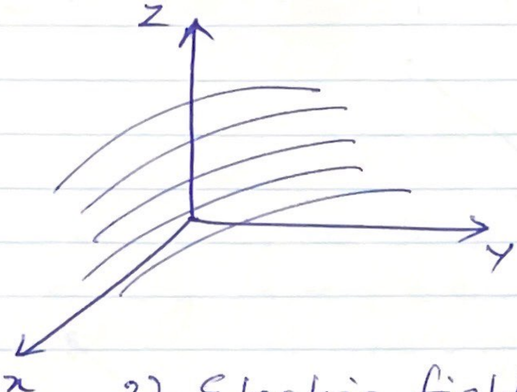
Sep 22, 2023

Vector Fields

Def. A vector field on $\mathbb{R}^n$ is a function $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$

Examples:

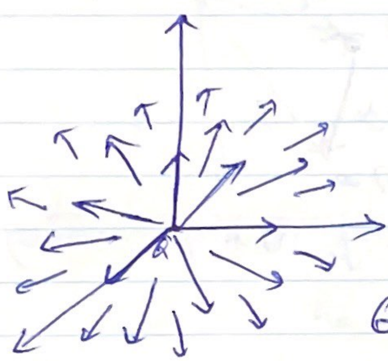
1) Velocity field of a fluid



Ex: $\vec{V}(x,y,z) = (x^2, xy, 3)$ (Steady flow of a fluid)

At any point (x,y,z), describes the velocity.

2) Electric field around a point charge.



The electrostatic field (force per unit charge) at $\vec{r}$ due the presence of (stationary) charge Q at O

$$\vec{E}(\vec{r}) = \frac{Q}{4\pi\varepsilon_0 r^3}\vec{r}$$

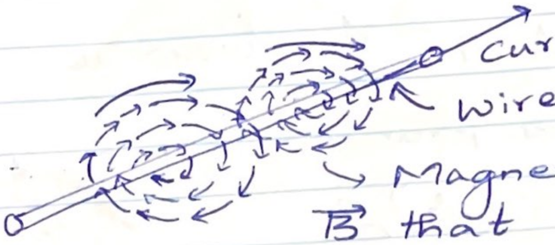
Above we have assumed that $Q > 0$. (convention)

Force on charge q at $\vec{r}$ is

$\vec{F} = q\vec{E}$ If $q < 0$, the $\vec{F}$ is attractive.

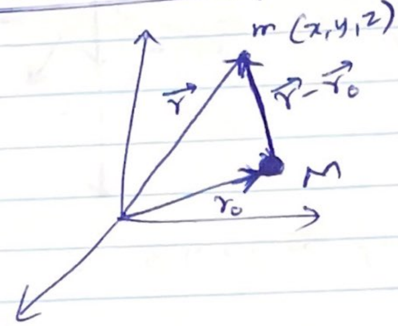
Electromagnetism

A moving charge generates a magnetic field:



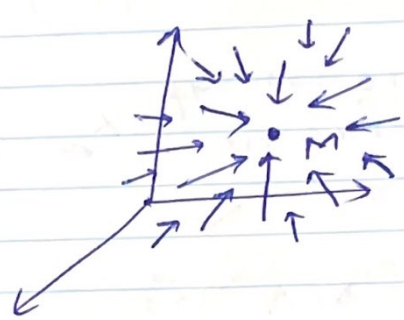
Magnetic field vectors $\vec{B}$ that encircle the current carrying wire.

Gravitational field



Force on mass m at $\vec{r}$ due to presence of M at $\vec{r}_0 = (x_0, y_0, z_0)$

$$\vec{F}(\vec{r}) = -\frac{GMm}{\|\vec{r}-\vec{r}_0\|^3}(\vec{r}-\vec{r}_0)$$



Contrast with scalar fields $f: \mathbb{R}^n \to \mathbb{R}$

For $f: \mathbb{R}^3 \to \mathbb{R}$, $\vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$\vec{\nabla} f$ is a vector field.

Ex: $f(x,y,z) = xy^2 + z$

Gradient drive flows:

eg: f - temperature at (x, y, z), $\vec{\nabla f}$ - heat transfer (Conduction)

f - Concentration of a solute, $\vec{\nabla f}$ - mass transfer (diffusion)

f - pressure (air/water), $\vec{\nabla f}$ - fluid flow

f - electrostatic potential, $\vec{\nabla f}$ - potential gradient (charge flow).

Self-exploration: If $\vec{\nabla f}(x_0, y_0, z_0) \neq 0$, the direction of the gradient is the direction in which the function increases most quickly from (x_0, y_0, z_0), and the magnitude of the gradient is the rate of increase in that direction

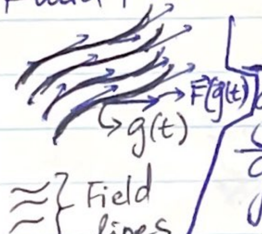
∇ - nabla symbol and pronounced "del".

But $\vec{\nabla f}$ for a scalar field f is called grad f conventionally.

