Oct 20/2023.

Summary :- "Integral of a derivative" Theorems
(Generalizations of Fundamental Thm. of Calculus)

FTC

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Fund. Thm of Line Integral 2D, 3D

$$\int_C \nabla\phi \cdot d\vec{x} = \phi(\vec{b}) - \phi(\vec{a})$$

ϕ - scalar potential for $\vec{F} = \nabla\phi$

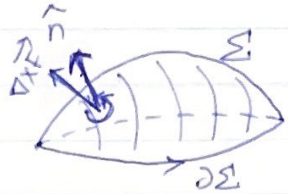
Stokes' Thm 3D

$$\iint_{\Sigma} (\nabla \times \vec{F}) \cdot \hat{n}\, dS = \underbrace{\oint_{\partial\Sigma} \vec{F} \cdot d\vec{x}}_{\text{Circulation of } \vec{F} \text{ around } \partial\Sigma}$$

Special case: Green's theorem

Σ → open surface

$\partial\Sigma$ → Closed boundary curve



$\vec{F}$ is C^1 on $\Sigma \cup \partial\Sigma$

Oriented - Right hand Rule

2D

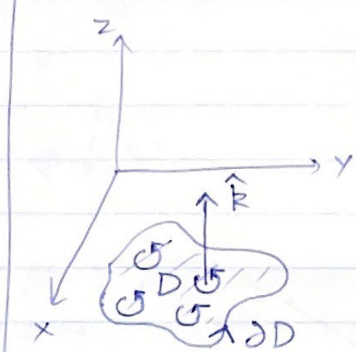
When $\vec{F} = (F_1, F_2, 0)$, $F_1 : \mathbb{R}^2 \to \mathbb{R}$, $F_2 : \mathbb{R}^2 \to \mathbb{R}$

$$\nabla \times \vec{F} = \left(0, 0, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$\hat{n}$ for any ~~curve~~ region D in xy-plane is $\vec{k}$

$\therefore$ Stokes' $\Rightarrow$ Green

$$\iint_D \underbrace{\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)}_{\substack{\text{Circulation density (or)}\\ \text{Vorticity at } (x,y)}} dA = \underbrace{\oint_{\partial D} \vec{F} \cdot d\vec{x}}_{\text{Circulation of } \vec{F} \text{ around } \partial D}$$

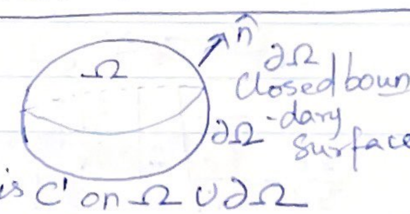


$\vec{F} = (F_1, F_2)$ is C^1 on $D \cup \partial D$

∂D - smooth, simple curve, closed curve oriented CCW

Divergence Thm 3D

$$\iiint_{\Omega} \underbrace{\nabla \cdot \vec{F}}_{\substack{\text{Flux density}\\ \text{at } (x,y,z)}}\, dV = \underbrace{\iint_{\partial\Omega} \vec{F} \cdot \hat{n}\, dS}_{\text{Flux of } \vec{F} \text{ outward through } \partial\Omega}$$



$\vec{F}$ is C^1 on $\Omega \cup \partial\Omega$

$$\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (F_1, F_2, F_3) = \underbrace{\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}_{\text{Scalar value at } (x,y,z)}$$

Scalar field

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix} = \underbrace{\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)}_{\text{Vector at } (x,y,z)}$$

vector field

Parallelism with Potentials for $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$

