Consider the set $\mathbb{C} = \{x+iy : x, y \in \mathbb{R}\}$ of complex numbers.

$= \{z : z \in \mathbb{C}\}$.

(a) For x_1+iy_1, $x_2+iy_2 \in \mathbb{C}$ & $\alpha \in \mathbb{R}$, define

$$(x_1+iy_1) \oplus (x_2+iy_2) = (x_1+x_2) + i(y_1+y_2)$$

$$\alpha \odot (x_1+iy_1) = \alpha x_1 + i\alpha y_1$$

Then $\mathbb{C}$ is a real vector space.

(b) For x_1+iy_1, $x_2+iy_2 \in \mathbb{C}$ & $\alpha+i\beta \in \mathbb{C}$, define

$$(x_1+iy_1) \oplus (x_2+iy_2) = (x_1+x_2) + i(y_1+y_2)$$

$$(\alpha+i\beta) \odot (x_1+iy_1) = (\alpha x_1 - \beta y_1) + i(\alpha y_1 + \beta x_1)$$

Then $\mathbb{C}$ forms a complex vector space.

$V = \mathbb{C}^n = \{(z_1, z_2, \ldots, z_n) : z_i \in \mathbb{C}, 1 \le i \le n\}$.

$$(z_1, z_2, \ldots, z_n) \oplus (w_1, w_2, \ldots, w_n) = (z_1+w_1, \ldots, z_n+w_n)$$

$$\alpha \odot (z_1, z_2, \ldots, z_n) = (\alpha z_1, \alpha z_2, \ldots, \alpha z_n)$$

(a) If the set $\mathbb{F}$ is the set $\mathbb{C}$ of complex numbers, then $\mathbb{C}^n$ is a complex vector space having n-tuple of complex numbers as its vectors.

(b) If the set $\mathbb{F}$ is the set $\mathbb{R}$ of real numbers, then $\mathbb{C}^n$ is a real vector space having n-tuple of complex numbers as its vectors.

Remark:- In (a), the scalars are complex numbers & hence $i(1,0) = (i,0)$. whereas in 7(b), the scalars are real numbers & hence we cannot write $i(1,0) = (i,0)$.

Subspace:

Assume V is a vector space over $\mathbb{F}$.

Definition: Let S be a nonempty subset of V.
S is said to be subspace of V if

$$\boxed{\alpha u + \beta v \in S \quad \text{whenever } u, v \in S \ \& \ \alpha, \beta \in \mathbb{F},}$$

where the vector addition & scalar multiplication are same as that of V.

Remark: 1) Any subspace is a vector space in its own right with respect to vector addition and scalar multiplication that is defined for V.

2) a) $S = \{0\}$, the set consisting of the zero vector.

b) $S = V$ are vector subspaces of V, are called TRIVIAL SUBSPACE.

Example: $V = \mathbb{R}^3$, $\mathbb{F} = \mathbb{R}$

$S = \{(x, y, z) \in \mathbb{R}^3 : x = y = z\} = \{x(1,1,1) \in \mathbb{R}^3 : x \in \mathbb{R}\}$

Then S is a subspace of $\mathbb{R}^3$.

Proof: Let $\alpha, \beta \in \mathbb{R}$, $u, v \in S$. i.e $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2)$ s.t $x_1 = y_1 = z_1$ & $x_2 = y_2 = z_2$

$$\begin{aligned}
\alpha u + \beta v &= \alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2) \\
&= (\alpha x_1, \alpha y_1, \alpha z_1) + (\beta x_2, \beta y_2, \beta z_2) \\
&= (\underbrace{\alpha x_1 + \beta x_2}_{\in \mathbb{R}}, \underbrace{\alpha y_1 + \beta y_2}_{\in \mathbb{R}}, \underbrace{\alpha z_1 + \beta z_2}_{\in \mathbb{R}})
\end{aligned}$$

Thus $\alpha u + \beta v \in S$ $\because$ $\alpha x_1 + \beta x_2 = \alpha y_1 + \beta y_2 = \alpha z_1 + \beta z_2$

Hence S is a subspace of $\mathbb{R}^3$.

Geometrically, S is a line through the origin and the point (1, 1, 1).

(b) Let $S = \{(x, y, z) \in \mathbb{R}^3 : x + y - 2z = 0\}$. Then S is a subspace of $\mathbb{R}^3$.

