Sequence
(Lecture-3)

Engineering Calculus

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Definition

A sequence of real numbers or a sequence in $\mathbb{R}$ is a function $f : \mathbb{N} \to \mathbb{R}$.

- We write a_n for $f(n),\ n \in \mathbb{N}$ and the notation for a sequence is $\{a_n\}_{n=1}^{\infty}$.

Examples

1. Constant sequence: $\{c, c, c, \cdots\}$, where $c \in \mathbb{R}$.
2. Sequence defined by listing: $\{1, 4, 8, 11, 52, \cdots\}$.
3. Sequence defined by rule: $\{a_n\}_{n=1}^{\infty}$, where $a_n = 3n^2$ for all $n \in \mathbb{N}$.
4. $\{\frac{(-1)^{n+1}}{n}\}_{n=1}^{\infty}$
5. $\{\frac{n-1}{n}\}_{n=1}^{\infty}$
6. $\{\sqrt{n}\}_{n=1}^{\infty}$

- What does **convergence** mean?
- Think of the examples: $\{2, 2, 2, \cdots\}$, $\{\frac{1}{n}\}_{n=1}^{\infty}$, $\{n^2 - 1\}_{n=1}^{\infty}$, $\{1, 2, 1, 2, \cdots\}$, $\{(-1)^n \frac{1}{n}\}_{n=1}^{\infty}$, $\{(-1)^n(1 - \frac{1}{n})\}_{n=1}^{\infty}$.

Convergence

Definition

A sequence $\{a_n\}_{n=1}^{\infty}$ **converges** to limit L if for every $\epsilon > 0$ (given) there exists a positive integer N such that $n \geq N \implies |a_n - L| < \epsilon$.

- Notation: $L = \lim\limits_{n\to\infty} a_n$ or $a_n \to L$.
- If $\{a_n\}_{n=1}^{\infty}$ is a sequence and if both $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} a_n = M$ holds, then $L = M$.

Examples

1. Constant sequence $\{c\}_{n=1}^{\infty}, c \in \mathbb{R}$, has c as it's limit.
2. Show that $\lim\limits_{n\to\infty} \frac{1}{n} = 0$.

Solution: Let $\epsilon > 0$ be given. To show that $1/n$ approaches 0, we must show that there exists an integer $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \epsilon.$$

But $1/n < \epsilon \Leftrightarrow n > 1/\epsilon$. Thus, if we choose $N \in \mathbb{N}$ such that $N > 1/\epsilon$, then for all $n \geq N$, $1/n < \epsilon$.

Example

Show that $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$.

Solution: For any $\epsilon > 0$,

$$\left|\frac{(-1)^n}{n} - 0\right| = \frac{1}{n} < \epsilon \; \forall \; n \geq N,$$

where N is a positive integer such that $N > \frac{1}{\epsilon}$. Thus, $\frac{(-1)^n}{n} \to 0$ as $n \to \infty$.

Example

Show that $\lim_{n\to\infty} \frac{n}{n+1} = 1$.

Solution: Note that $|a_n - 1| = \frac{1}{n+1} < \frac{1}{n}$. Thus, for any $\epsilon > 0$, take $N > \frac{1}{\epsilon}$, we get

$$\left|\frac{n}{n+1} - 1\right| = \frac{1}{1+n} < \frac{1}{n} < \epsilon \; \forall \; n \geq N.$$

Hence, $\frac{n}{1+n} \to 1$ as $n \to \infty$.

Theorem

Let $\{a_n\}_1^\infty$ and $\{b_n\}_1^\infty$ be two sequences such that $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = M$. Then

(i) $\lim\limits_{n\to\infty}(a_n + b_n) = L + M$.

(ii) $\lim\limits_{n\to\infty}(a_n - b_n) = L - M$.

(iii) $\lim\limits_{n\to\infty}(ca_n) = cL, \quad c \in \mathbb{R}$.

(iv) $\lim\limits_{n\to\infty}(a_n b_n) = LM$.

(v) $\lim\limits_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{L}{M}$ if $M \neq 0$.

Examples

Find the limit of the following sequences:

(i) $\left\{\dfrac{5}{n^2}\right\}_1^\infty$, (ii) $\left\{\dfrac{3n^2 - 6n}{5n^2 + 4}\right\}_1^\infty$, (iii) $\lim\limits_{n\to\infty}\left(\frac{n-1}{n}\right)$.

