Solution - Tut sheet 3

1. Given that $P(X > \frac{1}{2}) = \frac{7}{8}$.

So that

$$\int_{1/2}^{\infty} f_X(x)\,dx = \frac{7}{8}$$

$$\int_{1/2}^{1} ax\,dx + \int_{1}^{2} (b-x)\,dx = \frac{7}{8}$$

$$\Rightarrow \boxed{3a + 8b = 19} \longrightarrow (A)$$

Also we have property of PDF :

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

$$\Rightarrow \int_{0}^{1} ax\,dx + \int_{1}^{2} (b-x)\,dx = \cancel{\tfrac{7}{8}}\ 1$$

$$\Rightarrow \boxed{a + 2b = 5} \longrightarrow (B)$$

Solving (A) & (B) we get

$$\boxed{a = 1,\ b = 2}$$

$$f_X(x) = \begin{cases} \cancel{a}\ 0 & x \le 0 \\ x & 0 \le x < 1 \\ 2-x & 1 \le x < 2 \\ 0 & 2 \le x \end{cases}$$

OR:

$$f_X(t) = \begin{cases} 0 , & t < 0 \\ t , & 0 \le t < 1 \\ 2-t , & 1 \le t < 2 \\ 0 , & 2 \le t \end{cases}$$

Distribution function : $F_X(x)$

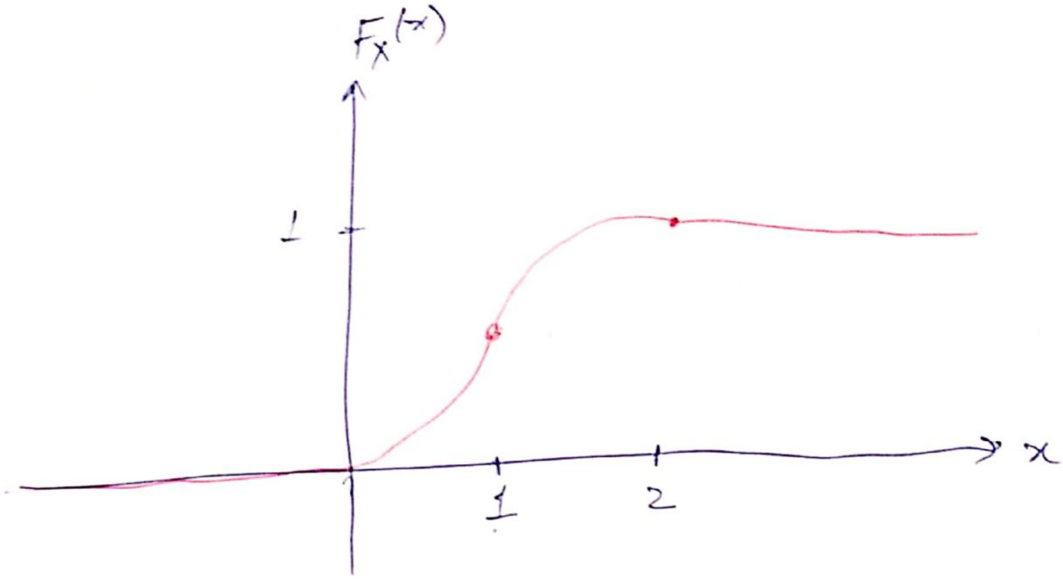
$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(t)\,dt$$

$$= \begin{cases} \int_{-\infty}^{x} 0\,dt , & x < 0 \\ \int_{-\infty}^{0} 0\,dt + \int_{0}^{x} t\,dt , & 0 \le x < 1 \\ \int_{-\infty}^{0} 0\,dt + \int_{0}^{1} t\,dt + \int_{1}^{x} (2-t)\,dt , & 1 \le x < 2 \\ \int_{-\infty}^{0} 0\,dt + \int_{0}^{1} t\,dt + \int_{1}^{2} (2-t)\,dt + \int_{2}^{x} 0\,dt , & 2 \le x \end{cases}$$

(solving we get)

$$= \begin{cases} 0 , & x < 0 \\ \frac{x^2}{2} , & 0 \le x < 1 \\ -\frac{x^2}{2} + 2x - 1 , & 1 \le x < 2 \\ 1 , & 2 \le x \end{cases}$$

Graph :



(2) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$\Rightarrow \int_{-\infty}^{1} 0\,dx + \int_{1}^{3}\left(\frac{x}{4}+a\right)dx + \int_{3}^{5}\left(-\frac{x}{4}+b\right)dx + \int_{5}^{\infty} 0\,dx = 1$

$\Rightarrow \int_{1}^{3}\left(\frac{x}{4}+a\right)dx + \int_{3}^{5}\left(-\frac{x}{4}+b\right)dx = 1$

$\Rightarrow \boxed{a+b=1}$ —— (*)

Using condition $P(X>2)=\frac{7}{8}$

$\Rightarrow \int_{2}^{3}\left(\frac{x}{4}+a\right)dx + \int_{3}^{5}\left(-\frac{x}{4}+b\right)dx = \frac{7}{8}$

