Differentiability
(Lecture 15 & 16)

Engineering Calculus

School of Engineering and Applied Sciences
Department of Mathematics
Bennett University

Definition

Let I be an interval which is not singleton and let f be a function defined on I. A function f is said to be differentiable at $x \in I$ if the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{exists in } \mathbb{R}.$$

- If the above limit exists, it is called the derivative of f at x and is denoted by $f'(x)$.
- $f : I \to \mathbb{R}$ is said to be differentiable if f is differentiable at each $x \in I$, then f' is a function on I.
- If f is differentiable at $c \in I$, then the derivative of f at c is

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

Example

If $f(x) = x^2$, then

$$f'(x) = \lim_{h \to 0} \frac{2xh + h^2}{h} = 2x.$$

Theorem (Differentiability implies continuity)

If $f(x)$ is differentiable at c, then it is continuous at c.

Proof: For $x \neq c$, we may write,

$$f(x) = (x-c)\frac{f(x)-f(c)}{(x-c)} + f(c).$$

Now taking the limit $x \to c$ and noting that $\lim_{x \to c}(x-c) = 0$ and $\lim_{x \to c} \frac{f(x)-f(c)}{(x-c)} = f'(c)$, we get the result.

Remark

The continuity of $f : I \to \mathbb{R}$ at a point does not assure the existence of the derivative at that point. For example, if $f(x) = |x|$ for $x \in \mathbb{R}$, then for $x \neq 0$

$$\frac{f(x)-f(0)}{x-0} = \frac{|x|}{x} = \begin{cases} 1 & x > 0 \\ -1 & x < 0. \end{cases}$$

Thus the limit at 0 does not exist and therefore the function is not differentiable at 0.

Definition

Let $I = [a, b]$ be an interval and a function $f : I \to \mathbb{R}$.

(a) f is said to be differentiable at a if $\lim_{x \to a^+} \frac{f(x)-f(a)}{x-a}$ exists. The derivative of f at a is denoted by $f'(a)$.

(b) f is said to be differentiable at b if $\lim_{x \to b^-} \frac{f(x)-f(b)}{x-b}$ exists. The derivative of f at b is denoted by $f'(b)$.

(c) If c is an interior point of I, then f is said to be differentiable at c if both the limits

$$\lim_{x \to c^+} \frac{f(x)-f(c)}{x-c} \quad \text{and} \quad \lim_{x \to c^-} \frac{f(x)-f(c)}{x-c}$$

exist and be equal. The derivative of f at c is denoted by $f'(c)$.

Example

Let $f : [0, 2] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 2 - x^2 & 1 < x \leq 2. \end{cases}$$

Then the derived function f' and its domain

$$f'(x) = \begin{cases} 1 & 0 \leq x < 1 \\ -2x & 1 < x \leq 2. \end{cases}$$

The domain of f' is $[0,1) \cup (1,2]$.

Theorem

Let f, g be differentiable at $c \in (a,b)$. Then $f \pm g$, fg, $\dfrac{f}{g}$ $(g(c) \neq 0)$ is also differentiable at c.

Proof: First note that

$$\frac{(fg)(x) - (fg)(c)}{x - c} = f(x)\frac{g(x) - g(c)}{x - c} + g(c)\frac{f(x) - f(c)}{x - c}.$$

Now taking the limit $x \to c$, we get the product formula

$$(fg)'(c) = f(c)g'(c) + f'(c)g(c).$$

Since $g(c) \neq 0$ and g is continuous, we get $g(x) \neq 0$ in a small interval around c. Therefore

$$\frac{f}{g}(x) - \frac{f}{g}(c) = \frac{g(c)f(x) - g(c)f(c) + g(c)f(c) - g(x)f(c)}{g(x)g(c)}$$

Hence

$$\frac{(f/g)(x) - (f/g)(c)}{x-c} = \left\{ g(c)\frac{f(x)-f(c)}{x-c} - f(c)\frac{g(x)-g(c)}{x-c} \right\} \frac{1}{g(x)g(c)}$$

Now taking the limit $x \to c$, we get

$$\left(\frac{f}{g}\right)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{g^2(c)}.$$

Theorem (Chain Rule)

Suppose $f(x)$ is differentiable at c and g is differentiable at $f(c)$, then $h(x) := g(f(x))$ is differentiable at c and $h'(c) = g'(f(c))\, f'(c)$.

