Sep 18, 2023

Polar co-ordinates

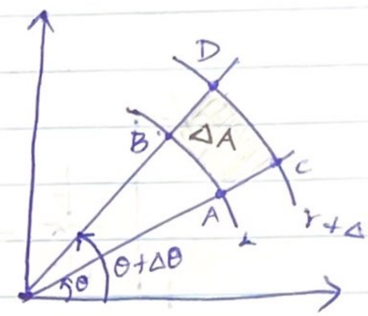
Let $x = r\cos\theta$, $y = r\sin\theta$. Then the Jacobian is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r.$$

$$\Rightarrow \quad dA = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta$$

$$= r\,dr\,d\theta$$



$AC = \Delta r$

$AB = r\Delta\theta$

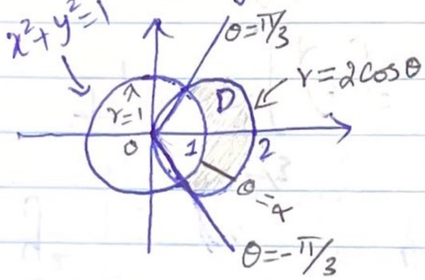
$\Delta A \sim r\,\Delta r\,\Delta\theta$

Ex. Evaluate $I = \iint_D \frac{x}{\sqrt{x^2+y^2}}\, dA$,

where D is the region inside $x^2+y^2 = 2x$ and outside $x^2+y^2=1$.

Soln. The region D is bounded by two circles:

$$x^2+y^2=1 \text{ and } \left(x^2+y^2 = 2x \Leftrightarrow (x-1)^2+y^2 = 1\right)$$



This region is much easier to describe in polar co-ordinates.

$$x^2+y^2=1 \Leftrightarrow r=1,$$

$$x^2+y^2 = 2x \Leftrightarrow r = 2\cos\theta$$

with $-\pi \le \theta \le \pi$

[Since $r = 2\cos\theta$, $r^2 = 2r\cos\theta$, we get $x^2+y^2 = 2x$]

$\Downarrow$

(Check graphs in Polar co-ordinates on Mobius)

Intersection Points

The curves $r=1$ and $r=2\cos\theta$ intersect in the 1st and 4th quadrant at points (r,θ) where

$$r=1 \ \& \ r=2\cos\theta, \ \theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

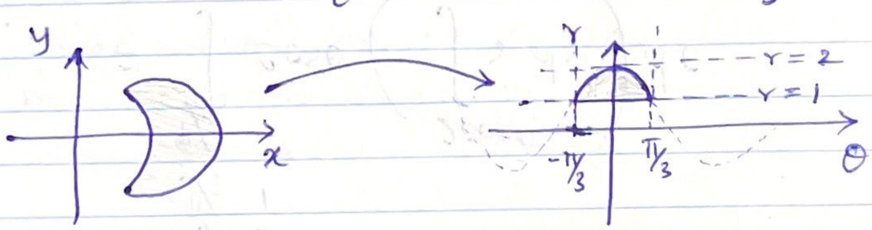
$$\Leftrightarrow 2\cos\theta = 1, \ \theta\in[-\pi/2,\pi/2]$$

$$\Leftrightarrow \cos\theta = \frac{1}{2}, \ \theta\in[-\pi/2,\pi/2] \Leftrightarrow \theta = -\pi/3, \pi/3.$$

Also, every ray $\theta=\alpha$, $-\pi/3 \le \alpha \le \pi/3$ intersects D in a line segment whose points have radius r: $1 \le r \le 2\cos\alpha$.

