MATH 137 LEC 010 (Mani Thamizhazhagan)
Week 6 Oct 24–28 Lecture Summary / Notes
(+ Week MT Oct 21 (Friday))

On Oct 21, Friday : We explored, intuitively, how derivatives can be viewed as instantaneous rates of change of a quantity. Then, to make this statement more precise mathematically, we appealed to the theory of limits that we have been seeing.

Read Chapter 3, Sections 3.1, 3.2 Pages 133–136.

In the same class, we did highlight an important relationship between continuity and differentiability, especially that Differentiability implies Continuity (Theorem 1, Page 139)

Now read subsection 3.2.2 Page 138–141.

Week 6 Monday: We looked at the definition of the derivative function, introduced Leibniz notation and saw higher derivatives' definition. We also computed derivatives of elementary functions using the following equivalence:

If f is diff. at $a \in I$, where $f: I \to \mathbb{R}$ then

$$\underbrace{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}_{①} = f'(a) = \underbrace{\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}}_{②}$$

For example, for $f(x) = x^2$, ① renders easy computation
for $f(x) = \sin x$, ② renders meaningful easy computation

Read Section 3.3, Pages 141 – 147.
& 3.4

A Short Note on the Derivative of e^x

Recall that in our lecture summary of Week 5, we defined the exponential function from the scratch and proved the following properties:

$\exp(x) : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$\exp(x) := \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$ has the

following properties:

i) $\exp(0) = 1$
ii) $e = \exp(1)$ & $2 < e < 3$
iii) $\exp(x) = e^x$ for all $x \in \mathbb{R}$
iv) $\exp(-x) = (\exp(x))^{-1}$ for all $x \in \mathbb{R}$
v) $\exp(x+y) = \exp(x).\exp(y)$ for all $x, y \in \mathbb{R}$
vi) exp is a continuous function

vii) For $|x| < 1$, $1 + x \leq \exp(x) \leq \frac{1}{1-x}$

Using vii), in class on Oct 24, we proved a fundamental exponential limit:

viii) $\lim_{h\to 0} \frac{\exp(h) - 1}{h} = 1 = \lim_{h\to 0} \frac{e^h - 1}{h}$

Note that the above limit is nothing but the derivative of e^x at $x=0$.

Proof of viii)

By vii), if $|h|<1$, then

$$1+h \le exp(h) \le \frac{1}{1-h}$$

$$\Rightarrow \quad h \le exp(h)-1 \le \frac{1}{1-h}-1 = \frac{h}{1-h}$$

when $h \neq 0$,

$$\Rightarrow \quad 1 \le \frac{exp(h)-1}{h} \le \frac{1}{1-h}$$

Now as, $\lim_{h\to 0} 1 = 1 = \lim_{h\to 0} \frac{1}{1-h}$

by Squeeze theorem,

$$\lim_{h\to 0} \frac{exp(h)-1}{h} = 1 = exp(0) = exp'(0).$$

$\blacksquare$

Let's compute $exp'(a)$ for any $a \neq 0$.

$$\begin{aligned} exp'(a) &= \lim_{h\to 0} \frac{exp(a+h)-exp(a)}{h} \\ &= \lim_{h\to 0} \frac{exp(a)exp(h)-exp(a)}{h} \quad (\text{by v)}) \\ &= \lim_{h\to 0} exp(a)\left(\frac{exp(h)-1}{h}\right) \\ &= exp(a) \lim_{h\to 0}\left(\frac{exp(h)-1}{h}\right) = exp(a) \quad (\text{by viii)}) \end{aligned}$$

∴ The derivative function of $\exp(x)$ (or) e^x is e^x (itself).
(Very fundamental)

<u>Week 6 Wednesday</u>: We introduced a simple, yet subtle characterization of the differentiability of a function at "a".

