Tutorial Sheet 2

System of Linear Equations

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1. Find the row echelon form and rank of the following matrices:

$$\text{(a)} \begin{bmatrix} 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix} \quad (b) \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 2 & 1 \\ 0 & 4 & 0 & 1 & 1 \end{bmatrix} \quad (c) \begin{bmatrix} 0 & 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

   Also, reduce them in the row reduced echelon form.

2. Write a system of linear equations consisting of three equations in three unknowns with (a) no solutions (b) exactly one solution (c) infinitely many solutions.

3. Find the general solution the following system using Gauss elimination method as well as Gauss Jordon method, whenever the solution exist:

   (a) $x + 2y + z = 1,\ 3x + 7y + 6z = 5,\ -2x - y + 7z = 4.$

   (b) $x + 4y - 2z + 3u = 0,\ x + 6y - z + u = 0,\ x + 2y - z + 2u = 0.$

   (c) $x+2y-3z-2u+4v = 1,\ 2x+5y-8z-u+6v = 4,\ x+4y-7z+5u+2v = 8.$

4. Solve the following systems of linear equations :
   (a) $2x_1 + x_2 + x_3 = 0, \quad x_1 + x_3 = 0, \quad 2x_2 + x_3 = 0$ over $\mathbb{Z}_3$.
   (b) $2x_1 + x_2 + x_3 = 2, \quad x_2 + x_3 = 1, \quad x_3 = 4$ over $\mathbb{Z}_7$.

5. Find the values of a, b, c such that the graph of the polynomial $p(x) = ax^2 + bx + c$ passes through the points $(1, 2)$, $(-1, 6)$, and $(2, 3)$.

6. The following table gives the number of milligrams of vitamins A, B, C contained in one gram of each of the foods F_1; F_2; F_3; F_4. A mixture is to be prepared containing precisely 14 mg. of A, 29 mg. of B and 23 mg. of C. Find the greatest amount of F_2 that can be used in the mixture.

	F_1	F_2	F_3	F_4
A	1	1	1	1
B	1	3	2	1
C	4	0	1	1

7. Construct the augmented matrix for each of the following linear systems of equations. In each case determine the rank of the coefficient matrix, the rank of the augmented matrix, whether or not the system has any solutions, no solution and the number of free parameters in the general solution.

   (a) $2x_1 + 3x_2 = 3,\ 4x_1 + 5x_2 = 5.$

   (b) $2x_1 + 3x_2 + 4x_3 = 3,\ 2x_1 + x_2 - x_3 = 1,\ 6x_1 + 5x_2 + 2x_3 = 5.$

   (c) $2x_1 + x_2 + x_3 = 2,\ 2x_1 + 2x_2 - x_3 = 1,\ 6x_1 + 4x_2 + x_3 = 4.$

   (d) $x_1 - x_2 + x_3 + 2x_4 + x_5 = -1,\ -x_1 + 3x_2 + 2x_3 + x_4 + x_5 = 2,$
   $2x_1 + 5x_3 + 7x_4 + 4x_5 = -1,\ -x_1 + 5x_2 + 5x_3 + 4x_4 + 3x_5 = 3.$

8. Find the values of a and b for which the following systems of linear equations :

   (1) $ax + z = 2,\ ax + ay + 4z = 4,\ ay + bz = 1.$

   (2) $x + ay = 4,\ ax + 9y = b.$

   (3) $x_1+x_2+2x_3 = b, x_2+x_3+2x_4 = 0, x_1+x_2+3x_3+3x_4 = 0, 2x_2+5x_3+ax_4 = 3$

   have (a) no solution (b) a unique solution (c) a one or two -parameter solution. Also find the solutions whenever they exist.

9. Prove the following:

   (a) Show that if x_1 and x_2 are any two solutions of a linear system of equations $Ax = b$, then $x_1 - x_2$ is a solution of the associated homogeneous system $Ax = 0$.

   (b) Show that, if x_1 is any solution of the original system $Ax = b$, every solution is of the form $x_1 + x_2$, where x_2 is a solution of the associated homogeneous system.

   (c) Show that if x_1 and x_2 are both solutions of a given homogeneous system of equations $Ax = 0$ and if c_1 and c_2 are numbers then $c_1x_1 + c_2x_2$ is also a solution.

   (d) Show that if $x_1 + x_2 + \cdots + x_k$ are all solutions of a given homogeneous system of equations and if $c_1, c_2, \cdots, c_k$ are numbers then $c_1x_1 + c_2x_2 + \cdots + c_kx_k$ is also a solution.