

AutoML: Gaussian Processes

Gaussian Processes

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Weight-Space View

- So far, we have considered a hypothesis space \mathcal{H} of parameterized functions $f(\mathbf{x} \mid \boldsymbol{\theta})$ (in particular, the space of linear functions).
- Using Bayesian inference, we derived distributions for $\boldsymbol{\theta}$ after having observed data $\mathcal{D}_{\text{train}}$.
- Prior beliefs about the parameter are expressed via a prior distribution $q(\boldsymbol{\theta})$, which is updated according to Bayes' rule

$$\underbrace{p(\boldsymbol{\theta} \mid \mathbf{X}, \mathbf{y})}_{\text{posterior}} = \frac{\overbrace{p(\mathbf{y} \mid \mathbf{X}, \boldsymbol{\theta})}^{\text{likelihood}} \overbrace{q(\boldsymbol{\theta})}^{\text{prior}}}{\underbrace{p(\mathbf{y} \mid \mathbf{X})}_{\text{marginal}}}.$$

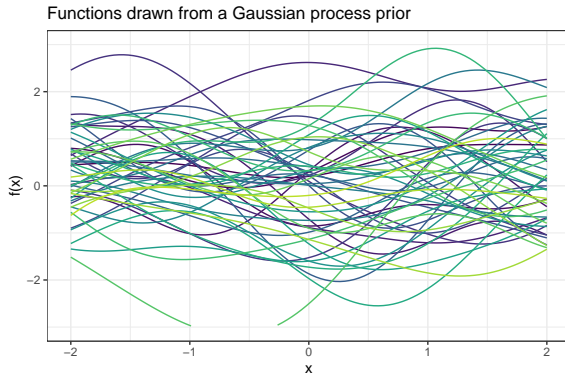
Function-Space View I

Let us change our point of view:

- Instead of “searching” for a parameter θ in the parameter space, we directly search in a space of “allowed” functions \mathcal{H} .
- We will still use Bayesian inference, but instead of specifying a prior distribution over a parameter, we will specify a prior distribution **over functions** and will update it according to the data points that we observe.

Function-Space View II

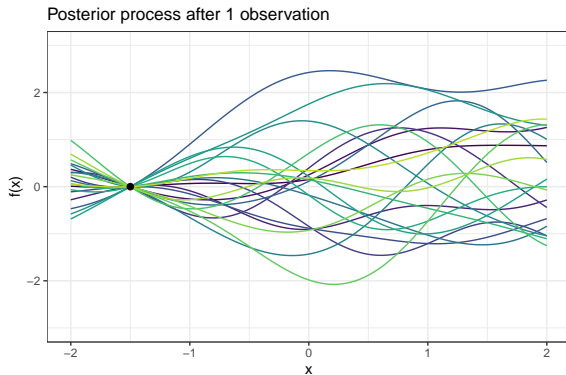
Intuitively, imagine we could draw a huge number of functions from some prior distribution over functions ^(*).



^(*) We will see in a minute how distributions over functions can be specified.

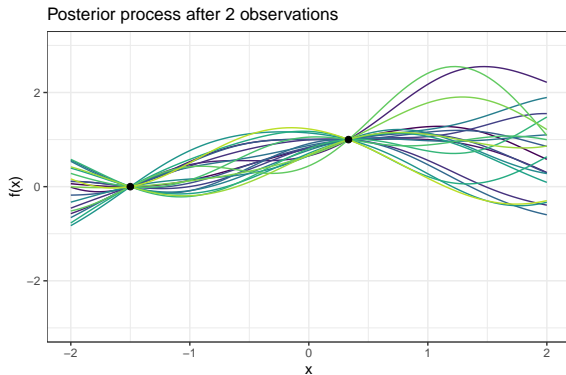
Function-Space View III

After observing some data points, we are allowed to sample only those functions that are consistent with the data.



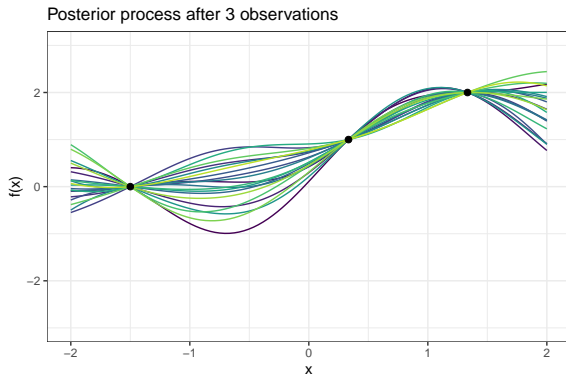
Function-Space View IV

After observing some data points, we are allowed to sample only those functions that are consistent with the data.