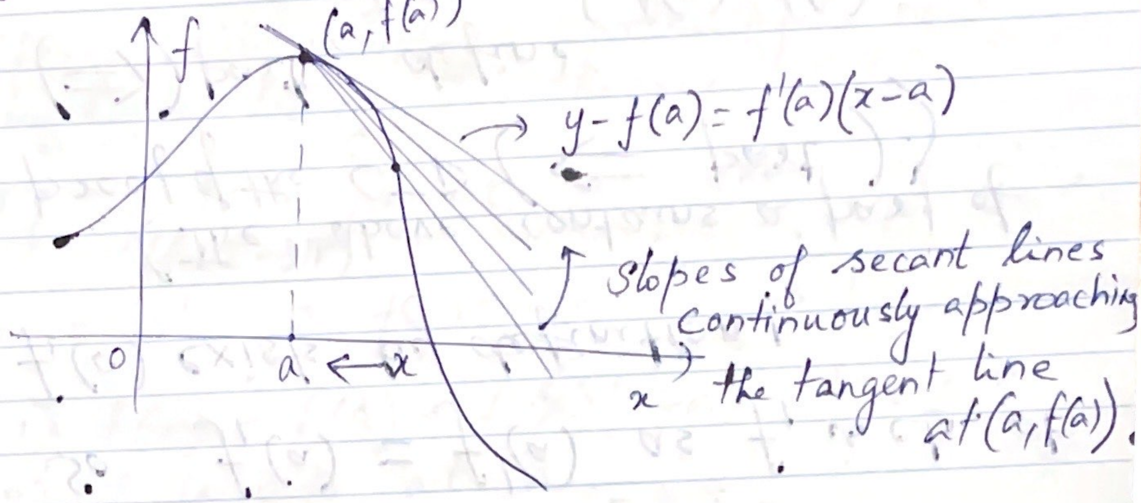
This characterization encompasses a very important geometric consequence about the existence of the derivative.

Assume that for $f: I \longrightarrow \mathbb{R}$ with $a \in I$, $f'(a)$ exists. Here, I is an (open) interval

$$f'(a) \text{ exists} \Rightarrow \lim_{x \to a} \frac{f(x)-f(a)}{x-a} \text{ exists}$$

The following characterization shifts this existence of limit at "a" to an existence of a function that is continuous at "a" and this continuity at "a" guarantees the existence of $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$.

Geometrically, $\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = f'(a)$ says that the slopes of the secant lines through $(a, f(a))$ and $(x, f(x))$ continuously approach the slope of the tangent line passing through $(a, f(a))$ with slope $= f'(a)$.



The following theorem characterizing the above is not in our course book.

<u>Carathéodory Theorem on derivatives</u>: (CTD)

Let $f: I \to \mathbb{R}$ be given. Then f is diff. at $a \in I$ iff there exists a function $f_1: I \to \mathbb{R}$ satisfying the following two conditions:

1) We have $f(x) = f(a) + f_1(x)(x-a)$ for $x \in I$

2) f_1 is continuous at a.

In such a case, $f'(a) = f_1(a)$.

Remark (Proof)

1) Note that, if we indeed get a function f_1 s.t it satisfies

1) $f(x) = f(a) + f_1(x)(x-a) \quad \forall x \in I$

then

$f_1(x)$ must be equal to.

$$f_1(x) = \frac{f(x)-f(a)}{x-a} \quad \text{when } x \neq a.$$

Now, 2) f_1 cts. at a implies

$$f_1(a) = \lim_{x \to a} f_1(x) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

So $f_1(a) = f'(a)$ as f_1 is cts. at a, $f_1(a)$ exists by definition!

(The above contains a part of the proof of the CTD ($\Leftarrow$ part))

For ($\Rightarrow$) part, define

$$f_1(x) = \begin{cases} \frac{f(x)-f(a)}{x-a}, & x \neq a \\ f'(a) ; & x = a \end{cases}$$

Check that f_1 satisfies ① and ②.

2) Differentiability implies continuity is now immediate from this theorem CTD.

If f is diff. at a, then, by CTD,
$(f: I \to \mathbb{R})$

there exists a function f_1 s.t.

1) $f(x) = f(a) + f_1(x)(x-a) \quad \forall x \in I$

2) f_1 is cts. at a.

So RHS of ① is a function that is cts. at a. LHS of ① is simply $f(x)$.

$\Rightarrow$ f is cts. at a.

3) In spite of its simplicity, CTD is a very powerful characterization of differentiability at a point. We illustrate its use in the next few examples. Recall how our characterization of continuity using sequences reduced the proofs of arithmetic rules of cts. fns. to that of arithmetic rules of convergent sequences. Similarly, CTD above will reduce the problem of proving the arithmetic rules of differentiable functions to the arithmetic rules of cts. fns.

Examples

A) Let $f(x) = x^n$. Let $a \in \mathbb{R}$

$\forall x$, $f(x) - f(a) = x^n - a^n = (x-a)(x^{n-1} + ax^{n-2} + \cdots + a^{n-1})$. $\longrightarrow$ ★

$\Rightarrow$ if we define

$f_1(x) = x^{n-1} + ax^{n-2} + \cdots + a^{n-1}$, then

1) $f(x) = f(a) + f_1(x)(x-a) \quad \forall x$

2) f_1 being a polynomial, of course, is cts. at a.

