Oct 6, 2023

The Divergence Theorem (aka Gauss-Ostragradskii Theorem)

Suppose Ω is a region in $\mathbb{R}^3$ with closed boundary surface $\partial\Omega$ that has outward unit normal $\hat{n}$ and that $\vec{F}$ is C^1 on Ω and $\partial\Omega$. Then

$$\underbrace{\iint_{\partial\Omega} \vec{F}\cdot\hat{n}\,dS}_{\text{Total flux through the surface } \partial\Omega} = \iiint_{\Omega} \underbrace{\nabla\cdot\vec{F}}_{\text{Flux per unit volume}}\,dV .$$

Note:

- applies to only closed surfaces such a sphere (where there is no edge or boundary curve)
- yet another example of an ('integral of a derivative thm')
- provides an alternate way to compute the flux through a (closed:) surface in terms of triple integral evaluation (or) vice versa.

Exercises:

1) Find the flux of the vector field

$$\vec{F} = (\underbrace{x+y}_{F_1}, \underbrace{x+ye^{yz}}_{F_2}, \underbrace{-ze^{yz}}_{F_3})$$ outward through the ellipsoid $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$

Solution:

Note that the ellipsoid is a closed surface so the Divergence Theorem applies.

Let $\partial\Omega$ denote the boundary surface of the region inside the ellipsoid Ω with outward normal $\hat{n}$.

(i.e) $\Omega = \{(x,y,z) : x^2 + y^2/4 + z^2/9 \le 1\}$

Then,

$$\iint_{\partial\Omega} \vec{F} \cdot \vec{n} \, dS = \iiint_{\Omega} \nabla \cdot \vec{F} \, dV$$

Let's compute $\nabla \cdot \vec{F}$:

$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= 1 + (zye^{yz} + e^{yz}) + (-yze^{yz} - e^{yz})$$

$$= 1.$$

$$\Rightarrow \iint_{\partial\Omega} \vec{F}\cdot\vec{n}\, dS = \iiint_{\Omega} 1\, dv = Vol(\Omega)$$

Computing $Vol(\Omega)$:

Let's do a change of variables:

$$u = x, \quad v = \frac{y}{2}, \quad w = \frac{z}{3}$$

$$\Rightarrow x = u, \quad y = 2v, \quad w = 3z \quad \text{and}$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 1\cdot 2\cdot 3 = 6$$

Under these change of variables, the region inside the given ellipsoid, $x^2 + \frac{y^2}{4} + \frac{z^2}{9} \le 1$ becomes the region inside the sphere, $u^2 + v^2 + w^2 \le 1$. whose volume is $\frac{4}{3}\pi$

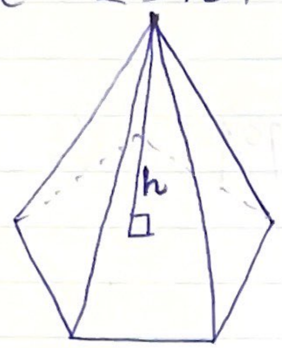
$$\therefore \iiint_{\Omega_{xyz}} dx\,dy\,dz = \iiint_{\Omega_{uvw}} \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

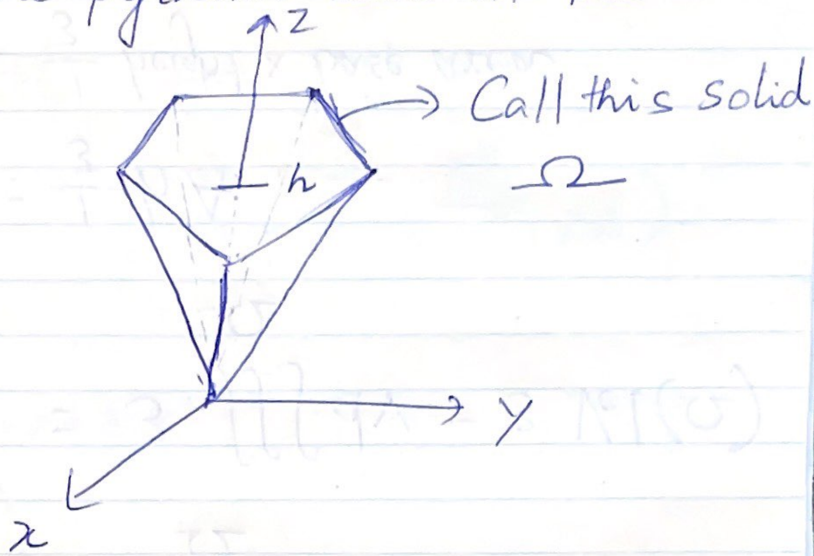
$$= 6\left(\frac{4}{3}\pi\right)$$

$$= 8\pi.$$

Example 2 (Volume of a Pyramid)