Proof: Let $X = (x_1, y_1, z_1) \in S \Rightarrow x_1 + y_1 - 2z_1 = 0$

$Y = (x_2, y_2, z_2) \in S \Rightarrow x_2 + y_2 - 2z_2 = 0$

Then for $\alpha, \beta \in \mathbb{R}$

$$\alpha X + \beta Y = (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$$

Consider.

$$\alpha x_1 + \beta x_2 + \alpha y_1 + \beta y_2 - 2\alpha z_1 - 2\beta z_2$$

$$= \alpha(x_1 + y_1 - 2z_1) + \beta(x_2 + y_2 - 2z_2)$$

$$= \alpha 0 + \beta 0 = 0.$$

i.e. $\alpha X + \beta Y \in S$.

Hence S is a subspace of $\mathbb{R}^3$.

Geometrically, S is a plane passes through the origin.

(c) $S = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 1\}$. Then S is not a subspace of $\mathbb{R}^3$.

Soln: $(0, 0, 0) \notin S$ as $0 + 0 + 0 \neq 1$.

Geometrically, S is a plane in $\mathbb{R}^3$ which does not pass through origin.

The vector space $\mathbb{P}_n(\mathbb{R})$ is a subspace of the vector space $\mathbb{P}(\mathbb{R})$

Let $S = \{f \in C([-1,3]) : f(2) = 0\}$. Then S is subspace of $C([-1,3])$

Solⁿ: Let $\alpha, \beta \in \mathbb{R}$ & $f, g \in C([-1,3])$. Then $f(2) = 0$
$g(2) = 0$

To show:- $\alpha f + \beta g \in S$.
i.e need to show $\alpha f + \beta g \in C([-1,3])$ &
$(\alpha f + \beta g)(2) = 0$
i.e $\alpha f(2) + \beta g(2) = 0$.

Now, f is cts, g is cts. $\therefore$ $\alpha f + \beta g$ is cts.

As. $f(2) = 0$, $g(2) = 0$ $\Rightarrow$ $\alpha f(2) + \beta g(2) = 0$
$\Rightarrow (\alpha f + \beta g)(2) = 0$

i.e $\alpha f + \beta g \in C([-1,3])$

Hence S is a subspace of $C([-1,3])$

$S = \{A \in M_{n\times n}(\mathbb{R}) : \text{trace} A = 0\}$. Then S is a subspace of $M_{n\times n}(\mathbb{R})$.

Solⁿ: Verify! (Hint - $\text{trace}(\alpha A + \beta B) = \alpha\, \text{trace}(A) + \beta\, \text{trace}(B) = \alpha \cdot 0 + \beta \cdot 0 = 0$)

$S = \{X \in \mathbb{R}^{n\times 1} : AX = 0,\ A \in M_{m\times n}\}$ is a subspace of $\mathbb{R}^{n\times 1}$.

$X \in \mathbb{R}^{n\times 1}$ i.e $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

The following are the examples of subspaces of $M_{n\times n}(\mathbb{R})$ ...

1) Set of all upper triangular matrices

2) set of all lower triangular matrices

3) set of all diagonal matrices

4) set of all symmetric matrices

5) set of all skew-symmetric matrices.

Sol^n:- TRY YOURSELF !

2) Which of the following are subspace of $\mathbb{R}^n(\mathbb{R})$?

(a) $U=\{(x_1,x_2,--,x_n): x_1 \geq 0\}$ (No, $\because \alpha=-1$, Then in U, $(\alpha x_1, ---, \alpha x_n)$, $-x_1 \leq 0$)

(b) $\{(x_1,x_2,--,x_n): x_1+3x_2=4x_4\}$ (YES)

(c) $\{(x_1,x_2,--,x_n): x_2$ is irrational$\}$ (No, scalar multiplication does not hold)

(d) $\{(x_1,x_2,--,x_n): x_1=x_4^2\}$ (No, not closed under addition.

(f) $\{(x_1,x_2,--,x_n): |x_1|\leq 1\}$ (No, scalar multiplication as well as addition don't hold).

LINEAR COMBINATION:-

Let V be a vector space over $\mathbb{F}$ (field).
A vector $v \in V$ is a linear combination of vectors $u_1, u_2, \ldots, u_m$ in V if there exist scalars $a_1, a_2, \ldots, a_m \in \mathbb{F}$ such that

$$v = a_1 u_1 + a_2 u_2 + \cdots + a_m u_m .$$

Alternatively, v is a linear combination of $u_1, u_2, \ldots, u_m$ if there is a solution to the vector eq^n

$$v = x_1 u_1 + x_2 u_2 + \cdots + x_m u_m ,$$

where $x_1, x_2, \ldots, x_m$ are unknown scalars.

