Parametric Surfaces Sep 11, 2023

Defn. General form of a parametric surface.

The vector form of a parametric surface is $\vec{X} = \vec{g}(u,v)$, where $\vec{g} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is given by

$$\vec{g}(u,v) = (x(u,v), y(u,v), z(u,v)), \text{ and } (u,v) \in D_{uv}$$
$$\hookrightarrow ⊛ \qquad (D_{uv} \subseteq \mathbb{R}^2)$$

Grid curves

For each fixed v, the parametrization ⊛ defines a curve (in general). Thus, as we vary v, we produce a family of such curves which, in general, will generate a surface.

Defn. The grid curves of a parametric surface $\vec{X} = \vec{g}(u,v)$ are the curves: $\vec{g}(u_0, v)$, where u_0 runs through the domain of u, and $\vec{g}(u, v_0)$, where v_0 runs through the domain of v.

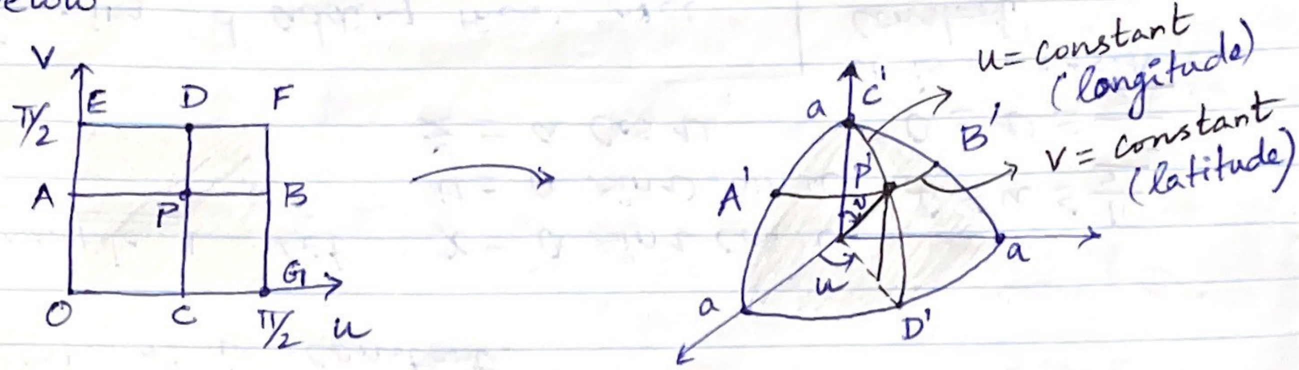
These are the curves on which u is constant (or) v is constant.

Example 1 Let
$$x = a \sin v \cos u \qquad 0 \le u \le \frac{\pi}{2}$$
$$y = a \sin v \sin u \qquad 0 \le v \le \frac{\pi}{2}$$
$$z = a \cos v$$

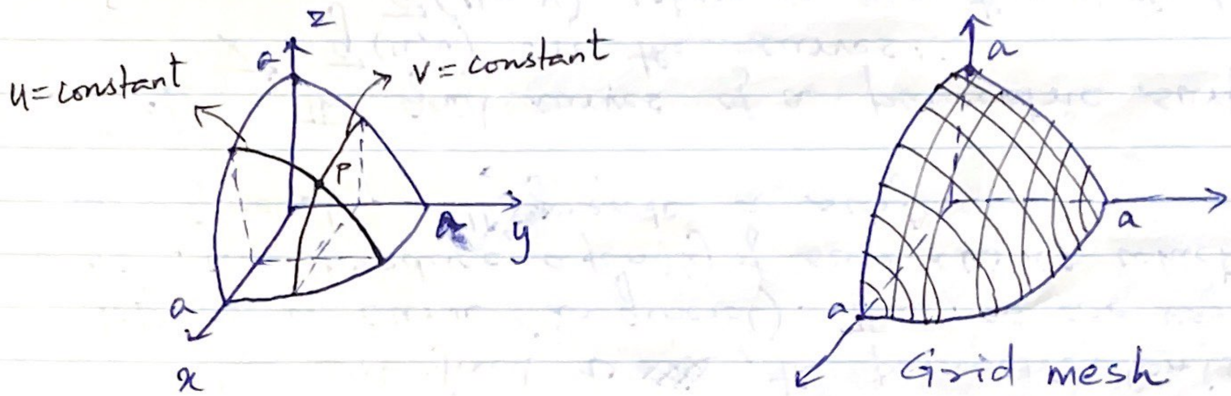
a is a positive constant.

