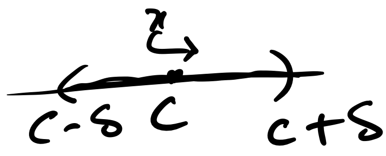
Limits

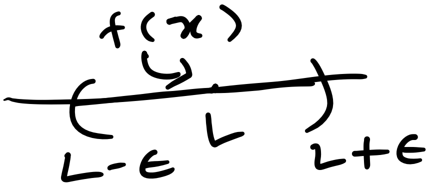
$f : (a,b) \to \mathbb{R}, \quad c \in (a,b)$

$\lim_{x \to c} f(x) = L$ if $\epsilon > 0 \; \exists \; \delta > 0$ s.t

$$|x - c| < \delta \Rightarrow |f(x) - L| < \epsilon .$$

$c - \delta < x < c + \delta \qquad L - \epsilon < f(x) < L + \epsilon$





Ex1: $\lim_{x \to 1} \left(\frac{3x}{2} - 1\right) = \frac{1}{2}$

$\epsilon > 0$ find $\delta > 0$ s.t

$$|x - 1| < \delta \Rightarrow |f(x) - L| < \epsilon$$

$$\left|\left(\frac{3x}{2} - 1\right) - \frac{1}{2}\right| = \left|\frac{3x}{2} - \frac{3}{2}\right| = \frac{3}{2}|x - 1|$$

$\left|\left(\frac{3x}{2} - 1\right) - \frac{1}{2}\right| = \frac{3}{2}|x - 1| < \epsilon$ whenever $|x - 1| < \frac{2\epsilon}{3}$

choose $\delta = \frac{2\epsilon}{3} > 0$

$\epsilon > 0 \; \exists \; \delta = \frac{2\epsilon}{3} > 0$ s.t $|x - 1| < \delta$

$$\Rightarrow |f(x) - \tfrac{1}{2}| < \epsilon$$

Ex2: $\lim_{x \to 0} \sqrt{x} = 0, \; x \geq 0$

$\epsilon > 0$ find $\delta > 0$ s.t $|x - 0| = x < \delta$

$$\Rightarrow |f(x) - 0| = \sqrt{x} < \epsilon$$

$|f(x) - 0| = \sqrt{x} < \epsilon$ whenever $x < \epsilon^2$

choose $\delta = \epsilon^2$

$\epsilon > 0, \; \delta = \epsilon^2 > 0$ s.t $x < \delta \Rightarrow \underset{|f(x) - 0|}{\underset{\parallel}{\sqrt{x}}} < \epsilon$

Sequential criteria of limits

$\lim_{x\to c} f(x) = L \iff$ Every sequence $\{x_n\}$, $x_n \to c$ $\Rightarrow f(x_n) \to L$ as $n\to\infty$

Ex:- $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist.

$$x_n = \frac{1}{n\pi}, \quad y_n = \frac{1}{2n\pi + \frac{\pi}{2}}$$

$$x_n \to 0, \quad y_n \to 0$$

$$f(x_n) = \sin\left(\frac{1}{x_n}\right) \to 0$$

$$f(y_n) = \sin\left(\frac{1}{y_n}\right) \to 1$$

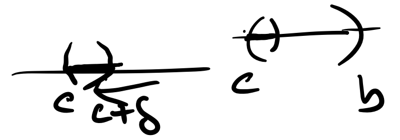
Result: suppose f is bounded in an iterval containing c, $\lim_{x\to c} g(x) = 0$

$$\Rightarrow \lim_{x\to c} f(x)g(x) = 0$$

not true if $\lim_{x\to c} g(x) \neq 0$.

Ex: $\lim_{x\to 0} x\cos\frac{1}{x} = 0$

$\lim_{x\to 0} x\sin\frac{1}{x^2} = 0$, $\lim_{x\to 0} x\sin x = 0$



One sided limits:

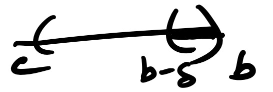
$f : (c, b) \to \mathbb{R}$,

R.H.L of f at c, $\lim_{x\to c^+} f(x) = L$

$\epsilon > 0$ $\exists$ $\delta > 0$ s.t $c < x < c+\delta$ $\Rightarrow |f(x) - L| < \epsilon$

L.H.L of f at b, $\lim_{x\to b^-} f(x) = L$

$\epsilon > 0$ $\exists$ $\delta > 0$ s.t $b - \delta < x < b \Rightarrow |f(x) - L| < \epsilon$



Result: $\lim_{x\to c} f(x) = L \iff \lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = L$

$c \in (a, b)$

$\overset{(\quad c \quad)}{a \qquad b}$

Limits at infinity

$\lim_{x\to\infty} f(x) = L$ if $\epsilon > 0\ \exists\ M > 0$

s.t $x > M \Rightarrow |f(x) - L| < \epsilon$

$\lim_{x\to -\infty} f(x) = L$, if $\epsilon > 0\ \exists\ M > 0$ s.t

$x < -M \Rightarrow |f(x) - L| < \epsilon$

Ex 1: $\lim_{x\to\infty} \frac{1}{x} = 0$, $f(x) = \frac{1}{x}$

$\epsilon > 0\ \exists\ M > 0$ s.t $x > M \Rightarrow |f(x) - 0| < \epsilon$

$\frac{1}{x} < \epsilon \Rightarrow \frac{1}{\epsilon} < x$

choose, $M = \frac{1}{\epsilon} > 0$

$\epsilon > 0\ \exists\ M = \frac{1}{\epsilon} > 0$ s.t $x > M \Rightarrow \frac{1}{x} < \epsilon$.