# Multivariable Calculus

(Lecture-11)

Department of Mathematics
Bennett University

$21^{st}$ November, 2018

Multiple Integration

of

(Scalar Valued Function of Vector Variable)

(Scalar Field)

$$F : R \subseteq \mathbb{R}^n \rightarrow \mathbb{R}, \; n = 2, 3$$

(Continuation....)

# Learning Outcome of this lecture

In the last lecture, we have learnt double integral over rectangular region.
In this lecture, we learn double integral over simple and bounded region $\mathcal{R}$.

- Double Integral of $f : \mathcal{R} \subset \mathbb{R}^2 \to \mathbb{R}$ where $\mathcal{R}$ is a Bounded region in $\mathbb{R}^2$.
- Double Integral of $f : \mathcal{R} \subset \mathbb{R}^2 \to \mathbb{R}$ where $\mathcal{R}$ is a Simple region in $\mathbb{R}^2$.
- Iterated Integral of $f$ and Fubini's Theorem for Simple Regions
- Applications of Double Integrals

## Double Integral of $f$ over Non-Rectangular region $\mathcal{D}$

Let $\mathcal{D}$ be a bounded set in $\mathbb{R}^2$.
Let $f$ be a bounded, real valued function on $\mathcal{D}$.
Take a rectangular region $\mathcal{R}$ such that $\mathcal{D} \subset \mathcal{R}$.
Define a function $\tilde{f} : \mathcal{R} \to \mathbb{R}$ by

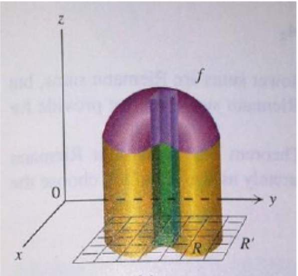
$$\tilde{f}(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in \mathcal{D}, \\ 0 & \text{if } (x,y) \in \mathcal{R} \setminus \mathcal{D} \end{cases}$$

If the double integral of $\tilde{f}$ over the rectangular region $\mathcal{R}$ exists then the double integral of $f$ over the region $\mathcal{D}$ is defined by

$$\iint_{\mathcal{D}} f(x,y)dA = \iint_{\mathcal{R}} \tilde{f}(x,y)dA.$$

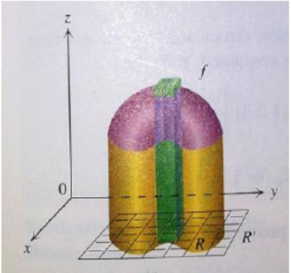
# Picture: Integration over non-rectangular region

In the following pictures,
$\mathcal{R}$ is a non-rectangular region.
Therefore, a rectangular region $\mathcal{R}'$ is considered such that $\mathcal{R} \subset \mathcal{R}'$.




(a)

Lower Sum
(Inscribed Parallelepiped)




(b)

Upper Sum
(Circumscribed Parallelepiped)

Note: $f$ is 0 on the set $\mathcal{R}' \setminus \mathcal{R}$.

Note:

If $\mathcal{D}$ is a bounded set with some arbitrary shape then computing the double/iterated integral of $f$ may become difficult.

So, we look for simple / elementary region $\mathcal{D}$ on which evaluation of the iterated integral of $f$ becomes easier.

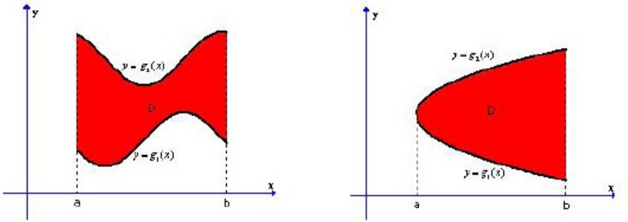
## Simple / Elementary Regions in $\mathbb{R}^2$

**Simple / Elementary Regions in $\mathbb{R}^2$**

**and**

**Double Integrals over Simple Regions**

# Vertically Simple / Type-I / $y$-simple Regions/ $y$-regular Regions

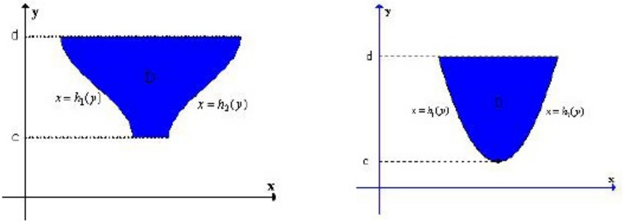



- Vertically Simple/Type-I Region:

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \ : \ x \in [a, b] \text{ and } g_1(x) \leq y \leq g_2(x)\},$$

where $g_1(x)$ and $g_2(x)$ are continuous functions on $[a, b]$ and $g_1(x) \leq g_2(x)$ for all $x \in [a, b]$.

