Sep 29, 2023

Last time:

Fundamental theorem of Line Integrals:

If there exists a function ϕ s.t $\vec{F} = \vec{\nabla}\phi$, with C a curve that starts at $\vec{a}$ and ends at $\vec{b}$, then:

$$\int_C \vec{F} \cdot d\vec{x} = \phi(\vec{b}) - \phi(\vec{a})$$

Immediate: If C is a simple, closed curve and $\vec{F} = \vec{\nabla}\phi$ (called conservative fields), then

$$\oint_C \vec{F} \cdot d\vec{x} = 0 \qquad (\text{since } \vec{a} = \vec{b} \text{ here})$$

Also, in $\mathbb{R}^2$, if $\vec{F} = \vec{\nabla}\phi$, then $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$

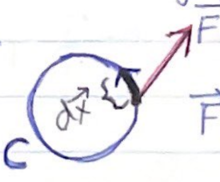
in $\mathbb{R}^3$, if $\vec{F} = (F_1, F_2, F_3) = \underbrace{\left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right)}_{\vec{\nabla}\phi}$, then

$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}, \quad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$$

Now, for general fields $\vec{F}$,

$\oint_C \vec{F} \cdot d\vec{x}$ can be intrepreted as the circulation of $\vec{F}$ around C, because this is a measurement of how much the vector field tends to circulate around the curve.

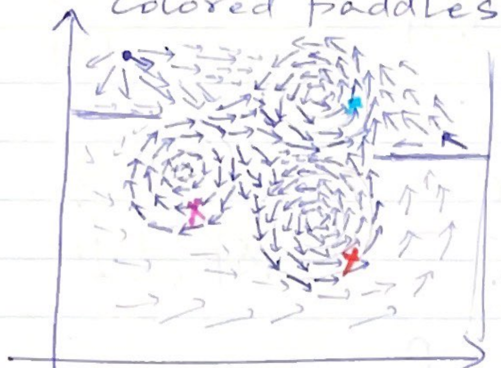
(Think it as Infinitesimal (Contribution of $\vec{F}$ along the tangents to the curve)

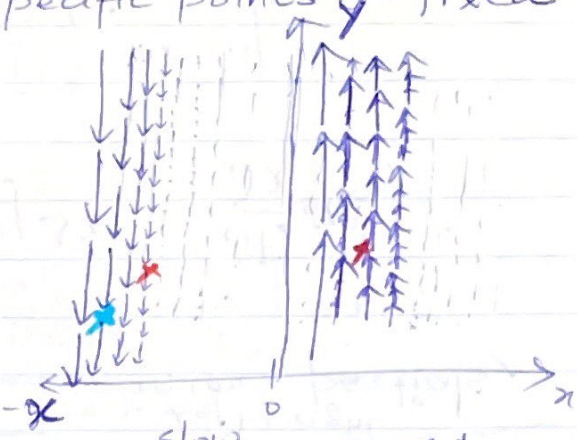


$\vec{F} \cdot d\vec{x}$

$\underline{\vec{F}(\vec{g}(t)) \cdot \vec{g}\,'(t)\, dt}$

Suppose that ~~the~~ vector field looks as below with colored paddles at specific points → fixed





Imagine this as small paddles rotating because of the vector field

× CW
× CCW
× CCW
} local rotations

× slow CCW
× Fast CCW

× CW

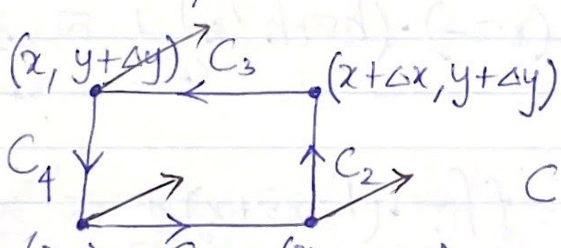
CW – Clockwise

CCW – Counter-clockwise

Now, let's come up with a quantity that measure this spinning at a point (x,y)

Consider an infinitesimal rectangle at (x,y)

$\vec{F} = (F_1, F_2)$



$C = C_1 \cup C_2 \cup C_3 \cup C_4$

We shall calculate the circulation of $\vec{F}$ around C

$$\oint_C \vec{F} \cdot d\vec{x} = \oint_{C_1} \vec{F} \cdot d\vec{x} + \oint_{C_2} \vec{F} \cdot d\vec{x} + \oint_{C_3} \_ + \oint_{C_4} \_$$

Since we are considering an infinitesimal Δx and Δy,

$$\oint_{C_1} \vec{F}\cdot d\vec{x} \simeq \vec{F}(x,y)\cdot \Delta x\, \hat{i} = F_1(x,y)\,\Delta x$$

$$\oint_{C_2} \vec{F}\cdot d\vec{x} \simeq \vec{F}(x+\Delta x,y)\cdot \Delta y\, \hat{j} = F_2(x+\Delta x,y)\,\Delta y$$

$$\oint_{C_3} \vec{F}\cdot d\vec{x} \simeq \vec{F}(x,y+\Delta y)\cdot (-\Delta x)\, \hat{i} = -F_1(x,y+\Delta y)\,\Delta x$$

