Lecture-6 (ODE)

Linear Algebra

Gram Schmidt Orthogonalization Process :

$$\underbrace{\{u_1, u_2, \text{---}, u_n\}}_{\downarrow} \rightarrow \text{L.I Set}$$

$$\{v_1, v_2, \text{---}\ v_n\} \rightarrow \text{Orthogonal Set}$$

$$v_1 = u_1$$

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2$$

$$\vdots$$

$$v_k = u_k - \frac{\langle u_k, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle u_k, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2$$

$$- \quad - \frac{\langle u_k, v_{k-1} \rangle}{\langle v_{k-1}, v_{k-1} \rangle} v_{k-1}$$

Exact Equations

$$\underbrace{\overbrace{M(x,y)\,dx + N(x,y)\,dy}^{dF} = 0,}_{} \quad ①$$

$$M(x,y)\,dx + N(x,y)\,dy = d(F)$$

$$\boxed{\frac{\partial F}{\partial x} = M(x,y) \quad \Big| \quad \frac{\partial F}{\partial y} = N(x,y)}$$

The solⁿ of ⓪ is $\underline{F(x,y) = C}$.

Alternative method to find the solⁿ of an Exact DE:

If $M(x,y)\,dx + N(x,y)\,dy = 0$

is exact, then the solⁿ is given by

$$\int_{y\ \text{constant}} M\,dx + \int \left(\text{terms of } N \text{ not containing } x\right) dy = C$$

Example: Solve the DE

$$(y\cos x + 2xe^{y})\,dx + (\sin x + x^{2}e^{y} - 1)\,dy = 0 \quad \text{——— ①}$$

Comparing ① with $M\,dx + N\,dy = 0$,

$M = y\cos x + 2xe^{y}$

$N = \underline{\sin x + x^{2}e^{y} - 1}$

$\frac{\partial M}{\partial y} = \cos x + 2xe^{y}$, $\qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\frac{\partial N}{\partial x} = \cos x + 2xe^{y} \qquad \Rightarrow$ ① is exact

Sol^n of ① is given by

$$\int_{y\text{ constant}} M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = C$$

$$\Rightarrow \int_{y=\text{constant}} (y\cos x + 2xe^y)\,dx + \int (-1)\,dy = C$$

$$\Rightarrow y\sin x + \frac{2x^2}{2}e^y - y = C$$

$$\Rightarrow \boxed{y\sin x + x^2 e^y - y = C}$$

Integrating factors!

Suppose $M\,dx + N\,dy = 0$ —— ①
is not exact.

but if you multiply ① with $\mu(x,y)$,
that is

$$\mu M\,dx + \mu N\,dy = 0,$$

then it becomes exact.

Then $\mu(x,y)$ is called I.F of ①

Example: $y\,dx - x\,dy = 0$ is not exact.

but if you will multiply the above eqn with $\frac{1}{y^2}$, then it becomes exact.

$$\boxed{\frac{y}{y^2}\,dx - \frac{x}{y^2}\,dy} = 0$$

$$\Rightarrow \quad d\left(\frac{x}{y}\right) = 0 \quad \Rightarrow \boxed{\frac{x}{y} = c}$$

Rules for finding Integrating factors

(i) If $\frac{M_y - N_x}{N} = f(x)$, then

$$\boxed{I.F = e^{\int f(x)\,dx}} \checkmark$$

(ii) If $\frac{N_x - M_y}{M} = f(y)$, then

$$I.F = e^{\int f(y)\,dy}$$

Example: $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0$ ——— ①

—

$M = y^4 + 2y, \quad N = xy^3 + 2y^4 - 4x$

$\frac{\partial M}{\partial y} = 4y^3 + 2, \quad \frac{\partial N}{\partial x} = y^3 - 4$

$\Rightarrow \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\Rightarrow$ ① is not exact.

Consider $\frac{N_x - M_y}{M} = \frac{y^3 - 4 - (4y^3 - 2)}{y^4 + 2y}$

$= \frac{-3y^3 - 6}{y(y^3+2)} = \frac{-3(y^3+2)}{y(y^3+2)}$

$= -\frac{3}{y} = f(y)$

$I \cdot F = e^{\int f(y)\,dy} = e^{\int -\frac{3}{y}\,dy}$

$= \frac{1}{y^3}$

Multiplying ① with $\frac{1}{y^3}$, we get

$$\left(\frac{y^4 + 2y}{y^3}\right) d\cancel{y}x + \left(\frac{xy^3 + 2y^4 - 4x}{y^3}\right) dy = 0$$

$$\Rightarrow \quad \underbrace{\left(y + \frac{2}{y^2}\right) dx + \left(x + 2y - \frac{4x}{y^3}\right) dy = 0}$$

find the sol^n yourself.

Linear Differential Equations:

A first order linear DE is of the form

$$\underbrace{a_0(x) \frac{dy}{dx} + a_1(x)\, y = g(x)} \quad \text{—} ①$$

where $a_0(x) \neq 0$

If $g(x) = 0$, then ① is called a homogeneous linear DE otherwise non-homogeneous.

