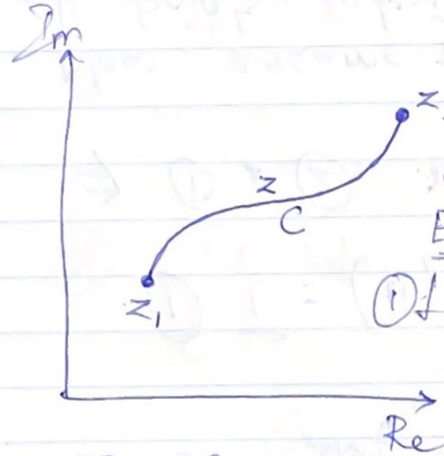
Contour Integration

$\int_C f(z)\,dz$ – the integral of the complex function $f(z)$ along a contour (or path) C in $\mathbb{C}$.



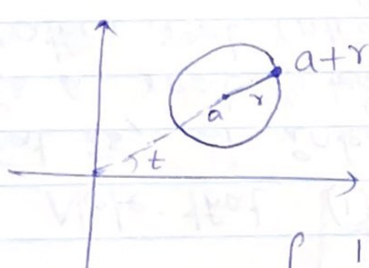
C – a curve from z_1 to z_2

Examples

① Line segment connecting z_1 to z_2

$C: z = g(t) = z_1 + t(z_2 - z_1),\ 0 \le t \le 1$

② Circle centered at a, and radius r, (CCW)



$C: z = g(t) = a + re^{it},\ 0 \le t \le 2\pi$

For $f(z) = \frac{1}{z-a}$, $\int_C f(z)\,dz = \int_0^{2\pi} f(g(t))\,g'(t)\,dt$

$$\int_C \frac{1}{z-a}\,dz = \int_0^{2\pi} \frac{1}{re^{it}}\, ire^{it}\,dt$$

$$= 2\pi i$$

Note that ~~for~~ $\int_{C_1} \frac{1}{z}\,dz = 4\pi i$ for $C_1: 0 + re^{i2t}$, $0 \le t \le 2\pi$ (CCW)

$\int_{C_2} \frac{1}{z}\,dz = -2\pi i$ for $C_2: 0 + re^{-it}$, $0 \le t \le 2\pi$ (CW)

Another way to realize $\int_C f(z)\,dz$:

Since $f(z) = u + iv$, $dz = dx + i\,dy$,

$$\int_C f(z)\,dz = \int_C (u+iv)(dx + i\,dy)$$

$$= \underbrace{\int_C (u\,dx - v\,dy)}_{①} + i \underbrace{\int_C (v\,dx + u\,dy)}_{②} \rightarrow (*)$$

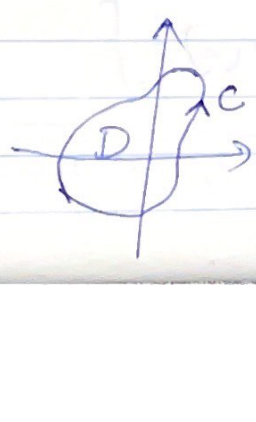
Note that ① & ② are real integrals.
That is, an integral of a complex-valued function is a real integral added to i multiplied by another real integral.

Similar to, when $\vec{F} = (F_1, F_2)$, $\int_C \vec{F} \cdot d\vec{x} = \int_C F_1\,dx + F_2\,dy$.

① $\vec{F} = (u, -v)$ ② $\vec{F} = (v, u)$

$\Rightarrow$ ① & ② real-valued line integrals.

Now, assume that $f(z)$ is analytic at all points interior to and on a simple closed curve C (oriented CCW). Then the partial derivatives u_x, v_x, u_y, v_y are all C^1 on $D \cup \partial D$, where D is the region enclosed by C and $\partial D = C$.



~~By~~ By Green's theorem,

$$\oint_C \vec{F}\cdot d\vec{x} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

Apply Green's theorem to (1) & (2)

(1): $F_1 = u$, $F_2 = -v$

(2): $F_1 = v$, $F_2 = u$

(*) : $$\int_C f(z)\,dz = \iint_D (-v_x - u_y)\,dA + i\iint_D (u_x - v_y)\,dA$$

Since f is analytic on D, CRE holds everywhere in D

$\therefore$ $u_x = v_y$ and $u_y = -v_x$

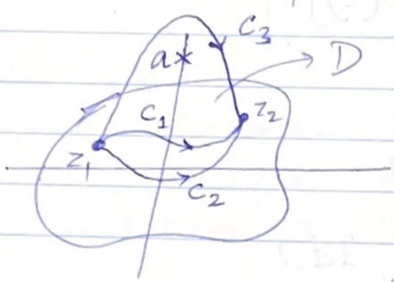
$$\therefore \int_C f(z)\,dz = 0.$$

This is called Cauchy-Goursat theorem:

> If a function $f(z)$ is analytic at all points interior to and on a simple closed curve C, then
>
> $$\oint_C f(z)\,dz = 0$$

Note: In ~~some~~ this sense, analytic functions captures the place of conservative vector fields.

