$$\frac{\partial u}{\partial t} = D\nabla^2 u$$

Analyst A : $u^{l+1} - u^l = \frac{r}{2}\left(3\delta^2 u^l - \delta^2 u^{l-1}\right)$

Analyst B : $u^{l+1} - u^l = \frac{r}{12}\left(5\delta^2 u^{l+1} + 8\delta^2 u^l - \delta^2 u^{l-1}\right)$

$r = \frac{D\Delta t}{h^2}$, $\delta^2 \to$ CD operator , $l \to$ time level

Which one is superior ? [1D space-time analysis]

Are they better than Crank-Nicholson ?

(a) Consistency

$\boxed{A}$ FD Equation : [i = x-index , h = x-spacing , $\Delta t, k$ = t-spacing]

$$\underbrace{\frac{u_i^{l+1} - u_i^l}{\Delta t}} = \frac{D}{2h^2}\Big[\underbrace{3\left(u_{i+1}^l - 2u_i^l + u_{i-1}^l\right)}_{①} - \underbrace{\left(u_{i+1}^{l-1} - 2u_i^{l-1} + u_{i-1}^{l-1}\right)}_{②}\Big]$$

$$= \frac{\partial u}{\partial t} + \frac{\Delta t}{2}\frac{\partial^2 u^{l'}}{\partial t^2}$$

Taylor Expand Each Term at u_i^l.

For space derivative :

①
$$u_{i+1}^l = u_i^l + h\frac{\partial u_i^l}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u_i^l}{\partial x^2} + \frac{h^3}{3!}\frac{\partial^3 u_i^l}{\partial x^3} + \frac{h^4}{4!}\frac{\partial^4 u_i^l}{\partial x^4} + \dots$$

$$-2\left[u_i^l = u_i^l\right.$$

$$u_{i-1}^l = u_i^l - h\frac{\partial u_i^l}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u_i^l}{\partial x^2} - \frac{h^3}{3!}\frac{\partial^3 u_i^l}{\partial x^3} + \frac{h^4}{4!}\frac{\partial^4 u_i^l}{\partial x^4} + \dots$$

$$\delta^2 u_i^l = h^2\frac{\partial^2 u_i^l}{\partial x^2} + \frac{2h^4}{4!}\frac{\partial^4 u_i^{l'}}{\partial x^4}$$ ← Truncation Error

$$\left[\times \frac{3}{2h^2}\right]$$

$$\frac{3}{2h^2}\delta^2 u_i^\ell = \frac{3}{2}\frac{\partial^2 u_i^\ell}{\partial x^2} + \frac{3h^2}{4!}\frac{\partial^4 u_i^{\ell'}}{\partial x^4}$$

(2)

$$u_{i+1}^{\ell-1} = u_i^\ell + h\frac{\partial u_i^\ell}{\partial x} - k\frac{\partial u_i^\ell}{\partial t} - kh\frac{\partial^2 u_i^\ell}{\partial t\partial x} + \frac{h^2}{2}\frac{\partial^2 u_i^\ell}{\partial x^2} + \frac{k^2}{2}\frac{\partial^2 u_i^\ell}{\partial t^2} + $$

$$-2\left[u_i^{\ell-1} = u_i^\ell - k\frac{\partial u_i^\ell}{\partial t} + \frac{k^2}{2}\frac{\partial^2 u_i^\ell}{\partial t^2} + \right.$$

$$u_{i-1}^{\ell-1} = u_i^\ell - \underbrace{h\frac{\partial u_i^\ell}{\partial x}}_{} \underbrace{- k\frac{\partial u_i^\ell}{\partial t}}_{} + \underbrace{kh\frac{\partial^2 u_i^\ell}{\partial t\partial x}}_{} + \frac{h^2}{2}\frac{\partial^2 u_i^\ell}{\partial x^2} + \underbrace{\frac{k^2}{2}\frac{\partial^2 u_i^\ell}{\partial t^2}}_{} + $$

ODD h TERMS CANCEL OUT

ALL TERMS WITH JUST TIME DERIVATIVE CANCEL OUT

$\partial^4 u_i^\ell$ CROSS TERMS:

$\substack{\ell-1 \\ i+1} \rightarrow \cancel{-kkkh}, \; kkhh, \; \cancel{-khhh}$

