Oct 16, 2023.

Recall Stokes' Theorem

Suppose that Σ is a(n open) surface with closed boundary curve $\partial\Sigma$, with Σ and $\partial\Sigma$ oriented according to the right hand rule, and $\vec{F}$ is C^1 on $\Sigma \cup \partial\Sigma$. Then,

$$\iint_{\Sigma} (\nabla \times \vec{F}) \cdot \hat{n} \, dS = \oint_{\partial\Sigma} \vec{F} \cdot d\vec{x}$$

~~Exercise~~ Implication of Stokes':

Suppose that $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$. The following three statements are equivalent.

1) $\vec{F}$ is conservative (i.e ϕ exists s.t $\vec{F} = \nabla\phi$)
2) $\vec{F}$ is irrotational (i.e $\nabla \times \vec{F} = \vec{0}$)
3) For any closed curve in $\mathbb{R}^3$,

$$\oint_C \vec{F} \cdot d\vec{x} = 0$$

Proof:

$3) \not\Rightarrow 1)$ is beyond the scope!

$1) \Rightarrow 2)$: PS 4 #10a) curl of a gradient identity says that

$$\nabla \times \vec{F} = \nabla \times (\nabla\phi) = \vec{0}$$

$2) \Rightarrow 3)$: Apply Stokes' to get $\oint_C \vec{F} \cdot d\vec{x} = \iint_{\Sigma} (\nabla \times \vec{F}) \cdot \hat{n} \, dS = 0$.

(Here, $C = \partial\Sigma$)

Vector Potential and Surface Independence

PS 4 #10 b) proves that

$$\nabla\cdot(\nabla\times\vec{F}) = 0 \quad \forall \vec{F} \text{ that are } C^2$$

(divergence of a curl identity)

It turns out that the converse is also true.

Thm.
If a vector field $\vec{F}$ satisfies $\nabla\cdot\vec{F} = 0$, then there exists another vector field $\vec{A}$ such that $\vec{F} = \nabla\times\vec{A}$.

$\vec{A}$ is called ~~the~~ a vector potential for $\vec{F}$

In short,
Irrotational fields possess a scalar potential.
Solenoidal fields possess a vector potential.

(General Theory: If $\vec{F}$ is neither solenoidal nor irrotational, it can decomposed into a gradient and a curl) (under some nice conditions on the region where $\vec{F}$ is considered).

Remark: Vector potentials are not unique.
if $\vec{F} = \nabla\times\vec{A}$, then $\vec{F} = \nabla\times(\vec{A}+\nabla f)$ for any C^2 scalar field $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$.

Advantage of a vector potential

open

Suppose we have the surface Σ with boundary $\partial\Sigma$. Given the flux integral

$\iint_{\Sigma} \vec{F}\cdot\hat{n}\,ds$, one can check if $\nabla\cdot\vec{F}=0$ and

if so, attempt to find $\vec{A}$ such that $\vec{F}=\nabla\times\vec{A}$

Then, we can replace $\vec{F}$ in the surface integral and apply Stokes' Theorem:

$$\iint_{\Sigma} \vec{F}\cdot\hat{n}\,dS = \iint_{\Sigma}(\nabla\times\vec{A})\cdot\hat{n}\,dS = \oint_{\partial\Sigma}\vec{A}\cdot d\vec{x}$$

Key point: Replace a surface integral with line integral

Trade-off: We have to find $\vec{A}$ (which is non-trivial since $\vec{A}=(A_1,A_2,A_3)$ so, we must find all of A_1, A_2, A_3).

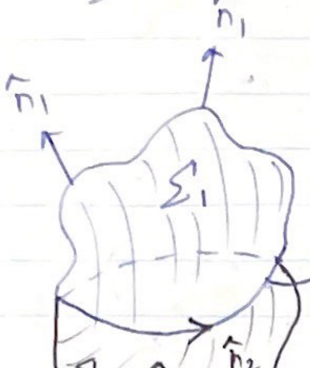
Trick: Due to the fact that $\vec{A}$ is not unique, if you choose $f:\mathbb{R}^3\to\mathbb{R}$ s.t $\frac{\partial f}{\partial z}=-A_3$, we can assume that

$\vec{A}=(A_1,A_2,0)$ by replacing $\vec{A}+\nabla f$ with $\vec{A}$!

