Finding Residues at isolated singularities $\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{2\pi i}\int_C f(z)\,dz$

where C is positively oriented simple closed contour in $0<|z-z_0|<\varepsilon$ s.t z_0 is inside C and f is analytic throughout $0<|z-z_0|<\varepsilon$

① Looking at the $\frac{b_1}{z-z_0}$ in the Laurent series of f at z_0.

$\operatorname*{Res}_{z=z_0} f(z) = b_1$

(for essential singularity, this is the only option)

② For a simple pole at $z=z_0$

ⓐ $\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0} [(z-z_0)f(z)]$

ⓑ $\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$ if p, q are analytic around z_0, $q(z_0)=0$ and $q'(z_0)\neq 0$.

③ For a pole of order $m>1$ at $z=z_0$:

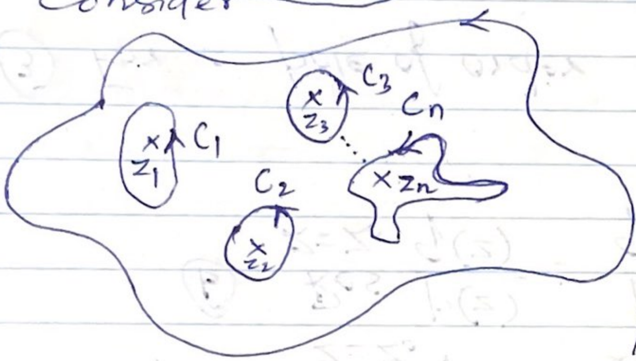
$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]\Bigg|_{z=z_0}.$$

The Residue theorem

Let C be a positively oriented simple closed contour, and let f be analytic inside and on C except at finitely many isolated singular points $z_1, z_2, \dots, z_n$ inside C. Then

$$\boxed{\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).}$$

Idea: Consider



Let C, C_k $(k=1,2,\dots,n)$ be +vely oriented. f is analytic in C but outside C_k. So by principal of deformation of paths and Cauchy-Goursat theorem,

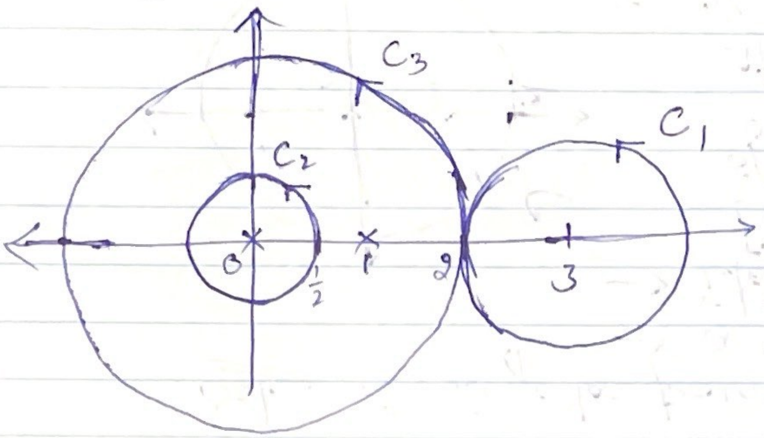
$$\oint_C f(z)\,dz = \sum_{k=1}^{n} \underbrace{\oint_{C_k} f(z)\,dz}_{2\pi i \operatorname*{Res}_{z=z_k} f(z)}$$

$\blacksquare$

Example

(1) Evaluate $\displaystyle\oint_C \frac{\exp(z)}{z(z-1)}\,dz$

where (i) $C = C_1 : |z-3| = 1$, (ii) $C = C_2 : |z| = \frac{1}{2}$,
(iii) $C = C_3 : |z| = 2$. (all oriented +vely)



Singular points at $z=0$ and $z=1$.
for $f(z) = \dfrac{\exp(z)}{z(z-1)}$

(i) f is analytic inside C_1 and on C_1. So by Cauchy-Goursat theorem, $\oint_{C_1} f(z)\,dz = 0$

(ii) C_2 encloses $z=0$! By Residue theorem

$$\oint_{C_2} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=z_0} f(z) = 2\pi i \operatorname*{Res}_{z=z_0} \frac{\exp(z)}{z(z-1)}$$

f has simple pole at $z=0$!

$$\Rightarrow \operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to 0} \cancel{z} \cdot \frac{\exp(z)}{\cancel{z}(z-1)}$$

$$= \frac{e^0}{-1} = -1.$$

$$\Rightarrow \oint_{C_2} f(z)\,dz = -2\pi i.$$

(iii) C_3 encloses both $z_1 = 0$ and $z_2 = 1$.

