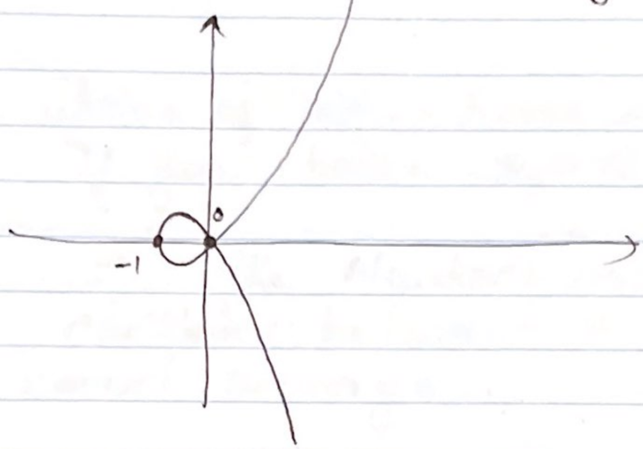
MATH 137 LEC 010 (Mani Thamizhazhagan)
Week 8 Nov 7-11 Lecture Summary/Notes.

On Monday, we covered Implicit differentiation. Please read Section 3.12 Pages 189-196. The following note contains ~~an~~ a brief explanation of when it is valid to consider implicit differentiation.

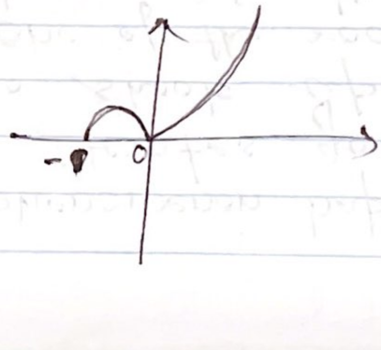
Consider the curve defined by $y^2 = x^2(x+1)$

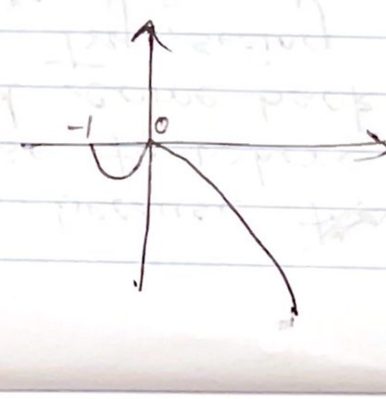


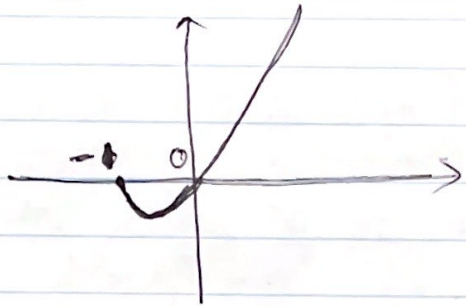
Solving for y as a function of x, we would try
$$y = \pm\sqrt{x^2(x+1)}$$
$$= \pm |x|\sqrt{x+1}$$

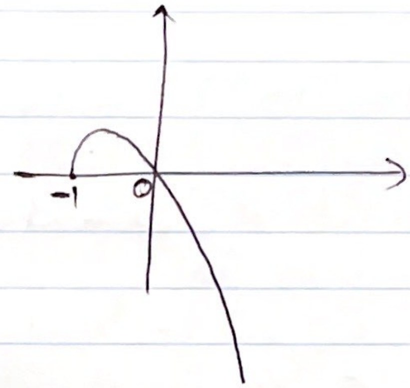
One of the ways to visualize the above implicit curve is as a combination of functions.

For example, $f_1(x) = |x|\sqrt{x+1}$ $\quad f_2(x) = -|x|\sqrt{x+1}$

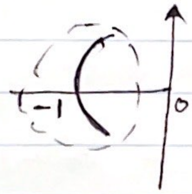




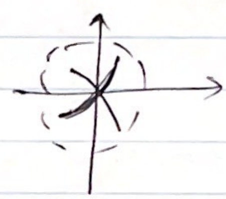




But it is not possible to express the above implicit curve as a function of x around -1 (or) 0 that captures how locally the curve looks like because



(or)



doesn't represent the graph of a function.

The theorem that addresses when implicit differentiation is valid is infact a one that concerns about a function of atleast two variables, that needs what is differentiability for such a function means.

Now consider $f(x,y) = y^2 - x^2(x+1)$.