Calculating Field Lines

Defn. The field lines of $\vec{F}$ are defined by the equation:

Fluid flow:



$$\vec{g}'(t) = \vec{F}(\vec{g}(t))$$

$\vec{g}(t)$ - parametrization of a field line

$\vec{g}'(t)$ - tangent vector

$F(g(t))$ - F evaluated at $\vec{g}(t)$, by defn., this is also tangent

Consider $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$, thus $\vec{F} = (p(x,y), q(x,y))$

(For $\vec{g}(t) = (x(t), y(t))$)

~~Let~~ $x = x(t)$, $y = y(t)$ be a parametrization of a field line.

Then
$$x'(t) = p(x(t), y(t))$$
$$y'(t) = q(x(t), y(t))$$

(i.e)
$$\frac{dx}{dt} = p(x,y)$$
$$\frac{dy}{dt} = q(x,y)$$

Using Chain rule, if y were a function of x,

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Thus, we obtain a differential equation for the field lines of $\vec{F}$

$$\boxed{\frac{dy}{dx} = \frac{q(x,y)}{p(x,y)}}$$

<u>Example</u>: Determine the field lines of $\vec{F}(x,y) = (-y, x)$, and sketch the field potrait.

Soln.

Field lines obey DE : $\frac{dy}{dx} = \frac{+x}{-y}$ (separable DE)

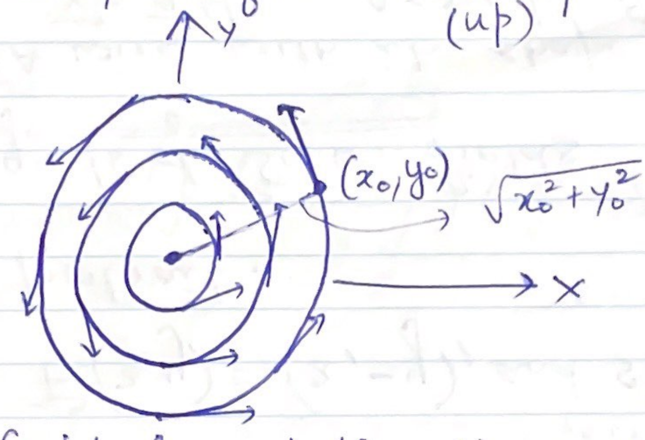
$$\Rightarrow \int y\,dy = -\int x\,dx$$

$$\Rightarrow \frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$$

$$\Rightarrow x^2 + y^2 = D$$ where D is a constant.

Direction:

	I^{st} quadrant		2^{nd} quadrant $(x<0, y>0)$
$(x, y>0)$	X-Component of $\vec{F} = -y < 0$	(left)	$-y<0$ (left)
	Y-Component of $\vec{F} = x > 0$	(up)	$x<0$ (down)



Any field line of the given vector field is a circle centred on the origin. The circles are traversed counterclockwise.

Note that the field line through a given point (x_0, y_0) is a circle of radius $\sqrt{x_0^2 + y_0^2}$.
- specifying a point on the field line fixes the constant value.

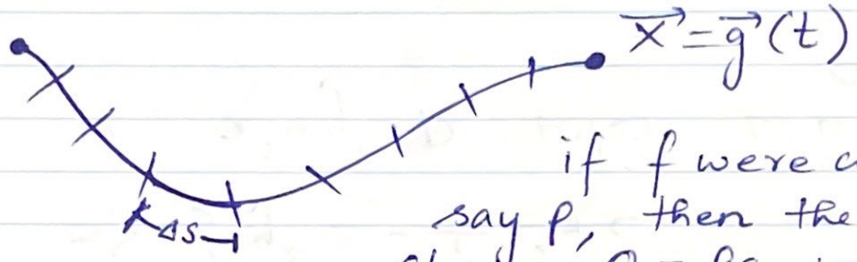
Different vector fields can have same field lines

Find the field lines of $\vec{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ on $\mathbb{R}^2 - \{(0,0)\}$ and sketch the portrait.

Find the field lines of $\vec{F}(x,y) = (x,-y)$, and sketch the portrait.

Line integrals of Scalar fields

Setup: A wire with its shape parametrized by $\vec{x} = \vec{g}(t)$, $a \le t \le b$. Suppose that the charge density on the wire is not a uniform, but instead $f(\vec{x})$. Find total charge



if f were constant, say ρ, then the total Charge $Q = \rho s$, where s is the length of the wire.

On the arc length element Δs, approximate the charge density as being constant, get a charge element $\Delta Q \approx f \Delta s$

Summing up over all subintervals, $Q = \int_C f(\vec{x})\,ds$

Recall that $\Delta s \simeq \|\vec{g}\,'(t)\|\Delta t$.

