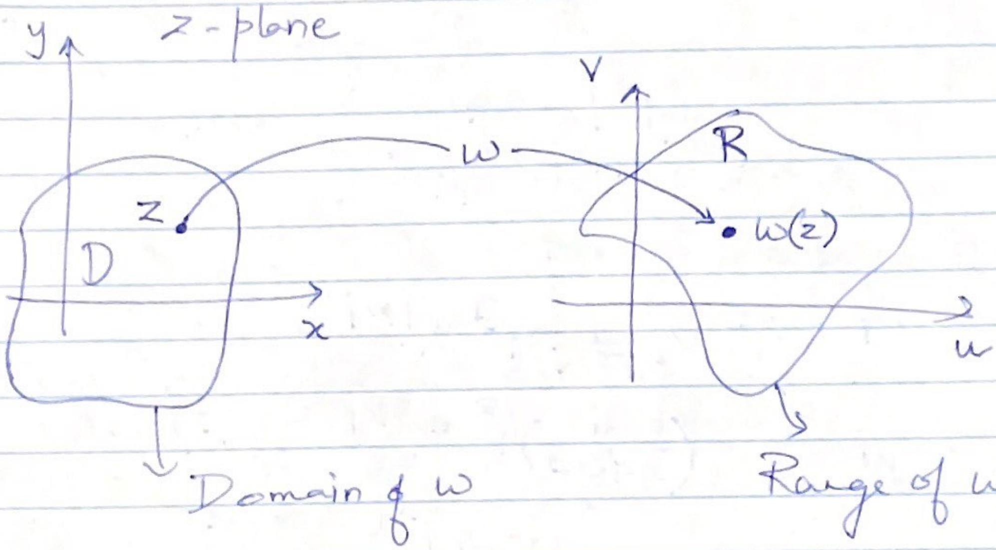
Def^n. We define a complex function $f(z)$ to have the form:

$$w = f(z) = u(x,y) + i\,v(x,y) \quad \text{where}$$

$z = x + iy$, u, v are real functions $u, v : \mathbb{R}^2 \to \mathbb{R}$.

<u>Polar form</u> $w = f(z) = f(re^{i\theta}) = u(r,\theta) + i\,v(r,\theta)$

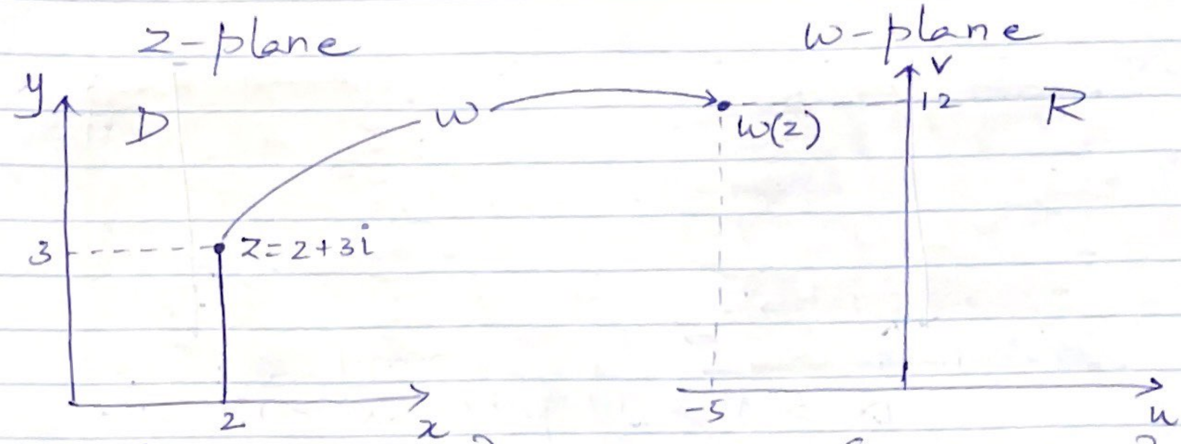
Polar co-ordinates.



1) Because both D and R are 2-D regions, the "graph of w" would require a plot in 4-D.

2) Hard to understand the behaviour of w by plotting the surfaces given by $u(x,y)$ and $v(x,y)$.

Let's look at a more appealing and commonly used device to understand ~~the~~ a mapping w in the next example.



