Nov 10, 2023

Derivative of a Complex function

Defn. The derivative of a complex function f at a point z_0 is

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$

(or) equivalently, $f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$

If this limit exists, we say f is differentiable at z_0.

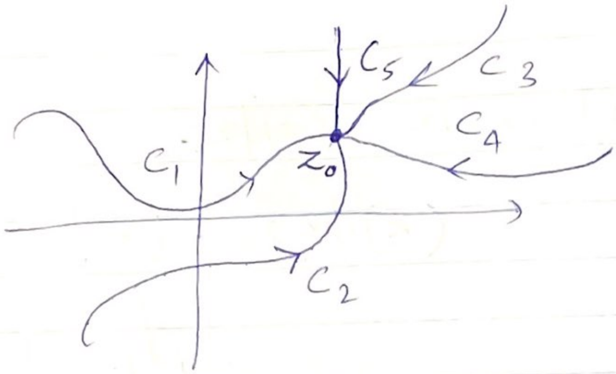
Defn. A complex function f is said to be analytic at a point z_0 if f is differentiable at z_0 and at every point in some neighborhood of z_0.

A function is analytic in a domain D if it is analytic at every point in D.

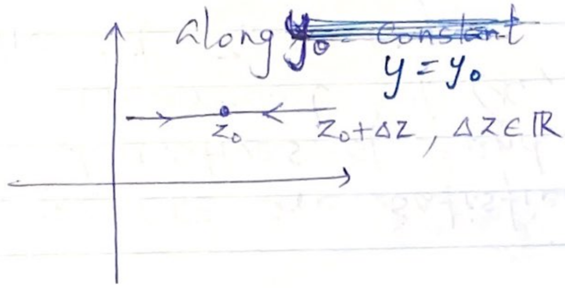
We will see an example of a function that is differentiable at only one point and hence not analytic!

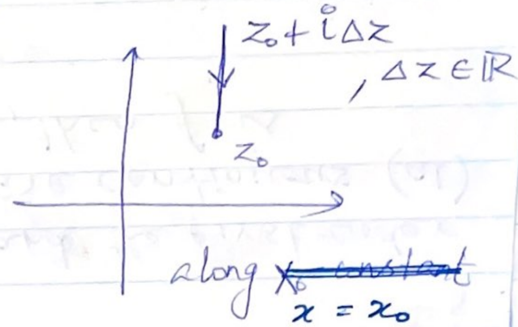
Cauchy-Riemann Equations

In the definition of derivative, there are an infinite number of paths over which z can approach z_0. Some of these are shown below. For the derivative to exist, all paths must give the same value.



We want $\Delta z \to 0$. Let's look at the following special directions $\quad z_0 = x_0 + iy_0$





① along ~~y_0 = constant~~ $y = y_0$

$$f'(z_0) = \lim_{\substack{\Delta z \to 0 \\ \Delta z \in \mathbb{R}}} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\substack{\Delta z \to 0 \\ \Delta z \in \mathbb{R}}} \frac{[u(x_0+\Delta z, y_0) + i\,v(x_0+\Delta z, y_0)] - [u(x_0,y_0) + i\,v(x_0,y_0)]}{\Delta z}$$

$$= u_x(x_0,y_0) + i\,v_x(x_0,y_0) \longrightarrow ⓐ$$

$$\left(= \frac{\partial u}{\partial x}(x_0,y_0) + i\,\frac{\partial v}{\partial x}(x_0,y_0) \right)$$

② along ~~x_0 = constant~~, $x = x_0$,

$$f'(z_0) = \frac{1}{i}\left(u_y(x_0,y_0) + i\,v_y(x_0,y_0)\right) = v_y(x_0,y_0) - i\,u_y(x_0,y_0)$$

$\hookrightarrow$ ⓑ

If $f'(z_0)$ exists, then ⓐ = ⓑ. And we get Cauchy-Riemann Equations (CRE)

Suppose that $f(z) = u(x,y) + i v(x,y)$ and that $f'(z_0)$ exists, where $z_0 = x_0 + iy_0$. Then at z_0, the partial derivatives of u and v must satisfy:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} , \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$u_x = v_y \qquad\qquad v_x = -u_y$$

and $f'(z_0) = u_x + i v_y = v_y - i u_y$.

Thm.

If the CRE are satisfied and the first-order partial derivatives of u and v are continuous (at) in a neighborhood of (x_0, y_0), then f is analytic at $z_0 = x_0 + iy_0$.
(differentiable)

Ex: For each of the following, where is f diff.? analytic? At all those points where f' exists, find f'.

1) $f(z) = \exp(z)$.

$$\exp(z) = \exp(x+iy) = \exp(x)\exp(iy)$$

$$\therefore \exp(z) = \underbrace{\exp(x)\cos y}_{u} + i\,\underbrace{\exp(x)\sin y}_{v} \rightarrow (A)$$

So, $u = e^x \cos y$
$v = e^x \sin y$

Check CRE:

$u_x = e^x \cos y$
$v_y = e^x \cos y$ $\Rightarrow u_x = v_y$

$u_y = -e^x \sin y$
$v_x = e^x \sin y$ $\Rightarrow u_y = -v_x$

Also, all partials are continuous on $\mathbb{R}^2$.

