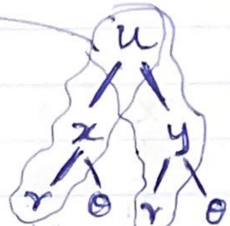
<u>CRE - Polar form:</u>

Given $u_x = v_y$, $u_y = -v_x$ for an analytic function, convert these to polar form using

$x = r\cos\theta$ i.e., express CRE in terms
$y = r\sin\theta$ of partials w.r.t r, θ.

<u>Soln</u>.

$$u_r = u_x x_r + u_y y_r$$



$$= u_x \cos\theta + u_y \sin\theta$$

Now, do the same for u_θ, v_r, v_θ & use CRE - cartesian form to derive (PS 7 Q 6.)

$$\boxed{r u_r = v_\theta \qquad r v_r = -u_\theta}$$

Note that

To compute $u_\theta = u_x x_\theta + u_y y_\theta$, Note we would need arg to be a continuous function. So Choose a branch cut and our f must be defined on appropriate domain where arg is continuous!

$u + iv$

<u>PS 7 Q7 (a)</u>: If $f(z) = u(r,\theta) + i v(r,\theta)$ is holomorphic (analytic) in a domain that does not include the origin. Then using CRE-polar form, you show that $u(r,\theta)$ satisfies the PDE: $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$ (Polar form of Laplace eq.)
($v(r,\theta)$ also satisfies the same equation).

Recall that if $f = u(x,y) + iv(x,y)$ is analytic in a domain D, then both u and v are harmonic in D. (i.e) $\nabla^2 u = u_{xx} + u_{yy} = 0$

$$\nabla^2 v = v_{xx} + v_{yy} = 0.$$

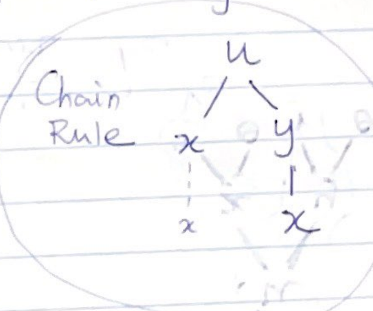
Given an harmonic function u in D, we call v the harmonic conjugate of u if $u + iv$ is analytic in D.

(Guaranteed to exist such v if D is connected and simply connected).

FACT: The level curves of $u(x,y)$ and $v(x,y)$ are orthogonal at their intersections.

Idea: Differentiate $u(x,y) = c_1$ implicitly w.r.t x:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0$$



