Riemann Integral
(Lecture 21 & 22)

Engineering Calculus

School of Engineering and Applied Sciences
Department of Mathematics
Bennett University

- Let $f : [a, b] \to \mathbb{R}$ be a bounded real valued function on the closed, bounded interval $[a, b]$. Also let m, M be the infimum and supremum of $f(x)$ on $[a, b]$, respectively.
- A partition P of $[a, b]$ is an ordered set $P := \{a = x_0, x_1, x_2, ..., x_n = b\}$ such that $x_0 < x_1 < \cdots < x_n$.
- Let m_k and M_k be the infimum and supremum of $f(x)$ on the subinterval $[x_{k-1}, x_k]$, respectively.

Definition

Lower sum: The Lower sum, denoted with $L(P,f)$ of $f(x)$ with respect to the partition P is given by

$$L(P,f) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}).$$

Upper sum: The Upper sum, denoted with $U(P,f)$ of $f(x)$ with respect to the partition P is given by

$$U(P,f) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}).$$

- For a given partition P, $U(P,f) \geq L(P,f)$.

Refinement of a Partition: A partition Q is called a refinement of the partition P if $P \subseteq Q$.

Lemma

If Q is a refinement of P, then

$$L(P,f) \leq L(Q,f) \quad \text{and} \quad U(P,f) \geq U(Q,f).$$

Proof: Let $P = \{x_0, x_1, x_2, ..., x_{k-1}, x_k, ..., x_n\}$ and $Q = \{x_0, x_1, x_2, ..., x_{k-1}, z, x_k, ..., x_n\}$. Then

$$\begin{aligned} L(P,f) &= m_0(x_1 - x_0) + ... + m_k(x_k - x_{k-1}) + ... + m_{n-1}(x_n - x_{n-1}) \\ &\leq m_0(x_1 - x_0) + ... + m_k^{'}(x_k - z) + m_k^{''}(z - x_{k-1}) + ... + m_{n-1}(x_n - x_{n-1}) \\ &= L(Q,f) \end{aligned}$$

where $m_k^{'} = \inf\limits_{[z,x_k]} f(x)$ and $m_k^{''} = \inf\limits_{[x_{k-1},z]} f(x)$.

Lemma

If P_1 and P_2 be any two partitions, then $L(P_1,f) \leq U(P_2,f)$.

Proof: Let $Q = P_1 \cup P_2$. Then Q is a refinement of both P_1 and P_2. So by above Lemma, we have $L(P_1,f) \leq L(Q,f) \leq U(Q,f) \leq U(P_2,f)$.

Definition

Let $\mathcal{P}$ be the collection of all possible partitions of $[a,b]$. Then upper integral of f is defined as

$$\overline{\int_a^b} f = \inf\{U(P,f) : P \in \mathcal{P}\}$$

and lower integral of f is defined as

$$\underline{\int_a^b} f = \sup\{L(P,f) : P \in \mathcal{P}\}.$$

- For a bounded function $f : [a,b] \to \mathbb{R}$, we have $\underline{\int_a^b} f \leq \overline{\int_a^b} f$.
- **Riemann integrability:** $f : [a,b] \to \mathbb{R}$ is said to be Riemann integrable if $\underline{\int_a^b} f = \overline{\int_a^b} f$

  and the value of the limit is denoted with $\int_a^b f(x)dx$. We say $f \in \mathcal{R}[a,b]$.

Example 1

Consider $f(x) = x$ on $[0, 1]$ and the sequence of partitions $P_n = \{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, \frac{n}{n}\}$. Then

$$\begin{aligned} L(P_n, f) &= 0 \cdot \frac{1}{n} + \frac{1}{n} \cdot \frac{1}{n} + \cdots + \frac{n-1}{n} \cdot \frac{1}{n} \\ &= \frac{1}{n^2}[1 + 2 + ... + (n-1)] \\ &= \frac{n(n-1)}{2n^2} \end{aligned}$$

Thus $\lim_{n\to\infty} L(P_n, f) = \frac{1}{2}$. Hence from the definition $\underline{\int_0^1} f(x)dx \geq \frac{1}{2}$. Similarly

$$\begin{aligned} U(P_n, f) &= \frac{1}{n} \cdot \frac{1}{n} + \frac{2}{n} \cdot \frac{1}{n} + \cdots + \frac{n}{n} \cdot \frac{1}{n} \\ &= \frac{1}{n^2}[1 + 2 + \cdots + n] \\ &= \frac{n(n+1)}{2n^2} \end{aligned}$$

Hence $\lim_{n\to\infty} U(P_n, f) = \frac{1}{2}$. Again from the definition $\overline{\int_0^1} f(x)dx \leq \frac{1}{2}$.