Res. Thm: $\oint_{C_3} f(z)\,dz = 2\pi i \sum_{k=1}^{2} \operatorname{Res}_{z=z_k} f(z)$.

$= 2\pi i \left(\operatorname{Res}_{z=0} f(z) + \underbrace{\operatorname{Res}_{z=1} f(z)}_{\text{Simple pole at } z=1} \right)$

From before, $\operatorname{Res}_{z=0} f(z) = -1$.

$\operatorname{Res}_{z=1} f(z) = \lim_{z\to 1} \cancel{(z-1)} \dfrac{\exp(z)}{z\cancel{(z-1)}} = \dfrac{e}{1} = e$.

$\therefore \oint_{C_3} f(z)\,dz = 2\pi i(-1+e)$

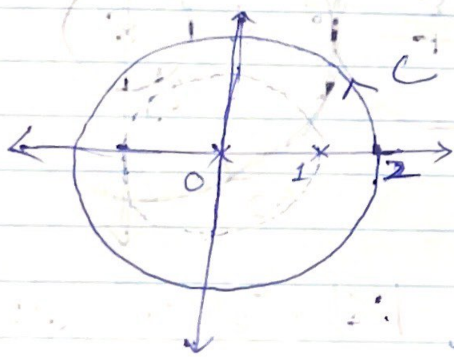
(2) Evaluate: $\oint_C \dfrac{e^{\pi/z}}{(z-1)^2}\,dz$, where $C: |z| = 2$ (ccw)

Soln. 2 singular points:

→ a pole of order 2 at $z=1$.

→ identify singularity at $z=0$!

[Need Laurent expansion of $f(z) = \dfrac{e^{\pi/z}}{(z-1)^2}$ centered at $z=0$ in $0<|z|<1$]



$\therefore \oint_C \dfrac{e^{\pi/z}}{(z-1)^2}\,dz = 2\pi i \left(\operatorname{Res}_{z=0} f(z) + \operatorname{Res}_{z=1} f(z) \right)$

$$\frac{e^{\pi/z}}{(z-1)^2} = \frac{e^{\pi/z}}{(1-z)^2}$$

$\Bigg[$ Recall, for $|z|<1$, $\frac{1}{1-z} = 1+z+z^2+\cdots$

Diff. on b.s, we get $\frac{1}{(1-z)^2} = 1+2z+3z^2+\cdots \Bigg]$

$$= \left(\cdots + \frac{\pi^3}{z^3 3!} + \frac{\pi^2}{z^2 2!} + \frac{\pi}{z} + 1\right)\left(1+2z+3z^2+\cdots\right)$$

$$= \cdots + \frac{\pi^3}{z^3 3!} + \frac{\pi^2}{z^2 2!} + \frac{\pi}{z} + 1$$

$$+ \cdots + \frac{2\pi^3}{z^2 3!} + \frac{2\pi^2}{2! z} + 2\pi + 2z$$

$$+ \cdots + \frac{3\pi^3}{3! z} + \frac{3\pi^2}{2!} + 3\pi z + 3z^2$$

(Principal part of f at $z=0$)

$\Rightarrow$ $P(z)$ of $\frac{e^{\pi/z}}{(z-1)^2}$ has infinitely many terms and so, $z=0$ is an essential singularity

$$\operatorname*{Res}_{z=0} \frac{e^{\pi/z}}{(z-1)^2} = b_1$$

Just the $\frac{1}{z}$ terms : $\frac{\pi}{z} + \frac{\pi^2}{z} + \frac{\pi^3}{z \cdot 2!} + \frac{\pi^4}{z \cdot 3!} + \cdots$

$$= \frac{\pi}{z}\left(1 + \frac{\pi}{1} + \frac{\pi^2}{2!} + \frac{\pi^3}{3!} + \cdots\right)$$