$$\Rightarrow \left.\frac{dy}{dx}\right|_u = -\frac{u_x}{u_y}$$

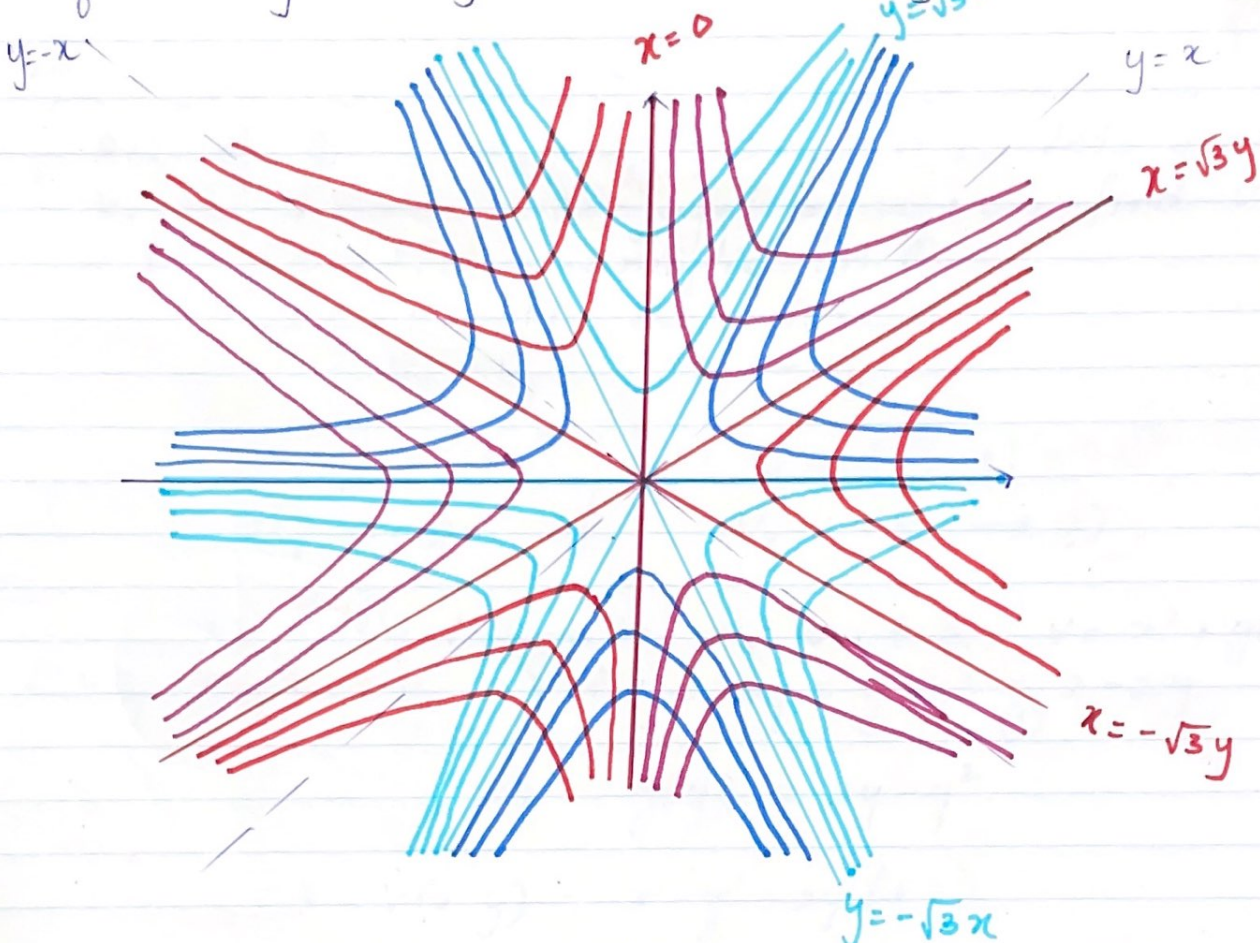
$\hookrightarrow$ this notation specifies that we mean the slope of the u level curves.

$\parallel^{ly}$ for $v(x,y) = c_2$, we get

$$\left.\frac{dy}{dx}\right|_v = -\frac{v_x}{v_y} \underset{\text{CRE}}{\overset{\downarrow}{=}} \frac{u_y}{u_x} = -\frac{1}{\left.\frac{dy}{dx}\right|_u}$$

(i.e) Slopes are negative reciprocals and hence the family of curves for u and v are orthogonal.

Recall the example from the previous lecture where we found that the harmonic conjugate of $u = y^3 - 3x^2y$ is $v = x^3 - 3xy^2$.



$v = +$ve constant
$v = -$ve constant
$u = -$ve constant
$u = +$ve constant

One can use the above fact to find orthogonal trajectories to a given family of curves, provided they are level curves of a harmonic fn. (PS 7 Q 4)

(An application of complex analysis in hind sight)

Ex. Find a family of curves which intersects the family $2x(1-y) = c$ orthogonally at all points of intersection.

Soln.

Let $u = 2x(1-y)$. Then u is harmonic on all of $\mathbb{C}$. ($u_{xx} + u_{yy} = 0 + 0 = 0$). Let's find v, the harmonic conjugate of u i.e., find v s.t $f = u + iv$ is analytic on $\mathbb{C}$.

