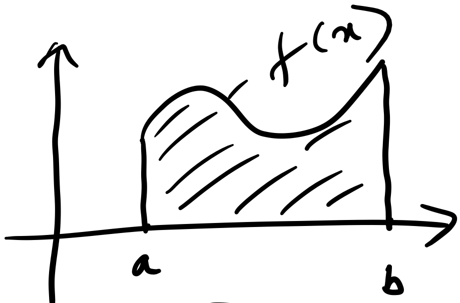
Riemann Integral

$$\int_a^b f(x)\,dx$$



<u>Ex</u> :- $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [0,1] \\ 0, & x \in \mathbb{Q}^c \cap [0,1] \end{cases}$

$\int_0^1 f(x)\,dx$, No, f is not integrable

- Let $f : [a,b] \to \mathbb{R}$ is bounded on the closed bounded interval $[a,b]$.

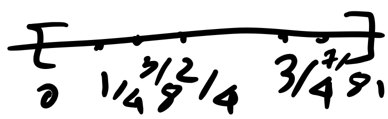
$$m = \inf_{x \in [a,b]} f(x) \quad , \quad M = \sup_{x \in [a,b]} f(x) .$$

- <u>partition</u> :- A partition P of $[a,b]$ is $P = \{x_0, x_1, x_2, \dots, x_n\}$ s.t

$$a = x_0 < x_1 < x_2 < \dots < x_n = b .$$

P divides $[a,b]$ on sub-interval $[a, x_1], [x_1, x_2], \dots, [x_{n-1}, b]$.

<u>Ex</u>:- $P = \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$ is partition of $[0,1]$



✓ $Q = P = \{0, \frac{1}{4}, \frac{3}{8}, \frac{2}{4}, \frac{3}{4}, \frac{7}{8}, 1\}$ is partition of $[0,1]$

$P \subseteq Q$.

$[a,b]$, $b = a + nh$, $n \to$ no. of partition
$h \to$ length of "

$[0,1]$, $n = 8$, $\frac{b-a}{n} = \frac{1-0}{8} = h$

$x_i = x_0 + ih$, $\Rightarrow h = \frac{1}{8}$

$P = \{0, \frac{1}{8}, \frac{2}{8}, \dots, 1\}$

$P = \{x_0, x_1, x_2, \ldots, x_n\}$ of $[a,b]$.

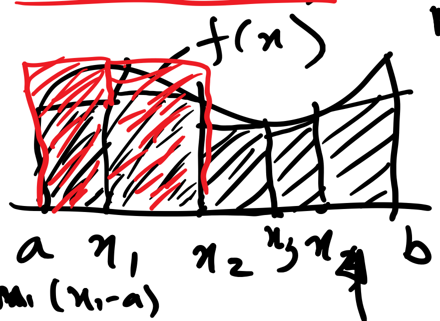
$$m_k = \inf_{x \in [x_{k-1}, x_k]} f(x), \quad M_k = \sup_{x \in [x_{k-1}, x_k]} f(x)$$

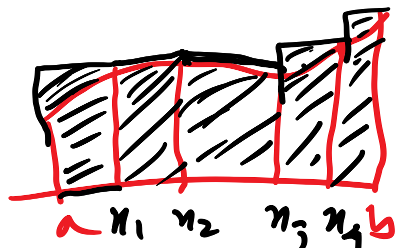
Lower Sum: The Lower Sum $L(P,f)$ of function f with respect to some partition P is

$$L(P,f) = \sum_{k=1}^{n} m_k (x_k - x_{k-1})$$
$$= m_1(x_1 - x_0) + m_2(x_2 - x_1) + \cdots + m_n(x_n - x_{n-1}).$$

Upper Sum: $U(P,f) = \sum_{k=1}^{n} M_k (x_k - x_{k-1})$

$$= \underline{M_1(x_1 - x_0)} + \underline{M_2(x_2 - x_1)} + \cdots + M_n(x_n - x_{n-1})$$

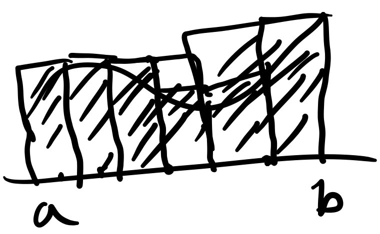


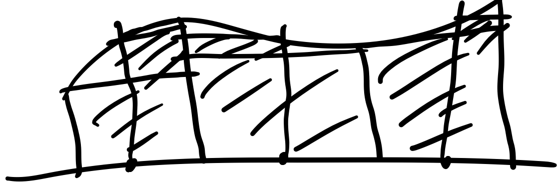


- For any P, $L(P,f) \leq U(P,f)$

Refinement of a partition: A partition Q is called a refinement of P if $P \subseteq Q$.

