Elementary Row operation (Elementary operations)

The following row operation are called elementary row operation (elementary operation) :-

(i) Interchange of two rows (equations) R_i & R_j

Notation – $R_i \leftrightarrow R_j$

(ii) Multiply a row (equation) by a non zero constant "c"

Notation – $R_i \longrightarrow c R_i$

(iii) Add a multiple of a Row (equation) R_j to another row (equation) R_i.

Notation :- $R_i \longrightarrow R_i + c R_j$.

Equivalent linear system :- Two linear systems are said to be equivalent if one can be obtained from the others by a finite no. of elementary operation.

Thm :- Two equivalent systems have the same set of solutions.

Def^n :- (Row Equivalent Matrices) Two matrices are said to be row equivalent if one can be obtained from the other by a finite no. of elementary row operations.

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Example :- ① The three matrices given below are row equivalent-

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix}.$$

$R_3 \to R_3 - 2R_{\cancel{1}}$

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ 0 & -1 & -4 & -1 \\ 1 & 1 & 1 & 3 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix}$$ is not row equivalent to $$\begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix}$$

Ex 2:- Apply row elementary operation to reduce the following matrix in upper triangular form.

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 4 \end{bmatrix}$$

Using
$R_3 \to R_3 - R_1$
$R_4 \to R_4 - R_1$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

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Row Echelon Matrix :-

A matrix A is called a row echelon matrix if the following two conditions hold.

(i) All zero rows, if any, are at the bottom of the matrix.

(ii) Each "leading" non zero entry in a row is to the right of the leading non zero entry in the preceding row.

(* A leading non zero element of a row of A is the first non zero element in the row)

Example :- The following matrices are in row echelon form.

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 & 4 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

The following are Not in Row Echelon form :-

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 & -1 & 5 \\ 1 & 0 & 5 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Remark: (1) If a matrix "A" is in row echelon form then in each column of "A" containing a leading entry, the entries below that leading entry are zero.

(2) Using the elementary row operation, every matrix can be reduce in row echelon form.

(3) Row echelon form of a matrix is not unique.

Ex:- Transform the following matrix to row echelon form:

1) $$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 1 & 0 & 2 & 1 \end{bmatrix}$$

Solⁿ: $R_3 \to R_1$

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_1$

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 \end{bmatrix}$$

(2) $$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

Solⁿ: $R_3 \to R_2$

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(3) $$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 2 & 1 & 3 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

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Gauss Elimination Method :-

Defⁿ :- Gauss Elimination Method is a method of solving a linear system $Ax = b$ (Consisting of m equations in n unknown) by bringing the augmented matrix

$$[A \vdots b] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

to an upper triangular form

$$\left[\begin{array}{cccc|c} c_{11} & c_{12} & \cdots & c_{1n} & d_1 \\ 0 & c_{22} & \cdots & c_{2n} & d_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & c_{mn} & d_m \end{array}\right].$$

This elimination process is also called the forward elimination procedure.

<u>Remark</u> :- To solve a linear system, $Ax = b$, one need to apply only the "<u>Elementary Row Operation</u>" to the augmented Matrix $[A|b]$

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Gauss Elimination Method :

STEPS TO SOLVE A SYSTEM OF EQUATIONS $AX=b$, with n variables and m equations

① Write the augmented Matrix $[A|b]$

② Use elementary row operation to reduce $[A|b]$ to row echelon form.

③ By back subtitution methods solve (find) the value of unknown variable.

Note that : ① The variable corresponding to leading elements in the first "n" column of Row echelon form of matrix are called "leading variables".

② The variables which are not corresponding to leading are called "free variable."

③ The system $Ax=b$ can either have a unique solution, infinitely many solutions or no solution.

4) If the system $AX=b$ has some solution then it is called a "consistent system."
Otherwise, it is called an "Inconsistent system."

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Ex. Find all solutions, if any, of the system

$$x_1 + 2x_2 + x_3 + 3x_4 = 4$$
$$2x_1 + x_2 + 3x_3 + 2x_4 = 1$$
$$2x_2 + x_3 + x_4 = 3$$
$$3x_1 + x_2 + 3x_3 + 4x_4 = 2$$

over $\mathbb{Z}_5$.

Sol^n: We have,

$$[A : b] = \left[\begin{array}{cccc:c} 1 & 2 & 1 & 3 & 4 \\ 2 & 1 & 3 & 2 & 1 \\ 0 & 2 & 1 & 1 & 3 \\ 3 & 1 & 3 & 4 & 2 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$, $R_4 \to R_4 - 3R_1$

$$\left[\begin{array}{cccc:c} 1 & 2 & 1 & 3 & 4 \\ 0 & -3 & 1 & -4 & -7 \\ 0 & 2 & 1 & 1 & 3 \\ 0 & -5 & 0 & 5 & -10 \end{array}\right] \Big/ \mathbb{Z}_5 = \left[\begin{array}{cccc:c} 1 & 2 & 1 & 3 & 4 \\ 0 & 2 & 1 & 1 & 3 \\ 0 & 2 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\left[\begin{array}{cccc:c} 1 & 2 & 1 & 3 & 4 \\ 0 & 2 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$R_2 \to 3R_2$

$$\left[\begin{array}{cccc:c} 1 & 2 & 1 & 3 & 4 \\ 0 & 1 & 3 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0. \end{array}\right] \Rightarrow \begin{array}{l} x_1 + 2x_2 + x_3 + 3x_4 = 4 \\ x_2 + 3x_3 + 3x_4 = 4 \end{array}$$

Here x_1, x_2 are leading variable & x_3, x_4 are free variable.

