MATH-4101(3)-001 Assignment 7 Due: Mar 15, 2022 (before class)

Warning: The textbook uses Log for principle value of log i.e., $\log_\pi$. Our course $\text{Log}(z)$ is the set of all complex numbers w such that $\exp(w) = z$ for $z \neq 0$.

1. Let $w_i \in \text{Log}(z_i)$ for $i = 1, 2$. Show that $w_1 + w_2 \in \text{Log}(z_1 z_2)$ and $-w_1 \in \text{Log}(1/z_1)$.

2. Find the values of $\text{Log}(-1), \text{Log}(i)$, and $\text{Log}(\exp(z))$.

3. Show that $(-1)^i = \{\exp([2n+1]\pi) : n \in \mathbb{Z}\}$.

4. Find all solutions of $z^{1+i} = 4$.

   Hint: Apply the definition to z^{1+i} to get $\exp((1+i)\text{Log}(z)) = 4 = 4\exp(i2\pi k), k \in \mathbb{Z}$ and substitute $\text{Log}(z) = \ln(|z|) + it, t \in arg(z)$. Now get for each $k \in \mathbb{Z}$, a system of linear equations in $\ln(|z|)$ and t, by comparing the polar representations on either side and solve them (OR) describe the set $4^{1/1+i}$ and explain that it is exactly the solutions of $z^{1+i} = 4$.

5. Recall that we defined for $\lambda \in \mathbb{C}$ and $z \neq 0$, $p_\alpha^\lambda(z) = \exp(\lambda \log_\alpha(z))$, for $z \in \mathbb{C}\backslash L_\alpha$ to get a single-valued function for $z \to z^\lambda$. Show that for $\lambda = \frac{1}{2}$,

   (a) $(p_\alpha^{\frac{1}{2}}(z))^2 = z$ for $z \neq 0$.

   (b) $p_{\alpha+2\pi}^{\frac{1}{2}}(z) = -p_\alpha^{\frac{1}{2}}(z)$.
   Hint: Substitute $p_\alpha^\lambda(z) = \exp(\lambda \log_\alpha(z))$ and apply properties of exp and observe that $\log_{\alpha+2\pi}(z) = \log_\alpha(z) + 2\pi$.

   Overall, you have shown that each of $\pm p_\alpha^{\frac{1}{2}}$ is a square root.

6. We say a continuous function $g : U \to \mathbb{C}$ is **a** logarithm on U if we have $\exp(g(z)) = z$ for all $z \in U$. Show that if g is a logarithm on U, then $g'(z) = 1/z$ for $z \in U$.

   [Example: If $U := B(1+i, 1)$] and if we take as g the restriction of any one of $\log_0, \log_{\pi/2}, \log_\pi$ and $\log_{3\pi/2}$ to U, then g is a logarithm on U.]

7. [Bonus] For any $z \in \mathbb{C}$, show that $n \log_\pi(1 + z/n)$ is defined for all large values of n and that it tends to z as $n \to \infty$. Hence deduce that $(1 + z/n)^n \to \exp(z)$ as $n \to \infty$.

   Hint: Use the power series defined for $\log_\pi(1 + z)$ for $|z| < 1$. It may be helpful to change the index in the power series, say to 'k', because the question has already an 'n'.