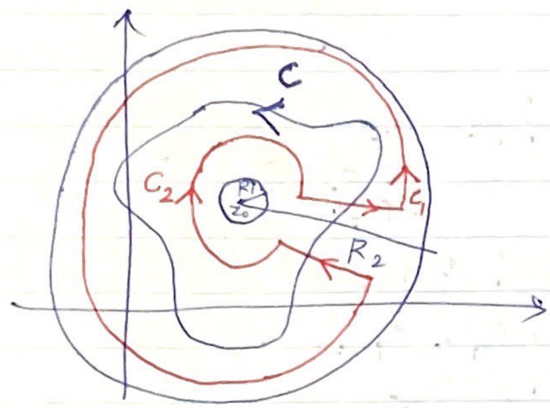
Recall, Laurent's theorem:

Suppose f is analytic throughout the annulus $D: R_1 < |z-z_0| < R_2$, with C a positively oriented simple closed curve in the domain D.



Then $f(z)$ can be represented by the Laurent series

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{Analytic part of } f \text{ at } z_0} + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}}_{\text{singular or principal part of } f \text{ at } z_0}$$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz, \quad b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{-n+1}}\, dz$$

Significance: With $n=1$, $\oint_C f(z)\, dz = 2\pi i\, b_1$

b_1 is called the residue of f at z_0.

If we know b_1 (from the series expansion), we can evaluate the integral.

[b_1 - the co-efficient of $\frac{1}{z-z_0}$ (or) $(z-z_0)^{-1}$ term]

For instance,

the Laurent series of $f(z) = \frac{\sin(2z)}{z^4}$ in $0<|z|<\infty$:

$$\frac{\sin(2z)}{z^4} = \frac{1}{z^4} \sum_{n=0}^{\infty} \frac{(-1)^n (2z)^{2n+1}}{(2n+1)!} = \frac{2}{z^3} - \frac{8}{3!\, z} + \frac{2^5}{5!} z - \frac{2^7}{7!} z^3 + \dots$$

$$= \frac{2}{z^3} - \frac{8}{6z} + \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+5}}{(2n+5)!} z^{2n+1}$$

$$\Rightarrow b_1 = -\frac{8}{6} \Rightarrow \oint_{C:|z|=1} \frac{\sin(2z)}{z^4}\, dz = -\frac{8\pi i}{3}$$ [Recall, from Lec 19]

On the other hand, we could compute the Laurent series' Co-efficients using C.G-Th and GCIF.

Take again $f(z) = \frac{\sin(2z)}{z^4}$ in $0<|z|<\infty$: & $C: z=e^{it}, t\in[0,2\pi]$

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^{n+1}}\,dz = \frac{1}{2\pi i}\oint_C \frac{\sin(2z)}{z^{n+5}}\,dz$$

$$= \frac{1}{(n+4)!} \times \left((n+4)^{th} \text{ derivative of } \sin(2z)\right)\Big|_{z=0}$$

$$= \begin{cases} (-1)^k \dfrac{2^{(2k+1)+4}}{((2k+1)+4)!} \cos(2z)\Big|_{z=0} & ; n=2k+1 \\ (-1)^k \dfrac{2^{2k+4}}{(2k+4)!} \sin(2z)\Big|_{z=0} & ; n=2k \end{cases}$$

$$= \begin{cases} (-1)^k \dfrac{2^{2k+5}}{(2k+5)!} & ; n=2k+1 \\ 0 & ; n=2k \end{cases}.$$

$$b_n = \frac{1}{2\pi i}\oint_C \frac{\sin(2z)}{z^{-n+5}}\,dz = \frac{1}{2\pi i}\oint_C \sin(2z)z^{n-5}\,dz$$

for $n \geq 5$, $b_n = 0$ as $\sin(2z)z^{n-5}$ would be analytic everywhere! (C.G.Th)

$$b_1 = \frac{1}{2\pi i}\oint_C \frac{\sin(2z)}{z^4}\,dz = +\frac{1}{2\pi i} \times \frac{-8\pi i}{3} = -\frac{8}{6} \quad \text{(From Lec 19)}$$

$$b_3 = \frac{1}{2\pi i}\oint_C \frac{\sin(2z)}{z^2}\,dz = \sin'(2z)\big|_{z=0} = 1 \quad \text{(G.C.I.F)}$$

$b_2 = b_4 = 0$! $\therefore$ $\frac{\sin(2z)}{z^4} = \frac{1}{z^3} - \frac{8}{6z} + \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+5}}{(2k+5)!} z^{2k+1}$.

Example: $f(z) = \frac{e^z}{z}$; $0 < |z| < \infty$ $\quad C = e^{it} : t \in [0, 2\pi]$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}}\,dz = \frac{1}{2\pi i} \oint \frac{e^z}{z^{n+2}}\,dz \quad \text{) GCIF}$$

$$= \frac{1}{(n+1)!}\left((n+1)^{th} \text{ derivative of } e^z\right)\Big|_{z=0}$$

$$= \frac{1}{(n+1)!}$$

$$b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{-n+1}}\,dz = \frac{1}{2\pi i} \oint_C \frac{e^z}{z^{-n+2}}\,dz$$

$$= \begin{cases} 0, & n \geq 2 \text{ (as } e^z z^{-2+n} \text{ is analytic everywhere)} \\ e^0 = 1, & n = 1 \text{ (CIF)} \end{cases}$$