$\substack{\ell-1 \\ i-1} \rightarrow \cancel{kkkh}, \; kkhh, \; \cancel{khhh}$

↳ TERM PERSISTS

(EVEN) hhhh TERM PERSISTS

$\partial^3 u_i^\ell$ CROSS TERMS:

$\substack{\ell-1 \\ i+1} \rightarrow \cancel{+kkh}, \; -khh$

$\substack{\ell-1 \\ i-1} \rightarrow \cancel{-kkh}, \; -khh$

↳ TERM PERSISTS

DOMINATED

$$\delta^2 u_i^{\ell-1} = h^2\frac{\partial^2 u_i^\ell}{\partial x^2} + \frac{2h^4}{4!}\frac{\partial^4 u_i^{\ell'}}{\partial x^4} + \cancel{\frac{2k^2h^2}{4!}\frac{\partial^4 u_i^{\ell'}}{\partial t^2\partial x^2}} - \frac{2kh^2}{3!}\frac{\partial^3 u_i^{\ell'}}{\partial x^2\partial t}$$

$\left[x - \frac{1}{2h^2}\right]$

$$-\frac{\delta^2 u_i^{\ell-1}}{2h^2} = -\frac{1}{2}\frac{\partial^2 u_i^\ell}{\partial x^2} - \frac{h^2}{4!}\frac{\partial^4 u_i^{\ell'}}{\partial x^4} - \frac{k}{3!}\frac{\partial^3 u_i^{\ell'}}{\partial t\,\partial x^2}$$

$$\Rightarrow \frac{3}{2h^2}\delta^2 u_i^\ell - \frac{1}{2h^2}\delta^2 u_i^{\ell-1} = \frac{\partial^2 u_i^\ell}{\partial x^2} + \frac{2}{4!}h^2\frac{\partial^4 u_i^{\ell'}}{\partial x^4} - \frac{k}{3!}\frac{\partial^3 u_i^{\ell'}}{\partial t\,\partial x^2}$$

↳ Might be different $\frac{\partial^4 u_i^{\ell'}}{\partial x^4}$ BUT CLUBBED AS TERM → 0 AS $k, h \rightarrow 0$ ANYWAYS

$\ast$ $\boxed{\therefore \text{ FD EQUATION } \boxed{A} \text{ CONSISTENT WITH PDE}}$

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} \underbrace{- \frac{k}{2}\frac{\partial^2 u'}{\partial t^2} + D\left[\frac{2h^2}{4!}\frac{\partial^4 u'}{\partial x^4} - \frac{k}{3!}\frac{\partial^3 u'}{\partial t\,\partial x^2}\right]}_{}$$

$\boxed{O(k+h^2)}$ TRUNCATION ERROR TERMS $\longrightarrow 0$

AS $k, h \longrightarrow 0$ ✓

$\boxed{B}$ FD EQ:

$$\underbrace{\frac{u_i^{l+1} - u_i^l}{\Delta t}} = \frac{D}{12h^2}\left(\underbrace{5\delta^2 u^{l+1}}_{①} + \underbrace{8\delta^2 u^l}_{②} - \underbrace{\delta^2 u^{l-1}}_{③} \right)$$

$$\frac{\partial u_i^l}{\partial t} + \frac{k}{2}\frac{\partial^2 u_i^{l'}}{\partial t^2}$$

② $$\frac{8}{12h^2}\delta^2 u_i^l = \frac{2}{3}\frac{\partial^2 u_i^l}{\partial x^2} + \frac{8}{12}\cdot\frac{2h^2}{4!}\frac{\partial^4 u_i^l}{\partial x^4}$$

$$\frac{2}{3h^2}\delta u_i^l = \frac{2}{3}\frac{\partial^2 u_i^l}{\partial x^2} + \frac{h^2}{18}\frac{\partial^4 u_i^l}{\partial x^4}$$

③ $$-\frac{1}{12h^2}\delta^2 u_i^{l-1} = -\frac{1}{12}\frac{\partial^2 u_i^l}{\partial x^2} - \frac{h^2}{6\cdot 4!}\frac{\partial^4 u_i^l}{\partial x^4} - \frac{k}{6\cdot 3!}\frac{\partial^3 u_i^l}{\partial x^2 \partial t}$$

