MATH-4101(3)-001 **Assignment 4** Due: Feb 10, 2022 (before class)

1. Assume the following Theorem (Abel's Test): Let $(c_n)$ be a monotone decreasing sequence of nonnegative reals with $\lim_n c_n = 0$. Let $(b_n)$ be a bounded sequence in $\mathbb{C}$. Then

$$\sum_n (b_{n+1} - b_n)c_n$$

is convergent. If one takes $b_n = (-1)^n/2$ in the above, what you get is Alternating Series Test.

Now show that $\sum_1^\infty (z^n/n)$ converges for all $z$ with $|z| = 1$ and $z \neq 1$. Use Abel's test and the fact that both the sums in Assignment 2, 2(b) are bounded provided $0 < \theta < 2\pi$.

2. For $|a_n| \leq 1$, show that $\sum a_n z^n$ is absolutely convergent for all $|z| < 1$. If $f(z) := \sum a_n z^n$ for $|a_n| < 1$, $z \in B(0,1)$, show that

$$|f(z)| \leq \frac{1}{1 - |z|}.$$

3. Find the radius of convergence of the power series $\sum_n a_n z^n$, whose $n$-th coefficient $a_n$ is given below:

   (a) $e^{in\pi}/n$

   (b) $n^n$

   (c) $\left(\frac{n}{\log n}\right)^n$

   (d) $\frac{(2n)!}{n!}$

   (e) $(a^n + b^n), a > b > 0$

   (f) $4^{n(-1)^n}$

4. Find the radius of convergence of each of the following series. (Caution: Note that $a_k = 0$ for infinitely many $k$.)

   (a) $\sum_{n=0}^\infty \frac{1}{n!} z^{2n+1}$

   (b) $\sum_{n=0}^\infty n^2 z^{n^n}$

For questions on showing continuity, only use $\epsilon - \delta$ argument. For showing $f$ is not continuous at $a \in \mathbb{C}$, it is enough to find a sequence $(z_n)$ converging to $a$ such that $(f(z_n))$ doesn't converge to $f(a)$.

5. Show that the function $f : \mathbb{C} \to \mathbb{C}$ by $f(z) := |z|$ is continuous.

6. Let $f : \mathbb{C} \to \mathbb{C}$ be continuous at $a \in \mathbb{C}$. Assume that $f(a) \neq 0$. Show that there exists $r > 0$ such that $|f(z)| > |f(a)|/2$ for all $z \in B(a, r)$. Particularly, what does this say?

7. Let $a \in \mathbb{C}$ and $r > 0$. For any $b \in B(a, r)$, show that $\exp_{|B(a,r)}$ (exp with its domain restricted to the ball) is continuous at $b \in \mathbb{C}$. In your $\epsilon - \delta$ argument, you must get a $\delta$ that is independant of $b$!

8. [Bonus](Abel's Theorem on Boundary Behaviour) Let the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$ be 1. Assume that $\sum_{n=0}^{\infty} a_n$ is convergent. If $f(z) := \sum_{n=0}^{\infty} a_n z^n$ for $z \in B(0, 1)$, show that

$$\lim_{x \to 1-} f(x) = \sum_{n=0}^{\infty} a_n.$$

(*Hint:* Show first that it is sufficient to consider the case in which $\sum a_n = 0$.) Let $s_n = \sum_0^n a_k$. Show that

$$f(z) = (1 - z) \sum_{n=0}^{\infty} s_n z^n, \quad z \in B(0, 1).$$

Given $\epsilon > 0$, choose $N$ such that $|s_n| < \epsilon$ for $n \geq N$. Now split the series into two parts and deduce the result.