Thus, in Polar co-ordinates,

$$D_{r\theta} = \left\{ \begin{array}{c} -\pi/3 \le \theta \le \pi/3 \\ 1 \le r \le 2\cos\theta \end{array} \right\}$$



$$\Rightarrow I = \iint_D \frac{x}{\sqrt{x^2+y^2}}\, dA = \iint_{D_{xy}} \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy$$

Abs. value of Jacobian!

$$= \iint_{D_{r\theta}} \frac{r\cos\theta}{\sqrt{r^2}}\, r\, dr\, d\theta$$

$$\therefore\ I = \int_{-\pi/3}^{\pi/3}\int_{1}^{2\cos\theta} \frac{r\cos\theta}{r}\, r\,dr\,d\theta$$

$$= \int_{-\pi/3}^{\pi/3}\int_{1}^{2\cos\theta} r\cos\theta\, dr\,d\theta$$

$$= \int_{-\pi/3}^{\pi/3} \cos\theta\left(\int_{1}^{2\cos\theta} r\,dr\right)d\theta$$

$$= \int_{-\pi/3}^{\pi/3} \cos\theta\left[\frac{r^2}{2}\right]_{r=1}^{r=2\cos\theta} d\theta$$

$$= \int_{-\pi/3}^{\pi/3} \cos\theta\,(2\underbrace{\cos^2\theta}_{1-\sin^2\theta} - 1/2)\,d\theta$$

$$= \int_{-\pi/3}^{\pi/3} 2(1-\sin^2\theta)\cos\theta\,d\theta - \frac{1}{2}\int_{-\pi/3}^{\pi/3}\cos\theta\,d\theta$$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

θ	u
$-\pi/3$	$-\sqrt{3}/2$
$\pi/3$	$\sqrt{3}/2$

$$= \int_{-\sqrt{3}/2}^{\sqrt{3}/2} 2(1-u^2)\,du - \frac{1}{2}\left[\sin\theta\right]_{-\pi/3}^{\pi/3}$$

$$= 2u - \frac{2u^3}{3}\Big|_{u=-\sqrt{3}/2}^{u=\sqrt{3}/2} - \frac{1}{2}\left(\frac{\sqrt{3}}{2} - \left(-\frac{\sqrt{3}}{2}\right)\right)$$

$$= \underline{\underline{\sqrt{3}}}\,.$$

Give a read on Mobius content about the Definition of Triple integrals, intrepretations and properties. You will familiarity with that of double integral.

Iterated Integrals for Triple integrals

Let $D \subseteq \mathbb{R}^3$ defined by

$$\underset{\substack{\nearrow \\ \text{lower surface}}}{z_l(x,y)} \le z \le \underset{\substack{\uparrow \\ \text{upper surface}}}{z_u(x,y)} \quad \text{and } (x,y) \in D_{xy}$$

where z_l and z_u are continuous functions on D_{xy}, and D_{xy} is a closed bounded subset in $\mathbb{R}^2$, whose boundary is a piecewise smooth closed curve.

Also assume that the boundary Σ of D is a piecewise smooth surface. [[Smooth surface means continuously turning normal, piecewise means surfaces arranged edge to edge (finite number of)] Here D_{xy} is the projection of D onto xy-plane.

If $f(x,y,z)$ is cts. on D, then

$$\iiint_D f(x,y,z)\,dV = \iint_{D_{xy}} \int_{z_l(x,y)}^{z_u(x,y)} f(x,y,z)\,dA$$

$$\Rightarrow \int_{x_l}^{x_u}\int_{y_l(x)}^{y_u(x)}\int_{z_l(x,y)}^{z_u(x,y)} f(x,y,z)\, dz\, dy\, dx$$

Note: z doesn't have to be inner integral. Could have

$$y_l(x,z) \le y \le y_u(x,z) \ \&$$

$$\iint_{D_{xz}} \int_{y_l(x,z)}^{y_u(x,z)} f(x,y,z)\, dy\, \underline{\underline{dA}} \rightarrow$$ two ways of computing subsequent double integral

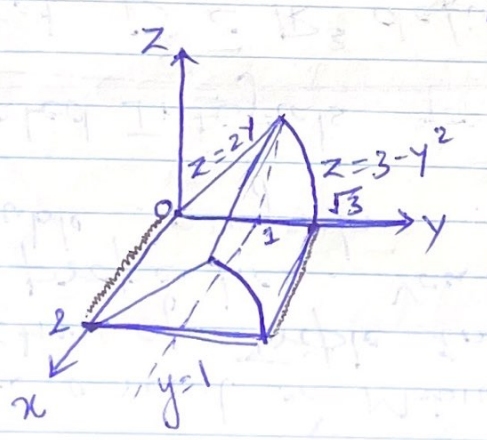
So there are $\underline{\underline{6}}$ orders of integration.

