# Inner Product Spaces, Orthogonality :-

Recall that, the concept of "length" & "Orthogonality" in $\mathbb{R}^n$ or $\mathbb{C}^n$.

1) length of a vector $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ is defined as $\sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$.

(ii) Orthogonality :- Let $X = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$

$Y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$.

Then $X$ is orthogonal to $Y$ if

$$X \cdot Y = XY^t = x_1 y_1 + x_2 y_2 + \dots + x_n y_n = 0$$

Now, we generalize the notion of length & Orthogonality in arbitrary Vector space.

For this, we first, need to study Inner Product Spaces.

**Def:-** Let $V$ be a real vector space.

Let $\langle \cdot \rangle : V \times V \longrightarrow \mathbb{R}(\mathbb{C})$ be a function.

$\langle \cdot \rangle$ is called an "Inner product" on $V$ if it satisfies the following axioms :-

(i) Linear Properties :- $\langle au + bv, w \rangle = a\langle u, w \rangle + b\langle v, w \rangle$

(ii) Symmetric Property :- $\langle u, w \rangle = \langle w, u \rangle$

(iii) Positive Definite Property :- $\langle u, u \rangle \geq 0$ &

$\langle u, u \rangle = 0$ iff $u = 0$.

The vector space V with an Inner Product is called a real (complex) inner product space.

Note that For Real Inner product space.

**Remark**:- ~~From~~ (i) State that an inner product function is linear in the first position.

Using (i) & (ii), we obtain

$$\langle u, c\cancel{w} + dw \rangle = \langle cv + dw, u \rangle$$
$$= c\langle v, u \rangle + d\langle w, u \rangle$$
$$= c\langle u, v \rangle + d\langle u, w \rangle$$

i.e, The inner product function is also linear in its second position.

Combining these two properties, yield in general

$$\langle \sum_i a_i u_i, \sum_j b_j v_j \rangle = \sum_i \sum_j a_i b_j \langle u_i, v_j \rangle$$

i.e The inner product of linear combination of vectors is equal to the linear combination of the inner product of vectors.

Example:- (i) $\langle 2u-5v, 4u+6v\rangle = 8\langle u,u\rangle + 12\langle u,v\rangle - 20\langle v,u\rangle - 30\langle v,v\rangle$

$= 8\langle u,u\rangle - 8\langle v,u\rangle - 30\langle v,v\rangle$.

Remark:- Axiom (i) implies $\langle 0,0\rangle = \langle 0v, 0\rangle = 0\langle v,0\rangle = 0$

Thus (i), (ii) & (iii) are equivalent to

(i), (ii) & (iii)$'$ — if $u \neq 0$ Then $\langle u,u\rangle > 0$.

i.e A function satisfying (i), (ii) & (iii)$'$ defines an inner product.

Norm of a Vector:-

By (iii) axioms, $\langle u,u\rangle$ is non-negative for any vector $u$.

Thus, its positive square root exists.

We use the notation

$$\|u\| = \sqrt{\langle u,u\rangle}$$

The non-negative number is called the norm or length of $u$.

**Remark:** (1) Frequently, we used $\|u\|^2 = \langle u, u \rangle$.

2) If $\|u\| = 1$ or equivalently, $\langle u, u \rangle = 1$. Then $u$ is called unit vector and it is said to be normalized.

3) Every non-zero vector $v \in V$ can be multiplied by the reciprocal of its length to obtain the unit vector.

$$\hat{v} = \frac{1}{\|v\|} v$$ , which is a positive multiple of $v$.

This process is called normalizing $v$.

**Examples:** Euclidean $n$-Space $\mathbb{R}^n$:-

Consider the vector space $\mathbb{R}^n$. The dot product or scalar product in $\mathbb{R}^n$ is defined by

$$u \cdot v = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n ,$$

where $u = (u_1, u_2, \ldots, u_n)$ , $v = (v_1, v_2, \ldots, v_n)$.