Example 2

Consider $f(x) = x^2$ on $[0, 1]$ and the sequence of partitions $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n}\}$. Then

$$\begin{aligned} U(P_n, f) &= \frac{1}{n^2} \cdot \frac{1}{n} + \left(\frac{2}{n}\right)^2 \cdot \frac{1}{n} + \cdots + \left(\frac{n}{n}\right)^2 \cdot \frac{1}{n} \\ &= \frac{1}{n^3}[1 + 2^2 + \cdots + n^2] \\ &= \frac{n(n+1)(2n+1)}{6n^3} \end{aligned}$$

Thus $\lim_{n\to\infty} U(P_n, f) = \frac{1}{3}$. Similarly

$$\begin{aligned} L(P_n, f) &= 0 \cdot \frac{1}{n} + \left(\frac{1}{n}\right)^2 \cdot \frac{1}{n} + \cdots + \left(\frac{n-1}{n}\right)^2 \cdot \frac{1}{n} \\ &= \frac{1}{n^3}[1 + 2^2 + \cdots + (n-1)^2] \\ &= \frac{n(n-1)(2n-1)}{6n^3} \end{aligned}$$

Therefore, $\lim_{n\to\infty} L(P_n, f) = \frac{1}{3}$. Hence from the definition $\underline{\int_a^b} f \geq \frac{1}{3}$ and $\overline{\int_a^b} f \leq \frac{1}{3}$.

Example 3

On $[0, 1]$, define $f(x) = \begin{cases} 1, & x \in Q, \\ 0, & x \notin Q. \end{cases}$

Let P be a partition of $[0, 1]$. In any sub interval $[x_{k-1}, x_k]$, there exists a rational number and irrational number. Then the supremum in any subinterval is 1 and infimum is 0. Therefore, $L(P,f) = 0$ and $U(P,f) = 1$. Hence $\underline{\int_0^1} f \neq \overline{\int_0^1} f$.

Necessary and sufficient condition for integrability

A bounded function $f \in \mathcal{R}[a, b]$ if and only if for every $\epsilon > 0$, there exists a partition P_ϵ such that $U(P_\epsilon,f) - L(P_\epsilon,f) < \epsilon$.

- The functions considered in Example 1 and Example 2 are integrable. For any $\epsilon > 0$, we can find n (large) and P_n such that $\frac{1}{n} < \epsilon$. Then

$$U(P_n,f) - L(P_n,f) = \frac{1}{2n^2}(n(n+1) - n(n-1)) = \frac{1}{n} < \epsilon.$$

Similarly we can choose n in Example 2.

Remark: $f : [a, b] \to \mathbb{R}$ is integrable if and only if there exists a sequence $\{P_n\}$ of partitions of $[a, b]$ such that $\lim_{n\to\infty} U(P_n,f) - L(P_n,f) = 0$.

Remark

Let $S(P,f) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}),\ \xi_i \in [x_{i-1}, x_i]$. Then we have the following

$$m(b-a) \leq L(P,f) \leq S(P,f) \leq U(P,f) \leq M(b-a).$$

Darboux theorem

Let $f : [a,b] \to \mathbb{R}$ be a Riemann integrable function. Then for a given $\epsilon > 0$, there exists $\delta > 0$ such that for any partition P with $\|P\| := \max_{1 \leq i \leq n} |x_i - x_{i-1}| < \delta$, we have

$$\left| S(P,f) - \int_a^b f(x)dx \right| < \epsilon.$$

Result

If $f \in \mathcal{R}[a,b]$, then for any sequence of partitions $\{P_n\}$ with $\|P_n\| \to 0$, we have $L(P_n,f) \to \underline{\int_a^b} f(x)dx$ and $U(P_n,f) \to \overline{\int_a^b} f(x)dx$.

