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On Relaxation Systems in Network and Systems Theory

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Declaration

This thesis is the result of my own work and includes nothing which is the outcome of work done in collaboration except where specifically indicated in the text. This dissertation meets the 15 000 word requirement.

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Abstract

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The analysis of nonlinear conductance-based models in biological systems is a fundamental imperative in neuromorphic engineering. However, recent work has highlighted a discrepancy between the state-space model and empirical input-output data of a component of one of the most renowned conductance-based models in biology: the Hodgkin-Huxley model of an excitable cell [1–4]. In an effort to address this issue, we propose and motivate an input-output framework for modeling and nonlinear circuit analysis predicated on *relaxation systems*.

Herein, we provide a primer in linear networks and dissipativity theory to contextualize the linear relaxation systems as the impedance functions of resistor-inductor and resistor-capacitor circuits. When viewed as functions on the right half-plane, it is found that relaxation systems are positive functions, mapping families of cones to their interior.

When viewed as Hankel operators on the domain of finite energy signals, relaxation systems are positive on families of exponentials, mapping signals decaying in the past to signals decaying in the future. Furthermore, the Hankel operators of relaxation systems exhibit cyclic monotonicity, hence are subgradients of a unique convex functional up to an additive constant. It is shown this functional specifies the energy stored internally by a relaxation system as predicted by dissipativity theory.

The thesis characterizes numerous input-output properties of linear relaxation systems. We conclude by arguing that the same properties, namely cone-invariance and cyclic monotonicity, readily extend to nonlinear systems and provides a foundation for which a nonlinear theory of relaxation may develop.

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Second, I want to express my gratitude to my mentor, colleague and friend, Tom Chaffey. When I joined Rodolphe's group, Tom had recently completed his PhD and accepted a research fellowship at Pembroke College. While Rodolphe was away in Leuven, Tom spent many hours with me at the whiteboard going through circuit analysis and monotone operator theory. Importantly, he taught me how to survey a technical field without getting lost in the mathematics, and that every good problem should be guided by a concrete example.

On the fourth floor of the Department of Engineering sits the Controls Group, and its members significantly improved my academic experience. To the group members: Jeff, Tony, Daniel, Andrea, Thomas, Thiago, Octave, Raphael and Charlie, I want to thank you for your friendship and discussions. Prior to Cambridge and meeting you, I never would have imagined myself reading about neuroscience and biological control in my spare time; these exciting topics now inspire my current research interests. To the academics: Fulvio Forni, Glenn Vinnicombe, Tim O'Leary, Somenath Bakshi, Malcolm Smith and Keith Glover, thank you for your exciting lectures and discussions in robust, nonlinear and biological control. I would be remiss if I failed to mention tea on Thursdays, which provided grounds for lively discussions ranging from hiking to fading memory operators.

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Contents

CHAPTER 1

Introduction

1.1 Context and Motivation

Of fundamental concern in linear circuit theory is the network synthesis problem: given an impedance function $Z(s)$ describing the desired external behavior of a passive one-port network, how does one construct an internally passive realization consistent with the desired response, if it exists? A key result in network theory states the driving-point impedance $Z(s)$ of a linear (lumped) passive one-port network is realizable by a network containing only resistors, inductors and capacitors if and only if $Z(s)$ is positive-real. Given a positive-real impedance function, construction of an internally passive network realization is afforded by one of the various synthesis techniques developed by Brune, Bott, Duffin and others [5–7].

The synthesis problem is well understood for linear circuits. However, for circuits constructed out of nonlinear elements, there exists no satisfactory answer to the synthesis problem. This gap in the theory was arguably ignored because of the power of feedback control. Indeed, one of control's first successes was in H.S. Black's use of high gain in negative feedback to force linear behavior of a nonlinear amplifier. Yet, many modern problems, motivated by the neuroscience and neuromorphic engineering communities, require a theory of nonlinear synthesis.

For instance, one of the most renowned examples of nonlinear circuit theory applied to biological systems is the Hodgkin-Huxley (HH) model describing the action potential of an excitable cell. One component of this conductance-based model relates current through the potassium ion channel to voltage across the cellular membrane by the set of nonlinear differential equations

$$\begin{aligned} \dot{\eta} &= \alpha(V)(1-\eta) - \beta(V)\eta, \quad \eta(0) = 0 \\ I &= g_K \eta^4 (V - V_K) \end{aligned} \tag{1.1}$$

where $\eta : \mathbb{R} \to \mathbb{R} \in [0,1]$ denotes the dimensionless gating (state) variable, $V : \mathbb{R} \to \mathbb{R}$ the input voltage across the cellular membrane and $I : \mathbb{R} \to \mathbb{R}$ the output current density of the potassium ion channel. Nonlinear functions $\alpha(\cdot), \beta(\cdot)$ explicitly depend on the input voltage V. Real-valued constants V_K, g_K denote the reversal potential and maximal conductance, respectively [1].

From a series of voltage-clamp experiments and subsequent curve fitting procedures, Hodgkin and Huxley obtained the following model parameters for Equation 1.1:

$$\begin{aligned} \alpha(V) &= 0.01\frac{V+10}{e^{\frac{V+10}{10}}-1} \\ \beta(V) &= 0.125e^{\frac{V}{80}} \end{aligned} \tag{1.2}$$

with $V_K = 12$ and $g_K = 36$, respectively. A reconstruction of the input-output data from the state-space model of Equation 1.1 with the identified parameters of Equation 1.2 is shown in Figure 1.1 for different constant voltage-clamp experiments with V ranging from -109 to -6 [mV]. As demonstrated, the input-output mapping from voltage to current suggests a monotone (non-increasing) admittance operator. Yet, it was recently shown that this proposed state-space model is *not monotone* [2–4]. This example demonstrates the prescribed nonlinear relationship by Hodgkin and Huxley between voltage and current is not realizable by the chosen nonlinear circuit elements.

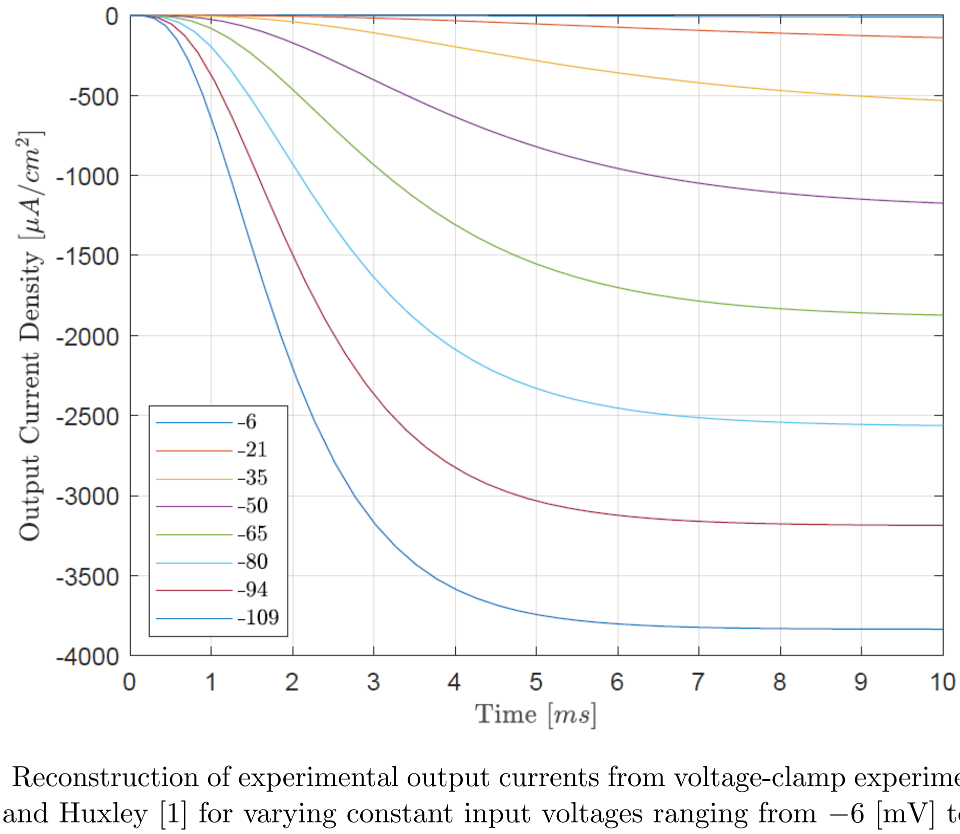


Figure 1.1: Reconstruction of experimental output currents from voltage-clamp experiments measured by Hodgkin and Huxley [1] for varying constant input voltages ranging from -6 [mV] to -109 [mV].

