SOLUTIONS (MAJOR EXAM).

① The following statements are true/false.

(a) $W = \{A \in M_{n\times n}(\mathbb{R}) : A \text{ is non-singular}\}$ is a subspace of $M_{n\times n}(\mathbb{R})$.

Ans:- False, as A = zero matrix does not belongs to W

$\because \det A = 0$.

(b) If the eigenvalues of a 3×3 matrix are $2, i$. Then trace $A = 3$, $\det A = -2$.

Sol^n:- False, Eigenvalues are $2, i, -i$

trace $A = 2 + i - i = 2$

$\det A = 2 \times (i)(-i) = 2$.

(c) Let $T : \underbrace{M_{3\times 4}(\mathbb{R})}_{V} \longrightarrow \underbrace{M_{2\times 3}(\mathbb{R})}_{W}$ be a linear transformation which is onto, Then $\dim \text{Null}(T) = 4$.

Sol^n:- False. By Rank nullity Thm, we know that

$\dim V = \dim R(T) + \dim N(T)$

$12 = \dim W + \dim N(T)$ $\quad (\because T \text{ is on-to})$

$\Rightarrow 12 = 6 + \dim N(T)$

$\Rightarrow \dim N(T) = 6$

(d) The vectors $v_1 = (2,1,0,1)$ and $v_2 = (-1,2,i,1)$ in $C^4(\mathbb{R})$ are orthogonal.

Sol^n: False. Here $v_1 = (x_1, x_2, x_3, x_4)$, $v_2 = (y_1, y_2, y_3, y_4)$

$$\langle v_1, v_2 \rangle = x_1\bar{y}_1 + x_2\bar{y}_2 + x_3\bar{y}_3 + x_4\bar{y}_4$$

$$= -2 + 2 + 0 + 1 = 1.$$

$\Rightarrow$ The given vectors are not orthogonal.

(e) If f & g are cts fun. on $[0,1]$ then

$$\int_0^1 fg \geq \left(\int f^2\right)^{1/2} \left(\int g^2\right)^{1/2}.$$

Sol^n: False.

Using Cauchy Schwarz inequality, we have

$$\int_0^1 fg \leq \left(\int f^2\right)^{1/2} \left(\int g^2\right)^{1/2}$$

(2) $W = \{p(x) \in P_3(\mathbb{R}) : p(0) = p(1) = 0\}$, where

$$\langle p, q \rangle = \int_{-1}^{1} p(x)\, q(x)\, dx .$$

<u>Soln</u>: <u>Step-1</u>:

$$W = \{p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 : p(0) = 0 = p(1)\}$$

$$= \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 : p(0) := a_0 = 0, \; p(1) := a_1 + a_2 + a_3 = 0\}$$

$$= \{-(a_2 + a_3)x + a_2 x^2 + a_3 x^3 : a_2, a_3 \in \mathbb{R}\}$$

$$= \{a_2 (x^2 - x) + a_3 (x^3 - x) : a_2, a_3 \in \mathbb{R}\}$$

$$= \operatorname{span}\{(x^2 - x), (x^3 - x)\}$$

Also, the set $\{(x^2 - x), (x^3 - x)\}$ is linearly independent set.

$\dim W = 2$,

$$\text{Basis } W = \{\underbrace{(x^2 - x)}_{v_1}, \underbrace{(x^3 - x)}_{v_2}\}$$

<u>Step 2</u>:- Find an orthogonal basis:

$$w_1 = v_1 = x^2 - x$$

$$w_2 = v_2 - \frac{\langle v_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$$\langle v_2, v_1 \rangle = \int_{-1}^{1} v_2 v_1 \, dx = \int_{-1}^{1} (x^2 - x)(x^3 - x) = \int_{-1}^{1} x^5 - x^4 - x^3 + x^2$$

$$= \int_{-1}^{1} -x^4 + x^2 = 2\int_{0}^{1} -x^4 + x^2$$

$$= 2\left(\frac{-1}{5} + \frac{1}{3}\right) = \frac{4}{15}$$

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$$\langle v_1, v_1 \rangle = \int v_1 v_1$$

