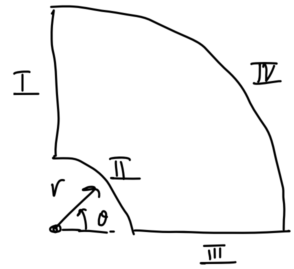
$\boxed{12c}$

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$



Boundary

$$\text{I}: \quad u(r, \tfrac{\pi}{2}) = 0$$

$$\text{II}: \quad \frac{\partial u}{\partial r}(a,\theta) = 0$$

$$\text{III}: \quad \frac{\partial u}{\partial \theta}(r,0) = 0$$

$$\text{IV}: \quad u(R,\theta) = -\frac{I_0 R}{\sigma}\cos(k\theta)$$

where $a = 0.1$, $R = 1$, $\frac{I_0}{\sigma} = 1$, $k = 3$

a) Verify $u = -\frac{I_0}{\sigma}R\left(\frac{r^k + a^{2k}r^{-k}}{R^k + a^{2k}R^{-k}}\right)\cos(k\theta)$

$$\nabla^2 u = \nabla^2\left(-\frac{I_0 R}{\sigma}\left(\frac{r^k + a^{2k}r^{-k}}{R^k + a^{2k}R^{-k}}\right)\cos(k\theta)\right)$$

$$\left[\text{Let } \beta = -\frac{I_0 R}{\sigma}\cdot\frac{1}{R^k + a^{2k}R^{-k}}\right]$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(\cancel{r}\,\beta\cos(k\theta)\left(k\frac{r^k}{\cancel{r}} - a^{2k}\cdot k\cdot\frac{r^{-k}}{\cancel{r}}\right)\right) - \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\beta(r^k + a^{2k}r^{-k})\sin(k\theta)\cdot k\right.$$

$$= \frac{1}{\cancel{r}}\beta\cos(k\theta)\,k\left(k\frac{r^k}{\cancel{r}} - a^{2k}\cdot k\cdot\frac{r^{-k}}{\cancel{r}}\right) - \frac{1}{r^2}\beta(r^k + a^{2k}r^{-k})\cdot k\cdot\cos(k\theta)\cdot k$$

$$= \underline{\underline{0}}$$

Plugging in parameters into analytical solution to simplify,

$$u = -\left(\frac{r^3 + 0.1^6 r^{-3}}{1 + 0.1^6}\right)\cos(3\theta)$$

Checking B.Cs,

I : $u(r, \frac{\pi}{2}) = 0$ ✓ as $\cos(\frac{3\pi}{2}) = 0$

II : $\frac{\partial u}{\partial r}(a, \theta) = -\beta k \cos(k\theta)\,\cancel{(a^k - a^{2k-k})}^{\to 0} = 0$ ✓

III : $\frac{\partial u}{\partial \theta}(r, 0) = -\beta\,(r^k + a^{2k} \cdot r^{-k}) \cdot k \cdot \cancel{\sin(k \cdot 0)}^{\to 0} = 0$ ✓

IV : $u(R, \theta) = -\frac{I_0}{\sigma} R \left(\cancel{\frac{R^k + a^{2k} R^{-k}}{R^k + a^{2k} R^{-k}}}^{\to 1} \right) \cos(k\theta) = -\frac{I_0}{\sigma} R \cos(k\theta)$ ✓

Thus, verified given expression for u is a solution of $\nabla^2 u = 0$ with given B.C.s

b. Solns plotted in Matlab

c. FD Molecule :

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Let $\Delta r = h$, $\Delta\theta = k$. Let i be the radial index, j be the angular index.

Finite differencing :

$$\frac{\partial^2 u}{\partial r^2} \simeq \frac{u_{i+1,j} - 2u_{ij} + u_{i-1j}}{h^2}$$

$$\frac{\partial^2 u}{\partial \theta^2} \simeq \frac{u_{ij+1} - 2u_{ij} + u_{ij-1}}{k^2}$$

$$\frac{\partial u}{\partial r} \simeq \frac{u_{i+1j} - u_{i-1j}}{2h}$$

$$u_{i+1j} - 2u_{ij} + u_{i-1j} + (u_{ij+1} - 2u_{ij} + u_{ij-1})\frac{\beta}{r^2} + \frac{h}{2r}(u_{i+1j} - u_{i-1,j}) = 0 \qquad \beta = \frac{h^2}{k^2}$$

- INTERIOR NODE

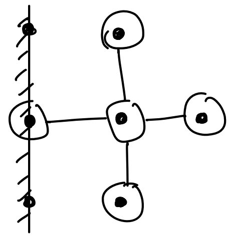
$\theta \uparrow \rightarrow r$

$$
\begin{array}{cccc}
 & & \frac{\beta}{r^2} & \\
j+1 & & | & \\
j & \left(1-\frac{h}{2r}\right) & - \left(-2-\frac{2\beta}{r^2}\right) - & \left(1+\frac{h}{2r}\right) \\
j-1 & & | & \\
 & & \frac{\beta}{r^2} & \\
 & i-1 & i & i+1
\end{array}
= 0
$$

- Boundary I [Stop one short as Type I]