$D = \{(x,y) : x, y > 0\}$
First quadrant

$R = \{(u,v) : v > 0\}$
Upper Half-plane

What can be done, easily, to provide some form of graphical display of w?

Points get mapped to points (Laborious work to select some and display)

Regions get mapped to regions (Did so with Domain)

(especially when studying conformal mappings)
an useful method, is to display the images of certain representative curves. For eg. where does the line $x=1$ get mapped to under $w = z^2$? $(0<y<\infty)$

$$w = z^2 = \underbrace{x^2-y^2}_{u} + i\,\underbrace{2xy}_{v}$$

With $x=1$, $u = 1-y^2$, $v = 2y$ $(0<y<\infty)$
Eliminating y: $u = 1-\left(\frac{v}{2}\right)^2$ (a parabola)

Eg: Let $w(z) = z^2$, defined on the ~~domain~~ first quadrant of the z-plane : $0 < x < \infty, 0 < y < \infty$.

Then
$$\begin{aligned} w(z) &= (x+iy)^2 \\ &= x^2 - y^2 + 2ixy \\ &= \underbrace{x^2 - y^2}_{u(x,y)} + i\, \underbrace{2xy}_{v(x,y)} \qquad \text{[Cartesian form]} \end{aligned}$$

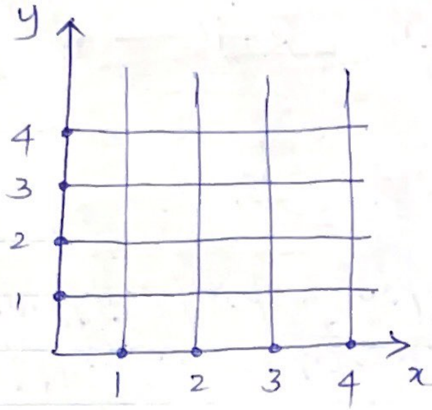
$$\left(\begin{aligned} w(re^{i\theta}) &= (re^{i\theta})^2 \qquad \left[(e^{i\theta})^n = e^{in\theta},\ n \in \mathbb{Z} \quad \text{De Moivre's Thm} \right] \\ &= r^2 e^{i2\theta} \\ &= \underbrace{r^2 \cos 2\theta}_{u(r,\theta)} + i\, \underbrace{r^2 \sin 2\theta}_{v(r,\theta)} \qquad \text{[Polar form]} \end{aligned} \right)$$

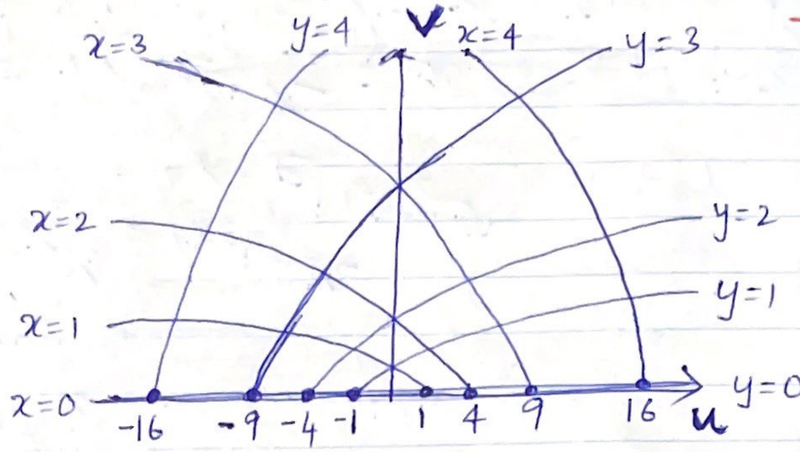
Since $0 < x < \infty$ and $0 < y < \infty$
[in polar co-ordinates, $0 < r < \infty$ and $0 < \theta < \pi/2$],

it follows from either form of u and v that

$-\infty < u < \infty$ and $0 < v < \infty$.