Rearrange and rotate the polygonal pyramid such that its apex is at origin and the base of the pyramid lies in the plane $z = h$.





Let A be the area of the base.

Let $\vec{F} = \vec{r} = (x, y, z)$

Then $$\iint_{\partial\Omega_{side}} \vec{F} \cdot \hat{n} \, dS = 0$$ (why? Close your eyes and imagine)

$\hat{n}$ for $\partial\Omega_{top}$ is $\hat{k}$ and z-co-ordinate of $\vec{F}$ for $\partial\Omega_{top}$ is h $\Rightarrow$ $$\iint_{\partial\Omega_{top}} \vec{F} \cdot \hat{n} \, dS = \iint_{\partial\Omega_{top}} h \, dS$$
$$= hA.$$

$$\therefore \iint_{\partial\Omega} \vec{F}\cdot\hat{n}\,ds = \iint_{\partial\Omega_{side}} \vec{F}\cdot\hat{n}\,ds + \iint_{\partial\Omega_{top}} \vec{F}\cdot\hat{n}\,ds$$

$$= hA$$

As $\partial\Omega$ is a closed surface and $\vec{F}$ is C^1 on $\Omega \cup \partial\Omega$, by the divergence theorem,

$$hA = \iint_{\partial\Omega} \vec{F}\cdot\hat{n}\,dS = \iiint_{\Omega} \nabla\cdot\vec{F}\,dV$$

$$= 3\iiint_{\Omega} dV = 3\,\mathrm{Vol}(\Omega)$$

$$\Rightarrow \mathrm{Vol}(\Omega) = \frac{1}{3} hA.$$

$$= \frac{1}{3}\text{ height} \times \text{base area}$$

Example 3 (Moving the surface)

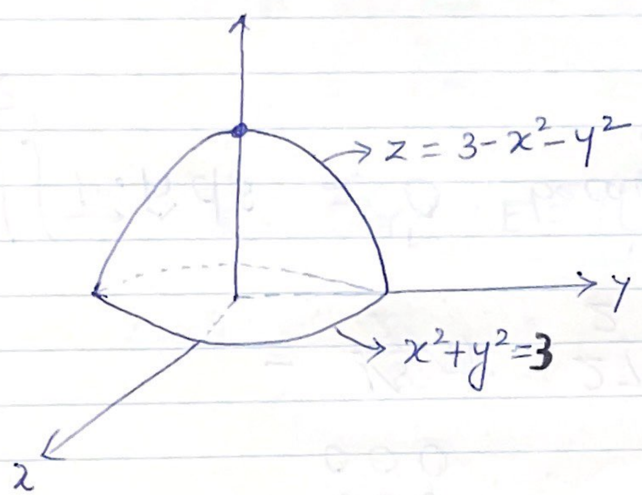
R is Graph of $f(x,y)=3-x^2-y^2$ defined on the disk of radius $\sqrt{3}$.

Let $R=\{(x,y,z): x^2+y^2\le(\sqrt{3})^2, z=3-x^2-y^2\}$, with the outward unit normal $\hat{n}$

Compute

$$\iint_R \vec{F}\cdot\hat{n}\,dS \quad \text{for}$$

$$\vec{F}=\left(\cos(yz),\, e^{\frac{x^2}{1+z^2}},\, 3z\right)$$



Solution:

Let $\Omega=\{(x,y,z): x^2+y^2\le 3,\ 0\le z\le 3-x^2-y^2\}$ with the unit normal oriented outward on $\partial\Omega$.

Thus Ω is the region between R and the xy-plane, and $\partial\Omega=R\cup R'$, where $R'=\{(x,y,0): x^2+y^2\le 3\}$

So the outward unit normal on R' is $-\hat{k}$!

