WEEK-10 LEC 010 (Mani Thamizhazhagan)
Nov 21-25 Lecture Summary / Notes

This week, we interpreted the second derivative, f'' as a measure of how quickly the slopes of the tangent lines of f change and introduced the concept of concavity. Also we developed the first and second derivative tests to classify the critical points and had enough tools to sketch the graph of a function.

Read section 4.2.5, 4.2.6 Pages 226 — 230 (Concavity)

Read section 4.2.7 Pages 232 — 236 (Classifying CPs)

Read section 2.11 Pages 130–132 (Curve Sketching Part 1)
4.4 Pages 247–258 (Curve Sketching Part 2)

Some Proofs about Concavity and interpretation of its definitions:

Look at the following pictures.

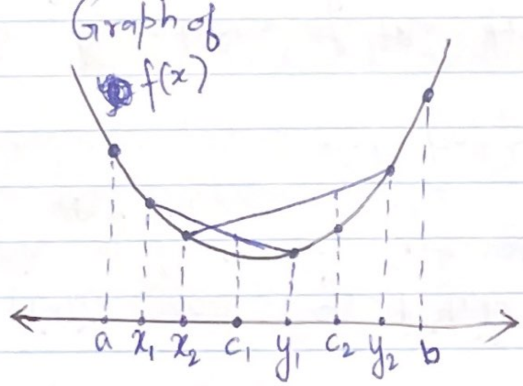


Fig. 1

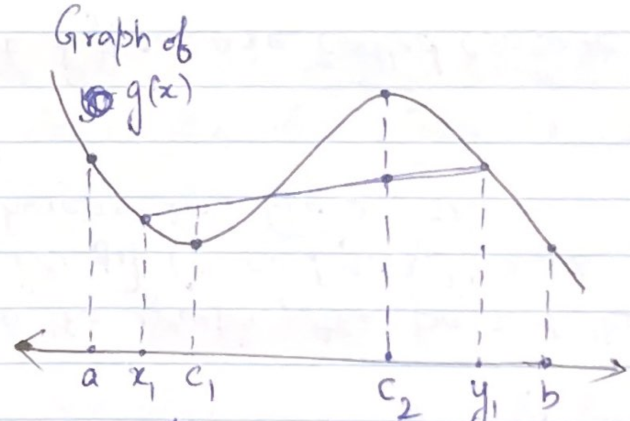


Fig. 2

In figure 1, if we draw "any" secant line joining $(x_1, f(x_1))$ and $(y_1, f(y_1))$ on the graph, the part of the graph of f on the interval $[x_1, y_1]$ (or on $[y_1, x_1]$) lies below the secant line. Whereas in Fig 2, this phenomenon does not happen.

The functions of the first type are called concave up functions.

We need a few definitions:

The line segment joining the two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is given by

$$(1-t)(x_1, f(x_1)) + t(x_2, f(x_2)) \quad \text{for } t \in [0,1]$$

[This is a parametric version of the equation for the line joining two points in $\mathbb{R}^2$]

because if (x,y) is a point on the line, then we can solve for t so that

$$(x,y) = (1-t)(x_1, f(x_1)) + t(x_2, f(x_2)).$$

Note that $x = (1-t)x_1 + t x_2$

$y = (1-t)f(x_1) + t f(x_2)$ and we have

$$x - x_1 = t(x_2 - x_1) \text{ and } y - f(x_1) = t(f(x_2) - f(x_1))$$

We get $t = \dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - f(x_1)}{f(x_2) - f(x_1)}$

If $x_1 < x < x_2$, then $0 < t < 1$, and

since $\dfrac{f(x_2)-f(x_1)}{x_2-x_1} = \dfrac{y-f(x_1)}{x-x_1}$ is the slope of the line, we get $\dfrac{x-x_1}{x_2-x_1} = \dfrac{y-f(x_1)}{f(x_2)-f(x_1)}$!

Now, as $x = (1-t)x_1 + t(x_2)$, the condition for the concave up of f is that for each $t \in [0,1]$, the y-co-ordinate of $(x, f(x))$ must be less than (or) equal to the y-co-ordinate of $\left(x=(1-t)x_1+t(x_2)\right)$ $\left((1-t)f(x_1)+t\,f(x_2)\right)$. We have thus arrived at the following definition.

