Answer of Tutorial Sheet 4 :

Ques 1: $P_2(\mathbb{R})$ — consisting of all polynomials of degree less than or equal to 2 with coefficient from $\mathbb{R}$.

Show that $S = \{1-x,\ 1+x,\ x^2\}$ forms a basis for $P_2(\mathbb{R})$.

Sol: **Step 1:-** To show S is linearly independent.

$$\alpha(1-x) + \beta(1+x) + \gamma x^2 = 0 = \text{zero poly.}$$

$$(\alpha+\beta) + x(\beta-\alpha) + \gamma x^2 = 0$$

comparing the like powers of x, we obtain

$$\left.\begin{aligned} \alpha+\beta &= 0 \\ \beta-\alpha &= 0 \end{aligned}\right\} \Rightarrow \beta = 0,\ \alpha = 0$$

$$\gamma = 0$$

Thus, $\alpha = 0,\ \beta = 0,\ \gamma = 0$

$\Rightarrow$ S is linearly independent.

Step 2: To show S span $P_2(\mathbb{R})$.

Every element of $P_2(\mathbb{R})$ can be expressed as linear combinati[on] of elements of S.

i.e $\quad a_0 + a_1 x + a_2 x^2 = \alpha(1-x) + \beta(1+x) + \gamma x^2$

$\Rightarrow a_0 + a_1 x + a_2 x^2 = (\alpha+\beta) + (\beta-\alpha)x + \gamma x^2$

comparing like powers of x, we obtain

$$\gamma = a_2, \quad \begin{aligned} \alpha+\beta &= a_0 \\ -\alpha+\beta &= a_1 \end{aligned} \Rightarrow \beta = \frac{a_0+a_1}{2},\ \alpha = \frac{a_0-a_1}{2}$$

Hence, $$\boxed{a_0 + a_1 x + a_2 x^2 = \left(\frac{a_0-a_1}{2}\right)(1-x) + \left(\frac{a_0+a_1}{2}\right)(1+x) + a_2 x^2.}$$

Q. Let $S = \{(1,0,0,2,3), (0,1,1,0,0), (1,1,1,2,3)\}$
Then find the basis for $L(S)$ and extend it to the basis of $\mathbb{R}^5$.

Sol^n: Let $A = \begin{bmatrix} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 & 3 \end{bmatrix}$

Step 1: Write down the row echelon form of A

$R_3 \to R_3 - R_1$ $\begin{bmatrix} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$

$R_3 \to R_3 - R_2$ $\begin{bmatrix} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Basis for $L(S) = \{(1,0,0,2,3), (0,1,1,0,0)\}$.

let $v_3 = (0,0,1,0,0)$, $v_4 = (0,0,0,1,0)$, $v_5 = (0,0,0,0,$

Then $\{(1,0,0,2,3), (0,1,1,0,0), v_3, v_4, v_5\}$

form a basis for $\mathbb{R}^5$.

(3) Let $\mathbb{P}_4(\mathbb{R}) = \{ a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 : a_0, a_1, a_2, a_3, a_4 \in \mathbb{R}$

$$W = \{ p(x) \in \mathbb{P}_4(\mathbb{R}) : p(1) = p(-1) = 0 \}$$

$$= \{ p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 : \quad p(1) = a_0 + a_1 + a_2 + a_3 + a_4 = 0 \quad -(1)$$

$$p(-1) = a_0 - a_1 + a_2 - a_3 + a_4 = 0 \} \quad -(2)$$

$$\& \ a_i \in \mathbb{R}, \ 0 \leq i \leq 4$$

$$= \{ a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 : a_0 + a_2 + a_4 = 0 \quad (\text{by adding } (1) \,\&\, (2))$$

$$a_1 + a_3 = 0 \quad (\text{by subtracting } (1) \,\&\, (2))$$

$$\& \ a_i \in \mathbb{R} : 0 \leq i \leq 4 \}.$$

$$= \{ a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 : \quad a_0 = -(a_2 + a_4), \ a_3 = -a_1 \}$$

$$= \{ -(a_2 + a_4) + a_1x + a_2x^2 - a_1x^3 + a_4x^4 : \quad a_i \in \mathbb{R}, \ i = 1, 2, 4 \}$$

$$= \{ a_1(x - x^3) + a_2(x^2 - 1) + a_4(x^4 - 1) \quad : \quad a_1, a_2, a_4 \in \mathbb{R} \}$$

$$= \text{span} \{ \underbrace{x - x^3, \ x^2 - 1, \ x^4 - 1}_{S} \}$$

Span (S) is always a subspace of $\mathbb{P}_4(\mathbb{R})$.
Infact, it is a smallest subspace containg S.