Example.

Evaluate the volume of the region D bounded by $z=2y$, $z=3-y^2$, $z=0$, $x=0$ and $x=2$.

Solution.

$$Vol(D) = \iiint_D 1 \, dV$$



Method 1: (Take vertical Cross Sections)

We see that z is bounded by the surfaces whose equations involve z:

$$\left.\begin{array}{l} z=2y \\ z=3-y^2 \end{array}\right\} \rightarrow \text{Upper surfaces}$$

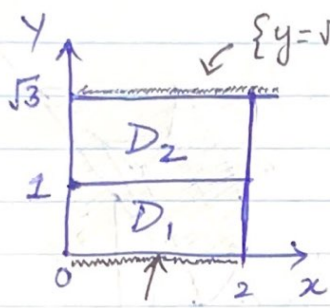
$z=0 \rightarrow$ Lower surface

Also, the upper bound changes on the line of intersection of the upper surfaces:

$$z = 2y \;\&\; z = 3 - y^2 \iff 2y = 3 - y^2$$
$$\iff y^2 + 2y - 3 = 0$$
$$\iff y = -3, 1$$

$\boxed{y = 1}$ relevant to the region D.

* Bounds of x & y: Project D onto xy-plane.

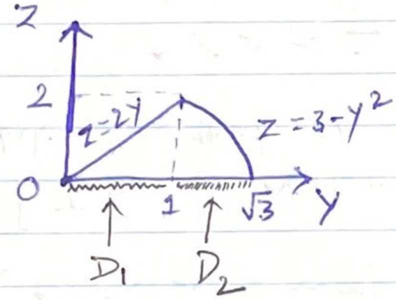


$\{y = \sqrt{3}\} = \{z = 0\} \cap \{z = 3 - y^2\}$

$\{y = 0\} = \{z = 0\} \cap \{z = 2y\}$

$$D_1 = \left\{ \begin{array}{l} 0 \le x \le 2 \\ 0 \le y \le 1 \end{array} \right\} \text{ and } D_2 = \left\{ \begin{array}{l} 0 \le x \le 2 \\ 1 \le y \le \sqrt{3} \end{array} \right\}$$

* Bounds for z: (take a vertical cross-section



for all $(x, y, z) \in D$,
if $(x, y) \in D_1$, $0 \le z \le 2y$
if $(x, y) \in D_2$, $0 \le z \le 3 - y^2$

So $$D = \left\{ \begin{array}{l} 0 \le x \le 2 \\ 0 \le y \le 1 \\ 0 \le z \le 2y \end{array} \right\} \cup \left\{ \begin{array}{l} 0 \le x \le 2 \\ 1 \le y \le \sqrt{3} \\ 0 \le z \le 3 - y^2 \end{array} \right\}$$

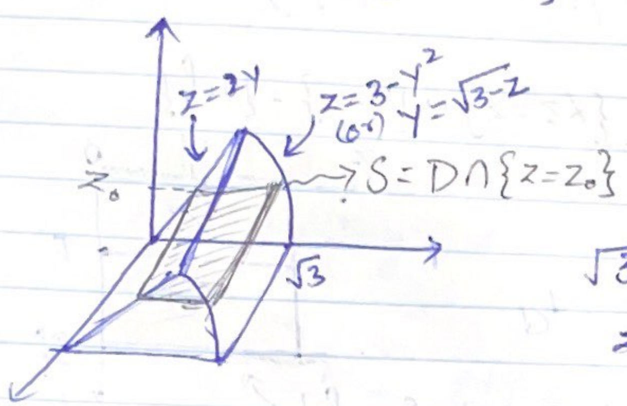
$$\Rightarrow Vol(D) = \int_0^2\int_0^1\int_0^{2y} 1\,dz\,dy\,dx + \int_0^2\int_1^{\sqrt{3}}\int_0^{3-y^2} 1\cdot dz\,dy\,dx$$

