MATH 137 LEC 010 (Mani Thamizhazhagan)
Week 1: Sep 6 – 9 Lecture Notes

Notations:

$\{\,\}$, ϕ – the empty set

$\mathbb{N}$: $\{1, 2, 3, 4, 5, \dots\}$ the set of natural numbers

$\mathbb{Z}$: $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ the set of integers

$\mathbb{Q}$: $\{\frac{a}{b} ; \ a, b \in \mathbb{Z} \text{ and } b \neq 0\}$ the set of rational numbers. These are decimals that repeat or terminate. For example, $0.25 = \frac{25}{100} = \frac{1}{4}$ and $0.\overline{3} = 0.3333\dots = \frac{1}{3}$.

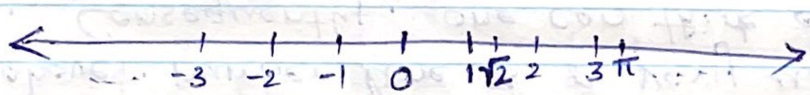
Note that, $\underset{\text{(Subset)}}{\mathbb{N} \subset} \mathbb{Z} \subset \mathbb{Q}$

Irrational numbers are defined as any number that cannot be written as a ratio of two integers. Non-terminating decimals that do not repeat are irrational. For example,

$$\pi = 3.14159\dots \text{ and } \sqrt{2} = 1.41421\dots$$

$\mathbb{R}$: the set of all real numbers is defined as the set of all rational numbers combined with the set of all irrational numbers.

A real number line, (or) simply number line, allows us to visually display real numbers by associating them with unique points on a line.



The above number line is a very useful model to have. Consequently, one can think of the non-zero real numbers with a sign, + or −, depending on the corresponding point's orientation with respect to (w.r.t) O, and a magnitude that represents the distance that the point is from O. This magnitude for $x \in \mathbb{R}$ is called the absolute value of x and is denoted by $|x|$.

It is common to think of it as a mechanism that simply drops negative signs:

$$|2| = 2, \quad |-3| = 3, \quad |-\pi| = \pi, \text{ etc.}$$

However, if x was some unknown quantity, would $|-x| = x$?

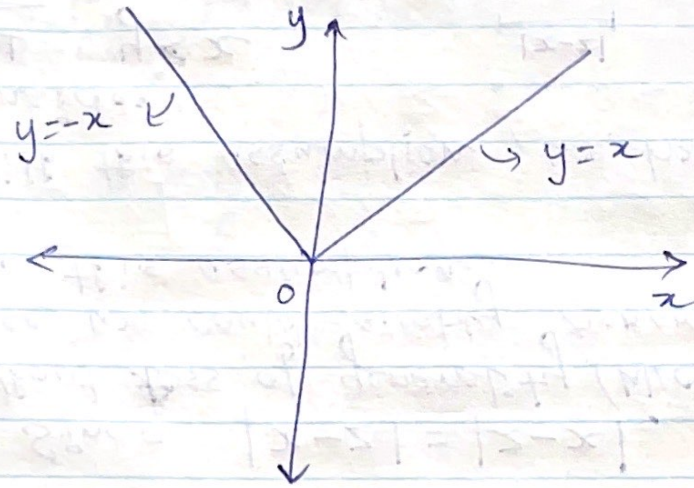
NO, take $x = -5$, then $|-(-5)| = 5 \neq x$

So, let's see the definition to avoid ambiguity:

For each $x \in \mathbb{R}$, define the <u>absolute value</u> of x by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Graphically, the absolute value function looks as below:



<u>Remarks:</u> For each $x \in \mathbb{R}$,

i) $|x| \geq 0$

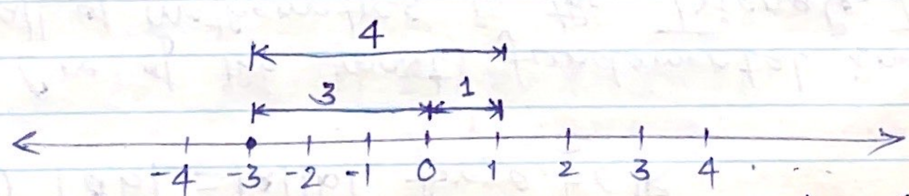
ii) $|-x| = \begin{cases} -x & \text{if } -x \geq 0 \\ -(-x) & \text{if } -x < 0 \end{cases}$ by definition

$= \begin{cases} -x & \text{if } x \leq 0 \\ x & \text{if } x > 0 \end{cases} \rightarrow$ This is the definition of $|x|$

$\therefore |-x| = |x|$

iii) Given any two points a, b on the number line, the distance from a to b is given by $|b-a|$.

by ii), $|b-a| = |-(b-a)| = |a-b|$ as well.



