Oct 2, 2023

Line Integrals - Summary

3-techniques to compute $\int_C \vec{F} \cdot d\vec{x}$

① definition (parametrize)

② Fundamental Thm. of line integrals

→ Check for existence of ϕ s.t $\vec{F} = \vec{\nabla}\phi$

→ Use a "simpler" path instead

③ Green's Thm:

D - a region with smooth, simple, oriented CCW boundary curve ∂D. If $\vec{F} = (F_1, F_2)$ is C^1 on $D \cup \partial D$, then

$$\underbrace{\oint_{\partial D} \vec{F} \cdot d\vec{x}}_{\substack{\text{Circulation of} \\ \vec{F} \text{ around } \partial D}} = \iint_D \underbrace{\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)}_{\substack{\text{Circulation} \\ \text{Density / vorticity} \\ [\text{Curl in 2-D}]}} dA$$

→ applied to closed curves

→ Can find A(D), the area of D (or) if the area is known, one can "move the curve"

→ Convert difficult line integrals to computable double integrals.

Consult Problem sets for more examples!

Surface Integrals of Scalar fields $f: \mathbb{R}^3 \to \mathbb{R}$

Recall: For the surface Σ in $\mathbb{R}^3$ parametrized (key part) by $\vec{x} = \vec{g}(u,v)$, the surface area is

$$S = \iint_{D_{uv}} \underbrace{\|\vec{g}_u \times \vec{g}_v\|}_{\text{Scaling factor}} \, du\,dv$$

Now suppose that $f(\vec{x})$ is the surface charge (or mass) density of Σ.

Qn: What is the total charge (or mass) on the surface?

Idea: The ~~surface~~ charge element ΔQ on the surface element ΔS: $\Delta Q \approx f \, \Delta S \approx f \, \|\vec{g}_u \times \vec{g}_v\| \Delta u \, \Delta v$

Summing it up,

$$Q = \iint_{D_{uv}} f(\vec{g}(u,v)) \|\vec{g}_u \times \vec{g}_v\| \, du\,dv$$

Definition.

The surface integral of the scalar field f over the surface $\Sigma: \vec{x} = \vec{g}(u,v), (u,v) \in D_{uv}$ is defined by

$$\iint_{\Sigma} f \, dS = \iint_{D_{uv}} f(\vec{g}(u,v)) \|\vec{g}_u \times \vec{g}_v\| \, du\,dv.$$

Surface Integrals & Flux of a vector field

The surface integral of a vector field can describe how various physical quantities are transported.

1) Mass transfer by a fluid

Given a fluid flowing through a (fixed) surface Σ, the mass flux is the mass transported through Σ in unit time.

Depends on

i) The mass flux vector $\vec{F} = \rho \vec{V}$ — $\vec{V}$: fluid velocity (vector field); ρ: mass density (scalar field)

ii) the surface normal $\hat{n}$

to get flux element (normal component)

$$\vec{F}\cdot\hat{n}$$

(Try Mobius demo)

2) Heat flux vector: $\vec{F} = -k\vec{\nabla}T$, where T is the temperature at (x,y,z)

k - thermal conductivity

3) Electric / Magnetic Flux, dictated by Gauss' Law, Gauss' Law for Magnetism, Faraday's Law, Ampere's Law.

The surface integral of $\vec{F}$ over Σ is denoted

$$\iint_{\Sigma} \vec{F} \cdot \hat{n} \, dS$$ and is called

the flux (or rate of flow) of $\vec{F}$ through Σ.

Gauss Law: $\iint_{\Sigma} \vec{E} \cdot \hat{n} \, dS = \frac{Q}{\varepsilon_0}$, where

Q is the net charge enclosed by a closed surface Σ. Here ε_0 - permittivity of free space.

Derivation of Formula:

The flux of $\vec{F}$ through a small surface element ΔS is

$$\Delta(\text{flux}) \approx (\vec{F} \cdot \hat{n}) \Delta S$$

Recall that, $\Delta S \approx \|\vec{g}_u \times \vec{g}_v\| \Delta u \, \Delta v$

and $\hat{n} = \dfrac{\vec{g}_u \times \vec{g}_v}{\|\vec{g}_u \times \vec{g}_v\|}$

Sub these in $\Delta(\text{flux})$, we get

$$\Delta(\text{flux}) \approx \vec{F}(\vec{g}(u,v)) \cdot \vec{g}_u \times \vec{g}_v \, \Delta u \, \Delta v$$

Summing over ΔS and take limit:

$$\text{flux} = \iint_{D_{uv}} \vec{F}(\vec{g}(u,v)) \cdot (\vec{g}_u \times \vec{g}_v)\, du\, dv$$

Definition. The surface integral of a vector field $\vec{F}$ over the surface

$$\Sigma : \vec{X} = \vec{g}(u,v) \ , \ (u,v) \in D_{uv} \text{ with}$$

outward unit normal $\hat{n}$ is

$$\iint_{\Sigma} \vec{F} \cdot \hat{n}\, dS = \iint_{D_{uv}} \vec{F}(\vec{g}(u,v)) \cdot (\vec{g}_u \times \vec{g}_v)\, du\, dv.$$

Comments on

PS4 Q5: Calculate the mass flux of the fluid flow with velocity $\vec{V} = (0, 0, kz)$ ($k > 0$ constant) and constant mass density ρ_0 through the cylinder $x^2 + z^2 = b^2$ for $-l \le y \le l$

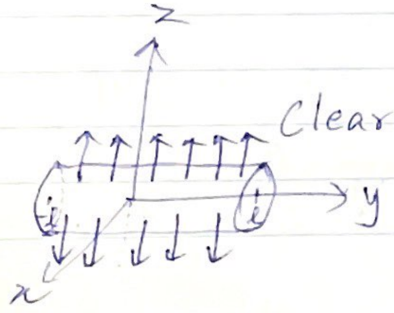
mass flux vector $\vec{F} = \rho \vec{V} = \rho_0 (0, 0, kz)$

1) Parametrize the surface

$$\vec{X} = \vec{g}(u,v) = (b\cos u, v, b\sin u)$$

2) Compute $\vec{g}_u \times \vec{g}_v$

Here $\vec{g}_u \times \vec{g}_v = (-b\cos u, 0, -b\sin u)$



Clearly, there is an outward flow of fluid so positive flux.

