

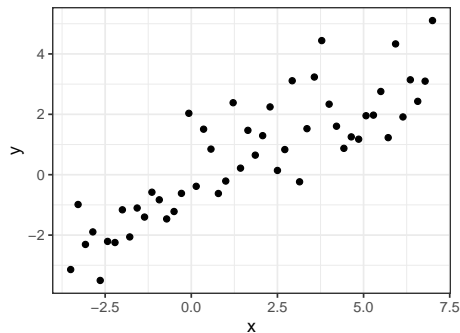
AutoML: Gaussian Processes

The Bayesian Linear Model

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Review: The Bayesian Linear Model I

Let $\mathcal{D}_{\text{train}} = \{(\mathbf{x}^{(1)}, y^{(1)}), \dots, (\mathbf{x}^{(n)}, y^{(n)})\}$ be a training set of i.i.d. observations from some unknown distribution.



Let $\mathbf{y} = (y^{(1)}, \dots, y^{(n)})^\top$ and $\mathbf{X} \in \mathbb{R}^{n \times p}$ be the design matrix where the i-th row contains vector $\mathbf{x}^{(i)}$.

Review: The Bayesian Linear Model II

The linear regression model is defined as

$$y^{(i)} = f(\mathbf{x}^{(i)}) + \epsilon^{(i)} = \boldsymbol{\theta}^\top \mathbf{x}^{(i)} + \epsilon^{(i)}, \text{ for all } i \in \{1, \dots, n\}.$$

The observed values $y^{(i)}$ differ from the function values $f(\mathbf{x}^{(i)})$ by some additive noise, which is assumed to be i.i.d. Gaussian

$$\epsilon^{(i)} \sim \mathcal{N}(0, \sigma^2).$$

Review: The Bayesian Linear Model III

- Let us assume we have **prior beliefs** about the parameter θ that are represented in a prior distribution $\theta \sim \mathcal{N}(\mathbf{0}, \tau^2 \mathbf{I}_p)$.
- Whenever data points are observed, we update the parameters' prior distribution according to Bayes' rule

$$\underbrace{p(\theta \mid \mathbf{X}, \mathbf{y})}_{\text{posterior}} = \frac{\overbrace{p(\mathbf{y} \mid \mathbf{X}, \theta)}^{\text{likelihood}} \overbrace{q(\theta)}^{\text{prior}}}{\underbrace{p(\mathbf{y} \mid \mathbf{X})}_{\text{marginal}}}.$$

Review: The Bayesian Linear Model IV

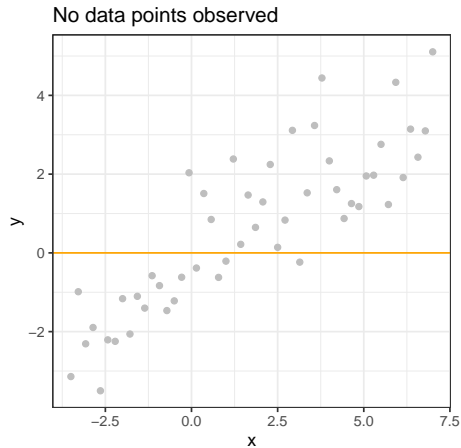
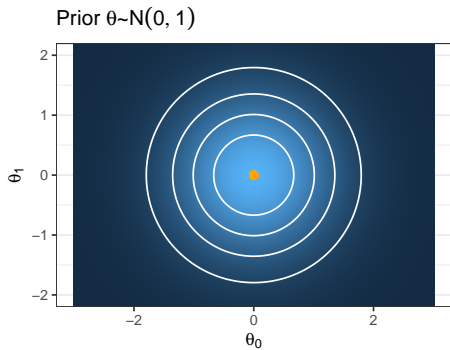
The posterior distribution of the parameter θ is again normal distributed (the Gaussian family is self-conjugate):

$$\theta \mid \mathbf{X}, \mathbf{y} \sim \mathcal{N}(\sigma^{-2} \mathbf{A}^{-1} \mathbf{X}^\top \mathbf{y}, \mathbf{A}^{-1}), \text{ where } \mathbf{A} := \sigma^{-2} \mathbf{X}^\top \mathbf{X} + \frac{1}{\tau^2} \mathbf{I}_p.$$

