MATH 137 LEC 010 (Mani Thamizhazhagan)
Week 3: Sep 19 – 23 Lecture Summary.

<u>Monday</u>: Previously we have seen how to show certain limits of sequences using formal definition. We agreed that it would be laborious to use the definition to figure out always. So we saw some natural rules of arithmetic for sequences. As a representative proof, we prove that <u>the</u> limit of the product of two <u>converging</u> sequences is <u>the</u> ~~limit~~ product of their limits.
(Why can we write "the" before limit?)
Notice the underlined words: In order to apply the rules, some care must be taken to meet all of the underlying conditions.
For example: We cannot apply the product rule for limits to the sequence

$$a_n = \frac{\sin(n)}{n}$$

like
$$\lim_{n\to\infty} a_n = \left(\lim_{n\to\infty} \sin(n)\right)\cdot\left(\lim_{n\to\infty}\frac{1}{n}\right)$$
$$= \left(\lim_{n\to\infty} \sin(n)\right)\cdot 0 = 0$$
is NOT VALID because $\{\sin(n)\}$ does not converge.
(Ponder rigorously why!)

<u>PRODUCT RULE</u>: Let $\{a_n\}$ and $\{b_n\}$ be sequences.
Assume that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$ where $L, M \in \mathbb{R}$.

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Then $\lim_{n\to\infty} a_n b_n = LM$.

<u>Proof</u>:

"Cinnamon/Bay leaf trick"

We add them in our cooking, take the essence out and throw it out later.

<u>Understanding</u>:

To show: for all $\varepsilon > 0$, there exist $N \in \mathbb{N}$ s.t

$$|a_n b_n - LM| < \varepsilon \text{ for all } n \geq N.$$

Mathematically, we are going to transform the task of finding a cutoff N for given $\varepsilon > 0$ for $\{a_n b_n\}$ to finding a cutoff that's common for the sequences $\{a_n\}$ and $\{b_n\}$ for a different tolerance related to given ε (Something constant times ε). We know how to find the cutoff for the latter because $\{a_n\}$ & $\{b_n\}$ converges and the definition spits out.

Let's start with

$$|a_n b_n - LM| = |a_n b_n \underbrace{- a_n M + a_n M}_{\substack{\text{Cinnamon} \\ \text{(adding and throwing out)}}} - LM|$$

$$(\triangle^{gle} \text{ inequality}) \Rightarrow \leq |a_n (b_n - M)| + |(a_n - L) M|$$

$$= |a_n||b_n - M| + |a_n - L||M|.$$

Since $\{a_n\}$ is convergent, there exists an $A \in \mathbb{R}$ s.t

$$|a_n| \leq A \text{ for each } n$$ (Refer to Week 2 lecture summary)

Now $|a_nb_n - LM| \leq |a_n||b_n - M| + |a_n - L||M|$ implies that if we find ε_1 and ε_2 s.t

$$|a_n||b_n - M| + |a_n - L||M| \leq A\varepsilon_1 + |M|\varepsilon_2 = \varepsilon$$

and find the appropriate cutoffs for when

$|b_n - M| < \varepsilon_1$ and $|a_n - L| < \varepsilon_2$ happens we are done.

For $\varepsilon_1 = \frac{\varepsilon}{2A}$, find $N_1 \in \mathbb{N}$ s.t $|b_n - M| < \frac{\varepsilon}{2A}$ for all $n \geq N_1$

For $\varepsilon_2 = \frac{\varepsilon}{2|M|}$, find $N_2 \in \mathbb{N}$ s.t $|a_n - L| < \frac{\varepsilon}{2|M|}$ for all $n \geq N_2$

By definition, since $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$ such N_1 and N_2 exists.

For Given $\varepsilon > 0$, Choose $N = \max\{N_1, N_2\}$. Then for all $n \geq N$, we have

$$|a_nb_n - LM| \leq |a_n||b_n - M| + |a_n - L||M|$$
$$< A\frac{\varepsilon}{2A} + \frac{\varepsilon}{2|M|}|M| = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\lim_{n\to\infty} a_n b_n = LM$.

$\blacksquare$

Now back to the example $a_n = \frac{\sin(n)}{n}$.

Read Section 1.3 Squeeze Theorem from Pages 37 – 39 in our Course book / Notes

Squeeze theorem comes in handy in figuring out the limits of certain oscillating sequences such as ones involving sin/cos (or) $\frac{(-1)^n}{n}$.

<u>Squeeze Theorem</u>: Assume that $a_n \le b_n \le c_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$.

Then $\{b_n\}$ converges and $\lim_{n\to\infty} b_n = L$.