(Exercise : Compute)

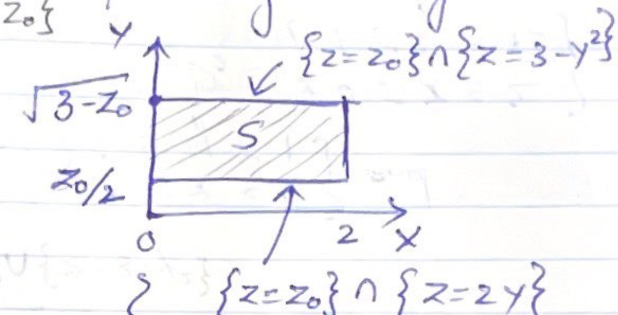
Method 2: [Take a horizontal slice]
We see that in the region D,

$$0 \le z \le 2.$$

For a fixed $z_0 \in [0,2]$, take the horizontal slice $S = D \cap \{z = z_0\}$



Then the projection of S is the following rectangle:



$$D = \begin{Bmatrix} 0 \le z \le 2 \\ 0 \le x \le 2 \\ \frac{z}{2} \le y \le \sqrt{3-z} \end{Bmatrix} \Leftarrow S = \begin{Bmatrix} 0 \le x \le 2 \\ \frac{z_0}{2} \le y \le \sqrt{3-z_0} \end{Bmatrix}$$

$$\Rightarrow Vol(D) = \int_0^2\int_0^2\int_{z/2}^{\sqrt{3-z}} 1\cdot dy\,dx\,dz$$

Change of Variable Thm

Let $x = f(u,v,w)$, $y = g(u,v,w)$, $z = h(u,v,w)$ be a one-to-one mapping of D_{uvw} onto D_{xyz}, with f, g, h having continuous partials, and

Jacobian in $\mathbb{R}^3 \longrightarrow \dfrac{\partial(x,y,z)}{\partial(u,v,w)} \neq 0$ on the interior of D_{uvw}

If $G(x,y,z)$ is continuous on D_{xyz}, then

$$\iiint_{D_{xyz}} G(x,y,z)\,\underbrace{dV}_{dxdydz} = \iiint_{D_{uvw}} G\big(f(u,v,w), g(u,v,w), h(u,v,w)\big) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw.$$

Cylindrical co-ordinates (Useful for regions with symmetry about the z-axis)

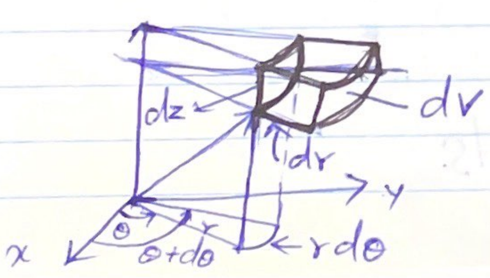
$x = r\cos\theta \quad y = r\sin\theta \quad z = z.$

Then

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

Since $r \geq 0$,

$$dV = r\,dr\,d\theta\,dz.$$



$2 \quad 1 \quad 2y$ $\qquad$ $2 \quad \sqrt{2} \quad 2-y^2$

Spherical Co-ordinates. (Useful for regions with symmetry about origin).

