Method of Undetermined Coefficients :

As we have seen that the general solution of

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \text{---} + a_n y = f(x) \quad \text{on } I \qquad \text{——— (1)}$$

can be written as

$$y = y_c + y_p ,$$

where y_c is called the complementary function (or general solution of corresponding homogeneous equation $a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \text{---} + a_n y = 0$).

and y_p is called the particular solution of (1).

Till now, we have learnt how to calculate $y_c(x)$ Complementary function (or solution of corresponding homogeneous DE).

Now, we will learn how to calculate $y_p(x)$ (Particular solution of (1)) using the method of undetermined coefficients in the following particular cases of $f(x)$:

1. $f(x) = k e^{\alpha x}$, $k \neq 0$, α a real constant.
2. $f(x) = e^{\alpha x}(k_1 \cos\beta x + k_2 \sin\beta x)$, $k_1, k_2, \alpha, \beta \in \mathbb{R}$.
3. $f(x) = x^n$

Case-I : $f(x) = k e^{\alpha x}$, $k \neq 0$, α real constant.

If α is not a root of the characteristic equation (auxiliary eq^n) (ie $p(\alpha) \neq 0$).

[The auxiliary equation (characteristic equation) corresponding to ② is given by
$$p(m) = a_0 m^n + a_1 m^{n-1} + \cdots + a_n = 0$$]

Then we assume the particular solution of the form
$$y_p = A e^{\alpha x},$$
where A, an unknown, is an undetermined coefficient.

Since y_p satisfies ①. Therefore by substituting the values of y_p, (y_p') in ①, we get the value of A.

If α is a root of the characteristic equation with multiplicity 'r', (ie $p(\alpha) = p'(\alpha) = \cdots = p^{r-1}(\alpha) = 0$ and $p^{r}(\alpha) \neq 0$),

then we take y_p of the form
$$y_p = A x^r e^{\alpha x}.$$
Then by substituting the value of y_p in ①, we get the value of A.

Example : Find the particular solution of $y'' - 4y = 2e^x$.

Solution: Here $f(x) = 2e^x$, here $\alpha = 1$, $k = 2$.

The characteristic equation is $p(m) = m^2 - 4 = 0 \rightarrow m = \pm 2$.

$\Rightarrow \alpha = 1$ is not a root of $p(m) = 0$.

Thus we assume $y_p = Ae^x$.

Substituting the value of y_p in ①, we get

$$Ae^x - 4Ae^x = 2e^x$$

$$\Rightarrow \quad -3Ae^x = 2e^x.$$

$$\Rightarrow \quad A = \frac{-2}{3}.$$

Thus the particular solution is

$$y_p = -\frac{2}{3}e^x.$$

Example-2: Find the particular solution of

$$y''' - 3y'' + 3y' - y = 2e^x. \qquad \text{———} ①$$

Solution: The auxiliary equation is

$$p(m) = m^3 - 3m + 3m - 1 = 0$$

$$\Rightarrow \quad (m-1)^3 = 0$$

$$\Rightarrow \quad m = 1, 1, 1$$

Here $\alpha = 1$.

Clearly, $p(\alpha) = p(1) = 0$.

$\Rightarrow$ $\alpha = 1$ is a root of the auxiliary equation of multiplicity $r = 3$.

Thus, we assume particular solution of the form

$$y_p = Ax^3e^x.$$

Substituting the value of y_p in ①, we get

$$Ae^x(x^3+9x^2+18x+6) - 3Ae^x(x^3+6x^2+6x)$$
$$+ 3Ae^x(x^3+3x^2) - Ax^3e^x = 2e^x.$$

Solving for A, we get

$A = \frac{1}{3}$, and thus the particular solution is

$$y_p = \frac{x^3 e^x}{3}.$$

Example-3: Find the particular solution of

$$y''' - y' = e^{2x} \qquad \text{——— (1)}$$

and hence obtain the general solution.

Solution: The characteristic (auxiliary) equation is

$$p(m) = m^3 - m = 0$$
$$\Rightarrow \quad m(m^2-1) = 0$$
$$\Rightarrow \quad m = 0, \pm 1.$$
$$\Rightarrow \quad y_c(x) = c_1 + c_2 e^x + c_3 e^{-x}$$

Here $\alpha = 2$; which is not the root of the characteristic equation.

