Multivariable Calculus

(Lecture-14)

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Triple Integration

of

(Scalar Valued Function of Vector Variable)

(Scalar Field)

$$F : R \subseteq \mathbb{R}^3 \to \mathbb{R}$$

Learning Outcome of this lecture

In this lecture, we learn triple integral over closed and bounded region $\mathcal{R}$ in $\mathbb{R}^3$.

- Triple integral of $f : \mathcal{R} \subset \mathbb{R}^3 \to \mathbb{R}$ where $\mathcal{R}$ is a closed and bounded region in $\mathbb{R}^3$.
- Fubini's Theorem for triple integrals
- Change of Variables: Spherical coordinates, Cylindrical coordinates

Partition of a rectangular cube in $\mathbb{R}^3$

Let $\mathcal{R}$ be a parallelpiped(rectangular cube) in $\mathbb{R}^3$ such that

$$\begin{aligned}\mathcal{R} &= [a,b]\times[c,d]\times[p,q]\\ &= \{(x,y,z)\in\mathbb{R}^3 \,:\, a\leq x\leq b,\ c\leq y\leq d,\, p\leq z\leq q\}.\end{aligned}$$

Consider a partition P_x of $[a,b]$ given by

$$P_x:\ a=x_0<x_1<x_2<\cdots<x_{m-1}<x_m=b.$$

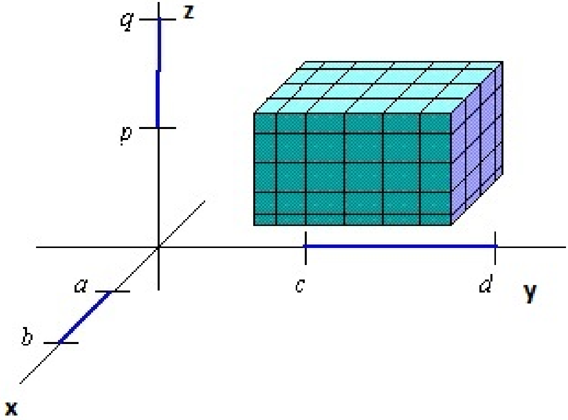
Consider a partition P_y of $[c,d]$ given by

$$P_y:\ c=y_0<y_1<y_2<\cdots<y_{n-1}<y_n=d.$$

Consider a partition P_z of $[p,q]$ given by

$$P_z:\ p=z_0<z_1<z_2<\cdots<z_{s-1}<z_s=q.$$

Picture: Partition of a rectangular cube



Norm of the partition of a rectangular cube

Then $\mathcal{P} = (P_x, P_y, P_z)$ partitions $\mathcal{R}$ into *mns* subrectanglular cubes as follows. Set for $1 \leq i \leq m$, $1 \leq j \leq n$ and $1 \leq k \leq s$,

$$R_{ijk} = \{(x, y, z) \in \mathcal{R} \ : \ x_{i-1} \leq x \leq x_i,\ y_{j-1} \leq y \leq y_j,\ z_{k-1} \leq z \leq z_k\}.$$

$$\mathcal{P} = \{R_{ijk} \ : \ 1 \leq i \leq m \ 1 \leq j \leq n \text{ and } 1 \leq k \leq s\}.$$

Then P is a partition of the rectangular cube $\mathcal{R}$.

- Volume of the subrectangular cube R_{ijk} is

$$|R_{ijk}| = (x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1}).$$

- norm of the partition $\mathcal{P}$ is

$$\|\mathcal{P}\| = \max\{|R_{ijk}| \ : \ 1 \leq i \leq m,\ 1 \leq j \leq n,\ 1 \leq k \leq s\}.$$

Reimann triple integral

Let $\mathcal{R}$ be rectangular cube in $\mathbb{R}^3$ and $f : \mathcal{R} \to \mathbb{R}$ be bounded. Let

$$\|\mathcal{P}\| = \max\{|R_{ijk}| \;:\; 1 \leq i \leq m,\; 1 \leq j \leq n,\; 1 \leq k \leq s\}$$

be a partition of $\mathcal{R}$. Let $(x_i, y_j, z_k) \in R_{ijk}$. Then

Riemann sum:

$$S(\mathcal{P},f) = \sum_{k=1}^{s}\sum_{j=1}^{n}\sum_{i=1}^{m} f(x_i, y_j, z_k)|R_{ijk}|$$

Riemann triple integral:

$$\iiint_{\mathcal{R}} f(x,y,z)dV = \iiint_{\mathcal{R}} f = \lim_{\|\mathcal{P}\|\to 0} S(\mathcal{P},f).$$

If $\iiint_{\mathcal{R}}$ is finite then f is called Riemann integrable or integrable over R.