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

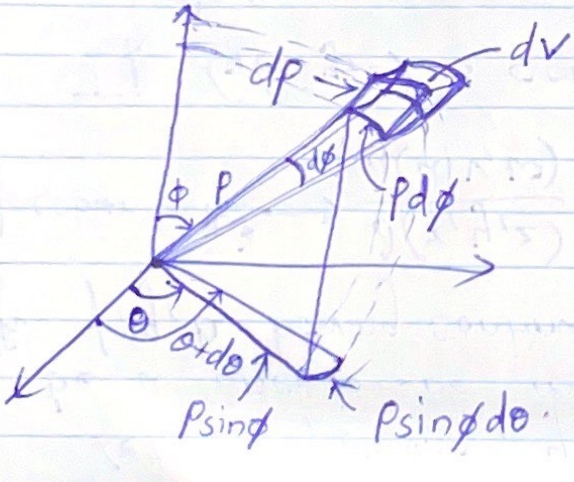
$\rho \geq 0, \quad 0 \leq \phi \leq \pi$
$\qquad 0 \leq \theta \leq 2\pi.$

So $$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} x_\rho & x_\phi & x_\theta \\ y_\rho & y_\phi & y_\theta \\ z_\rho & z_\phi & z_\theta \end{vmatrix}$$

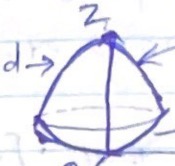
$$= \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

$$= \rho^2 \sin\phi$$

$$\therefore \; dv = \rho^2 |\sin\phi| \, d\rho \, d\phi \, d\theta$$



Examples. Suppose a solid is shaped like the region D below $z=4-x^2-y^2$ and inside $x^2+y^2+z^2=4z$, and the density (mass per unit volume) at (x,y,z) is given by $f(x,y,z)=2$[redacted]. Determine the total mass of the solid.



Soln.

Total mass $M=\iiint\limits_D (2$[redacted]$)\,dV$

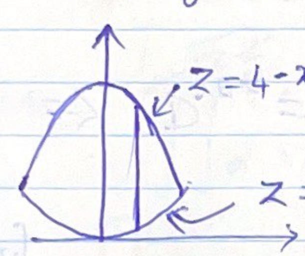
Jacobian

$= \iiint\limits_{D_{r\theta z}} (2$[redacted]$)\; r\,dz\,dr\,d\theta$

$x^2+y^2+z^2=4z$
$\Leftrightarrow x^2+y^2+(z-2)^2=4$
Sphere with centre $(0,0,2)$ & radius 2.

Method 1:

Bounds of z. (Take a vertical cross-section)



Sphere.
$(z-2)^2=4-(x^2+y^2)$
$\Rightarrow z-2=\sqrt{4-(x^2+y^2)}$ is upper sphere
$z-2=-\sqrt{4-(x^2+y^2)}$ is lower sphere.

$\Rightarrow$

$2-\sqrt{4-r^2}\le z\le 4-r^2$

Bounds for r,θ: Project D onto xy-plane.

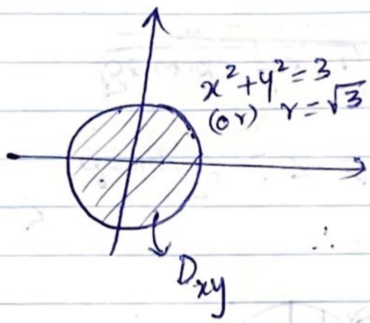
$\to$ get a disc bounded by the circle of intersection of the paraboloid $z=4-x^2-y^2$ and the sphere

$x^2+y^2+z^2=4z \Leftrightarrow x^2+y^2=4-z$ & $x^2+y^2=4z-z^2$

$\Rightarrow 4-z=4z-z^2 \Leftrightarrow z^2-5z+4=0 \Leftrightarrow \underline{z=1,4}$

$\leadsto$ get circle $x^2+y^2=\underset{\underset{z=1}{\uparrow}}{3}$ (or) $\underset{\underset{z=4}{\uparrow}}{0}$

$\Rightarrow$ $\boxed{x^2+y^2=3}$ at height $z=1$.