Defn. Let I be an interval in $\mathbb{R}$. A function $f: I \longrightarrow \mathbb{R}$ (or) a function f on I is said to be concave up if for all $t \in [0,1]$ and for all $a, b \in I$, the following inequality holds:

$$f(ta+(1-t)b) \le t f(a) + (1-t) f(b) \quad (*)$$

If for $0 < t < 1$ and for all $a \neq b \in I$, strict inequality holds in $(*)$, then the function is said to be strictly concave up on I.

[This is what our book calls as concave up, but it is not very clear].

We say that f is concave down on I, if the reverse inequality holds in $(*)$.

Thm: [Derivative Test for Concavity]

Assume that $f:[a,b]\to\mathbb{R}$ is cts. and diff. on (a,b). If f' is non-decreasing on (a,b), then f is concave up on $[a,b]$.

Proof: This is yet another application of MVT.

Let $x<y\in[a,b]$. For any $z\in(x,y)\subseteq[a,b]$,

apply MVT on $[x,z]$ and $[z,y]$ to get $r\in(x,z)$ and $s\in(z,y)$ s.t

$$\frac{f(z)-f(x)}{z-x}=f'(r) \quad \& \quad \frac{f(y)-f(z)}{y-z}=f'(s).$$

Now since $x<r<z<s<y$, $f'(r)\le f'(s)$ by the given hypothesis (non-decreasing f')

$$\Rightarrow \quad \frac{f(z)-f(x)}{z-x}\le\frac{f(y)-f(z)}{y-z}$$

$$\Rightarrow \quad (f(z)-f(x))(y-z)\le(f(y)-f(z))(z-x)$$

as $y-z>0$, $z-x>0$

$$\Rightarrow \quad f(z)(y-z+z-x)\le f(y)(z-x)+f(x)(y-z)$$

$$\Rightarrow \quad f(z)(y-x)\le f(y)(z-x)+f(x)(y-z)$$

$\longrightarrow (\star)$

Now as $x < z < y$, there exists $t \in (0,1)$ s.t

$$z = tx + (1-t)y$$

$$\Rightarrow (z-x) = (1-t)(y-x) \;\&\; (y-z) = t(y-x)$$

Substituting this in ⊛, we get

$$(y-x)\left[f(tx+(1-t)y)\right] \le \left[(1-t)f(y) + t f(x)\right](y-x)$$

$$\therefore \quad f(tx+(1-t)y) \le t f(x) + (1-t) f(y)$$

as x and y are arbitrary,

$$f(tx+(1-t)y) \le t f(x) + (1-t) f(y) \text{ for } t \in [0,1] \;\&\; \text{for all } x, y \in [a,b]$$

$\therefore$ f is concave up on $[a,b]$.

Corollary: (Remarks)

1) If f' were increasing on (a,b), then we would get strict concave up of f.

2) If $f''(x) > 0$ for all $x \in (a,b)$, f' is increasing on (a,b). Therefore, f is (strictly) Concave up on $[a,b]$.

3) $f''(x) \geq 0 \;\forall x \in (a,b) \iff f$ is concave up on $[a,b]$ for a twice diff. function on $[a,b]$.

($\Leftarrow$: Idea of Proof: Using concavity of f show that for any $p < x < y < q \in [a,b]$, $\frac{f(x)-f(p)}{x-p} \le \frac{f(q)-f(y)}{q-y}$
Take limits as $p \to x^-$ & $q \to y^+$ to get $f'(x) \le f'(y)$)

Remark about second derivative test

Theorem 13 in Page 236

We don't need the assumption of f'' being cts. at $x = c$.

Suppose that $f'(c) = 0$ and $f''(c) > 0$!

Then by defn., $\lim_{h \to 0} \frac{f'(c+h) - f'(c)}{h} = f''(c) > 0$

$$\Rightarrow \lim_{h \to 0} \frac{f'(c+h)}{h} > 0$$

So when $h \to 0^-$ as h is negative, $f'(c+h)$ must be negative for some $h \in (-\delta_1, 0)$, $\delta_1 > 0$

III^{ly} $f'(c+h)$ must be positive for some $h \in (0, \delta_2)$, $\delta_2 > 0$ when $h \to 0^+$

$$\Rightarrow f'(x) < 0 \text{ for } x \in (c - \delta_1, c)$$
$$f'(x) > 0 \text{ for } x \in (c, c + \delta_2)$$

$\therefore$ By first derivative test, f has a local minimum at c.

