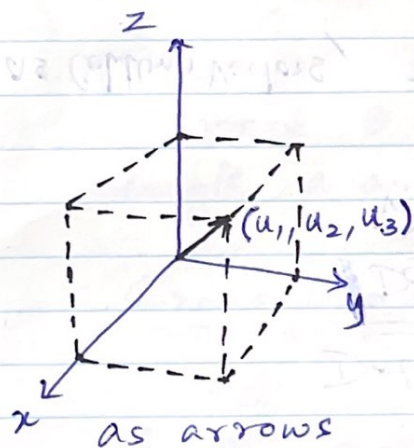


Linear Algebra Review

On paper/Chalkboard, we typically write  $\vec{v}$  instead of boldfaced as you would see in Mobius Content (or) assignments (or) exams.

Defn: A (real)  $n$ -dimensional vector  $\vec{v}$  is an  $n$ -tuple  $\vec{v} = (v_1, v_2, \dots, v_n)$  of real numbers  $v_1, v_2, \dots, v_n \in \mathbb{R}$ . The collection of all real  $n$ -dimensional vectors is denoted as  $\mathbb{R}^n$ .

 $\mathbb{R}^3$  - visualizations / realizations

$u = (u_1, u_2, u_3)$   
 $(x, y, z)$   
 $(1, 0, 2)$   
 etc  
 as ordered  
 lists

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

Column  
vectors

$$u_1 \hat{i} + u_2 \hat{j} + u_3 \hat{k}$$

$$\hat{i} + 2\hat{k}$$

$\hat{i}\hat{j}\hat{k}$ -notation

$$\hat{i} = (1, 0, 0)$$

$$\hat{j} = (0, 1, 0)$$

$$\hat{k} = (0, 0, 1)$$

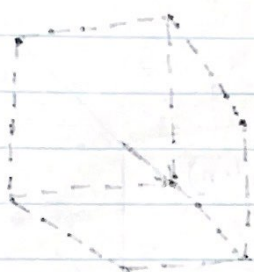
One important vector we must distinguish in  $\mathbb{R}^3$  is the zero vector (or the origin), denoted by

$$\vec{0} = (0, 0, 0).$$

Defining features of  $\mathbb{R}^3$  as a vector space are

(For any  $\vec{u}, \vec{v} \in \mathbb{R}^3$  with  $\vec{u} = (u_1, u_2, u_3)$  and any scalar  $a \in \mathbb{R}$ , we have)

- 1) Scalar multiplication :  $a\vec{u} = (au_1, au_2, au_3)$
- 2) Vector addition :  $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$



### Dot Product, norm, and angle

Defn. The dot product of vectors  $\vec{u}$  and  $\vec{v}$  is defined as

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$$

as Column Vectors,  $\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = (u_1, u_2, u_3) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

Norm of  $\vec{v} \in \mathbb{R}^3$  :  $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$   
 $= \sqrt{v_1^2 + v_2^2 + v_3^2}$

### Properties:

- 1)  $\|a\vec{v}\| = |a| \|\vec{v}\|$  for all  $a \in \mathbb{R}$  and  $\vec{v} \in \mathbb{R}^3$
- 2)  $\|\vec{v}\| \geq 0$  for all  $\vec{v} \in \mathbb{R}^3$  (and  $\|\vec{v}\| = 0 \Leftrightarrow \vec{v} = \vec{0}$ )
- Triangle inequality  $\rightarrow$  3)  $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$  for all  $\vec{u}, \vec{v} \in \mathbb{R}^3$
- 4)  $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$  for all  $\vec{u}, \vec{v} \in \mathbb{R}^3$   
 $\hookrightarrow$  (Cauchy-Schwarz inequality)

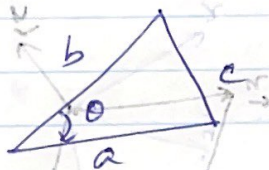
Angle between two non-zero vectors in  $\mathbb{R}^3$   
 $\rightarrow$  unique number  $\theta \in [0, \pi]$  that satisfies

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta.$$

i.e.,  $\theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}\right)$

Defn. Two vectors  $\vec{u}, \vec{v} \in \mathbb{R}^3$  are said to be orthogonal if  $\vec{u} \cdot \vec{v} = 0$ . Write  $\vec{u} \perp \vec{v}$ .

Exercise: Prove the law of cosines. If  $a, b, c$  are lengths of the sides of any triangle,



where  $\theta$  is the angle between the sides of length  $a$  and  $b$ , then  $c^2 = a^2 + b^2 - 2ab \cos \theta$

### Cross Product

Defn. The cross product of vectors  $\vec{u}$  and  $\vec{v}$  in  $\mathbb{R}^3$  is the vector  $\vec{u} \times \vec{v}$  defined as

$$\vec{u} \times \vec{v} = (u_2 v_3 - v_2 u_3) \hat{i} + (u_3 v_1 - u_1 v_3) \hat{j} + (u_1 v_2 - u_2 v_1) \hat{k}$$

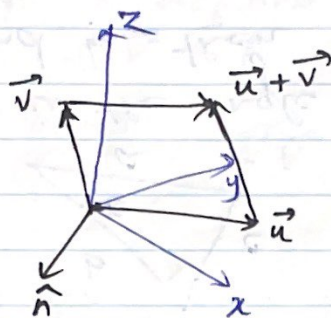
(Idea: use a determinant)

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \hat{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \hat{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \hat{k}$$

Properties for all  $\vec{u}, \vec{v} \in \mathbb{R}^3$ ,

- 1)  $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
- 2)  $\vec{u} \times \vec{u} = \vec{0}$
- 3)  $\vec{u} \times \vec{v} \perp \vec{u}, \vec{u} \times \vec{v} \perp \vec{v}$
- 4)  $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin \theta$ , where  $\theta$  is the angle between  $\vec{u}$  and  $\vec{v}$

Any two vectors  $\vec{u}$  and  $\vec{v}$  in  $\mathbb{R}^3$  define a parallelogram



$$\hat{n} = \frac{\vec{u} \times \vec{v}}{\|\vec{u} \times \vec{v}\|}$$

is the normal unit vector that is perpendicular to the plane of the parallelogram.

Area of the parallelogram defined by  $\vec{u}$  and  $\vec{v}$  in  $\mathbb{R}^3$  is given by  $\|\vec{u} \times \vec{v}\|$

True or False

1)  $\vec{u} \cdot \vec{v} = 0 \Leftrightarrow \vec{u} = \vec{0}, \vec{v} = \vec{0}$  (or) both

False  $(1, -1, 0) \cdot (-1, -1, 0) = 0$

perpendicular (or) orthogonal vectors.

2)  $\vec{u} \times \vec{v} = \|\vec{u}\| \|\vec{v}\| \sin \theta$

False:  $\vec{u} \times \vec{v}$  is a vector

$\|\vec{u}\| \|\vec{v}\| \sin \theta$  is a scalar.

3) The equation  $\hat{n} \cdot (\vec{x} - \vec{x}_0) = 0$  represents the equation of a plane passing through  $\vec{x}_0 = (x_0, y_0, z_0)$  with normal vector  $\hat{n}$   
True

4) The plane  $ax + by + cz = d$  has unit normal vector  $(a, b, c)$   
False.  $\|(a, b, c)\|$  need not be 1.