Continuity

Def^n:- $f: D \to R$, $c \in D$, f is cont at $x=c$ if $\lim\limits_{x \to c} f(x) = f(c)$.

Ex:- $f(x) = \begin{cases} x^2 \sin \frac{1}{x} , & x \neq 0 \\ 0 , & x = 0 \end{cases}$

show that f is continuous at 0.

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \sin \frac{1}{x} = 0 = f(0)$$

Ex:- $f(x) = \begin{cases} x \cos \frac{1}{x^2} , & x \neq 0 \\ 0 , & x = 0 \end{cases}$

Def^n:- ($\epsilon - \delta$ defination)

f is continuous at $x=c$, if $\epsilon > 0$ $\exists \delta > 0$ s.t $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$

$\underset{c-\delta}{(}\ \underset{c}{\cdot}\ \underset{c+\delta}{)}$ $\quad$ $\underset{f(c)-\epsilon}{(}\ \underset{f(c)}{\cdot}\ \underset{f(c)+\epsilon}{)}$

Ex:- $f(x) = \begin{cases} x^2 \sin \frac{1}{x} , & x \neq 0 \\ 0 , & x = 0 \end{cases}$ continuous at $x=0$

we have to show, $\epsilon > 0$ $\exists$ $\delta > 0$ s.t

$$|x - 0| < \delta \Rightarrow |f(x) - f(0)| < \epsilon.$$

$$|f(x) - f(0)| = |x^2 \sin \tfrac{1}{x}| \leq x^2 < \epsilon$$

whenever $x < \sqrt{\epsilon}$

choose $\delta = \sqrt{\epsilon}$.

$\epsilon > 0$ $\exists$ $\delta = \sqrt{\epsilon} > 0$

s.t $|x - 0| < \underset{= \delta}{\sqrt{\epsilon}} \Rightarrow |f(x) - 0| < \epsilon$.

Ex:- $f(x) = \sin x$, $c \in R$, show that $\sin x$ continuous at $x = c$.

show: $\epsilon > 0 \ \exists \ \delta > 0$ s.t $|x - c| < \delta$
$\Rightarrow |\sin x - \sin c| < \epsilon$.

$$|\sin x - \sin c| = \left|2 \cos \frac{x+c}{2} \sin \frac{x-c}{2}\right|$$

$$\leq 2 \left|\cos \frac{x+c}{2}\right| \left|\sin \frac{x-c}{2}\right|$$

$(\because |\cos x| \leq 1)$ $\leq 2 \left|\sin \frac{(x-c)}{2}\right|$

$(\because |\sin x| \leq |x|)$ $\leq 2 \cdot \frac{|x-c|}{2} = |x - c|$

$|\sin x - \sin c| < \epsilon$ whenever $|x - c| < \underset{\delta}{\overset{\shortparallel}{\epsilon}}$
choose $\delta = \epsilon$.

Def^n:- $f : D \to R$ is call continuous on D if it is conti at each point $c \in D$.

✓ $\sin x$ is conti on $\mathbb{R}$.

Ex:- $f(x) = x$, $x \in R$, conti on R.
show that.

Sequential criteria of continuity

f is conti. at $c \Leftrightarrow$ for every $\{x_n\}$
$x_n \to c \Rightarrow f(x_n) \to f(c)$ as $n \to \infty$

Ex:- $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ show that f is conti. at 0.

take, $\{x_n\}$, $x_n \to 0$ as $n \to \infty$
$\Rightarrow f(x_n) \to f(0)$ as $n \to \infty$

$|f(x_n)| = |x_n^2 \sin \frac{1}{x_n}| \le x_n^2 \rightarrow 0$ as $n \to \infty$

$x_n \to 0 \Rightarrow f(x_n) \to f(0) = 0$ as $n \to \infty$

Ex:- $f(x) = \begin{cases} \frac{1}{x} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ it is not conti at 0.

$$x_n = \frac{1}{2n\pi + \frac{\pi}{2}} \rightarrow 0 \text{ as } n \to \infty$$

$$f(x_n) = \frac{1}{x_n} \sin\left(\frac{1}{x_n}\right) = \frac{1}{x_n} \rightarrow \infty \text{ as } n \to \infty$$

$\therefore f$ is not conti at 0.

Discontinuous function

not conti $\Rightarrow$ discontinuous.

Types of discontinuity

① Removable discontinuity :-

$f: D \to R$, $c \in D$ $\lim_{x \to c} f(x) \neq f(c)$

Ex:- $f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$\lim_{x \to 0} f(x) = 1 \neq f(0) \rightarrow f$ has R.D at 0.

$f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases} \rightarrow$ conti

Ex:- $f(x) = \begin{cases} \frac{x^2 - 4}{x - 2}, & x > 2 \\ 10, & x = 2 \end{cases}$ at $x = 2$

$(2, \infty)$