The distance between -3 and 1 is $|1-(-3)| = |-3-1| = 4$

iv) $|ab| = |a||b|$ for $a, b \in \mathbb{R}$

One of the most fundamental inequalities in all of mathematics is the Triangle Inequality. In two-dimensional, this inequality corresponds to familiar statement that "the sum of the length of any two sides of a triangle exceeds the length of the third".

Thm : Theorem.

<u>Thm</u> Triangle Inequality:

For any real numbers x, y, and z,

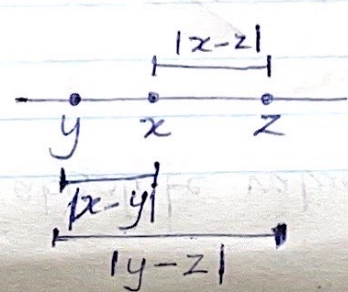
$$|x-z| \leq |x-y| + |y-z|$$

(This says that the distance from x to z is less than or equal to the sum of the distance from x to y and the distance from y to z)

<u>Proof</u>: Since $|x-z| = |z-x|$, We assume "without loss of generality (WLOG)" that $x \leq z$, because we could simply rename the points to achieve this assumption.

With this assumption, we have three cases to consider:

<u>Case 1</u>: $y < x$

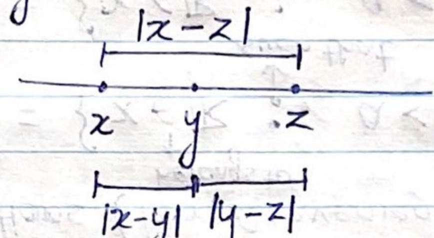


Clearly, $|x-z| \le |y-z|$

$\therefore$ $|x-z| \le |x-y| + |y-z|$, (as $|x-y| \geq 0$)

(Therefore,)

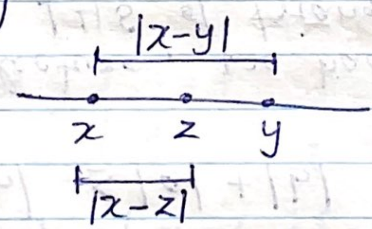
<u>Case 2</u>: $x \le y \le z$



Here, $|x-z| = |x-y| + |y-z|$

$\therefore$ $|x-z| \le |x-y| + |y-z|$ (unobjectionably)

<u>Case 3</u>: $z < y$



Clearly, $|x-z| \le |x-y|$

$\therefore$ $|x-z| \le |x-y| + |y-z|$.

Since we have covered all possible cases, we have verified the inequality. ▨

(end of the proof)

An application: For any real number x,

$$
\begin{aligned}
|x-1| &= |x-3+3-1| \\
&\le |x-3| + 2 .
\end{aligned}
$$

Thm: Triangle Inequality II (An useful variant)

For $a, b \in \mathbb{R}$,
$$|a+b| \leq |a| + |b|$$

Proof: Apply triangle inequality with

$x = a$, $z = -b$, and, $y = 0$ to get

$$|x-z| = |a+b| \leq |x-y| + |y-z| = |a-0| + |0+b|$$
$$= |a| + |b|$$

So $|a+b| \leq |a| + |b|$ $\blacksquare$

Remark: In our lecture, we had $|x-y|$ instead of $|x-z|$ in LHS of triangle inequality. It is intentionally changed to make you train in achieving some flexibility when abstractly mathematics is written. (Sorry!)