Squaring and adding these three equations gives $x^2 + y^2 + z^2 = a^2$, which represents a spherical surface of radius a, centered at origin.

More precisely, it is one-eighth of a sphere as below.



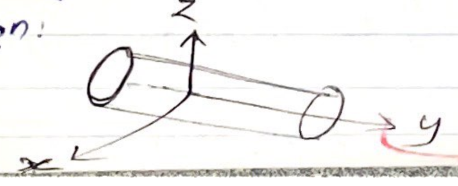
<u>Remark</u>: If the same surface is parametrized by $x = u$, $y = v$, $z = \sqrt{a^2 - u^2 - v^2}$, the grid curves through a given point P is as below.



Have a look at MOBIUS Demo.

<u>Exercise</u>: Find a parametric description of the cylinder $x^2 + z^2 = b^2$ with $-l \le y \le l$. Describe the grid curves

<u>Solution</u>:



Start with $x^2+z^2=b^2$

$\Rightarrow \quad x = b\cos u \qquad u \in [0,2\pi]$

$z \;\cancel{x} = b\sin u$:

About y ? $\Rightarrow$ $y = v$ (simplest option).
$v \in [-l, l]$

$$\therefore \quad \vec{X} = \vec{g}(u,v) = (b\cos u,\ b\sin u,\ v), \quad \begin{array}{l} u \in [0,2\pi] \\ v \in [-l,l] \end{array}$$

Grid curves:

① $u = u_0$ $\qquad \left.\begin{array}{l} x = b\cos u_0 \\ z = b\sin u_0 \end{array}\right\}$ constants

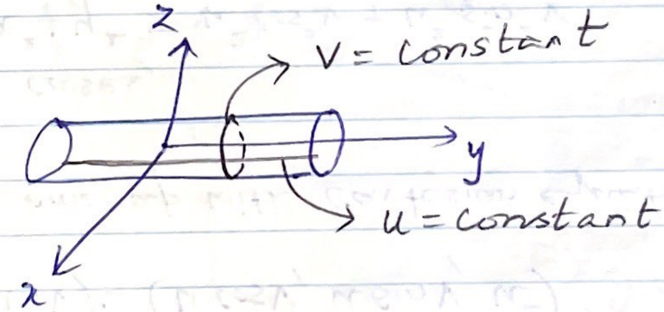
(Intersection of two planes)

So these are lines in the y-direction

② $v = v_0$ $\qquad y = v_0 = \text{constant}$

$x^2+z^2=b^2$ $\quad$ Eliminating u from parametric eqns to get Cartesian form.

These are circles in $y = \text{constant}$ planes.



Exercise: Identify the surfaces:

1) $\vec{x} = \vec{g}(u,v) = (u\cos v, u\sin v, u)$

2) $\vec{x} = \vec{g}(u,v) = (u\cos v, u\sin v, u^2)$

Idea: Come up with cartesian equation.

Solution:

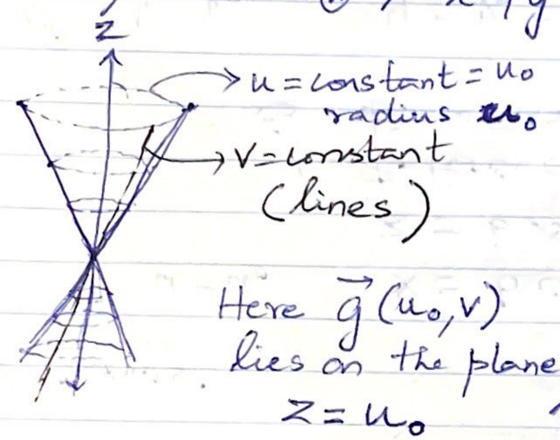
In both cases,

$$x^2+y^2 = u^2\cos^2 v + u^2 \sin^2 v$$
$$= u^2 \quad ⊛$$

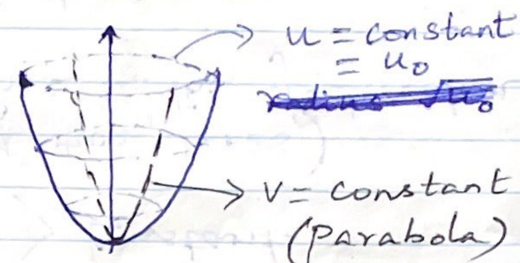
In 1) $z = u$ ⊛ $\Rightarrow x^2+y^2 = z^2$ determines the surface