# Horizontally Simple / Type-II / $x$-simple Regions / $x$-Regular Regions

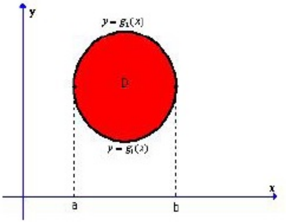


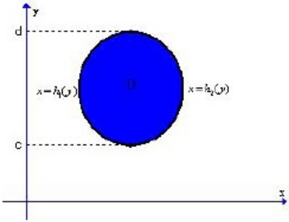

- **Horizontally Simple/Type-II Region**:

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \;:\; h_1(y) \le x \le h_2(y) \text{ and } y \in [c, d]\},$$

where $h_1(y)$ and $h_2(y)$ are continuous functions on $[c, d]$ and $h_1(y) \le h_2(y)$ for all $y \in [c, d]$.

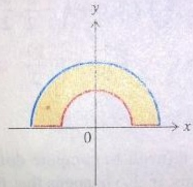
# Simple Regions (Both Type-I and Type-II)



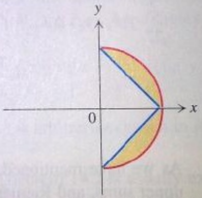




- Simple Region: If a region is both vertically simple and horizontally simple then it is said to be a simple region.

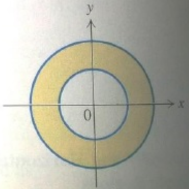
# More Examples




A vertically simple but not horizontally simple region

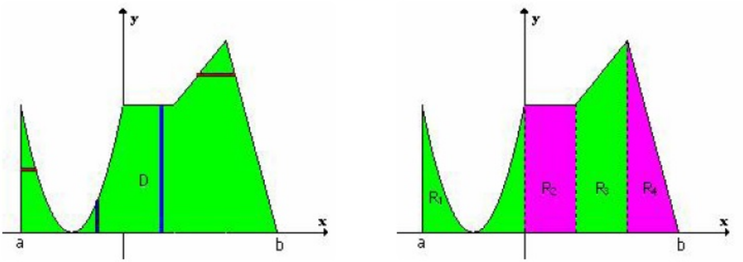



A horizontally simple but not vertically simple region

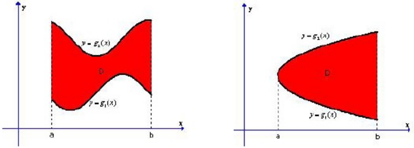



A region that is neither vertically nor horizontally simple

# Some non-simple region can be divided into union of simple regions



# Fubini's Theorem for Vertically Simple Regions



## Theorem

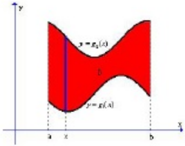
If $f : \mathcal{R} \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous on a vertically simple region

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \ : \ x \in [a, b] \text{ and } g_1(x) \leq y \leq g_2(x)\},$$

where $g_1(x)$ and $g_2(x)$ are continuous functions on $[a, b]$ and $g_1(x) \leq g_2(x)$ for all $x \in [a, b]$. Then

$$\iint_{\mathcal{R}} f(x, y)dA = \int_{x=a}^{x=b} \left( \int_{y=g_1(x)}^{y=g_2(x)} f(x, y)dy \right) dx.$$

# Finding Integral Limits for Vertically Simple Regions




- Inner most Integral: Inner most integral is with respect to the variable $y$. Draw a Vertical Strip or (Arrow headed Line Parallel to $y$-axis) over the region $\mathcal{R}$.
- $y$-Limits: Where the vertical strip enters the region (Bottom Curve of the Region $\mathcal{R}$)? Where the vertical strip leaves the region (Top Curve of the Region $\mathcal{R}$)?
- $x$-Limits: Slide the vertical strip over the region $\mathcal{R}$ from left to right. While vertical strip is sliding over the region $\mathcal{R}$, What is the starting value of $x$ (Left most value of $x$)? What is the ending value of $x$ (Right most value of $x$)?

