Parametric Curves

Sep 8, 2023

Curves in $\mathbb{R}^3$

There are three ways a curve can be described.

1) Implicit form

Unit Circle: $x^2+y^2=1$

Ellipse: $\left(\frac{x}{2}\right)^2+\left(\frac{y}{3}\right)^2=1$

Advantage: Single equation

Disadvantage: Not necessarily functions so can't apply calculus without pulling some major theorem and appropriate restrictions on the domain.

2) Explicit form

Semicircle: $y=\sqrt{1-x^2}$

Advantage: This is a function because we restrict the domain.

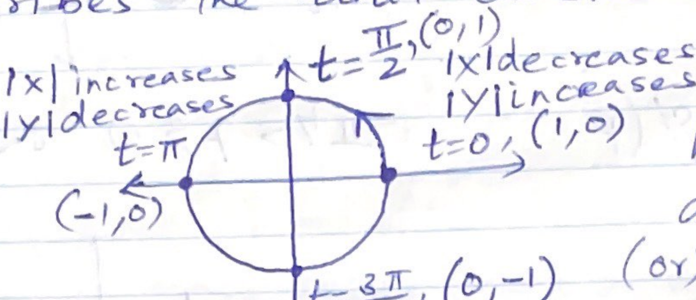
To describe a curve such as the unit circle, we need two separate functions $\pm\sqrt{1-x^2}$

3) Parametric form

Eg: $x = \cos(t)$, $y = \sin(t)$ $\quad 0 \le t \le 2\pi$

Here t is called the parameter.

Since
$x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1$, the above describes the unit circle



being traced anti-clockwise (or) counter clockwise.

Advantage: an orientation (or) direction of motion

So helpful to imagine a parametric curve as being the path of a particle in the plane (or) in space.

Definition: General form of a parametric curve

Let $x(t), y(t), z(t)$ be continuous functions of a real parameter t over a closed interval $[a, b]$, that is, over $a \le t \le b$.

Now, the vector function $\vec{g} : \mathbb{R} \to \mathbb{R}^3$ given by $\vec{X} = \vec{g}(t) = (x(t), y(t), z(t))$ where $a \le t \le b$ is the parametric form (or) parametrization of the curve specified by the points $x = x(t)$, $\cancel{y(t) =}$, $y = y(t)$, $z = z(t)$, $a \le t \le b$.

If the endpoints coincide, the curve is called a closed curve.

<u>Examples</u> Suppose that

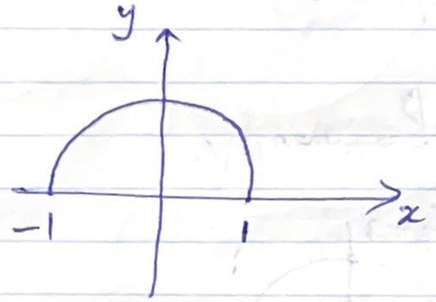
$$x = \beta t + \cos t, \quad y = \sin t, \quad z = 0,$$

where β is a constant.

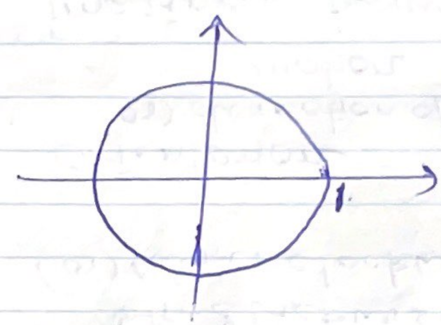
Since $z=0$, the curve lies in a plane (the xy-plane in this case) and is therefore called a plane curve.

Choosing different values of β and different t intervals, we obtain different curves.