① $$\delta^2 u^{l+1} = u_{i+1}^{l+1} - 2u_i^{l+1} + u_{i-1}^{l+1}$$

$$u_{i+1}^{l+1} = u_i^l + k\frac{\partial u}{\partial t} + h\frac{\partial u}{\partial x} + \frac{kh}{2}\frac{\partial u}{\partial t\partial x} + \frac{k^2}{2}\frac{\partial^2 u}{\partial t^2} + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2} + \ldots$$

$$-2[\,u_i^{l+1} = u_i^l + k\frac{\partial u}{\partial t} + \frac{k^2}{2}\frac{\partial^2 u}{\partial t^2} + \ldots$$

$$u_{i-1}^{l+1} = u_i^l + \underline{k\frac{\partial u}{\partial t}} - \underline{h}\frac{\partial u}{\partial x} - \frac{kh}{\underline{2}}\frac{\partial u}{\partial t\partial x} + \underline{\frac{k^2}{2}\frac{\partial^2 u}{\partial t^2}} + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}$$

ODD h TERMS CANCEL

JUST k TERMS CANCEL

$l+1$, $i+1$ $\rightarrow$ ~~kh~~, khh

$l+1$, $i-1$ $\rightarrow$ ~~$-kh$~~, khh

TERMS LEFT $\rightarrow$ h^2, h^4, kh^2

$$\delta^2 u^{l+1} = h^2\frac{\partial^2 u}{\partial x^2} + \frac{2h^4}{4!}\frac{\partial^4 u'}{\partial x^4} + \frac{2}{3!}kh^2\frac{\partial^3 u'}{\partial t\partial x^2}$$

$[\times \frac{5}{12h^2}]$

$$\frac{5}{12h^2}\delta^2 u^{l+1} = \frac{5}{12}\frac{\partial^2 u}{\partial x^2} + \frac{5}{6\cdot 4!}h^2\frac{\partial^4 u'}{\partial x^4} + \frac{5k}{6\cdot 3!}\frac{\partial^3 u'}{\partial t\partial x^2}$$

$$\Rightarrow \frac{8}{12h^2}\delta^2 u_i^l + \frac{5}{12h^2}\delta^2 u_i^{l+1} - \frac{1}{12}\delta^2 u_i^{l-1} = \frac{\partial^2 u}{\partial x^2} + h^2\frac{\partial^4 u'}{\partial x^4}\left(\frac{8-1+5}{6\cdot 4!}\right) + k\frac{\partial^3 u}{\partial t\partial x^2}\left(\frac{5}{6\cdot 3!} - \frac{1}{6\cdot 3!}\right)$$

FD EQUATION :

$$\frac{\partial u_i^l}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + \underbrace{D\left[\frac{h^2}{12}\frac{\partial^4 u'}{\partial x^4} + \frac{k}{9}\frac{\partial^3 u'}{\partial t\partial x^2}\right] - \frac{k}{2}\frac{\partial^2 u'}{\partial t^2}}_{\text{TRUNC ERROR} \to 0 \text{ AS } k,h\to 0 \checkmark}$$

$\boxed{O(k+h^2)}$

* $\boxed{\text{FD EQUATION } \boxed{B} \text{ IS CONSISTENT W/ PDE}}$

(b) Von Neumann Analysis

Assume $u = Ae^{\alpha t}e^{j\sigma x}$ as Parabolic Equation

$\boxed{A}$

$$u_i^{l+1} - u_i^l = \frac{r}{2}\left(3\delta^2 u^l - \delta^2 u^{l-1}\right)$$

$$\boxed{\begin{aligned} \delta^2 &= u_{i+1} - 2u_i + u_{i-1} \\ &= \left(e^{j\sigma h} - 2 + e^{-j\sigma h}\right)u_i \\ &= \left(\cos\sigma h + j\sin\sigma h - 2 + \cos\sigma h - j\sin\sigma h\right)u_i \\ &= \left(2\cos\sigma h - 2\right)u_i \\ u^{l+1} &= e^{\alpha k}u^l \equiv \gamma u^l \quad \text{WANT TO BOUND} \\ u^{l-1} &= e^{-\alpha k}u^l = \gamma^{-1}u^l \end{aligned}}$$

$$(\gamma - 1)\cancel{u_i^\ell} = \frac{r}{2}\left(3(2\cos\sigma h - 2)\cancel{u_i^\ell} - \gamma^{-1}(2\cos\sigma h - 2)\cancel{u_i^\ell}\right)$$

$$\gamma - 1 = \frac{3r}{2}\underbrace{(2\cos\sigma h - 2)}_{\equiv \psi} - \frac{r}{2\gamma}(2\cos\sigma h - 2)$$

$$\gamma^2 + \gamma\left(-1 - \frac{3r}{2}\psi\right) + \frac{r\psi}{2} = 0$$

FOR STABILITY, $|\gamma| \le 1 \rightarrow$ IFF $\frac{c}{a} \le 1$ AND $|b| \le a + c$

$$\frac{r\psi}{2} \le 1$$

[shortest waves $-4 \le \psi \le 0$ longest waves]