Theorem: If f is analytic on a region D, then the integral along any curve inside D is independent of the path (in D)



$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

But we can't say anything about $\int_{C_3} f(z)dz$ using this theorem.

as f may not be analytic outside D (say at a!)

Thm: Let $f : D \to \mathbb{C}$ be continuous. Assume that there exists an $F : D \to \mathbb{C}$ s.t $F' = f$ on D. Let C be a curve inside D, from z_1 to z_2. Then

$$\int_C f(z)dz = F(z_2) - F(z_1)$$

$$\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1)$$

and $\oint_C f(z)dz = 0$ when C is a closed path in D.

Example

Recall, $\sin(x+iy) = \sin x \cosh(y) + i\sinh(y)\cos(x)$

$$\int_{-\pi/2+i}^{\pi+i} \cos z \, dz = \sin z \Big|_{-\pi/2+i}^{\pi+i}$$

$$= \sin(\pi+i) - \sin(-\pi/2+i)$$

$$= (\sin\pi \cosh(1) + i\cos\pi \sinh(1))$$

$$- (\sin(-\pi/2)\cosh(1) + i\cos(-\pi/2)\sinh(1))$$

$$= -i\sinh(1) + \cosh(1)$$

Verify using the line segment connecting $-\pi/2+i$ to $\pi+i$, Recall $\cos(x+iy) = \cos x \cosh y - i\sin x \sinh y$

$$C: g(t) = -\frac{\pi}{2}+i + t\left(\pi+i - (-\pi/2+i)\right) \quad 0 \le t \le 1$$

$$= \frac{3\pi}{2}t - \frac{\pi}{2} + i, \quad 0 \le t \le 1$$

$$\int_C \cos z \, dz = \int_0^1 \underbrace{\cos\left(\left(\frac{3\pi}{2}t - \frac{\pi}{2}\right) + i\right)}_{\cos(g(t))} \underbrace{\frac{3\pi}{2}}_{g'(t)} dt$$

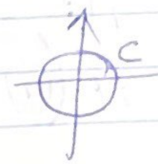
$$= \frac{3\pi}{2}\int_0^1 \left(\cos\left(\frac{3\pi}{2}t - \frac{\pi}{2}\right)\cosh(1) - i\sin\left(\frac{3\pi}{2}t - \frac{\pi}{2}\right)\sinh(1)\right) dt$$

$$= \left[\cosh(1)\sin\left(\frac{3\pi}{2}t - \frac{\pi}{2}\right) + i\sinh(1)\cos\left(\frac{3\pi}{2}t - \frac{\pi}{2}\right)\right]_0^1$$

$$= \cosh(1) - i\sinh(1) \quad \text{(Compute upper limit - lower limit evaluation)}$$

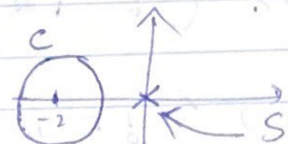
Example: (Cauchy-Goursat theorem)

① $\oint_C z^2 dz = 0$ where $C : |z| = 1$



analytic inside and on C!