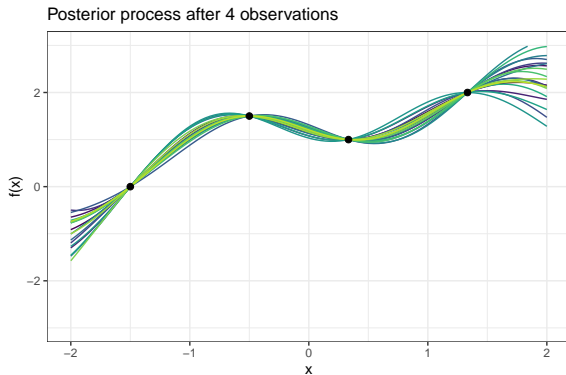
Function-Space View V

After observing some data points, we are allowed to sample only those functions that are consistent with the data.



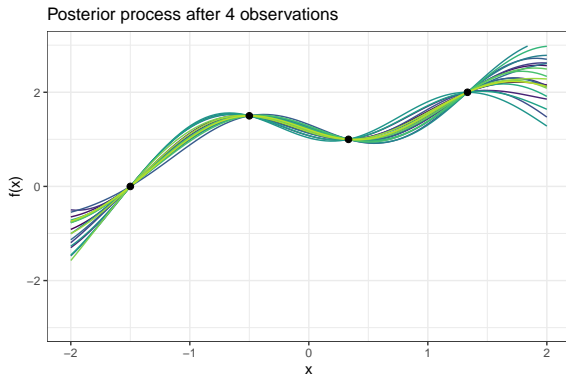
Function-Space View VI

As we observe more and more data points, the number of functions that consistent with the data shrinks.



Function-Space View VII

Intuitively, there is something like the “mean” and “variance” of a distribution over functions.



Weight-Space View vs. Function-Space View

Weight-Space View

Parameterize functions

Example: $f(\mathbf{x} \mid \boldsymbol{\theta}) = \boldsymbol{\theta}^\top \mathbf{x}$

Define distributions on $\boldsymbol{\theta}$

Inference in parameter space Θ

Function-Space View

Define distributions on f

Inference in function space \mathcal{H}

Next, we will see how we can define distributions over functions mathematically.

Distributions on Functions

Discrete Functions I

For simplicity, we will firstly consider functions with finite domains.

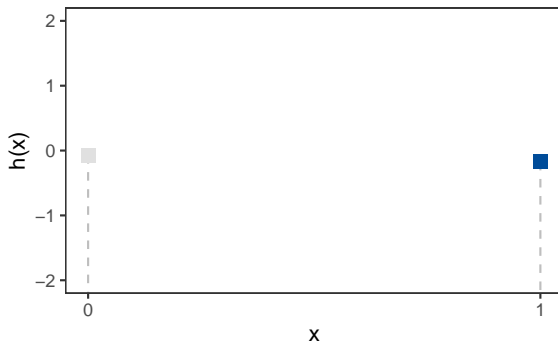
- Let $\mathcal{X} = \{\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}\}$ be a finite set of elements and \mathcal{H} the set of all functions $h : \mathcal{X} \rightarrow \mathbb{R}$.
- Since the domain of any $h(\cdot) \in \mathcal{H}$ has only n elements, we can represent the function $h(\cdot)$ compactly as a n -dimensional vector

$$\mathbf{h} = \left[h\left(\mathbf{x}^{(1)}\right), \dots, h\left(\mathbf{x}^{(n)}\right) \right].$$

Discrete Functions II

Example 1: Consider function $h : \mathcal{X} \rightarrow \mathcal{Y}$ where the input space consists of **two** points $\mathcal{X} = \{0, 1\}$.

Examples for functions that live in this space:



Discrete Functions III

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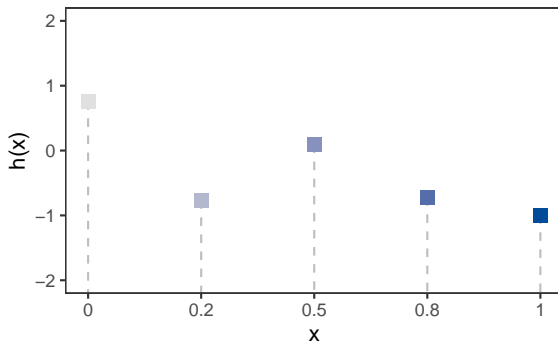
Examples for functions that live in this space:



Discrete Functions V

Example 2: Consider $h : \mathcal{X} \rightarrow \mathcal{Y}$ where the input space consists of **five** points $\mathcal{X} = \{0, 0.25, 0.5, 0.75, 1\}$.

