Limit comparison test:-

$\{a_n\}, \{b_n\}$, $a_n, b_n > 0$.

① $\lim_{n \to \infty} \frac{a_n}{b_n} = c > 0$ Then $\Sigma a_n, \Sigma b_n$ conv or div together.

②. $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$ and Σb_n conv

$\Rightarrow \Sigma a_n$ conv

3. $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty$ and Σb_n div

$\Rightarrow \Sigma a_n$ div

Ex:- $\sum_{n=1}^{\infty} \frac{1}{n(n+1)^2}$, $a_n = \frac{1}{n(n+1)^2}$

$b_n = \frac{1}{n}$. $\lim \frac{a_n}{b_n} = 0$ $\Rightarrow$ no conclusion

$b_n = \frac{1}{n^2}$, $\lim \frac{a_n}{b_n} = 0$, $\Sigma \frac{1}{n^2}$ conv

$\Rightarrow \Sigma a_n$ conv

$b_n = \frac{1}{n^3}$, $\lim \frac{a_n}{b_n} = \lim \frac{1}{n(n+1)^2} \cdot n^3 = 1 > 0$

$\Sigma \frac{1}{n^3}$ conv $\Rightarrow \Sigma a_n$ conv

Ex2: $\sum_{n=1}^{\infty} \frac{n+2}{(n+1)^2}$, $a_n = \frac{n+2}{(n+1)^2}$

$b_n = \frac{1}{n}$

$\lim \frac{a_n}{b_n} = 1 > 0$, $\Sigma \frac{1}{n}$ div

$\Rightarrow \Sigma a_n$ div

EX-3 $\sum_{n=1}^{\infty} \frac{e^{-n}}{n^2}$, $a_n = \frac{e^{-n}}{n^2}$

$b_n = \frac{1}{n^2}$, $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} e^{-n}$
$= 0$

$\sum b_n = \sum \frac{1}{n^2}$ conv

$\Rightarrow \sum a_n$ conv

EX4 $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

EX5 $\sum_{n=1}^{\infty} \frac{1}{n} \log\left(1+\frac{1}{n}\right)$

EX6:- $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$

Alternating series: $\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$

Leibniz's test $\{a_n\}$, $a_n > 0$

① $a_n \geq a_{n+1}$ $\forall n \in \mathbb{N}$

② $\lim_{n\to\infty} a_n = 0$ Then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ conv

Ex1: $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n}$, $a_n = \frac{1}{n}$

① $a_n \geq a_{n+1}$ $\forall n \in \mathbb{N}$

② $\lim a_n = 0$

$\therefore \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ conv

Ex2: $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{\log n}$, $a_n = \frac{1}{\log n}$

$\Rightarrow \sum (-1)^{n+1} \cdot \frac{1}{\log n}$ conv.

Def^n:- ① If $\sum_{n=1}^{\infty} |a_n|$ converges, then we say $\sum_{n=1}^{\infty} a_n$ converges absolutely

② If $\sum_{n=1}^{\infty} a_n$ conv but $\sum_{n=1}^{\infty} |a_n|$ diverges we say, $\sum_{n=1}^{\infty} a_n$ conv conditionally.

<u>Ex 1</u>: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges absolutely.

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges}$$

<u>Ex 2</u>:- $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$ conv. absolutely.

<u>Ex 3</u>:- $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ conditionally conv.

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n} \text{ div.}$$

✓

<u>Ex 4</u>: $\sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{2n-1}$ conditionally conv.

<u>Result</u>: If $\sum_{n=1}^{\infty} a_n$ absolutely conv

$$\Rightarrow \sum_{n=1}^{\infty} a_n \text{ conv}$$

<u>Ratio test</u>:- $\sum_{n=1}^{\infty} a_n$ series

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = l$$

① $l<1 \Rightarrow \Sigma a_n$ absolutely con
$\Rightarrow \Sigma a_n$ conv.

② $l>1 \Rightarrow \Sigma a_n$ div.

3. $l=1 \Rightarrow$ no conclusion

① $a_n = \frac{1}{n}$
$\lim \frac{a_{n+1}}{a_n} = 1$
$\Sigma \frac{1}{n}$ div ✓

② $a_n = \frac{1}{n^2}$
$\lim \frac{a_{n+1}}{a_n} = 1$, $\Sigma \frac{1}{n^2}$ conv ✓

Ex 1 $\sum_{n=1}^{\infty} \frac{2n-1}{n!}$, $\lim \left|\frac{a_{n+1}}{a_n}\right| = 0<1$

$a_n = \frac{2n-1}{n!}$

$$= \lim_{n\to\infty} \left| \frac{2n+2-1}{n+1!} \cdot \frac{n!}{2n-1} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2n+1}{2n-1} \cdot \frac{1}{n+1} \right|$$

$$= \lim_{n\to\infty} \left| \frac{1+\frac{1}{2n}}{1-\frac{1}{2n}} \cdot \frac{1}{n+1} \right|$$

$$= 0<1$$

Ex 2 :- $\sum_{n=1}^{\infty} \frac{x^n}{n!}$, $x \in \mathbb{R}$,

$a_n = \frac{x^n}{n!}$, $\lim \left|\frac{a_{n+1}}{a_n}\right| = 0<1$

$\Rightarrow \sum_{n=1}^{\infty} \frac{x^n}{n!}$ conv

Ex 3 : $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges.

$$\boxed{\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e}$$

$\circledast$ $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = A$.

$\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = a$

1. if $A<1 \Rightarrow \sum a_n$ conv
2. if $a>1 \Rightarrow \sum a_n$ div
3. $a<1<A \Rightarrow$ no conclusion

Root test : $\sum a_n$, $\lim |a_n|^{1/n} = l$

1. $l<1$, $\Rightarrow \sum a_n$ conv
2. $l>1$, $\Rightarrow \sum a_n$ div
3. $l=1$, $\Rightarrow$ no conclusion

Ex-1 $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$, $a_n = \frac{1}{(\log n)^n}$

$\lim |a_n|^{1/n} = 0 < 1$

Ex 2: $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$, $x \in R$, $\Rightarrow$ conv.

$\lim_{n\to\infty} |a_n|^{1/n} = 0$

$= \lim \left|\frac{x^n}{n^n}\right|^{1/n} = 0 < 1$

Ex 3:- $\sum_{n=1}^{\infty} \frac{x^n}{n}$, $x \in R$

show that

it conv if $|x| < 1$

div if $|x| > 1$

$\lim |a_n|^{1/n} = l < 1$

$= |x| < 1$

$\lim |a_n|^{1/n}$ = not exist, $\limsup |a_n|^{1/n} = l$

$l > 1 \Rightarrow \sum a_n$ div

$l < 1 \Rightarrow \sum a_n$ conv