Sep 25, 2023

Recall,

The line integral of a vector field $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ over a curve $C: \vec{x} = \vec{g}(t),\ a \le t \le b$ is defined by

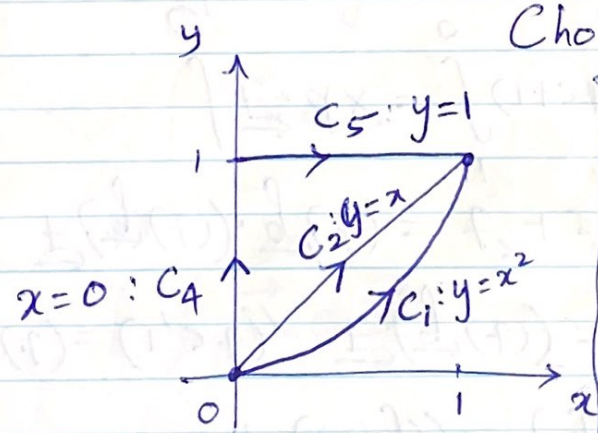
$$\int_C \vec{F} \cdot d\vec{x} = \int_a^b \vec{F}(\vec{g}(t)) \cdot \vec{g}\,'(t)\, dt$$

dot product

Example:

Evaluate the line integral $\vec{F} = (y, kx)$ (k constant) over the curves defined as follows:

1. C_1 is the parabola $y = x^2$ from $(0,0)$ to $(1,1)$
2. C_2 is the line $y = x$ from $(0,0)$ to $(1,1)$
3. $C_3 = C_4 \cup C_5$, where C_4 is the y-axis from $(0,0)$ to $(0,1)$ and C_5 is the line $y = 1$ from $(0,1)$ to $(1,1)$.



Choice of Parametrizations

Curves	$\vec{g}(t)$
$C_1: y = x^2$	(t, t^2) $0 \le t \le 1$
$C_2: y = x$	(t, t) $0 \le t \le 1$
$C_4: x = 0$	$(0, t)$ $0 \le t \le 1$
$C_5: y = 1$	$(t, 1)$ $0 \le t \le 1$

Curves $\vec{g}(t)$

i) $\vec{F}((x,y)) = (y, kx)$; k constant

1) $\vec{g}'(t) = (1, 2t)$, $\vec{F}(\vec{g}(t)) = (t^2, kt)$ and

$$\vec{F}(\vec{g}(t)) \cdot \vec{g}'(t) = t^2 + 2kt^2 = (1+2k)t^2$$

$$\therefore \int_{C_1} \vec{F} \cdot d\vec{x} = \int_0^1 (1+2k)t^2 dt = \frac{1}{3}(1+2k)$$

2) $\vec{g}'(t) = (1,1)$, $\vec{F}(\vec{g}(t)) = (t, kt)$ and

$$\vec{F}(\vec{g}(t)) \cdot \vec{g}'(t) = t + kt = (1+k)t$$

$$\int_{C_2} \vec{F} \cdot d\vec{x} = \int_0^1 (1+k)t\,dt = \frac{1}{2}(1+k)$$

3) A property of line integrals states that if $C_3 = C_4 \cup C_5$, then

$$\int_{C_3} \vec{F} \cdot d\vec{x} = \int_{C_4} \vec{F} \cdot d\vec{x} + \int_{C_4} \vec{F} \cdot d\vec{x}$$

$C_4: \vec{g}(t) = (0,t) \; ; \; 0 \le t \le 1.$

$\therefore \; \vec{g}'(t) = (0,1), \; \vec{F}(\vec{g}(t)) = (t,0)$ and

$\vec{F}(\vec{g}(t)) \cdot \vec{g}'(t) = 0.$ Thus,

$$\int_{C_4} \vec{F} \cdot d\vec{x} = 0.$$

(<u>Note</u>: the vector field $\vec{F}$ is of the form $(y,0)$ on the y-axis, i.e it acts horizontally. The path is vertical and thus the path and force are perpendicular, meaning that no work is done.)

