Lecture-8th (ODE)

$$x^2+y^2=a^2$$
$$\downarrow$$
$$y=cx$$

$$F(x,y,c)=0$$

$$G(x,y,c)=0$$

Suppose F and G are identical, then F and G are self orthogonal.

Example: $y^2=4a(x+a)$ is self orthogonal

Existence and Uniqueness of Solution of an IVP:

$$\frac{dy}{dx}=\frac{2y}{x}, \quad y(2)=4 \qquad \text{——— (1)}$$

$$\frac{dy}{2y}=\frac{dx}{x}$$

$$\Rightarrow \log y = 2\log x + \log c$$

$$\Rightarrow y = cx^2 \Rightarrow$$

$$y=cx^2,$$

$y(2)=4 \Rightarrow 4=c\cdot 4 \Rightarrow c=1.$

$y=x^2$ is the solⁿ of (1)

$\Rightarrow$ unique solⁿ.

$$\frac{dy}{dx}=\frac{2y}{x}, \quad y(0)=4$$

$y = cx^2, \quad y(0) = 4$

$\Rightarrow 4 = c \cdot 0 = 0$ (Not possible)

$\Rightarrow$ No solⁿ

$\frac{dy}{dx} = \frac{2y}{x}, \quad y(0) = 0$

$y = cx^2$

$\underline{0 = c \cdot 0}$

$\Rightarrow$ infinitely many solⁿs.

An IVP can have unique solⁿ, infinitely many solⁿ or no solⁿ.

When an IVP will have a solⁿ and when it will be unique?

The answer to such problems are given by Picard's Existence Theorem and Picard's Existence & Uniqueness Theorem.

Bounded function: $f(x,y)$ is said to be bounded in the region R of the xy plane if

positive real constant

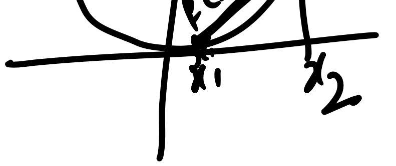
$$|f(x,y)| \leq M \quad \forall (x,y) \in R.$$

Lipschitz Condition: A function $f(x,y)$ is said to satisfy Lipschitz condition w.r.t. y in the region R of xy plane if $\exists$ a constant $K > 0$ such that

$$|f(x,y_1) - f(x,y_2)| \leq K|y_1 - y_2|$$

$$\Rightarrow \frac{|f(x,y_1) - f(x,y_2)|}{|y_1 - y_2|} \leq K$$

$K \rightarrow$ Lipschitz Constant



For functions of one variable,

$$|f(x_1) - f(x_2)| \leq K|x_1 - x_2|$$

Example, $f(x) = x^2$, $x \in [-1, 4]$

Let $x_1, x_2 \in [-1, 4]$ $\Rightarrow |x_1| \le 4$ & $|x_2| \le 4$

$$|f(x_1) - f(x_2)| = |x_1^2 - x_2^2|$$

$$= |x_1 - x_2| \, |x_1 + x_2|$$

$$\le (|x_1| + |x_2|) \, |x_1 - x_2|$$

$$\le (4+4) \, |x_1 - x_2|$$

$$\le 8 |x_1 - x_2|$$

$$\Rightarrow |f(x_1) - f(x_2)| \le 8 |x_1 - x_2|$$

$K = 8$

$\Rightarrow$ $f(x) = x^2$ is L.C. in $[-1, 4]$

$f(x) = x^2$, $x \in \mathbb{R}$ is not L.C.

$f(x) = \sin x$ is Lipschitz continuous on $\mathbb{R}$.

Is Lipschitz continuous function continuous?

$$|f(x_1) - f(x_2)| \le K \, |x_1 - x_2|$$

$$\le K\delta \quad \text{when } |x_1 - x_2| < \delta$$

$$= \varepsilon$$

$\Rightarrow |f(x_1) - f(x_2)| < \varepsilon$ whenever $|x_1 - x_2| < \delta$.

$\Rightarrow$ Lipschitz continuity $\Rightarrow$ continuity

but continuity $\not\Rightarrow$ Lipschitz continuity.

Sufficient Condition to check Lipschitz continuity

If $\frac{\partial f}{\partial y}$ exists and is bounded $\forall (x, y) \in R$, where R is the domain of f, then $f(x, y)$ is Lipschitz continuous w.r.t. y in R and Lipschitz constant is

$$K = \sup_{(x,y)\in R} \left|\frac{\partial f}{\partial y}\right|$$

$$\left[\begin{aligned} &|f(x, y_1) - f(x, y_2)| \leq K|y_1 - y_2| \\ \Rightarrow\ &\frac{|f(x, y_1) - f(x, y_2)|}{|y_1 - y_2|} \leq K \\ &\text{As } y_1 \to y_2, \\ &\left|\frac{\partial f}{\partial y}\right| \leq K \end{aligned}\right]$$

Example: $f(x,y) = 1+y^2$, $R: |x| \le 1, |y| \le 2$

$$\left|\frac{\partial f}{\partial y}\right| = |2y| \le 2(2) = 4 = K$$

$\Rightarrow$ f is Lipschitz continuous w.r.t. y in R.