More specifically, the HH model highlights two broader challenges in nonlinear circuit theory: (i) a characterization of solutions to the nonlinear synthesis problem and (ii) a discrepancy between nonlinear input-output operators and their corresponding state-space models. Neuromorphic engineering promises to develop machines which perform computations akin to how biological systems process information. If such systems compute with nonlinear circuit elements, a nonlinear circuit theory addressing these challenges is needed for neuromorphic engineering to deliver on its promise.

The empirical results from Hodgkin and Huxley demonstrate a monotonic relationship between voltage and current which relaxes toward a solution over a sufficiently long time-horizon. Such dynamics are reminiscent of the *relaxation systems*, a class of (marginally) stable linear dynamical systems which derive from monotonic impulse responses. In network theory, they correspond to the resistor-inductor (RL) and resistor-capacitor (RC) circuits, hence contain only one type of energy storage element. Since the HH model consists of one capacitor in parallel with nonlinear conductors (resistors), it is plausible that developing a conductance-based model like Hodgkin and Huxley's via a nonlinear relaxation system would rectify the aforementioned difficulties, thereby partially addressing challenge (i).

Yet, a nonlinear theory of relaxation is absent in the literature. In fact, no agreed upon definition of what constitutes a nonlinear relaxation system exists. More surprisingly, it appears a complete theory of linear relaxation systems is lacking and unexplored. For instance, an interesting remark by Jan Willems in his landmark dissipativity paper [8] states:

> Relaxation systems ... have the very interesting property that for such systems one may always deduce the storage function from input/output experiments, i.e., the storage function is uniquely determined by the constitutive equations and by the qualitative assumptions that the system is externally and internally of the relaxation type.

In general dynamical systems, the state-dependent storage functional from dissipativity theory describes the energy stored by the system with respect to an input-output-dependent supply rate, which describes the instantaneous power supplied to the system. Via these two quantities, dissipativity theory links the external, input-output behavior of a system to its internal, state-space description [9]. However, since relaxation systems have a storage function determinable from input-output data alone, one obviates the need for a state-space description entirely, thereby addressing challenge (ii) above. This remark by Willems is not proven nor discussed further in [8], and the exploitation of this remarkable property of relaxation systems appears relatively under-utilized.

The similarities between linear relaxation systems and the nonlinear model developed by Hodgkin and Huxley suggests that developing a theory of nonlinear relaxation is a promising route forward in advancing tractable conductance-based models for applications in neuromorphic engineering. As a first step toward this goal, this thesis seeks to characterize linear relaxation systems purely in terms of their input-output characteristics. Input-output properties are sought because the current state-space characterizations of relaxation do not readily generalize to the nonlinear setting [8, 10]. Moreover, in observance of the discrepancies that exist in the HH model between the nonlinear state-space representation and its empirical data, we choose to focus on properties the data support. Herein, it will be argued that many of the input-output properties enjoyed by linear relaxation systems readily extend to the nonlinear setting, hence providing a foundation for which a nonlinear theory may develop.

1.2 Outline

This thesis is organized as follows. Chapter 2 discusses the preliminaries of linear network theory and positive-real functions. The relevant linear analysis techniques for signals and systems is then described with particular emphasis on square-integrable functions. Finally, passivity and dissipativity theory are briefly introduced along with their connections to the relaxation systems. Chapter 3 introduces Type-1 and Type-2 relaxation systems in the context of passive electrical networks, corresponding to resistor-inductor and resistor-capacitor circuits, respectively. Two characterizations are given, one by a Stieltjes-type integral and its rational approximation, and the other by a cone invariance property in the right half-plane. Furthermore, the convex structure of the set of relaxation functions within the set of positive-real functions is elucidated. Chapter 4 proves the remark of Willems, which claims the storage functional of a relaxation system is determined by input-output data. This necessitates a discussion of Hankel operators and cyclic monotonicity, where it is demonstrated that linear relaxation systems derive from convex functionals and hence may be regarded as gradient systems. Finally, Chapter 5 includes a brief discussion of the results herein and advocates how the results in the linear case can extend to nonlinear systems. We conclude with a discussion of possible generalizations of relaxation systems to the nonlinear setting, which is reserved for future work.

CHAPTER 2

Background

A primer on linear networks, passivity, positive-realness and their relation to the classical synthesis problem is presented. What follows is a survey of the fundamental mathematical tools for signals and systems needed throughout this thesis, with particular emphasis on the Lebesgue and Hardy spaces. Finally, the basics of dissipativity theory is presented to contextualize the results discussed in Chapter 4.

2.1 Linear Networks and Passivitiy

2.1.1 Preliminaries of Passive Electrical Networks

This section is concerned with one-port networks describing linear passive electrical circuits. The canonical example of a one-port is shown in Figure 2.1 where voltage $v(t)$ is measured across the terminals with current $i(t)$ entering one terminal and exiting the other. The driving-point *impedance* (sometimes referred to only as "impedance") is defined as $Z(s) = v(s)/i(s)$ with complex frequency variable $s = \sigma + j\omega$ where σ, ω are real-valued and j the imaginary unit. The driving-point *admittance* is similarly defined as $Y(s) = Z^{-1}(s) = i(s)/v(s)$. By viewing $i(s)$ as the input (output) and $v(s)$ as the output (input), the impedance (admittance) $Z(s)$ ($Y(s)$) is representative of a classical single-input, single-output (SISO) transfer function from control theory. Unless otherwise stated, this thesis will only consider SISO systems.

One-port networks consisting of resistors, inductors and capacitors which satisfy Kirchoff's current and voltage laws will be classified as RLC networks. Unless stated otherwise, this thesis will consider linear, lumped, time-invariant and passive RLC elements (see Figure 2.2). *Linearity* arises from constraining the port variables (i, v) by a linear relation. *Lumped* elements refer to those whose port variables obey either a memoryless transformation or an ordinary differential equation. *Time-invariance* refers to the constancy of the reactance R, inductance L, and capacitance C respectively. The impedances of the fundamental network elements are

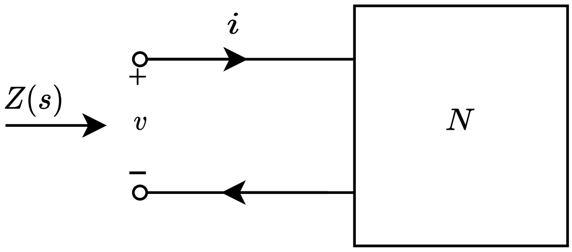


Figure 2.1: Exemplar one-port circuit network N with driving-point impedance $Z(s)$.

R, sL and $1/(Cs)$ respectively [11]. Passivity of an electrical component formalizes the notion that the total energy delivered to the component in a finite time interval is always non-negative. Therefore, a passive component cannot supply energy to the environment.

Definition 2.1.1 (Passivity [12], Def. 5)**.** Denote (u, y) as the input-output trajectory of a(n) network or electrical component. Let $t_0 \in \mathbb{R}$ denote an arbitrary time. Then a network (component) is *passive* if for all (u, y) and t_0, there exists a $K \in \mathbb{R}$ such that if $(\hat{u}, \hat{y})$ is also an input-output trajectory of the network (component) and $(\hat{u}(t), \hat{y}(t)) = (u(t), y(t))$ for all $t < t_0$, then

$$-\int_{t_0}^{t_1} \hat{u}^T(t)\hat{y}(t)dt \leq K \tag{2.1}$$

for all $t_1 \geq t_0$

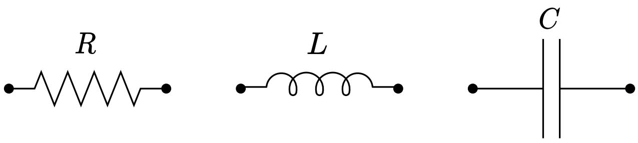


Figure 2.2: Fundamental passive circuit elements: resistor, inductor, and capacitor.