$\Rightarrow$ CRE holds everywhere

$$u_x = v_y \qquad v_x = -u_y.$$

$$\left.\begin{aligned} u_x &= 2(1-y) \\ u_y &= -2x \end{aligned}\right\} \Rightarrow \begin{aligned} v_y &= 2(1-y) \longrightarrow (1) \\ v_x &= 2x \longrightarrow (2) \end{aligned}$$

Partially integrate v_x w.r.t x: $v = x^2 + g(y)$

Partially Diff. v w.r.t y: $v_y = g'(y) \underset{(1)}{=} 2 - 2y$

$$\Rightarrow \quad g(y) = 2y - y^2$$

$$\Rightarrow \quad v(x,y) = x^2 - y^2 + 2y \; (+k)$$

$\therefore$ the family of curves orthogonal to $2x(1-y) = c$ is :

$$x^2 - y^2 + 2y = C_1 \quad (\text{or}) \quad x^2 - (y-1)^2 = C_2$$

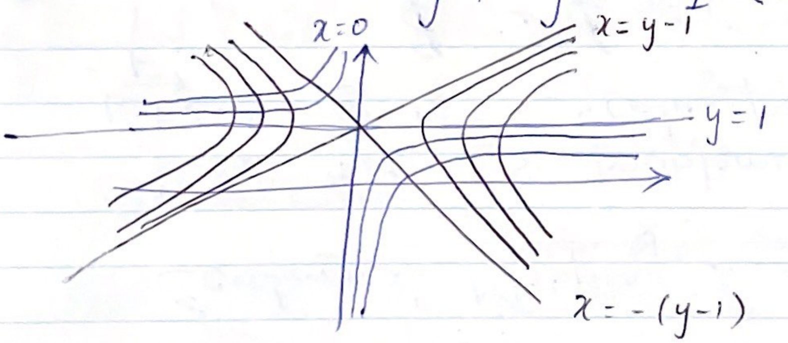


Illustration of
$u = +\text{ve}$ constants
$v = +\text{ve}$ constants

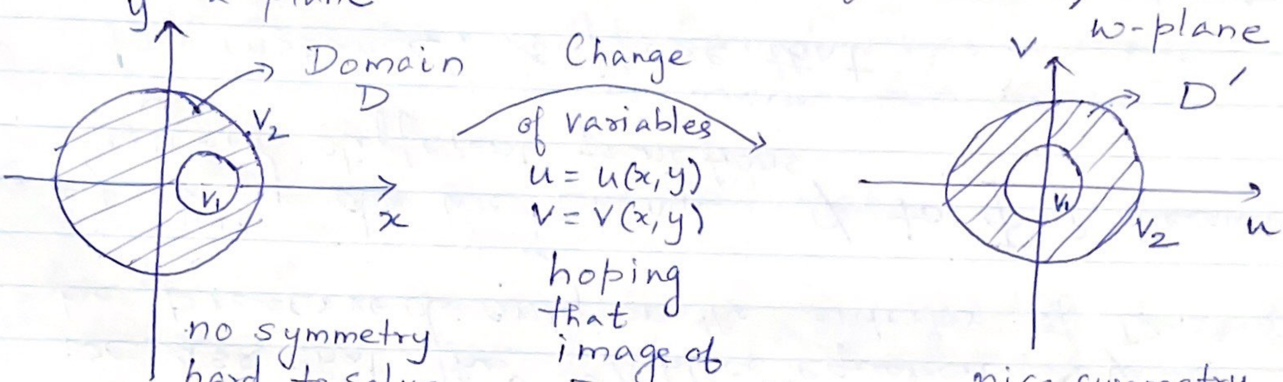
Conformal mapping & Electrostatics Problems

Recall that ES problems satisfy $\nabla^2\phi = 0$, where ϕ is a scalar potential.

$$\phi_{xx} + \phi_{yy} = 0 \quad (\text{Cartesian form of } \nabla^2\phi = 0)$$

$$\phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} = 0 \quad \text{on } \mathbb{R}^2 - \{(0,0)\}$$

$$(\text{Polar form of } \nabla^2\phi = 0)$$



hoping that image of D is simpler to work with

no symmetry
hard to solve $\nabla^2\phi$ in D given the boundary values of ϕ

nice symmetry
easier to solve the corresponding complex potential

$$\Phi(u,v) = \phi(x(u,v), y(u,v))$$

depends only on the radius!

