MATH137 LEC 010 (Mani Thamizhazhagan)
Week 2: Sep 12-16 Lecture Notes.

(In our course Notes by (B. Forrest)^2)

Monday: Read through Pages 7-10, Section 1.2.3-1.2.4
i.e. Pages 16-23. Wherever you read the word "observation", please observe

Tuesday: Started with three problems.
Show the following:

i). $\lim_{n\to\infty} \frac{1}{n^2} = 0$.

Solution: Recall that $\lim_{n\to\infty} a_n = L$, $L \in \mathbb{R}$ if for all $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ (depending on ε) s.t

$|a_n - L| < \varepsilon$ for all $n \geq N$.

Here our $a_n = \frac{1}{n^2}$ and $L = 0$.

Let $\varepsilon > 0$. Choose $N > \boxed{1/\sqrt{\varepsilon}}$.

We will fill this box after aside calculation.

Then for all $n \geq N$, we have

$$\left|\frac{1}{n^2} - 0\right| = \frac{1}{n^2} \leq \frac{1}{N^2} < \varepsilon.$$

$\therefore$ By definition, as ε is arbitrary,

$$\lim_{n\to\infty} \frac{1}{n^2} = 0.$$

Aside
Want N s.t
$\frac{1}{N^2} < \varepsilon$
$\Rightarrow N > \frac{1}{\sqrt{\varepsilon}}$.
Now fill the box above.
No need to show this aside calculation

2) $\lim_{n\to\infty} \frac{7n+3}{4n+2} = \frac{7}{4}$.

Solution: Let $\varepsilon > 0$. Choose $N > \boxed{}$.

Then for all $n \geq N$, we have

$$\left|\frac{7n+3}{4n+2} - \frac{7}{4}\right| = \left|\frac{28n+12-28n-14}{4(4n+2)}\right| = \left|\frac{-2}{4(4n+2)}\right|$$

$$= \frac{1}{2(4n+2)}$$

You can start your aside calculation at any of these three steps.

$$= \frac{1}{4(2n+1)} \to \text{I}$$

$$< \frac{1}{2n+1} \to \text{II}$$

$$< \frac{1}{2n} \to \text{III}$$

Aside

I Want N s.t

$\frac{1}{4(2n+1)} < \varepsilon$ for all $n \geq N$.

$\frac{1}{4\varepsilon} < 2n+1$

$\Rightarrow n > \frac{1}{2}\left[\frac{1}{4\varepsilon} - 1\right]$

So choose $N > \frac{1}{2}\left[\frac{1}{4\varepsilon} - 1\right]$ (any natural)

II Want N s.t

$\frac{1}{2n+1} < \varepsilon$ for all $n \geq N$

$\Rightarrow n > \frac{1}{2}\left[\frac{1}{\varepsilon} - 1\right]$

Choose $N > \frac{1}{2}\left[\frac{1}{\varepsilon} - 1\right]$

III Want N s.t

$\frac{1}{2n} < \varepsilon$ for all $n \geq N$

$\Rightarrow n > \frac{1}{2\varepsilon}$

Choose any $N > \frac{1}{2\varepsilon}$.

Fill the box above.

We will rewrite the solution in each choice!

(I) Let $\varepsilon > 0$. Choose $N > \frac{1}{2}\left[\frac{1}{4\varepsilon} - 1\right]$

Then for all $n \geq N$, we have

$$\left|\frac{7n+3}{4n+2} - \frac{7}{4}\right| = \frac{1}{4(2n+1)} \leq \frac{1}{4(2N+1)} < \underbrace{\frac{1}{4\left(2\left[\frac{1}{2}\left(\frac{1}{4\varepsilon}-1\right)\right]+1\right)}}_{\text{Simplifies to } \varepsilon} = \varepsilon$$

(II) Let $\varepsilon > 0$. Choose $N > \frac{1}{2}\left[\frac{1}{\varepsilon} - 1\right]$

Then for all $n \geq N$, we have

$$\left|\frac{7n+3}{4n+2} - \frac{7}{4}\right| = \underbrace{\frac{1}{4(2n+1)} < \frac{1}{2n+1}}_{\text{Reduction}} \leq \frac{1}{2N+1} < \varepsilon.$$

(III) Let $\varepsilon > 0$. Choose $N > \frac{1}{2\cdot\varepsilon}$.