By CTD, f is differentiable at a

and $f'(a) = f_1(a) = a^{n-1} + a^{n-1} + \cdots + a^{n-1}$
$= n a^{n-1}$

B) Let $f(x) = x^{1/n}$ for a fixed $n \in \mathbb{N}$ and $f: (0,\infty) \longrightarrow (0,\infty)$. Let $a > 0$.

We would like to guess f_1 by looking at $x^{1/n} - a^{1/n}$. ~~We set $t = x^{1/n}$ and $s = a^{1/n}$ and get,~~ By ★ above, get for $x > 0$,

$$(x-a) = (x^{1/n})^n - (a^{1/n})^n$$
$$= (x^{1/n} - a^{1/n})\left(x^{\frac{n-1}{n}} + a^{\frac{1}{n}} x^{\frac{n-2}{n}} + \cdots + a^{\frac{n-1}{n}}\right) \text{ (by ★)}$$

$\Rightarrow$ For $x > 0$,

$$f(x) - f(a) = x^{1/n} - a^{1/n} = \left[\frac{1}{x^{\frac{n-1}{n}} + a^{\frac{1}{n}} x^{\frac{n-2}{n}} + \cdots + a^{\frac{n-1}{n}}} \right] (x-a)$$

$\Rightarrow$ if we define

$$f_1(x) = \frac{1}{x^{\frac{n-1}{n}} + a^{\frac{1}{n}} x^{\frac{n-2}{n}} + \cdots + a^{\frac{n-1}{n}}} \quad \text{for } x > 0,$$

then

1) $f(x) = f(a) + f_1(x)(x-a) \quad \forall\, x > 0$

& 2) As the denominator is nonzero, $f_1(x)$ is cts. at a

By CTD, f is diff. at a and

$$f'(a) = f_1(a)$$

sub $x = a$ in $f_1(x)$

$$= \frac{1}{a^{\frac{n-1}{n}} + a^{\frac{1}{n}} a^{\frac{n-2}{n}} + \cdots + a^{\frac{n-1}{n}}}$$

$$= \frac{1}{n a^{1-1/n}}$$

$$= \frac{1}{n} a^{\frac{1}{n} - 1}$$

c) Let $f(x) = \frac{1}{x}$ for $x \neq 0$. Let $a \in \mathbb{R} \setminus \{0\}$.

Then for all $x \neq 0$,

$$f(x) - f(a) = \frac{1}{x} - \frac{1}{a} = \frac{a-x}{ax}$$

$$= \left[-\frac{1}{ax}\right](x-a)$$

This suggests, if we define

$f_1(x) = -\frac{1}{ax}$ for $x \neq 0$, then

1) $f(x) - f(a) = f_1(x)(x-a) \quad \forall x \neq 0$

2) As $x \neq 0$, $a \neq 0$, f_1 is cts. at a!

By CTD, f is diff. at a and

$f'(a) = f_1(a) = -\frac{1}{a^2}$ at any $a \neq 0$.

<u>Remark</u> Like Continuity, differentiability is also a local concept. That is, to check whether a function $f: I \to \mathbb{R}$ is diff. at $a \in I$ (or) not, we need to know f only on a small interval $(a-\delta, a+\delta)$ around a. This is evident if you unravel the limit of difference quotient (or) using "local"ness of $f_1(x)$ given by CTD.

Week 6 Friday:

Have this understanding always!

Again our $f_1(x)$ given by CTD, satisfies $f_1(x) = \dfrac{f(x)-f(a)}{x-a}$ when $x \neq a$

& $\lim\limits_{x\to a} f_1(x) = \lim\limits_{x \to a} \dfrac{f(x)-f(a)}{x-a} = f'(a)$

So Continuity of f_1 at $x=a$. says $f_1(a) = f'(a)$.

For example, for $f(x) = x^2$

$$f_1(x) = \begin{cases} x+a, & x \neq a \\ 2a, & x = a \end{cases}$$

(i.e) $f_1(x) = x+a$ satisfies ① & ② of CTD.

and $f'(a) = f_1(a) = 2a$!

Arithmetic Rules for derivatives

Let f, g be diff. at $a \in I$, where $f : I \to \mathbb{R}$
$g : I \to \mathbb{R}$.

By CTD, there are functions $f_1, g_1 : I \to \mathbb{R}$
s.t $\forall x \in I$,

$$f(x) = f(a) + f_1(x)(x-a) \longrightarrow \text{I}$$

$$g(x) = g(a) + g_1(x)(x-a) \longrightarrow \text{II}$$

& f_1, g_1 are cts. at $x = a$

Moreover, $f_1(a) = f'(a)$ & $g_1(a) = g'(a)$.

