NEGATION OF QUANTIFIERS & METHODS OF PROOFS

NEGATION OF QUANTIFIERS

$\neg\exists x P(x)$ equivalent to $\forall x \neg P(x)$

- True when P(x) is false for every x
- False if there is an x for which P(x) is true.

$\neg \forall x P(x)$ is equivalent to $\exists x \neg P(x)$

- True if there exists an x for which P(x) is false.
- False if P(x) is true for every x.

QUANTIFICATION OF TWO VARIABLES
(READ LEFT TO RIGHT)

$\forall x \forall y P(x,y)$ or $\forall y \forall x P(x,y)$

- True when P(x,y) is true for every pair x,y.
- False if there is a pair x,y for which P(x,y) is false.

$\exists x \exists y P(x,y)$ or $\exists y \exists x P(x,y)$

True if there is a pair x,y for which P(x,y) is true.

False if P(x,y) is false for every pair x,y.

QUANTIFICATION OF TWO VARIABLES

$\forall x \exists y P(x,y)$

- True when for every x there is a y for which P(x,y) is true.
  (in this case y can depend on x)
- False if there is an x such that P(x,y) is false for every y.

$\exists y \forall x P(x,y)$

- True if there is a y for which P(x,y) is true for every x.
  (i.e., true for a particular y regardless (or independent) of x)
- False if for every y there is an x for which P(x,y) is false.

Note that order matters here

In particular, if $\exists y \forall x P(x,y)$ is true, then $\forall x \exists y P(x,y)$ is true.

However, if $\forall x \exists y P(x,y)$ is true, it is not necessary that $\exists y \forall x P(x,y)$ is true.

BASIC NUMBER THEORY DEFINITIONS

- Z = Set of all Integers
- Z+ = Set of all Positive Integers
- N = Set of Natural Numbers (Z+ and Zero)
- R = Set of Real Numbers
- Addition and multiplication on integers produce integers.

  $(a,b \in Z) \rightarrow [(a+b) \in Z] \wedge [(ab) \in Z]$

NUMBER THEORY DEF. CONTD…

- n is **even** is defined as $\exists k \in Z \ni n = 2k$
- n is **odd** is defined as $\exists k \in Z \ni n = 2k+1$
- x is **rational** is defined as $\exists a,b \in Z \ni x = a/b, b \neq 0$
- x is **irrational** is defined as $\neg\exists a,b \in Z \ni x = a/b, b \neq 0$ or $\forall a,b \in Z, x \neq a/b, b \neq 0$
- $p \in Z+$ is **prime** means that the only positive factors of p are p and 1. If p is not prime we say it is **composite**.

$\ni$ = "such that"

METHODS OF PROOF

$p \rightarrow q$ (Example: if n is even, then n^2 is even)

- **Direct proof**: Assume p is true and use a series of previously proven statements to show that q is true.
- **Proof by contradiction**: Assume negation of what you are trying to prove ($p \wedge \neg q$). Show that this leads to a contradiction.

DIRECT PROOF

Prove: $\forall n \in Z$, if n is even, then n^2 is even.

Tabular-style proof:

n is even	*hypothesis*
$n=2k$ for some $k \in Z$	*definition of even*
$n^2 = 4k^2$	*algebra*
$n^2 = 2(2k^2)$ which is 2*(an integer)	*algebra and mult of integers gives integers*
n^2 is even	*definition of even*

SAME DIRECT PROOF

Prove: $\forall n \in Z$, if n is even, then n^2 is even.

Sentence-style proof:

Assume that n is even. Thus, we know that $n = 2k$ for some integer k. It follows that $n^2 = 4k^2 = 2(2k^2)$. Therefore n^2 is even since it is 2 times $2k^2$, which is an integer.

STRUCTURE OF A DIRECT PROOF

Prove: $\forall n \in Z$, if n is even, then n^2 is even.

Proof:

Assume that n is even. Thus, we know that $n = 2k$ for some integer k. It follows that $n^2 = 4k^2 = 2(2k^2)$. Therefore n^2 is even since it is 2 times $2k^2$ which is an integer.

EXAMPLE: PROOF BY CONTRADICTION

Prove: The sum of an irrational number and a rational number is irrational.

Proof: Let q be an irrational number and r be a rational number. **Assume that their sum is rational, i.e., q+r=s where s is a rational number.** Then q = s-r. But by our previous proof the sum of two rational numbers must be rational, so we have an irrational number on the left equal to a rational number on the right. This is a contradiction. Therefore q+r can't be rational and must be irrational.

STRUCTURE OF PROOF BY CONTRADICTION

- Basic idea is to assume that the opposite of what you are trying to prove is true and show that it results in a violation of one of your initial assumptions.
- In the previous proof we showed that assuming that the sum of a rational number and an irrational number is rational and showed that it resulted in the impossible conclusion that a number could be rational and irrational at the same time. (It can be put in a form that implies $n \wedge \neg n$ is true, which is a contradiction.)

USING CASES

Prove: $\forall n \in Z$, $n^3 + n$ is even.
Separate into cases based on whether n is even or odd. Prove each separately using direct proof.
Proof: We can divide this problem into two cases. n can be even or n can be odd.

<u>Case 1: n is even</u>. Then $\exists k \in Z \ni n = 2k$.
$n^3+n = 8k^3 + 2k = 2(4k^3+k)$ which is even since $4k^3+k$ must be an integer.

CASES (CONT.)

Case 2: n is odd. Then $\exists k \in Z \ni n = 2k+1$.

$n^3 + n = (8k^3 + 12k^2 + 6k + 1) + (2k + 1) = 2(4k^3 + 6k^2 + 4k + 1)$ which is even since $4k^3 + 6k^2 + 4k + 1$ must be an integer.

Therefore $\forall n \in Z$, $n^3 + n$ is even

PROOF?

Prove if n^3 is even then n is even.

Proof: Assume n^3 is even.

Then $\exists k \in Z \ni n^3 = 8k^3$ for some integer k. It follows that $n = \sqrt[3]{8k^3} = 2k$. Therefore n is even.

Statement is true but *argument* is false.

Argument assumes that n is even in making the claim $n^3=8k^3$, rather than $n^3 = 2k$. This is circular reasoning.