$\Rightarrow \boxed{4a+8b=9}$ —— (**)

Solving (*) and (**) for a and b, we get

$\boxed{a=-\frac{1}{4},\ b=\frac{5}{4}}$

(3) Given $E(X) = -1. \Rightarrow \int_{-\infty}^{\infty} x f_X(x)\,dx = -1$

$Var(X) = 2$

$E(X^2) - (E(X))^2 = 2$

$E(X^2) - (-1)^2 = 2$

$E(X^2) = 3 \Rightarrow \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = 3$

Also, by property of PDF $\Rightarrow \int_{-\infty}^{\infty} f_X(x)\,dx = 1$

Using above three integrals:

$$\int_{-\infty}^{\infty} x f_X(x)\,dx = -1$$

$$\Rightarrow \int_{-1}^{1} ax^2\,dx + \int_{1}^{2} (bx^2 + cx)\,dx = -1$$

$$\Rightarrow \boxed{\frac{2a}{3} + \frac{7b}{3} + \frac{3c}{2} = -1} \quad \text{—— (A)}$$

$$\int_{-\infty}^{\infty} x^2 f_X(x)\,dx = 3$$

$$\Rightarrow \int_{-1}^{1} ax^3\,dx + \int_{1}^{2} (bx^3 + cx^2)\,dx = 3$$

$$\Rightarrow \boxed{\frac{15b}{4} + \frac{7c}{3} = 3} \quad \text{—— (B)}$$

And $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$$\int_{-1}^{1} ax\,dx + \int_{1}^{2} (bx+c)\,dx = 1$$

$$\boxed{\frac{3b}{2} + c = 1} \quad \text{——— (C)}$$

Solving (A), (B) and (C) we get:

$$\boxed{a = \frac{-49}{18}, \; b = \frac{8}{3}, \; c = -3}$$

~~Rest~~ Find $F_X(x)$ & it's graph similar to questions (1) & (2).

(4)

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$= \int_{2}^{7} x \frac{1}{5}\,dx$$

$$= 4.5 .$$

[For uniform distribution between values (a, b) the average is $\frac{a+b}{2}$.].

(5) (a) $E(X) = \sum_{k=0}^{\infty} k \cdot P(X=k)$

$$= \sum_{k=0}^{\infty} k \frac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda}\sum_{k=1}^{\infty} \frac{k\lambda^k}{k!}$$

$$= e^{-\lambda}\sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} = \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$$

$$= \lambda e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right)$$

$$= \lambda e^{-\lambda} e^{\lambda} = \lambda = 50 \text{ customers}$$

(b) $P(X=0) = \frac{e^{-50} 50^0}{0!} = e^{-50}$

(c) $P(X \geq 5) = 1 - P(X<5)$

$$= 1 - [P(X=0) + P(X=1) + \cdots + P(X=4)]$$

$$= 1 - \left[\frac{e^{-50}50^0}{0!} + \frac{e^{-50}50}{1!} + \cdots + \frac{e^{-50}50^4}{4!}\right]$$

$=$

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2. The life of a light bulb in months is denoted by a random variable X with following PDF (Exponential distribution):

$$f_X(x) = 0.25e^{-0.25x}, \quad x > 0.$$

(2.A) What is the average life of light bulb?

(2.B) What is the average life of the bulb given that $X \geq 1$?

Example 3 contd.

(3.d) The 10 pieces of the bulb are put under observation independently for 6 months. Let Y denotes the number of working bulbs after 6 months of inspection.

Find $P(Y = 3)$? What is distribution function and PMF of random variable Y?

(3.e) The 5 pieces of the bulb are put in a series and wired together. (A series system works if all of its component are working.) Let Z denotes the life of this series system in months.

Find $P(Z \leq 2.5)$. What is distribution function and PDF of random variable Z?

EXAMPLE 3

(3) The life of a light bulb in months is denoted by a random variable X with following PDF (Exponential distribution):

$$f_X(x) = 0.25e^{-0.25x}, \quad x > 0.$$

- (3.A) What is the probability that the life of light bulb will be more than 2 months?
- (3.B) What is the probability that the life of light bulb will be less than 45 days?
- (3.C) Given that the bulb was working for last 1 month, what is the probability that it will be working for next 3 months?

(2) Given that $f_X(x) = .25\,e^{-0.25x}$, $x>0$

(2.A) Average life of bulb

$= E(X)$

$= \int_{-\infty}^{\infty} x\, f_X(x)\, dx$

$= .25 \int_{0}^{\infty} \underset{\substack{\downarrow \\ \text{1st} \\ \text{function}}}{x}\, \underset{\substack{\downarrow \\ \text{2nd} \\ \text{function}}}{e^{-0.25x}}\, dx$

(Using ILATE Rule)

$= .25 \left[\frac{x}{-0.25} e^{-0.25} \Big|_{x=0}^{x=\infty} - \int_0^{\infty} \left(\frac{-1}{0.25}\right) e^{-0.25x}\, dx \right]$

$= \frac{1}{.25} = 4$ months

(2.B) $E(X/X \geq 1) = \dfrac{\int_1^{\infty} x\, f_X(x)\, dx}{P(X \geq 1)}$

$= \dfrac{\int_1^{\infty} x\, f_X(x)\, dx}{\int_1^{\infty} f_X(x)\, dx}$

(3)

$$= \frac{.25 \int_1^{\infty} x e^{-0.25x} dx}{0.25 \int_1^{\infty} \cancel{0.25} e^{-0.25x} dx}$$

$= 5$ months

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(3)

(3.A) $P(X \geq 2) = \int_2^{\infty} f_X(x)\,dx = 0.25 \int_2^{\infty} e^{-0.25x} dx$

$= e^{-.5}$

(3.B) 45 days $= 1.5$ months.