$u(r, \frac{\pi}{2}) = 0$

$\theta \uparrow \rightarrow r$



$$
\begin{array}{ccc}
\left(1-\frac{h}{2r}\right) & - \left(-2-\frac{2\beta}{r^2}\right) - & \left(1+\frac{h}{2r}\right) \\
 & | & \\
 & \frac{\beta}{r^2} &
\end{array}
= 0
$$

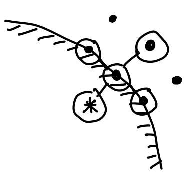
u_{ij-1} on Boundary

- Boundary II [Go to boundary as Type II]

$\frac{\partial u}{\partial r}(a, \theta) = 0$

$\theta \uparrow \rightarrow r$

Approximate $\frac{\partial u}{\partial r}$ at $r=a$ $\simeq \frac{u_{i+1j} - u_{i-1j}}{2h} = 0$



$u_{i-1j} = u_{i+1j}$

$$
\begin{array}{cc}
\frac{\beta}{r^2} & \\
| & \\
\left(-2-\frac{2\beta}{r^2}\right) & - \left(2+\frac{h}{r}\right) \\
| & \\
\frac{\beta}{r^2} &
\end{array}
= 0
$$

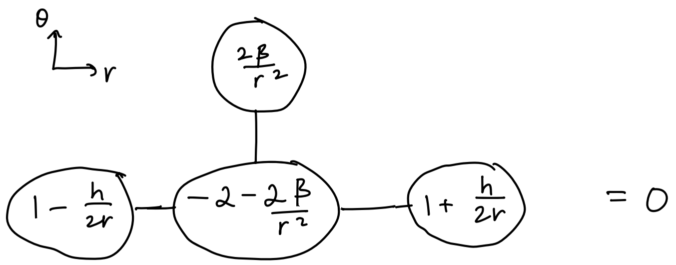
u_{ij} on Boundary

- Boundary III

$$\frac{\partial u}{\partial \theta}(r,0) = 0 \approx \frac{u_{ij+1} - u_{ij-1}}{2k}$$

[Type II] $\therefore u_{ij-1} = u_{ij+1}$

u_{ij} on Boundary

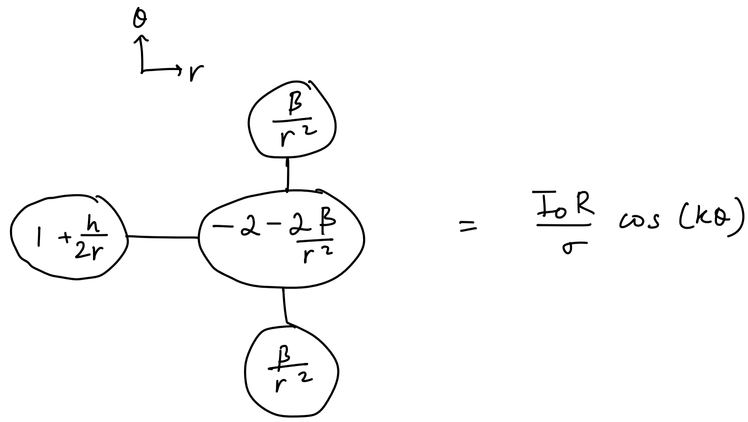


- Boundary IV

$$u(R,\theta) = -\frac{I_o R}{\sigma}\cos(k\theta)$$

[Type I]

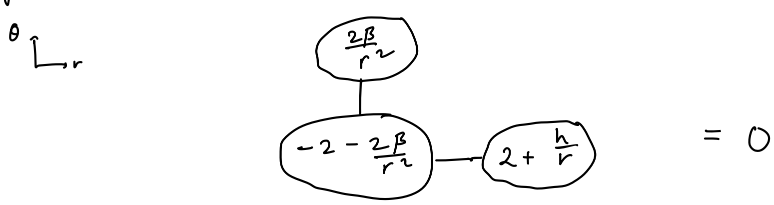
u_{i+1j} on Boundary



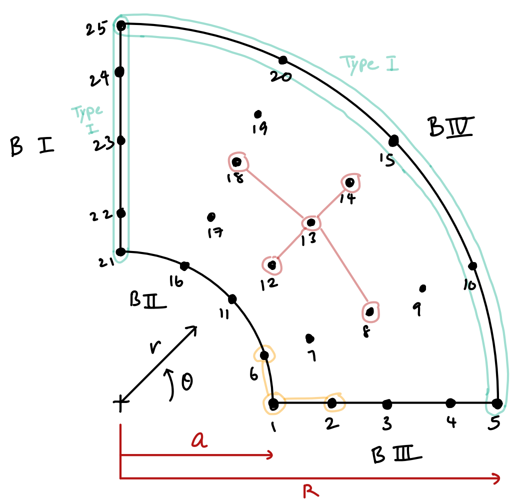
Corners of Boundary I + II , III + IV , IV + I $\longrightarrow$ Respective Type I condition dominates

Corner of Boundary II + III $\longrightarrow$ Both Type II

u_{ij} on Boundary corner



NODE MAP



Let $N = 5$

[FD Molecule on Corners Drawn]