Solution: (i)

$$\lim_{n\to\infty} \frac{5}{n^2} = \lim_{n\to\infty} 5 \cdot \frac{1}{n} \cdot \frac{1}{n} = 5 \cdot \lim_{n\to\infty} \frac{1}{n} \cdot \lim_{n\to\infty} \frac{1}{n} = 5 \cdot 0 \cdot 0 = 0.$$

(ii) Notice that

$$\frac{3n^2 - 6n}{5n^2 + 4} = \frac{3 - 6/n}{5 + 4/n^2}.$$

Now

$$\lim_{n\to\infty} (3 - 6/n) = 3 - 6 \lim_{n\to\infty} 1/n = 3 - 6 \cdot 0 = 3$$

and

$$\lim_{n\to\infty} (5 + 4/n^2) = 5 + 4 \lim_{n\to\infty} 1/n^2 = 5 + 4 \cdot 0 = 5.$$

Therefore,

$$\lim_{n\to\infty} \frac{3n^2 - 6n}{5n^2 + 4} = \lim_{n\to\infty} \frac{3 - 6/n}{5 + 4/n^2} = \frac{3}{5}.$$

(iii)

$$\lim_{n\to\infty} \left(\frac{n-1}{n} \right) = \lim_{n\to\infty} \frac{1 - 1/n}{1} = 1 - \lim_{n\to\infty} \frac{1}{n} = 1 - 0 = 1.$$

Sandwich theorem for sequences

Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three sequences such that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$. If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Proof: Let $\epsilon > 0$ be given. As $\lim_{n\to\infty} a_n = L$, there exists $N_1 \in \mathbb{N}$ such that

$$n \geq N_1 \implies |a_n - L| < \epsilon. \tag{1}$$

Similarly as $\lim_{n\to\infty} c_n = L$, there exists $N_2 \in \mathbb{N}$

$$n \geq N_2 \implies |c_n - L| < \epsilon. \tag{2}$$

Let $N = \max\{N_1, N_2\}$. Then, $L - \epsilon < a_n$ (from (1)) and $c_n < L + \epsilon$ (from (2)). Thus

$$L - \epsilon < a_n \leq b_n \leq c_n < L + \epsilon.$$

Thus $|b_n - L| < \epsilon$ for all $n \geq N$. Hence the proof.

Examples

Using Sandwich theorem, prove the following:

(i) $\lim_{n\to\infty} \frac{\cos n}{n} = 0.$

(ii) $\lim_{n\to\infty} \frac{1}{2^n} = 0.$

(iii) $\lim_{n\to\infty} (-1)^n \frac{1}{n} = 0.$

(iv) If $0 < b < 1$, then $\lim_{n\to\infty} b^n = 0.$

(v) $\lim_{n\to\infty} \sqrt[n]{n} = 1.$

Solution: (i) Consider the sequence $\left\{\frac{\cos n}{n}\right\}_{n=1}^{\infty}$. Then $\frac{-1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$. Hence by Sandwich theorem $\lim_{n\to\infty} \frac{\cos n}{n} = 0.$

(ii) As $0 \leq \frac{1}{2^n} < \frac{1}{n}$ and $\frac{1}{n} \to 0$ as $n \to \infty$, $\frac{1}{2^n}$ also converges to 0 by Sandwich theorem.

(iii) As $\frac{-1}{n} \leq (-1)^n \frac{1}{n} \leq \frac{1}{n}$ for all $n \geq 1$ and $\frac{1}{n} \to 0$ as $n \to \infty$, $(-1)^n \frac{1}{n}$ also converges to 0 by Sandwich theorem.

(iv) Since $0 < b < 1$, we can write $b = \frac{1}{1+a}$, where $a := \frac{1}{b} - 1$ so that $a > 0$. Also we have $(1+a)^n \geq 1 + na$. Hence

$$0 < b^n = \frac{1}{(1+a)^n} \leq \frac{1}{1+na} < \frac{1}{na}.$$

So, by sandwich Theorem, we conclude that $\lim_{n\to\infty} b^n = 0$.

(v) Let $a_n = n^{\frac{1}{n}} - 1$. Then $0 \leq a_n < 1$ for all $n \in \mathbb{N}$. Further,

$$n = (1+a_n)^n \geq \frac{n(n-1)}{2} a_n^2.$$

Thus $0 \leq a_n \leq \sqrt{\frac{2}{(n-1)}}$ $(n \geq 2)$. As $\sqrt{\frac{2}{(n-1)}} \to 0$ as $n \to \infty$, by Sandwich theorem, $a_n \to 0$, i.e., $n^{\frac{1}{n}} \to 1$ as $n \to \infty$.

Definition

The sequence $\{a_n\}$ is bounded if there exists $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$. Otherwise $\{a_n\}$ is called unbounded (not bounded).

Examples: (i) $\left\{\frac{3n+2}{2n+5}\right\}$, (ii) $\{1, 2, 1, 3, 1, 4, \cdots\}$.

Result: Every convergent sequence is bounded. So, not bounded implies not convergent.


Thank You