By the divergence theorem,

$$\iiint_\Omega \nabla\cdot\vec{F}\,dV=\iint_{\partial\Omega}\vec{F}\cdot\hat{n}\,dS=\iint_R\vec{F}\cdot\hat{n}\,dS+\iint_{R'}\vec{F}\cdot\hat{n}\,dS$$

$$\therefore \iint\limits_{R} \vec{F}\cdot\hat{n}\,ds = \iiint\limits_{\Omega} \nabla\cdot\vec{F}\,dV - \iint\limits_{R'} \vec{F}\cdot\hat{n}\,ds$$

Note that, $\displaystyle\iiint\limits_{\Omega} \nabla\cdot\vec{F}\,dV = \iiint\limits_{\Omega} 3\,dV = 3\,Vol(\Omega)$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^{3-r^2} 3r\,dz\,dr\,d\theta \quad \left(\begin{array}{c}\text{In Cylindrical} \\ \text{Co-ordinates}\end{array}\right)$$

$$= \cancel{\frac{18}{2}\pi} \; \frac{27}{2}\pi$$

$\displaystyle\iint\limits_{R'} \vec{F}\cdot\hat{n}\,ds = 0$ because $\vec{F}\cdot\hat{n} = -3z$ on R' and $z=0$ on R'

$$\therefore \iint\limits_{R} \vec{F}\cdot\hat{n}\,ds = \cancel{\frac{18}{2}\pi} \; \frac{27}{2}\pi$$

Stokes' Theorem

Suppose that Σ is an open) surface with closed boundary curve $\partial\Sigma$, with Σ and $\partial\Sigma$ oriented according to the right hand rule, and $\vec{F}$ is C^1 on $\Sigma \cup \partial\Sigma$. Then,

$$\iint_{\Sigma} (\nabla \times \vec{F}) \cdot \hat{n} \, dS = \oint_{\partial\Sigma} \vec{F} \cdot d\vec{x}$$

Notes

1) In order for Σ to possess a boundary curve $\partial\Sigma$, Σ must be an open surface.

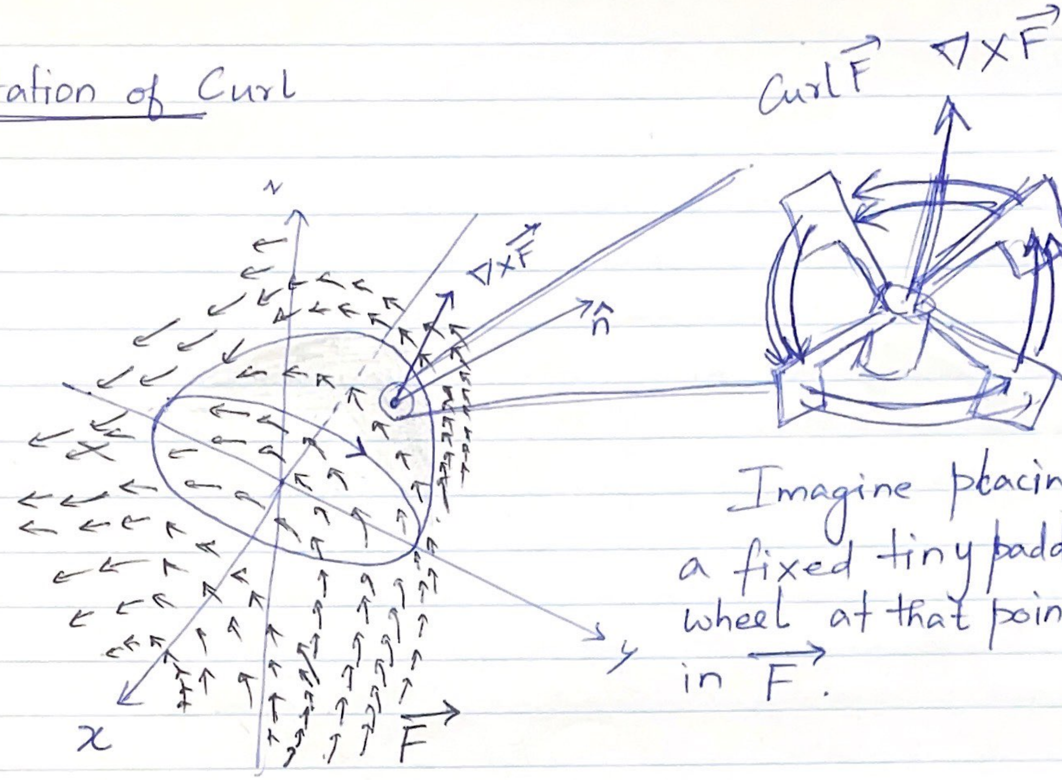
2) Special Case $\Sigma : D \subset \mathbb{R}^2$, $\vec{F} = (F_1, F_2, 0)$
Then $\hat{n} = \hat{k}$ and $\nabla \times \vec{F} = \left(0, 0, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$ [F_1, F_2 are functions of (x, y)]

