Recall Cauchy-Integral formula

Let f be analytic within and on C (simple, closed)
For z_0 inside C, we have

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz$$

The above is true for any z_0 inside C,
So, if we let our z_0 vary over all the points interior to C, we get a different realization of our analytic function f inside C (i.e) let w replace z_0 in CIF and let it be any point inside C, then

$$\text{(★)} \quad f(w) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - w}\, dz \quad \text{for } w \text{ inside } C.$$

Now, recall the examples of analytic everywhere (i.e) entire functions we have seen so far,

for all $z \in \mathbb{C}$ (ie) $|z| < \infty$

Polynomials : $a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$, $a_i \in \mathbb{C}$

Exponential & its associates : $\exp(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$

$$\cos(z) = \frac{\exp(iz) + \exp(-iz)}{2}$$

$$= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots$$

for all $z \in \mathbb{C}$ (or) $|z| < \infty$

$$\sin(z) = \frac{\exp(iz) - \exp(-iz)}{2i}$$

$$= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

$$\cosh(z) = \frac{\exp(z) + \exp(-z)}{2}$$

$$= 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots$$

$$\sinh(z) = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots = \frac{\exp(z) - \exp(-z)}{2}$$

Geometric Series: $\frac{1}{1-z} = 1 + z + z^2 + \cdots$ $\boxed{|z| < 1}$

$\hookrightarrow$ locally, a power series!

So, More specifically, the above series are Taylor series centered at $0 \in \mathbb{C}$ (Maclaurin series).

So, is every analytic function f in a domain D, locally, a power series centered at any point $a \in D$? Yes.

Let's generate new Taylor series from the known ones.

$$\frac{1}{1+z} = \frac{1}{1-(-z)} = 1 - z + z^2 - z^3 + \cdots$$

when $|-z| < 1$

(i.e) $|z| < 1$

centered at 0!

$f(z) = \frac{1}{z}$ is not analytic at 0, but it is analytic on $\mathbb{C} - \{0\}$. For any $a \neq 0$,

write $$\frac{1}{z} = \frac{1}{z-a+a}$$

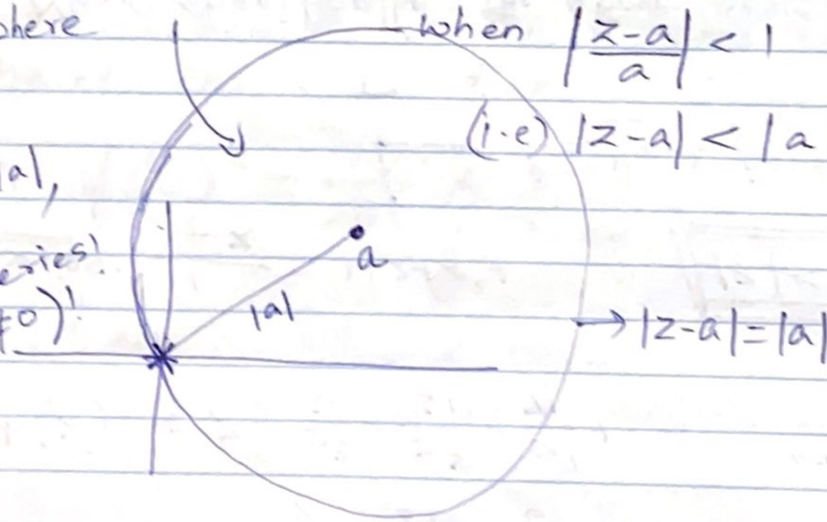
$$= \frac{1}{a}\left(\frac{1}{1+\frac{z-a}{a}}\right)$$

$$= \frac{1}{a}\sum_{n=0}^{\infty}\left(-\frac{(z-a)}{a}\right)^n$$

when $\left|\frac{z-a}{a}\right| < 1$

(i.e) $|z-a| < |a|$

Everywhere inside this circle $|z-a| = |a|$, it is a power series! (centered at $a \neq 0$)!



