

MATH-4101(3)-001 Assignment 5 Due: Feb 17, 2022 (before class)

WARNING: Don't use "e" for exp. $e := \exp(1)$ and we haven't defined or seen what complex powers of a number means. Nevertheless, it is true that when exp is restricted to \mathbb{R} , it is the real exponential function. This follows from the power series representation of exp and the fact that \mathbb{R} is closed in \mathbb{C} .

1. Show that for any $n \in \mathbb{N}$ and $x \in \mathbb{R}$, $x^{-n} \exp(x) \rightarrow \infty$, i.e., given $M > 0$, there exists $R > 0$ such that for all $x > R$, we have $x^{-n} \exp(x) > M$.
(Remark: Loosely speaking, this problem says that $\exp(x)$ goes to infinity as $x \rightarrow \infty$ much faster than any power of x and hence much faster than any polynomial in x .)
2. For all $z, w \in \mathbb{C}$, show that the following are true:
 - (a) $\exp(iz) = \cos(z) + i \sin(z)$.
 - (b) $\cos(z+w) = \cos(z)\cos(w) - \sin(z)\sin(w)$. In particular, $\cos(2z) = \cos^2(z) - \sin^2(z)$.
 - (c) $\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$.
 - (d) The function cos is not bounded on \mathbb{C} .
3. Let $f(z) := z \sin(1/z)$ for $z \neq 0$ and $f(0) = 0$. Is f continuous at 0?
4. The hyperbolic functions cosh and sinh are defined as follows:

$$\cosh(z) := [\exp(z) + \exp(-z)]/2$$

$$\sinh(z) := [\exp(z) - \exp(-z)]/2.$$

Show that the following hold:

- (a) $\cosh^2 z - \sinh^2 z = 1$.
 - (b) $\sin(x + iy) = \sin(x) \cosh(y) + i \cos(x) \sinh(y)$ for $z = x + iy \in \mathbb{C}$.
5. Find all solutions of $\sin(z) = 2$ using 4b.
 6. Show that $\sin : \mathbb{C} \rightarrow \mathbb{C}$ maps the lines parallel to the x -axis into ellipses and the lines parallel to the y -axis into hyperbola (*Hint*: Use 4b).