For RLC networks, passivity is achieved if and only if $R, L, C \geq 0$. Letting the input-output trajectories $(u(t), y(t))$ be the driving-point current $i(t)$ and voltage $v(t)$, the one-port network has the passivity property if the port variables (i, v) satisfy Definition 2.1.1 [11]. If $t_1 = \infty$, convergence of Equation 2.1 requires that $i(t)$ and $v(t)$ are Lebesgue square integrable. We will properly discuss Lebesgue spaces in the next section.

2.1.2 RL and RC Impedances

Classical network synthesis arguably began with Foster's work in characterizing the impedance functions of passive LC networks [13]. Cauer expanded Foster's work to RL and RC networks and provided forms of the allowable impedance functions of each [7, 14].

Theorem 2.1.1. The driving-point impedance of a passive network which contains only resistors and inductors, a (RL) network, takes the form

$$Z_1(s) = k_1 \frac{(s+p_0)(s+p_2)\cdots(s+p_{2n})}{(s+p_1)(s+p_3)\cdots(s+p_{2n\pm 1})} \tag{2.2}$$

with $k_1 \geq 0$, $0 \leq p_0 < p_1 < \cdots$ and $n \geq 0$. The RL impedances physically realizable are depicted in Figure 2.3 a), b) through a partial fraction expansion of $s^{-1}Z(s)$ or $Y(s)$.

Note that for a RL network, the impedance may be proper or improper. Alternatively, we may decompose Equation 2.2 into a sum of first-order terms, which is permissible because each pole is simple. To do so, we first factor out an s term (if $Z_1(s)$ is improper), then perform a partial fraction expansion.

$$Z_1(s) = A + A_0 s + \sum_{i=1}^{n} \frac{sA_i}{s+p_{2i\pm 1}} \quad A, A_0, A_i \geq 0 \tag{2.3}$$

Theorem 2.1.2. The driving-point impedance of a passive network which contains only resistors and capacitors, a (RC) network, takes the form

$$Z_2(s) = k_2 \frac{(s+p_1)(s+p_3)\cdots(s+p_{2n\pm 1})}{(s+p_0)(s+p_2)\cdots(s+p_{2n})} \tag{2.4}$$

with $k_2 \geq 0$, $0 \leq p_0 < p_1 < \cdots$ and $n \geq 0$. The RC impedances physically realizable are depicted Figure 2.3 c), d) through a partial fraction expansion of $Z(s)$ or $s^{-1}Y(s)$.

For a RC network, the impedance may be proper or strictly proper. As before, we may decompose Equation 2.4 into a sum of first-order terms. To do so, we first write $Z_2(s)$ as a constant and a strictly proper rational function. Partial fractions then yields

$$Z_2(s) = A + \sum_{i=0}^{n} \frac{A_i}{s+p_{2i}} \quad A, A_0, A_i \geq 0 \tag{2.5}$$

It is emphasized that for RL and RC impedances, $p_i \neq p_j$ for $i \neq j$. Furthermore, by the strict monotonic increase of p_i, the zeros and poles of both impedances are interlaced: a pole's neighbor's (if they exist) on a pole-zero plot cannot be other poles, and vice-versa for zeros. As evident from Equations 2.2 and 2.4 above, a simple inversion will map functions from one representation to the other, assuming the zero function is excluded. This is a manifestation of the duality principle in network theory and is listed below as a corollary.

Corollary 2.1.1 (Inversion of RL and RC Impedances)**.** Suppose $Z(s) \neq 0$. If $Z(s)$ is the driving point impedance of a RL network, then $Z^{-1}(s)$ has the form of a driving-point impedance of a RC network, and vice-versa.

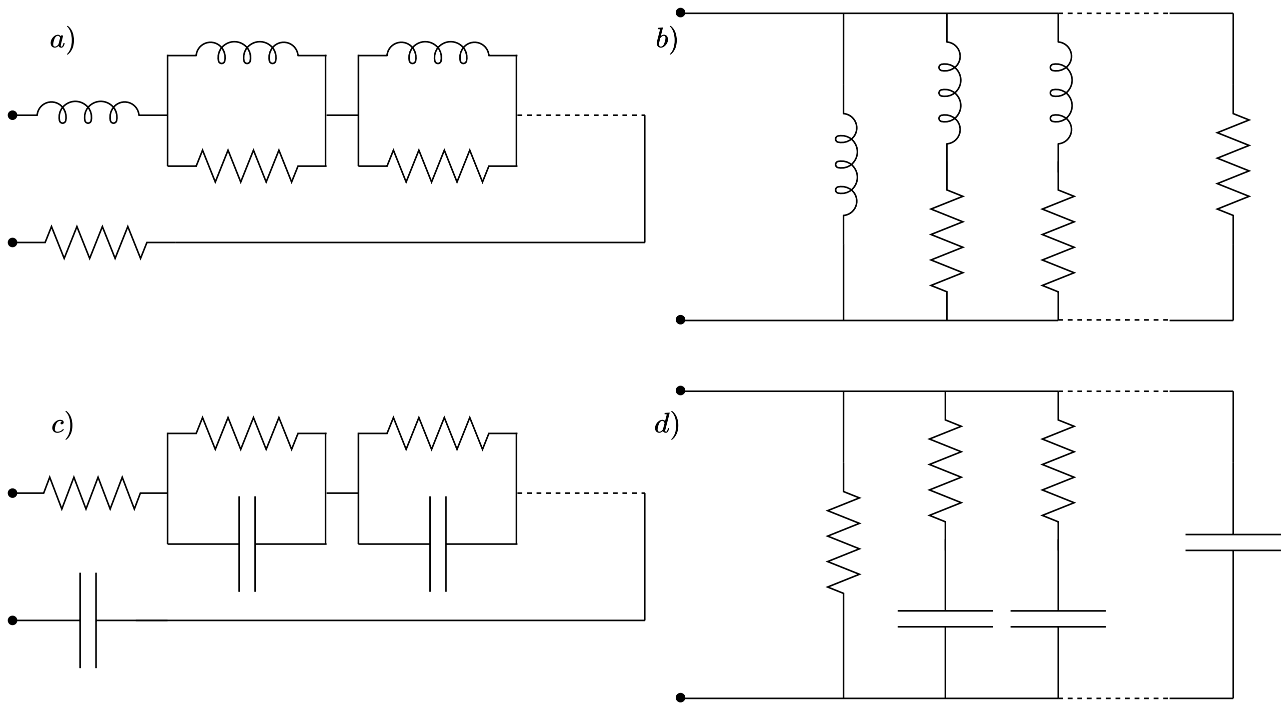


Figure 2.3: Canonical forms for RL and RC circuits.

2.1.3 Positive-Real Analytic Functions

So far, we have characterized the driving point impedances of RL and RC networks from considerations of the underlying circuit: this is called *network analysis*. The converse is the *network synthesis question*: given a prescribed impedance, construct an internally passive realization consistent with the desired response, if it exists. Necessary and sufficient conditions for realizability of a prescribed impedance function were given by Brune and Cauer, later simplified by Bott and Duffin by removing the need for transformers [5, 6]. We present the simplified result in the following theorem.

Theorem 2.1.3 (Realizability Theorem)**.** The driving-point impedance $Z(s)$ of a linear passive one-port network is realizable by a network consisting only of resistors, inductors and capacitors if and only if $Z(s)$ is positive real.

The key properties of positive-real functions $Z(s)$ are that *i*) their real parts are nonnegative for all frequencies and *ii*) the function's phase magnitude is less than equal to the input's phase magnitude when evaluated in the right half-plane [5, 7], i.e.,

$$|\arg Z(s)| \leq |\arg(s)| \text{ for all } |\arg(s)| \leq \frac{\pi}{2} \tag{2.6}$$

More formally, positive-realness is defined as follows.

