**Solutions for Problem Sheet 2**

1. Electric field due to the point charge

$$\vec{\boldsymbol{E}} = \frac{Q}{4\pi\epsilon_0}\frac{(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_0)}{|(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_0)|^3}$$

where

$$\vec{\boldsymbol{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\widehat{\boldsymbol{k}}$$
$$\vec{\boldsymbol{r}}_0 = x_0\hat{\boldsymbol{\imath}} + y_0\hat{\boldsymbol{\jmath}} + z_0\widehat{\boldsymbol{k}}$$

Thus

$$\vec{\boldsymbol{E}} = \frac{Q}{4\pi\epsilon_0}\frac{(x - x_0)\hat{\boldsymbol{\imath}} + (y - y_0)\hat{\boldsymbol{\jmath}} + (z - z_0)\widehat{\boldsymbol{k}}}{[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{3/2}}$$

2. (a) Fields at a point with coordinates $(x, y, z)$ due to the two point charges are given by

$$\vec{\boldsymbol{E}}_+ = \frac{Q}{4\pi\epsilon_0}\frac{(x - a)\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\widehat{\boldsymbol{k}}}{[(x - a)^2 + y^2 + z^2]^{3/2}}$$

$$\vec{\boldsymbol{E}}_- = -\frac{Q}{4\pi\epsilon_0}\frac{(x + a)\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\widehat{\boldsymbol{k}}}{[(x + a)^2 + y^2 + z^2]^{3/2}}$$

Total field is

$$\vec{\boldsymbol{E}} = \vec{\boldsymbol{E}}_+ + \vec{\boldsymbol{E}}_-$$

(b)

$$\nabla.\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$\frac{\partial E_x}{\partial x} = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{[(x - a)^2 + y^2 + z^2]^{3/2}} - \frac{3}{2}\frac{2(x - a)(x - a)}{[(x - a)^2 + y^2 + z^2]^{5/2}} - \frac{1}{[(x + a)^2 + y^2 + z^2]^{3/2}} + \frac{3}{2}\frac{2(x + a)(x + a)}{[(x + a)^2 + y^2 + z^2]^{5/2}}\right]$$

One can similarly calculate $\frac{\partial E_y}{\partial y}$ and $\frac{\partial E_z}{\partial z}$. Adding all the three terms we will get

$$\nabla.\vec{E} = 0$$

at all points except at the points where the two point charges are placed. Remember that this is nothing but the differential form of Gauss's law.

(c) The electrostatic potential is

$$V(x, y, z) = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{[(x - a)^2 + y^2 + z^2]^{1/2}} - \frac{1}{[(x + a)^2 + y^2 + z^2]^{1/2}}\right]$$

$$\nabla V = \hat{\imath}\frac{\partial V}{\partial x} + \hat{\jmath}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}$$

Now

$$\frac{\partial V}{\partial x} = \frac{Q}{4\pi\epsilon_0}\left[-\frac{1}{2}\frac{2(x - a)}{[(x - a)^2 + y^2 + z^2]^{3/2}} + \frac{1}{2}\frac{2(x + a)}{[(x + a)^2 + y^2 + z^2]^{3/2}}\right]$$

Similarly one can evaluate $\frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial z}$. We then obtain

$$\nabla V = -\frac{Q}{4\pi\epsilon_0}\left[\frac{(x - a)\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\widehat{\boldsymbol{k}}}{[(x - a)^2 + y^2 + z^2]^{3/2}} - \frac{(x + a)\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\widehat{\boldsymbol{k}}}{[(x + a)^2 + y^2 + z^2]^{3/2}}\right] = -\vec{\boldsymbol{E}}$$

5. (a) and (b) Since the field inside the conductor must be zero, the total charge induced in the inner surface must be equal and opposite to the charge placed within the cavity. Due to spherical symmetry the charge on the inner surface will be uniformly distributed throughout the inner surface. Thus the charge on the inner surface will be -10 mC and equally distributed across the surface.

