MATH-4101(3)-001 **Assignment 6** Due: Mar 3, 2022 (before class)

1. Evaluate the following limits if they exist; if they do not exist state why. Show work.

   a) $\lim_{z\to -i} \frac{z^3+z}{z+i}$ b) $\lim_{z\to\infty} \frac{7z^2}{5+2z+3z^2}$ c) $\lim_{z\to 0} \frac{3}{2+\bar{z}}$

2. Prove the following:

   a) $\lim_{z\to 0} \frac{\sin z}{z} = 1$ b) $\lim_{z\to 0} \frac{\sinh z}{iz} = -i$

   [*Hint*: Recall that $\frac{f(z)-f(0)}{z} \to f'(0)$.]

3. Show that $f(z) = |z|$ is not differentiable at 0. Show that $g(z) = |z|^2$ is differentiable at $z = 0$ and nowhere else. Thus, g is differentiable at 0 but not holomorphic in any open disk containing 0.

4. Consider

$$f(z) = \frac{xy^2(x+iy)}{x^2+y^4}(z = x+iy \neq 0) \quad \text{and } f(0) = 0.$$

   Verify that $\lim_{z\to 0} \frac{f(z)}{z} = 0$ as $z \to 0$ along any straight line, $z = (a+ib)t, t \in \mathbb{R}$. Now, by considering $z \to 0$ along the path $z(t) = t^2 + it$, show that f is not differentiable at 0.

5. Assume U is open and connected. Let $f : U \to \mathbb{C}$ be given.

   (a) If f is holomorphic on U and assumes only real (resp. purely imaginary) values, then f is a constant.

   (b) If f and $\bar{f}$ are both holomorphic on U, then f is a constant.

6. Let $f(z) : \sum_{n=0}^{\infty} a_n z^n$ with $R > 0$ as its radius of convergence. Assume that $f' = f$ on $B(0, R)$ and that $f(0) = 1$. Find a_n explicitly and hence f.

   [*Hint*: Use the formula for a_n given by infinite differentiability of power series. Do you see that you need the result on uniqueness of power series?]

7. Let $f : U \to \mathbb{C}$ be holomorphic. Let $\overline{U} := \{\bar{z} : z \in U\}$. Define $g(z) := \overline{f(\bar{z})}$ for $z \in \overline{U}$. Show g is holomorphic on $\overline{U}$ and find $g'(z)$ for $z \in \overline{U}$ in terms of f'. Also, write briefly why we can't conclude this using the converse of Cauchy-Riemann equations implying holomorphicity theorem.

   [*Hint*: Carefully apply the proposition to g immediate to definition of differentiability (Recall the notation $f'(z) = f_1(z)$) and convince the reader why it gives us the result required. It may be helpful to do it for polynomials and wonder why it is true!]