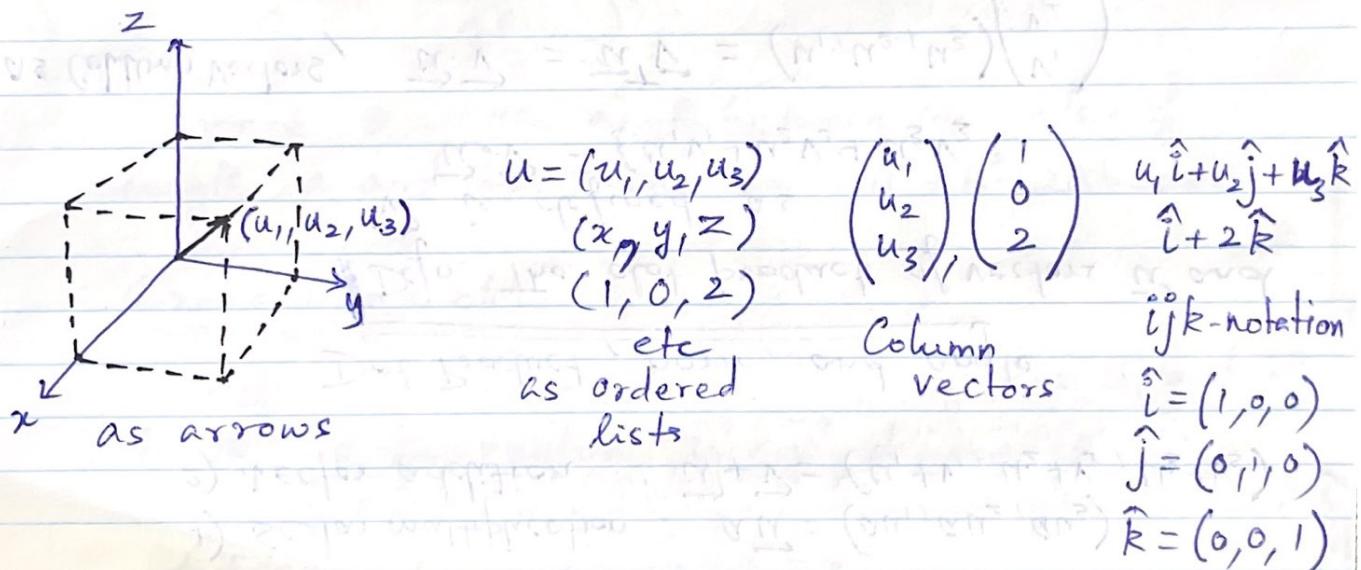


Linear Algebra Review

On paper/Chalkboard, we typically write \vec{v} instead of boldfaced as you would see in Mabin's content (or) assignments (or) exams.

Defn: A (real) n -dimensional vector \vec{v} is an n -tuple $\vec{v} = (v_1, v_2, \dots, v_n)$ of real numbers $v_1, v_2, \dots, v_n \in \mathbb{R}$. The collection of all real n -dimensional vectors is denoted as \mathbb{R}^n .

 \mathbb{R}^3 - visualizations / realizations

One important vector we must distinguish in \mathbb{R}^3 is the zero vector (or the origin), denoted by

$$\vec{0} = (0, 0, 0).$$

Defining features of \mathbb{R}^3 as a vector space are

(For any $\vec{u}, \vec{v} \in \mathbb{R}^3$ with $\vec{u} = (u_1, u_2, u_3)$ and any scalar $a \in \mathbb{R}$, we have)

1) Scalar multiplication : $a\vec{u} = (au_1, au_2, au_3)$

2) Vector addition : $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$

Dot Product, norm, and angle

Defn. The dot product of vectors \vec{u} and \vec{v} is defined as

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

as Column Vectors, $\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = (u_1, u_2, u_3) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

Norm of $\vec{v} \in \mathbb{R}^3$: $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$
 $= \sqrt{v_1^2 + v_2^2 + v_3^2}$

Properties:

1) $\|a\vec{v}\| = |a| \|\vec{v}\|$ for all $a \in \mathbb{R}$ and $\vec{v} \in \mathbb{R}^3$

