Mid	20	1 hr
End	40	2 hrs

[QUIZ-1, CR-203
Thursday, 4-30 – 5.30]

Continuity

Result: ① If f is conti. on $[a, b]$ then f is uni. conti on $[a, b]$

② If f has removable discontinuity in $[a, b]$. Then $\tilde{f}$, extension fun. is uni. cont. on $[a, b]$

Ex:- $f(x) = \dfrac{\sin x}{x}, \quad x \in [0, 1]$

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}, \quad f(x) = \begin{cases} \dfrac{\sin x}{x}, & [0,1] \\ 0, & x = 0 \end{cases}$$

$$\tilde{f} = \begin{cases} \dfrac{\sin x}{x}, & x \in (0, 1] \\ 1, & x = 0 \end{cases}$$

$\therefore \tilde{f}$ is uni conts on $[0, 1]$

Ex:- $f(x) = \dfrac{x^2 - 4}{x - 2}, \quad x \in [2, 5]$

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} (x + 2) = 4$$

$$\tilde{f}(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \in (2, 5] \\ 4, & x = 2 \end{cases}$$

$\tilde{f}$ is uni conts on $[2, 5]$

Q: f is uni conts $(2, 5)$

$\tilde{f} = f$ on $(2, 5)$

$\Rightarrow f$ is uni conts $(2, 5)$

Result: If f is uni. conts & $\{x_n\}$ is any cauchy sequence
$\Rightarrow \{f(x_n)\}$ is also cauchy seq^n.

Ex:- $f(x) = \frac{1}{x^2}$ is not uni conts on $(0,1)$

$x_n = \frac{1}{n} \to 0$, $\{x_n\}$ is c.s. in $(0,1)$

$f(x_n) = n^2 \to \infty$, $\{f(x_n)\}$ is not a c.s.

$\Rightarrow f$ is not uni. conts $_{(0,1)}$

Ex:- $f(x) = \frac{1}{x}$, $x \in (0,1)$, f is not uni. conts on $(0,1)$.

$n \in \mathbb{N}$ $\quad x_n = \frac{1}{n+1} \to 0$, $\{x_n\}$ is c.s $(0,1)$

$f(x_n) = n+1 \to \infty$, $\{f(x_n)\}$ is not c.s.

$\therefore f$ is not uni. conts.

Ex:- $f(x) = \sin\left(\frac{1}{x}\right)$, $x \in (0,1)$.
is not uni. conts on $(0,1)$.

$x_n = \frac{2}{n\pi}$, $\to 0$.

$f(x_n) =$??

Differentiability

Def^n:- $f: I \to R$, $x \in I$ Then f is diff. at x if $\lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists.

$f: I \to R$ is diff. on I. if f is diff at each $x \in I$, $= f'(x)$.

$f': I \to R$.

f is diff. at $c \in I$. $x-c=h$

$$f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$$
$$= \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

Ex:- $f(x)=x^2$, $x \in \mathbb{R}$, $c \in \mathbb{R}$.

$$f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} = \lim_{x \to c} x+c$$
$$= 2c.$$

Ex:- $f(x) = \sqrt{x}$, $x \in [0,\infty)$, $c \in [0,\infty)$

$$f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} = \frac{1}{2\sqrt{c}}.$$

Diff $\Rightarrow$ continuity.

If f is diff at c $\underset{\nLeftarrow}{\Rightarrow}$ f is conti at c.

proof:- $\lim_{x \to c} \frac{f(x)-f(c)}{x-c} = f'(c)$ ✓

we show, $\lim_{x \to c} f(x) = f(c)$ ✓

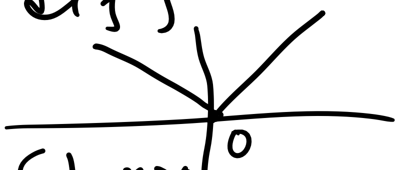
$$f(x) = (x-c)\frac{f(x)-f(c)}{x-c} + f(c)$$

$$\lim_{x \to c} f(x) = \lim_{x \to c} (x-c) \cdot f'(c) + f(c)$$
$$= f(c) \Rightarrow f \text{ is conts at } c.$$

if f is not conts $\Rightarrow$ f is not diff

f is conti $\nRightarrow$ f is diff.

$f(x) = |x|$, $x \in \mathbb{R}$,



$$\lim_{x \to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0} \frac{|x|}{x} = \begin{cases} 1, & x>0 \\ -1, & x<0. \end{cases}$$

Def^n:- $I = [a, b]$. $f : [a, b] \to R$.

① $x = a$, $\displaystyle\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} = f'(a)$.

② $x = b$, $\displaystyle\lim_{x \to b^-} \frac{f(x) - f(b)}{x - b} = f'(b)$.

③ $c \in (a, b)$, $\displaystyle\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$

$$= \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = f'(c).$$

Ex:- $f(x) = \begin{cases} x, & \underline{0 \le x \le 1} \\ 2 - x^2, & \underline{1 < x \le 2}. \end{cases}$ $f : [0, 2] \to R$. find $f'(x)$.

$f'(x) = \begin{cases} 1, & 0 \le x < 1 \\ -2x, & 1 < x \le 2. \end{cases}$ $f' : [0, 1) \cup (1, 2] \to R$.