MATH 137 LEC 010 (Mani Thamizhazhagan)
Week 7 Oct 31 – Nov 4 Lecture Summary / Notes

<u>Monday</u>: We saw why a naive approach to prove chain rule won't work and is incorrect and also proved Inverse Function theorem using CDT.
Read Sections 3·10 – 3·11 Pages 177 – 189 in our Course book.

<u>Incorrectness of Naive Approach to prove Chain Rule</u>

<u>Chain Rule</u>: Let $f: I \to \mathbb{R}$ be diff. at a and $f(I) \subseteq I_1$, I_1 being an interval. Let $g: I_1 \to \mathbb{R}$ be diff. at $f(a)$. Then $g \circ f$ is diff. at a with

$$(g \circ f)'(a) = g'(f(a))\, f'(a).$$

(WRONG) Proof:

First note that since g is diff. at $f(a)$, g is continuous at $f(a)$.

$\Rightarrow$ $\forall$ sequences $\{y_n\} \subseteq I_1$ s.t $y_n \to f(a)$, we have

$$\lim_{n\to\infty} g(y_n) = g(f(a))$$

In particular, as f is also cts. at a, for any sequence $\{x_n\} \subseteq I$ s.t $x_n \to a$, we have

$$\lim_{n\to\infty} f(x_n) = f(a).$$

Convince yourself of the validity of these!

$$\text{So} \quad \lim_{x\to a} \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} = \lim_{x_n \to a} \frac{g(f(x_n)) - g(f(a))}{f(x_n) - f(a)}$$
$$= \lim_{y \to f(a)} \frac{g(y) - g(f(a))}{y - f(a)}$$
$$= g'(f(a))$$

Consider $\lim_{x \to a} \frac{g(f(x))-g(f(a))}{x-a}$ (Multiply and divide by $f(x)-f(a)$)

$$= \lim_{x \to a} \frac{g(f(x))-g(f(a))}{f(x)-f(a)} \cdot \frac{f(x)-f(a)}{x-a}$$

$$= \lim_{x \to a} \frac{g(f(x))-g(f(a))}{f(x)-f(a)} \cdot \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

$$= g'(f(a)) \cdot f'(a).$$

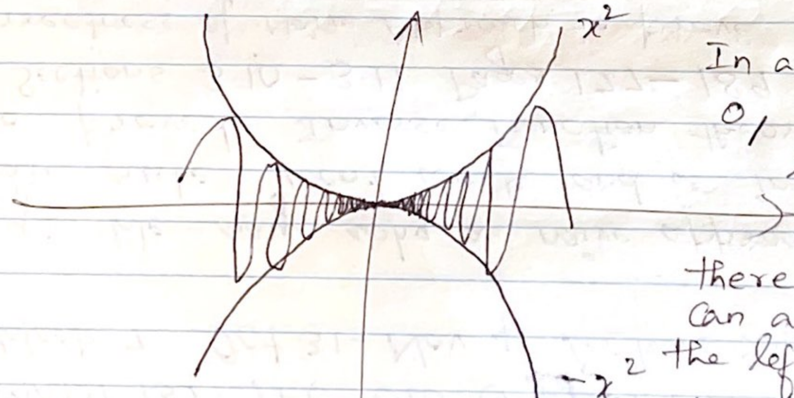
→ This is wrong because,

we can only divide by something that is non-zero. There are functions f such that f is differentiable at a, still $f(x)=f(a)$ for infinitely many x ~~around~~ in any interval ~~etg~~ containing a.

For example,

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

is diff. at $x=0$.



In any interval containing 0, $f(x)=0$ for infinitely many x.

So in the above, there is no way we can approach 0 from the left (or) right s.t we avoid $f(x)=0$!

Inverse Function theorem: (Theorem 10, Page 180)

Assume that $y=f(x)$ is continuous and invertible on $[c,d]$ with inverse $x=g(y)$, and f is differentiable at $a \in (c,d)$, then g is diff. at $b=f(a)$, and

$$g'(b) = \frac{1}{f'(a)} = \frac{1}{f'(g(b))}, \text{ if } f'(a) \neq 0.$$

Moreover, L_a^f is also invertible and

$$(L_a^f)^{-1}(x) = L_b^g(x) = L_{f(a)}^g(x).$$

Proof:

Note that, $g(f(x)) = x$ for all $x \in [c,d]$.

Also remember that differentiability is a local concept!

By CDT, since f is differentiable at a,

See below.

$\exists f_1$ s.t 1) $f(x) = f(a) + f_1(x)(x-a) \; \forall x \in I$

2) f_1 is cts. at a with $f_1(a) = f'(a)$.

Now since $f_1(a) = f'(a) \neq 0$, $f_1(x) \neq 0$ in a small interval say I containing a.

