Linear Independence :

Definition:- (Linear Independence and Dependence)

Let $S = \{v_1, v_2, \dots, v_m\}$ be a non-empty set of vectors in a vector space V.

If there exist some non-zero α_i's, $1 \le i \le m$ such that

$$\boxed{\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_m v_m = 0}$$

Then the set 'S' is called a "Linearly Dependent set".

Otherwise, the set S is called "linearly independent".

Example :- ① The most basic linearly independent set in $\mathbb{R}^n$ is the set of standard unit vectors

$e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, 0, \dots)$, ..., $e_n = (0, 0, \dots, \underset{n\text{th place}}{1})$

Sol^n: $\alpha_1 e_1 + \alpha_2 e_2 + \dots + \alpha_n e_n = \underset{\hookrightarrow \text{ vector in } \mathbb{R}^n}{0}$

$\Rightarrow (\alpha_1, \alpha_2, \dots, \alpha_n) = (0, 0, \dots, 0)$

$\Rightarrow \alpha_1 = 0, \alpha_2 = 0, \dots, \alpha_n = 0$

Thus, $e_1, e_2, \dots, e_n$ are linearly independent.

② The set $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is linearly independent.

Sol^n:- $\alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \alpha_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \implies \alpha_i = 0 \;\; \forall \; 1 \le i \le 4.$

3) In. $P_n(t)$,

The set $\{1, t, t^2, \ldots, t^n\}$ is a linearly independent set.

$\underline{\underline{Sol^n}}$:- $\alpha_0 \cdot 1 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 + \ldots + \alpha_n t^n = 0$ → zero polynomial.

i.e $\alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 + \ldots + \alpha_n t^n = 0.1 + 0.t + \ldots + 0.t^n$

comparing the like powers of t, we obtain.

$\alpha_0 = 0, \alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_n = 0$

Thus the set $\{1, t, t^2, \ldots, t^n\}$ is linearly independent.

4) Determine whether the vectors $v_1 = (1, -2, 3)$, $v_2 = (5, 6, -1)$, $v_3 = (3, 2, 1)$ are linearly indepenent or linearly dependent in $\mathbb{R}^3$.

$\underline{\underline{Sol^n}}$:- Consider $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$

$\alpha_1 (1, -2, 3) + \alpha_2 (5, 6, -1) + \alpha_3 (3, 2, 1) = 0$

i.e
$$\alpha_1 + 5\alpha_2 + 3\alpha_3 = 0$$
$$-2\alpha_1 + 6\alpha_2 + 2\alpha_3 = 0$$
$$3\alpha_1 - \alpha_2 + \alpha_3 = 0$$

Thus, we need to check that whether the given system have trivial sol^n or not.

One can easily check that the system has non-trivial sol^n.

i.e $\alpha_1 = -\frac{t}{2}$, $\alpha_2 = -t/2$, $\alpha_3 = t$

This means that the vectors $\{v_1, v_2, v_3\}$ are linearly dependent.

Determine whether the vectors

$v_1 = (1, 2, 2, -1)$, $v_2 = (4, 9, 9, -4)$, $v_3 = (5, 8, 9, -5)$ are linearly dependent or linearly independent in $\mathbb{R}^4$.

$\underline{\text{Sol}^n}$: Consider

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$$

$$\Rightarrow \quad \alpha_1 + 4\alpha_2 + 5\alpha_3 = 0$$
$$2\alpha_1 + 9\alpha_2 + 8\alpha_3 = 0$$
$$2\alpha_1 + 9\alpha_2 + 9\alpha_3 = 0$$
$$-\alpha_1 - 4\alpha_2 - 5\alpha_3 = 0$$

$$\begin{bmatrix} 1 & 4 & 5 \\ 2 & 9 & 8 \\ 2 & 9 & 9 \\ -1 & -4 & -5 \end{bmatrix}$$

$$\begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 + R_1 \end{array} \sim \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_3 \to R_3 - R_2 \quad \sim \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, we obtain $\alpha_1 = 0 = \alpha_2 = \alpha_3$

Hence $\{v_1, v_2, v_3\}$ are linearly independent.

Ex :- Determine whether the polynomials

$p_1 = 1-x$, $p_2 = 5+3x-2x^2$, $p_3 = 1+3x-x^2$

are linearly independent or linearly dependent in $\mathbb{P}_2$.

Sol^n :- consider

$$\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 = 0$$

$$\text{i.e. } \alpha_1 p_1(x) + \alpha_2 p_2(x) + \alpha_3 p_3(x) = 0(x) \quad \forall x.$$

$$\alpha_1(1-x) + \alpha_2(5+3x-2x^2) + \alpha_3(1+3x-x^2) = 0+0x+0x^2$$

$$\Rightarrow \quad \begin{aligned} \alpha_1 + 5\alpha_2 + \alpha_3 &= 0 \\ -\alpha_1 + 3\alpha_2 + 3\alpha_3 &= 0 \\ -2\alpha_2 - \alpha_3 &= 0. \end{aligned}$$

We note that linearly independente or linearly dependence hinges on whether the above system has a trivial sol^n or non-trivial sol^n.

