# Sequence
# (Lecture-4)

Engineering Calculus

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## Result

Let $a_n \neq 0$ for all $n \in \mathbb{N}$ and let $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exist.
(i) If $L < 1$, then $a_n \to 0$.
(ii) If $L > 1$, then $\{a_n\}$ is divergent.

**Examples:** (i) $\left\{\frac{\alpha^n}{n!}\right\}$, $\alpha \in \mathbb{R}$, (ii) $\left\{\frac{2^n}{n^4}\right\}$.

## Theorem

Let $\{a_n\}$, $\{b_n\}$ are two convergent sequences such that $a_n \to a$ as $n \to \infty$ and $b_n \to b$ as $n \to \infty$. Then

(i) if $a_n \geq 0$ for all $n \in \mathbb{N}$ then $a = \lim_{n\to\infty} a_n \geq 0$.

(ii) if $a_n \leq b_n$ for all $n \in \mathbb{N}$ then $a \leq b$.

(iii) if $k \in \mathbb{N}$, then $(a_n)^k \to a^k$ as $n \to \infty$, converse is not true.

(iv) if $a_n \geq 0$ then $\sqrt{a_n} \to \sqrt{a}$, converse is not true.

## Divergence

### Definition

Let $\{a_n\}$ be a sequence of real numbers. We say that $a_n$ approaches **infinity or diverges to infinity**, if for any real number $M > 0$, there is a positive integer $N$ such that $n \geq N \Rightarrow a_n \geq M$.

- If $a_n$ approaches infinity, then we write $a_n \to \infty$ as $n \to \infty$.
- A similar definition is given for the sequences diverging to $-\infty$. In this case we write $a_n \to -\infty$ as $n \to \infty$.

### Examples

(i) The sequence $\{\log(1/n)\}_1^\infty$ diverges to $-\infty$.
(ii) The sequence $\{a_n\}$ with $a_n = \frac{n^2}{n+1}$ diverges to $\infty$.

**Solution:** (i) For any $M > 0$, we must produce a $N \in \mathbb{N}$ such that

$$\log(1/n) < -M, \ \ \forall\, n \geq N.$$

But this is equivalent to saying that $n > e^M$, $\forall\, n \geq N$. Choose $N > e^M$. Then, for this choice of $N$,

$$\log(1/n) < -M, \ \ \forall\, n \geq N.$$

Thus $\{\log(1/n)\}_1^\infty$ diverges to $-\infty$.

(ii) Notice that $\dfrac{n^2}{n+1} \geq \dfrac{n}{2}$ for all $n \in \mathbb{N}$, so that for any $M > 0$, $a_n > M$ whenever $n > 2M$. Hence, taking a positive integer $N$ such that $N > 2M$, we have the relation $a_n > M$ for all $n \geq N$. Thus $\dfrac{n^2}{n+1} \to \infty$ as $n \to \infty$.

- Consider the sequence $\{(-1)^{n+1}n\}_1^\infty$. This is not a convergent sequence. Also it does not approach to $\infty$ or $-\infty$.
- The sequence $\{(-1)^n\}$ is also an example of the previous type.

### Definition

If a sequence $\{a_n\}$ does not converge to a value in $\mathbb{R}$ and also does not diverge to $\infty$ or $-\infty$, we say that $\{a_n\}$ oscillates.

### Theorem

Let $\{a_n\}$ and $\{b_n\}$ be two sequences.

(i) If $\{a_n\}$ and $\{b_n\}$ both diverge to $\infty$, then the sequences $\{a_n + b_n\}$ and $\{a_n b_n\}$ also diverge to $\infty$.

(ii) If $\{a_n\}$ diverges to $\infty$ and $\{b_n\}$ converges then $\{a_n + b_n\}$ diverges to $\infty$.

## Remark

Difference/Division of two diverging sequences may converge.

## Example

Consider the sequences $\{\sqrt{n+1}\}_{n=1}^{\infty}$, $\{\sqrt{n}\}_{n=1}^{\infty}$. Then both the sequences $\sqrt{n+1}$ and $\sqrt{n}$ diverge to $\infty$. But the sequence $\{\sqrt{n+1}-\sqrt{n}\}_{n=1}^{\infty}$ converges to 0 and the sequence $\{\frac{\sqrt{n+1}}{\sqrt{n}}\}$ converges to 1.