Then the above curve is the pre-image of 0 under $f(x,y)$ (i.e) $\{(x,y) \in \mathbb{R}^2 : f(x,y) = 0\}$.

The set of points in the plane that satisfies $y^2 = x^2(x+1)$.

The following theorem might be hard to digest. Only those who are interested go through!

Implicit Function Theorem for $\mathbb{R}^2$:

Suppose we have a nice (differentiable in $\mathbb{R}^2$ and derivative is cts.) function $f: \mathbb{R}^2 \to \mathbb{R}$ and a point $(x_0, y_0) \in \mathbb{R}^2$ so that $f(x_0, y_0) = 0$. Let the slope of the tangent line to the curve defined by $f(x,y) = 0$ be m. If m is well-defined, then there is a neighborhood of (x_0, y_0) so that whenever x is sufficiently close to x_0, there is a unique y so that $f(x,y) = 0$. Moreover, this assignment ~~is~~ makes y a continuous function of x.

Now back to $y^2 = x^2(x+1)$.

Differentiating b.s w.r.t x,

$$2y \frac{dy}{dx} = 2x(x+1) + x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x(x+1) + x^2}{2y}$$

$\frac{dy}{dx}$ DNE when $y = 0$.

and $\{(x,y) \in \mathbb{R}^2 : y^2 = x^2(x+1)\} \neq \emptyset$

So by the above theorem, when $y \neq 0$, the implicit differentiation is valid because it is "locally" expressible as a function of x and we are taking derivative of that function.

(Page 195)
Now ponder on Example 27, why we can't apply the above theorem.

Another important technique of implicit differentiation is logarithmic differentiation.

Example For $x \geq 0$, $e^x \geq 1 + x + x^2/2$ [Correction!]

Consider $y = \left(e^x - 1 - x - \frac{x^2}{2}\right)^{\cos(x)}$

This is a valid function only when $x \geq 0$

When $x > 0$;

$$\ln y = \cos x \ln\left(e^x - 1 - x - \frac{x^2}{2}\right).$$

Diff. b.s w.r.t x

$$\frac{1}{y}\frac{dy}{dx} = -\sin x \ln\left(e^x - 1 - x - \frac{x^2}{2}\right) + \frac{\cos x\,(e^x - 1 - x)}{e^x - 1 - x - \frac{x^2}{2}}$$

$$\Rightarrow \frac{dy}{dx} = \left(e^x - 1 - x - \frac{x^2}{2}\right)^{\cos x}\left[-\sin(x)\ln\left(e^x - 1 - x - \frac{x^2}{2}\right) + \frac{\cos(x)\,(e^x - 1 - x)}{e^x - 1 - x - \frac{x^2}{2}}\right]$$

$\frac{dy}{dx}$ DNE when $x = 0$!

Remark: While taking $\ln$ always ensure that the argument of $\ln$ is a positive real number.

On Wednesday, we covered Local extrema theorem and introduced critical points. Also we discussed a strategy to find global maximum and global minimum of a continuous function defined on a closed interval.

Please read Section 3.13 Pages 196 − 202.
Also read Section 4.28 Pages 236 − 238.

On Friday, we proved Mean Value Theorem using Rolle's theorem and as a consequence derived the constant function theorem.

The key to understanding the global implications of the derivative is the Mean Value Theorem. For example, we will see next week that if $f'(x) > 0$ for all x in some interval I, we can conclude that the function is increasing on I.

Please read Chapter 4, Section 4.1 Pages 208 − 218 (until the Example 3 list in page 217−218)

Some Exercises using MVT

1) Let $f : [0,3] \to \mathbb{R}$ be given by $f(x) := \sqrt{3x - x^2}$. Show that f satisfies the conditions of Rolle's theorem. Find a c s.t $f'(c) = 0$.

2) Show that $f(x) = x^3 - 3x^2 + 17$ is not one-one on the interval $[-1, 1]$.

3) Show that $\cos x = x^3 + x^2 + 4x$ has exactly one root in $[0, \pi/2]$.

4) Show that on the graph of any quadratic polynomial f, the line joining the points $(a, f(a))$ and $(b, f(b))$ is parallel to the tangent line at the midpoint of a and b.

5) Tough Problem: Assume that $f : (a,b) \to \mathbb{R}$ is differentiable on (a,b) except possibly at $c \in (a,b)$. Assume that $\lim_{x \to c} f'(x)$ exists.

Prove that $f'(c)$ exists and f' is continuous at c.

(MVT and Squeeze ; Consider $\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$ and $\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}$ separately)