MATH 137 LEC 010 (Mani Thamizhazhagan)
Week 5 : Oct 3 – 7, Lecture Summary & Notes

Monday: Read through Section 2.7 Pages 86 – 102

In particular, we covered a fundamental log limit namely $\lim_{x\to\infty} \frac{\ln x}{x} = 0$.

This emphasize that the growth of $\ln$ is much slower than that of any polynomial. Further, growth of any polynomial is much slower than that of exponential e^x.

More notes about rational functions on page 92.

for a rational function $f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + \dots + a_1 x + a_0}{b_m x^m + \dots + b_1 x + b_0}$,

$$\lim_{x\to\pm\infty} f(x) = \begin{cases} \text{DNE}, & \text{if } n > m \\ 0, & \text{if } n < m \\ \frac{a_n}{b_n}, & \text{if } n = m \end{cases}$$

Exercise: Show the above for $n<m$ and $n=m$.

Now, we will show why $\lim_{x\to\pm\infty} f(x)$ DNE if $n>m$.

Rewrite $f(x) = \frac{x^{n-m}\left(a_n x^m + \dots + a_1 x^{m-n+1} + a_0 x^{m-n}\right)}{\left(b_m x^m + \dots + b_1 x + b_0\right)}$

Convince yourself of why $\lim_{x\to\infty} \frac{a_n x^m + \dots + a_1 x^{m-n+1} + a_0 x^{m-n}}{b_m x^m + \dots + b_1 x + b_0}$

$$= \frac{a_n}{b_m} \,!$$

Assume $\frac{a_n}{b_m} > 0$!

Now, since $\lim\limits_{x\to\infty} \underbrace{\frac{a_n x^m + \cdots + a_1 x^{m-n+1} + a_0 x^{m-n}}{b_m x^m + \cdots + b_1 x + b_0}}_{\text{Call this } g(x)} = \frac{a_n}{b_m}$,

for $\varepsilon = \frac{1}{2}\frac{a_n}{b_m}$, we can find $N_1 \in \mathbb{R}$ s.t

if $x > N_1$, then $\left|\frac{a_n x^m + \cdots + a_1 x^{m-n+1} + a_0 x^{m-n}}{b_m x^m + \cdots + b_1 x + b_0} - \frac{a_n}{b_m}\right| < \varepsilon$

(i.e)

$$x > N_1 \overset{(\text{i.e})}{\Rightarrow} \left|g(x) - \frac{a_n}{b_m}\right| < \frac{1}{2}\frac{a_n}{b_m}$$

$$\Rightarrow \frac{a_n}{2b_m} < g(x) < \frac{3}{2}\frac{a_n}{b_m}.$$

Now $f(x) = x^{n-m} g(x)$, $n > m$.

To show, $\lim\limits_{x\to\infty} x^{n-m} g(x) = \infty$, let $M > 0$.

Choose $N = \max\left\{N_1, \left(\frac{2b_m}{a_n}M\right)^{1/n-m}\right\}$.

Then for $x > N$, we have

$$f(x) = \underbrace{x^{n-m} g(x) > \frac{a_n}{2b_m} x^{n-m}}_{\substack{\text{Since} \\ x > N_1}} > \left(\frac{a_n}{2b_m}\right)\left(\frac{2b_m}{a_n}M\right)^{\frac{n-m}{n-m}} = M.$$

Aside

$$x^{n-m} g(x) > \frac{a_n}{2b_m} x^{n-m} > M$$

$$\Rightarrow x > \left(\frac{2b_m M}{a_n}\right)^{1/n-m}$$

$\therefore \lim\limits_{x\to\infty} f(x) = \infty$ if $n > m$ and $\frac{a_n}{b_m} > 0$.

Wednesday ~~Tuesday~~: Read Section 2.8 pages 103–104 &
pages 108 – until Example 26.

Friday: Read Section 2.8.1 pages 105–107,
Section 2.8.3 pages 110 – 113.

A paramount importance is to understand that continuity is a "local" concept.

A function $(f: \mathbb{R} \to \mathbb{R})$ is said to be continuous if f is cts. at $\underbrace{\text{each} \quad x = a \in \mathbb{R}}_{\text{"Local"}}$

Now we will define exponential function from the scratch and show it is continuous.

