Ordinary Differential Equations(EMAT102L)
(Lecture-12)

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Outline of the Lecture

We will learn

- Method of Reduction of Order

Method of Reduction of Order

According to this method, if one non-zero solution of a second order homogeneous linear differential equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \quad (1)$$

is known, then by making the appropriate transformation we may reduce the given equation to another homogeneous linear equation that is one order lower than the original.

Consider the second order homogeneous linear differential equation

$$a_0(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)y=0 \tag{1}$$

Derivation

- Suppose $f(x)$ is a known non-trivial solution of the given equation, then we can determine a solution $g(x)$ using this method such that $\{f,g\}$ forms a fundamental set of solutions of (1). i.e, $f(x)$ and $g(x)$ are linearly independent.
- For this, use the transformation $g(x)=f(x)v$, where f is the known solution of (1) and v is a function of x to be determined.
- Then differentiating, we obtain

$$g'=f(x)v'+f'(x)v$$

$$g''=f(x)v''+2f'(x)v'+f''(x)v.$$

Since g is a solution of (1), therefore substituting the values of g, g', g'' in (1), we obtain

$$\Rightarrow a_0(x)(v''f(x) + 2f'(x)v' + f''(x)v) + a_1(x)(f(x)v' + f'(x)v) + a_2(x)f(x)v = 0$$

$$a_0(x)f(x)v'' + \{2a_0(x)f'(x) + a_1(x)f(x)\}v' + \{a_0(x)f''(x) + a_1(x)f'(x) + a_2(x)f(x)\}v = 0$$

Since f is a solution of (1), the coefficient of v is zero. So, the last equation becomes

$$a_0(x)f(x)v'' + \{2a_0(x)f'(x) + a_1(x)f(x)\}v' = 0.$$

Put $w = \dfrac{dv}{dx}$, then the above equation becomes

$$a_0(x)f(x)\frac{dw}{dx} + [2a_0(x)f'(x) + a_1(x)f(x)]w = 0$$

$$\Rightarrow \frac{w'}{w} = -\frac{2a_0(x)f'(x) + a_1(x)f(x)}{a_0(x)f(x)} = -\frac{2f'(x)}{f(x)} - \frac{a_1(x)}{a_0(x)}$$

- Integrating, we obtain

$$\log|w| = -\log[f(x)]^2 - \int \frac{a_1(x)}{a_0(x)}dx + \log|c|$$

$$\Rightarrow w = \frac{e^{-\int \frac{a_1}{a_0}dx}}{[f(x)]^2}$$

Since $w = \dfrac{dv}{dx}$, so the above equation becomes

$$v = \int \frac{e^{-\int \frac{a_1}{a_0}dx}}{[f(x)]^2}dx.$$

- Therefore the new solution is

$$g(x) = f(x)v(x) = f(x)\frac{e^{-\int \frac{a_1}{a_0}dx}}{[f(x)]^2}$$

Also, the original known solution $f(x)$ and the new solution $g(x) = v(x)f(x)$ are linearly independent. Since

$$\begin{aligned} W(f,g)(x) &= W(f,fv)(x) = f(x)(f(x)v)' - f'(x)(f(x)v) \\ &= f(x)(v'(x)f(x) + v(x)f'(x)) - v(x)f(x)f'(x) \\ &= [f(x)]^2 v' \\ &= [f(x)]^2 \frac{e^{-\int \frac{a_1(x)}{a_0(x)}dx}}{[f(x)]^2} \\ &= e^{-\int \frac{a_1}{a_0}dx} \neq 0. \end{aligned}$$

- The general solution of the ODE (1) is

$$y(x) = c_1 f(x) + c_2 g(x), c_1, c_2 \text{ are arbitrary constants}$$

Hypothesis

Let f be a known non-trivial solution of the second order homogeneous linear differential equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \qquad (1)$$

then we can determine a solution $g(x)$ by the **Method of reduction of order** such that $\{f, g\}$ forms a fundamental set of solutions of (1). i.e, $f(x)$ and $g(x)$ are linearly independent.

- **Conclusion 1.** The transformation $g(x) = f(x)v$ reduces the equation (1) to the first order homogeneous linear order differential equation

$$a_0(x)f(x)\frac{dw}{dx} + [2a_0(x)f'(x) + a_1(x)f(x)]w = 0 \qquad (2)$$

  in the dependent variable w, where $w = v'$.

- **Conclusion 2.** The particular solution

$$w = \frac{e^{-\int \frac{a_1}{a_0} dx}}{[f(x)]^2}$$

of equation (2) gives rise to the function v, where

$$v = \int \frac{e^{-\int \frac{a_1}{a_0} dx}}{[f(x)]^2} dx.$$

The function g defined by $g(x) = f(x)v(x)$ is then a solution of the second order equation (1).

- **Conclusion:** The original known solution $f(x)$ and the new solution $g(x) = v(x)f(x)$ are linearly independent. Hence, the general solution of (1) can be expressed as

$$y(x) = c_1 f(x) + c_2 g(x).$$

Example

Given that $y = x$ is a solution of

$$(x^2+1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

find a linearly independent solution by reducing the order.

Solution: Given differential equation is

$$(x^2+1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

Observe that $y = x$ satisfies the given differential equation.

- Let us use the transformation $g(x) = f(x)v = xv$.
- Here

$$v(x) = \int \frac{e^{-\int \frac{a_1}{a_0}dx}}{[f(x)]^2}dx = \int \frac{e^{-\int \frac{-2x}{x^2+1}dx}}{x^2}dx$$
$$= \int \frac{x^2+1}{x^2}dx = \int \left(1 + \frac{1}{x^2}\right)dx = x - \frac{1}{x}.$$

Example(cont.)

- $g(x) = f(x)v(x) = x\left(x - \frac{1}{x}\right) = x^2 - 1.$
  Also f and g are linearly independent, since $W(x, x^2 - 1) \neq 0$.
- Thus the general solution of the ODE is

$$y(x) = c_1 f(x) + c_2 g(x) = c_1 x + c_2(x^2 - 1), c_1, c_2 \text{ are arbitrary constants}$$

Example

If e^x is one of the solutions of homogeneous equation

$$x\frac{d^2y}{dx^2} - (2x-1)\frac{dy}{dx} + (x-1)y = 0$$

find a linearly independent solution by reducing the order.


Thank You