Row Space of a Matrix :

Let $A = [a_{ij}]$ be an arbitrary $m \times n$ matrix over a field K. Then row of A,

$R_1 = (a_{11}, a_{12}, \ldots, a_{1n})$, $R_2 = (a_{21}, a_{22}, \ldots, a_{2n}), \ldots, R_m = (a_{m1}, a_{m2}, \ldots, a_{mn})$
may be viewed as vector in K^n.

Hence they span a subspace of K^n called row space of A. denoted by "rowsp(A)".

i.e. $\text{row sp}(A) = \text{span}(R_1, R_2, \ldots, R_m)$

Analagously, the column of A may be viewed as vectors in K^m called column space of A and denoted by colsp(A).

We observe that $\text{colsp}(A) = \text{rowsp}(A^T)$
$\text{rowspace}(A) = \text{col sp}(A^t)$

Result : Row equivalent matrices have the same rowspace.

Result : Suppose A & B are in row-reduced echelon form. Then A & B have same row space iff they have same non-zero rows.

Result : Elementary row operations do not change the row space of a matrix.

Remark : The elementary row operations CHANGE the column space of a matrix -

Example

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

Row echelon form of A OR (Use elementary row operation)

$R_2 \to R_2 - 2R_1$

$\sim \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} = B$ (say).

Column space of A = set of all scalar multiple of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Column space of B = set of all scalar multiple of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

and cl space (A) $\neq$ cl. sp. (B).

But We have the following relation :

Thm :- Let A & B are row equivalent matrices, Then :

(a) A given set of column vectors of A is linearly independent if and only if the corresponding column vectors of B are linearly independent.

(b) A given set of column vectors of A form a basis for the column space of A if and only if the corresponding column vectors of B form a basis for the column space of B.

Example: Consider the following two sets of vectors in $\mathbb{R}^4$.

$u_1 = (1, 2, -1, 3)$, $u_2 = (2, 4, 1, -2)$, $u_3 = (3, 6, 3, -7)$

$w_1 = (1, 2, -4, 11)$, $w_2 = (2, 4, -5, 14)$

Let $U = \text{span}(u_i)$, $V = \text{span}(w_i)$

$\boxed{\text{Show that } U = V}$

Sol^{n}: There are two ways to show $\boxed{U = V}$

(a) Show that each u_i is a linear combination of w_1 & w_2.
and show that each w_i is a linear combination of u_1, u_2, u_3.

Observe that, we have to solve the six system of linear eqn. are consistent.

(b) From the matrix A, whose rows are u_1, u_2, u_3 and row reduce A to row reduced echelon form, and form the matrix B whose rows are w_1 & w_1 and reduce B to row reduced echelon form.

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 6 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 1/3 \\ 0 & 0 & 1 & -8/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & -4 & 11 \\ 2 & 4 & -5 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -4 & 11 \\ 0 & 0 & 3 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 1/3 \\ 0 & 0 & 1 & -8/3 \end{bmatrix}$$

Because, the non-zero rows of the matrices in row-reduced form are identical. The row space of A & B are equal.

$\therefore$ $U = W$.

Rank of Matrixes :-

Definition : The row rank of a matrix A, $\text{rank}(A)$, is equal to the maximum no. of l.i rows of A.

or equivalently, the dimension of the row space of A.

Thm : The maximum no. of linearly independent rows of any matrix is equal to the maximum number of linearly independent column of A.

Thus, $\dim \text{rowsp}(A) = \dim \text{colsp}(A)$

→ **Basis finding Problem** :

Here, one will see how an echelon form of any matrix A gives us the solution to certain problems about A itself.

Specifically, let A & B be the following matrices, where the echelon matrix B (whose pivot are circled) is an echelon form A :-

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 1 & 2 \\ 2 & 5 & 5 & 6 & 4 & 5 \\ 3 & 7 & 6 & 11 & 6 & 9 \\ 1 & 5 & 10 & 8 & 9 & 9 \\ 2 & 6 & 8 & 11 & 9 & 12 \end{bmatrix} \quad \& \quad B = \begin{bmatrix} 1 & 2 & 1 & 3 & 1 & 2 \\ 0 & 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis of the row space of A.

Ans :- We are given that A & B are row equivalent. So, they have same row space.

Moreover, "B" is in echelon form, so its non-zero rows are linearly indepentent & hence form a basis of the row space of B.

