$E[X+Y] = E[X] + E[Y]$

even if X and Y are dependent

E.g. 5 card hand (# of aces)

$X_i = 1$ if i^{th} card is an ace

$E[X] = E[X_1 + \dots X_5]$

$= E[X_1] + E[X_2] + \dots E[X_5]$

$= 5 E[X_1]$

$= 5 \times \frac{1}{13} = \frac{5}{13}$

Geom(p) : independent Bern(p) trials

$(1-p)^{n-1} \times p$

PMF $(P(X=k)) = (1-p)^{k} \times p$

k failures before success

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$E[X] = \sum_{k=0}^{\infty} k p q^k$

$= p \sum_{k=0}^{\infty} k q^k$ — (1)

$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$.

$\sum_{k=0}^{\infty} k q^{k-1} = \frac{1}{(1-q)^2}$.

or $\sum_{k=1}^{\infty} q^{k-1} \times k = \frac{1}{(1-q)^2}$.

or $\sum_{k=1}^{\infty} q^k \times K = \frac{q}{p^2}$. — (2)

Using (1) and (2), we get

$E[X] = p \times \frac{q}{p^2} = \frac{q}{p}$.

Proof of linearity.

let $T = X+Y$

proove $E(T) = E[X] + E[Y]$.

$\sum_{\forall t} t P(T=t) \overset{?}{=} \sum_{\forall x} x P(X=x) + \sum_{\forall y} y P(Y=y)$

$P(T=t) = \sum_{\forall x} P(T=t \mid X=x) \times P(X=x)$.

$=$ (stuck

$P(T=t) = \sum_{\forall s} (X+Y)(s)\, p\{s\}$

$= \sum_{\forall s} \big(X(s) + Y(s)\big)\, p\{s\}$

$= \sum_{\forall s} X(s)\, p\{s\} + \sum_{\forall s} y(s)\, p\{s\}$

$= E[X] + E[Y]$.

$r = 5$ (success)
$n = 11$ (failure)

1000100~~0~~10000100~~1~~

Negative binomial r, p

$$\binom{n+r-1}{r-1} p^r (1-p)^n = P(X=n)$$

Here X = # of failure before r success.

Random Variable → One particular choice.

Distribution → Maps for different choices

Poisson Distribution
(Important)

$P(X=k) = e^{-\lambda} \lambda^k / k!$

$k \geq 0; k \in \mathbb{I}$ (Integer)

λ is the scale parameter. $\lambda > 0$

$$PMF = \sum_{x \geq k} P(x=k) = 1$$

$$= \sum_{k=0}^{\infty} \frac{e^{-\lambda} \lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} (e^{\lambda})$$ → from Taylor series

Taylor series of a function $f(x)$ at a is

$$f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(x-a)^2}{2!} + \ldots$$

if $f(x) = e^x$; $a=0$, then

$$\frac{1}{1} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = e^x$$

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$$

$$E[X] = \sum k \frac{e^{-\lambda} \lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=1}^{\infty} \frac{k \lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!}$$

$$= e^{-\lambda} \times \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \qquad \boxed{\begin{array}{c} k-1 = j \\ j = 0 \\ \infty \end{array}}$$

$$= e^{-\lambda} \times \lambda \times e^{\lambda} = \lambda$$

This distribution is used to count # of "success" where each success has a small probability of success.

e.g. no. of emails

no. of insects in your food.

· no. of earthquakes.

no. of hacking attacks

Say we have j events

$A_1, A_2 \ldots A_n$ $P(A_j) = p_j$ and n is very large

of A_j's that occur is approximately $Pois(\lambda)$

$$\lambda = E[X] \quad X \sim Pois(\lambda)$$
$$= \sum p_j$$

Example

↳ Hospital 1.8 births/hour

1) $P(X=4) = \frac{e^{-\lambda}\lambda^k}{k!} = \frac{e^{-1.8}(1.8)^4}{4!} = 0.07$

2) $P(X \geq 2) = P(X=2) + P(X=3) \ldots$

$= 1 - P(X < 2)$

$= 1 - P(X=0) - P(X=1)$

$= 1 - \frac{e^{-1.8} \times (1.8)^0}{0!} - \frac{e^{-1.8} (1.8)}{1}$

$= 0.537$

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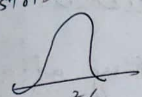
Normal Distribution.

$N(0,1)$ 

$$f(z) = c\, e^{-z^2/2}.$$

$$= \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

↳ Normalizing constant.

$$\int_{-\infty}^{\infty} e^{-z^2/2}\, dz = \int_{-\infty}^{\infty} f(z)\, dz = I.$$

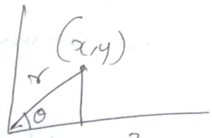
$$I^2 = \int_{-\infty}^{\infty} e^{-z^2/2}\, dz \int_{-\infty}^{\infty} e^{-z^2/2}\, dz$$

$$= \int_{-\infty}^{\infty} e^{-x^2/2}\, dx \int_{-\infty}^{\infty} e^{-y^2/2}\, dy$$ (Notation change)

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2/2} e^{-y^2/2}\, dx\, dy.$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2/2)}\, dx\, dy.$$

$$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2/2} \times \underbrace{r}_{\text{Jacobian}}\, dr\, d\theta$$



$r^2 = x^2 + y^2$

$$= 2\pi$$

$$\Rightarrow I = \sqrt{2\pi}.$$

CDF

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\, dt$$

Easy to compute on computer systems.

Exponential Distribution.

→ One parameter λ. (Rate at which some events occur)

$X \sim Exp(\lambda)$ has PDF $\lambda e^{-\lambda x}, x > 0$.
otherwise 0.

PDF
$$P(X=x) = \lambda e^{-\lambda x}.$$

CDF
$$F(x) = \int_0^x \lambda e^{-\lambda x} dx = 1 - e^{-\lambda x}.$$

We know $\frac{dF(x)}{dx} = f(x)$.

One good thing

if $Y = \lambda X$

$Y \sim Exp(1)$

$$P(Y \le y) = P(\lambda X \le y) = P(X \le y/\lambda)$$
$$= 1 - e^{-\lambda \cdot y/\lambda}$$
$$= 1 - e^{-y}.$$
$$\Rightarrow \lambda = 1.$$

Say $Y \sim Exp(1)$.

$$E[Y] = \int_0^\infty y e^{-y} dy.$$
$$= y(-e^{-y})\Big|_0^\infty + \int_0^\infty e^{-y} dy.$$
$$= 0 + (-e^{-y})\Big|_0^\infty$$
$$= 0 + 1 = 1.$$

$$Var(Y) = E[Y^2] - (E[Y])^2$$
$$= \int_0^\infty y^2 e^{-y} dy - (1)^2$$
$$= y^2(-e^{-y})\Big|_0^\infty + 2\left[\int_0^\infty y e^{-y} dy\right] - 1.$$
$$= 0 + 2 - 1 = 1.$$

$E[X] = \frac{1}{\lambda}$

$Var(X) = \frac{1}{\lambda^2}$

since $X = \frac{Y}{\lambda}$
and $E[Y] = 1$
$Var[Y] = 1$

∴ using properties of Expectation and Variance, we get the result.

Memoryless property.

$P(X \geq t+s / \underbrace{X \geq s}_{\text{we have like waited } s \text{ minutes}}) = P(X \geq t)$.

$P(X \geq s) = 1 - P(X \leq s)$.
$= e^{-\lambda s}$

$P(X \geq t+s / X \geq s) = \dfrac{P(X \geq t+s, \underbrace{X \geq s}_{\text{redundant}})}{P(X \geq s)}$.

coz if we r waiting for $t+s$, we have already waited for s minute

$= \dfrac{e^{-\lambda(t+s)}}{e^{-\lambda s}}$

$= e^{-\lambda t}$

$= P(X \geq t)$

$\Rightarrow P(X \geq s+t / X \geq s) = P(X \geq t)$.