$\nabla \times \vec{F} = 0$	$\nabla \cdot \vec{F} = 0$
$\Rightarrow \vec{F}$ is conservative (or) irrotational	$\Rightarrow \vec{F}$ is solenoidal (or) incompressible
(i.e) There exists a scalar potential $\phi : \mathbb{R}^3 \to \mathbb{R}$ for $\vec{F}$	there exists a vector potential $\vec{A} : \mathbb{R}^3 \to \mathbb{R}^3$ for $\vec{F}$
(i.e) $\vec{F} = \nabla\phi$	(i.e) $\vec{F} = \nabla \times \vec{A}$
• For C, a curve from $\vec{a}$ to $\vec{b}$	• For Σ, open surface with boundary $\partial\Sigma$ (oriented)
$\int_C \vec{F} \cdot d\vec{x}$ is path independent	$\iint_\Sigma \vec{F} \cdot \hat{n}\, dS$ is surface independent
For any other curve C' from $\vec{a}$ to $\vec{b}$	For any other open Σ' with same boundary $\partial\Sigma$ (with orientation same as above)
$\int_C \vec{F} \cdot d\vec{x} = \int_{C'} \vec{F} \cdot d\vec{x}$ $= \phi(\vec{b}) - \phi(\vec{a})$	$\iint_\Sigma \vec{F} \cdot \hat{n}\, dS = \iint_{\Sigma'} \vec{F} \cdot \hat{n}\, dS$ $= \oint_{\partial\Sigma} \vec{A} \cdot d\vec{x}$
(Curves must share same end-points)	

A Special kind of field:

Suppose that $\vec{F}$ is conservative as well as solenoidal. Then

$\vec{F} = \nabla\phi$ for some ϕ (that is, C^2)
and $\nabla\cdot\vec{F} = 0$ (i.e) $\nabla\cdot(\nabla\phi) = 0$. (**)

Now,
$$\nabla\cdot\nabla\phi = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$
$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

Sub. this (**) gives a PDE

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \quad \text{(Laplace Equation)}$$

Conversely, if a scalar field ϕ obeys the Laplace Equation, then ϕ is the potential of a Conservative vector field.

Notation:
$$\nabla^2 := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$ is the Laplace operator (or Laplacian)

More concisely, (in 2D and 3D), Laplace Equation can be written as $\nabla^2\phi = 0$

Divergence Thm, Stokes' Thm & Conservation Laws

\# One can use Div. Thm. to derive fundamental equations governing:

→ Fluid mechanics
(Continuity eqn. – See Mobius)

→ Electromagnetism

Two Math Results

1) If $F(t) = \int_{x=a}^{x=b} g(x,t)\,dx$ and g is C^1, then

$$F'(t) = \frac{d}{dt}F(t) = \frac{d}{dt}\int_{x=a}^{x=b} g(x,t)\,dx$$

$$= \int_{x=a}^{x=b} \frac{\partial g}{\partial t}(x,t)\,dx$$

Moving the derivative inside the integral, as long as the differentiation and integration variables are different.

2) If $\int_{x_1}^{x_2} f(x) = 0 \;\; \forall x_1, x_2 \in [a,b]$, and f is continuous, then

$$f(x) = 0 \quad \forall x \in [a,b]$$

If an integral is zero regardless of the bounds of integration, the function itself must be zero.

The generalized form of 2) is known as the du Bois-Reymond Lemma.

Both 1) and 2) hold true for higher dimensions ((i.e) they apply to double and triple integrals)

$$\iiint_{\Omega} f \, dV = 0 \quad \forall \Omega \Rightarrow f = 0$$

Often, you will see "Ω is an arbitrary region in $\mathbb{R}^3$"

Deriving Gauss law for electricity

Gauss law : Flux of $\vec{E}$ through a closed surface $= \dfrac{(\text{Charge inside})}{\varepsilon_0}$

Let Ω be an arbitrary region with closed boundary surface $\partial\Omega$

$$\iint_{\partial\Omega} \vec{E} \cdot \hat{n} \, ds = \frac{Q}{\varepsilon_0}$$

$\varepsilon_0 \rightarrow$ permittivity

Here, Q is the total charge inside $\partial\Omega$.