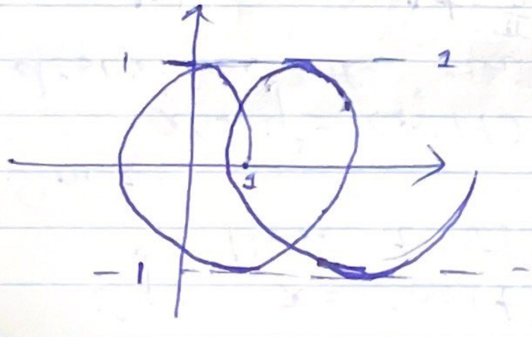
a) $\beta = 0, \quad 0 \le t \le \pi$



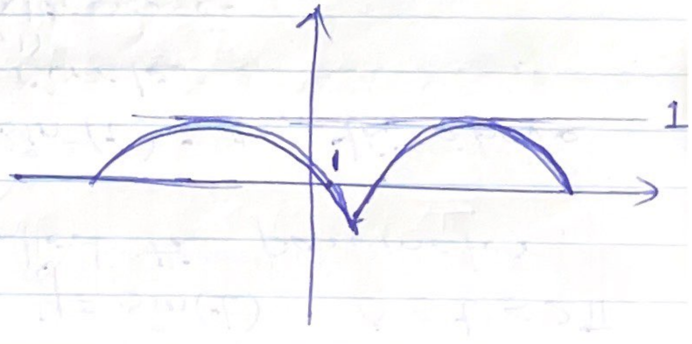
b) $\beta = 0; \quad 0 \le t \le 2\pi$



c) $\beta = 0.2, \quad 0 \le t \le 4\pi$



d) $\beta = -1, \quad -2\pi \le t \le \pi$



Example

What is the curve given by

$$\vec{x} = \vec{g}(t) = (\underbrace{b\cos(t)}_{x}, \underbrace{b\sin(t)}_{y}, \underbrace{kt}_{z}) \text{ where } b, k > 0 \text{ are constants?}$$

Soln.

Starting with x, y

$$x^2+y^2 = b^2\cos^2(t) + b^2\sin^2(t)$$
$$= b^2$$

↳ represents a circle of radius b in $\mathbb{R}^2$

but represents a cylinder in $\mathbb{R}^3$ (surface)

$\Rightarrow$ curve lies on cylinder

Spiral (or) helix

$z = kt \Rightarrow z$ increases w.r.t t.

Example: Determine a parametric form for the curve that is the intersection of the sphere $x^2+y^2+z^2=1$ and the plane $z=\sqrt{3}y$

Solution. Plug $z=\sqrt{3}y$ into $x^2+y^2+z^2=1$ to get

$$x^2+y^2+(\sqrt{3}y)^2 = 1 \Rightarrow \underbrace{x^2+4y^2=1}$$

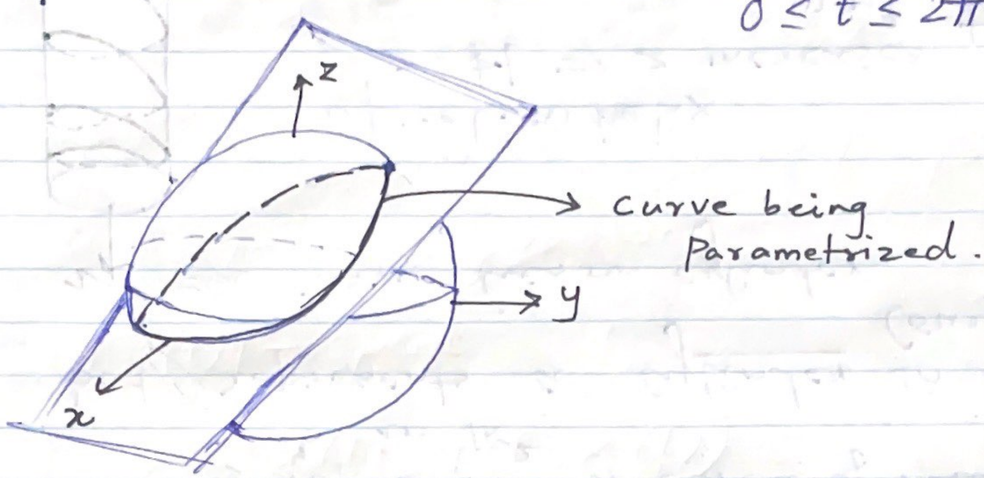
This is the projection of the circle of intersection by the given sphere and the plane onto xy-plane

As such this projection is an ellipse, which we can parametrize by $x = \cos t$, $y = \frac{1}{2}\sin t$ for $0 \le t \le 2\pi$.

Putting back into $z = \sqrt{3}y$, we get $z = \frac{\sqrt{3}}{2}\sin t$.

Thus, our parametric form is:

$$\vec{x} = \vec{g}(t) = \left(\cos t, \frac{1}{2}\sin t, \frac{\sqrt{3}}{2}\sin t\right), \quad 0 \le t \le 2\pi$$



<u>Note</u>: parametrizations are not unique.
e.g. we could have taken

$$x = \sin(t), \quad y = \frac{1}{2}\cos(t), \quad z = \frac{\sqrt{3}}{2}\cos(t), \quad 0 \le t \le 2\pi$$

in the above example.

