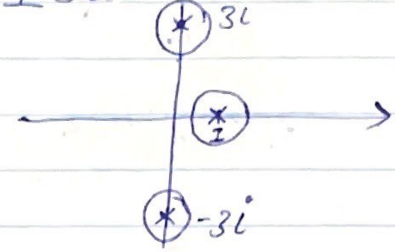
Dec 1, 2023

Singularities and Residues

Defn. If f fails to be analytic at z_0, but is analytic in a neighborhood of z_0, (i.e) $0 < |z - z_0| < \varepsilon$ for some $\varepsilon > 0$, then f has an isolated singularity at z_0.

E.g. $f(z) = \dfrac{1}{(z^2+9)(z-1)}$ has three isolated singularities at $z = 1, \pm 3i$.



whereas $f(z) = \dfrac{1}{\sin\left(\frac{1}{z}\right)}$ has singularities at $z = 0$ and

$$z = \frac{1}{n\pi}, \quad n = \pm 1, \pm 2, \ldots$$

On any neighborhood of 0, one can find some $m \in \mathbb{Z}$ s.t it contains $\frac{1}{m\pi}$ $\Rightarrow$ $z = 0$ is not an isolated singularity.

whereas $z = \frac{1}{n\pi}$ is for any $n \in \mathbb{Z}$.

IN THIS COURSE, WE WILL ONLY DEAL WITH ISOLATED SINGULARITIES.

Consider the three functions:

$$\left.\begin{aligned} &1)\ f(z) = \frac{\sin z}{z}, \quad z \neq 0 \\ &2)\ g(z) = \frac{\cos z}{z}, \quad z \neq 0 \\ &3)\ h(z) = e^{1/z}, \quad z \neq 0 \end{aligned}\right\}$$

$z = 0$ is an isolated singularity (as it is the only singularity here).

On a closer look,

$$f(z) = \frac{\sin z}{z} = \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!} = 1 + \sum_{n=1}^{\infty}\frac{(-1)^n z^{2n}}{(2n+1)!}$$

$\Rightarrow$ as $z \to 0$, $f(z) \to 0 + 1 = 1$

if we redefine f as

$$f(z) = \begin{cases} \frac{\sin z}{z}, & z \neq 0 \\ 1, & z = 0 \end{cases}$$

In fact, not only now f is continuous at $z=0$, but is also analytic at $z=0$! Thus by defining a suitable value for the function at 0, we have been able to remove the singularity. Accordingly, we call $z=0$ as a removable singularity of $f(z) = \frac{\sin z}{z}$ defined on $\mathbb{C} \setminus \{0\}$.

On the other hand, the singularities at $z=0$ of $g(z) = \frac{\cos z}{z}$ and $h(z) = e^{1/z}$ are not removable.

$$g(z) = \frac{\cos z}{z} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!}$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty}(-1)^n \frac{z^{2n-1}}{(2n)!}$$

$$\lim_{z\to 0}\frac{\cos z}{z} = \infty \ !$$

$$h(z) = e^{1/z}$$

$$= \sum_{n=0}^{\infty}\frac{1}{z^n n!}$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2 2!} + \frac{1}{z^3 3!} + \cdots$$

$\lim_{z\to 0} e^{1/z}$ does not exist.

$$\lim_{x\to 0^+} e^{1/z} = \lim_{x\to 0^+} e^{1/x} = \infty$$

$$\lim_{x\to 0^-} e^{1/z} = \lim_{x\to 0^-} e^{1/x} = 0$$

Defn: Assume that f has an isolated singularity at z_0.
If

1) $\lim_{z \to z_0} f(z)$ exists (as a finite complex number)

then z_0 is called a removable singularity.

2) $\lim_{z \to z_0} f(z) = \infty$, then z_0 is called a pole of f

3) $\lim_{z \to z_0} f(z)$ DNE, then z_0 is called an essential singularity of f.

The intimate connection between the nature of isolated singularity of f at z_0 and the Laurent series of f at z_0 is given by the following:

[Since f is analytic in $0 < |z - z_0| < \varepsilon$ for some $\varepsilon > 0$, f has Laurent expansion at z_0 in $0 < |z - z_0| < \varepsilon$ of the form

$$\underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{analytic part of } f \text{ at } z_0} + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}}_{P(z) \text{ - the principal part of } f \text{ at } z_0}$$]

i) f has a removable singularity at $z_0 \Leftrightarrow P(z) = 0$ (i.e) $b_n = 0 \ \forall n$.

ii) f has a pole at $z_0 \Leftrightarrow P(z)$ has finite number of terms. There exists m (i.e) s.t $b_n = 0$ for $n > m$ and $b_m \neq 0$ [Co-efficient of largest -ve power]

[$m = 1$: simple pole
$m \neq 1$: pole of order m]

iii) f has an essential singularity at z_0

$\Longleftrightarrow P(z)$ has infinite number of terms

(i.e) $b_n \neq 0$ for infinitely many $n \in \mathbb{N}$.