Definition 2.1.2 (Positive-Real [11])**.** A function $G(s)$, not-necessarily rational, is *positive-real* if

1. $G(s)$ is real for s real and positive

2. $G(s)$ is analytic in the right half-plane, i.e., analytic for $Re(s) > 0$

3. $Re(G(s)) \geq 0$ for $Re(s) > 0$

Remark 2.1.1. Some authors present the third condition of positive-realness as $G^*(s)+G(s) \geq 0$ for $Re(s) > 0$, where * denotes the Hermitian (complex conjugate) transpose [11].

If a positive-real function $G(s)$ is bounded away from the imaginary axis for all $s = j\omega$, then we say it is strictly positive-real.

Definition 2.1.3 (Strictly Positive-Real)**.** A function $G(s)$ is *strictly positive-real* if $G(s - \epsilon)$ is positive-real for some $\epsilon > 0$.

With respect to the realizability theorem, positive-realness is necessary because, otherwise, one runs into contradictions. Indeed, suppose that $Z(s)$ were not positive-real. Then there are two possibilities: either a real-valued current produces a complex-valued voltage, which has no physical meaning, or the network can generate energy, violating the passivity assumption. Sufficiency of positive-realness is proved by Brune, where it is shown that one can find a passive network realization given any positive-real impedance [5].

Analyticity of positive-real functions in the right half-plane ($\mathbb{C}^+$) will play a prominent role later in this work. Loosely speaking, a function analytic in the right half-plane has no poles in that region, hence is marginally stable from a control-theoretic sense. The first and third conditions above constrain positive-real functions to have phase lags less than 90 degrees and enforce the relative difference between number of zeros to number of poles to be at most one. A useful result from Brune's work is that if $G(s)$ is positive-real, its inverse $G^{-1}(s)$ is also positive-real, a manifestation of the inverse relationship between impedance and admittance [5]. Finally, positive-realness requires all poles and zeros to exist in the closed left half-plane ($\overline{\mathbb{C}^-}$). A proof of this requirement is given in [5] page 19. To demonstrate positive-realness, consider the following basic example.

Example 2.1 ([11], 2.7.2)**.** Let $G(s) = 1/s$ be the impedance of a capacitor. It is readily apparent that $G(s)$ is real for s real and positive and is analytic in the right half-plane. If $Re(s) > 0$, one has

$$G^*(s) + G(s) = \frac{1}{s^*} + \frac{1}{s} = \frac{2Re(s)}{|s|^2} \geq 0$$

Thus, $G(s)$ is positive-real.

Thus far, we have discussed properties of positive-real functions. The form such functions take was given by Cauer, where he represented them by a Stieltjes-like integral [15].

Proposition 2.1.1 (Cauer's Representation)**.** A function $G(s)$ is *positive-real* if and only if it permits a representation of the form

$$G(s) = Cs + \int_0^\infty \frac{s}{s^2 + t} d\alpha(t) \tag{2.7}$$

for $Re(s) > 0$, where $C \geq 0$ and $\alpha(t)$ is a non-decreasing (monotonically increasing) real-valued function.

Approximating Equation 2.7 by a finite sum of terms results in a *rational* positive-real function, thereby yielding a circuit realization consisting of finitely many electrical elements [16].

Definition 2.1.4 (Rational Positive-Real [11, 17])**.** A *rational* function $G(s)$,

$$G(s) = \frac{p(s)}{q(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \tag{2.8}$$

where $|m - n| \leq 1$, is *positive-real* if

1. $G^*(j\omega) + G(j\omega) \geq 0$ for all $\omega \in \mathbb{R}$ with $j\omega$ not a pole of $G(s)$
2. $G(s)$ is analytic in the right half-plane, i.e., analytic for $Re(s) > 0$
3. All poles on the extended imaginary axis are simple and have residue which is positive semidefinite Hermitian. For a pole at $j\omega_0$, the residue is $\lim_{s \to j\omega_0}(s - j\omega_0)G(s)$ and for a pole at infinity, the residue is $\lim_{s \to \infty}(s^{-1})G(s)$.

   imaginary axis are simple. If $j\omega_0$ is a pole, then the residue $\lim_{s \to j\omega_0}(s - j\omega_0)G(s)$ is positive semidefinite Hermitian.

The coefficients of $p(s)$ and $q(s)$ in Equation 2.8 are non-negative for a positive-real function. Otherwise, $Re(G(s))$ may be negative for $Re(s) > 0$. Furthermore, a complex pole must be accompanied by its complex conjugate pair, if one exists. Otherwise the sum $G^*(j\omega) + G(j\omega)$ may be complex-valued.

In summary, for realizability of a linear passive system, it is necessary and sufficient that its impedance function be positive-real. Rational approximations of the analytic positive-real function of Equation 2.7 immediately give realizable circuit representations. Basic classes of scalar (real-rational) positive-real transfer functions include the following, which represent various RL, RC and RLC networks, respectively [18]:

1. $G(s) = as + b, \quad a, b \geq 0$
2. $G(s) = 1/(s + a), \quad a \geq 0$
3. $G(s) = (s + c)/(s^2 + as + b), \quad a \geq c \geq 0, b \geq 0$

This thesis will not cover synthesis techniques as it would take us too far afield. Interested readers are recommended to read the classical works of Brune [5], Bott and Duffin [6], and Darlington [19]. Comprehensive treatments of passive network synthesis are found in [20] and [21]; a modern perspective is presented by Morelli and Smith [22].

2.2 Linear Analysis

One of the prevailing techniques in systems theory, pioneered by George Zames, is treatment of dynamical systems as operators mapping signals to signals [23, 24]. This perspective implicitly requires the theorist to be conversant in functional analysis. In the following sections, we provide a refresher on the relevant definitions and theorems of signals and system spaces utilized throughout this work. The developments herein primarily follow the works of Paganini (Chapter 3) with supplements from Partington (Chapter 1) [25, 26].

2.2.1 Signal Spaces

One of the most important signal spaces used in systems theory are the Lebesgue spaces, consisting of equivalence classes of functions which agree everywhere except on sets of measure zero. They naturally generalize the p-norm for finite-dimensional spaces.

Definition 2.2.1 (Lebesgue Spaces)**.** For $p \geq 1$, let $L_p(-\infty, \infty) : \mathbb{R} \to \mathbb{C}$ denote the *Lebesgue* vector space of signals u that satisfy

$$\int_{-\infty}^{\infty} |u(t)|^p \, dt < \infty \tag{2.9}$$

where $|\cdot|$ denotes the modulus of a complex number. Such signals consequently have finite norm defined as

$$\|u\|_p := \left(\int_{-\infty}^{\infty} |u(t)|^p \, dt \right)^{\frac{1}{p}} \tag{2.10}$$

When $p = \infty$, the L_∞ space consists of signals u such that

$$\|u\|_\infty := \operatorname*{ess\,sup}_{t \in \mathbb{R}} |u(t)| \tag{2.11}$$

Throughout this thesis, we primarily focus on $L_p(-\infty, \infty)$ restricted to either the non-negative or non-positive axes.

Definition 2.2.2. Denote $L_p[0, \infty) : \mathbb{R} \to \mathbb{C}$ as

$$L_p[0, \infty) := \{u(t) \in L_p(-\infty, \infty) : u(t) = 0 \text{ for } t < 0\} \tag{2.12}$$

Likewise, denote $L_p(-\infty, 0] : \mathbb{R} \to \mathbb{C}$ as

$$L_p(-\infty, 0] \coloneqq \{u(t) \in L_p(-\infty, \infty) : u(t) = 0 \text{ for } t > 0\} \tag{2.13}$$

Alternatively, one could define both subspaces via the *projection operator* $P_\pm$, where $L_p[0, \infty) = P_+ (L_p(-\infty, \infty))$ and $L_p(-\infty, 0] = P_- (L_p(-\infty, \infty))$.

The interpretation is that the subspaces $L_p(\infty, 0]$ and $L_p[0, \infty)$ have support in the "past" and "future." We are interested in causal signals, hence will almost always utilize the non-negative real-axis as our domain. For this reason, we will refer to L_p as $L_p(-\infty, \infty)$ or $L_p[0, \infty)$: the distinction between the two should be clear based upon the context. Of particular importance in systems theory is the L_2 space because of all the L_p spaces, they are the only Hilbert space (complete inner product space). Due to ambiguity of which argument is conjugate linear for inner products over complex vector spaces, we give the definition of the inner product.