This function defines an inner product on $\mathbb{R}$.

The norm $\|u\|$ of the vector $u = (u_i)_i$ in the space is as follows:

$$\|u\| = \sqrt{u \cdot u} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

On the other hand, by the Phthagorean Thm, The distant from origin "$\underline{O}$" in $\mathbb{R}^3$ to a pt $P(a,b,c)$ is given by $\sqrt{a^2+b^2+c^2}$.

This is precisely the same as the above defined norm of the vector $V = (a,b,c) \in \mathbb{R}^3$.

* Because, the Phythogorean theorem is a consequence of the axioms of Euclidean geometry.

$\therefore$ The vector space $\mathbb{R}^n$ with the above inner product and norm is called Euclidean $n$-space.

This inner product is called the usual (or standard) inner product on $\mathbb{R}^n$.

Remark:- If the vectors in $\mathbb{R}^n$ is represented by column vector i.e $n \times 1$ column matrix. Then

$\langle u, v \rangle = u^T v$ defines the usual inner product on $\mathbb{R}^n$.

2) Function space $C[a,b]$ and Polynomial space $P(t)$ :

$C[a,b]$ — $\{ f : [a,b] \to \mathbb{R} \ : \ f \text{ is cts} \}$

$C[a,b]$ is a Vector space with respect to the following addition & Scalar multiplication.

$$(f+g)(x) = f(x) + g(x)$$

$$(\alpha f)x = \alpha f(x)$$

We define inner product on $C[a,b]$ as

$$\langle f, g \rangle = \int_a^b f(x)\, g(x)\, dx.$$

Then $(C[a,b], \langle \cdot \rangle)$ is an inner product space.

Also $\|f\|^2 = \langle f, f \rangle$

Verification :- 1) $\langle \alpha f + \beta h, g \rangle = \int_a^b (\alpha f + \beta h)(x)\, g(x)\, dx$

$$= \int_a^b (\alpha f(x) + \beta h(x))\, g(x)\, dx$$

$$= \alpha \int_a^b f(x)\, g(x)\, dx + \beta \int_a^b h(x)\, g(x)\, dx$$

$$= \alpha \langle f, g \rangle + \beta \langle h, g \rangle$$

(ii) $\langle f, g \rangle = \int_a^b f(x)\, g(x)\, dx = \int_a^b g(x)\, f(x)\, dx = \langle g, f \rangle$

(iii) $\langle f, f \rangle = \int_a^b f^2(x) \geq 0$

$\langle f, f \rangle = 0 \Rightarrow \int_a^b f^2 \, dx = 0 \Rightarrow \underline{\underline{f = 0}}.$

Hence, $\langle \cdot \rangle$ defines an inner product on $C[a,b]$.

<u>Remark</u>:- The vector space $P(t)$ – {set of all polynomial}. is a subspace of $C[a,b]$ for any interval $[a,b]$, & hence, it also defines an inner product on $P(t)$.

<u>Ex</u>:- Consider $f(t) = 3t-5$, $g(t) = t^2$ in $P(t)$ with inner product

$$\langle f, g \rangle = \int_0^1 f \, g \, dx$$

Compute $\langle f, g \rangle$, $\|f\|$, $\|g\|$

$$\langle f, g \rangle = \int_0^1 f g = \int_0^1 (3t-5) t^2 = \frac{3}{4} t^4 - \frac{5}{3} t^3 \Big|_0^1 = \frac{3}{4} - \frac{5}{3} = \frac{9-20}{12} = \frac{-11}{12}$$

$$\|f\| = \langle f, f \rangle^{1/2} = \left( \int_0^1 (3t-5)^2 dt \right)^{1/2} = \left[ \frac{1}{9} (3t-5)^3 \Big|_0^1 \right]^{1/2} = $$

$$= \left[ \frac{-8}{9} + \frac{125}{9} \right]^{1/2} = \left( \frac{117}{9} \right)^{1/2} = \sqrt{13}$$

$$\|g\| = \langle g, g \rangle^{1/2} = \left( \int_0^1 t^4 \right)^{1/2} = \frac{1}{\sqrt{5}}$$

3) Matrix Space $M = M_{m \times n}$

$$\langle A, B \rangle = \text{trace}\left(B^t A\right) = \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} b_{ij}$$

$\langle A, B \rangle$ is the sum of the products of the corresponding entries in A & B.

$$\langle A, A \rangle = \text{trace}\left(A^t A\right) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2$$

= sum of the square of the entries of A.

