Limits
(Lecture-12)

Engineering Calculus

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Definition

Let $f(x)$ be defined on (a, b) except possibly at $c \in (a, b)$. We say that $\lim_{x \to c} f(x) = L$ if, for every real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that

$$0 < |x - c| < \delta \implies |f(x) - L| < \epsilon.$$

Example 1

$\lim_{x \to 1} \left(\frac{3x}{2} - 1 \right) = \frac{1}{2}.$

Solution: Let $\epsilon > 0$. Then we have to find $\delta > 0$ such that

$$|x - 1| < \delta \implies |f(x) - L| = \left| \left(\frac{3x}{2} - 1 \right) - \frac{1}{2} \right| = \frac{3}{2} |x - 1| < \epsilon.$$

Now, we have

$$|f(x) - L| = \frac{3}{2}|x - 1| < \epsilon \text{ whenever } |x - 1| < \delta = \frac{2}{3}\epsilon.$$

Example 2

Prove that $\lim_{x \to 2} f(x) = 4$, where $f(x) = \begin{cases} x^2 & x \neq 2 \\ 1 & x = 2. \end{cases}$

Solution: Let $\epsilon > 0$ be given. Then we have to find a $\delta > 0$ such that

$$0 < |x-2| < \delta \implies |f(x) - L| < \epsilon.$$

Now, $|x^2 - 4| = |x+2||x-2| = |x-2||x+2+2-2| < |x-2|(|x-2|+4) < \delta(\delta+4) < 5\delta$. Choose $\delta = \frac{\epsilon}{5}$ and we are done.

Theorem

If limit exists, then it is unique.

Theorem (Sequential criteria of limits)

$\lim_{x \to c} f(x) = L$ if and only if for any sequence $\{x_n\}$ with $x_n \to c$, we have $f(x_n) \to L$ as $n \to \infty$.

Example

Show that $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist.

Solution: Consider the sequences $\{x_n\} = \left\{\frac{1}{n\pi}\right\}, \{y_n\} = \left\{\frac{1}{2n\pi + \frac{\pi}{2}}\right\}$. Then it is easy to see that $x_n, y_n \to 0$ and $\sin\left(\frac{1}{x_n}\right) \to 0$, $\sin\left(\frac{1}{y_n}\right) \to 1$.

Example

Let $f(x) = \frac{1}{x}$. Then $\lim_{x \to 0} f(x)$ does not exist.

Solution: Consider the sequence $\{x_n\}$ with $x_n = \frac{1}{n}$. Then $x_n \to 0$ but $f(x_n)$ diverges to infinity. Therefore, $\lim_{x \to 0} f(x)$ does not exist.

Theorem

Suppose $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, then

(a) $\lim_{x \to c} (f(x) \pm g(x)) = L \pm M$.

(b) $f(x) \leq g(x)$ for all x in an open interval containing c. Then $L \leq M$.

(c) (i) $\lim_{x \to c} (fg)(x) = LM$ and (ii) when $M \neq 0$, $\lim_{x \to c} \frac{f}{g}(x) = \frac{L}{M}$.

(d) (Sandwich) Suppose that $h(x)$ satisfies $f(x) \leq h(x) \leq g(x)$ in an interval containing c, and $L = M$. Then $\lim_{x \to c} h(x) = L$.

(e) If $\lim_{x \to c} f(x) = L$ then $\lim_{x \to c} |f(x)| = |L|$. But converse is not true.

Example

(a) Consider $f : [0, 1] \to [-1, 1]$ as $f(x) = -1$ if $0 \leq x < 1/2$ and $f(x) = 1$ if $1/2 \leq x < 1$. Then $\lim_{x \to \frac{1}{2}} f(x)$ does not exist.

(b) $\lim_{x \to 0} x^m = 0$ $(m > 0)$.

(c) $\lim_{x \to 0} x \sin x = 0$.

Theorem

Suppose $f(x)$ is bounded in an interval containing c and $\lim_{x \to c} g(x) = 0$. Then $\lim_{x \to c} f(x)g(x) = 0$. but result does not hold if $\lim_{x \to c} g(x) \neq 0$.

Proof: Let $\epsilon > 0$, then there exist $\delta > 0$ such that

$$|x - c| < \delta \implies |g(x)| < \epsilon.$$

Also, there exist $M > 0$ such that $|f(x)| \leq M$. Now, for $|x - c| < \delta$ we have

$$|fg| \leq M.\frac{\epsilon}{M} = \epsilon.$$

Example

$\lim_{x \to 0} |x| \sin \frac{1}{x} = 0.$

Example

Show that (i) $\lim_{x \to 0} \sin x = 0$, and (ii) $\lim_{x \to 0} \cos x = 1$.