To show: $\{ x - x^3, \ x^2 - 1, \ x^4 - 1 \}$ is linearly independent.

$$\alpha(x - x^3) + \beta(x^2 - 1) + \gamma(x^4 - 1) = 0$$

$$\Rightarrow (-\beta - \gamma) + \alpha x + \beta x^2 - \alpha x^3 + \gamma x^4 = 0$$

comparing the like power of x, we obtain

$\alpha = 0$, $\beta = 0$, $\gamma = 0$

Thus, the set S is linearly independent.

So, S forms a basis for W.

$\dim W = 3$.

4) $V = \{(x, y, z, w) \in \mathbb{R}^4 : \begin{array}{l} x+y-z+w = 0 \; -① \\ x+y+z+w = 0 \; -② \end{array}\}$

$= \{(x, y, z, w) \in \mathbb{R}^4 :$ $z = 0$ (by subtracting ① from ②)

$x+y+w = 0$ (by adding ① & ②) $\}$.

$= \{(x, y, z, w) \in \mathbb{R}^4 : z = 0, \quad w = -(x+y)\}$

$= \{(x, y, 0, -(x+y))\}$

$= \{x(1, 0, 0, -1) + y(0, 1, 0, -1)\}$

$= \text{span}\{(1, 0, 0, -1), (0, 1, 0, -1)\}$

Also, $\{(1,0,0,-1), (0,1,0,-1)\}$ is linearly independent

$\therefore$ The set $\{(1,0,0,-1), (0,1,0,-1)\}$ forms a basis for V.

$\dim V = 2$.

$W = \{(x, y, z, w) \in \mathbb{R}^4 : x-y-z+w = 0 \; -①, \; x+2y-w = 0 \; -② \}$,

$= \{(x, y, z, w) \in \mathbb{R}^4 : z = x-y+w, \quad w = x+2y\}$.

$= \{(x, y, z, w) \in \mathbb{R}^4 : z = x-y+x+2y, \quad w = x+2y\}$

$= \{(x, y, z, w) \in \mathbb{R}^4 : z = 2x+y, \quad w = x+2y\}$

$= \{(x, y, 2x+y, x+2y)\}$

$= \{x(1, 0, 2, 1) + y(0, 1, 1, 2)\}$

$= \text{span}\{(1, 0, 2, 1), (0, 1, 1, 2)\}$

Also, $\{(1,0,2,1), (0,1,1,2)\}$ is l. i. set.

Therefore, form a basis of W.

$\boxed{\dim W = 2}$

$$V \cap W = \left\{ (x,y,z,w) \in \mathbb{R}^4 : \begin{array}{l} x+y-z+w=0 \\ x+y+z+w=0 \\ x+2y-w=0 \\ x-y-z+w=0 \end{array} \right\}$$

$$= \{ (x,y,z,w) \in \mathbb{R}^4 : z=0,\ w=-(x+y),\ z=2x+y,\ w=x+2y \}$$

$$= \{ (x,y,z,w) \in \mathbb{R}^4 : z=0,\ x=0,\ y=0,\ w=0 \}$$

$$= \{(0,0,0,0)\}$$

$\therefore$ $\dim(V \cap W) = 0$.

Thus $\dim(U+W) = \dim U + \dim W - \dim(U \cap W$

$= 2+2 = 4$

i.e $\dim(U+W) = 4$.

Thus, $U+W = \mathbb{R}^4$.

(5) Let W = upper triangular matrix of 4×4.

$$= \left\{ \begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{bmatrix} : a, b, c, d, e, f, g, h, i, j \in \mathbb{R} \right\}.$$

Show that W is a subspace of $M_{4\times4}$.

Sol^n: Ans, Let $A, B \in W \Rightarrow A, B$ both are upper triangular matrix.

Then $A+B$ is an upper triangular matrix.

Also, for $\alpha \in \mathbb{R}$, αA is an upper triangular matrix.

Thus, W is a subspace of $M_{4\times4}$.

Now

$$W = a\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{v_1} + b\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{v_2} + c\underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{v_3}$$

$$+ d\underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{v_4} + e\underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{v_5} + f\underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{v_6}$$

$$+ g\underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{v_7} + h\underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{v_8} + i\underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{v_9}$$

$$+ j\underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{v_{10}}$$

$W = \{\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_{10} v_{10}\}$.

$\dim W = 10$. The set $\{v_1, v_2, \cdots, v_{10}\}$ form a basis for W.

6 Sol^n: We know that $\dim(U+W) = \dim U + \dim W - \dim(U \cap$

$$8 = 5 + 3 - \dim(U \cap W)$$

$$\Rightarrow \dim(U \cap W) = 0$$

$$\Rightarrow U \cap W = \{0\}.$$