If $\rho(\vec{x}, t)$ denotes the charge density in Ω,

then $\quad Q = \iiint_{\Omega} \rho \, dV$

$$\therefore \quad \iint_{\partial\Omega} \vec{E} \cdot \hat{n} \, ds = \frac{1}{\varepsilon_0} \iiint_{\Omega} \rho \, dV$$

Apply Divergence Thm to LHS: (when $\vec{E}$ is C^1 on $\Omega \cup \partial\Omega$)

$$\iint_{\partial\Omega} \vec{E} \cdot \hat{n}\, dS = \iiint_{\Omega} \nabla \cdot \vec{E}\, dV$$

$$\Rightarrow \iiint_{\Omega} \nabla \cdot \vec{E}\, dV = \frac{1}{\varepsilon_0} \iiint_{\Omega} \rho\, dV$$

$$\Rightarrow \iiint_{\Omega} \left[\nabla \cdot \vec{E} - \frac{\rho}{\varepsilon_0} \right] dV = 0$$

As Ω is arbitrary,

$$\nabla \cdot \vec{E} - \frac{\rho}{\varepsilon_0} = 0$$

(i.e) $\boxed{\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}}$ (ME 1)

Partial differential equation (PDE) for $\vec{E}\ (=(E_1, E_2, E_3))$ each E_i is function of $(\vec{x}, t)$, $\vec{x} \in \mathbb{R}^3$

In what follows, Σ is an arbitrary open surface with closed boundary curve $\partial\Sigma$

Ω is an arbitrary region with closed boundary surface $\partial\Omega$.

Assume $\vec{E}$ and $\vec{B}$, magnetic field are C^1 on wherever we apply Div. Thm (or) Stokes' Thm

Faraday's Law: $$\oint_{\partial\Sigma} \vec{E}\cdot d\vec{x} = -\frac{d}{dt}\iint_{\Sigma} \vec{B}\cdot\hat{n}\, ds$$

Stokes' ↓   ↓ Move the derivative inside

$$\iint_{\Sigma} (\nabla\times\vec{E})\cdot\hat{n}\, ds = -\iint_{\Sigma} \frac{\partial\vec{B}}{\partial t}\cdot\hat{n}\, ds$$

$$\Rightarrow \iint_{\Sigma}\left(\nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t}\right)\cdot\hat{n}\, ds = 0$$

↓ du Bois-Reymond lemma as Σ is arbitrary

(ME 2) $$\nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \quad \text{(or)} \quad \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$\parallel^{ly}$ in PS 5,

(ME 3) $\nabla\cdot\vec{B} = 0$ from $\underbrace{\iint_{\partial\Omega} \vec{B}\cdot\hat{n}\, ds = 0}_{\text{Gauss law for Magnetism}}$ and

(ME 4) $\nabla\times\vec{B} = \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} + \mu_0\vec{J}$ from $$\underbrace{\oint_{\partial\Sigma}\vec{B}\cdot d\vec{x} = \mu_0\varepsilon_0\frac{d}{dt}\iint_{\Sigma}\vec{E}\cdot\hat{n}\, ds + \mu_0\iint_{\Sigma}\vec{J}\cdot\hat{n}\, ds}_{\text{Ampere's Law}}$$

Here $\vec{J}(\vec{x},t)$ is the electric current density at $(\vec{x},t)$.

(ME 4) can be rewritten as

$$\frac{1}{\mu_0 \varepsilon_0} \nabla \times \vec{B} = \frac{\partial \vec{E}}{\partial t} + \frac{\vec{J}}{\varepsilon_0}$$

Let $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$

(ME 4') $$c^2 (\nabla \times \vec{B}) = \frac{\partial \vec{E}}{\partial t} + \frac{\vec{J}}{\varepsilon_0}$$

Now take divergence on ~~the~~ both sides

$$c^2 \; \nabla \cdot (\nabla \times \vec{B}) = \nabla \cdot \frac{\partial \vec{E}}{\partial t} + \frac{\nabla \cdot \vec{J}}{\varepsilon_0}$$

$$0 = \frac{\partial}{\partial t} \nabla \cdot \vec{E} + \frac{\nabla \cdot \vec{J}}{\varepsilon_0}$$

$$0 = \frac{1}{\varepsilon_0} \frac{\partial \rho}{\partial t} + \frac{\nabla \cdot \vec{J}}{\varepsilon_0}$$

$$\Rightarrow \quad \nabla \cdot \vec{J} = - \frac{\partial \rho}{\partial t}$$

In Global form, by Divergence Thm,

$$\iint_{\partial \Omega} \vec{J} \cdot \hat{n} \, dS = - \frac{d}{dt} \iiint_{\Omega} \rho \, dV = - \frac{dQ}{dt}$$

Flux of current from a closed surface $\partial\Omega$ is the decrease of the charge inside $\partial\Omega$ [Conservation of Electric Charge] any flow of charge must come from some supply.

