Department of Physics, Bennett University
EPHY105L (I Semester 2018-2019)
Solution to Problem Sheet 7

1. A parallel plate capacitor with plate separation of 0.6 mm and filled with free space has an applied peak voltage of 25 V at a frequency of 100 MHz. Find the peak value of displacement current density. [Ans: ~ 231.7 A/m^2]

   Solution
   The displacement current density is defined as, $J_d = \frac{\epsilon_0 V_0 \omega}{d}\cos\omega t$. The peak value of displacement current density can be obtained for $\cos\omega t = 1$. Given, $d = 0.6\ \mathrm{mm}, V_0 = 25\ \mathrm{V}, f = 100\ \mathrm{MHz}$. However, we know that $2\pi f = \omega$. Hence, inserting all the values,
   $J_d = \frac{(8.85\times10^{-12} Fm^{-1})\times(25V)\times(2\pi\times100\ MHz)}{6\times10^{-4} m} \simeq 231.7\ \mathrm{A.m^{-2}}$.

2. Consider a parallel plate capacitor with circular plates having a radius of 5 cm and plate separation of 0.5 mm and filled with free space. A peak voltage of 20 V at a frequency of 20 MHz is applied across the plates. Neglecting end effects in the capacitor calculate
   a) The peak value of displacement current density [Ans: 44.5 A/m^2]
   b) The magnetic field at the mid plane between the capacitor plates at a distance of 2 cm from the axis. [Ans: ~ 5.6x10^{-7} T]
   c) The magnetic field at the mid plane between the capacitor plates at a distance of 10 cm from the axis. [Ans: ~ 7x10^{-7} T]
   d) At what distance from the axis will the magnetic field be highest?

   Solution
   a) As we did in the previous problem, the peak value of the displacement current density will be, $J_d = \frac{\epsilon_0 V_0 \omega}{d} = \frac{(8.85\times10^{-12} Fm^{-1})\times(20V)\times(2\pi\times20MHz)}{5\times10^{-4} m} \simeq 44.5\ \mathrm{A.m^{-2}}$.
   b) Considering an Amperean loop of radius r, we can write, $B(2\pi r) = \mu_0 J_d \times \pi r^2$. Given, *r* = 2 cm. Hence, $B \times (2\pi \times 2 \times 10^{-2}\ \mathrm{m}) = (4\pi\times10^{-7}\ \mathrm{H.m^{-1}}) \times (44.5\ \mathrm{A.m^{-2}}) \times (\pi\times4\times10^{-4}\ \mathrm{m^2}) \Rightarrow B \simeq 5.6\times10^{-7}\ \mathrm{T}$.
   c) We have already calculated the displacement current density which reads $J = 44.5\ \mathrm{A.m^{-2}}$. So, the total displacement current for a parallel plate capacitor of radius (R) 5 cm will be, $I_d = j_d \times \pi R^2 = 0.3493\ \mathrm{A}$. Now considering an Amperean loop of radius 10cm, the magnetic field 10cm away from the center of the midplane will be, $B \times (2\pi\times10\times10^{-2}\mathrm{m}) = (4\pi\times10^{-7}\mathrm{H.m^{-1}}) \times (0.3493\ \mathrm{A}) \Rightarrow B \simeq 7\times10^{-7}\ \mathrm{T}$.
   d) If the distance from the axis is less than the radius of the circular disk (r<R), then the magnetic field $B = \frac{\mu_0}{2} J_d r = \frac{\mu_0 I_d}{2\pi R^2} r$, which suggests a linear increase of the magnetic field. If the distance is greater than the radius of the disk (r>R), then, $B = \mu_0 \frac{J_d \pi R^2}{2\pi r} = \frac{\mu_0}{2}\frac{I_d}{\pi r}$,this suggests that the magnetic field varies inversely with the distance. Hence, magnetic field will me maximum when r=R.

3. Consider an infinitely long air core tightly wound straight solenoid having *N* turns per unit length and carrying a current given by $I = I_0 \sin\omega t$.
   a) Obtain the induced electric field within the interior of the solenoid.
   b) Calculate the displacement current density within the solenoid.