Ex :- $A = \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

$\langle A, B \rangle = 2 \cancel{+} 2 + 4 + 12 = 16$

$\|A\|^2 = \langle A, A \rangle = 1 + 4 + 16 + 36 = 57$.

$\|B\|^2 = \langle B, B \rangle = 4 + 1 + 1 + 4 = 10$

# Cauchy-Schwarz Inequality, Application:

**Thm:** For any vectors $u$ & $v$ in an inner product space $V$,

$$\langle u, v\rangle^2 \le \langle u, u\rangle \langle v, v\rangle$$

or

$$|\langle u, v\rangle| \le \|u\| \, \|v\|.$$

**Ex:** a) Let $u = (u_1, u_2, \cdots, u_n)$ , $v = (v_1, v_2, \cdots, v_n) \in \mathbb{R}^n$.

$$|u_1 v_1 + u_2 v_2 + \cdots + u_n v_n| \le (u_1^2 + u_2^2 + \cdots + u_n^2)^{1/2} (v_1^2 + v_2^2 + \cdots + v_n^2)^{1/2}$$

or

$$\left(\sum_{i=1}^{n} u_i v_i\right)^2 \le \left(\sum_{i=1}^{n} u_i^2\right)\left(\sum_{i=1}^{n} v_i^2\right)$$

i.e. $(u \cdot v)^2 \le \|u\|^2 \cdot \|v\|^2$

(b) $$\left[\int_0^1 f(x)\, g(x)\right]^2 \le \left(\int f^2(x)\, dx\right)\left(\int g^2(x)\, dx\right)$$

i.e. $$\langle f, g\rangle^2 \le \langle f, f\rangle \langle g, g\rangle$$
$$= \|f\|^2 \, \|g\|^2$$

## Angles b/w Vectors :-

For non-zero vectors $u, v$ in an inner product space $V$,

The angle b/w $u$ & $v$ is defined to be the angle $\theta$ s.t $0 \le \theta \le \pi$,

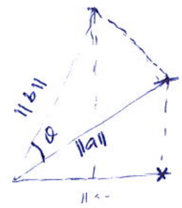
$$\cos\theta = \frac{\langle u, v\rangle}{\|u\| \, \|v\|}$$

in $\mathbb{R}^n$

$a \cdot b = \|a\| \, \|b\| \cos\theta$

$\cos\theta = \dfrac{a \cdot b}{\|a\| \, \|b\|}$

By Cauchy schwartz inequality,




$$|\cos\theta| \le 1$$

$\Rightarrow \quad -1 \le \cos\theta \le 1$ & so angle exist & unique

Ex:- a) $u = (2, 3, 5)$, $v = (1, 4, 3) \in \mathbb{R}$ Then

$\langle u, v\rangle = 5$, $\|u\| = \sqrt{38}$, $\|v\| = \sqrt{26}$

The angle $\theta$ b/w $u$ & $v$ is

$$\cos\theta = \frac{5}{\sqrt{38}\sqrt{26}}.$$

(b) Let $f(t) = 3t - 5$, $g(t) = t^2$.

$\langle f, g\rangle = \int f(t) g(t) \, dt$.