## Theorem

Every convergent sequence is bounded.

**Proof:** Let $\{a_n\}$ be a convergent sequence and $\lim\limits_{n\to\infty} a_n = L$. Let $\epsilon = 1$. Then there exists $N \in \mathbb{N}$ such that $|a_n - L| < 1$ for all $n \geq N$. Further,

$$|a_n| = |a_n - L + L| \leq |a_n - L| + |L| < 1 + |L|, \forall\, n \geq N.$$

Let $M = \max\{|a_1|, |a_2|, ..., |a_{N-1}|, 1 + |L|\}$. Then $|a_n| \leq M$ for all $n \in \mathbb{N}$. Hence $\{a_n\}$ is bounded.

## Remark

The condition given in previous result is necessary but not sufficient. For example, the sequence $\{(-1)^n\}$ is a bounded sequence but not convergent sequence.

**Question: Boundedness + (??) $\Longrightarrow$ Convergence.**

## Monotone sequences

A sequence $\{a_n\}$ of real numbers is called a **nondecreasing** sequence if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$ and $\{a_n\}$ is called a **nonincreasing** sequence if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence that is nondecreasing or nonincreasing is called a **monotone sequence**.

**Examples:**

1. The sequences $\{1 - 1/n\}$, $\{n^3\}$ are nondecreasing sequences.
2. The sequences $\{1/n\}$, $\{1/n^2\}$ are nonincreasing sequences.
3. The sequences $\{(-1)^n\}$, $\left\{\cos\left(\frac{n\pi}{3}\right)\right\}$, $\{(-1)^n n\}$, $\left\{\frac{(-1)^n}{n}\right\}$ and $\left\{n^{1/n}\right\}$ are not monotonic sequences.

## Result

(i) A nondecreasing sequence which is not bounded above diverges to $\infty$.

(ii) A nonincreasing sequence which is not bounded below diverges to $-\infty$.

**Examples:** (i) If $b > 1$, then the sequence $\{b^n\}_1^\infty$ diverges to $\infty$.

(ii) Let $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Then $a_{n+1} = a_n + \frac{1}{n+1} > a_n$, we see that $\{a_n\}$ is an increasing sequence. Now, we show that the sequence $\{a_n\}$ is not bounded above.

$$\begin{aligned} a_{2^n} =& 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \cdots + \left(\frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n}\right) \\ >& 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \cdots + \left(\frac{1}{2^n} + \cdots + \frac{1}{2^n}\right) \\ =& 1 + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} \\ =& 1 + \frac{n}{2}. \end{aligned}$$

Since $\{a_n\}$ is unbounded, therefore by above result $\{a_n\}$ is divergent.

## Theorem

(i) A nondecreasing sequence which is bounded above is convergent. Or suppose $\{a_n\}$ is a bounded above and increasing sequence. Then the least upper bound of the set $\{a_n : n \in \mathbb{N}\}$ is the limit of $\{a_n\}$.

(ii) A nonincreasing sequence which is bounded below is convergent. Or suppose $\{a_n\}$ is a bounded below and decreasing sequence. Then the greatest lower bound of the set $\{a_n : n \in \mathbb{N}\}$ is the limit of $\{a_n\}$.

## Example

If $0 < b < 1$, then the sequence $\{b^n\}_1^\infty$ converges to 0.

**Solution:** We write $b^{n+1} = b^n b < b^n$. Hence $\{b^n\}$ is nonincreasing. Since $b^n > 0$ for all $n \in \mathbb{N}$, the sequence $\{b^n\}$ is bounded below. Hence, by the above theorem, $\{b^n\}$ converges. Let $L = \lim_{n\to\infty} b^n$. Further, $\lim_{n\to\infty} b^{n+1} = \lim_{n\to\infty} b \cdot b^n = b \cdot \lim_{n\to\infty} b^n = b \cdot L$. Thus the sequence $\{b^{n+1}\}$ converges to $b \cdot L$. On the other hand, $\{b^{n+1}\}$ is a subsequence of $\{b^n\}$. Hence $L = b \cdot L$ which implies $L = 0$ as $b \neq 1$.

