Series

$$\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, \dots\}$$

$$a_1 + a_2 + \dots + a_n + \dots = \sum_{k=1}^{\infty} a_k$$

$a_n \downarrow$ nth term of the series

$$S_n = a_1 + a_2 + \dots + a_n = \sum_{k=1}^{n} a_k$$

$\{S_n\}_{n=1}^{\infty} \to$ sequence of partial sums of the series

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$\vdots$$
$$S_n = a_1 + a_2 + \dots + a_n$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \sum_{k=1}^{\infty} a_k$$

$\lim_{n\to\infty} S_n = L$, $\sum_{k=1}^{\infty} a_k = L$

$\lim_{n\to\infty} S_n = \infty$, $\sum_{k=1}^{\infty} a_k$ diverges

Ex 1 $1 + 2 + 3 + \dots + n + \dots \to$ diverges

$$S_n = 1 + 2 + \dots + n = \frac{n(n+1)}{2}$$

$$\lim_{n\to\infty} S_n = \infty$$

Ex 2: $1 + \frac{1}{2} + \frac{1}{2^2} + \dots = 2$

$$S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}}$$

$$= \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} = 2\left(1 - \frac{1}{2^n}\right)$$

$$\lim_{n\to\infty} S_n = 2$$

EX 3 $\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$

$$S_n = \sum_{k=1}^{n} \log\left(\frac{k+1}{k}\right)$$

$$= \sum_{k=1}^{n} \log(k+1) - \log k$$

$$= \log 2 - \log 1 + \log 3 - \log 2 + \log 4 - \log 3 + \cdots + \log(n+1) - \log n$$

$$= \log(n+1) \to \infty \text{ as } n \to \infty$$

Ex: $0 < x < 1$ $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

Ex:- $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$ (Telescopic series)

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}$$

$$= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1} \to 1 \text{ as } n \to \infty$$

Necessary condition

If $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow \lim_{n\to\infty} a_n = 0$

$\not\Leftarrow$

Ex: $\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$ diverges, $\lim_{n\to\infty} \log\left(\frac{n+1}{n}\right) = 0$

if $\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ diverges.

Ex: $\sum_{n=1}^{\infty} \frac{n}{n+2}$, $a_n = \frac{n}{n+2}$, $\lim_{n\to\infty} a_n = 1 \neq 0$

↳ diverges

Ex:- $\sum_{n=1}^{\infty} \frac{(n^2+1)}{(n+3)(n+4)}$, $a_n = \frac{n^2+1}{(n+3)(n+4)}$

$\lim_{n\to\infty} a_n = 1 \neq 0$

↓

diverges.

Ex: $x > 1$, $\sum_{n=0}^{\infty} x^n$ diverges.

Necessary & sufficient condition

$a_n \geq 0$, $\sum_{n=1}^{\infty} a_n$ converges $\Leftrightarrow \{S_n\}$ is bounded above

Ex:- $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$S_n = \sum_{k=1}^{n} \frac{1}{k^2} = 1 + \sum_{k=2}^{n} \frac{1}{k^2}$$

$$< 1 + \sum_{k=2}^{n} \frac{1}{k^2-k}$$

$$= 1 + \sum_{k=2}^{n} \left(\frac{1}{k(k-1)}\right)$$

$$= 1 + \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k}\right)$$

$$= 1 + (1 - \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{3} + \cdots + \tfrac{1}{n-1} - \tfrac{1}{n})$$

$$= 2 - \frac{1}{n} < 2$$

$\Rightarrow \{S_n\}$ is bounded above

$\Rightarrow \sum a_n$ is convergent.

<u>Ex</u>: $\sum_{n=1}^{\infty} \frac{1}{n}$, $S_n = \sum_{k=1}^{n} \frac{1}{k}$

<u>diverges</u>

$\{S_{2^n}\}$ $S_{2^n} > 1 + \frac{n}{2}$

<u>Result</u> $\sum_{n=1}^{\infty} a_n$ converges $\Longleftrightarrow \sum_{n=p}^{\infty} a_n$ converges

$p \geq 1$