Definition 2.2.3 (Inner Product [27])**.** An *inner product* on a complex vector space X is a map

$$\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C} \tag{2.14}$$

such that for all $x, y, z \in X$, $\lambda, \mu \in \mathbb{C}$

$$\begin{aligned}
&\langle x, \lambda y + \mu z\rangle = \lambda\langle x, y\rangle + \mu\langle x, z\rangle \text{ (linear in the second argument)}\\
&\langle y, x\rangle = \overline{\langle x, y\rangle} \text{ (Hermitian symmetric)}\\
&\langle x, x\rangle \geq 0 \text{ with equality if and only if } x = 0 \text{ (positive semi-definite)}
\end{aligned}$$

The first two conditions of an inner product imply $\langle \lambda x + \mu y, z\rangle = \overline{\lambda}\langle x, z\rangle + \overline{\mu}\langle x, y\rangle$. Hence, in this thesis, the inner product is *antilinear* or *conjugate linear* in the first argument [27]. Some authors define conjugate linearity with respect to the second argument. Now we introduce the L_2 space.

Definition 2.2.4 (L_2 Space)**.** With $L_2(-\infty, \infty)$ defined as above, denote the inner product between $u, y \in L_2$ as

$$\langle u, y\rangle_2 \coloneqq \int_{-\infty}^{\infty} u^*(t) y(t) dt \tag{2.15}$$

where * denotes the Hermitian conjugate.

We explicitly labeled the space for which the inner product is defined via the subscript. Throughout this rest of this work, we may neglect this notation when the space will be evident.

It is often advantageous to study problems in the frequency-domain for which computation is easier or conceptually clearer. In particular, the L_2 space is relevant because they are easily treated in the frequency domain as we explain below.

Definition 2.2.5. Denote $\hat{L}_2(j\mathbb{R})$ as the complex inner product space consisting of functions $\hat{u} : j\mathbb{R} \to \mathbb{C}^n$ such that $\langle \hat{u}, \hat{u} \rangle_2^2 < \infty$. The inner product between $\hat{u}, \hat{y} \in \hat{L}_2(j\mathbb{R})$ is defined as

$$\langle \hat{u}, \hat{y} \rangle_2 := \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}^*(j\omega)\hat{y}(j\omega)d\omega \tag{2.16}$$

We will utilize the same notation for the norm and inner product on $\hat{L}_2$ as L_2 which should be clear whether the frequency- or time-domains are utilized. We now introduce the Fourier and Laplace transforms of L_2 signals.

Definition 2.2.6 (Fourier Transform)**.** Let $u \in L^2(-\infty, \infty) : \mathbb{R} \to \mathbb{C}^n$. The *Fourier transform* $\Phi : L^2(-\infty, \infty) \to \hat{L}_2(j\mathbb{R})$ of u is defined as

$$\hat{u}(j\omega) = \Phi u(t) = \int_{-\infty}^{\infty} u(t)e^{-j\omega t}d\omega \tag{2.17}$$

The *inverse Fourier transform* $\Phi^{-1} : \hat{L}_2(j\mathbb{R}) \to L^2(-\infty, \infty)$ is defined as

$$u(t) = \Phi^{-1}\hat{u}(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(j\omega)e^{j\omega t}d\omega \tag{2.18}$$

It can be shown that $u(t) = \Phi^{-1}(\Phi u)(t)$ for almost every t and $\hat{u}(j\omega) = \Phi(\Phi^{-1}\hat{u})(j\omega)$ for almost every ω. A key result linking the L_2 spaces in the time- and frequency-domains is the Parseval-Plancherel theorem.

Theorem 2.2.1 (Parseval-Plancherel)**.** If $u, y \in L^2(-\infty, \infty)$, then

$$\langle u, y \rangle_2 = \langle \Phi u, \Phi y \rangle_2 \tag{2.19}$$

and if $\hat{u}, \hat{y} \in \hat{L}^2(j\mathbb{R})$, then

$$\langle \hat{u}, \hat{y} \rangle_2 = \langle \Phi^{-1}\hat{u}, \Phi^{-1}\hat{v} \rangle_2 \tag{2.20}$$

Thus, the Fourier Transform is a unitary operator between $L^2(-\infty, \infty)$ and $\hat{L}^2(j\mathbb{R})$, hence the two spaces isomorphic. In particular, the time- and frequency-domain norms of L_2 are preserved under the Fourier operator. It is also of interest to investigate signals on the imaginary axis which analytically extend to the right half-plane $\mathbb{C}^+$. We now recall the Laplace transform.

Definition 2.2.7 (Laplace Transform)**.** Let $u \in L^2[0, \infty)$ and denote complex variable $s = \sigma + j\omega$. The *Laplace transform* $\Lambda : L^2[0, \infty) \to H_2(\mathbb{C}^+)$ of u is defined as

$$\hat{u}(s) := \lim_{T \to \infty} \int_0^T e^{-st}u(t)dt = \int_0^{\infty} e^{-st}u(t)dt \tag{2.21}$$

when the limit exists and set $\hat{u}(s) = 0$ otherwise. The *inverse Laplace transform* $\Lambda^{-1} : H_2(\mathbb{C}^+) \to L^2[0, \infty)$ is defined as

$$u(t) = \Lambda^{-1}\hat{u}(s) = \frac{1}{2\pi} \int_0^{\infty} e^{st}\hat{u}(s)d\omega \tag{2.22}$$

The Laplace integral transformation converges absolutely when $Re(s) > 0$, which motivates introduction of the Hardy space $H_2(\mathbb{C}^+)$, and more generally the $H_p(\mathbb{C}^+)$ spaces.

2.2.1.1 Hardy Spaces on the Right Half-Plane

Definition 2.2.8 (Hardy Spaces [26])**.** For $1 \leq p < \infty$, the *Hardy space* $H_p(\mathbb{C}^+)$ on the right half-plane is defined as the set of analytic functions $\hat{u} : \mathbb{C}^+ \to \mathbb{C}$ such that

$$\|\hat{u}\|_p = \left(\sup_{\sigma > 0} \int_{-\infty}^{\infty} |\hat{u}(\sigma + j\omega)|^p \, d\omega \right)^{\frac{1}{p}} < \infty \tag{2.23}$$

Definition 2.2.9 (Hardy Space $H_2(\mathbb{C}^+)$)**.** When $p = 2$ in Definition 2.2.8, the corresponding Hardy space $H_2(\mathbb{C}^+)$ is defined as the set of functions $\hat{u}$ such that

$$\|\hat{u}\|_2 = \left(\sup_{\sigma > 0} \int_{-\infty}^{\infty} |\hat{u}(\sigma + j\omega)\,|^2 d\omega \right)^{\frac{1}{2}} < \infty \tag{2.24}$$

Functions in Hardy space $H_2(\mathbb{C}^+)$ are not a-priori defined on the imaginary axis but have boundary values $\tilde{u}(j\omega) = \lim_{\sigma \to 0} \hat{u}(\sigma + j\omega)$ almost everywhere. In fact, a non-trivial result from functional analysis says that $\sqrt{2\pi}\|\tilde{u}\|_{\hat{L}_2} = \|\hat{u}\|_{H_2}$, which allows us to identify $\tilde{u}$ and $\hat{u}$ [26]. Therefore, $H_2(\mathbb{C}^+)$ is regarded as a closed subspace of $\hat{L}_2(j\omega)$ and inherits the inner product from $\hat{L}_2(j\mathbb{R})$:

$$\langle \hat{u}, \hat{v} \rangle_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{u}^*(j\omega)\tilde{v}(j\omega) d\omega \tag{2.25}$$

The Laplace transform $\Lambda : L_2[0, \infty) \to H_2(\mathbb{C}^+)$ gives an isometric isomorphism between the two spaces.

