Random Variable → One particular choice.

Distribution → Maps for different choice

Poisson Distribution
(Important)

$P(X=k) = e^{-\lambda} \lambda^k / k!$

$k \geq 0$; $k \in I$ (Integer)

λ is the scale parameter $\lambda > 0$

$$PMF = \sum_{x=k} P(x=k) = 1$$

$$= \sum_{k=0}^{\infty} \frac{e^{-\lambda} \lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} e^{\lambda}$$ → from Taylor series

Taylor series of a function $f(x)$ at a is

$$f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(x-a)^2}{2!} + \cdots$$

if $f(x) = e^x$; $a=0$, then

$$\frac{1}{1} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$$

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$$

$$E[X] = \sum k \frac{e^{-\lambda} \lambda^{k}}{k!}$$

$$= e^{-\lambda} \sum_{k=1}^{\infty} \frac{k \lambda^{k}}{k!}$$

$$= e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k}}{(k-1)!}$$

$$= e^{-\lambda} \times \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$$

$$\boxed{\begin{matrix} k-1 = j \\ j = 0 \\ \infty \end{matrix}}$$

$$= e^{-\lambda} \times \lambda \times e^{\lambda} = \lambda$$

This distribution is used to count # of "success" where each success has a small probability of success

e.g. no of emails

no. of insects in your food.

· no. of earthquakes.
no of ~~at~~ hacking attacks

Say we have j events

$A_1, A_2 \cdots A_n$ $P(A_j) = p_j$ and n is very large

of A_j's that occur is approximately

$Pois(\lambda)$

$\lambda = E[X]$ $X \sim Pois(\lambda)$

$= \sum p_j$

Example

↳ Hospital 1.8 births/hour

1) $P(X=4) = \frac{e^{-\lambda}\lambda^k}{k!} = \frac{e^{-1.8}(1.8)^4}{4!} = 0.07$

2) $P(X \geq 2) = P(X=2) + P(X=3) \cdots$

$= 1 - P(X<2)$

$= 1 - P(X=0) - P(X=1)$

$= 1 - \frac{e^{-1.8} \times (1.8)^0}{0!} - \frac{e^{-1.8} (1.8)}{1}$

Normal Distribution

$N(0,1)$

$$f(z) = c\, e^{-z^2/2}$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}}}_{\text{Normalizing constant}} e^{-z^2/2}$$

$$\int_{-\infty}^{\infty} e^{-z^2/2}\, dz = \int_{-\infty}^{\infty} f(z)\, dz = I$$

$$I^2 = \int_{-\infty}^{\infty} e^{-z^2/2}\, dz \int_{-\infty}^{\infty} e^{-z^2/2}\, d\cancel{x}z$$

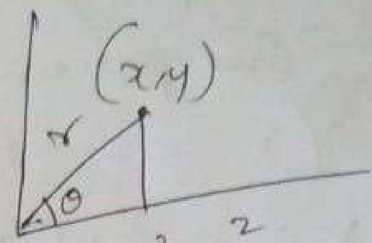
$$= \int_{-\infty}^{\infty} e^{-x^2/2}\, dx \int_{-\infty}^{\infty} e^{-y^2/2}\, dy \qquad \text{(Notation change)}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2/2}\, e^{-y^2/2}\, dx\, dy$$

$$= \int_{-\infty}\int_{-\infty} e^{-(x^2+y^2/2)}\, dx\, dy$$

$$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2/2} \times r \; dr\, d\theta$$

Jacobian



$r^2 = x^2 + y^2$

$= 2\pi$

$\Rightarrow I = \sqrt{2\pi}$

CDF

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\, dt$$

Easy to compute on computer systems

Exponential Distribution.

→ One parameter λ (Rate at which some events occur)

$X \sim Exp(\lambda)$ has PDF $\lambda e^{-\lambda x}$, $x > 0$.
otherwise 0.

PDF $P(X=x) = \lambda e^{-\lambda x}$.

CDF $F(x) = \int_0^x \lambda e^{-\lambda x} dx = 1 - e^{-\lambda x}$.

We know $\frac{dF(x)}{dx} = f(x)$.

One good thing

if $Y = \lambda X$

$Y \sim Exp(1)$

$P(Y \leq y) = P(\lambda X \leq y) = P(X \leq y/\lambda)$.

$= 1 - e^{-\lambda \times y/\lambda}$

$= 1 - e^{-y}$.