Standard Normal

$Z \sim N(0,1)$.

$-Z \sim N(0,1)$. (Symmetry).

$E[Z] = 0$

$E[Z^2] = 1$

$E[Z^k] = \begin{cases} 0 & k = \text{odd} \\ 1 & k = \text{even} \end{cases}$

let $X = \mu + \sigma Z$, $\mu \in \mathbb{R}$ (mean)
$\sigma > 0$ (S.D.) $\Rightarrow$ This is the scale.

then $X \sim N(\mu, \sigma^2)$.

$E[X] = E[\mu + \sigma Z]$
$= E\mu + \sigma E[Z]$
$= \mu$

$Var(\mu + \sigma Z) =$

Note

$Var[X+Y] \neq Var[X] + Var[Y]$.

if X, Y are independent

$Var[X+Y] = Var[X] + Var[Y]$.

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$Var(\mu+\sigma Z) = Var(\sigma Z)$
$= \sigma^2 Var[Z]$.

or $Z = \frac{X-\mu}{\sigma}$. [standardization].

Find PDF of $N(\mu, \sigma^2)$.

CDF: $P(X \le x) = P\left(\frac{X-\mu}{\sigma} \le x\right)$.
$= P\left(\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right)$.
$= \Phi\left(\frac{x-\mu}{\sigma}\right)$.
$= F\left(\frac{x-\mu}{\sigma}\right)$

$f\left(x = \frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma}}$

$f(x) = \frac{dF(x)}{dx}$.

$-X = -\mu + \sigma(-Z) \sim N(-\mu, \sigma^2)$.

---

if $X_j \sim N(\mu_j, \sigma_j^2)$ independent
then $X_1 + X_2 \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$
$X_1 - X_2 \sim N(\mu_1-\mu_2, \sigma_1^2+\sigma_2^2)$

Rule of thumb => We cannot evaluate $F(x)$ for normal.

$P(|X-\mu| \le \sigma) \approx 0.68$
$P(|X-\mu| \le 2\sigma) \approx 0.95$
$P(|X-\mu| \le 3\sigma) \approx 0.97$.

$Var[X]$ if $X \sim pois(\lambda)$.

$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{\lambda}$

Diff. b/s

$\sum_{k=0}^{\infty} \frac{k\lambda^{k-1}}{k!} = e^{\lambda}$

Mult. b/s by λ

$\sum_{k=0}^{\infty} \frac{k \times \lambda^k}{k!} = \lambda e^{\lambda}$

Diff. b/s

$\sum_{k=1}^{\infty} \frac{k^2 \lambda^{k-1}}{k!} = \lambda e^{\lambda} + e^{\lambda}$
$= e^{\lambda}(\lambda+1)$.

$Var[X] = E[X^2] - (E[X])^2$

$= \sum_{k=0}^{\infty} k^2 \frac{e^{-\lambda}\lambda^k}{k!} - \lambda^2$

$= \cancel{\lambda^2} + \lambda - \cancel{\lambda^2}$

$= \lambda$.

$$\sum_{k=0}^{\infty} \frac{k^2 \lambda^k}{k!} = \lambda e^{\lambda}(\lambda+1)$$

$$\sum_{k=0}^{\infty} \frac{k^2 e^{-\lambda} \lambda^k}{k!} = \lambda e^{-\lambda} \times e^{\lambda}(\lambda+1)$$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{k^2 e^{-\lambda} \lambda^k}{k!} = \lambda^2 + \lambda.$$

Strong Law of large numbers

Assume we have $X_1, X_2, X_3 \dots X_j$ iid samples with mean μ and Var σ^2.

$$X_n = \frac{\sum_{i=1}^{n} X_i}{n}$$

SLLN says as $n \to \infty$

$X_n \to \mu$

with prob 1.

E.g. $X_j \sim \text{Bern}(p)$

$$\frac{X_1 + X_2 \dots X_n}{n} \to p \text{ with prob 1.}$$

E.g.

Application: Coin toss.

say biased coin (p) don't know p

If we toss the coin many times, we can even estimate p.

Weak law of large number (WLLN)

For any $c > 0$ $P(|\bar{X} - \mu| > c) \to 0$ as $n \to \infty$.

Proof: Using Chebyshev's inequality

$$P(|\bar{X}_n - \mu| > c) \le \frac{\text{Var}(\bar{X}_n)}{c^2}$$

$$= \frac{\frac{1}{n^2} n\sigma^2}{c^2}$$

$$= \frac{\sigma^2}{c^2 n}$$

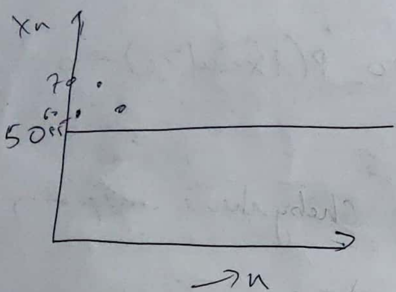
$$= 0 \quad n \to \infty.$$

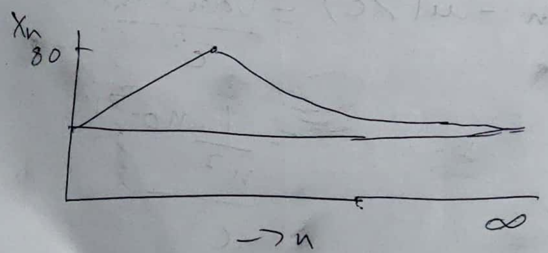
X = # of heads in 100 tosses

$$X_n = \frac{60 + 70 + 55 + \dots X_n}{n}$$

as $n \to \infty$ $X_n \to 50$.

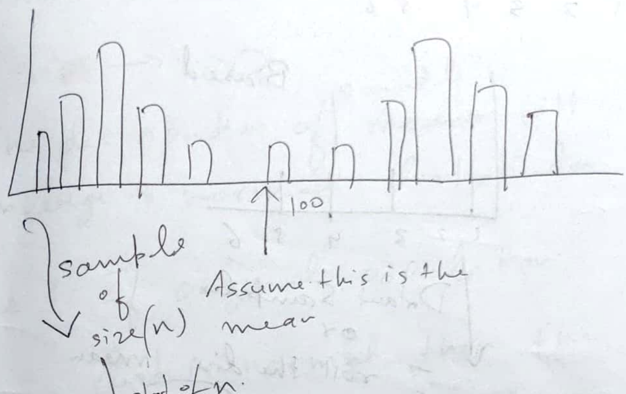
$E[X] \equiv 50$

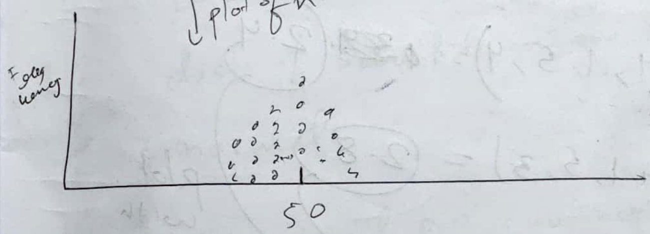


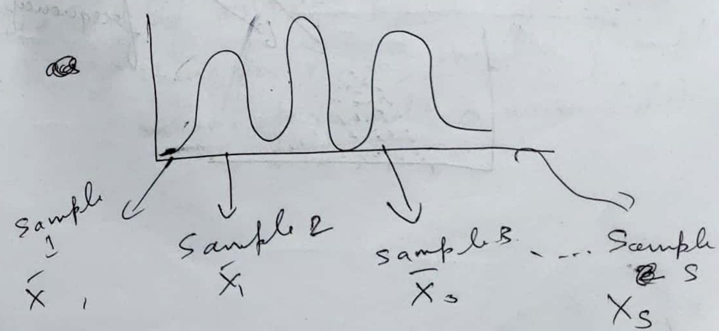


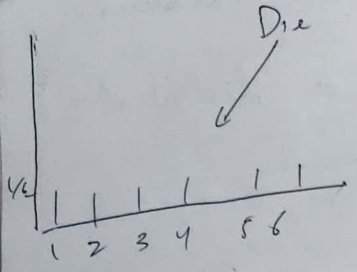
Central limit theorem.

