Multivariable Calculus

(Lecture-15 & 16)

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Learning Outcome of this lecture

In this lecture, we learn

- Positive/Negative orientation of closed curve, Simply connected domains
- Green's Theorem (Tangential Form) for Simply Connected Domains
- Extending Green's Theorem for Multiply Connected Domains

Orientation of a Curve

Definition

(Natural) Orientation of a Curve: Let $\mathcal{C}$ be a curve that is parametrized by $R(t)$ for $t \in [a, b]$. As t increases from a to b, the points $R(t)$ moves continuously from $R(a)$ to $R(b)$ in a specific direction which we indicate by drawing arrows along the curve. This direction is called the orientation(or natural orientation) of the curve induced by the parametrization $R(t)$ for $t \in [a, b]$.

Definition

Opposite Curve: Consider the curve $\mathcal{C}$ having parametrization $R(t)$ for $a \leq t \leq b$. The opposite curve, denoted by $-\mathcal{C}$, traces out the same set of points but in the reverse order, and it has the parametrization

$$R^*(t) = R(-t) \quad \text{for} \quad -b \leq t \leq -a.$$

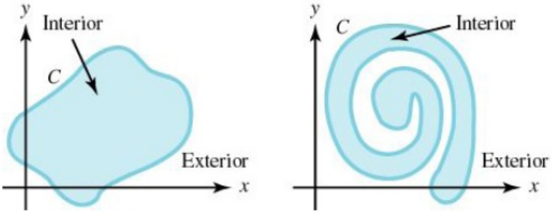
Jordan Curve Theorem

Theorem

Jordan Curve Theorem:

The points on any simple close curve (Jordan curve) $\mathcal{C}$ are boundary points of two disjoint open and connected sets,

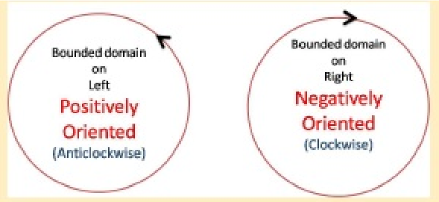
- one of which is the interior of $\mathcal{C}$ and is bounded,
- the other, which is the exterior of $\mathcal{C}$ is unbounded.

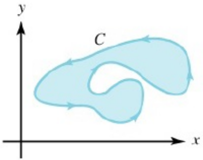


Definition

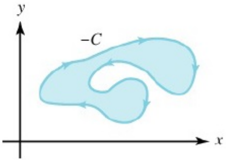
Let $\mathcal{C}$ be a simple closed (piecewise) smooth curve in $\mathbb{R}^2$.

- If $\mathcal{C}$ is parametrized so that the interior bounded domain of $\mathcal{C}$ is kept on the left as $R(t)$ moves around $\mathcal{C}$, then we say that $\mathcal{C}$ is oriented in the positive(counterclockwise or anticlockwise) direction.
- If $\mathcal{C}$ is parametrized so that the interior bounded domain of $\mathcal{C}$ is kept on the right as $R(t)$ moves around $\mathcal{C}$, then we say that $\mathcal{C}$ is oriented in the negative(clockwise) direction.





(a) A positively oriented contour.



(b) A negatively oriented contour.

Note:

- If a simple closed curve $\mathcal{C}$ is positively oriented, then the opposite curve $-\mathcal{C}$ is negatively oriented.
- If the orientation (or parametrization) of a simple closed curve $\mathcal{C}$ is not given, then it is understood that the simple closed curve $\mathcal{C}$ is oriented positively.

Examples:

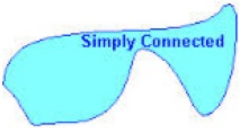
The circle $\mathcal{C}\ :\ R(t) = 2\cos(t)i + 2\sin(t)j$ for $0 \leq t \leq 2\pi$ is oriented positively.

The circle $-\mathcal{C} : R^*(t) = 2\cos(t)i - 2\sin(t)j$ for $t \in [-2\pi, 0]$ is oriented negatively.

Simply Connected Region

Definition

- A connected set S is said to be a simply connected set if every simple closed (piecewise) smooth curve C lying inside S encloses only points of S.
- A connected set S that is not simply connected is called a multiply connected set.



An open and connected set in $\mathbb{R}^2$ is called a domain in $\mathbb{R}^2$. A domain, together with some, none, or all of its boundary points, is called a region.

