Solutions – Tutorial Sheet-9

Problem-1: Determine the largest interval in which the given IVP is certain to have a unique solution.

(a) $x^2y'' + 4y = x, \quad y(1) = 1, \quad y'(1) = 2.$

Solution: Given DE is

$$x^2y'' + 4y = x, \quad y(1) = 1, \quad y'(1) = 2.$$

Comparing the given equation with

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\,y = F(x), \text{ we get}$$

$$a_0(x) = x^2, \quad a_1(x) = 0, \quad a_2(x) = 4$$

Since $a_0(x) = x^2$, which is zero at $x=0$.

So, the largest interval containing 1 on which $a_0(x) \neq 0$ and $a_0(x)$, $a_1(x)$, $a_2(x)$, $F(x)$ are continuous is $(0, \infty)$.

$\Rightarrow$ The largest interval in which the given IVP has unique solution is $(0, \infty)$.

Existence & Uniqueness Theorem for Second Order Linear Nonhomogeneous IVP:

If $a_0(x) \neq 0$ and $a_0(x)$, $a_1(x)$, $a_2(x)$, $F(x)$ are continuous in (a, b), then the IVP (Initial Value Problem)

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\,y = F(x)$$

with the initial conditions $y(x_0) = y_1$, $y'(x_0) = y_2$

has unique solution in (a, b), where $x_0 \in (a, b)$.

(b) $(x-3)y'' - 3xy' + 4y = \sin x, \quad y(-2) = 2, \quad y'(-2) = 1.$

Solution: Comparing the given DE with

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = F(x), \text{ we get}$$

$$a_0(x) = x-3, \quad a_1(x) = -3x, \quad a_2(x) = 4, \quad F(x) = \sin x.$$

Since $a_0(x) = x-3$ is zero at $x = 3$.

So, the largest interval containing -2 on which $a_0(x) \neq 0$ and $a_0(x)$, $a_1(x)$, $a_2(x)$, $F(x)$ are continuous is $(-\infty, 3)$.

Thus by existence and uniqueness theorem for second order non homogeneous linear ODE, the largest interval in which the given IVP has unique solution is $(-\infty, 3)$.

Problem 2: Consider the DE $x^2y'' - 4xy' + 6y = 0$.

(a) Verify that the functions $y_1(x) = x^3$ and $y_2(x) = x^2|x|$ are linearly independent solutions of the given DE on $(-\infty, \infty)$.

(b) Show that y_1 and y_2 are linearly dependent on $(-\infty, 0)$ and on $(0, \infty)$.

(c) Although y_1 and y_2 are linearly independent, show that $W(y_1, y_2) = 0$ $\forall$ $x \in (-\infty, \infty)$. Does this violate the fact that $W(y_1, y_2) = 0$ for every $x \in (-\infty, \infty)$ implies that y_1 are y_2 are linearly dependent?.

Solution: (a) $y_1(x) = x^3$ and $y_2(x) = x^2|x|$ satisfy the given DE.

$$\left[y_2(x) = \begin{cases} x^3, & x \geq 0 \\ -x^3, & x < 0 \end{cases} \right]$$

<u>To check whether they are linearly independent or not in $(-\infty, \infty)$:</u>

We have $y_1(x) = x^3$, $y_2(x) = \begin{cases} x^3, & x \geq 0 \\ -x^3, & x < 0 \end{cases}$

Suppose $c_1 y_1(x) + c_2 y_2(x) = 0$ $\forall$ $x \in (-\infty, \infty)$

<u>for $x \in (0, \infty)$</u>, we have

$$c_1 x^3 + c_2 x^3 = 0 \qquad \text{——— (1)}$$

and for $x \in (-\infty, 0)$, we have

$$c_1 x^3 - c_2 x^3 = 0 \qquad \text{——— (2)}$$

Solving (1) and (2), we get

$$2c_1 x^3 = 0 \Rightarrow c_1 = 0 \qquad (\text{as } x \neq 0)$$

from (2), $c_2 = 0$

Thus, we have

$$c_1 = 0, \quad c_2 = 0.$$

So, the only constants c_1, c_2 for which

$$c_1 y_1(x) + c_2 y_2(x) = 0 \text{ for every } x \in (-\infty, \infty).$$

are $c_1 = 0, \; c_2 = 0$.

Hence $y_1(x)$ and $y_2(x)$ are linearly independent on $(-\infty, \infty)$.

(b) To show: y_1 and y_2 are linearly dependent on $(-\infty, 0)$, and on $(0, \infty)$.