(Assuming F is continuous, F has almost same value on the top)

$$\oint_{C_4} \vec{F}\cdot d\vec{x} \simeq \vec{F}(x,y)\cdot (-\Delta y)\, \hat{j} = -F_2(x,y)\,\Delta y$$

So
$$\oint_{C} \vec{F}\cdot d\vec{x} \simeq -\overbrace{\left(F_1(x,y)+F_1(x,y+\Delta y)\right)}^{\text{Top + bottom}}\Delta x + \underbrace{\left(F_2(x+\Delta x,y)-F_2(x,y)\right)}_{\text{Left + right}}\Delta y$$

Assuming F has continuous partials,

$$\oint_{C} \vec{F}\cdot d\vec{x} \simeq -\frac{\partial F_1}{\partial y}\Delta y\,\Delta x + \frac{\partial F_2}{\partial x}\Delta x\,\Delta y$$

$$\underbrace{\oint_{C} \vec{F}\cdot d\vec{x}}_{\text{Circulation around a small rectangle } C} \simeq \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\underbrace{\Delta x\,\Delta y}_{\text{Area of } C}$$

Circulation ~~density~~ around a small rectangle C

Now, if we consider the limit:

$$\lim_{\left\{\substack{\text{area of}\\ \text{rectangle } C}\right\}\to 0} \frac{\oint_C \vec{F}\cdot d\vec{x}}{\text{area of } C}$$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\Delta x \Delta y}{\Delta x \Delta y}$$

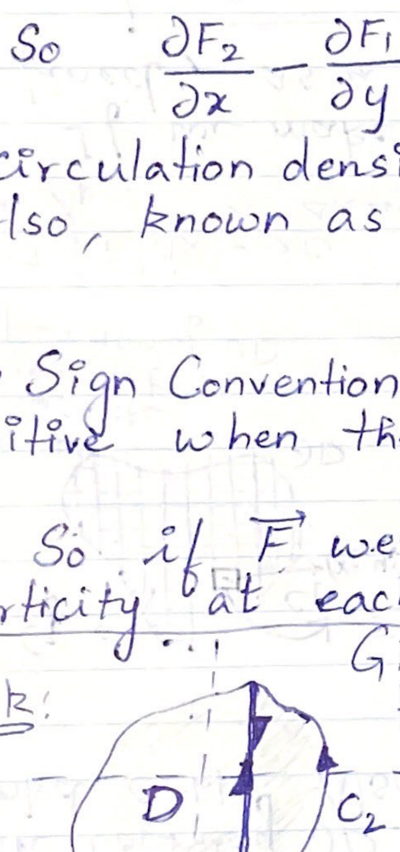
$$= \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

So $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$ at (x,y) is called the circulation density (circulation per unit area). Also, known as vorticity at (x,y).
(from vortex)

- Sign Convention: Circulation/vorticity are positive when there is CCW rotation.

So if $\vec{F}$ were conservative, then the vorticity at each (x,y) is 0.

Remark:



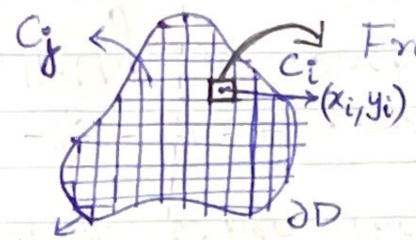
cut D to get C_1, C_2 CCW oriented

Given a region D and $\vec{F}=(F_1,F_2)$
For $\partial D = C_1 \cup C_2$,

$$\oint_{\partial D} \vec{F}\cdot d\vec{x} = \underbrace{\oint_{C_1} \vec{F}\cdot d\vec{x} + \oint_{C_2} \vec{F}\cdot d\vec{x}}_{\text{because the line integral cancels on the cut}}$$

Let's consider a region D in the plane with boundary ∂D, which is a smooth, simple curve oriented CCW. Also assume $\vec{F} = (F_1, F_2)$ is C^1 on $D \cup \partial D$

(Continuous partials of order 1)



From previous calculations,

$$\oint_{\substack{\text{Small}\\ \text{rectangle}\\ C_i}} \vec{F} \cdot d\vec{x} \approx \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)_{(x_i, y_i)} \Delta x \Delta y$$

A_k be the curves that share the portion of ∂D

also,

$$\oint_{\partial D} \vec{F} \cdot d\vec{x} = \sum_{\substack{\text{Small}\\ \text{rectangles}\\ C_i}} \oint_{C_i} \vec{F} \cdot d\vec{x} + \sum_k \oint_{A_k} \vec{F} \cdot d\vec{x}$$

$$\sum_k \oint_{A_k} \vec{F} \cdot d\vec{x} \longrightarrow 0 \quad \text{as } \Delta x \Delta y \to 0$$

If we make the partition more and more refined, as a limiting process, we get

$$\boxed{\oint_{\partial D} \vec{F} \cdot d\vec{x} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)_{(x,y)} \underbrace{\boxed{dx\,dy}}_{dA}}$$

This is $\boxed{\text{Green's Theorem}}$

(Another "integral of a derivative" theorem)

Immediately, this implies that if $\frac{\partial F_2}{\partial x} - \frac{\partial F}{\partial y} = 0$, then the line integral around the closed curve is zero. (Aligns with consequences (properties) of conservative fields)

Applications of Green's Theorem: Example 1: Finding Area of a region.

