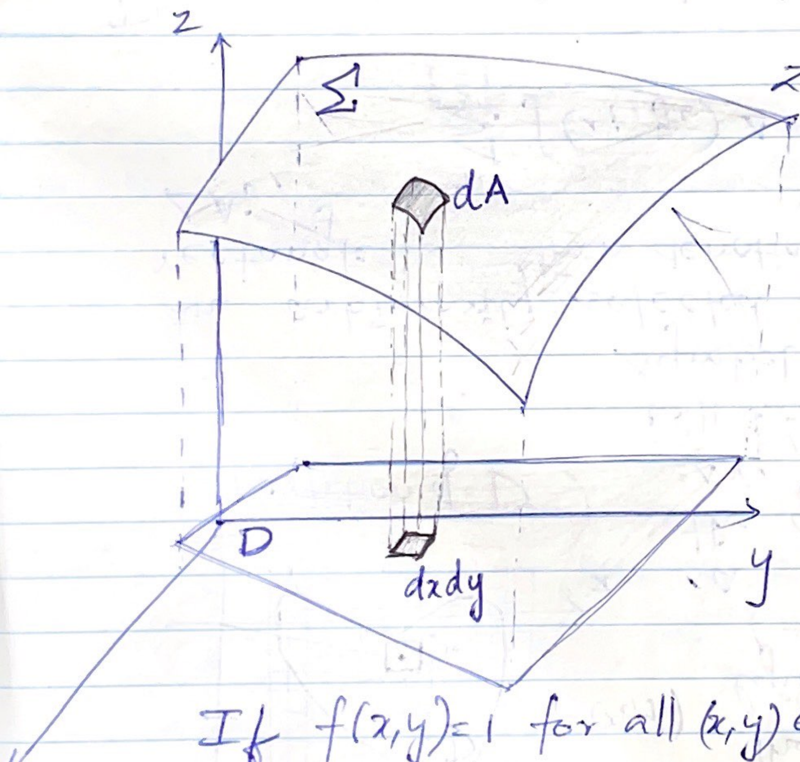
Sep 15, 2023

Double Integrals:

Consider the double integral

$$I = \iint_D f(x,y)\, dA,$$

Where D is a region in the x,y plane. Assume that D is a closed (contain its boundary points), bounded region in the x,y plane, that its boundary is a closed piecewise smooth curve, and that $f(x,y)$ is defined on D.



If $f(x,y) \geq 0$ on D, Can intrepret I as volume of ~~V(Σ)~~ the ~~region~~ solid defined by

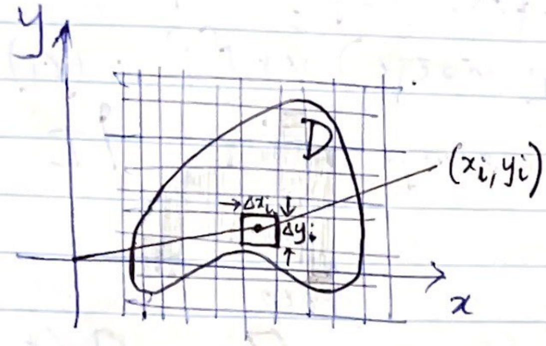
$$\{(x,y,z) : 0 \leq z \leq f(x,y),\ (x,y) \in D\}$$

∴ Check out the Mobius interactive applet.

If $f(x,y) = 1$ for all $(x,y) \in D$, then area of D

$$A(D) = \iint_D 1\, dA \text{ (Area Intrepretation)}$$

Other intrepretations: Total Charge (or) Mass given charge density (or) mass density over a surface.



Partition of D

The n-rectangles lying entirely within D (the shaded ones) constitute a partition P of D and the greatest of the dimensions Δx_i, Δy_i for $i = 1, 2, \ldots, n$ will be denoted by the symbol $|\Delta P|$. Let (x_i, y_i) be an arbitrarily selected point in the i^{th} partition rectangle. Then denoting the area $\Delta x_i \Delta y_i$ by ΔA_i,

$$\sum_{i=1}^{n} f(x_i, y_i)\,\Delta A_i \quad \text{is called the}$$

Riemann sum corresponding to the chosen partition and (x_i, y_i) points. The idea, in principle at least, is to compute the Riemann sum, then to introduce a finer partition (i.e. one with a smaller norm) and compute the new Riemann sum, and so on, such that the norm of the partitions $|\Delta P| \longrightarrow 0$.

