① Find an inner product in $\mathbb{R}^2$ s.t the following condition hold

$$\|(1,2)\| = \|(2,-1)\| = 1 \quad \& \quad \langle (1,2), (2,-1) \rangle = 0 .$$

Sol^n: We know that if $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is a symmetric matrix Then for $x, y \in \mathbb{R}^2$, $x = (x_1, x_2)$, $y = (y_1, y_2)$.

$\langle x, y \rangle = y^t A x = a x_1 y_1 + b x_2 y_1 + b x_1 y_2 + c x_2 y_2$ defines an inner product in $\mathbb{R}^2$.

$\Rightarrow \langle x, x \rangle = x^t A x = a x_1^2 + 2b x_1 x_2 + c x_2^2$

We know that $\langle x, x \rangle = \|x\|^2$, Here $x = (x_1, x_2)$

$$\|(1,2)\| = 1 \Rightarrow \|(1,2)\|^2 = 1 \Rightarrow \langle (1,2), (1,2) \rangle = 1$$

$$\langle (1,2), (1,2) \rangle = 1 \Rightarrow a + 4b + 4c = 1$$

Illy $\langle (2,-1), (2,-1) \rangle = 1 \Rightarrow 4a - 4b + c = 1$

$$\langle (1,2), (2,-1) \rangle = 0 \Rightarrow 2a + 3b - 2c = 0$$

After solving the system of equations, $a = 1/5$, $b = 0$, $c = 1/5$

Thus, $\boxed{\langle x, y \rangle = y^t A x = \frac{1}{5} x_1 y_1 + \frac{1}{5} x_2 y_2}$

Ques 1: Method 2 :-

We can define the weighted inner product on $\mathbb{R}^2$ as

$$\langle x, y \rangle = a x_1 y_1 + b x_2 y_2 , \quad \text{where } x = (x_1, x_2)$$

$$y = (y_1, y_2)$$

$$a, b > 0$$

Then $\|(1,2)\| = 1 \Rightarrow a + 4b = 1$

$\|(2,-1)\| = 1 \Rightarrow 4a + b = 1$

$\langle (1,2), (2,-1) \rangle = 0 \Rightarrow 2a - 2b = 0$

$\Rightarrow \boxed{a = b}$

$a + 4b = 1 \Rightarrow 5a = 1 \Rightarrow \boxed{a = 1/5}$

$a + 4b = 1 \Rightarrow \boxed{b = 1/5}$

Thus

$$\langle x, y \rangle = \frac{1}{5} x_1 y_1 + \frac{1}{5} x_2 y_2$$

(2) Let $x, y \in \mathbb{R}^n$. Then we have the following

(a) $\langle x, y \rangle = 0$ iff $\|x-y\|^2 = \|x\|^2 + \|y\|^2$

Solⁿ: Suppose $\boxed{\langle x, y \rangle = 0 \text{. Then } \langle y, x \rangle = 0}$

Now

$$\|x-y\|^2 = \langle x-y, x-y \rangle$$
$$= \langle x, x \rangle - \langle y, x \rangle - \langle x, y \rangle + \langle y, y \rangle$$
$$= \|x\|^2 - \langle y, x \rangle - \langle x, y \rangle + \langle \|y\|^2$$
$$= \|x\|^2 + \|y\|^2$$

Conversely, suppose $\|x-y\|^2 = \|x\|^2 + \|y\|^2$.

Then,

$$\langle x-y, x-y \rangle = \langle x, x \rangle + \langle y, y \rangle$$

$$\Rightarrow \langle x, x \rangle - \langle y, x \rangle - \langle x, y \rangle + \langle y, y \rangle = \|x\|^2 + \|y\|^2$$

$$\Rightarrow \|x\|^2 + \|y\|^2 - 2\langle x, y \rangle = \|x\|^2 + \|y\|^2 \quad (\because \langle x, y \rangle = \langle y, x \rangle)$$