For $\vec{F}$ s.t $\nabla \cdot \vec{F} = 0$, additionally, suppose two surfaces Σ_1 and Σ_2 share the same boundary curve $\partial\Sigma$ $(= \partial\Sigma_1 = \partial\Sigma_2)$. Then

$$\iint_{\Sigma_1} \vec{F} \cdot \hat{n}_1 \, dS = \iint_{\Sigma_1} (\nabla \times A) \cdot \hat{n}_1 \, dS$$

↕ Stokes' Thm to Σ_1 with $\hat{n}_1$



$$= \oint_{\partial\Sigma_1 = \partial\Sigma_2 = \partial\Sigma} \vec{A} \cdot d\vec{x}$$

↕ Stokes' Thm to Σ_2 with $\hat{n}_2$

$$= \iint_{\Sigma_2} (\nabla \times \vec{A}) \cdot \hat{n}_2 \, dS = \iint_{\Sigma_2} \vec{F} \cdot \hat{n}_2 \, dS$$

The surface integral of a solenoidal vector field is <u>independent</u> of surface!

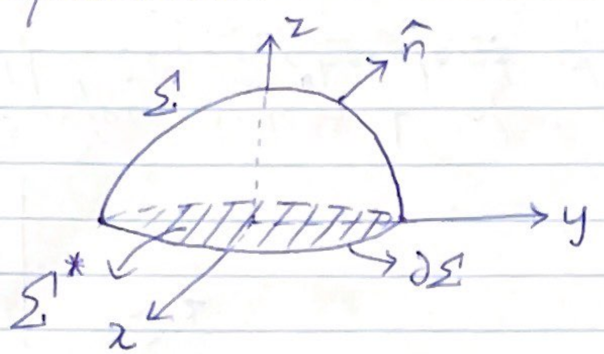
(i.e) Given a surface integral, we can choose another surface that shares its boundary curve and compute the integral over that instead.

<u>Thm</u>: If Σ is a piecewise smooth closed surface in $\mathbb{R}^3$ and $\vec{A}$ is a vector field that is C^1 near Σ then $\iint_{\Sigma} (\nabla \times \vec{A}) \cdot \hat{n} \, dA = 0$. with outward unit normal $\hat{n}$.

Example-1) Using Surface Independence

Let $\vec{F} = (-x, y, 1)$ and define $\Sigma: z = \sqrt{1-x^2-y^2}$.

Compute the flux of $\vec{F}$ upward through Σ by finding a new surface over which to do the computation.



Note that $\partial\Sigma: x^2+y^2 = 1$ and $\nabla\cdot\vec{F} = -1+1$
$= 0$.

(Good idea is to use planar surfaces when you can because $\hat{n}$ is easy to compute and is a constant)

So Choose $\Sigma^*: \{(x,y,z): x^2+y^2 \le 1, z=0\}$

$\hat{n}^*$ for Σ^* is $(0,0,1) \Rightarrow \vec{F}\cdot\hat{n}^* = 1$

$$\iint_{\Sigma} \vec{F}\cdot\hat{n}\,dS = \iint_{\Sigma^*} \vec{F}\cdot\hat{n}^*\,dS = \iint_{\Sigma^*} 1\,dS = \text{Area}(\Sigma^*)$$
$$= \pi$$

(Compare this to parametrization of the hemisphere).

Example 2) When do we need to find vector potential?

Find the portion of the unit sphere's surface through which the flux of $\vec{F} = (-x, -y, 2(z+1))$ has the maximum value.

Soln.