Curve Sketching Problems done in the Class:

1) $f(x) = x^4 - 4x^2 + 1$, $f'(x) = 4x(x^2-2)$, $f''(x) = 12x^2 - 8$.

2) $f(x) = \dfrac{1}{x^2+1}$, $f'(x) = -\dfrac{2x}{(x^2+1)^2}$, $f''(x) = \dfrac{2(3x^2-1)}{(x^2+1)^3}$

3) $f(x) = \dfrac{x^4}{x^2-3}$, $f'(x) = \dfrac{2x^5 - 12x^3}{(x^2-3)^2} = \dfrac{2x^3(x^2-6)}{(x^2-3)^2}$

$$f''(x) = \frac{2x^2(x^4 - 9x^2 + 54)}{(x^2-3)^3}$$

$$= \frac{2x^2\left(\left(x^2 - 9/2\right)^2 + 135/4\right)}{(x^2-3)^3}$$

4) $f(x) = \dfrac{x^3}{x^2-4}$, $f'(x) = \dfrac{x^2(x^2-12)}{(x^2-4)^2}$, $f''(x) = \dfrac{8x(x^2+12)}{(x^2-4)^3}$

[Exercise, Check for the asymptotic line!]

Details:

1) $f(x) = x^4 - 4x^2 + 1$

Domain : $\mathbb{R}$

Intercepts: $f(0) = 1$, $x^4 - 4x^2 + 1 = 0$

$$\Rightarrow (x^2-2)^2 - 3 = 0$$

$$\Rightarrow x^2 = 2 \pm \sqrt{3}$$

$$\Rightarrow x = \pm\sqrt{2+\sqrt{3}}, \pm\sqrt{2-\sqrt{3}}.$$

Asymptotes: None

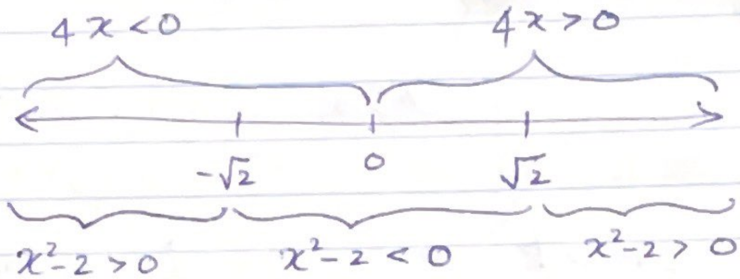
$$\lim_{x \to \pm\infty} f(x) = \infty$$

Note that $f(-x) = f(x)$. So f is symmetric about y-axis.

Intervals of Increasing / Decreasing:

$f'(x) = 4x(x^2-2)$ is cts. on $\mathbb{R}$, So the critical points for f are those $x \in \mathbb{R}$ s.t $f'(x) = 0$.

$\Rightarrow \quad x = 0, \pm\sqrt{2}$ are critical points



$\therefore \; f' > 0$ on $(-\sqrt{2}, 0) \cup (\sqrt{2}, \infty)$
$f' < 0$ on $(-\infty, -\sqrt{2}) \cup (0, \sqrt{2})$

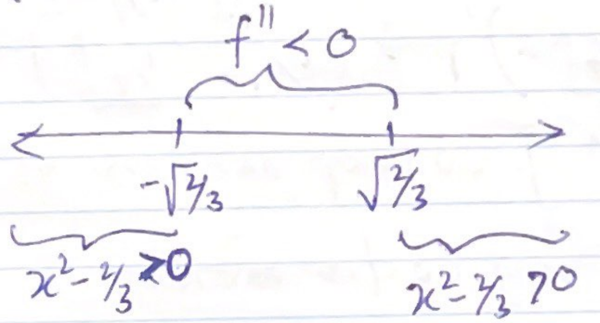
$\Rightarrow$ f is increasing on $[-\sqrt{2}, 0]$ & $[\sqrt{2}, \infty)$
f is decreasing on $(-\infty, -\sqrt{2}]$ & $[0, \sqrt{2}]$

$\Rightarrow$ f has local max at 0 and local min at $\pm\sqrt{2}$.