<u>Proof</u>: Refer to Theorem 9 proof in our Course book (or) notes.

<u>Remark</u>.

Observe for $a_n = 1 - \frac{1}{n}$ and $c_n = 3 + \frac{1}{n}$, $\lim_{n\to\infty} a_n = 1$ and $\lim_{n\to\infty} c_n = 3$.

Let $b_n = \begin{cases} 2 & \text{if } n \text{ is even} \\ 2.5 & \text{if } n \text{ is odd}. \end{cases}$

Then $a_n \le b_n \le c_n$ for all $n \in \mathbb{N}$.

$\lim_{n\to\infty} a_n \neq \lim_{n\to\infty} c_n$. $\{b_n\}$ does not converge.

where as $d_n = 2 + \frac{1}{n}$ would converge sitting between a_n and c_n.

<u>Example</u>: Find $\lim\limits_{n\to\infty} \frac{n \sin(7n^3+8n^2+3n+1)+8n}{n^3}$

Let $a_n = \frac{7n}{n^3}$, $b_n = \frac{9n}{n^3}$.

as $\quad -1 \leq \sin(7n^3+8n^2+3n+1) \leq 1$ for all n

$$\left.\begin{aligned}
\Rightarrow \quad -n &\leq n \sin(7n^3+8n^2+3n+1) \leq n \\
\frac{-n+8n}{n^3} &\leq \frac{n \sin(7n^3+8n^2+3n+1)+8n}{n^3} \leq \frac{n+8n}{n^3} \\
\frac{7n}{n^3} &\leq \frac{n \sin(7n^3+8n^2+3n+1)+8n}{n^3} \leq \frac{9n}{n^3} \\
\frac{7}{n^2} &\leq \frac{n \sin(7n^3+8n^2+3n+1)+8n}{n^3} \leq \frac{9}{n^2}
\end{aligned}\right\} \text{for all } n.$$

as $\lim\limits_{n\to\infty} \frac{7}{n^2} = 0 = \lim\limits_{n\to\infty} \frac{9}{n^2}$, by squeeze thm,

$$\lim_{n\to\infty} \frac{n \sin(7n^3+8n^2+3n+1)+8n}{n^3} = 0.$$

<u>Wednesday</u>: We saw a fundamental axiom of real number line called Least upper bound axiom. Axioms can neither be proved nor disproved, nevertheless we assume it to be true and build a system. Without LUB axiom, there won't be a line model for real numbers.

LUB axiom is the actually the theoretical foundation for all of the most fundamental results of calculus. We will visit this throughout the course.

We proved Monotone convergence theorem (MCT) for non-decreasing sequences. (Theorem 11)

Read Section 1.4 from Pages 39 — 44 in our Course book / Notes.

Friday:

Recall your Quiz 2, Long answer 2.

Assume, $a_n > 0$ for all $n \in \mathbb{N}$ and $L > 0$. Prove that, if $\lim_{n\to\infty} a_n = L$ then $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{L}$.

Understanding:

The following computation

$$|\sqrt{a_n} - \sqrt{L}| = \left|\frac{(\sqrt{a_n} - \sqrt{L})(\sqrt{a_n} + \sqrt{L})}{\sqrt{a_n} + \sqrt{L}}\right|$$