$$\Rightarrow -2\langle x, y \rangle = 0$$

$$\Rightarrow \langle x, y \rangle = 0.$$

② (b) $\|x\| = \|y\| \iff \langle x+y, x-y\rangle = 0.$

Sol^n: Suppose $\|x\| = \|y\|$. Then

$$\langle x+y, x-y\rangle = \langle x,x\rangle + \langle y,x\rangle - \langle x,y\rangle - \langle y,y\rangle$$

$$= \|x\|^2 + \langle x,y\rangle - \langle x,y\rangle - \|y\|^2 \quad (\because \langle x,y\rangle = \langle y,$$

$$= \|x\|^2 - \|x\|^2 \quad (\because \|x\| = \|y\|).$$

$$= 0$$

$\Rightarrow \langle x+y, x-y\rangle = 0$

Conversely: suppose $\langle x+y, x-y\rangle = 0$

$\Rightarrow \|x\|^2 - \langle x,y\rangle + \langle y,x\rangle - \|y\|^2 = 0$

$\Rightarrow \|x\|^2 = \|y\|^2$

$\Rightarrow \|x\| = \|y\|$

(C) $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + \|y\|^2$

Sol^n: $\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + \|y\|^2 + 2\langle x,y\rangle$ — ①

$\|x-y\|^2 = \langle x-y, x-y\rangle = \|x\|^2 + \|y\|^2 - 2\langle x,y\rangle$ — ②

Adding ① & ②, we obtain

$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$

(d) $\|x+y\|^2 - \|x-y\|^2 = 4\langle x,y\rangle.$

Sol^n: $\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x,y\rangle$

$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x,y\rangle$

Subtracting ② from ①, we obtain,

$$\|x+y\|^2 - \|x-y\|^2 = 4\langle x,y\rangle.$$

(e) If $x,y \in \mathbb{C}^n(\mathbb{C})$. Then

$$4\langle x,y\rangle = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$$

Sol^n:

$$\boxed{\|x+y\|^2 = \|x\|^2 + \|y\|^2 + \langle x,y\rangle + \langle y,x\rangle} \quad \text{— ①}$$

$$\boxed{\|x-y\|^2 = \|x\|^2 + \|y\|^2 - \langle x,y\rangle - \langle y,x\rangle} \quad \text{— ②}$$

$$\begin{aligned}
\|x+iy\|^2 &= \langle x+iy, x+iy\rangle \\
&= \langle x,x\rangle + \langle iy,x\rangle + \langle x,iy\rangle + \langle iy,iy\rangle \\
&= \langle x,x\rangle + \langle iy,x\rangle + \overline{\langle iy,x\rangle} + i\bar{i}\langle y,y\rangle \\
&= \|x\|^2 + \langle iy,x\rangle + \overline{\langle iy,x\rangle} + \|y\|^2
\end{aligned}$$

i.e $\boxed{\|x+iy\|^2 = \|x\|^2 + \|y\|^2 + \langle iy,x\rangle + \langle x,iy\rangle}$ — ③

$$\begin{aligned}
\|x-iy\|^2 &= \langle x-iy, x-iy\rangle \\
&= \langle x,x\rangle - \langle iy,x\rangle - \langle x,iy\rangle + \langle iy,iy\rangle \\
&= \|x\|^2 - \langle iy,x\rangle - \langle x,iy\rangle + i\bar{i}\langle y,y\rangle \\
&= \|x\|^2 + \|y\|^2 - \langle iy,x\rangle - \langle x,iy\rangle
\end{aligned}$$

i.e $\boxed{\|x-iy\|^2 = \|x\|^2 + \|y\|^2 - \langle iy,x\rangle - \langle x,iy\rangle}$ — ④

$$\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$$

$$= 2\langle x, y\rangle + 2\langle y, x\rangle + 2i\langle iy, x\rangle + 2i\langle x, iy\rangle$$

$$= 2\langle x, y\rangle + 2\cancel{\langle y, x\rangle} - 2\cancel{\langle y, x\rangle} + 2i\bar{i}\langle x, y\rangle$$

$$= 2\langle x, y\rangle + 2\langle x, y\rangle$$

$$= 4\langle x, y\rangle.$$

③ (a) Let $\{u_1 = (1,-1,1,1),\ u_2 = (1,0,1,0),\ u_3 = (0,1,0,1)\}$ be linearly independent set in $\mathbb{R}^4(\mathbb{R})$.