$$= \int_{-1}^{1} (x^2 - x)(x^2 - x)\, dx$$

$$= \int_{-1}^{1} x^4 - x^3 - x^3 + x^2 \, dx$$

$$= 2 \int_{0}^{1} x^4 + x^2 \, dx$$

$$= 2\left(\frac{1}{5} + \frac{1}{3}\right) = \frac{16}{15}$$

$$\omega_2 = x^3 - x - \frac{\cancel{4} \times \cancel{15}}{\cancel{15} \times \cancel{16}_{4}} (x^2 - x)$$

$$= x^3 - x - \frac{x^2}{4} + \frac{x}{4}$$

$$= x^3 - \frac{x^2}{4} - \frac{3x}{4}$$

3 Solve the differential equation

$$a\left(\frac{dy}{dx}\right) + by = ke^{-dx},$$

where a, b and k are positive constants and d is a nonnegative constant.

Also show that

(a) If $d=0$, then every solution approaches to $\frac{k}{b}$ as $x \to \infty$.

(b) If $d>0$, every solution approaches to 0 as $x \to \infty$.

Solution: Given DE is

$$a\left(\frac{dy}{dx}\right) + by = ke^{-dx} \quad \text{——— (1)}$$

$$\Rightarrow \frac{dy}{dx} + \frac{b}{a}y = \frac{ke^{-dx}}{a}$$

[Comparing it with $\frac{dy}{dx} + Py = Q$, we have $P(x) = \frac{b}{a}$, $Q(x) = \frac{k}{a}e^{-dx}$]

$$I.F = e^{\int \frac{b}{a}dx} = e^{\frac{b}{a}x}$$

Thus the solution of (1) is

$$y \times I.F = \int Q(x) \times I.F \, dx + C$$

$$\Rightarrow y \cdot e^{\frac{b}{a}x} = \int \frac{k}{a} e^{-dx} \cdot e^{\frac{b}{a}x} dx + c$$

$$\Rightarrow y \cdot e^{\frac{b}{a}x} = \frac{k}{a} \int e^{\left(\frac{b}{a}-d\right)x} dx + c$$

$$\Rightarrow y \cdot e^{\frac{b}{a}x} = \frac{k}{a} \frac{e^{\left(\frac{b}{a}-d\right)x}}{\frac{b}{a}-d} + C$$

$$\Rightarrow y = \frac{k}{a} e^{-\frac{b}{a}x} \cdot \frac{e^{\left(\frac{b}{a}-d\right)x}}{\frac{b}{a}-d} + c\, e^{-\frac{b}{a}x}$$

$$\Rightarrow y = \frac{k}{a} \frac{e^{-dx}}{\frac{b}{a}-d} + c\, e^{-\frac{b}{a}x}$$

(a) If $d=0$, then

$$y = \frac{k}{a} \cdot \frac{1}{\left(\frac{b}{a}-0\right)} + c e^{-\frac{b}{a}x}$$

$$\Rightarrow y = \frac{k}{a} \cdot \frac{a}{b} + c e^{-\frac{b}{a}x}$$

$$\Rightarrow y = \frac{k}{b} + c e^{-\frac{b}{a}x}.$$

As $x \to \infty$, $y \to \frac{k}{b}$.

(b) If $d > 0$, then $y = \frac{k}{a} \frac{e^{-dx}}{\frac{b}{a}-d} + c e^{-\frac{b}{a}x}$

which tends to zero as $x \to \infty$.

$\Rightarrow$ every solution approaches to 0 as $x \to \infty$.

4 Attempt any four parts

(a) Find the value of c for which the following DE is exact.

$$(4xe^{2y}+3y)\,dx + (cx^2e^{2y}+3x)\,dy = 0.$$

Solution: Comparing the given DE with

$$M(x,y)\,dx + N(x,y)\,dy = 0, \text{ we get}$$

$M(x,y) = 4xe^{2y}+3y$ and $N(x,y) = cx^2e^{2y}+3x$.

$\Rightarrow \frac{\partial M}{\partial y} = 4x \cdot 2e^{2y}+3$ and $\frac{\partial N}{\partial x} = 2cx\,e^{2y}+3$

Condition for exactness is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\Rightarrow 8xe^{2y}+3 = 2cxe^{2y}+3 \Rightarrow 2c = 8 \Rightarrow \boxed{c=4}$ Ans

(b) Let y_1 and y_2 be any two linearly independent solutions of

$$y'' + a(x)\,y = 0, \quad x \in (a,b),$$

where $a(x)$ is continuous on (a,b). Find $W(y_1, y_2)$.