Rules:

1) $h = f + g$ is diff. at a and $h'(a) = f'(a) + g'(a)$

2) $h = cf$ is diff. at a and $h'(a) = cf'(a)$, where $c \in \mathbb{R}$ is some constant

3) $h = fg$ is diff. at a and $h'(a) = f'(a)g(a) + g'(a)f(a)$.

4) If f is diff. at a with $f(a) \neq 0$, then, $h := 1/f$ is differentiable at a with

$$h'(a) = -\frac{f'(a)}{(f(a))^2}.$$

Strategy: In each of the proofs, we exploit the existence of f_1 and g_1 given by CTD and manipulate the expressions to find the required auxiliary function h_1 s.t

(i) $h(x) = h(a) + h_1(x)(x-a) \quad \forall x \in \text{Domain}(h)$

(ii) h_1 is cts. at a.

Then by CTD again, h is diff. at a and

$$h'(a) = h_1(a).$$

Proof of (1): $\forall x \in I$,

By (I) & (II), $h(x) = (f+g)(x) = f(x) + g(x)$

$= f(a) + f_1(x)(x-a) + g(a) + g_1(x)(x-a)$

$= f(a) + g(a) + [f_1(x) + g_1(x)](x-a)$

$h(x) = h(a) + [f_1(x) + g_1(x)](x-a)$

So take $h_1(x) = f_1(x) + g_1(x) \quad \forall x \in I$

h_1 satisfies (i) by choice above

as f_1 and g_1 are cts. at a,

$h_1 = f_1 + g_1$ is cts. at a.

By CTD, h is diff. at a and

$h'(a) = h_1(a) = f_1(a) + g_1(a) = f'(a) + g'(a)$ $\square$

Proof of ②

Here $h = cf$, for some constant $c \in \mathbb{R}$

Multiply (I) by c to get

$$cf(x) = cf(a) + cf_1(x)(x-a) \quad \forall x \in I$$

(i.e) $h(x) = h(a) + cf_1(x)(x-a) \ \forall x \in I$

So take $h_1(x) = cf_1(x) \ \forall x \in I$

as f_1 is cts. at a, $h_1 = cf_1$ is cts. at a.

$\therefore$ By CTD, h is diff at a and

$$h'(a) = h_1(a) = cf_1(a) = cf'(a)$$

Proof of ③

Here $h = fg$.

$\forall x \in I$ Multiply (I) & (II) to get,

$$\begin{aligned} h(x) &= f(x)g(x) \\ &= [f(a) + f_1(x)(x-a)][g(a) + g_1(x)(x-a)] \\ &= f(a)g(a) + f(a)g_1(x)(x-a) + f_1(x)g(a)(x-a) + f_1(x)g_1(x)(x-a)^2 \\ &= h(a) + [f(a)g_1(x) + f_1(x)g(a) + f_1(x)g_1(x)(x-a)](x-a) \end{aligned}$$

So take $h_1(x) = \overbrace{f(a)g_1(x)}^{A} + \overbrace{f_1(x)g(a)}^{B} + \overbrace{f_1(x)g_1(x)(x-a)}^{C}$
$\forall x \in I$

as f_1 and g_1 are cts. at a,

h_1 is cts. at a because each function represented by A, B and C are cts. at a by arithmetic rules of continuous fn.

$\therefore$ By CTD, h is diff. at a and

$$h'(a) = h_1(a) = f(a)g_1(a) + f_1(a)g(a) + f_1(a)g_1(a)\cdot 0$$
$$= f(a)g'(a) + f'(a)g(a).$$

$\blacksquare$

Proof of 4

Before we prove this, let us make some preliminary remarks. Since we assume f is differentiable at a, it is continuous at a.

Since $f(a) \neq 0$, $\exists\ \delta > 0$ s.t $|x-a| < \delta \Rightarrow f(x) \neq 0$

(i.e) f is non-zero on some $(a-\delta, a+\delta) \subseteq I$.

(Hint: Use $\varepsilon = \frac{|f(a)|}{3}$ and find a δ by continuity of f at a.)