On $(-\infty, 0)$, $y_1(x) = x^3$, $y_2(x) = -x^3$

In otherway,

$-x^3 = (-1)(x^3)$

$\Rightarrow x^3$ and $-x^3$ are linearly dependent on $(-\infty, 0)$

Consider

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} x^3 & -x^3 \\ 3x^2 & -3x^2 \end{vmatrix}$$

$$= -3x^5 + 3x^5 = 0$$

Note! Here Wronskian $=0$

$\Rightarrow y_1$ & y_2 are L.D

as $a_0(x) = x^2 \neq 0$ on $(-\infty, 0)$ and on $(0, \infty)$

$\Rightarrow \quad W(y_1, y_2)(x) = 0 \quad \forall x$

$\Rightarrow \quad y_1$ and y_2 are linearly dependent on $(-\infty, 0)$.

On $(0, \infty)$, $y_1(x) = x^3$, $y_2(x) = x^3$

Consider $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^3 & x^3 \\ 3x^2 & 3x^2 \end{vmatrix} = 0$

$\Rightarrow \quad W(y_1, y_2)(x) = 0 \quad \forall x$

$\Rightarrow \quad y_1$ and y_2 are linearly dependent on $(0, \infty)$.

(c) On $(-\infty, \infty)$,

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$\Rightarrow \quad W(y_1, y_2) = \begin{cases} x^3(3x^2) - (3x^2)\, x^3, & x \geq 0 \\ x^3(-3x^2) + x^3(3x^2), & x < 0 \end{cases}$$

$$\Rightarrow \quad W(y_1, y_2) = 0 \quad \forall \; x \in (-\infty, \infty).$$

But this doesnt violate the fact that $W(y_1, y_2) = 0 \;\; \forall \, x \in (-\infty, \infty)$

$\Rightarrow$ y_1 and y_2 are linearly dependent.

because here $a_0(x) = x^2 = 0$ at $x = 0 \in (-\infty, \infty)$.

Ques-3: Verify that $y_1(x) = e^x$ and $y_2(x) = xe^x$ are solutions of

$$y'' - 2y' + y = 0 \text{ for } x \in \mathbb{R}. \quad \text{---} (1)$$

Do they constitute a fundamental set of solutions?

Solution: Here $y_1(x) = e^x$, $y_2(x) = xe^x$ (Verify that y_1 and y_2 are solutions of (1))

$\boxed{\text{Here } a_0(x) = 1 \neq 0 \text{ for } x \in \mathbb{R}.}$

So,
$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix}$$

$$= e^x(xe^x + e^x) - xe^{2x}$$
$$= e^{2x} \neq 0$$

$\Rightarrow$ $W(y_1, y_2) \neq 0$

$\Rightarrow$ y_1 and y_2 are linearly independent solutions of

$$y'' - 2y' + y = 0 \text{ for } x \in \mathbb{R}.$$

$\Rightarrow$ y_1 and y_2 constitute a fundamental set of solutions.

Ques-4 If y_1 and y_2 are linearly independent solutions of

$$xy'' + 2y' + xe^x y = 0, \quad x \in (0, \infty).$$

and if $W(y_1, y_2)(1) = 2$, find the value of $W(y_1, y_2)(5)$.

Solution: Given DE is

$$xy'' + 2y' + xe^x y = 0, \quad x \in (0, \infty)$$

Here $a_0(x) = x$, $a_1(x) = 2$, $a_2(x) = xe^x$

By Abel's formula, we have

$$W(y_1, y_2)(x) = C e^{-\int \frac{a_1(x)}{a_0(x)} dx},$$

where y_1 and y_2 are the solutions of the given DE.

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Ques-3: Verify that $y_1(x) = e^x$ and $y_2(x) = xe^x$ are solutions of

$$y'' - 2y' + y = 0 \quad \text{for } x \in \mathbb{R}. \qquad \text{——— (1)}$$

Do they constitute a fundamental set of solutions?

Solution: Here $y_1(x) = e^x$, $y_2(x) = xe^x$ (Verify that y_1 and y_2 are solutions of (1))

Here $a_0(x) = 1 \neq 0$ for $x \in \mathbb{R}$.

So,
$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix}$$

$$= e^x(xe^x + e^x) - xe^{2x}$$

$$= e^{2x} \neq 0$$

$\Rightarrow$ $W(y_1, y_2) \neq 0$

$\Rightarrow$ y_1 and y_2 are linearly independent solutions of

$$y'' - 2y' + y = 0 \quad \text{for } x \in \mathbb{R}.$$

$\Rightarrow$ y_1 and y_2 constitue a fundamental set of solutions.

Ques-4 If y_1 and y_2 are linearly independent solutions of

$$xy'' + 2y' + xe^x y = 0, \quad x \in (0, \infty).$$

and if $W(y_1, y_2)(1) = 2$, find the value of $W(y_1, y_2)(5)$.

Solution: Given DE is

$$xy'' + 2y' + xe^x y = 0, \quad x \in (0, \infty)$$

Here $a_0(x) = x$, $a_1(x) = 2$, $a_2(x) = xe^x$

By Abel's formula, we have

$$W(y_1, y_2)(x) = C e^{-\int \frac{a_1(x)}{a_0(x)} dx},$$

where y_1 and y_2 are the solutions of the given DE.