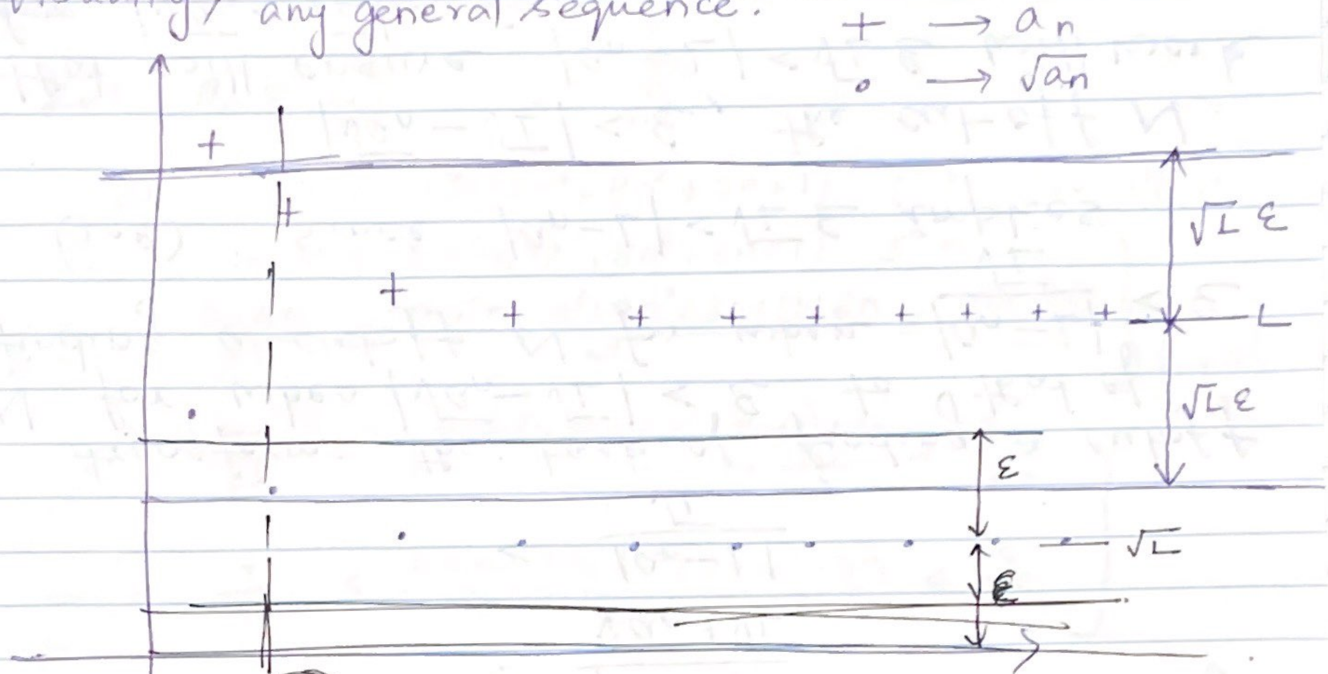
$$= \frac{|a_n - L|}{\sqrt{a_n} + \sqrt{L}}$$

$$< \frac{|a_n - L|}{\sqrt{L}},$$

transforms the task of finding a cutoff N for when $|\sqrt{a_n} - \sqrt{L}| < \varepsilon$ to that of finding a cutoff N for when $\frac{|a_n - L|}{\sqrt{L}} < \varepsilon$.

(i.e) Since $|a_n - L| < \sqrt{L}\,\varepsilon$ implies $|\sqrt{a_n} - \sqrt{L}| < \varepsilon$, the cut-off N that will ensure $|a_n - L| < \sqrt{L}\,\varepsilon$ will work for $|\sqrt{a_n} - \sqrt{L}| < \varepsilon$.

Visually, This is the best we can do as a_n's are any general sequence.



N given by when

$$|a_n - L| < \sqrt{L}\,\varepsilon$$

works for when

$$|\sqrt{a_n} - \sqrt{L}| < \varepsilon \quad \text{for all } n \geq N.$$

This N maynot be the threshold after which $\sqrt{a_n}$ gets into the ε-band of $\sqrt{L}$ but a sufficient N that ensures that for all $n \in \mathbb{N}$ that are $\geq N$, $|\sqrt{a_n} - \sqrt{L}| < \varepsilon$.

We will now find how to determine the limits of certain recursively defined sequences. Idea is to set the stage to be able to apply MCT to ensure that such sequences are bounded and monotonic.

Before we can use the MCT, we need to understand one proof technique called the Principle of Mathematical Induction.

Let $P(n)$ be a statement for each $n \in \mathbb{N}$.
If we can

1) prove $P(1)$ is true. (Base case) and
2) prove: $\underbrace{\text{If } P(k) \text{ is true for some } k}_{\text{Inductive hypothesis}}$, then $\underbrace{P(k+1) \text{ is true}}_{\text{Inductive step.}}$

Then we can conclude that $P(n)$ is true for all $n \in \mathbb{N}$.

Steps to find limits of certain recursive sequences.

1) Prove the sequence is monotonic.
2) Prove the sequence is bounded (above (or) below).
3) Conclude the sequence converges by MCT.
4) Find the limit by using the rule

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} a_{n+1}.$$

Remark:

In step 4, we can apply the rule only when step 3 concludes that the sequence converges to some $L \in \mathbb{R}$.

