MATH 137 LEC 010 (Mani Thamizhazhagan)
Week MT (Monday, Wednesday) Oct 17, 19 Lecture Summary.

On Monday, we started with definition of a function continuous on an interval (open as well as closed). Section 2.8.4 Pages 113 — 114.

A short note on why continuity over a closed interval is a "global" concept as opposed to "local", we have been seeing so far.

Recall how we figured $\lim_{x \to 2} x^2 = 4$. We restricted first to a smaller interval first, say $(1,3)$ (or) $(-1,5)$, (or) $(-2,6)$ all containing 2 and appropriately chose $\delta > 0$ for a given $\varepsilon > 0$ s.t

$$|x-2| < \delta \Rightarrow |x^2 - 4| < \varepsilon.$$

for $\underset{\{|x-2|<1\}}{x \in (1,3)}$, $|x+2| < 5$ so $\delta = \min\{1, \varepsilon/5\}$ works

for $\underset{\{|x-2|<3\}}{x \in (-1,5)}$, $|x+2| < 7$ so $\delta = \min\{3, \varepsilon/7\}$ works

for $\underset{\{|x-2|<4\}}{x \in (-2,6)}$, $|x+2| < 8$ so $\delta = \min\{4, \varepsilon/8\}$ works.

We had to restrict to a smaller interval because $f(x) = x^2$ was given to us as a function $f: \mathbb{R} \to \mathbb{R}$.
(smaller interval relative to $\mathbb{R}$)

Now consider $f(x) = x^2$ but as a function $f : [1,3] \to \mathbb{R}$.

Let's show f is cts. at $x = 2$ using $\varepsilon - \delta$ defn.

First understanding: $x \in [1,3]$ that's our domain.

Let $\varepsilon > 0$. Choose $\delta = \boxed{\varepsilon/6}$ will figure out

Then for all $|x-2| < \delta$,

$$|x^2 - 4| = |x-2||x+2| < 6|x-2| < \varepsilon \quad \boxed{\text{as } \delta = \varepsilon/5}.$$

(as $x \in [1,3]$, $|x+2| \le |x| + 2 \Rightarrow |x+2| < 6$)

Note that δ does not depend on $x = 2$ at all.

In fact, f is cts. at any $x = a \in [1,3]$. Same delta works.

because
$$\begin{aligned} |x^2 - a^2| &= |x-a||x+a| \\ &\le |x-a|(|x| + |a|) \\ &\le |x-a|(3+3) \quad \text{as } x, a \in [1,3] \\ &\le 6|x-a| \qquad |x| < 3, \ |a| < 3 \end{aligned}$$

So $\delta = \frac{\varepsilon}{6}$ works for given $\varepsilon > 0$.

The moment, the restriction is already given to you, the "local" (δ depends on the point) has become a "global" property (δ does not depend on the point)

This latter property is called uniform continuity!

Then we saw Intermediate Value Theorem (IVT) Section 2.9 Pages 115–126. Particularly, we discussed bisection method for approximating solutions of equations.

Exercises:

1) Prove that $ce^c = 1$ for some $c \in (0,1)$

Solution 1:

Let $f(x) = xe^x$. Then f is cts. on $[0,1]$ and $f(0) = 0 < 1 < e = f(1)$. Therefore, by IVT, there exists $c \in (0,1)$ s.t $f(c) = ce^c = 1$.

Solution 2:

Let $f(x) = xe^x - 1$. Then f is cts. on $[0,1]$ and $f(0) = -1 < 0$ and $f(1) = e - 1 > 0$. Therefore, by IVT, there exists $c \in (0,1)$ s.t $f(c) = 0$ (i.e) $ce^c - 1 = 0$ (i.e) $ce^c = 1$ for some $c \in (0,1)$.

2) Are there non-constant continuous functions $f: \mathbb{R} \to \mathbb{R}$ s.t $f(x) \in \mathbb{Z}$ for all $x \in \mathbb{R}$?

3) Let $f: [0, 2\pi] \longrightarrow [0, 2\pi]$ be cts s.t $f(0) = f(2\pi)$. Show that there exists $c \in [0, 2\pi]$ s.t $f(c) = f(c+\pi)$. (Hint: Either $f(0) = f(\pi)$ (or) $f(0) < f(\pi)$ (or) $f(0) > f(\pi)$ is true).

4) Does the converse of IVT true?

