Lecture 5 - ODE (Exact Differential Equations)

$$\boxed{M(x,y)\,dx + N(x,y)\,dy = 0} \rightarrow Mdx + Ndy = d(F(x,y))$$

$$dF = 0 \Rightarrow \boxed{F(x,y) = c}$$

Condition for exactness $\rightarrow$ $\boxed{\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}}$

Exact DE:

$$M(x,y)\,dx + N(x,y)\,dy = dF$$

$$\Rightarrow M(x,y)\,dx + N(x,y)\,dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$\Rightarrow \boxed{M(x,y) = \frac{\partial f}{\partial x}, \quad N(x,y) = \frac{\partial f}{\partial y}}$$

\# Solve the eq^n

$$(y\cos x + 2xe^y)\,dx + (\sin x + x^2e^y - 1)\,dy = 0 \quad \text{—— (1)}$$

Comparing (1) with $Mdx + Ndy = 0$,

$$M = y\cos x + 2xe^y, \quad N = \sin x + x^2e^y - 1$$

$$\frac{\partial M}{\partial y} = \cancel{y(\sin}\ \cos x + 2xe^y,$$

$$\frac{\partial N}{\partial x} = \cos x + 2xe^y$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{(1) is exact.}$$

We need to find $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M(x,y) \quad \Bigg| \quad \frac{\partial F}{\partial y} = N(x,y)$$

$$\Rightarrow \frac{\partial F}{\partial x} = y\cos x + 2xe^y$$

$$\Rightarrow F(x,y) = \int (y\cos x + 2xe^y)\,dx + \phi(y)$$

$$\Rightarrow F(x,y) = y\sin x + \frac{2x^2}{2}e^y + \phi(y)$$

$$= y\sin x + x^2 e^y + \phi(y) \quad (2)$$

$$\frac{\partial F}{\partial y} = \sin x + x^2 e^y + \phi'(y) = N(x,y)$$

$$\Rightarrow \cancel{\sin x} + \cancel{x^2 e^y} + \phi'(y) = \cancel{\sin x} + \cancel{x^2 e^y} - 1$$

$$\Rightarrow \phi'(y) = -1$$

$$\Rightarrow \boxed{\phi(y) = -y + C_1}$$

$$\Rightarrow F(x,y) = y\sin x + x^2 e^y - y + C_1$$

The Sol^n of (1) is given by $F(x,y) = C$

$$\Rightarrow y\sin x + x^2 e^y - y + C_1 = C$$

$$\Rightarrow \boxed{y\sin x + x^2 e^y - y = C_0}$$

$(C_0 = C - C_1)$

Example: Solve the DE by the method of inspection

$$y + x\frac{dy}{dx} = 0$$

$$\Rightarrow \quad y\,dx + x\,dy = 0$$

$$\Rightarrow \quad d(xy) = 0$$

$$\Rightarrow \quad \boxed{xy = c_1}$$

Example: $(2x+y^2)\,dx + 2xy\,dy = 0$

$$2x\,dx + y^2\,dx + 2xy\,dy = 0$$

$$\Rightarrow \quad d(x^2) + d(xy^2) = 0$$

$$\Rightarrow \quad d(x^2 + xy^2) = 0$$

$$\Rightarrow \quad \boxed{x^2 + xy^2 = c.}$$

Example: $(3x^2 + 4xy)\,dx + (2x^2 + 2y)\,dy = 0$

$$\Rightarrow \quad 3x^2\,dx + 4xy\,dx + 2x^2\,dy + 2y\,dy = 0$$

$$\Rightarrow \quad d(x^3) + d(2x^2y) + d(y^2) = 0$$

$$\Rightarrow \quad d(x^3 + 2x^2y + y^2) = 0$$

$$\Rightarrow \quad \boxed{x^3 + 2x^2y + y^2 = c}$$

Example: $y\,dx - x\,dy = 0$ —— (1)

$M = y, \quad N = -x$

$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = -1$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\Rightarrow$ not exact

Multiplying (1) with $\frac{1}{y^2}$, we get

$$\frac{y}{y^2}dx - \frac{x}{y^2}dy = 0 \Rightarrow \underbrace{\frac{1}{y}dx - \frac{x}{y^2}dy}=0$$

This eq^n is exact. $\left[\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\right]$

If $M(x,y)\,dx + N(x,y)\,dy = 0$ —— (1)

is not exact but if you will multiply (1) with $\mu(x,y)$, then (1) becomes exact, that is

$$\mu(x,y)\,M(x,y)\,dx + \mu(x,y)\,N(x,y)\,dy = 0$$

is exact.

Then such a factor $\mu(x,y)$ is called integrating factor.

Method to find integrating factor:

Suppose that

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \text{---} (1)$$

is not exact and $\mu(x,y)$ is an integrating factor of (1)

$$\Rightarrow \quad \underbrace{\mu(x,y)\,M(x,y)}\,dx + \underbrace{\mu(x,y)\,N(x,y)}\,dy = 0 \quad \text{---} (2)$$

is exact.