(Similarly)
$\hookrightarrow$ |||ly, the image of $y=1$ $(0<x<\infty)$ is given parametrically by $u=x^2-1$, $v=2x$, $0<x\leq\infty$
(or) by the parabola $u=\left(\frac{v}{2}\right)^2-1$

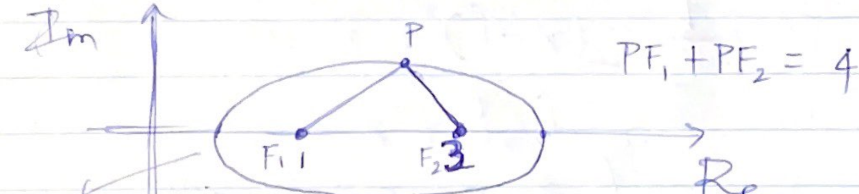




Find the domain of

1) $f(z)=\dfrac{1}{4-|z-1|-|z-3|}$

f is undefined when
$4-|z-1|-|z-3|=0$
(ie) $|z-1|+|z-3|=4$
$\{z\in\mathbb{C}: |z-1|+|z-3|=4\}$ $\rightarrow$ all those points in z-plane such that the sum of their distances from two fixed points (1 and 3) is constant $=4$.



Centre (2,0)
$a=2$
$b=\sqrt{3}$

For $z=x+iy$
$|z-1|+|z-3|=4 \iff |x-1+iy|+|x-3+iy|=4$
$\iff \sqrt{(x-1)^2+y^2}=4-\sqrt{(x-3)^2+y^2}$
$\iff \dfrac{(x-2)^2}{4}+\dfrac{y^2}{3}=1$

2) $f(z) = \dfrac{z}{|z|^2 - \operatorname{Im} z}$ is undefined when

$$|z|^2 = \operatorname{Im} z.$$

$$|z|^2 = \operatorname{Im} z \iff x^2 + y^2 = y$$
$$\iff x^2 + y^2 - y = 0$$
$$\iff x^2 + (y - \tfrac{1}{2})^2 = \tfrac{1}{4}$$

Circle with centre $(0, \tfrac{1}{2})$ and radius $\tfrac{1}{2}$

One of the mysteries of Complex analysis is that there are multi-valued functions.

The function $w = z^2$ is an example of a single-valued function. For each z, there is a unique w. Or, for each (x, y), there is a unique pair (u, v).

Whereas $w = \sqrt{z}$ is an example of a multivalued function.

$$w = \sqrt{z} = z^{1/2} = (re^{i\theta})^{1/2}$$
$$= r^{1/2} e^{i\left(\frac{\theta + 2n\pi}{2}\right)}, \quad n = 0, 1.$$

$$n = 0: \quad w_1 = \sqrt{r}\, e^{i\theta/2} = \sqrt{r}\left(\cos\theta/2 + i\sin\theta/2\right)$$
$$n = 1: \quad w_2 = \sqrt{r}\, e^{i\left(\frac{\theta}{2} + \pi\right)} = \sqrt{r}\left(\cos\left(\tfrac{\theta}{2} + \pi\right) + i\sin\left(\tfrac{\theta}{2} + \pi\right)\right)$$

These two functions are called the <u>branches</u> of $w = \sqrt{z}$.

Any root functions are multivalued, as are the complex logarithm and complex inverse trignometric functions.

The Exponential Function (and its associates)

Defn. The complex exponential function is defined as

$$\exp(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

(For all $z \in \mathbb{C}$, this series converges)

Properties

i) $\exp(z_1 + z_2) = \exp(z_1)\exp(z_2)$ for all $z_1, z_2 \in \mathbb{C}$.

ii) $\exp(z) \neq 0$ and $\exp(-z) = 1/\exp(z)$

iii) $\exp(\bar{z}) = \overline{\exp(z)}$

iv) exp, when restricted to $\mathbb{R}$, is positive and increasing!

v) If y is real, then $|\exp(iy)| = 1$ and

$$\exp(iy) = \cos y + i \sin(y)$$ [Check in the above power series].

Note that,

$$\exp(z + 2\pi i) = \exp(z)\exp(2\pi i) = \exp(z).$$

So exp when extended to $\mathbb{C}$ is not 1-1 furthermore it is periodic with period $2\pi i$. (imaginary)

$$\exp(x+iy) = \exp(x)\exp(iy) = \exp(x)(\cos y + i\sin y)$$

Additionally, since cos and sin on real inputs can be positive (or) negative, exp(z) can be negative.

