Ordinary Differential Equations(EMAT102L)
(Lecture-5)

Department of Mathematics
Bennett University, India

Outline of the Lecture

We will learn

- Exact Differential Equation(cont.)
- How to convert a non-exact DE to an exact DE?
- Integrating Factors
- Examples

Examples-Exact Differential Equation

Example-1

Solve the DE by method of inspection

$$y + x\frac{dy}{dx} = 0$$

Solution:

$$d(xy) = ydx + xdy = 0.$$

$\Rightarrow xy = c$ is the solution of the given DE.

Example-2

Solve the DE by method of inspection

$$(2x + y^2)dx + 2xydy = 0$$

Solution:

$$(2x + y^2)dx + 2xydy = 0$$

$$\Rightarrow 2xdx + (y^2dx + 2xydy) = 0$$

$$\Rightarrow d(x^2) + d(xy^2) = 0$$

$$\Rightarrow x^2 + xy^2 = c.$$

Example

Solve the equation
$$(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0$$

Solution:
To check whether the equation is exact or not:
Comparing with $Mdx + Ndy = 0$, we get

$$M = (3x^2 + 4xy) \ and \ N = (2x^2 + 2y)$$

$$\frac{\partial M}{\partial y} = 4x = \frac{\partial N}{\partial x}$$

So, the given DE is exact.
Solution of exact differential equation: We need to find $F(x, y)$ such that

$$\frac{\partial F}{\partial x} = M(x, y) = (3x^2 + 4xy)$$

$$\frac{\partial F}{\partial y} = N(x, y) = (2x^2 + 2y)$$

Example(cont.)

Step 1. Integrate $\frac{\partial F}{\partial x} = M(x,y)$ with respect to x.

$$F(x,y) = \int M(x,y)dx + \phi(y)$$

$$F(x,y) = \int (3x^2 + 4xy)dx + \phi(y)$$

$$\Rightarrow F(x,y) = x^3 + 2x^2y + \phi(y).$$

Step 2. Find the unknown function $\phi(y)$ using the condition $\frac{\partial F}{\partial y} = N(x,y)$.

$$\frac{\partial F}{\partial y} = 2x^2 + \frac{d\phi(y)}{dy} = 2x^2 + 2y$$

$$\frac{d\phi(y)}{dy} = 2y \Rightarrow \phi(y) = y^2 + c_0$$

where c_0 is an arbitrary constant.

$$So, F(x,y) = x^3 + 2x^2y + y^2 + c_0.$$

Step 3. Hence a one parameter family of solutions is $F(x,y) = c_1$ or

$$x^3 + 2x^2y + y^2 + c_0 = c_1$$

Combining the constant c_1 and c_0, we get

$$x^3 + 2x^2y + y^2 = c$$

where $c = c_1 - c_0$ is an arbitrary constant.

The same differential equation can be solved by the method of grouping also.

Solve the differential equation by the method of grouping

$$(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0$$

Solution: Writing the given equation in the form

$$3x^2dx + (4xydx + 2x^2dy) + 2ydy = 0$$

We can write this as

$$d(x^3) + d(2x^2y) + d(y^2) = d(c)$$

where c is an arbitrary constant.

$$\Rightarrow d(x^3 + 2x^2y + y^2) = d(c)$$

$$x^3 + 2x^2y + y^2 = c$$

is the required solution.

Example

Solve the differential equation

$$(y\cos x + 2xe^y)dx + (\sin x + x^2e^y - 1)dy = 0.$$

Solution: Comparing with $Mdx + Ndy = 0$, we get

$$M = (y\cos x + 2xe^y) \ and \ N = (\sin x + x^2e^y - 1)$$

Check whether the given equation is exact or not:

$$\frac{\partial M}{\partial y} = \cos x + 2xe^y = \frac{\partial N}{\partial x}$$

So, the given DE is exact.
Solution of exact differential equation:
We need to find $F(x, y)$ such that

$$\frac{\partial F}{\partial x} = M(x, y) = y\cos x + 2xe^y$$

$$\frac{\partial F}{\partial y} = N(x, y) = y\sin x + x^2e^y - 1$$

Step 1. Integrate $\dfrac{\partial F}{\partial x} = M(x,y)$ with respect to x.

$$F(x,y) = \int M(x,y)dx + \phi(y)$$

$$F(x,y) = \int (y\cos x + 2xe^y)dx + \phi(y)$$

$$\Rightarrow F(x,y) = y\sin x + x^2e^y + \phi(y).$$

Step 2. Find $\phi(y)$ using the condition $\dfrac{\partial F}{\partial y} = N(x,y)$.

$$\frac{\partial F}{\partial y} = \sin x + x^2e^y + \phi'(y) = \sin x + x^2e^y - 1$$

$$\phi'(y) = -1 \Rightarrow \phi(y) = -y + c_0$$

So,

$$F(x,y) = y\sin x + x^2e^y - y + c_0.$$

Example(cont.)

