Multivariable Calculus

(Lecture-3)

Department of Mathematics
Bennett University
India

23^{rd} October, 2018

Learning Outcome of the Lecture

We learn

- Limits of Functions $F : S \subseteq \mathbb{R}^2 \to \mathbb{R}$
- Algebra of Limits
- How to show the limit exists?
- How to show the limit does not exist?
- Limit and Iterated Limits
- Continuity of Functions $F : S \subseteq \mathbb{R}^2 \to \mathbb{R}$
- Properties of continuous functions

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- Vector field: Vector Potential Function
  $V(x, y) = (x^2 + 2, e^x + 3y)$
  for $|x| \leq 1, |y| \leq 1.$

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- If $m = 1$, then the component function of F and the function F are same.

Example

Let $F : \mathbb{R}^4 \to \mathbb{R}^5$ be the function defined by

$$F(x_1, x_2, x_3, x_4) = (x_1 + x_2, x_1^2, x_3 e^{x_1}, 5x_2, x_4)$$

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The functions $f_i (1 \leq i \leq 5)$ are called the component functions or simply components of F.

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In other words, we say that $f(x)$ approaches b as x approaches a or $f(x)$ has the limit b as x tends to a.

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Moral

Concept of the limit of functions in multivariable calculus

is essentially same as that of

Concept of the limit of functions in single variable calculus.

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- That is, Along each path on which X approaches A, the function $F(X)$ is always approaching the same point/value L.

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Example

Let $f(x,y) = x\sin(1/y) + y\sin(1/x)$ if $xy \neq 0$, and $f(x,y) = 0$ if $xy = 0$. Examine the existence of $\lim_{(x,y)\to(0,0)} f(x,y)$.

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$$|f(x,y) - 0| \leq |x| + |y| \quad \text{for } (x,y) \in N(0,0) \setminus (0,0).$$

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$$\implies |f(x, y) - 0| \leq |x| + |y| \leq 2\sqrt{x^2 + y^2} < 2 \times \frac{\epsilon}{2} = \epsilon.$$

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  Consider the path $\gamma_1 : \; y = x, \; x \neq 0$ and $x \to 0$.
  As $(x,y) \to (0,0)$ along γ_1 , $f(x,y) = \frac{x^2}{x^2+x^2} = \frac{1}{2} \to \frac{1}{2}$.

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  Since f approaches two different values as $(x,y) \to (0,0)$ along two different paths, we conclude that $\lim_{(x,y)\to(0,0)} f(x,y)$ does NOT exist.

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Algebra of Limits

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Let $F : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ and $G : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ be two functions. Let X_0 be a limit point of S.
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- $\lim_{X \to X_0} |F(X)| = |\lim_{X \to X_0} F(X)|$, provided the limit in the R.H.S. exists.
- If $\lim_{X \to X_0} G(X) \neq 0$, then

$$\lim_{X \to X_0} \left(\frac{F(X)}{G(X)} \right) = \frac{\lim_{X \to X_0} F(X)}{\lim_{X \to X_0} G(X)}.$$