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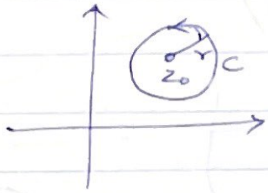
Recall that

→ $\int_C \frac{1}{z-z_0}\,dz = 2\pi i$ where $C = z_0 + re^{it}$ $\quad 0 \le t \le 2\pi$



→ Analytic functions on a simply-connected ~~doma~~ region D has an antiderivative in D. Therefore, the integral along any curve in D is path-independent.

(Note this region need not be the domain of the function) Eg: $f(z) = \frac{1}{z}$ is analytic on $\mathbb{C} - \{0\}$. But $\mathbb{C} - \{0\}$ is not simply-connected. $\mathbb{C} | L_\alpha$ is simply-connected. ($L_\alpha = \{te^{i\alpha} : t \ge 0\}$) So, it has anti-derivative on $\mathbb{C} | L_\alpha$ for any $\alpha \in \mathbb{R}$. What is it? $\log_\alpha z$!

→ Cauchy-Goursat Theorem:

If a function $f(z)$ is analytic at all points interior to and on a simple closed curve C, then

$$\oint_C f(z)\,dz = 0.$$

→ Principle of deformation of paths:

Let C_1 and C_2 be simple, closed curves oriented positively. If f is analytic on C_1, C_2 and the region in between, then $\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$

Cauchy's Integral Formula

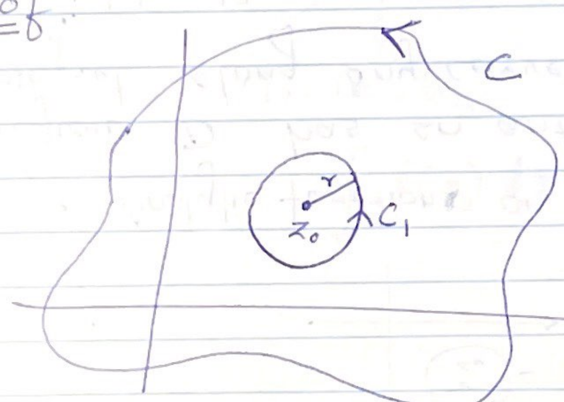
Let f be analytic everywhere within and on a simple closed contour C. If z_0 is any point interior to C, then

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz$$

Rewritten as $$\oint_C \frac{f(z)}{z - z_0}\, dz = 2\pi i\, f(z_0)$$

This is remarkable formula because the value of any analytic function inside a closed contour is completely determined by its values on the contour. On the other hand, this also provides a way to evaluate integrals over closed contours.

Proof:



Since f is analytic everywhere within C, at z_0, there is a neighborhood within C on which f is diff. So Surround z_0 by a small circle of radius r, (Say C_1)

Orient C and C_1 (CCW).

By the principle of deformation of paths,

$$\oint_C \frac{f(z)}{z-z_0}\,dz = \oint_{C_1} \frac{f(z)}{z-z_0}\,dz \quad (\star)$$

as $\dfrac{f(z)}{z-z_0}$ is analytic between C and C_1.

Now let $C_1 : z = z_0 + re^{it}$ $\quad 0 \le t \le 2\pi$.

Then $dz = rie^{it}$ and

$$\oint_{C_1} \frac{f(z)}{z-z_0}\,dz = \int_0^{2\pi} \frac{f(z_0+re^{it})}{re^{it}}\, rie^{it}\,dt$$

$$(\star) \Rightarrow \oint_C \frac{f(z)}{z-z_0}\,dz = i\int_0^{2\pi} f(z_0+re^{it})\,dt$$

for really small values of $r > 0$!