Thus we assume y_p of the form

$$y_p(x) = A e^{2x}.$$

Substituting the value of y_p in (1), we get

$$y_p''' - y_p' = e^{2x}$$
$$\Rightarrow \quad 8Ae^{2x} - 2Ae^{2x} = e^{2x}$$
$$\Rightarrow \quad 6A e^{2x} = e^{2x}$$
$$\Rightarrow \quad 6A = 1$$
$$\Rightarrow \quad A = \frac{1}{6}.$$

Thus the particular solution is

$$y_p = \frac{1}{6} e^{2x}.$$

The general solution of (1) is

$$y = y_c(x) + y_p(x)$$

$$\Rightarrow \boxed{y = c_1 + c_2 e^{x} + c_3 e^{-x} + \frac{1}{6} e^{2x}}$$ Ans.

Case-II If $f(x) = e^{\alpha x}(k_1 \cos(\beta x) + k_2 \sin(\beta x))$; $k_1, k_2, \alpha, \beta \in \mathbb{R}$.

We first assume that $\alpha + i\beta$ is not a root of the characteristic equation i.e. $p(\alpha + i\beta) \neq 0$.

In this case, we assume that y_p is of the form

$$y_p = e^{\alpha x}(A \cos \beta x + B \sin \beta x)$$

Then by substituting the values of y_p in (1), we get the values of A and B.

If $\alpha + i\beta$ is a root of the characteristic equation, i.e., $p(\alpha + i\beta) = 0$, with multiplicity r, then we assume a particular solution as

$$y_p = x^r e^{\alpha x}(A \cos(\beta x) + B \sin(\beta x)),$$

and then by substituting the values of y_p in (1) and by comparing coefficients, we get the values of A and B.

Example: Find the particular solution of

$$y'' + 2y' + 2y = 4e^{x} \sin x \qquad \text{——— (1)}$$

Solution: The characteristic equation is

$$p(m) = m^2 + 2m + 2 = 0$$

$$\Rightarrow \quad m = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$$

Here $\alpha = 1$ and $\beta = 1$.

Thus $\alpha + i\beta = 1 + i$, which is not a root of the characteristic equation $p(m) = m^2 + 2m + 2 = 0$.

Thus, let us assume $y_p = e^{x}(A \sin x + B \cos x)$

Since y_p satisfies the given DE.

Substituting the value of y_p in (1), we get

$$(-4B + 4A)\, e^{x} \sin x + (4B + 4A)\, e^{x} \cos x = 4e^{x} \sin x.$$

Comparing the coefficients of $e^{x}\cos x$ and $e^{x} \sin x$, we get

$$A - B = 1 \quad \text{and} \quad A + B = 0.$$

On solving for A and B, we get $A = -B = \frac{1}{2}$.

So, the particular solution is $\boxed{y_p = \frac{e^{x}}{2}(\sin x - \cos x)}$ Ans.

Example-2 Find a particular solution of

$$y'' + y = \sin x \quad \text{——— (1)}$$

and hence find the general solution.

Solution: Here the characteristic equation is

$$p(m) = m^2 + 1 = 0$$

$$\Rightarrow m = \pm i.$$

$$\therefore y_c(x) = c_1 \cos x + c_2 \sin x.$$

Here $f(x) = \sin x$.

$\Rightarrow \alpha = 0$ and $\beta = 1$. Thus $\alpha + i\beta = i$, which is a root of the characteristic equation with multiplicity $r = 1$;

So, let $y_p = x(A\cos x + B\sin x)$.

Since y_p satisfies the given DE.

$\Rightarrow$ Substituting the values of y_p in (1), we get

$$\boxed{\begin{aligned} y_p' &= x(-A\sin x + B\cos x) \\ &\quad + (A\cos x + B\sin x) \\ \Rightarrow y_p'' &= x(-A\cos x - B\sin x) \\ &\quad + (-A\sin x + B\cos x) \\ &\quad + (-A\sin x + B\cos x) \end{aligned}}$$

$$y_p'' + y_p = \sin x$$

$$\Rightarrow x(-A\cos x - B\sin x) - 2A\sin x + 2B\cos x$$
$$+ x(A\cos x + B\sin x) = \sin x$$

$$\Rightarrow -2A\sin x + 2B\cos x = \sin x$$

$$\Rightarrow -2A = 1 \text{ and } 2B = 0 \quad \left(\text{Comparing the coefficients of } \cos x \text{ and } \sin x\right)$$

$$\Rightarrow \boxed{A = -\frac{1}{2} \text{ and } B = 0}$$

Thus the particular solution is $y_p = -\frac{1}{2}x\cos x$.