$\langle f, g\rangle = -\frac{11}{12}$, $\|f\| = \sqrt{13}$, $\|g\| = 1/\sqrt{5} = \frac{\sqrt{5}}{5}$

$$\cos\theta = \frac{-55}{12\sqrt{13}\sqrt{5}}$$

$\theta$ is an obtuse angle $\because$ $\cos\theta$ is negative

## Cauchy-Schwarz Inequality, Application :→

**Thm:-** For any vector $u$ & $v$ in an inner product space $V$, we have

$$\langle u, v\rangle^2 \leq \langle u, u\rangle \langle v, v\rangle$$

or

$$|\langle u, v\rangle| \leq \|u\| \, \|v\|$$

→ equality occurs iff $u$ & $v$ are l. dependent.

**Example :-** 1. Consider any real no's $u=(a_1, a_2, \dots, a_n)$, $v=(b_1, \dots, b_n)$. Then, by Cauchy Schwarz inequality,

$$(a_1 b_1 + a_2 b_2 + \dots + a_n b_n)^2 \leq (a_1^2 + a_2^2 + \dots + a_n^2)(b_1^2 + b_2^2 + \dots + b_n^2)$$

i.e
$$\langle u, v\rangle^2 \leq \|u\|^2 \|v\|^2$$
$$|\langle u, v\rangle| \leq \|u\| \, \|v\|$$

2) Let $f$ & $g$ be continuous function on the unit interval $[0,1]$. Then, by Cauchy Schwarz inequality,

$$\left[\int_0^1 f(t) g(t)\, dt\right]^2 \leq \int_0^1 f^2(t)\, dt \int_0^1 g^2(t)\, dt$$

$$\langle f, g\rangle^2 \leq \|f\|^2 \|g\|^2$$

OR
$$|\langle f, g\rangle| \leq \|f\| \, \|g\|$$

## Angle b/w Vectors :-

For any nonzero vector $u$ & $v$ in an inner product space $V$, the angle b/w $u$ & $v$ is defined to be the angle $\theta$ s.t $0 \le \theta \le \pi$ &

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\| \, \|v\|}$$

By the cauchy-Schwarz inequality, $-1 \le \cos\theta \le 1$.

So, the angle exists & is unique.

Ex :- let $u = (2,3,5)$, & $v = (1,-4,3)$ in $\mathbb{R}^3$

$\langle u, v\rangle = 5$ , $\|u\| = \sqrt{4+9+25} = \sqrt{38}$

$\|v\| = \sqrt{1+16+9} = \sqrt{26}$

Then, the angle $\theta$ b/w $u$ & $v$ is given by

$$\cos\theta = \frac{5}{\sqrt{38}\,\sqrt{26}}.$$

(b) $f(t) = 3t - 5$ , $g(t) = t^2$, $\langle f, g\rangle = \int_0^1 f(t)\, g(t)\, dt$

$$\cos\theta = \frac{-11/12}{(\sqrt{13})\left(\frac{\sqrt{5}}{5}\right)} = -\frac{55}{12\sqrt{13}\sqrt{5}}$$

Note that $\theta$ is an Obtuse angle, because $\theta$ is -ve.

## Orthogonality :-

Let $V$ be an inner product space. The vectors $u, v \in V$ are said to be orthogonal and $u$ is said to be orthogonal to $v$ if

$$\boxed{\langle u, v \rangle = 0}$$

1. The relation is symmetric. i.e if $\langle u, v \rangle = 0$ then $\langle v, u \rangle = 0$. i.e if $u$ is orthogonal to $v$ then $v$ is orthogonal to $u$.

**Remark** :- 1. $0 \in V$ is orthogonal to every vector $v \in V$; because

$$\langle 0, v \rangle = \langle 0v, v \rangle = 0\langle v, v \rangle = 0.$$

Conversely, if $u$ is orthogonal to every $v \in V$.

Then $\langle u, v \rangle = 0 \quad \forall \; v \in V$

In particular $\langle u, u \rangle = 0 \Rightarrow \|u\|^2 = 0 \Rightarrow \boxed{u = 0}$.