If the sequence of values of the Riemann sum thus generated converges to a unique limit, independent of the choice of the partition sequence and (x_i, y_i) points, that limit is, by definition, the double integral. That is,

$$\iint_D f(x,y)\, dA = \lim_{|\Delta P| \to 0} \sum_P f(x_i, y_i)\, \Delta A_i$$

It can be shown that the limit will indeed exist if f is continuous in the closed region D, that condition being sufficient, not necessary. If the limit does exist, we say that the integral <u>converges</u> (or) <u>exists</u>, and that f is <u>integrable</u> on D, Riemann - integrable to be more specific.

<u>Properties</u>: $D \subseteq \mathbb{R}^2$ Closed and bounded f, g - integrable on D

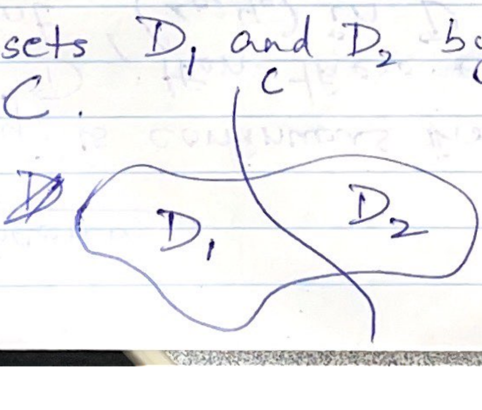
(Linearity) 1) $$\iint_D [\alpha f(x,y) + \beta g(x,y)]\, dA = \alpha \iint_D f(x,y)\, dA + \beta \iint_D f(x,y)\, dA$$

for α, β any constants.

2) $$\iint_D f(x,y)\, dA = \iint_{D_1} f(x,y)\, dA + \iint_{D_2} f(x,y)\, dA$$

(Decomposition)

D is decomposed into two closed and bounded subsets D_1 and D_2 by a piecewise smooth curve C.



$D = D_1 \cup D_2$

3) Mean Value theorem.

If $f(x,y)$ is continuous throughout the closed region D, then there exists at least one point (x_0, y_0) in D such that

$$\iint_D f(x,y)\,dA = f(x_0,y_0)\,A,$$

where $A = \iint_D dA$ is the area of D.

Check Mobius lesson for Basic inequality and Absolute value inequality theorem on the Definition of Double integral page.

Double integrals can be evaluated approximately by using a computer to evaluate a suitable Riemann sum subject to the acuracy dependant on how fine a partition was chosen.

How do we evaluate a double integral analytically? Iterated Integral.

Simplest case: D is rectangular (i.e) $x_1 \le x \le x_2$
$y_1 \le y \le y_2$

Then

$$\iint_D f(x,y)\,dA = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y)\,dx\,dy$$

$$= \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)\,dy\,dx$$

(i.e) Switching order of integration is straightforward only when D is rectangular.

Property: If $f(x,y) = g(x)h(y)$ & D is rectangular, then

$$\iint_D f(x,y)\,dA = \int_{x_1}^{x_2} g(x)\,dx \cdot \int_{y_1}^{y_2} h(y)\,dy$$

Example.

$$I = \iint_D x^2y^2\,dA \quad D: \{0 \le x \le 1,\ 0 \le y \le 2\}$$

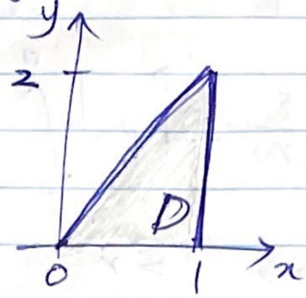
Soln: By above, $I = \int_0^1 x^2\,dx \int_0^2 y^2\,dy = \left.\frac{x^3}{3}\right]_0^1 \left[\frac{y^3}{3}\right]_0^2$

$$= \frac{1}{3}\cdot\frac{8}{3}$$

$$= \frac{8}{9}$$

Now, how can we compute

$$\iint_D x^2y^2\,dA$$ where D is the triangular region depicted below



Let D be a region in the xy-plane and let f be a function s.t $f(x,y) \geq 0$ for all $(x,y) \in D$. If V denotes the volume of the solid above D and below the surface $z = f(x,y)$, then we have

$$V = \iint_D f(x,y)\,dA$$

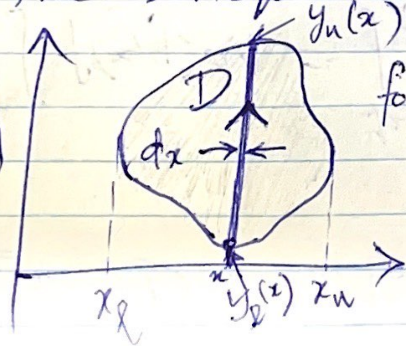
Assume that the region D lies between vertical lines $x = x_l$ and $x = x_u$ with $x_l < x_u$ and has top curve $y = y_u(x)$ and bottom curve $y = y_l(x)$. That is, D is described by the inequalities

$$x_l \leq x \leq x_u \;\&$$

for each $(x,y) \in D$, $y_l(x) \leq y \leq y_u(x)$

(X-simple regions)
(i.e) constant bounds on x.