Then for all $n \geq N$, we have

$$\left|\frac{7n+3}{4n+2} - \frac{7}{4}\right| = \underbrace{\frac{1}{4(2n+1)} < \frac{1}{2n+1} < \frac{1}{2n}}_{\text{Reduction}} \leq \frac{1}{2N} < \varepsilon.$$

One can use any of the above choices, but your further calculations should reflect your choice (i.e. Show all the reductions especially in (II) and (III) if at all you prefer these)

3) $\lim_{n\to\infty} \frac{n^2}{(2n-1)^2+3} = \frac{1}{4}$

Solution: Let $\varepsilon > 0$. Choose $N > \boxed{1/4\varepsilon}$

Then for all $n \geq N$, we have

$$\left|\frac{n^2}{(2n-1)^2+3} - \frac{1}{4}\right| = \left|\frac{4n^2-(4n^2-4n+4)}{4(4n^2-4n+4)}\right|$$

$$= \left|\frac{4(n-1)}{16(n^2-n+1)}\right|$$

$$= \frac{1}{4}\,\frac{n-1}{n(n-1)+1}$$

$$< \frac{1}{4}\,\frac{n-1}{n(n-1)} \quad \text{as } n(n-1)+1 > n(n-1).$$

$$= \frac{1}{4n} \leq \frac{1}{4N}$$

$$< \varepsilon.$$

Aside

Want $\frac{1}{4N} < \varepsilon$

$\Rightarrow N > \frac{1}{4\varepsilon}$.

By definition, as ε is arbitrary (General unspecified value),

$$\lim_{n\to\infty} \frac{n^2}{(2n-1)^2+3} = \frac{1}{4}.$$

Remarks:

In using formal definition to show limits, don't write $|a_n - L| < \varepsilon$ in your solution at the start. ε comes right at the end of ~~th~~your calculations.

Now in the course Notes, read from Observation on Page 23 to Important Note after Theorem 3.

Question: Can a sequence have more than one limit?

Example: Consider $\{(-1)^n\}_{n=1}^{\infty}$. This sequence takes -1 and 1 alternatively.

Claim: $\{(-1)^n\}_{n=1}^{\infty}$ has no limit.

We will make use of following equivalence mentioned in Theorem 3, repeatedly. So make sure you understand and imagine the below often.

$\lim_{n\to\infty} a_n = L \iff$ Every interval (a,b) containing L contains a tail of $\{a_n\}$.

(i.e) for some N, $a_n \in (a,b)$ for all $n \geq N$.

[Read 23 to 24 page transition] again.

Proof of the claim:

Assume to the contrary, $\{(-1)^n\}_{n=1}^{\infty}$ converges to some limit L. Then the interval $(L-\frac{1}{4}, L+\frac{1}{4})$ must contain a tail of $\{(-1)^n\}_{n=1}^{\infty}$.

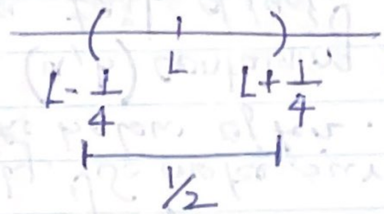
That is, for some $N \in \mathbb{N}$,

$$a_n \in \left(L - \frac{1}{4}, L + \frac{1}{4}\right) \text{ for all } n \geq N.$$

Now, note that $|a_N - a_{N+1}| = 2$ (because it is either $|1-(-1)|$ or $|-1-1|$)

and the length of the interval $\left(L - \frac{1}{4}, L + \frac{1}{4}\right)$ is

$$\left|L + \frac{1}{4} - \left(L - \frac{1}{4}\right)\right| = \frac{1}{2}$$



So, both a_N and a_{N+1} cannot belong to $\left(L - \frac{1}{4}, L + \frac{1}{4}\right)$. This is the contradiction to our assumption that $\{(-1)^n\}$ has a limit L.

$\therefore$ $\{(-1)^n\}$ doesn't have a limit. $\blacksquare$

Note on how proof by contradiction works.

1) The proposition to be proved, P, is assumed to be false. That is, $\neg P$ is true.
(negation of P)

2) It is then shown that $\neg P$ implies two mutually contradictory assertions, Q and $\neg Q$.
(Above, Q: $\left(L - \frac{1}{4}, L + \frac{1}{4}\right)$ contains a tail of $\{(-1)^n\}$)

3) Since Q and $\neg Q$ cannot both be true, the assumption that P is false must be wrong, So P must be true.

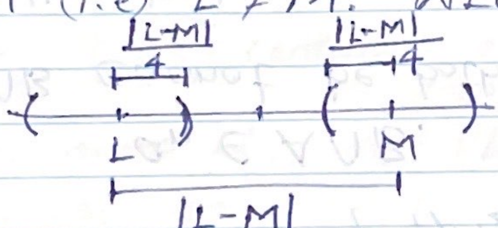
The argument in the previous example can also be modified to prove the following theorem that says that limit of sequences are unique.