Hence the general solution is $\boxed{y = c_1\cos x + c_2\sin x - \frac{1}{2}x\cos x}$ Ans.

Case-III If $f(x) = x^n$.

Suppose $m=0$ is not a root of the characteristic equation $p(m) = 0$, then we assume that

$$y_p = A_n x^n + A_{n-1} x^{n-1} + \text{—} + A_0 .$$

and then by substituting the value of y_p in the given equation we can obtain the values of A_i for $0 \le i \le n$.

If $m=0$ is a root of the characteristic equation, i.e, $p(0) = 0$, with multiplicity r, then we assume a particular solution as

$$y_p = x^r (A_n x^n + A_{n-1} x^{n-1} + \text{—} + A_0)$$

and then (~~compare the~~ coefficients ~~of~~) by substituting the values of y_p in the given equation, we can obtain the value of A_i, for $0 \le i \le n$.

Example: Find the particular solution of

$$y''' - y'' + y' - y = x^2 . \quad \text{———} (1)$$

Solution: The characteristic equation is

$$m^3 - m^2 + m - 1 = 0$$

$$\Rightarrow m^2(m-1) + 1(m-1) = 0$$

$$\Rightarrow (m-1)(m^2+1) = 0$$

$$\Rightarrow m = 1, \pm i .$$

Here $f(x) = x^2$ and $m=0$ is not a root of the characteristic equation. (i.e $p(0) \neq 0$).

Thus, we assume

$$y_p = A_2x^2 + A_1x + A_0.$$

Since y_p satisfies (1). By substituting the value of y_p in (1), we get

$$y_p''' - y_p'' + y_p' - y_p = x^2$$

$$\Rightarrow \quad 0 - 2A_2 + 2A_2x + A_1 - A_2x^2 - A_1x - A_0 = x^2$$

$$\Rightarrow \quad -A_2x^2 + (2A_2 - A_1)x - (2A_2 - A_1 + A_0) = x^2$$

$$\boxed{\begin{aligned} y_p &= A_2x^2 + A_1x + A_0 \\ \Rightarrow y_p' &= 2A_2x + A_1 \\ \Rightarrow y_p'' &= 2A_2 \\ \Rightarrow y_p''' &= 0 \end{aligned}}$$

Comparing the coefficients of x^2, x and constant terms, we get

$-A_2 = 1,$	$2A_2 - A_1 = 0,$	$-2A_2 + A_1 - A_0 = 0$
$\Rightarrow A_2 = -1,$	$A_1 = 2A_2$	$-2(-1) - 2 - A_0 = 0$
	$\Rightarrow A_1 = 2(-1) = -2$	$\Rightarrow A_0 = 0$

$$\Rightarrow \quad A_1 = -2, \quad A_2 = -1, \quad A_0 = 0.$$

Thus the particular solution is

$$\boxed{y_p = -(x^2 + 2x).}$$

Ans.

Note: If y_{p_1} is a particular solution of

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \text{———} + a_n y = f_1(x)$$

and y_{p_2} is a particular solution of

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \text{———} + a_n y = f_2(x).$$

Then the particular solution of

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \text{———} + a_n y = k_1 f_1(x) + k_2 f_2(x)$$

is given by

$$\boxed{y_p = k_1 y_{p_1} + k_2 y_{p_2}}.$$

In view of this, one can use this method of undetermined coefficients for the cases, where $f(x)$ is a linear combination of the functions described above.

Example: Find the particular solution of

$$y'' + y = 2\sin x + \sin 2x. \quad \text{———} (1)$$

Solution: We can divide the problem into two problems:

$$y'' + y = 2\sin x \quad \text{———} (2)$$

and $$y'' + y = \sin 2x. \quad \text{———} (3)$$

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for the first problem, the particular solution is

$$y_{p_1} = -x\cos x$$

and for the second problem, the particular solution is

$$y_{p_2} = -\frac{1}{3}\sin 2x \,.$$

Thus the particular solution of the given problem is

$$y_p(x) = y_{p_1}(x) + y_{p_2}(x)$$

$$\Rightarrow \boxed{y_p(x) = -x\cos x - \frac{1}{3}\sin 2x} \quad \text{Ans.}$$