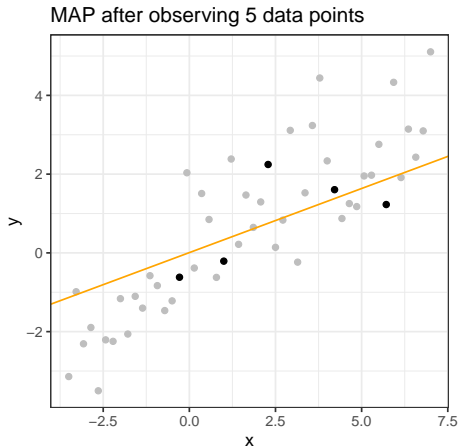
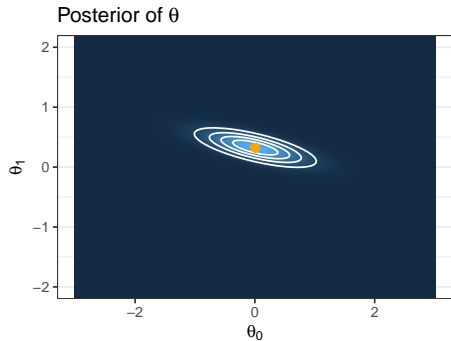
Note: If the posterior distributions $p(\theta \mid \mathbf{X}, \mathbf{y})$ are in the same probability distribution family as the prior $q(\theta)$, the prior and posterior are then called **conjugate distributions**, and the prior is called a **conjugate prior** for the likelihood function $p(\mathbf{y} \mid \mathbf{X}, \theta)$.

Note: The Gaussian family is **self-conjugate** with respect to a Gaussian likelihood function: choosing a Gaussian prior for a Gaussian likelihood ensures that the posterior is also Gaussian.

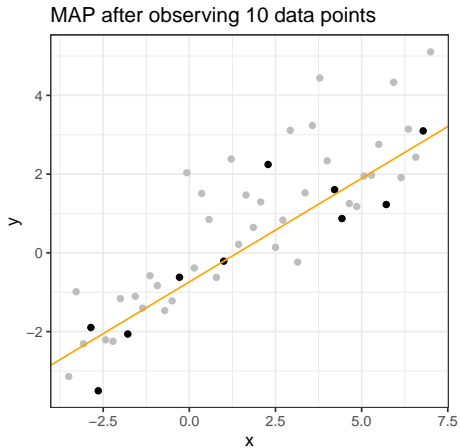
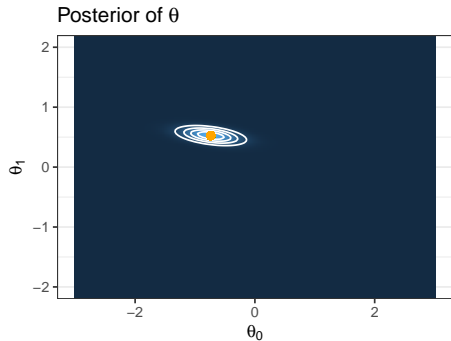
Review: The Bayesian Linear Model V



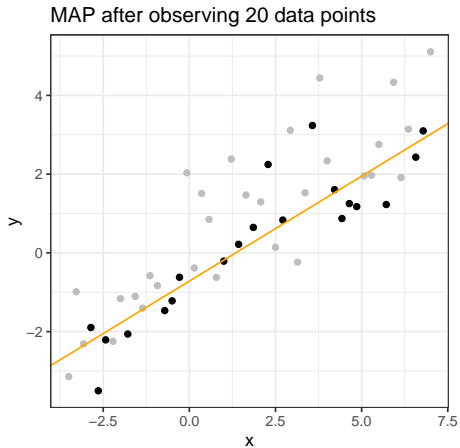
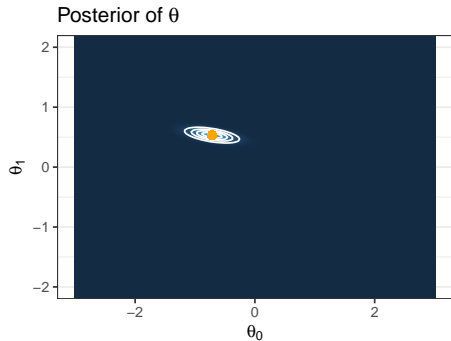
Review: The Bayesian Linear Model VI



Review: The Bayesian Linear Model VII



Review: The Bayesian Linear Model VIII



Review: The Bayesian Linear Model IX

Theorem:

- For a Gaussian prior on $\boldsymbol{\theta} \sim \mathcal{N}(\mathbf{0}, \tau^2 \mathbf{I}_p)$ and a Gaussian likelihood $y \mid \mathbf{X}, \boldsymbol{\theta} \sim \mathcal{N}(\mathbf{X}^\top \boldsymbol{\theta}, \sigma^2 \mathbf{I}_n)$, the resulting posterior is Gaussian: $\mathcal{N}(\sigma^{-2} \mathbf{A}^{-1} \mathbf{X}^\top \mathbf{y}, \mathbf{A}^{-1})$, with $\mathbf{A} := \sigma^{-2} \mathbf{X}^\top \mathbf{X} + \frac{1}{\tau^2} \mathbf{I}_p$.