Find the orthonormal set $\{v_1, v_2, v_3\}$ s.t $L(u_1, u_2, u_3) = L(v_1, v_2, v_3)$.

Solution:- Using Gram Schmidt Orthogonalization Process on (u_1, u_2, u_3) we obtain orthogonal vector (w_1, w_2, w_3),

where $w_1 = u_1$

$$w_2 = u_2 - \frac{\langle u_2, w_1 \rangle w_1}{\langle w_1, w_1 \rangle}$$

$$w_3 = u_3 - \frac{\langle u_3, w_1 \rangle w_1}{\langle w_1, w_1 \rangle} - \frac{\langle u_3, w_2 \rangle w_2}{\langle w_2, w_2 \rangle}$$

Then Orthonormal vectors are $v_1 = \frac{w_1}{\|w_1\|}$, $v_2 = \frac{w_2}{\|w_2\|}$, $v_3 = \frac{w_3}{\|w_3\|}$

$$\langle w_1, w_1 \rangle = w_1 \cdot w_1 = 1+1+1+1 = 4$$

Thus, $w_1 = (1,-1,1,1)$

$$w_2 = u_2 - \frac{\langle u_2, w_1 \rangle w_1}{\langle w_1, w_1 \rangle} = (1,0,1,0) - \frac{\langle (1,0,1,0)\cdot(1,-1,1,1) \rangle w_1}{4}$$

$$= (1,0,1,0) - \frac{2(1,-1,1,1)}{4}$$

$$= (1,0,1,0) - (\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2})$$

$$= (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2})$$

$$w_3 = u_3 - \frac{(u_3, w_1) w_1}{\langle w_1, w_1 \rangle} - \frac{(u_3, w_2) w_2}{\langle w_2, w_2 \rangle}$$

$$= (0,1,0,1) - \frac{\langle (0,1,0,1), (1,-1,1,1) \rangle w_1}{\langle w_1, w_1 \rangle} - \frac{\langle (0,1,0,1), (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}) \rangle}{\langle w_2, w_2 \rangle}$$

$$= (0,1,0,1) - \frac{(0-1+0+1)\,w_1}{\langle w_1, w_1\rangle} - \frac{(1/2 - 1/2)\,w_2}{\langle w_2, w_2\rangle}$$

$$= (0,1,0,1)$$

$$\Rightarrow w_3 = (0,1,0,1) \quad , \quad \|w_3\| = \sqrt{\langle w_3, w_3\rangle} = \sqrt{2}$$

$$v_1 = \frac{w_1}{\|w_1\|} = \frac{(1,-1,1,1)}{\sqrt{4}} = \frac{(1,-1,1,1)}{2}.$$

$$v_2 = \frac{w_2}{\|w_2\|} = \frac{(\frac{1}{2}, \frac{1}{2}, 1/2, -1/2)}{\sqrt{(\frac{1}{2})^2 + (\frac{1}{2})^2 + (\frac{1}{2})^2 + (-\frac{1}{2})^2}} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -1/2\right) = \frac{1}{2}(1,1,1,-1)$$

$$v_3 = \frac{w_3}{\|w_3\|} = \frac{(0,1,0,1)}{\sqrt{2}} = \left(0, 1/\sqrt{2}, 0, 1/\sqrt{2}\right)$$

Ex :- Find an orthonormal basis of $P_2(\mathbb{R})$, where the inner product
(b) is given by $\langle p, q\rangle = \int_{-1}^{1} p(x)\, q(x)\, dx$.

