EPHY 105L / EPHY 103L ELECTROMAGNETICS

SOLUTIONS FOR MID-TERM TEST

Q.1. a) For the field to represent an electrostatic field we must have $\nabla \times \vec{E} = 0$. Now

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (12x^2 - y^2) & -2xy & 0 \end{vmatrix} z$$

$$= \hat{x}(0-0) + \hat{y}(0-0) + \hat{z}(-2y+2y)$$

$$= 0$$

Hence the given vector field can represent an electrostatic field.

b) $$\vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{[(x-x_0)\hat{x} + (y-y_0)\hat{y} + (z-z_0)\hat{z}]}{[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{3/2}}$$

$$= \frac{Q}{4\pi\epsilon_0} \frac{[(0-0)\hat{x} + (2-0)\hat{y} + (0-2)\hat{z}]}{[0^2 + 2^2 + 2^2]^{3/2}}$$

$$= \frac{Q}{4\pi\epsilon_0} \frac{[2\hat{y} - 2\hat{z}]}{8^{3/2}} = \frac{10^{-6} \times 9\times10^9 \times 2(\hat{y} - \hat{z})}{8^{3/2}}$$

$$= 795.5 (\hat{y} - \hat{z})$$

c) According to Gauss's law

$$\nabla \cdot \vec{E} = \rho/\epsilon_0$$

For the given $\vec{E}$, we have

$$\nabla \cdot \vec{E} = \frac{A}{R}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(-r^2\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta)\right]$$

$$= \frac{A}{R}\left[-\frac{2}{r}\cos\theta + \frac{2}{r}\cos\theta\right] = 0$$

Hence $\rho = 0$

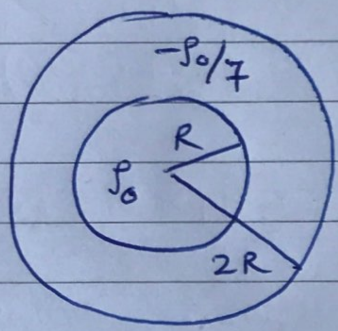
d) Inside the conducting plate $\vec{E}=0$. Thus the surface charges must produce an electric field $\vec{E}_s = -E_0\hat{z}$. If charge density is σ_f, then

$$E_0 = \frac{\sigma_f}{\epsilon_0}$$

or $\sigma_f = \epsilon_0 E_0$

2)

a) Gauss's law



$$\oint \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$$

For $r<R$

- $\vec{E}$ is radial and independent of θ and ϕ.
- Hence for $r<R$ $\quad \oint \vec{E}\cdot d\vec{a} = 4\pi r^2 E$
- $Q_{enc} = \frac{4\pi}{3} r^3 \rho_0$.

Hence
$$4\pi r^2 E = \frac{4\pi}{3}\rho_0 r^3$$

or
$$E = \frac{\rho_0 r}{3\epsilon_0}$$

Hence
$$\vec{E} = \frac{\rho_0 r}{3\epsilon_0}\hat{r} = \frac{\rho_0 \vec{r}}{3\epsilon_0}$$

- For $R<r<2R$

$$\oint \vec{E}\cdot d\vec{a} = 4\pi r^2 E$$

$$Q_{enc} = \frac{4\pi}{3} R^3 \rho_0 + \frac{4\pi}{3}\left(r^3 - R^3\right)\left(-\frac{\rho_0}{7}\right)$$

$$= \frac{4\pi}{3}\rho_0\left(\frac{8R^3}{7} - \frac{r^3}{7}\right)$$

Hence $4\pi r^2 E = \dfrac{4\pi \rho_0}{3\epsilon_0}\left(\dfrac{8R^3}{7} - \dfrac{r^3}{7}\right)$

$$\vec{E} = \frac{\rho_0}{3\epsilon_0 r^2}\left(\frac{8R^3}{7} - \frac{r^3}{7}\right)\hat{r}$$