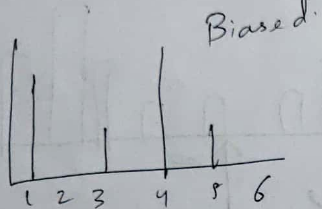
Say we have a distribution of data as







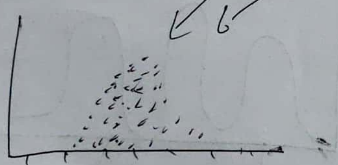




Draw Samples
or
I roll the die mean

(1, 1, 1, 5, 4) = ~~3~~ 2.4
⋮
(1, 4, 1, 5, 3) = 2.8
⋮

plot
with
frequency



As the number of samples increase
the plot will approximate the
↳ frequency
normal distribution.

$S \to \infty$
the distribution of sample means will
converge to normal distribution.

Note. I: If original population
dist is normal, then the
dist. of sample mean is
normal.

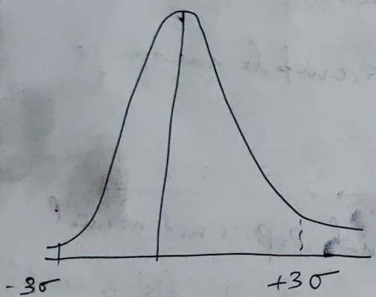
Note 2: If original pop is not normal,
then for $n > 30$, the dist. of sample
means approximate a normal dist.

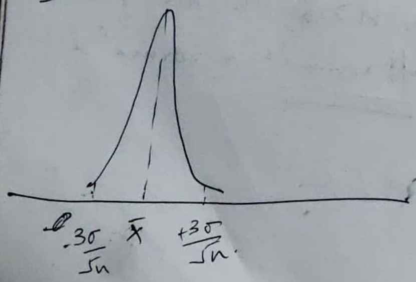
$\bar{X}_1, \bar{X}_2, \bar{X}_3 \dots \bar{X}_{z}$

$$\bar{x} = \frac{\bar{x}_1 + \bar{x}_2 \dots \bar{X}_z}{n}$$

$\bar{x} \approx \mu$.

$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$ → sample size.



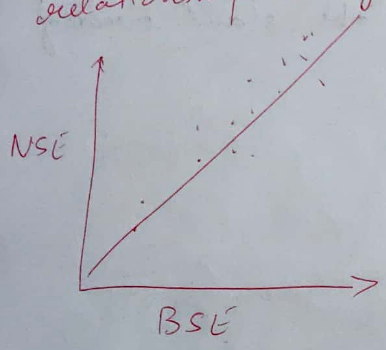


$\frac{\sigma}{\sqrt{n}}$ is the std. dev of the samples distribution of the sample ~~the~~ mean

Also called as the stand. error of means.

Covariance and correlation

→ They form a part of bivariate relationship analysis.



→ Look at the data points
↳ pattern looks like linear relationship

→ When one rises, other rises in the same way.

→ In this case, relationship is strong.

In other words, there is a strong positive relationship.

We can quantify this relationship via covariance i.e how they co-vary.

Note: Covariance is only one measure to quantify this idea.

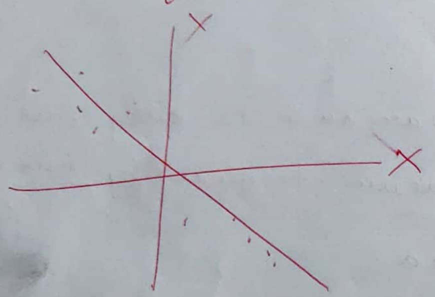
Others include: correlation

Covariance: A descriptive measure of the relationship b/w two variables.

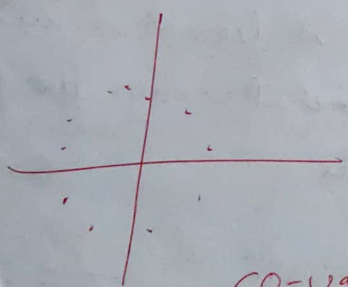
Emphasis is on direction.

+ve indicates increasing R/P
-ve " decreasing R/P

Note Covariance is used to identify linear relationship



-ve covariance



CO-variance = 0

$$\mathrm{cov}(X,Y) = E[(X - E[X])(Y - E[Y])]$$

Co-variance:

$$S_{xy} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{n-1}$$

for sample

$$\sigma_{x,y} = \frac{\sum_{i=1}^{n}(x_i - \cancel{\mu_x})(y_i - \mu_y)}{n}$$

For population

Example: x : workers — 10 random samples from new building

y : no. of walls painted

$\bar{x} = 21.3$ $\bar{y} = 41.2$

$x\ (s=6.48)$	$y\ (s=16.69)$	$x_i - \bar{x}$	$y - \bar{y}$	$\sum(x_i-\bar{x})(y_i-\bar{y})$
12	20	-9.3	-21.2	197.16
30	60	8.7	18.8	
15	27	-6.3	-14.2	
24	50	2.7	8.8	
14	21	-7.3	-20.2	
18	30	-3.3	-11.2	
28	61	6.7	19.8	
26	54	4.7	12.8	
19	32	-2.3	-9.2	
27	57	5.7	15.8	

$\Sigma = 962.4$

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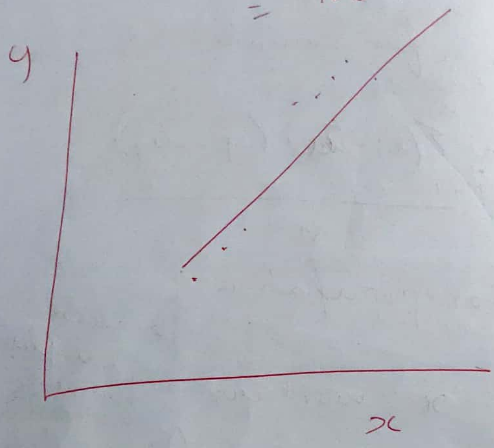
$$Cov(x,y) = S_{xy} = \frac{962.4}{10-1}$$

$$= 106.93$$



Positive linear relationship.

Focus only on sign

Correlation

Covariance provide the direction

Correlation provide direction and strength.

Covariance → No upper and lower bound.

Correlation → Between (-1, 1).

Covariance is dependent on scale

Correlation is independent of scale.

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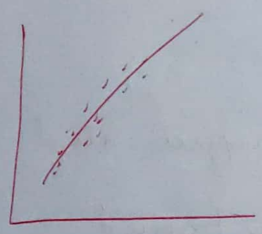
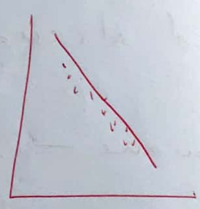
Drawback: Covariance and correlation are applicable to linear relationships.



Near +1



Close to -1

Close to 0.

Correlation Coefficient
(r)

$$r_{xy} \text{ (or } \rho_{xy}) = \frac{cov(X,Y)}{\sigma_x \sigma_y}$$

$$cov(X,Y) = E[(X-\mu)(Y-\mu)]$$

From wikipedia -

If (x_i, y_i) (for $i = 1$ to n) can be realized with probability p_i, then

$$cov(X,Y) = \sum_{i=1}^{n} p_i \left(x_i - E[X]\right)\left(y_i - E[Y]\right)$$

$$cov(X,a) = 0$$

$$cov(X,X) = var(X)$$

$$cov(X,Y) = cov(Y,X)$$

$$cov(aX,bY) = ab\, cov(X,Y)$$

$$cov(X,Y) = E[XY] - E[X]E[Y]$$

$$\rho_{xy} = \frac{cov(X,Y)}{\sigma_x \sigma_y}$$

$$= \frac{106.93}{6.48 \times 16.69} = 0.989$$

Rule of thumb

if $|r| \geq \frac{2}{\sqrt{n}}$

then relation exists

Not Universal

Important point:

Correlation does not imply causation.