Let's get back to ⊛.

$$f(w) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-w}\,dz \text{ for all } w \text{ inside } C.$$

~~[illegible]~~ ~~[illegible]~~

Let z_0 be inside C and z on C fixed.

Since z_0 is inside C, there is a $r>0$ s.t the disk centered at z_0 of radius r is completely inside C.

So $$|z-z_0| > r \Rightarrow \frac{1}{|z-z_0|} < r$$

Now, ~~if we choose~~ for any w s.t $|w-z_0| < r/3$,

we have $$\left|\frac{w-z_0}{z-z_0}\right| < \frac{1}{3} < \frac{1}{2} < 1 \,!$$

Now,
$$\frac{f(z)}{z-w} = \frac{f(z)}{z-z_0+z_0-w}$$
$$= \frac{f(z)}{(z-z_0)-(w-z_0)} = \frac{1}{z-z_0}\frac{f(z)}{\left(1-\frac{w-z_0}{z-z_0}\right)}$$
$$= \frac{1}{z-z_0} f(z)\left(1+\left(\frac{w-z_0}{z-z_0}\right)+\left(\frac{w-z_0}{z-z_0}\right)^2+\cdots\right)$$
$$\frac{f(z)}{z-w} = \sum_{n=0}^{\infty} \frac{f(z)}{(z-z_0)^{n+1}}(w-z_0)^n$$

So, by ⊛,

$$f(w) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-w}\,dz$$

$$= \frac{1}{2\pi i}\int_C \sum_{n=0}^{\infty} \frac{f(z)}{(z-z_0)^{n+1}}(w-z_0)^n\,dz$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz\right)(w-z_0)^n$$

for $|w-z_0| < \frac{r}{3}$.

(i.e) f is locally a power series centered at z_0

$$f(w) = \sum_{n=0}^{\infty} a_n (w-z_0)^n \quad \text{s.t}$$

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \longrightarrow ①$$

→ A fantastic thing, when we have a power series (or) polynomial centered at z_0, (for f) is

that the co-efficients encode the details of any order derivative of f at z_0!

More specifically,

$$a_n = \frac{f^{(n)}(z_0)}{n!} \longrightarrow ②$$

From ① & ②, we get Generalized C.I.F

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} \, dz$$

When f is (merely) assumed to be analytic inside and on C and z_0 any point inside C.

A phenomenal Consequence

If f is analytic, its derivatives exists.
In fact, f is analytic at $z_0 \Rightarrow f^{(n)}$ is analytic at z_0 for any $n \in \mathbb{N}$

These formulas allow us to compute integrals of the form:

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}} \, dz$$

Example.

(1) Let C be $|z|=1$. Evaluate $\oint_C \frac{\sin(2z)}{z^4}\,dz$

Recognize $z_0 = 0$, $n = 3$ in $\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$

$f(z) = \sin(2z)$

By GCIF,

$$\oint_C \frac{\sin(2z)}{z^4}\,dz = \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

$\left(= 2\pi i\, a_n \right.$

$= 2\pi i\, a_3$

Co-efficient of $(z-0)^3$ in Taylor series of $\sin(2z)\Big)$

$f' = 2\cos(2z)$

$f'' = -4\sin(2z)$

$f''' = -8\cos(2z) \Rightarrow f^{(3)}(0) = -8$

$$\therefore \oint_C \frac{\sin(2z)}{z^4}\,dz = \frac{2\pi i}{3!}(-8) = -\frac{8\pi i}{3}$$

Note that for $\oint_C \frac{\sin(2z)}{z^{21}}\,dz$ we would need $f^{(20)}(0)$!

