Improper Integrals

Gamma function :-

For $p > 0$, $\Gamma(p) = \int_0^\infty x^{p-1} e^{-x}\, dx$.

convergence: $\int_0^\infty x^{p-1} e^{-x}\, dx$

$= \int_0^1 x^{p-1} e^{-x}\, dx + \int_1^\infty x^{p-1} e^{-x}\, dx$

$= I_1 + I_2$

convergence of I_1: $f(x) = x^{p-1} e^{-x}$

$g(x) = x^{p-1}$

$\lim_{x \to 0} \frac{f(x)}{g(x)} = 1 > 0$, and $\int_0^1 x^{p-1}\, dx$ conv.

$= \lim_{\epsilon \to 0} \int_\epsilon^1 x^{p-1}\, dx$

$= \lim_{\epsilon \to 0} \left[\frac{x^p}{p} \right]_\epsilon^1 = \frac{1}{p}$

conv of I_2 :- $f(x) = x^{p-1} e^{-x}$, $g(x) = \frac{1}{x^2}$

$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{x^{p-1} e^{-x}}{1/x^2} = \lim_{x \to \infty} x^{p+1} e^{-x} =$

and $\int_1^\infty \frac{1}{x^2}\, dx$ conv $= \lim_{x \to \infty} \frac{x^{p+1}}{e^x}$ ($\frac{\infty}{\infty}$ form)

$= 0$

$\Rightarrow I_2$ is conv.

$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x}\, dx$ conv $p > 0$.

Beta function: $p > 0$, $q > 0$

$B(m,n) = \int_0^1 x^{p-1} (1-x)^{q-1}\, dx$.

convergence $\int_0^1 x^{p-1}(1-x)^{q-1}\,dx$

$= \int_0^{1/2} x^{p-1}(1-x)^{q-1}\,dx + \int_{1/2}^1 x^{p-1}(1-x)^{q-1}\,dx$

$= I_1 + I_2$

convergence of I_1: $f(x) = x^{p-1}(1-x)^{q-1}$

$g(x) = x^{p-1}$

$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} (1-x)^{q-1} = 1 > 0$ and $\int_0^{1/2} x^{p-1}\,dx$ conv.

$\Rightarrow I_1$ is conv.

convergence of I_2: $f(x) = x^{p-1}(1-x)^{q-1}$

$g(x) = (1-x)^{q-1}$

$\lim_{x\to 1} \frac{f(x)}{g(x)} = 1$ and $\int_{1/2}^1 (1-x)^{q-1}\,dx$ conv.

$\Rightarrow I_2$ is conv.

$\therefore\ B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx$ conv $p,q>0$.

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① $\Gamma(1) = \int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx = 1$

② $\Gamma(p+1) = p\cdot\Gamma(p)$.

$\Gamma(p+1) = \int_0^\infty x^p e^{-x}\,dx = \lim_{b\to\infty}\int_0^b x^p e^{-x}\,dx$.

$= p\cdot\Gamma(p)$.

$\Gamma(p) = \Gamma(\underline{p-1}+1)$

$= (p-1)\cdot\Gamma(p-1)$.

$\Gamma(p-1) = (p-2)\cdot\Gamma(p-2)$.

$m\in\mathbb{N}$, $\Gamma(m+1) = m\cdot\Gamma(m)$

$= m\cdot(m-1)\cdot\Gamma(m-1)$

$= m(m-1)(m-2)\,\Gamma(m-2)$

$= m(m-1)(m-2)\cdots\cdots 1\cdot\Gamma(1)$

$= m!$

③ $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

④ $\beta(m,n) = \beta(n,m)$

⑤ $\beta(m,n) = \dfrac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$.

Ex:- $\displaystyle\int_0^\infty x^{2/3} e^{-\sqrt{x}}\, dx$.