For all $x \in \mathbb{R}$, define $\exp(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$

The following inequality will be used frequently

Bernoulli's "$\leq$" (BI) : For $y > -1$, $y \in \mathbb{R}$,

$$(1+y)^n \geq 1+ny \text{ for all } n \in \mathbb{N}.$$

(Prove this using PMI)
Induction.

For each $x \in \mathbb{R}$, first we have to ensure that $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$ exists. So fix $x \in \mathbb{R}$ and

let $e_n(x) = \left(1+\frac{x}{n}\right)^n$.

Then $\dfrac{e_{n+1}(x)}{e_n(x)} = \dfrac{\left(1+\frac{x}{n+1}\right)^{n+1}}{\left(1+\frac{x}{n}\right)^{n}}$

$$= \left(\frac{1+\frac{x}{n+1}}{1+\frac{x}{n}}\right)^{n+1}\left(1+\frac{x}{n}\right)$$

$$= \left(\frac{(n+1+x)n}{(n+1)(n+x)}\right)^{n+1}\left(1+\frac{x}{n}\right)$$

$$= \left(1-\frac{x}{(n+1)(n+x)}\right)^{n+1}\left(1+\frac{x}{n}\right)$$

For $n>|x|$, $\dfrac{-x}{(n+1)(n+x)} > -1$. So apply BI with $y = \dfrac{-x}{(n+1)(n+x)}$ to get,

$$\frac{e_{n+1}(x)}{e_n(x)} = \left(1-\frac{x}{(n+1)(n+x)}\right)^{n+1}\left(1+\frac{x}{n}\right) \geq \left(1-\frac{x}{n+x}\right)\left(1+\frac{x}{n}\right) = 1.$$

$\Rightarrow e_{n+1}(x) \geq e_n(x)$ for all $n>|x|$.

So $e_n(x)$ is eventually non-decreasing sequence for a given $x\in\mathbb{R}$.

Now, if $x<0$, $0<e_n(x)<1$ for all $n>|x|$.

Let's show $e_n(x)$ is bounded above for $x>0$ as well.

When $x=1$,

$$e_n(x) = \left(1+\frac{1}{n}\right)^n, \text{ by binomial theorem,}$$

$$= \binom{n}{0} + \binom{n}{1}\left(\frac{1}{n}\right) + \binom{n}{2}\left(\frac{1}{n}\right)^2 + \cdots + \binom{n}{n-1}\left(\frac{1}{n}\right)^{n-1} + \binom{n}{n}\left(\frac{1}{n}\right)^n$$

$$= 1+1+\frac{\left(1-\frac{1}{n}\right)}{2!} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3!} + \cdots + \frac{1}{n^n}.$$

$$\leq 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} \quad \left(\text{as } \left(1-\frac{i}{n}\right)<1 \text{ for } 1\leq i\leq n-1 \text{ and } n!<n^n\right)$$

$$< 1+1+\frac{1}{2!}+\frac{1}{3!}\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-3}}\right)$$

Verify.

$$= 1+1+\frac{1}{2!}+\frac{1}{3!}\left(\frac{1-\left(\frac{1}{2}\right)^{n-2}}{1-\frac{1}{2}}\right)$$

$$< 2+\frac{1}{2!}+\frac{1}{3!}\left(\frac{1}{\left(\frac{1}{2}\right)}\right) \quad \text{as } 1-\left(\frac{1}{2}\right)^{n-2}<1$$

$$< 2+\frac{1}{2}+\frac{1}{3}$$

$$< 3.$$

as $e_n(1)$ is monotonic, $2<e_n(1)<3$ for all n

$\therefore$ By MCT, $\lim_{n\to\infty} e_n(1) = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ exists

The corresponding limit is the real number e

$$e = \exp(1) = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

Now for $m > 1$ in naturals,

$$e_n(m) = \left(1+\frac{m}{n}\right)^n = \left(\frac{n+m}{n}\right)^n$$

$$= \left(\frac{n+1}{n}\cdot\frac{n+2}{n+1}\cdot\frac{n+3}{n+2}\cdots\frac{n+m}{n+m-1}\right)^n$$

$$= \left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n+1}\right)^n\left(1+\frac{1}{n+2}\right)^n\cdots\left(1+\frac{1}{n+m-1}\right)^n \quad \bigstar$$

Each term except the first in the product on the right is of the form

$$\left(1+\frac{1}{n+i}\right)^n \text{ where } i \text{ is a natural independent of } n.$$

Rewrite $\left(1+\frac{1}{n+i}\right)^n = \dfrac{\left(1+\frac{1}{n+i}\right)^{n+i}}{\left(1+\frac{1}{n+i}\right)^{i}}$.

