Department of Mathematics, Bennett University
Engineering Calculus (EMAT101L)
Solutions for Tutorial Sheet 7

---

1. (a) Consider the partition $P_\epsilon = \{0, 1-\epsilon, 1\}$. The upper and lower sum with this partition is $U(P_\epsilon, f) = 1(1-\epsilon) + 2\epsilon = 1+\epsilon$ and $L(P_\epsilon, f) = 1(1-\epsilon) + 1\epsilon = 1$. Therefore, $\lim_{\epsilon \to 0}(U(P_\epsilon, f) - L(P_\epsilon, f)) = 0$. Hence f is integrable.

   2nd Method: f is continuous on $[0,1]$ except $x = 1$. Therefore f is Riemann integrable.

   (b) Consider the sequence of partitions $\{P_n\}$, where $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n}{n}\}$. Then $\|P_n\| \to 0$ as $n \to \infty$ and $U(P_n, f) = \sum_{i=1}^{n}(1+\frac{i}{n})\frac{1}{n} = \frac{1}{n}(n + \frac{n+1}{2})$, $L(P_n, f) = \sum_{i=1}^{n}(1-\frac{i}{n})\frac{1}{n} = \frac{1}{n}(n - \frac{n-1}{2})$. So $\lim_{n\to\infty}(U(P_n, f) - L(P_n, f)) = 1$. Hence f is not integrable.

   (c) Let $\epsilon > 0$. Choose N such that $\frac{1}{N} < \epsilon$ and $\frac{1}{N} < \frac{\pi}{4}$. Note that f has only finite discontinuities in $[\frac{1}{N}, 1]$. Hence integrable in $[\frac{1}{N}, 1]$. Now for any partition of $[0, \frac{1}{N}]$, $U(P, f) - L(P, f) \leq \sum(M_i - m_i)\Delta x_i \leq \sum \Delta x_i = \frac{1}{N} < \epsilon$. Thus f is integrable on $[0, \frac{1}{N}]$ also.

   2nd Method: Note that f has finitely many discontinuities in $[\frac{\pi}{4}, 1]$. Hence integrable in $[\frac{\pi}{4}, 1]$. Now for other half, $\xi_1, \xi_1, \xi_2, \cdots$ be the infinitely many discontinuities of f in $[0, \frac{\pi}{4}]$. Then $\xi_i = \frac{1}{N+i}$ for some N. Consider the partitions $P_\epsilon = \left\{\frac{\pi}{4}, \xi_1 - \frac{\epsilon}{2^{N+1}}, \xi_1 + \frac{\epsilon}{2^{N+1}}, \cdots, \xi_r - \frac{\epsilon}{2^{N+r}}, \xi_r + \frac{\epsilon}{2^{N+r}}, \cdots, 0\right\}$. Then

$$U(P_\epsilon, f) - L(P_\epsilon, f) = \sum_{i=1}^{\infty}(M_i - m_i)\triangle x_i \leq \sum_{i=1}^{\infty} M_i \triangle x_i \leq \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} \leq \epsilon C,$$

   for some $C > 0$. Hence f is integrable.

   3rd Method: f is continuous except $x = \frac{1}{n}$ such that $n \in \mathbb{N}$. Thus set of points of discontinuity has a limit point 0. Therefore f is Riemann integrable.

   (d) Note that f is discontinuous at $x = 1, 2, 3, 4, 5$, which are finite in number. Therefore f is Riemann integrable.

   Or, f is continuous except $x = 1, 2, 3, 4, 5$. Therefore f is Riemann integrable.

2. (a) Hint 1: f is bounded and monotonic increasing on $[0,1]$. Hence f is integrable.

   Hint 2: f is continuous on $[0,1]$, except at the set of points $0, \frac{1}{2}, \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \cdots$, which have only one limit point, say 0. Hence f is integrable.

   (b) Let $F(x) = f(x) - g(x) = 0$ except at a finite number of points of $[a,b]$ so that $F(x)$ has a finite number of points of discontinuity on $[a,b]$. Thus F is integrable. Hence $F + g = f$ is integrable.

(c) Let $h = f - g$. Then h is also continuous. Then by mean value theorem, there exists $\xi \in [a, b]$ such that $\int_a^b h(x)dx = h(\xi)(b - a)$. But as $\int_a^b f(x)dx = \int_a^b g(x)dx$. So $\int_a^b h(x)dx = 0$. Hence $h(\xi) = 0 \Rightarrow f(\xi) = g(\xi)$.

3. (a) Take $f(x) = \frac{1}{x+1}$ then f is integrable on $[0, 1]$ and $\int_0^1 f(x)dx = \ln 2$. Take $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n}{n}\}$. Then $S(P_n, f) = \sum_{i=1}^{n} f(\frac{i}{n})\frac{1}{n} = \sum_{i=1}^{n} \frac{1}{i+n}$. Hence $\lim_{n\to\infty} S(P_n, f) = \ln 2$.

   (b) Take $f(x) = \sin \pi x$. Then f is continuous on $[0, 1]$ and hence integrable on $[0, 1]$ and $\int_0^1 f(x)dx = \frac{2}{\pi}$. Take $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n}{n}\}$. Then $S(P_n, f) = \sum_{i=1}^{n} f(\frac{i}{n})\frac{1}{n} = \sum_{i=1}^{n} \sin(\frac{i\pi}{n})\frac{1}{n}$. Hence $\lim_{n\to\infty} S(P_n, f) = \lim_{n\to\infty} \sum_{i=1}^{n} \sin(\frac{i\pi}{n})\frac{1}{n} = \frac{2}{\pi}$.

4. The function $f(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x}, & 0 < x < 1 \\ 0, & x = 0 \end{cases}$ is not continuous on $[0, 1]$, but it is bounded and continuous on $]0, 1]$. Therefore f is Riemann integrable.

   The function $g(x) = \begin{cases} x^2\sin\frac{1}{x}, & 0 < x < 1 \\ 0, & x = 0 \end{cases}$ is differentiable on $[0, 1]$ and satisfies $g'(x) = f(x)$ for all $x \in [0, 1]$. Thus $\int_0^1 \left(2x\sin\frac{1}{x} - \cos\frac{1}{x}\right) dx = g(1) - g(0) = \sin 1$.

5. (a) $f'(x) = 2x\cos\left(\frac{\pi}{x^2}\right) + \frac{2\pi}{x}\sin\left(\frac{\pi}{x^2}\right)$ if $0 < x \le 1$ and $f'(0) = 0$, which is not bounded. Therefore $\int_0^1 f'(x)dx$ is not integrable. Hence the given equation fails to hold.

   (b) $f'(x) = \frac{1}{(x-1)^2}$, which is not bounded. Therefore $\int_0^2 f'(x)dx$ is not integrable.

   (c) $f'(x) = \frac{1}{\sqrt{x}}$, which is not bounded. Therefore $\int_0^1 f'(x)dx$ is not integrable.