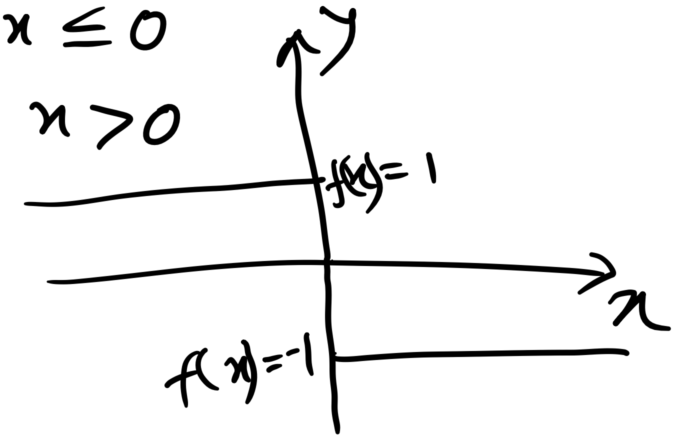
continuity

Types of discontinuity :-

② Jump discontinuity :- (discontinuity of 1st kind)

$$\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$$

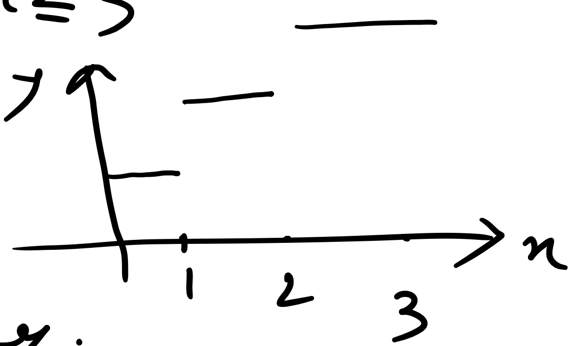
Ex:- $f(x) = \begin{cases} 1, & x \leq 0 \\ -1, & x > 0 \end{cases}$



$$\left.\begin{aligned} \lim_{x \to 0^+} f(x) &= -1 \\ \lim_{x \to 0^-} f(x) &= 1 \end{aligned}\right\} \neq$$

Ex:- $f(x) = \begin{cases} 1, & 0 \leq x < 1 \\ 3, & 1 \leq x < 2 \\ 5, & 2 \leq x \leq 3 \end{cases}$

$x = 1, 2$ points of jump.



Infinite discontinuity :-

$$\lim_{x \to a^+} f(x) = \infty \text{ or } -\infty$$

$$\lim_{x \to a^-} f(x) = \infty \text{ or } -\infty$$

Ex:- $f(x) = \frac{1}{x}$, $\lim_{x \to 0^+} f(x) = \infty$

$\lim_{x \to 0^-} f(x) = -\infty$

Discontinuity of 2nd kind:-

$\lim_{x\to a^+} f(x)$ or $\lim_{x\to a^-} f(x)$ does not exist.

Ex:- $f(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ 1, & x \notin \mathbb{Q} \end{cases}$

$c \in \mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$, let $c \in \mathbb{Q}$.

$\lim_{x\to c^+} f(x)$

$\lim_{x\to c^-} f(x)$ does not exist.

$\{c + \frac{1}{n}\} \in \mathbb{Q}$. $\quad c+\frac{1}{n} \to c$, $f(c+\frac{1}{n}) = 0 \;\forall n$

$\{c + \frac{\pi}{n}\} \in \mathbb{Q}^c$ $\quad c+\frac{\pi}{n} \to c$, $f(c+\frac{\pi}{n}) = 1 \;\forall n$

$f(c+\frac{1}{n}) \to 0$, $f(c+\frac{\pi}{n}) \to 1$ as $n \to \infty$

$\Rightarrow \lim_{x\to c^+} f(x)$ does not exist.

$\{c - \frac{1}{n}\} \in \mathbb{Q}$, $c-\frac{1}{n} \to c$, $f(c-\frac{1}{n}) = 0$

$\{c - \frac{\pi}{n}\} \in \mathbb{Q}^c$, $c-\frac{\pi}{n} \to c$, $f(c-\frac{\pi}{n}) = 1$

$f(c-\frac{1}{n}) \to 0$, $f(c-\frac{\pi}{n}) \to 1$, $n \to \infty$

$\lim_{x\to c^-} f(x)$ does not exist

Ex:- $c \in \mathbb{Q}^c$, $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^-} f(x)$ does not exist.

Result: (1) continuous function on closed and bounded interval is bounded.