Sol^n: We know that $\{\underset{u_1}{1}, \underset{u_2}{x}, \underset{u_3}{x^2}\}$ is a standard basis of $P_2(\mathbb{R})$

We apply Gram-Schmidt Procedure to obtain orthonormal basis

$$u_1 = 1$$

$$\|1\|^2 = \int_{-1}^{1} 1^2\, dx = 2$$

$$\Rightarrow \|1\| = \sqrt{2}. \qquad \text{Then } w_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{2}}$$

$$\langle x, \tfrac{1}{\sqrt{2}}\rangle = \int_{-1}^{1} x \cdot \frac{1}{\sqrt{2}} = 0$$

$$v_2 = u_2 - \underbrace{\langle u_2, w_1\rangle}_{\shortparallel} w_1$$

$$= x - \langle x, \tfrac{1}{\sqrt{2}}\rangle \frac{1}{\sqrt{2}} = x$$

We also have $\|v_2\|^2 = \langle v_2, v_2\rangle = \int_{-1}^{1} x \cdot x = \frac{2}{3}$

$$\Rightarrow \|v_2\| = \sqrt{2/3}$$

$$w_2 = \frac{v_2}{\|v_2\|} = x\sqrt{2/3}$$

$$v_3 = u_3 - \langle u_3, w_1\rangle w_1 - \langle u_3, w_2\rangle w_2$$

$$= x^2 - \langle x^2, \tfrac{1}{\sqrt{2}}\rangle \frac{1}{\sqrt{2}} - \langle x^2, x\sqrt{2/3}\rangle x\sqrt{2/3}.$$

$$= x^2 - \left(\int_{-1}^{1} \frac{x^2}{\sqrt{2}}\right)\frac{1}{\sqrt{2}} - \left(\int_{-1}^{1} x^3 \frac{\sqrt{2}}{\sqrt{3}}\right) \cdot x\sqrt{2/3} = x^2 - \frac{1}{3}$$

$$\|v_3\|^2 = \langle v_3, v_3\rangle = \int_{-1}^{1} (x^2 - 1/3)^2 = 8/45 \Rightarrow \|v_3\| = \sqrt{8/45}$$

$$w_3 = \frac{v_3}{\|v_3\|} = \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right).$$

(4) (a) $V = \mathbb{R}^3$.

$$W = \{(x,y,z) : x+y-z=0\}$$
$$= \{(x,y,z) : x+y=z\}$$
$$= \{(x,y,x+y)\}$$
$$= \{x(1,0,1) + y(0,1,1)\}$$
$$= \text{span}\{(1,0,1), (0,1,1)\}$$

$$\boxed{\text{Basis } W = \{(1,0,1), (0,1,1)\}, \quad \dim W = 2}$$

$$W^\perp = \{v \in V : \langle v, w\rangle = 0 \ \forall w \in W\}$$
$$= \{(x,y,z) \in V : \langle (x,y,z), (1,0,1)\rangle = 0, \ \langle (x,y,z), (0,1,1)\rangle = 0\}$$

$$= \{(x,y,z) \in V : x+z=0, \ y+z=0\}$$
$$= \{(x,y,z) \in V : x=y=-z\}$$
$$= \{(x,x,-x) : x \in \mathbb{R}\}$$
$$= \{x(1,1,-1) : x \in \mathbb{R}\} = \text{span}\{(1,1,-1)\}$$

$$\boxed{\dim W^\perp = 1, \quad \text{Basis } W^\perp = \{(1,1,-1)\}}$$

(b) We know that if V is finite dimensional vector space. Then for any subspace W of V, we have

$$V = W \oplus W^{\perp}.$$

In this case, $\dim V = \dim(W) + \dim(W^{\perp})$

We know that $\dim(W) = \dfrac{n(n+1)}{2}$. [$\because$ W = set of sym. matri

$$\dim V = n^2$$

Then $\dim W^{\perp} = \dim V - \dim W$

$$= n^2 - \left(\frac{n(n+1)}{2}\right)$$

$$= \frac{n(n-1)}{2}.$$

Extra:- One can identify $W^{\perp}$ = set of all skew sym. matrix.

Basis: $\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & & \vdots \\ 0 & \cdots & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ \vdots & \cdots & & & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & \cdots & & \\ -1 & \cdots & & \ddots & 0 \\ \vdots & \cdots & & & \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & \ddots & & & & \vdots \\ 0 & & \ddots & & & \vdots \\ -1 & & & & & 0 \\ 0 & \cdots & & & & \end{bmatrix}$

One can write the basis for skew sym.