(Units of light bulb is in months.)

$P(X < 1.5) = \int_{-\infty}^{1.5} f_X(x)\,dx = \int_0^{1.5} 0.25\, e^{-0.25x} dx.$

$= 1 - e^{.375}$

(3.C) $P(X \geq 4 / X \geq 1) = \dfrac{P(\{X \geq 4\} \cap \{X \geq 1\})}{P(X \geq 1\}}$

Pl.

$$= \frac{P(X \geq 4)}{P(X \geq 1)}$$

[Intersection of $\{x \geq 4\}$ and $\{x \geq 1\}$ is smaller set $\{x \geq 4\}$]

$$= \frac{\int_4^{\infty} f_X(x)\,dx}{\int_1^{\infty} f_X(x)\,dx}$$

$$= \frac{0.25 \int_4^{\infty} e^{-0.25x}\,dx}{0.25 \int_1^{\infty} e^{-0.25x}\,dx}$$

$$= e^{-.75}$$

3.D Let us denote by p the following probability -

$p = P$ (A bulb is working after 6 months)

$$= P(X \geq 6)$$

$$= \int_6^{\infty} f_X(x)\,dx.$$

$$= 0.25 \int_6^{\infty} e^{-0.25x}\,dx$$

$$= e^{-1.5}$$

$P(Y=3) = P($ Out of 6 bulbs under inspection 3 are working after 6 months.$)$

$= {}^6C_3\, p^3 (1-p)^{6-3}$ —— (**)

[Here, ${}^6C_3$ = Any combination of 3 bulbs from 6 may be working.

p = Probability that bulb is working after 6 months.

$1-p$ = Probability that bulb is not working.]

Distribution function and PMF of Y

First see what are the possible values of Y:

$Y \in \{0, 1, 2, 3, 4, 5, 6\}$

Now for any $k = 0, 1, 2, 3, 4, 5, 6$;

$P(Y=k) = P($ Out of 6 bulbs k are working after 6 months.$)$

$$= {}^{6}C_{k}\, p^{k} (1-p)^{6-k}$$

[As in (**)]

Thus PMF is

$$P(Y=k) = {}^{6}C_{k}\, p^{k}(1-p)^{6-k}, \quad k = 0,1,2,3,4,5,6$$

~~That~~ where $p = e^{-1.5}$ (As given by (*))

$\Rightarrow$ $Y \sim B(6, p)$ ~~[Bino~~

~~Distribo~~ Y is Binomially distributed with parameters 6 and $e^{-1.5}$.

Distribution function

$$F_Y(y) = P(Y \leq y) = \begin{cases} 0, & y < 0 \\ \sum_{k \leq y} P(Y=k), & y \geq 0 \end{cases}$$

(3.E) Z is a continuous random variable. (7)

$P(Z \le 2.5) = 1 - \underline{P(Z > 2.5)}$ → It's easier to calculate as compared to $P(Z \le 2.5)$

$= 1 - P($life of the series system is greater than 2.5 months.$)$

$= 1 - P($life of all the 5 bulbs in the series system is greater than 2.5 months.$)$

$= 1 - \left[P(X > 2.5)\right]^5$

(As all 5 bulbs are independent, so working of one bulb is independent to other. That is why above probability got multiplied 5 times.]

$= 1 - \left[\int_{2.5}^{\infty} f_X(x)\,dx\right]^5$

$= 1 - \left[e^{-0.625}\right]^5$

$= 1 - e^{-3.125}$

Distribution function and PDF of Z

Possible values of $Z = (0, \infty)$

For any $z \in (0, \infty)$

$$F_Z(z) = P(Z \le z) = 1 - P(Z > z)$$

$$= 1 - P(\text{life of the series system is greater than } z \text{ months})$$

$$= 1 - P(\text{life of } \underline{\text{all}} \text{ the 5 bulbs in series is greater than } z \text{ months})$$

$$= 1 - [P(X > z)]^5$$

$$= 1 - \left[\int_z^\infty 0.25 e^{-0.25x} dx\right]^5$$

$$= 1 - e^{-1.25z}$$

Thus Distribution function

$$= F_Z(z) = 1 - e^{-1.25z} \; ; \; z > 0$$

PDF $f_Z(z) = \frac{d}{dz} F_Z(z) = 1.25 e^{-1.25z} \; ; \; z > 0$

Thus $Z \sim \text{Exp}(1.25)$.

Z is exponentially distributed with parameter 1.25.