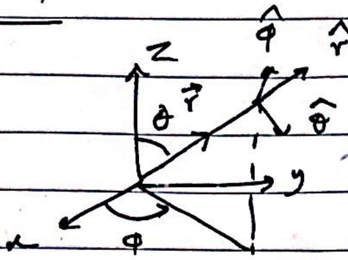
EPHY105L

Solutions to Problem Sheet 1



1. $\hat{r} = \hat{x}\sin\theta\cos\varphi + \hat{y}\sin\theta\sin\varphi + \hat{z}\cos\theta$

$\hat{\theta} = \hat{x}\cos\theta\cos\varphi + \hat{y}\cos\theta\sin\varphi - \sin\theta\,\hat{z}$

$\hat{\varphi} = -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y}$

$\hat{x} = \hat{r}\sin\theta\cos\varphi + \hat{\theta}\cos\theta\cos\varphi - \hat{\varphi}\sin\varphi$

$\hat{y} = \hat{r}\sin\theta\sin\varphi + \hat{\theta}\cos\theta\sin\varphi + \hat{\varphi}\cos\varphi$

$\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$

2. $\hat{r} = \hat{x}\cos\varphi + \hat{y}\sin\varphi$

$\hat{\varphi} = -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y}$

$\hat{z} = \hat{z}$

$\hat{x} = \hat{r}\cos\varphi - \hat{y}\sin\varphi$

$\hat{y} = \hat{r}\sin\varphi + \hat{y}\cos\varphi$

$\hat{z} = \hat{z}$

3. $r = \sqrt{x^2+y^2+z^2}$; $\theta = \cos^{-1}\left(\frac{z}{r}\right)$; $\varphi = \tan^{-1}\left(\frac{y}{x}\right)$

a) $r = \sqrt{10^2+0^2+0^2} = 10$; $\theta = \cos^{-1}\left(\frac{0}{10}\right) = \frac{\pi}{2}$, $\varphi = \tan^{-1}\left(\frac{0}{10}\right) = 0$

b) $r = \sqrt{0^2+0^2+5^2} = 5$; $\theta = \cos^{-1}\left(\frac{5}{5}\right) = 0$; $\varphi =$ ~~tan~~ undefined as the point is on z-axis.

c) $r = \sqrt{29}$; $\theta = \cos^{-1}\left(\frac{0}{\sqrt{29}}\right) = \frac{\pi}{2}$; $\varphi = \tan^{-1}\left(\frac{2}{5}\right)$.

d) $r = 3\sqrt{2}$; $\theta = \cos^{-1}\left(\frac{3}{3\sqrt{2}}\right) = \frac{\pi}{4}$; $\varphi = \tan^{-1}\left(\frac{3}{0}\right) = \frac{\pi}{2}$

$$\hat{r} = \hat{x}\sin\theta\cos\varphi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$$

a) $\hat{r} = \hat{x}\sin\frac{\pi}{2}\cos 0 + \hat{y}\sin\frac{\pi}{2}\sin 0 + \hat{z}\cos\frac{\pi}{2} = \hat{x}$

b) $\hat{r} = \hat{x}\sin 0\cos\phi + \hat{y}\sin 0\sin\varphi + \hat{z}\cos 0 = \hat{z}$

c) $\hat{r} = \hat{x}\sin\frac{\pi}{2}\cos\left[\tan^{-1}\left(\frac{2}{3}\right)\right] + \hat{y}\sin\frac{\pi}{2}\sin\left[\tan^{-1}\left(\frac{2}{3}\right)\right] + \hat{z}\cos\frac{\pi}{2}$

$= \hat{x}\cos\left[\tan^{-1}\left(\frac{2}{3}\right)\right] + \hat{y}\sin\left[\tan^{-1}\left(\frac{2}{3}\right)\right]$

d) $\hat{r} = \hat{x}\sin\frac{\pi}{4}\cos\frac{\pi}{2} + \hat{y}\sin\frac{\pi}{4}\sin\frac{\pi}{2} + \hat{z}\cos\frac{\pi}{4}$

$= \frac{1}{\sqrt{2}}(\hat{y} + \hat{z})$

4) $x = r\sin\theta\cos\varphi$; $y = r\sin\theta\sin\phi$; $z = r\cos\theta$

a) $x = 5\cos\frac{\pi}{4}\sin\frac{\pi}{2} = 5/\sqrt{2}$

$y = 5\sin\frac{\pi}{2}\sin\frac{\pi}{4} = 5/\sqrt{2}$

$z = 5\cos\pi/2 = 0$

Cartesian Coordinates : $\left(\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}}, 0\right)$

b) $x = 3\sin\frac{\pi}{4}\cos 0 = 3/\sqrt{2}$

$y = 3\sin\frac{\pi}{4}\sin 0 = 0$

$z = 3\cos\pi/4 = 3/\sqrt{2}$

Cartesian Coordinates $\left(\frac{3}{\sqrt{2}}, 0, \frac{3}{\sqrt{2}}\right)$

c) $x = 8\sin\frac{\pi}{2}\cos\pi = -8$

$y = 8\sin\frac{\pi}{2}\sin\pi = 0$

$z = 8\cos\frac{\pi}{2} = 0$

Cartesian Coordinates $(-8, 0, 0)$

5) $$\vec{\nabla} f = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z}$$

a) $\vec{\nabla} f = \hat{x}\cdot 2x + \hat{y}\, 2y + \hat{z}\cdot 2z$

$\vec{\nabla} f$ at $(2,-1,2) = 4\hat{x} - 2\hat{y} + 4\hat{z}$

b) $\vec{\nabla} g = \hat{x}\cdot 2x + \hat{y}\, 2z - 1\hat{z}$

$\vec{\nabla} g$ at $(2,-1,2) = 4\hat{x} - 2\hat{y} - \hat{z}$

Angle between the surfaces is the same as the angle between the gradients. Hence angle θ is

