Continuity
(Lecture 13 & 14)

Engineering Calculus

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Definition

A real valued function $f(x)$ is said to be continuous at $x = c$ if

(i) $c \in domain(f)$,

(ii) $\lim_{x \to c} f(x)$ exists,

(iii) the limit in (ii) is equal to $f(c)$.

Definition

Let $D(\neq \emptyset) \subseteq \mathbb{R}$ and let $f : D \to \mathbb{R}$. We say that f is continuous at $c \in D$ if for each $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ for all $x \in D$ satisfying $|x - c| < \delta$.
We say that $f : D \to \mathbb{R}$ is continuous if f is continuous at each $c \in D$.

Example

Show that $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is continuous at 0.

Solution: Let $\epsilon > 0$. Then $|f(x) - f(0)| \leq |x^2|$. So it is enough to choose $\delta = \sqrt{\epsilon}$.

Theorem (Sequential criteria of continuity)

A function f is continuous at c if and only if for every sequence $x_n \to c$, we must have $f(x_n) \to f(c)$ as $n \to \infty$.

Example

Show that $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is continuous at 0.

Solution: We note that $|f(x)| \leq |x^2|$. Therefore, $f(x_n) \to f(0)$ whenever $x_n \to 0$. This proves that f is continuous at $x = 0$.

Example

Show that $f(x) = \begin{cases} \frac{1}{x} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is not continuous at 0.

Solution: Choose $\frac{1}{x_n} = \frac{\pi}{2} + 2n\pi$. Then $\lim\limits_{n\to\infty} x_n = 0$ and $f(x_n) = \frac{1}{x_n} \to \infty$.

Theorem

Suppose f and g are continuous at c. Then

1. $f \pm g$ is also continuous at c.
2. fg is continuous at c.
3. $\dfrac{f}{g}$ is continuous at c if $g(c) \neq 0$.
4. $|f|$ is also continuous at c and $\lim\limits_{x \to c} |f(x)| = |f(c)|$.

Result

(a) Composition of continuous functions is also continuous i.e., if f is continuous at c and g is continuous at $f(c)$ then $g(f(x))$ is continuous at c.

(b) If $f(x)$ is continuous at c, then $|f|$ is also continuous at c.

(c) If f, g are continuous at c, then $\max(f, g)$ and $\min(f, g)$ are continuous at c. Also $\lim\limits_{x \to c} \max(f, g) = \max\{f(c), g(c)\}$ and $\lim\limits_{x \to c} \min(f, g) = \min\{f(c), g(c)\}$.

Proof: (c) Proof follows from the relation

$$\max(f, g) = \frac{1}{2}(f + g) + \frac{1}{2}|f - g| \qquad \min(f, g) = \frac{1}{2}(f + g) - \frac{1}{2}|f - g|.$$

Discontinuous Function

• A function which is not continuous is called discontinuous function.

Types of discontinuities:

Removable discontinuity

$f(x)$ is defined every where in an interval containing a except at $x = a$ and limit exists at $x = a$ OR $f(x)$ is defined also at $x = a$ and limit is NOT equal to function value at $x = a$. Then we say that $f(x)$ has removable discontinuity at $x = a$. These functions can be extended as continuous functions by defining the value of f to be the limit value at $x = a$.

Example

$f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Here limit as $x \to 0$ is 1. But $f(0)$ is defined to be 0.

Jump discontinuity

The left and right limits of $f(x)$ exists but not equal. This type of discontinuities are also called discontinuities of first kind.

Discontinuous Function

Example

$f(x) = \begin{cases} 1 & x \leq 0 \\ -1 & x > 0 \end{cases}$. Easy to see that left and right limits at 0 are different.

Infinite discontinuity

Left or right limit of $f(x)$ is ∞ or $-\infty$.

Example

$f(x) = \frac{1}{x}$ has infinite discontinuity at $x = 0$.

Discontinuity of second kind

If either $\lim_{x \to c^-} f(x)$ or $\lim_{x \to c^+} f(x)$ does not exist, then c is called discontinuity of second kind.