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$$\Rightarrow \quad W(y_1, y_2)(x) = c\,e^{-\int \frac{2}{x}dx} = c\,e^{-2\log x} = c\,e^{\log x^{-2}}$$

$$\Rightarrow \quad W(y_1, y_2)(x) = \frac{c}{x^2}$$

Since $W(y_1, y_2)(1) = \frac{c}{1} = c$

$$\Rightarrow \quad 2 = c$$

Thus $W(y_1, y_2)(x) = \frac{2}{x^2}$

$$\Rightarrow \boxed{W(y_1, y_2)(5) = \frac{2}{(5)^2} = \frac{2}{25}}$$

Ans.

Problem-5 Find the second linearly independent solution of the following problems using the method of reduction of order. Hence find the general solution.

(a) $(2x+1)y'' - 4(x+1)y' + 4y = 0, \quad y_1 = e^{2x}$. —— (1).

Solution:

Here $y_1 = e^{2x}$ is the given solution of (1).

Let $y_1 = f(x) = e^{2x}$.

Let us suppose $g(x) = f(x)\,v$, where

$$v = \int \frac{e^{-\int \frac{a_1(x)}{a_0(x)}dx}}{[f(x)]^2}\,dx$$

$$= \int \frac{e^{-\int \frac{-4(x+1)}{(2x+1)}dx}}{(e^{2x})^2}\,dx$$

$$= \int \frac{e^{\int \left(\frac{4x+4}{2x+1}\right)dx}}{e^{4x}}\,dx$$

$$= \int \frac{e^{\int \left(2 + \frac{2}{2x+1}\right)dx}}{e^{4x}}\,dx$$

$$= \int \left(\frac{e^{2x + 2\frac{\log(2x+1)}{2}}}{e^{4x}}\right)dx$$

$$= \int \frac{e^{2x}\cdot(2x+1)}{e^{4x}}\,dx = \int \left(\frac{2x+1}{e^{2x}}\right)dx$$

$$= \int (2x+1)\, e^{-2x}\, dx$$

$$= (2x+1)\frac{e^{-2x}}{-2} - \int 2\left(\frac{e^{-2x}}{-2}\right) dx$$

$$= -\frac{(2x+1)e^{-2x}}{2} + \int e^{-2x}\, dx$$

$$= \frac{-(2x+1)\, e^{-2x}}{2} + \frac{e^{-2x}}{-2}$$

$$= -\frac{e^{-2x}}{2}\left(1 + 2x+1\right) = -\frac{e^{-2x}}{2}(2x+2) = -e^{-2x}(x+1)$$

Thus $v(x) = -e^{-2x}(x+1)$.

So, $g(x) = f(x)\cdot v = e^{2x}\cdot\left(-e^{-2x}\cdot(x+1)\right) = -(x+1)$

$$\Rightarrow \boxed{g(x) = -(x+1)}$$

Thus the another linearly independent solution of ① is

$$g(x) = -(x+1).$$

The general solution is

$$y(x) = c_1 f(x) + c_2\, g(x)$$

$$\Rightarrow \boxed{y(x) = c_1 e^{2x} - c_2(x+1)}$$

Ans

(b) $9y'' - 12y' + 4y = 0, \quad y_1 = e^{\frac{2x}{3}}$ ——— (1)

Here $y_1 = e^{\frac{2x}{3}}$ is the given solution of (1).

Let $f(x) = y_1 = e^{\frac{2x}{3}}$.

Let us suppose $g(x) = f(x) \cdot v = e^{\frac{2x}{3}} \cdot v$,

where $$v(x) = \int \frac{e^{-\int \frac{a_1(x)}{a_0(x)} dx}}{[f(x)]^2} dx$$

$$= \int \frac{e^{-\int \frac{-12}{9} dx}}{\left(e^{\frac{2x}{3}}\right)^2} dx$$

$$= \int \frac{e^{\frac{4}{3}x}}{e^{4x/3}} dx = x$$

$\Rightarrow \quad v(x) = x$.

Thus $g(x) = f(x) \cdot v = e^{\frac{2x}{3}} \cdot x = x\, e^{\frac{2x}{3}}$.

So, we have another linearly independent solution is

$$g(x) = x e^{2x/3}$$

Hence, the general solution is

$$y(x) = c_1 f(x) + c_2 g(x)$$
$$= c_1 e^{2x/3} + c_2 x e^{2x/3}$$

$$\Rightarrow \boxed{y(x) = e^{2x/3}(c_1 + c_2 x)}$$

Ans

(C) $x^2y'' - xy' + 2y = 0, \quad y_1 = x\sin(\log x)$. ——— ①

Solution: Let $y_1 = f(x) = x\sin(\log x)$.