$$= \frac{\pi}{z}\left(e^{\pi}\right) \Rightarrow \boxed{b_1 = \pi e^{\pi}}$$

$$\operatorname*{Res}_{z=1} f(z) = \operatorname*{Res}_{z=1} \frac{e^{\pi/z}}{(z-1)^2}$$

$$= \frac{1}{(2-1)!} \frac{d}{dz}\left((z-1)^2 \cdot \frac{e^{\pi/z}}{(z-1)^2}\right)\Bigg|_{z=1}$$

$$= \frac{-\pi e^{\pi/z}}{z^2}\Bigg|_{z=1} = -\pi e^{\pi}$$

$$\therefore \oint_C \frac{e^{\pi/z}}{(z-1)^2}\, dz = 2\pi i\left(\pi e^{\pi} - \pi e^{\pi}\right) = 0.$$

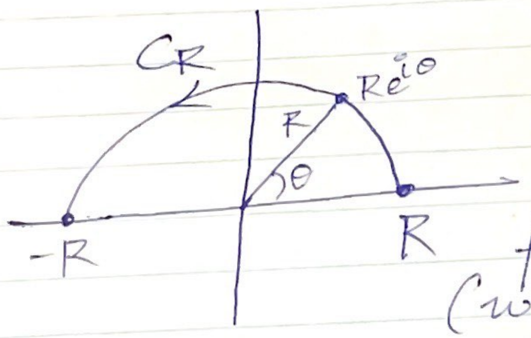
Applications of Contour Integration

(Type 2) Improper integrals of the type $\int_{-\infty}^{\infty} f(x)\,dx$

(where $|f(x)| \leq \frac{c}{|x|^2}$ for $|x|$ very large)

For $R>0$ very large,

Replace x by z and consider the closed contour C which consists of the semicircle of radius R : C_R, and the part of the x-axis with $-R \leq x \leq R$ ($: [-R, R]$) as below:



On $C_R : z = Re^{i\theta}$ assume, $0 \le \theta \le \pi$

i) $f(z)$ is analytic in the upper half-plane except for a finite number of poles (which do not lie on the x-axis)

ii) $|f(z)| \le \dfrac{M}{|z|^2} = \dfrac{M}{R^2}$ for $|z| \ge R$ (where $M > 0$ is a constant)

Eg: $f(z) = \dfrac{1}{z^2+1}$

In this case $\displaystyle\oint_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz$

by residue thm. $\displaystyle = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z),$ (z_k inside C)

Note: $\displaystyle\int_{[-R,R]} f(z)\,dz = \int_{-R}^{R} f(x)\,dx$ $\left[\because z = x \Rightarrow dz = dx,\ -R \le x \le R\right]$

Since $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$, then

as long as we have $\displaystyle\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0,$

we get $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$, where

$z_1, z_2, \dots, z_k$ are singularities of f in upper half-plane

Condition (ii) : Why do we need?

$$\lim_{R\to\infty}\left|\int_{C_R} f(z)\,dz\right| = \lim_{R\to\infty}\left|\int_0^{\pi} f(\underbrace{Re^{i\theta}}_{z})\cdot\underbrace{iRe^{i\theta}d\theta}_{dz}\right|$$

$$\leq \lim_{R\to\infty}\int_0^{\pi}|f(Re^{i\theta})|\,R\,d\theta$$

(by a property of integrals)

$\because |iRe^{i\theta}| = R$

$|f(z)| \leq \frac{M}{R^2}$ for $|z| = R$

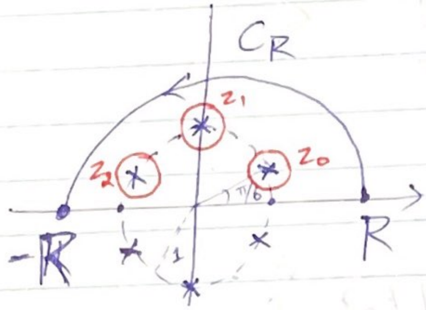
$$\leq \lim_{R\to\infty}\int_0^{\pi}\frac{M}{R^2}R\,d\theta$$

$$= \lim_{R\to\infty}\frac{\pi M}{R} = 0.$$

Example:

① Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^6+1}\,dx$

Solution: Consider $\displaystyle\oint_C \frac{1}{z^6+1}\,dz$



Poles of $\frac{1}{z^6+1}$ are at sixth roots of -1

$z_k = e^{i\frac{\pi}{6}(2n+1)}$, $k = 0,1,2,\dots,5$

From previous class,

$$\operatorname*{Res}_{z=z_k} f(z) = \operatorname*{Res}_{z=z_k} \frac{1}{z^6+1} = \frac{1}{6z_k^5}$$

Condition I: Only three of z_k's are in $\{z : Im(z) > 0\}$ upper-half plane

Condition II: $|f(z)| \le \frac{1}{R^6-1} < \frac{1}{R^2}$ for $|z| = R$ (R very large)

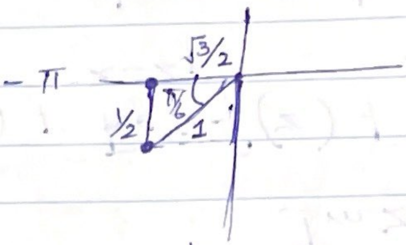
$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{x^6+1}\,dx = 2\pi i \sum_{Im\, z>0} Res\, f(z).$$

$$= 2\pi i \left[\operatorname*{Res}_{z=z_0} f(z) + \operatorname*{Res}_{z=z_1} f(z) + \operatorname*{Res}_{z=z_2} f(z)\right]$$

$$= 2\pi i \left[\frac{1}{6z_0^5} + \frac{1}{6z_1^5} + \frac{1}{6z_2^5}\right]$$

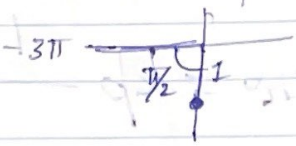
$$= \frac{2\pi i}{6}\left[e^{-i5\pi/6} + e^{-i5\pi/2} + e^{-i25\pi/6}\right]$$

$$e^{-i5\pi/6} = e^{i(-\pi+\pi/6)}$$



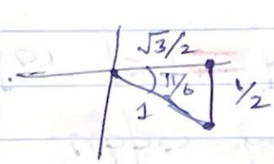
$$= \frac{-\sqrt{3}-i}{2}$$

$$e^{-i5\pi/2} = e^{i(-3\pi+\pi/2)}$$



$$= -i$$

$$e^{-i25\pi/6} = e^{-i\left(\frac{24\pi+\pi}{6}\right)} = e^{-i\pi/6}$$



$$= \frac{\sqrt{3}-i}{2}$$

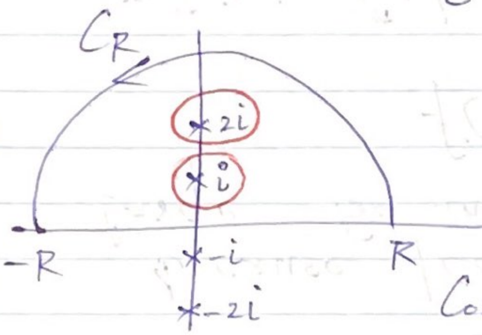
$$\int_{-\infty}^{\infty} \frac{1}{x^6+1}\,dx = \frac{\pi i}{3}\left[\frac{-\sqrt{3}-i}{2} - i + \frac{\sqrt{3}-i}{2}\right]$$

$$= +\frac{2\pi}{3}$$

$$\Rightarrow \int_{0}^{\infty} \frac{1}{x^6+1}\,dx = \frac{\pi}{3}$$

② $\int^{\infty} \frac{dx}{(x^2+1)^2(x^2+4)} = \frac{\pi}{18}$.

Soln. Consider $\oint_C \frac{1}{(z^2+1)^2(z^2+4)}\,dz$



Poles: $z = \pm i$ - order 2

$z = \pm 2i$ - simple pole

Condition I: finite no. of poles in $\{z : \operatorname{Im} z > 0\}$: $i, 2i$

Condition II: for large R, if $|z| = R$,

$$|f(z)| < \frac{1}{(R^2-1)^2(R^2-4)} \leq \frac{1}{R^2 \cdot R} < \frac{1}{R^2}.$$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2(x^2+4)} = 2\pi i \sum_{\operatorname{Im} z > 0} \operatorname{Res} f(z)$$

$$= 2\pi i \left[\operatorname{Res}_{z=i} f(z) + \operatorname{Res}_{z=2i} f(z) \right]$$

