WEEK 9 LEC 010 (Mani Thamizhazhagan)
Nov 14-18 Lecture Summary / Notes

We covered several practical as well as theoretical applications of MVT this week.
Please read sections 4.2.2 - 4.2.4 Pages 218 - 226
L'Hôpital's Rule: Please read section 4.3 Pages 239-247

Some remarks & Exercises:

Remark:
Theorem 6 Page 220: If $f' > 0$ $\boxed{\text{on } I}$, then f is increasing on I.
This must be over an interval to conclude.

Counter-Example:
Let $f(x) = \begin{cases} x + 2x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

Show that $f'(0) = 1 > 0$ but f is not monotonic in any interval around 0.
→ Use definition to find $f'(0)$
→ Find using arithmetics, $f'(x)$ for $x \neq 0$.
→ Show that in any interval $(-\delta, \delta)$, $\delta > 0$, $\exists\, x, y$ s.t $f'(x) > 0$, $f'(y) < 0$.

Exercises:
1) S.T $e^t > 1 + t$ for $t \neq 0$.
Solution: Let $f(x) = e^x$. Suppose $t > 0$.
Then f is cts. on $[0, t]$ and diff. on $(0, t)$
So by MVT, $\exists\, c \in (0, t)$ s.t
$$\frac{f(t) - f(0)}{t} = f'(c) = e^c \Rightarrow e^t - 1 = e^c t.$$

Since $c \in (0,t)$, $e^c > 1$

$\therefore$ $e^t - 1 > t$

$e^t > 1 + t$ for $t > 0$.

If $t < 0$, consider $[t, 0]$!

Sub-exercise: Use Comparing Functions Theorem (or) Increasing Function Theorem to solve this.

2) Show $\frac{b-a}{b} < \ln\left(\frac{b}{a}\right) < \frac{b-a}{a}$, $0 < a < b$.

Solution:

$f(x) = \ln(x)$ is cont. on $[a,b]$ and diff. on (a,b). By MVT, $\exists\, c \in (a,b)$ s.t

$$\frac{\ln(b) - \ln(a)}{b-a} = f'(c) = \frac{1}{c}$$

$$\Rightarrow \quad \ln\left(\frac{b}{a}\right) = \frac{b-a}{c}$$

as $a < c < b$, $\frac{1}{b} < \frac{1}{c} < \frac{1}{a}$.

$$\therefore \frac{b-a}{b} < \ln\left(\frac{b}{a}\right) < \frac{b-a}{a}.$$

3) Use the above exercise to show that if $e \le a < b$, then $a^b > b^a$

Solution:

We have $\frac{b-a}{b} < \ln\left(\frac{b}{a}\right) < \frac{b-a}{a}$

In particular, $a \ln\left(\frac{b}{a}\right) < b-a$.

$$\Rightarrow \ln\left(\frac{b}{a}\right)^a < b-a$$

as e^x is an increasing fn on $\mathbb{R}$,

$$\Rightarrow e^{\ln\left(\frac{b}{a}\right)^a} < e^{b-a}$$

$$(\text{i.e}) \quad \left(\frac{b}{a}\right)^a < e^{b-a}$$

$$\Rightarrow b^a < e^{b-a} \cdot a^a$$

as $e \leq a$, $e^t \leq a^t$ for $t \geq 0$.

$$\therefore b^a < e^{b-a} \cdot a^a \leq a^{b-a} \cdot a^a = a^b.$$

$$\therefore \boxed{b^a < a^b} \text{ if } e \leq a < b.$$

Exercises:

1) Prove that $e^x > ex$ for $x \in \mathbb{R}$.
2) Prove that $\frac{x}{1+x} < \ln(1+x) < x$ for $x > 0$.
3) Prove that $n(b-a)a^{n-1} < b^n - a^n < n(b-a)b^{n-1}$ for $0 < a < b$.
4) Show that $\sin x \leq x$ for $x > 0$.
5) Show that $0 < \frac{1}{x}\ln\left(\frac{e^x - 1}{x}\right) < 1$ for $x > 0$
6) Let $0 < a < b$. Show that $b^{1/n} - a^{1/n} < (b-a)^{1/n}$
(Hint: Consider $f(x) = x^{1/n} - a^{1/n} - (x-a)^{1/n}$ and Show that f is increasing for $x \geq a$)
(Then $f(b) > f(a)$ if $b > a$)

MVT given in our course book is Lagrange's form of MVT. It is the single most important result in the theory of differentiation.