$$= \lim_{r\to 0} i \int_0^{2\pi} f(z_0+re^{it})\,dt$$

$$= i \int_0^{2\pi} \lim_{r\to 0} f(z_0+re^{it})\,dt$$

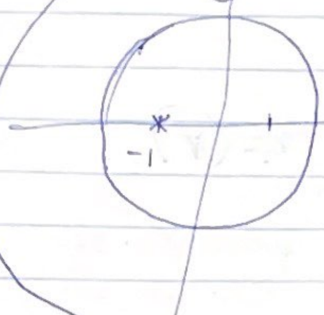
$$= i \int_0^{2\pi} f(z_0)\,dt = 2\pi i f(z_0).$$

Examples

1) Evaluate $\oint_C \frac{\cos z}{z+1}\,dz$ where $C : |z| = 2$

Recognize $\oint_C \frac{\cos z}{z+1}\,dz$ as $\oint_C \frac{f(z)}{z-z_0}\,dz$

So $z_0 = -1$, $f(z) = \cos z$ which is analytic everywhere within C.



(As such $\frac{\cos z}{z+1}$ has a singularity at $z=-1$. So we cannot apply Cauchy-Goursat theorem)

Nevertheless, we can apply C.I.F to $f(z) = \cos z$ as it is analytic everywhere inside C.

$$\therefore \oint_C \frac{\cos z}{z+1}\,dz = 2\pi i f(z_0) = 2\pi i f(-1)$$

$$= 2\pi i \cos(-1)$$

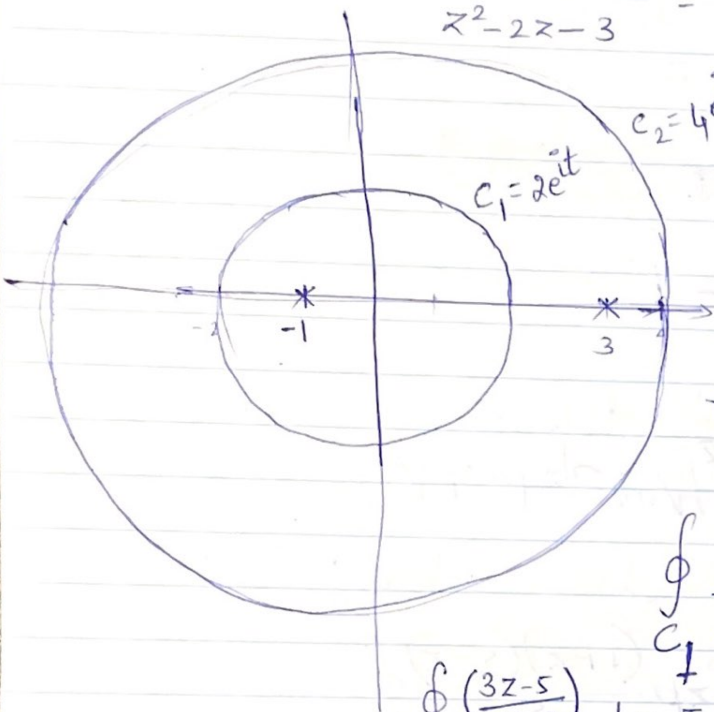
$$= 2\pi i \cos(1)$$

(since cos is an even function on $\mathbb{R}$)

Example. Evaluate $\oint_C \frac{3z-5}{z^2-2z-3}\,dz$ i) $C = C_1 : |z| = 2$

ii) $C = C_2 : |z| = 4$

Note that $\frac{3z-5}{z^2-2z-3} = \frac{3z-5}{(z-3)(z+1)}$



(i)- Only -1 lies inside C_1. So if we take $f(z) = \frac{3z-5}{z-3}$, then f is analytic everywhere within C. So, by C.I.F

$$\oint_{C_1} \frac{f(z)}{z-(-1)}\,dz = 2\pi i\, f(-1)$$

$$\oint_{C_1} \frac{\left(\frac{3z-5}{z-3}\right)}{z-(-1)}\,dz = \oint_{C_1} \frac{3z-5}{(z-3)(z+1)}\,dz = 2\pi i \left(\frac{-3-5}{-1-3}\right)$$

$$= 4\pi i$$

ii) $\frac{3z-5}{(z-3)(z+1)}$ has singular points at -1 and 3 inside C_2

Can't apply Cauchy's theorem (or)

find $f(z)$ which is analytic inside C_2 to apply C.I.F!