(Check Problem set 4 #2)

Recall that if $\vec{F} = (F_1, F_2)$, then

$$\oint_C \vec{F} \cdot d\vec{x} = \int_a^b \vec{F}(\vec{g}(t)) \cdot \vec{g}\,'(t)\, dt$$

where $\vec{x} = \vec{g}(t) = (x(t), y(t))$ is a parametrization of the curve C, $a \le t \le b$.

$$= \int_a^b F_1(x(t), y(t))\, x'(t)\, dt + \int_a^b F_2(x(t), y(t))\, y'(t)\, dt$$

$$= \oint_C F_1\, dx + \oint_C F_2\, dy$$

$$\left(\text{Understanding } \vec{F} = \underbrace{(F_1, 0)}_{\vec{\tilde{F}}_1} + \underbrace{(0, F_2)}_{\vec{\tilde{F}}_2} \right)$$

Then Green's theorem implies that,

for $\vec{F} = (-y, x)$ $\quad \tilde{F}_1 = (-y, 0)$ $\quad \tilde{F}_2 = (0, x)$

$$\oint_{\partial D} x\, dy = -\oint_{\partial D} y\, dx = \frac{1}{2} \oint_{\partial D} (x\,dy - y\,dx)$$

$$= \iint_D 1\, dx\, dy = \text{Area}(D).$$

Example 2: Difficult line integrals, hmm?

Let D be the region in the first quadrant bounded by the parabola $y = 2 - x^2 + 2x$, and compute

$$\oint_{\partial D} \vec{F} \cdot d\vec{x}, \quad \text{where } \vec{F} = (\underbrace{xy + 3x^2 e^{x^3} y}_{F_1}, \underbrace{e^{x^3} + y^4}_{F_2})$$

Note that
$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 3x^2 e^{x^3} - (x + 3x^2 e^{x^3})$$
$$= -x$$

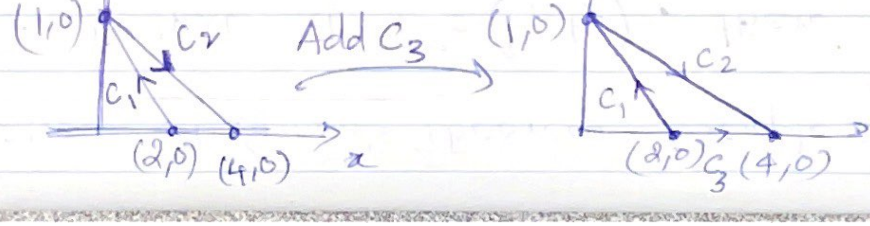
Thus,

$$\oint_{\partial D} \vec{F} \cdot d\vec{x} = -\underbrace{\iint_D x \, dA}$$

This is relatively easy to compute!

Example 3: "Moving the curve".

Let C be the curve consisting of 2 line segments: C_1 – segment from $(2,0)$ to $(0,1)$
C_2 – segment from $(0,1)$ to $(4,0)$



Compute $\int_C \vec{F}\cdot d\vec{x}$ where

$$\vec{F} = \Big(\underbrace{y + \sin(e^{y^2})}_{F_1},\ \underbrace{2xye^{y^2}\cos(e^{y^2})}_{F_2}\Big)$$

Too hard to compute the line integral!

look above

Now let C_3 be the line segment from $(2,0)$ to $(4,0)$ and D be the region inside the triangle with vertices $(2,0)$, $(4,0)$ and $(1,0)$ with ∂D traversed in CCW direction.

Pay attention to the orientation of C_1 and C_3

Then $\partial D = C_3 \cup (-C_2) \cup (-C_1)$

$$\therefore \oint_{\partial D} \vec{F}\cdot d\vec{x} = \int_{C_3} \vec{F}\cdot d\vec{x} - \int_{C_2} \vec{F}\cdot d\vec{x} - \int_{C_1} \vec{F}\cdot d\vec{x}$$

(Because Green's theorem considers CCW circulation along ∂D)

Now, $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} = 2ye^{y^2}\cos(e^{y^2}) - \left(1 + 2ye^{y^2}\cos(e^{y^2})\right)$

$= -1$

By Green's Theorem, $\displaystyle\oint_{\partial D} \vec{F}\cdot d\vec{x} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$

Examble 2: Difficult line integrals, hmm?

$$\therefore \oint_{\partial D} \vec{F} \cdot d\vec{x} = \iint_{D} (-1)\, dx\,dy = -(\text{Area of triangle}) = -1$$

(Check)

$$\Rightarrow + \int_{C_1} \vec{F} \cdot d\vec{x} + \int_{C_2} \vec{F} \cdot d\vec{x} = +1 + \int_{C_3} \vec{F} \cdot d\vec{x}$$