Green's Theorem

for

Simply Connected Regions

Green's Theorem for Simply Connected Regions

Theorem

Green's Theorem (Tangential/Circulation Form):
Let $F(x,y) = M(x,y)\,\hat{i} + N(x,y)\,\hat{j}$ be a continuously differentiable vector field on an open set S in $\mathbb{R}^2$.
Let $\mathcal{C}$ be a positively oriented, (piecewise) smooth, simple, closed curve in S such that the interior bounded domain D enclosed by $\mathcal{C}$ lies entirely inside S . Set the region $\mathcal{R} = D \cup C$. Note that $\mathcal{R} \subset S$ and $\mathcal{R}$ is simply connected.

$$\int_{\mathcal{C}} F \bullet dR = \int_{\mathcal{C}} F \bullet T ds = \iint_{\mathcal{R}} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$$

Example: Applying Green's Theorem

Determine the work done by the force field $F(x,y) = (x - xy)\,\hat{i} + y^2\,\hat{j}$, in moving a particle one complete round counterclockwise along the rectangle $\mathcal{C}$ with vertices $(0,0)$, $(4,0)$, $(4,6)$ and $(0,6)$.

Answer: Here $M(x,y) = x - xy$, and $N(x,y) = y^2$.

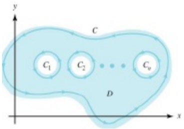
Thus, $\frac{\partial M}{\partial y} = -x$ and $\frac{\partial N}{\partial x} = 0$. By Green's theorem,

$$\int_{\mathcal{C}} F \bullet dR == \iint_{\mathcal{R}} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dy\, dx$$

$$= \int_{x=0}^{4} \int_{y=0}^{6} x dy dx = \int_{x=0}^{4} 6x dx = 48.$$

Green's Theorem
for
Multiply Connected Regions

Green's theorem for multiply connected regions



In $\mathbb{R}^2$, suppose that

- $\mathcal{C}$ is a simple closed piecewise smooth curve positively oriented.
- $\mathcal{C}_k$ $(k = 1, 2, \ldots, n)$ denotes a finite number of simple closed piecewise smooth curves, all positively oriented, that are interior to $\mathcal{C}$ and whose interiors have no points in common.

Let $\mathcal{D}$ denotes closed region consisting of all points within and on $\mathcal{C}$ except for the points interior to each $\mathcal{C}_k$ (See:Blue color region).
Let $F(x, y) = M(x, y)\,\hat{i} + N(x, y)\,\hat{j}$ be a continuously differentiable vector field on an open set S containing the region $\mathcal{D}$. Then

$$\int_{\mathcal{C}} F \bullet dR - \sum_{k=1}^{n} \int_{\mathcal{C}_k} F \bullet dR == \iint_{\mathcal{D}} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy.$$

Example

Let $F(x,y) = \frac{-y\hat{i}+x\hat{j}}{x^2+y^2}$ for $(x,y) \in \mathcal{D}^* = \mathbb{R}^2 \setminus \{(0,0)\}$. Let $\mathcal{C}$ be any simple closed, piecewise smooth curve in $\mathcal{D}^*$ which encloses the origin $(0,0)$. Find $\int_{\mathcal{C}} F \bullet dR$.

Done in the class.

Answer: 2π.

Same Example, but the curve C does not enclose the origin

Let $F(x,y) = \frac{-y\hat{i}+x\hat{j}}{x^2+y^2}$ for $(x,y) \in \mathcal{D}^* = \mathbb{R}^2 \setminus \{(0,0)\}$. Let $\mathcal{C}$ be any simple closed, piecewise smooth curve in $\mathcal{D}^*$ which does NOT enclose the origin $(0,0)$. Find $\int_{\mathcal{C}} F \bullet dR$.

Answer: Here $M(x,y) = -\frac{y}{x^2+y^2}$ and $N(x,y) = \frac{x}{x^2+y^2}$.

$$\frac{\partial M}{\partial y} = \frac{y^2 - x^2}{(x^2+y^2)^2} = \frac{\partial N}{\partial x}.$$

Let $\mathcal{C}$ be any simple closed, piecewise smooth curve in $\mathcal{D}^*$ which does NOT enclose the origin $(0,0)$, Then by Green's theorem

$$\int_{\mathcal{C}} F \bullet dR = \iint_{\mathcal{D}} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy = \iint_{\mathcal{D}} 0 dxdy = 0.$$

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