2) $\|\vec{v}\| \geq 0$ for all $\vec{v} \in \mathbb{R}^3$ (and $\|\vec{v}\| = 0 \iff \vec{v} = 0$)

Triangle inequality \Rightarrow 3) $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$ for all $\vec{u}, \vec{v} \in \mathbb{R}^3$

4) $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$ for all $\vec{u}, \vec{v} \in \mathbb{R}^3$

↳ (Cauchy-Schwarz inequality)

Angle between two non-zero vectors in \mathbb{R}^3

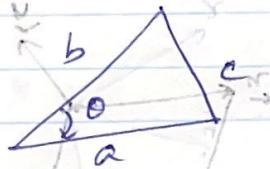
→ unique number $\theta \in [0, \pi]$ that satisfies

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta.$$

$$\text{i.e., } \theta = \arccos \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \right)$$

Defn. Two vectors $\vec{u}, \vec{v} \in \mathbb{R}^3$ are said to be orthogonal if $\vec{u} \cdot \vec{v} = 0$. Write $\vec{u} \perp \vec{v}$.

Exercise: Prove the law of cosines. If a, b, c are lengths of the sides of any triangle,



where θ is the angle between the sides of length a and b , then $c^2 = a^2 + b^2 - 2ab \cos \theta$

Cross Product

Defn. The cross product of vectors \vec{u} and \vec{v} in \mathbb{R}^3 is the vector $\vec{u} \times \vec{v}$ defined as

$$\vec{u} \times \vec{v} = (u_2 v_3 - v_2 u_3) \hat{i} + (u_3 v_1 - v_3 u_1) \hat{j} + (u_1 v_2 - v_1 u_2) \hat{k}$$

(Idea use a determinant)

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = u_2 v_3 \hat{i} - u_1 v_3 \hat{j} + u_1 v_2 \hat{k}$$

Properties for all $\vec{u}, \vec{v} \in \mathbb{R}^3$,

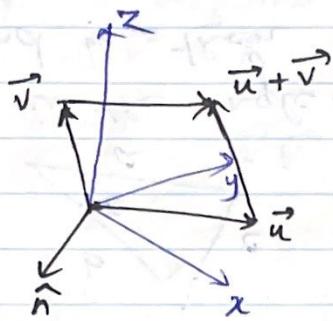
$$1) \vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$$

$$2) \vec{u} \times \vec{u} = 0$$

$$3) \vec{u} \times \vec{v} \perp \vec{u}, \vec{u} \times \vec{v} \perp \vec{v}$$

$$4) \|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin \theta, \text{ where } \theta \text{ is the angle between } \vec{u} \text{ and } \vec{v}$$

Any two vectors \vec{u} and \vec{v} in \mathbb{R}^3 define a parallelogram



$$\hat{n} = \frac{\vec{u} \times \vec{v}}{\|\vec{u} \times \vec{v}\|}$$

\hat{n} is the normal unit vector that is perpendicular to the plane of the parallelogram.

Area of the parallelogram defined by \vec{u} and \vec{v} in \mathbb{R}^3 is given by $\|\vec{u} \times \vec{v}\|$

True or False

$$1) \vec{u} \cdot \vec{v} = 0 \Leftrightarrow \vec{u} = 0, \vec{v} = 0 \text{ or both}$$

False $(1, -1, 0) \cdot (-1, -1, 0) = 0$

perpendicular (\Rightarrow) orthogonal vectors.

$$2) \vec{u} \times \vec{v} = \|\vec{u}\| \|\vec{v}\| \sin \theta$$

False: $\vec{u} \times \vec{v}$ is a vector

$\|\vec{u}\| \|\vec{v}\| \sin \theta$ is a scalar.

3) The equation $\hat{n} \cdot (\vec{x} - \vec{x}_0) = 0$ represents
the equation of a plane passing through
 $\vec{x}_0 = (x_0, y_0, z_0)$ with normal vector \hat{n}

True

4) The plane $ax+by+cz=d$ has unit
normal vector (a, b, c)
False. $\|(a, b, c)\|$ need not be 1.