$\Rightarrow \lambda = 1$

Say

$E[Y$

Var(

$\cancel{\times}$ E

Va

Say $Y \sim Exp(1)$

$$E[Y] = \int_0^\infty y e^{-y}\, dy$$

$$= \underbrace{y(-e^{-y})\Big|_0^\infty}_{0} + \int_0^\infty e^{-y}\, dy$$

$$= 0 + \left(-e^{-y}\right)\Big|_0^\infty$$

$$= 0 + 1 = 1.$$

$$Var(Y) = E[Y^2] - \left(E[Y]\right)^2$$

$$= \int_0^\infty y^2 e^{-y}\, dy - (1)^2$$

$$= y^2(-e^{-y})\Big|_0^\infty + 2\left[\int_0^\infty y e^{-y}\, dy\right] - 1$$

$$= 0 + 2 - 1 = 1.$$

$$E[X] = \frac{1}{\lambda} \qquad \left\{ \begin{array}{l} \text{since } X = \dfrac{Y}{\lambda} \\ \text{and } E[Y] = 1 \\ \quad Var[Y] = 1 \end{array} \right.$$

$$Var(X) = \frac{1}{\lambda^2}$$

$\therefore$ using properties of Expectation and Variance, we get the result.

Memoryless property.

$$P(X \geq t+s \mid \underbrace{X \geq s}_{\text{we have like waited s minutes}}) = P(X \geq t)$$

$$P(X \geq s) = 1 - P(X \leq s)$$
$$= e^{-\lambda s}$$

$$P(X \geq t+s \mid X \geq s) = \frac{P(X \geq t+s, \overbrace{X \geq s}^{\text{redundant}})}{P(X \geq s)}$$

coz if we r waiting for t+s, we have already waited for s minutes

$$= \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}}$$
$$= e^{-\lambda t}$$
$$= P(X \geq t)$$

$$\Rightarrow P(X \geq s+t \mid X \geq s) = P(X \geq t)$$

Standard Normal

$Z \sim N(0,1)$.

$-Z \sim N(0,1)$. (Symmetry).

$E[Z]=0$

$E[Z^2]=1$

$$E[Z^k] = \begin{cases} 0 & k = \text{odd} \\ 1 & k = \text{even} \end{cases}$$

Let $X = \mu + \sigma Z$, $\mu \in R$ (mean)

$\sigma > 0$ (S.D.) $\Rightarrow$ This is the scale.

then $X \sim N(\mu + \sigma^2)$.

$E(X) = E[\mu + \sigma Z]$

$= E\mu + \sigma E[Z]$

$= \mu$

$Var(\mu + \sigma Z) =$

Note

$Var[X+Y] \neq Var[X] + Var[Y]$.

~~Var~~ if X, Y are independent

$Var[X+Y] = Var[X] + Var[Y]$.

$Var(\mu+\sigma Z) = Var(\sigma Z)$

$= \sigma^2 Var[Z].$

or $Z = \frac{X-\mu}{\sigma}$. [standardization].

Find PDF of $N(\mu, \sigma^2)$.

CDF $P(X \leq x) = P\left(\frac{X-\mu}{\sigma} \leq x\right)$

$= P\left(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right)$

$= \Phi\left(\frac{x-\mu}{\sigma}\right)$

$= F\left(\frac{x-\mu}{\sigma}\right)$

$$f\left(X = \frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{2\sigma}\right)^2}$$

$$f(x) = \frac{dF(x)}{dx}$$

$-X = -\mu + \sigma(-Z) \sim N(-\mu, \sigma^2).$

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if $X_j \sim N(\mu_j, \sigma_j^2)$ independent

then $X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

$X_1 - X_2 \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$

Rule of thumb $\Rightarrow$ We cannot evaluate $F(x)$ for normal.

$P(|X-\mu| \leq \sigma) \approx 0.65$

$P(|X-\mu| \leq 2\sigma) \approx 0.95$

$P(|X-\mu| \leq 3\sigma) \approx 0.97$

$Var[X]$ if $X \sim pois(\lambda)$.

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{\lambda}$$

Diff. b/s

$$\sum_{k=0}^{\infty} \frac{k\lambda^{k-1}}{k!} = e^{\lambda}$$

Mult. b/s by λ

$$\sum_{k=0}^{\infty} \frac{k \times \lambda^k}{k!} = \lambda e^{\lambda}$$

Diff. b/s

$$\sum_{k=1}^{\infty} \frac{k^2 \lambda^{k-1}}{k!} = \lambda e^{\lambda} + e^{\lambda} = e^{\lambda}(\lambda+1)$$

