Ordinary Differential Equations(EMAT102L) (Lecture-3)

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Outline of the Lecture

We will learn

- Equations reducible to Separable Equations
- Homogeneous Equations
- Equations Reducible to Homogeneous Equation

Recall that

Definition

Separable Equation: A first order differential equation of the form

$$\frac{dy}{dx} = g(x)h(y)$$

is called **separable** or to have **separable variables**.

Such ODEs can be solved by direct integration: Write $\frac{dy}{dx} = g(x)h(y)$ as $\frac{dy}{h(y)} = g(x)dx$ and then integrate both sides, we get

$$\int \frac{dy}{h(y)} = \int g(x)dx + c$$

$\Rightarrow H(y) = G(x) + c$, where c is a constant of integration.

Consider the differential equation

$$\frac{dy}{dx} = f(x,y)$$

If $f(x,y)$ **is of the form** $g(ax+by+c)$, then by putting $r = ax+by+c$ we get

$$\frac{dr}{dx} = a + b\frac{dy}{dx} = a + b.g(r)$$

$$\Rightarrow \qquad \frac{dr}{a+bg(r)} = dx$$

On integrating both sides and replacing r in terms of x and y, we get the solution.

Example

Solve $\frac{dy}{dx} = (4x + y + 1)^2$

Solution: Put $r = 4x + y + 1$, then

$$\frac{dr}{dx} = 4 + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{dr}{dx} - 4$$

So, from the given equation, we get

$$\frac{dr}{dx} - 4 = r^2$$

$$\int \frac{dr}{4 + r^2} = \int dx + c \Rightarrow \frac{1}{2}\tan^{-1}\left(\frac{r}{2}\right) = x + c$$

$$\Rightarrow 4x + y + 1 = 2\tan(2x + c_1)$$

where c_1 is an arbitrary constant.

Example

Solve $\dfrac{dy}{dx} = x\tan(y-x)+1$.

Put $y - x = r \Rightarrow \dfrac{dy}{dx} - 1 = \dfrac{dr}{dx}$.
Then from the given equation, we get

$$1 + \frac{dr}{dx} = x\tan r + 1 \Rightarrow \frac{dr}{dx} = x\tan r$$

$$\Rightarrow \frac{dr}{\tan r} = xdx$$

$$\Rightarrow \int \cot r dr = \int xdx + c$$

$$\Rightarrow \log|\sin r| = \frac{x^2}{2} + c$$

$$\Rightarrow \log|\sin(y-x)| = \frac{x^2}{2} + c$$

A class of differential equations can be reduced to separable equations by using change of variables.

Definition

A function $f(x,y)$ is said to be **homogeneous** of degree n if $f(kx,ky) = k^n f(x,y)$ for all (x,y) in the domain and for all $k \in \mathbb{R}$.

Examples

1. $f(x,y) = x^2 + y^2$ is homogeneous of degree 2 as $f(kx,ky) = (kx)^2 + (ky)^2 = k^2(x^2+y^2) = k^2 f(x,y)$
2. $f(x,y) = tan^{-1}(\frac{y}{x})$ is homogeneous of degree 0.
3. $f(x,y) = \dfrac{x(x^2+y^2)}{y^2}$ is homogeneous of degree 1.
4. $f(x,y) = x^2 + xy + 1$ is NOT homogeneous.

Definition

A first order DE of the form

$$M(x,y)dx + N(x,y)dy = 0 \; or \; \frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$$

is said to be **homogeneous** if both $M(x,y)$ and $N(x,y)$ are homogeneous functions of the same degree.
Such equations can be reduced to **separable equations** by transformation

$$y = vx$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting it in the above equation, we obtain,

$$v + x\frac{dv}{dx} = -\frac{M(x,vx)}{N(x,vx)}$$

We can solve this by separable method. Put $v = \dfrac{y}{x}$ to obtain the required solution.

Example

Find the general solution of

$$2xyy' - y^2 + x^2 = 0$$

Solution:

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

Put $y = vx$, then

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v}$$

$$x\frac{dv}{dx} = \frac{v^2 - 1}{2v} - v = \frac{-v^2 - 1}{2v}$$

$$\frac{2v}{v^2 + 1}dv = -\frac{1}{x}dx$$

Example(cont.)

On integrating,

$$\log|v^2+1| = -\log|x| + \log c$$
$$v^2+1 = \frac{c}{x}$$

Put $v = \dfrac{y}{x}$, we get

$$y^2 + x^2 = cx$$

This can be rewritten as

$$\left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4}$$

This represent a family of circles with centre $\left(\dfrac{c}{2}, 0\right)$ and radius $\dfrac{c}{2}$.

Example

Solve $x^2 y dx - (x^3 + y^3) dy = 0$.

Solution: The given differential equation can be rewritten as $\dfrac{dy}{dx} = \dfrac{x^2 y}{x^3 + y^3}$.

Let $y = vx$, then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$.
Putting this in the given equation, we get

$$v + x\frac{dv}{dx} = \frac{v}{1 + v^3}.$$

Or in other words,

$$\left(\frac{1 + v^3}{v^4}\right) dv = -\frac{dx}{x}$$

which is now in separable variables form.

DE reducible to homogeneous DE

For solving differential equation of the form

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

where a_1,a_2, b_1,b_2, c_1,c_2 are constants.