[Ans: $E = \mu_0 I_0 N r\omega/2\cos\omega t; J_d = I_0 N r\omega^2/2c^2 \sin\omega t$]

Solution

a) The induced electric field is $E = \frac{\mu_0 N}{2}\frac{dI}{dt}r = \frac{\mu_0 N}{2} I_0 \omega r \cos\omega t$.

b) The displacement current density is defined as, $J_d = \frac{dD}{dt} = \epsilon_0 \frac{dE}{dt} = -\frac{\epsilon_0 \mu_0 I_0 N \omega^2 r}{2} \sin\omega t = -\frac{N\omega^2}{2c^2} I_0 r \sin\omega t$.

4. An electromagnetic wave propagating in free space is described by the following expression for the electric field:

$$\vec{\boldsymbol{E}} = \vec{\boldsymbol{E}} \exp[-i(3\times10^6 x - 4\times10^6 y - \omega t)]$$

a) What is the value of ω?

b) What is the wavelength of the wave?

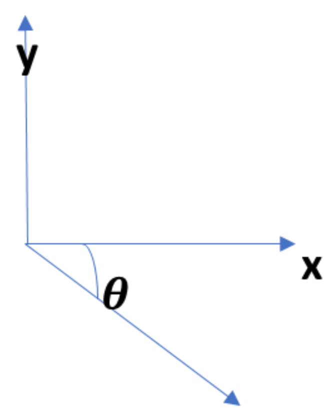
c) Write down the unit vector along the propagation direction of the wave.

Solution

a) Let us consider $\vec{E} = \vec{E}\exp\left[i\left(-k_x x + k_y y + \omega t\right)\right]$. Now the classical wave equation reads, $\nabla^2 E = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$. Hence, applying the given electric field in the classical wave equation, $\left(k_x^2 + k_y^2\right)E = \frac{\omega^2}{c^2}E \Rightarrow \frac{\omega^2}{c^2} = (3^2+4^2)\times10^{12} \Rightarrow \frac{\omega}{c} = 5\times10^6 m^{-1} \Rightarrow \omega = 15\times10^{14}\ \mathrm{s}^{-1}$

b) Now $\frac{\omega}{c} = \frac{2\pi}{\lambda} = 5\times10^6\ \mathrm{m}^{-1} \Rightarrow \lambda = \frac{2\pi}{5\times10^6} m \simeq 1.25\times10^{-6}\ \mathrm{m}$.

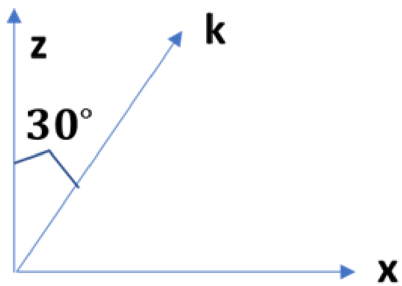
c) The direction of propagation is in the x-y plane since the wave has $k_x = 3\times10^6\ \mathrm{m}^{-1}, k_y = 4\times10^6\ \mathrm{m}^{-1}, k_z = 0$. If the angle made with the x axis is θ, then $\tan\theta = \frac{k_y}{k_x} = \frac{4}{3} \Rightarrow \theta = 53.13°$ as described in the figure.



5. Write down an expression for the electric field $\vec{\boldsymbol{E}}$ of a plane wave propagating in free space along a direction making an angle of 30° with the z-axis and lying in the $x - z$ plane.