$Var[X] = E[X^2] - (E[X])^2$

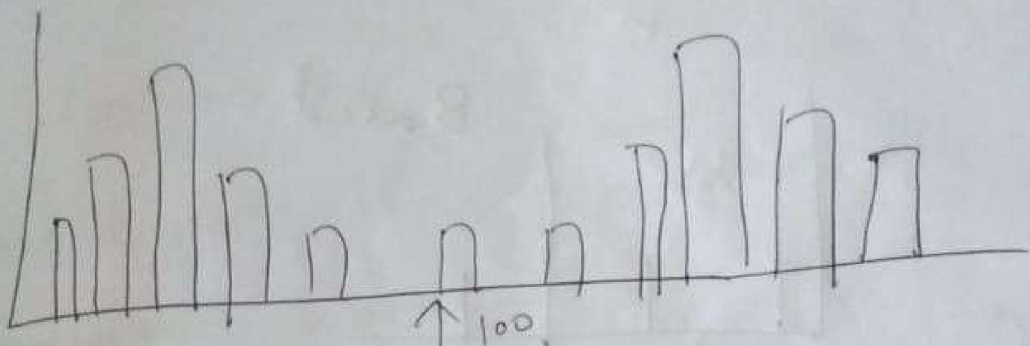
$$= \sum_{k=0}^{\infty} \frac{k^2 e^{-\lambda} \lambda^k}{k!} - \lambda^2$$

$$= \cancel{\lambda^2} + \lambda - \cancel{\lambda^2}$$

$$= \lambda$$

Central limit theorem.

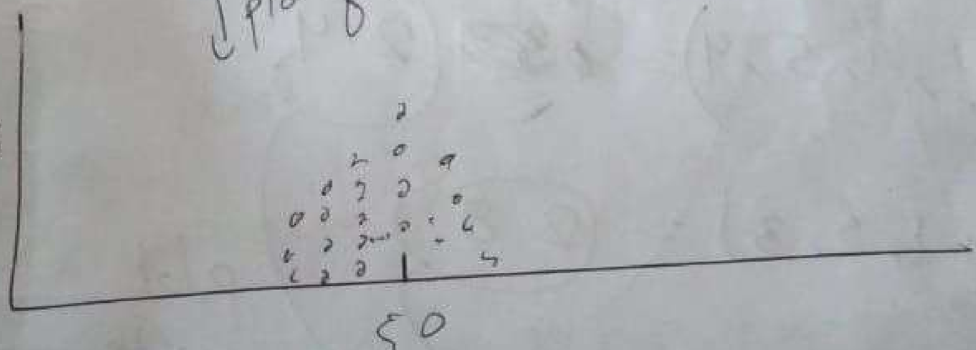
Say we have a distribution of data as

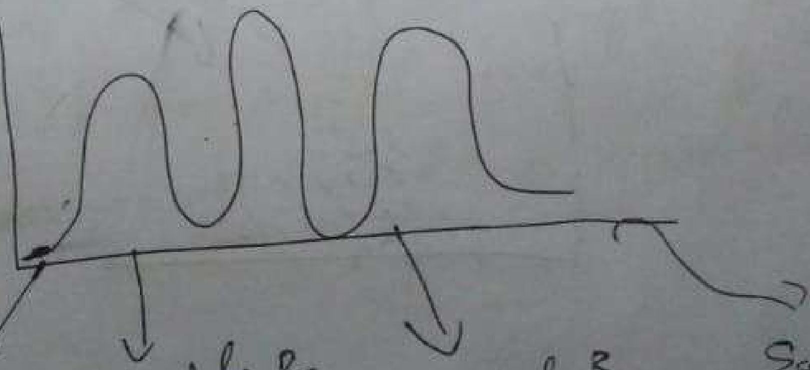


sample of size(n)

Assume this is the mean

plot of n



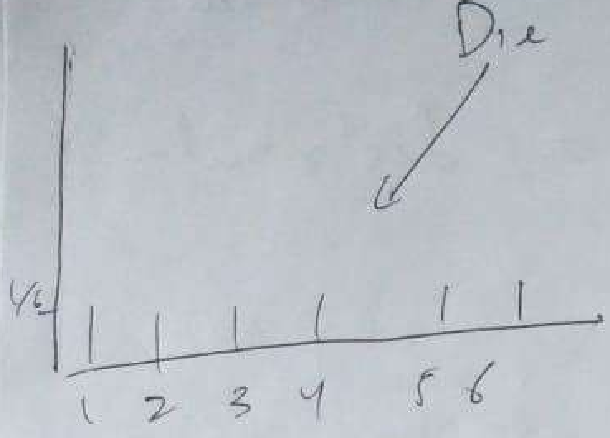


Sample 1 $\bar{X}$,

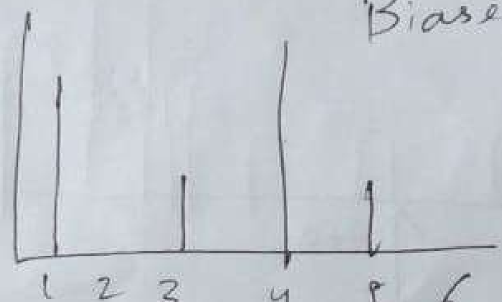
Sample 2 $\bar{X}_1$

Sample 3 $\bar{X}_3$ . . .