- If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, then use the substitution $x = X + h$ and $y = Y + k$, where h and k are choosen such that

  $$a_1h + b_1k + c_1 = 0$$

  $$a_2h + b_2k + c_2 = 0$$

  This condition changes the given differential equation into homogeneous equation in X and Y.

  $$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$$

  Now consider $Y = VX$ and solve as before.
- If $\frac{a_1}{a_2} = \frac{b_1}{b_2}$, then use the substitution $z = a_1x + b_1y$. This transformation reduces the given DE to a separable equation in the variables x and z.

DE reducible to homogeneous DE(cont.)

Solve
$$\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$$

Solution: Observe that this DE is of the form $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ where $\frac{1}{2} = \frac{a_1}{a_2} \neq \frac{b_1}{b_2} = 2.$

Put $x = X + h$, $y = Y + k$, where h and k are constants to be determined. Then we have $dx = dX$, $dy = dY$ and
$$\frac{dY}{dX} = \frac{X+h+2(Y+k)-3}{2(X+h)+Y+k-3}$$
$$\frac{dY}{dX} = \frac{X+2Y+(h+2k-3)}{2X+Y+(2h+k-3)}$$

Choose h and k such that
$$h+2k-3=0, 2h+k-3=0$$
$$\Rightarrow h=1, k=1$$
$$\Rightarrow x = X+1, y = Y+1$$

So, the given equation becomes
$$\frac{dY}{dX} = \frac{X+2Y}{2X+Y}$$

which is a Homogeneous differential equation.

Example(cont.)

Put $Y = VX$, we get

$$\frac{dY}{dX} = V + X\frac{dV}{dX}$$

$$V + X\frac{dV}{dX} = \frac{1+2V}{2+V} \Rightarrow X\frac{dV}{dX} = \frac{1-V^2}{2+V}$$

Separating the variables, we obtain

$$\frac{dX}{X} = \frac{2+V}{1-V^2}dV$$

$$\Rightarrow \log X = \log\left(\frac{1+V}{1-V}\right) - \frac{1}{2}\log(1-V^2) + \log c$$

$$\log\left(\frac{X}{c}\right) = \log\left(\frac{X+Y}{X-Y}\right) - \log\left(\frac{\sqrt{X^2-Y^2}}{X}\right) = \log\left(\frac{X\sqrt{X+Y}}{(X-Y)^{3/2}}\right)$$

$$\frac{X}{c} = \frac{X\sqrt{X+Y}}{(X-Y)^{3/2}}$$

$$\Rightarrow \frac{X}{c} = \frac{X\sqrt{X+Y}}{(X-Y)^{3/2}}$$

$$\Rightarrow (X-Y)^{3/2} = c\sqrt{X+Y}$$

$$\Rightarrow (x-1-y+1)^{3/2} = c(x-1+y-1)^{1/2}$$

Example

Solve

$$\frac{dy}{dx} = \frac{x+y+4}{x+y-6}$$

Solution: Observe that this DE is of the form $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ where $\frac{a_1}{a_2} = \frac{b_1}{b_2}$.
Use the substitution $x+y=z$. Then we have,

$$1 + \frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{dy}{dx} = \frac{dz}{dx} - 1$$

Substituting the value of $x+y$ and $\frac{dy}{dx}$ in the given equation, we get

$$\Rightarrow \frac{dz}{dx} - 1 = \frac{z+4}{z-6}$$

$$\Rightarrow \frac{dz}{dx} = \frac{2(z-1)}{z-6}$$

$$\Rightarrow 2dx = \frac{z-6}{z-1}dz = \left(1 - \frac{5}{z-1}\right)dz$$

Example(cont.)

$$\Rightarrow 2x = z - 5\log(z-1) + c$$
$$\Rightarrow 2x = x + y - 5\log(x+y-1) + c$$
$$\Rightarrow 5\log(x+y-1) = y - x + c$$

Example

Solve $\dfrac{dy}{dx} = \dfrac{x+y-4}{x-y-6}$.

Solution: Observe that this DE is of the form $\dfrac{dy}{dx} = \dfrac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ where $1 = \dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2} = -1$. Put $x = X+h$, $y = Y+k$, where h and k are constants to be determined. Then we have $dx = dX$, $dy = dY$ and

$$\frac{dY}{dX} = \frac{X+Y+(h+k-4)}{X-Y+(h-k-6)} \tag{1}$$

If h and k are such that $h+k-4=0$ and $h-k-6=0$, then (1) becomes

$$\frac{dY}{dX} = \frac{X+Y}{X-Y}$$

which is a homogeneous DE. We can easily solve the system

$$h+k=4$$

$$h-k=6$$

of linear equations to determine the constants h and k.

Example

Solve $\dfrac{dy}{dx} = \dfrac{x+y-4}{3x+3y-5}$.

Solution: Observe that this DE is of the form $\dfrac{dy}{dx} = \dfrac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ where $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2}$. Use the substitution $z = x+y$. Then we have

$$\frac{dz}{dx} = 1 + \frac{dy}{dx}.$$

Putting these in the given DE, we get

$$\frac{dz}{dx} - 1 = \frac{z-4}{3z-5},$$

or in other words,

$$\frac{3z-5}{4z-9}dz = dx.$$

This equation is now in variable separable form.


Thank You