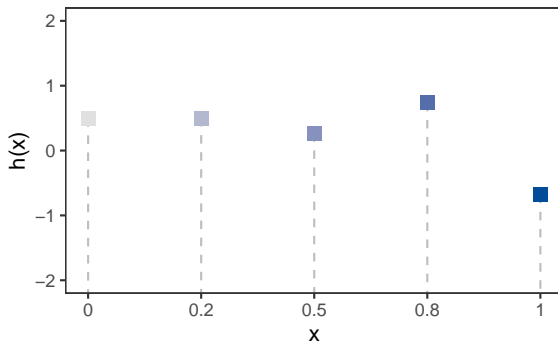
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Discrete Functions VI

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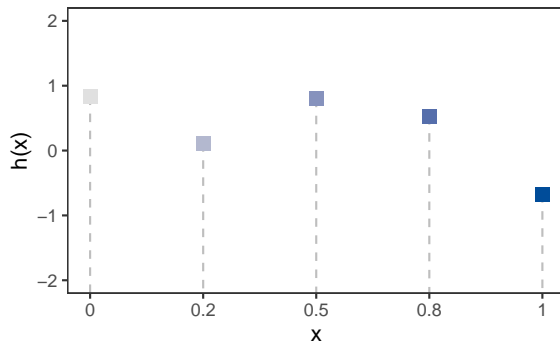
Examples for functions that live in this space:



Discrete Functions VII

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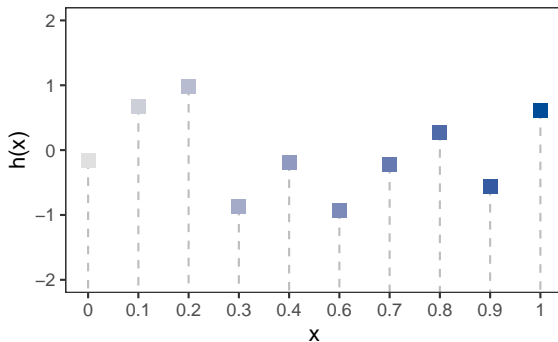
Examples for functions that live in this space:



Discrete Functions VIII

Example 3: Consider $h : \mathcal{X} \rightarrow \mathcal{Y}$ where the input space consists of **ten** points.

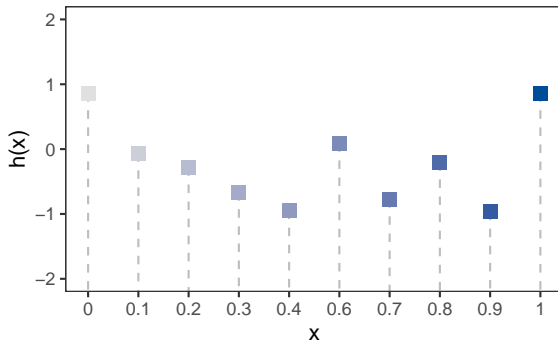
Examples for functions that live in this space:



Discrete Functions IX

Example 3: Consider $h : \mathcal{X} \rightarrow \mathcal{Y}$ where the input space consists of **ten** points.

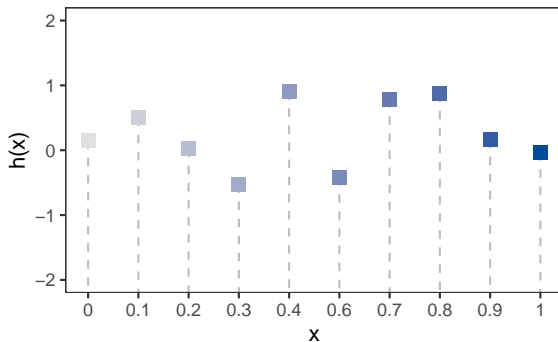
Examples for functions that live in this space:



Discrete Functions X

Example 3: Consider $h : \mathcal{X} \rightarrow \mathcal{Y}$ where the input space consists of **ten** points.

Examples for functions that live in this space:



Distributions on Discrete Functions I

- One natural way to specify a probability distribution on a discrete function $h \in \mathcal{H}$ is to use the vector representation of the function:

$$\mathbf{h} = \left[h\left(\mathbf{x}^{(1)}\right), h\left(\mathbf{x}^{(2)}\right), \dots, h\left(\mathbf{x}^{(n)}\right) \right].$$

- Let us consider \mathbf{h} as a n -dimensional random variable. We will further assume the following normal distribution:

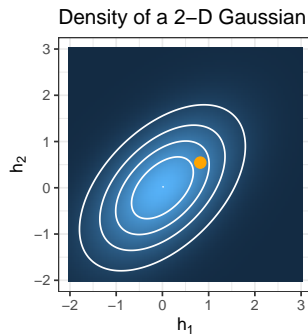
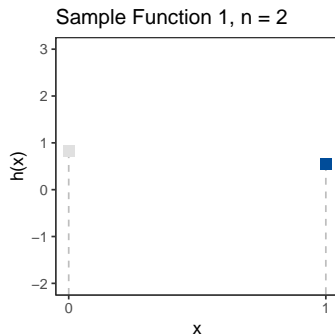
$$\mathbf{h} \sim \mathcal{N}(\mathbf{m}, \mathbf{K}).$$

Note: For now, we set $\mathbf{m} = \mathbf{0}$ and take the covariance matrix \mathbf{K} as given. We will see later how they are chosen / estimated.