Sample S X_S



Biased



Draw Samples
or
I roll the die

mean

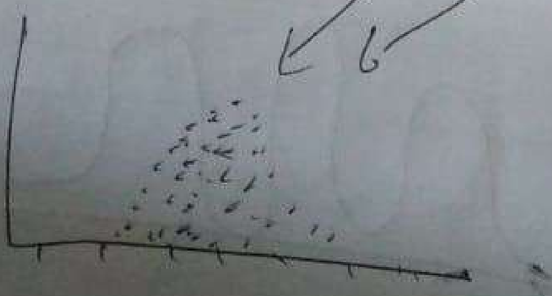
(1, 1, 1, 5, 4) = ~~3.9~~ 2.4

⋮

(1, 4, 1, 5, 3) = 2.8

⋮

plot with frequency



As the
the
norm
s
the
conve
Note 1
Note 2

$\frac{\sigma}{\sqrt{n}}$ is the std. dev of the samplin

distribution of the sample ~~the~~ mean

Alsocalled as the stand. error of

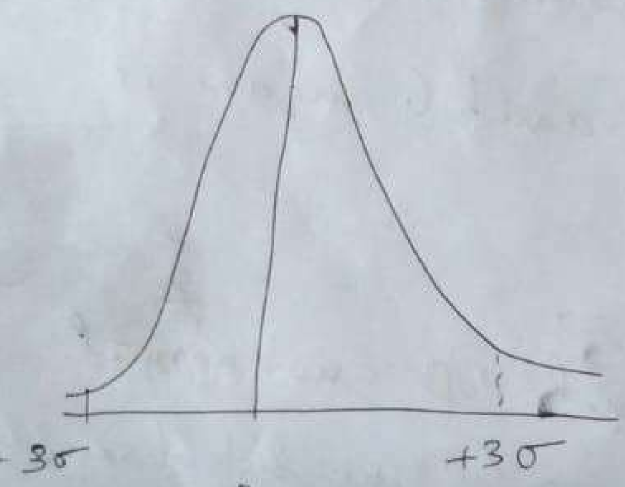
means.

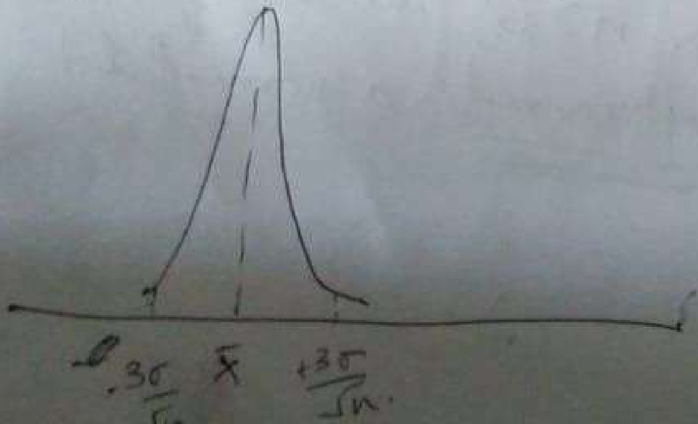
$\bar{x}_1, \bar{x}_2, \bar{x}_3 \ldots \bar{x}_z$

$$\bar{x} = \frac{\bar{x}_1 + \bar{x}_2 \ldots \bar{x}_z}{n}$$

$\bar{x} \approx \mu$

$\sigma_{\bar{x}} = \dfrac{\sigma}{\sqrt{n}}$ → sample size





As the number of samples increase the plot will approximate the frequency normal distribution.

$s \rightarrow \infty$

the distribution of sample means will converge to normal distribution.

Note 1: If original population dist is normal, then the dist. of sample mean is normal.

Note 2: If original pop is not normal, then for $n>30$, the dist. of sample means approximate a normal dist.

$$\sum_{k=0}^{\infty} \frac{k^2 \lambda^k}{k!} = \lambda e^{\lambda}(\lambda+1)$$

$$\sum_{k=0}^{\infty} \frac{k^2 e^{-\lambda} \times \lambda^k}{k!} = \lambda \times e^{-\lambda} \times e^{\lambda}(\lambda+1)$$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{k^2 e^{-\lambda} \times \lambda^k}{k!} = \lambda^2 + \lambda.$$

Strong Law of large numbers

Assume we have $X_1, X_2, X_3 \dots X_j$ iid samples with mean μ and Var σ^2.

$$X_n = \frac{\sum_{i=1}^{n} X_i}{n}$$

SLLN says as $n \to \infty$

$$X_n \to \mu$$

with prob 1.

E.g. $X_j \sim \text{Bern}(p)$

$$\frac{X_1 + X_2 \dots X_n}{n} \to p. \text{ with prob } 1.$$