Remark

From the above theorem, we note that if there exists a sequence of partition $\{P_n\}$ such that $\|P_n\| \to 0$ and $U(P_n,f) - L(P_n,f) \not\to 0$ as $n \to \infty$, then f is not integrable.

Problem 1

Show that the function $f : [0, 1] \to \mathbb{R}$

$$f(x) = \begin{cases} 1 + x & x \in \mathbb{Q} \\ 1 - x & x \notin \mathbb{Q} \end{cases}$$

is not integrable.

Solution: Consider the sequence of partitions $P_n = \{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n}{n} = 1\}$. Then

$$\begin{aligned} U(P_n,f) &= \left(1 + \frac{1}{n}\right)\frac{1}{n} + \left(1 + \frac{2}{n}\right)\frac{1}{n} + \cdots + \left(1 + \frac{n}{n}\right)\frac{1}{n} \\ &= 1 + \frac{1}{n^2}(1 + 2 + \cdots + n) \to \frac{3}{2} \text{ as } n \to \infty. \end{aligned}$$

Now using the fact that infimum of f on $[0, \frac{1}{n}]$ is $1 - \frac{1}{n}$, though it is not achieved, we get

$$L(P_n,f) = \left(1 - \frac{1}{n}\right)\frac{1}{n} + \left(1 - \frac{2}{n}\right)\frac{1}{n} + ... + \left(1 - \frac{n}{n}\right)\frac{1}{n} \to \frac{1}{2} \text{ as } n \to \infty.$$

Hence f is not integrable.

Problem 2

Consider $f(x) = \frac{1}{x}$ on $[1, b]$. Divide the interval in geometric progression and compute $U(P_n, f)$ and $L(P_n, f)$ to show that $f \in \mathcal{R}[1, b]$.

Solution: Let $P_n = \{1, r, r^2, ..., r^n = b\}$ be a partition on $[1, b]$. Then

$$\begin{aligned} U(P_n, f) &= f(1)(r-1) + f(r)(r^2 - r) + \cdots + f(r^{n-1})(r^n - r^{n-1}) \\ &= (r-1) + \frac{1}{r} r(r-1) + \cdots + \frac{1}{r^{n-1}} r^{n-1}(r-1) = n(r-1) = n(b^{\frac{1}{n}} - 1). \end{aligned}$$

Therefore $\lim\limits_{n\to\infty} U(P_n, f) = \lim\limits_{n\to\infty} \dfrac{b^{\frac{1}{n}} - 1}{\frac{1}{n}} = \lim\limits_{n\to\infty} \dfrac{b^{\frac{1}{n}} \ln b (\frac{-1}{n^2})}{\frac{-1}{n^2}} = \ln b$. Similarly

$$\begin{aligned} L(P_n, f) &= f(r)(r-1) + f(r^2)(r^2 - r) + \cdots + f(r^n)(r^n - r^{n-1}) \\ &= \frac{1}{r}(r-1) + \cdots + \frac{1}{r^n} r^{n-1}(r-1) \\ &= \frac{n}{r}(b^{\frac{1}{n}} - 1) = n(1 - \frac{1}{b^{\frac{1}{n}}}) = \frac{b^{\frac{1}{n}} - 1}{\frac{1}{n} \cdot b^{\frac{1}{n}}} \to \ln b \text{ as } n \to \infty. \end{aligned}$$

Result

Suppose f is a continuous function on $[a, b]$. Then $f \in \mathcal{R}[a, b]$.

