \alpha = 0 = \beta$.

$\therefore \dim W = 2$

$$U \cap W = \{ (0, y, 0) : y \in \mathbb{R} \}.$$

$$= \{ y(0,1,0) \}$$

$$= \text{span}\{ (0,1,0) \}$$

$\dim U \cap W = 1$

Also, one can easily ~~verify~~ Calculate $\dim(U \cup W)$, We know that $\dim(U \cup W) = \dim U + \dim W - \dim(U \cap W)$

$= 2 + 2 - 1 = 3$

i.e $\dim(U \cup W) = 3$

Remark: $L(U \cup W) = U + W$, is a subspace of V.

Miscellaneous Problems :

(1) $\{1, i\}$ is linearly independent over $\mathbb{R}$ but linearly dependent over $\mathbb{C}$.

Sol^n: $\alpha \cdot 1 + \beta \cdot i = 0$; $\alpha, \beta \in \mathbb{R}$

$\Rightarrow \alpha = 0, \beta = 0.$

$\Rightarrow \alpha, \beta$ is l. i.

if $i = \alpha \in \mathbb{C}$ Then $i \cdot 1 = i$. i.e i is multiple of 1 and vice-versa also.

$-i\,i = 1.$

Thus $1, i$ is linear dependent over $\mathbb{C}$.

2) Find the dimension of $\mathbb{C}(\mathbb{R})$, $\mathbb{C}(\mathbb{C})$.

Sol^n:- $\dim(\mathbb{C}(\mathbb{R})) = 2$ as $a + ib = a \cdot 1 + i \cdot b$

$\{1, i\}$ is l.i over $\mathbb{R}$ and span $\{1, i\}$ is $\mathbb{C}$.

$\dim(\mathbb{C}(\mathbb{C})) = 1$. as $1 \cdot (a+ib) = a + ib$

$\{1\}$ is l.i over $\mathbb{C}$.

& $\{1\}$ span $\mathbb{C}$.

3) Find the dimension of $\mathbb{C}^2(\mathbb{C})$ & $\mathbb{C}^2(\mathbb{R})$

Sol^n: $\mathbb{C}^2 = \{(\alpha, \beta) : \alpha = a + ib, \beta = c + id, a, b, c, d \in \mathbb{R}\}$

$= \{(a+ib, c+id)\}$

$$= \begin{cases} (a+ib)(1,0) + (c+id)(0,1) & \text{over } \mathbb{C} \\ a(1,0) + b(i,0) + c(0,1) + d(0,i) & \text{over } \mathbb{R}. \end{cases}$$

Thus, one can check that $\{(1,0), (0,1)\}$ forms a basis for $\mathbb{C}^2$ over $\mathbb{C}$

and the set $\{(1,0), (i,0), (0,1), (0,i)\}$ forms a basis for $\mathbb{C}^2$ over $\mathbb{R}$.

Thus $\dim(\mathbb{C}^2(\mathbb{R})) = 4$, $\dim(\mathbb{C}^2(\mathbb{C})) = 2$.

4) Find the dim W, dim U, $\dim(U \cap W)$, $\dim(W+U)$, where $U = \{(a,b,c) : a = 2b, b = c\}$, $a, b, c \in \mathbb{R}$.

$W = \{(a,b,c) : a + 2b = 0, b = c\}$.

Sol^n: $U = \{(a,b,c) : a = 2b, b = c\}$

$= \{(2c, c, c) : c \in \mathbb{R}\}$

$= \{c(2,1,1) : c \in \mathbb{R}\}$

$= \text{span}\{(2,1,1)\}$

Also $\{(2,1,1)\}$ is linearly independent set. $\therefore \dim U = 1$.

$W = \{(a,b,c) : a + 2b = 0, b = c\}$

$= \{(-2c, c, c) : c \in \mathbb{R}\}$

$= \{c(-2,1,1) : c \in \mathbb{R}\}$

$= \text{span}\{(-2,1,1)\}$

Also, $\{(-2,1,1)\}$ is linearly independent.

$\therefore \dim W = 1$.

$U \cap W = \{(a,b,c) : \underline{a+2b=0}, \underline{a-2b=0}, b=c\}$

$= \{(a,b,c) : , 4b=0, : , b=c\}$

$= \{(a,b,c) : a=0, b=0, c=0\}$

$\Rightarrow U \cap W = \{(0,0,0)\}$

$\dim(U \cap W) = 0.$

$\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$

$= 1 + 1 - 0$

$= 2.$

Problem:- Let $U = \{(x,y) : x = y\}$

$W_1 = \{(x,0) : x \in \mathbb{R}\}$

$W_2 = \{(0,y) : y \in \mathbb{R}\}$

Verify that:- $(U \cap W_1) + (U \cap W_2) \neq U \cap (W_1 + W_2)$

i.e. $L((U \cap W_1) \cup (U \cap W_2)) \neq L(U \cap L(W_1 \cup W_2))$

Sol^n: $U \cap W_1 = \{(0,0)\}$

$U \cap W_2 = \{(0,0)\}$

$(U \cap W_1) \cup (U \cap W_2) = \{(0,0)\}$

$L(W_1 \cup W_2) = W_1 + W_2$

$= \{(x,y) : x, y \in \mathbb{R}\}$

$U \cap (W_1 + W_2) = \{(x,y) : x = y\}$

Prob:- Determine the dimension of vector space V of the following n-square matrices:-

as $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & a_{23} \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ a_{1n} & a_{2n} & & a_{nn} \end{bmatrix}$

1) Symmetric Matrix . . dim $= \frac{n(n+1)}{2}$

$\begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ -a_{12} & \ddots & & \\ -a_{1n} & & & 0 \end{bmatrix}$

2) Skew symmetric matrix, dim $\frac{n(n-1)}{2}$

$\begin{bmatrix} a_{11} & & 0 \\ & a_{22} & \\ 0 & & \ddots \; a_{nn} \end{bmatrix}$

3) diagonal, dim $= n$

$a_{11}\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

4) Scalar. dim $= 1$.

5) Diagonal matrix s.t trace is equal to zero.

$$W = \left\{ \begin{bmatrix} a_{11} & & 0 \\ & a_{22} \ddots & \\ 0 & & a_{nn} \end{bmatrix} : a_{11} + a_{22} + \cdots + a_{nn} = 0 \right\}$$

dim $W = (n-1)$.