$$\Rightarrow z = \pm\sqrt{x^2+y^2} \quad \text{(Cone)}$$

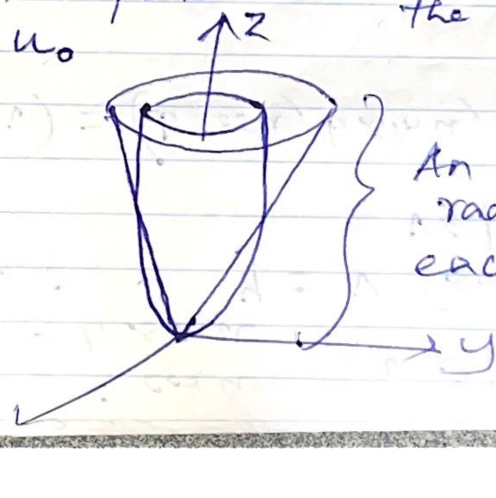
In 2) $z = u^2$ ⊛ $\Rightarrow x^2+y^2 = z$ paraboloid.



Here $\vec{g}(u_0, v)$ lies on the plane $z = u_0$



Here $\vec{g}(u_0,v)$ lies on the plane $z = u_0^2$



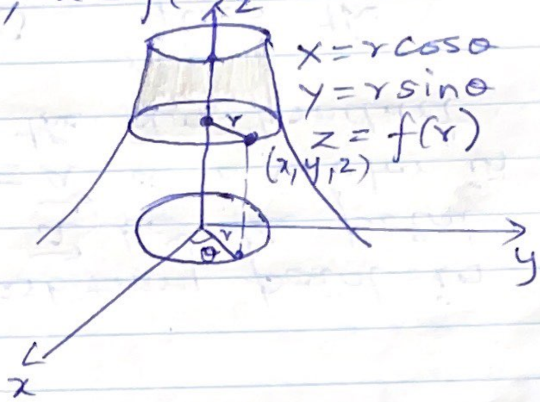
An idea how radii vary at each z-level.

Remark:

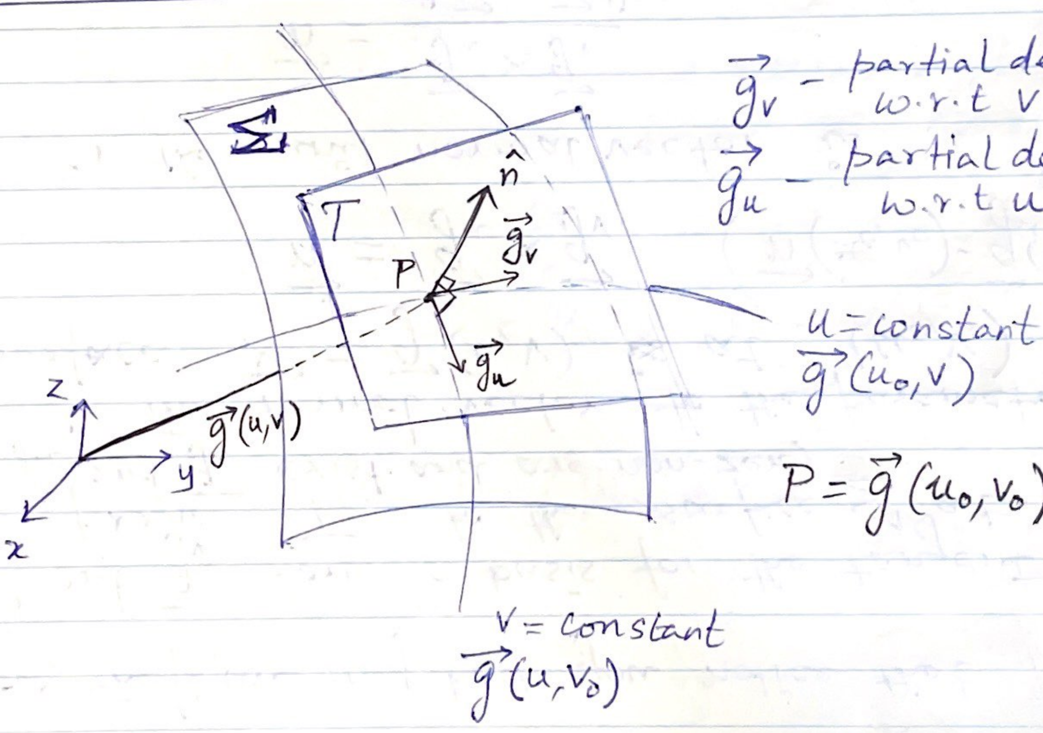
Every surface of revolution $z = f(\sqrt{x^2+y^2})$ can be parametrized as $x = x,\ y = y,\ z = f(\sqrt{x^2+y^2})$. However, if a projection of a part, one may consider on the xy-plane produces a disc (or) some part of it, polar co-ordinates are much suitable.