Example

1) $$\frac{\sin(z)}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$$

$P(z) = 0 \Rightarrow$ removable singularity at $z = 0$.

2) $$\frac{\cos(z)}{z} = \frac{1}{z} - \frac{+z}{2!} + \frac{z^3}{4!} - \frac{z^5}{6!} + \cdots$$

$P(z) = \frac{1}{z}$ (i.e) $b_1 \neq 0$ and $b_n = 0$ for $n > 1$.

$\Rightarrow$ simple pole at $z = 0$.

3) $$\frac{\cos(z-\pi)}{(z-\pi)^3} = \frac{1}{(z-\pi)^3} - \frac{1}{(z-\pi)\,2!} + \frac{(z-\pi)}{4!} - \frac{(z-\pi)^3}{6!} + \cdots$$

$$P(z) = \frac{1}{(z-\pi)^3} - \frac{1}{2(z-\pi)}$$ (i.e) $b_3 \neq 0$ and $b_n = 0$ for $n > 3$.

$\Rightarrow$ pole of order 3!

4) $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots$

$P(z)$ has infinitely many terms

$\Rightarrow z=0$ is an essential singularity.

5) $\frac{\exp^{3z^2} - 1}{z^3} = \frac{1}{z^3}\left(3z^2 + \frac{(3z^2)^2}{2!} + \frac{(3z^2)^3}{3!} + \dots\right)$

$= \frac{3}{z} + \frac{9z}{2!} + \frac{9z^3}{2} + \dots$

$P(z) = \frac{3}{z} \Rightarrow z=0$ is a simple pole.

6) $\frac{1-\cos(z^2)}{z^5} = \frac{1}{z^5}\left(1 - \left(1 - \frac{(z^2)^2}{2!} + \frac{(z^2)^4}{4!} - \frac{(z^2)^6}{6!} + \dots\right)\right)$

$= \frac{1}{2z} - \frac{z^3}{4!} + \frac{z^7}{6!} - \frac{z^{11}}{8!} + \dots$

$P(z) = \frac{1}{2z} \Rightarrow z=0$ is a simple pole.

7) $\frac{1-\sin z}{z^4} = \frac{1}{z^4}\left(1 - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right)\right)$

$= \underbrace{\frac{1}{z^4} - \frac{1}{z^3} + \frac{1}{3!z}}_{P(z)} + \frac{z}{5!} - \frac{z^3}{7!} + \dots$

$\Rightarrow z=0$ is a pole of order 4!

Definition

The co-efficient b_1 in the Laurent series is called the residue of f at the isolated singular point z_0, and is denoted by $\operatorname{Res}_{z=z_0} f(z)$.

Recall that,

$$\operatorname{Res}_{z=z_0} f = b_1 = \frac{1}{2\pi i}\int_C f(z)\,dz$$

as long as C contains just z_0 as a singular point inside!

Simple poles

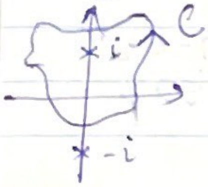
$$f(z) = \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

$$\Rightarrow (z-z_0)f(z) = b_1 + a_0(z-z_0) + a_1(z-z_0)^2 + \cdots$$

when z_0 is a simple pole of f.

$$\boxed{\lim_{z\to z_0}(z-z_0)f(z) = b_1 = \operatorname{Res}_{z=z_0} f(z)}$$

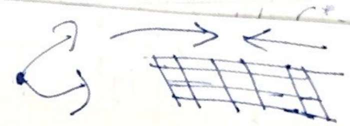
Example: $$\operatorname{Res}_{z=i}\frac{e^{-iz}}{z^2+1} = \lim_{z\to i}(z-i)\frac{e^{-iz}}{(z-i)(z+i)} = \frac{e}{2i}$$



$$\frac{1}{2\pi i}\int_C \frac{e^{-iz}}{z^2+1}\,dz = \frac{1}{2\pi i}\int \frac{e^{-iz}/z+i}{z-i}\,dz = \frac{e}{2i} \quad (\text{CIF})$$

Defn. A compact quantum group is a pair (A, Δ) ... C^*-algebra,

$$\frac{\cos(z-\pi/2+\pi/2)}{(z-\pi/2)}$$



Title: Second Duals of cb multiplier closure of Fourier algebra $A(G)$.

