(Tools to compute contour integrals $\oint_C$)

Recall, If f is analytic

- on a simply-connected domain, (and connected) $\int_{z_1}^{z_2} f(z)dz$ is

  independent of path (from z_1 to z_2) $\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1)$

  has an antiderivative F s.t $F' = f \Rightarrow$

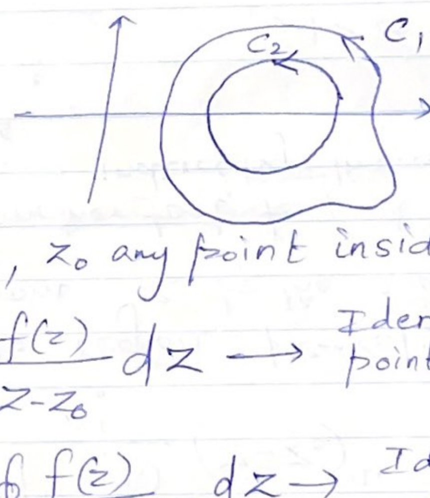
- within and on simple, closed contour C,

$$\oint_C f(z)dz = 0$$

- on the region between C_1 and C_2, on C_1 and C_2,

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz \quad \text{provided}$$

C_1 and C_2 are both simple, closed, oriented CCW contour.



CIF

For f analytic inside C, z_0 any point inside C,

$$\text{(CIF)}: \; f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz \longrightarrow \text{Identify } z_0 \text{ (singular point)}, f(z) \text{ analytic}$$

$$\text{(GCIF)}: \; f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz \rightarrow \text{Identify } z_0, f(z), n$$

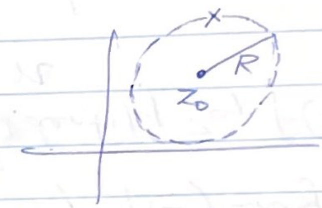
$\longrightarrow$ May have to deform path if multiple singular points (in which case, different $f(z)$ on each integral)

$\oint_C f = \oint_{C_1} f + \oint_{C_2} f$

Taylor's Theorem

If $f(z)$ is analytic within a circle of radius R and centre z_0, then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n$$



converges within the region $|z-z_0|<R$ where R is the distance from z_0 to the nearest singularity. $\hookrightarrow$ (Yet another tool to find integrals indirectly through GCIF)

Taylor (Maclaurin) series:

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad , \quad |z| < \infty$$

$$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \quad , \quad |z| < \infty$$

$$\cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \quad , \quad |z| < \infty$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad , \quad |z| < 1$$

$$\frac{1}{z} = \frac{1}{a} \sum (-1)^n \frac{(z-a)^n}{a^n} \quad , \quad |z-a| < |a|$$

for $a \neq 0 \in \mathbb{C}$

Power series allow term-by-term differentiation and integration within its radius of convergence:

Recall that

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots \qquad |z|<1$$

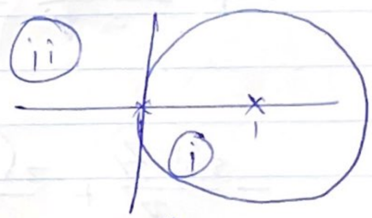
$$\Rightarrow \frac{d}{dz}\left(\frac{1}{1+z}\right) = \frac{-1}{(1+z)^2} = -1 + 2z - 3z^2 + 4z^3 - \cdots$$

$\hookrightarrow$ (A) $\qquad |z|<1$

analytic in $|z|<1$, so derivative exists and is analytic in $|z|<1$

Same function in (2) $f(z) = \dfrac{1}{z^2(1-z)}$

Now, centre it at $z=1$



(i) $0<|z-1|<\infty$

(i) $0<|z-1|<1$

(ii) $1<|z-1|<\infty$

$$\frac{1}{z^2(1-z)} = \frac{-1}{(z-1)\left[1+(z-1)\right]^2}$$

$$= +\frac{1}{z-1}\left(-1 + 2(z-1) - 3(z-1)^2 + \cdots\right)$$

$$= -\frac{1}{z-1} + 2 - 3(z-1) + 4(z-1)^2 - \cdots$$

(ii) $1 < |z-1| < \infty$

$$\frac{1}{z^2(1-z)} = \frac{-1}{(z-1)\left(1+(z-1)\right)^2}$$

$$= \frac{-1}{(z-1)^3\left(1+\frac{1}{z-1}\right)^2}$$

$\because \left|\frac{1}{z-1}\right| < 1$, by (A)

$$= -\frac{1}{(z-1)^3}\left(-1 + \frac{2}{z-1} - \frac{3}{(z-1)^2} + \frac{4}{(z-1)^3} - \cdots\right)$$