$$\operatorname{Res}_{z=i} f(z) = \frac{d}{dz}\left((z-i)^2 \frac{1}{(z-i)^2(z+i)^2(z^2+4)} \right)\Bigg|_{z=i}$$

$$= \frac{d}{dz}\left(\frac{1}{(z+i)^2(z^2+4)} \right)\Bigg|_{z=i}$$

$$= \left[\frac{-2}{(z+i)^3(z^2+4)} - \frac{2z}{(z^2+4)^2(z+i)^2} \right]_{z=i}$$

$$= \frac{-2}{-8i(3)} - \frac{2i}{9(4)i^2} = \frac{1}{12i} - \frac{1}{18i} = \frac{1}{36i}$$

$p(z) = \frac{1}{(z^2+1)^2} \quad q(z) = z^2+4$

$$\operatorname*{Res}_{z=2i} f(z) = \left.\frac{\frac{1}{(z^2+1)^2}}{2z}\right|_{z=2i} = \frac{1}{36i}$$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2(x^2+4)} = 2\pi i\left(\frac{1}{36i} + \frac{1}{36i}\right) = \frac{\pi}{9}$$

$$\therefore \int_{0}^{\infty} \frac{dx}{(x^2+1)^2(x^2+4)} = \frac{\pi}{18}.$$

Remark: $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}$ exists, for $a>0$.

The obvious choice $f(z) = \frac{\cos z}{z^2+a^2}$ is bad because $|\cos z|$ becomes large when $\operatorname{Im} z$ is large. So instead choose

$$f(z) = \frac{e^{iz}}{z^2+a^2}.$$ Let C_R be as earlier. Only pole in the upper half plane is $z=ia$, ~~when~~ since $a>0$.

$$\operatorname*{Res}_{z=ia} \frac{e^{iz}}{z^2+a^2} = \frac{e^{-a}}{2ia}$$

Then $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2} dx = 2\pi i \left(\frac{e^{-a}}{2ia}\right) = \frac{\pi e^{-a}}{a}$

provided $\lim_{R \to \infty} \int_{C_R} f(z)\,dz \longrightarrow 0$

$$\left| \int_{C_R} f(z)\,dz \right| = \left| \int_{C_R} \frac{e^{iz}}{z^2+a^2} dz \right|$$

$$\leq \int_{C_R} \left| \frac{e^{iz}}{z^2+a^2} \right| |dz|$$

$$\leq \int_0^{\pi} \frac{e^{-y}}{R^2-a^2} \cdot R\,d\theta \quad \text{for } R > a$$

$$\leq \frac{\pi R}{R^2-a^2} \quad \text{for } R > 0 \quad \because e^{-y} < 1 \text{ for } y > 0!$$

$$\downarrow \text{ as } R \longrightarrow \infty$$

$$0$$

So, consider improper integrals of the form

$$\int_{-\infty}^{\infty} f(x) \cos kx\,dx \text{ (or) } \int_{-\infty}^{\infty} f(x) \sin kx\,dx \text{ (or)}$$

$$\int_{-\infty}^{\infty} f(x) e^{ikx}\,dx \quad , k > 0!$$

Same contour as above

Condition ① finite no. of poles in $\{z : \text{Im } z > 0\}$
(none on Re axis)

Condition ② $|f(z)| \leq \frac{M}{R}$ on $|z| = R$.
[$\lim_{z \to \infty} f(z) = 0$ is enough]

Then $\int_{C_R} f(z) e^{ikz} \longrightarrow 0$ as $R \longrightarrow \infty$ [Jordan's Lemma]

Idea. $\left| \int_{C_R} f(z) e^{ikz} dz \right| \leq R \int_0^{\pi} \frac{M}{R} e^{-kR\sin t} dt = 2M \int_0^{\pi/2} e^{-kR\sin t} dt$ ⊛

Jordan's inequality: $\sin t \geq \frac{2}{\pi} t$ for $0 \leq t \leq \pi/2$.
$\left[\frac{2}{\pi} \leq \frac{\sin t}{t} \leq 1 \text{ for } 0 < t \leq \pi/2 \right]$
Observe the concavity of
$\sin t - 2t/\pi$ on $(0, \pi/2)$.