But $\iint_{\Sigma} \vec{F} \cdot \hat{n}\, dS = -2\pi k \rho_0 b^2 l$

The sign is $-ve$ (since all the constants are $+ve$)

What is the contradiction?

$\longrightarrow$ must be careful of the direction of $\hat{n}$

$\vec{g_u} \times \vec{g_v} = (-b\cos u, 0, -b\sin u)$

$\uparrow$ This is inward normal. towards y-axis.

Fix: 1) Either switch u and v in our parametrization

2) Or say we need outward normal and take out negative sign in $\vec{g_u} \times \vec{g_v}$.

$(\vec{g_v} \times \vec{g_u} = -(\vec{g_u} \times \vec{g_v}))$

Differentiating a vector field:

Given $f: \mathbb{R} \longrightarrow \mathbb{R}$, $f' = \frac{d}{dx}(f)$

differential operator (acting on a function)

We have

vector differential operator in $\mathbb{R}^3$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

(a vector operator)

For f, a scalar field in $\mathbb{R}^3$,

$$\vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

How does ∇ act on a vector field?

say $\vec{F}: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$\vec{F} = (F_1, F_2, F_3)$

1) $$\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (F_1, F_2, F_3)$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$ (a scalar field)

divergence of $\vec{F}$

If $\nabla \cdot \vec{F} = 0$, $\vec{F}$ is called incompressible (or) solenoidal

2) $$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

(a vector field)

curl of $\vec{F}$ (if $\nabla \times \vec{F} = 0$, $\vec{F}$ is called irrotational)

Note: if $F_3 = 0$, then $\nabla \times \vec{F} = \left(0, 0, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$

Circulation density

As the name suggests, the curl measures a local circulation.

Note: $\vec{F}\cdot\nabla$ (or) $\vec{F}\times\nabla$ are not defined!

Radial Fields:

The general radial vector field has the form: $\vec{F} = f(r)\vec{r}$, where

$\vec{r} = (x, y, z)$, $r = \|\vec{r}\| = \sqrt{x^2+y^2+z^2}$, and f is some arbitrary function.

Two properties of such fields:

1) Depend only on the distance to the origin (r), and
2) They point directly toward (or) away from the origin.

Eg: Gravitational field $\vec{F} = -\frac{GMm}{r^3}\vec{r}$

Electrostatic field $\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^3}\vec{r}$

A few results: ~~For~~ $\vec{r} = (x, y, z)$. Then $r = \sqrt{x^2+y^2+z^2}$

1) $\nabla\cdot\vec{r} = 3$
2) $\nabla\times\vec{r} = 0$
3) $\nabla r = \frac{\vec{r}}{r} = \hat{r}$, a unit vector in the direction of $\vec{r}$

To understand divergence and curl of general radial fields, we need the following identities:

Product rules for Divergence and Curl

Let f be a scalar field, and $\vec{G}$ be a vector field. Then

$$\nabla \cdot (f\vec{G}) = \nabla f \cdot \vec{G} + f(\nabla \cdot \vec{G})$$

$$\nabla \times (f\vec{G}) = \nabla f \times \vec{G} + f(\nabla \times \vec{G})$$

Notes: 1) $\nabla \cdot f$, $\nabla \times f$ are meaningless as f is a scalar field.

2) $f\vec{G} = (fG_1, fG_2, fG_3)$ for $\vec{G} = (G_1, G_2, G_3)$.

For $\vec{F} = f(r)\vec{r}$,

$$\nabla \cdot \vec{F} = \nabla \cdot (f(r)\vec{r})$$

$$= \nabla f(r) \cdot \vec{r} + f(\nabla \cdot \vec{r})$$

$$= \nabla f(r) \cdot \vec{r} + 3f$$

What is $\nabla f(r)$?

We know that

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

when f is a function of x, y, z.

But our f is a function of r ! and

Our r is a function of x, y, z.

So we need chain rule!

$$\frac{\partial f}{\partial x} = \frac{df}{dr}\frac{\partial r}{\partial x}, \quad \frac{\partial f}{\partial y} = \frac{df}{dr}\frac{\partial r}{\partial y}, \quad \frac{\partial f}{\partial z} = \frac{df}{dr}\frac{\partial r}{\partial z}$$

Recall that $\nabla r = \frac{\vec{r}}{r}$.

$$\boxed{\text{Chain rule for } \nabla: \quad \nabla f = \frac{df}{dr}\nabla r}$$

Back to $\nabla \cdot \vec{F} = \left(\frac{df}{dr}\frac{\vec{r}}{r}\right) \cdot \vec{r} + 3f$

$$= \left(\frac{1}{r}\frac{df}{dr}\right)\vec{r}\cdot\vec{r} + 3f$$

$$= r\frac{df}{dr} + 3f \quad \text{as } \vec{r}\cdot\vec{r} = r^2.$$

Exercise: 1) Find the form of $f(r)$ s.t $\vec{\nabla}\cdot\vec{F} = 0$

2) Show any radial field is irrotational.