2. Observe that $u$ & $v$ are orthogonal iff $\cos\theta = 0$, where $\theta$ is an angle b/w $u$ & $v$. Also, this is true if $u$ & $v$ are perpendicular i.e $\theta = \pi/2$ or $90^\circ$.

Ex:- $u = (1,1,1)$, $v = (1,2,-3)$, $w = (1,-4,3)$ in $\mathbb{R}^3$.

Then $u \perp v, w$ but $v$ is not orthogonal to $w$.

(b) [Con]sider the function {sin t & cos t} in the vector space $C[-\pi, \pi]$ of cts fun. on the closed interval $[-\pi, \pi]$.

Then

$$\langle \sin t, \cos t \rangle = \int_{-\pi}^{\pi} \sin t \cos t = \frac{1}{2} \sin^2 t \Big|_{-\pi}^{\pi} = 0 - 0 = 0.$$

(c) A vector $w = (x_1, x_2, \ldots, x_n)$ is orthogonal to $u = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ if

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$$

Ex: Find a non-zero vector $w$ that is orthogonal to $u_1 = (1,2,1)$ & $u_2 = (2,5,4)$ in $\mathbb{R}^3$.

Solⁿ: Let $w = (x, y, z)$. Then $\langle u_1, w \rangle = 0 = \langle u_2, w \rangle$

$$\Longrightarrow \quad \begin{aligned} x + 2y + z &= 0 \\ 2x + 5y + 4z &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} x + 2y + z &= 0 \\ y + 2z &= 0 \end{aligned}$$

$z$ is only the free variable in the echelon system.

Set $z = 1$, $y = -2$, $x = 3$

Thus, $w = (3, -2, 1)$ is a desired non-zero vector orthogonal to $u_1$ & $u_2$.

Any multiple of $w$ is also orthogonal to $u_1$ & $u_2$

Normalizing $w$, we obtain unit vector orthogonal to $u_1$ & $u_2$.

$$\hat{w} = \frac{w}{\|w\|} = \left( \frac{3}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{1}{\sqrt{14}} \right).$$

## Orthogonal Complement :

Let $S$ be a subset of an I.P.S $V$.

The orthogonal complement of $S$, denoted by $S^{\perp}$ (read as "S perp") consists of those vectors in $V$ that are orthogonal to every vector $u \in S$.

i.e $\boxed{S^{\perp} = \{v \in V : \langle u, v \rangle = 0 \ \forall \ u \in S\}}$

In particular, for a given vector $u \in V$, we have

$$u^{\perp} = \{v \in V : \langle v, u \rangle = 0\}$$

**Remark:-** $S^{\perp}$ is a subspace of $V$.

Clearly, $0 \in S$, because $0$ is orthogonal to every vector in $V$.

Now, suppose, $v, w \in S^{\perp}$

Then for any scalar $\alpha, \beta$ & for any vector $u \in S$, we have

$$\langle \alpha v + \beta w, u \rangle = \alpha \langle v, u \rangle + \beta \langle w, u \rangle = \alpha . 0 + \beta . 0 = 0.$$

Thus, $\alpha v + \beta w \in S^{\perp}$

& $\therefore$ $S^{\perp}$ is a subspace of $V$.

**Remark :-** 1) Suppose $u$ is a non-zero vector in $\mathbb{R}^3$.

Then Geometrical interpertation of $u^{\perp}$ is a plane in $\mathbb{R}^3$ through the origin $O$ and perpendicular to the vector $u$.

2) Let $W$ be the solⁿ space of an $m \times n$ homogeneous system $AX = 0$, where $A = [a_{ij}]$ & $X = [x_j]$

**Recall that :-** $W$ may be viwed as the kernel of the linear mapping $A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

Now, we can give another interpretation of $W$ using the orthogonality.

Specifically, each solution vector $w = (x_1, x_2, \cdots x_n)$ is orthogonal to each row of $A$.