$C_5: \vec{g}(t) = (t,1) \; ; \; 0 \le t \le 1$

$\therefore \; \vec{g}'(t) = (1,0), \; \vec{F}(\vec{g}(t)) = (1,kt)$ and

$\vec{F}(\vec{g}(t)) \cdot \vec{g}'(t) = 1.$ Thus,

$$\int_{C_5} \vec{F} \cdot d\vec{x} = \int_0^1 1 \, dt = 1$$

$$\Rightarrow \quad \int_{C_3} \vec{F} \cdot d\vec{x} = 1.$$

<u>Observe</u> for $\vec{F}((x,y)) = (y, kx)$; k constant

Curves C	$\int_C \vec{F}\cdot dx$
C_1	$\frac{1}{3}(1+2k)$
C_2	$\frac{1}{2}(1+k)$
C_3	1

→ different values, except when $k=1$!

And this is a special phenomenon.
For all of the paths, the start and end points are same.

Apparently when $k=1$, for $\vec{F} = (y, x)$, the value of the line integral will always be 1 irrespective of the path taken from $(0,0)$ to $(1,1)$

Such line integrals and vector field are called path independent. In contrast, when $k \neq 1$, for example, $\vec{F} = (y, 2x)$ (or) $\vec{F} = (y, -x)$. is ~~not~~ path dependent.

Apparently, for $\vec{F} = (y, x)$, there exists a scalar function $\phi(x,y) = xy$ s.t $\vec{F} = \vec{\nabla}\phi$.

We call such ϕ, the <u>scalar potential</u>

Any vector field $\vec{F}$ with this property, (ie) $\vec{F} = \vec{\nabla}\phi$, where ϕ is a scalar function has the path independence property.

Terminologies for such vector fields

$\longrightarrow$ gradient fields

$\longrightarrow$ Conservative fields.

The scalar potential $\longleftrightarrow$ potential energy function

Path independence $\longleftrightarrow$ Energy is conserved in the system.

Exam

1) $\vec{F}(\vec{r}) = -\dfrac{GMm}{\|\vec{r}\|^3}\vec{r}$ (The force of gravity exerted by a mass M on another mass m)

$\vec{r}$ - position vector from M to m

$\phi(\vec{r}) = +\dfrac{GMm}{\|\vec{r}\|}$ and $\vec{F} = \vec{\nabla}\phi$

2)
$$\vec{E}(\vec{r}) = \frac{Q}{4\pi\varepsilon_0 \|\vec{r}\|^3}\vec{r}$$
Here $\vec{r} = (x, y, z)$

(Electrostatic field generated by the presenke of a charge Q at origin)

Electrostatic potential
$$\phi(\vec{r}) = -\frac{Q}{4\pi\varepsilon_0 \|\vec{r}\|}$$

The following theorem says why line integrals of a gradient field is path-independent.

FUNDAMENTAL THEOREM OF LINE INTEGRALS:-
If there exist a function ϕ such that $\vec{F} = \vec{\nabla}\phi$, with C a curve that starts at $\vec{a}$ and ends at $\vec{b}$, then:
$$\int_C \vec{F} \cdot d\vec{x} = \phi(\vec{b}) - \phi(\vec{a})$$

Idea of a proof

Let $\vec{x} = \vec{g}(t)$, $p \leq t \leq q$ be a parametrization of C. Then
$$\int_C \vec{F} \cdot d\vec{x} = \int_p^q \vec{F}(\vec{g}(t)) \cdot \vec{g}'(t)\, dt$$

Since $\vec{F} = \vec{\nabla}\phi$, $\vec{g}(p) = \vec{a}$ and $\vec{g}(q) = \vec{b}$,

$$\int_C \vec{F} \cdot d\vec{x} = \int_p^q \vec{\nabla}\phi(\vec{g}(t)) \cdot \vec{g}\,'(t)\, dt$$

