Multivariable Calculus

(Lecture-6)

Department of Mathematics
Bennett University
India

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Differentiation

of

Scalar Valued Function of Vector Variable

(Scalar Field)

$$F : S \subseteq \mathbb{R}^2 \to \mathbb{R}$$

Learning Outcome of the lecture

In this lecture, We learn Differentiation of $F : S \subseteq \mathbb{R}^2 \to \mathbb{R}$, where S is an open set of $\mathbb{R}^2$.

- Partial Derivatives
- Partial Derivatives versus Continuity

Differential Calculus for $F : S \subseteq \mathbb{R}^2 \to \mathbb{R}$

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Questions:

- What does it mean to say that F is differentiable ?
- How to define differentiability of F at a point $X_0 = (x_0, y_0)$?
- How to determine $F'(X_0)$?

Partial Derivatives of a Scalar Field F

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Definition

The partial derivative of F with respect to the variable x at the point $X_0 = (x_0, y_0)$ is denoted by $\frac{\partial F}{\partial x}(x_0, y_0)$ and is defined by

$$\frac{\partial F}{\partial x}(x_0, y_0) := \lim_{h \to 0} \frac{F(x_0 + h, y_0) - F(x_0, y_0)}{h}$$

provided the limit exists.

Partial Derivatives of a Scalar Field F

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The partial derivative of F with respect to the variable y at the point $X_0 = (x_0, y_0)$ is denoted by $\frac{\partial F}{\partial y}(x_0, y_0)$ and is defined by

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Interpretation of Partial Derivatives:

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Interpretation of Partial Derivatives:

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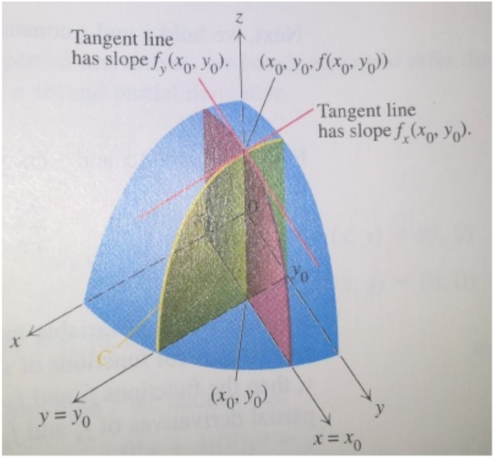
- $\frac{\partial F}{\partial y}(x_0, y_0)$ is the slope of the tangent to the curve $C_2 : z = F(x, y)$ in the plane $x = x_0$ at the point $(x_0, y_0, F(x_0, y_0))$.

Picture explaining Partial Derivatives of F at (x_0, y_0)



Further explanation on Partial Derivatives of F at the point $X_0 = (x_0, y_0)$

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- $\frac{\partial F}{\partial x}(x_0, y_0)$ is basically the rate of change of F in the direction of $\hat{i} = (1, 0) = e_1$ at (x_0, y_0).

$$\frac{\partial F}{\partial x}(x_0, y_0) := \lim_{t \to 0} \frac{F(X_0 + te_1) - F(X_0)}{t}$$

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Notations for Partial Derivatives

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- If $z = F(x, y)$ then $\frac{\partial z}{\partial x}$ is used to denote $\frac{\partial F}{\partial x}$ and $\frac{\partial z}{\partial y}$ is used to denote $\frac{\partial F}{\partial y}$.

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Example-1: Partial derivatives of F exist & F is continuous

Let $F(x, y) = xy$ for $(x, y) \in \mathbb{R}^2$. Let $X_0 = (x_0, y_0)$ be an arbitrary point in $\mathbb{R}^2$.

Example-1: Partial derivatives of F exist & F is continuous

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Question: Examine the existence of (first order) partial derivatives of F at X_0. Also examine the continuity of F at X_0.

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Question: Examine the existence of (first order) partial derivatives of F at X_0. Also examine the continuity of F at X_0.

Answer:
$\frac{\partial F}{\partial x}(X_0) = y_0$ and $\frac{\partial F}{\partial y}(X_0) = x_0$.

The function $F(x, y) = xy$ is continuous at X_0.

Example-2: Partial derivatives of F exist $\&F$ is not continuous

Let $F(x,y) = \frac{xy}{x^2+y^2}$ if $(x,y) \neq (0,0)$. And $F(0,0) = 0$.

Example-2: Partial derivatives of F exist $\&F$ is not continuous

Let $F(x,y) = \frac{xy}{x^2+y^2}$ if $(x,y) \neq (0,0)$. And $F(0,0) = 0$.

Question: Examine the existence of (first order) partial derivatives of F at $(0,0)$. Also examine the continuity of F at $(0,0)$.

Example-2: Partial derivatives of F exist $\&F$ is not continuous

Let $F(x,y) = \frac{xy}{x^2+y^2}$ if $(x,y) \neq (0,0)$. And $F(0,0) = 0$.

Question: Examine the existence of (first order) partial derivatives of F at $(0,0)$. Also examine the continuity of F at $(0,0)$.

Answer:
$\frac{\partial F}{\partial x}(0,0) = 0$ and $\frac{\partial F}{\partial y}(0,0) = 0$.

The function $F(x,y) = \frac{xy}{x^2+y^2}$ is not continuous at $(0,0)$.

Details are worked out in the class.

Example-3: F is continuous & Some partial derivatives do not exist

Let $F(x,y) = x \sin \frac{1}{x} + y$ if $x \neq 0$.. And $F(x,y) = y$, if $x = 0$.

Example-3: F is continuous & Some partial derivatives do not exist

Let $F(x,y) = x\sin\frac{1}{x} + y$ if $x \neq 0$.. And $F(x,y) = y$, if $x = 0$.

Question: Examine the existence of (first order) partial derivatives of F at $(0,0)$. Also examine the continuity of F at $(0,0)$.

Example-3: F is continuous & Some partial derivatives do not exist

Let $F(x, y) = x \sin \frac{1}{x} + y$ if $x \neq 0$.. And $F(x, y) = y$, if $x = 0$.

Question: Examine the existence of (first order) partial derivatives of F at $(0, 0)$. Also examine the continuity of F at $(0, 0)$.

Answer:
$\frac{\partial F}{\partial x}(0, 0)$ does not exist and $\frac{\partial F}{\partial y}(0, 0) = 1$.

The function $F(x, y) = x \sin \frac{1}{x} + y$ is continuous at $(0, 0)$.

Details are worked out in the class.