Department of Mathematics, Bennett University
Engineering Calculus (EMAT101L)
Solutions for Tutorial Sheet 4

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1. (a) Choose $\{x_n = \frac{1}{n\pi}\}$, then $x_n \to 0$, but $\cos\left(\frac{1}{x_n}\right) = (-1)^n$ does not converge.
   (b) Choose $\left\{x_n = \frac{1}{n}\right\}$, then $x_n \to 0$, but $f(x_n) = n \to \infty$.
   (c) Choose $\left\{x_n = \frac{1}{(n\pi)^k} + a\right\}$ and $\left\{y_n = \frac{1}{(2n\pi+\frac{\pi}{2})^k} + a\right\}$, then $x_n, y_n \to a$ but $f(x_n) \to 0$ and $f(y_n) \to 1$.

2. (a) $x = 2$ is the point of infinite discontinuity.
   (b) $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{\cos x}{x - \frac{\pi}{2}} = \lim\limits_{x\to\frac{\pi}{2}} \dfrac{-\sin x}{1} = -1$. Therefore $x = \frac{\pi}{2}$ is a removable discontinuity.
   (c) There are no points of discontinuity.

3. (a) $x = 2$ is a vertical asymptote. (b) $x = 1$ is a vertical asymptote.
   (c) $x = 0$ is a vertical asymptote.

4. (a) Define a new function $\tilde{f}(x) = \begin{cases} e^{x^2}\sin x^2, & x \neq 0, 1 \\ 0, & x = 0 \\ e\sin 1, & x = 1. \end{cases}$

   Then $\tilde{f}$ is uniformly continuous on $[0, 1]$. Now note that $\tilde{f} = f$ on $(0, 1)$. Hence f is uniformly continuous on $(0, 1)$.
   (b) Using the inequality: $||x| - |y|| \leq |x - y|$ for all x, y and mean value theorem, one can show that $||\sin x| - |\sin y|| \leq |\sin x - \sin y| \leq |x - y|$ for all x, y. Hence we can choose $\delta = \epsilon$.
   (c) Similar to (a).

5. (a) $f(x) \equiv x^5 - 3x^2 + 1,\ x \in [0, 1],\ f(0) = 1,\ f(1) = -1$ and f is continuous. Now apply IVT.
   (b) $f(0) = -1, f(\frac{\pi}{2}) = 2,\ f$ is continuous. Now apply IVT.
   (c) Similar as (a) and (b).

6. (a) Let f be the function defined by $f(x) = 1$ if x is rational, and $f(x) = 0$ if x is irrational. Then f is discontinuous at every point of $\mathbb{R}$.
   (b) Let $f(x) = x$ if x is rational, and $f(x) = 0$ if x is irrational. Then f is continuous only at $x = 0$.
   (c) Let $f(x) = \sin\pi x$ if x is rational, and $f(x) = 0$ if x is irrational. As $\sin\pi x = 0$ if and only if $x \in \mathbb{Z}$, the function f is continuous only at the integers.

(d) Let f be the function defined by $f(x) = 0$ if x is irrational and $f(x) = \frac{1}{b}$ if x is the rational number $\frac{a}{b}$. Then f is discontinuous at every rational point, but continuous at every irrational point.

(e) Constant function, polynomial function, cosine function are continuous everywhere.