Interval notations (for convenience): $a, b \in \mathbb{R}$

Open Interval: $(a, b) = \{x \underset{\text{belongs to}}{\in} \mathbb{R} \underset{\text{such that}}{:} a < x < b\}$

Closed Interval: $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$

Half open Intervals:

$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$

$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$

$$(a,\infty) = \{x \in \mathbb{R} : \quad x > a\}$$
$$(-\infty,b) = \{x \in \mathbb{R} : \quad x < b\}$$

With these notations, observe that $(-\infty,\infty) = \mathbb{R}$.

Note $-\infty$, (or) ∞ is not a real number.

Exercise: Solve the following equations/inequalities. i.e (that is) find the real numbers x that satisfy the following equations/inequalities. Use intervals whenever you can.

1) $|x-4| < 5$

Solution: Note that by definition,

$$|x-4| = \begin{cases} x-4 & \text{if } x-4 \geq 0 \\ 4-x & \text{if } x-4 < 0 \end{cases}$$

$$\text{(ie) } |x-4| = \begin{cases} x-4 & \text{if } x \geq 4 \\ 4-x & \text{if } x < 4 \end{cases}$$

It is often useful to recognize this as sign change at 4. Consider case wise.

Case i) When $x \geq 4$ (i.e) $x \in [4,\infty)$.

In this case, $|x-4| = x-4$

So $|x-4| < 5 \underset{\text{(implies)}}{\Longrightarrow} x-4 < 5$

So $x < 9$ (i.e) $x \in (-\infty, 9)$

Always remember to restrict your case wise solution to case's assumption.

So for case i) $x \in [4,\infty) \cap (-\infty, 9)$.

(ie) $x \in [4,9)$.

<u>Case ii)</u>: When $x < 4$ (i.e) $x \in (-\infty, 4)$.

In this case, $|x-4| = 4-x$.

So $|x-4| < 5 \Longrightarrow 4-x < 5$

Recall that multiplying both sides of an inequality by a negative value flips the inequality.

Multiply b.s of $4-x<5$ by -1 to get

$$x-4 > -5$$

$$\Rightarrow \quad x > -1$$

(i.e) $x \in (-1, \infty)$

So for case ii) $\underset{\text{Case's assumption}}{(-\infty, 4)} \cap \underset{\text{Case's solution}}{(-1,\infty)}$, i.e. $x \in (-\infty,4) \cap (-1,\infty)$

(i.e) $x \in (-1,4)$.

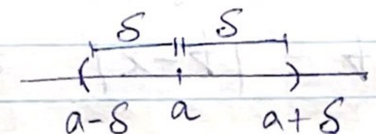
So when $x \in (-1,4)$ (or) $[4,9)$, we have $|x-4| < 5$. and viceversa (ie) $x \in (-1,4) \cup [4,9) \overset{\text{if and only if}}{\Longleftrightarrow} |x-4| < 5$.

$x \in (-1,9) \Longleftrightarrow |x-4| < 5$.

<u>Remark:</u> Sometimes, from a given set, you might to find the corresponding inequality.

For example, consider $x \in (a-\delta, a+\delta)$, a is fixed real no. $\delta > 0$

δ - Greek letter called delta



Pictorially $(a-\delta, a+\delta)$ is the set of all real numbers x whose distance away from 'a' is less than δ.

So $x \in (a-\delta, a+\delta) \Longleftrightarrow |x-a| < \delta$.

Convince yourselves of the following:

$x \in [a-\delta, a+\delta] \Longleftrightarrow |x-a| \le \delta$

$x \in (a-\delta, a+\delta) \setminus \{a\} \Longleftrightarrow 0 < |x-a| < \delta$.

(Punctured open interval at a)

2) $1 < |x-4|$

Observe that the expression $x-4$ changes sign at 4.

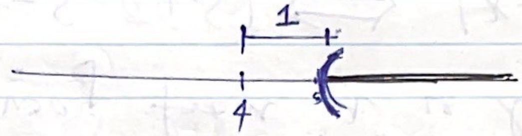
Case i) : When $x \geq 4$ (ie) $x \in [4, \infty)$.