Let us ~~sa~~ suppose $g(x) = f(x)\cdot v$, where

$$v(x) = \int \frac{e^{-\int \frac{a_1(x)}{a_0(x)}dx}}{[f(x)]^2}dx = \int \frac{e^{-\int \frac{-x}{x^2}dx}}{x^2\cdot \sin^2(\log x)}dx$$

$$= \int \frac{e^{\log x}}{x^2 \sin^2(\log x)}dx = \int \frac{x}{x^2\sin^2(\log x)}dx$$

$$= \int \frac{1}{x\cdot \sin^2(\log x)}dx$$

Put $\log x = t \Rightarrow \frac{1}{x}dx = dt$

$$= \int \frac{dt}{\sin^2 t} = \int \operatorname{cosec}^2 t \, dt$$

$$= -\cot t$$

$$\Rightarrow \boxed{v(x) = -\cot(\log x)}$$

Thus, we have $g(x) = f(x)\cdot v = x\sin(\log x)\cdot(-\cot(\log x))$

$$= -x\cos(\log x)$$

$$\Rightarrow g(x) = -x\cos(\log x).$$

Thus we have another linearly independent solution is $g(x) = -x\cos(\log x)$.

Hence, the general solution is

$$y(x) = c_1 f(x) + c_2 g(x)$$

$$\Rightarrow \boxed{y(x) = c_1 x \sin(\log x) - c_2 x \cos(\log x)}$$ Ans.

(d) $(1-x^2)y'' + 2xy' = 0, \quad y_1 = 1.$ ——— (1)

Let $f(x) = y_1 = 1$.

Let us suppose $g(x) = f(x) \cdot v = 1 \cdot v$,

where $v(x) = \displaystyle\int \frac{e^{-\int \frac{a_1(x)}{a_0(x)} dx}}{[f(x)]^2} dx$

$$= \int \frac{e^{-\int \frac{2x}{1-x^2} dx}}{1^2} dx$$

$$= \int e^{\log(1-x^2)} dx = \int (1-x^2)\, dx$$

$$= x - \frac{x^3}{3}$$

$$\Rightarrow v(x) = x - \frac{x^3}{3}.$$

So, we have $g(x) = f(x) \cdot v = (1)\left(x - \frac{x^3}{3}\right) = x - \frac{x^3}{3}$.

Hence the general solution is

$$y(x) = c_1 f(x) + c_2 g(x)$$

$$= c_1(1) + c_2\left(x - \frac{x^3}{3}\right)$$

$$\Rightarrow \boxed{y(x) = c_1 + c_2\left(x - \frac{x^3}{3}\right)}$$ Ans.

(e) $(x-1)y'' - xy' + y = 0, \quad y_1 = e^x$.

<u>Solution:</u> Here $y_1 = f(x) = e^x$.

Let us suppose $g(x) = f(x)\cdot v = e^x \cdot v$,

where $$v(x) = \int \frac{e^{-\int \frac{a_1(x)}{a_0(x)}dx}}{[f(x)]^2}$$

$$= \int \frac{e^{-\int\left(\frac{-x}{x-1}\right)dx}}{(e^x)^2}\,dx$$

$$= \int \frac{e^{\int\left(1+\frac{1}{x-1}\right)dx}}{e^{2x}}\,dx$$

$$= \int \left(\frac{e^{x+\log(x-1)}}{e^{2x}}\right)dx$$

$$= \int \frac{e^x \cdot e^{\log(x-1)}}{e^{2x}}\,dx$$

$$= \int \frac{x-1}{e^x}\,dx$$

$$= \int (x-1)\,e^{-x}\,dx$$

$$= (x-1)\frac{e^{-x}}{-1} - \int \left(\frac{e^{-x}}{-1}\right)dx$$

$$= -(x-1)e^{-x} - e^{-x} = -xe^{-x}$$

$$\Rightarrow \boxed{v(x) = -xe^{-x}}$$

Thus, we have $g(x) = f(x)\cdot v = e^{x}(-xe^{-x}) = -x$.

So, we have another linearly independent solution is

$$g(x) = -x.$$

Hence the general solution is

$$y(x) = c_1 f(x) + c_2 g(x)$$

$$\Rightarrow \boxed{y(x) = c_1 e^{x} - c_2 x}$$

Ans.

Problem-6 Find the general solution of the following DE's.

(a) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$.

Solution: The auxiliary equation is

$$m^2 + m + 1 = 0$$

$$\Rightarrow m = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}\,i}{2}.$$

Thus the roots of auxiliary equation are complex. (i.e. of the form $\alpha \pm i\beta$, where $\alpha = \frac{-1}{2}$, $\beta = \frac{\sqrt{3}}{2}$).