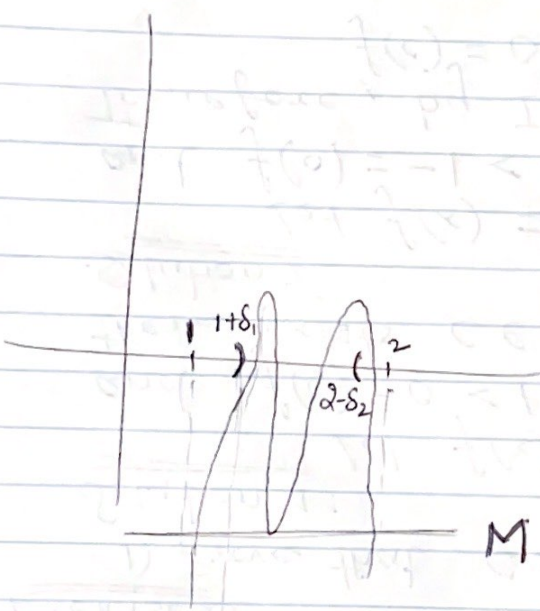
(i.e) If f is a real-valued function such that for every $a < b$ in the domain of f, and every choice of real number α between $f(a)$ and $f(b)$, there exists $c \in [a,b]$ that is in the domain of f s.t $f(c) = \alpha$, then f is a continuous function.

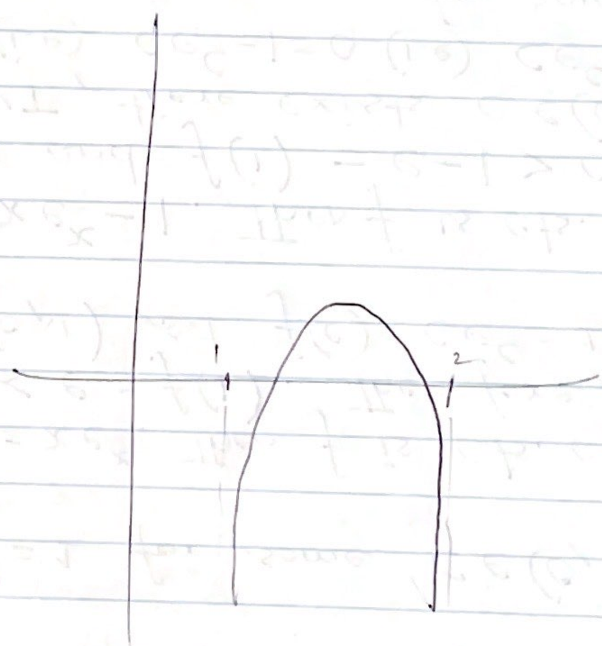
On Wednesday, we saw Extreme Value Theorem. Section 2.10 Pages 127—129.

Please go through all the examples of this section. These theorems are existence theorems and it is often difficult to apply these in a concrete setting. This is because these sit in between a chain of theorems that evolve to a bigger theory. Anyways look at the following technically tough problem:

<u>Exercise</u>: Suppose that f is continuous on $(1, 2)$ and $\lim_{x \to 1^+} f(x) = -\infty = \lim_{x \to 2^-} f(x)$. Show that there exists a global maximum for f.

<u>Draw</u> couple of examples of such function.





Proof:

For $M = \boxed{\min\{f(\tfrac{3}{2}), -1\}} < 0$, figure this out later

since $\lim_{x \to 1^+} f(x) = -\infty = \lim_{x \to 2^-} f(x)$, we can find a δ_1 and δ_2 both positive such that

$$\left.\begin{array}{l} 1 < x < 1+\delta_1 \\ \& \ 2-\delta_2 < x < 2 \end{array}\right\} \Longrightarrow f(x) < M .$$

Furthermore, we can find the above δ_1 and δ_2 s.t $\delta_1 < \frac{1}{4}$ and $\delta_2 < \frac{1}{4}$ (Replace δ_1 by $\min\{\frac{1}{4}, \delta_1\}$ δ_2 by $\min\{\frac{1}{4}, \delta_2\}$)

Now, $[1+\delta_1, 2-\delta_2] \subset (1,2)$ and f is cts. on $[1+\delta_1, 2-\delta_2]$. So by EVT, there exists $c \in [1+\delta_1, 2-\delta_2]$

s.t $f(x) \leq f(c)$ for all $x \in [1+\delta_1, 2-\delta_2]$.

Note that $\frac{3}{2} \in [1+\delta_1, 2-\delta_2]$

Infact $[1+\frac{1}{4}, 2-\frac{1}{4}] = [\frac{5}{4}, \frac{7}{4}] \subseteq [1+\delta_1, 2-\delta_2]$

So $f(\frac{3}{2}) \leq f(c)$.

for any $y \in [\frac{5}{4}, \frac{7}{4}]$,

one can choose $M = f(y)$

In particular, we choose $M = f(\frac{3}{2})$

(One can replace M by $\min\{-1, f(\frac{3}{2})\}$ for a negative value)

So for $\left.\begin{array}{l} 1 < x < 1+\delta_1 \\ \& \ 2-\delta_2 < x < 2 \end{array}\right\} \Rightarrow f(x) < f\left(\frac{3}{2}\right) \leq f(c)$

And when $x \in [1+\delta_1, 2-\delta_2]$, $f(x) \leq f(c)$ by EVT.

$\therefore$ c is a global maximum for f.