Differentiability

Defn:- $f: I \to \mathbb{R}$, Then

① f is strictly increasing on I if $x, y \in I$, $x < y \Rightarrow f(x) < f(y)$.

② f is strictly decreasing on I if $x, y \in I$ $x < y \Rightarrow f(x) > f(y)$.

Result:- If f is diff on (a,b). Then

① f is strictly increasing in (a,b) if $f'(x) > 0$ $\forall x \in (a,b)$.

② f is S. decreasing in (a,b) if $f'(x) < 0$ $\forall x \in (a,b)$

Ex:- Show that $\sin x < x < \tan x$, $x \in (0, \frac{\pi}{2})$

$f(x) = x - \sin x$, $x \in (0, \frac{\pi}{2})$

$f'(x) = 1 - \cos x > 0$, $\forall x \in (0, \frac{\pi}{2})$

i.e, $f(x) > f(0)$ $\forall x \in (0, \frac{\pi}{2})$

i.e, $x - \sin x > 0$ $\forall x \in (0, \frac{\pi}{2})$

$\Rightarrow x > \sin x$ $\forall x \in (0, \frac{\pi}{2})$

$g(x) = \tan x - x$, $x \in (0, \frac{\pi}{2})$

$g'(x) = \sec^2 x - 1 = \tan^2 x > 0$

$x < \tan x$, $x \in (0, \frac{\pi}{2})$

L' Hospital's Rule:-

$\lim_{x \to c} f(x) = A$, $\lim_{x \to c} g(x) = B$, $B \neq 0$

Then $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{A}{B}$.

1. $A \neq 0, B = 0, \frac{A}{B} = \infty$

2. $A = 0, B = 0, \frac{0}{0} \rightarrow$ indeterminate

$\frac{\infty}{\infty} \nearrow$

Ex:- $f(x) = \alpha x,\ g(x) = x,\ \alpha \in \mathbb{R}$.

$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \underbrace{\frac{\alpha x}{x}}_{(\frac{0}{0}\text{ form})} = \alpha$.

Result :- $f, g : [a,b] \rightarrow \mathbb{R}$, $f(a) = g(a) = 0$

$g(x) \neq 0,\ x \in (a,b]$.

If f, g are diff (a,b), & $g'(a) \neq 0$.

Then $\boxed{\lim_{x \to a^+} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}}$.

Ex:- $f(x) = x + 17,\ g(x) = 2x + 3$.

$\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{17}{3}$, $\frac{f'(0)}{g'(0)} = \frac{1}{2}$

Ex:- $\lim_{x \to 0} \frac{x^2 + x}{\sin 2x}$ $\left(\frac{0}{0}\text{ form}\right)$.

$= \frac{1}{2}$

Ex:- $\lim_{x \to 1} \left[\frac{\ln x}{x - 1}\right] = 1$

Ex, $\lim_{x \to 0} \left[\frac{1 - \cos x}{x^2}\right] = \frac{1}{2}$, $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

Result: ① If f, g are diff $(a, \infty), a > 0$

$\lim_{x \to \infty} f(x) = 0 = \lim_{x \to \infty} g(x)$. and $\lim_{x \to \infty} \frac{f'(x)}{g'(x)}$

exist. Then $\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$.

(2) f, g are diff (a,b), $x_0 \in (a,b)$.

$\lim_{x \to x_0} f(x) = \infty = \lim_{x \to x_0} g(x)$, $\lim_{x \to x_0} \frac{f'(x)}{g'(x)}$ exists

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

EX:- $\lim_{x \to \infty} \frac{\ln x}{x} = 0$

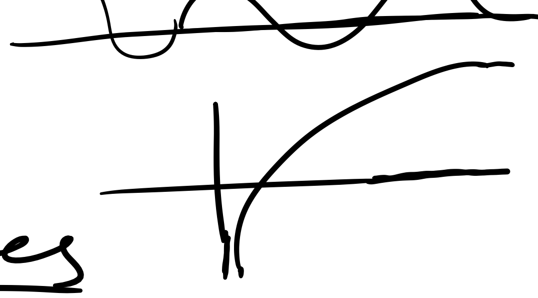
EX:- $\lim_{x \to \infty} e^{-x} \cdot x^2 = \lim_{x \to \infty} \frac{x^2}{e^x}$ ($\frac{\infty}{\infty}$ form) $= 0$

$= \lim_{x \to \infty} \frac{2x}{e^x}$

$= \lim_{x \to \infty} \frac{2}{e^x} = 0$

EX:- $\lim_{x \to 0^+} \frac{\ln(\sin x)}{\ln x}$ ($\frac{\infty}{\infty}$ form)

$= 1$



Power series

Defn.:- $\{a_n\}_{n=0}^{\infty}$, $\sum_{n=0}^{\infty} a_n (x-c)^n$ is call

power series with center c.

$x = c$, power series converges.

$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ if series converges for that x.

$c = 0$, $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$

$x' = x - c$, $\sum_{n=0}^{\infty} a_n x'^n \longrightarrow$ power series at 0.

① $\sum_{n=0}^{\infty} x^n \longrightarrow \{x \in \mathbb{R} : |x| < 1\}$

② $\sum_{n=0}^{\infty} \frac{x^n}{n!} \rightarrow$ conv. on $\mathbb{R} \rightarrow$ everywhere conv.

③ $\sum_{n=0}^{\infty} n! \cdot x^n \rightarrow$ conv. $\{0\} \rightarrow$ nowhere conv.

Result: ① $\sum_{n=0}^{\infty} a_n x^n$ conv. at $x = b$. then series conv for $|x| < |b|$

② $\sum_{n=0} a_n x^n$ div. at $x = n$. then series div. for $|x| > |b|$