Ex. Solve

$$\exp(z) = -1.$$

Solution: Write $\exp(z) = \exp(x+iy)$

$$= \exp(x)\exp(iy)$$

Write $-1 = e^{i\pi} = e^{i(\pi+2n\pi)}$, $n \in \mathbb{Z}$

$$\Rightarrow \exp(x)\exp(iy) = 1\, e^{i(\pi+2n\pi)},\ n\in\mathbb{Z}$$

$$\Rightarrow \begin{aligned} \exp(x) &= 1 \\ \exp(iy) &= e^{i(\pi+2n\pi)} \end{aligned}$$

$$\Rightarrow x = 0,\quad y = \pi(2n+1)\quad n\in\mathbb{Z}.$$

$$z = i\pi(2n+1)\ ;\ n\in\mathbb{Z}.$$

Visualizing exp(z) in w-plane

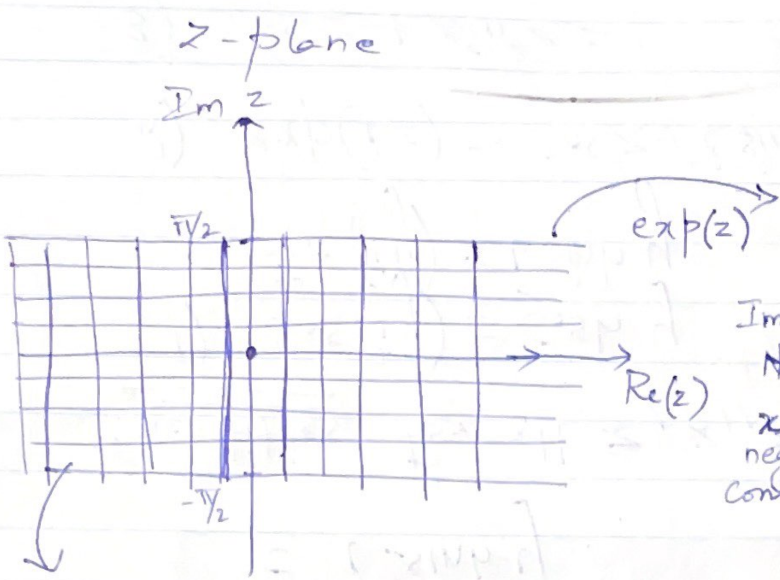
$$w = \exp(z) = \exp(x)\exp(iy)$$

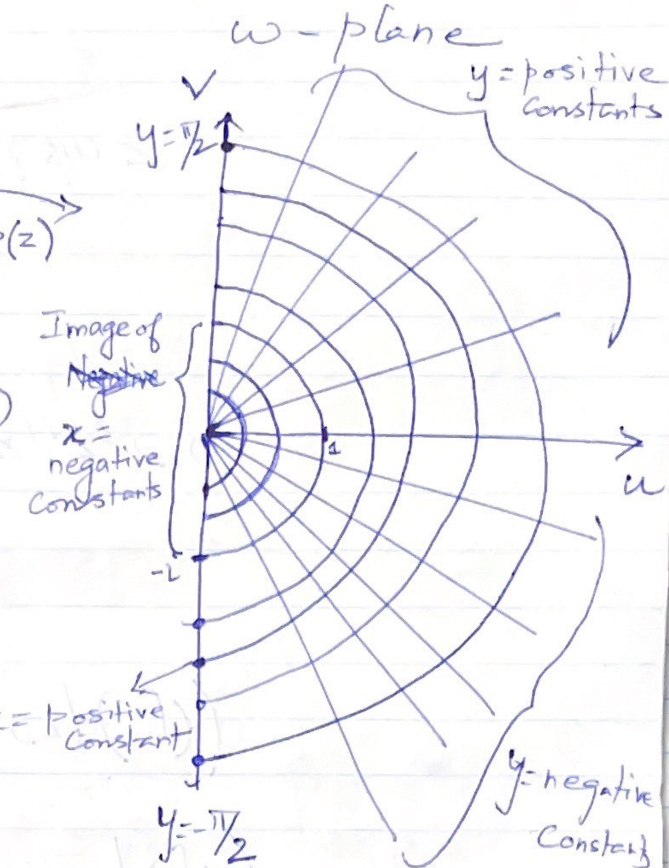
$$= \exp(x)(\cos y + i\sin y)$$

$$= \underbrace{\exp(x)\cos y}_{u} + i\,\underbrace{\exp(x)\sin(y)}_{v}$$

Note that for $x = x_0$ (fixed), $u^2+v^2 = \exp^2(x_0)$ (circle)