- If Q is a refinement of P, then $\underline{L(P,f) \leq L(Q,f)}$, $U(P,f) \geq U(Q,f)$





Def^n:- Let $\mathcal{P}$ be the collection of all possible partition of $[a,b]$. Then

Upper integral of f

$$\overline{\int_a^b} f = \inf\{U(P,f) : P \in \mathcal{P}\}$$

Lower integral of f

$$\underline{\int_a^b} f = \sup\{L(P,f) : P \in \mathcal{P}\}$$

- For a bounded function f, $\underline{\int_a^b} f \le \overline{\int_a^b} f$.

- Riemann integrability: $f : [a,b] \to \mathbb{R}$ is Riemann integrable if $\boxed{\underline{\int_a^b} f = \overline{\int_a^b} f} = \underline{\int_a^b f}$

we say $f \in \mathcal{R}[a,b]$

$\to$ Riemann integrable on $[a,b]$

EX:- $f(x) = c$, $x \in [a,b]$. Show that $f \in \mathcal{R}[a,b]$.

f is bounded function $[a,b]$

Take a partition P of $[a,b]$

$$P = \{x_0, x_1, \dots, x_n\}$$

$$m_k = \inf_{x \in [x_{k-1}, x_k]} f(x), \quad M_k = \sup_{x \in [x_{k-1}, x_k]} f(x)$$

$$m_k = c, \quad M_k = c$$

$$L(P,f) = m_1(x_1 - x_0) + m_2(x_2 - x_1) + \dots + m_n(x_n - x_{n-1})$$

$$= \underline{c\,(b-a)} \quad = c(\cancel{x_1} - x_0 + \cancel{x_2} - \cancel{x_1} + x_3 - \cancel{x_2} \quad = c(x_n - x_0))$$

$$U(P,f) = M_1(x_1 - x_0) + M_2(x_2 - x_1) + \dots + M_n(x_n - x_{n-1})$$

$$= c(b-a)$$

$\mathcal{P}$ be the all possible partition of $[a,b]$

$$\{L(P,f) : P \in \mathcal{P}\} = \{c(b-a)\}$$

$$\underline{\int_a^b} f = \sup\{L(P,f) : P \in \mathcal{P}\} = c(b-a)$$

$$\{U(P,f) : P \in \mathcal{P}\} = \{c(b-a)\}$$

$$\overline{\int_a^b} f = \inf\{U(P,f) : P \in \mathcal{P}\} = c(b-a)$$

$$\therefore \underline{\int_a^b} f = \overline{\int_a^b} f = c(b-a)$$

$$= \int_a^b f$$

$$\Rightarrow f \in R[a,b].$$

$f : [0,1] \to \mathbb{R}$

Ex: $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0,1] \\ 0, & x \in \mathbb{Q}^c \cap [0,1] \end{cases}$

$P = \{x_0, x_1, \ldots, x_n\}$ of $[0,1]$

$$m_k = \inf_{x \in [x_{k-1}, x_k]} f(x) = 0, \quad M_k = \sup_{x \in [x_{k-1}, x_k]} f(x) = 1$$

$$L(P,f) = 0, \quad U(P,f) = 1$$

$$\left\{\begin{array}{l} \{L(P,f) : P \in \mathcal{P}\} = \{0\} \\ \{U(P,f) : P \in \mathcal{P}\} = \{1\} \end{array}\right.$$

$$\underline{\int_0^1} f = \sup\{L(P,f) : P \in \mathcal{P}\} = 0$$

$$\overline{\int_0^1} f = \inf\{U(P,f) : P \in \mathcal{P}\} = 1$$

$$\underline{\int_0^1} f \neq \overline{\int_0^1} f \Rightarrow f \text{ is not Riemann inte.}$$

Norm of a partition:

$P = \{x_0, x_1, \ldots, x_n\}$

$\|P\| =$ max length of subinterval

$= \max\{|x_1 - x_0|, |x_2 - x_1|, \ldots, |x_n - x_{n-1}|\}$

$P = \{0, \frac{1}{8}, \frac{2}{8}, \ldots, \frac{7}{8}, 1\}$

$\|P\| = \frac{1}{8}.$