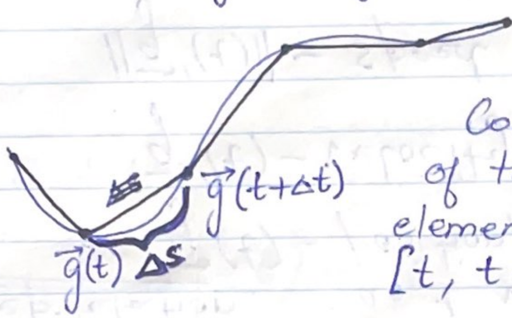
(or)

$$x = \sin(t^2), \quad y = \frac{1}{2}\cos(t^2), \quad z = \frac{\sqrt{3}}{2}\cos(t^2), \quad 0 \le t \le \sqrt{2\pi}$$

Arc length:

Given a parametric curve $C : \vec{x} = \vec{g}(t)$ on $a \le t \le b$, how do we find the length of C?

We know what we mean by the length of a straight line, but what is the length of a curved arc? As is done so often in mathematics, we use the limit concept to define this "new" quantity as the limit of a sequence of "old" quantities. Specifically, we approximate the curve by a system of linear segments.



Consider a short segment of the curve with arc length element Δs on the time interval $[t, t+\Delta t]$.

If the interval is sufficiently small, then

$$\Delta s \approx \| \vec{g}(t+\Delta t) - \vec{g}(t) \|$$

Recall that the derivative provides a linear approximation:

$$\vec{g}(t+\Delta t) - \vec{g}(t) \approx \vec{g}'(t)\Delta(t).$$

Thus $\Delta s \approx \|\vec{g}'(t)\|\Delta t$

Thus, the limiting process of summation described above leads to (i.e) summing over all the elements and taking the limit as the subinterval length approaches zero ($\Delta t \to 0$) (so the $\Delta \to d$), we obtain $s = \int ds$,) we get

Arc length formula

The arc length of a curve $C: \vec{x} = \vec{g}(t)$, $a \le t \le b$ is given by:

$$\boxed{s = \int_a^b \|g'(t)\| \, dt.}$$

Intrepretation: as t varies,

$\vec{g}(t)$ - position/displacement of a particle

$\vec{g}'(t)$ - velocity vector

$\|\vec{g}'(t)\|$ - speed

$$\vec{g}'(t) = (x'(t), y'(t), z'(t))$$

where $x'(t) = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}$

$\vec{g}''(t)$ is the acceleration of the particle at time t.

distance travelled by particle is the integral of the speed function.

Example: What is the arc length of

$$\vec{x} = \vec{g}(t) = (\cos t, \tfrac{1}{2}\sin t, \tfrac{\sqrt{3}}{2}\sin t),$$

$$0 \le t \le 2\pi$$

Soln.

Formula $S = \int_0^{2\pi} \|\vec{g}'(t)\| dt$.

Computing $\vec{g}'(t) = \left(-\sin t, \frac{1}{2}\cos t, \frac{\sqrt{3}}{2}\cos t\right)$

$$\|\vec{g}'(t)\| = \sqrt{(-\sin t)^2 + \left(\frac{1}{2}\cos t\right)^2 + \left(\frac{\sqrt{3}}{2}\cos t\right)^2}$$

$$= \sqrt{\sin^2 t + \cos^2 t}$$

$$= 1$$

$\therefore \quad S = \int_0^{2\pi} dt = 2\pi$ (It's a great circle of radius 1 on the sphere example above)

Example.

Find the arc length of

$$\vec{x} = \vec{g}(t) = (b\cos t, b\sin t, kt)$$

for $0 \leq t \leq 4\pi$?

Soln. To Compute: $S = \int_0^{4\pi} \|\vec{g}'(t)\| dt$

$$\vec{g}'(t) = (-b\sin t, b\cos t, k)$$

$$\Rightarrow \|\vec{g}'(t)\| = \sqrt{b^2\sin^2 t + b^2\cos^2 t + k^2}$$

$$= \sqrt{b^2 + k^2}$$

$$\Rightarrow \quad S = \int_0^{4\pi} \sqrt{b^2+k^2}\, dt = \sqrt{b^2+k^2} \int_0^{4\pi} dt = 4\pi\sqrt{b^2+k^2}.$$