Hence, the function $h = 1/f$ is defined on the interval $(a-\delta, a+\delta) \subseteq I$

Now for $x \in (a-\delta, a+\delta)$, as $f(x) \neq 0$

$$h(x) - h(a) = \frac{1}{f(x)} - \frac{1}{f(a)} = -\frac{(f(x)-f(a))}{f(x)f(a)}$$

$$= -\frac{f_1(x)(x-a)}{f(x)f(a)} \quad (\text{by } \textcircled{I})$$

$$\Rightarrow h(x) = h(a) + \left[\frac{-f_1(x)}{f(x)f(a)}\right](x-a)$$

So take $h_1(x) = \dfrac{-f_1(x)}{f(x)f(a)}$ $\forall x \in (a-\delta, a+\delta)$

as f_1 and f are cts. at a,

h_1 is cts. at a.

$\therefore$ By CTD, h is diff. at a and

$$h'(a) = h_1(a) = \frac{-f_1(a)}{f(a)f(a)} = \frac{-f'(a)}{(f(a))^2}$$

$\boxtimes$

Quotient rule is a combination of the Product rule and reciprocal rule:
if $g(a) \neq 0$ where g is diff at a and f is diff at a, then
$\frac{f}{g}$ is diff at a and $\left(\frac{f}{g}\right)'(a) = \dfrac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}$.

Chain Rule: Let $f : I \to \mathbb{R}$ be diff. at a and $f(I) \subseteq I_1$, I_1 being an interval. Let $g : I_1 \to \mathbb{R}$ be diff. at $f(a)$. Then $g \circ f$ is diff. at a with

$$(g \circ f)'(a) = \underbrace{g'(f(a))} \cdot \underbrace{f'(a)}$$

First Derive the outer function and evaluate it at a for inner function & Derive the inner function.

Proof:

Since f is diff. at a, there exists $f_1 : I \to \mathbb{R}$ s.t

III : $f(x) = f(a) + f_1(x)(x-a)$ & $f_1(a) = f'(a)$
f_1 is cts at a.

Similarly, as g is diff. at $f(a)$, there exists $g_1 : I_1 \to \mathbb{R}$ s.t

IV : $g(y) = g(f(a)) + g_1(y)(y - f(a)) \ \forall y \in I_1$
g_1 is cts. at $f(a)$ & $g_1(f(a)) = g'(f(a))$.

Now let $h(x) = g \circ f(x) = g(f(x))$.

Substitute $y = f(x)$ in IV,

$$g(f(x)) = g(f(a)) + g_1(f(x))(f(x) - f(a)) \ \forall x \in I$$
$$= g(f(a)) + g_1(f(x)) f_1(x)(x-a) \text{ by III}$$

Now for $x \in (a-\delta, a+\delta)$. as $f(x)$ [illegible]

So take $h_1(x) = g_1(f(x))f_1(x)$

Then $h(x) = h(a) + h_1(x)(x-a) \ \forall x \in I$

as g_1 is cts. at $f(a)$ and f is cts. at a,

$g_1(f(x))$ is cts. at a.

$\therefore$ h_1 is cts. at a.

By CTD, h is diff. at a and

$$h'(a) = h_1(a) = g_1(f(a))f_1(a)$$
$$= g'(f(a))f'(a)$$

▨

Now read 3.2.1 (Subsection) Pages 137–138 with the understanding of the function f_1 given by CTD.

Derivative of a^x

Assume $a > 0$ and that $f(x) = a^x$.

Then $f(x) = e^{\ln(a)x} = e^{g(x)}$ where $g(x) = \ln(a)x$

By Chain rule, $f'(x) = \exp'(g(x)) \cdot g'(x)$

$$= \exp(g(x)) \cdot \ln(a)$$
$$= e^{\ln(a)x} \cdot \ln(a)$$
$$= \ln(a) \cdot a^x$$

Read Section 3.7, 3.8 Pages 167 – 175.

On Section 3.9, Derivatives of other trig. functions are computed using the above rules.

For example: $\tan(x) = \frac{\sin x}{\cos x}$ is differentiable whenever $\cos x \neq 0$ (ie) when $x \neq \frac{2k+1}{2}\pi$, $k \in \mathbb{Z}$

By Quotient Rule:

$$\frac{d}{dx}(\tan(x)) = \frac{\frac{d}{dx}(\sin(x))\cos x - \sin x \frac{d}{dx}(\cos x)}{\cos^2(x)}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \quad \left(= 1 + \tan^2(x)\right)$$

$$= \sec^2(x)$$

Read pages 175 — 177.

Exercise: Let $f: \mathbb{R} \to \mathbb{R}$ be diff. at $x=0$. Define $g(x) = f(x^2)$. Show that g is diff. at 0

i) by CTD

ii) by Chain Rule.