$$\boxed{\begin{array}{l} \beta = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| , \quad R = \frac{1}{\beta} , \quad \underline{0<\beta<\infty} \\ ① \ \sum a_n x^n \text{ conv } \ |x|<R \\ ② \ \sum a_n x^n \text{ div } \ |x|>R. \end{array}}$$

$\beta = 0$, ⇒ series conv everywhere

$\beta = \infty$ ⇒ series nowhere conv.

Taylor's Series

<u>Taylor's Theorem</u>: $f : I \to \mathbb{R}$ s.t $f, f', \dots, f^{(n)}$ are conti- on I and. f^{n+1} is exist in nbd of a point $x=a\in I$

Then for any $x \in I$ $\exists$ $c \in (a, x)$ or (x, a)

s.t

$$f(x) = P_n(x) + R_n(x)$$

$$= \underline{\sum_{k=0}^{n} \frac{f^k(a)}{k!}(x-a)^k} + \underline{\frac{f^{n+1}(c)}{n+1!}(x-a)^{n+1}}$$

$$= f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \dots + \frac{f^n(a)}{n!}(x-a)^n + \frac{f^{n+1}(c)}{n+1!}(x-a)^{n+1}$$

$R_n \to$ Error term or Remainder term.

<u>Ex</u>:- $f(x) = e^x$. $\quad f(x) = P_n(x) + R_n(x)$

$$f(x) = f(a) + f^{(1)}(a)(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$c \in (a, x)$ or (x, a) $\quad + \frac{f^{(n+1)}(c)}{n+1!}(x-a)^{n+1}$

$a=0.$

$$f(x)=f(0)+f'(0)(x)+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{n}(0)}{n!}x^n+\frac{f^{n+1}(c)}{n+1!}x^{n+1}$$

$$\boxed{\begin{array}{l} f^{n}(x)=e^{x} \quad \forall n\in\mathbb{N} \\ f^{n}(0)=1 \quad \forall n\in\mathbb{N} \end{array}}$$

$$=\underline{1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}}+\underline{\frac{e^c}{n+1!}x^{n+1}}$$

$c\in(0,x)$
$(x,0)$

Ex:- $f(x)=\sin x,\quad a=0.$

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\frac{f^{n+1}(c)}{n+1!}x^{n+1}$$

$f'(x)=\cos x=\sin(\frac{\pi}{2}+x)$

$f''(x)=-\sin x=\sin(2\cdot\frac{\pi}{2}+x)$

$f'''(x)=-\cos x=\sin(3\cdot\frac{\pi}{2}+x)$

$$f^{n}(x)=\sin(\frac{n\pi}{2}+x),\quad \forall n\in\mathbb{N}.$$

$$f(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}\cdots+\frac{x^n}{n!}\sin(\frac{n\pi}{2})+\frac{x^{n+1}}{n+1!}\sin(\frac{(n+1)\pi}{2}+c)$$

Ex:- $f(x)=\cos x,\quad a=0$

$$f(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots+\frac{x^{n+1}}{n+1!}\cos(\frac{(n+1)\pi}{2}+c)$$

$c\in(0,x)$ or $(x,0)$.

Problem:- Find the order n of Taylor poly P_n about $x=\underline{0}$ to app. e^x in $(-1,1)$ so that the error is not more than 0.005.

$e^x \qquad P_n(x)=1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}$

$$R_n(x)=\frac{x^{n+1}}{n+1!}e^c \qquad I=(-1,1)$$

$c\in(0,1)$ or $(-1,0)$.

$$\boxed{|R_n(x)|\le\frac{|x|^{n+1}e^c}{n+1!}\le\frac{e^c}{n+1!}\le\frac{e}{n+1!}}$$

$\frac{e}{n+1!} \leq 0.005 \Rightarrow \boxed{n \geq 5}$.

Prob 2 :- Find the interval of validity when we app. $\cos x$ with 2nd order Taylor's poly with error tolerance 10^{-4}.

Soln :- $P_2(x) = 1 - \frac{x^2}{2!}$, $R_2(x) = \frac{x^3}{3!} \sin c$.

$$|R_2(x)| = \left|\frac{x^3}{3!} \sin c\right| \leq \left|\frac{x^3}{3!}\right| \leq 10^{-4}$$

$$\Rightarrow |x| < 0.084.$$

Taylor's series :-

If f is infinitely diff at a. and if $R_n(x) \to 0$ as $n \to \infty$. then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

$\hookrightarrow$ taylor series of f about a.

$$R_n(x) = \frac{f^{n+1}(c)}{n+1!}(x-a)^{n+1}$$ ✓

if $C = C(x) > 0$ s.t $\boxed{|f^{n+1}(x)| < C(x)}$ then

$R_n \to 0$ as $n \to \infty$ if $\boxed{\lim_{n\to\infty} \frac{|x-a|^{n+1}}{n+1!} = 0}$ ✓

$\underbrace{}_{a_n}$ by Ratio test.

Ex :- $e^x = P_n(x) + R_n(x)$.

$$= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{n+1!} e^c.$$

$$R_n(x) = \frac{x^{n+1}}{n+1!} e^c$$

$$\lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{n+1!} e^c\right| = e^c \cdot \underbrace{\lim_{n\to\infty} \left|\frac{x^{n+1}}{n+1!}\right|}_{=0}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \to \text{Taylor's series of } e^x$$

Ex:- $f(x) = \sin x$, $R_n(x) = \frac{x^{n+1}}{n+1!} \sin\left(c + \frac{(n+1)\pi}{2}\right)$

$$\lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{n+1!} \sin\left(c + \frac{(n+1)\pi}{2}\right) \right|$$

$$\leq \lim_{n\to\infty} \left| \frac{x^{n+1}}{n+1!} \right| = 0$$

$$\Rightarrow \lim_{n\to\infty} R_n(x) = 0$$

$\because \left| \sin\left(c + \frac{(n+1)\pi}{2}\right) \right| \leq 1$

Remark:— If $a=0$, Taylor formula is call as Maclaurin's formula and Taylor series is call " series.

Ex:- find the taylor's series $f(x) = \tan^{-1} x$ and find the domain of conv.

Soln:- $\tan^{-1} x = \int \frac{dx}{1+x^2} = \int (1 - x^2 + x^4 - \dots) dx$

$(1+x^2)^{-1} = 1 - x^2 + x^4 - \dots$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{2n-1} \rightarrow \text{Taylor series.}$$

$R = 1$, $(-1, 1)$

$1 - \frac{1}{3} + \frac{1}{5} - \dots = \tan^{-1}(1) = \frac{\pi}{4}$

$x = -1$, $= -\frac{\pi}{4}$

$[-1, 1]$ domain of conv.