Writing $f(\vec{x}) = f(\vec{g}(t))$, we get

$$Q = \int_a^b f(\vec{g}(t))\|\vec{g}\,'(t)\|\,dt$$

Defn. The line integral of a scalar field f over the curve $\vec{x} = \vec{g}(t)$, $a \le t \le b$ is defined by

$$\int_C f\,ds = \int_a^b f(\vec{g}(t))\|\vec{g}\,'(t)\|\,dt.$$

Example. Compute $\int_C f(x,y)\,ds$ where $f(x,y) = x^2+y^2$

~~Let~~ 1) C - straight line from $(0,0)$ to $(1,1)$

Let $x(t) = t$ $\quad 0 \le t \le 1$.
$y(t) = t$

$\Rightarrow \vec{g}(t) = (t,t)$ & $\vec{g}\,'(t) = (1,1)$ so that

$\|\vec{g}\,'(t)\| = \sqrt{2} \Rightarrow ds = \sqrt{2}\,dt$.

Evaluating f over the curve

$$f(\vec{g}(t)) = f((t,t)) = t^2+t^2 = 2t^2$$

$$\therefore \int_C f\,ds = \int_0^1 2t^2\sqrt{2}\,dt = 2\sqrt{2}\int_0^1 t^2\,dt = \frac{2\sqrt{2}}{3}$$

2) C - straight line from $(0,0)$ to $(1,1)$ with parametrization

$$x(t) = t^2 \qquad 0 \le t \le 1.$$
$$y(t) = t^2$$

$$\Rightarrow \vec{g}(t) = (t^2,t^2) \quad \& \quad \vec{g}'(t) = (2t,2t)$$

$$\& \quad \|\vec{g}'(t)\| = \sqrt{4t^2+4t^2} = 2\sqrt{2}\,t$$

$$\Rightarrow \quad ds = 2\sqrt{2}\,dt.$$

Also $f(\vec{g}(t)) = f((t^2,t^2)) = 2t^4$

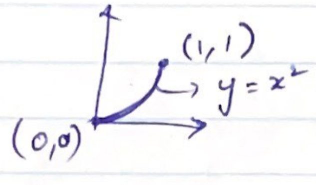
$$\therefore \int_C f\,ds = \int_0^1 2t^4(2\sqrt{2})t = 4\sqrt{2}\int_0^1 t^5\,dt$$

$$= \frac{2\sqrt{2}}{3}.$$

We arrive at the same result!

3) Consider the curve C from $(0,0)$ to $(1,1)$,

$$x(t) = t, \quad y(t) = t^2, \quad 0 \le t \le 1$$



Exercise! $\displaystyle\int_C f\,ds = \frac{349}{768} - \frac{7}{512}\ln\left(1+\frac{\sqrt{5}}{2}\right) - \frac{7}{512}\ln 2$.

Line integrals of vector fields

In engineering science applications, it is more common to meet line integrals in a different form, namely, in the form

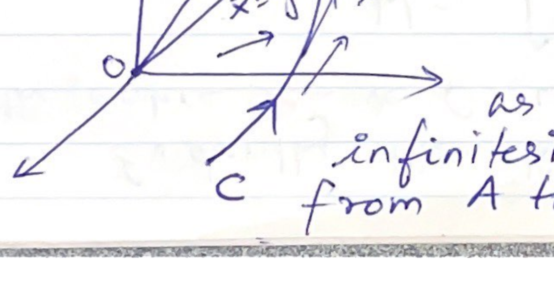
$$\int_C \vec{F} \cdot d\vec{x}$$

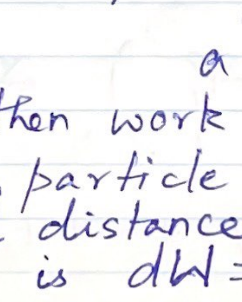
line integral over a curve.

$\cdot$ ← dot product

$\vec{F}$ ← vector field

(or) $\displaystyle\int_C P\,dx + Q\,dy + R\,dz$ where $\vec{F} = (P, Q, R)$





If a particle is subjected to this Force field $\vec{F}$ as it moves along a curve C, then then work dW done by $\vec{F}$, as the particle moves an infinitesimal distance along C, say from A to B is $dW = \vec{F} \cdot \vec{AB} = \vec{F} \cdot d\vec{x}$.

Evidently, then the net work done in traversing the entire curve C is

$$W = \int_C \vec{F}\cdot d\vec{x} \quad \left(\begin{aligned} &\text{if } \vec{x} = \vec{g}(t), \\ &d\vec{x} = \vec{g}(t+dt) - \vec{g}(t) \\ &\qquad \simeq \vec{g}'(t)\,dt \end{aligned} \right)$$

For instance, $\vec{F}(x,y,z)$ might be a gravitational force field induced by a point or distributed systems of mass. Since the curve C is actually traversed by the particle, we sometimes call it a <u>path</u> instead of a curve.

Defn.

The line integral of a vector field $F: \mathbb{R}^3 \to \mathbb{R}^3$ over the curve $C: \vec{x} = \vec{g}(t),\ a \le t \le b$ is defined by

$$\int_C \vec{F}\cdot d\vec{x} = \int_a^b \vec{F}(\vec{g}(t))\cdot \vec{g}'(t)\,dt$$

<u>Example:</u>