Static Phenomena: Suppose that there is no time dependence, i.e., any t derivatives are zero. The equations then read:

$$\left.\begin{aligned}\nabla\cdot\vec{E} &= \frac{\rho}{\varepsilon_0} \quad (1)\\ \nabla\times\vec{E} &= 0 \quad (2)\end{aligned}\right\} \text{Electrostatics}$$

$$\left.\begin{aligned}\nabla\cdot\vec{B} &= 0 \quad (3)\\ \nabla\times\vec{B} &= \mu_0\vec{J} \quad (4)\end{aligned}\right\} \text{Magnetostatics}$$

Notice that the electric and magnetic effects are decoupled.

(1) & (2) Electrostatics

$\nabla\times\vec{E} = 0 \Rightarrow \vec{E}$ is conservative

$\exists\, \phi$ s.t $\vec{E} = \nabla\phi$

$$(1) \Rightarrow \nabla\cdot(\nabla\phi) = \frac{\rho}{\varepsilon_0}$$

$$\nabla^2\phi = \frac{\rho}{\varepsilon_0} \quad \text{[Poisson Equation]}$$

∇^2: Laplacian operator

If $\rho = 0$, we get $\underbrace{\nabla^2\phi = 0}_{\text{Laplace equation}}$

By solving for ϕ, we can find $\vec{E}$ by taking $\vec{E} = \nabla\phi$.

Magnetostatics - (3) & (4)

$\nabla\cdot\vec{B} = 0 \Rightarrow$

$\exists\, \vec{A}$ s.t $\vec{B} = \nabla\times\vec{A}$

$$(4) \Rightarrow \nabla\times(\nabla\times\vec{A}) = \mu_0\vec{J} \quad (*)$$

$$\nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} = \nabla\times(\nabla\times\vec{A})$$

(Curl of a curl identity)

$\vec{A}$ is not unique. We can choose $\vec{A}$ s.t

$\nabla\cdot\vec{A} = 0$.

Then $(*)$ becomes

$$\nabla^2\vec{A} = -\mu_0\vec{J} \quad \text{(Vector Poisson equation)}$$

$$(\nabla^2 A_1, \nabla^2 A_2, \nabla^2 A_3) = (-\mu_0 J_1, -\mu_0 J_2, -\mu_0 J_3)$$

3 Poisson equations

((or) Laplace if $\vec{J} = \vec{0}$)

Once we know $\vec{A}$, compute $\vec{B} = \nabla\times\vec{A}$

Part-2: Complex Analysis

Analysis on the functions of a complex variable.

- Two important applications

Differentiation

→ Use of conformal mapping to solve boundary-value problems in two-dimensional potential theory (i.e. governed by the 2-D Laplace Equation)

Contour Integration

→ Evaluation of a wide class of definite integrals (even along the real axis)

→ Integral transforms (i.e) Fourier, Laplace transforms

→ Stability of Control Systems. (Poles and zeroes of some complex function)

Remark: The proper setting to study the power series is complex variable theory.

Consider

$$f(x) = \frac{1}{x^2+1}.$$ This function is

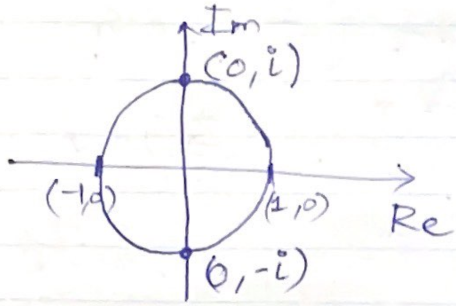
infinitely differentiable and its Taylor series at $0 \in \mathbb{R}$ is

$$\frac{1}{x^2+1} = 1 - x^2 + x^4 - x^6 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

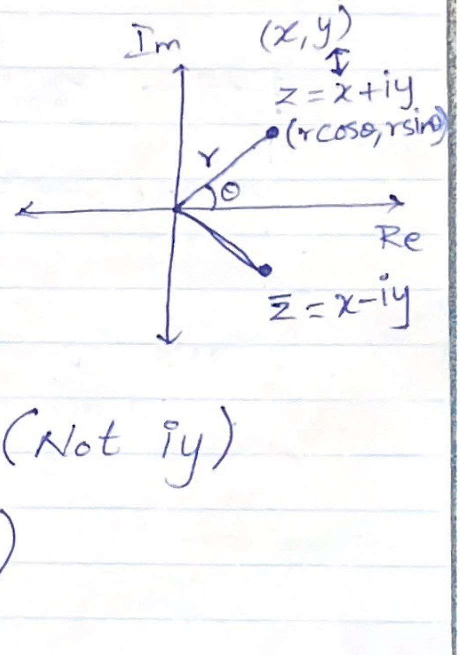
holds only for $|x| < 1$.

There is nothing in the behaviour of f <u>in</u> $\mathbb{R}$ that accounts for this restriction on the radius of convergence. However, if we try to extend the domain of the function to $\mathbb{C}$ (i.e) $f(z) = \frac{1}{1+z^2}$

where $z = x + iy$, $i^2 = -1$, f is well-behaved except at the points $z^2 + 1 = 0 \Rightarrow z = \pm i$



There is a link for review on Complex numbers in Mobius. Please go through.