Solution: From the graph of the function $\sin x$, it is clear that

$$0 < x < \frac{\pi}{2} \Longrightarrow 0 < \sin x < x, \quad \text{and} \quad -\frac{\pi}{2} < x < 0 \Longrightarrow 0 < |\sin x| < |x|.$$

Hence $\lim_{x \to 0} |\sin x| = 0$. Thus $\lim_{x \to 0} \sin x = 0$. Also, since $\cos x = 1 - 2\sin^2(x/2)$ and $\lim_{x \to 0} \sin(x/2) = 0$. Therefore $\lim_{x \to 0} \cos x = 1$.

Result

Suppose $\lim_{x \to c} f(x) = b$ and $\lim_{y \to b} g(y) = a$. Then $\lim_{x \to c} g(f(x)) = a$.

Definition

Let $f(x)$ is defined on (c, b). The right hand limit of $f(x)$ at c is denoted by $\lim_{x\to c^+} f(x) = L$ and defined by: for given $\epsilon > 0$, there exists $\delta > 0$, such that

$$0 < x - c < \delta \implies |f(x) - L| < \epsilon.$$

Similarly, the left hand limit of $f(x)$ at b is denoted by $\lim_{x\to b^-} f(x) = L$ and defined by: for given $\epsilon > 0$, there exists $\delta > 0$, such that

$$b - \delta < x < b \implies |f(x) - L| < \epsilon.$$

Theorem

$\lim_{x\to a} f(x) = L$ exists $\iff \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L.$

Definition

$f(x)$ has limit L as x approaches $+\infty$, if for any given $\epsilon > 0$, there exists $M > 0$ such that

$$x > M \implies |f(x) - L| < \epsilon.$$

Similarly, $f(x)$ has limit L as x approaches $-\infty$, if for any given $\epsilon > 0$, there exists $M > 0$ such that

$$x < -M \implies |f(x) - L| < \epsilon.$$

Example

(a) $\lim\limits_{x\to\infty} \frac{1}{x} = 0.$

Solution: For every $\epsilon > 0$, there exist $M = \frac{1}{\epsilon}$ such that $x > M \Rightarrow \frac{1}{x} < \epsilon$.

(b) $\lim\limits_{x\to-\infty} \frac{1}{x} = 0.$

Solution: For every $\epsilon > 0$, there exist $M = \frac{1}{\epsilon}$ such that $x < -M \Rightarrow \left|\frac{1}{x}\right| < \epsilon$.

(c) $\lim\limits_{x\to\infty} \sin x$ does not exist.

Solution: Choose $x_n = n\pi$ and $y_n = \frac{\pi}{2} + 2n\pi$. Then $x_n, y_n \to \infty$ and $\sin x_n = 0$, $\sin y_n = 1$. Hence the limit does not exist.

Definition (Horizontal Asymptote)

A line $y = b$ is a horizontal asymptote of $y = f(x)$ if either $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$.

Example

$y = 1$ is a horizontal asymptote for $f(x) = 1 + \frac{1}{x+1}$.

Definition (Infinite Limits)

A function $f(x)$ approaches ∞ ($f(x) \to \infty$) as $x \to c$ if, for every real $B > 0$, there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \implies f(x) > B.$$

Similarly, A function $f(x)$ approaches $-\infty$ ($f(x) \to -\infty$) as $x \to c$ if, for every real $B > 0$, there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \implies f(x) < -B.$$

Example

$\lim_{x\to 0} \frac{1}{x^2} = \infty$.

Solution: For given $B > 0$, we can choose $\delta \leq \frac{1}{\sqrt{B}}$. such that $|x| < \delta \implies \frac{1}{x^2} > B$.

Example

$\lim_{x \to 0} \frac{1}{x^2} \sin\left(\frac{1}{x}\right)$ does not exist.

Solution: Choose a sequence $\{x_n\}$, $\{y_n\}$ such that $\frac{1}{x_n} = \frac{\pi}{2} + 2n\pi$ and $\frac{1}{y_n} = n\pi$. Then $x_n, y_n \to 0$ as $n \to \infty$ but $\lim_{n \to \infty} f(x_n) = \frac{1}{x_n^2} \to \infty$ and $\lim_{n \to \infty} f(y_n) = 0$.

Definition (Vertical Asymptote)

A line $x = a$ is a vertical asymptote of $y = f(x)$ if either $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$.

Example

$x = -2$ is a vertical asymptote and $y = 1$ is a horizontal asymptote of $f(x) = \frac{x+3}{x+2}$.


Thank You