Solution:

$$W(y_1, y_2) = c\, e^{-\int \frac{a_1(x)}{a_0(x)}\,dx}$$

Here $a_0(x) = 1, \quad a_1(x) = 0, \quad a_2(x) = a(x)$

[Comparing the given equation with

$$a_0(x)\,y'' + a_1(x)\,y' + a_2(x)\,y = 0, \text{ we get}$$

$$a_0(x) = 1, \quad a_1(x) = 0, \quad a_2(x) = a(x)]$$

Thus $W(y_1, y_2) = c\,e^{-\int \frac{0}{1}\,dx} = c \Rightarrow \boxed{W(y_1, y_2) = c}$ Ans.

Thus Wronskian is constant in this case.

(c) If the two roots of a cubic auxiliary equation with real coefficients are $m_1 = 0$, $m_2 = 5+i$, then what is the corresponding homogeneous DE?

Solution:

Since the roots of cubic A.E are $0, 5+i$.

$\Rightarrow$ the third root must be $5-i$

as complex roots occur in conjugate pairs.

$\therefore$ The A.E is

$$(m-0)\,(m-(5+i))\,(m-(5-i)) = 0$$

$$\Rightarrow m\,((m-5)^2 - i^2) = 0$$

$$\Rightarrow m\,(m^2+25-10m+1) = 0$$

$$\Rightarrow m(m^2 - 10m + 26) = 0$$

$\Rightarrow m^3 - 10m^2 + 26m = 0$

Thus the corresponding homogeneous DE is

$$\boxed{\frac{d^3y}{dx^3} - 10\frac{d^2y}{dx^2} + 26\frac{dy}{dx} = 0}$$

Ans

(d) Find the inverse Laplace transform of $\frac{1}{s(s+5)}$.

Solution:

$$\frac{1}{s(s+5)} = \frac{1}{5s} - \frac{1}{5(s+5)}$$

$$\Rightarrow \frac{1}{s(s+5)} = \frac{1}{5}\left(\frac{1}{s} - \frac{1}{s+5}\right)$$

Thus $L^{-1}\left(\frac{1}{s(s+5)}\right) = \frac{1}{5}L^{-1}\left(\frac{1}{s}\right) - \frac{1}{5}L^{-1}\left(\frac{1}{s+5}\right)$

$$= \frac{1}{5}(1) - \frac{1}{5}e^{-5t}$$

$\left[\because L^{-1}\left(\frac{1}{s}\right) = 1 \text{ and } L^{-1}\left(\frac{1}{s-a}\right) = e^{at}\right]$

$$\Rightarrow \boxed{L^{-1}\left(\frac{1}{s(s+5)}\right) = \frac{1}{5}\left(1 - e^{-5t}\right)}$$

Ans

(e) Check whether the function $f(x,y) = \cos x + y^2$ satisfies Lipschitz condition or not in the region $R: |x| \le 1, |y| \le 1$.

Solution:

Here $f(x,y) = \cos x + y^2$

$$\frac{\partial f}{\partial y} = 2y$$

$$\Rightarrow \left|\frac{\partial f}{\partial y}\right| = |2y| \le 2(1) = 2 = K$$

$\Rightarrow$ partial derivative of f w.r.t. 'y' is bounded.

$\Rightarrow$ $f(x,y)$ satisfies Lipschitz condition w.r.t. 'y' in R.

Alternative:

Here $f(x,y) = \cos x + y^2$

$$|f(x,y_1) - f(x,y_2)| = |\cos x + y_1^2 - \cos x - y_2^2|$$
$$= |y_1^2 - y_2^2|$$
$$= |y_1 - y_2|\,|y_1 + y_2|$$
$$\leq (|y_1| + |y_2|)\,|y_1 - y_2|$$
$$\leq ((1) + (1))\,|y_1 - y_2|$$
$$\leq 2\,|y_1 - y_2|$$

$$\Rightarrow |f(x,y_1) - f(x,y_2)| \leq 2\,|y_1 - y_2|.$$

Here Lipschitz constant $K = 2$.

Thus $f(x,y)$ satisfies Lipschitz condition w.r.t. 'y' in R.