Proof:

Plugging in Bayes' rule and multiplying out yields

$$\begin{aligned} p(\boldsymbol{\theta} \mid \mathbf{X}, \mathbf{y}) &\propto p(\mathbf{y} \mid \mathbf{X}, \boldsymbol{\theta}) q(\boldsymbol{\theta}) \propto \exp \left[-\frac{1}{2\sigma^2} (\mathbf{y} - \mathbf{X}\boldsymbol{\theta})^\top (\mathbf{y} - \mathbf{X}\boldsymbol{\theta}) - \frac{1}{2\tau^2} \boldsymbol{\theta}^\top \boldsymbol{\theta} \right] \\ &= \exp \left[-\frac{1}{2} \left(\underbrace{\sigma^{-2} \mathbf{y}^\top \mathbf{y}}_{\text{doesn't depend on } \boldsymbol{\theta}} - 2\sigma^{-2} \mathbf{y}^\top \mathbf{X}\boldsymbol{\theta} + \sigma^{-2} \boldsymbol{\theta}^\top \mathbf{X}^\top \mathbf{X}\boldsymbol{\theta} + \tau^{-2} \boldsymbol{\theta}^\top \boldsymbol{\theta} \right) \right] \\ &\propto \exp \left[-\frac{1}{2} \left(\sigma^{-2} \boldsymbol{\theta}^\top \mathbf{X}^\top \mathbf{X}\boldsymbol{\theta} + \tau^{-2} \boldsymbol{\theta}^\top \boldsymbol{\theta} - 2\sigma^{-2} \mathbf{y}^\top \mathbf{X}\boldsymbol{\theta} \right) \right] \\ &= \exp \left[-\frac{1}{2} \boldsymbol{\theta}^\top \underbrace{\left(\sigma^{-2} \mathbf{X}^\top \mathbf{X} + \tau^{-2} \mathbf{I}_p \right)}_{:=\mathbf{A}} \boldsymbol{\theta} + \sigma^{-2} \mathbf{y}^\top \mathbf{X}\boldsymbol{\theta} \right] \end{aligned}$$

This expression resembles a normal density - except for the term in red!

Review: The Bayesian Linear Model X

Note: We need not worry about the normalizing constant since its mere role is to convert probability functions to density functions with a total probability of one.

We subtract a (not yet defined) constant c while compensating for this change by adding the respective terms (“adding 0”), emphasized in green:

$$\begin{aligned} p(\boldsymbol{\theta}|\mathbf{X}, \mathbf{y}) &\propto \exp\left[-\frac{1}{2}(\boldsymbol{\theta}-\mathbf{c})^\top \mathbf{A}(\boldsymbol{\theta}-\mathbf{c}) - \mathbf{c}^\top \mathbf{A} \boldsymbol{\theta} + \underbrace{\frac{1}{2} \mathbf{c}^\top \mathbf{A} \mathbf{c}}_{\text{doesn't depend on } \boldsymbol{\theta}} + \sigma^{-2} \mathbf{y}^\top \mathbf{X} \boldsymbol{\theta}\right] \\ &\propto \exp\left[-\frac{1}{2}(\boldsymbol{\theta}-\mathbf{c})^\top \mathbf{A}(\boldsymbol{\theta}-\mathbf{c}) - \mathbf{c}^\top \mathbf{A} \boldsymbol{\theta} + \sigma^{-2} \mathbf{y}^\top \mathbf{X} \boldsymbol{\theta}\right] \end{aligned}$$

If we choose c such that $-\mathbf{c}^\top \mathbf{A} \boldsymbol{\theta} + \sigma^{-2} \mathbf{y}^\top \mathbf{X} \boldsymbol{\theta} = 0$, the posterior is normal with mean c and covariance matrix \mathbf{A}^{-1} . Taking into account that \mathbf{A} is symmetric, this is if we choose

$$\begin{aligned} \sigma^{-2} \mathbf{y}^\top \mathbf{X} &= \mathbf{c}^\top \mathbf{A} \\ \Leftrightarrow \sigma^{-2} \mathbf{y}^\top \mathbf{X} \mathbf{A}^{-1} &= \mathbf{c}^\top \\ \Leftrightarrow \mathbf{c} &= \sigma^{-2} \mathbf{A}^{-1} \mathbf{X}^\top \mathbf{y} \end{aligned}$$

as claimed.