for $y = y_0$ (fixed), $\left(\frac{v}{u}\right) = \tan(y_0)$





$D := \{x + iy : |y| \le \pi/2\}$

$R := \{u + iv : u \geq 0\}$

Trigonometric Functions: We define the cosine and sine functions on $\mathbb{C}$ by

$$\cos z := \frac{\exp(iz) + \exp(-iz)}{2}$$

$$\sin z := \frac{\exp(iz) - \exp(-iz)}{2i}$$

for $z \in \mathbb{C}$.

Recall the hyperbolic functions

$$\cosh y = \frac{1}{2}(e^{y}+e^{-y}) \quad \sinh y = \frac{1}{2}(e^{y}-e^{-y})$$

Put $z = iy$ into definition of $\cos(z)$ and $\sin(z)$:

$$\cos(iy) = \frac{1}{2}\left(\exp(i(iy)) + \exp(-i(iy))\right)$$

$$= \frac{1}{2}\left(\exp(-y) + \exp(y)\right) = \cosh y$$

$$\sin(iy) = \frac{1}{2i}\left(\exp(i(iy)) - \exp(-i(iy))\right)$$

$$= \frac{1}{2i}\left(\exp(-y) - \exp(y)\right)$$

$$= \frac{i}{2}\left(\exp(y) - \exp(-y)\right)$$

$$= i \sinh y.$$

Identities For all $z, z_1, z_2 \in \mathbb{C}$

1) $\cos(iy) = \cosh y$
   $\sin(iy) = i \sinh y$

2) $\exp(iz) = \cos z + i \sin z$

3) $\cos^2 z + \sin^2 z = 1$

4) $\cos(z_1+z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

$\sin(z_1+z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$

Particularly, $\cos(2z) = \cos^2 z - \sin^2 z$

$\sin(2z) = 2\sin z \cos z$

5) $\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$, $\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$

Using ① ② ④, with $z_1 = x$, $z_2 = iy$ so that $z_1 + z_2 = x + iy = z$, we get

$\cos(z) = \cos x \cosh y - i \sin x \sinh y$ (*)

$\sin(z) = \sin x \cosh y + i \sinh y \cos x$ (**)

The importance of these expressions is that the complex trig functions are now expressed in the form $u(x,y) + i v(x,y)$.

Now, observe $|\sin z|^2 = \sin^2 x \cosh^2 y + \sinh^2 y \underbrace{\cos^2 y}_{1-\sin^2 y}$

$= \sin^2 x \underbrace{(\cosh^2 y - \sinh^2 y)}_{1} + \sinh^2 y$

$= \sin^2 x + \sinh^2 y$

Ill^{ly} $|\cos z|^2 = \cos^2 x + \sinh^2 y$.

Sinch sinh function can be arbitrarily large, unlike their real counterparts, $\sin z$ and $\cos z$ are not bounded by ± 1.

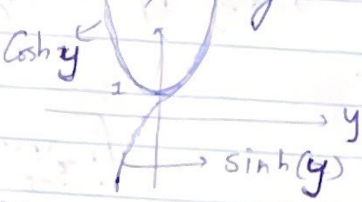
Consequently, we are faced with new equations:

Ex. Solve $\sin z = 2$

Using $(**)$, equating the Re/Im parts, we get

$\sin x \cosh(y) = 2 \to ①$

$\cos x \sinh(y) = 0 \to ②$



② : $\sinh(y) = 0$ iff $y = 0$ iff $\cosh(y) = 1$.