Theorem 2.2.2 (Paley-Weiner)**.** Suppose $u \in L_2[0, \infty)$. Then $\Lambda u \in H_2(\mathbb{C}^+)$. Conversely, if $\hat{u} \in H_2(\mathbb{C}^+)$, then there exists a $u \in L_2[0, \infty)$ such that $\Lambda u = \hat{u}$.

The norm on Hardy space $H_2(\mathbb{C}^+)$ defines a valid inner product, and since $H_2(\mathbb{C}^+)$ is complete, it is a *complex* Hilbert space. One may restrict this space to the subspace consisting only of analytic functions which assume real values for real arguments. This will be advantageous for a result in Chapter 4 and in defining the rational subsets below.

Definition 2.2.10 (Real-Valued Hardy Space $H_2(\mathbb{C}^+)$)**.** The set of functions in $H_2(\mathbb{C}^+)$ which are real for real arguments, denoted by $H_2^{\mathbb{R}}(\mathbb{C}^+)$ is defined as

$$H_2^{\mathbb{R}}(\mathbb{C}^+) \coloneqq \{\hat{u} \in H_2(\mathbb{C}^+) : \hat{u}(s) \in \mathbb{R} \text{ for } s \in \mathbb{R}\} \tag{2.26}$$

Under this restriction, $H_2^{\mathbb{R}}(\mathbb{C}^+)$ inherits the structure of a *real* Hilbert space [28].

The Hardy space of square integrable functions enjoys many remarkable properties. For instance, Hardy space $H_2(\mathbb{C}^+)$ is a reproducing kernel Hilbert space (RKHS). Loosely speaking, a RKHS $\mathcal{H}$ is a space in which if two bounded linear functionals $f, g \in \mathcal{H}$ converge in norm $\|f - g\|$, then $|f(x) - g(x)|$ converges point-wise for all $x \in \mathcal{H}$ [29]. Discussing the intricacies of RKHS theory would take us too far afield. However, a result which we will make use of later is the following:

Proposition 2.2.1 (Linear Span of Partial Kernel Evaluations)**.** If $\mathcal{H}$ is RKHS over space X, then $\mathcal{H} = \overline{\text{span}(k_x : x \in X)}$

The functions k_x are called the *partial evaluation kernels*, where a kernel is a unique positive definite function $K : X \times X \to \mathbb{C}$. For $H_2(\mathbb{C}^+)$, the kernel is called the Szegő kernel:

$$k_\lambda(\cdot) = \frac{1}{2\pi} \frac{1}{(\cdot) + \overline{\lambda}} \tag{2.27}$$

For us, this means that any element $G \in H_2(\mathbb{C}^+)$ can be written as follows

$$G(s) = \frac{1}{2\pi} \sum_j \alpha_j k_{\lambda_j}(s) = \sum_j \frac{\beta_j}{s + \overline{\lambda_j}} \tag{2.28}$$

where $\alpha_j \in \mathbb{C}$. Furthermore, for any number $\epsilon > 0$, there exists an integer $M >> 1$ such that

$$\left| G(s) - \sum_{j=1}^{M} \frac{\beta_j}{s + \overline{\lambda_j}} \right| < \epsilon \tag{2.29}$$

By Proposition 2.2.1, the sum in Equation 2.29 will converge to $G(s)$ as M tends to infinity.

The main benefit of utilizing the RKHS structure induced by $H_2(\mathbb{C}^+)$ is it provides a succinct representation for any element belonging to the space. One interpretation of $H_2(\mathbb{C}^+)$ is that it consists of signals which can be written as the limit of sums of Laplace-transformed complex exponentials with distinct poles. The reader should note the sum in Equation 2.28 is unordered and may consist of countably many terms, a result due to the infinite-dimensionality of the Hilbert space. In practice, one is often interested in a finite-dimensional representation; for example, network theorists utilize rational positive-real functions when designing and analyzing LTI lumped circuit models. Hence, it is of interest to characterize the set of rational functions in the Hardy space.

Definition 2.2.11 (Rational Hardy Space $H_2(\mathbb{C}^+)$)**.** The set of rational functions in $H_2(\mathbb{C}^+)$, denoted as $RH_2(\mathbb{C}^+)$, is defined as

$$RH_2(\mathbb{C}^+) := \left\{ \hat{u} \in H_2(\mathbb{C}^+) : \hat{u}(s) = \frac{p(s)}{q(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \right\} \tag{2.30}$$

where integers $m < n$ and each coefficient a_i, b_j are complex.

Clearly, $RH_2(\mathbb{C}^+)$ consists of strictly proper signals with no poles in the closed right half-plane $\overline{\mathbb{C}^+}$. Likewise, its readily apparent that $RH_2(\mathbb{C}^+) \subset H_2(\mathbb{C}^+)$.

A few more important remarks are in order. Firstly, the rationality assumption asserts every element $u \in RH_2(\mathbb{C}^+)$ may be represented by a finite number of terms by the Szegő kernel in Equation 2.28. Secondly, $RH_2(\mathbb{C}^+)$ is dense in $H_2(\mathbb{C}^+)$, so every element in $H_2(\mathbb{C}^+)$ is the limit of a sequence of functions in $RH_2(\mathbb{C}^+)$. While $RH_2(\mathbb{C}^+)$ is an inner product space with the inner product inherited from $H_2(\mathbb{C}^+)$, it is not a Hilbert space because it is not complete. Hence, an arbitrary limiting sequence of elements in $RH_2(\mathbb{C}^+)$ may converge to a non-rational element [25].

Before concluding our discussion on Hardy space $H_2(\mathbb{C}^+)$, we introduce one more subset. In signal processing applications, real-rational functions are utilized because we require real-valued signals in design and analysis; for example, a complex current $\hat{i}(s) = 1/(s + (1 + j)) \in RH_2(\mathbb{C}^+)$ yields a complex-valued time-domain signal of $i(t) = e^{-(1-j)t} = e^{-t}\left(cos(t) - jsin(t)\right)$ which is non-physical. In practice, one would add together combinations of such signals to produce a real-valued function. Hence, it is pragmatic to work with real-rational functions from the onset, motivating the following set.

Definition 2.2.12 (Real-Rational Hardy Space $H_2(\mathbb{C}^+)$)**.** The set of real-rational functions in $H_2(\mathbb{C}^+)$, denoted as $RH_2^{\mathbb{R}}(\mathbb{C}^+)$, is defined as

$$RH_2^{\mathbb{R}}(\mathbb{C}^+) \coloneqq \left\{ \hat{u} \in H_2(\mathbb{C}^+) : \hat{u}(s) = \frac{p(s)}{q(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \right\} \tag{2.31}$$

where integers $m < n$ and each coefficient a_i, b_j are real.

The restriction to real coefficients of the polynomials $p(s)$ and $q(s)$ enforces functions in $RH_2^{\mathbb{R}}(\mathbb{C}^+)$ to take on real values for real arguments, hence inherits the structure of a *real* inner product space [28]. Similar to before, one has that $RH_2^{\mathbb{R}}(\mathbb{C}^+) \subset H_2^{\mathbb{R}}(\mathbb{C}^+)$.

In comparison to $RH_2(\mathbb{C}^+)$, all poles of a signal $\hat{u}(s)$ off the negative-real axis necessarily come in complex-conjugate pairs. It is readily seen that this set is not dense in $H_2(\mathbb{C}^+)$, nor complete either. However, it is an inner product space [30].

A real-rational function $\hat{u}(s) = p(s)/q(s)$ can still be factored over the complex numbers by the Fundamental Theorem of Algebra. This is in line with the Szegő kernel representation of a rational function in $H_2(\mathbb{C}^+)$. For example,

$$G(s) = \frac{b_1 s + b_0}{a_2 s^2 + a_1 s + a_0} = \frac{2s + (4 - 2\sqrt{2})}{s^2 + 4s + 6} = \frac{1 + j}{s + (2 - j\sqrt{2})} + \frac{1 - j}{s + (2 + j\sqrt{2})}$$

In summary, by means of the unitary Fourier transform operator, the time-domain space of $L_2(-\infty, \infty)$ is isomorphic to $L_2(j\mathbb{R})$. By choosing the subspace of signals that are 0 for negative time, we may identify the natural subspace $L_2[0, \infty)$ of $L_2(-\infty, \infty)$. The Hardy space $H_2(\mathbb{C}^+)$

arises naturally via the Laplace transformation or Fourier transform with analytic extension to the right half-plane [25]. $H_2(\mathbb{C}^+)$ is a RKHS and enjoys a rich theory in interpolation and approximation [29]; we will utilize a few of these results for proofs in the subsequent sections.