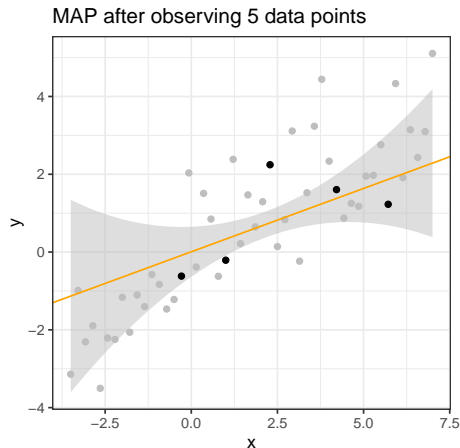
Review: The Bayesian Linear Model XI

- Based on the posterior distribution, $\boldsymbol{\theta} \mid \mathbf{X}, \mathbf{y} \sim \mathcal{N}(\sigma^{-2} \mathbf{A}^{-1} \mathbf{X}^\top \mathbf{y}, \mathbf{A}^{-1})$, we can derive the predictive distribution for a new observations \mathbf{x}_* .
- The predictive distribution for the Bayesian linear model, i.e. the distribution of $\boldsymbol{\theta}^\top \mathbf{x}_*$, is

$$y_* \mid \mathbf{X}, \mathbf{y}, \mathbf{x}_* \sim \mathcal{N}(\sigma^{-2} \mathbf{y}^\top \mathbf{X} \mathbf{A}^{-1} \mathbf{x}_*, \mathbf{x}_*^\top \mathbf{A}^{-1} \mathbf{x}_*).$$

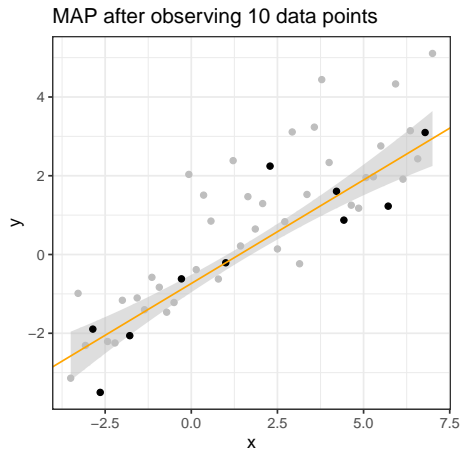
Note: This can be obtained by applying the rules for linear transformations of Gaussians.

Review: The Bayesian Linear Model XII



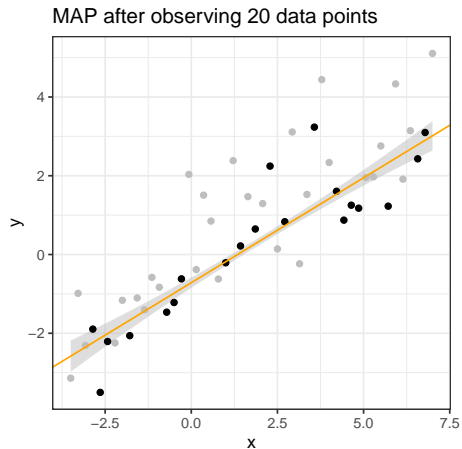
For every test input x_* , we get a distribution over the prediction y_* . In particular, we get a posterior mean (orange) and a posterior variance (the grey region, which equals \pm two times the standard deviation).

Review: The Bayesian Linear Model XIII



For every test input x_* , we get a distribution over the prediction y_* . In particular, we get a posterior mean (orange) and a posterior variance (the grey region, which equals \pm two times the standard deviation).

Review: The Bayesian Linear Model XIV



For every test input x_* , we get a distribution over the prediction y_* . In particular, we get a posterior mean (orange) and a posterior variance (the grey region, which equals \pm two times the standard deviation).

Summary: The Bayesian Linear Model

- By switching to a Bayesian perspective, we have not only point estimation for the parameter θ but also whole **distributions**.
- From the posterior distribution of θ , we can derive a predictive distribution for $y_* = \theta^\top \mathbf{x}_*$.
- We can perform online updates: the **posterior distribution** of θ can be updated whenever new datapoints are observed.
- In the next step, we would like go beyond the linear functions and develop a theory for functions with general shapes.