$\Rightarrow f(z) = \exp(z)$ is analytic on all of $\mathbb{C}$

(Such functions are called entire)

Compute $f'(z)$

$$\begin{aligned} f'(z) &= u_x + i v_x \\ &= e^x \cos y + i e^x \sin y \\ &= u + iv \\ &= f(z) \\ &= \exp(z). \end{aligned}$$

2) $f(z) = z^2 = \underbrace{(x^2-y^2)}_{u} + i\,\underbrace{2xy}_{v}$

CRE: $u_x = 2x$, $v_y = 2x$ $\Rightarrow u_x = v_y$ $\qquad$ $u_y = -2y$, $v_x = 2y$ $\Rightarrow u_y = -v_x$

$\Rightarrow$ CRE holds for all x, y

Partials are cts. for all x, y

$\Rightarrow f(z)$ is analytic on $\mathbb{C}$ (entire).

$f'(z) = u_x + i v_x = 2x + i2y = 2(x+iy)$
$= 2z$.

(3) $f(z) = x^2 - iy^3$

$u = x^2 \quad v = -y^3$

CRE:

$u_x = 2x$, $v_y = -3y^2$ $\qquad$ $u_y = 0$, $v_x = 0$ $\Rightarrow u_y = v_x$ for all x, y

$u_x = v_y \Leftrightarrow 2x = -3y^2 \Leftrightarrow x = -\frac{3}{2}y^2$

a parabola that is open left

$\therefore$ CRE holds for all z s.t. $z = x+iy$ with $x = -3/2\, y^2$.

Partials are continuous at above z's but not f is not differentiable in any neighborhood of those z's. Therefore, f is not analytic anywhere.

Let $z \in \{(x+iy) : x = -\frac{3}{2}y^2\}$

$$f'(z) = u_x + i v_x \qquad \text{(or)} \qquad f'(z) = v_y - i u_y$$

$$= 2x \qquad\qquad = -3y^2$$

$$= 2\,\mathrm{Re}\,z \qquad\qquad = -3(\mathrm{Im}\,z)^2$$

List of derivatives:

It can be shown that the following formulas for complex derivatives hold on their domains:
(So analytic on their domains)

$$\frac{d}{dz}(\exp(z)) = \exp(z)$$

$$\frac{d}{dz}(z^n) = nz^{n-1}$$

$$\frac{d}{dz}(\sin z) = \cos z$$

$$\frac{d}{dz}(\cos z) = -\sin z$$

$$\frac{d}{dz}(\log_\alpha(z)) = \frac{d}{dz}(\ln_\alpha(z)) = \frac{1}{z}$$

($\mathbb{C} \setminus L_\alpha$ is domain)

$$\frac{d}{dz}(a^z) = \frac{d}{dz}(\exp(z \log_\alpha a))$$

$$= \log_\alpha a \exp(z \log_\alpha a)$$

$$= a^z \log_\alpha a$$

(Choose $\log_\alpha$ s.t $\log_\alpha a$ is defined)

If $\sinh z = \frac{1}{2}(\exp(z) - \exp(-z))$
$\cosh z = \frac{1}{2}(\exp(z) + \exp(-z))$,

$$\frac{d}{dz}(\sinh z) = \cosh z$$

$$\frac{d}{dz}(\cosh z) = \sinh z$$

Harmonic functions

Defn. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is harmonic if

$$f_{xx} + f_{yy} = 0$$

i.e., if f solves the Laplace equation $\nabla^2 f = 0$.

Thm. If $f = u(x,y) + i\, v(x,y)$ is analytic in a domain D, then both u and v are harmonic in D.

Proof:

To show: $u_{xx} + u_{yy} = 0$ & $V_{xx} + V_{yy} = 0$.

f is analytic $\Rightarrow$ CRE hold

$$\therefore \quad u_x = V_y \rightarrow (1)$$
$$u_y = -V_x \rightarrow (2)$$

Diff. (1) w.r.t x $\quad u_{xx} = V_{yx}$

Diff. (2) w.r.t y $\quad u_{yy} = -V_{xy}$

(It's a fact that u & v are infinitely differentiable because of analyticity).

So $\quad V_{yx} = V_{xy} \Rightarrow u_{xx} + u_{yy}$

$$= V_{yx} - V_{xy}$$
$$= 0.$$

Apply the same argument to $-if$ to get that v is harmonic. $\boxtimes$

For $u: D \to \mathbb{R}$ harmonic,
We call v the harmonic conjugate of u if $f = u + iv$ is analytic on D.
Given a harmonic function, we can find the harmonic conjugate as follows.

Example: Find the harmonic conjugate of $u(x,y) = y^3 - 3x^2 y$. Then find $f(z)$ corresponding to u and v.

Soln. If $v: \mathbb{C} \to \mathbb{R}$ is one such, then by CRE,

$$\begin{cases} v_y(x,y) = u_x = -6xy \\ -v_x = +u_y = 3y^2 - 3x^2 \Rightarrow v_x = 3x^2 - 3y^2 \end{cases}$$

Find v from these equations.
(Recall where similar calculations from before in this course we did: finding scalar potentials)
Partially integrating w.r.t y

$$v(x,y) = \int v_y \, dy = -3xy^2 + g(x) \text{ for some } g.$$

Partially diff v w.r.t x

$$\Rightarrow -3y^2 + g'(x) = 3x^2 - 3y^2 \Rightarrow g(x) = x^3 + C$$

for constant C.

Therefore, $V(x,y) = x^3 - 3xy^2 + C$

$$
\begin{aligned}
f(z) &= u + iv \\
&= y^3 - 3x^2y + i(x^3 - 3xy^2) + iC \\
&= i(z^3 + C)
\end{aligned}
$$