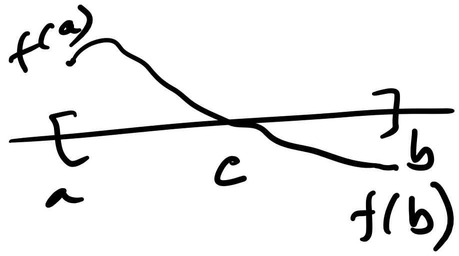
Ex:- $f(x) = \frac{1}{x}$ on $(0,1)$

$f(x) = x$, $x \in \mathbb{R}$.

Result:- if f is continuous on $[a,b]$
Then $\max f$, $\min f$ are achieved in $[a,b]$

Result: Let f is continuous on $\mathbb{R}$, let
$a, b \in \mathbb{R}$, s.t $f(a)\cdot f(b) < 0$. Then
There exist $c \in (a,b)$ s.t $f(c) = 0$.

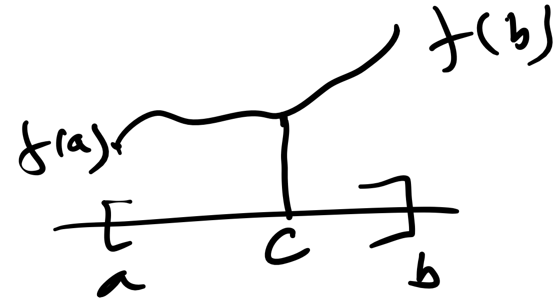
Ex:- $f(x) = x^2 - 2$, $x \in (1,2)$



$f(1) = -1$, $f(2) = 2$
$c \in (1,2)$, $f(c) = 0$.

Intermediate value theorem:-

If f is continuous on $[a,b]$. and
$f(a) < y < f(b)$. Then $\exists\, c \in (a,b)$
s.t $f(c) = y$.



Ex:- $f(x) = x^3 - 2$,
$x \in [1, 5]$.

$f(1) = \underset{<0}{-1}$, $f(5) = 123 > 0$.

$f(1) = -1 < 100 < f(5) = 123$

$c \in (1,5)$ s.t $f(c) = 100$.

Uniformly continuous functions:-

A function f is Uni. conts on S.
if for any $\epsilon > 0$ $\exists\, \delta > 0$ s.t $\forall\, x, y \in S$
$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$.

Ex:- $f(x) = \frac{1}{x}$, $x \in [1,\infty)$ then f is uni. conts on $[1,\infty)$

To Show:-

$\epsilon > 0$, $\exists\, \delta > 0$ s.t $\forall x, y \in [1,\infty)$ s.t.

$$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon.$$

$x, y \in [1,\infty)$, $x \geq 1$, $y \geq 1$, $\Rightarrow xy \geq 1$

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{x-y}{xy}\right|$$
$$\leq |x-y|$$

$$\underset{\delta > 0}{|x-y| < \underset{\parallel}{\epsilon}} \Rightarrow |f(x) - f(y)| < \epsilon$$

Ex:- $f(x) = x^2$ is uni. conts on $[a,b]$.

Result:- If f is uni conti. $\Longleftrightarrow$
for any $\{x_n\}, \{y_n\}$ s.t $|x_n - y_n| \to 0$
then $|f(x_n) - f(y_n)| \to 0$ as $n \to \infty$

$\epsilon > 0$ $\exists\, \delta > 0$ s.t $\forall x, y \in [a,b]$
$$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

$$|f(x) - f(y)| = |x^2 - y^2| = |(x+y)(x-y)|$$
$$\leq (|x| + |y|)\, |x-y|$$
$$= 2b\,|x-y|$$

$y, x \in [a,b]$
$a \leq x \leq b$
$a \leq y \leq b$

$|f(x) - f(y)| < \epsilon$ whenever $2b|x-y| < \epsilon$
$$\Rightarrow |x-y| < \frac{\epsilon}{2b} = \delta$$

Ex:- $f(x) = \frac{1}{x}$, $x \in (0,1)$ not uni. conts on $(0,1)$

$x_n = \frac{1}{n+1}$, $y_n = \frac{1}{n}$, $n \geq 2$
$$|x_n - y_n| = \frac{1}{n(n+1)} \to 0 \text{ as } n \to \infty$$

$$|f(x_n) - f(y_n)| = 1 \quad \forall n$$
$$\not\to 0. \Rightarrow f \text{ is not uni conts on } (0,1)$$

Ex:- $f(x) = x^2$ is not uni cont. on $\mathbb{R}$.

$$x_n = n + \frac{1}{n}, \quad y_n = n$$
$$|x_n - y_n| = \frac{1}{n} \to 0 \text{ as } n \to \infty$$
$$|f(x_n) - f(y_n)| = 2 + \frac{1}{n^2} \to 2 \text{ as } n \to \infty.$$