ALWAYS TRUE ✓

$$\left|-1 - \frac{3r}{2}\psi\right| \le 1 + \frac{r\psi}{2}$$

$$-1 - \frac{3r}{2}\psi \le 1 + \frac{r\psi}{2}$$

$$-\frac{3r}{2}\psi - \frac{r\psi}{2} \le 2$$

$$2r\psi \ge -2$$

$$r\psi \ge -1$$

$\boxed{\psi = 0}$ ALWAYS TRUE

$\boxed{\psi = -4}$ $\boxed{r \le \frac{1}{4}}$

$$-1 - \frac{3r}{2}\psi \ge -1 - \frac{r\psi}{2}$$

$$3r\psi \le r\psi$$

$$2r\psi \le 0$$

$$r\psi \le 0$$

$\boxed{\psi = 0}$ ALWAYS TRUE

$\boxed{\psi = -4}$ ALWAYS TRUE AS r IS POSITIVE

FD $\boxed{A} \longrightarrow$ $\boxed{\text{CONDITIONALLY STABLE}}$

$\boxed{B}$

$$u_i^{\ell+1} - u_i^\ell = \frac{r}{12}\left[5\delta^2 u_i^{\ell+1} + 8\delta^2 u_i^\ell - \delta^2 u_i^{\ell-1}\right]$$

$$(\gamma - 1) = \frac{r}{12}\left[5\gamma\psi + 8\psi - \frac{1}{\gamma}\psi\right]$$

$$\gamma^2 - \gamma - \frac{5r}{12}\gamma^2\psi - \frac{8r}{12}\gamma\psi + \frac{\psi r}{12} = 0$$

$$\gamma^2\left(1 - \frac{5r}{12}\psi\right) + \gamma\left(-1 - \frac{8r}{12}\psi\right) + \frac{\psi r}{12} = 0$$

FOR $|\gamma| \le 1$, $\frac{c}{a} \le 1$ AND $|b| \le a + c$

$$\frac{\frac{\psi r}{12}}{1 - \frac{5r}{12}\psi} \le 1$$

$\boxed{\psi = 0}$ TRUE ✓

$\boxed{\psi = -4}$ $-\frac{r}{3} \le 1 + \frac{5}{3}r$

$-2r \le 1$

$r \ge -\frac{1}{2}$

ALWAYS TRUE ✓

$$\left|-1 - \frac{8r}{12}\psi\right| \le 1 - \frac{5r}{12}\psi + \frac{\psi r}{12}$$

$-1 - \frac{8r}{12}\psi \le 1 - \frac{4r}{12}\psi$

$\frac{4r}{12}\psi \ge -2$

$r\psi \ge -6$

$\boxed{\psi = 0}$ ALWAYS TRUE

$\boxed{\psi = -4}$ $\boxed{r \le \frac{3}{2}}$

$-1 - \frac{8r}{12}\psi \ge -1 + \frac{4r}{12}\psi$

$-2r\psi \ge r\psi$

$r\psi \le 0$

$\boxed{\psi = 0}$ ALWAYS TRUE

$\boxed{\psi = -4}$ $-4r \le 0$

$r \ge 0$

ALWAYS TRUE

FD $\boxed{B}$ $\longrightarrow$ $\boxed{\text{UNCONDITIONALLY UNSTABLE}}$

C. Accuracy

Study $\underbrace{\frac{\text{Numerical Amplification factor}}{\text{Analytical Amplification factor}}} = \frac{\gamma}{\gamma_0}$

Analytical γ_0 :

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} \quad w/ \quad u = A e^{\alpha t} e^{j\sigma x}$$

$$\alpha = -D\sigma^2$$

$$\gamma_0 = e^{\alpha \Delta t} = e^{-D\sigma^2 k} = e^{-r(\sigma h)^2} \quad \left[\text{as} \quad r = \frac{Dk}{h^2}\right]$$