$f(x,y) = x^2|y|$, $R: |x| \le 1, |y| \le 2$

$$|f(x,y_1) - f(x,y_2)| = \left| x^2|y_1| - x^2|y_2| \right|$$

$\Big[\, \big||y_1| - |y_2|\big| \le |y_1 - y_2| \,\Big]$

$$\le |x|^2 \, |y_1 - y_2|$$

$$\le (1) \, |y_1 - y_2|$$

$$\Rightarrow |f(x,y_1) - f(x,y_2)| \le (1)|y_1 - y_2|$$

$K = 1$

$\Rightarrow$ f satisfies Lipschitz condition w.r.t. y in R.

[But $\frac{\partial f}{\partial y}$ does not exist in R.]

Picard's Existence Theorem:

Let R be a rectangle and (x_0, y_0) be an interior point of R.

(i) Let $f(x,y)$ be continuous $\forall$ points (x,y) in R.

$$R: |x-x_0| \le a, \ |y-y_0| \le b$$

(ii) $f(x,y)$ is bounded in R.

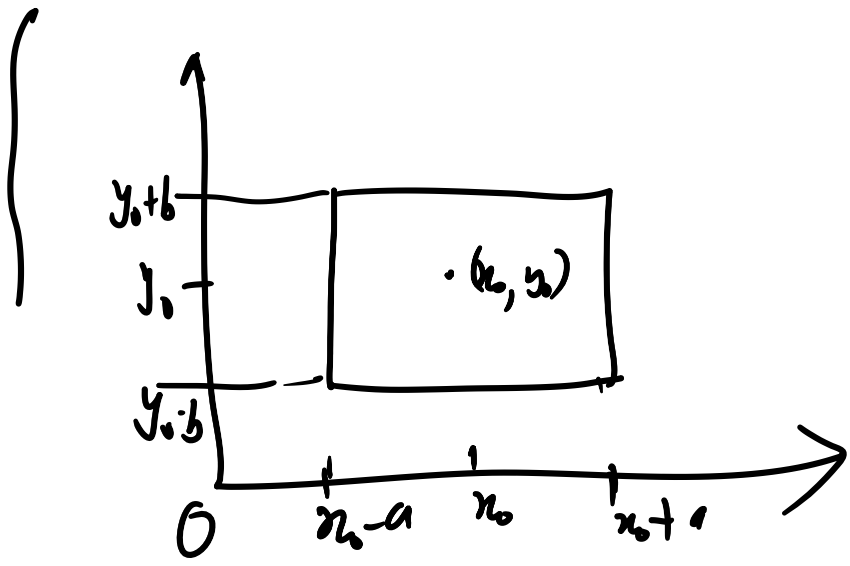
$$|f(x,y)| \le M$$

Then $\exists$ atleast one sol^n of IVP

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$

in the region ~~$|x|$~~ $|x-x_0| \le h$,

where $h = \min\left(a, \frac{b}{M}\right)$



Example: find the interval of existence of IVP

$$\frac{dy}{dx} = \underbrace{2x^2 + 3y^2}_{f(x,y)}, \quad R: \; |x| \le 1, \; |y-1| \le 1,$$

a, b

$-1 \le y-1 \le 1$

$$y(\underset{x_0}{0}) = \underset{y_0}{1}$$

Sol^n

$$f(x,y) = 2x^2 + 3y^2, \quad x_0 = 0, \; y_0 = 1$$

$$a = 1, \; b = 1$$

(i) $f(x,y)$ is continuous in R. as it is a poly. function

(ii)
$$|f(x,y)| = |2x^2 + 3y^2|$$
$$\le 2|x|^2 + 3|y|^2$$
$$\le 2(1)^2 + 3(2)^2$$
$$\le 2 + 12 = 14$$

$$\Rightarrow |f(x,y)| \le 14 = M.$$

Thus $\exists$ atleast one sol^n of the given IVP in the region

$$|x - x_0| \le h, \text{ where } h = \min\left(a, \frac{b}{M}\right)$$

$$\Rightarrow |x| \le h, \quad h = \min\left(1, \frac{1}{14}\right)$$

$$= \frac{1}{14}$$

$$\Rightarrow \boxed{|x| \leq \frac{1}{\pi\varphi}}\,.$$