Thus, they also form a basis of the row space of A.

i.e

Basis of rowsp(A): $(1,2,1,3,1,2)$, $(0,1,3,1,2,1)$, $(0,0,0,1,1,2)$

(b) Find each column C_k of A that is a l. combination of preceding columns of A.

Sol^n:- (i) C_3 is a l.c. of C_1, C_2

(ii) C_5, C_6 are l. combination of preceding column of A.

Let $M_k = [C_1, C_2, \dots, C_k]$

$\text{Rank}(M_2) = \text{rank}(M_3) = 2$

$\text{Rank}(M_4) = \text{rank}(M_5) = \text{rank}(M_6) = 3.$

(c) Find the basis of the column space of A.

Since the column C_1, C_2, C_4 are l.i. Thus, they form a basis.

i.e basis of colsp(A): $[1,2,3,1,2]^t$, $[2,5,7,5,6]^t$, $[3,6,11,8,11]^t$

Observe that, C_1, C_2, C_4 may also be characterized as those column of A that contains the pivot in any echelon form of A.

(4) Find the rank of A :-

Here, we see that three definitions of the rank A yield same value.

1. There are three pivot in B, which is an echelon form of A.
2. The three pivot in B correspond to the non-zero rows of B, which form a basis of the row space of A.
3) The three pivot in B correspond to the column of A, which form a basis of the column space of A.

Thus, rank (A) = 3.

Ex :- Let W be a subspace of $\mathbb{R}^5$ spanned by the following vectors :-

$u_1 = (1,2,1,3,2)$, $u_2 = (1,3,3,5,3)$, $u_3 = (3,8,7,13,18)$

$u_4 = (1,4,6,9,7)$, $u_5 = (5,13,13,25,19)$

Find a basis of W consisting the original given vectors and find dim W.

Ans:- Form a matrix whose column are given vectors, reduce M to echelon form

$$M = \begin{bmatrix} 1 & 1 & 3 & 1 & 5 \\ 2 & 3 & 8 & 4 & 13 \\ 1 & 3 & 7 & 6 & 13 \\ 3 & 5 & 13 & 9 & 25 \\ 2 & 3 & 18 & 7 & 19 \end{bmatrix} \quad N \quad \begin{bmatrix} 1 & 1 & 3 & 1 & 5 \\ 0 & 1 & 2 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\{C_1, C_2, C_4\}$ forma basis.

dim W = 3.

Given a matrix $A_{m \times n}$, One can define the linear transformation

$$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$T(x) = Ax^t \quad , \quad x = (x_1, x_2, \ldots, x_n)$$

or

$$T(x) = Ax \quad \text{if } x = [x_1 \; x_2 \; \ldots \; x_n]^t$$

Now,

$$Ax = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$Ax \in$ span(column of A)

Range of T = Span (column of A)

→ Converse is also true. We will see later.
i.e Every linear map can be represented by a matrix.

→ We denote this map by A itself as $A(x) = Ax$

Again, if $A \in M_{n\times m}(\mathbb{F})$, Then $A^t \in M_{n\times m}(\mathbb{F})$

Thus, $A^t : \mathbb{R}^m \longrightarrow \mathbb{R}^n$
$\mathbb{F}^m \longrightarrow \mathbb{F}^n$.

Again, we can talk about the range(A^t),
For $y \in \mathbb{R}^m$
$A^t(y) \in$ span (column of A^t)
$\in$ span (rows of A)

range (A^t) = span (rows of A).

→ Since, A matrix $A \in M_{n\times m}$ can be represented by a linear transformation.
So, we can talk about null space of A.

- Null space of A, denoted null(A), is a subspace of $\mathbb{F}^n$, consisting of the sol^{n}s of the homogeneous linear system $AX = 0$.

i.e null$(A) = \{ x \in \mathbb{F}^n : Ax = 0 \}$

- Column space of A, denoted col(A), is a subspace of $\mathbb{R}^m$ spanned by columns of A.

i.e col(A) = range of A = span (columns of A)

Method of finding Basis for the row space, the nullspace and the column space of a matrix :-

- Let "A" be a given matrix and let R be the reduced row echelon form of A.

*1) Use the non-zero row of R to form a basis of Row sp(A).

(2) Use the column of A that corresponds to the column's of R containing the leading 1's to form a basis of col sp(A).