Example

Consider the following function $f : [0,1] \to \mathbb{R}$,

$$f(x) = \begin{cases} 1, & x \neq \frac{1}{2} \\ 0, & x = \frac{1}{2} \end{cases}$$

Clearly $U(P,f) = 1$ for any partition P. We notice that $L(P,f)$ will be less than 1. We can try to isolate the point $x = \frac{1}{2}$ in a subinterval of small length. Consider the partition

$$P_\epsilon = \{0, \frac{1}{2} - \frac{\epsilon}{2}, \frac{1}{2} + \frac{\epsilon}{2}, 1\}.$$

Then

$$L(P_\epsilon, f) = \left(\frac{1}{2} - \frac{\epsilon}{2}\right) + \left(1 - \frac{1}{2} - \frac{\epsilon}{2}\right) = 1 - \epsilon.$$

Therefore, for given $\epsilon > 0$ we have

$$U(P_\epsilon, f) - L(P_\epsilon, f) = \epsilon.$$

Hence f is integrable.

Theorem

Suppose $f : [a,b] \to \mathbb{R}$ be a bounded function which has finitely many discontinuities. Then $f \in \mathcal{R}[a,b]$.

Properties of definite integral

(a) For a constant $c \in \mathbb{R}$, $\int_a^b cf(x)dx = c\int_a^b f(x)dx$.

(b) Let $f_1,\ f_2 \in \mathcal{R}[a,b]$. Then

$$\int_a^b (f_1+f_2)(x)dx = \int_a^b f_1(x)dx + \int_a^b f_2(x)dx.$$

(c) If $f(x) \leq g(x)$ on $[a,b]$. Then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.

(d) If $f \in \mathcal{R}[a,b]$ then $|f| \in \mathcal{R}[a,b]$ and $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f|(x)dx$.

(e) Let f be bounded on $[a,b]$ and let $c \in (a,b)$. Then f is integrable on $[a,b]$ if and only if f is integrable on $[a,c]$ and $[c,b]$. In this cases

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Example 1

Consider the following function $f : [0, 1] \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & x = \frac{1}{n}, \quad \text{for some } n \in \mathbb{N},\ n \geq 2 \\ 0 & x \neq \frac{1}{n}. \end{cases}$$

Then f is Riemann integrable.

Solution: Let $\epsilon > 0$. Choose N such that $\frac{1}{N} < \frac{\epsilon}{2}$. Note that $f(x)$ has only finitely many discontinuities in $[\frac{1}{N}, 1]$ say $\xi_1, \xi_2, ..., \xi_r$. Define the partition P_ϵ as

$$P_\epsilon = \{0, \frac{1}{N}, \xi_1 - \frac{\epsilon}{4r}, \xi_1 + \frac{\epsilon}{4r}, ..., \xi_r - \frac{\epsilon}{4r}, \xi_r + \frac{\epsilon}{4r}, 1\}.$$

Since ξ_r is the last discontinuity, $f = 0$ in $[\xi_r + \frac{\epsilon}{4r}, 1]$. Now $L(P_\epsilon, f) = 0$ and

$$\begin{aligned} U(P_\epsilon, f) &= 1 \cdot \frac{1}{N} + \frac{\epsilon}{2r} + \frac{\epsilon}{2r} + \cdots + \frac{\epsilon}{2r} + 0 \cdot (1 - \xi_r - \frac{\epsilon}{4r}) \\ &= \frac{1}{N} + \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

Example 2

Consider the following function $f : [0, 1] \to \mathbb{R}$.

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ \sin \frac{1}{x} & x \notin \mathbb{Q}. \end{cases}$$

Then f is not Riemann integrable.

Solution: Consider f on the subinterval $I_1 = [\frac{2}{\pi}, 1]$. Clearly $L(P, f) = 0$ for any partition P of I_1 because $f(x) \geq 0$ in the subinterval I_1. Let M_k be the supremum of f on subintervals $[x_{k-1}, x_k]$ of I_1. Also the minimum of M_k's is $\sin 1$. Therefore,

$$U(P, f) = \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) > \left(1 - \frac{2}{\pi}\right) \sin 1.$$

Hence $U(P, f) - L(P, f)$ cannot be made less than ϵ for any $\epsilon < (1 - \frac{2}{\pi}) \sin 1$.

