ENGS 105 Midterm

Winter 2024

Rules of Engagement: All work must be performed individually without consulting classmates. Any questions should be directed to Prof Paulsen.

Materials that may be used: Course textbook (including reserve texts, but no other books or internet materials), materials from the course website, your class notes, your previous problems sets and codes, calculators, computers and software you have been using so far.

Dates: Post your solutions BY NOON on Friday, February 9. Scan any handwritten work so you can post it as a pdf.

GOOD LUCK

The Thermal Thomas Company (TTC) is an engineering firm that specializes in computational solutions to complex heat transfer problems. You are head of the software development division, which is constantly evaluating algorithms for solving the classic diffusion equation

$$\frac{\partial U}{\partial t} = D\nabla^2 U$$

A dispute has broken out between two of your best numerical analysts: Analyst A has proposed the following finite difference (FD) scheme

$$U^{l+1} - U^l = \frac{r}{2}(3\delta^2 U^l - \delta^2 U^{l-1})$$

whereas Analyst B says her finite difference method is much better

$$U^{l+1} - U^l = \frac{r}{12}(5\delta^2 U^{l+1} + 8\delta^2 U^l - \delta^2 U^{l-1})$$

where $r = D\Delta t/h^2$, $\delta^2 =$ centered difference operator, and l is the time level.

Both Analysts claim their method is superior to Crank-Nicholson and that TTC should invest in developing commercial software based on their schemes. As the leader of the division, you need to resolve the disagreement. Fortunately, you remember taking an outstanding numerical methods class from a superb professor during your graduate studies where you learned that analyzing consistency, stability and accuracy of FD molecules is critical for evaluating their behavior. You also recall that 1D space-time analysis is sufficient for making a comparison between the two methods your employees have proposed. You also remember

(a) consistency of the FD molecule with the PDE of interest and the order of the expected convergence, including the specific forms of the leading terms in the truncation errors, is important.

(b) stability is key, and can be determined using the Von Neumann method.

(c) Accuracy is also critical and can be evaluated by studying the nature of numerical amplification and/or propagation factors as a function of dimensionless wavenumber compared to the analytic behavior. Long and short wavelength components of a numerical solution including any oscillations that might occur, their period, whether they decay with time or display any type of hyperbolic (i.e. propagating) character should be considered.

(d) Plots that illustrate your points are always persuasive.

(e) Solving a simple 1D problem with the methods proposed by your Analysts that validate your theoretical findings is convincing and using a 1D domain of length $L = 10$, $D = 0.5$, $h = 0.1$, $\Delta t = 0.05$, the boundary conditions $U(0, t) = U(L, t) = 0$ and initial conditions $U(-\Delta t, x) = U(0, x) = \exp[-(x - x_o)^2/2\sigma^2]$ where $x_o = 5$ and $\sigma = 0.1$ is a good start. Exploring the numerical behavior by considering variations in the parameters, D, σ, Δt, and h (the impact of both smaller and larger values of each) might be a good idea, and in some cases plotting the solution as a function of x at early (i.e $t = \Delta t, 2\Delta t, 3\Delta t$) and late (i.e. t=$20\Delta t$, $30\Delta t$, etc) times might be informative. The goal is to show how well the theoretical analysis holds in practice and you don't want to bore your Analysts with every plot you end up creating, but only show those which illustrate the key results which summarize your computational experience with the two molecules and support your theoretical analyses.

Your final report should evaluate the pros and cons of the two methods, and make recommendations about the superiority/inferiority of one method relative to the other.