EPHY105L (Fall Semester 2018-2019)
Solutions to Problem Sheet 4

1. (a) $$\oint \vec{E}d\vec{a} = \frac{Q_{enclosed}}{\varepsilon_o} = \frac{q_2}{\varepsilon_o},$$

$$\oint Dd\vec{a} = Q_{free\ (enclosed)} = q_2$$

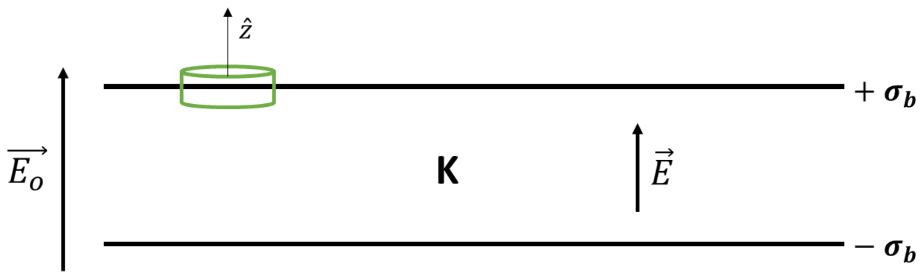
(b) $\oint \vec{E}d\vec{l} = 0,$

(c) At point A, $\vec{\nabla} \cdot \vec{D} = 0$

(d) At point B, $\vec{\nabla} \cdot \vec{E} = 0$

2. The electric field inside the dielectric can be calculated using Gauss's law involving $\vec{\boldsymbol{D}}$:

$$\oint \vec{D}.\overrightarrow{da} = q_{f,encl} \qquad (1)$$



Let us consider a cylindric pillbox of area A half way submerged into the surface of a dielectric. The electric field inside ($\vec{\boldsymbol{E}}$) and outside the dielectric ($\overrightarrow{\boldsymbol{E_0}}$) are shown in the Figure; the $\vec{D}$ will be parallel to the $\vec{E}$ with $D_0 = \epsilon_0 E_0$ and $D = \epsilon_0 KE$. Since there are no free charges enclosed by the Gaussian surface we must have

$$\oint \vec{D}.\overrightarrow{da} = 0 \quad (2)$$

Since $\vec{D}$ is normal to the flat surfaces of the pillbox, we obtain

$$D = D_0$$

Thus we have

$$E = \frac{E_0}{K}$$

Now, the polarization vector is obtained from

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}\,. \qquad (4)$$

Since $\vec{\boldsymbol{D}} = \epsilon \vec{\boldsymbol{E}}$, therefore

$$\vec{P} = (\epsilon - \epsilon_0)\,\vec{E}. \qquad (5)$$

Then the bound surface charge density is

$$\sigma_b = \vec{P}.\hat{n} = (\epsilon - \epsilon_0)E = \frac{(K-1)}{K}E_0$$

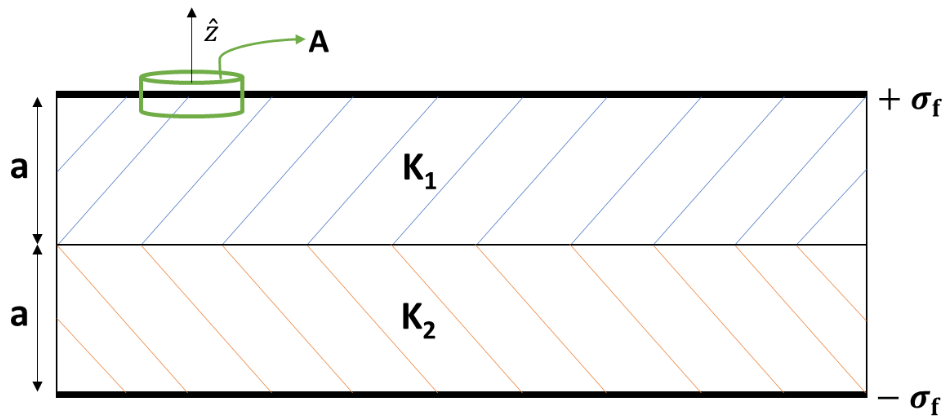
The surface charge density on the lower surface would be $-\sigma_b$.