Therefore, the general solution of the given DE is

$$y = e^{\frac{-x}{2}}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right).$$

[$\because$ If the roots of the auxiliary equation are complex (i.e. $m = \alpha \pm i\beta$), then the general solution of

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0, \quad a_0 \neq 0 \quad \left(\begin{array}{c}a_0, a_1, a_2 \text{ are} \\ \text{constants}\end{array}\right)$$

is

$$y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$$]

(b) $\frac{d^5y}{dx^5} - 2\frac{d^4y}{dx^4} + \frac{d^3y}{dx^3} = 0$.

Solution: The auxiliary equation is

$$m^5 - 2m^4 + m^3 = 0$$

$$\Rightarrow m^3(m^2 - 2m + 1) = 0$$

$$\Rightarrow m^3(m-1)^2 = 0$$

$$\Rightarrow m = 0, 0, 0, 1, 1.$$

$\Rightarrow$ The roots of the auxiliary equation are real and repeated.

Thus the general solution is

$$y = (c_1 + c_2x + c_3x^2)e^{0x} + (c_4 + c_5x)e^x$$

$$\Rightarrow \boxed{y = (c_1 + c_2x + c_3x^2) + (c_4 + c_5x)e^x}$$ Ans.

[$\because$ If the roots of the auxiliary equation are real and repeated (i.e. $m = m_1, m_1$), then the general solution of

$$a_0\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = 0, \quad a_0 \neq 0 \quad \left(a_0, a_1, a_2 \text{ are constants}\right)$$

is

$$y = (c_1 + c_2x)e^{m_1x}$$]

(c) $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0$.

Solution: The auxiliary equation is

$$m^3 - m^2 + m - 1 = 0$$

$$\Rightarrow m^2(m-1) + 1(m-1) = 0$$

$$\Rightarrow (m-1)(m^2+1) = 0$$

$$\Rightarrow m = 1, \pm i.$$

Thus the general solution is

$$\boxed{y = c_1 e^{x} + (c_2 \cos x + c_3 \sin x)}$$ Ans.

Ques-7 Find a linear DE with constant coefficients and of order 3 which admits the following solutions.

(a) $\cos x$, $\sin x$ and e^{-3x} (b) e^{x}, e^{2x}, e^{3x}.

Sol^n: (a) The solutions of the linear DE are

$$\cos x, \sin x, e^{-3x}.$$

$\Rightarrow$ The roots of the auxiliary equation are

$$\pm i, -3.$$

$\Rightarrow$ The auxiliary equation is

$$(m^2+1)(m+3) = 0$$

$$\Rightarrow \quad m^3 + m + 3m^2 + 3 = 0$$

$$\Rightarrow \quad m^3 + 3m^2 + m + 3 = 0.$$

Thus the corresponding linear DE is

$$\boxed{\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + \frac{dy}{dx} + 3y = 0}$$ Ans

(b) e^{x}, e^{2x}, e^{3x}

Sol^n: The solutions of the linear DE are

$$e^{x}, e^{2x}, e^{3x}.$$

$\Rightarrow$ The roots of the auxiliary equation are 1, 2, 3.

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$\Rightarrow$ The auxiliary equation whose roots are 1, 2, 3 is

$$(m-1)(m-2)(m-3) = 0$$

$$\Rightarrow (m^2-3m+2)(m-3) = 0$$

$$\Rightarrow m^3-3m^2+2m-3m^2+9m-6 = 0$$

$$\Rightarrow m^3-6m^2+11m-6 = 0 .$$

Therefore, the corresponding DE is

$$\boxed{\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0}$$

Ans.

Ques-8: Find the condition on d for which all the solutions of

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - dy = 0$$

tends to zero as $x \to \infty$.

Solution: Given DE is

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - dy = 0. \quad \text{——— (1)}$$

The auxiliary equation is

$$m^2+2m-d = 0 \quad \text{——— (2)}$$

Let m_1, m_2 be the roots of the above equation.

Then the solution of (1) can be written as

$$y = c_1 e^{m_1x} + c_2 e^{m_2x}.$$

According to the statement, $y \to 0$ as $x \to \infty$,

which is possible only if $m_1, m_2 < 0$.

Also, sum of the roots of ② is

$$m_1 + m_2 = -2$$

and product of roots of ② is

$$m_1 \cdot m_2 = -d$$

Since $m_1, m_2 < 0$

$$\Rightarrow m_1 \cdot m_2 > 0$$

$$\Rightarrow -d > 0$$

$$\Rightarrow \boxed{d < 0}$$

which is the condition on d for which the solutions of ① tends to zero as $x \to \infty$.

Ques-9: Find the particular solution of the following DE's using the method of undetermined coefficients and hence find the general solution.

(a) $y'' + 2y' + 2y = 4e^x \sin x$. ——— ①

Solution: Here the auxiliary equation is

$$m^2 + 2m + 2 = 0$$

$$\Rightarrow m = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2}$$

$$\Rightarrow m = -1 \pm i.$$

Thus $y_c(x) = e^{-x}(c_1 \cos x + c_2 \sin x)$.