2.2.2 Operators

Operators map signals to signals and provide a powerful mathematical abstraction of a dynamical system. Since signals are often represented as finite-dimensional vectors, operators take the form of matrix-valued functions. One of the first operators one studies is systems theory is the multiplication operator.

Definition 2.2.13. Denote $\hat{L}_\infty(j\mathbb{R})$ as the space of matrix valued-functions $j\mathbb{R} \to \mathbb{C}^{m\times n}$ such that

$$\|G\|_\infty := \operatorname*{ess\,sup}_{\omega\in\mathbb{R}} \overline{\sigma}\left(G(j\omega)\right) < \infty \tag{2.32}$$

where $\overline{\sigma}$ denotes the singular value.

It turns out that $\hat{L}_\infty(j\mathbb{R})$ represents the set of bounded linear operators on $\hat{L}_2(j\mathbb{R})$.

Theorem 2.2.3. Every function $\hat{G} \in \hat{L}_\infty(j\mathbb{R})$ defines a bounded linear operator $M_{\hat{G}} : \hat{L}_2(j\mathbb{R}) \to \hat{L}_2(j\mathbb{R})$ via the relationship

$$(M_{\hat{G}}\hat{u})(j\omega) = \hat{G}(j\omega)\hat{u}(\omega) \tag{2.33}$$

and $\|M_{\hat{G}}\|_{\hat{L}_2\to\hat{L}_2} = \|\hat{G}\|_\infty$.

The operator $M_{\hat{G}}$ is the *multiplication operator* and $\hat{G}(j\omega)$ is the *frequency response*. A well-known result is that every multiplication operator defines a time-invariant operator $G = \Phi^{-1}M_{\hat{G}}\Phi$. In fact, the set of time-invariant operators $G : L_2(-\infty,\infty) \to L_2(-\infty,\infty)$ exactly corresponds to the set of functions in $\hat{L}_\infty(j\mathbb{R})$.

We are often concerned with causal systems, which informally are systems whose past input affects future output. In the time-domain, a time-invariant operator $G : L_2(-\infty,\infty) \to L_2(-\infty,\infty)$ is causal if and only if it maps $L_2[0,\infty)$ to $L_2[0,\infty)$. By the isometric isomorphism of the Fourier Transform, it follows that a time-variant operator $G = \Phi^{-1}M_{\hat{G}}\Phi$ is causal if and only if $M_{\hat{G}}$ maps $H_2(\mathbb{C}^+)$ to $H_2(\mathbb{C}^+)$. To characterize the causal, time-invariant operators in the frequency domain, we must introduce $H_\infty(\mathbb{C}^+)$.

Definition 2.2.14. The $H_\infty(\mathbb{C}^+)$ space is a closed subspace in $\hat{L}_\infty(j\mathbb{R})$ with functions that are analytic in the open right half-plane and bounded on the imaginary axis. A function $\hat{G}$ exists in $H_\infty(\mathbb{C}^+)$ if

$$\|\hat{G}\|_\infty := \sup_{Re(s)>0} \overline{\sigma}\left(\hat{G}\right) = \operatorname*{ess\,sup}_{\omega\in\mathbb{R}} \overline{\sigma}\left(\hat{G}(j\omega)\right) < \infty \tag{2.34}$$

Similar as before, we may restrict ourselves to rational and real-rational matrix-valued functions in $H_\infty(\mathbb{C}^+)$, denoted as $RH_\infty(\mathbb{C}^+)$ and $R_{\mathbb{R}}H_\infty(\mathbb{C}^+)$, respectively. Such matrices have elements consisting of proper functions with no poles in the closed right half-plane. We now present the final theorem for causal LTI operators in the frequency domain: all LTI operators on $L_2[0, \infty)$ are represented by $H_\infty(\mathbb{C}^+)$ functions.

Theorem 2.2.4. Every $\hat{G} \in H_\infty(\mathbb{C}^+)$ defines a causal, time invariant operator G on $L_2[0, \infty)$, where $y = Gu$ is defined as $\hat{y}(j\omega) = \hat{G}(j\omega)\hat{u}(j\omega)$. Conversely, if the bounded linear operator G over $L_2[0, \infty)$ is time-invariant, then there exists $\hat{G} \in H_\infty(\mathbb{C}^+)$ such that $y = Gu$ satisfies $\hat{y}(j\omega) = \hat{G}(j\omega)\hat{u}(j\omega)$ for all $u \in L_2[0, \infty)$

For the remainder of this thesis, we will be loose with notation of G vs. $\hat{G}$ for the time and frequency domains when the distinction is clear. Note that in the single-input-single-output (SISO) setting, an operator is a scalar-valued function instead of matrix-valued.

2.3 Passivity and Dissipativity

Equipped with the relevant mathematical tools, we now are set to embark on a discussion of passivity and dissipativity theory in systems and control, both of which utilize the *state-space* modeling and analysis framework. This approach adopts ideas from nonlinear mechanics and variational calculus, leading to well known stability techniques like Lyapunov stability. Informally, the output of a *causal* dynamical system depends not only on the present input, but also past inputs. The *states* of a system are a set of non-unique variables that represent this memory of past inputs. Herein, we investigate LTI state-space systems of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{2.35}$$

where $x(t) \in \mathbb{R}^n$ denotes the *state*, $u(t) \in \mathbb{R}^m$ denotes the input and $y(t) \in \mathbb{R}^p$ denotes the output. Matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$ and $D \in \mathbb{R}^{p \times m}$ describe the dynamics of the system. In network theory, these constitute Kirchoff's current and voltage laws. The system is called *stable* if A is Hurwitz and *minimal* if (A, B) and (A, C) are controllable and observable, respectively [31, 32]. Using complex variable s and the state-space equations, one obtains the transfer function of the system $G(s) = C(sI - A)^{-1} + D$, which is the Laplace transform of the system's impulse response $g(t) = Ce^{At}B + D\delta(t)$, where $\delta(t)$ is the Dirac delta function.

In contrast, the notions of inputs $u(s)$ and outputs $y(s)$, transfer functions $G(s) = y(s)/u(s)$, passivity, positive-realness and realizability from network theory were central in developing the frequency-based, *input-output* control techniques of Nyquist and Bode: collectively, these methods form what is normally referred to as classical control. By construction, they are readily applicable to linear time-invariant (LTI) dynamical systems.

For several years during the 1950's, control theory was largely divided among these two disparate approaches. Introduction of passivity by Popov in the Luŕe problem brought the two traditions closer together, but it was not until the celebrated Kálmán-Yakubovich-Popov (KYP) Lemma that the connection was made explicit. For linear systems, the KYP Lemma posits the equivalence of a positive-real transfer function (passive system) and a quadratic Lyapunov function of the system's state, where a Lyapunov function is defined as follows:

Definition 2.3.1 (Lyapunov Function)**.** Let X denote the state-space. A function $V : X \to \mathbb{R}$ is called a *Lyapunov function* in the neighborhood of equilibrium point x_{eq} if

1. V is continuous at x_{eq}.
2. there exists a continuous function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ with $\alpha(\sigma) > 0$ for $\sigma > 0$ such that $V(x) - V(x_{eq}) \geq \alpha(\|x - x_{eq}\|)$ for all x in a neighborhood of x_{eq}.
3. V is monotone non-increasing along solutions in the neighborhood of x_{eq}.

A Lyapunov function may be interpreted as an energy functional of the system which decreases along trajectories. Crucially, Lyapunov functions apply to *closed* dynamical systems. If one considers *open* systems, i.e., systems with inputs and outputs, one can generalize the Lyapunov function to a *storage function*. From this perspective, passivity of a state-space model is available.