$X = x_1 \; x_2 \; x_3$

$cov(X) \rightarrow$ covariance matrix

	X_1	X_2	X_3
X_1	$var(X_1)$	$cov(X_1, X_2)$	$cov(X_1, X_3)$
X_2	$cov(X_2, X_1)$	$Var(X_1)$	$cov(X_2, X_3)$
X_3	$cov(X_3, X_1)$	$cov(X_3, X_2)$	$Var(X_1)$

Suppose we have a random variable $X \sim N(\mu, \sigma^2)$.

To convert this normal distribution into standard normal, we follow the standardizing procedure. i.e.

$$Z = \frac{X-\mu}{\sigma}$$

Here $Z \sim N(0,1)$.

E.g. $X \sim N(162.2, \sigma = 6.8)$

Say what is the prob. that an average girl is taller than 170.5

$P(X > 170.5)$

$$P\left(\frac{X-162.5}{6.8} > \frac{170-162.5}{6.8}\right)$$

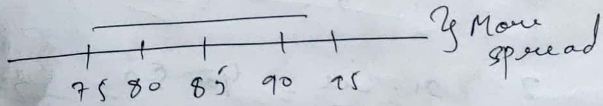
or, $P(Z > 1.22)$.

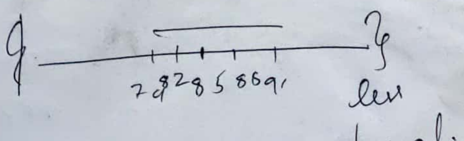
Discussion on reading z-table

z-score.

Suppose, we have the following data

85	88
95	79
75	91
80	85
90	82
$\mu = 85$	$\mu = 85$





Z-score is a statistical measure that describes how far a given data point is from the mean.
↳ It describes the distance in terms of standard deviation.

Q) Suppose a data sample has mean 50 and σ of 5. A value of 55 will have z-score of 1. and 40 will have z-score of -2

$$z = \frac{x-\mu}{\sigma}$$

$$= \frac{55-50}{5} = 1$$

$$z = \frac{x-\mu}{\sigma} = \frac{40-50}{5} = -2.$$

Q) Suppose in a test a person scored ~~500~~ 630 marks. Mean marks is 500, sd is 150.

$$z = \frac{x-\mu}{\sigma} = 630$$

$$= \frac{630-500}{150} = 0.83$$

Q) ~~US~~ ICSE board a person ~~had~~ got 87% with others having an average 80% with S.D. of 5.

CBSE board a person got 82% with class avg. 73 and s.d. of 6

$$z_1 = \frac{87-80}{5} \quad ; \quad z_2 = \frac{82-73}{6}$$

$$z_1 = 1.4 \quad ; \quad z_2 = 1.5$$

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Suppose IQ of people follow normal distribution with $\mu = 100$ and $\sigma = 15$

What % of people are i) stupid
ii) smart
iii) average

stupid: 50 or less

smart: 140 or more

avg: 50-140

Stupid $= \dfrac{50-100}{15} = -3.333$

~~.005~~ % 0.004

= ~~0.5~~ % 0.4%

smart: $\dfrac{140-100}{15} = \dfrac{40}{15} = 2.66$

$= .9961$

$1 - 0.9961$

.0039

0.4%

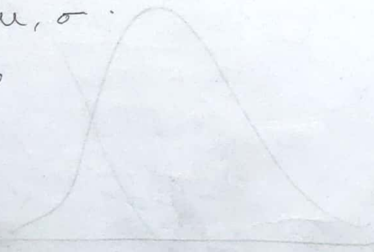
avg = 99.2%

Inferential Statistics: We draw statistical inference when complete information is not available.

Population μ, σ

sample

$\bar{x}, s$



Goal is maintain 32 points all time time (for students)

Student	Points
1	31.93
2	31.24
3	31.53
4	31.45
5	30.93
6	34.66
7	33.55
8	29.79
9	31.82
10	32.27
11	32.56
12	33.32
13	32.04
14	32.82
15	31.70

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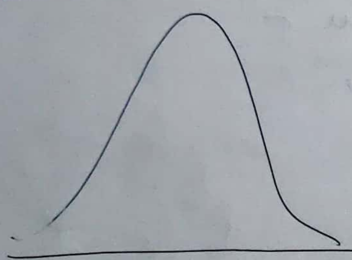
$\bar{x} = 32.10$

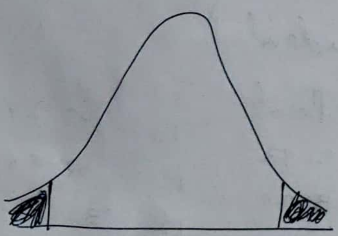
$s = 1.17$

Confidence Interval:

Point estimate $\pm$ margin of error



sampling distribution of mean



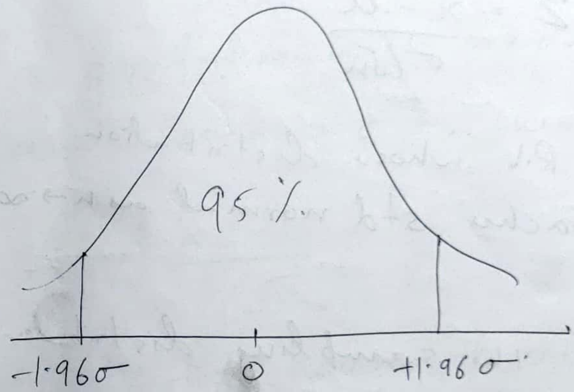
95% probability interval

Point estimate is a single number calculated from sample data for which we have some assurance that it is close to the parameter it is supposed to estimate.

95% of all sample means will be in the unshaded region

α - area in the tails is called as alpha.

α - area of shaded region.



We know standard deviation of the population is $\frac{\sigma}{\sqrt{n}}$ $(\sigma_{\bar{x}})$

So the interval is

$\bar{x} \pm 1.96\,\sigma_{\bar{x}}$. $\left(\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$.

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Sampling Distribution.

1) If $\bar{x}$ is the mean of a random sample of size n taken from a population having mean μ and variance σ^2, then

$$Z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$$

is a R.V. whose distribution approaches std normal as $n \to \infty$.

This is the sampling distribution of the mean with known σ

2) Sampling distribution of mean with unknown σ.

If $\bar{X}$ is the mean of a random sample of size n taken from a population with mean μ and variance σ^2, then

$$t = \frac{\bar{X} - \mu}{S / \sqrt{n}}$$

is the R.V. having t distribution with parameter $V = n - 1$.

here $s = \text{var}(\text{samples})$

$$= \frac{\sum_{i=1}^{n} (x_i - \bar{x})}{n - 1}$$

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3) Sampling distribution of variance.

If S^2 is the variance of a random sample of size n taken from normal population with variance σ^2, then

$$\chi^2 = \frac{(n-1)S^2}{\sigma^2}$$

It has chi-square distribution with $v = n-1$.

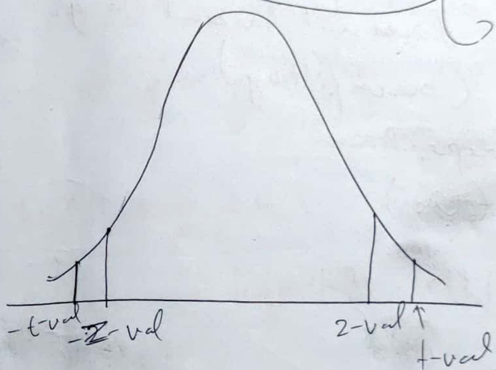
$$\frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{\pi v}\,\Gamma\left(\frac{v}{2}\right)}\left(1+\frac{t^2}{v}\right)^{-\frac{v+1}{2}}$$ = t-distribution

$$\frac{x^{v/2-1}\, e^{-x/2}}{2^{v/2}\,\Gamma\left(\frac{v}{2}\right)},$$ $x > 0$; 0 otherwise

Confidence interval estimation of the variance of the population is not known. Margin of error

$$\bar{x} \pm t\text{-value} \times \frac{s}{\sqrt{n}}$$

std. error.