Since the conductor is neutral an equal amount of charge must exist at the outer surface. Thus the total charge on the outer surface would be + 10 mC. Again due to spherical symmetry the charge will be uniformly distributed across the outer surface.

(c) If the point charge is not placed at the center of the cavity then the charge of -10mC will not be uniformly distributed on the inner surface. On the other hand the charge of +10 mC will be uniformly distributed on the outer surface independent of the position of the point charge within the cavity.

6. Due to spherical symmetry in the problem the electric field will be pointing radially; for a positive charge distribution the electric field will point away from the center while if the charge were negative it will point radially inward.

   We now use Gauss's law in integral form

$$\oiint \vec{\boldsymbol{E}}.d\vec{\boldsymbol{a}} = \frac{Q_{enc}}{\epsilon_0}$$

   Taking a spherical Gaussian surface with its centre coinciding with the centre of the configuration and of radius $r < R_1$, we have $Q_{enc} = 0$ and since the electric field is radial and uniform on the Gaussian surface we have in the region $r < R_1$

$$\vec{\boldsymbol{E}} = 0$$

   We now take a Gaussian surface of radius $R_1 < r < R_2$. In this case the charge enclosed by the surface would be the charge lying between radii $R_1$ and $r$ which is given by

$$Q_{enc} = \frac{4\pi}{3}\rho(r^3 - R_1^3)$$

   Again due to spherical symmetry we have

$$\oiint \vec{\boldsymbol{E}}.d\vec{\boldsymbol{a}} = 4\pi r^2 E$$

   Thus we have

$$\vec{\boldsymbol{E}} = \frac{1}{4\pi r^2}\frac{4\pi}{3\epsilon_0}\rho(r^3 - R_1^3)\hat{\boldsymbol{r}} = \frac{\rho}{3\epsilon_0}\left(r - \frac{R_1^3}{r^2}\right)\hat{\boldsymbol{r}}$$

   For points with $r > R_2$, the charge enclosed will the total charge in the system which is

$$Q_{enc} = \frac{4\pi}{3}\rho(R_2^3 - R_1^3)$$

   And the electric field is given by

$$\vec{\boldsymbol{E}} = \frac{1}{4\pi r^2}\frac{4\pi}{3\epsilon_0}\rho(R_2^3 - R_1^3)\hat{\boldsymbol{r}} = \frac{\rho}{3\epsilon_0 r^2}(R_2^3 - R_1^3)\hat{\boldsymbol{r}}$$

   In the region $r < R_1$, $\nabla.\vec{\boldsymbol{E}} = 0$

   In the region $R_1 < r < R_2$, we have

$$\nabla.\vec{\boldsymbol{E}} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) = \frac{\rho}{\epsilon_0}$$

   In the region $r > R_2$ we have

$$\nabla.\vec{\boldsymbol{E}} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) = 0$$

   consistent with the differential form of Gauss's law.

7. Since the charge distribution is spherical, the electric field will be in the radial direction along the unit vector $\hat{\boldsymbol{r}}$.

   For $r < R$, we take a spherical Gaussian surface with its center coincident with the center of the spherical charge distribution and apply Gauss's law. The charge contained will be

$$Q_{enc} = 4\pi\int_0^r \rho(r)r^2 dr = 4\pi r^3\left(\frac{\rho_0}{3} + \frac{\alpha r}{4}\right)$$

   Also

$$\oiint \vec{\boldsymbol{E}}.d\vec{\boldsymbol{a}} = 4\pi r^2 E$$

Thus the magnitude of the electric field is given by

$$E(r < R) = \frac{r}{\epsilon_0}\left(\frac{\rho_0}{3} + \frac{\alpha r}{4}\right)$$

For points with $r > R$, the charge enclosed is the total charge which is given by

$$Q_{enc} = 4\pi \int_0^R \rho(r) r^2 dr = 4\pi r R^3 \left(\frac{\rho_0}{3} + \frac{\alpha R}{4}\right)$$