Intervals of Concave up / down:

$f''(x) = 12x^2 - 8 = 12(x^2 - 2/3)$ is cts. on $\mathbb{R}$

$f''(x) = 0$ when $x = \pm\sqrt{2/3}$



$f'' < 0$ on $(-\sqrt{2/3}, \sqrt{2/3})$

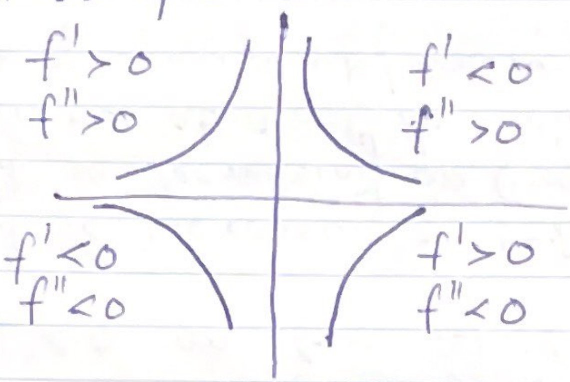
$\Rightarrow$ $f'' > 0$ on $(-\infty, -\sqrt{2/3}) \cup (\sqrt{2/3}, \infty)$

$\therefore$ f is concave up on $(-\infty, -\sqrt{2/3}]$ & $[\sqrt{2/3}, \infty)$

f is concave down on $[-\sqrt{2/3}, \sqrt{2/3}]$

$\Rightarrow$ $\left(\sqrt{2/3}, -11/9\right)$ and $\left(-\sqrt{2/3}, -11/9\right)$ are points of inflection for f.

Now connect the dots using the following general shape

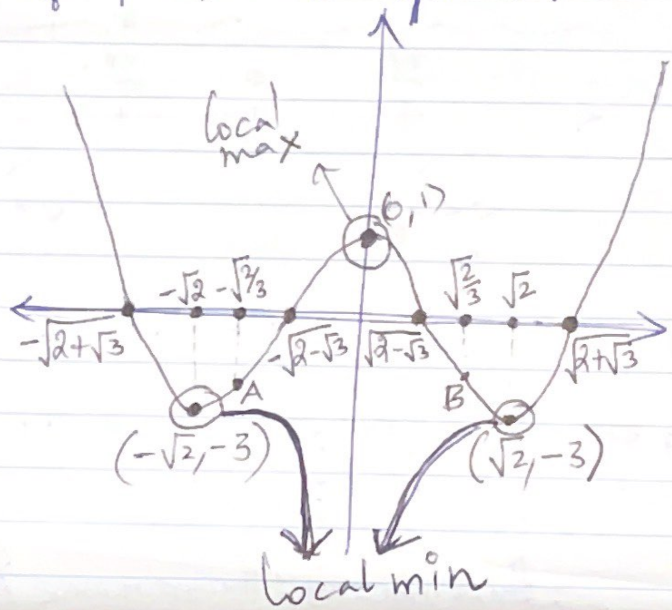


$\sqrt{2+\sqrt{3}} \simeq 2$

$\sqrt{2-\sqrt{3}} = \dfrac{1}{\sqrt{2+\sqrt{3}}} \simeq \dfrac{1}{2}$

$\sqrt{\frac{2}{3}} \simeq 0.8$

Sketch of $f(x) = x^4 - 4x^2 + 1$



A: $\left(-\sqrt{2/3}, -11/9\right)$

B: $\left(\sqrt{2/3}, -11/9\right)$

are pts of inflection

2) $f(x) = \frac{1}{x^2+1}$, $f(-x) = f(x)$ So symmetric about y-axis.

Note that $f(x) > 0$ for all $x \in \mathbb{R}$ as $x^2+1 > 0$.

Domain: $\mathbb{R}$

Intercepts: $f(0) = 1$. No x-intercepts

Asymptotes: $\lim_{x \to \pm\infty} f(x) = 0$ So $y = 0$ is a horizontal asymptote. No V.A.s.