② $\oint_C \frac{1}{z} dz$, where $C : |z+2| = 1$



Singular point $z=0$

$\frac{1}{z}$ is analytic inside and on C

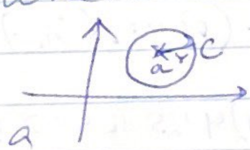
So $\oint_C \frac{1}{z} dz = 0$

Contrast this with earlier example

$\oint_C \frac{1}{z-a} dz = 2\pi i$ where $C : |z-a| = r$

$a + re^{it}$

$0 \le t \le 2\pi$



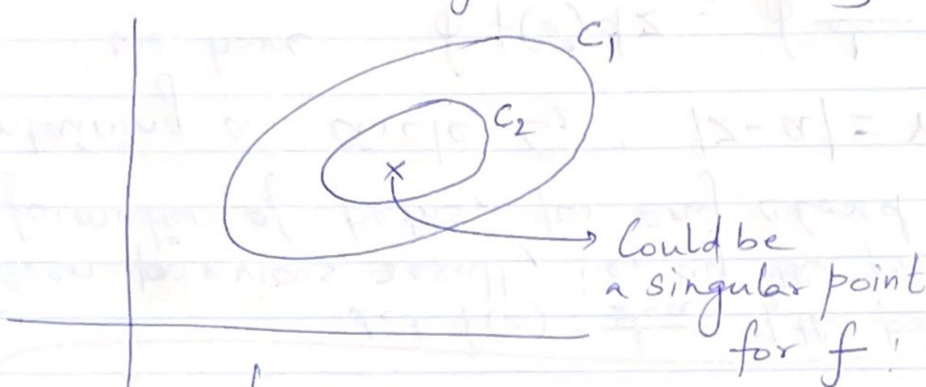
Singular pt. at $z = a$

$\frac{1}{z-a}$ is not analytic at a, which is inside C.

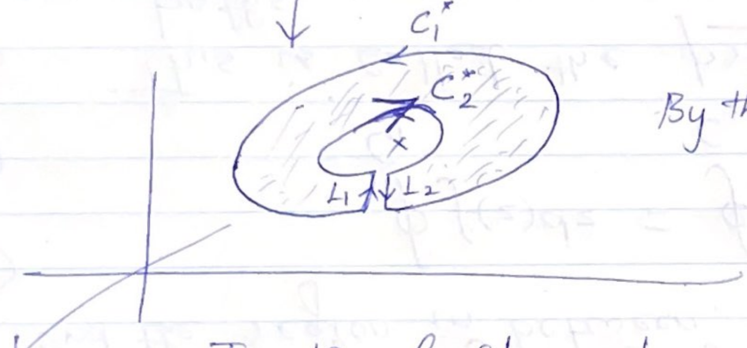
Deformation of Paths

An important consequence of the Cauchy-Goursat theorem is the following:

Let $f(z)$ be analytic on C_1, C_2 and the region between them (though not necessarily inside C_2)



Form



doesn't enclose any singular point!

By the theorem on $C_1^* \cup L_1 \cup C_2^* \cup L_2$,

$$\int_{C_1^*} f + \int_{L_1} f + \int_{C_2^*} f + \int_{L_2} f = 0$$

In the limit where L_1 and L_2 coincide and $C_1^* \longrightarrow C_1$, $C_2^* \longrightarrow C_2$,

$$\int_{L_1} f + \int_{L_2} f \longrightarrow 0$$

(opposite orientation, ~~same curve~~ in the limit)

f is analytic on $C_1^* \cup L_1 \cup C_2^* \cup L_2$ and inside this curve!

$$\int_{C_1} f + \int_{C_2} f = 0$$

But notice that these have opposite orientation
C_1^* and C_2^*

(CCW – positively oriented)

Swap the orientation on C_2, we have the result:

Let C_1 and C_2 be simple, closed curves oriented positively. If f is analytic on C_1, C_2 and the region in between, then

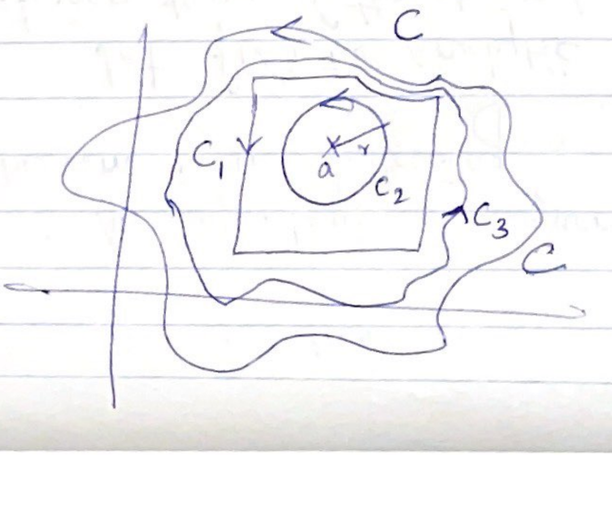
$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz.$$

This is called the principle of deformation of paths

Let $f(z) = \frac{1}{z-a}$ of the previous example.

Given previous result, i.e, by the principle of deformation of paths, for any closed contour C containing a circle C_2 : $|z-a| = r$: $a + re^{it}$
$0 \le t \le 2\pi$

we have $$\oint_C f(z)\,dz = \oint_{C_2} \frac{1}{z-a}\,dz = 2\pi i$$



$$\int_{C_3} f = \int_{C_1} f = \int_{C_2} f = 2\pi i$$