Example

Suppose that f is defined by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Using the product rule and the chain rule we obtain

$$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x} \quad \text{for} \quad x \neq 0.$$

Derivative of f at $x = 0$

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin \frac{1}{x}}{x} = \lim_{x \to 0} x \sin \frac{1}{x} = 0.$$

Hence, the derivative f' of f exists at all $x \in \mathbb{R}$. But $\lim_{x \to 0} f'(x)$ does not exist, since $\lim_{x \to 0} \cos \frac{1}{x}$ does not exist. Therefore f' is discontinuous at $x = 0$. Thus, a function f that is differentiable at every point of $\mathbb{R}$ need not have a continuous derivative f'.

Local extremum

A point $x = c$ is called **local maximum** of $f(x)$, if there exists $\delta > 0$ such that

$$c - \delta < x < c + \delta \implies f(c) \geq f(x).$$

Similarly, one can define **local minimum**: $x = b$ is a local minimum of $f(x)$ if there exists $\delta > 0$ such that

$$b - \delta < x < b + \delta \implies f(b) \leq f(x).$$

Theorem

Let $f(x)$ be a differentiable function on (a, b) and let $c \in (a, b)$ is a local maximum or a local minimum of f. Then $f'(c) = 0$.

Proof: Suppose f has a local maximum at $c \in (a, b)$. Let δ be as in the above definition. Then

$$x \in (c, c + \delta) \implies \frac{f(x) - f(c)}{x - c} \leq 0$$

$$x \in (c - \delta, c) \implies \frac{f(x) - f(c)}{x - c} \geq 0.$$

Now taking the limit $x \to c$, we get $f'(c) = 0$.

Remark

Previous theorem is not valid if c is a or b. For example, if we consider the function $f : [0, 1] \to \mathbb{R}$ such that $f(x) = x$, then f has maximum at 1 but $f'(x) = 1$ for all $x \in [0, 1]$.

Rolle's Theorem

Let $f(x)$ be a continuous function on $[a, b]$ and differentiable on (a, b) such that $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Problem

Show that the equation $x^{13} + 7x^3 - 5 = 0$ has exactly one(real) root.

Solution: Let $f(x) = x^{13} + 7x^3 - 5$. Then $f(0) < 0$ and $f(1) > 0$. By the IVP, there is at least one positive root of $f(x) = 0$. If there are two distinct positive roots then by Rolle's theorem there is some $x_0 > 0$ such that $f'(x_0) = 0$, which is not true. Moreover, we observe that $f'(x) > 0$ for all x means that f is strictly increasing.

Problem

Let f and g be functions, continuous on $[a,b]$, differentiable on (a,b) and let $f(a)=f(b)=0$. Prove that there is a point $c \in (a,b)$ such that $g'(c)f(c)+f'(c)=0$.

Solution: Define $h(x)=f(x)e^{g(x)}$. Here h is a continuous function on $[a,b]$ and differentiable on (a,b) such that $h(a)=h(b)=0$, by Rolle's theorem, there exist $c \in (a,b)$ such that $h'(c)=0$. Since

$$h'(x)=[g'(x)f(x)+f'(x)]e^{g(x)} \quad \text{and} \quad e^{\alpha} \neq 0$$

for any $\alpha \in \mathbb{R}$, we see that $g'(c)f(c)+f'(c)=0$.

Question

If the value of f at the end points a and b are not same, is it true that there is some $c \in [a,b]$ such that the tangent line at c is parallel to the line connecting the endpoints of the curve?

- The answer is yes and this is essentially the **Mean Value Theorem**.

Mean-Value Theorem (MVT)

Let f be a continuous function on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a).$$

Proof: Let $l(x)$ be a straight line joining $(a, f(a))$ and $(b, f(b))$. Consider the function $g(x) = f(x) - l(x)$. Then $g(a) = g(b) = 0$. Hence by Rolle's theorem

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

Corollary

If f is a differentiable function on (a, b) and $f' = 0$, then f is constant.

Proof: By mean value theorem $f(x) - f(y) = 0$ for all $x, y \in (a, b)$.