[Proof: As $\lim_{x\to a} f_1(x) = f_1(a)$, for $\varepsilon = \frac{|f_1(a)|}{2} > 0$,

$\exists \delta > 0$ s.t $x \in (a-\delta, a+\delta) \Rightarrow f_1(x) \in \left(f_1(a) - \frac{|f_1(a)|}{2}, f_1(a) + \frac{|f_1(a)|}{2}\right)$

Check that $f_1(x) \neq 0$, for $x \in I = (a-\delta, a+\delta)$]

Therefore, in ①, $f_1(x) \neq 0 \ \forall x \in I$ and rewrite ① as

$$f(x) - f(a) = f_1(x)(x-a)$$

Now let $y = f(x)$, $b = f(a)$, then $x = g(y)$, $a = g(b)$.

$$\Rightarrow \quad y - b = f_1(g(y))(g(y) - g(b)) \quad \forall y \in f(I)$$

$$\Rightarrow \quad g(y) - g(b) = \frac{1}{f_1(g(y))}(y-b)$$

and as f_1 is cts at $g(b) = a$ and $f_1(x) \neq 0 \ \forall x \in I$,

$\frac{1}{f_1(g(y))}$ is cts. at b.

$\therefore$ By CDT,

g is diff at $b = f(a)$ and

$$g'(b) = \frac{1}{f_1(g(b))} = \frac{1}{f_1(a)} = \frac{1}{f'(a)} = \frac{1}{f'(g(b))}$$

Exercise: Show that the tangent line $L_a^f(x)$ is invertible and $(L_a^f)^{-1}(x) = L_b^g(x) = L_{f(a)}^g(x)$

Derivative of $\ln(x)$

$g(x) = \ln(x)$ is the inverse function of $e^x = f(x)$. By IFT,

g is diff. at $b = f(a)$ and
$((\text{i.e}) \ b = e^a)$

$$g'(b) = \frac{1}{f'(a)} = \frac{1}{e^a} = \frac{1}{b} \quad \forall b > 0.$$

$$\therefore \frac{d(\ln(x))}{dx} = \frac{1}{x}$$

Derivative of x^α, $x > 0$, $\alpha \in \mathbb{R}$

Definition of $x^\alpha := e^{\alpha \ln(x)}$.

$\therefore$ $f(x) = e^{\alpha \ln(x)}$ is a function from $(0, \infty) \to \mathbb{R}$.

By Chain Rule,

$$f'(x) = e^{\alpha \ln x} \cdot \left(\frac{\alpha}{x}\right)$$

$$= x^\alpha \cdot \left(\frac{\alpha}{x}\right)$$

$$= \alpha x^{\alpha - 1}.$$

Please go through the derivatives of inverse trig. functions from section 3.11.

On Wednesday, we discussed tangent lines and linear approximation. We also saw a theorem about bounding the error in linear approximation.

Read Section 3.5 Pages 150–160.
(Theorem 6, Page 157 is the main result).

Exercise

Find $(26.97)^{1/3}$ approximately and bound the error in linear approximation (by choosing a suitable point closer to 26.97 that renders the calculation minimal)

Solution:

We choose $a = 27$ and consider $f(x) = x^{1/3}$ because $(27)^{1/3} = 3$ is readily available.

$$\text{If } f(x) = x^{1/3},\ f'(x) = \frac{1}{3}x^{-2/3} \text{ for } x \neq 0.$$

$$\therefore f'(27) = \frac{1}{3(27)^{2/3}} = \frac{1}{27}.$$

$\therefore$ $L_{27}^{f}(x)$ approximates $f(x)$ in a smaller interval around 27.

$$\therefore (26.97)^{1/3} \underset{\substack{\uparrow \\ \text{approximately}}}{\simeq} L_{27}^{f}(26.97)$$

$$L_{27}^{f}(x) = f(27) + f'(27)(x-27)$$
$$= 3 + \frac{1}{27}(x-27)$$

$$\therefore L_{27}^{f}(26.97) = 3 + \frac{1}{27}(26.97-27)$$
$$= 3 - \frac{0.03}{27}$$
$$= 3 - \frac{1}{900}$$
$$= 3 - 0.00111\overline{1}$$
$$= 2.99888\overline{8}.$$

Check that $(26.97)^{1/3} = 2.99888847711$

$$\left| L_{27}^{f}(26.97) - f(26.97) \right| \cong 0.00000041177$$

Now, let's apply theorem 6 on $I = [26, 28]$ and

check if $\left| L_{27}^{f}(x) - f(x) \right| \leq \frac{M}{2}(x-27)^2$, where

$$|f''(x)| \leq M \text{ on } I$$

$$f''(x) = \frac{1}{3}\left(-\frac{2}{3}\right) x^{-5/3}$$

$$= -\frac{2}{9} x^{-5/3}$$

$$|f''(x)| = \left| -\frac{2}{9x^{5/3}} \right| = \frac{2}{9x^{5/3}} \text{ for } x \in [26, 28]$$

$$\leq \frac{2}{9(26)^{5/3}}$$

$$\cong 0.00097386 < \frac{2}{9}$$

Max value of $(x-27)^2$ is 1 if $x \in [26, 28]$.

$$\therefore \left| L_{27}^{f}(x) - f(x) \right| \leq \frac{0.00097386}{2}(x-27)^2 \leq 0.000487.$$

So, for sure, $L_{27}^{f}(x)$ approximates $f(x)$ in $[26, 28]$ accurately to three decimal places!