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Here, $\text{rank}(A) = \text{rank}(A \mathbin{\vdots} b) = 2$.

Setting $x_3 = s$, $x_4 = t$ with $s, t \in \mathbb{Z}_5$;

all sol^n are given by

$$x_2 = 4 - 3x_3 - 3x_4 = 4 + 2s + 2t.$$

$$\begin{aligned} x_1 &= 4 - 2x_2 - x_3 - 3x_4 \\ &= 4 + 3x_2 + 4x_3 + 2x_4 \\ &= 4 + 3(4 + 2s + 2t) + 4s + 2t \\ &= 4 + 12 + 6s + 6t + 4s + 2t \\ &= 16 + 10s + 8t \\ &= 1 + 3t \end{aligned}$$

Since $\mathbb{Z}_5$ is a finite set.

$\therefore$ It has a finite no. of sol^n.

i.e.

$$\left.\begin{aligned} x_1 &= 1 + 3t \\ x_2 &= 4 + 2s + 2t \\ x_3 &= s \\ x_4 &= t \end{aligned}\right\} \underline{\underline{\text{Ans}}}$$

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Def^n: (Rank of a matrix)

The rank of a matrix A is the number of non-zero rows in its row echelon form.

It is denoted by "rank(A)" or "$\rho(A)$."

Remark:- The total number of "free variables" in the consistent system $Ax=b$ of n variables is equal to "n − rank A".

Result: Let $Ax=b$ be a system of equations with n variables. Then

1. If rank(A) $\neq$ rank([A|b]). Then the system $Ax=b$ is inconsistent. i.e "The system $Ax=b$ has no solution".

2) If rank(A) = rank([A|b]) $= n$. Then the system $Ax=b$ has a unique sol^n. i.e System is consistent

3) If rank(A) = rank([A|b]) $< n$. Then the system $Ax=b$ is consistent. i.e The system $Ax=b$ has infinitely many solution.

Result: Let $Ax=0$ be a Homogeneous system of equations with n variables.

If (1) rank(A) $= n$. Then the system has only zero solution.

(2) rank(A) $< n$. Then the system has infinitely many solution.

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Prob:- Consider the system

$$x+2y+z=3$$
$$ay+5z=10$$
$$2x+7y+az=b.$$

(a) Find those values of "a" for which the system has a unique so

(b) Find those pairs of value (a,b) for which the system has more than one solution.

Sol^n: Consider the Augmented Matrix, and reduce it in Row echelon-form.

$$[A\,|\,b]=\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & a & 5 & 10 \\ 2 & 7 & a & b \end{array}\right]$$

$R_3 \to R_3 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & a & 5 & 10 \\ 0 & 3 & a-2 & b-6 \end{array}\right]$$

$R_3 \to aR_3 - 3R_2.$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & a & 5 & 10 \\ 0 & 0 & a^2-2a-15 & ab-6a-30 \end{array}\right]$$

Thus,

$$(a^2-2a-15)\,z = ab-6a-30$$

(a) The system has a unique sol^n iff $\rho(A)=\rho([A|b])=3$

i.e $a^2-2a-15 \neq 0. \Rightarrow \boxed{a \neq 5 \text{ and } a \neq -3}$

$\&\ b \in \mathbb{R}.$

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(b) The system has more than one solution if and only if $P(A) = P(A|b) < 3$

(i.e) (ii) both sides are zero,

i.e $a^2 - 2a - 15 = 0$, $\boxed{ab - 6a - 30 = 0}$ — (2)

i.e either $a = -3$ or $a = 5$

Thus if $a = 3$, we obtain from (2), $-3b = 6a + 30$

$-3b = 12$

$b = -4$

if $a = 5$, then $5b = 60 \Rightarrow \boxed{b = 12}$

Thus $(3, -4)$ & $(5, 12)$ are the pairs for which the system has more than one solution.

(ii) If $a = 0$, $b \in \mathbb{R}$.

Then $[A|b] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 0 & 5 & 10 \\ 0 & 0 & -15 & -30 \end{array}\right]$

$R_3 \to 3R_2 + R_3 \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 0 & 5 & 10 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$P(A) = P([A|b]) = 2.$

Thus on $\{(0,b) : b \in \mathbb{R}\}$, system has more than two solution.

In this case Sol^n are $\to 5z = 10 \Rightarrow z = 2$, $y = k$, where k is any scalar

$x = z - 2y = 2(1-k)$.

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