Using the condition of exactness,

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x} \qquad \left[\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\right]$$

$$\Rightarrow \quad \mu_y M + \mu M_y = \mu_x N + \mu N_x$$

$$\Rightarrow \quad (\mu_y M - \mu_x N) + \mu(M_y - N_x) = 0 \quad (3)$$

Thus eqn (3) is a PDE and we are not in this position to solve a PDE.

That means, we try to reduce (3) into ODE.

Case-I: If μ is a fn of x alone (i.e. $\mu = \mu(x)$)

$$\Rightarrow \quad \mu_y = 0$$

from (3),

$$-\mu_x N + \mu(M_y - N_x) = 0$$

$$\Rightarrow \quad \mu(M_y - N_x) = \mu_x N$$

$$\Rightarrow \quad \frac{d\mu}{dx} N = \mu(M_y - N_x)$$

1 $\rightarrow$ $$\boxed{\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu}$$

If further, $\frac{M_y - N_x}{N}$ is a f^n of x alone

i.e $\frac{M_y - N_x}{N} = f(x)$

$$\Rightarrow \frac{d\mu}{dx} = f(x)\,\mu \Rightarrow \frac{d\mu}{\mu} = f(x)\,dx$$

$$\Rightarrow \log \mu = \int f(x)\,dx + \log c$$

$$\Rightarrow \boxed{\mu = c\, e^{\int f(x)\,dx}} = c\, e^{\int\left(\frac{M_y - N_x}{N}\right)dx}$$

If $\frac{M_y - N_x}{N} = f(x)$, then $\mu = e^{\int\left(\frac{M_y - N_x}{N}\right)dx}$

Case II : If μ is a f^n of y alone.
(i.e $\mu = \mu(y)$)

$$\Rightarrow \mu_x = 0$$

from (3), $\mu_y M + (M_y - N_x)\mu = 0$

$$\mu = e^{\int\left(\frac{N_x - M_y}{M}\right)dy}$$

Integrating factors!

(i) If $\dfrac{M_y - N_x}{N} = f(x)$, then

$$\text{I.F.} \leftarrow \boxed{\mu = e^{\int \left(\frac{M_y - N_x}{N}\right) dx}}$$

(ii) If $\dfrac{N_x - M_y}{M} = f(y)$, then

$$\underline{\text{I.F}} \leftarrow \mu = e^{\int \left(\frac{N_x - M_y}{M}\right) dy}$$

<u>Example</u>: Solve the DE

$$(2x^2 + y)\,dx + (x^2 y - x)\,dy = 0 \quad (1)$$

$$M = 2x^2 + y, \qquad N = x^2 y - x$$

$$M_y = \frac{\partial M}{\partial y} = 1, \qquad N_x = \frac{\partial N}{\partial x} = 2xy - 1$$

$$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$$ (1) is not an exact DE.

Consider $\dfrac{M_y - N_x}{N} = \dfrac{1 - (2xy - 1)}{x^2 y - x} = \dfrac{2 - 2xy}{x(xy - 1)}$

$$= \frac{2(1 - xy)}{-x(1 - xy)} = \frac{-2}{x}$$

$$\Rightarrow \frac{M_y - N_x}{N} = \frac{-2}{x} = \underline{f(x)}$$

$$I.F = e^{\int \left(\frac{M_y - N_x}{N}\right) dx}$$

$$= e^{\int -\frac{2}{x} dx} = e^{-2 \log x} = \frac{1}{x^2}$$

Multiplying ① with $\frac{1}{x^2}$, we get

$$\left(\frac{2x^2+y}{x^2}\right) dx + \left(\frac{x^2y - x}{x^2}\right) dy = 0$$

$$\Rightarrow \left(2 + \frac{y}{x^2}\right) dx + \left(y - \frac{1}{x}\right) dy = 0$$

$$\Rightarrow M = 2 + \frac{y}{x^2}, \quad N = y - \frac{1}{x}$$

$$\frac{\partial M}{\partial y} = \frac{1}{x^2}, \quad \frac{\partial N}{\partial x} = \frac{1}{x^2}$$

We need to find $f(x,y)$ such that

$$\frac{\partial f}{\partial x} = M(x,y) \quad \Big| \quad \frac{\partial f}{\partial y} = N(x,y)$$

find <u>Solⁿ yourself.</u>

<u>Example</u>: Solve $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0$

$$M = y^4 + 2y, \quad N = xy^3 + 2y^4 - 4x$$

$$\frac{\partial M}{\partial y} = 4y^3 + 2, \quad \frac{\partial N}{\partial x} = y^3 - 4$$

Consider $\dfrac{N_x - M_y}{M} = \dfrac{y^3 - y - 4y^3 - 2}{y^4 + 2y} = \dfrac{-3y^3 - 6}{y(y^3+2)}$

$$= \frac{-3(y^3+2)}{y(y^3+2)} = \frac{-3}{y}$$

$$= f(y)$$

$$I.F = e^{\int -\frac{3}{y} dy} = e^{-3\log y} = \frac{1}{y^3}$$

$$\left(\frac{dy}{dx}\right)^2 + y^2 + 4 = 0$$