& hence $W$ is the orthogonal complement of the row space of $A$

**Ex :-** Find a basis for the subspace $u^{\perp}$ of $\mathbb{R}^3$, where $u = (1, 3, -4)$

$$u^{\perp} = \{ v = (x, y, z) \in \mathbb{R}^3 : \langle u, v \rangle = 0 \}$$

$$= \{ (x, y, z) : x + 3y - 4z = 0 \}$$

Set $y = 1$, $z = 0$, we obtain $w_1 = (-3, 1, 0)$

set $y = 0$, $z = 1$, we obtain $w_2 = (4, 0, 1)$.

Thus, $w_1$ & $w_2$ form a basis for the solⁿ space of eqn.

& hence a basis for $u^{\perp}$.

* → Suppose $W$ is a subspace of $V$. Then both $W$ and $W^{\perp}$ are subspace of $V$.

Then $V = W \oplus W^{\perp}$.

i.e $V$ is the direct sum of $W$ & $W^{\perp}$.
iff $V = W + W^{\perp}$ & $W \cap W^{\perp} = \{0\}$, So, $\dim V = \dim W + \dim W^{\perp}$

Orthogonal Set & Bases :-

Let $S = \{u_1, u_2, \ldots, u_r\}$ be set of non-zero vector in I.P.S V.

Then $S$ is called

(i) Orthogonal if $\langle u_i, u_j \rangle = 0 \quad \forall\, i \neq j$

(ii) Orthonormal if $\langle u_i, u_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

Thm:- Suppose $S = \{u_1, u_2, \ldots, u_r\}$ is an orthogonal set of non-zero vector.
Then $S$ is linearly independent.

Solⁿ:- Let $\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n = 0$ — (1)

Then we need to show $\alpha_i = 0 \ \forall\, i$.

$\langle \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n, u_i \rangle = \alpha_i \langle u_i, u_i \rangle = \alpha_i \|u_i\|^2$ — (2)

Using (1) & (2), we obtain
$\alpha_i \|u_i\|^2 = 0$

Since $u_i \neq 0 \Rightarrow \alpha_i = 0 \ \forall\, i$

Thus, The set $\{u_1, u_2, \ldots, u_r\}$ is l.i

(2) Suppose $\{u_1, u_2, \ldots, u_r\}$ is an orthogonal set of vectors. Then

$$\|u_1 + u_2 + \cdots + u_n\|^2 = \|u_1\|^2 + \|u_2\|^2 + \cdots + \|u_r\|^2.$$

This is known as Phythagoras Thm.

(3) a) Let $E = \{e_1, e_2, e_3\}$, $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$ basis of $\mathbb{R}^3$.

Then $E$ is an orthonormal basis of $\mathbb{R}^3$.

(b) $E = \{1, \cos t, \cos 2t, \ldots, \sin t, \sin 2t, \sin 3t, \ldots\}$

(Let $V = C[-\pi, \pi]$ be the vector space of cts fun. on the interval $-\pi \le t \le \pi$ with inner product defined by

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t)\, g(t)\, dt.$$

Then $E \subset V$ is an orthogonal set in $V$.

This set plays a fundamental role in the theory of Fourier series.

$\longrightarrow$

## Orthogonal Basis and linear Combination, Fourier Co-efficient:-

Let $S$ consist of the following three vectors in $\mathbb{R}^3$

$$u_1 = (1,2,1), \quad u_2 = (2,1,-4), \quad u_3 = (3,-2,1).$$

Then one can verify that $u_1, u_2$ & $u_3$ are orthogonal.

So, They are linearly independent.

Thus, $S$ is an orthogonal basis of $\mathbb{R}^3$.

**Next**, suppose we want to write $v = (7,1,9)$ as a linear combination of $u_1, u_2, u_3$.

$$v = x_1 u_1 + x_2 u_2 + x_3 u_3$$

$$(7,1,9) = x_1(1,2,1) + x_2(2,1,-4) + x_3(3,-2,1)$$

Method 1:- Gauss Elimination Method :-

$$x_1 + 2x_2 + 3x_3 = 7$$
$$2x_1 + x_2 - 2x_3 = 1$$
$$x_1 - 4x_2 + x_3 = 9$$

We obtain $x_1 = 3, x_2 = -1, x_3 = 2$.