Note that $\nabla \cdot \vec{F} = 0 \Rightarrow \vec{F}$ has a vector potential, say $\vec{A}$ of the form $(A_1, A_2, 0)$

For $\iint_{\Sigma} \vec{F} \cdot \hat{n}\, ds = \iint_{\Sigma} (-x, -y, 2(z+1)) \cdot (x, y, z)\, ds$

$$= \iint_{\Sigma} \left(-x^2 - y^2 + 2z(z+1)\right) ds$$ to

be maximum, we need to find

$$\Sigma = \{(x, y, z) : x^2 + y^2 + z^2 = 1 \text{ and } -x^2 - y^2 + 2(z^2 + z) \geq 0\}$$

Need $\quad -x^2 - y^2 + 2z^2 + 2z \geq 0 \;\cap\; x^2 + y^2 + z^2 = 1$

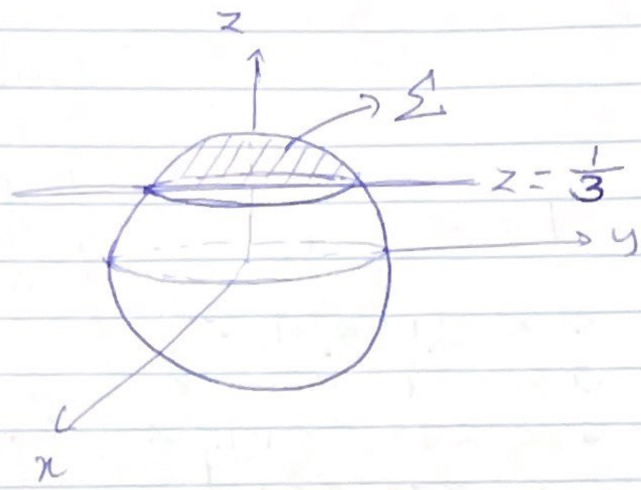
i.e., $\quad z^2 - 1 + 2z^2 + 2z \geq 0$

$3z^2 + 2z - 1 \geq 0$

$(3z - 1)(z + 1) \geq 0$

$\Rightarrow \quad \frac{1}{3} \leq z < 1 \quad$ Discard $z \leq -1$ Not in the surface of unit sphere.

if we find this Σ, then since $-x^2 - y^2 + 2z(z+1) \geq 0$ on this Σ, we are adding up all possible positive flux which will give the maximum.



If we find $\vec{A}$, then the desired value is $\oint_{\partial\Sigma} \vec{A}\cdot d\vec{x}$, where $\partial\Sigma = \{(x^2+y^2+z^2=1) \cap z=\frac{1}{3}\}$

$= \{x^2+y^2=\frac{8}{9},\ z=\frac{1}{3}\}$

Since $\vec{A}=(A_1, A_2, 0)$ & $\nabla\times\vec{A}=\vec{F}$,

$$-\frac{\partial A_2}{\partial z}=F_1,\ \frac{\partial A_1}{\partial z}=F_2,\ \frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}=F_3$$

where $F_1=-x,\ F_2=-y,\ F_3=2(z+1)$

$$-\frac{\partial A_2}{\partial z}=-x \Rightarrow A_2 = xz + C_2(x,y)$$

$$\frac{\partial A_1}{\partial z}=-y \Rightarrow A_1 = -yz + C_1(x,y)$$

$$\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y} = z+\frac{\partial C_2}{\partial x}-\left(-z+\frac{\partial C_1}{\partial y}\right)=2(z+1)$$

$$\Rightarrow \quad \frac{\partial C_2}{\partial x}-\frac{\partial C_1}{\partial y}=2$$