$\Rightarrow (\nabla \times \vec{F}) \cdot \hat{n} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$. Also $\iint_{\Sigma} dS$ becomes $\iint_{D} dx\,dy$.

That is, in the two dimensional case, Stokes' Thm is Green's Thm.

3) Yet another "integral of a derivative theorem". The derivative (in this case curl) appears on the side with two integral symbols, while just the field itself appears with a single integral symbol.

Interpretation of Curl



Imagine placing a fixed tiny paddle wheel at that point in $\vec{F}$.

$\nabla \times \vec{F}$ - the component of the curl in the $\hat{n}$ direction is the circulation per unit area around a circle (or a rectangle, a triangle, a square, (or) any piecewise smooth closed curve) lying in the plane orthogonal (i.e perpendicular) to $\hat{n}$!

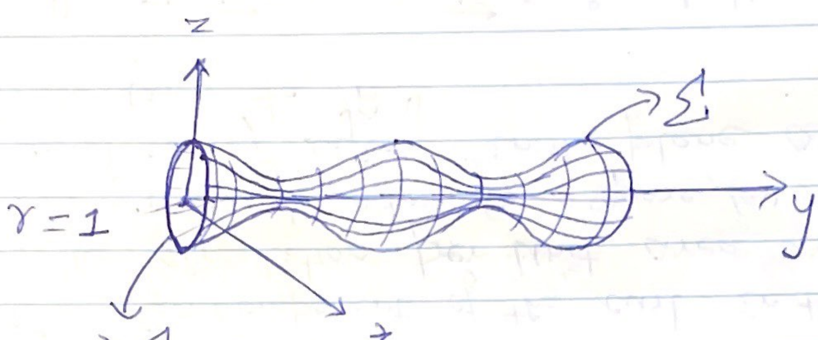
Recall $\nabla \times \vec{F} = 0 \Rightarrow \vec{F}$ is irrotational.

Thus, Stoke's Thm says that the global circulation around the boundary curve of the surface is equal to the sum (surface integral) of the circulation elements on the surface.

Example 1) Use Stokes' theorem to calculate surface integral

$$\iint_{\Sigma} \nabla \times \vec{F} \cdot \hat{n}\, ds \quad \text{where}$$

$\vec{F} = (z, 2xy, x+y)$ and Σ is the surface as shown below



$\partial\Sigma$: $\cancel{x^2+y^2}$

Solution:

Notice that $\partial\Sigma$ is the unit circle on the xz-plane $= \{(x,0,z) : x^2+z^2=1\}$

Parametrize $\partial\Sigma$ as $\vec{g}(t) = (\cos t, 0, \sin t)$, $0 \le t \le 2\pi$
Then, by Stokes' theorem,

$$\iint_{\Sigma} \nabla \times \vec{F} \cdot \hat{n}\, ds = \oint_{\partial\Sigma} \vec{F} \cdot d\vec{x}$$

$$= \oint_{0}^{2\pi} \vec{F}(\vec{g}(t)) \cdot \vec{g}\,'(t)\,dt$$

$$= \int_{0}^{2\pi} (\sin t, 0, \cos t) \cdot (-\sin t, 0, \cos t)\,dt$$

$$= \int_{0}^{2\pi} (\cos^2 t - \sin^2 t)\,dt$$

$$= \int_{0}^{2\pi} \cos 2t\,dt$$

$$= \left.\frac{\sin 2t}{2}\right|_{0}^{2\pi}$$

$$= 0.$$