Here $f(x) = 4e^x \sin x$, which is of the form ~~$e^{\alpha x} \sin \beta x$~~

$$e^{\alpha x}(k_1 \cos \beta x + k_2 \sin \beta x).$$

$$\Rightarrow \alpha = 1, \ \beta = 1$$

$\Rightarrow \alpha + i\beta = 1 + i$, which is not the root of the auxiliary equation.

Thus, let us assume $y_p = e^x(A\sin x + B\cos x)$.

Since y_p satisfies (1), thus substituting the value of y_p in (1), we get

$$(-4B+4A)e^x\sin x + (4B+4A)e^x\cos x = e^x\sin x.$$

Comparing the coefficients of $e^x\cos x$ and $e^x\sin x$ on both sides, we get

$$A-B = 1 \quad \text{and} \quad A+B=0.$$

On solving for A and B, we get $A=-B=\frac{1}{2}$.

Thus the particular solution is

$$y_p(x) = \frac{e^x}{2}(\sin x - \cos x).$$

The general solution of (1) is

$$y(x) = y_c(x) + y_p(x)$$

$$\Rightarrow \boxed{y(x) = e^{-x}(c_1\cos x + c_2\sin x) + \frac{e^x}{2}(\sin x - \cos x)}$$ Ans

9(b) Given DE is

$$y'' + y = 2\sin x + \sin 2x \qquad \text{——— (1)}$$

The auxiliary equation is

$$m^2 + 1 = 0$$

$$\Rightarrow m = \pm i$$

Thus $y_c(x) = c_1\cos x + c_2\sin x$.
(Complementary function)

To find the particular integral, we divide the given problem into two problems

$$y'' + y = 2\sin x \quad \text{———} \quad (2)$$

and $$y'' + y = \sin 2x \quad \text{———} \quad (3)$$

First, we find the particular integral of (2).

For (2), $f(x) = 2\sin x$, which is of the form $e^{\alpha x}(P_1 \cos\beta x + P_2 \sin\beta x)$ with $\alpha = 0$, $\beta = 1$.

$\Rightarrow$ $\alpha + i\beta = i$, which is a root of the characteristic equation with multiplicity $r = 1$.

So, let $y_{p_1} = x(A\cos x + B\sin x)$. (Particular integral for (1))

Since y_p satisfies the equation (2).

$\Rightarrow$ Substituting the values of y_p in (2), we get

$$y_{p_1}'' + y_{p_1} = 2\sin x$$

$$\Rightarrow x(-A\cos x - B\sin x) - 2A\sin x + 2B\cos x + x(A\cos x + B\sin x) = 2\sin x$$

$$\Rightarrow -2A\sin x + 2B\cos x = 2\sin x$$

$$\Rightarrow -2A = 2, \quad 2B = 0$$

$$\Rightarrow A = -1, \quad B = 0$$

Thus $y_{p_1} = -x\cos x$

Now, for the equation (3), we find the particular integral.

$$y'' + y = \sin 2x.$$

Here $f(x) = \sin 2x$, which is of the form $e^{\alpha x}(P_1\cos\beta x + P_2 \sin\beta x)$

with $\alpha = 0, \; \beta = 2$.

$\Rightarrow \; \alpha + i\beta = 2i$, which is not the root of the characteristic equation.

Thus, we assume $y_{p_2} = A\cos 2x + B\sin 2x$ (particular integral for ③)

Since y_p satisfies the DE ③.

$\Rightarrow$ Substituting the value of y_p in ③, we get

$$y_{p_2}'' + y_{p_2} = \sin 2x$$

$$\Rightarrow \; -4A\cos 2x - 4B\sin 2x + A\cos 2x + B\sin 2x = \sin 2x$$

$$\Rightarrow \; -3A\cos 2x - 3B\sin 2x = \sin 2x$$

Comparing the coefficients, we get

$$-3A = 0 \quad \Rightarrow \quad A = 0$$

and $$-3B = 1 \quad \Rightarrow \quad B = -\frac{1}{3}$$

Thus $A = 0$ and $B = -\frac{1}{3}$

$$\Rightarrow \; y_{p_2} = -\frac{1}{3}\sin 2x$$

Thus $\boxed{y_{p_2} = -\frac{1}{3}\sin 2x}$

So, we have $y_{p_1} = -x\cos x$ and $y_{p_2} = -\frac{1}{3}\sin 2x$.

