MATH-4101(3)-001 Assignment 10 Due: Apr 7, 2022 (before class)

1. (Mean Value Property). Let $f \in H(U)$ and $B(a,R) \subset U$ and $\gamma_r(t) = a + re^{it},\ t \in [0, 2\pi],\ 0 < r < R$. Show using CIF that
$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{it})dt.$$
[Note: The right side of the above equation may be considered as the "average" or "mean" of f on the circle γ_r. Thus this exercise shows that the mean value of f on *any* circle γ_r centered at a (for $0 < r < R$) is $f(a)$.]

2. Find the values of the integrals
$$\int_{|z|=2} \frac{\sin z}{z^4} dz, \text{ and } \int_{|z|=3} \frac{\sin z}{(z^4 + 8z^2 + 16)} dz$$
[Hint: Use CIF-D. Write $z^4 + 8z^2 + 16 = (z^2 + 4)^2 = (z + 2i)^2(z - 2i)^2$.]

3. (a) Let f be entire with $|\mathrm{Re} f(z)| \leq M$ for all $z \in \mathbb{C}$. Show that f is a constant. [Hint: Consider e^f.]
   (b) Let f be entire with $|f(z)| \leq A + B|z|^{5/2}$. Show that f is a polynomial of degree at most 2. [Hint: Use Cauchy's estimates for high-order derivatives.]

4. Using Identity theorem, is it possible to construct $f \in H(B(0,1))$ such that $f(1/n) = z_n$ where $(i) z_n = (-1)^n$, and $(ii) z_n = n/(n+1)$.

5. Find the Laurent expansion of $e^{1/z}$ in $0 < |z| < \infty$ and hence show that
$$\frac{1}{2\pi}\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta - n\theta)d\theta = \frac{1}{n!}, n \in \mathbb{N}.$$

6. Using partial fractions and geometric series formula, find the Laurent expansion of the following functions in the indicated annulus:
   (a) $\dfrac{2z-2}{(z+1)(z-3)}, \quad 1 < |z| < 3.$
   (b) $\dfrac{1}{(z-1)(z-2)}, \quad |z| > 2.$
   (c) $\dfrac{1}{z^2}, \quad 1 < |z - i| < \infty.$