Distributions on Discrete Functions II

Example 1 (continued): Let $h : \mathcal{X} \rightarrow \mathcal{Y}$ be a function that is defined on **two** points \mathcal{X} . We sample functions by sampling from a two-dimensional normal variable

$$\mathbf{h} = [h(1), h(2)] \sim \mathcal{N}(\mathbf{m}, \mathbf{K}).$$

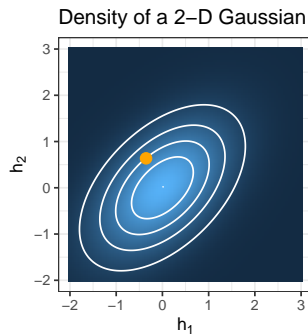
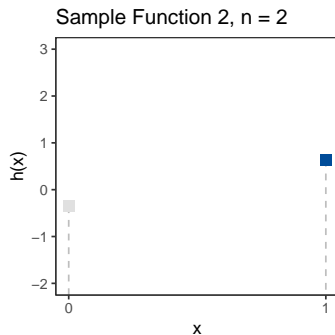


In this example, $\mathbf{m} = (0, 0)$ and $\mathbf{K} = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$.

Distributions on Discrete Functions III

Example 1 (continued): Let $h : \mathcal{X} \rightarrow \mathcal{Y}$ be a function that is defined on **two** points \mathcal{X} . We sample functions by sampling from a two-dimensional normal variable

$$\mathbf{h} = [h(1), h(2)] \sim \mathcal{N}(\mathbf{m}, \mathbf{K}).$$

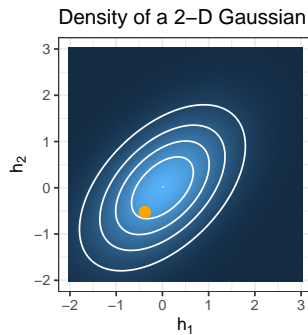
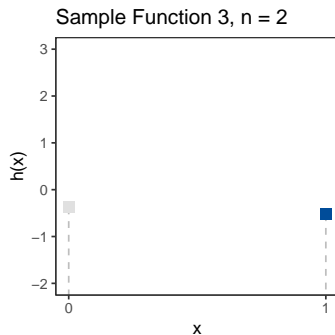


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Distributions on Discrete Functions IV

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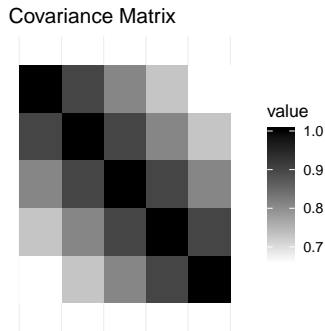
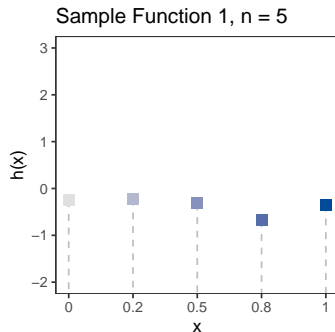


In this example, $\mathbf{m} = (0, 0)$ and $\mathbf{K} = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$.

Distributions on Discrete Functions V

Example 2 (continued): Let us consider $h : \mathcal{X} \rightarrow \mathcal{Y}$ where the input space consists of **five** points. We sample functions by sampling from a five-dimensional normal variable

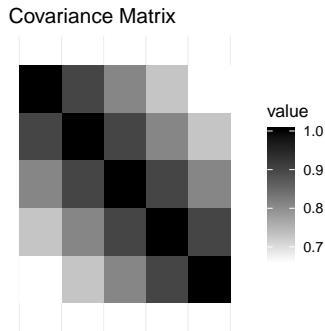
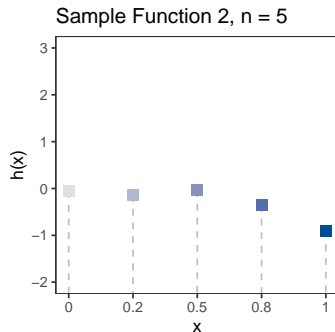
$$\mathbf{h} = [h(1), h(2), h(3), h(4), h(5)] \sim \mathcal{N}(\mathbf{m}, \mathbf{K}).$$



