MATH-4101(3)-001 Assignment 9 Due: Mar 31, 2022 (before class)

1. (Fresnel's Integrals). Assume that $\int_{\mathbb{R}} e^{-x^2} dx = 2\int_0^\infty e^{-x^2} dx = \sqrt{\pi}$. For $a \in \mathbb{R}$ with $|a| < 1$, show that

$$\int_0^\infty e^{-(1+ia)^2 t^2} dt = \frac{\sqrt{\pi}}{2} \left(\frac{1}{1+ia} \right) \quad \cdots (\star)$$

by the following strategy:

- Assume $0 \le a < 1$. Define $f(z) = e^{-z^2}$. Let $\gamma := \gamma_1 * \gamma_2 * \tilde{\gamma_3}$ where

$$\begin{aligned} \gamma_1(t) &= t, & t &\in [0, R] \\ \gamma_2(t) &= R + it, & t &\in [0, aR] \\ \gamma_3(t) &= (1+ia)t, & t &\in [0, R]. \end{aligned}$$

  Draw this contour and convince yourself that it is a triangle with vertices $0, R, R + iaR$ and γ winds around in anti-clockwise direction. Here $\tilde{\gamma_3}$ denotes the opposite or reverse path of γ_3.

- Apply Cauchy theorem, to get

$$\int_{\gamma_3} f = \int_{\gamma_1} f + \int_{\gamma_2} f.$$

- As done in the class, show that one of the integrals goes to 0 as $R \to \infty$ and rest will give the required answer as $R \to \infty$.

- Now separate the real and imaginary parts of $(\star)$, to find the values of following integrals:

$$\int_0^\infty e^{(a^2-1)t^2} \cos(2at^2) dt = \frac{1}{2(1+a^2)} \sqrt{\pi}, \quad 0 \le a < 1$$
$$\int_0^\infty e^{(a^2-1)t^2} \sin(2at^2) dt = \frac{a}{2(1+a^2)} \sqrt{\pi}, \quad 0 \le a < 1$$

- Conclude by stating why the above integrals continue to be true for $|a| \le 1$.

2. Evaluate $\int_\gamma \frac{3z-5}{z^2-2z-3} dz$, where $\gamma(t) = 2e^{it}, t \in [0, 4\pi]$. Use partial fractions.

3. Show that $\int_0^{2\pi} \frac{1}{2+\cos(\theta)} d\theta = 2\pi/\sqrt{3}$.

   Hint: Observe that $\frac{1}{2+\cos(\theta)} = \frac{2z}{z^2+4z+1}$ when $z = e^{i\theta}$. Carefully choose your f and you may not need to do partial fractions trick if you choose the appropriate domain such as $B(0,2)$.

4. Read through sections 7.4 - 7.8 of our textbook and do the exercise 7.9.22 (Computing winding number by eye for each contour in Figure 7.11). If you don't have the textbook, do email me.