Real Number System
(Lecture 1 & 2)

Engineering Calculus

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- The set of **natural numbers** $\mathbb{N} := \{1, 2, 3, \cdots\}$.
- The set of **integers** $\mathbb{Z} := \{0, 1, -1, 2, -2, \cdots\}$.
- The set of **rational numbers** $\mathbb{Q} := \{\frac{m}{n} : m, n \in \mathbb{Z}$ and $n \neq 0\}$.
- The set of **real numbers** $\mathbb{R}$.
- Solve $x^2 - 2 = 0$. The roots are $x = \pm\sqrt{2}$.

Theorem

Suppose that $a_0, a_1, ..., a_n (n \geq 1)$ are integers such that $a_0 \neq 0, a_n \neq 0$ and that r satisfies the equation

$$a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0 = 0.$$

If $r = \frac{p}{q}$ where p, q are integers with no common factors and $q \neq 0$. Then q divides a_n and p divides a_0.

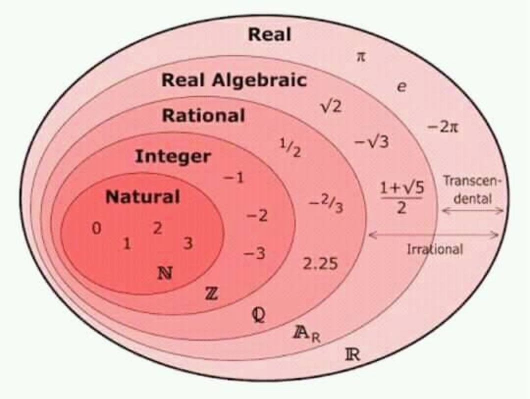
- This theorem tells us that only rational candidates for solutions of the above equation have the form $\frac{p}{q}$ where p divides a_0 and q divides a_n.

Question

Can we have a number system without these gaps?

Answer: Yes. The complete number system with out these gaps is the real line $\mathbb{R}$.

- The elements of the set $\mathbb{R}$ are called real numbers and $\mathbb{R}$ is closed with respect to addition and multiplication. That is, given any $a, b \in \mathbb{R}$, the sum $a + b$ and the product ab also represent real numbers.



Definitions

Let S be a non-empty subset of $\mathbb{R}$. Then we give the following definitions:

- If S contains a largest element s^0, then we call s^0 the **maximum** of S.
- If S contains a smallest element s_0, then we call s_0 the **minimum** of S.
- A subset S is said to be **bounded above** if there is an element $s^0 \in \mathbb{R}$ such that $s \leq s^0$ for all $s \in S$. Such an element s^0 is called an **upper bound** of S.
- A subset S is said to be **bounded below** if there is an element $s_0 \in \mathbb{R}$ such that $s_0 \leq s$ for all $s \in S$. Such an element s_0 is called an **lower bound** of S.
- S is said to be **bounded** if there exist $s_0, s^0 \in \mathbb{R}$ such that $s_0 \leq s \leq s^0$ for all $s \in S$.

- If S is bounded above and S has least upper bound, then we call it the **supremum** of S. In other words, an upper bound s^0 of S is said to least upper bound (l.u.b) or supremum (sup) of S if whenever t is an upper bound of S, $s^0 \leq t$.
- If S is bounded below and S has greatest lower bound, then we call it as **infimum** of S. In other words, an lower bound s_0 of S is said to greatest lower bound (g.l.b) or infimum (inf) of S if whenever t is an lower bound of S, $t \leq s_0$.
- Unlike maximum and minimum, $\sup S$ and $\inf S$ may not belong to the set S.

Example

Consider the sets $A := \{x \in \mathbb{R} : 0 < x < 1\}$, $B := \{x \in Q : 0 \leq x \leq 1\}$, $C := \{1 - 1/n : n \in \mathbb{N}\}$. Then

1. All the sets A, B, C are bounded as bounded below by 0 and bounded above by 1.
2. 1 is the l.u.b of A, B, C such that $1 \notin A, C$ and $1 \in B$.
3. 0 is the inf of A, B, C such that $0 \notin A$ and $0 \in B, C$.

Examples

1. The set of Natural number is bound below by 1 but not bounded above.
2. Any finite set is bounded.
3. Each of the following interval is bounded: $[a,b]$, $[a,b)$, $(a,b]$, (a,b).
4. Any bounded subset of Natural numbers has maximum and minimum.

Completeness Property

- **Least upper bound property:** Every non-empty subset S of $\mathbb{R}$ which is bounded above has a least upper bound i.e., $\sup S$ exists and is a real number.
- **Greatest lower bound property:** Every non-empty subset S of $\mathbb{R}$ which is bounded below has a greatest lower bound i.e., $\inf S$ exists and is a real number.

Remark

The completeness property does not holds for $\mathbb{Q}$ i.e., every non-empty subset of $\mathbb{Q}$ that is bounded above by a rational number need not have a rational least upper bound. For example $\{r \in \mathbb{Q} : 0 \leq r^2 < 2\}$.