Here $|x-4| = x-4$

So $|x-4| > 1 \Rightarrow x-4 > 1$

(ie) $x > 5$

So for case i), $x \in [4, \infty) \cap (5, \infty)$
$= (5, \infty)$



Case ii) : When $x < 4$ (i.e) $x \in (-\infty, 4)$

Here $|x-4| = 4-x$

So $|x-4| > 1 \Rightarrow 4-x > 1$

$\Rightarrow x-4 < -1$

(i.e) $x < 3$

So for case ii), $x \in (-\infty, 4)$ and
$x \in (-\infty, 3)$

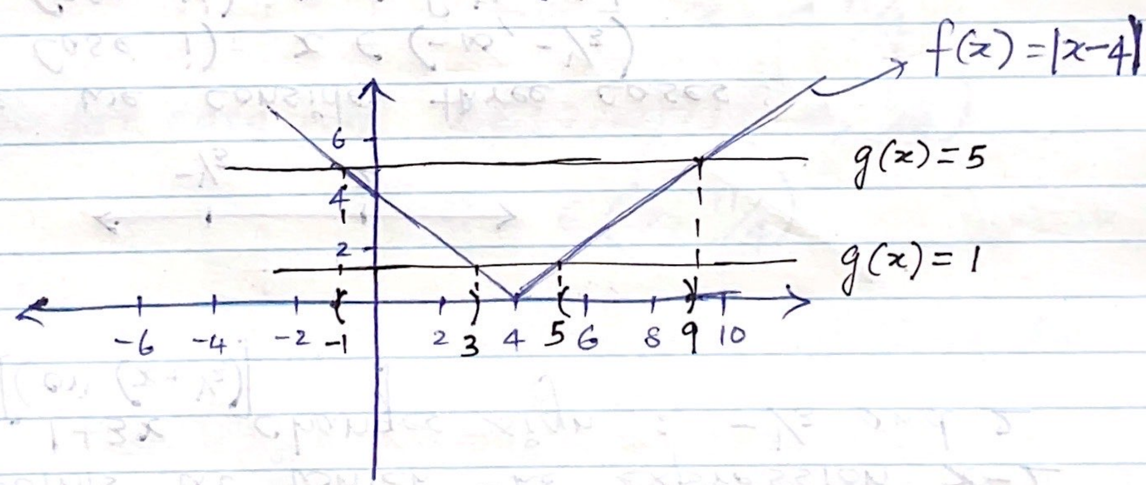
(i.e) $x \in (-\infty, 4) \cap (-\infty, 3)$
$= (-\infty, 3)$

"and" corresponds to intersection $\cap$
"or" corresponds to union $\cup$

$\therefore$ The solution for $1 < |x-4|$ is
$x \in (5, \infty)$ or $(-\infty, 3)$
(i.e.) $x \in (-\infty, 3) \cup (5, \infty)$

3) $1 < |x-4| < 5$

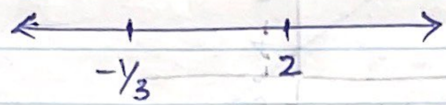
$$1<|x-4|<5 \iff 1<|x-4| \text{ and } |x-4|<5$$
$$\iff x\in(-\infty,3)\cup(5,\infty) \text{ and } x\in(-1,9)$$
$$\iff x\in[(-\infty,3)\cup(5,\infty)]\cap(-1,9)$$
$$\iff x\in[(-\infty,3)\cap(-1,9)]\cup[(5,\infty)\cap(-1,9)]$$
$$\iff x\in(-1,3)\cup(5,9)$$



4) $|x-2| + |1+3x| < 10$

Rewrite $|1+3x| = 3|x+1/3|$

As with the other problems, determine the points at which the expression $x-2$ and $1+3x$ [(or) $(x+1/3)$] changes sign : $-1/3$ and 2



Here we consider three cases :

Case i) $x \in (-\infty, -1/3)$

Case ii) $x \in [-1/3, 2]$

Case iii) $x \in (2, \infty)$

Case i) : $x \in (-\infty, -1/3)$

In this case $|x-2| = 2-x$ and $|1+3x| = -3x-1$.

So $|x-2| + |1+3x| < 10 \Rightarrow 2-x-3x-1 < 10$

(i.e) $1-4x < 10$

$\Rightarrow 4x-1 > -10$

$\therefore x > -9/4$

So for case i) $x \in (-\infty, -1/3) \cap (-9/4, \infty)$

(i.e) $x \in (-9/4, -1/3)$.