$$\therefore \quad D_{r\theta} = \left\{ \begin{array}{l} 0 \le r \le \sqrt{3} \\ 0 \le \theta \le 2\pi \end{array} \right\}$$

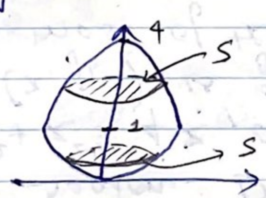
$$\Rightarrow \quad D_{r\theta z} = \left\{ \begin{array}{l} 0 \le r \le \sqrt{3} \\ 0 \le \theta \le 2\pi \\ 2-\sqrt{4-r^2} \le z \le 4-r^2 \end{array} \right\}$$

$$\therefore \quad M = \int_0^{\sqrt{3}} \int_0^{2\pi} \int_{2-\sqrt{4-r^2}}^{4-r^2} (2\cancel{+z}) \cdot r\,dz\,d\theta\,dr$$

$$= \dots = \frac{37\pi}{3} \text{ (Exercise)}$$

Method 2: In D, $0 \le z \le 4$.

For $z_0 \in [0,4]$, let $S = D \cap \{z=z_0\}$. Then S is always a disc:



$\Rightarrow$ if $0 \le z_0 \le 1$, the disc is bounded by the circle of intersection of $z=z_0$ with the sphere $x^2+y^2+z^2=4z$

(i.e) $x^2+y^2+z_0^2=4z_0$

$\iff x^2+y^2=4z-z_0^2$

$\Rightarrow r=\sqrt{4z-z_0^2}$

Projection of

$$\therefore S = \left\{ \begin{array}{l} 0 \le r \le \sqrt{4z_0-z_0^2} \\ 0 \le \theta \le 2\pi \end{array} \right\}$$

III^{ly} if $1 \le z_0 \le 4$, the disc is bounded by the circle of intersection of $z = z_0$ with the paraboloid $z = 4 - x^2 - y^2$:

$$\Rightarrow z_0 = 4 - x^2 - y^2 \Leftrightarrow x^2 + y^2 = 4 - z_0$$

$$\Rightarrow r = \sqrt{4 - z_0}$$

Projection of S $= \begin{cases} 0 \le r \le \sqrt{4 - z_0} \\ 0 \le \theta \le 2\pi \end{cases}$

$$\therefore D = \begin{Bmatrix} 0 \le z \le 1 \\ 0 \le r \le \sqrt{4z - z^2} \\ 0 \le \theta \le 2\pi \end{Bmatrix} \cup \begin{Bmatrix} 1 \le z \le 4 \\ 0 \le r \le \sqrt{4 - z} \\ 0 \le \theta \le 2\pi \end{Bmatrix}$$

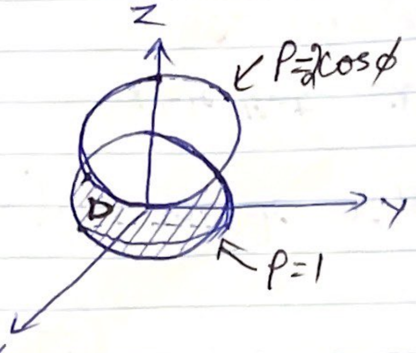
$$\therefore M = \int_0^1 \int_0^{\sqrt{4z - z^2}} \int_0^{2\pi} 2 \cdot r \, d\theta \, dr \, dz + \int_1^4 \int_0^{\sqrt{4 - z}} \int_0^{2\pi} 2 \cdot r \, d\theta \, dr \, dz$$

<u>Example:</u> Evaluate $\iiint_D z \, dv$, where D is the region inside $x^2 + y^2 + z^2 = 1$ and outside $x^2 + y^2 + z^2 = 2z$.

<u>Soln.</u>

Note that $x^2 + y^2 + z^2 = 2z \Leftrightarrow x^2 + y^2 + (z-1)^2 = 1$

Sphere centered at $(0,0,1)$ and radius 1.