We can easily see that the system has non-trivial sol^n.
($\because$ det of cofficient matrix is zero).

Thus, $\{p_1, p_2, p_3\}$ is linearly dependent.

Results :-

1) Let $S = \{u_1, u_2, --, u_m\} \subseteq \mathbb{R}^n$ and consider the $n \times m$ matrix $A = [u_1 \quad u_2 \quad --- \quad u_m]$.

Then S is Linearly independent iff the system $AX = 0$ has trivial solution.

2) A set S with two or more vector is

(a) linearly dependent iff at least one of the vectors in S is expressible as a linear combination of the other vectors in S.

(b) Linearly independent iff no vector in S is expressible as a linear combination of the other vectors in S.

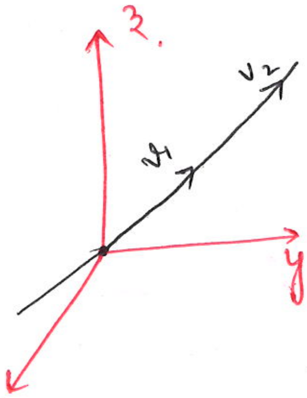
3) A set that contains "0" is linearly dependent.

4) A set with exactly one vector is linearly independent iff that vector is not 0.

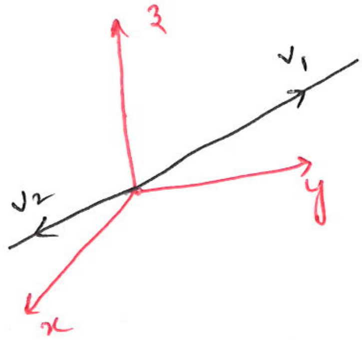
5) A set with exactly two vectors is l.i. iff neither vector is a scalar multiple of other.

6) Let $S = \{v_1, v_2, --, v_r\}$ be a set of vectors in $\mathbb{R}^n$.
If $r > n$, the S is l.d.

Geometric Interpretation of Linear Independence / Dependence



(linearly dependent)



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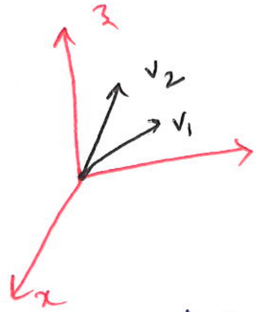
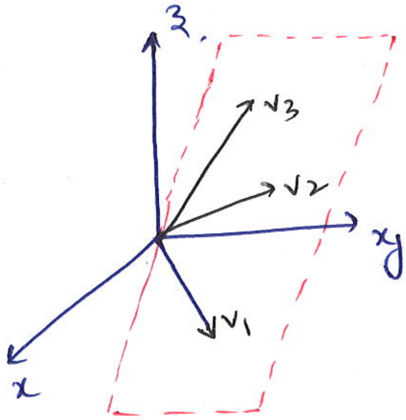


Linearly independent.

\# Three vectors in $\mathbb{R}^3$ are l.i iff they don't lie in the same plane when they have their initial pts at origin

Otherwise, at least one would be a linear combination of other two.



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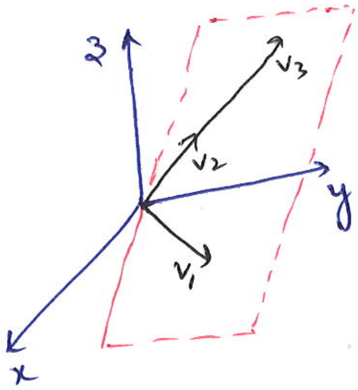

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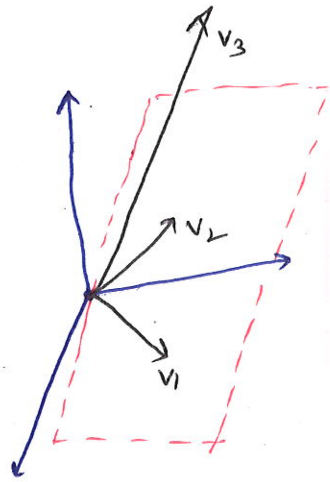

(L.I)

BASIS OF A VECTOR SPACE :-

Definition :- If V is any vector space and $S = \{v_1, v_2, \ldots, v_n\}$ is a finite set of vectors in V.

Then S is called a basis for V if the following two conditions hold :-

(i) S is linearly independent.

(ii) S span V.

