Tutorial Sheet 3
Rank, Inverse And Determinants

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1. Let A and B be two matrices. Then we have

   (a) if $A + B$ is defined, then $rank(A + B) \leq rank(A) + rank(B)$,

   (b) if AB is defined, then $rank(AB) \leq rank(A)$ and $rank(AB) \leq rank(B)$.

2. If A and B are two $n \times n$ non-singular matrices, are the matrices $A + B$ and $A - B$ non-singular? Justify your answer.
   Solution: The matrices $A + B$ and $A - B$ may or may not be non-singular.
   If $A = I$, $B = I$ then $A + B = 2I$ and $A - B = 0$ which implies $A + B$ is non-singular but $A - B$ is singular.
   If $A = I$, $B = -I$ then $A - B = 2I$ and $A + B = 0$ which implies $A - B$ is non-singular but $A + B$ is singular.

3. Let A be an $n \times n$ matrix. If the system $A^2x = 0$ has a non trivial solution then show that $Ax = 0$ also has a non trivial solution.
   Solution: Suppose the system $A^2x = 0$ has a non trivial solution. Then $det(A^2) = 0$ which also implies that $det(A) = 0$. Thus the system $Ax = 0$ has a non trivial solution.

4. State whether each of the following statements is true or false. In each case give a brief reason.

   a) If A is an arbitrary matrix such that the system of equations $Ax = b$ has a unique solution for $b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ then the system has a unique solution for any three component column vector b.

   b) Any system of 47 homogeneous equations in 19 unknowns whose coefficient matrix has rank greater than or equal to 8 always has at least 11 independent solutions.

   Solution. a) This statement is true . The system $Ax = b$ has a unique solution for the given b if and only if both the coefficient matrix A and augmented matrix $[A|b]$ have rank 3. But then both the coefficient matrix A and the augmented matrix $[A|b]$ have rank 3 for any b.

b) This statement is false . The coefficient matrix could have rank as large as 19. In this event the system has a unique solution, namely $x = 0$.

5. If A is a symmetric matrix, is the matrix A^{-1} symmetric?
   Solution: True. We have $A^t = A$. Therefore $(A^{-1})^t = (A^t)^{-1} = A^{-1}$.

6. . Let A be a 1×2 matrix and B be a 2×1 matrix having positive entries. Which of BA or AB is invertible? Give reasons.
   Solution: AB is invertible as it is 1×1 positive entries and $det(AB) \neq 0$.
   BA may or may not be invertible.
   Example: Let $A = B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ then $BA = A^2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ and $detBA = 0$.
   Let $B = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Then $BA = \begin{pmatrix} 3 & 5 \\ 2 & 3 \end{pmatrix}$ and $detBA \neq 0$

7. Show that a triangular matrix A is invertible if and only if each diagonal entry of A is non-zero.
   Solution: Using $det(A) = a_{11}a_{22} \cdots a_{nn}$, we obtain the result.

8. Let A be an $n \times n$ matrix such that $det(A) = 4$ then what is the $det(5A)$?
   Solution: As we know $|kA| = k^n|A|$. Therefore $det(5A) = 5^n det(A)$.

9. Let A be an $n \times n$ matrix. then $det(adj(A)) = (det(A))^{n-1}$.
   Solution: We know that $Aadj(A) = det(A)I_n$. Take the determinant both sides, we get
   $$det(A) \cdot det(adj(A)) = (det(A))^n det(I_n) = (det(A))^n.$$

10. Let A be an $n \times n$ matrix. Then show that A is invertible $\Leftrightarrow$ $Adj(A)$ is invertible.
    Solution: From above problem, we can obtain the result.

11. Let A be an $n \times n$ matrix. then show that $det(adj(adjA)) = (detA)^{(n-1)^2}$.
    Solution: Replacing A with adj in problem 13, we obtain
    $$det(adj(adjA)) = (det(adjA))^{n-1}$$
    and use $det(adj(A)) = (det(A))^{n-1}$ in above, we obtain
    $$det(adj(adjA)) = (det(A))^{(n-1)^2}.$$

12. Let A and B be invertible matrices. Prove that $Adj(AB) = Adj(B)Adj(A)$.

    Solution: Try yourself!

13. Using the Gauss Jordon Method, Find A^{-1}, whenever exist

    1) $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$, 2) $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$,

    3) $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix}$

    (Hint 3): Inverse donot exists.)