$t = \sqrt{x}$, $dt = \dfrac{1}{2\sqrt{x}}\, dx$

$\Rightarrow 2t\, dt = dx$.

$$= \int_0^\infty t^{4/3} e^{-t} \cdot 2t\, dt$$

$$= 2\int_0^\infty t^{7/3} e^{-t}\, dt = 2\int_0^\infty t^{\frac{10}{3}-1} e^{-t}\, dt$$

$$= 2\cdot\Gamma\left(\tfrac{10}{3}\right).$$

$$= 2\cdot\tfrac{7}{3}\cdot\Gamma\left(\tfrac{7}{3}\right)$$

$$= 2\cdot\tfrac{7}{3}\cdot\tfrac{4}{3}\,\Gamma\left(\tfrac{4}{3}\right)$$

$$= 2\cdot\tfrac{7}{3}\cdot\tfrac{4}{3}\cdot\tfrac{1}{3}\,\Gamma\left(\tfrac{1}{3}\right).$$

Ex: $\displaystyle\int_0^1 x^{3/2}(1-\sqrt{x})^{1/2}\, dx$

$t = \sqrt{x}$.

$$= 2\int_0^1 t^3 (1-t)^{1/2}\, t\, dt$$

$$= 2\int_0^1 t^4 (1-t)^{1/2}\, dt$$

$$= 2\int_0^1 t^{5-1}(1-t)^{3/2-1}\, dt. = 2\,\beta(5, 3/2)$$

$$= 2\cdot\frac{\Gamma(5)\cdot\Gamma(3/2)}{\Gamma(5+3/2)}$$

$\boxed{\Gamma(m+1) = m!}$

$\boxed{\Gamma(p+1) = p\cdot\Gamma(p)}$

$\boxed{\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}}$

$$= \frac{2\cdot 4!\ \tfrac{1}{2}\Gamma(1/2)}{\Gamma\left(\tfrac{13}{2}\right)}$$

$$= \frac{2\cdot 4!\ \tfrac{1}{2}\cdot\sqrt{\pi}}{\ } = \boxed{\frac{512}{3465}}$$

$\Gamma\left(\tfrac{13}{2}\right) = \tfrac{11}{2}\Gamma\left(\tfrac{11}{2}\right)$

$= \tfrac{11}{2}\cdot\tfrac{9}{2}\,\Gamma\left(\tfrac{9}{2}\right)$

$= \tfrac{11}{2}\cdot\tfrac{9}{2}\cdot\tfrac{7}{2}\,\Gamma(7/2) = \dfrac{11\cdot 9\cdot 7}{2\cdot 2\cdot 2}\cdot\tfrac{5}{2}\,\Gamma\left(\tfrac{5}{2}\right)$

$= \dfrac{11\cdot 9\cdot 7\cdot 5}{2^4}\cdot\tfrac{3}{2}\,\Gamma(3/2)$

$= \dfrac{11\cdot 9\cdot 7\cdot 5\cdot 3}{2^5}\cdot\tfrac{1}{2}\,\Gamma(1/2)$

Integrals dependent on a parameter

$$I(\alpha) = \int_a^b f(x,\alpha)\,dx.$$

① If $f(x,\alpha)$, $\frac{\partial f}{\partial \alpha}(x,\alpha)$ both cont's on $[a,b]$.

② $|f(x,\alpha)| \le A(x)$, $\left|\frac{\partial f}{\partial \alpha}(x,\alpha)\right| \le B(x)$

s.t $A(x)$, $B(x)$ both integrable on $[a,b]$

If interval is unbounded, then $\int_a^b A(x)\,dx$ and $\int_a^b B(x)\,dx$ both imp. int conv.

Then $I(\alpha)$ is diff and

$$I'(\alpha) = \int_a^b \frac{df}{d\alpha}(x,\alpha)\,dx.$$

Ex:- $I(\alpha) = \int_0^\infty e^{-x} \frac{\sin \alpha x}{x}\,dx.$

$$I'(\alpha) = \int_0^\infty \frac{d}{d\alpha}\left(e^{-x} \frac{\sin \alpha x}{x}\right)dx$$

$$= \int_0^\infty e^{-x} \cos \alpha x\,dx$$

$$= \frac{1}{1+\alpha^2}$$

by int, $I(\alpha) = \tan^{-1}\alpha + c.$

$$I(0) = \int_0^\infty e^{-x} \frac{\sin 0}{x}\,dx = 0 \Rightarrow c = 0$$

$$I(\alpha) = \tan^{-1}\alpha.$$

Newton-Leibnitz Formula:-

$$h(x) = \int_{a(x)}^{b(x)} f(x,t)\,dt.$$

$$h'(x) = \int_{a(x)}^{b(x)} \frac{df}{dx}(x,t)\,dt + f(x,b(x))\cdot b'(x) - f(x,a(x))\cdot a'(x).$$

Ex:- $\frac{d}{dt}\int_0^1 (2x+t^3)^2\,dx = 6t^2 + 6t^5.$