Thus the particular integral of

$$y'' + y = 2\sin x + \sin 2x$$

is $$y_p = y_{p_1} + y_{p_2} = -x\cos x - \frac{1}{3}\sin 2x.$$

The general solution is

$$y = y_c(x) + y_p(x)$$

$$\Rightarrow \boxed{y = c_1 \cos x + c_2 \sin x - x \cos x - \frac{1}{3} \sin 2x}$$ Ans

(c) $y''' - y'' + y' - y = x^2$

Solution: The characteristic equation is

$$p(m) = m^3 - m^2 + m - 1 = 0$$

$$\Rightarrow m^2(m-1) + 1(m-1) = 0$$

$$\Rightarrow (m^2+1)(m-1) = 0$$

$$\Rightarrow m = 1, \pm i$$

Here $f(x) = x^2$ and '0' is not a root of the characteristic equation. (i.e $p(0) \neq 0$).

Thus we assume $y_p = A_2 x^2 + A_1 x + A_0$.

Then by substituting the values of y_p into the given equation, we get

$$A_2 = -1, \quad A_1 = -2, \quad A_0 = 0.$$

$$\Rightarrow y_p = -(x^2 + 2x).$$

and $y_c(x) = c_1 e^x + c_2 \cos x + c_3 \sin x$.

Thus $y(x) = y_c(x) + y_p(x)$

$$\Rightarrow \boxed{y(x) = c_1 e^x + c_2 \cos x + c_3 \sin x - (x^2 + 2x)}$$ Ans

Quest(10) By using the method of variation of parameters, find the general solution of the following differential equations

(a) $y'' + y = \sec x$.

Solution: Given differential equation is

$$y'' + y = \sec x. \qquad \text{——— (1)}$$

Comparing the given equation with

$$a_0(x)\, y'' + a_1(x)\, y' + a_2(x)\, y = F(x), \text{ we get}$$

$$a_0(x) = 1, \quad a_1(x) = 0, \quad a_2(x) = 1, \quad F(x) = \sec x.$$

The corresponding Homogeneous equation is

$$y'' + y = 0. \qquad \text{——— (2)}$$

The auxiliary equation is

$$m^2 + 1 = 0 \Rightarrow m = \pm i.$$

Thus the complementary function is

$$y_c(x) = c_1 \cos x + c_2 \sin x.$$

$\Rightarrow$ $y_1(x) = \cos x$, $y_2(x) = \sin x$ are two linearly independent solutions of the corresponding Homogeneous differential equation.

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Now, $W(y_1, y_2) = W(\cos x, \sin x)$

$$= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$\Rightarrow \quad W(y_1, y_2) = 1.$

Let us take $y_p(x) = A(x)\, y_1 + B(x)\, y_2$

$\Rightarrow \quad y_p(x) = A(x) \cos x + B(x) \sin x,$ ——— (3)

where $A(x) = -\displaystyle\int \frac{y_2\, F(x)}{a_0(x)\, W(y_1, y_2)}\, dx$

and $B(x) = \displaystyle\int \frac{y_1\, F(x)}{a_0(x)\, W(y_1, y_2)}\, dx.$

Thus $A(x) = -\displaystyle\int \frac{\sin x \cdot \sec x}{(1)\ (1)}\, dx$

$$= -\int \frac{\sin x}{\cos x}\, dx = \log(\cos x)$$

$\Rightarrow \quad \boxed{A(x) = \log(\cos x)}$

and $B(x) = \displaystyle\int \frac{\cos x \cdot \sec x}{(1).(1)}\, dx = \int (1)\, dx = x.$

Substituting the values of $A(x)$ and $B(x)$ in (3), we get

$$\boxed{y_p(x) = \log(\cos x) \cdot \cos x + x \sin x}$$

Thus the general solution of the given DE is

$$y = y_c + y_p$$

$$\Rightarrow \boxed{y = c_1 \cos x + c_2 \sin x + \log(\cos x)\cdot \cos x + x \sin x}$$ Ans.

10 (b) $y'' + 4y = 3 \operatorname{cosec} 2x$.

Solution: Given D.E is

$$y'' + 4y = 3 \operatorname{cosec} 2x \quad \text{——— (1)}$$

Comparing the given DE with

$a_0(x)y'' + a_1(x)y' + a_2(x)y = F(x)$, we get

$a_0(x) = 1$, $a_1(x) = 0$, $a_2(x) = 4$, $F(x) = 3 \operatorname{cosec} 2x$.

The corresponding Homogeneous DE of (1) is

$$y'' + 4y = 0 \quad \text{——— (2)}$$

The auxiliary equation is

$$m^2 + 4 = 0$$

$$\Rightarrow m = \pm 2i.$$

Thus the complementary function is

$$y_c(x) = c_1 \cos 2x + c_2 \sin 2x.$$

$\Rightarrow$ $y_1(x) = \cos 2x$, $y_2(x) = \sin 2x$ are two linearly independent solutions of the corresponding Homogeneous DE.

Now, $W(y_1, y_2) = W(\cos 2x, \sin 2x) = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$

$$= 2\cos^2 2x + 2\sin^2 2x = 2.$$

$\Rightarrow$ $W(y_1, y_2) = 2$.