<u>Intervals of increasing/decreasing:</u>

$f'(x) = -\frac{2x}{(x^2+1)^2}$ is cts. on $\mathbb{R}$. So $x = 0$ is the only critical point of f.

Also $f' < 0$ for $x \in (0, \infty)$

$f' > 0$ for $x \in (-\infty, 0)$

$\therefore$ f is increasing on $(-\infty, 0]$

f is decreasing on $[0, \infty)$

$\therefore$ f has local max at $x = 0$.

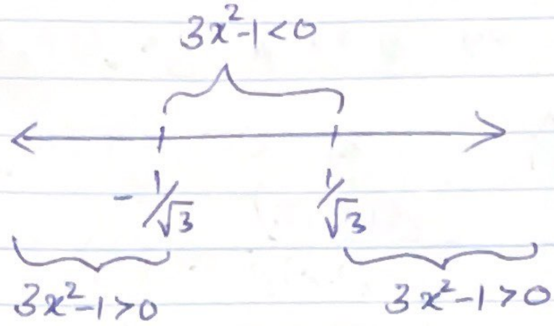
(In fact, this is the global max for f)

<u>Intervals of concave up/down:</u>

$f''(x) = \frac{2(3x^2-1)}{(x^2+1)^3}$ is cts. on $\mathbb{R}$.

$f''(x) = 0 \Rightarrow x = \pm 1/\sqrt{3}$.

$(x^2+1)^3 > 0$ $\forall x \in \mathbb{R}$, So $3x^2-1$ determines the sign of f''.



$\Rightarrow$ $f'' > 0$ on $(-\infty, -1/\sqrt{3}) \cup (+1/\sqrt{3}, \infty)$

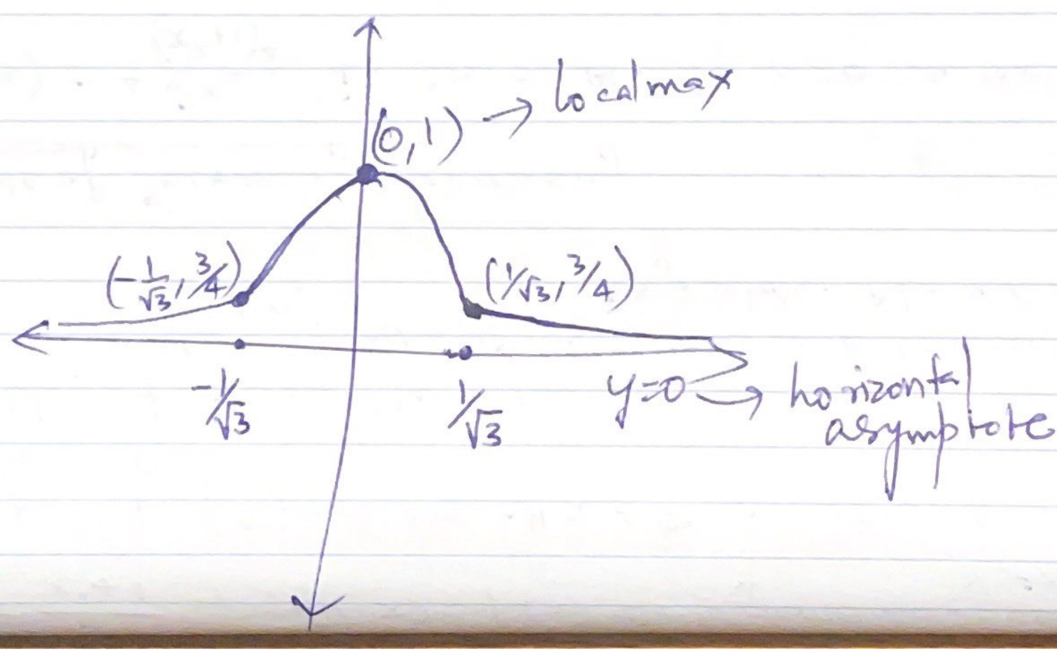
$f'' < 0$ on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$\therefore$ f is concave up on $(-\infty, -1/\sqrt{3}]$ & $[1/\sqrt{3}, \infty)$

f is concave down on $[-1/\sqrt{3}, 1/\sqrt{3}]$.