5(a) Find the general solution of $\frac{d^4y}{dx^4} - a^4y = 0$

Solution: Given DE is

$$\frac{d^4y}{dx^4} - a^4y = 0$$

Auxiliary equation is

$$m^4 - a^4 = 0$$
$$\Rightarrow (m^2 - a^2)(m^2 + a^2) = 0$$
$$\Rightarrow m = \pm a, \pm ai$$

Thus the general solution is

$$\boxed{y(x) = c_1e^{ax} + c_2e^{-ax} + c_3\cos ax + c_4\sin ax}.$$

(5) (b) Find a matrix whose null space consists of all multiple of $(2,3,4,1)$.

Sol^n: Let $A = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -4 \end{bmatrix}$

Then null space of $A = \left\{ X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} : AX = 0 \right\}$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} : \begin{array}{l} x_1 - 2x_4 = 0 \\ x_2 - 3x_4 = 0 \\ x_3 - 4x_4 = 0 \end{array} \right\}$$

x_4 is free vatriable. Take $x_4 = t$

Then $x_1 = 2t$, $x_2 = 3t$, $x_3 = 4t$.

Thus $\text{Null}(A) = \left\{ t \begin{pmatrix} 2 \\ 3 \\ 4 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\}$

5(b) OR Let $y_1(x)$ and $y_2(x)$ be two solutions of

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (\sec x)y = 0$$

with Wronskian $W(x)$. If $y_1(0)=1$, $y_1'(0)=0$ and $W\left(\frac{1}{2}\right)=\frac{1}{3}$, then find $y_2'(0)$?

Solution:

Given DE is

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (\sec x)y = 0.$$

Here $a_0(x) = 1-x^2$, $a_1(x) = -2x$, $a_2(x) = \sec x$.

$$W(x) = c\,e^{-\int \frac{a_1(x)}{a_0(x)}dx} = c\,e^{-\int \left(\frac{-2x}{1-x^2}\right)dx}$$

$$= c\,e^{-\log(1-x^2)}$$

$$= c\,e^{\log(1-x^2)^{-1}}$$

$$= \frac{c}{1-x^2}$$

$$\Rightarrow W(x) = \frac{c}{1-x^2}$$

Since

$$W(y_1,y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

$$\Rightarrow W(y_1,y_2)(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix}$$

$$\Rightarrow \frac{c}{1-0} = \begin{vmatrix} 1 & y_2(0) \\ 0 & y_2'(0) \end{vmatrix} \Rightarrow \boxed{y_2'(0) = c}$$

Since $W\left(\frac{1}{2}\right) = \frac{1}{3}$

$$\Rightarrow \frac{c}{1-\left(\frac{1}{2}\right)^2} = \frac{1}{3} \qquad \left[\because W(x) = \frac{c}{1-x^2}\right]$$

$$\Rightarrow \frac{c}{1-\frac{1}{4}} = \frac{1}{3}$$

$$\Rightarrow \frac{4c}{3} = \frac{1}{3}$$

$$\Rightarrow \boxed{c = \frac{1}{4}}$$

$$\Rightarrow \boxed{y_2'(0) = \frac{1}{4}}$$

Ans.

5(c) Find the first three approximations using Picards iteration method.

$$\frac{dy}{dx} = xy, \quad y(0) = 1.$$

Solution:

$$\frac{dy}{dx} = xy, \quad y(0) = 1$$

Here $y_0 = y(0) = 1$

$$y_1 = y_0 + \int_0^x f(s, y_0(s))\, ds$$

$$= 1 + \int_0^x f(s, 1)\, ds$$

$$= 1 + \int_0^x s(1)\, ds = 1 + \frac{x^2}{2}$$

$$\Rightarrow y_1 = 1 + \frac{x^2}{2}$$

$$y_2 = y_0 + \int_0^x f(s, y_1(s))\, ds$$

$$= 1 + \int_0^x s \cdot y_1(s)\, ds$$

$$= 1 + \int_0^x s \cdot \left(1 + \frac{s^2}{2}\right) ds$$

$$= 1 + \int_0^x \left(s + \frac{s^3}{2}\right) ds$$

$$= 1 + \left|\frac{s^2}{2} + \frac{s^4}{8}\right|_0^x$$

$$\Rightarrow y_2 = 1 + \frac{x^2}{2} + \frac{x^4}{8}$$

Thus, the first three iterates are

$$y_0 = 1, \quad y_1 = 1 + \frac{x^2}{2}, \quad y_2 = 1 + \frac{x^2}{2} + \frac{x^4}{8}.$$

6. If $y_1 = x^a$ is a solution of $x^2y'' - (2a-1)xy' + a^2y = 0, (x>0, a \neq 0)$, then find the second linearly independent solution using the method of reduction of order. Hence find the general solution.