Problem

Show that $|\cos x - \cos y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

Solution: Let $x, y \in \mathbb{R}$. By the Mean-Value theorem, $\cos x - \cos y = -\sin c\ (x - y)$ for some c between x and y. Using the fact that $|\sin x| \leq 1$, we obtain that $|\cos x - \cos y| \leq |x - y|$.

- Show that $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

Problem

If $f(x)$ is differentiable and $\sup |f'(x)| < C$ for some C. Then, f is uniformly continuous.

Solution: Apply mean value theorem to get $|f(x) - f(y)| \leq C|x - y|$ for all x, y.

Definition

A function $f(x)$ is **strictly increasing** on an interval I, if for $x, y \in I$ with $x < y$ we have $f(x) < f(y)$. We say f is **strictly decreasing** if $x < y$ in I implies $f(x) > f(y)$.

Theorem

A differentiable function f is

(a) increasing (respectively strictly increasing) in (a, b) if $f'(x) \geq 0$ (resp. $f'(x) > 0$) for all $x \in (a, b)$.

(b) decreasing (respectively strictly decreasing) in (a, b) if $f'(x) \leq 0$ (resp. $f'(x) < 0$) for all $x \in (a, b)$.

(c) one-one (i.e, $f(x) \neq f(y)$ whenever $x \neq y$) if $f'(x) \neq 0$ for all $x \in (a, b)$.

Proof: (a) Choose x, y in (a, b) such $x < y$. Then by MVT, for some $c \in (x, y)$

$$\frac{f(x) - f(y)}{x - y} = f'(c) > 0.$$

Hence $f(x) < f(y)$.
(b) Choose x, y in (a, b) such $x < y$. Then by MVT, for some $c \in (x, y)$

$$\frac{f(x) - f(y)}{x - y} = f'(c) < 0.$$

Hence $f(x) > f(y)$.
(c) Choose x, y in (a, b) such $x \neq y$. Then by MVT, for some $c \in (x, y)$

$$\frac{f(x) - f(y)}{x - y} = f'(c) \neq 0.$$

Hence $f(x) \neq f(y)$.

Intermediate value property of derivatives

Let $f : I \to \mathbb{R}$ be differentiable and let $a, b \in I$ with $a < b$. If $f'(a) < k < f'(b)$, then there exists $c \in (a, b)$ such that $f'(c) = k$.

Example

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable such that $f(-1) = 5,\ f(0) = 0$ and $f(1) = 10$. Then there exist $c_1, c_2 \in (-1, 1)$ such that $f'(c_1) = -3$ and $f'(c_2) = 3$.

Solution: By the mean value theorem, there exist $\alpha \in (-1, 0)$ and $\beta \in (0, 1)$ such that

$$f'(\alpha) = \frac{f(0) - f(-1)}{0 - (-1)} = -5 \quad \text{and} \quad f'(\beta) = \frac{f(1) - f(0)}{1 - 0} = 10.$$

Hence by the intermediate value property of derivatives, there exist $c_1, c_2 \in (\alpha, \beta)$ (and so $c_1, c_2 \in (-1, 1)$) such that $f'(c_1) = -3$ and $f'(c_2) = 3$.

• Let $\lim_{x\to c} f(x) = A$ and $\lim_{x\to c} g(x) = B$. If $B \neq 0$ then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{A}{B}.$$

• If $B = 0$ and $A \neq 0$, then the limit is infinite.
• If $B = 0$ and $A = 0$, then the limit is said to be **indeterminate**. In this case the limit may not exist or may be any real value, depending on f, g. The symbolism $\frac{0}{0}$ is used to refer this situation. Another indeterminate form $\frac{\infty}{\infty}$.
• **Example:** Let $\alpha \in \mathbb{R}$, and $f(x) = \alpha x,\ g(x) = x$, then

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{\alpha x}{x} = \lim_{x\to 0} \alpha = \alpha.$$

Thus the indeterminate form $\frac{0}{0}$ can lead to any real number α as a limit.

Theorem

Let f and g be defined on $[a, b]$, let $f(a) = g(a) = 0$ and let $g(x) \neq 0$ for $a < x < b$. If f and g are differentiable at a and if $g'(a) \neq 0$, then the limit of f/g at a exists and is equal to $f'(a)/g'(a)$. Thus $\lim_{x\to a^+} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$.