Case ii) : $x \in [-1/3, 2]$

Now, $|x-2| = 2-x$ and $|1+3x| = 1+3x$

So $|x-2| + |1+3x| < 10 \Rightarrow 2-x+1+3x < 10$

(i.e) $3+2x < 10$

$\Rightarrow x < 7/2$

So for case ii) $x \in [-1/3, 2] \cap (-\infty, 7/2)$

(i.e) $x \in [-1/3, 2]$ as $7/2 > 2$

Case iii) : $x \in (2, \infty)$

Now, $|x-2| = x-2$ and $|1+3x| = 1+3x$

So $|x-2| + |1+3x| < 10 \Rightarrow x-2+1+3x < 10$

(i.e) $4x-1 < 10$

$\Rightarrow x < 11/4$

So for case iii), $x \in (2, \infty) \cap (-\infty, 11/4)$

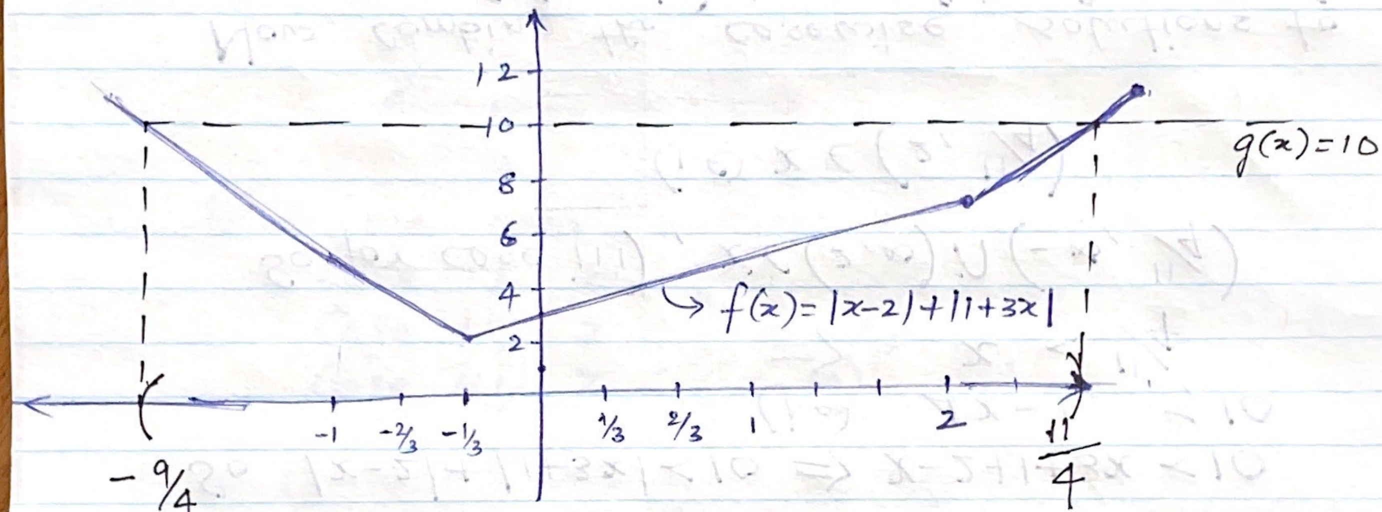
(i.e) $x \in (2, 11/4)$

Now combine the casewise solutions to get $x \in (-9/4, -1/3)$ (or) $x \in [-1/3, 2]$ (or) $x \in (2, 11/4)$

(i.e) $x \in (-9/4, -1/3) \cup [-1/3, 2] \cup (2, 11/4) = (-9/4, 11/4)$

Graphically,

$$|x-2|+|1+3x| = \begin{cases} -4x+1, & x \in (-\infty, -\tfrac{1}{3}) \\ 3+2x, & x \in [-\tfrac{1}{3}, 2] \\ 4x-1, & x \in (2, \infty) \end{cases}$$



Graphical analysis quickly reveals the solution for case ii)

If $3+2x$ were to extend and hit $g(x)=10$, it would at $x = 7/2$.