At 95% confidence
z-value = ±1.96
t-value

16	9	± 2.262
30	29	± 2.045
75	74	± 1.993
100	99	± 1.984

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Hypothesis Testing:

Example: Railway IRCTC Neer bottled water 1ltr.

IRCTC does not want to upset its customer (underfilling)

IRCTC does not want to go bankrupt (overfilling).

Two perspectives

Customer (Us).

Vol $\geq$ 1ltr

IRCTC.

Vol = 1ltr

Us: Are we getting (on an average) atleast 1ltr of water.

IRCTC: Are we supplying exactly 1ltr of water

∴ lets ~~the~~ test this using hypothesis

Definition: A supposition or proposed explanation made on the basis of limited evidence as a starting point for further investigation.

Null hypothesis

Alternate hypothesis.

By definition Null and Alternate hypothesis are opposite to each other.

We can start either with null or alternate hypothesis.

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Null
↳ Accepted as true (so test it)
↳ Assumed one.

Alternate
↳ Unknown as true
↳ It is a claim.

Null : H_0

Alternate : H_a.

Logically speaking (common sense)

$H_0 =$	$H_0 \leq$	$H_0 \geq$
$H_a \neq$	$H_a >$	$H_a <$

Some points

We do not reject the null hypothesis

We say the data supports the alternate hypothesis

Examples : IRCTC example

Null $H_0 =$ 1 ltr

$H_a \neq$ 1 ltr

Errors

	Null Hypothesis	
	True	False
Fail of reject	True positive ✓	False Negative × Type II error
Reject	False positive × Type I error	True Negative ✓

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Actual Reality

Decision ↓	$\mu = 1\,ltr$	$\mu \neq 1\,ltr$
Don't reject	correct	Type II error
Reject	Type I error	correct

$H_0: \mu = 1\,ltr$ $\quad H_a: \mu \neq 1\,ltr$

Say, we take 100 bottles

$\bar{X} \approx 0.999\,ltr$

Don't reject

Sample 2: 200 bottles
(ridiculous sample)

$\bar{X} = 900\,ml$

reject the null hypothesis

Say, in case 2, we get a weird sample.

$\bar{X} = 899\,ml$

You will reject the null.

However, if this sample is flawed you rejected the null. The reality however is opposite. Bottles are filled correctly.

$\Downarrow$

Classic Type I error.

$\Downarrow$

Sample had the problem.

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Say the average IQ of a population is 100 with std deviation of 15. We now believe the value has changed. We took a sample of 75 people and avg IQ was 105. Do you think the IQ has changed?

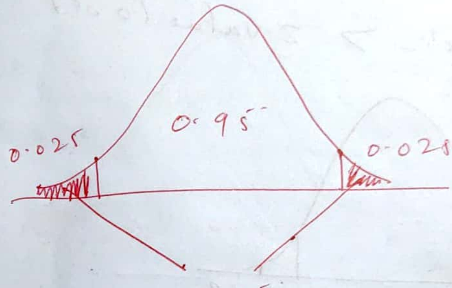
Steps:

1) Null and Alternate hypothesis
2) Choose significance level (α)
3) Find critical values
4) Find test statistic
5) Conclusion

$H_0: \mu = 100$

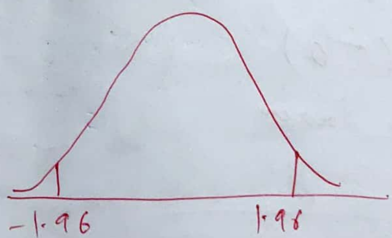
$H_a: \mu \neq 100$

It is a two tailed test.



$\sigma = 15$

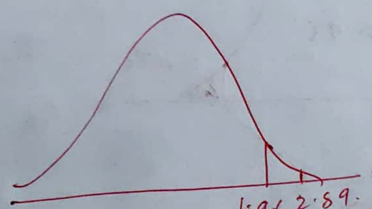
Z-value(0.05) = 1.96.



Test statistic

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{105 - 100}{15/\sqrt{75}} = 2.89.$$

Test statistic > z-value (0.05)

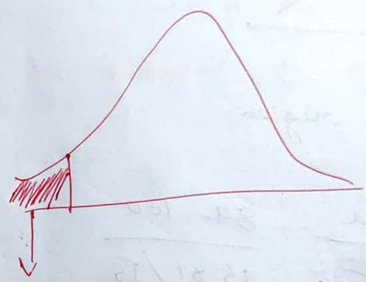


⇒ Conclusion : The data supports the alternate hypothesis

Q2. $\mu = 100$ (IQ)
IQ is now lower
Sample IQ
69, 79, 89, 99, 109.
s.d = 15.81.

1) $H_0 : \mu = 100$ (1 tailed test)
$H_a : \mu < 100$

2) $\alpha = 0.05$ (suppose)



We have chosen this side as we are testing for lower values. i.e lower than the mean.

3) This time we don't know population mean
⇒ We will use t-distribution thus the t-test.

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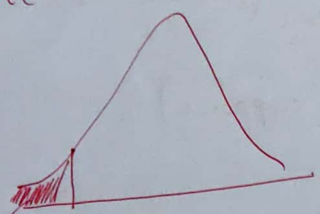
$t(0.05, 4) = 2.132$

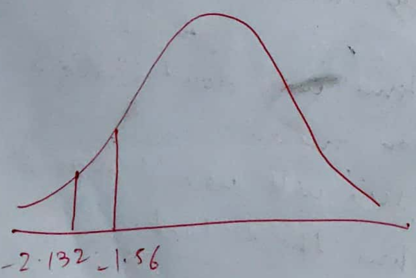


-2.132

→ rejection region.

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{89 - 100}{15.81/\sqrt{5}}$$

$$= -1.56$$



~~The~~

Conclusion: The data supports the null hypothesis.

Say we took a survey of one of the subjects in 2^{nd} year.

1 - 2 - - - 5

poor excellent

previous study said

$\mu = 3$

$\sigma = 1.5$

say we want to test the teachers capability now.

Sample size = 225

$\bar{x} = 3.25$

1) $H_0: \mu \leq 3$

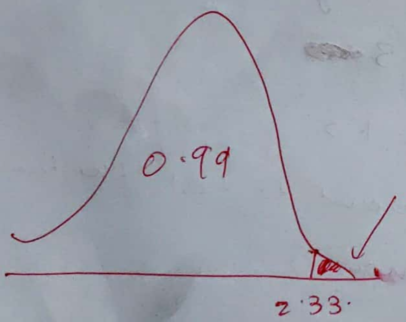
$H_a: \mu > 3$

(We are interested in better than average making of leaders)

2) Confidence level now 99%

$\Rightarrow \alpha = 0.01$

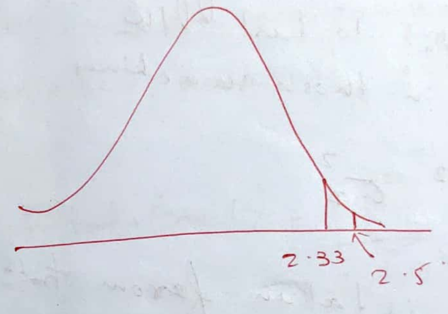
$z(\alpha) = 2.33$.



3) Test - statistic

$$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{3.25 - 3}{1.5/\sqrt{225}}$$

$$= 2.5$$



test statistic is in the rejection region.