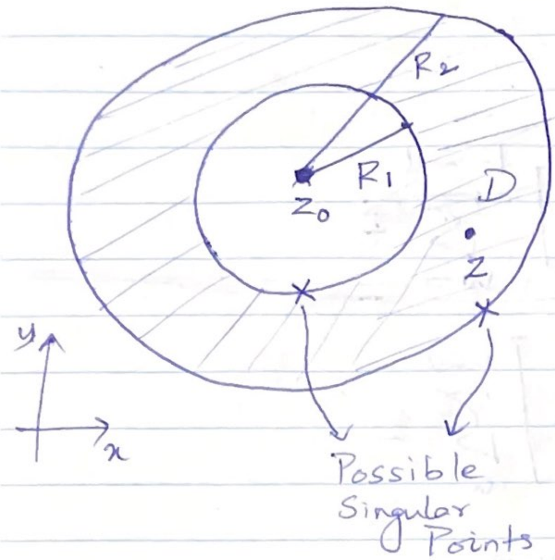
$$= \frac{1}{(z-1)^3} - \frac{2}{(z-1)^4} + \frac{3}{(z-1)^5} - \frac{4}{(z-1)^6} + \cdots$$

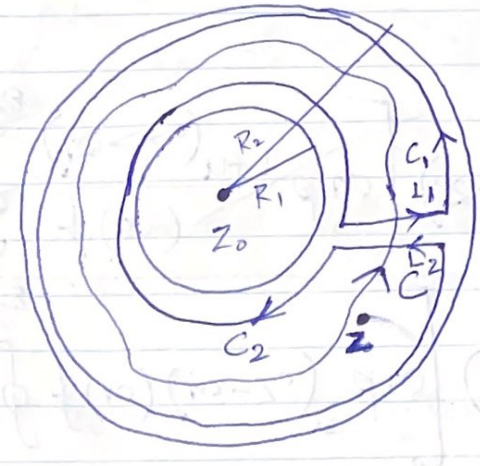
Laurent's Theorem

Suppose f is analytic throughout the annulus $D: R_1 < |z-z_0| < R_2$, with C a positively oriented simple closed curve in D. Then $f(z)$ can be represented by the Laurent series:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \quad \text{where}$$

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \text{ and } b_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{-n+1}}\,dz$$





By C.I.F, & deformation of Path

$$(\spadesuit): f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{w-z}\,dw + \cancel{\frac{1}{2\pi i}\int_{L_1} \text{—}} + \frac{1}{2\pi i} \oint_{C_2} \frac{f(w)}{w-z}\,dw + \cancel{\frac{1}{2\pi i}\int_{L_2} \text{—}}$$

Since w varies on C_1, $\left|\frac{z-z_0}{w-z_0}\right| < 1$

$$\frac{1}{w-z} = \frac{1}{w-z_0+z_0-z} = \frac{1}{w-z_0\left(1-\frac{z-z_0}{w-z_0}\right)}$$

$$= \frac{1}{w-z_0}\left(1 + \frac{z-z_0}{w-z_0} + \left(\frac{z-z_0}{w-z_0}\right)^2 + \cdots\right)$$

$$\Rightarrow \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{w-z}\,dw = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

where $a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{(w-z_0)^{n+1}}\,dw$

Now, consider C_3 integral in $\spadesuit$, as w varies over C_3,

$$\left|\frac{w-z_0}{z-z_0}\right| < 1$$

$$\therefore \frac{1}{w-z} = \frac{1}{w-z_0+z_0-z}$$

$$= \frac{-1}{z-z_0}\left(\frac{1}{1-\frac{w-z_0}{z-z_0}}\right)$$

$$= -\frac{1}{z-z_0}\sum_{n=0}^{\infty}\left(\frac{w-z_0}{z-z_0}\right)^n \left(\because \left|\frac{w-z_0}{z-z_0}\right|<1\right)$$

$$\therefore \frac{1}{2\pi i}\oint_{C_3}\frac{f(w)}{w-z}\,dw$$

$$= -\frac{1}{2\pi i}\oint_{C_3}\frac{f(w)}{z-z_0}\sum_{n=0}^{\infty}\frac{(w-z_0)^n}{(z-z_0)^n}\,dw$$

$$= \sum_{n=0}^{\infty}\left[-\frac{1}{2\pi i}\oint_{C_3} f(w)\,(w-z_0)^n\,dw\right](z-z_0)^{-(n+1)}$$

$$= \sum_{n=1}^{\infty} \left[\frac{1}{2\pi i} \oint_{-C_3} \frac{f(w)}{(w-z_0)^{-n+1}} \, dw \right] (z-z_0)^{-n}$$

$$= \sum_{n=1}^{\infty} b_n (z-z_0)^{-n} = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

where $b_n = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w-z_0)^{-n+1}} \, dw$

Principle of deformation of paths.

So, (♠) $f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{analytic part of } f \text{ at } z_0} + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}}_{\text{Principal part}}$

where analytic part of f at z_0; Principal part (Converges for $|z-z_0| > R_1$) (or) singular part of f at z_0

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w-z_0)^{n+1}} \, dw$$

$$b_n = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w-z_0)^{-n+1}} \, dw$$

Significance with $n=1$, $b_1 = \frac{1}{2\pi i} \oint_C \frac{f(w)}{1} \, dw$

$$\Rightarrow \oint_C f(w)\, dw = 2\pi i \, b_1$$