Contractive Sequence

$\{a_n\}$ $\exists\ 0 < \alpha < 1$ s.t.

$$|a_{n+2} - a_{n+1}| \leq \alpha\, |a_{n+1} - a_n|$$

$\forall n \in \mathbb{N}$

Ex1: $a_1 = 1$, $a_{n+1} = 1 + \frac{1}{a_n}$ $\quad a_n \geq 1\ \forall n$

$\to a_{n+1} = \frac{a_n + 1}{a_n}$

$\Rightarrow a_{n+1} a_n = \underbrace{a_n + 1}_{\geq 2}$

$$|a_{n+2} - a_{n+1}|$$

$$= \left| 1 + \frac{1}{a_{n+1}} - 1 - \frac{1}{a_n} \right|$$

$$= \left| \frac{1}{a_{n+1}} - \frac{1}{a_n} \right| = \left| \frac{a_n - a_{n+1}}{a_{n+1} a_n} \right|$$

$$= \left| \frac{a_{n+1} - a_n}{a_{n+1} a_n} \right|$$

$$\leq \frac{1}{2} |a_{n+1} - a_n|$$

$\forall n \in \mathbb{N}$

$a_{n+1} \cdot a_n \geq 2$

$\frac{1}{a_{n+1} a_n} \leq \frac{1}{2}$

Ex2: $a_1 > 0$, $a_{n+1} = \frac{1}{2 + a_n}$

$$|a_{n+2} - a_{n+1}| = \left| \frac{1}{2 + a_{n+1}} - \frac{1}{2 + a_n} \right|$$

$$= \left| \frac{a_n - a_{n+1}}{(2 + a_{n+1})(2 + a_n)} \right|$$

$$\leq \frac{1}{4} |a_{n+1} - a_n|$$

$\frac{a_n \geq 0\ \forall n}{2 + a_n \geq 2}$

$\frac{1}{2 + a_n} \leq \frac{1}{2}$

$a_{n+1} + 2 = \frac{1}{2 + a_n} + 2 > 2$, $a_1 > 0$

Result: $a_n > 0$. $\underline{\lim_{n\to\infty} \frac{a_{n+1}}{a_n}} = L$

$= \underline{\lim_{n\to\infty} a_n^{1/n}}$

EX: $\lim_{n\to\infty} n^{1/n} = 1$, $a_n = n$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 = \lim a_n^{1/n}$
$= \lim n^{1/n}$

Result: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$, $L<1$, $a_n \to 0$
$L>1$, $a_n \to \infty$

$\lim_{n\to\infty} a_n^{1/n} = L$, $L<1$, $a_n \to 0$
$L>1$, $a_n \to \infty$

$L=1$, no conclusion.

$a_n = n$, $\lim n^{1/n} = 1$, $n \to \infty$

$a_n = \frac{1}{n}$, $\underline{\lim \frac{a_{n+1}}{a_n} = \lim a_n^{1/n}} = 1$, $\frac{1}{n} \to 0$

EX $a_n = \frac{n^\alpha}{(1+p)^n}$, $\alpha > 0$, $p > 0$.

$\lim_{n\to\infty} a_n = 0$?? $\lim_{n\to\infty} a_n^{1/n} = \lim \left(\frac{n^\alpha}{(1+p)^n}\right)^{\frac{1}{n}}$

$= \lim \frac{n^{\frac{\alpha}{n}}}{1+p} = \frac{1}{1+p}$
< 1

EX: $\lim_{n\to\infty} n^\alpha \cdot x^n = 0$?? $|x| < 1$, $\alpha \in \mathbb{R}$

$a_n = n^\alpha \cdot x^n$, $\lim \left|\frac{a_{n+1}}{a_n}\right| < 1$

EX: $0<a<1$, $b>1$ ① $\{n^2 \cdot a^n\}$

② $\left\{\frac{b^n}{n^2}\right\}$, ③ $\left\{\frac{b^n}{n!}\right\}$

④ $\left\{\frac{n!}{n^n}\right\}$

lim sup & lim inf

$\{a_n\} \to$ bounded sequence. $\sup\{a_m : m \geq n\} = B_n$

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \left(\sup_{m\geq n} a_m\right)$$

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \left(\inf_{m\geq n} a_m\right)$$

$\{a_n\}$, $n=1$, $\sup\{a_1, a_2, a_3, \dots\} = B_1$

$n=2$, $\sup\{a_2, a_3, \dots\} = B_2$

$n=3$, $\sup\{a_3, a_4, \dots\} = B_3$

$\downarrow$

B_n

$B_n = \sup\{a_m : m \geq n\}$

$\lim B_n = \limsup a_n$

EX1 : $\{a_n\} = \{0, 1, 0, 1, \dots\}$

$B_n = \sup\{a_m : m \geq n\}$

$n=1$, $\sup\{a_1, a_2, \dots\} = 1$

$n=2$, $\sup\{a_2, a_3, \dots\} = 1$

$\{B_n\} = \{1, 1, 1, \dots\}$

$\lim B_n = 1$. $\limsup a_n = 1$

$n=1$, $\inf\{a_1, a_2, \dots\} = 0$

$\alpha_n = \inf\{a_m : m \geq n\}$

$\{\alpha_n\} = \{0, 0, 0, \dots\} \to 0$

$\liminf a_n = 0$.

Ex 2: $a_n = (-1)^n\left(1+\frac{1}{n}\right)$.

$-2 \quad -\frac{4}{3} \rightarrow 0 \leftarrow \frac{5}{4} \quad \frac{3}{2}$

$\beta_1 = \frac{3}{2}$

$\beta_2 = \frac{5}{4}$, $\beta_n = \left(1+\frac{1}{2n}\right)$

$\lim \beta_n = 1 = \limsup a_n$

$\alpha_n = \inf\{a_m : m \geq n\}$

$\alpha_1 = -2,\ \alpha_2 = -\frac{4}{3},\ \ldots$

$\alpha_n = -\left(1+\frac{1}{2n-1}\right) \rightarrow -1$

$\lim \alpha_n = -1 = \liminf a_n$

Result: ① $\{a_n\}$, $\lim a_n = L = \liminf a_n$
$= \limsup a_n$

② $\{a_n\}$ bounded, $\limsup a_n = \liminf a_n$
$= L$

$\Rightarrow \lim a_n = L$