Choose $C_1(x,y) = -2y$ and $C_2(x,y) = 0$

$\Big($ (or) $C_2(x,y) = 2x$ and $C_1(x,y) = 0$

(or) $C_2(x,y) = x$ and $C_1(x,y) = -y$ $\Big)$

$$\vec{A} = (-y(z+2),\ xz,\ 0)$$

$$\partial\Sigma :\ \vec{g}(t) = \left(\sqrt{8/9}\cos t,\ \sqrt{8/9}\sin t,\ 1/3\right)$$

$$0 \le t \le 2\pi$$

$$\oint_{\partial\Sigma} \vec{A}\cdot d\vec{x} = \int_0^{2\pi} \vec{A}(\vec{g}(t))\cdot \vec{g}\,'(t)\,dt$$

$$= \int_0^{2\pi} \left(-\sqrt{8/9}\sin t\left(\frac{7}{3}\right),\ \frac{1}{3}\sqrt{\frac{8}{9}}\cos t,\ 0\right) \cdot \left(-\sqrt{8/9}\sin t,\ \sqrt{8/9}\cos t,\ 0\right) dt$$

$$= \int_0^{2\pi} \left(\frac{7}{3}\left(\frac{8}{9}\right)\sin^2 t + \frac{1}{3}\left(\frac{8}{9}\right)\cos^2 t\right) dt$$

$$= \frac{1}{3}\left(\frac{8}{9}\right)\int_0^{2\pi} (7\sin^2 t + \cos^2 t)\,dt$$

$$= \frac{1}{3}\left(\frac{8}{9}\right)\int_0^{2\pi} (1 + 6\sin^2 t)\,dt$$

$$= \frac{1}{3}\left(\frac{8}{9}\right)\int_0^{2\pi} (1 + 3 - 3\cos 2t)\,dt = \frac{64}{27}\pi$$

Would have been easier if you choose
$C_2(x,y) = x$ and $C_1(x,y) = -y$.

Note that, $\partial\Sigma$ is the curve on which the work done by $\vec{A}$ is maximum!

(One can use the surface independence and calculate the flux through the disc of radius $\sqrt{8/9}$ at $z = \frac{1}{3}$)

$z = \frac{1}{3}$ on this disc.

$$\text{So} \quad \iint_{\substack{\text{Disc} \\ \text{of radius} \\ \sqrt{8/9}}} \vec{F}\cdot(0,0,1)\,dS = \iint \left(-x,-y,\frac{8}{3}\right)\cdot(0,0,1)\,dS$$

$$= \frac{8}{3}\left(\text{Area of Disc of radius } \sqrt{8/9}\right)$$

$$= \frac{8}{3}\pi\left(8/9\right)$$

$$= \frac{64\pi}{27}$$

What if the boundary curve were a non-planar curve? Look at the next example. (Same question with a different field)

Find the portion of the unit sphere's surface with outward normal $\hat{n}=(x,y,z)$ through which the flux of $\vec{F}=(-x,y,1)$ has maximum value.

<u>Soln.</u>

Note that $\nabla\cdot\vec{F}=0$

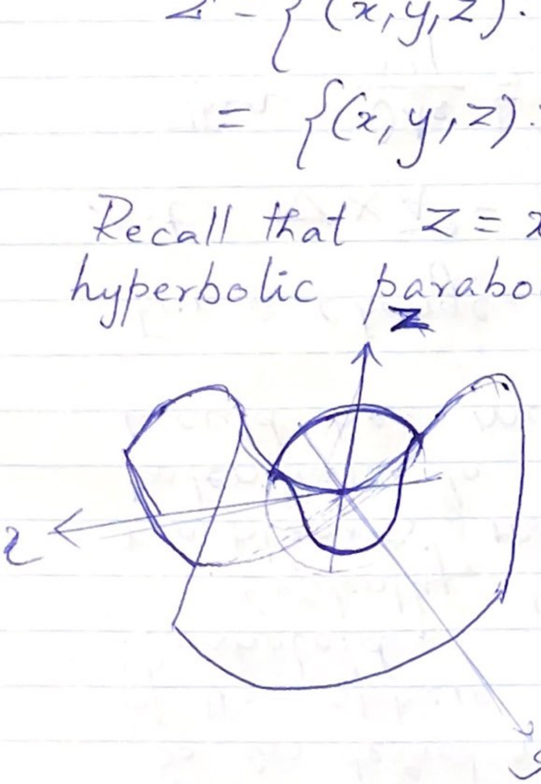
For $\displaystyle\iint_{\Sigma}\vec{F}\cdot\hat{n}\,dS=\iint_{\Sigma}(-x,y,1)\cdot(x,y,z)\,dS$