The following form of MVT is useful in proving L'Hôpital's Rule.

Cauchy's form of MVT: Let f, g be cts. on $[a,b]$ and diff. on (a,b). Assume that $g'(x) \neq 0$ for any $x \in (a,b)$. Then there exists $c \in (a,b)$ s.t

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Strategy: As in the proof of MVT, the basic idea is to use Rolle's thm. (i.e) we wish to get a function $h(x)$ satisfying conditions of Rolle's theorem and that $h'(c)=0$ gives $f'(c)/g'(c)$. Look at $h(x) = f(x) - \lambda g(x)$ and find λ so that $h(a) = h(b)$

Proof:

Note that $g(a) \neq g(b)$. (O/w, by Rolle's Thm, $\exists\, c \in (a,b)$ s.t $g'(c)=0$. This contradicts $g'(x) \neq 0$)

Let $h(x) = f(x) - \lambda g(x)$ where $\lambda \in \mathbb{R}$ is chosen so that $h(a) = h(b)$. [Exercise: S.T $\lambda = \frac{f(b)-f(a)}{g(b)-g(a)}$]

Now h satisfies the condition of Rolle's thm on $[a,b]$. So $\exists\, c \in (a,b)$ s.t $h'(c) = 0 \Rightarrow \lambda = \frac{f'(c)}{g'(c)}$.

$\therefore \exists\, c \in (a,b)$ s.t $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(c)}{g'(c)}$.

$\blacksquare$

<u>Remark</u>:

If $g(x)=x$ in the above, Cauchy's MVT reduces to Lagrange's MVT.

As an application to the Cauchy MVT, we prove L'Hôpital's rules. (Only for those really interested)

<u>L'Hôpital's Rule</u>: Let I be an open interval. Let either $a \in I$ (or) a is an endpoint of I. (Note that it may happen that $a = \pm\infty$!) Assume that

i) f, g are diff. on $I \setminus \{a\}$.

ii) $g(x) \neq 0$, $g'(x) \neq 0$ for $x \in I \setminus \{a\}$, and

iii) $A = \lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, where A is either 0 (or) ∞.

Assume that $B := \lim_{x\to a} \dfrac{f'(x)}{g'(x)}$ exists either in $\mathbb{R}$ (or) $B = \pm\infty$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = B.$$

<u>Strategy</u>: The trick is to bring the expression $\dfrac{f(x)}{g(x)}$ to the form $\dfrac{f(x)-f(d)}{g(x)-g(d)}$ so that Cauchy's MVT can be applied.

The proof below shows a clever way of achieving this.

<u>Proof</u>: <u>Simple case</u>: $A=0$, $a\in\mathbb{R}$, and $B\in\mathbb{R}$.

Set $f(a)=0=g(a)$. Then f and g are continuous on I. Let $\{x_n\}$ be a sequence in I s.t either $x_n>a$ (or) $x_n<a$ for all $n\in\mathbb{N}$ and $x_n\to a$. By Cauchy's MVT, there exists c_n between a and x_n s.t

$$\frac{f(x_n)-f(a)}{g(x_n)-g(a)} = \frac{f'(c_n)}{g'(c_n)}$$

as $x_n\to a$, then $c_n\to a$.