Theorem

Let $f(x)$ be a continuous function on $[a, b]$. Then there exists $\xi \in [a, b]$ such that

$$\int_a^b f(x)dx = f(\xi)(b-a).$$

Proof: Let $m = \min_{x \in [a,b]} f(x)$ and $M = \max_{x \in [a,b]} f(x)$. Then

$$m(b-a) \leq \int_a^b f \leq M(b-a) \implies m \leq \frac{1}{(b-a)} \int_a^b f \leq M.$$

Now since $f(x)$ is continuous, it attains all values between it's maximum and minimum values. Therefore there exists $\xi \in [a, b]$ such that $f(\xi) = \frac{1}{(b-a)} \int_a^b f$.

Fundamental theorem

Let $f(x)$ be a continuous function on $[a, b]$ and let $\phi(x) = \int_a^x f(s)ds$. Then ϕ is differentiable and $\phi^{'}(x) = f(x)$.

Proof: As

$$\frac{\phi(x + \Delta x) - \phi(x)}{\Delta x} = \frac{1}{\Delta x} \int_x^{x+\Delta x} f(s)ds,$$

by Mean value theorem, there exists $\xi \in [x, x + \Delta x]$ such that

$$\int_x^{x+\Delta x} f(s)ds = \Delta x f(\xi).$$

Therefore

$$\lim_{\Delta x \to 0} \frac{\phi(x + \Delta x) - \phi(x)}{\Delta x} = \lim_{\Delta x \to 0} f(\xi).$$

Since f is continuous,

$$\lim_{\Delta x \to 0} f(\xi) = f(\lim_{\Delta x \to 0} \xi) = f(x).$$

Thus $\phi^{'}(x) = f(x)$.

Remark

It is always not true that $\displaystyle\int_a^b f'(x)dx = f(b) - f(a)$.

Example: Let $f(x) = x^2 \sin \frac{1}{x^2}$ for $x \neq 0$ and $f(0) = 0$. Then f is differentiable on $[0, 1]$ and $f'(x) = 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2}$ for $x \in (0, 1)$ and $f'(0) = 0$. Hence f' is not bounded and so not integrable.

- A function $F(x)$ is called anti-derivative of $f(x)$, if $F'(x) = f(x)$.

Second fundamental theorem

Suppose $F(x)$ is an anti- derivative of continuous function $f(x)$. Then $\int_a^b f(x)dx = F(b) - F(a)$.

Proof: By first fundamental theorem, we have

$$\frac{d}{dx}\int_a^x f(s)ds = f(x).$$

Also $F'(x) = f(x)$. Hence $\displaystyle\int_a^x f(s)ds = F(x) + c$ for some constant $c \in \mathbb{R}$. Taking $x = a$, we get $c = -F(a)$. Now taking $x = b$ we get $\displaystyle\int_a^b f(x)dx = F(b) - F(a)$.

Theorem

Let $u(t),\ u'(t)$ be continuous on $[a,b]$ and f is a continuous function on the interval $u([a,b])$. Then

$$\int_a^b f(u(x))\ u'(x)dx = \int_{u(a)}^{u(b)} f(y)dy.$$

Proof: Note that $u([a,b])$ is a closed and bounded interval. Since f is continuous, it has primitive F i.e.,

$$F(x) = \int_a^x f(t)dt.$$

Then by chain rule

$$\frac{d}{dt}F(u(t)) = f(u(t))\ u'(t).$$

i.e., $F(u(t))$ is the primitive of $f(u(t))u'(t)$ and by Newton-Leibnitz formula, we get

$$\int_a^b f(u(t))u'(t)dt = F(u(b)) - F(u(a)).$$

On the other hand, for any two points in $u([a,b])$, we have

$$\int_A^B f(y)dy = F(B) - F(A). \quad \text{Hence} \quad B = u(b) \ \text{ and } \ A = u(a).$$

Problem

Evaluate $\int_0^1 x\sqrt{1+x^2}dx$.

Solution: Taking $u = 1 + x^2$, we get $u' = 2x$ and $u(0) = 1, u(1) = 2$. Then

$$\int_0^1 x\sqrt{1+x^2}dx = \frac{1}{2}\int_1^2 \sqrt{u}du = \frac{1}{3}\left[u^{\frac{2}{3}}\right]_{u=1}^2 = \frac{1}{3}(2^{\frac{2}{3}} - 1).$$


Thank You