MATH-4101(3)-001 Assignment 8 Due: Mar 24, 2022 (before class)

1. Let $\lambda = a + ib \in \mathbb{C}^*$. Evaluate $\int_0^1 e^{\lambda s} ds$. Equating the real parts, show that
$$(a^2 + b^2) \int_0^1 e^{as} \cos(bs) ds = e^{at}[a \cos b + b \sin b] - a.$$
Hint: Fundamental Theorem of Calculus for a complex-valued function of a real variable.

2. Apply Parseval's identity to the function
$$f(z) = \begin{cases} \dfrac{z^n - 1}{z - 1} = 1 + z + \cdots + z^{n-1} & z \neq 1 \\ n & z = 1 \end{cases}$$
to show that
$$\int_0^{2\pi} \left(\frac{\sin(\frac{1}{2}nt)}{\sin(\frac{1}{2}t)} \right)^2 dt = 2\pi n$$
Hint: $f(z)$ is a polynomial hence a power series (about 0) that is defined everywhere on $\mathbb{C}$. Do the estimation on the unit circle, i.e, compute $\int_0^{2\pi} |f(e^{it})|^2 dt$ and use Parseval's identity.

3. Compute the path integrals $\int_\gamma f(z)dz$:
   (a) $f(z) = |z|^2$ and γ is the line segment from 2 to $3 + i$.
   (b) $f(z) = \bar{z}$ and γ is the semicircle from 1 to -1 passing through i.
   (c) $f(z) = \bar{z}$ and γ is the semicircle from 1 to -1 passing through $-i$.
   (d) $f(z) = \exp(z)$ and γ is the line segment $[-1, 1] = \ell_{-1,1}$.
   (e) $f(z) = \log_\pi(z)$ and γ is the semicircle connecting $-i$ to i in the right half-plane: Im $z \geq 0$.

4. [Bonus] Assume that f is holomorphic on U, f' is continuous on U and $f(U) \subset \mathbb{C}\backslash L_0$. Show that $\int_\gamma (f'/f) = 0$ for any closed contour γ in U.

   Hint: Find an antiderivative for f'/f, explain why it is well-defined on U and apply the fundamental theorem of contour integration.

5. Using ML inequality or estimation lemma, establish the following:

(a) $|\int_\gamma \frac{1}{z^2+4} dz| \leq \frac{\pi R}{R^2-4}$ where $\gamma(t) = Re^{it}$ for $t \in [0, \pi]$ and $R > 2$.

(b) Show that $\lim_{R\to\infty} \int_{S(0,R)} \frac{z}{z^5+9} dz = 0$.

[Here $S(0, R) = \{z \in \mathbb{C} : |z| = R\}$ i.e., the circle of radius R and centre 0. Use the standard parametrization if you would like.]