Since the polarization vector has a constant magnitude, therefore bound volume charge density $\rho_b = -\boldsymbol{\nabla}.\vec{\boldsymbol{P}} = \boldsymbol{0}$ and bound surface charge density on the upper surface is $\sigma_b = \vec{P}.\hat{n} = (K-1)\,\epsilon_0\frac{E_0}{K}$.

The total bound charge will be zero.

3. (a) Gauss theorem in terms of electric displacement vector $\vec{D}$ is

$$\oint \vec{D}.\overrightarrow{da} = q_{free}$$



where q_{free} is enclosed free charge. To obtain electric displacement vector within the dielectric slabs 1 and 2, we again consider the cylindrical pillbox Gaussian surface with surface area A. The free charge enclosed within the Gaussian surface is A*σ_f. Electric field and displacement vector would be pointing down (i.e., towards positive to negatively charged plate). Therefore

$$\oint \vec{D}.\overrightarrow{da} = -D * A = \mathrm{A} * \sigma_f$$

or

$$\vec{D} = \sigma_f\,(-\hat{z}).$$

The displacement vector depends only on free charge on the metallic plates and is same for both dielectrics.

(b) The electric polarization is obtained using

$$\vec{D} = \epsilon_0\vec{E} + \vec{P}$$

or

$$\vec{P} = \epsilon\vec{E} - \epsilon_0\vec{E} = (\epsilon - \epsilon_0)\vec{E}.$$

The electric field in each of the dielectric can be obtained using relation

$$\vec{D} = \epsilon\vec{E}.$$

Using the displacement vector from the solution of part (a) as

$$\vec{D} = \sigma_f\,(-\hat{z}) = \epsilon\vec{E},$$

the electric field inside the dielectric 1 and 2 is

$$\overrightarrow{E_1} = \frac{\sigma_f}{\epsilon_1}(-\hat{z}) \;\; and\; \overrightarrow{E_2} = \frac{\sigma_f}{\epsilon_2}(-\hat{z})\,.$$

Using $K_1 = \epsilon_1/\epsilon_0$ and $K_2 = \epsilon_2/\epsilon_0$, we obtain

$$\vec{P}_1 = \frac{(K_1-1)}{K_1}\,\sigma_f(-\hat{z}) \quad \text{and}\; \vec{P}_2 = \frac{(K_2-1)}{K_2}\,\sigma_f(-\hat{z}).$$

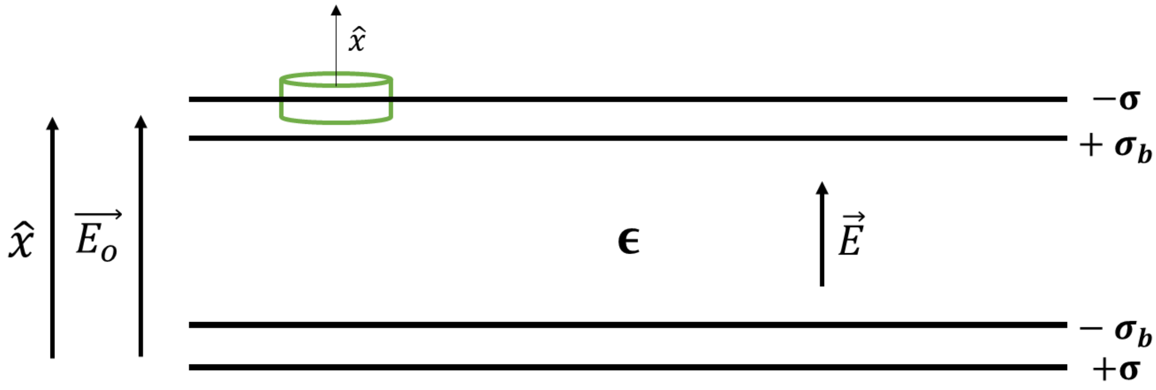
(c) Since the polarization vector has a constant magnitude, therefore volume bound charge density

$$\rho_b = -\nabla.\vec{P} = 0.$$

The surface bound charge density can be obtained using $\sigma_b = \vec{P}.\hat{n}$, where $\hat{n}$ is the outward vector normal to the surface of dielectric.