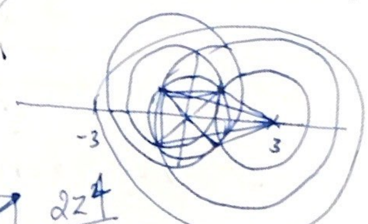
$$-\frac{\sin(z-\pi/2)}{(z-\pi/2)} = -\frac{\sum(-1)^n (z-\pi/2)^{2n}}{(z-\pi/2)}$$

Title: $\cos(z-\pi/2)\cos(\pi/2) - \sin(z-\pi/2)\sin(\pi/2)$

$$\frac{1}{(z^4+1)(z^2-9)}$$

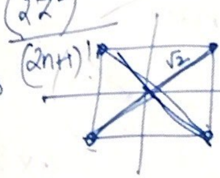
C17,

$\pi/4$

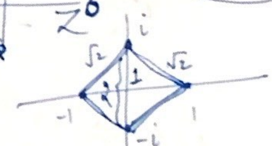


$-7\pi/4$ $\sum_{n=0}^{\infty} \frac{(2z^2)^{2n+1}}{(2n+1)!}$

$9\pi/4$, $\pi/4$, $-7\pi/4$



$$\frac{2z^4}{z^6}$$



$$r\frac{df}{dr} + 3f$$

$$r\frac{df}{dr} = -3f$$

$$\frac{df}{f} = -3\frac{dr}{r}$$

$$\ln f = -3\ln r$$

$$\Rightarrow f = \frac{1}{r^3}$$

$(1+i)^i$

$\exp(i \ln(1+i))$

$\exp(i(\ln\sqrt{2} + i\pi/4)) = e^{-\pi/4}\exp(\ln\sqrt{2})$

$e^{\frac{\sin z}{z}}$

$1 + \left(\frac{\sin z}{z}\right) + \frac{(\sin z)^2}{z^2}$

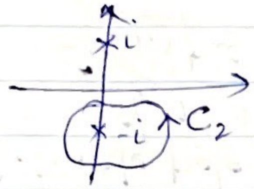
$$\frac{e^{2z}+1-2e^z}{z^2}$$

$1 + (2z) + \frac{4z^2}{2!} +$

$$\frac{e^{2z}}{z^2} + \frac{1}{z^2} - \frac{2e^z}{z^2}$$

$1 + z + z^2$

$$\operatorname*{Res}_{z\to -i} \frac{e^{-iz}}{z^2+1} = \lim_{z\to -i} (z+i)\frac{e^{-iz}}{(z-i)(z+i)} = -\frac{e^{-1}}{2i}$$



$$\Rightarrow \frac{1}{2\pi i}\int_{C_2} \frac{e^{-iz}}{z^2+1}\,dz = -\frac{e^{-1}}{2i}$$ [Verify using CIF].

<u>Another method to compute residues at simple poles</u>

Suppose $f(z) = \dfrac{p(z)}{q(z)}$ where p, q analytic around z_0 and z_0 is a simple ~~pole~~ zero of q with $p(z_0) \neq 0$ $\big($(i.e) $q(z_0)=0$ but $q'(z_0)\neq 0\big)$

Then $$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \lim_{z\to z_0}\left[(z-z_0)\frac{p(z)}{q(z)}\right]$$

$$= \lim_{z\to z_0} \frac{p(z)}{\dfrac{q(z)}{z-z_0}}$$

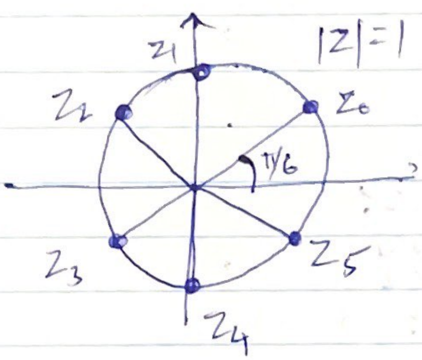
$$= \lim_{z\to z_0} \frac{p(z)}{\dfrac{q(z)-q(z_0)}{z-z_0}}$$

$$\boxed{\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}}$$

when z_0 is a simple zero of q and $p(z_0)\neq 0$

Eg. Consider $f(z) = \frac{1}{z^6+1}$. Then f has 6 simple poles: $z^6+1 \Rightarrow z = (-1)^{1/6} = \left(e^{i(\pi+2n\pi)}\right)^{1/6}$

$$z_n = e^{\frac{i\pi(2n+1)}{6}}, \quad n = 0, 1, 2, \dots, 5.$$



$$z_0 = e^{i\pi/6},\ \underset{z_1}{e^{i\pi/2}},\ \underset{z_2}{e^{i5\pi/6}},\ \underset{z_3}{e^{i7\pi/6}},\ \underset{z_4}{e^{i3\pi/2}},\ \underset{z_5}{e^{i11\pi/6}}$$