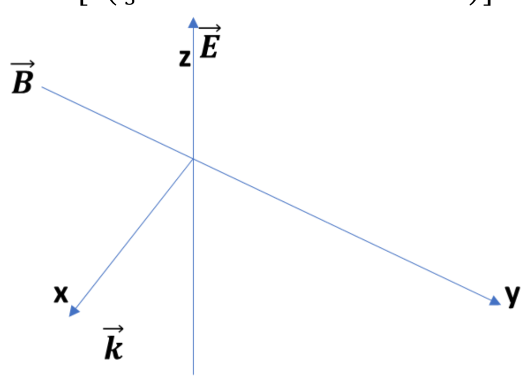
Solution

Since the wave is propagating in the x-z plane therefore $k_y = 0$. Also $k_z = k\cos 30° = \frac{\sqrt{3}}{2}k$ and $k_x = k\sin 30° = \frac{1}{2}k$ (as described in the figure). Thus, $\vec{E} = \overrightarrow{E_0}\exp[i(\overrightarrow{k_x}\cdot\vec{x} + \overrightarrow{k_z}\cdot\vec{z} - \omega t] = \overrightarrow{E_0}\exp[ik(\frac{1}{2}x + \frac{\sqrt{3}}{2}z - \omega t]$.



6. Consider a plane electromagnetic wave having a frequency of 1 GHz propagating along the $x-$direction in a medium with a dielectric permittivity $\epsilon = 2\epsilon_0$ and $\mu = \mu_0$ with its electric field pointing along the $z-$direction. Write down complete expressions for the electric and magnetic fields of the wave.

   Solution
   We know that $\frac{1}{\sqrt{\epsilon\mu}} = \frac{c}{n}$, where n is the refractive index of the medium. Since, $\epsilon = 2\epsilon_0$ and $\mu = \mu_0$, thus $n = \sqrt{2}\left(c = \frac{1}{\sqrt{\epsilon_0\mu_0}}\right)$. Given frequency (f) is 1GHz, thus $\omega = 2\pi \times 10^9 s^{-1}$. So, the electric field can be written as, $\vec{E} = E\exp[i(kx - \omega t)]\hat{z}$ and magnetic field, which is mutually perpendicular to the propagation direction (x – direction) and the direction of the electric field, is $\vec{B} = -B\exp[i(kx - \omega t)]\hat{y}$ (see the figure below). Applying the electric field in the classical wave equation, $k^2 = \frac{\omega^2}{v^2}$, where $v = \frac{c}{n}$. Thus $k = \sqrt{2}\frac{\omega}{c}$. So $\vec{E} = E\exp\left[i\left(\frac{\sqrt{2}}{3} \times 2\pi \times 10^1 x - 2\pi \times 10^9 t\right)\right]\hat{z}$ and $\vec{B} = -B\exp\left[i\left(\frac{\sqrt{2}}{3} \times 2\pi \times 10^1 x - 2\pi \times 10^9 t\right)\right]\hat{y}$.



7. On the surface of the earth, we receive about 1.33 kW of energy per square meter from the Sun. Calculate the electric and magnetic fields associated with the sunlight.

   Solution

   Given, $\langle|\vec{S}|\rangle = 1.33 \times 10^3\ \mathrm{W.m^{-2}} = \frac{c\epsilon_0}{2}E_0^2 \Rightarrow E_0 = \left[\frac{2\times1.33\times10^3}{3\times10^8\times8.85\times10^{-12}}\right]^{\frac{1}{2}}\mathrm{V.m^{-1}} = 1.0009 \times 10^3\ \mathrm{V.m^{-1}}$. The magnetic field is, $B_0 = \frac{E_0}{c} = 3.33 \times 10^{-6}\ \mathrm{T}$.

8. Consider a laser beam having a power of 5 mW. The beam is focused to a spot of radius 5 μm. Calculate the peak value of electric field generated at the focus.

Solution

If P denotes the power of the laser and A is the area of the focused spot then, $\langle|\vec{S}|\rangle = \frac{\langle P\rangle}{A} = \frac{5\times10^{-3} W}{\pi\times25\times10^{-12}\ m^2} = 63.6 \times 10^6\ \mathrm{W.m^{-2}}$. Hence, electric field is, $E_0 = \sqrt{\frac{2\langle|\vec{S}|\rangle}{c\epsilon_0}} \simeq 218.8 \times 10^3\ \mathrm{V.m^{-1}}$.