Since $f(a)=0=g(a)$, it follows that

$$\frac{f(x_n)}{g(x_n)} = \frac{f(x_n)-f(a)}{g(x_n)-g(a)} = \frac{f'(c_n)}{g'(c_n)}$$

By hypothesis, $\lim\limits_{n\to\infty}\frac{f'(c_n)}{g'(c_n)} = B$ and hence

$\lim\limits_{n\to\infty}\frac{f(x_n)}{g(x_n)} = B$. (Now, the result follows after a technical realization! Try).

<u>Next Case</u>: when $A=\infty$, $a\in\mathbb{R}$, $B\in\mathbb{R}$.

Write $h(x)=f(x)-Bg(x)$ for $x\in I\setminus\{a\}$.

Then $h'(x)=f'(x)-Bg'(x)$ so that

$$\lim_{x\to a}\frac{h'(x)}{g'(x)} = 0.$$

Now if we show that $\lim_{x \to a} \frac{h(x)}{g(x)} = 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = B$ follows!

To Show: $\lim_{x \to a} \frac{h(x)}{g(x)} = 0$:

(The following calculations are worth going through a couple of times to master the tricks of analysis).

Since $\lim_{x \to a} \frac{h'(x)}{g'(x)} = 0$, given $\varepsilon > 0$, $\exists \delta_1 > 0$ s.t

$$(\bigstar) \quad g(x) > 0 \text{ and } \left|\frac{h'(x)}{g'(x)}\right| < \frac{\varepsilon}{2} \text{ for } x \in (a, a+\delta_1]$$

(as $\lim_{x \to a} g(x) = \infty$)

Now if $x \in (a, a+\delta_1)$, then by Cauchy's MVT,

$$(\ast) \quad \frac{h(x) - h(a+\delta_1)}{g(x) - g(a+\delta_1)} = \frac{h'(c_x)}{g'(c_x)} \text{ for some } c_x \in (x, a+\delta_1).$$

Combining $(\bigstar)$ and $(\ast)$, we get

$$(\blacklozenge) \quad \left|\frac{h(x) - h(a+\delta_1)}{g(x) - g(a+\delta_1)}\right| < \frac{\varepsilon}{2} \text{ for } x \in (a, a+\delta_1).$$

Since $\lim_{x\to a} g(x) = \infty$, $\exists\, \delta_2 < \delta_1$ s.t

(♥) $\quad g(x) > g(a+\delta_1)$ for $x \in (a, a+\delta_2)$.

Deduce now that $0 < g(x) - g(a+\delta_1) < g(x)$
from (★) & (♥) $\quad$ for $x \in (a, a+\delta_2)$.

From (◆) and above,

$$\frac{|h(x) - h(a+\delta_1)|}{(g(x))} < \frac{|h(x) - h(a+\delta_1)|}{g(x) - g(a+\delta_1)} < \frac{\varepsilon}{2}, \text{ for } x \in (a, a+\delta_2)$$

Now choose $\delta_3 < \delta_2$ so that

$$\frac{|h(a+\delta_1)|}{g(x)} < \frac{\varepsilon}{2} \text{ for } x \in (a, a+\delta_3)$$

[why ?]

Then $\dfrac{h(x)}{g(x)} = \dfrac{h(x) - h(a+\delta_1)}{g(x)} + \dfrac{h(a+\delta_1)}{g(x)}$

If $x \in (a, a+\delta_3)$, we have

$$\left|\frac{h(x)}{g(x)}\right| \leq \left|\frac{h(x) - h(a+\delta_1)|}{g(x)} + \frac{|h(a+\delta_1)|}{g(x)} < \varepsilon.$$

$\Rightarrow \lim_{x\to a} \dfrac{h(x)}{g(x)} = 0$, the result follows

since $\dfrac{f(x)}{g(x)} = \dfrac{h(x)}{g(x)} + B$.