Let $f = \phi \circ \vec{g} : [p, q] \to \mathbb{R}$. Then

$$\frac{df}{dt} = \nabla\phi(\vec{g}(t)) \cdot \vec{g}\,'(t)$$

dot product

(Generally a matrix multiplication

$$\left[\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\bigg|_{\vec{g}(t)}\right] \cdot \begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix}$$

where $\vec{g}(t) = (x(t), y(t), z(t))$)

By Fundamental theorem of Calculus,

$$\int_p^q \frac{df}{dt}\, dt = f(q) - f(p)$$

$$\Rightarrow \int_p^q \vec{\nabla}\phi(\vec{g}(t)) \cdot \vec{g}\,'(t)\, dt$$

$$= f(q) - f(q)$$

$$= \phi \circ \vec{g}(q) - \phi \circ \vec{g}(p)$$

$$= \phi(\vec{b}) - \phi(\vec{a}).$$

$\blacksquare$

Remarks:

1) Pro: To apply the theorem, only endpoints are used, and no parametrization is required
Con: Must determine ϕ s.t $\vec{F} = \vec{\nabla}\phi$.

2) Generalization of the FTC:

$$\int_{\vec{a}}^{\vec{b}} \vec{\nabla}\phi \cdot d\vec{x} = \phi(\vec{b}) - \phi(\vec{a})$$

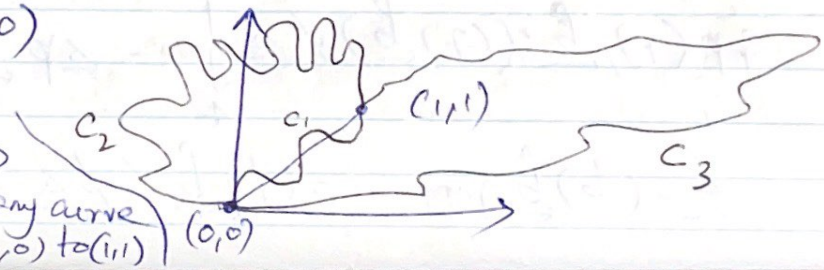
ϕ plays the role of an antiderivative of the vector field $\vec{F}$ and ϕ is a multi-variable function so its gradient vector is its "derivative".

Instead of integrating along an interval, we are integrating along a curve in $\mathbb{R}^3$.

Example revisited

We saw that $\vec{F} = (y, x)$ has a scalar potential $\phi = xy$ so that $\vec{F} = \vec{\nabla}\phi$. By the theorem,

$$\int_{(0,0)}^{(1,1)} \vec{F} \cdot d\vec{x} = \phi(1,1) - \phi(0,0) = 1 - 0 = 1.$$



Could be any curve that connects $(0,0)$ to $(1,1)$

QUESTION: Given a vector field $\vec{F}$, how do we determine whether ϕ exists such that $\vec{F} = \vec{\nabla}\phi$?

HEURISTICS:

Consider for now $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\vec{F} = (F_1, F_2)$ where

$F_1 : \mathbb{R}^2 \to \mathbb{R}$

$F_2 : \mathbb{R}^2 \to \mathbb{R}$

If $\vec{F} = \vec{\nabla}\phi$ for some $\phi : \mathbb{R}^2 \to \mathbb{R}$, let's see the consequences:

$$\vec{F} = \vec{\nabla}\phi \Rightarrow (F_1, F_2) = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right)$$

$$\frac{\partial \phi}{\partial x} = F_1, \quad \frac{\partial \phi}{\partial y} = F_2$$

$$\Rightarrow \underbrace{\frac{\partial^2 \phi}{\partial y \partial x}}_{\phi_{yx}} = \frac{\partial F_1}{\partial y}, \quad \underbrace{\frac{\partial^2 \phi}{\partial x \partial y}}_{\phi_{xy}} = \frac{\partial F_2}{\partial x}$$