<u>One option</u>: Partial Fraction decomposition.

$$\frac{3z-5}{(z-3)(z+1)} = \frac{A}{z-3} + \frac{B}{z+1}$$

$$= \frac{(A+B)z + A-3B}{(z-3)(z+1)}$$

$\Rightarrow$ $A+B=3$ $\quad A-3B = -5$

$\Rightarrow$ $A=1,\ B=2$

$$\therefore \oint_{C_2} \frac{3z-5}{(z-3)(z+1)}\,dz = \oint_{C_2} \frac{1}{z-3}\,dz + \oint_{C_2} \frac{2}{z+1}\,dz$$

Advantage: Apply C.I.F to $f_1(z)=1$ $\quad f_2(z)=2$

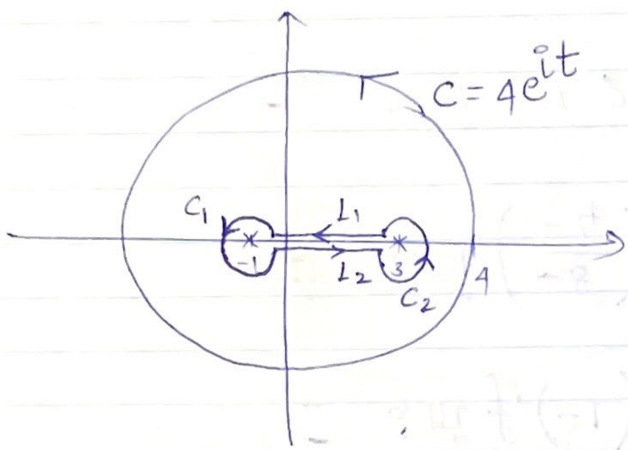
$$= 2\pi i f_1(3) + (2\pi i) f_2(-1)$$

~~$6\pi i$~~

$$= 2\pi i(1) + 2\pi i(2)$$

$$= 6\pi i$$

Second Option: Using the principle of deformation of paths.



Deform C into dumb-bell shaped as shown (keeping same orientation)

(i.e.) C and $C_1 \cup L_2 \cup C_2 \cup L_1$ are oriented CCW (+ve) and for f that is analytic on these two curves and the region between them,

$$\oint_C f(z)\,dz = \int_{C_1} f + \int_{L_2} f + \int_{C_2} f + \int_{L_1} f$$

(by the principle of deformation)

$$= \int_{C_1} f + \int_{C_2} f$$

(as we narrow down the width between L_1 and L_2 in the limit)

L_1 and L_2 will become a single line and their contributions cancel.

So, we can replace the integral around C which has 2 singular points inside with 2 integrals each having 1 singular point.

$$\oint_{C=4e^{it}} \frac{3z-5}{(z+1)(z-3)}\,dz = \oint_{C_1} \frac{3z-5}{(z+1)(z-3)}\,dz + \oint_{C_2} \frac{3z-5}{(z+1)(z-3)}\,dz$$

$C_1 = -1 + re^{it}$

r very small

(around $z=-1$)

$z_0 = -1$

$f_1(z) = \frac{3z-5}{z-3}$

analytic inside C_1

$\downarrow$ C.I.F to f_1

$C_2 = 3 + re^{it}$

r - very small

(around $z = 3$)