Example:- (Linear combination in $\mathbb{R}^n$)
Write $v = (3, 7, -4) \in \mathbb{R}^3$ as a linear combination of vectors $u_1 = (1, 2, 3)$, $u_2 = (2, 3, 7)$, $u_3 = (3, 5, 6)$

i.e. we find the scalars x_1, x_2, x_3 s.t. $v = x_1 u_1 + x_2 u_2 + x_3 u_3$

$$\begin{bmatrix} 3 \\ 7 \\ -4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}$$

(For notational convenience, we have written the vector in $\mathbb{R}^3$ as columns, since it is easier to find the equivalent system of equations.)

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 3 \\ 2x_1 + 3x_2 + 5x_3 &= 7 \\ 3x_1 + 7x_2 + 6x_3 &= -4 \end{aligned}$$

Consider the Augumented matrix,

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 2 & 3 & 5 & 7 \\ 3 & 7 & 6 & -4 \end{array}\right].$$

$$\begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & -3 & -13 \end{array}\right]$$

$$R_3 \to R_3 + R_2 \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & -4 & -12 \end{array}\right]$$

Notation
$x_1 = x$
$x_2 = y$
$x_3 = z$.

By back subtitution, $-4z = -12 \Rightarrow z = 3$

$-y - z = -12 \Rightarrow y = -4$

$x + 2y + 3z = 3 \Rightarrow x = 2$

Thus $\boxed{v = 2u_1 - 4u_2 + 3u_3.}$

Generally speaking :-

Given a vector "$\underline{v}$" in $V = \mathbb{R}^n$ or $\mathbb{C}^n$ as a linear combination of vectors $u_1, u_2, \ldots, u_m$ in V is equivalent to solving the system $AX = B$ of linear equations, where, v is the column B of constants and the u's are the columns of the coefficient matrix A.

Such a system may have a unique solution, many solutions, or no solution.

In the last case - no solution - means that v can not be written as a linear combination of the u's.

Example:- Express the polynomial $V = 3t^2+5t-5$ as a linear combination of the polynomials

$p_1 = t^2+2t+1$, $p_2 = 2t^2+5t+4$, $p_3 = t^2+3t+6$.

Solution: $v = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3$

Method 1: $3t^2+5t-5 = \alpha_1(t^2+2t+1) + \alpha_2(2t^2+5t+4) + \alpha_3(t^2+3t+6)$

$\Rightarrow 3t^2+5t-5 = (\alpha_1+2\alpha_2+\alpha_3)t^2 + (2\alpha_1+5\alpha_2+3\alpha_3)t + (\alpha_1+4\alpha_2+6\alpha_3)$

Comparing the like power of t, we obtain

$$\alpha_1+2\alpha_2+\alpha_3 = 3$$
$$2\alpha_1+5\alpha_2+3\alpha_3 = 5$$
$$\alpha_1+4\alpha_2+6\alpha_3 = -5.$$

Solve the system of eq^n:- $\left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 2 & 5 & 3 & 5 \\ 1 & 4 & 6 & -5 \end{array}\right]$.

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$ $\sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & 5 & -8 \end{array}\right]$

$R_3 \to R_3 - 2R_2$ $\sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 3 & -6 \end{array}\right]$

$3\alpha_3 = -6 \Rightarrow \alpha_3 = -2$

$\alpha_2 + \alpha_3 = -1 \Rightarrow \alpha_2 = 1$

$\alpha_1 + 2\alpha_2 + \alpha_3 = 3 \Rightarrow \alpha_1 = -2.$

Thus, $v = 3p_1 + p_2 - 2p_3.$

Method 2:-

$$v = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3$$

i.e $\quad v = \alpha_1 p_1(t) + \alpha_2 p_2(t) + \alpha_3 p_3(t) \quad \forall\ t \in \mathbb{R}. \quad$ — (1)

$$3t^2 + 5t - 5 = \alpha_1 (t^2 + 2t + 1) + \alpha_2 (2t^2 + 5t + 4) + \alpha_3 (t^3 + 3t + 6)$$

We can obtain three equations in the unknowns α_1, α_2 and α_3 by setting t equal to any three values.

For example :- $t = 0$, in (1), we obtain $\quad \alpha_1 + 4\alpha_2 + 6\alpha_3 = -5$

$t = 1$, " , $\quad 4\alpha_1 + 11\alpha_2 + 10\alpha_3 = 3$

$t = -1$, " $\quad \alpha_2 + 4\alpha_3 = -7.$

Reducing this system to echelon form, solving by back subtitution, we obtain

$$x = 3, \ y = 1, \ z = -2.$$

Definition:- Let V be a vector space over $\mathbb{F}$ and let $S = \{u_1, u_2, \dots, u_n\}$ be a non-empty subset of V.