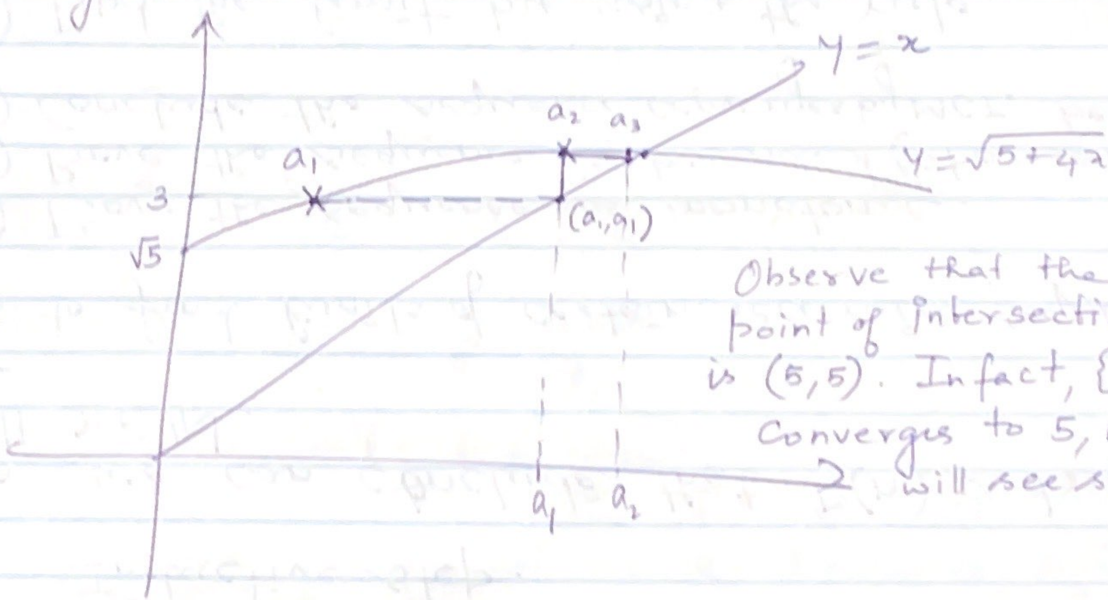
It might be possible to combine Step 1 and Step 2.

Examples:

1) Find the limit for $a_{n+1} = \sqrt{5+4a_n}$,
$a_1 = 3$

Let's see how we can visualize recursive sequences.

Graph the function $f(x) = \sqrt{5+4x}$ and $g(x) = x$.



Observe that the point of intersection is $(5,5)$. In fact, $\{a_n\}$ converges to 5, we will see shortly

i) Prove that $a_n \le a_{n+1} < 6$ for each n.

We will use PMI to prove that

$P(n)$: $a_n \le a_{n+1} < 6$ is true for all n.

Base case:

$a_1 = 3$, $a_2 = \sqrt{17}$

$\therefore$ $a_1 \le a_2 < 6$ is true

(i.e) $P(1)$ is true.

(IH) Inductive hypothesis: Assume $P(k)$ is true for some k
(i.e) $a_k \le a_{k+1} < 6$ is true for some k.

Inductive Step: Show $P(k+1)$ is true.

Let's start with our IH.

$a_k \le a_{k+1} < 6$ is true

$\Rightarrow \quad 4a_k \le 4a_{k+1} < 24$ is true

$\Rightarrow \quad 5+4a_k \le 5+4a_{k+1} < 29$ is true

$\Rightarrow \quad \sqrt{5+4a_k} \le \sqrt{5+4a_{k+1}} < \sqrt{29} < 6$ is true

$\Rightarrow \quad a_{k+1} \le a_{k+2} < 6$ is true

$\therefore$ $P(k+1)$ is true.

$\therefore$ By PMI, $P(n)$ is true for all n.

$\therefore$ $a_n \le a_{n+1} < 6$ for each n.

$\Rightarrow$ $\{a_n\}$ is non-decreasing and bounded above.

$\therefore$ By MCT, $\{a_n\}$ converges to some $L \in \mathbb{R}$.

Now, take $a_{n+1} = \sqrt{5+4a_n}$

As $\{a_n\}$ is convergent, by a rule,

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L.$$

$$\Rightarrow L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{5+4a_n}$$

$$= \sqrt{\lim_{n\to\infty}(5+4a_n)}$$

(By Quiz 2, LA2) Square root and limit are interchangeable That's the understanding

$$= \sqrt{5+4\lim_{n\to\infty} a_n}$$

$$L = \sqrt{5+4L}$$

$$\Rightarrow L^2 = 5+4L$$

$$\Rightarrow L^2-4L-5=0$$

$$\Rightarrow (L-5)(L+1)=0$$

$L=5$ (or) $L=-1$

as $a_n > 0$ for all n, $L>0$! $L \neq -1$

$$\therefore \boxed{L=5}$$

Exercise: Find the limit of $\{a_n\}$ where $a_{n+1} = \sqrt{5+4a_n}$, $a_1 = 10$.

i) Prove that $4 < a_{n+1} \leq a_n$ for each n

Then invoke MCT to conclude $\{a_n\}$ converges to some $L \in \mathbb{R}$. Now find L!