Lecture - 13th (ODE)

Non Hom. nth Order Linear DE

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_n(x)y = F(x) \quad \text{--- (NH)}$$

The corresponding Hom. Linear DE is

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_n(x)y = 0 \quad \text{--- (H)}$$

If $y_c(x)$ is the solⁿ of (H) and $y_p(x)$ is the particular solⁿ of (NH).

Then $y_c(x) + y_p(x)$ is the solⁿ of (NH).

Proof: Since $y_c(x)$ is the solⁿ of (H).

$$\Rightarrow a_0(x)\frac{d^n y_c}{dx^n} + a_1(x)\frac{d^{n-1}y_c}{dx^{n-1}} + \cdots + a_n(x)y_c = 0 \quad \text{--- (1)}$$

and $y_p(x)$ is the particular solⁿ of (NH).

$$\Rightarrow a_0(x)\frac{d^n y_p}{dx^n} + a_1(x)\frac{d^{n-1}y_p}{dx^{n-1}} + \cdots + a_n(x)y_p = F(x) \quad \text{--- (2)}$$

Adding (1) & (2),

$$a_0(x)\frac{d^n}{dx^n}(y_c+y_p) + a_1(x)\frac{d^{n-1}}{dx^{n-1}}(y_c+y_p) + \cdots + a_n(x)(y_c+y_p) = F(x)$$

$\Rightarrow$ $y_c + y_p$ is the solⁿ of (NH).

$\underline{y_c}$ $\longrightarrow$ Complementary function (Solⁿ of corresponding Hom. DE).

y_p $\longrightarrow$ particular solⁿ of (NH) (particular integral)

To find y_p

(i) Method of undetermined coeff

(ii) Variation of parameter

$$\underbrace{a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_n(x)y = 0}_{} \quad \text{--- (H)}$$

If $y_1, y_2, \ldots, y_n$ are n L-I solⁿs of (H), then $\boxed{y(x) = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n}$

Solⁿ of Homogeneous linear DE with constant coefficients:

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_n y = 0 \quad \text{--- (1)}$$

where $a_0 \neq 0$, $a_0, a_1, a_2, \ldots, a_n$ are constants

Let us start with second order

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0 \quad \text{--- (SH)}$$

where $a_0 \neq 0$, a_0, a_1, a_2 are constants.

Which f^n can be the sol^n of (SH).

Let us start with $\boxed{y = e^{mx}}$

Suppose $y = e^{mx}$ is the sol^n of (SH).

$$\Rightarrow \quad a_0 m^2 e^{mx} + a_1 m e^{mx} + a_2 e^{mx} = 0$$

$$\Rightarrow \quad (a_0 m^2 + a_1 m + a_2) e^{mx} = 0$$

(Characteristic eq^n) or (auxiliary eq^n)

Since $e^{mx} \neq 0 \Rightarrow \underbrace{a_0 m^2 + a_1 m + a_2 = 0}$ ②

$\Rightarrow y = e^{mx}$ is the sol^n of (SH) iff m is sol^n of ②.

Since ② $(a_0 m^2 + a_1 m + a_2 = 0)$ is a quadratic eq^n and it will have 2 roots.

Now, three cases arise

Case-I: The roots are real and distinct

Case-II: The roots are real & equal

Case-III: The roots are complex

Case-I: If the roots are real & distinct

Let us suppose $a_0 m^2 + a_1 m + a_2 = 0$

has two distinct roots m_1, m_2.

$\Rightarrow e^{m_1 x}, e^{m_2 x}$ are the sol^{n}s of (SH).

Also $W(e^{m_1 x}, e^{m_2 x}) = \begin{vmatrix} e^{m_1 x} & e^{m_2 x} \\ m_1 e^{m_1 x} & m_2 e^{m_2 x} \end{vmatrix}$

$\neq 0$

$\Rightarrow e^{m_1 x}, e^{m_2 x}$ are L-I sol^{n}s of (SH).

Thus $\boxed{y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$

Example: $\dfrac{d^2 y}{dx^2} - 5\dfrac{dy}{dx} + 6y = 0$ ——— (1)

The Characteristic eq^n is

$m^2 - 5m + 6 = 0$ $\quad y = e^{mx}$

$\Rightarrow (m-2)(m-3) = 0$

$\Rightarrow \boxed{m = 2, 3}$

$\Rightarrow$ The roots are real & distinct.

$\Rightarrow e^{2x}, e^{3x}$ are L.I sol^{n}s of (1)

$$\boxed{y(x) = c_1 e^{2x} + c_2 e^{3x}}$$

Case-II: If the roots are real & equal:

Suppose A.E $a_0 m^2 + a_1 m + a_2 = 0$ has real & equal roots

$m = m_1, m_2,$

$m_1 = m_2$.

$m = \underline{m_1, m_1}$ (two equal roots.

$\Rightarrow \; y(x) = c_1 e^{m_1 x} + c_2 e^{m_1 x}$

$= (c_1 + c_2) e^{m_1 x}$

$\Rightarrow \; \underline{y(x) = c e^{m_1 x}}$

$\Rightarrow$ The only L.I solⁿ is $e^{m x}$.

$\Rightarrow$ we have to apply reduction of order to find another L.I solⁿ.

The another L.I solⁿ is $\underline{x e^{m_1 x}}$. (?)