Dimension :- $\boxed{\text{No. of vectors in a basis for } V}$ $\boxed{\text{denoted by } \dim(V)}$

Example :- (1) The standard Basis for $\mathbb{R}^n$:-

The standard unit vectors

$e_1 = (1,0,\ldots,0)$, $e_2 = (0,1,\ldots,0)$, $\ldots$, $e_n = (0,0,\ldots,1)$

span $\mathbb{R}^n$ and are linearly independent.

$$\Big(\text{As, } \quad v = (x_1, x_2, \ldots, x_n) = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n \quad (\text{span})$$

$$\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n = \bar{0}$$

$$\Rightarrow (\alpha_1, \alpha_2, \ldots, \alpha_n) = (0,0,\ldots,0)$$

$$\Rightarrow \alpha_1 = 0 = \alpha_2 = \cdots = \alpha_n$$

Thus $e_1, e_2, \ldots e_n$ are l.i $\Big)$

Hence $\{e_1, e_2, \ldots, e_n\}$ form a basis of $\mathbb{R}^n$. (called standard basis of $\mathbb{R}^n$)

$\boxed{\dim(\mathbb{R}^n) = n.}$

2) Show that $\{(1,1), (1,0)\}$ form a basis of $\mathbb{R}^2$.

Sol^n: Check Linearly independent :-

$$\alpha_1(1,1) + \alpha_2(1,0) = 0$$
$$\Rightarrow \alpha_1 + \alpha_2 = 0, \quad \alpha_2 = 0$$
$$\Rightarrow \alpha_1 = 0, \quad \alpha_2 = 0$$

Spaning - $(a,b) = \alpha(1,1) + \beta(1,0)$

$$\Rightarrow b = \alpha, \quad \beta = a - b$$

i.e $(a,b) = (a-b)(1,0) + b(1,1)$

Thus $\{(1,1), (1,0)\}$ form a basis of $\mathbb{R}^2$.

Show that the vectors $V_1 = (1,2,1)$, $V_2 = (2,9,0)$ & $V_3 = (3,3,4)$ form a basis of $\mathbb{R}^3$.

Sol^n: We must show that the vectors V_1, V_2, V_3 are linearly independent and span $\mathbb{R}^3$.

(i) To prove linear independence, we must show that

$$\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 = 0.$$

has only the trivial sol^n i.e $\alpha_1 = 0 = \alpha_2 = \alpha_3$.

$$\alpha_1 (1,2,1) + \alpha_2 (2,9,0) + \alpha_3 (3,3,4) = (0,0,0)$$

$$\Rightarrow \quad \begin{aligned} \alpha_1 + 2\alpha_2 + 3\alpha_1 &= 0 \\ 2\alpha_1 + 9\alpha_2 + 3\alpha_3 &= 0 \\ \alpha_1 + 4\alpha_3 &= 0. \end{aligned} \quad , \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix}$$

Solve the above system, we obtain $\alpha_1 = \alpha_2 = \alpha_3 = 0$ ($\because \det A \neq 0$).

(ii) $\{V_1, V_2, V_3\}$ span $\mathbb{R}^3$.

i.e let $(a,b,c) \in \mathbb{R}^3$ then $(a,b,c) = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3$

i.e we need to solve, $(a,b,c) = \alpha_1 (1,2,1) + \alpha_2 (2,9,0) + \alpha_3 (3,3,4)$

Solve the system of eq^n

$$\begin{aligned} \alpha_1 + 2\alpha_2 + 3\alpha_3 &= a \\ 2\alpha_1 + 9\alpha_2 + 3\alpha_3 &= b \\ \alpha_1 + 4\alpha_3 &= c \end{aligned}$$

One can check that the above system have a unique sol^n.

Thus, the vectors $\{V_1, V_2, V_3\}$ form a basis for $\mathbb{R}^3$.

$\dim(\mathbb{R}^3) = 3$.

Ex:- The Standard Basis for P_n.

Show that $S = \{1, x, x^2, \ldots, x^n\}$ is a basis for the vector space P_n of polynomials of degree less than or equal to "n". Find the dimension of P_n.

Sol^n: Clearly S span P_n as $p(x) \in P_n$ can be written as

$p(x) = a_0 + a_1 x + \cdots + a_n x^n$: Then

$a_0 + a_1 x + \cdots + a_n x^n = a_0 \cdot 1 + a_1 x + \cdots + a_n x^n$.

Then S span P_n.

(ii) S is l.i. For this, consider

$a_0 + a_1 x + \cdots + a_n x^n = 0$ = zero polynomial

Comparing the like powers of x, we obtain

$a_0 = 0 = a_1 = \cdots = a_n$.

dim P_n = No. of elements (vectors) in S = $n+1$.

Ex:- Show that the matrices

$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

form a basis for the vector space M_{22} of 2×2 matrices

Sol^n:- Clearly $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = aM_1 + bM_2 + cM_3 + dM_4$

i.e $\{M_1, M_2, M_3, M_4\}$ span M_{22}.

For l.i, consider $\alpha_1 M_1 + \alpha_2 M_2 + \alpha_3 M_3 + \alpha_4 M_4 = 0$

$$\Rightarrow \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \alpha_i = 0 \quad \forall\ 1 \le i \le 4.$$

Thus, $\{M_1, M_2, M_3, M_4\}$ is l.i. & $\dim(\{M_1, M_2, M_3, M_4\}) = 4$.