Theorem: Uniqueness of Limits for sequences.

Let $\{a_n\}$ be a sequence. If it has a limit L, then the limit is unique.

Proof:

Suppose that $\{a_n\}$ has two different limits L and M (i.e) $L \neq M$. WLOG, assume $L < M$.



Now, $\left(L - \frac{|L-M|}{4}, L + \frac{|L-M|}{4}\right)$ contains L and

call this interval A.

&

$\left(M - \frac{|L-M|}{4}, M + \frac{|L-M|}{4}\right)$ contains M and

call this interval B.

It is clear that $A \cap B = \phi$ (empty)

But as $\lim_{n\to\infty} a_n = L$, A contains a tail of sequence $\{a_n\}$

(i.e). for some N_1, $a_n \in A$ for all $n \geq N_1$

and as $\lim_{n\to\infty} a_n = M$; B contains a tail of sequence $\{a_n\}$

(i.e) for some N_2, $a_n \in B$ for all $n \geq N_2$.

$\Rightarrow$ for all $n \geq \max\{N_1, N_2\}$,

$a_n \in A \cap B$.

Now, as $A \cap B$ cannot be both empty and non-empty, our assumption that L and M are different is false.

Therefore $L = M$.

$\boxtimes$

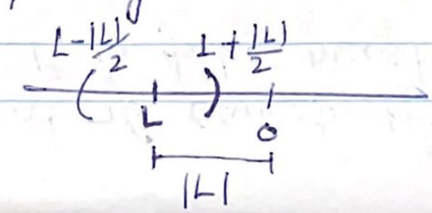
Proposition: Let $\{a_n\}$ be a sequence with $a_n \geq 0$ for each $n \in \mathbb{N}$. Assume that $L = \lim_{n\to\infty} a_n$;

Then $L \geq 0$.

Proof: Assume to the contrary that $L < 0$. Then the interval

$$\left(L - \frac{|L|}{2}, L + \frac{|L|}{2}\right)$$ contains L

and lies completely inside $(-\infty, 0)$



Since $\lim_{n\to\infty} a_n = L$, $\left(L-\frac{|L|}{2}, L+\frac{|L|}{2}\right)$ contains a tail of $\{a_n\}$. But $a_n \geq 0$ for each n.

So $\left(L-\frac{|L|}{2}, L+\frac{|L|}{2}\right)$ cannot contain a tail of $\{a_n\}$.

Contradiction to our assumption $L<0$.

$\therefore L \geq 0$.

$\blacksquare$

The following proposition is not in the course notes.

Proposition: Assume the sequence $\{a_n\}$ has the limit L. Then a_n's are bounded (i.e) there exists a real number M such that $|a_n| \leq M$ for all n.

Proof:

Since $\lim_{n\to\infty} a_n = L$,

$(L-1, L+1)$ contains a tail of $\{a_n\}$.

(i.e) for some N, $a_n \in (L-1, L+1)$ for all $n \geq N$.

$\Rightarrow |a_n| \leq L+1$ for all $n \geq N$.

Now we have to find a bound only for finitely many elements $\{a_1, a_2, \dots, a_{N-1}\}$.

So choose $M = \max\{a_1, a_2, \dots, a_{N-1}, L+1\}$.

Then $|a_n| \leq M$ for all n.

$\blacksquare$

Now read Pages 27 — 37 just prior to 1.3.

A remark about Theorem 7, v) [Page 29].

It says that $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$ if $M \neq 0$

where $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} b_n = M$, $L, M \in \mathbb{R}$.

We have to understand that $M \neq 0$ ensures a tail of the sequence $\left\{\frac{a_n}{b_n}\right\}_{n=1}^{\infty}$ is well-defined.

Since $M \neq 0$, $\left(M - \frac{|M|}{2}, M + \frac{|M|}{2}\right)$ does not contain 0.

also as $\lim_{n\to\infty} b_n = M$, $\left(M - \frac{|M|}{2}, M + \frac{|M|}{2}\right)$ contains a tail of $\{b_n\}$. (i.e) for some N,

$b_n \in \left(M - \frac{|M|}{2}, M + \frac{|M|}{2}\right)$ for all $n \geq N$.

$\Rightarrow$ $b_n \neq 0$ for all $n \geq N$.

If any of $b_1, b_2, \ldots, b_{N-1}$ were zero, just replace the zeroes by a non-zero value.

Changing finitely many terms of a sequence doesnot affect its limit.