Distributions on Discrete Functions VI

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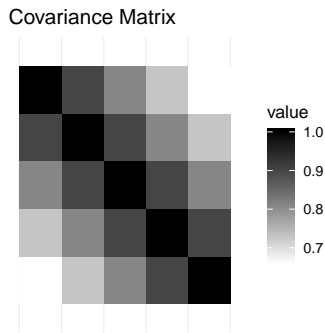
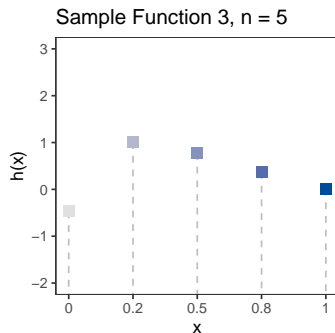
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Distributions on Discrete Functions VII

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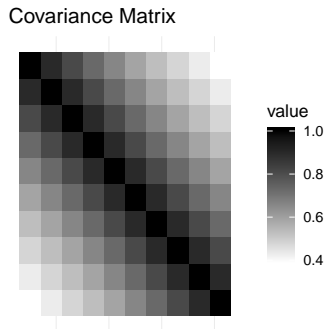
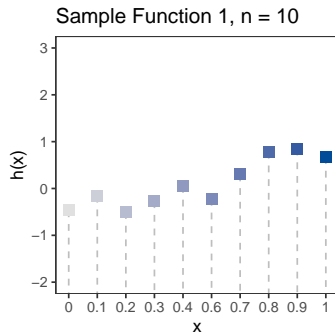
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Distributions on Discrete Functions VIII

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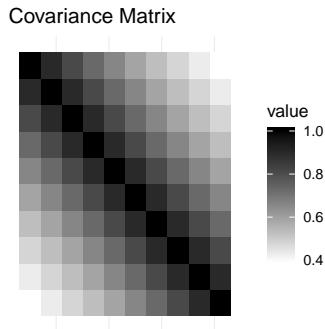
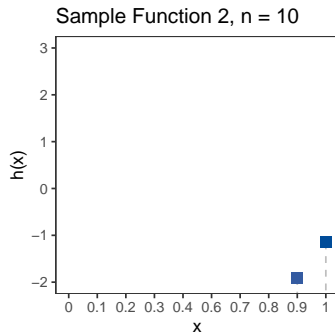
$$\mathbf{h} = [h(1), h(2), \dots, h(10)] \sim \mathcal{N}(\mathbf{m}, \mathbf{K}).$$



Distributions on Discrete Functions IX

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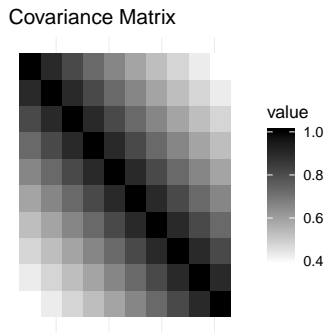
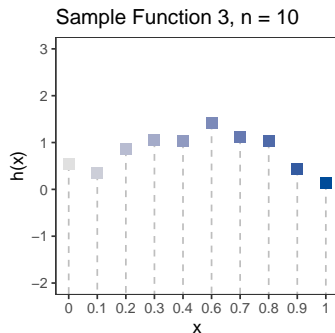
$$\mathbf{h} = [h(1), h(2), \dots, h(10)] \sim \mathcal{N}(\mathbf{m}, \mathbf{K}).$$



Distributions on Discrete Functions X

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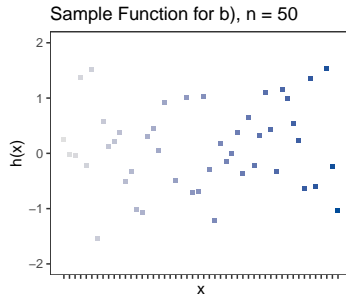
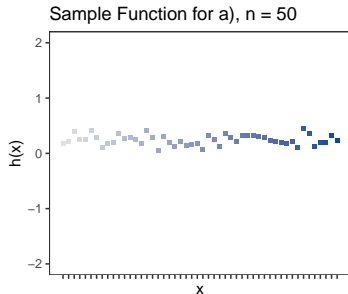
$$\mathbf{h} = [h(1), h(2), \dots, h(10)] \sim \mathcal{N}(\mathbf{m}, \mathbf{K}).$$



The Role of Covariance Function I

The covariance controls the “shape” of drawn functions. Consider two extreme cases where function values are:

- a) strongly correlated: $\mathbf{K} = \begin{pmatrix} 1 & 0.99 & \dots & 0.99 \\ 0.99 & 1 & \dots & 0.99 \\ 0.99 & 0.99 & \ddots & 0.99 \\ 0.99 & \dots & 0.99 & 1 \end{pmatrix}$
- b) uncorrelated: $\mathbf{K} = \mathbf{I}$.