OR Find a basis for rowspace (A^t) that will also be a basis for Col(A)

(3) For null space :-

1) Find free variables.
2) Solve the leading variables of $Rx=0$ in terms of the free variable.
3) Set the free variables equal to parameters, substitute back into x.
4) Write the result as a linear combination of k vectors (where k is the no. of free variables)

These k vectors form a basis for null A.

Ex: Find a ba[sis for] the row space, column space, and the null space of the matrix given below

$$A = \begin{bmatrix} 3 & 4 & 0 & 7 \\ 1 & -5 & 2 & -2 \\ -1 & 4 & 0 & 3 \\ 1 & -1 & 2 & 2 \end{bmatrix}$$

Sol^n:- The row reduced form of A is $R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

* basis for Rowsp(A) is the non-zero rows of R

i.e $\{(1,0,0,1), (0,1,0,1), (0,0,1,1)\}$

* basis for colsp(A) – The column, which contains leading coefficent 1.

i.e $\{(3\ 1\ -1\ 1)^t, (4\ -5\ 4\ -1)^t, (0\ 2\ 0\ 2)^t\}$

* null space:-

let $X = (x_1, x_2, x_3, x_4)$, consider the system $Rx = 0$.

(i) find free variable :- x_4 is the only free variable.

(ii) Solve other variable interms of free variable $Rx = 0 \Rightarrow x_1 = -x_4$, $x_2 = -x_4$, $x_3 = -x_4$

(iii) let $x_4 = k$, where k is a parameter. Thus we have

$$(x_1, x_2, x_3, x_4) = (-k, -k, -k, k) = k(-1, -1, -1, 1)$$

Thus, basis of null A $= \{(-1, -1, -1, 1)\}$

$$\text{Row sp}(A) = \text{Range}(A^t) = \text{span}(\text{Row of } A)$$

→ Dimension of the Rowsp(A) is called the row rank of A.

→ dimension of col sp (A) is called the column rank of A

Thm: ① Let $T: V \longrightarrow W$ be a Linear Transformation.

B, B

$$T(x, y, z) = (x+y, \ y+z, \ z+x)$$

$$T(1,0,0) = (1,0,1) = 1e_1 + 0e_2 + 1e_3$$

$$T(0,1,0) = (1,1,0) = 1e_1 + 1e_2 + 0e_3$$

$$T(0,0,1) = (0,1,1) = 0e_1 + 1e_2 + 1e_3.$$

$$[T]_{B,B}(x,y,z) = \begin{array}{ccc} Te_1 & Te_2 & Te_3 \end{array} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

② $T: V \longrightarrow W$

$$T(x, y, z) = (2x, \ y+z, \ 0)$$

Let $B = \{(1,0,0), (0,1,0), (1,1,1)\}$ form a basis of V & W

$[T]_{B,B}$.

$$T((1,0,0)) = (2,0,0) = \alpha_1(1,0,0) + \beta_1(0,1,0) + \gamma_1(1,1,1) \quad -①$$

$$T((0,1,0)) = (0,1,0) = \alpha_2(1,0,0) + \beta_2(0,1,0) + \gamma_2(1,1,1) \quad -②$$

$$T((1,1,1)) = (2,2,0) = \alpha_3(1,0,0) + \beta_3(0,1,0) + \gamma_3(1,1,1) \quad -③$$

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

① $\Rightarrow$

$2 = \alpha_1 + \gamma_1 \quad \Rightarrow \alpha_1 = 2$

$0 = \beta_1 + \gamma_1 \quad \Rightarrow \boxed{\beta_1 = 0}$

$0 = \gamma_1 \quad \Rightarrow \boxed{\gamma_1 = 0}$

② $\Rightarrow$

$0 = \alpha_2 + \gamma_2 \quad \Rightarrow \boxed{\alpha_2 = 0}$

$1 = \beta_2 + \gamma_2 \quad \Rightarrow \boxed{\beta_2 = 1}$

$0 = \gamma_2 \quad \Rightarrow \boxed{\gamma_2 = 0}$

③ $\Rightarrow$

$2 = \alpha_3 + \gamma_3 \quad \Rightarrow \boxed{\alpha_3 = 2}$

$2 = \beta_3 + \gamma_3 \quad \Rightarrow \boxed{\beta_3 = 2}$

$0 = \gamma_3 \quad \Rightarrow \boxed{\gamma_3 = 0}$