$\Rightarrow$ The data supports the alternate hypothesis.

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Analysis of variance

F-test

Say we have a bottle filling machine (5ltr).

Say, we have two different machines and we want to test the variation in those machines

i.e. $\sigma_1^2 = \sigma_2^2$

A sample is taken from both the machines

S_1

$n = 25$, $\bar{x} = 5.0392$; $s^2 = 0.1130$, $s = 0.338$

$n = 22$; $\bar{x} = 4.9808$ $s^2 = 0.0137$, $s = 0.1171$

Clearly, there is diff is std dev.

However we are dealing with samples, hence we have to test them.

Here, we are comparing two sample variances, hence we use F-test

$$\text{F-ratio (or F)} = \frac{s_1^2}{s_2^2}$$

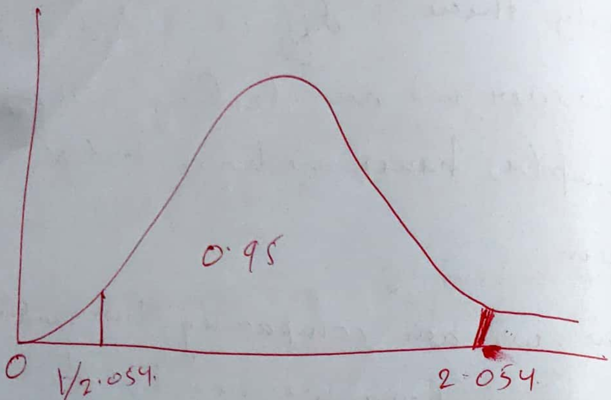
S_1 = larger sample variance

S_2 = smaller "

N_r has $n_1 - 1$ degree of freedom

D_r " $n_2 - 1$ " " "

$df_1 = 24$; $df_2 = 21$



$Ho: \sigma_1^2 = \sigma_2^2$

$Ha: \sigma_1^2 \neq \sigma_2^2$

$F = \frac{0.1130}{0.0137} = 8.248$

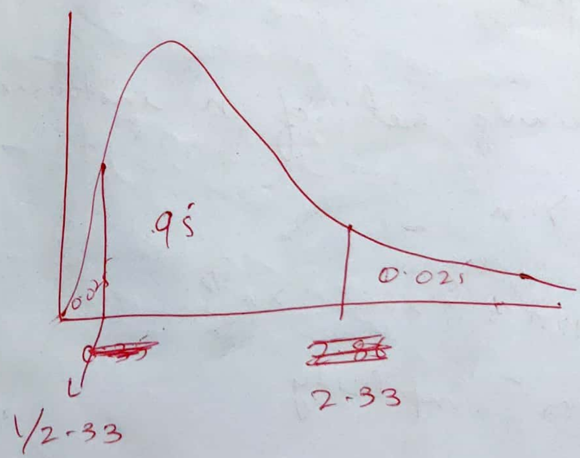
One more example

A	B	
21	16	n
2.5	1.8	σ
100	95	$\bar{x}$

$Ho = \sigma_1^2 = \sigma_2^2$

$Ha = \sigma_1^2 \neq \sigma_2^2$

Two tailed



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$F = \frac{6.25}{3.24} = 1.92$

F falls in the acceptance region.

Linear Regression.

→ Supervised learning

→ Assumption: There is a linear relationship between input & output.

Training set of n instances

$X_{1:n}, Y_{1:n}$

$X_{1:n} = \{x_1 \dots x_n\}$.

For simplicity

$x_i \in R$; $y_i \in R$.

Wind	People inside BU	Energy
		1
100	2	5
50	42	25
45	31	22
60	35	18

In general

$x_i \in R^d$

1 Training

Data

$[X_{1:n}, Y_{1:n}] \longrightarrow \boxed{\text{Learner}}$

↓

Model parameters

(θ)

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(2) Testing

Prediction

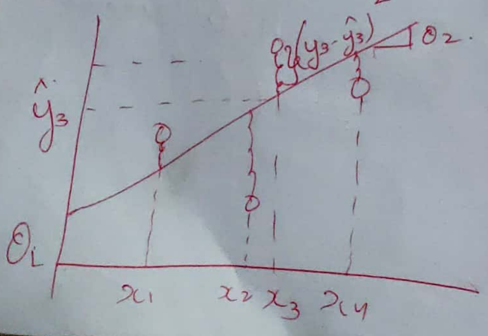
$x_{n+1}, \theta \longrightarrow$ Prediction Engine $\longrightarrow \hat{Y}_{n+1}$

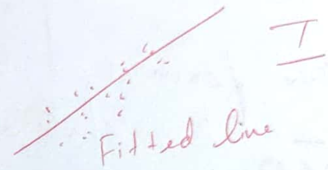
For simplicity.

$$y(x_i) = \theta_1 + x_i\theta_2$$

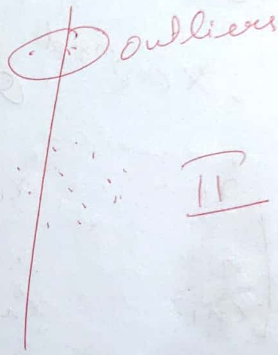
$$\underbrace{J(\theta) = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2}_{\text{Objective function}}$$

$$= \sum_{i=1}^{n}(y_i - \theta_1 - x_i\theta_2)^2$$





However, outliers



The outliers in case II destroyed the fitted line

Multiple Regression (generalized)

$$\hat{y}_i = \sum_{j=1}^{d} x_{ij}\theta_j$$

$$= x_{i1}\theta_1 + x_{i2}\theta_2 + x_{i3}\theta_3 \cdots$$

$$\cdots \theta_{d+1}$$

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θ_{d+1} is the bias

In matrix form.

$$\hat{y} = \bar{X}\theta$$

$$\hat{y} = R^{n\times 1} \;;\; X \in R^{n\times d} \quad \theta \in R^{d\times 1}$$

$$\begin{bmatrix} \hat{y}_1 \\ \vdots \\ \hat{y}_n \end{bmatrix} = \begin{bmatrix} x_{11} & \cdots & x_{1d} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nd} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_n \end{bmatrix}$$

For the BU example

$$y = \begin{bmatrix} 5 \\ 25 \\ 22 \\ 18 \end{bmatrix} \quad X = \begin{bmatrix} 1 & 100 & 2 \\ 1 & 50 & 42 \\ 1 & 45 & 31 \\ 1 & 60 & 35 \end{bmatrix}$$

$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

Hypothetically,

$$\theta = \begin{bmatrix} 1 & 0 & 0.5 \end{bmatrix}^T$$

$$\Rightarrow Y = X\theta$$

$$= \begin{bmatrix} 1 & 100 & 2 \\ 1 & 50 & 42 \\ 1 & 45 & 31 \\ 1 & 60 & 35 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 22 \\ 16.5 \\ 18.5 \end{bmatrix}$$

Say $x = [50 \quad 20]$

$$\hat{y}(x) = [1 \quad 50 \quad 20] \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix}$$

$$= 1 + 0 + 10 = 11$$

Find the parameters.

$$J(\theta) = \sum_{i=1}^{n} (y_i - \hat{y})^2$$

Partially differentiate $J(\theta)$ w.r.t. ~~b~~ all parameters

$$\frac{\partial J(\theta)}{\partial \theta_1} = g_1(\cdot)$$

$$\frac{\partial J(\theta)}{\partial \theta_2} = g_2(\cdot).$$

You will normal equations.

You then solve the normal equations.