For simplicity, we look at the case with no charges ($\rho = 0$ and no currents ($\vec{J} = 0$))
Then the equations read:

$$\nabla \cdot \vec{E} = 0 \qquad (1)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (2)$$

$$\nabla \cdot \vec{B} = 0 \qquad (3)$$

$$\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad (4)$$

Take curl of (4),

$$\underbrace{\nabla \times (\nabla \times \vec{B})}_{\text{Curl of a Curl identity}} = \nabla \times \left(\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

switch order of "$\nabla \times$" and "$\frac{\partial}{\partial t}$"

$$\underbrace{\nabla(\nabla \cdot \vec{B})}_{=0 \ (3)} - \nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\nabla \times \vec{E})$$

$$\Big\downarrow (2)$$

$$-\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial}{\partial t}\left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$\Rightarrow \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t} - \nabla^2 \vec{B} = 0 \quad \left(\text{as } c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}\right)$$

$$\Rightarrow \boxed{\frac{\partial^2 \vec{B}}{\partial t} = c^2 \nabla^2 \vec{B}}$$

a well-known second order PDE equation called "wave equation"

More importantly, for waves travel with speed c,
This c is nothing but the speed of light (in free space)

Static Phenomena: Suppose that there is no time dependence, i.e., any t derivatives are zero. The equations then read:

$$\left.\begin{aligned} \nabla\cdot\vec{E} &= \frac{\rho}{\varepsilon_0} \quad (1) \\ \nabla\times\vec{E} &= 0 \quad (2) \end{aligned}\right\} \text{Electrostatics}$$

$$\left.\begin{aligned} \nabla\cdot\vec{B} &= 0 \quad (3) \\ \nabla\times\vec{B} &= \mu_0\vec{J} \quad (4) \end{aligned}\right\} \text{Magnetostatics}$$

Notice that the electric and magnetic effects are decoupled.

(1) & (2) Electrostatics

$\nabla\times\vec{E}=0 \Rightarrow \vec{E}$ is conservative

$\exists\, \phi$ s.t $\vec{E}=\nabla\phi$

$$(1) \Rightarrow \nabla\cdot(\nabla\phi) = \frac{\rho}{\varepsilon_0}$$

$$\underbrace{\nabla^2}_{\text{Laplacian operator}}\phi = \frac{\rho}{\varepsilon_0} \quad \left[\text{Poisson Equation}\right]$$

If $\rho=0$, we get $\underbrace{\nabla^2\phi=0}_{\text{Laplace equation}}$

By solving for ϕ, we can find $\vec{E}$ by taking $\vec{E}=\nabla\phi$.

Magnetostatics - (3) & (4)

$\nabla\cdot\vec{B}=0 \Rightarrow$

$\exists\, \vec{A}$ s.t $\vec{B}=\nabla\times\vec{A}$

$$(4) \Rightarrow \nabla\times(\nabla\times\vec{A}) = \mu_0\vec{J} \quad (*)$$

$$\nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} = \nabla\times(\nabla\times\vec{A})$$

(Curl of a curl identity)

$\vec{A}$ is not unique. We can choose $\vec{A}$ s.t

$\nabla\cdot\vec{A}=0$.

Then $(*)$ becomes

$$\nabla^2\vec{A} = -\mu_0\vec{J} \quad \text{(Vector Poisson equation)}$$

$$(\nabla^2 A_1, \nabla^2 A_2, \nabla^2 A_3) = (-\mu_0 J_1, -\mu_0 J_2, -\mu_0 J_3)$$

3 Poisson equations ((or) Laplace if $\vec{J}=\vec{0}$)

Once we know $\vec{A}$, compute $\vec{B}=\nabla\times\vec{A}$