Solution: Given DE is

$$x^2y'' - (2a-1)xy' + a^2y = 0, \quad (x>0, a\neq 0)$$

Here $f(x) = x^a \, (= y_1)$, then the second linearly independent solution is $g(x) = f(x) \cdot v$, where

$$v = \int \frac{e^{-\int \frac{a_1(x)}{a_0(x)} dx}}{[f(x)]^2} dx$$

$$\Rightarrow v = \int \frac{e^{-\int \frac{-(2a-1)x}{x^2} dx}}{(x^a)^2} dx$$

$$= \int \frac{e^{(2a-1)\int \frac{1}{x} dx}}{x^{2a}} dx$$

$$= \int \frac{e^{(2a-1)\log x}}{x^{2a}} dx$$

$$= \int \frac{x^{2a-1}}{x^{2a}} dx$$

$$= \int \frac{1}{x} dx$$

$$= \log x$$

$\Rightarrow v = \log x$

Thus $g(x) = x^a . \log x$.

The general solution is

$$y(x) = c_1 f(x) + c_2 g(x)$$

$$\boxed{y(x) = c_1 x^a + c_2 x^a \log x}$$

Ans

Problem-7 Solve the DE $y'' - 4y = \sin x + e^{-2x}$.

Solution: Given DE is

$$y'' - 4y = \sin x + e^{-2x} \quad \text{——— (1)}$$

The auxiliary equation is

$$m^2 - 4 = 0$$

$$\Rightarrow \quad m = \pm 2.$$

$$\therefore \quad y_c(x) = c_1 e^{2x} + c_2 e^{-2x}.$$

To calculate $y_p(x)$, we use the method of undetermined coefficients.

for this, we find the particular integral for the following DE's.

$$y'' - 4y = \sin x \quad \text{——— (2)}$$

and $$y'' - 4y = e^{-2x} \quad \text{——— (3)}$$

Consider $y'' - 4y = \sin x$.

Here $f(x) = \sin x$ (which is of the form $e^{\alpha x}(k_1 \cos\beta x + k_2 \sin\beta x)$)

$\Rightarrow \quad \alpha = 0, \ \beta = 1.$

$\Rightarrow \quad \alpha + i\beta = i$, which is not the root of A.E.

Thus $$y_{p_1} = A\cos x + B\sin x,$$

where A and B are undetermined coefficients.

$$y_{p_1}' = -A\sin x + B\cos x,$$

$$y_{p_1}'' = -A\cos x - B\sin x$$

Substituting the values of y_{p_1}'', y_{p_1} in (2), we get

$$y_{p_1}'' - 4y_{p_1} = \sin x$$

$$\Rightarrow -A\cos x - B\sin x - 4(A\cos x + B\sin x) = \sin x.$$

$$\Rightarrow -5A\cos x - 5B\sin x = \sin x$$

Comparing the coefficients of $\sin x$, $\cos x$, we get

$$-5B = 1 \quad \text{and} \quad -5A = 0$$

$$\Rightarrow B = \frac{-1}{5} \quad \text{and} \quad A = 0.$$

Thus $\boxed{y_{p_1} = -\frac{1}{5}\sin x}$

Now, we calculate particular integral y_{p_2} for (3).

$$y'' - 4y = e^{-2x}.$$

Here $f(x) = e^{-2x}$,

which is of the for $ke^{\alpha x}$.

$\Rightarrow$ $\alpha = -2$, which is the root of A.E of multiplicity 1.

Thus we assume $y_{p_2} = Cxe^{-2x}$,

where C is an undetermined coefficient.