Fubini's theorem for triple integrals

Theorem

Let $f : \mathcal{R} \subset \mathbb{R}^3 \to \mathbb{R}$ be continuous. Then

$$\iiint_{\mathcal{R}} f(x,y,z)\, dV = \int_{x=a}^{b} \int_{y=c}^{d} \int_{z=p}^{q} f(x,y,z) dz\, dy\, dx.$$

$\iiint_{\mathcal{R}} f$ is independent of the order of the integration between x, y and z.

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Triple Integral of f over closed and bounded region(non rectangular cube) $\mathcal{D}$

Let $\mathcal{D}$ be a closed and bounded region in $\mathbb{R}^3$.
Let f be a bounded, real valued function on $\mathcal{D}$.
Take a rectangular cube $\mathcal{R}$ such that $\mathcal{D} \subset \mathcal{R}$.
Define a function $\tilde{f} : \mathcal{R} \to \mathbb{R}$ by

$$\tilde{f}(x, y, z) = \begin{cases} f(x, y, z) & \text{if } (x, y, z) \in \mathcal{D}, \\ 0 & \text{if } (x, y, z) \in \mathcal{R} \setminus \mathcal{D} \end{cases}$$

If the triple integral of $\tilde{f}$ over the rectangular cube $\mathcal{R}$ exists then the triple integral of f over the region $\mathcal{D}$ is defined by

$$\iiint_{\mathcal{D}} f(x, y, z) dV = \iiint_{\mathcal{R}} \tilde{f}(x, y, z) dV.$$

Example 1: Evaluating the triple Integrals

Find the volume of the region $\mathcal{D}$ enclosed by the surface $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

The volume is

$$V = \iiint_{\mathcal{D}} dz\, dy\, dx,$$

the integral of $F(x, y, z) = 1$ over $\mathcal{D}$.

How to evaluate this integral?

How to find the limits of integration?

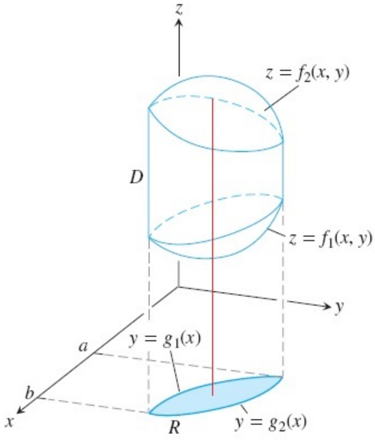
Limits of integration

$$I = \iiint_{\mathcal{D}} f(x,y,z)dz\,dy\,dx$$

- Draw a sketch of the region $\mathcal{D}$ and its shadow (vertical projection) $\mathcal{R}$ of $\mathcal{D}$ in the xy-plane.
- Draw a line parallel to z-axis that passes through the point (x,y) of $\mathcal{R}$.
- z-limit: Identify the lower surface $z = f_1(x,y)$ and upper surface $z = f_2(x,y)$ through which this line passes at most once.
- This gives the following integration

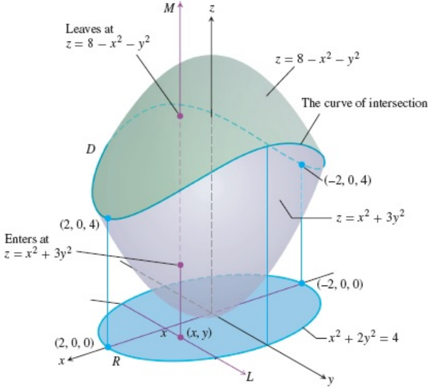
$$\iiint_{\mathcal{D}} f(x,y,z)dz\,dy\,dx = \iint_{\mathcal{R}} \left(\int_{z=f_1(x,y)}^{z=f_2(x,y)} f(x,y,z)\,dz \right) dy\,dx.$$

This can be solved by using idea of double integration over region $\mathcal{R}$ in xy-plane.


z
$z = f_2(x, y)$
D
$z = f_1(x, y)$
y
$y = g_1(x)$
a
b
x
R
$y = g_2(x)$

Example-1

Find the volume of the region $\mathcal{D}$ enclosed by the surface $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

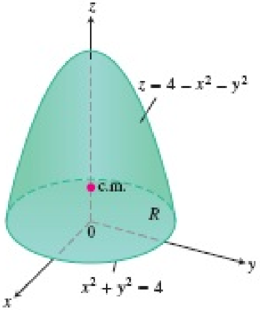


$V = \iiint_{\mathcal{D}} f(x, y, z) dz\, dy\, dx = \iint_{\mathcal{R}} \left(\int_{z=x^2+3y^2}^{z=8-x^2-y^2} dz \right) dy\, dx$, where $\mathcal{R}$ is an ellipse in the xy-plane with the equation $x^2 + 2y^2 = 4$.