∎

Exercises

1) $\lim_{x \to 3} \dfrac{x^3 - 2x^2 - 3x}{x^2 - 5x + 6}$

Let $f(x) = x^3 - 2x^2 - 3x$

$g(x) = x^2 - 5x + 6$.

$$\lim_{x \to 3} \frac{f'(x)}{g'(x)} = \lim_{x \to 3} \frac{3x^2 - 4x - 3}{2x - 5}$$

$$= 12.$$

$\therefore$ By LHR,

$$\lim_{x \to 3} \frac{x^3 - 2x^2 - 3x}{x^2 - 5x + 6} = 12.$$

Only in retrospect, we know we can have equality

$$\lim_{x \to 3} \frac{x^3 - 2x^2 - 3x}{x^2 - 5x + 6} = \lim_{x \to 3} \frac{3x^2 - 4x - 3}{2x - 5}$$

because the latter limit exists.

2) $\lim_{x \to -3} \dfrac{\sin(\pi x)}{x + 3}$

$$\lim_{x \to -3} \pi \cos(\pi x) = -\pi.$$

$\therefore$ By LHR,

$$\lim_{x \to -3} \frac{\sin(\pi x)}{x + 3} = -\pi.$$

3) $\lim_{x\to\infty} \frac{\ln x}{x}$

$\overset{\text{"LHR"}}{=} \lim_{x\to\infty} \frac{1}{x} = 0$

4) $\lim_{x\to\infty} \frac{x^n}{e^x}$

$$\overset{LHR}{=} \lim_{x\to\infty} \frac{n x^{n-1}}{e^x} \overset{LHR}{=} \lim_{x\to\infty} \frac{n(n-1)x^{n-2}}{e^x} \overset{LHR}{=} \cdots \overset{LHR}{=} \underbrace{\lim_{x\to\infty} \frac{n!}{e^x}}_{} = 0$$

Because this exists, the prior equalities hold by LHR (in retrospect)

5) Recall a T/F question on your midterm:

$$\lim_{x\to 0^+} f(1/x) = \lim_{x\to\infty} f(x) \quad (\text{Why?})$$

So (4) $\Rightarrow \lim_{x\to 0^+} \frac{(1/x)^n}{e^{1/x}} = \lim_{x\to 0^+} x^{-n} e^{-1/x} = 0.$

This gives rise to an interesting example.

Consider $f(x) = \begin{cases} e^{-1/x}, & x > 0 \\ 0, & x \le 0 \end{cases}$

Then $f'(0) = \lim_{x\to 0^+} \frac{e^{-1/x} - 0}{x} = 0.$

Indeterminate forms

$\frac{0}{0}$, $\pm\frac{\infty}{\infty}$, $(0)(\pm\infty)$, $\infty-\infty$, 1^∞, ∞^0, and 0^0.

Why is 1^∞ indeterminate?

Though $\lim_{n\to\infty} 1^n = 1$, if f and g are functions such that $\lim_{n\to\infty} f(n) = 1$ and $\lim_{n\to\infty} g(n) = \infty$, it is $\boxed{\text{not}}$ necessarily true that

$$\lim_{n\to\infty} f(n)^{g(n)} = 1 \longrightarrow (*)$$

For eg. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

While $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n^2} = \infty$ and

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{\sqrt{n}} = 0.$$

So a limit of the form (1) always has to be evaluated on its own merit; the limits of f and g don't by themselves determine its value.

0^0 Indeterminate:

$$\lim_{x\to 0^+} x^0 = 1$$

$$\lim_{x\to 0^+} 0^x = 0$$

$\lim_{x\to\infty} x^{1/x} = 1$, but $\lim_{x\to\infty} (x^x)^{1/x} = \infty$ $\lim_{x\to\infty} x^{1/\ln(x)} = e.$	∞^0

Thus, in general, knowing that $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ is not sufficient to evaluate $\lim_{x\to a} f(x)^{g(x)}$.