$\left(1+\frac{1}{n+i}\right)^{n+i} \longrightarrow e$ as $n \longrightarrow \infty$ (as this is a subsequence of $\{e_n(1)\}$)

and $\left(1+\frac{1}{n+i}\right)^{i} \longrightarrow 1$ as $n \longrightarrow \infty$.

$$\therefore \lim_{n\to\infty}\left(1+\frac{1}{n+i}\right)^n = e.$$

So in ⊛, since the number of such terms is m, it follows that

$$\exp(m) = \lim_{n\to\infty} e_n(m) = e^m.$$

Now, for $x>0$, find $m \in \mathbb{N}$ s.t $x<m$.

Then $e_n(x) = \left(1+\frac{x}{n}\right)^n < \left(1+\frac{m}{n}\right)^n < e^m$

as $\{e_n(m)\}$ is a increasing seq. eventually.

For $x=0$, $\exp(0)=1$.

$\therefore$ We have shown that for each $x \in \mathbb{R}$, $\{e_n(x)\}$ is eventually monotonically increasing and bounded above. So by MCT, $\lim_{n\to\infty} e_n(x)$ exists and we call it $\exp(x)$. Meanwhile, we have shown also that

$$2 < \exp(1) = e < 3.$$

$$\exp(m) = e^m \text{ for any } m \in \mathbb{N}.$$

For any $k \in \mathbb{N}$, $\left(e_n\left(\frac{1}{k}\right)\right)^k = \left(1+\frac{1}{nk}\right)^{nk}$

$$\Rightarrow \lim_{n\to\infty} \left(e_n\left(\tfrac{1}{k}\right)\right)^k = e$$

$$\Rightarrow \exp\left(\tfrac{1}{k}\right)^k = e$$

$$\therefore \exp\left(\tfrac{1}{k}\right) = e^{1/k}$$

$\therefore \exp(p/q) = e^{p/q}$ for all rational numbers $p/q > 0$.

Now for $x > 0$,

$$e_n(x)\, e_n(-x) = \left(1 - \frac{x^2}{n^2}\right)^n$$

For all $n > |x|$, by BI,

$$\left(1 - \frac{x^2}{n}\right) \le \left(1 - \frac{x^2}{n^2}\right)^n < 1$$

Apply Squeeze thm. to get

$$\exp(x)\exp(-x) = 1.$$

$$\Rightarrow \exp(-x) = \frac{1}{\exp(x)}$$

Also,

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \exp(x) = \frac{1}{\exp(-x)} = \lim_{n\to\infty}\left(1-\frac{x}{n}\right)^{-n}$$

$\Rightarrow \exp(p/q) = e^{p/q}$ for all $p/q < 0$, simultaneously. $p, q \in \mathbb{N}$ as well.

Now, let us show that

$$\exp(x+y) = \exp(x)\cdot\exp(y), \text{ for } x, y \in \mathbb{R}.$$

If $xy = 0$, it is apparent.

Assume $xy > 0$ (for $xy < 0$, the following $\leq$'s will be reversed).

$$\exp(x)\exp(y) = \lim_{n\to\infty} e_n(x)\, e_n(y).$$

$$e_n(x)\,e_n(y) = \left(1+\frac{x}{n}\right)^n\left(1+\frac{y}{n}\right)^n$$

$$= \left(1+\frac{x+y+\frac{xy}{n}}{n}\right)^n$$

Then for all $n \geq N$ for some $N \in \mathbb{N}$, (Choose any N)

$$e_n(x+y) = \left(1+\frac{x+y}{n}\right)^n \leq \left(1+\frac{x+y+\frac{xy}{n}}{n}\right)^n \leq \left(1+\frac{x+y+\frac{xy}{N}}{n}\right)^n$$

as $n\to\infty \implies \underbrace{\exp(x+y) \leq \exp(x)\exp(y)}_{*} \leq \exp\left(x+y+\frac{xy}{N}\right)$

