Sequence
(Lecture-7)

Engineering Calculus

School of Engineering and Applied Sciences
Department of Mathematics
Bennett University

Definition

A sequence $\{a_n\}$ is called a **Cauchy sequence** if for any given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < \epsilon \text{ for all } n, m \geq N.$$

Example

Show that the sequence $\{\frac{1}{n}\}$ is a Cauchy sequence.

Solution: Let $\epsilon > 0$ be given, we choose a natural number N such that $N > 2/\epsilon$. Then if $m, n \geq N$, we have $\frac{1}{n} \leq \frac{1}{N} < \frac{\epsilon}{2}$ and similarly $\frac{1}{m} < \frac{\epsilon}{2}$. Therefore, it follows that if $m, n \geq N$, then

$$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since, $\epsilon > 0$ is arbitrary, we conclude that $\{\frac{1}{n}\}$ is a Cauchy sequence.

Cauchy sequence

Theorem

Every convergent sequence is a Cauchy sequence.

Proof: Let $\{a_n\}$ be a sequence such that $\{a_n\}$ converges to L (say). Let $\epsilon > 0$ be given. Then there exists $N \in \mathbb{N}$ such that

$$|a_n - L| < \frac{\epsilon}{2} \; \forall \; n \geq N.$$

Now, for $n, m \geq N$, we have

$$|a_n - a_m| \leq |a_n - L| + |a_m - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $\{a_n\}$ is a Cauchy sequence.

Theorem

If $\{a_n\}$ is a Cauchy sequence, then $\{a_n\}$ is bounded.

Theorem

If $\{a_n\}$ is a Cauchy sequence, then $\{a_n\}$ is convergent.

Cauchy's criterion for convergence

A sequence $\{a_n\}$ converges if and only if for every $\epsilon > 0$, there exists N such that

$$|a_n - a_m| < \epsilon, \ \ \forall \ \ m, n \geq N.$$

Theorem

Let $\{a_n\}$ be a sequence such that $|a_{n+2} - a_{n+1}| < \alpha |a_{n+1} - a_n|$ for all $n \geq N$ for some N and $0 < \alpha < 1$. Then $\{a_n\}$ is a Cauchy sequence.

Definition

We say that a sequence of real numbers $\{a_n\}$ is **contractive** if there exists a constant α, $0 < \alpha < 1$ such that $|a_{n+2} - a_{n+1}| \leq \alpha |a_{n+1} - a_n|$ for all $n \in \mathbb{N}$. The number α is called the constant of the contractive sequence.

- Every contractive sequence is a Cauchy sequence, and therefore is convergent.

Example

Let $\{a_n\}$ be defined as $a_1 = 1, a_{n+1} = 1 + \frac{1}{a_n}$. The show that $\{a_n\}$ is a Cauchy sequence and hence convergent sequence.

Solution: Note that $a_n > 1$ and $a_n a_{n+1} = a_n + 1 > 2$. Then

$$|a_{n+2} - a_{n+1}| = \left| \frac{a_{n+1} - a_n}{a_n a_{n+1}} \right| \leq \frac{1}{2} |a_{n+1} - a_n|, \ \forall \, n \geq 1.$$

Hence $\{a_n\}$ is a contractive sequence. Thus $\{a_n\}$ is a Cauchy sequence and hence convergent sequence.

Theorem

For any sequence $\{a_n\}$ with $a_n > 0$

$$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

provided the limit on the right side exists.

Result

Let $a_n > 0$ and $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = L$.
(i) If $L < 1$, then $\lim\limits_{n\to\infty} a_n = 0$.
(ii) If $L > 1$, then $a_n \to \infty$.

Remark

If $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = L = 1$, we cannot make any conclusion. For example, consider the sequence $\{n\}$, $\{\frac{1}{n}\}$ and $\{\frac{2+n}{n}\}$.

Examples

(i) Let $a_n = \frac{n}{2^n}$. Then $a_n \to 0$ as $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{2}$.
(ii) Let $a_n = ny^{n-1}$ for some $y \in (0,1)$. Since $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = y$, $a_n \to 0$.
(iii) Let $a_n = \frac{n^\alpha}{(1+p)^n}$ for some $\alpha > 0$ and $p > 0$. Then $a_n \to 0$.
(iv) $\lim\limits_{n\to\infty} n^\alpha x^n = 0$, if $|x| < 1$ and $\alpha \in \mathbb{R}$.
Hint: If $x \neq 0$, take $a_n = n^\alpha x^n$, then $\lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim\limits_{n\to\infty} (1 + \frac{1}{n})^\alpha |x| = |x|$.

Limit superior and limit inferior

Definition

Let $\{a_n\}$ be a bounded sequence. Then limit superior of the sequence $\{a_n\}$, denoted by $\limsup\limits_{n\to\infty} a_n$, is defined as

$$\limsup_{n\to\infty} a_n := \lim_{k\to\infty}\left(\sup_{n\geq k}\ a_n\right).$$

Similarly limit inferior of the sequence $\{a_n\}$, denoted by $\liminf\limits_{n\to\infty} a_n$, is defined as

$$\liminf_{n\to\infty} a_n := \lim_{k\to\infty}\left(\inf_{n\geq k} a_n\right).$$

Examples

Consider the sequence $\{a_n\} = \{0, 1, 0, 1, ...\}$. Then $\beta_n = \sup\{a_m : m \geq n\} = 1$ and $\alpha_n = \inf\{a_m : m \geq n\} = 0$. Therefore, $\liminf a_n = 0$, $\limsup a_n = 1$.

Theorem

(i) If $\{a_n\}$ is a bounded sequence, then $\limsup_{n\to\infty} a_n \geq \liminf_{n\to\infty} a_n$.

(ii) If $\{a_n\}$ and $\{b_n\}$ are bounded sequences of real numbers and if $a_n \leq b_n$ for all $n \in \mathbb{N}$, then

$$\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n \quad \text{and} \quad \liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n.$$

(iii) Let $\{a_n\}$ and $\{b_n\}$ are bounded sequences of real numbers. Then

$$\limsup_{n\to\infty}(a_n+b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \quad \text{and} \quad \liminf_{n\to\infty}(a_n+b_n) \geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n.$$

Example

Consider the sequences $\{(-1)^n\}$ and $\{(-1)^{n+1}\}$. Here $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$. Also $\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} b_n = 1$. But $a_n + b_n = 0$ for all $n \in \mathbb{N}$ and hence $\limsup_{n\to\infty} (a_n + b_n) = 0$. Thus a strict inequality may hold in (iii) the above theorem.

Theorem

If $\{a_n\}$ is a bounded sequence, then there exists subsequences $\{a_{n_k}\}$ and $\{b_{n_k}\}$ such that

$$\limsup a_n = \lim a_{n_k} \quad \text{and} \quad \liminf a_n = \lim b_{n_k}.$$

Theorem

If $\{a_n\}$ is a convergent sequence, then $\liminf\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_n = \limsup\limits_{n\to\infty} a_n$.

Theorem

If $\{a_n\}$ is a bounded sequence and if $\limsup\limits_{n\to\infty} a_n = \liminf\limits_{n\to\infty} a_n = L$, $L \in \mathbb{R}$, then $\{a_n\}$ is a convergent sequence.


Thank You