Sol $\Rightarrow$ $y_{p_2}' = C\left[x(-2e^{-2x}) + e^{-2x}\right]$

$$y_{p_2}' = -2Cxe^{-2x} + Ce^{-2x}$$

and $y_{p_2}'' = -2Cx(-2e^{-2x}) - 2Ce^{-2x} - 2Ce^{-2x}$

$$\Rightarrow y_{p_2}'' = 4Cxe^{-2x} - 4Ce^{-2x}$$

Substituting the values of y_{p_2} and y_{p_2}'' in ③, we get

$$y_{p_2}'' - 4y_{p_2} = e^{-2x}$$

$$\Rightarrow \quad 4cxe^{-2x} - 4ce^{-2x} - 4cxe^{-2x} = e^{-2x}$$

$$\Rightarrow \quad -4ce^{-2x} = e^{-2x}$$

$$\Rightarrow \quad -4c = 1$$

$$\Rightarrow \quad \boxed{c = -\frac{1}{4}}$$

Thus $\boxed{y_{p_2} = -\frac{1}{4}xe^{-2x}}$

Since y_{p_1} is the particular integral of $y''-4y = \sin x$ and y_{p_2} is the particular integral of $y''-4y = e^{-2x}$.

$\Rightarrow$ $y_{p_1}+y_{p_2}$ is the particular integral of $y''-4y = \sin x+e^{-2x}$.

$\Rightarrow$ $-\frac{1}{5}\sin x - \frac{1}{4}xe^{-2x}$ is the particular integral of $y''-4y = \sin x+e^{-2x}$.

$$\therefore \boxed{y_p(x) = -\frac{1}{5}\sin x - \frac{1}{4}xe^{-2x}}$$

The general solution of ① is

$$y(x) = y_c(x) + y_p(x)$$

$$\Rightarrow \boxed{y(x) = c_1e^{2x} + c_2e^{-2x} - \frac{1}{5}\sin x - \frac{1}{4}xe^{-2x}} \quad \underline{\text{Ans}}$$

Alternative : Given DE is

$$y'' - 4y = \sin x + e^{-2x}.$$

Here $a_0(x) = 1,\quad a_1(x) = 0,\quad a_2(x) = -4,$

$$f(x) = \sin x + e^{-2x}.$$

The auxiliary equation is

$$m^2 - 4 = 0$$

$$\Rightarrow \quad m = \pm 2.$$

$$\therefore \quad y_c(x) = c_1 e^{2x} + c_2 e^{-2x}.$$

$$\Rightarrow \quad y_1(x) = e^{2x} \quad \text{and} \quad y_2(x) = e^{-2x}.$$

To calculate $y_p(x)$, we use the method of variation of parameters.

$$y_p(x) = A(x)e^{2x} + B(x)\, e^{-2x},$$

where $A(x) = -\int \dfrac{f(x)\, y_2}{a_0(x)\, W}\, dx.$

and $B(x) = \int \dfrac{f(x)\, y_1}{a_0(x)\, W}\, dx.$

Here $W = W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$= \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix}$$

$$= -2 - 2 = -4$$

$$\Rightarrow W(y_1, y_2) = -4.$$

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Thus $A(x) = -\int \frac{f(x)\, y_2}{a_0(x) \cdot W} dx$

$$= -\int \frac{(\sin x + e^{-2x}) \cdot e^{-2x}}{(1)\,(-4)} dx$$

$$= \frac{1}{4} \int \left(e^{-2x} \sin x + e^{-4x}\right) dx$$

$$= \frac{1}{4} \int e^{-2x} \sin x \, dx + \frac{1}{4} \int e^{-4x} \, dx$$

$$= \frac{1}{4} \left(\frac{e^{-2x}}{(-2)^2 + (1)^2} \left[-2 \sin x - \cos x \right] \right) + \frac{1}{4} \frac{e^{-4x}}{-4}$$

$$= \frac{1}{4} \frac{e^{-2x}}{5} \left(-2 \sin x - \cos x \right) - \frac{1}{16} e^{-4x}$$

$$= \frac{e^{-2x}}{20} \left(-2 \sin x - \cos x \right) - \frac{1}{16} e^{-4x}$$

$$\boxed{A(x) = -\frac{e^{-2x}}{20} \left(2 \sin x + \cos x \right) - \frac{1}{16} e^{-4x}}$$

and $B(x) = \int \frac{f(x)\, y_1}{a_0(x)\, W} dx$

$$= \int \frac{(\sin x + e^{-2x})\, e^{2x}}{(1)\,(-4)} dx$$

$$= -\frac{1}{4} \left[\int e^{2x} \sin x \, dx + \int (1) \, dx \right]$$

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$$= \frac{-1}{4}\left[\frac{e^{2x}}{(2)^2+(1)^2}\left(2\sin x - \cos x\right) + x\right]$$

$$= \frac{-1}{4}\left[\frac{e^{2x}}{5}\left(2\sin x - \cos x\right) + x\right]$$

$$\Rightarrow B(x) = \frac{-e^{2x}}{20}\left(2\sin x - \cos x\right) - \frac{x}{4}.$$