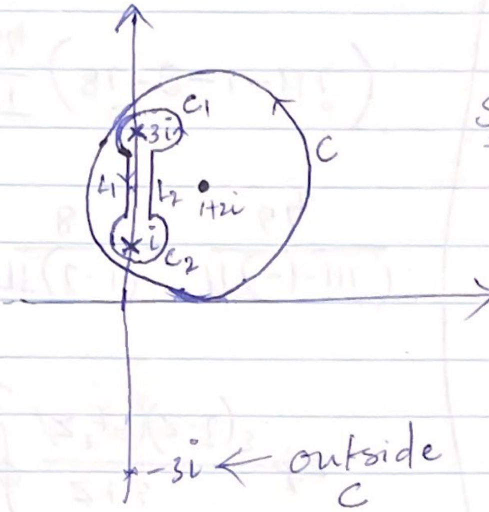
This is where Taylor series proves to be useful.

$$\sin(2z) = \sum_{n=0}^{\infty} \frac{(-1)^n (2z)^{2n+1}}{(2n+1)!} \Rightarrow \text{all co-efficients of even power are } 0$$

$$\therefore \oint_C \frac{\sin(2z)}{z^{21}}\,dz = 2\pi i\, a_{20} = 0 \quad \left(\text{So, } \oint_C \frac{\sin(2z)}{z^{2k+1}}\,dz = 0 \right)$$

② Evaluate $\oint_C \frac{z+3}{(z^2+9)(z-i)^3}\,dz$ where $C: |z-(1+2i)| = 2$

Circle centered at $1+2i$ with radius 2



Soln: Combo of path deformation, CIF & GCIF.

Singular points of $\frac{z+3}{(z^2+9)(z-i)^3}$ are $\pm 3i, i$ $\left(z^2+9 = (z+3i)(z-3i)\right)$

But only $3i$ and i are inside C.

Deform path C and limiting,

$$\oint_C = \oint_{C_1} + \oint_{C_2}$$

C_1: $\oint_{C_1} \frac{\frac{z+3}{(z+3i)(z-i)^3}}{z-3i}\,dz$ Use CIF with $z_0 = 3i$, $f_1(z) = \frac{z+3}{(z+3i)(z-i)^3}$ analytic on C_1 and inside C_1

$= 2\pi i\, f_1(z_0)$

$= 2\pi i \, \frac{3+3i}{(6i)(-8i)} = -\frac{\pi i(1+i)}{8i^2} = +\frac{\pi}{8}(1+i)i$

$= \frac{\pi}{8}(i-1)$

C_2: $\displaystyle\oint_{C_2} \frac{\left(\frac{z+3}{z^2+9}\right)}{(z-i)^3}\,dz$

$= 2\pi i \dfrac{f^{(2)}(i)}{2!}$

$= \pi i \dfrac{(-7i-9)}{64}$

$= \dfrac{\pi(7-9i)}{64}$

$\therefore \displaystyle\oint_C \frac{z+3}{(z^2+9)(z-i)^3}\,dz$

$= \dfrac{\pi(i-1)}{8} + \dfrac{\pi(7-9i)}{64}$

$= \dfrac{\pi}{64}(8i-8+7-9i)$

$= \dfrac{\pi}{64}(-1-1i)$

$= \cancel{-\dfrac{3\pi}{64}(3+i)} = -\dfrac{\pi}{64}(1+i)$

Use GCIF with

$z_0 = i$, $n=2$,

$f(z) = \dfrac{z+3}{z^2+9}$

$f' = \dfrac{(z^2+9)-(z+3)(2z)}{(z^2+9)^2}$

$= \dfrac{z^2+9-2z^2-6z}{(z^2+9)^2}$

$= \dfrac{9-z^2-6z}{(z^2+9)^2}$

$f'' = \dfrac{(z^2+9)^2(-2z-6)-(9-z^2-6z)(z^2+9)4z}{(z^2+9)^4}$

$f^{(2)}(i) = \dfrac{(8)^2(-2i-6)-(9+1-6i)(8)(4i)}{(8)^4}$

$= \dfrac{-2i-6}{8^2} - \dfrac{(10-6i)i}{8^2(2)}$

$= \dfrac{-2i-6-5i+3i^2}{8^2}$

$= \dfrac{-7i-9}{64}$