Multivariable Calculus

(Lecture-1)

Department of Mathematics
Bennett University
India

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Books

- Maurice D. Weir and Joel Hass, “Thomas Calculus”, 12th Edition, Pearson Education India, 2016.
- T. M. Apostol, “Calculus - Vol. 2”, 2nd Edition, Wiley India, 2003.

The space

$$\mathbb{R}^n = \{X = (x_1, \ldots, x_n) \; : \; x_i \in \mathbb{R} \;\text{ for } 1 \leq i \leq n\}.$$

Properties of $\mathbb{R}^n$

Let $\vec{A} = (a_1, \ldots, a_n)$ and $\vec{B} = (b_1, \ldots, b_n)$ be two vectors in $\mathbb{R}^n$

- Vector addition:

$$\vec{A} + \vec{B} = (a_1 + b_1, \ldots, a_n + b_n).$$

- Scalar multiplication: Let λ be a scalar (from the real field $\mathbb{R}$)

$$\lambda\vec{A} = (\lambda a_1, \ldots, \lambda a_n).$$

$\mathbb{R}^n$ forms a vector space over the real field $\mathbb{R}$ with respect to the above mentioned vector addition and scalar multiplication.

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What is vector space? (You will learn in the next semester course.)

Roughly for $\mathbb{R}^n$: for any $\vec{A}, \vec{B} \in R^n$ and $\lambda \in \mathbb{R}$

$$\vec{A} + \lambda\vec{B} \in \mathbb{R}^n.$$

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Sequences

- Sequences of Real Numbers: Convergence, Bounded, montonically increasing/ decreasing, Cauchy sequence, Subsequences, Bolzano-Weierstrass theorem.

Single Variable Calculus versus Multivariable Calculus

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Single Variable Calculus versus Multivariable Calculus

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- Series in $\mathbb{R}^n$: Convergence.

Limit

- Limits of Functions in $\mathbb{R}$: Properties of Limits.

Single Variable Calculus versus Multivariable Calculus

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Continuity

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Single Variable Calculus versus Multivariable Calculus

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- Differentiability in Multivariable Calculus:
  - Vector valued function of Scalar variable
    $F : (a, b) \subseteq \mathbb{R} \to \mathbb{R}^m$ where $m > 1$ - Differentiation (and we also do Integration)

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    $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ where $n > 1$ - Partial Derivatives, Directional Derivatives, Total Derivative, Gradient Vector, MVT, Maxima and Minima, Taylor's theorem for functions of two variables.

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    $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ where $n > 1$ and $m > 1$ - Partial Derivatives and Directional Derivatives of Component Functions f_j, $1 \leq j \leq m$, Total Derivative of F , Jacobian matrix, Necessary condition for differentiability, Sufficient conditions for differentiability.

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Single Variable Calculus versus Multivariable Calculus

- Riemann Integration in Single Variable Calculus:
  Sufficient condition for integrability $U(P,f) - L(P,f) < \epsilon$, Properties of Riemann Integration, Application of Definite Riemann Integration in finding area and volume, Improper Integrals.

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  - Three Dimension $n = 3$: Triple integral of f over D if D is a rectangular cube or D is a simple solid region, Iterated Integrals, Fubini's theorem. Change of variables in integration (Cylindrical, Spherical, Other coordinate systems), Application of Triple Integrals.

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  - Higher Dimension $n > 3$: Idea how to generalize the Riemann Integration in higher dimensional space.

New kind of integrals in Multivariable Calculus

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- Surface Integrals (Integration over two-sided, Smooth / Piecewise Smooth Surfaces in $\mathbb{R}^3$)
  - Scalar Surface Integrals: Integration of Scalar Field $f : S \subseteq \mathbb{R}^3 \to R$ over the piecewise smooth non oriented surface S.
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- Greens Theorem, Gauss Theorem, Stokes Theorem.

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What can go wrong in higher dimensional situation ?

- Let $f(x,y) = \frac{x^2}{x^2+y^2}$ for $(x,y) \neq (0,0)$.

$$\begin{aligned} \lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) &= \lim_{x\to 0} 1 = 1 \\ \lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) &= \lim_{y\to 0} 0 = 0 \end{aligned}$$

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Let $f(x,y) = \frac{xy^3}{x^2+y^2}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$, Compute the second order mixed partial derivatives of f at $(0,0)$.

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\Big|_{(0,0)} &= 1 \\
\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\Big|_{(0,0)} &= 0
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Then

$$\frac{\partial^2 f(0,0)}{\partial x \partial y} = 1 \neq 0 = \frac{\partial^2 f(0,0)}{\partial y \partial x}.$$

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$$\frac{F(X) - F(X_0)}{X - X_0}.$$

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Challenges:

- Numerator is $(F(X) - F(X_0)) \in \mathbb{R}^m$ which is a vector quantity.
- Denominator is $(X - X_0) \in \mathbb{R}^n$ which is a vector quantity.
- We are unable to define the quantity $\frac{F(X)-F(X_0)}{X-X_0}$.

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So, How to overcome this difficulty in order to define the differentiability of F ?

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- Concept of Limits of Functions is same.
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- Differentiation and Integration can be taken as such to the functions $F : (a, b) \subseteq \mathbb{R} \to \mathbb{R}^n$ where $n > 1$.
- Riemann Integration can be taken as such to the functions

$$f : D \subseteq \mathbb{R}^n \to \mathbb{R},$$

where $n > 1$ and $D = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ x_i \in [a_i, b_i], \ 1 \leq i \leq n\}$.

Notations

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$$\langle U, V\rangle = U \cdot V = x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$