$z_0 = 3$

$f_2(z) = \frac{3z-5}{z+1}$

analytic inside C_2

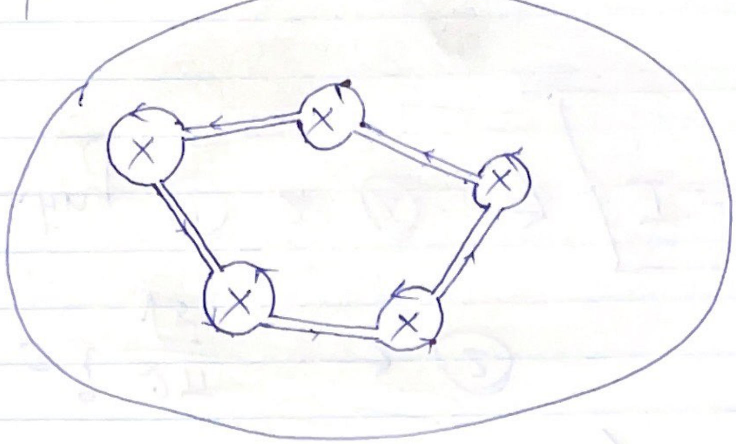
$\downarrow$ C.I.F to f_2

$$= 2\pi i\, f_1(-1) + 2\pi i\, f_2(3)$$

$$= 2\pi i\left(\frac{-8}{-4}\right) + 2\pi i\left(\frac{9-5}{4}\right)$$

$$= 4\pi i + 2\pi i = \underline{\underline{6\pi i}}$$

More singular points: Same idea → Deform the path



We will not consider for now
- the singular points on C
- line of singular points:

Evaluate: $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = I$ (say)

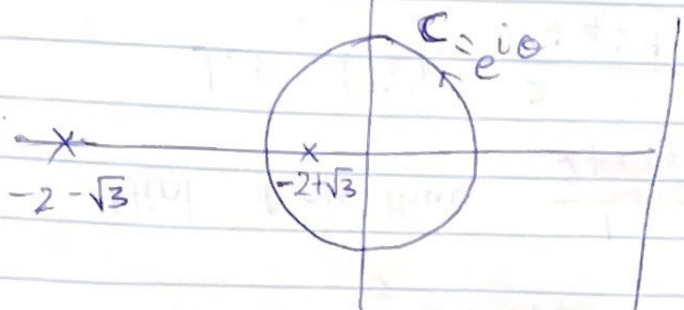
Hint: Note that $\dfrac{1}{2+\cos\theta} = \dfrac{2z}{z^2+4z+1}$ for $z = e^{i\theta}$

Let $f(z) = \dfrac{2}{z^2+4z+1} = \dfrac{2}{\left(z-(-2-\sqrt{3})\right)\left(z-(-2+\sqrt{3})\right)}$

We have $\displaystyle\oint_{C=e^{i\theta}} f(z)\,dz = \int_0^{2\pi} f(e^{i\theta}).\,ie^{i\theta}\,d\theta = i\int_0^{2\pi} \frac{2e^{i\theta}}{e^{2i\theta}+4e^{i\theta}+1}\,d\theta = iI$

$\longrightarrow (1)$

$$\oint_{C=e^{i\theta}} f(z)\,dz = \oint_C \frac{2}{\left(z-(-2-\sqrt{3})\right)\left(z-(-2+\sqrt{3})\right)}\,dz$$



$\dfrac{2}{z-(-2-\sqrt{3})}$ is analytic within C

C.I.F applied to

$$g(z) = \frac{2}{z-(-2-\sqrt{3})} = \frac{2}{z+2+\sqrt{3}}$$

$$z_0 = -2+\sqrt{3}$$

$$= 2\pi i\, g(z_0)$$

$$= 2\pi i \left(\frac{2}{-2+\sqrt{3}+2+\sqrt{3}}\right)$$

$$= i\,\frac{2\pi}{\sqrt{3}} \longrightarrow (2)$$

Equating (1) & (2) $\Rightarrow$ $\boxed{I = \dfrac{2\pi}{\sqrt{3}}}$