Thus $$y_p(x) = \left(\frac{-e^{-2x}}{20}\left(2\sin x + \cos x\right) - \frac{1}{16}e^{-4x}\right)e^{2x}$$

$$+ \left(\frac{-e^{2x}}{20}\left(2\sin x - \cos x\right) - \frac{x}{4}\right)e^{-2x}$$

$$= -\frac{1}{20}\left(2\sin x + \cos x\right) - \frac{1}{16}e^{-2x} - \frac{1}{20}\left(2\sin x - \cos x\right) - \frac{x}{4}e^{-2x}$$

$$= -\frac{2(\sin x)}{10} - \frac{1}{16}e^{-2x} - \frac{x}{4}e^{-2x}$$

$$= -\frac{2\sin x}{10} - \left(\frac{1}{16} + \frac{x}{4}\right)e^{-2x}$$

$$= -\frac{1}{5}\sin x - \frac{1}{16}e^{-2x} - \frac{x}{4}e^{-2x}$$

$$\Rightarrow y_p(x) = \frac{-1}{5}\sin x - \frac{x}{4}e^{-2x} - \frac{1}{16}e^{-2x}$$

Thus the general solution is

$$y = y_c(x) + y_p(x)$$

$$\boxed{\Rightarrow y(x) = C_1 e^{2x} + C_2 e^{-2x} - \frac{1}{5}\sin x - \frac{x}{4}e^{-2x} - \frac{1}{16}e^{-2x}}$$

Ans

8. Let $x(t)$ be the solution of the initial value problem

$$\frac{d^2x}{dt^2} + x = 6\cos 2t + t^2 e^{2t}, \quad x(0) = 3, \ x'(0) = 1.$$

Let the Laplace transform of $x(t)$ be $X(s)$. Then find the value of $X(1)$.

Solution: Given DE is

$$\frac{d^2x}{dt^2} + x = 6\cos 2t + t^2 e^{2t}, \qquad x(0) = 3, \ x'(0) = 1.$$

Take the Laplace transform on both the sides, we get

$$L[x''(t)] + L[x(t)] = L[6\cos 2t] + L[t^2 e^{2t}]$$

$$\Rightarrow s^2 L[x(t)] - s\,x(0) - x'(0) + L[x(t)] = 6\cdot\frac{s}{s^2+4} + \frac{2}{(s-2)^3}$$

$$\Rightarrow (s^2+1)L[x(t)] - s(3) - 1 = \frac{6s}{s^2+4} + \frac{2}{(s-2)^3}$$

$$\Rightarrow (s^2+1)L[x(t)] = \frac{6s}{s^2+4} + \frac{2}{(s-2)^3} + 3s+1$$

$$\Rightarrow L[x(t)] = \frac{6s}{(s^2+1)(s^2+4)} + \frac{2}{(s^2+1)(s-2)^3} + \frac{3s+1}{s^2+1}$$

$$\Rightarrow X(s) = \frac{6s}{(s^2+1)(s^2+4)} + \frac{2}{(s^2+1)(s-2)^3} + \frac{3s+1}{s^2+1}$$

$$L[\cos at] = \frac{s}{s^2+a^2}$$

$$L[e^{2t}] = \frac{1}{s-2}$$

$$\Rightarrow L[t^2 e^{2t}] = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s-2}\right)$$

$$= \frac{2}{(s-2)^3}$$

$$\Rightarrow X(1) = \frac{6(1)}{(1+1)(1+4)} + \frac{2}{(1+1)(-1)^3} + \frac{4}{2}$$

$$= \frac{6}{(2)(5)} + \frac{2}{(2)(-1)} + 2$$

$$= \frac{3}{5} - 1 + 2$$

$$= \frac{3}{5} + 1$$

$$= \frac{8}{5}$$

$$\Rightarrow \boxed{X(1) = \frac{8}{5}}$$

Ans.