Department of Mathematics, Bennett University
Engineering Calculus (EMAT101L)
Solutions for Tutorial Sheet 5

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1. (a) $\lim\limits_{h\to 0} \dfrac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0} \dfrac{f(h)}{h} = \begin{cases} \lim\limits_{h\to 0} \frac{h}{h} = 1, & h \in \mathbb{Q} \\ \lim\limits_{h\to 0} \frac{\sin h}{h} = 1, & h \notin \mathbb{Q}. \end{cases}$ Thus $f'(0) = 1$.

   (b) $\lim\limits_{h\to 0} \dfrac{f(h)-f(0)}{h} = \lim\limits_{h\to 0} \dfrac{\sin \frac{1}{h}}{\sqrt{h}}$ doesn't exist. So f is not differentiable at $x = 0$.

   (c) $\lim\limits_{h\to 0} \dfrac{f(h)-f(0)}{h} = \lim\limits_{h\to 0} h \cos \dfrac{1}{h} = 0$. Therefore f is differentiable at $x = 0$.

   (d) $\lim\limits_{h\to 0} \dfrac{f(h)-f(0)}{h} = \lim\limits_{h\to 0} \dfrac{e^{-\frac{1}{h^2}}}{h} = \lim\limits_{k\to \infty} \dfrac{k}{e^{k^2}} = 0$. Thus f is differentiable at 0.

   (e) $\lim\limits_{h\to 0} \dfrac{f(h)-f(0)}{h} = \lim\limits_{h\to 0} \cos \dfrac{1}{h}$ doesn't exist. So f is not differentiable at 0.

   (f) $\lim\limits_{h\to 0} \dfrac{f(h)-f(0)}{h} = \lim\limits_{h\to 0} \dfrac{e^{-|h|}-1}{h}$ doesn't exist. So f is not differentiable at 0.

2. (a) $\lim\limits_{h\to 0} \dfrac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0} \dfrac{h^3 \sin \frac{1}{h}}{h} = \lim\limits_{h\to 0} h^2 \sin \dfrac{1}{h} = 0$. Thus f is differentiable at 0 and $f'(0) = 0$. Now $f'(x) = 3x^2 \sin \frac{1}{x} - x \cos \frac{1}{x}$. So $\lim\limits_{x\to 0} f'(x) = 0 = f'(0)$. Therefore f' is continuous at $x = 0$.

   (b) $f'(0) = \lim\limits_{h\to 0} \dfrac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0} \dfrac{h^2 \cos \frac{1}{h}}{h} = 0$. Therefore f is differentiable at 0 and $f'(0) = 0$. Now $f'(x) = 2x \cos \frac{1}{x} + \sin \frac{1}{x}$, $x \neq 0$. So limit does not exist as $x \to 0$. Thus f' is not continuous at $x = 0$.

   (c) For $x > 0$, $f'(x) = 2x \ln \frac{1}{x} - x$ and $\lim\limits_{x\to 0^+} f'(x) = 0$. Also for $x < 0$, $f'(x) = 2x \ln \frac{1}{|x|} - x$ and $\lim\limits_{x\to 0^-} f'(x) = 0$. As $f'(0) = \lim\limits_{h\to 0} h \ln \frac{1}{|h|} = 0$, thus f' is continuous at 0.

3. Use $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$.

4. Let $x_0 \in (a, b)$. Then by mean value theorem, we have $\frac{f(x)-f(x_0)}{x-x_0} = f'(d) = c$ (given) where $a < d < x_0$. From this we have, $f(x) = cx + f(x_0) - cx_0$. Hence $f(x) = cx + k$, where $k = f(x_0) - cx_0$.

5. Now $(f-g)'(x) \leq 0$. Then $f - g$ is decreasing function. Therefore $(f-g)(x) \leq (f-g)(0)$ for all $x \geq 0$. As $f(0) - g(0) = 0$, we have $f(x) \leq g(x)$ for all $x \geq 0$.

6. By mean value theorem, we have $f(x) = f\left(\frac{1}{2}\right) + (x - \frac{1}{2})f'(c)$ for some $c \in (0, 1)$. Then
$$|f(x)| \leq \left| f\left(\frac{1}{2}\right) \right| + \left| \left(x - \frac{1}{2}\right) \right| |f'(c)| \leq \frac{1}{2} + \frac{1}{2}\alpha < \frac{1}{2} + \frac{1}{2} = 1.$$

7. Use L'Hospital rule. $(a)\ \frac{1}{2}$, $(b)\ -\frac{1}{24}$, $(c)\ -1$.