The Role of Covariance Function II

- On a numeric space \mathcal{X} , “meaningful” functions may be characterized by the following spatial property:

If $\mathbf{x}^{(i)}$ and $\mathbf{x}^{(j)}$ are close in the \mathcal{X} -space, their function values $f(\mathbf{x}^{(i)})$ and $f(\mathbf{x}^{(j)})$ should be close in \mathcal{Y} -space.

- 💡 In other words, if two data points are close in \mathcal{X} -space, their corresponding values should be **correlated**!

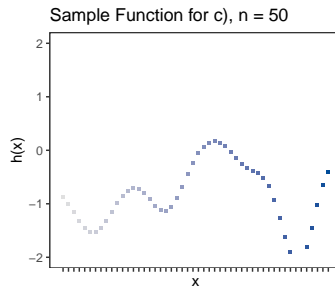
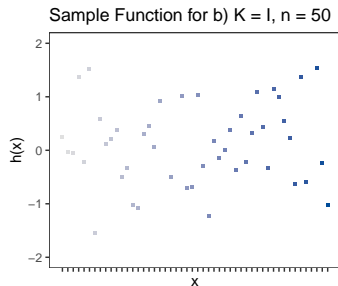
- 💡 We can enforce this condition by choosing a covariance function for which,

K_{ij} is high, if $\mathbf{x}^{(i)}$ and $\mathbf{x}^{(j)}$ are close.

The Role of Covariance Function III

We can compute the entries of the covariance matrix by a function that is based on the distance between $\mathbf{x}^{(i)}$ and $\mathbf{x}^{(j)}$. For example:

c) spatial correlation: $K_{ij} = k(\mathbf{x}^{(i)}, \mathbf{x}^{(j)}) = \exp\left(-\frac{1}{2} \left|\mathbf{x}^{(i)} - \mathbf{x}^{(j)}\right|^2\right)$



Note: $k(\cdot, \cdot)$ is known as the **covariance function** or **kernel**. It will be studied in more detail later on.

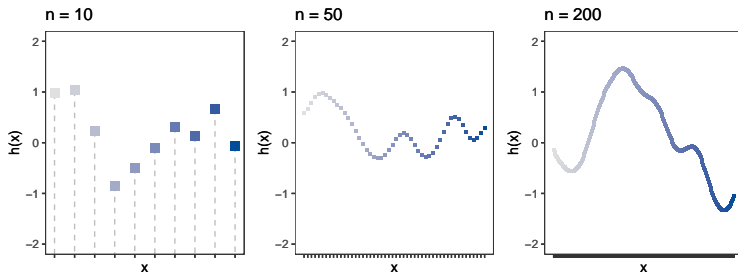
Gaussian Processes

From Discrete to Continuous Functions

- We have already considered distributions on functions with discrete domain. We did so, by defining Gaussian distributions on the vector of the respective function values

$$\mathbf{h} = [h(\mathbf{x}^{(1)}), h(\mathbf{x}^{(2)}), \dots, h(\mathbf{x}^{(n)})] \sim \mathcal{N}(\mathbf{m}, \mathbf{K}).$$

- We can generalize this idea for $n \rightarrow \infty$.



Gaussian Processes: Intuition I

- No matter how large n is, we consider functions with discrete domains.
- But, how can we extend our definition to functions with **continuous** domains $\mathcal{X} \subset \mathbb{R}$?
- Intuitively, a function f drawn from a **Gaussian process** can be understood as an “infinite” long Gaussian random vector.
- It is unclear how to handle an “infinite” long Gaussian random vector!

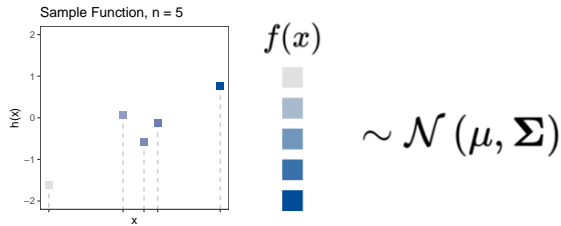


Gaussian Processes: Intuition II

- Thus, it is required that for **any finite set** of inputs $\{\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}\} \subset \mathcal{X}$, the vector \mathbf{f} has a Gaussian distribution with \mathbf{m} and \mathbf{K} being calculated by a mean function $m(\cdot)$ and a covariance function $k(\cdot, \cdot)$:

$$\mathbf{f} = \left[f(\mathbf{x}^{(1)}), \dots, f(\mathbf{x}^{(n)}) \right] \sim \mathcal{N}(\mathbf{m}, \mathbf{K}).$$

- This property is called the **Marginalization Property**.