Examine $\left(\frac{\gamma}{\gamma_0}\right)^N$ where $N = \frac{T}{k}$ (# timesteps to advance soln by T)

$$N = \frac{1}{D\sigma^2 k} = \frac{1}{r(\sigma h)^2}$$

Define $T = \left(\frac{\gamma}{\gamma_0}\right)^N = \frac{\gamma^{\frac{1}{r(\sigma h)^2}}}{\left(e^{-r(\sigma h)^2}\right)^{\frac{1}{r(\sigma h)^2}}} = \gamma^{\frac{1}{r(\sigma h)^2}} \cdot e$

"Propagation Factor"

Find T for various r values used

Plot T vs $\frac{\lambda}{h} \left(= \frac{2\pi}{\sigma h}\right)$

$= 1$ is perfect. Need to be as close as possible

(e)

1D Problem : $\frac{\partial u}{\partial t} = D \frac{\partial^2 U}{\partial x^2}$

$L = 10$

$D = 0.5$

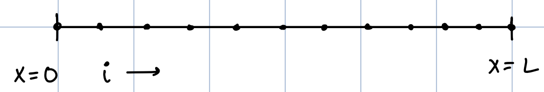
$\Delta x = 0.1 \equiv h$

$\Delta t = 0.05 \equiv k$

$\boxed{\text{B.C.s :}}$ $U(0,t) = U(L,t) = 0$

$\boxed{\text{I.C.s :}}$ $U(-\Delta t, x) = U(0, x) = e^{-\frac{(x-x_0)^2}{2\sigma^2}}$

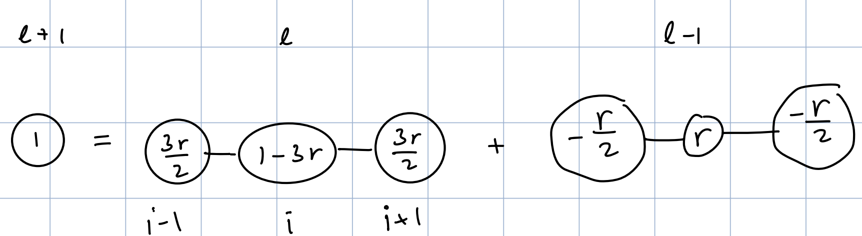
where $x_0 = 5$, $\sigma = 0.1$



$\boxed{A}$ FD Molecule :

$$u_i^{l+1} - u_i^l = \frac{r}{2}\left[3u_{i+1}^l - 6u_i^l + 3u_{i-1}^l - u_{i+1}^{l-1} + 2u_i^{l-1} - u_{i-1}^{l-1}\right]$$

$$u_i^{l+1} = \left(\frac{3r}{2}\right)u_{i+1}^l + (1-3r)\,u_i^l + \left(\frac{3r}{2}\right)u_{i-1}^l + \left(-\frac{r}{2}\right)u_{i+1}^{l-1} + (r)\,u_i^{l-1} + \left(-\frac{r}{2}\right)u_{i-1}^{l-1}$$



JUST NEEDS MATRIX MULTIPLICATION

NO NEED FOR SOLVING $\underline{\underline{A}}\,\underline{x} = \underline{b}$

$\boxed{\text{Computational Effort for time increment : } O(N)}$

$\boxed{B}$ FD Molecule :

$$u_i^{l+1} - u_i^{l} = \frac{5r}{12}\left(u_{i+1}^{l+1} + u_{i-1}^{l+1}\right) - \frac{5r}{6}u_i^{l+1} + \frac{2r}{3}\left(u_{i+1}^{l} + u_{i-1}^{l}\right) - \frac{4r}{3}u_i^{l} - \frac{r}{12}\left(u_{i+1}^{l-1} + u_{i-1}^{l-1}\right) + \frac{r}{6}u_i^{l-1}$$

$$\overbrace{\left(-\frac{5r}{12}\right)u_{i-1}^{l+1} + u_i^{l+1}\left(1 + \frac{5r}{6}\right) + \left(-\frac{5r}{12}\right)u_{i+1}^{l+1}}^{l+1} = \overbrace{\frac{2r}{3}u_{i-1}^{l} + \left(1 - \frac{4r}{3}\right)u_i^{l} + \frac{2}{3}r\,u_{i+1}^{l}}^{l} + \overbrace{u_{i-1}^{l-1}\left(-\frac{r}{12}\right) + \frac{r}{6}u_i^{l-1} + u_{i+1}^{l-1}\left(-\frac{r}{12}\right)}^{l-1}$$

$$\overbrace{-\frac{5r}{12} \;-\; 1+\frac{5r}{6} \;-\; -\frac{5r}{12}}^{l+1} \quad = \quad \underbrace{\overbrace{\frac{2}{3}r \;-\; 1-\frac{4r}{3} \;-\; \frac{2}{3}r}^{l} \;+\; \overbrace{-\frac{r}{12} \;-\; \frac{r}{6} \;-\; -\frac{r}{12}}^{l-1}}_{b_{ij}}$$

