* **Invertible Matrix :** A square matrix A of order "n" is said to be invertible if $\exists$ some $n \times n$ matrix B such that $\boxed{AB = BA = I_n.}$

In that case, B is called the inverse of A and is denoted by A^{-1}.

* **Properties of invertible matrices :**

Let A, B, C are $n \times n$ invertible matrices. Then the following hold:

(i) AB is invertible & $(AB)^{-1} = B^{-1}A^{-1}$

(ii) A^t is invertible & $(A^t)^{-1} = (A^{-1})^t$.

(iii) $(A^{-1})^{-1} = A$.

Prob:- If A and B are square matrices of same order such that A^{-1} exist and B^{-1} does not exist. Does $(AB)^{-1}$ exist ?

Sol^n: $(AB)^{-1}$ does not exist.

Exam:- $A = I_2$.

$$B = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$, B^{-1} does not exist as $|B| = 0$

$AB = B$, not exist.

BLOCK MATRICES :-

Using a system of horizontal and vertical (dashed) lines, we can partition a matrix A into submatrices called "block of A".

→ Given a matrix may be divided into blocks in different ways.

Ex:-

$$\left[\begin{array}{cc|cc|c} 1 & -2 & 0 & 1 & 3 \\ 2 & 3 & 5 & 7 & 2 \\ \hline 3 & 1 & 4 & 5 & 9 \\ 4 & 6 & -3 & 1 & 8 \end{array}\right]$$

$$\left[\begin{array}{ccc|cc} 1 & -2 & 0 & 1 & 3 \\ 2 & 3 & 5 & 7 & 2 \\ \hline 3 & 1 & 4 & 5 & 9 \\ 4 & 6 & -3 & 1 & 8 \end{array}\right]$$

$$\left[\begin{array}{c|cc|cc} 1 & -2 & 0 & 1 & 3 \\ \hline 2 & 3 & 5 & 7 & 2 \\ 3 & 1 & 4 & 5 & 9 \\ \hline 4 & 6 & -3 & 1 & 8 \end{array}\right]$$

and so on ...

Block diagonal Matrices :- Let $A = [A_{ij}]$ be a square block matrix such that the non diagonal block are all zero matrices. Then A is called block diagonal Matrix.

Denote → $A = \operatorname{diag}(A_{11}, A_{22}, \ldots A_{rr})$

Ex:-

Determine which of the following block matrices are upper diagonal, lower diagonal or diagonal :-

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 4 & 0 \\ 5 & 0 & 6 & 0 \\ 0 & 7 & 8 & 9 \end{bmatrix}$$

Ans- Upper triangular (blocks below the diagonal are zero)

Lower Triangular ($\because$ block above the diagonal are zero).

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 5 \\ 0 & 6 & 7 \end{bmatrix}$$

$\downarrow$ diagonal block Matrices

neither upper triangular nor lower triangular.

Ex-2.

1) $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$; 2) $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 4 & 5 \end{bmatrix}$

compute A^2, A^3.

1) $A^2 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 9 & 8 & 0 \\ 0 & 4 & 9 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$, In short form $A^2 = \text{diag}([4], [9, 8; 4 9], [9])$.

$A^3 = \begin{bmatrix} 8 & 0 & 0 & 0 \\ 0 & 25 & 44 & 0 \\ 0 & 22 & 25 & 0 \\ 0 & 0 & 0 & 27 \end{bmatrix}$ OR $A^3 = \text{dia}([8], [25, 44; 22, 25], [27])$.

(2) $A^2 = \text{diag}([3, 4 ; 8, 11], [9, 12 ; 24, 33])$

$A^3 = \text{diag}([11, 15 ; 30, 41], [57, 58 ; 156, 213])$

<u>Example</u>:- Let $A = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$. If P, Q, R and S are sym. what can you say about A?

<u>Sol^n</u>:- A may not be symmetric.

$$A = \left[\begin{array}{cc|cc} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 3 \\ \hline 2 & 0 & 4 & 0 \\ 0 & 2 & 0 & 4 \end{array}\right],$$ which is not symmetric.

but $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, $R = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $S = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

are symmetric.

Submatrix of a Matrix :- A matrix obtained by deleting some of its rows and/or columns of a matrix is said to be a submatrix of the given matrix.

Ex: $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.

A few submatrix of A are

$[1]$, $[4]$, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$, $\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$, A.

But the matrices $\begin{bmatrix} 1 & 5 \\ 4 & 6 \end{bmatrix}$, $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ are not the submatrices of A.

Remark:- suppose A and B are Block matrices.

$A = \begin{bmatrix} P & R \\ Q & S \end{bmatrix}$, $B = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$.

Then the matrices P, Q, R, S and E, F, G, H, are called blocks of the matrices A & B respectively.

* Even if A+B is defined, the order of P & E may not be same & hence, we may not be able to defined addition in A & B in block form.

But if A + B & P+E & others are defined.

Then $A+B = \begin{bmatrix} P+E & R+F \\ Q+G & S+H \end{bmatrix}$

Ex :-

$$A = \begin{array}{c} P \\ \\ Q \\ \\ \end{array} \left[\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ \hline 2 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 2 & 3 \end{array}\right] \begin{array}{c} R \\ S \\ \\ \\ \end{array}, \qquad B = \begin{array}{c} E \\ \\ G \\ \\ \end{array} \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} F \\ \\ H. \\ \\ \end{array}$$

$A+B$ is defined but $P+E$ is not defined

$R+F$ is not defined

$Q+G$ " "

$S+H$ " " .

Thus, addition is not defined blockwise.

similarly, if the product AB is defined, the product PE need not be defined.

Therefore, we can talk of matrix product AB as block product of matrices.

If the product AB, PE,all are defined then.

$$AB = \begin{bmatrix} PE + RG & PF + RH \\ QE + SG & QF + SH \end{bmatrix}.$$

Exercise.-

$$A = \begin{array}{c} P \\ \\ Q \\ \\ \end{array} \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \begin{array}{c} R \\ \\ S \\ \\ \end{array}, \qquad B = \begin{array}{c} E \\ \\ G \\ \\ \end{array} \left[\begin{array}{cc|cc} 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 1 \\ \hline 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{array}\right] \begin{array}{c} F \\ \\ H. \\ \\ \end{array}$$

Block wise product,

$$AB = \begin{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix} & \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix} \\ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} & \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 3 & 1 & 2 \\ 1 & 3 & 2 & 3 \\ \hline 2 & 2 & 3 & 2 \\ 0 & 2 & 1 & 2 \end{array}\right]$$