On Friday, We explored Newton's method of finding roots of certain nice functions.

Read Section 3.6 Pages 161 – 166.

On page 163, look at the first few lines that says "for most nice functions and reasonable choices of initial point, the sequence $\{x_n\}$ converges very rapidly to a number c with $f(c)=0$".

We shall indicate an instance of such nice functions:

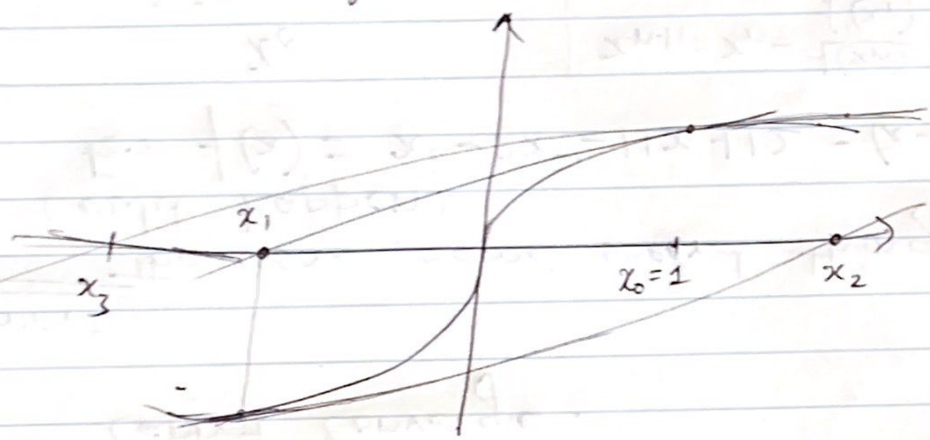
> Let $f:[a,b]\to\mathbb{R}$ be twice differentiable s.t f'' is also continuous. Let c be a simple root of f in (a,b). [(i.e) $f(c)=0$, but $f'(c)\neq 0$]
>
> Then the Newton's method defined by
>
> $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$ converges to c
>
> for any x_0 (initial point) sufficiently close to c.

As such, one can apply Newton's method as long as sequences are well-defined and check if it converges for any differentiable function.

Failures of Newton's method

1) Look at the following silly example:

Use $x_0 = 1$ to find the approximation to the root of $f(x) = x^{1/3}$.



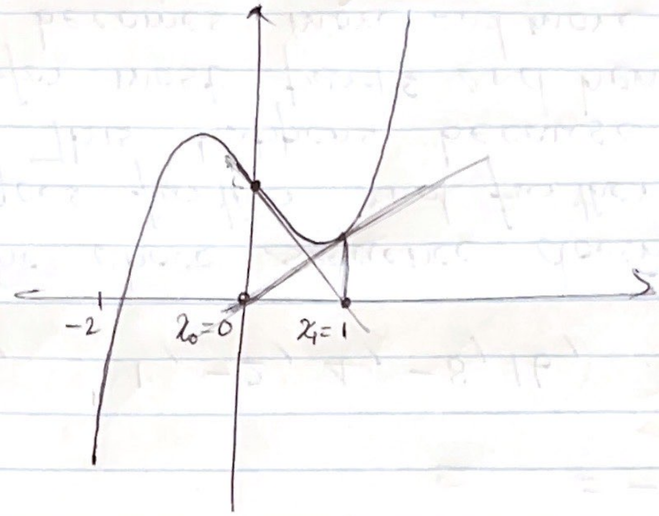
Clearly the root is at 0. but let's see what happens in Newton's method.

$$x_{n+1} = x_n - \frac{x_n^{1/3}}{\frac{1}{3} x^{-2/3}} = x_n - 3x_n$$
$$= -2x_n.$$

So, $1, -2, 4, -8, 16, \ldots$

The above sequence doesnot converge and goes further and further away from zero. This happens because the graph is flat for most parts and hence the tangent lines becomes more and more flatter (slopes being close to zero).

2) Let $f(x) = x^3 - 2x + 2$.



Choice of initial point matters!

If you choose $x_0 = 0$, then $x_1 = 1, x_2 = 0, \ldots$

$\Rightarrow$ the Newton's method sequences oscillates between 0 and 1 and hence cannot converge.

<u>Remark</u>:

There are some weird things that could happen!

For $f(x) = x^3 - 2x^2 - 11x + 12 = (x-4)(x-1)(x+3)$

x_0	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ converges to
2.35287527	4
2.35284172	-3
2.35283735	4
2.352836327	-3
2.352836323	1

The above phenomenon happens because ~~the~~ some of the iterates go close to where $f'(x) = 0$ so shoots up and come back later to one of the roots! Try seeing about 40 to 50 iterations through a programming!