Lecture - 12th

$$f(x, y) = x|\sin y|$$

$\times$ $\boxed{\left|\frac{\partial f}{\partial y}\right|}$

$$|f(x, y_1) - f(x, y_2)| = \big|x|\sin y_1| - x|\sin y_2|\big|$$

$$= |x| \, \big| |\sin y_1| - |\sin y_2| \big|$$

$$\leq |x| \, |\sin y_1 - \sin y_2|$$

$$\left[\, \big||a| - |b|\big| \leq |a - b| \right.$$

Reduction of Order

$y(x) = c_1 y_1 + c_2 y_2$

$\{y_1, y_2\}$ → L.I.

$$a_0(x) \frac{d^2y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x) y = 0 \quad \text{—— (1)}$$

If we know one linearly independent solⁿ of (1), then you can find another L.I. solⁿ of (1).

Let $y = f(x)$ be ~~one~~ the L.I. solⁿ of (1).

Now, we want to find another L.I. solⁿ $g(x)$ of (1) such that $\{f, g\}$ form fundamental set of solⁿs.

$g''(x) = f''(x)v + 2f'(x)v' + f(x)v''$

Let us suppose

$$\underline{g(x) = f(x)\cdot v},$$

$g'(x) = f'(x)v + f(x)v'$

$g''(x) = f''(x)v + 2f'(x)v' + f(x)v''$

where v is a f^n of x to be determined

Since $g(x)$ is sol^n of ①,

$$\Rightarrow a_0(x)\, \underline{g''(x)} + a_1(x)\, \underline{g'(x)} + a_2(x)\, \underline{g(x)} = 0$$

$$\Rightarrow a_0(x)\left(f''(x)v + 2f'(x)v' + f(x)v''\right) + a_1(x)\left(f(x)v' + f'(x)v\right) + a_2(x)\, f(x)\, v = 0$$

$$\Rightarrow a_0(x) f(x) v'' + \left(2a_0(x)f'(x) + a_1(x)f(x)\right)v' + \boxed{\left(a_0(x) f''(x) + a_1(x) f'(x) + a_2(x) f(x)\right)} v = 0 \quad ②$$

Since f is the sol^n of ①

$$\Rightarrow \underline{a_0(x) f''(x) + a_1(x) f'(x) + a_2(x) f(x) = 0}$$

Using this in ②,

$$a_0(x) f(x) v'' + \left(2a_0(x) f'(x) + a_1(x) f(x)\right) v' = 0$$

Put $v' = w$

$$\Rightarrow a_0(x) f(x) w' + \left(2a_0(x) f'(x) + a_1(x) f(x)\right) w = 0$$

$$\Rightarrow \int \frac{w'}{w} = \int \left(-\frac{2f'(x)}{f(x)} - \frac{a_1(x)}{a_0(x)} \right) dx$$

$$\Rightarrow \log w = -2 \log f(x) - \int \frac{a_1(x)}{a_0(x)} dx$$

$$\Rightarrow \log w + \log [f(x)]^2 = -\int \frac{a_1(x)}{a_0(x)} dx$$

$$\Rightarrow \log w (f(x))^2 = -\int \frac{a_1(x)}{a_0(x)} dx$$

$$\Rightarrow w = \frac{e^{-\int \frac{a_1(x)}{a_0(x)} dx}}{[f(x)]^2}$$

$$\Rightarrow v' = \frac{e^{-\int \frac{a_1(x)}{a_0(x)} dx}}{[f(x)]^2}$$

$$\Rightarrow \boxed{v = \int \frac{e^{-\int \frac{a_1(x)}{a_0(x)} dx}}{[f(x)]^2}}$$

$$\Rightarrow g(x) = f(x) \cdot v(x) = f(x) \cdot \int \frac{e^{-\int \frac{a_1(x)}{a_0(x)} dx}}{[f(x)]^2} dx$$

$$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$$

$$= \begin{vmatrix} f & fv \\ f' & f'v + fv' \end{vmatrix}$$

$$= f(f'v + fv') - ff'v$$

$$= f^2 v'$$

$$= \cancel{f^2} \cdot \frac{e^{-\int \frac{a_1(x)}{a_0(x)} dx}}{\cancel{(f(x))^2}}$$

$$\Rightarrow W(f,g) = e^{-\int \frac{a_1(x)}{a_0(x)} dx} \neq 0$$

$\Rightarrow$ f and g are L-I sol^n's of (1).