Remark

The hypothesis that $f(a) = g(a) = 0$ is essential. For example, if $f(x) = x + 17$ and $g(x) = 2x + 3$ for $x \in \mathbb{R}$, then

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{17}{3}, \quad \text{while} \quad \frac{f'(0)}{g'(0)} = \frac{1}{2}.$$

Example

$$\lim_{x \to 0} \frac{x^2 + x}{\sin 2x} = \frac{2 \cdot 0 + 1}{2 \cos 0} = \frac{1}{2}.$$

Theorem

Let $-\infty \leq a < b \leq \infty$ and let f, g be differentiable on (a, b) such that $g'(x) \neq 0$ for all $x \in (a, b)$. Suppose that $\lim\limits_{x \to a^+} f(x) = 0 = \lim\limits_{x \to a^+} g(x)$. If $\lim\limits_{x \to a^+} \frac{f'(x)}{g'(x)} = L$, then $\lim\limits_{x \to a^+} \frac{f(x)}{g(x)} = L$.

Examples

Evaluate (i) $\lim_{x\to 0}\left[\frac{1-\cos x}{x^2}\right]$, (ii) $\lim_{x\to 0}\frac{e^x-1}{x}$, (iii) $\lim_{x\to 1}\left[\frac{\ln x}{x-1}\right]$.

Solution: (i)

$$\begin{aligned}
& \lim_{x\to 0}\left[\frac{1-\cos x}{x^2}\right] && \left(\frac{0}{0}\text{ form}\right)\\
= & \lim_{x\to 0}\frac{\sin x}{2x} && \left(\frac{0}{0}\text{ form}\right)\\
= & \lim_{x\to 0}\frac{\cos x}{2}\\
= & \frac{1}{2}.
\end{aligned}$$

(ii) $\lim_{x\to 0}\frac{e^x-1}{x}\quad\left(\frac{0}{0}\text{ form}\right)=\lim_{x\to 0}\frac{e^x}{1}=1.$

(iii) $\lim_{x\to 1}\left[\frac{\ln x}{x-1}\right]\;\left(\frac{0}{0}\text{ form}\right)=\lim_{x\to 1}\frac{(1/x)}{1}=1.$

L'Hospital's Rules

Theorem

Suppose f and g are differentiable at every point in (a, ∞) for some $a > 0$. Suppose $\lim_{x\to\infty} f(x) = 0 = \lim_{x\to\infty} g(x)$ and $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ exists. Then $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ exists and

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

Theorem

Suppose f and g are continuous functions on $[a, b]$ which are differentiable at every point in (a, b), except possibly at $x_0 \in [a, b]$. Suppose $\lim_{x\to x_0} f(x) = \infty = \lim_{x\to x_0} g(x)$ and $\lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ exists. Then $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ exists and

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}.$$

Examples

Evaluate (i) $\lim_{x\to\infty} \dfrac{\ln x}{x}$, (ii) $\lim_{x\to\infty} e^{-x}x^2$, (iii) $\lim_{x\to 0+} \dfrac{\ln\sin x}{\ln x}$.

Solution: (i) $\lim_{x\to\infty} \dfrac{\ln x}{x} \quad \left(\dfrac{\infty}{\infty} \text{ form}\right) = \lim_{x\to\infty} \dfrac{(1/x)}{1} = 0.$

(ii)

$$\begin{aligned}
\lim_{x\to\infty} e^{-x}x^2 &= \lim_{x\to\infty} \frac{x^2}{e^x} \quad \left(\frac{\infty}{\infty} \text{ form}\right) \\
&= \lim_{x\to\infty} \frac{2x}{e^x} \quad \left(\frac{\infty}{\infty} \text{ form}\right) \\
&= \lim_{x\to\infty} \frac{2}{e^x} = 0.
\end{aligned}$$

(iii) $\lim_{x\to 0+} \dfrac{\ln\sin x}{\ln x} \quad \left(\dfrac{\infty}{\infty} \text{ form}\right) = \lim_{x\to 0+} \dfrac{(\cos x/\sin x)}{(1/x)} = \lim_{x\to 0+} \left[\dfrac{x}{\sin x}\right] \cdot \lim_{x\to 0+} \cos x = 1.$


Thank You