Since $\sqrt{x^2+y^2}$ is r in polar co-ordinates, the surface can be parametrized as follows

$$x = r\cos\theta,\quad y = r\sin\theta,\quad z = f(r)$$



Tangent plane and normal



$\vec{g}_v$ - partial derivative w.r.t v

$\vec{g}_u$ - partial derivative w.r.t u

$P = \vec{g}(u_0, v_0)$

Now, the figure may help you notice that

$\vec{g}_u$ and $\vec{g}_v$ form a basis for the tangent plane T to the surface Σ at (u_0, v_0)
(if $\vec{g}_u$ and $\vec{g}_v$ exist and are non-zero)

$\therefore$ The normal vector to the parametric surface $\vec{x} = \vec{g}(u,v)$ ~~is~~ at (u_0, v_0)

$$\vec{n} = \vec{g}_u \times \vec{g}_v \qquad \left(\vec{n}(u_0,v_0) = \vec{g}_u(u_0,v_0) \times \vec{g}_v(u_0,v_0)\right)$$

or the unit normal vector is

$$\vec{n} = \frac{\vec{g}_u \times \vec{g}_v}{\|\vec{g}_u \times \vec{g}_v\|}$$

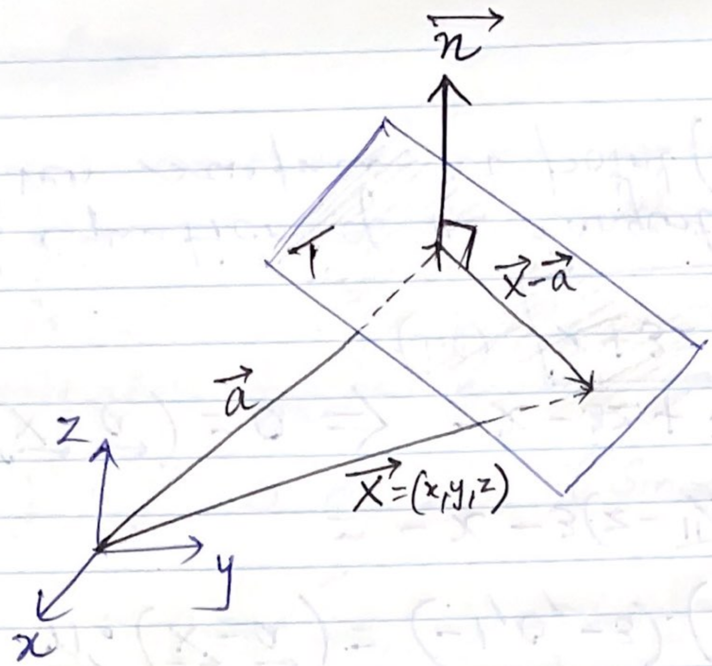
Tangent Plane:

Let $\vec{x}$ be an arbitrary point in the tangent plane T, and $\vec{a}$ be the point of tangency so that $\vec{x} - \vec{a}$ is a vector in the tangent plane. Then the dot product

$$(\vec{x} - \vec{a}) \cdot \vec{n} = 0$$ gives the equation of the tangent plane at $\vec{a}$.

Notes.

- $\vec{a}$ is the Cartesian point corresponding to (u_0, v_0), i.e. $\vec{a} = \vec{g}(u_0, v_0)$. Often times you will be given $\vec{a}$, e.g. find the equation of the tangent plane at $(1,2,3)$ and you will need to determine the values of u, v corresponding to $(1,2,3)$,



Exercise: Find the normal vector $\vec{n}$ and the equation of the tangent plane for the surface $\vec{g}(u,v) = (v\cos u, v\sin u, u)$ at the point $(0, 3, \pi/2)$ (This is a helicoid)

Soln.

Key step: Find u, v corresponding to $\vec{a} = (0, 3, \pi/2)$

So $v\cos u = 0$

$v \sin u = 3$

$u = \pi/2 \Rightarrow v \sin \pi/2 = 3$

$\Rightarrow \boxed{v = 3}$

So $(u,v) = (\pi/2, 3)$.