Thus the general sol^n is

$$y(x) = c_1 f(x) + c_2 g(x)$$

$$\boxed{y(x) = c_1 f(x) + c_2 (f(x) v(x))}$$

\# $a_0(x)\dfrac{d^2y}{dx^2} + a_1(x)\dfrac{dy}{dx} + a_2(x)\,y = 0 \quad \text{--- (1)}$

Suppose $y = f(x)$ is given L.I. solⁿ of (1).

(another L.I. solⁿ of (1)) $g(x) = f(x)\,v(x)$,

where
$$\boxed{v(x) = \int \frac{e^{-\int \frac{a_1(x)}{a_0(x)}dx}}{[f(x)]^2}\,dx}$$

The general solⁿ of (1) is

$$\boxed{y(x) = c_1 f(x) + c_2 g(x)}$$

Example: Given that $y = x$ is a solⁿ of

$$(x^2+1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0.$$

find another L.I. solⁿ by reducing the order. and hence obtain the general solⁿ.

Solⁿ:
$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\,y = 0,$$

$a_0(x) = x^2+1, \quad a_1(x) = -2x, \quad a_2(x) = 2$

Here $\underline{f(x) = x}$.

$$g(x) = f(x) \cdot v(x)$$

where $v(x) = \displaystyle\int \frac{e^{-\int \frac{a_1(x)}{a_0(x)} dx}}{[f(x)]^2} dx$

$$= \int \frac{e^{-\int \frac{-2x}{x^2+1} dx}}{x^2} dx$$

$$= \int \frac{e^{\int \frac{2x}{x^2+1} dx}}{x^2} dx$$

$$= \int \frac{e^{\log(x^2+1)}}{x^2} dx$$

$$= \int \left(\frac{x^2+1}{x^2}\right) dx$$

$$= \int \left(1 + \frac{1}{x^2}\right) dx$$

$$\Rightarrow \boxed{g(x) = x - \frac{1}{x}}$$

$\Rightarrow \quad g(x) = f(x) \cdot v(x) = x \cdot \left(x - \frac{1}{x} \right) = x^2 - 1$

The general solⁿ of given DE is

$$y(x) = c_1 f(x) + c_2 g(x)$$

$$\boxed{y(x) = c_1 x + c_2 (x^2 - 1)}$$

Example: If e^x is one of the solⁿs of Hom. eqⁿ

$$x \frac{d^2y}{dx^2} - (2x-1) \frac{dy}{dx} + (x-1) y = 0$$

find the another L.I solⁿ by reducing the order and hence find the general solⁿ.

$$\boxed{\begin{aligned} g(x) &= e^x \log x \\ y(x) &= c_1 f(x) + c_2 g(x) \\ &= c_1 e^x + c_2 e^x \log x \end{aligned}}$$

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nth order linear Homogeneous DE:

$$a_0(x) \frac{d^n y}{dx^n} + a_1(x) \frac{d^{n-1} y}{dx^{n-1}} + a_2(x) \frac{d^{n-2} y}{dx^{n-2}} + \cdots$$

$- \quad + a_n(x)\, y = 0$ ——— (1)

where $a_0(x) \not\equiv 0$

The set of sol^{n}s of an nth order linear Hom. DE forms a vector space (V) over $\mathbb{R}$.

$$\dim(V) = n$$

$\Rightarrow$ Any set consisting n L.I sol^{n}s is going to form the basis of the set of sol^{n}s of (1).

If we know n L.I sol^{n}s $y_1, y_2, - y_n$ of (1), then

$$\boxed{y(x) = c_1 y_1 + c_2 y_2 + - + c_n y_n}$$

\# $f_1(x) = \cos^2 x$, $f_2(x) = \sin^2 x$, $f_3(x) = \sec^2 x$, $f_4(x) = \tan^2 x$

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) + c_4 f_4(x) = 0$$

$$\Rightarrow \underline{c_1 \cos^2 x + c_2 \sin^2 x} + c_3 \sec^2 x + c_4 \tan^2 x = 0$$

$$\boxed{\text{If } c_1 = 1,\ c_2 = 1,\ c_3 = -1,\ c_4 = 1}$$

$\Rightarrow \{f_1, f_2, f_3, f_4\}$ are L-D on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

e^x, e^{2x}, e^{3x} are the sol's of

$$\underbrace{y''' - 6y'' + 11y' - 6y = 0}$$

$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$W(e^x, e^{2x}, e^{3x}) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}$$

$$= 2e^{2x} \neq 0$$

$$\boxed{W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}}$$

$\Rightarrow e^x, e^{2x}, e^{3x}$ are L.I.

$\Rightarrow \boxed{y(x) = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}}$