Thus, $v = 3u_1 - u_2 + 2u_3$.

Method 2:- This method uses the fact that the basis vector are orthogonal.

$$\langle v, u_i \rangle = \langle x_1u_1 + x_2u_2 + x_3u_3, u_i \rangle$$

$$\langle v, u_i \rangle = x_i \langle u_i, u_i \rangle \quad \text{or} \quad x_i = \frac{\langle v, u_i \rangle}{\langle u_i, u_i \rangle}$$

$$x_1 = \frac{\langle v, u_1 \rangle}{\langle u_1, u_1 \rangle} = \frac{7+2+9}{1+4+1} = \frac{18}{6} = 3, \quad x_2 = \frac{\langle v, u_2 \rangle}{\langle u_2, u_2 \rangle} = -1$$

$$x_3 = \frac{\langle v, u_3 \rangle}{\langle u_3, u_3 \rangle} = 2$$

**Thm:-** Let $\{u_1, u_2, \dots, u_n\}$ be an orthogonal basis of $V$. Then, for any $v \in V$,

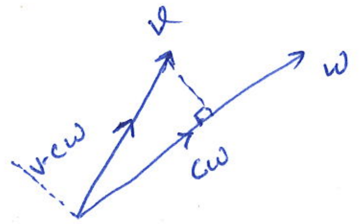
$$v = \frac{\langle v, u_1\rangle}{\langle u_1, u_1\rangle} u_1 + \frac{\langle v, u_2\rangle}{\langle u_2, u_2\rangle} u_2 + \dots + \frac{\langle v, u_n\rangle}{\langle u_n, u_n\rangle} u_n .$$

**Remark:-** The scalar $k_i = \dfrac{\langle v, u_i\rangle}{\langle u_i, u_i\rangle}$ is called the Fourier co-efficient of $v$ w.r.t $u_i$.

## PROJECTIONS:-

Let $V$ be an I.P.S. Suppose $w \neq 0 \in V$ and $v$ be another vector.

We find proj of $v$ along $w$.




$= cw$ s.t

$v - cw \perp w$

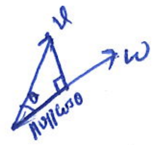
i.e $\langle v - cw, w\rangle = 0$

$$c = \frac{\langle v, w\rangle}{\langle w, w\rangle}.$$

Thus $\quad \text{Proj}_w v = cw = \dfrac{\langle v, w\rangle}{\langle w, w\rangle} . w$

Such, a scalar is unique.

OR




$\|u^*\| = \|v\| \cos\theta$

$$= \|v\| . \frac{v.w}{\|v\|\,\|w\|}$$

$$u^* = \|u^*\| . \frac{w}{\|w\|} = \frac{v.w}{\cancel{\|v\|}} \frac{\cancel{\|v\|}\, w}{\|w\|\,\|w\|}$$

$$= \frac{v.w}{\|w\|^2} . w$$

$$= \frac{\langle v, w\rangle}{\|w\|^2} . w$$

This can be generalized as the following Thm,

<u>Thm</u>: Suppose $w_1, w_2, \dots, w_n$ form an orthogonal set of non-zero vectors in $V$.

Let $v$ be any vector in $V$.

Define $v' = v - \underbrace{(c_1 w_1 + c_2 w_2 + \dots + c_n w_n)}$

where $c_1 = \dfrac{\langle v, w_1 \rangle}{\langle w_1, w_1 \rangle}$, $c_2 = \dfrac{\langle v, w_2 \rangle}{\langle w_2, w_2 \rangle}$, $\dots$, $c_n = \dfrac{\langle v, w_n \rangle}{\langle w_n, w_n \rangle}$

Then $v'$ is orthogonal to $w_1, w_2, \dots, w_n$.