## Example

Show that the sequence $\{a_n\}$, where $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$, for all $n \in \mathbb{N}$ is convergent.

**Solution:** Now $a_{n+1} - a_n = \dfrac{1}{2n+1} + \dfrac{1}{2n+2} - \dfrac{1}{n+1} = \dfrac{1}{2(n+1)(2n+1)} > 0$ for all $n$.
Therefore the sequence $\{a_n\}$ is monotonically increasing. Again

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} < \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n} = 1.$$

i.e. $0 < a_n < 1$. Therefore, the sequence $a_n$ is bounded. Hence the sequence being bounded and monotonically increasing, is convergent.

## Subsequence

Let $\{a_n\}$ be a sequence and $\{n_1, n_2, ...\}$ be a sequence of positive integers such that $i > j$ implies $n_i > n_j$. Then the sequence $\{a_{n_i}\}_{i=1}^{\infty}$ is called a subsequence of $\{a_n\}$.

## Example

$\left\{\frac{1}{k^2}\right\}_{k=1}^{\infty}$ and $\left\{\frac{1}{2^k}\right\}_{k=1}^{\infty}$ are subsequences of $\{\frac{1}{n}\}$, where $n_k = k^2$ and $n_k = 2^k$.

## Theorem

If the sequence of real numbers $\{a_n\}_1^{\infty}$, is convergent to $L$, then any subsequence of $\{a_n\}$ is also convergent to $L$.

## Remark

Sequences $(1, 1, 1, \cdots)$ and $(0, 0, 0, \cdots)$ are both subsequences of $(1, 0, 1, 0, \cdots)$. From this we see that a given sequence may have convergent subsequence though the sequence itself is not convergent.

- If $\{a_n\}$ has two subsequences converging to two different limits, then $\{a_n\}$ cannot be convergent.
- Let $\{a_n\}$ be a sequence such that $a_{2n} \to \ell$ and $a_{2n-1} \to \ell$. Then $a_n \to \ell$.
  **Example:** The sequence $\{1, \frac{1}{2}, 1, \frac{2}{3}, 1, \frac{3}{4}, \cdots\}$ converges to 1.
- Every sequence has a monotone subsequence.

## Bolzano-Weierstrass Theorem

Every bounded sequence has a convergent subsequence.

## Theorem

Let $\{a_n\}$ be a sequence such that $|a_{n+1} - a| \leq r|a_n - a|$ for all $n \in \mathbb{N}$, for some $a \in \mathbb{R}$ and for some $r$ with $0 < r < 1$. Then $a_n \to a$.

**Proof:** For each $n \in \mathbb{N}$, we have

$$|a_{n+1} - a| \leq r|a_n - a| \leq \cdots \leq r^n|a_1 - a|.$$

Since $0 < r < 1$, $r^n \to 0$ as $n \to \infty$ so by Sandwich theorem $a_n \to a$.

### Question

If $\{a_n\}$ is such that $|a_{n+1} - a| < |a_n - a|$ for all $n \in \mathbb{N}$ for some $a \in \mathbb{R}$, then $a_n \to a$?

**Not Necessary!** Consider $\{a_n\}$ with $a_n = \dfrac{n+1}{n}$, $n \in \mathbb{N}$. Since $\dfrac{n+2}{n+1} < \dfrac{n+1}{n}$ for all $n \in \mathbb{N}$, taking $a = 0$, we have $|a_{n+1} - a| < |a_n - a|$ for all $n \in \mathbb{N}$. But $\{a_n\}$ does not converges to 0. In fact $a_n \to 1$.

### Question

If the condition on $a_n$ in the previous Theorem can be replaced by $|a_{n+2} - a_{n+1}| \leq r|a_{n+1} - a_n|$ for all $n \in \mathbb{N}$ for some $r$ with $0 < r < 1$, then $\{a_n\}$ is convergent?

- The answer is affirmative.


Thank You