(find yourself!

Thus $y(x) = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$

$$\boxed{y(x) = (c_1 + c_2 x) e^{m_1 x}}$$

Example: $\underbrace{\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = 0}$

The A.E is

$$m^2 - 2m + 1 = 0$$
$$\Rightarrow \quad (m-1)^2 = 0$$
$$\Rightarrow \quad \boxed{m = 1, 1}$$

$$y(x) = (c_1 + c_2 x)\, e^{x}$$

Case-III: If the roots are complex conjugate:

Suppose the A.E

$$a_0 m^2 + a_1 m + a_2 = 0$$

has the complex root $\alpha + i\beta$.

$\Rightarrow$ $\alpha - i\beta$ must be the root of A.E.

(Complex roots occur in conjugate pairs)

$$\Rightarrow \quad m = \alpha \pm i\beta$$
$$= \underbrace{\alpha + i\beta}_{m_1},\ \underbrace{\alpha - i\beta}_{m_2}$$

$$y(x) = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}$$

$$= c_1 e^{\alpha x}(e^{i\beta x}) + c_2 e^{\alpha x}\cdot(e^{-i\beta x})$$

$$= e^{\alpha x}\Big[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)\Big]$$

$$\left[\ e^{i\theta} = \cos\theta + i\sin\theta\right]$$

$$= e^{\alpha x}\Big[(c_1+c_2)\cos\beta x + (ic_1 - ic_2)\sin\beta x\Big]$$

$$y(x) = e^{\alpha x}\left[c_3\cos\beta x + c_4\sin\beta x\right]$$

$$c_3 = c_1 + c_2, \quad c_4 = ic_1 - ic_2$$

If the A·E has complex roots $\alpha \pm i\beta$,

$$\boxed{y(x) = e^{\alpha x}\left(c_1\cos\beta x + c_2\sin\beta x\right)}$$

Example:
$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

The A·E is $m^2 + m = 0$

$$\Rightarrow m(m+1) = 0$$

$$\Rightarrow m = 0, -1$$

$\therefore\ y(x) = c_1 e^{0x} + c_2 e^{-x} = c_1 + c_2 e^{-x}.$

<u>Example</u>: $\dfrac{d^2y}{dx^2} + y = 0$

The A.E is $m^2 + 1 = 0$

$\Rightarrow m = \pm i$

$(\alpha \pm i\beta)$

$\alpha = 0,\ \beta = 1$

$$y(x) = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$$
$$= e^{0x}(c_1 \cos x + c_2 \sin x)$$

$$\boxed{y(x) = c_1 \cos x + c_2 \sin x}$$

If the roots of the A.E are

$\underbrace{1, 1, 1}, \underbrace{2, 3, 4}, \underbrace{5 \pm i}$

Then what is the general solⁿ of the corresponding DE.

$$y(x) = (c_1 + c_2 x + c_3 x^2) e^{x} + c_4 e^{2x} + c_5 e^{3x}$$
$$+ c_6 e^{4x} + e^{5x}(c_7 \cos x + c_8 \sin x)$$

If the roots are repeated twice

$$m = m_1, m_2$$

$$y = (C_1 + C_2 x) e^{m_1 x}$$

If the roots of A.E are repeated thrice

$$m = m_1, m_2, m_3$$

$$y = (C_1 + C_2 x + C_3 x^2) e^{m_1 x}$$

If the roots of A.E are repeated 4 times

$$m = m_1, m_2, m_3, m_4$$

$$y = (C_1 + C_2 x + C_3 x^2 + C_4 x^3) e^{m_1 x}$$

If the roots of A.E are repeated k times, $(m = m_1, m_2, \ldots m_k)$

$$y = (C_1 + C_2 x + C_3 x^2 + \ldots + C_k x^{k-1}) e^{mx}$$

Q: If the roots of A.E are $1 \pm 2i$, $1 \pm 2i$, then what is the general sol^n of corresponding DE?

$$y(x) = e^{x}\left[(c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x\right]$$

If $\alpha \pm i\beta$ is repeated k-times

$$y(x) = e^{\alpha x}\left[(c_1 + c_2 x + \cdots + c_k x^{k-1})\cos\beta x + (d_1 + d_2 x + \cdots + d_k x^{k-1})\sin\beta x\right]$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left(x + \sqrt{x^2+a^2}\right) + C$$

$$= \sinh^{-1}\frac{x}{a}$$

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$

$|x|$ is not diff —

$x|x|$ is diff.

$x^2|x|$ is differentiable

$$x|x| = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$$

$$\begin{cases} 2x, & x \geq 0 \\ -2x, & x < 0 \end{cases}$$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

If $W(y_1, y_2) \neq 0 \Rightarrow y_1$ & y_2 are L-I.

If y_1 and y_2 are L-D $\Rightarrow$ ~~y_1 & y_2~~ $W(y_1, y_2) = 0$

If y_1 and y_2 are sol^n's of

$$a_0(x) \frac{d^2y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x)\, y = 0, \quad x \in I$$

where $a_0(x) \neq 0$, $a_0(x), a_1(x), a_2(x)$ are cts on I,

then

$W(y_1, y_2) = 0 \iff y_1$ & y_2 are L.D

$W(y_1, y_2) \neq 0 \iff y_1$ & y_2 are L.I.