Solutions – Quiz Test

Problem-1 Find the orthogonal trajectory of $x^2 = ce^y$, where c is an arbitrary constant.

Solution: Given family of curves is $x^2 = ce^y$ ——— (1)

Diff. (1) w.r.t. 'x', we get $2x = ce^y \cdot \frac{dy}{dx}$

$$\Rightarrow \quad c = \frac{2x}{e^y} \cdot \frac{1}{\frac{dy}{dx}}$$

$\therefore$ from (1), we get $x^2 = \frac{2x}{e^y} \cdot \frac{1}{\frac{dy}{dx}} \cdot e^y$

$$\Rightarrow \quad x^2 = \frac{2x}{\frac{dy}{dx}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2}{x}$$

Replacing $\frac{dy}{dx}$ by $\frac{-1}{\frac{dy}{dx}}$, we get

$$\frac{-1}{\frac{dy}{dx}} = \frac{2}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-x}{2}$$

$$\Rightarrow \quad \boxed{y = \frac{-x^2}{4} + C}$$

which is the family of orthogonal trajectories to the given family of curves.

Problem-2 : Solve the differential equation

$$(xy^2 - \cos x)\,dx - (\sin y - x^2 y)\,dy = 0 \quad \text{——} (1)$$

Solⁿ: Here $M = xy^2 - \cos x$ and $N = -\sin y + x^2 y$

$$\Rightarrow \frac{\partial M}{\partial y} = 2xy \qquad \frac{\partial N}{\partial x} = 2xy$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\Rightarrow$ (1) is an exact DE.

The solution of (1) is given by

$$\int_{y=\text{Constant}} M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = C$$

$$\Rightarrow \int_{y=\text{Constant}} (xy^2 - \cos x)\,dx + \int -\sin y\,dy = C$$

$$\Rightarrow \boxed{\frac{x^2 y^2}{2} - \sin x + \cos y = C}$$

Ans.

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Problem-3: Suppose $y = f(x)$ is a solution of the DE $\frac{dy}{dx} = y(a-by)$, where a and b are positive constants. Find the intervals on y-axis for which the function $y = f(x)$ is strictly decreasing, without solving the DE.

Solution: Using the given DE, we get that $y = f(x)$ is strictly decreasing if

$$\frac{dy}{dx} < 0$$

$$\Rightarrow \quad y(a-by) < 0$$

Two cases arise:

Case-I: $y < 0$ and $a - by > 0$

$$\Rightarrow \quad y < 0 \text{ and } y < \frac{a}{b}$$

$$\Rightarrow \quad y < 0$$

Case-II: $y > 0$ and $a - by < 0$

$$\Rightarrow \quad y > 0 \text{ and } y > \frac{a}{b}$$

$$\Rightarrow \quad y > \frac{a}{b}$$

(as a and b are positive constants.)

Thus the solution $y = f(x)$ is strictly decreasing on the intervals

$\boxed{y < 0 \quad \text{or} \quad y > \frac{a}{b}}$ on the y-axis.

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Problem-4 : Discuss the existence and uniqueness of the following IVP in $R: |x| \le 2, |y| \le 1$.

$$\frac{dy}{dx} = 2y^{2/3} + x, \quad y(0) = 0. \quad \text{——— (1)}$$

Solution: Here $f(x,y) = 2y^{2/3} + x$, which is continuous in R.

Also, $|f(x,y)| = |2y^{2/3} + x| \le 2|y|^{2/3} + |x| \le 2(1) + (2) = 4 = M$.

$\Rightarrow$ By Picard's Existence theorem, $\exists$ solution of (1) in $|x| \le h$,

where $h = \min\left(2, \frac{1}{4}\right) = \frac{1}{4}$

$\Rightarrow$ $\boxed{|x| \le \frac{1}{4}}$ (Interval of existence of solution).

But $f(x,y)$ doesnot satisfy Lipschitz condition in R as for $y_1 > 0$ and $y_2 = 0$, we have

$$\frac{|f(x,y_1) - f(x,0)|}{|y_1 - 0|} = \frac{|2y_1^{2/3} + x - \cancel{x}|}{|y_1|} = \frac{2}{y_1^{1/3}}$$

which is unbounded in the neighbourhood of origin.

$\Rightarrow$ $f(x,y)$ doesnot satisfy Lipschitz condition in R.

$\Rightarrow$ uniqueness of the solution is not guaranteed.

Problem-5: Let $y=\phi(x)$ and $y=\psi(x)$ be the solutions of

$$y'' - 2xy' + (\sin x^2)y = 0$$

such that $\phi(0)=1$, $\phi'(0)=1$ and $\psi(0)=1$, $\psi'(0)=2$.

Then find the value of Wronskian $W(\phi,\psi)$ at $x=1$.

Solution: As $W(\phi,\psi)(x) = c\,e^{-\int \frac{a_1(x)}{a_0(x)}dx} = c\,e^{-\int -2x\,dx} = c\,e^{x^2}$.

$$\Rightarrow \quad W(\phi,\psi)(0) = ce^0 = c$$

$$\Rightarrow \quad \begin{vmatrix} \phi(0) & \psi(0) \\ \phi'(0) & \psi'(0) \end{vmatrix} = c \qquad \left[\because W(y_1,y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}\right]$$

$$\Rightarrow \quad \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = c$$

$$\Rightarrow \quad c = 1$$

Thus $W(\phi,\psi)(x) = e^{x^2}$

$$\Rightarrow \quad \boxed{W(\phi,\psi)(1) = e^1 = e} \quad \text{Ans.}$$

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