$$\int_0^{\pi/2} e^{-kR\sin t} dt \leq \int_0^{\pi/2} e^{-\frac{2kRt}{\pi}} dt = \frac{\pi}{2kR}\left(1 - e^{-kR}\right)$$

$$\frac{\pi M}{kR}\left(1 - e^{-kR}\right) \longrightarrow 0 \quad \text{as } R \to \infty$$

$$\therefore \int_{-\infty}^{\infty} f(x)e^{ikx}\,dx = 2\pi i \sum_{\operatorname{Im} z>0} \operatorname{Res} f(z)e^{ikz}$$

Ex. $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin 3x}{x^2+9}\,dx = \frac{\pi}{e^9}$

Consider $f(z) = \dfrac{ze^{3iz}}{z^2+9}$. Single pole in $\{z : \operatorname{Im} z > 0\}$ at $z = 3i$!

$$2\pi i \operatorname*{Res}_{z=3i} f(z) = \frac{2\pi i}{2e^9} = \frac{\pi i}{e^9}$$

$\therefore$

Inverse Fourier Transforms

The Fourier Transform of a function $f(x)$ is

$$F(w) = \int_{-\infty}^{\infty} f(x)e^{-iwx}\,dx$$

The inverse Fourier transform is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)e^{iwx}\,dx$$

Treat w as a complex variable and use contour integration.

For $x>0$, the variable x plays the role of k in Jordan's lemma,

$\therefore$ for $x>0$,

$$f(x) = \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega x}\, d\omega = \int_{-R}^{R} F(\omega) e^{i\omega x}\, d\omega$$

$$= 2\pi i \sum_{\operatorname{Im}\omega>0} \operatorname{Res}\left(F(\omega) e^{i\omega x}\right)$$

When there are finite Poles in lower half plane

for $x<0$, Consider C_R in lower half plane ccw.

$$f(x) = \int_{-\infty}^{\infty} F(\omega) e^{i\omega x}\, d\omega = \int_{R}^{-R} F(\omega) e^{i\omega x}\, d\omega$$

$$= -2\pi i \sum_{\operatorname{Im}\omega>0} \operatorname{Res}\left(F(\omega) e^{i\omega x}\right).$$

<u>Example</u> Find $\mathcal{F}^{-1}$ for $F(\omega) = \dfrac{1}{a+i\omega}$

<u>Solution</u>:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega x}}{a+i\omega}\, d\omega.$$

Pole at $\omega = -\dfrac{a}{i} = ia$ (upper half plane)

For $x>0$, $\operatorname*{Res}_{\omega=ia} \dfrac{e^{i\omega x}}{i(\omega-ia)} = \dfrac{e^{-ax}}{i} \Rightarrow f(x) = e^{-ax}$

For $x<0$, $F(\omega) e^{i\omega x}$ is analytic $\therefore f(x) = 0$

$$\therefore f(x) = \begin{cases} e^{-ax} &, x > 0 \\ 0 &, x < 0 \end{cases}$$

(or) $e^{-ax} H(x)$

Heaviside step function.

<u>Example</u>. Find $\mathcal{F}^{-1}$ for $F(\omega) = \dfrac{1}{\omega^2 + i\omega + 2}$

<u>Soln</u>. Note $\omega^2 + i\omega + 2 = (\omega - i)(\omega + 2i)$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega x}}{\omega^2 + i\omega + 2}\, d\omega$$

$+i \rightarrow$ Pole in upper half plane

$-2i \rightarrow$ Pole in lower half plane

For $x > 0$, $\operatorname*{Res}_{\omega = i} F(\omega) e^{i\omega x} = \left.\dfrac{e^{i\omega x}}{2\omega + i}\right|_{\omega = i} = \dfrac{e^{-x}}{3i}$

$$\therefore f(x) = \frac{e^{-x}}{3}.$$

For $x < 0$, $\operatorname*{Res}_{\omega = -2i} F(\omega) e^{i\omega x} = \left.\dfrac{e^{i\omega x}}{2\omega + i}\right|_{\omega = -2i} = \dfrac{e^{2x}}{-3i}$

$$\therefore f(x) = \frac{e^{2x}}{3}.$$

$$\therefore f(x) = \frac{1}{3} \begin{cases} e^{2x} &, x < 0 \\ e^{-x} &, x > 0 \end{cases}$$