By $*$, (Sneaky) $\exp\left(x+y+\frac{xy}{N}\right) \leq \exp(x+y)\exp\left(\frac{xy}{N}\right)$

So, we have

$$\text{I}: \exp(x+y) \leq \exp(x)\cdot\exp(y) \leq \exp(x+y)\exp\left(\frac{xy}{N}\right)$$

for any $N \in \mathbb{N}$

Note that earlier we showed that

$$e_{n+1}(x) \geq e_n(x) \text{ for all } n > |x|$$

Therefore for $-\frac{1}{2} < x < \frac{1}{2}$,

$$e_{n+1}(x) \geq e_n(x) \text{ for all } n \geq 1.$$

$$\Rightarrow \quad \exp(x) \geq 1 + x = e_1(x) \text{ for } -\tfrac{1}{2} < x < \tfrac{1}{2}$$

Also $\exp(-x) \geq 1 - x$ for $-\frac{1}{2} < x < \frac{1}{2}$.

So if $x \in (-\frac{1}{2}, \frac{1}{2})$, then since $\exp(x)\exp(-x) = 1$,

$$\exp(x) \leq \frac{1}{1-x}.$$

$$\Rightarrow \text{II}: \ 1 + x \leq \exp(x) \leq \frac{1}{1-x} \text{ for } x \in (-\tfrac{1}{2}, \tfrac{1}{2})$$

Now, back to I:

$$1 + \frac{xy}{N} \leq \exp\left(\frac{xy}{N}\right) \leq \frac{1}{1 - \frac{xy}{N}} \text{ for large enough } N.$$

Apply Squeeze thm, to get $\lim_{N\to\infty} \exp\left(\frac{xy}{N}\right) = 1$.

Apply Squeeze thm, let $N\to\infty$ on (I), to get

$$\exp(x+y) \le \exp(x)\exp(y) \le \exp(x+y)$$

$$\Rightarrow \exp(x+y) = \exp(x)\exp(y) \quad \text{for } xy>0.$$

(|||ly do it for $xy<0$ reversing the $\le$'s).

$\therefore$ $\exp(x+y) = \exp(x)\exp(y)$ for all $x,y\in\mathbb{R}$.

Now, we are well-equipped to show that $\exp(x)$ is cts. at any $a\in\mathbb{R}$.

$$\exp(x) - \exp(a) = \exp(a)\left(\exp(x-a) - 1\right)$$

Restrict x s.t $|x-a| < 1/2$

By (II) $\exp(a)(x-a) \le \exp(x) - \exp(a) \le \exp(a)\dfrac{x-a}{1-(x-a)}$

Apply Squeeze thm, to get

$$\lim_{x\to a} \exp(x) = \exp(a).$$

(Or) Choose $\delta = \min\left\{\frac{1}{2}, \frac{\varepsilon}{2\exp(a)}\right\}$ for a given $\varepsilon>0$.

$\left(\text{If } |x-a| < \frac{1}{2}, \text{ then } \dfrac{|x-a|}{|1-(x-a)|} < 2|x-a|\right)$

Thus, exp is a continuous function on $\mathbb{R}$.

A way how the existence of e^x for x an irrational number is done by choosing a rational sequence $\{x_n\}$ that converges to x and define

$$e^x = \lim_{n\to\infty} e^{x_n} \quad \text{(where } e \text{ is an irrational number s.t } 2<e<3\text{)}$$

(This depends on LUB axiom)

Irrespective of what rational sequence you would choose that converges to x,

(i.e) $\lim_{n\to\infty} e^{y_n} = \lim_{n\to\infty} e^{x_n}$ for any $x_n \to x$, $y_n \to x$

So this corresponding unique limit for x an irrational number is defined as e^x.

Now, as exp is cts. and $\exp(p/q) = e^{p/q}$ for any rational p/q,

it follows that $\exp(x) = e^x$ for all $x \in \mathbb{R}$.

One can define $\ln(x)$ as the inverse function of $\exp(x)$, now.

Now read Theorem 12 in page 110.