Here $y_u(x)$ & $y_l(x)$ are continuous functions of x on the interval $[x_l, x_u]$



Then

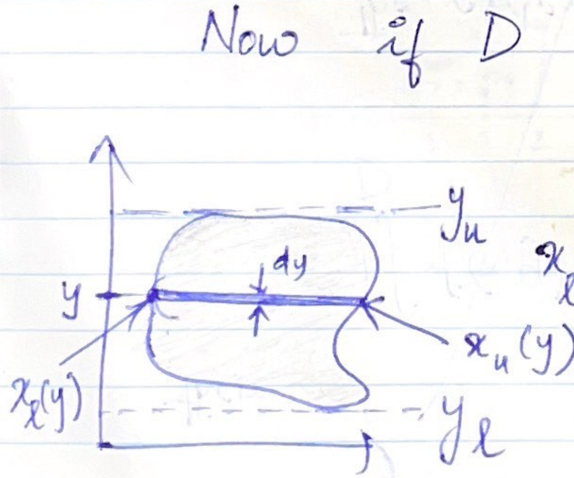
$$\iint_D f(x,y)\,dA = \int_{x_l}^{x_u} \underbrace{\left\{ \int_{y_l(x)}^{y_u(x)} f(x,y)\,dy \right\}}\,dx$$

We integrate across the (bold) vertical sliver from $y_l(x)$ on the bottom to $y_u(x)$ on the top and then we sweep the sliver from left $(x = x_l)$ to right $(x = x_u)$.

Observe that, the "inner" integral $\int_{y_l(x)}^{y_u(x)} f(x,y)\,dy$ is a function of x, say $F(x)$; the subsequently carried out "outer" integration is of the form

$$\int_{x_l}^{x_u} F(x)\,dx$$ where x_l and x_u are constants.

Now if D is described by the inequalities



$y_l \le y \le y_u$ — y_l, y_u are constants

$x_l(y) \le x \le x_u(y)$ — $x_l(y)$ & $x_u(y)$ are continuous functions of y on $[y_l, y_u]$

(Y-simple regions)

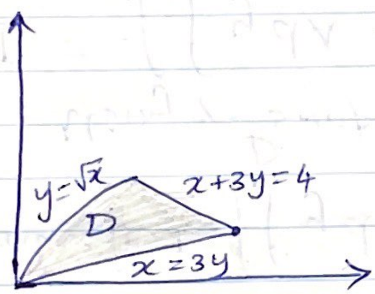
then

$$\iint_D f(x,y)\,dA = \int_{y_l}^{y_u} \int_{x_l(y)}^{x_u(y)} f(x,y)\,dx\,dy$$

The order of integration is important, and switching order is not trivial.

Ex.

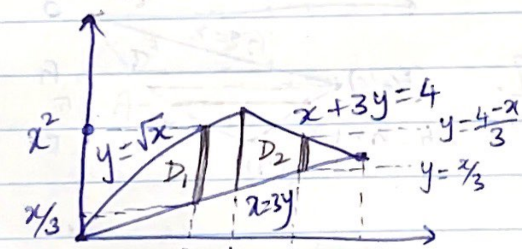
0) Express D shown below in two different ways.



X- simple regions

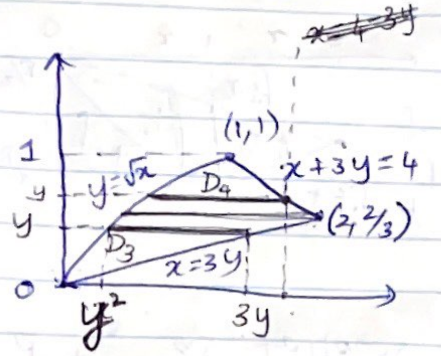
As x varies, there are two different top curves. So we would decompose D into D_1 and D_2 as follows.

$$D = D_1 \cup D_2$$



D_1	D_2
$0 \le x \le 1$	$1 \le x \le 2$
$x/3 \le y \le x^2$	$x/3 \le y \le \frac{4-x}{3}$

Y- simple regions



As y varies, there are two different right curves. So we would decompose D into D_3 and D_4 as follows.