$\Rightarrow \sin(x) = 2$ in ① which is impossible

So $\sinh(y) \neq 0$ and hence $\cos(x) = 0$

$\Rightarrow \quad x \in \{(2k+1)\frac{\pi}{2} : k \in \mathbb{Z}\}$

① : Now, since $\cosh(y) > 0$ for all y, $\sin(x)$ must be positive in ①. Therefore,

$x = \frac{4k+1}{2}\pi$, $k \in \mathbb{Z}$ and $y = \pm\cosh^{-1}(2)$

(so $\sin(x) = 1$)

(if z satisfies $\sin(z) = 2$, then $\bar{z}$ also does : $2 = \overline{\sin(z)} = \sin(\bar{z})$)

$$\therefore z = \frac{4k+1}{2}\pi \pm i\cosh^{-1}(2), \; k \in \mathbb{Z}$$

Visualizing $w = \sin z$ in w-plane.

Using $(**)$: when $x = x_0$ (fixed), $\sin z = \underbrace{\sin(x_0)\cosh(y)}_{u} + i\underbrace{\cos x_0 \sinh(y)}_{v}$

$$1 = \cosh^2(y) - \sinh^2(y) = \frac{u^2}{\sin^2(x_0)} - \frac{v^2}{\cos^2(x_0)}$$

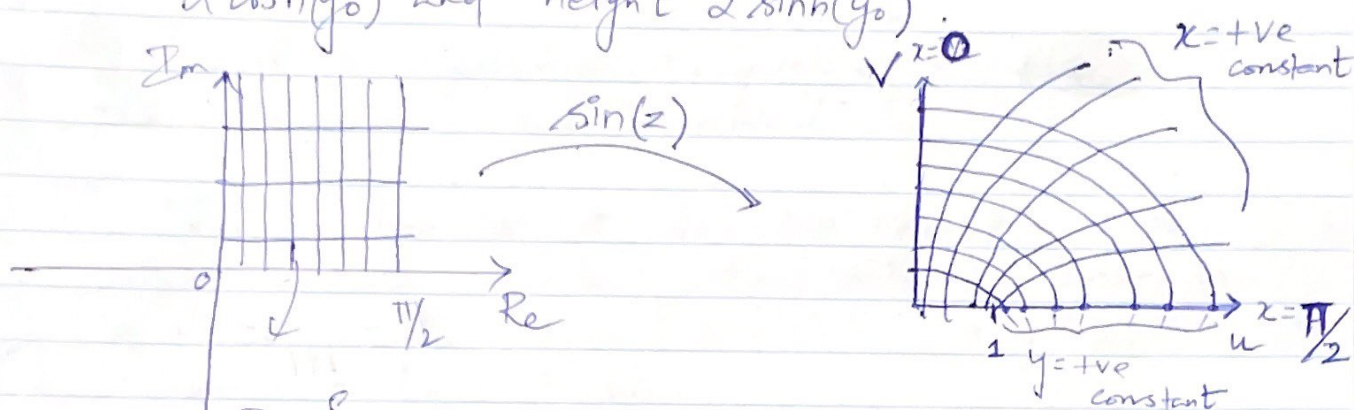
This (here $\sin$) is a standard hyperbola with vertices

$(\pm \sin(x_0), 0)$ and its conjugate hyperbola has $(0, \pm\cos(x_0))$ as vertices.

When $y = y_0$ (fixed), $\sin z = \underbrace{\sin(x)\cosh(y_0)}_{u} + i\underbrace{\cos x \sinh(y_0)}_{v}$

$$1 = \sin^2 x + \cos^2 x = \frac{u^2}{\cosh^2(y_0)} + \frac{v^2}{\sinh^2(y_0)}$$

This (lies in) is a standard ellipse with width $2\cosh(y_0)$ and height $2\sinh(y_0)$



$D: \{z = x+iy : 0 \le x \le \pi/2,\ 0 \le y < \infty\}$ $\qquad R := \{u+iv : u, v \geq 0\}$

Note that the image of $x = \pi/2$ under $\sin(z)$ is $[1, \infty)$!

Hence $\sin(z)$ maps $\{-\pi/2 < x\ \pi/2,\ y > 0\}$ onto the upper half-plane $\{\operatorname{Im}(z) > 0\}$.

<u>Exercise</u>: Let $U_1 := \{z \in \mathbb{C} : |\operatorname{Re}(z)| < \pi/2\}$

$U_2 := \{z \in \mathbb{C} : 0 < \operatorname{Re} z < \pi\}$

$V := \mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}$

Show that sin (respectively, cos) maps U_1 (respectively, U_2) bijectively onto V.