In spherical co-ordinates,

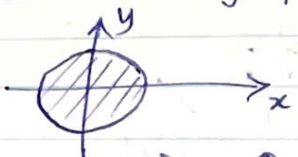
$x^2+y^2+z^2=1 \iff \rho = 1$

$x^2+y^2+z^2=2z \iff \rho^2 = 2\rho\cos\phi$

$\iff \rho = 0$ (or) $\boxed{\rho = 2\cos\phi}$

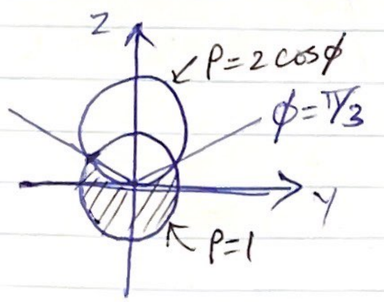
discard only gives $(0,0,0)$ which is obtained from $\rho = \cos\phi$ at $\phi = \frac{\pi}{2}$

Bounds for θ: Project D onto xy-plane



$\Rightarrow \quad 0 \le \theta \le 2\pi$

Bounds for ρ & ϕ: Take a (radially) vertical cross-section. Note that the points in D have ϕ at most π and atleast equal to the angle of intersection of the 2-spheres: $\rho = 1$ & $\rho = 2\cos\phi$



$\iff 1 = 2\cos\phi \iff \cos\phi = 1/2$

$\iff \phi = \pi/3$

since $\phi \in [0, \pi]$.

$\Rightarrow \quad \pi/3 \le \phi \le \pi$

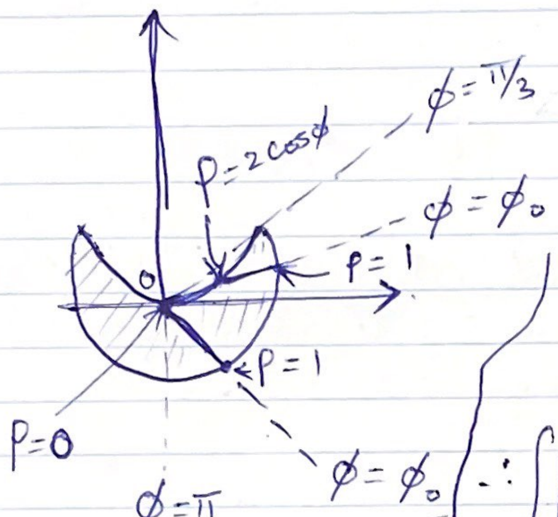
for points in D.

Furthermore, any radial ray corresponding to an angle $\pi/3 \le \phi_0 \le \pi$ intersects D as follows:

if $\pi/3 \le \phi_0 \le \frac{\pi}{2}$, then $2\cos\phi \le \rho \le 1$

if $\pi/2 \le \phi_0 \le \pi$, then $0 \le \rho \le 1$.

(lower bound change)



So

$$D_{\rho\phi\theta} = \left\{\begin{array}{l} 0 \le \theta \le 2\pi \\ \pi/3 \le \phi \le \pi/2 \\ 2\cos\phi \le \rho \le 1 \end{array}\right\} \cup \left\{\begin{array}{l} 0 \le \theta \le 2\pi \\ \pi/2 \le \phi \le \pi \\ 0 \le \rho \le 1 \end{array}\right\}$$

$$\therefore \iiint_D z \, dx\,dy\,dz$$

Jacobian

$$= \int_0^{2\pi}\int_{\pi/3}^{\pi/2}\int_{2\cos\phi}^{1} \overbrace{(\rho\cos\phi)}^{z} \rho^2\sin\phi \, d\rho\, d\phi\, d\theta$$

$$+ \int_0^{2\pi}\int_{\pi/2}^{\pi}\int_0^1 \rho\cos\phi \, \rho^2\sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \cdots = -5\pi/24.$$