The linear span of S is defined as

$$L(S) = \{\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n : \alpha_i \in F,\ 1 \le i \le n\}.$$

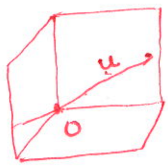
Remark:- 1) If $S = \{\phi\}$. Then $L(S) = \{0\}$.

(ii) $L(S)$ or $\text{span}(S)$ is a subspace of V that contains S.

(ii) If W is a subspace of V containing S, then $\text{span}(S) \subseteq W$.

i.e $L(S)$ is the "smallest" subspace of V containing S.

Ex:- Let $u \neq 0 \in \mathbb{R}^3$. Then find $\text{span}(u)$.



$L(u) = \{\alpha u : \alpha \in \mathbb{R}\}$.

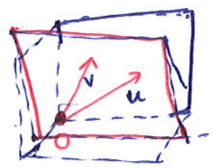
i.e set consisting of all scalar multiple of u.

Geometrically, $\text{span}(u)$ is the line through the origin and the end pts of u.

(2) Let u & v be vectors in $\mathbb{R}^3$ that are not multiple of each other. $S = \{u, v\}$

Then $\text{span}(S) = \{\alpha_1 u + \alpha_2 v : \alpha_1, \alpha_2 \in \mathbb{R}\}$

= "plane through the origin O and the endpoints of u & v.



Spanning Sets:-

Let V be a vector space over K.

Vectors $u_1, u_2, \ldots, u_m \in V$ are said to "span V" or to form a spanning set of V if every $v \in V$ is a linear a linear combination of the vector $u_1, u_2, \ldots, u_m$.

i.e $\exists$ scalars $a_1, a_2, \ldots, a_m$ in K such that

$$v = a_1 u_1 + a_2 u_2 + \cdots + a_m u_m.$$

Remark:- 1) Suppose $u_1, u_2, \ldots, u_m$ span V. Then, for any vector w, the set $w, u_1, u_2, \ldots, u_m$ also span V.

2) Suppose $u_1, u_2, \ldots, u_m$ span V and suppose u_k is a linear combination of some of the other u's. Then the u's with u_k also span V.

3) Suppose $u_1, u_2, \ldots, u_m$ span V and suppose one of the u's is the zero vector. Then the u's without the zero vector also span V.

Example:- (1) $V = \mathbb{R}^3$.

(a) $e_1 = (1,0,0)$, $e_2 = (0,1,0)$; $e_3 = (0,0,1)$ span $\mathbb{R}^3$.

or e_1, e_2, e_3 form a spanning set of $\mathbb{R}^3$.

Sol^n:- let $(a,b,c) \in \mathbb{R}^3$ Then

$$(a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1).$$

i.e $$(a,b,c) = a e_1 + b e_2 + c e_3.$$

(b) $w_1 = (1,1,1)$, $w_2 = (1,1,0)$, $w_3 = (1,0,0)$ span $\mathbb{R}^3$.

Sol^n: $v \in \mathbb{R}^3$,

$$v = (a,b,c) = c w_1 + (b-c) w_2 + (a-b) w_3.$$

As,
$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$Z = c$, $\quad x + y = b \Rightarrow y = b - x = b - c$

$x + y + z = a \Rightarrow z = a - x - y = a - b.$

(c) One can check that $v = (2,7,8)$ can not be written as a linear combination of vectors

$u_1 = (1,2,3)$, $u_2 = (1,3,5)$, $u_3 = (1,5,9)$

i.e u_1, u_2, u_3 donot span $\mathbb{R}^3$.

i.

Ex:- $V = P_n(t)$ consisting of all polynomials of degree $\le n$.

(a) Clearly every polynomial in $P_n(t) = a_0 + a_1 t + \cdots + a_n t^n$ can be expressed as a linear combination of the $(n+1)$ polynomials $1, t, t^2, \cdots, t^n$.

(b) Show that for any scalar c, the following $(n+1)$ powers of $t-c$,

$$1 = (t-c)^0, \quad (t-c), \quad (t-c)^2, \quad \cdots \quad (t-c)^n.$$

also form a spanning set for $P_n(t)$.

(c) Consider the vector space $M = M_{2\times 2}$ - consisting of all 2×2 matrices.

Then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

i.e any element of $M_{2\times 2}$ can be written as a linear combination of

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$