Tutorial Sheet 1
Matrices

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1. Find the 4×4 matrix $A = [a_{ij}]$ whose entries satisfy the stated condition.

   (a) $a_{ij} = i - j$ (b) $a_{ij} = (-1)^j ij$ (c) $a_{ij} = \begin{cases} 0 & \text{if } |i-j| \geq 1 \\ 1 & \text{if } |i-j| < 1. \end{cases}$

2. Find the value of a, b, c and d such that $\begin{pmatrix} 3 & a \\ 1 & a+b \end{pmatrix} = \begin{pmatrix} b & c-2d \\ c+2d & 0 \end{pmatrix}$.

3. Show that if a square matrix A satisfies $A^3 + 4A^2 - 2A + 7I = 0$ then so does A^T.

4. Show that if $p(\lambda) = \lambda^2 - (a+d)\lambda + (ad - bc)$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $p(A) = 0$.

5. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then calculate determinant of $\lambda I - A$, where I is an identity matrix, λ is any scalar.

6. Show that there are no $n \times n$ matrices A and B with $AB - BA = cI_n$, where I is an identity matrix, $0 \neq c$ is any scalar.

7. Let A and B be two $m \times n$ matrices and let x be an $n \times 1$ column vector.

   (a) Prove that if $Ax = 0$ for all x, then A is the zero matrix.

   (b) Prove that if $Ax = Bx$ for all x, then $A = B$.

8. Show that if a square matrix A satisfies the equation $A^2 + 5A - 2I = 0$, then $A^{-1} = \frac{1}{2}(A + 5I)$.

9. Find the inverse of the following 3×3 matrices, where a, b, c are all non zero.

$$(a) \begin{pmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{pmatrix} \quad (b) \begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix}$$

10. Find the inverse of $A = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}$ over $\mathbb{Z}_5$.

11. Find the inverse of $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$ over $\mathbb{Z}_3$. Does A have an inverse over $\mathbb{Z}_5$.

12. If x_0 is a solution of the non-homogeneous system $Ax = b$ and y_0 is a solution of the homogeneous system $Ax = 0$ then show that $x_0 + y_0$ is a solution of $Ax = b$. Moreover, if x_1 is any solution of the system $Ax = b$ then show that there is a solution y_1 of the system $Ax = 0$ such that $x_1 = x_0 + y_1$.

Answer:

10. $A^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 4 & 2 & 0 \end{pmatrix}$ over $\mathbb{Z}_5$.

11. $A^{-1} = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$ over $\mathbb{Z}_3$.