$$\theta = \cos^{-1}\left[\frac{\vec{\nabla} f\cdot\vec{\nabla} g}{|\vec{\nabla} f|\,|\vec{\nabla} g|}\right] = \cos^{-1}\left[\frac{16+4-4}{\sqrt{36}\,\sqrt{21}}\right]$$

$$= \cos^{-1}\left[\frac{8}{3\sqrt{21}}\right]$$

6) Maximum directional derivative $= |\vec{\nabla} f|$

Now $\vec{\nabla} f = \hat{x}\, 2xyz^3 + \hat{y}\, x^2 z^3 + 3\hat{z}\, x^2 y z^2$

At the point $(2,1,-1)$

$$\vec{\nabla} f = -4\hat{x} - 4\hat{y} + 12\hat{z}$$

Maximum directional derivative at $(2,1,-1)$ $= \sqrt{16+16+144} = \sqrt{176} = 13.27$

7) a) $\vec{\nabla}\cdot\vec{F_1} = \frac{\partial F_{1x}}{\partial x} + \frac{\partial F_{1y}}{\partial y} + \frac{\partial F_{1z}}{\partial z} = 1-1 = 0$

b) $\vec{\nabla}\cdot\vec{F_2} = \frac{\partial}{\partial x}0 + \frac{\partial}{\partial y}0 + \frac{\partial}{\partial z}z = 1$

c) $\vec{\nabla}\cdot\vec{F_3} = \frac{\partial}{\partial x}(\alpha x) + \frac{\partial}{\partial y}(\alpha y) + \frac{\partial}{\partial z}(\alpha z) = 3\alpha$

d) $\vec{\nabla}\cdot\vec{F_4} = \frac{\partial}{\partial x}\left[\frac{\beta x}{(x^2+y^2+z^2)^{3/2}}\right] + \frac{\partial}{\partial y}\left[\frac{\beta y}{(x^2+y^2+z^2)^{3/2}}\right] + \frac{\partial}{\partial z}\left[\frac{1^2}{(x^2+y^2+z^2)^{3/2}}\right]$

$$= \beta\left[\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3x^2}{(x^2+y^2+z^2)^{5/2}} + \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3y^2}{(x^2+y^2+z^2)^{5/2}}\right.$$

$$\left. + \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3z^2}{(x^2+y^2+z^2)^{5/2}}\right]$$

$$= \beta\left[\frac{3}{(x^2+y^2+z^2)^{3/2}} - \frac{3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}}\right]$$

$$= 0$$

8) a) $\vec{\nabla}\times\vec{F_1} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \alpha y & 0 & 0 \end{vmatrix} = \hat{x}\,(0-0) + \hat{y}\left(\frac{\partial}{\partial z}\alpha y - 0\right)$

$+\hat{z}\left(0 - \frac{\partial \alpha y}{\partial y}\right)$

$= -\alpha\hat{z}$

b) $\nabla\times\vec{F_2} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \alpha x & \beta y^2 & 0 \end{vmatrix}$

$= \hat{x}\left[0 - \frac{\partial}{\partial z}(\beta y^2)\right] + \hat{y}\left[\frac{\partial(\alpha x)}{\partial z} - 0\right] + \hat{z}\left[\frac{\partial}{\partial x}(\beta y^2) - \frac{\partial}{\partial y}(\alpha x)\right]$

$= 0$

c) $\nabla\times\vec{F_3} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & 3xz^2 & -2xz \end{vmatrix}$

$= \hat{x}\left[\frac{\partial}{\partial y}(-2xz) - \frac{\partial}{\partial z}(3xz^2)\right] + \hat{y}\left[\frac{\partial}{\partial z}x^2 - \frac{\partial}{\partial x}(-2xz)\right]$

$+\hat{z}\left[\frac{\partial}{\partial x}(3xz^2) - \frac{\partial}{\partial y}x^2\right]$

$= \hat{x}\,(-6xz) + \hat{y}\,(+2z) + \hat{z}\,(3z^2)$

$= -6xz\hat{x} + 2z\hat{y} + 3z^2\hat{z}$

9 $\quad \nabla f = \hat{x}\frac{\partial}{\partial x}(\alpha x y^2) + \hat{y}\frac{\partial}{\partial y}(\alpha x y^2) + \hat{z}\frac{\partial}{\partial z}(\alpha x y^2)$

$= \alpha y^2 \hat{x} + 2\alpha x y \hat{y}$

$$\nabla \times (\nabla f) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \alpha y^2 & 2\alpha x y & 0 \end{vmatrix}$$

$= \hat{x}(0-0) + \hat{y}(0-0) + \hat{z}(2\alpha y - 2\alpha y)$

$= 0$

10. $\nabla \times \vec{G} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & 3xz^2 & -2x^2 \end{vmatrix}$

$= \hat{x}(0 - 6xz) + \hat{y}(0 + 2z) + \hat{z}(3z^2 - 0)$

$= -6xz\,\hat{x} + 2z\,\hat{y} + 3z^2\,\hat{z}$

$\vec{\nabla}\cdot(\nabla\times\vec{G}) = \frac{\partial}{\partial x}(-6xz) + \frac{\partial}{\partial y}(2z) + \frac{\partial}{\partial z}(3z^2)$

$= -6z + 0 + 6z$

$= 0$