D_3	D_4
$0 \le y \le 2/3$	$2/3 \le y \le 1$
$y^2 \le x \le 3y$	$y^2 \le x \le 4-3y$

b) Evaluate $I = \iint_D y \, dA$. Note that $f(x,y) = y$ is cts. on D.

Using y-simple regions

$$I = \iint_{D_3} y\,dA + \iint_{D_4} y\,dA$$

$$= \underbrace{\int_0^{2/3}\int_{y^2}^{3y} y\,dx\,dy}_{I_1} + \underbrace{\int_{2/3}^{1}\int_{y^2}^{4-3y} y\,dx\,dy}_{I_2}$$

$$\int_{y^2}^{3y} y\,dx = y\left\{x\Big|_{x=y^2}^{x=3y}\right\} = y(3y-y^2) = 3y^2-y^3$$

$$\int_{y^2}^{4-3y} y\,dx = y\left(4-3y-y^2\right) = 4y-3y^2-y^3$$

$$I_1 = \int_0^{2/3} \left(3y^2-y^3\right) dy$$

$$= y^3 - \frac{y^4}{4}\Bigg|_0^{2/3} = \frac{8}{27} - \frac{16}{324}$$

$$= \frac{80}{324} = \frac{20}{81}$$

$$I_2 = \int_{2/3}^{1} \left(4y-3y^2-y^3\right) dy = 2y^2 - y^3 - \frac{y^4}{4}\Bigg|_{2/3}^{1}$$

$$\cancel{= \frac{3}{4}}$$

$$= \frac{67}{324}$$

$$\therefore\ I = \frac{20}{81} + \frac{67}{324} = \frac{147}{324}$$

Ex. Verify this using x-simple regions!

Example

Show that

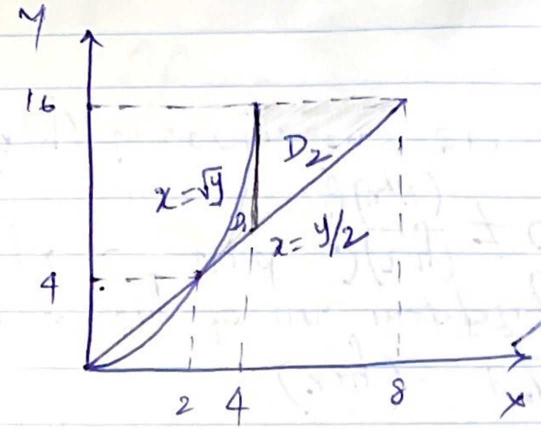
$$\int_4^{16}\int_{y/2}^{\sqrt{y}} \frac{1}{x} e^{y/x}\,dx\,dy = -6e^2 + e^4 + \int_4^8 e^{16/x}\,dx$$

Solution:

In the previous example, we were given a region D from which we deduced the integration limits. In the present example, we are given the limits. That's all good, but it is difficult to get started because $\exp(y/x)$ is a formidable function of x to integrate. Whereas, $\exp(y/x)$ is a simple function of y, so it is more promising to invert the order of integration. Thus, consider instead

$$I = \int_?^?\int_?^? \frac{1}{x} e^{y/x}\,dy\,dx$$

What are the new limits? One cannot just switch! We need a picture of the region D



To integrate on y and then on x, we must break D into subregions into D_1 and D_2 because the top boundaries of D_1 and D_2 are different, $y=x^2$ for D_1 and $y=16$ for D_2.

Thus

$$I = \iint_{D_1} \frac{1}{x} e^{y/x}\,dy\,dx + \iint_{D_2} \frac{1}{x} e^{y/x}\,dy\,dx$$

$$= \int_2^4 \int_{2x}^{x^2} \frac{1}{x} e^{y/x}\,dy\,dx + \int_4^8 \int_{2x}^{16} \frac{1}{x} e^{y/x}\,dy\,dx$$

$$= \int_2^4 \left\{ e^{y/x} \Big|_{2x}^{x^2} \right\} dx + \int_4^8 \left\{ e^{y/x} \Big|_{2x}^{16} \right\} dx$$

$$= \int_2^4 (e^x - e^2)\,dx + \int_4^8 (e^{16/x} - e^2)\,dx$$

$$= \left. e^x - xe^2 \right]_2^4 + \int_4^8 e^{16/x}\,dx - \left[xe^2\right]_4^8 = e^4 + \int_4^8 e^{16/x}\,dx - 6e^2.$$

Remark: Be sure to see that the functional nature of the integrand was not relevant in our search for the new limits. Rather, the original limits described the region D and D described the new limits.