Step 3. Hence a one parameter family of solutions is $F(x,y) = c_1$ or

$$F(x,y) = c_1$$

$$y\sin x + x^2e^y - y + c_0 = c_1$$

$$y\sin x + x^2e^y - y = c_1 - c_0 = c$$

$$y\sin x + x^2e^y - y = c$$

Solution of an exact differential equation

If the equation

$$M(x,y)dx + N(x,y)dy = 0$$

is exact, then the solution of this exact differential equation is given by

$$\underset{\text{treating y constant}}{\int} M(x,y)dx + \int (\text{ terms of N not containing } x)dy = c$$

where c is an arbitrary constant.

Problem 1.

Solve the differential equation

$$x(1+2y)+(x^2-y)\frac{dy}{dx}=0.$$

Problem 2.

Find the values of l and m such that the equation

$$ly^2+mxy\frac{dy}{dx}=0$$

is exact. Also find its general solution.

Converting a first order non-exact DE to exact DE

Consider the following example:

Example

The first order DE $ydx - xdy = 0$ is clearly not exact.

But observe that if we multiply both sides of this DE by $\frac{1}{y^2}$, the resulting ODE becomes

$$\frac{dx}{y} - \frac{x}{y^2}dy = 0$$

which is exact.

Definition

It is sometimes possible that even though the original first order DE

$$M(x,y)dx + N(x,y)dy = 0$$

is not exact, but we can multiply both sides of this DE by some function (say, $\mu(x,y)$) so that the resulting DE

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

becomes exact. Such a function/factor $\mu(x,y)$ is known as an **integrating factor** for the original DE $M(x,y)dx + N(x,y)dy = 0$.

Example

Consider the differential equation

$$(3y + 4xy^2)dx + (2x + 3x^2y)dy = 0$$

Here

$$M = (3y + 4xy^2) \; and \; N = (2x + 3x^2y)$$

$$\frac{\partial M}{\partial y} = 3 + 8xy \neq 2 + 6xy = \frac{\partial N}{\partial x}$$

So, the given DE is not exact. But if we multiply the given equation by $\mu(x, y) = x^2y$, then the given equation becomes

$$(3x^2y^2 + 4x^3y^3)dx + (2x^3y + 3x^4y^2)dy = 0$$

Now, this equation is exact, Since

$$\frac{\partial(\mu M)}{\partial y} = 6x^2y + 12x^3y^2 = \frac{\partial(\mu N)}{\partial x}$$

Hence $\mu(x, y) = x^2y$ is an **integrating factor** for the given DE.

Integrating Factors

Suppose the equation

$$M(x,y)dx + N(x,y)dy = 0 \tag{1}$$

is not exact and that $\mu(x,y)$ is an **integrating factor** of it.
Then the equation

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0 \tag{2}$$

is exact.
Now, using the criterion for exactness, equation (2) is exact iff

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

Thus

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x$$

That is, $\mu(x,y)$ satisfies the differential equation.

$$(\mu_y M - \mu_x N) + (M_y - N_x)\mu = 0 \tag{3}$$

Hence $\mu(x,y)$ is an integrating factor of given differential equation (1) iff it is a solution of the DE (3).
This is a PDE. So, we are in no position to attempt such an equation.

Let us instead attempt to determine integrating factors of certain special types.
Case I: Suppose μ is a function of x alone. That is, $\mu = \mu(x)$, $\mu_y = 0$. Then, the DE above reduces to

$$\mu_x N = (M_y - N_x)\mu$$

Thus,

$$\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu$$

If further, $\dfrac{M_y - N_x}{N}$ is a function of x, $i.e$, $\dfrac{M_y - N_x}{N} = f(x)(say)$, then the above DE is separable. We try to solve it to find $\mu(x)$.

$$\mu(x) = e^{\int f(x)dx}$$

Case II: Suppose μ is a function of y alone in the DE

$$(\mu_y M - \mu_x N) + (M_y - N_x)\mu = 0$$

That is, $\mu = \mu(y)$, $\mu_x = 0$. Then the differential equation reduces to

$$\frac{d\mu}{dy} = \left(\frac{N_x - M_y}{M}\right)\mu$$

If further, $\dfrac{N_x - M_y}{M}$ is a function of y, $i.e$, $\dfrac{N_x - M_y}{M} = f(y)(say)$, then the above DE is separable. We try to solve it to find $\mu(y)$.

$$\mu(y) = e^{\int f(y)dy}$$

Consider the DE $M(x,y)dx + N(x,y)dy = 0$ (1)

Rule 1

If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$(function of x-alone), then $e^{\int f(x)dx}$ is an integrating factor for the given differential equation.

