Lecture - 10th (ODE)

Picard's Iteration Method (Method of Successive Approximation)

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0 \quad \text{——} \quad (1)$$

$$\frac{dy}{dx} = f(x,y)$$

$$\Rightarrow \quad dy = f(x,y)\, dx$$

Integrate on both the sides from x_0 to x, we get

$y(x)\Big|_{y_0}^{y}$

$$\Rightarrow \int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)\, dx$$

$\Rightarrow y(x) - y_0$

$\Rightarrow y(x) - y(x_0)$

$$\Rightarrow \Big| y(x) \Big|_{y_0}^{y} = \int_{x_0}^{x} f(x,y)\, dx$$

$$\Rightarrow y(x) - y(x_0) = \int_{x_0}^{x} f(s, y(s))\, ds$$

$$\Rightarrow y(x) = y(x_0) + \int_{x_0}^{x} f(s, y(s))\, ds$$

$$\Rightarrow \boxed{y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))\, ds} \quad (2)$$

(2) is the corresponding integral eqn to (1).

Let $y_0(x) = y_0$

$$y_1(x) = y_0 + \int_{x_0}^{x} f(s, y_0(s))\, ds$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(s, y_1(s))\, ds$$

$$\vdots$$

$$y_n(x) = y_0 + \int_{x_0}^{x} f(s, y_{n-1}(s))\, ds .$$

$y_0, y_1, y_2, \ldots y_n$ are called Picard's iterations or seqⁿ of approximations.

This seqⁿ of approximations will converge (Under the assumption of Picard Existence & Uniqueness Thⁿ) to the solⁿ $y(x)$ of (1).

$$\boxed{y(x) = \lim_{n \to \infty} y_n(x)}$$

Example: $\dfrac{dy}{dx} = -y, \quad y(0) = 1.$

$$\boxed{\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0}$$

Here $f(x,y) = -y$, $x_0 = 0$, $y_0 = 1$.

$f(s,1) = -1$

$$y_0(x) = y_0 = 1.$$

$$y_1(x) = \underline{y_0} + \int_{x_0}^{x} f(s, y_0(s))\, ds$$

$$= 1 + \int_0^x f(s, 1)\, ds$$

$$y_1(x) = 1 + \int_0^x (-1)\, ds = \underline{1 - x}$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(s, \underline{y_1(s)})\, ds$$

$f(x,y) = -y$

$f(s, 1-s) = \underline{-(1-s)}$

$$= 1 + \int_0^x f(s, 1-s)\, ds$$

$$= 1 + \int_0^x -(1-s)\, ds$$

$$= 1 - \int_0^x (1-s)\, ds$$

$$= 1 - \left(s - \frac{s^2}{2}\right)_0^x$$

$$y_2(x) = 1 - x + \frac{x^2}{2!}$$

$$y_n(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^n}{n!}$$

$$y(x) = \lim_{n\to\infty} y_n(x) = \lim_{n\to\infty}\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + \frac{(-1)^n x^n}{n!}\right)$$

$$= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} \cdots$$

$$\left(e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} \cdots\right)$$

$$= e^{-x}$$

$$\boxed{y(x) = e^{-x}}$$

Solve

$$\frac{dy}{dx} = xy, \qquad y(0) = 1$$

by Picard's Iteration Method

Summary of first Order ODE :

Linear ODE

Nonlinear ODE

$\downarrow$

Bernoulli's Eq^n (Reducible to linear)

Separation of Variables

Reducible to Separable

Homogeneous Eq^{n}s

Reducible to Hom..

Exact Eq^{n}s

Reducible to exact
(Integrating Factor)

Picard's Existence & Uniqueness Theorem
Picard's Iteration Method

Second Order ODE :

The general form of a linear second order ODE is

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\,y = F(x) \quad \text{(1)}$$

where $a_0(x) \neq 0$.

If $F(x) = 0$, then homogeneous

If $F(x) \neq 0$, then non-hom.

Example : $y'' - y = \sin x$ (Linear Non Hom)

$y'' - y = \sin y$
Non-Linear

(i) e^x, e^{-x} are sol^ns of $y'' - y = 0$

(ii) $\sin x, \cos x$ are sol^ns of $y'' + y = 0$

Homogeneous Second Order ODE:

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\,y = 0 \qquad \text{—— (1)}$$

y_1, y_2 are sol^n's of (1),

then $c_1 y_1 + c_2 y_2$, $c_1, c_2 \in \mathbb{R}$ is also a sol^n of (1).

$$\left.\begin{aligned} a_0(x)\,y_1'' + a_1(x)\,y_1' + a_2(x)\,y_1 &= 0 \\ a_0(x)\,y_2'' + a_1(x)\,y_2' + a_2(x)\,y_2 &= 0 \end{aligned}\right]$$

Consider

$$a_0(x)\left(c_1 y_1 + c_2 y_2\right)'' + a_1(x)\left(c_1 y_1 + c_2 y_2\right) + a_2(x)\left(c_1 y_1 + c_2 y_2\right)$$

$$\Rightarrow a_0(x)\left(c_1 y_1'' + c_2 y_2''\right) + a_1(x)\left(c_1 y_1 + c_2 y_2\right) + a_2(x)\left(c_1 y_1 + c_2 y_2\right)$$

$$\Rightarrow c_1\left(\underline{a_0(x)\,y_1'' + a_1(x)\,y_1' + a_2(x)\,y_1}\right) + c_2\left(\underline{a_0(x)\,y_2''} + a_1(x)\,y_2' + a_2(x)\,y_2\right)$$

$$= c_1(0) + c_2(0) = 0$$

$$= 0$$

If we know any two linearly independent sol^{n}s of (1), then any sol^n of (1) can be written as the linear combination of those two sol^{n}s.

i.e. $y(x) = c_1 y_1 + c_2 y_2$,

where y_1, y_2 are L.I sol^{n}s of (1).

Linearly Independent / Linearly Dependent f^{n}s!

$f(x)$ & $g(x)$ are said to be L.I on I, if

$$af(x) + b\,g(x) = 0 \quad \forall x \in I.$$

$$\Rightarrow \quad \underline{a = 0, \; b = 0}.$$

otherwise $f(x)$ & $g(x)$ will be called L.D.

Examples:

i) $f(x) = \sin 2x$,

$g(x) = \sin x \cos x$,

$(-\infty, \infty)$

$$f(x) = 2g(x)$$

$$\underline{\sin 2x = 2\sin x \cos x}$$

$$\Rightarrow \underline{f(x) - 2g(x)} = 0$$

$\Rightarrow \{f, g\}$ is LD on $(-\infty, \infty)$.

(ii) $f(x) = x$, $g(x) = |x|$ $(-\infty, \infty)$

$$\underline{\alpha f(x) + \beta g(x) = 0} \quad x \in (-\infty, \infty)$$

$$\alpha x + \beta |x| = 0, \Rightarrow \alpha = 0, \beta = 0$$

$\Rightarrow$ x & $|x|$ are L.I on $(-\infty, \infty)$.

But on $\underline{(-\infty, 0)}$, & $(0, \infty)$ x & $|x|$ are L.D.

$$\boxed{\underline{\text{Check}} \quad \underline{\underline{x|x|} \ \& \ x^2} \quad \text{on } (-\infty, \infty)}$$