Thus

$$E(r > R) = \frac{R^3}{\epsilon_0 r^2}\left(\frac{\rho_0}{3} + \frac{\alpha R}{4}\right)$$

Now for $r > R$,

$$\nabla.\vec{\boldsymbol{E}} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) = 0$$

For $r < R$ we have

$$\nabla.\vec{\boldsymbol{E}} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{r}{\epsilon_0}\left(\frac{\rho_0}{3} + \frac{\alpha r}{4}\right)\right) = \frac{(\rho_0 + \alpha r)}{\epsilon_0} = \frac{\rho(r)}{\epsilon_0}$$

consistent with the Gauss's law in differential form.

Now since the electric field has only the radial component, we have (see book by Griffiths, last two pages which contain formulas for various vector derivatives)

$$\nabla \times \vec{\boldsymbol{E}} = \frac{1}{r sin\theta}\frac{\partial E_r}{\partial \phi}\hat{\theta} + \frac{1}{r}\frac{\partial E_r}{\partial \theta}\hat{\phi}$$

Since $E_r$is independent of $\theta$ and $\phi$ we have

$$\nabla \times \vec{\boldsymbol{E}} = 0$$

everywhere as it indeed should.

8. The point P is equidistant from the entire charge distribution and is at a distance of $\sqrt{z^2 + R^2}$ where z is the distance of the point P from the center of the ring and R is the radius of the ring.

(a) Thus we have for the electrostatic potential at P

$$V = \frac{Q}{4\pi\epsilon_0}\frac{1}{\sqrt{z^2 + R^2}} = 83.5 \text{ V}$$

(b) Work done in moving a point charge from one point to another is the product of the charge and the potential difference between the two points. The potential difference between the center of the ring and the point P is

$$\Delta V = V(z = 5) - V(z = 0)$$

Now

$$V(z = 0) = \frac{Q}{4\pi\epsilon_0}\frac{1}{R} = 224.5 \text{ V}$$

Hence work done in moving a charge from the center of the ring to the point P will be

$$W = q\Delta V = 10 \times 10^{-9}(224.5 - 83.5) = -1.41 \text{ μj}$$

What is the significance of the negative sign?

(c) The electrostatic potential 5 m above and 5 m before the plane are the same. Hence the work done in moving a charge from one point to another is zero.

9. The potential difference between two points B and A is given by

$$V(A) - V(B) = \int_A^B \vec{\boldsymbol{E}}.d\vec{\boldsymbol{l}}$$

We can choose any path for integration to go from point A to point B. Starting from the origin (0, 0, 0) we can first move along the x-axis to the point (4, 0, 0). The potential difference is

$$\Delta V_1 = \int_0^4 \vec{\boldsymbol{E}}.\hat{\boldsymbol{\imath}}\,dx = \int_0^4 2(x + 4 \times 0)dx = 16\ \mathrm{V}$$

where we have used the fact that along the path $y = 0$. We now move from the point (4, 0, 0) to the point (4,2,0) and we have

$$\Delta V_2 = \int_0^2 \vec{\boldsymbol{E}}.\hat{\boldsymbol{\jmath}}\,dy = \int_0^2 8 \times 4dy = 64\ \mathrm{V}$$

Thus the potential difference between the point (0, 0, 0) and the point (4,2,0) will be 80 V.

Note: you can choose any path to go from the starting point to the end point. You may like to try other paths and show that the potential difference is independent of the path taken between the two points.

10. For a charge Q located at $(x_0, y_0, z_0\ )$ the potential at a point $(x, y, z\ )$ is given by

$$V = \frac{Q}{4\pi\epsilon_0} \frac{1}{[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{1/2}}$$

The potential at (2,2,3) due to the point charge at (2,3,3) is 10.8 V and at the point (-2,3,3) is 2.7V. Thus the potential difference is 8.1 V.