$\therefore$ $(-1/\sqrt{3}, 3/4)$ & $(1/\sqrt{3}, 3/4)$ are points of inflection for f.

Sketch of $f(x) = \dfrac{1}{x^2+1}$



3) $f(x) = \dfrac{x^4}{x^2-3}$ $\quad f(-x) = f(x)$ Symmetric about y-axis.

Domain: f is not-defined when $x = \pm\sqrt{3}$.

$\therefore \mathbb{R} \setminus \{\pm\sqrt{3}\}$ is the domain of f.

Intercepts: $f(x) = 0 \Leftrightarrow x = 0$.

Asymptotes: $x = -\sqrt{3}$ and $x = \sqrt{3}$ are vertical asymptotes because

$$\lim_{x \to \sqrt{3}^-} f(x) = -\infty = \lim_{x \to -\sqrt{3}^+} f(x)$$

$$\lim_{x \to \sqrt{3}^+} f(x) = \infty = \lim_{x \to -\sqrt{3}^-} f(x).$$

$\lim_{x \to \pm\infty} f(x) = \infty$. So there are no horizontal asymptotes.

But it may be helpful to see that for large values of x, $f(x) = \dfrac{x^2}{1 - 3/x^2} \cong x^2$.

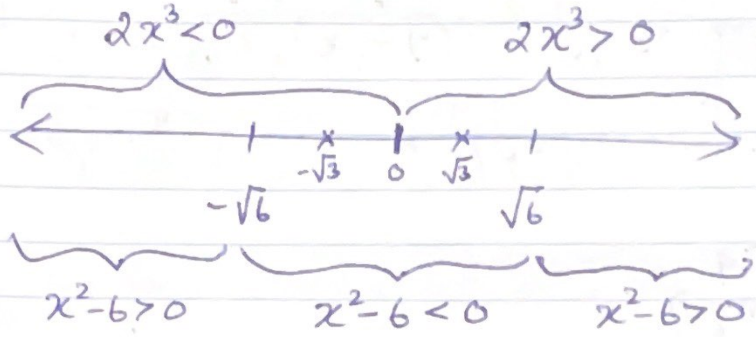
(Parabolic asymptote kind of deal)

Intervals of increasing/decreasing:

$f'(x) = \dfrac{2x^3(x^2-6)}{(x^2-3)^2}$, f' is undefined at $\pm\sqrt{3}$ but they are not in the domain.

So $x = 0, \pm\sqrt{6}$ are the critical points.

Note that $(x^2-3)^2 > 0 \quad \forall x \in \text{Domain}(f)$



$$\Rightarrow \quad f' < 0 \text{ on } (-\infty, -\sqrt{6}) \cup (0, \sqrt{6})$$
$$f' > 0 \text{ on } (-\sqrt{6}, 0) \cup (\sqrt{6}, \infty)$$

$\therefore$ f is increasing on $(-\infty, -\sqrt{6}]$ and $[0, \sqrt{6}]$
f is decreasing on $[-\sqrt{6}, 0]$ and $[\sqrt{6}, \infty)$.

$\therefore$ f has local min at $x = \pm\sqrt{6}$ and a local max at $x = 0$.

Intervals of Concave up/down:

$$f''(x) = \frac{2x^2\left((x^2-9/2)^2 + \frac{135}{4}\right)}{(x^2-3)^3}$$

$f''(x) \neq 0$ for any $x \in \text{Domain}(f) = \mathbb{R} \setminus \{\pm\sqrt{3}\}$

as $2x^2\left((x^2-9/2)^2 + 135/4\right) > 0$, $(x^2-3)^3$ determines the sign of f''.

$\therefore$ $(x^2-3)^3 < 0$ if $x \in (-\sqrt{3}, \sqrt{3})$

$(x^2-3)^3 > 0$ if $x \in (-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$

$\therefore$ f is concave down on $(-\sqrt{3}, \sqrt{3})$

& f is concave up on $(-\infty, -\sqrt{3})$ and $(\sqrt{3}, \infty)$

Note that $\pm\sqrt{3}$ cannot be x-co-ordinates of pt. of inflection as f is not defined there!

Sketch of $f(x) = \dfrac{x^4}{x^2-3}$