MATH 137 LEC 010 (Mani Thamizhazhagan)
Week 4 : Sep 26-30 Lecture Summary & Notes.

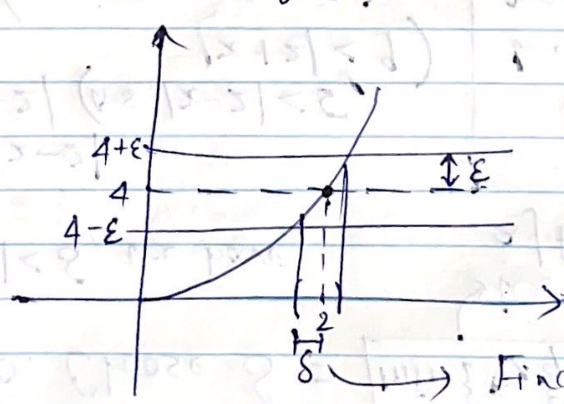
Monday: Read through Chapter 2, 2.1 from Pages 56-66.

We emphasized the "local" nature of the definition of the limit of a function at a point and realized how fundamental this "local" nature is when trying to show $\lim_{x \to 2} x^2 = 4$.

$\lim_{x \to 2} x^2 = 4$: Inorder to show this, for a given $\varepsilon > 0$, we need to find a $\delta > 0$ s.t

$$\underbrace{0 < |x-2| < \delta}_{\text{lim doesnot care about what is happening at } x=2, \text{ but only around } x=2.} \Rightarrow |x^2 - 4| < \varepsilon.$$

In order to show this, do we really need to know what the function is doing at say, $x = 100$, $x = -2$, $x = 0$, (or) any other value that is well away from $x=2$? That's why, this definition is of "local" in nature.



Find this δ for a given ε, s.t $(2-\delta, 2+\delta) \setminus \{2\}$ maps into ε-band of 4.

(Case II)

Solution: Let $\varepsilon > 0$. Choose $\delta = \boxed{\min\{5, \frac{\varepsilon}{9}\}}$.

We will fill this box after an aside calculation.

Then for $0 < |x-2| < \delta$, we have,

$$\begin{aligned}|x^2-4| &= |x+2||x-2| \\ &< 9|x-2| \quad (\text{as } |x-2|<5,\ |x+2|<9) \\ &< 9\delta \quad (\text{as } |x-2| < \tfrac{\varepsilon}{9}) \\ &\leq \varepsilon\end{aligned}$$

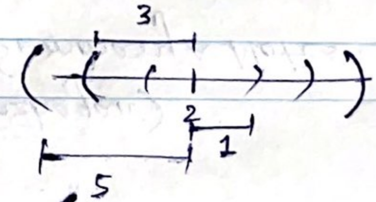
Aside

We want $\delta > 0$ s.t
$0 < |x-2| < \delta$
$\Rightarrow |x+2||x-2| < \varepsilon$.
$\Rightarrow |x-2| < \frac{\varepsilon}{|x+2|}$

We cannot choose $\delta = \frac{\varepsilon}{|x+2|}$ because as x varies, δ would vary. δ could only depend on ε and the point of interest $x = 2$.

Now the "local" nature kicks in.

Make it "local" around $x = 2$.
(i.e) $|x-2| < 3$,
(or)
$|x-2| < 5$
(or)
$|x-2| < 1$



Now under any of these considerations, we can find a $\delta > 0$.

I

$|x-2| < 3$
$\Rightarrow -1 < x < 5$
$\Rightarrow 1 < |x+2| < 7$

So under the consideration $|x-2| < 3$, we have $|x+2| < 7$

So,
$|x+2||x-2| < 7|x-2| < \varepsilon$ if $\delta \leq \frac{\varepsilon}{7}$

II

$|x-2| < 5$
$\Rightarrow -3 < x < 7$
$\Rightarrow |x+2| < 9$

Here, $|x+2||x-2| < 9\delta$ if $\delta \leq \frac{\varepsilon}{9}$ we get $|x^2-4| < \varepsilon$

III

$|x-2| < 1$
$\Rightarrow 1 < x < 3$
$\Rightarrow |x+2| < 5$

Here, $|x+2||x-2| < 5\delta$ if $\delta \leq \frac{\varepsilon}{5}$ we get $|x^2-4| < \varepsilon$.

Now which δ are you gonna choose?
Your choice of δ must reflect your consideration of how you localized.

For case I : Choose $\underbrace{\delta \leq 3}$ and $\delta \leq \frac{\varepsilon}{7}$

this localizes
to $|x-2|<3$
and further calculation above guarantee
$|x^2-4|<\varepsilon$ with
$\delta \leq \frac{\varepsilon}{7}$.

So $\delta \leq \min\{3, \frac{\varepsilon}{7}\}$.

For case II : Choose $\delta \leq \min\{5, \frac{\varepsilon}{9}\}$.

For case III : Choose $\delta \leq \min\{1, \frac{\varepsilon}{5}\}$.

One can do any localization (i.e) restricting to an interval around $x=2$, and carry forward the aside calculation. But your δ must show which localization you did.

· For this reason, the book always save you from all of these understandings and choose $\delta \leq 1$ initially.

Wednesday: Read through Section 2.2 - 2.3 Pages 67-75.

Friday: Read through Sections 2.4 - 2.6 Pages 76-86.

Exercises: Find $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$.

(Graph it and apply Squeeze theorem with appropriate functions)