Let us assume

$$y_p(x) = A(x)\, y_1 + B(x) \cdot y_2$$

$$\Rightarrow \quad y_p(x) = A(x) \cos 2x + B(x) \sin 2x \quad \text{——— (3)}$$

where $$A(x) = -\int \frac{y_2 \, F(x)}{a_0(x)\, W(y_1, y_2)} \, dx$$

and $$B(x) = \int \frac{y_1 \, F(x)}{a_0(x)\, W(y_1, y_2)} \, dx$$

Thus $$A(x) = -\int \frac{(\sin 2x) \cdot 3 \operatorname{cosec} 2x}{(1)\ (2)} \, dx$$

$$= -\frac{3}{2} \int \frac{\sin 2x}{\sin 2x} \, dx = -\frac{3}{2} x$$

and $$B(x) = \int \frac{(\cos 2x) \cdot 3 \operatorname{cosec} 2x}{(1)\ (2)} \, dx = \frac{3}{2} \int \frac{\cos 2x}{\sin 2x} \, dx$$

$$= \frac{3}{2} \frac{\log(\sin 2x)}{2} = \frac{3}{4} \log(\sin 2x).$$

Substituting the values of $A(x)$ and $B(x)$ in (3), we get

$$\boxed{y_p(x) = -\frac{3}{2} x \cos 2x + \frac{3}{4} \log(\sin 2x) \cdot \sin 2x}$$

Thus the general solution of the given DE is.

$$y = y_c + y_p$$

$$\Rightarrow \boxed{y = c_1 \cos 2x + c_2 \sin 2x - \frac{3}{2} x \cos 2x + \frac{3}{4} \log(\sin 2x) \cdot \sin 2x}$$

Ans

10(c) $y'' + 6y' + 9y = \frac{e^{-3x}}{x^3}$

Solution: Given DE is

$$y'' + 6y' + 9y = \frac{e^{-3x}}{x^3} \quad \text{——— (1)}$$

The corresponding homogeneous DE is

$$y'' + 6y' + 9y = 0. \quad \text{——— (2)}$$

The auxiliary equation is

$$m^2 + 6m + 9 = 0$$

$$\Rightarrow (m+3)^2 = 0$$

$$\Rightarrow m = -3, -3.$$

Thus the complementary function is

$$\boxed{y_c = (c_1 + c_2 x)\, e^{-3x}}$$

$\Rightarrow$ $y_1(x) = e^{-3x}$, $y_2(x) = x e^{-3x}$ are two linearly independent solutions of the given DE.

Now, $$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^{-3x} & x e^{-3x} \\ -3e^{-3x} & (-3x e^{-3x} + e^{-3x}) \end{vmatrix}$$

$$= -3x e^{-6x} + e^{-6x} + 3x e^{-6x}$$

$$= e^{-6x}$$

$$\Rightarrow \boxed{W(y_1, y_2) = e^{-6x}}$$

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Let us assume

$$y_p(x) = A(x)\, y_1 + B(x)\, y_2$$

$$\Rightarrow y_p(x) = A(x)\cdot e^{-3x} + B(x)\cdot x e^{-3x}, \qquad \text{—— (3)}$$

where $$A(x) = -\int \frac{y_2\ F(x)}{a_0(x)\cdot W(y_1,y_2)}\, dx$$

and $$B(x) = \int \frac{y_1\ F(x)}{a_0(x)\cdot W(y_1,y_2)}\, dx .$$

Thus $$A(x) = -\int \frac{x e^{-3x}\cdot e^{-3x}}{(1)\cdot x^3\cdot e^{-6x}}\, dx$$

$$= -\int \frac{x}{x^3}\, dx = -\int \frac{1}{x^2}\, dx$$

$$= \frac{-x^{-1}}{-1} = \frac{1}{x}$$

$$\Rightarrow A(x) = \frac{1}{x}.$$

and $$B(x) = \int \frac{e^{-3x}\cdot e^{-3x}}{(1)\cdot x^3\cdot e^{-6x}}\, dx$$

$$= \int \frac{1}{x^3}\, dx = \frac{x^{-2}}{-2} = \frac{-1}{2x^2}$$

$$\Rightarrow B(x) = \frac{-1}{2x^2}.$$

Substituting the values of $A(x)$ and $B(x)$ in (3), we get

$$y_p(x) = \frac{1}{x} \cdot e^{-3x} - \frac{1}{2x^2}\, x\, e^{-3x}$$

$$= \frac{e^{-3x}}{x}\left(1 - \frac{1}{2}\right) = \frac{1}{2}\frac{e^{-3x}}{x}.$$

$$\Rightarrow \boxed{y_p(x) = \frac{e^{-3x}}{2x}}$$

$\therefore$ Thus the general solution of the given DE is

$$y = y_c + y_p$$

$$\Rightarrow \boxed{y = (c_1 + c_2 x)\, e^{-3x} + \frac{e^{-3x}}{2x}}$$

Ans.