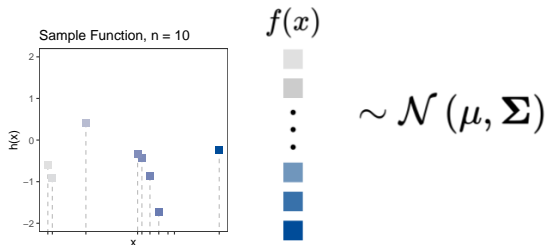


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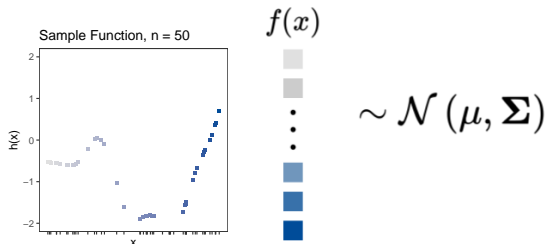


Gaussian Processes: Intuition IV

- Thus, it is required that for **any finite set** of inputs $\{\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}\} \subset \mathcal{X}$, the vector \mathbf{f} has a Gaussian distribution with \mathbf{m} and \mathbf{K} being calculated by a mean function $m(\cdot)$ and a covariance function $k(\cdot, \cdot)$:

$$\mathbf{f} = \left[f(\mathbf{x}^{(1)}), \dots, f(\mathbf{x}^{(n)}) \right] \sim \mathcal{N}(\mathbf{m}, \mathbf{K}).$$

- This property is called the **Marginalization Property**.



Gaussian Processes: Formal Definitions I

- The above intuitive explanation is formally defined as follows.

*A function $f(\mathbf{x})$ is generated by a Gaussian process \mathcal{G} if for **any finite** set of inputs $\{\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}\}$, the associated vector of function values has a Gaussian distribution:*

$$\mathbf{f} = \left(f(\mathbf{x}^{(1)}), \dots, f(\mathbf{x}^{(n)}) \right) \sim \mathcal{N}(\mathbf{m}, \mathbf{K}),$$

with

$$\mathbf{m} := \left(m(\mathbf{x}^{(i)}) \right)_i, \quad \mathbf{K} := \left(k(\mathbf{x}^{(i)}, \mathbf{x}^{(j)}) \right)_{i,j},$$

where $m(\mathbf{x})$ is called mean function and $k(\mathbf{x}, \mathbf{x}')$ is called covariance function.

Gaussian Processes: Formal Definitions II

- A GP is **completely specified** by its mean and covariance functions.
- The mean function $m(\mathbf{x})$ and the covariance function $k(\mathbf{x}, \mathbf{x}')$ of a real process $f(\mathbf{x})$ are defined as:

$$\begin{aligned}m(\mathbf{x}) &= \mathbb{E}[f(\mathbf{x})] \\k(\mathbf{x}, \mathbf{x}') &= \mathbb{E}\left[(f(\mathbf{x}) - \mathbb{E}[f(\mathbf{x})]) (f(\mathbf{x}') - \mathbb{E}[f(\mathbf{x}')])\right]\end{aligned}$$

- We denote a GP by

$$f(\mathbf{x}) \sim \mathcal{G}(m(\mathbf{x}), k(\mathbf{x}, \mathbf{x}'))$$

Note: For now, we assume $m(\mathbf{x}) \equiv 0$. This is not a drastic limitation. In fact, it is common to consider GPs with a zero mean function.

Sampling from a Gaussian Process Prior I

- We can draw functions from a Gaussian process prior. To do so, consider $f(\mathbf{x}) \sim \mathcal{G}(0, k(\mathbf{x}, \mathbf{x}'))$ with the squared exponential covariance function ^(*)

$$k(\mathbf{x}, \mathbf{x}') = \exp \left(-\frac{1}{2\ell^2} \|\mathbf{x} - \mathbf{x}'\|^2 \right), \quad \ell = 1.$$

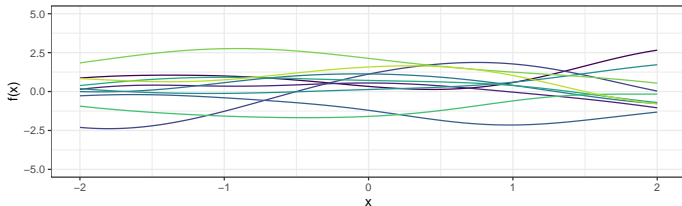
- This covariance function specifies the Gaussian process completely.

^(*) We will talk later about different choices of covariance functions.