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# Example-1

Let $\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \ : \ x \in [0, 1] \text{ and } x^3 \leq y \leq x\}$ and $f(x, y) = (1 - x)$ for $(x, y) \in \mathbb{R}^2$. Compute $\iint_{\mathcal{R}} f(x, y)dA$ ?

**Solution:**

$$\iint_{\mathcal{R}} f(x, y)dA = \int_{x=0}^{1} \left( \int_{y=x^3}^{x} (1 - x)dy \right) dx$$

$$= \int_{x=0}^{1} (1 - x) \left( \int_{y=x^3}^{x} dy \right) dx = \int_{x=0}^{1} (1 - x) \left( |y|_{x^3}^{x} \right) dx$$

$$= \int_{x=0}^{1} (1 - x)(x - x^3)dx = \int_{x=0}^{1} (x - x^2 - x^3 + x^4)dx$$

$$= \left| \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \right|_0^1$$

$$= \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} = \frac{7}{60}.$$

# Example-2

Let $\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \ : \ x \in [-2, 1] \text{ and } 0 \leq y \leq 1 - x\}$ and $f(x, y) = (4 - y)$ for $(x, y) \in \mathbb{R}^2$. Compute $\iint_{\mathcal{R}} f(x, y) dA$ ?

**Solution:**

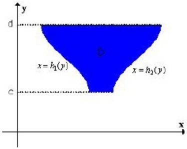
$$\iint_{\mathcal{R}} f(x, y) dA = \int_{x=-2}^{1} \int_{y=0}^{1-x} (4 - y) dy dx$$

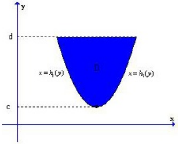
$$= \int_{x=-2}^{1} \left( \int_{y=0}^{1-x} (4 - y) dy \right) dx = \int_{x=-2}^{1} \left| 4y - \frac{y^2}{2} \right|_{0}^{1-x} dx$$

$$= \int_{x=-2}^{1} \left( 4(1 - x) - \frac{(1 - x)^2}{2} \right) dx$$

$$= \frac{27}{2}.$$

# Fubinis Theorem for Horizontally Simple Regions




## Theorem

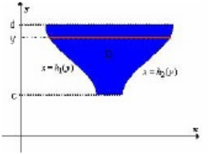
If $f : \mathcal{R} \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous on a horizontally simple region

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \;:\; h_1(y) \leq x \leq h_2(y) \text{ and } y \in [c, d]\},$$

where $h_1(y)$ and $h_2(y)$ are continuous functions on $[c, d]$ and $h_1(y) \leq h_2(y)$ for all $y \in [c, d]$. Then

$$\iint_{\mathcal{R}} f(x, y)dA = \int_{y=c}^{y=d} \left( \int_{x=h_1(y)}^{x=h_2(y)} f(x, y)dx \right) dy.$$

# Finding Integral Limits for Horizontally Simple Regions




- Inner most Integral: Inner most integral is with respect to the variable $x$. Draw a Horizontal Strip or (Arrow headed Line Parallel to $x$-axis) over the region $\mathcal{R}$.
- $x$-Limits: Where the horizontal strip enters the region (Leftmost/Bottom Curve of the Region $\mathcal{R}$)? Where the horizontal strip leaves the region (Rightmost/Top Curve of the Region $\mathcal{R}$)?
- $y$-Limits: Slide the horizontal strip over the region $\mathcal{R}$ from bottom to top. While horizontal strip is sliding over the region $\mathcal{R}$, What is the starting value of $y$ (Bottommost/Lowest value of $y$)? What is the ending value of $y$ (Topmost/highest value of $y$)?

# Example (Take same Example-2)

Let $\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \;:\; y \in [0, 3] \text{ and } -2 \leq x \leq 1 - y\}$ and $f(x, y) = (4 - y)$ for $(x, y) \in \mathbb{R}^2$. Compute $\iint_{\mathcal{R}} f(x, y)dA$ ?

**Solution:**

$$\iint_{\mathcal{R}} f(x, y)dA = \int_{y=0}^{3} \int_{x=-2}^{1-y} (4 - y)dxdy$$

$$= \int_{y=0}^{3} (4 - y) \left( \int_{x=-2}^{1-y} dx \right) dy = \int_{y=0}^{3} (4 - y) \left|x\right|_{-2}^{1-y} dy$$

$$= \int_{y=0}^{3} (4 - y)(3 - y)dy$$

$$= \frac{27}{2}.$$