Thm: [Preservation of Laplace Equation]

PS 7 Q8: $\Phi(u,v)$ is harmonic on D', where D' is the image of D in w-plane, under the mapping $w = f(z) = u + iv$.

The condition $|f'(z)| \neq 0$ for all $z \in D$ ensures that

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} u_x & -v_x \\ v_x & u_x \end{vmatrix} = u_x^2 + v_x^2 \neq 0.$$ So, we can

recover x and y from change the variables as $x(u,v)$ and $y(u,v)$ and find

$$\phi(x,y) = \Phi(u,v) \text{ by returning to z-plane}$$

[We will want a one-to-one correspondence between points in D and points D'].

Remarks:

1) $f'(z) \neq 0$ in D means at all interior points of D; $f'(z)$ need not be non-zero on the boundary of D. In other words, we ask that the Laplace equation $\nabla^2\phi = 0$ be preserved only on the interior of D.

2) Why do we change ϕ to Φ? Because they are different functions.

For instance, suppose that

$$\Phi(u,v) = u + 2v \text{ and } f(z) = z^2$$

Then $u = x^2 - y^2$, $v = 2xy$ so

$$\Phi(u,v) = u + 2v = x^2 - y^2 + 4xy = \phi(x,y)$$

Evidently, $\Phi(u,v) = u + 2v$ & $\phi(x,y) = x^2 - y^2 + 4xy$ are different functions

$\Phi(0,1) = 2$, $\phi(0,1) = -1$.

Mappings that preserve angle between curves and their orientation are said to be Conformal.

Thm. An analytic function f is conformal at every point z_0 at which $f'(z_0) \neq 0$.

Eg. The mapping $w = z^2$ is conformal at every point except $z = 0$.

Importatnt take-away: Conformal maps carry harmonic functions to harmonic functions.

An useful class of conformal maps

Fractional linear transformation ((or) bilinear (or) Möbius) $w = \dfrac{az+b}{cz+d}$ for $z \neq -d/c$

with $ad-bc \neq 0$

$a, b, c, d \in \mathbb{C}$.

Three classes of elementary Möbius mappings | $w' = \dfrac{ad-bc}{(cz+d)^2} \neq 0$ in its domain

τ_b: $z \mapsto z + b$ (translation)

M_a: $z \mapsto az$, $a \neq 0$ (Magnification by $|a|$ and CCW rotation by θ where $a = |a|e^{i\theta}$)

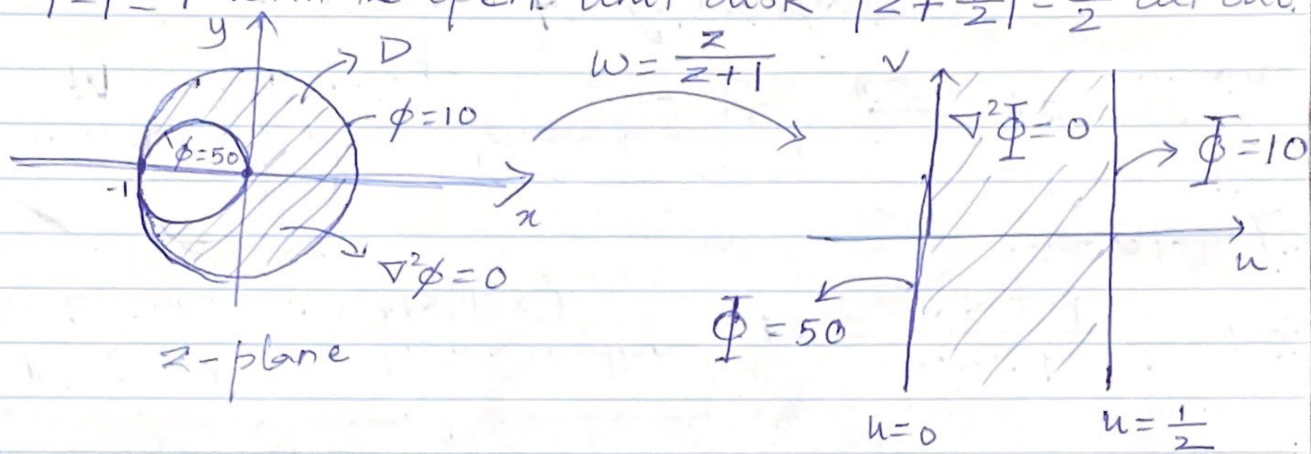
and $I: z \mapsto \dfrac{1}{z}$ (inversion)