The vector $\hat{n}$ for the top and bottom layers of the both dielectrics are in $\hat{z}$ and $-\hat{z}$ direction respectively. The polarization vector is always pointing in $-\hat{z}$ direction. Therefore, the surface charge density at the top and bottom layer of the upper dielectric (with dielectric constant K_1) is $\sigma_b = -P_1$ and $\sigma_b = +P_1$, whereas at the top and bottom layer of the lower dielectric (with dielectric constant K_2) is is $\sigma_b = -P_2$ and $\sigma_b = +P_2$.

4. The electric displacement vector $\vec{D}$ can be calculated using $\oint \vec{D}.\overrightarrow{da} = q_{free}$, which means that displacement vector $\vec{D} = \sigma\,(\hat{x})$



Electric filed in the slab is $\vec{\mathrm{E}} = \frac{\vec{\mathrm{D}}}{\epsilon} = \frac{\sigma}{\epsilon}\,(\hat{\mathrm{x}})$

To find bound surface charge and volume charge densities, we need to calculate polarization vector $\vec{\mathrm{P}}$ as follows

$$\vec{\mathrm{D}} = \epsilon_0\vec{\mathrm{E}} + \vec{\mathrm{P}} \implies \epsilon\vec{\mathrm{E}} = \epsilon_0\vec{\mathrm{E}} + \vec{\mathrm{P}} \Longrightarrow \vec{\mathrm{P}} = \frac{(\epsilon - \epsilon_0)\sigma}{\epsilon}(\hat{\mathrm{x}})$$

a) Volume bound charge density: $\rho_b = -\nabla.\vec{\mathrm{P}}$

$$= -\frac{\partial}{\partial \mathrm{x}}(\vec{\mathrm{P}}) = -\frac{\partial}{\partial \mathrm{x}}\left(1 - \frac{\epsilon_0}{\epsilon}\right)\sigma = \frac{\partial}{\partial \mathrm{x}}\left(\frac{\epsilon_0}{\epsilon}\right)\sigma$$

$$= \sigma\epsilon_0\frac{\partial}{\partial \mathrm{x}}\left(\frac{1}{\epsilon}\right) = -\frac{\sigma\epsilon_0}{\epsilon^2}\frac{\partial}{\partial \mathrm{x}}(\epsilon)$$

$$= -\frac{\sigma\epsilon_0}{\epsilon^2}\left(\frac{\epsilon_2 - \epsilon_1}{\mathrm{d}}\right)$$

b) Bound surface charge density: $\sigma_b = \vec{\mathrm{P}}.\hat{\mathrm{n}}$

$$= \frac{(\epsilon - \epsilon_0)\sigma}{\epsilon}$$

At bottom plate, $x = 0$, therefore $\epsilon = \epsilon_1$

At top plate, $x = d$, therefore $\epsilon = \epsilon_2$

Surface charge densities at bottom and top plates will be $\frac{(\epsilon_1 - \epsilon_0)\sigma}{\epsilon_1}$ and $\frac{(\epsilon_2 - \epsilon_0)\sigma}{\epsilon_2}$, respectively.