~~If~~ Dividing ① by $a_0(x)$, we get

$$\frac{dy}{dx} + \frac{a_1(x)}{a_0(x)}\, y = \frac{g(x)}{a_0(x)}$$

$$\underbrace{\frac{dy}{dx} + P(x)\, y = Q(x)} \quad \text{—} ②$$

② is called the standard form of ①.

Solⁿ of Linear first Order ODE:

$$\frac{dy}{dx} + P(x)\,y = Q(x) \quad \text{—} ①$$

$$\Rightarrow \quad dy + P(x)\,y\,dx = Q(x)\,dx$$

$$\Rightarrow \quad dy + \left(P(x)\,y - Q(x)\right)dx = 0 \quad \text{—} ②$$

$$\Rightarrow \quad \left(P(x)\,y - Q(x)\right)dx + dy = 0$$

Here $M = P(x)\,y - Q(x)$,

$N = 1$

$$\frac{\partial M}{\partial y} = P(x), \quad \frac{\partial N}{\partial x} = 0$$

$$\Rightarrow \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\Rightarrow$ ① is not an exact DE.

Consider $\dfrac{M_y - N_x}{N} = \dfrac{P(x) - 0}{1} = P(x)$

$$I.F = e^{\int P(x)\,dx}$$

Multiplying ① with $e^{\int P(x)\,dx}$, we get

$$\underbrace{e^{\int P(x)\,dx}\frac{dy}{dx} + P(x)\,y\,e^{\int P(x)\,dx}} = Q(x).\,e^{\int P(x)\,dx}$$

$$\Rightarrow \quad d\left(y \cdot \left(e^{\int P(x)\,dx}\right)\right) = Q(x) \cdot e^{\int P(x)\,dx}$$

$$\Rightarrow \quad y\, e^{\int P(x)\,dx} = \int Q(x)\, e^{\int P(x)\,dx}\, dx$$

$$+ \ C$$

$$\Rightarrow \boxed{y \times I \cdot f = \int Q(x) \times I \cdot f \ dx + C}$$

If $\dfrac{dy}{dx} + Py = Q$,

then $I \cdot f = e^{\int P(x)\,dx}$

and sol^n is

$$\boxed{y \times I \cdot f = \int Q(x) \times I \cdot f \cdot \ dx + C}$$

If you have an eq^n of the form

$$\frac{dx}{dy} + P(y)\,x = Q(y),$$

then $I \cdot f = e^{\int P(y)\,dy}$

Sol^n is

$$x \times \overset{I \cdot F}{\underbrace{e^{\int P(y)\, dy}}} = \int Q(y) \times \overset{I\cdot F}{\underbrace{e^{\int P(y)\, dy}}}\, dy + C$$

Example:

$$y^2 dx + (3xy-1)\, dy = 0$$

$$y^2\, dx = -(3xy-1)\, dy$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y^2}{3xy-1}$$

$$\Rightarrow \frac{dx}{dy} = \frac{3xy-1}{-y^2} = \frac{-3x}{y} + \frac{1}{y^2}$$

$$\Rightarrow \frac{dx}{dy} = \frac{-3}{y}x + \frac{1}{y^2}$$

$$\Rightarrow \frac{dx}{dy} + \frac{3}{y}x = \frac{1}{y^2}$$

$$I \cdot F = e^{\int \frac{3}{y} dy} = e^{3 \log y}$$

$$= y^3$$

Sol^n is

$$x \times I \cdot F = \int \frac{1}{y^2} \cdot \times I \cdot F\, dy + C$$

$$x y^3 = \int \frac{1}{y^2} \times y^3\, dy + C$$

$$\Rightarrow \boxed{xy^3 = \frac{y^2}{2} + C}$$

Bernoullis DE:

A DE of the form

$$\frac{dy}{dx} + P(x)\,y = \underline{Q(x)\cdot y^n} \quad ①$$

(where n is a real number). is called Bernoullis DE.

Dividing ① by y^n, we get

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{P(x)}{y^n}\cdot y = Q(x)$$

$$\Rightarrow \frac{1}{y^n}\frac{dy}{dx} + P(x)\cdot \boxed{y^{1-n}} = \underline{Q(x)} \quad ②$$

$$\left.\begin{aligned} &\text{Put } y^{1-n} = z \\ &\Rightarrow (1-n)\,y^{-n}\cdot\frac{dy}{dx} = \frac{dz}{dx} \end{aligned}\right\}$$

② becomes,

$$\frac{1}{(1-n)}\frac{dz}{dx} + P(x)\,z = Q(x)$$

$\frac{dy}{dx} + y = xy^3$ — ①

Dividing eqn ① by y^3, we get

$$\frac{1}{y^3}\frac{dy}{dx} + \frac{y}{y^3} = x$$

$$\Rightarrow \underline{\frac{1}{y^3}\frac{dy}{dx}} + \frac{1}{y^2} = x \quad \text{— ②}$$

Put $\frac{1}{y^2} = z$

$$\Rightarrow \frac{-2}{y^3}\frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow \frac{1}{y^3}\frac{dy}{dx} = \frac{-1}{2}\frac{dz}{dx}$$

② becomes.

$$\boxed{\frac{-1}{2}\frac{dz}{dx} + z = x}$$