Neighbourhood

Let $c \in \mathbb{R}$. A subset $S \subset \mathbb{R}$ is said to be a **neighbourhood** of c if there exists an open interval (a, b) such that $c \in (a, b) \subset S$. That is an open bounded interval containing the point c is a neighbourhood of c, and is denoted by $N(c)$. For $\delta > 0$, the open interval $(c - \delta, c + \delta)$ is said to be δ-neighbourhood of c and is denoted by $N(c, \delta)$.

Examples:

(i) For every $n \in \mathbb{N}$, $(-\frac{1}{n}, \frac{1}{n})$ is a neighbourhood of 0.

(ii) $1 \in [1, 3]$ but $[1, 3]$ is not a neighbourhood of 1.

Interior point

Let $S \subset \mathbb{R}$. A point $x \in S$ is said to be an **interior point** of S if there exists a neighbourhood $N(x)$ of x such that $N(x) \subset S$. The set of all interior point of S is said to be the interior of S and is denoted by $\text{int}\, S$.

Examples:

(i) Let $S = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$. Let $x \in S$. Every neighbourhood of x contains points not belonging to S. Therefore $\text{int}\, S = \phi$.

(ii) Let $S = \mathbb{N}$, or $\mathbb{Q}$. Then $\text{int}\, S = \phi$.

(iii) Let $S = \{x \in \mathbb{R} : 1 < x < 3\}$. Then $\text{int}\, S = S$.

Open set

Let $S \subset \mathbb{R}$. S is said to be an **open set** if each point of S is an interior point of S.

Examples:

(i) Let $S = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$. No point of S is an interior point of S. S is not an open set.

(ii) Let $S = \mathbb{Z}$, or $\mathbb{Q}$. No point of S is an interior point of S. S is not an open set.

(iii) Let $S = \{x \in \mathbb{R} : 1 < x < 3\}$. Each point of S is an interior point of S. S is an open set.

(iv) Let $S = \{x \in \mathbb{R} : 1 \leq x \leq 3\}$. $1, 3 \in S$ but they are not interior point of S. S is not an open set.

(v) Let $S = \mathbb{R}$. S is an open set.

Limit point

Let $S \subset \mathbb{R}$. A point $p \in \mathbb{R}$ is said to be a **limit point** (or an accumulation point, or a cluster point) of S if every neighbourhood of p contains a point of S other than p. Therefore p is a limit point of S if for each positive ϵ,

$$[N(p, \epsilon) - \{p\}] \cap S \neq \phi.$$

- A limit point of S may or may not belong to S.

Isolated point

Let $S \subset \mathbb{R}$. A point $x \in S$ is said to be an **isolated point** of S if x is not a limit point of S. i.e., there exists a neighbourhood $N(x)$ of x such that $N'(x) \cap S = \phi$ or $N(x) \cap S = \{x\}$ (since $x \in S$).

Examples:

(i) Let $S = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$. Every point of S is an isolated point of S. 0 is a limit point of S.

(ii) Let $S = \mathbb{Z}$. Every point of S is an isolated point of S. No point of S is a limit point of S.

(iii) Let $S = \mathbb{Q}$. No point of S is an isolated point of S. Every point $x \in \mathbb{R}$ is a limit point of $\mathbb{Q}$.

(iv) Let $S = \mathbb{R}$. No point of S is an isolated point of S. Every point x of $\mathbb{R}$ is a limit point of $\mathbb{R}$.

(v) The set $\mathbb{N}$ has no limit point.

Derived set

Let $S \subset \mathbb{R}$. The set of all limit points of S is said to be the **derived set** of S and is denoted by S'.

Examples:

(i) Let S be a finite set. Then $S' = \phi$.

(ii) Let $S = \mathbb{N}$, or $\mathbb{Z}$. Then $S' = \phi$.

(iii) Let $S = \mathbb{Q}$, or $\mathbb{R}$. Then $S' = \mathbb{R}$.

Closed set

Let $S \subset \mathbb{R}$. S is said to be a **closed set** if $S' \subset S$. (i.e., if S contains all its limit points.)

Examples:

(i) Let $S = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$. 0 is a limit point of S. As $0 \notin S$, S is not a closed set.

(ii) Let $S = \{x \in \mathbb{R} : 1 < x < 3\}$. Each point of S is a limit point of S. 1 and 3 are also limit points of S but $1 \notin S$, $3 \notin S$. Therefore S is not a closed set.

(iii) Let $S = \{x \in \mathbb{R} : 1 \leq x \leq 3\}$. Each point of S is a limit point of S. Here $S' = S$. As $S' \subset S$, S is a closed set.

(iv) Let $S = \mathbb{N}$, or $\mathbb{Z}$. Then $S' = \phi$. So $S' \subset S$ and S is a closed set.

(v) Let $S = \mathbb{Q}$. Then $S' = \mathbb{R}$. Here S' is not a subset of S. S is not a closed set. Note that $\mathbb{Q}$ is neither an open nor a closed set in $\mathbb{R}$.