<u>Remark</u>: The notion of the projection of a vector $v \in V$ along a subspace $W$ of $V$ is defined as follows, $(\text{proj}(v, W))$.,

if $W = \text{span}(w_1, w_2, \dots, w_n)$, $w_i$ form an orthogonal set.

then $\text{proj}(V, W) = c_1 w_1 + c_2 w_2 + \dots + c_n w_n$,

where $c_i = \dfrac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle}$ (This is called the component of $v$ along $w_i$ as above).

# GRAM-SCHMIDT ORTHOGONALIZATION PROCESS:-

Suppose $\{v_1, v_2, ---, v_n\}$ is a basis of an inner product space V. One can use this basis to construct an orthogonal basis $\{w_1, w_2, ---, w_n\}$ of V as follows:-

$$w_1 = v_1$$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1$$

$$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$$

$$\vdots$$

$$w_n = v_n - \frac{\langle v_n, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_n, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 - --- - \frac{\langle v_n, w_{n-1} \rangle}{\langle w_{n-1}, w_{n-1} \rangle} w_{n-1}$$

One can see that each "$w_k$" is orthogonal to the preceeding w's.

Thus, one can see that, $w_1, w_2, ---, w_n$ form an orthogonal basis for V.

Normalizing each $w_i$ (ie $\frac{w_i}{\|w_i\|}$) will form an orthonormal basis for V.

**Remark :**

**Ex 1 :→** Apply the Gram-Schmidt Orthogonalization Process to find an orthogonal basis & ~~then an~~ an orthonormal basis for the subspace $U$ of $\mathbb{R}^4$ spanned by

$$v_1 = (1,1,1,1) \quad , \quad v_2 = (1,2,4,5) \quad , \quad v_3 = (1,-3,-4,-2).$$

**Sol$^n$ :**

$$w_1 = v_1 = (1,1,1,1)$$

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1 = (1,2,4,5) - \frac{(1+2+4+5)}{1+1+1+1}\cdot(1,1,1,1)$$

$$= (1,2,4,5) - 3(1,1,1,1) = (-2,-1,1,2)$$

$$w_3 = v_3 - \frac{\langle v_3, w_1\rangle w_1}{\langle w_1, w_1\rangle} - \frac{\langle v_3, w_2\rangle w_2}{\langle w_2, w_2\rangle}$$

$$= v_3 - \left(\frac{-8}{4}\right) w_1 - \left(\frac{-7}{10}\right) w_2 = \left(\frac{8}{5}, \frac{-17}{10}, \frac{-13}{10}, \frac{7}{5}\right)$$

Clearly fraction, we obtain, $w_3 = (-6, -17, -13, 14)$

$$\|w_1\| = 2 \quad , \quad \|w_2\| = \sqrt{10} \quad , \quad \|w_3\| = \sqrt{910}.$$

Ex :-

In $\mathbb{R}^4$, let

$U = \text{span}\{[1\ 1\ 0\ 0]^t, [1\ 1\ 1\ 2]^t)$

Find $u \in U$ s.t $\| u - [1\ 2\ 3\ 4]^t \|$ is as small as possible.

Sol$^n$:- Find orthogonal or Orthonormal vector of U.

$u_1 = [\frac{1}{\sqrt{2}}\ \frac{1}{\sqrt{2}}\ 0\ 0]^t$, $\quad u_2 = [0\ 0\ \frac{1}{\sqrt{5}}\ \frac{2}{\sqrt{5}}]^t$

{Using Gram Smalimidth).

The closest point $u \in U$ to $\underset{v}{[1\ 2\ 3\ 4]}$ is

$$P_U([1\ 2\ 3\ 4]^t) = c_1 u_1 + c_2 u_2$$

$$c_i = \frac{\langle v, u_i \rangle}{\|u_i\|^2}$$

$$= \left[\frac{3}{2}\ \frac{3}{2}\ \frac{11}{5}\ \frac{22}{5}\right]^t$$