Example-2

Find the volume of the region $\mathcal{D}$ bounded below by the disk $\mathcal{R}:\ x^2+y^2\leq 4$ in the plane $z=0$ and above by the paraboloid $z=4-x^2-y^2$.

$$V=\iiint_{\mathcal{D}} dV=\iint_{\mathcal{R}}\left(\int_{z=0}^{z=4-x^2-y^2} dz\right) dxdy.$$



Change of Variables

in

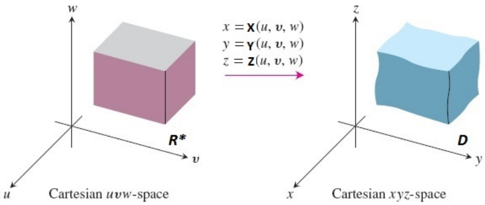
Triple Integrals

Transforming Triple Integrals from One System to Another System

Change of Variables $(x, y, z) \to (u, v, w)$

Suppose that a region $\mathcal{R}^*$ in the *uvw*-space transformed one-to-one into the region $\mathcal{D}$ in the *xyz*-space by equations

$$x = X(u, v, w), \quad y = Y(u, v, w) \quad \text{and} \quad z = Z(u, v, w).$$



Then $f(x, y, z)$ defined on $\mathcal{D}$ can be thought of as a function $f(X(u, v, w), Y(u, v, w), Z(u, v, w))$ on $\mathcal{R}^*$

Question: How is the integral of $f(x, y, z)$ over $\mathcal{D}$ related to the integral of $f(X(u, v, w), Y(u, v, w), Z(u, v, w))$ over $\mathcal{R}^*$?

Continuation of previous slide

Question: How is the integral of $f(x, y, z)$ over $\mathcal{D}$ related to the integral of $f(X(u, v, w), Y(u, v, w), Z(u, v, w))$ over $\mathcal{R}^*$?

Answer: If X, Y, Z, and f have continuous partial derivative and the "Jacobian" $J(u, v, w)$ is not zero for all $(u, v, w) \in \mathcal{R}^*$, where

$$J(u, v, w) = \frac{\partial(X, Y, Z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial X}{\partial u} & \frac{\partial X}{\partial v} & \frac{\partial X}{\partial w} \\ \frac{\partial Y}{\partial u} & \frac{\partial Y}{\partial v} & \frac{\partial Y}{\partial w} \\ \frac{\partial Z}{\partial u} & \frac{\partial Z}{\partial v} & \frac{\partial Z}{\partial w} \end{vmatrix}.$$

Then the formula for transforming triple integrals over the region $\mathcal{D}$ into triple integrals over the region $\mathcal{R}^*$ can be written as

$$\iiint_{\mathcal{D}} f(x, y, z)\, dV = \iiint_{\mathcal{R}^*} f(X(u, v, w), Y(u, v, w), Z(u, v, w)).|J(u, v, w)|\, dV,$$

Triple integral in cylindrical coordinates

$$xyz - \text{Cartesian coordinates} \;\to\; r\theta z - \text{Cylindrical coordinates}$$

In cylindrical coordinates,

$$x = X(r,\theta,z) = r\cos\theta \quad y = Y(r,\theta,z) = r\sin\theta \quad z = Z(r,\theta,z) = z$$

$$J = \frac{\partial(X,Y,Z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial X}{\partial r} & \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial z} \\ \frac{\partial Y}{\partial r} & \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial z} \\ \frac{\partial Z}{\partial r} & \frac{\partial Z}{\partial \theta} & \frac{\partial Z}{\partial z} \end{vmatrix} = r.$$

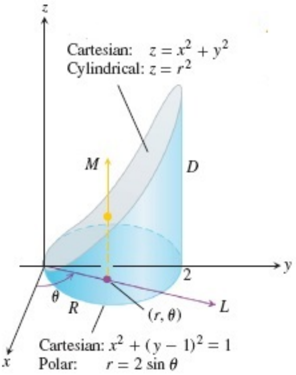
Then

$$\iint_{\mathcal{R}} f(x,y,z)\,dxdydz = \iint_{\mathcal{R}^*} f(r\cos\theta, r\sin\theta, z))dz\,r\,dr\,d\theta.$$

Example-3

Find the volume of the region $\mathcal{D}$ bounded below by the plane $z = 0$, laterally by the circular cylinder $x^2 + (y - 1)^2 = 1$, and bounded above by the paraboloid $z = x^2 + y^2$.

$$\iiint_{\mathcal{D}} dzdxdy = \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=2\sin\theta} \int_{z=0}^{z=r^2} dz\, r\, dr\, d\theta.$$



Note:Thomas Calculus Book (12 Edition): Refer Example 1 of Section 15.7 on Page Nos. 876 – 877.

Example-4

Find the volume of the region $\mathcal{D}$ enclosed by the cylinder $x^2 + y^2 = 4$, bounded above by the paraboloid $z = x^2 + y^2$ and below by the xy-plane.

$$\iiint_{\mathcal{D}} dzdxdy = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \int_{z=0}^{z=r^2} dz\, r\, dr\, d\theta.$$