$$w = \frac{az+b}{cz+d} = \frac{a}{c} + \frac{bc-ad}{c^2}\,\frac{1}{z+(d/c)}$$

$$\left[\; z \xmapsto[\tau_{d/c}]{} z+(d/c) \xmapsto[I]{} \frac{1}{z+d/c} \xmapsto[M_{\frac{bc-ad}{c^2}}]{} \frac{bc-ad}{c^2}\,\frac{1}{z+d/c} \xmapsto[\tau_{a/c}]{} \frac{a}{c} + \frac{bc-ad}{c^2}\,\frac{1}{z+d/c} \right.$$

<u>FACT</u>: Möbius maps takes circles (or) straight lines in $\mathbb{C}$ (z-plane) to circles (or) straight lines in w-plane.

<u>An application</u> Consider the following Laplace problem on the domain D which is the disk $|z| \le 1$ with the open unit disk $|z+\frac{1}{2}| = \frac{1}{2}$ cut out.



$$\Phi_{uu} + \Phi_{vv} = 0$$

$$w = \frac{z}{z+1} \Rightarrow w(z+1) = z \Rightarrow z = \frac{w}{1-w}$$

$\therefore$ $|z| = 1$ maps to $\left|\frac{w}{1-w}\right| = 1 \Rightarrow \underbrace{|w| = |1-w|}$

all points in w-plane that are equidistant from $u=0$ and $u=1$
$\therefore$ the line $u = \frac{1}{2}$

(Realize that $|z| < 1$ maps to $u < \frac{1}{2}$)

$$\left|z + \frac{1}{2}\right| = \frac{1}{2} \xrightarrow{\text{maps to}} \left|\frac{w}{1-w} + \frac{1}{2}\right| = 1 \Rightarrow \left|\frac{w+1}{1-w}\right| = 1$$

$\Rightarrow |w+1| = |1-w| \Rightarrow$ the line $u=0$

Furthermore, $w'(z) = \frac{1}{(z+1)^2} \neq 0$ for all $z \neq -1$ (particularly in the interior of D').

So w is conformal (in interior of D) and carries $\phi(x,y)$ to the harmonic function $\Phi(u,v)$ on $D' : \{u+iv : 0 \leq u \leq \frac{1}{2}\}$

Solving Φ on D' is easy because Φ varies with u but not v. So it is solving one-dimensional PDE: $\Phi_{uu} = 0$

$\Rightarrow \Phi(u,v) = A + Bu$ for some constants A and B.

$$\Phi(0,v) = 50 \Rightarrow \boxed{A = 50}$$

$$\Phi(\tfrac{1}{2}, v) = 10 \Rightarrow 50 + \frac{B}{2} = 10 \Rightarrow \boxed{B = -80}$$

$$\therefore \quad \Phi(u,v) = 50 - 80u$$

$$w = u + iv = \frac{z}{z+1} = \frac{x+iy}{(x+1)+iy} = \frac{x(x+1)+y^2}{(x+1)^2+y^2} + i\,\frac{y}{(x+1)^2+y^2}$$

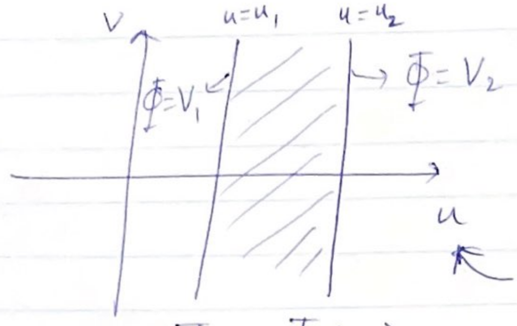
$$\therefore \phi(x,y) = \Phi\big(u(x,y), v(x,y)\big) = 50 - 80\,\frac{x(x+1)+y^2}{(x+1)^2+y^2} = 50 - 80\,\frac{|z+\frac{1}{2}|^2 - \frac{1}{4}}{|z+1|^2}$$

Plot the equipotentials for constants between 10 and 50.