Example

Consider the function $f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$ does not have left or right limit at any point c.
Indeed, $f(c + \frac{1}{n})$ and $f(c + \frac{\pi}{n})$ converges to different values.

Theorem

Continuous functions on closed, bounded interval is bounded.

Proof: Let $f(x)$ be continuous on $[a, b]$ and let $\{x_n\} \subset [a, b]$ be a sequence such that $|f(x_n)| > n$. Then $\{x_n\}$ is a bounded sequence and hence there exists a subsequence $\{x_{n_k}\}$ which converges to c. Then $f(x_{n_k}) \to f(c)$, a contradiction to $|f(x_{n_k})| > n_k$.

Theorem

Let $f(x)$ be a continuous function on closed, bounded interval $[a,b]$. Then maximum and minimum of functions are achieved in $[a,b]$.

Proof: Let $\{x_n\} \subset [a,b]$ be a sequence such that $f(x_n) \to \max f$. Then $\{x_n\}$ is bounded and hence by Bolzano-Weierstrass theorem, there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to c$ for some c. $a \leq x_n \leq b$ implies $c \in [a,b]$. Since f is continuous, $f(x_{n_k}) \to f(c)$. Hence $f(c) = \max f$. The attainment of minimum can be proved by noting that $-f$ is also continuous and $\min f = -\max(-f)$.

Remark

Closed and boundedness of the interval is important in the above theorem. Consider the examples $(i)\, f(x) = \frac{1}{x}$ on $(0,1)$ $\quad (ii)\, f(x) = x$ on $\mathbb{R}$.

Theorem

Let $f(x)$ be a continuous function on $[a, b]$ and let $f(c) > 0$ for some $c \in (a, b)$, Then there exists $\delta > 0$ such that $f(x) > 0$ in $(c - \delta, c + \delta)$.

Proof: Let $\epsilon = \frac{1}{2}f(c) > 0$. Since $f(x)$ is continuous at c, there exists $\delta > 0$ such that

$$|x - c| < \delta \implies |f(x) - f(c)| < \frac{1}{2}f(c)$$

i.e., $-\frac{1}{2}f(c) < f(x) - f(c) < \frac{1}{2}f(c)$. Hence $f(x) > \frac{1}{2}f(c)$ for all $x \in (c - \delta, c + \delta)$.

Theorem

Suppose a continuous functions $f(x)$ satisfies $\int_a^b f(x)\phi(x)dx = 0$ for all continuous functions $\phi(x)$ on $[a, b]$. Then $f(x) \equiv 0$ on $[a, b]$.

Proof: Suppose $f(c) > 0$. Then by above theorem $f(x) > 0$ in $(c - \delta, c + \delta)$. Choose $\phi(x)$ so that $\phi(x) > 0$ in $(c - \delta/2, c + \delta/2)$ and is 0 otherwise. Then $\int_a^b f(x)\phi(x) > 0$, which is a contradiction.

Theorem

Let $f(x)$ be a continuous function on $\mathbb{R}$ and let $f(a)f(b) < 0$ for some a, b. Then there exits $c \in (a, b)$ such that $f(c) = 0$.

Proof: Assume that $f(a) < 0 < f(b)$. Let $S = \{x \in [a, b] : f(x) < 0\}$. Then $[a, a + \delta) \subset S$ for some $\delta > 0$ and S is bounded. Let $c = \sup S$. We claim that $f(c) = 0$. Take $x_n = c + \frac{1}{n}$, then $x_n \notin S$, $x_n \to c$. Therefore, $f(c) = \lim f(x_n) \geq 0$. On the otherhand, taking $y_n = c - \frac{1}{n}$, we see that $y_n \in S$ for n large and $y_n \to c, f(c) = \lim f(y_n) \leq 0$. Hence $f(c) = 0$.

Intermediate Value Theorem

Let $f(x)$ be a continuous function on $[a, b]$ and let $f(a) < y < f(b)$. Then there exists $c \in (a, b)$ such that $f(c) = y$.

Proof: Consider $g(x) = y - f(x)$ and use above Theorem.

