Comparison test: $\{a_n\}, \{b_n\}$, $a_n > 0$, $b_n > 0$

$a_n \le b_n \quad \forall\, n \ge K, \; K \in \mathbb{N}$

① $\sum b_n$ converges $\Rightarrow \sum a_n$ converges
② $\sum a_n$ diverges $\Rightarrow \sum b_n$ diverges

Ex1:- $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$

$\frac{1}{(n+1)^2} \le \frac{1}{n^2} \quad \forall\, n \ge 1$

$\sum \frac{1}{n^2}$ converges

$\Rightarrow \sum \frac{1}{(n+1)^2}$ converges

Ex2: $\sum_{n=1}^{\infty} \frac{1}{2n^2 - n}$

$2n^2 - n \ge n^2 \quad \forall\, n \ge 1$

$$\boxed{\begin{array}{c} n \ge 1 \\ n^2 \ge n \\ n^2 + n^2 \ge n^2 + n \\ 2n^2 - n \ge n^2 \end{array}}$$

$\Rightarrow \frac{1}{2n^2 - n} \le \frac{1}{n^2}, \; \forall n \ge 1$

$\Rightarrow \sum \frac{1}{n^2}$ converges

$\Rightarrow \sum \frac{1}{2n^2 - n}$ converges

Ex3:- $\sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n}}$

$\frac{1}{n + \sqrt{n}} \ge \frac{1}{2n} \quad \forall\, n \ge 1$

$\sum \frac{1}{n}$ diverges

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n}}$ diverges

Ex3: $\sum_{n=1}^{\infty} \frac{7}{7n - 2}$

Ex4: $\sum_{n=1}^{\infty} \frac{1}{n!}$

Cauchy condensation test :-

$\{a_n\}$ decreasing, $a_n > 0 \ \forall n \in \mathbb{N}$.

$\sum_{n=1}^{\infty} a_n$ converges $\iff \sum_{n=1}^{\infty} 2^n a_{2^n}$ converges

Ex1 :- $\sum_{n=2}^{\infty} \frac{1}{n \cdot \log n}$, $a_n = \frac{1}{n \cdot \log n}$

$$\sum_{n=2}^{\infty} 2^n \cdot a_{2^n} = \sum_{n=2}^{\infty} \cancel{2^n} \cdot \frac{1}{\cancel{2^n} \cdot \log 2^n}$$

$$= \sum_{n=2}^{\infty} \frac{1}{n \cdot \log 2} = \frac{1}{\log 2} \sum_{n=2}^{\infty} \frac{1}{n}$$

$\downarrow$

diverges

$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n \cdot \log n}$ diverges.

Ex2 : $\sum_{n=1}^{\infty} \frac{1}{n^p}$, $p > 0$, $a_n = \frac{1}{n^p}$.

$$\sum_{n=1}^{\infty} 2^n \cdot a_{2^n} = \sum_{n=1}^{\infty} 2^n \cdot \frac{1}{(2^n)^p} = \sum_{n=1}^{\infty} \frac{1}{(2^n)^{p-1}}$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2^{p-1}}\right)^n$$

$$\boxed{\begin{array}{l} ① \ 0 < x < 1, \ \sum_{n=1}^{\infty} x^n \text{ converges} \\ ② \ x \geq 1, \ \sum_{n=1}^{\infty} x^n \text{ diverges} \end{array}}$$

① $p - 1 > 0$, $0 < \frac{1}{2^{p-1}} < 1$, $\sum_{n=1}^{\infty} \left(\frac{1}{2^{p-1}}\right)^n$ converges

② $p - 1 \leq 0$, $\frac{1}{2^{p-1}} \geq 1$, $\sum_{n=1}^{\infty} \left(\frac{1}{2^{p-1}}\right)^n$ diverges

$$\boxed{\begin{array}{l} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if } p > 1 \\ \qquad\qquad \text{diverges if } p \leq 1 \end{array}}$$

Limit comparison test :→

$\{a_n\}, \{b_n\}$, $a_n, b_n > 0$. then

① $\lim_{n\to\infty} \frac{a_n}{b_n} = l > 0$, $\Sigma a_n, \Sigma b_n$ conv or div together

②. $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, Σb_n conv $\Rightarrow \Sigma a_n$ conv

③ $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, Σb_n div $\Rightarrow \Sigma a_n$ div

Ex:- $\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$, $a_n = \frac{2n+1}{(n+1)^2}$

$b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{2n+1}{(n+1)^2}}{\frac{1}{n}} = 2 > 0$$

Σb_n div $\Rightarrow \Sigma a_n$ div.

② $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ $\qquad$ $\Sigma \frac{1}{n}$ div.

, $a_n = \frac{1}{2^n - 1}$

$b_n = \frac{1}{2^n}$, $\lim \frac{a_n}{b_n}$