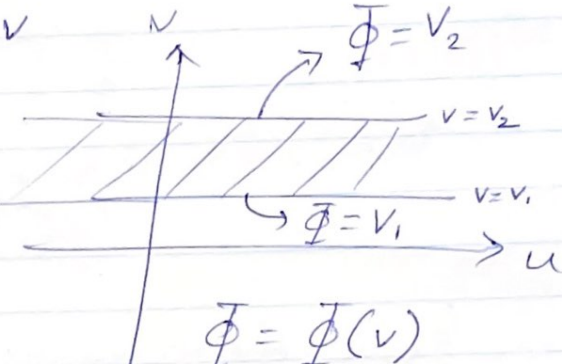
Following simplifications of the domains in w-plane are easy to solve $\nabla^2 \Phi = 0$

Cartesian Laplace: $\Phi_{uu} + \Phi_{vv} = 0$

$w = u + iv$



infinite Strips



$\Phi = \Phi(u)$

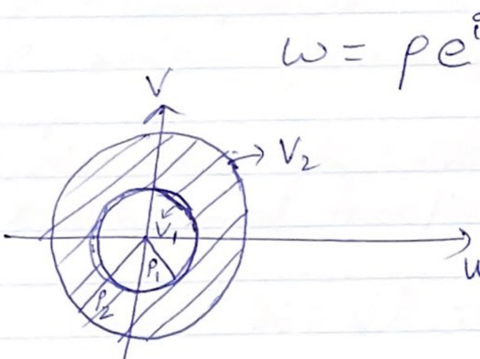
$\Rightarrow \Phi_{uu} = 0 \Rightarrow \Phi(u) = A + Bu$

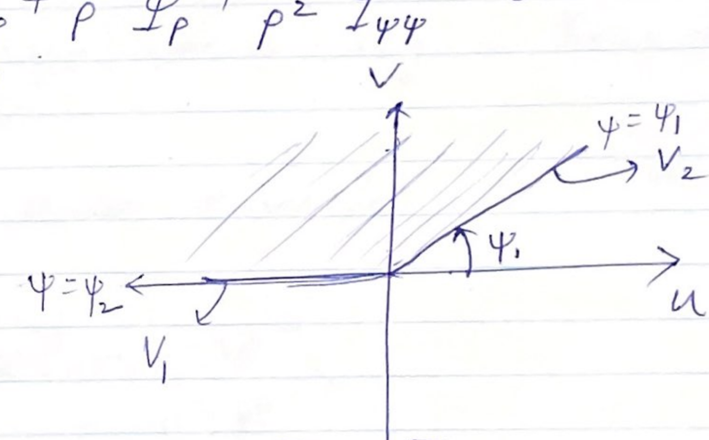
$\Phi = \Phi(v)$

$\Rightarrow \Phi(v) = A + Bv$

Polar Laplace: $\Phi_{\rho\rho} + \frac{1}{\rho}\Phi_\rho + \frac{1}{\rho^2}\Phi_{\varphi\varphi} = 0$

$w = \rho e^{i\varphi}$





$u = \rho\cos\varphi \quad v = \rho\sin\varphi$

$\Rightarrow \Phi = \Phi(\rho)$

$\Rightarrow \Phi_{\rho\rho} + \frac{1}{\rho}\Phi_\rho = 0$

$\Rightarrow \Phi = A + B\ln\rho$

So $\phi(x,y) = A + B\ln|f(z)|$

$\Rightarrow \Phi = \Phi(\varphi)$

$\Rightarrow \frac{1}{\rho^2}\Phi_{\varphi\varphi} = 0$

$\Rightarrow \Phi = A + B\varphi$

So $\phi(x,y) = A + B\tan^{-1}\left(\frac{v(x,y)}{u(x,y)}\right)$