For $r > 2R$

$$Q_{enc} = \frac{4\pi}{3} R^3 \rho_0 + \frac{4\pi}{3}\left(8R^3 - R^3\right)\cdot\left(-\frac{\rho_0}{7}\right)$$

$$= 0$$

Thus $\vec{E} = 0$ for $r > 2R$

b) $\nabla \cdot \vec{E}\Big|_{r = 3R/2} = \dfrac{\rho(3R/2)}{\epsilon_0} = -\dfrac{\rho_0}{7\epsilon_0}$

$\nabla \times \vec{E}\Big|_{r = 3R/2} = 0$

3) a) Charge on the conducting sphere resides on the surface.
Such a charge distribution produces zero field inside and outside for $r > R$

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$$

Potential at the surface

$$V(r = R) = \frac{Q}{4\pi\epsilon_0 R}$$

Since $\vec{E} = 0$ for $r < R$ we have

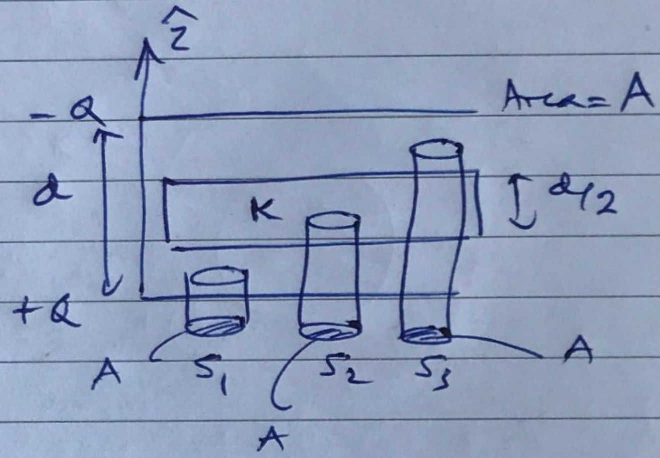
$$V(r < R) = \frac{Q}{4\pi\epsilon_0 R}$$

$$V(r > R) = \frac{Q}{4\pi\epsilon_0 r}$$

b) V depends only on radial coordinate r.
Hence work done in moving a unit positive charge will be

$$W = \Delta V = V(r=3R) - V(r=2R)$$
$$= \frac{Q}{4\pi\epsilon_0 (3R)} - \frac{Q}{4\pi\epsilon_0 (2R)}$$
$$= -\frac{Q}{4\pi\epsilon_0 \times 6R}$$

4)



a) Gauss's law in the presence of dielectric

$$\oint \vec{D} \cdot d\vec{a} = Q_{free}$$

$\vec{D}$ will be along $\hat{z}$
Taking surface S_1 we have

$$D.A = \sigma_f . A \Rightarrow D = \sigma_f \; ; \; \vec{D} = \sigma_f \hat{z}$$

Similarly

$$\vec{D} = \sigma_f \hat{z} \text{ inside dielectric}$$

Since $\vec{E} = \vec{D}/\epsilon$ inside dielectric
$= \vec{D}/\epsilon_0$ in free space
we get

$$\vec{E} = \frac{\sigma_f}{\epsilon_0} \hat{z} \quad \text{in air}$$
$$= \frac{\sigma_f}{\epsilon_0 K} \hat{z} \quad \text{in dielectric}$$

b) $\vec{P} = \epsilon_0 \chi \vec{E} = \epsilon_0 (K-1) \vec{E}$

$$= \frac{\epsilon_0 (K-1) \sigma_f}{\epsilon_0 K} \hat{z} = \frac{(K-1)\sigma_f}{K} \hat{z}$$

$\rho_b = -\vec{\nabla} \cdot \vec{P} = 0$

$\sigma_b = \vec{P} \cdot \hat{n}$

Upper surface of dielectric $\hat{n} = \hat{z}$ and
lower surface of dielectric $\hat{n} = -\hat{z}$.
Thus

$$\sigma_b = \frac{(K-1)\sigma_f}{K} \quad \text{at upper surface}$$

$$= -\frac{(K-1)\sigma_f}{K} \quad \text{at lower surface}$$