- Polar form $z = re^{i\theta}$
- Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$
- Conjugate $\bar{z} = x - iy = re^{-i\theta}$
- For $z = x + iy$,

$$\mathrm{Re}(z) = \frac{z + \bar{z}}{2} = x$$

$$\mathrm{Im}(z) = \frac{z - \bar{z}}{2i} = y \quad (\text{Not } iy)$$

$$\sqrt{z\bar{z}} = |z| = \sqrt{x^2 + y^2} \; (= r)$$

$$\cancel{\arg(z)} \quad |z_1 z_2| = |z_1||z_2|$$

$$|z_1 + z_2| \le |z_1| + |z_2|$$

$\{z \in \mathbb{C} : |z - \alpha| = r\}$ describes a circle with center α and radius r,
$(r > 0)$

True (or) False For $z = x+iy \in \mathbb{C}$

1) $|z| > 0$
2) $z\bar{z} \in \mathbb{R}$
3) $Re\left(\frac{z}{\bar{z}}\right) \neq 0$ for $z \neq 0$
4) $-1+i = \sqrt{2}\, e^{i3\pi/4}$
5) $2e^{-i5\pi/6} = 2(-\sqrt{3}-i)$
6) The n distinct roots of the equation $z^n = \alpha = |\alpha| e^{i\theta}$, for $\alpha \neq 0$ in $\mathbb{C}$, are given by

$$|\alpha|^{1/n} e^{i(\theta+2k\pi)/n}, \quad 0 \le k \le n-1$$

7) One of the roots of $z^3 + 8i = 0$ is $-\sqrt{3}-i$.

Solutions:

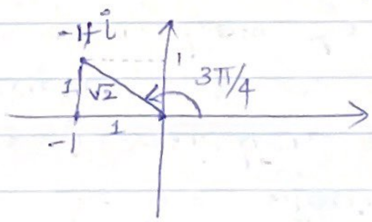
1) False: For $z=0$, $|z|=0$
2) True: $z\bar{z} = |z|^2 = x^2+y^2 \in \mathbb{R}$
3) False:

$$Re\left(\frac{z}{\bar{z}}\right) = \frac{x^2-y^2}{x^2+y^2} = 0 \text{ when } x = \pm y$$

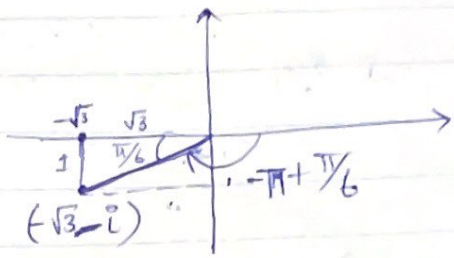
So $Re\left(\frac{z}{\bar{z}}\right)$ can be 0 for $z \neq 0$.

$$\left[\frac{z}{\bar{z}} = \frac{zz}{\bar{z}z} = \frac{z^2}{|z|^2} = \frac{x^2-y^2}{x^2+y^2} + i\frac{2xy}{x^2+y^2} \right.$$

4) True:

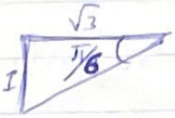


5) False:



$-\sqrt{3}-i = 2e^{-i\frac{5\pi}{6}}$

$\therefore 2(-\sqrt{3}-i) = 4e^{-i\frac{5\pi}{6}}$



$\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$

6) True: distinct

$$z^n = |\alpha| e^{i(\theta + 2k\pi)}$$

$$(z^n)^{1/n} = |\alpha|^{1/n} e^{i\left(\frac{\theta + 2k\pi}{n}\right)} \quad \text{(De-Moivre's Thm)}$$

$$z = |\alpha|^{1/n} e^{i\left(\frac{\theta + 2k\pi}{n}\right)}$$ will be distinct when $0 \le k \le n-1$

Equally spaced roots on the circle of radius $|\alpha|^{1/n}$.

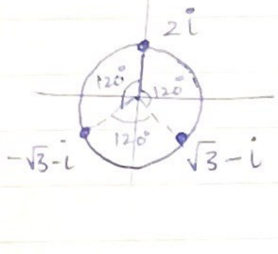
7) True

$$z^3 = -8i = 8e^{-i\pi/2}$$

$\Rightarrow$ the roots are $8^{1/3} e^{-i\left(\frac{\pi}{2} + \frac{2k\pi}{3}\right)}$, $0 \le k \le 2$

$\underbrace{2i}_{k=0}, \underbrace{2e^{-i5\pi/6}}_{k=1}, \underbrace{2e^{-i9\pi/6}}_{k=2}$

$2i, -\sqrt{3}-i, \sqrt{3}+i$



if you had considered $8e^{i3\pi/2}$ (or) roots as $8^{1/3} e^{i\left(\frac{-\pi/2 + 2k\pi}{3}\right)}$ $0 \le k \le 2$

You would get same set of roots!