Department of Mathematics
Bennett University
EMAT101L: July-December, 2018
Tutorial Sheet-1 Solution(Multivariable Calculus)

1) For each of the following sets in their mentioned spaces, find out whether the given set is (i) open, (ii) closed, (iii) bounded.

   (a) Space $= \mathbb{R}^2$, $S = \{(x,y) \in \mathbb{R}^2 : \ xy > 0\}$.

   (b) Space $= \mathbb{R}^2$, $S = \{(x,y) \in \mathbb{R}^2 : \ x^2 + y^2 \leq 1 \text{ and } \ y \geq 0\}$.

   (c) Space $= \mathbb{R}^2$, $S = \{(x,y) \in \mathbb{R}^2 : \ y < 1\}$.

   (d) Space $= \mathbb{R}^3$, $S = \left\{(\frac{1}{k}, k.0) \in \mathbb{R}^3 : \ k \in \mathbb{N}\right\}$.

   Hints: (i) Open, Not closed, Unbounded. (ii) Closed, Bounded, Not open. (iii) Open, Not closed, Unbounded. (iv) Not open, Closed, Unbounded.

2) Use the $\epsilon - \delta$ definition to show that the following function is continuous at $(0,0)$.

$$f(x,y) = \begin{cases} \frac{4xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

   Hints: $|f(x,y) - f(0,0)| = |\frac{4xy^2}{x^2+y^2} - 0| \leq 4|x| \leq 4\sqrt{x^2+y^2}$. Choose $\delta = \epsilon/4$. Then $||(x,y) - (0,0)|| = \sqrt{x^2+y^2} < \delta = \epsilon/4 \Rightarrow |f(x,y) - f(0,0)| \leq 4\sqrt{x^2+y^2} < 4 \times \epsilon/4 = \epsilon$.

3) Find the limit of the following function at $(0,0)$ using polar coordinates.

$$f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

   Hints: Putting $x = r\cos\theta, y = r\sin\theta$; we see that

$$|f(r,\theta)| = \left|\frac{r^3\cos^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta}\right| = |r\cos^3\theta| \leq r \to 0 \text{ as } r \to 0.$$

$$\text{Hence } \lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

4) Examine if the limit as $(x,y) \to (0,0)$ exists:

$$(i)\ f(x,y) = \begin{cases} \frac{x^3+y^3}{x^2-y^2} & \text{if } x \neq \pm y \\ 0 & \text{if } x = \pm y. \end{cases} \quad (ii)\ \frac{\sin(xy)}{x^2+y^2} \quad (iii)\ xy\frac{(x^2-y^2)}{x^2+y^2}.$$

   Hints: (i) Take

$$y = \sqrt{x^2 - mx^3}, \lim_{(x,y)\to(0,0)} f(x,y) \ \text{ depends on } m,$$

so limit does not exist at $(0,0)$.
(ii) Along the path, $y = mx, x \neq 0, x \to 0$.

$$\lim_{x\to 0} f(x, mx) = \lim_{x\to 0} \frac{\sin(mx^2)}{x^2 + m^2x^2} = \frac{m}{1+m^2},$$

hence limit does not exist at $(0,0)$.
(iii) $|f(x,y) - 0| = |xy\frac{(x^2-y^2)}{x^2+y^2}| \leq 2(x^2+y^2)$. Choose $\delta = \sqrt{\epsilon/2}$.
Then $0 < ||(x,y) - (0,0)|| = \sqrt{x^2+y^2} < \delta = \sqrt{\epsilon/2} \Rightarrow |f(x,y) - 0| \leq 2(x^2+y^2) < 2 \times \epsilon/2 = \epsilon$.

$$\text{Hence} \lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

5) Examine the continuity of $f : \mathbb{R}^2 \to \mathbb{R}$ at $(0,0)$:

$$(i)\ f(x,y) = \begin{cases} \frac{\sin^2(x-y)}{|x|+|y|} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases} \qquad (ii)\ f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Hints: (i)

$$|f(x,y) - f(0,0)| = \left|\frac{\sin^2(x-y)}{|x|+|y|} - 0\right| \leq \frac{|x-y|^2}{|x|+|y|} \leq \frac{(|x|+|y|)^2}{|x|+|y|} = |x| + |y| \leq 2\sqrt{x^2+y^2}.$$

Choose $\delta = \epsilon/2$. Then $||(x,y) - (0,0)|| = \sqrt{x^2+y^2} < \delta = \epsilon/2$
$\Rightarrow |f(x,y) - f(0,0)| \leq 2\sqrt{x^2+y^2} < 2 \times \epsilon/2 = \epsilon$. Hence, given function is continuous at $(0,0)$.
(ii) Take, $y = mx^2$, $x \neq 0$, $x \to 0$.

$$\lim_{x\to 0} f(x, mx^2) = \lim_{x\to 0} \frac{mx^4}{x^4 + m^2x^4} = \frac{m}{1+m^2},$$

hence limit does not exist at $(0,0)$.

6) For the functions $f : \mathbb{R}^2 \to \mathbb{R}$ given below which of the following limits exist:

$$\lim_{(x,y)\to(0,0)} f(x,y), \quad \lim_{x\to 0}\lim_{y\to 0} f(x,y), \quad \lim_{y\to 0}\lim_{x\to 0} f(x,y).$$

$$(i) f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases} \qquad (ii)\ f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} + y\sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

$$(iii)\ f(x,y) = \begin{cases} x + y\sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

(iv) $f(x,y) = \begin{cases} \frac{x^2y^2}{x^2y^2+(x-y)^2} & \text{when } x^2y^2+(x-y)^2 \neq 0 \\ 0 & \text{otherwise.} \end{cases}$

Hints: (i) Take, $y = mx$, $x \neq 0$, $x \to 0$.

$$\lim_{x\to 0} f(x, mx) = \lim_{x\to 0} \frac{x^2 - m^2x^2}{x^2 + m^2x^2} = \frac{1-m^2}{1+m^2}.$$

$$\text{Hence,} \quad \lim_{(x,y)\to(0,0)} f(x,y) \text{ does not exist.}$$

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = 1, \qquad \text{and} \quad \lim_{y\to 0}\lim_{x\to 0} f(x,y) = -1.$$

(ii)

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = 0, \qquad \text{and} \quad \lim_{y\to 0}\lim_{x\to 0} f(x,y) \text{ does not exist,}$$

$\because \lim_{x\to 0} \sin\frac{1}{x}$ does not exist. Take $x_n = \frac{1}{n\pi}$ and $y_n = \frac{1}{n}$, clearly $(x_n, y_n) \to (0,0)$ as $n \to \infty$. $f(x_n, y_n) = \frac{\frac{1}{n^2\pi}}{\frac{1}{n^2\pi^2}+\frac{1}{n^2}} + \frac{\sin n\pi}{n} = \frac{\pi}{1+\pi^2}$, $\forall\, n$. Now Take $x_n = 0$ and $y_n = \frac{1}{n}$, clearly $(x_n, y_n) \to (0,0)$ as $n \to \infty$, but $f(x_ny_n) = f(0, \frac{1}{n}) = 0$, $\forall\, n$.

$$\text{Hence,} \quad \lim_{(x,y)\to(0,0)} f(x,y) \text{ does not exist.}$$