Example

Solve the differential equation $(2x^2 + y)dx + (x^2y - x)dy = 0$.

Solution: Here $M = (2x^2 + y)$ and $N = (x^2y - x)$.
$\Rightarrow M_y = 1$ and $N_x = 2xy - 1$
So, the given equation is not exact.
We observe that

$$\frac{M_y - N_x}{N} = \frac{1 - 2xy + 1}{(x^2y - x)} = \frac{2 - 2xy}{x(xy - 1)} = \frac{-2}{x} = f(x)(say)$$

which depends upon x only, so integrating factor is

$$I.F. = e^{\int f(x)dx} = e^{\int \frac{-2}{x}dx} = \frac{1}{x^2}$$

Multiplying the given ODE by I.F., we get

$$\left(2 + \frac{y}{x^2}\right) dx + \left(y - \frac{1}{x}\right) dy = 0$$

which is an exact DE.

Solution of ODE:

$$\frac{\partial F}{\partial x} = 2 + \frac{y}{x^2}, \ \frac{\partial F}{\partial y} = y - \frac{1}{x}$$

$$\frac{\partial F}{\partial x} = 2 + \frac{y}{x^2} \Rightarrow F(x,y) = 2x - \frac{y}{x} + \phi(y)$$

To find unknown function $\phi(y)$, use the condition $\dfrac{\partial F}{\partial y} = N(x,y)$,

$$\frac{\partial F}{\partial y} = y - \frac{1}{x} \Rightarrow -\frac{1}{x} + \phi'(y) = y - \frac{1}{x} \Rightarrow \phi(y) = \frac{y^2}{2} + c_0$$

Solution of exact ODE is

$$2x - \frac{y}{x} + \frac{y^2}{2} = c$$

Consider the DE $M(x, y)dx + N(x, y)dy = 0 \quad (1)$

Rule 2

If $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y)$(function of y-alone), then $e^{\int f(y)dy}$ is an integrating factor for (1).

Example

Solve $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$.

Solution: Comparing the given equation with $Mdx + Ndy = 0$, we get that

$$M = (y^4 + 2y) \text{ and } N = (xy^3 + 2y^4 - 4x)$$
$$\therefore \frac{\partial M}{\partial y} = 4y^3 + 2 \text{ and } \frac{\partial N}{\partial x} = y^3 - 4$$

Thus $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$.

$\therefore$ the given equation is not exact.

Here

$$\frac{N_x - M_y}{M} = \frac{y^3 - 4 - 4y^3 - 2}{y^4 + 2y} = \frac{-3(y^3 + 2)}{y(y^3 + 2)} = \frac{-3}{y} = f(y)(say)$$

$\therefore$ the integrating factor is $e^{\int f(y)dy} = e^{\int \frac{-3}{y}dy} = e^{\log y^{-3}} = y^{-3} = \dfrac{1}{y^3}$.

Multiplying the given ODE by I.F., we get

$$\left(y + \frac{2}{y^2}\right) dx + \left(x + 2y - \frac{4x}{y^3}\right) dy = 0 \tag{4}$$

Now for this equation

$$\frac{\partial M}{\partial y} = 1 - \frac{4}{y^3} = \frac{\partial N}{\partial x}$$

The equation (4) is exact. Hence the required solution is

$$\left(y + \frac{2}{y^2}\right) x + y^2 = c$$

where c is an arbitrary constant.

Rules to remember (for finding integrating factors)

Consider the DE $M(x,y)dx + N(x,y)dy = 0 \qquad (1)$

Rule 1

If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$(function of x-alone), then $e^{\int f(x)dx}$ is an integrating factor for (1).

Rule 2

If $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y)$(function of y-alone), then $e^{\int f(y)dy}$ is an integrating factor for (1).

Problem

Solve the differential equation

$$y(2xy + e^x)dx - e^x dy = 0$$

Thank You