Sampling from a Gaussian Process Prior II

To visualize a sample function, we

- choose a large number of equidistant points: $\{\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}\}$,
- compute their corresponding covariance matrix by plugging in all pairs of $\mathbf{x}^{(i)}$ and $\mathbf{x}^{(j)}$ in $\mathbf{K} = (k(\mathbf{x}^{(i)}, \mathbf{x}^{(j)}))_{i,j}$,
- sample from a Gaussian $\mathbf{f} \sim \mathcal{N}(\mathbf{0}, \mathbf{K})$.



We draw 10 times from the Gaussian, to get 10 different samples. Since we specified the mean function to be zero, the drawn functions have a zero mean.

Gaussian Processes as an Indexed Family

Gaussian Processes as an Indexed Family

- A Gaussian process is a special case of a **stochastic process** which is defined as a collection of random variables indexed by some index set (also called an **indexed family**).
- What does it mean?
- An **indexed family** is a mathematical function (or “rule”) that maps indices $t \in T$ to objects in \mathcal{S} .

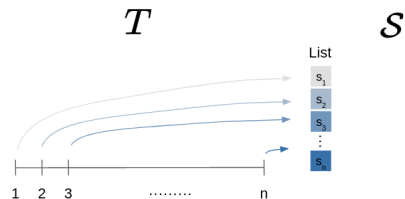
Definition: *an index family (or a family of elements in \mathcal{S} indexed by T) is a surjective function that is defined as follows:*

$$\begin{aligned}s : T &\rightarrow \mathcal{S} \\ t &\mapsto s_t = s(t)\end{aligned}$$

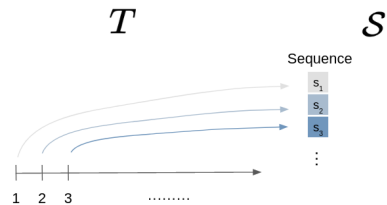
Index Family I

Some simple examples for indexed families are:

- Finite sequences (lists): $T = \{1, 2, \dots, n\}$ and $(s_t)_{t \in T} \in \mathbb{R}$



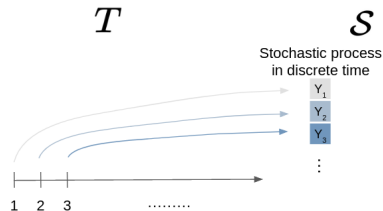
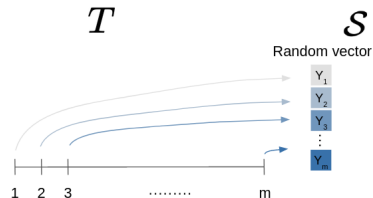
- Infinite sequences: $T = \mathbb{N}$ and $(s_t)_{t \in T} \in \mathbb{R}$



Index Family II

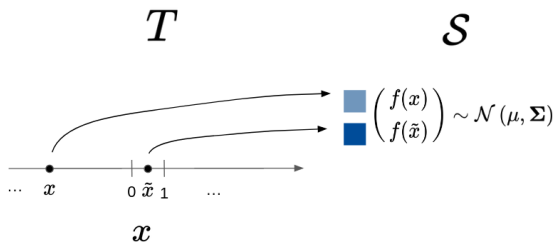
But the indexed set \mathcal{S} can be something more complicated, for example functions or **random variables** (RV):

- $T = \{1, \dots, m\}$, Y_t 's are RVs: Indexed family is a random vector.
- $T = \{1, \dots, m\}$, Y_t 's are RVs: Indexed family is a stochastic process in discrete time.
- $T = \mathbb{Z}^2$, Y_t 's are RVs: Indexed family is a 2D-random walk.



Index Family III

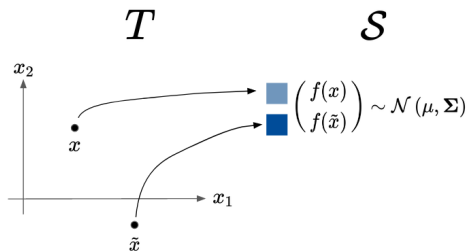
- A Gaussian process is also an indexed family, where the random variables $f(\mathbf{x})$ are indexed by the input values $\mathbf{x} \in \mathcal{X}$.
- Importantly, any indexed (finite) random vector has a multivariate Gaussian distribution (which comes with all the nice properties of Gaussianity!).



Visualization for a one-dimensional \mathcal{X} .

Index Family IV

- A Gaussian process is also an indexed family, where the random variables $f(\mathbf{x})$ are indexed by the input values $\mathbf{x} \in \mathcal{X}$.
- Importantly, any indexed (finite) random vector has a multivariate Gaussian distribution (which comes with all the nice properties of Gaussianity!).



Visualization for a two-dimensional \mathcal{X} .