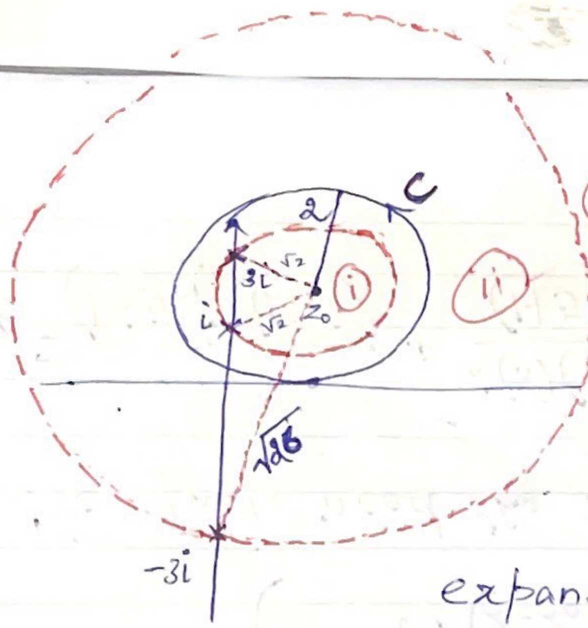
$$\therefore \frac{e^z}{z} = \frac{1}{z} + 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{z^{n-1}}{n!}$$

Lec 19: $\displaystyle\oint_C \frac{z+3}{(z^2+9)(z-i)^3}\,dz = -\frac{\pi}{64}(1+i)$ $\quad C : |z - (1+2i)| = 2$

How many different Laurent expansions are there for $f(z) = \frac{z+3}{(z^2+9)(z-i)^3}$ centered at $z_0 = 1+2i$

Singularities at $\pm 3i$, i



(i) $0 < |z-(1+2i)| < \sqrt{2}$

(ii) $\sqrt{2} < |z-(1+2i)| < \sqrt{26}$

(iii) $\sqrt{26} < |z-(1+2i)| < \infty$

Three different Laurent expansions for $f(z) = \dfrac{z+3}{(z^2+9)(z-i)^3}$ one each in (i), (ii) & (iii) centered at $z_0 = 1+2i$

Now, if one were to calculate

$$\oint_C \frac{z+3}{(z^2+9)(z-i)^3}\,dz$$ using Laurent series

One must choose the appropriate centre and annular domain in which C lies as positively oriented simple closed curve.

So, here $C: |z-(1+2i)| = 2$ lies in (ii)

So, use Laurent series of f valid in (ii)!

[Partial Fraction]

$$\frac{z+3}{(z^2+9)(z-i)^3} = \frac{1}{2}\left[\frac{1+i}{(z+3i)(z-i)^3} - \frac{1-i}{(z-3i)(z-i)^3}\right] \rightarrow \bigstar$$

$$\frac{1}{(z+3i)(z-i)^3} = \frac{1}{(z-(1+2i)+1+5i)\,(z-(1+2i)+1+i)^3}$$

Let $z_0 = 1+2i$

$$\therefore \frac{1}{(z+3i)(z-i)^3} = \frac{1}{(1+5i)(z-z_0)^3\left[1+\frac{z-z_0}{1+5i}\right]\left[1+\frac{1+i}{z-z_0}\right]^3}$$

Since C lies in domain (II)

(i.e) $\sqrt{2} < |z-z_0| < \sqrt{26}$

$$\left|\frac{z-z_0}{1+5i}\right| < 1 \text{ and } \left|\frac{1+i}{z-z_0}\right| < 1$$

$$\therefore \frac{1}{(z+3i)(z-i)^3} = \frac{1}{(1+5i)(z-z_0)^3}\left(1-\left(\frac{z-z_0}{1+5i}\right)+\left(\frac{z-z_0}{1+5i}\right)^2-\left(\frac{z-z_0}{1+5i}\right)^3+\cdots\right)$$
$$\frac{1}{2}\left(2-2(3)\left(\frac{1+i}{z-z_0}\right)+3(4)\left(\frac{1+i}{z-z_0}\right)^2-\cdots\right)$$

$$= \frac{1}{(1+5i)(2)}\left(\frac{1}{(z-z_0)^3}-\frac{(z-z_0)^{-2}}{(1+5i)}+\frac{(z-z_0)^{-1}}{(1+5i)^2}-\frac{1}{(1+5i)^3}+\frac{(z-z_0)}{(1+5i)^4}-\cdots\right)$$
$$\left(2-2(3)\left(\frac{1+i}{z-z_0}\right)+3(4)\left(\frac{1+i}{z-z_0}\right)^2-\cdots\right)$$

We just need the co-efficient of $(z-z_0)^{-1}$.

$$\therefore b_1 = \frac{1}{(1+5i)\cancel{(2)}}\left(\frac{2}{(1+5i)^2}+\frac{2(3)(1+i)}{(1+5i)^3}+\frac{3(4)(1+i)^2}{(1+5i)^4}+\cdots\right)$$

$$= \frac{1}{(1+5i)^3(2)}\left(2 + 2(3)\left(\frac{1+i}{1+5i}\right) + 3(4)\left(\frac{1+i}{1+5i}\right)^2 + \cdots\right)$$

$$= \frac{1}{(1+5i)^3}\left(\frac{1}{\left(1-\frac{1+i}{1+5i}\right)^3}\right) = \frac{1}{(4i)^3} = -\frac{1}{64i}$$

Co-efficient of $(z-z_0)^{-1}$ in Laurent expansion of $\frac{1}{(z-3i)(z-i)^3}$ would be zero because it would only have negative powers (i.e) $(z-z_0)^{-n}$ for $n \geq 3$!

Refer back to (*),

we get that the co-efficient of $(z-z_0)^{-1}$ in Laurent expansion of $f(z)$ at $z = z_0 = 1+2i$ is

$$b_1 = \frac{1+i}{2}\left(\frac{-1}{64i}\right)$$

$$\therefore \oint_C \frac{z+3}{(z^2+9)(z-i)^3}\,dz = 2\pi i\, b_1$$

$$= -\frac{\pi}{64}(1+i).$$