$O(N) \rightarrow$ NEED MATRIX MULTIPLICATION TO FIND b_{ij}

$\underbrace{O(N)}_{\downarrow} \rightarrow$ THEN MATRIX EQ SOLVER $\underline{\underline{A}}\,\underline{x} = \underline{b}$ to find next time step

AS TRIDIAG
$= O(N \cdot NB^2)$
$= O(N)$
USING LU DECOMP

$$\begin{bmatrix} 1+\frac{5r}{12} & -\frac{5r}{12} & & \\ -\frac{5r}{12} & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots \end{bmatrix} \underline{x} = b_{ij}$$

TRIDIAG

$\boxed{\text{Computational Effort : } O(N^2)}$

FOR BOTH

AT BOUNDARY :

$$\overset{l+1}{1} = \overset{l}{1}$$

APPENDIX

Analytical Solution for Comparison:

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}$$

Assume $u = X(x)\,T(t)$

$$X(x)\,\dot{T}(t) = D\,X''(x)\,T(t)$$

$$\frac{\dot{T}(t)}{D\,T(t)} = -\lambda = \frac{X''(x)}{X(x)}$$

Separation of Variables:

$$\frac{dT(t)}{dt} = -\lambda D\,T(t)$$

$$\dot{T}(t) = -\lambda D\,T(t)$$

$$\dot{T}(t) + \frac{n^2\pi^2}{L^2} D\,T(t) = 0$$

Solution: $T(t) = e^{-\lambda D t}$

$$X''(x) = -\lambda X(x)$$

$$X(x) = c_1 \cos(-\sqrt{\lambda}\,x) + c_2 \sin(-\sqrt{\lambda}\,x)$$

As $u(0,t) = u(L,t) = 0$,

$c_1 = 0$, $c_2 \rightarrow$ non-zero

$$c_2 \sin(-\sqrt{\lambda}\cdot L) = 0$$

$-\sqrt{\lambda}\cdot L = n\cdot\pi$ where $n \in \mathbb{Z}^+$

$$\lambda = \frac{n^2\pi^2}{L^2}$$

Plug into ODE for t

$$T(x,0) = f(x) = e^{-\frac{(x-x_0)^2}{2\sigma^2}} = \sum^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad n = \mathbb{Z}^+$$

$\hookrightarrow$ Plotted in Matlab

For PDEX1BC FUNCTION IN MATLAB:

$$p(x,t,u) + q(x,t)\,f\left(x,t,u,\frac{\partial u}{\partial x}\right) = 0$$

At $x = 0$,

$$u(0,t) = 0 \longrightarrow u + 0 \cdot \frac{\partial u}{\partial x} = 0$$

$p(0,t,u) = u$

$q(0,t) = 0$

At $x = L$,

$$u(L,t) = 0 \longrightarrow u + 0 \cdot \frac{\partial u}{\partial x} = 0$$

$p(L,t,u) = u$

$q(L,t,u) = 0$

Used float so roundoff errors can act for analysis perspective

Compare for spikier start — σ ↑ ↓

Show r range to show

↳ k change, change h, $\boxed{\text{change D}}$

↳ probably useless

↳ lower mean shorter waves exist for longer

1 → $r = 0.25$, $h = 0.01$, $k = 0.00005$

$r = 0.375$ $k = 0.000075$

$r = 0.25$ $h = 0.1$, $k = 0.005$

$r = 1$ $k = 0.0002$

5 → $r = 1.5$ $k = 0.0003$

2 $= 0.0004$

3 0.0006

4 0.0008

$\sigma = 0.01$ $r = 0.25$ $h = 0.01$

10 → $r = 0.375$

$r =$

$D \to 0.01$ $r = 0.25$ $h = 0.01$ 0.0025

13 → 10 0.000000025

$h = 0.$ 0.00025

$$\frac{0.0025 \cdot 10}{0.025}$$