(vi) Let $S = \mathbb{R}$. Then $S' = \mathbb{R}$. So $S' \subset S$ and S is a closed set.

(vii) Let $S = \phi$. Then $S' = \phi$. So $S' \subset S$ and S is a closed set.

Dense set

Let $S \subset \mathbb{R}$. A subset $T \subset S$ is said to be **dense in** S if $S \subset T'$. In particular, S is said to be dense in $\mathbb{R}$ if every point of $\mathbb{R}$ is a limit point of S.

Examples:

(i) The set $\mathbb{Q}$ is dense in $\mathbb{R}$, since $\mathbb{Q}' = \mathbb{R}$.

(ii) Let $S = \{x \in \mathbb{R} : 1 \leq x \leq 2\}$, $T = \{x \in \mathbb{R} : 1 < x < 2\}$. Then $S \subset T'$. T is dense in S.

Archimedean property

Theorem

For each $x \in \mathbb{R}$, there exists a natural number $n = n(x)$ such that $n > x$.

Proof: Assume by contradiction that this is not true. Then there is no $n \in \mathbb{N}$ such that $n > x$, i.e., $n \leq x$ for every $n \in \mathbb{N}$. This implies that x is an upper bound of the set $S := \{n : n \in \mathbb{N}\}$. By the completeness property, let M be the least upper bound of S. Then $n \leq M$ for all n and so $n + 1 \leq M$ for all $n \in \mathbb{N}$, this implies that $M - 1$ is an upper bound of S. Thus a number, $M - 1$ less than the supremum M(l.u.b) is an upper bound of S, which is a contradiction and so our assumption is wrong. Hence the theorem.

Corollary

1. The set of natural number $\mathbb{N}$ is unbounded.
2. If $x, y \in \mathbb{R}$ and $x > 0$, then there exist a positive integer n such that $nx > y$.
3. For any $\epsilon > 0$, there exists a positive integer n such that $1/n < \epsilon$.
4. If $y > 0$ be a real number, then there exists $n = n(y) \in \mathbb{N}$ such that $n - 1 \leq y < n$.

Well Ordering Principle

Every non-empty subset of Natural number has a minimal(least) element.

Theorem

Let x, y are real numbers such that $x < y$. Then there exists a rational number q such that $x < q < y$.

Proof: W. l. g. assume that $x > 0$. Since $y - x > 0$, there exist $n \in \mathbb{N}$ such that $y - x > \frac{1}{n}$, by *Archimedean Property*. Now consider the set

$$S = \{m \in \mathbb{N} : \frac{m}{n} > x\}.$$

Then S is non-empty (by Archimedean property). By well-ordering of $\mathbb{N}$, S has minimal element say m_0. Then $x < \frac{m_0}{n}$. By the minimality of m_0, we see that $\frac{m_0 - 1}{n} \leq x$. Then,

$$\frac{m_0}{n} \leq x + \frac{1}{n} < x + (y - x) = y.$$

Therefore,

$$x < \frac{m_0}{n} < y.$$

Hence the theorem.

Problem 1

Between any two distinct real numbers there is a irrational number.

Solution: Suppose $x, y \geq 0$, $y - x > 0$. Then $\frac{x}{\sqrt{2}} < \frac{y}{\sqrt{2}}$. By above Theorem, there exist a rational number r such that $x < r\sqrt{2} < y$.

Problem 2

Let $S = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then $w = \inf S = 0$.

Solution: We note that S is bounded below by 0. Let $\epsilon > 0$ be an arbitrary positive real number. By Archimedean property, there exists $n \in \mathbb{N}$ such that $n > \frac{1}{\epsilon}$. Then $0 \leq w \leq \frac{1}{n} < \epsilon$. Since ϵ is arbitrary, we have $w = 0$.

Problem 3

If $y > 0$ be a real number, then there exists $n = n(y) \in \mathbb{N}$ such that $n - 1 \leq y < n$.

Solution: Consider $S := \{m \in \mathbb{N} : m > y\}$. Then by **A.P**, there exist $m \in \mathbb{N}$ such that $m > y$. This shows that $S \neq \emptyset$. Also by well ordering Principle, S has a least element say n, i.e. $n \leq m$ for all $m \in S$. Since $n \in S$, we have $n > y$. If $n = 1$ then $0 < y < 1$ and if $n \neq 1$ then $n - 1 \in \mathbb{N}$. Also $n - 1 \notin S$, implies $n - 1 \leq y < n$.

Problem 4

Let $x, y \in \mathbb{R}$. Show that if $x < y + \frac{1}{n}$ for all $n \in \mathbb{N}$ then $x \leq y$.

Solution: Assume $x \leq y + \frac{1}{n}$ for all $n \in \mathbb{N}$ and $x > y$. Then $x - y > 0$ and by A. P, there exists $n_0 \in \mathbb{N}$ such that $n_0(x - y) > 1$ implies $x > y + \frac{1}{n_0}$, a contradiction.


Thank You