c) $\vec{\mathrm{E}} = \frac{\vec{\mathrm{D}}}{\epsilon} = \frac{\sigma}{\epsilon}\,(\hat{\mathrm{x}}) = \frac{\sigma}{\epsilon_1 + \left(\frac{\epsilon_2 - \epsilon_1}{\mathrm{d}}\right)\mathrm{x}}(\hat{\mathrm{x}})$

d) $\mathrm{V} = -\int_0^{\mathrm{d}} \vec{\mathrm{E}}.\mathrm{dx} = -\int_0^{\mathrm{d}} \frac{\sigma}{\epsilon_1 + \left(\frac{\epsilon_2 - \epsilon_1}{\mathrm{d}}\right)\mathrm{x}}.\mathrm{dx}$

$$= -\frac{\sigma\mathrm{d}}{\epsilon_2 - \epsilon_1}\ln\left[\epsilon_1 + \left(\frac{\epsilon_2 - \epsilon_1}{\mathrm{d}}\right)\mathrm{x}\right]_0^{\mathrm{d}}$$

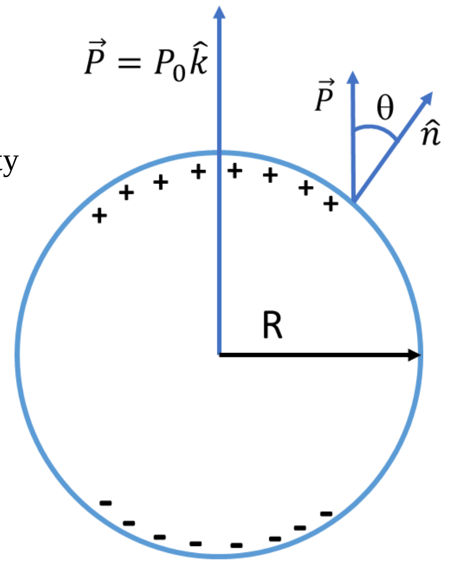
$$= -\frac{\sigma\mathrm{d}}{\epsilon_2 - \epsilon_1}[\ln(\epsilon_2) - \ln(\epsilon_1)]$$

$$= \frac{\sigma d}{\epsilon_2 - \epsilon_1}[\ln(\epsilon_1/\epsilon_2)]$$

5. Follow the answer of question 3 with following substitutions:

$$\sigma = 30\frac{mC}{m^2}; K_1 = 2 \; \& \; K_2 = 3$$

6.



$\vec{P}$ is constant in magnitude. So, bound volume charge density

$$\sigma_b = -\vec{\nabla}\cdot\vec{P} = 0$$

Bound surface charge density,

$$\sigma_b = \vec{P}\cdot\hat{n}$$

$$\sigma_b = P_0 cos\theta \,(\text{in spherical polar coordinate})$$

Total bound surface charge,

$$q_b = \int \sigma_b da \;\; (area\ element\ in\ spherical\ polar\ coordinate, da = \; r^2 sin\theta d\theta d\phi)$$

$$q_b = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} Pcos\theta r^2 \sin\theta d\theta d\phi$$

$$q_b = r^2 P 2\pi \int_{\theta=0}^{\pi} cos\theta \sin\theta d\theta d\phi = 0$$

7. For surface charge density, the polarization is given by:

$$\vec{P} = \vec{D} - \varepsilon_o\vec{E} \qquad (1)$$

To obtain electric displacement inside sphere, $\int \vec{D}\cdot d\vec{a} = q_{free}$

On the surface of radius r,

$$\vec{D}\cdot 4\pi r^2 = Q\hat{r}$$

Electric field inside the dielectric at radius r,

$$\vec{E} = \frac{Q}{4\pi\varepsilon r^2}\hat{r}$$

Substituting the values of E and D in equation 1,

$$\vec{P} = \frac{Q}{4\pi r^2}\left[1 - \frac{\varepsilon_o}{\varepsilon}\right]\hat{\mathrm{r}} = \frac{Q}{4\pi r^2}\left[\frac{K-1}{K}\right]\hat{\mathrm{r}}$$

At the surface of the sphere $r = R$,

$$\vec{P} = \frac{Q}{4\pi R^2}\left[\frac{K-1}{K}\right]\hat{\mathrm{r}}$$

Therefore, bound surface charge density,

$$\sigma_b = \vec{P} \cdot \hat{n} = \frac{Q}{4\pi R^2}\left[\frac{K-1}{K}\right]$$