Using matrix notation.

$$J(\theta) = (Y - X\theta)^T (Y - X\theta).$$

$$\frac{\partial J(\theta)}{\partial \theta} = 0$$

$$\Rightarrow \theta = (X^T X)^{-1} X^T Y.$$

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Polynomial regression.

$$y = \theta_0 + \theta_1 x + \theta_2 x^2 + \theta_3 x^3 + \epsilon$$

$$y_i = \theta_0 + \theta_1 x_i + \theta_2 x_i^2 + \theta_3 x_i^3 \dots$$

$$\dots \theta_m x_i^m$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^m \\ 1 & x_2 & x_2^2 & \dots & x_2^m \\ \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^m \end{bmatrix} X$$

$$[\theta_0 \;\; \theta_1 \;\; \theta_2 \;\dots\; \theta_m]^T.$$

$$Y = \theta^T X$$

$$\hat{\theta} = (X^T X)^{-1} X^T Y.$$

$$m < n.$$

Maximum likelihood:

Suppose $X \sim N(\mu, \sigma^2)$

$$\| \; \mu + N(0, \sigma^2).$$

Suppose we have 3 data points

$y_1 = 1$

$y_2 = 0.5$

$y_3 = 1.5$

Assume, all y_i's are iid and follow gaussian distribution.

$y \sim N(\theta, 1) = \theta + N(0, 1)$.

Find $\hat{\theta}$

Likelihood : It is a function of the parameters of a statistical model given some data.

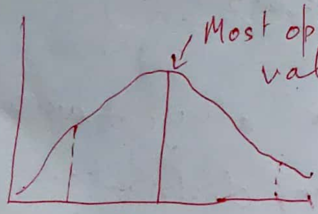
Let $X_1, X_2 \ldots X_n$ be random sample from PMF/PDF

$f(x|\theta)$.

and all X_i's are i.i.d.

Likelihood function

$L(\theta | x_1 \ldots x_n)$

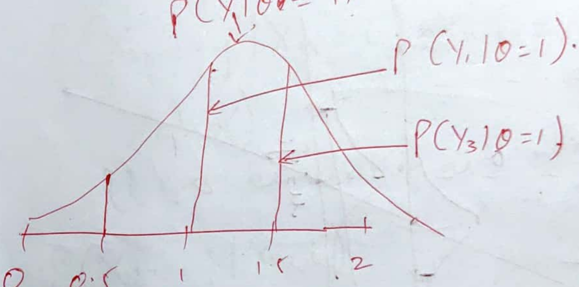


$L(\theta | x_1 \ldots x_n) = \prod_{i=1}^{n} f(x_i|\theta)$.

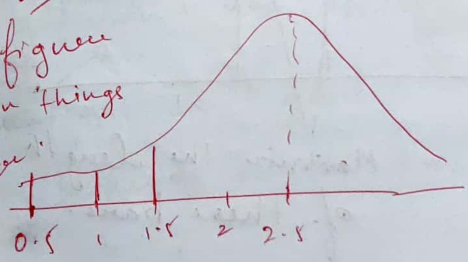
Going back to the example

$y_1 = 1 \quad y_2 = 0.5 \; ; \; y_3 = 1.5$

Guess $\theta = 1, 2.5$



This figure explain things better.

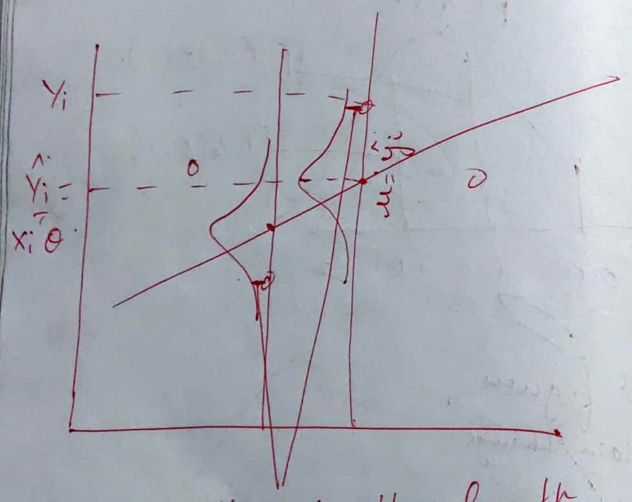


Assumption linear regression

$$y_i \sim N(x^T\theta, \sigma^2)$$

$$= x^T\theta + N(0, \sigma^2)$$

$$\cancel{p(Y|X,\theta,\sigma) = \prod_{i=1}^{n} p(y_i|x_i,\theta,\sigma)}$$



Maximize the length of these bars.

Markov Chains

$X_0 \dots X_n$: state of a system.

Suppose X_n is now, we want to predict X_{n+1}

$$P(X_{n+1}=j \mid X_n = i, X_{n-1}=z, X_{n-2}=a, \dots X_0=\alpha)$$

|| can be written as

$$P(X_{n+1}=j \mid X_n=i)$$

All old values are obsolete

Assumption: Is it good?

Depends on the problem.

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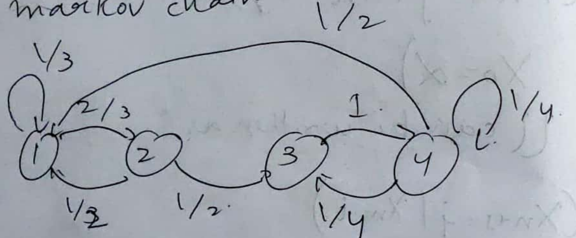
$$\underbrace{P(X_{n+1}=j \mid X_n = i) = q_{ij}}$$

⇓

Transition probability.

If q_{ij} is constant with time, the chain is called as homogeneous markov chain.



TM =

$$\begin{array}{c|cccc} Q & 1 & 2 & 3 & 4 \\ \hline 1 & 1/3 & 2/3 & 0 & 0 \\ 2 & 1/2 & 0 & 1/2 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 1/2 & 0 & 1/4 & 1/4 \end{array}$$

Note each row of Q sum to 1.

$q_{ij} \geq 0$.

$\forall j \sum q_{ij} = 1$.

First order Markov chain → Depends upon the current state.

⇓

Generalize to n^{th} order.

$$P(X_i \mid X_{i-1}, X_{i-2} \dots X_0) = P(X_i \mid X_{i-1} \dots X_{i-n})$$

Note: There is a debate on first order markov chain.

⇓

Applicable in real world?

Question: What's the probability ~~from~~ of going i to j in 2 steps?
or 1 step
or n steps.

We use Q to answer these questions

Current time : n.
" states: X_n.
Total states : m.

~~Xn~~ Distribution of $X_n = S$

$$S = \begin{array}{ccc} X_1 & \cdots & X_m \\ \left[\; \right. & & \left. \; \right] \end{array}$$

$$\begin{array}{ccc} X_1 & & X_m \\ \left[\; \right. & & \left. \; \right] \end{array}$$

$P(X_{n+1} = j) = ?$

Using LTP,

$$P(X_{n+1} = j) = \sum_i P(X_{n+1} = j \mid X_n = i)\, P(X_n = i)$$

$$= \sum_i q_{ij}\, s_i$$

$$= \sum_i s_i\, q_{ij}$$

$SQ \leftarrow$ i,j^{th} entry in SQ.

Or

~~for~~ $\forall j\ P(X_{n+1} = j) = SQ$.

Generalizing,

SQ^2 is the distribution at time $n+2$

SQ^k is the distribution at time $n+k$.

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Similarly,

obselete

$P(X_{n+1}=j \mid X_n=i) = q_{ij}$

$P(X_{n+2}=j \mid X_n=i) = \sum_k P(X_{n+2}=j \mid X_{n+1}=k, X_n=i)$

$\times P(X_{n+1}=k \mid X_n=i)$

$= \sum_k q_{kj} \times q_{ik}$

$\downarrow$ i,j entry in Q^2.

To generalize,

$P(X_{n+m}=j \mid X_n=i) \rightarrow i,j^{th}$ entry in Q^m.

Stationarity : or stationary distribution

$\Downarrow$

Idea

$\Downarrow$

You run something for a long time. Will the chain converge?