Recall that $\vec{n}(u,v) = \vec{g_u} \times \vec{g_v}$

Since, $\vec{g}(u,v) = (v\cos u, v\sin u, u)$

$$\vec{g_u} = (-v\sin u, v\cos u, 1)$$

$$\vec{g_v} = (\cos u, \sin u, 0)$$

$$\therefore \vec{g_u} \times \vec{g_v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -v\sin u & v\cos u & 1 \\ \cos u & \sin u & 0 \end{vmatrix}$$

$$= -\sin u\, \vec{i} + \vec{j} \cos u + \vec{k}\left(-v\sin^2 u - v\cos^2 u\right)$$

$$= (-\sin u, \cos u, -v)$$

$\therefore \vec{n}(\pi/2, 3) = (-1, 0, -3)$ Recall $\vec{a} = (0, 3, \pi/2)$

Compute $\vec{n} \cdot (\vec{x} - \vec{a}) = (-1, 0, -3) \cdot (x, y-3, z-\pi/2)$

$$= -x - 3(z - \pi/2)$$

$\vec{n} \cdot (\vec{x} - \vec{a}) = 0 \Rightarrow -x - 3z + 3\pi/2 = 0$

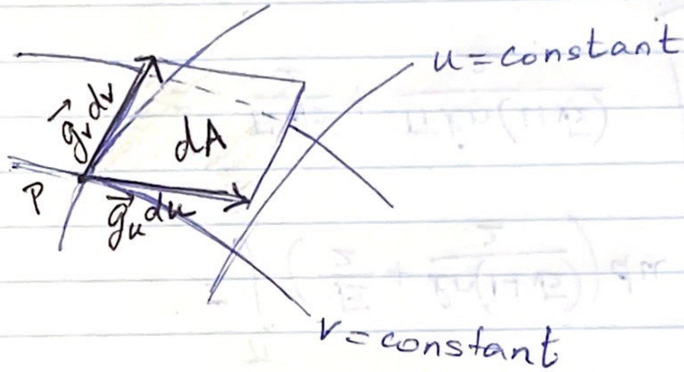
(i.e) $\boxed{x + 3z = \frac{3\pi}{2}}$ is

the equation of the tangent plane to the given surface at point $(0, 3, \pi/2)$.

Surface Area

For a surface Σ given parametrically by $\vec{g}(u,v) = (x(u,v), y(u,v), z(u,v))$, assume $\vec{g}$ is C^1 (i.e. $\vec{g}$ and its first-order partial derivatives $\vec{g}_u$ and $\vec{g}_v$ are continuous) and

$\vec{g}_u \times \vec{g}_v \neq 0$ on Σ. (Smooth surface)



The area element dA on Σ is approximately a parallelogram defined by the vectors $\vec{g}_u du$ and $\vec{g}_v dv$ lying in the tangent plane to Σ at P.

$$\therefore \quad dA = \|\vec{g}_u du \times \vec{g}_v dv\|$$
$$= \|\vec{g}_u \times \vec{g}_v\| \, du\, dv$$

As du and dv tend to zero, this plane area element lies closer and closer to Σ so that it seems reasonable to define the <u>surface area</u> of the curved surface Σ given by $\vec{g}(u,v)$, $(u,v) \in D_{uv}$ as

$$S = \iint_{D_{uv}} dA = \iint_{D_{uv}} \|\vec{g}_u \times \vec{g}_v\| \, du\, dv$$

Example. Find the area of the surface of the helicoid given by

$$\vec{x} = \vec{g}(u,v) = (v\cos u, v\sin u, u)$$

for $0 \le u \le \pi$ and $0 \le v \le 1$

Solution:

Recall that $\vec{g_u} \times \vec{g_v} = (-\sin u, \cos u, -v)$

$$\therefore \|\vec{g_u} \times \vec{g_v}\| = \sqrt{\sin^2 u + \cos^2 u + v^2}$$
$$= \sqrt{1+v^2}$$ and the

surface area

$$S = \int_0^{\pi}\int_0^{1} \sqrt{1+v^2}\, dv\, du$$

$$= \int_0^{\pi} \left(\frac{\sqrt{2}}{2} + \frac{\ln(1+\sqrt{2})}{2}\right) du$$

$$= \frac{\pi\sqrt{2}}{2} + \frac{\pi \ln(1+\sqrt{2})}{2}$$

$$\int \sqrt{1+x^2}\, dx = \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\ln\left|x+\sqrt{1+x^2}\right| + C$$