The transformations for applying L'Hôpital's Rule:

Indeterminate form	Conditions	Transformation to 0/0	Transformation to ∞/∞
$\frac{0}{0}$	$\lim_{x\to a} f(x)=0$ $\lim_{x\to a} g(x)=0$	—	$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{(1/g(x))}{(1/f(x))}$
$\frac{\infty}{\infty}$	$\lim_{x\to a} f(x)=\infty$ $\lim_{x\to a} g(x)=\infty$	$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{1/g(x)}{1/f(x)}$	—
$0\cdot\infty$	$\lim_{x\to a} f(x)=0$ $\lim_{x\to a} g(x)=\infty$	$\lim_{x\to a} f(x)g(x)=\lim_{x\to a}\frac{f(x)}{1/g(x)}$	$\lim_{x\to a} f(x)g(x)=\lim_{x\to a}\frac{g(x)}{(1/f(x))}$
$\infty-\infty$	$\lim_{x\to a} f(x)=\infty$ $\lim_{x\to a} g(x)=\infty$	$\lim_{x\to a} f(x)-g(x)$ $=\lim_{x\to a}\frac{1/g(x)-1/f(x)}{1/(f(x)g(x))}$	$\lim_{x\to a} f(x)-g(x)$ $=\ln\lim_{x\to a}\frac{e^{f(x)}}{e^{g(x)}}$
0^0	$\lim_{x\to a} f(x)=0^+$, $\lim_{x\to a} g(x)=0$	$\lim_{x\to a} f(x)^{g(x)}$ $=e^{\lim_{x\to a}\frac{g(x)}{1/\ln f(x)}}$	$\lim_{x\to a} f(x)^{g(x)}$ $=e^{\lim_{x\to a}\frac{\ln f(x)}{1/g(x)}}$
1^∞	$\lim_{x\to a} f(x)=1$ $\lim_{x\to a} g(x)=\infty$	$\lim_{x\to a} f(x)^{g(x)}$ $=e^{\lim_{x\to a}\frac{\ln f(x)}{1/g(x)}}$	$\lim_{x\to a} f(x)^{g(x)}$ $=e^{\lim_{x\to a}\frac{g(x)}{1/\ln f(x)}}$
∞^0	$\lim_{x\to a} f(x)=\infty$ $\lim_{x\to a} g(x)=0$	$\lim_{x\to a} f(x)^{g(x)}$ $=e^{\lim_{x\to a}\frac{g(x)}{1/\ln f(x)}}$	$\lim_{x\to a} f(x)^{g(x)}$ $=e^{\lim_{x\to a}\frac{\ln f(x)}{1/g(x)}}$.

Exercisos

6) $\lim_{x\to\infty} x\ln\left(1+\frac{100}{x}\right)$

$$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{100}{x}\right)}{1/x}$$

$$\overset{\text{"LHR"}}{=} \lim_{x\to\infty} \frac{\left(\frac{1}{1+\frac{100}{x}}\right)\left(\frac{-100}{x^2}\right)}{\left(-1/x^2\right)}$$

$$= \lim_{x\to\infty} \frac{1}{1+\frac{100}{x}} \cdot 100$$

$$= 100.$$

7) $\lim_{x\to \pi/2^-} \sec x - \tan x$

$$= \lim_{x\to\frac{\pi}{2}^-} \frac{1-\sin x}{\cos x}$$

$$\overset{\text{"LHR"}}{=} \lim_{x\to\frac{\pi}{2}^-} \frac{-\cos x}{-\sin x} = 0.$$

8) $\lim_{x\to\infty} \left(e^x+x^2\right)^{1/x}$

$$= e^{\lim_{x\to\infty} \frac{1}{x}\ln\left(e^x+x^2\right)}$$

$$\overset{\text{"LHR"}}{=} e^{\lim_{x\to\infty} \frac{e^x+2x}{e^x+x^2}} \overset{\text{"LHR"}}{=} e^{\lim_{x\to\infty} \frac{e^x+2}{e^x+2x}} \overset{\text{"LHR"}}{=} e^{\lim_{x\to\infty} \frac{e^x}{e^x+2}}$$

$$\overset{\text{"LHR"}}{=} e^{\lim_{x\to\infty} \frac{e^x}{e^x}} = e.$$