Beyond the scope of this class: if partials ϕ_{xy} and ϕ_{yx} are continuous near (a, b), then

$$\phi_{xy}(a,b) = \phi_{yx}(a,b)$$

So we will assume the equality of mixed partials and get

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

Theorem: If ϕ exists such that $\vec{F} = \vec{\nabla}\phi$, and F_1 & F_2 have continuous first order partials ((or) equivalenty ϕ_{xy} and ϕ_{yx} are cts.)

then

$$\boxed{\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}}$$

An Example on finding ϕ for $\vec{F}$

Determine a scalar potential ϕ for

$$\vec{F} = (\underbrace{y^2 - y\sin x}_{F_1}, \underbrace{\cos x + 2xy - 2y}_{F_2})$$

Soln.

Check : $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$

$$\frac{\partial F_1}{\partial y} = 2y - \sin x \quad \& \quad \frac{\partial F_2}{\partial x} = -\sin x + 2y$$

So $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$ holds.

Now if $\vec{F} = \vec{\nabla}\phi$, then

$$\frac{\partial \phi}{\partial x} = y^2 - y\sin x, \quad \frac{\partial \phi}{\partial y} = \cos x + 2xy - 2y$$

$\hookrightarrow (*)$

Trick: Do partial integration w.r.t appropriate variables

$$\Rightarrow \quad \phi = \int \frac{\partial \phi}{\partial x}\, dx = \int (y^2 - y\sin x)\, dx$$

$$= xy^2 + y\cos x + g(y) \quad (**)$$

$\Downarrow$

Crucial understanding

Here, g is some arbitrary function of y, because y is treated as constant.

(if we differentiate $(**)$ w.r.t x, we get $y^2 - y\sin x$!)

Now,

differentiate $(**)$ w.r.t y and equate to $\frac{\partial \phi}{\partial y}$ in $(*)$

$$\frac{\partial \phi}{\partial y} = 2xy + \cos x + g'(y) = \cos x + 2xy - 2y$$

$$\Rightarrow \quad g'(y) = -2y$$

$$\Rightarrow \quad g(y) = -y^2 + C \quad (C \text{ is a constant})$$

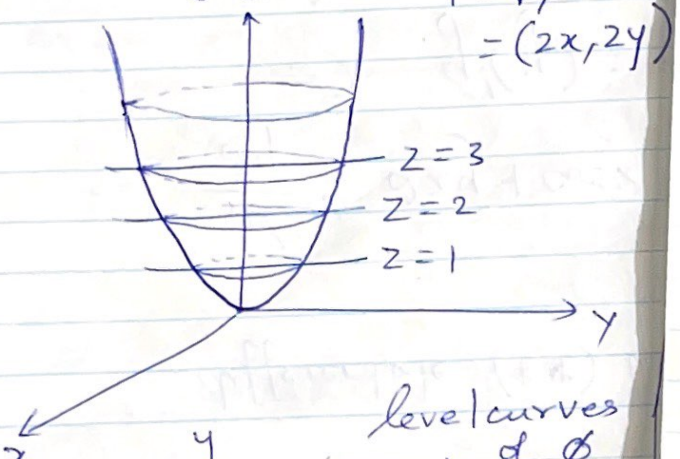
$\therefore \quad \phi = xy^2 + y\cos x - y^2 + C$ is our scalar potential.

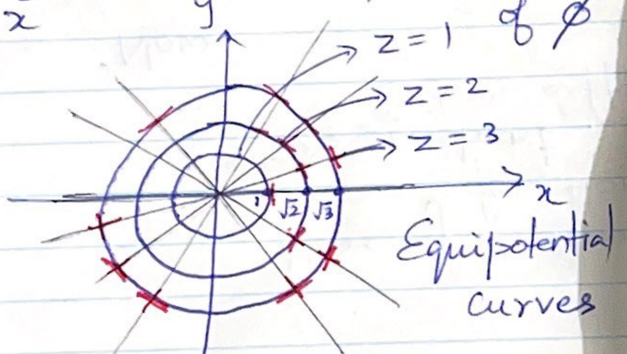
Equipotentials

In $\mathbb{R}^2$, the curves $\phi(x,y) = \text{constant}$ are known as the equipotential curves (or lines). These are the level curves of the scalar potential.