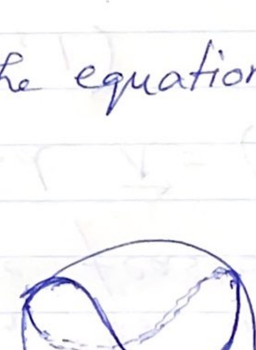
$$=\iint_{\Sigma}(-x^2+y^2+z)\,dS$$ to

be maximum, we need to find

$$\Sigma=\{(x,y,z): x^2+y^2+z^2=1 \text{ and } -x^2+y^2+z\geq 0\}$$
$$=\{(x,y,z): x^2+y^2+z^2=1 \text{ and } z\geq x^2-y^2\}$$

Recall that $z=x^2-y^2$ is the equation of a hyperbolic paraboloid.





So our desired region (or) the surface Σ is part of the sphere that's above the hyperbolic paraboloid.

Evidently, the boundary curve $\partial\Sigma$ is non-planar, maybe finding ~~the~~ vector potential to solve for the desired value would be much more feasible.

For a change, let's find $\vec{A} = (A_1, 0, A_3)$

s.t $\nabla \times \vec{A} = \vec{F} = (-x, y, 1)$

Then $\dfrac{\partial A_3}{\partial y} = -x$, $\dfrac{\partial A_1}{\partial z} - \dfrac{\partial A_3}{\partial x} = y$, $\dfrac{\partial A_1}{\partial y} = -1$

$\Rightarrow$ $A_3 = -xy + C_3(x, z)$ & $A_1 = -y + C_1(x, z)$

$$\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} = \frac{\partial C_1}{\partial z} + y - \frac{\partial C_3}{\partial x} = y$$

$\Rightarrow$ $C_1 = C_3 = 0$ works!

$\therefore$ $\vec{A} = (-y, 0, -xy)$ works.

<u>Now find $\partial\Sigma$:</u>

$$\partial\Sigma = \{x^2+y^2+z^2=1\} \cap \{z = x^2-y^2\}$$

Let's solve this in polar co-ordinates

$$\partial\Sigma = \{ r^2 + z^2 = 1 \} \cap \{ z = r^2\cos^2\theta - r^2\sin^2\theta \}$$
$$= \{ r^2 + z^2 = 1 \} \cap \{ z = r^2 \cos 2\theta \}$$

Sub $z = r^2 \cos 2\theta$ in $r^2 + z^2 = 1$

$$r^2 + r^4\cos^2 2\theta - 1 = 0$$

$$r^4 \cos^2 2\theta + r^2 - 1 = 0 \quad \text{(Quadratic in } r^2\text{)}$$

$$\Rightarrow \quad r^2 = \frac{-1 + \sqrt{1 + 4\cos^2 2\theta}}{2\cos^2 2\theta} \quad \text{(Disregarding -ve value)}$$

$$= \frac{2}{\sqrt{1 + 4\cos^2 2\theta} + 1} \quad \left(\text{Multiply \& divide by } \sqrt{1+4\cos^2 2\theta} + 1\right)$$

$$\Rightarrow \quad r = \sqrt{\frac{2}{\sqrt{1 + 4\cos^2 2\theta} + 1}} = r(\theta) \text{ (Call)}$$

$\therefore$

$$x = r(\theta)\cos\theta$$
$$y = r(\theta)\sin\theta$$
$$z = r^2(\theta)\cos^2\theta - r^2(\theta)\sin^2(\theta) \quad [z = x^2 - y^2]$$
$$= r^2(\theta)\cos 2\theta \quad , \; 0 \leq \theta \leq 2\pi \quad \text{is}$$

a parametrization of $\partial\Sigma$.

Solving further is tough, nevertheless it is atleast tangible and concrete to move further.