Remark

From the IVT, we can conclude that A continuous function assumes all values between its maximum and minimum.

Fixed point theorem

Let $f(x)$ be a continuous function from $[0, 1]$ into $[0, 1]$. Then show that there is a point $c \in [0, 1]$ such that $f(c) = c$.

Proof: Define the function $g(x) = f(x) - x$. Then $g(0) \geq 0$ and $g(1) \leq 0$. Now apply Intermediate Value Theorem, to get the result.

Example

Show that $f(x) = x^2 - 2$ has at least one root in $(1, 2)$.

Definition (Uniformly Continuous Functions)

A function $f(x)$ is said to be uniformly continuous on a set S, if for given $\epsilon > 0$, there exists $\delta > 0$ such that

$$x, y \in S, \quad |x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Here δ depends only on ϵ, not on x or y.

Uniformly continuity

Theorem

If $f(x)$ is uniformly continuous function $\iff$ for ANY two sequences $\{x_n\}, \{y_n\}$ such that $|x_n - y_n| \to 0$, we have $|f(x_n) - f(y_n)| \to 0$ as $n \to \infty$.

Example

(a) $f(x) = x^2$ is uniformly continuous on bounded interval $[a, b]$.
Solution: Note that $|x^2 - y^2| \leq |x+y||x-y| \leq 2b|x-y|$. So one can choose $\delta < \frac{\epsilon}{2b}$.

(b) $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1)$.
Solution: Take $x_n = \frac{1}{n+1}, y_n = \frac{1}{n}$, then for n large $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| = 1$.

(c) $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.
Solution: Take $x_n = n + \frac{1}{n}$ and $y_n = n$. Then $|x_n - y_n| = \frac{1}{n} \to 0$, but $|f(x_n) - f(y_n)| = 2 + \frac{1}{n^2} > 2$.

Remarks

(a) If f, g are uniformly continuous, then $f \pm g$ is also uniformly continuous.

(b) If f, g are uniformly continuous, then fg need not be uniformly continuous. This can be seen by noting that $f(x) = x$ is uniformly continuous on $\mathbb{R}$ but x^2 is not uniformly continuous on $\mathbb{R}$.

Theorem

A continuous function $f(x)$ on a closed, bounded interval $[a, b]$ is uniformly continuous.

Proof: Suppose not. Then there exists $\epsilon > 0$ and sequences $\{x_n\}$ and $\{y_n\}$ in $[a, b]$ such that $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| > \epsilon$. But then by Bolzano-Weierstrass theorem, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to c. Also $y_{n_k} \to c$. Now since f is continuous, we have $f(c) = \lim f(x_{n_k}) = \lim f(y_{n_k})$. Hence $|f(x_{n_k}) - f(y_{n_k})| \to 0$, a contradiction.

Result

Suppose $f(x)$ has only removable discontinuities in $[a, b]$. Then $\tilde{f}$, the extension of f, is uniformly continuous.

Example

Show that $f(x) = \frac{\sin x}{x}$ is uniformly continuous on $[0, 1]$.

Solution: f has a removable discontinuity at $x = 0$. We define $\tilde{f} : [0, 1] \to \mathbb{R}$ as

$$\tilde{f}(x) = \begin{cases} \frac{\sin x}{x} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0. \end{cases}$$

Uniformly continuity

Theorem

Let f be a uniformly continuous function and let $\{x_n\}$ be a cauchy sequence. Then $\{f(x_n)\}$ is also a Cauchy sequence.

Proof: Let $\epsilon > 0$. As f is uniformly continuous, there exists $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Since $\{x_n\}$ is a Cauchy sequence, there exists N such that

$$m, n > N \implies |x_n - x_m| < \delta.$$

Therefore $|f(x_n) - f(x_m)| < \epsilon$.

Example

$f(x) = \frac{1}{x^2}$ is not uniformly continuous on $(0, 1)$.

Solution: The sequence $x_n = \frac{1}{n}$ is Cauchy but $f(x_n) = n^2$ is not. Hence f cannot be uniformly continuous.

Thank You