Determinants :-

Notation :- For an $n\times n$ matrix A, by $A(i|j)$, we mean the submatrix B of A, which is obtained by deleting ith row and jth column.

Def^n :- (Determinant of a square matrix) :- Notation ($|A|$ or $\det(A)$)

Let A be a square matrix of order n.

Then

$$\det A \text{ or } |A| = \begin{cases} a & \text{if } A=[a],\ n=1 \\ \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A(i|j)) & \text{if } A=[a_{ij}]_{n\times n},\ n\geq 2. \end{cases}$$

Def^n :- (Minor, Cofactor of a Matrix) :- The number $\det(A(i|j))$ is called the (i,j)th minor of A.

We denote $A_{ij} = \det(A(i|j))$.

Then (i,j)th cofactor of A, denoted $C_{ij} = (-1)^{i+j} A_{ij}$

$$\boxed{\text{i.e. } \det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} A_{ij} \text{ or } \sum_{j=1}^{n} C_{ij}\, a_{ij}}$$

Def^n :- A matrix is said to be a "singular" matrix if $\boxed{\det A = 0}$. It is called "non-singular" if $\boxed{\det A \neq 0.}$

Ex :- Find the inverse of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Remark: 1) Many authors defines the determinant using permutations. It turns out that the way we defined determinant is usually called the expasion of the determinant along the first row.

2) One can also calculate the determinant by expanding along any row / column.

Hence, for $n \times n$ matrix A, for every $1 \le k \le n$, one also has

$$\det A = \begin{cases} \sum_{j=1}^{n} (-1)^{i+j} a_{kj} \det(A(k|j)) & \text{[Row expansion]} \\ \sum_{i=1}^{n} (-1)^{i+j} a_{ik} \det(A(i|k)) & \text{[Column expansion]} \end{cases}$$

3) For any $n \times n$ matrix A, it can be proved that $|\det A|$ is equal to the volume of n-dimensional parallelopiped.

(The actual proof is beyond the scope of this book).

4) One can easily check that the determinant of a triangular (upper triangular or lower triangular, diagonal) matrix is the "product of its diagonal entries".

5) If $\det A \neq 0 \iff \text{rank}(A) = n$. if A is $n \times n$ matrix.

Properties of determinant:

Let $A = [a_{ij}]$ be an $n \times n$ matrix Then

1) if all the elements of one row/column is zero. Then $\det A = 0$.

2) if A is a triangular matrix then $\det A = a_{11} a_{22} \cdots a_{nn}$.

3) if A is a square matrix having two rows equal then $\det A = 0$.

4) A is invertible iff $\det A \neq 0$

5) if B is obtained from A by multiplying a row by 'c' then $\boxed{\det B = c \det A}$.

6) if B is obtained by interchanging two rows then $\boxed{\det(B) = -\det(A)}$

7) If B is an $n \times n$ matrix. then $\det(AB) = \det A \cdot \det B$.

8) $\det(A) = \det(A^t)$, where A^t is the transpose of A.

9) if B is obtained from A by replacing the lth row by itself plus k times the mth row. for $l \neq m$.

(i.e $\underset{\text{in } B}{R_l} \to \underset{\text{in } A}{R_l} + k\underset{\text{in } A}{R_m}$)

Then $\boxed{\det B = \det A}$.

10) Let $B = [b_{ij}]$ & $C = [c_{ij}]$ be two matrices which differ from the matrix $A = [a_{ij}]$, only in the mth row for some m

If $c_{mj} = a_{mj} + b_{mj}$ for $1 \leq j \leq n$. Then $\boxed{\det(C) = \det(A) + \det(B)}$.

Adjoint of a matrix :-

Def^n :- Let A be a $n \times n$ matrix. The matrix $B = [b_{ij}]$ with $b_{ij} = C_{ji}$ for $1 \le i, j \le n$, $C_{ij} = (-1)^{i+j} \underbrace{A_{ij}}_{(i,j)\text{th minor of } A}$. is called the adjoint of A, denoted by $Adj(A)$.

\# $A(Adj\,A) = \det(A)\, I_n$.

Thus if $\det(A) \neq 0$ Then $A^{-1} = \frac{1}{\det A} Adj(A)$

\# 1) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}$. Compute $Adj(A)$.

Ans:→ $Adj\,A = \begin{bmatrix} 4 & 2 & -7 \\ -3 & -1 & 5 \\ 1 & 0 & -1 \end{bmatrix}$

2) $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$. Then compute A^{-1}.

Ans:- $\det A = -2$, $adj(A) = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -3 & 1 \end{bmatrix}$

$A^{-1} = \frac{1}{\det A} \cdot adj(A)$.

Cramer's Rule :-

Recall the following :-

The linear system $AX = b$ has a unique solution for every "b" if and only if A^{-1} exists.

A has an inverse iff $\det A \neq 0$.

Then the following method / Theorem gives a direct method of finding the solⁿ of the linear system $AX = b$, when $\det(A) \neq 0$.

Theorem :- (CRAMER'S RULE) :- Let $AX = b$ be a linear system with n equations in n unknowns. If $\det A \neq 0$ Then, the system has a unique solution given by

$$x_j = \frac{\det(A_j)}{\det(A)} \quad \text{for } j = 1, 2, \ldots, n,$$

where A_j is the matrix obtained from A by replacing the jth column of A by the column vector b.

Remark :- This method is used only for square matrix, which are invertible.

Ex :- Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Use Cramer's rule to find a vector x such that $Ax = b$.

Solⁿ: One can easily check $\det A = 1$.

$$\therefore \quad x_i = \frac{\det(A_i)}{\det A} = A_i$$

$$x_1 = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix} = -1$$

$$x_2 = \begin{vmatrix} 1 & 1 & 3 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 1$$

$$x_3 = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 0.$$

i.e. $$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$ Ans.