Department of Mathematics, Bennett University
Engineering Calculus (EMAT101L)
Solutions for Tutorial Sheet 8

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1. (a) $\int_0^\infty e^{-x}\cos x\, dx = \lim_{b\to\infty}\int_0^b e^{-x}\cos x\, dx$. Now take

$$I = \int_0^b e^{-x}\cos x\, dx = \frac{1}{2}(1 - e^{-b}\cos b + e^{-b}\sin b).$$

So $\lim_{b\to\infty} I = \frac{1}{2} \Rightarrow \int_0^\infty e^{-x}\cos x\, dx$ is convergent.

(b) $\int_1^\infty \frac{dx}{x^2(1+e^x)} \le \int_1^\infty \frac{dx}{x^2}$ which is convergent. Hence by comparison test given improper integral is convergent.

(c) $\int_1^\infty \frac{(x+1)}{x^{3/2}}\, dx = \int_1^\infty \frac{1}{\sqrt{x}}\, dx + \int_1^\infty x^{-3/2} dx$. The first integral on the right side diverges. Hence given integral diverges.

(d) $\int_0^\infty \frac{dx}{x^2+\sqrt{x}} \le \int_0^1 \frac{dx}{\sqrt{x}} + \int_1^\infty \frac{dx}{x^2}$. Both the integral on the right side are convergent hence the given integral is convergent.

2. (a) Take $\ln x = t$ then $x = e^t$ and the integral becomes $\int_0^{\ln 2} \frac{e^{t/2}}{t}\, e^t dt$. It is easy to see that integrand is $\ge \frac{1}{t}$ and the integral $\int_0^{\ln 2} \frac{1}{t}\, dt$ diverges.

(b) Let $f(x) = \frac{\sin(x^2)}{\sqrt{x}}$ and take $g(x) = \frac{1}{\sqrt{x}}$. Then $\lim_{x\to 0}\frac{f(x)}{g(x)} = 0$. Also $\int_0^1 \frac{dx}{\sqrt{x}}$ is convergent so $\int_0^1 \frac{\sin(x^2)}{\sqrt{x}}\, dx$ is convergent.

(c) Let $f(x) = \frac{\tan x}{x^{3/2}}$ and take $g(x) = \tan x$. Then $\lim_{x\to\frac{\pi}{2}}\frac{f(x)}{g(x)} \in (0,\infty)$. Also as $\int_1^{\frac{\pi}{2}} \tan x\, dx$ is divergent so $\int_1^{\frac{\pi}{2}} \frac{\tan x}{x^{3/2}}\, dx$ is divergent.

(d) $\int_2^3 \frac{\log x}{\sqrt{|2-x|}}\, dx = \int_2^3 \frac{\log x}{\sqrt{x-2}}\, dx$. Here 2 is a point of infinite discontinuity. Take $g(x) = \frac{1}{\sqrt{x-2}}$. Then $\lim_{x\to 2}\frac{f(x)}{g(x)} = \log 2$. Now $\int_2^3 g(x)\, dx$ converges. Therefore by limit test $\int_2^3 f(x)\, dx$ converges.

3. (a) $\int_0^\infty x^{-\frac{1}{2}} e^{x^2} dx = \int_0^1 \frac{e^{x^2}}{\sqrt{x}}\, dx + \int_1^\infty \frac{e^{x^2}}{\sqrt{x}}\, dx$. Now $\int_0^1 \frac{e^{x^2}}{\sqrt{x}}\, dx$ is convergent, since if we take $g(x) = \frac{1}{\sqrt{x}}$ then $\lim_{x\to 0} \frac{f(x)}{g(x)} = 1$ and $\int_0^1 \frac{dx}{\sqrt{x}}$ is convergent. But $\int_1^\infty \frac{e^{x^2}}{\sqrt{x}}\, dx$ is divergent, since if we take $g(x) = \frac{1}{\sqrt{x}}$ then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$ and $\int_1^\infty \frac{dx}{\sqrt{x}}$ is divergent. Hence the given integral is divergent.

(b) Let $g(x) = \frac{1}{1+x^2}$. Then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$. Since $\int_0^\infty g(x)\, dx$ converges, by limit test $\int_0^\infty g(x)\, dx$ converges.

4. (a) Let $f(t) = \int_0^\infty e^{-tx} \frac{\sin x}{x}\, dx$. Then

$$\frac{\partial f}{\partial t} = -\int_0^\infty e^{-tx} \sin x\, dx = -\frac{1}{1+t^2} \Rightarrow f(t) = -\arctan t + c.$$

By the second fundamental theorem

$$f(a) - f(0) = \int_0^a f'(t)\, dt = -\int_0^a \frac{1}{1+t^2}\, dt,$$

taking $a \to \infty$, $\lim_{a\to\infty} f(a) - f(0) = -\frac{\pi}{2}$. Also

$$|f(a)| = \left| \int_0^\infty e^{-ax} \frac{\sin x}{x}\, dx \right| \le C_1 \int_0^\infty e^{-ax} dx \to 0 \text{ as } \quad a \to \infty.$$

Therefore $\lim_{a\to\infty} f(a) = 0$. Using this we get $c = \frac{\pi}{2}$ and hence $f(t) = \frac{\pi}{2} - \arctan t$.

(b) Let $f(t) = \int_0^1 \frac{x^t - 1}{\ln x}\, dx$, then

$$\frac{\partial f}{\partial t} = \int_0^1 x^t dx = \frac{1}{t+1} \Rightarrow f(t) = \ln(t+1) + c.$$

Now $f(0) = 0 \Rightarrow c = 0 \Rightarrow \int_0^1 \frac{x^t - 1}{\ln x}\, dx = \ln(t+1)$.

5. (a) Let $I = \int_0^\infty e^{-x^2} dx$. Put $x^2 = t \Rightarrow 2xdx = dt$, then $I = \int_0^\infty \frac{1}{2} e^{-t} t^{-\frac{1}{2}} dt = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$.

(b) $\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{\tan x}\ dx = \int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}} x\ \ \cos^{-\frac{1}{2}} x\ dx = \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{4}\right).$

(c) Let $\displaystyle I = \int_0^1 x^m \left(\log\frac{1}{x}\right)^n dx$. Put $\displaystyle\log\frac{1}{x} = t \Rightarrow I = \int_0^\infty e^{-(m+1)t}\ t^n dt$. Now put $(m+1)t = y$ we get $\displaystyle I = \frac{1}{(m+1)^{(n+1)}}\Gamma(n+1)$.

(d) Let $\displaystyle I = \int_0^{\frac{\pi}{2}} \sin^4\theta\ \ \cos^6\theta\ d\theta = \frac{1}{2}\beta\left(\frac{5}{2}, \frac{7}{2}\right)$.