Change of Variables

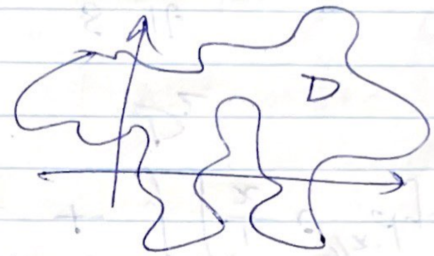
→ integrand hard to deal with, so make substitutions in integrals.

→ for multiple integrals, consider change of variables

1) complicated integrand

2) Complicated domain of integration

we map (x,y) to (u,v)



Change of Variable Theorem:

Let each of D_{uv} and D_{xy} be a closed bounded set whose boundary is a piecewise-smooth curve. Let

$$(x,y) = F(u,v) = (f(u,v),\ g(u,v))$$

be a one-to-one mapping of D_{uv} onto D_{xy}, with $f, g \in C^1$, and $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$ on the interior of D_{uv}. If $G(x,y)$ is continuous on D_{xy}, then

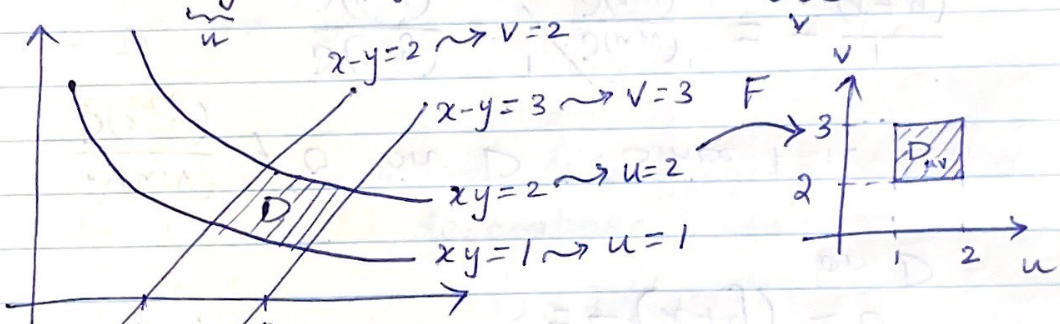
$$\iint_{D_{xy}} G(x,y)\,dx\,dy = \iint_{D_{uv}} G(f(u,v), g(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

where $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$, Jacobian

$$\Delta A_{xy} \approx \underbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}_{\text{scaling factor}} \Delta A_{uv}$$

Ex.

Find $\iint_D x+y\,dA$, where D is the region bounded by $\underbrace{xy=1,\ xy=2}_{u}$, $\underbrace{x-y=2,\ x-y=3}_{v}$, $x,y \geq 0$



Soln.

Let $u=xy$ and $v=x-y$. Then, since any point in D can be obtained by intersecting a unique hyperbola in the family $u=xy=k$, $1\leq k\leq 2$, with a unique line in the family $v=x-y=l$, $2\leq l\leq 3$, the mapping

$$F(x,y) = (xy, x-y) = (u,v) \text{ maps } D$$

in a one-to-one fashion onto

$$D_{uv} = \{(u,v) \mid 1 \le u \le 2,\ 2 \le v \le 3\}$$

Note that $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} y & x \\ 1 & -1 \end{vmatrix}$

$= -(x+y) < 0$ on D

and $\dfrac{\partial(u,v)}{\partial(x,y)} \neq 0$ on D. Since F is invertible on D, $\dfrac{\partial(x,y)}{\partial(u,v)} = 1 \Big/ \dfrac{\partial(u,v)}{\partial(x,y)} = -\dfrac{1}{(x+y)}$

$\Rightarrow \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = \dfrac{1}{x+y}$ since $x+y>0$ on D

Thus, by the Change of variable formula,

$$\iint_D x+y \, dA = \iint_{D_{uv}} (x+y)\cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \iint_{D_{uv}} (x+y)\cdot \frac{1}{(x+y)}\, du\,dv$$

$$= \int_2^3\int_1^2 1 \, du\,dv = 1.$$