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{6z^5}\Big|_{z=z_0} = \frac{1}{6e^{i5\pi/6}}$$

$$= \frac{1}{6} e^{-i5\pi/6}$$

Note that since $\frac{1}{z^6+1} = \frac{1}{(z-z_0)(z-z_1)\cdots(z-z_5)}$

$$\frac{1}{6}e^{-i5\pi/6} = \operatorname*{Res}_{z=z_0} \frac{1}{z^6+1} = \lim_{z\to z_0} (z-z_0)\left(\frac{1}{(z-z_0)(z-z_1)\cdots(z-z_5)}\right)$$

$$= \frac{1}{(z_0-z_1)(z_0-z_2)\cdots(z_0-z_5)}$$

Poles of order m:

In this case,

$$f(z) = \frac{b_m}{(z-z_0)^m} + \frac{b_{m-1}}{(z-z_0)^{m-1}} + \dots + \frac{b_1}{(z-z_0)} + \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{analytic part}}$$

Multiply by $(z-z_0)^m$:

$$(z-z_0)^m f(z) = b_m + b_{m-1}(z-z_0) + \dots + b_1 (z-z_0)^{m-1} + \sum_{n=0}^{\infty} a_n (z-z_0)^{n+m}$$

How do we get b_1 ?

Differentiate $(m-1)$ times and substitute $z = z_0$

$$\frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right] \Bigg|_{z=z_0} = (m-1)!\, b_1$$

$\Rightarrow$ Residue of a pole of order m:

$$\boxed{b_1 = \operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right] \Bigg|_{z=z_0}}$$

Example Find the residue of $f(z) = \dfrac{e^{3z}}{(z-2)^3}$ at $z=2$.

pole of order 3!

with $m = 3$

$$\operatorname*{Res}_{z=2} f(z)$$

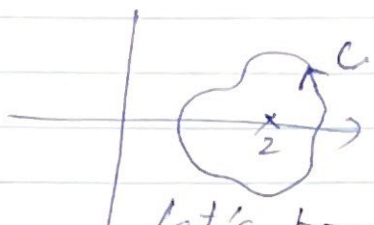
$$= \lim_{z \to 2} \frac{1}{2!} \frac{d^2}{dz^2} \left[(z-2)^3 \frac{e^{3z}}{(z-2)^3} \right]$$

$$= \frac{1}{2!} \frac{d^2}{dz^2} \left[e^{3z} \right] \Bigg|_{z=2} = \frac{9e^6}{2}$$

Recall that by GCIF,

$$b_1 = \frac{1}{2\pi i}\int_C \frac{e^{3z}}{(z-2)^3}\,dz = \frac{1}{2!}\left(2^{nd}\text{ derivative of } e^{3z}\right)\Big|_{z=2}$$

when C has 2 inside its trace.



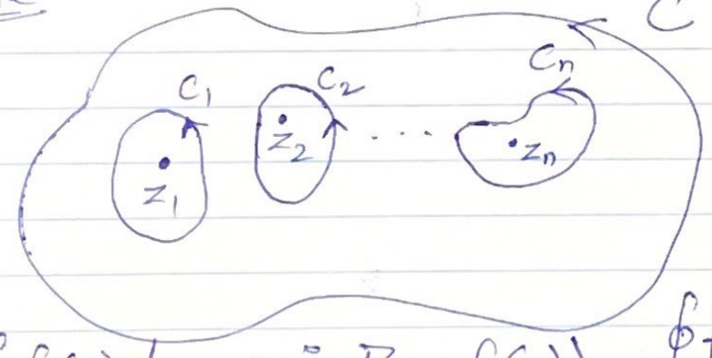
Let's prove a powerful generalization of Cauchy Integral Formulas.

<u>The Residue Theorem</u>:

Let C be a positively oriented simple closed contour, and let f be analytic inside and on C except at finitely many isolated singular points $z_1, \dots, z_n$ inside C. Then

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

<u>Consider</u>



f is analytic in C but outside C_k, so by Principal of path deformation & C·G·Th,

$$\oint_C f(z)\,dz = \sum_{k=1}^{n} \oint_{C_k} f(z)\,dz$$

and $\displaystyle\oint_{C_k} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=z_k} f(z)$