Consider

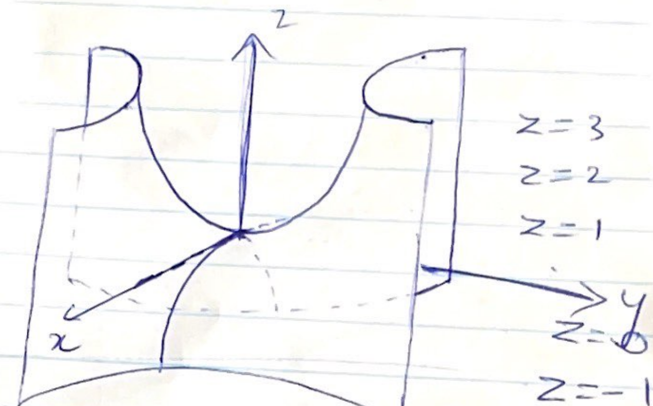
$$\phi(x,y) = x^2+y^2 \qquad \vec{F} = \vec{\nabla}\phi = (2x, 2y)$$



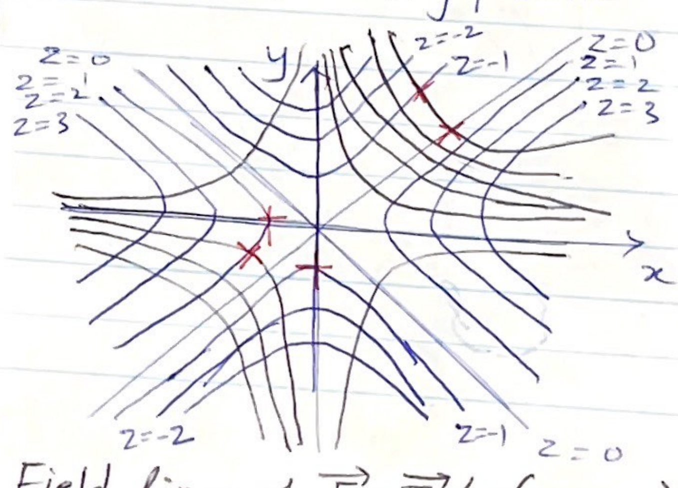


Field lines of $\vec{F} = (2x, 2y)$ are all lines passing through the origin.

$$\phi(x,y) = x^2 - y^2$$



Projecting z = constants planes onto x-y plane

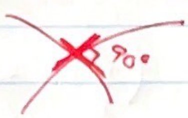


Field lines of $\vec{F} = \vec{\nabla}\phi = (2x, -2y)$

level curves of $\phi = x^2 - y^2$

A property of a gradient vector is that any point (a,b), $\nabla\phi(a,b)$ is perpendicular to the level curve passing through (a,b).

(Indicated by red above)



$\times 90°$

Thus if $\vec{F} = \vec{\nabla}\phi$, then the field lines of $\vec{F}$ will be perpendicular to the equipotentials!

(Two curves are perpendicular if the tangents are perpendicular at their points of intersection)

curves

In $\mathbb{R}^3$, $\phi(x,y,z) =$ constant represent the level surfaces of ϕ and are called equipotential surfaces.

Fantastic thing: Without solving a DE to find field lines, if $\vec{F} = \vec{\nabla}\phi$, then from level curves of ϕ, one can figure out the field lines.

Theorem.

If ϕ exists with $\vec{F} = \vec{\nabla}\phi$, and C is a simple, closed curve, then

$$\oint_C \vec{F} \cdot d\vec{x} = 0.$$

Notes:

- → $\oint$ – notation for an integral around a closed curve.
- → means that the line integral of a conservative field around any closed curve is zero.
- → Simple means doesnot intersect itself.