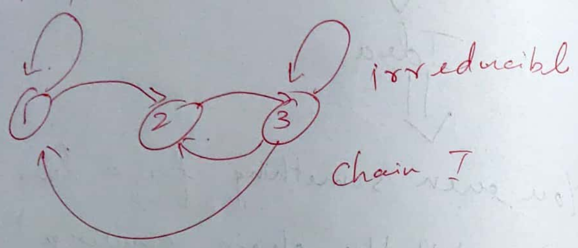
If $\underbrace{SQ}_{\text{Distribution 1 step ahead in time}} = S$, then S is stationary.

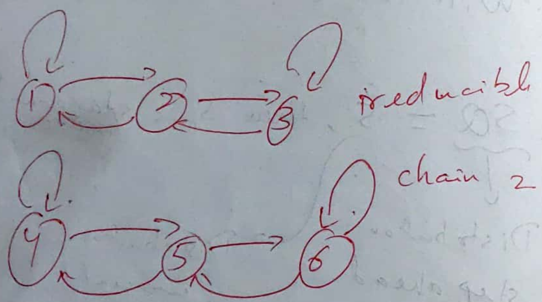
$S \rightarrow$ Distribution current!!

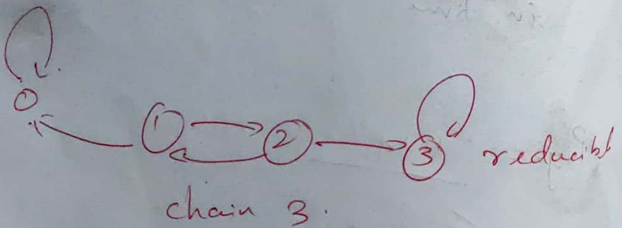
$$P_{ij} = \frac{n_{ij}}{\sum_{\forall k} n_{ik}}$$

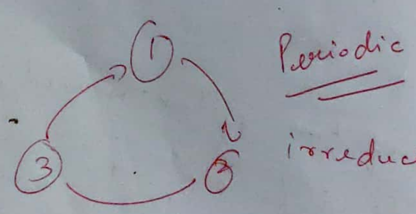
n_{ij} = no of time system moved from state "i to state j"

Markov chains: Similar to memoryless property.





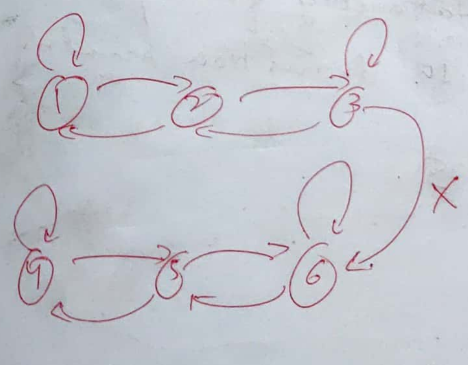




Chain is irreducible if you can get to any state from any state in finite steps with +ve probability.

A state is recurrent if ~~if~~ you start from a particular, then with ~~for~~ prob. 1, we will revisit that state.

Otherwise, the state is transient.



In the figure, 1,2,3 are recurrent if the system does not take X.

If X is taken, 1,2,3 will become

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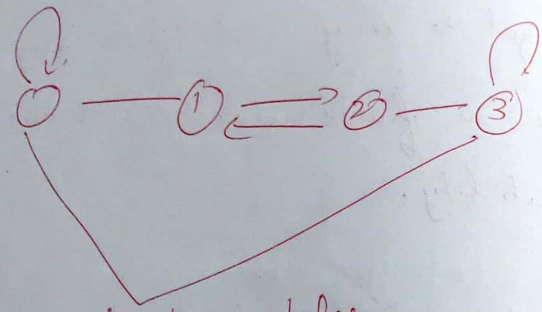
~~[illegible]~~ transient.



absorbing states

This example is similar to two people gambling.

s is stationary for a particular chain with transition matrix Q if $sQ = s$.

Note I:

→ For any irreducible Markov chain with k states, there is always a stationary distribution.

→ The stationary distribution is unique.

→ $s_i = \frac{1}{r_i}$

where, r_i is the avg time to return i starting from i.

$P(X_n = i) \to s_i$

as $n \to \infty$

and

$tQ^n \to s$ — It can be randomly chosen distribution.

as $n \to \infty$

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Reversible Markov chains (w.r.t. time)

Markov chain with TM Q is reversible, if there is a S such that

$$s_i q_{ij} = s_j q_{ji} \quad \text{— (A)}$$

$\forall i, j$

If (A) holds, then S is stationary.

It is like playing a video. You play it forward or backward it gives ~~the~~ only one kind of picture.

Proof

Let $s_i q_{ij} = s_j q_{ji}$ $\forall i, j$

$$\sum_{\forall i} s_i q_{ij} = \sum_i s_j q_{ji}$$

$$= s_j \sum_{\forall i} q_{ji} \; - 1.$$

$$= s_j$$

$$SQ = S$$

$\Rightarrow$ Reversibility implies stationarity.

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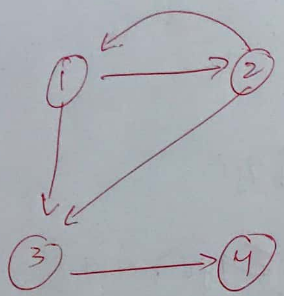
Google Pagerank



we use the same idea

$$s_j = \sum_{\forall i} s_i q_{ij}$$

$$Q = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1/2 & 1/2 & 0 \\ 2 & 1/2 & 0 & 1/2 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 1/4 & 1/4 & 1/4 & 1/4 \end{array}$$

$$G = \alpha Q + (1-\alpha) \frac{J}{M} \quad \text{— (A)}$$

M = no. of pages

$$J = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \vdots & & & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

Eq. (A) says that with prob α you follow the chain and $(1-\alpha)$ you randomly select any other page.

$\alpha = 0.85$

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Eqn. (A) guarantees reducibility.

No zeros in TM.

$tG, tG^2 \ldots tG^n$ → some random distribution

$\underbrace{tG^n}_{\text{Stationary distribution achieved.}}$

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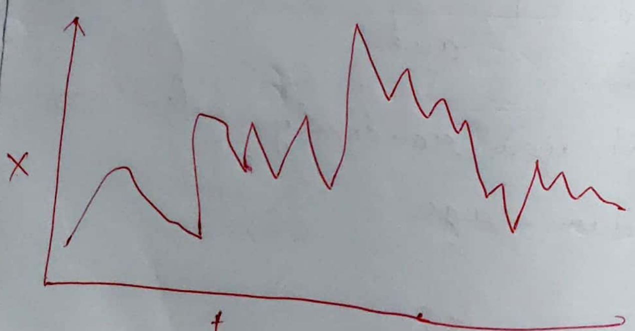
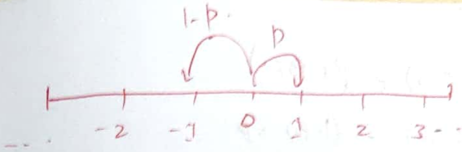
Random Walk

↓

Describes a sequence of successive random steps in some mathematical space.





P for $i = 0, \pm 1, \pm 2 \ldots$

and for number $p \in [0,1]$,

the transition probabilities are given by:

$P_{i,i+1} = p$

$P_{i,i-1} = 1-p$

Suppose a person starts at 0.

i) Calculate the prob. the person ends at same position in two steps.

$A = \{\overset{\text{Forward}}{F}\underset{}{\overset{\text{Backward}}{B}},\ BF\}$

$$P(A) = P(FB) + P(BF)$$
$$= p \times (1-p) + (1-p) \times p$$
$$= 2p(1-p).$$

2) Calculate the prob that the person ends one step forward in exactly three steps

$$A = \{FFB, BFF, FBF\}$$

$$\Rightarrow P(A) = 3p^2(1-p)$$

** $$tG = \alpha \underbrace{tQ}_{\text{Sparse matrix}} + (1-\alpha)\underbrace{(tI)}_{M}.$$

It is always $I^{1\times m}$

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