Definition 2.3.2 (Passivity, State-Space)**.** A state-space model, as in Equation 2.35, is termed *passive* if there exists a non-negative *storage function* $V : X \to \mathbb{R}$ such that

$$V(x(T)) - V(x(0)) \leq \int_0^T u(t)^T y(t) dt \tag{2.36}$$

Passivity of a state-space model therefore states that the change of storage is upper-bounded by the energy supplied. A well known result is that LTI models always admit quadratic storages of the form $V(x) = x^T P x$ where P is a symmetric positive definite matrix.

Generalizing Equation 2.36 further brings one into the realm of Jan Willems' *dissipativity theory* [8, 33]. Introduced in 1972, dissipativity generalizes the passivity property of circuit theory to dynamical systems which dissipate energy [9]. Standing at the intersection of controls engineering, physics and systems theory, it relates the external behavior of a system to its internal structure, and unified results from electrical networks, stability theory, optimal control and irreversible thermodynamics under a common framework. From a historical perspective, it is likely that [8] first introduced the linear matrix inequality (LMI) into systems theory [34].

Dissipativity theory relies on the notion of a storage functional and supply rate. Intuitively, the storage function represents stored energy, whereas the supply rate supplies energy to the system.

Definition 2.3.3 (Dissipativity)**.** Let X denote the state-space and $\mathcal{U}, \mathcal{Y}$ and input-output spaces, respectively. A dynamical system is called *dissipative* if, with respect to a state-dependent, non-negative *storage function* $S : X \to \mathbb{R}^+$ and an input-output-dependent *supply* *rate* $w : \mathcal{U} \times \mathcal{Y} \to \mathbb{R}$, the *dissipation inequality* holds:

$$S(x(T)) - S(x(0)) \leq \int_0^T w(u(t), y(t))dt \tag{2.37}$$

for all inputs $u \in \mathcal{U}$, $x_0 = x(0) \in X$ and times $T \geq 0$.

It is clear that the state-space definition of passivity from Equation 2.36 coincides with the dissipativity definition if $w(u, y) = u^T y$; this special supply rate is termed the *passive supply*. Therefore, a dissipative dynamical system's storage function represents its internal energy, which always is a fraction of total energy supplied. In other words, the Lie derivative of the storage is less than or equal to the supplied energy.

$$\dot{S}(x) \leq w(u, y) \tag{2.38}$$

From this perspective, dissipativity is the natural generalization of Lyapunov theory to open systems since one can show that the storage functional $S(x)$ satisfies the criteria of a valid Lyapunov function [31, 33, 34].

2.4 Summary

This section was concerned with discussing the preliminaries of linear network theory in the context of passive systems and positive-real functions. Functions which are analytic or rational positive-real have many constraints in relation to their pole-zero locations and properties when evaluated near the imaginary axis. The realizability theorem, providing necessary and sufficient conditions for network synthesis, demonstrated an equivalence between externally passive behavior and positive-realness of a network's impedance function for LTI systems. Basic results from functional analysis for signals and systems were presented with a focus on reproducing kernel Hilbert space $H_2(\mathbb{C}^+)$, which is isometrically isomorphic to the Lebesgue space of square integrable functions $L_2[0, \infty)$. Finally, classic results of passivity in systems and control theory were presented to provide context for dissipativity, the generalization of passivity beyond the passive supply rate and the Lyapunov theory of open systems.

2.5 Discussion

Energy functionals are paramount in both Lyapunov and dissipativity theory, and comparing the two warrants further discussion. In both frameworks, existence of a functional satisfying certain decay properties is sufficient to qualitatively describe the behavior of a dynamical system. While powerful as an analysis technique, finding such a functional is difficult and often relies on ad-hoc approaches or expert knowledge. Likewise, uniqueness of the functional is not guaranteed; indeed, it is well known that the set of valid storages forms a convex set [33].

However, for the *relaxation systems*, a class of (marginally) stable systems consisting of one type of energy storage element, a storage function is readily constructed from the system's input-output trajectories; indeed, Chaffey et al. demonstrate that for strictly stable relaxation systems, one may define a storage functional from a system's past inputs [35].

Definition 2.5.1 (External Storage Functional, Proposition 1, [35])**.** Consider the state-space representation of a system represented by Equation 2.35. Given an input $u \in L_2(\mathbb{R}, \mathbb{R}^m)$ and a time $t \in \mathbb{R}$, denote by u_t the truncation of u to the time-axis $(-\infty, t]$. A *external storage functional* is any functional S mapping a truncated signal u_t into $\mathbb{R}^+$ satisfying

$$\begin{aligned} S(u_{t_0}) - S(u_{t_1}) &\geq -\int_{t_0}^{t_1} u^T(t)y(t)dt \\ \dot{S}(u_t) &= \frac{dS}{dt}(u_t) \leq u^T(t)y(t) \end{aligned}$$

for all $t \in \mathbb{R}$, $t_1 \geq t_0$ and input output trajectories $(u(t), y(t))$ of the system.

The difference between the equations of Definition 2.5.1 and Equations 2.37 and 2.38 is one of input-dependency versus state-dependency. In recent years, Hughes showed that the external storage functional is indeed a valid storage function, in the context of dissipativity theory, with the passive supply rate [12, 36]; hence is passive in accordance with Definition 2.3.2.

Existence of an external storage for relaxation systems has ramifications for the remainder of this thesis and is discussed in detail in Chapter 4. To contextualize this result, a characterization of the relaxation systems is first needed, which is the basis of the next chapter.

CHAPTER 3

Characterizations of Relaxation Systems

An introduction to LTI relaxation systems of Type-1 and Type-2 is presented, along with a brief survey of related work. Next, analytic and real-rational descriptions of relaxation systems are given based upon Cauer's representation of a positive-real function. The convex structure of positive-real analytic functions is described with Type-1 and Type-2 relaxation systems as subsets. Finally, cone-invariance (positivity properties) of the relaxation systems in the right half-plane are delineated.

3.1 Introduction to Relaxation and Related Work

Relaxation systems characterize marginally stable systems containing only one type of energy storage element; in electrical circuits, the relaxation systems correspond to RL and RC networks, hence are positive-real. Originally studied in the context of thermodynamics and viscoelasticity (see Figure 3.1 for a prototypical relaxation system from the viscoelasticity field), there is renewed interest in relaxation systems for applications in soft-robotics, memristive devices, and synthetic biology [37].

One of the first rigorous investigations of relaxation systems was conducted by Meixner in the context of linear passive systems [38]. Motivated by various relaxation phenomena in viscoelasticity and acoustics, he sought to characterize which systems will produce a harmonic response for a give harmonic stimulus. In answering this question, Meixner mathematically described relaxation systems "of the first and second kind" via a Laplace-Stieltjes transform of a nowhere decreasing "spectral function." The transform yielded a completely monotonic function which was utilized to arrive at analytic descriptions (Equation 3.2 and 3.3 of [38]) for both kinds of relaxation functions. Complete monotonicity is a key characteristic of relaxation systems and is a recurrent motif of this chapter. We remark that Meixner likened the relaxation

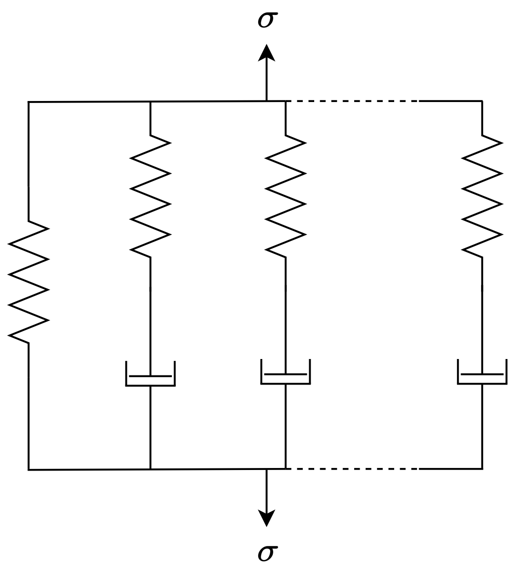


Figure 3.1: Exemplar relaxation system: the Maxwell–Wiechert model of linear viscoelasticity, comprised of an arbitrary number of spring-dashpot interconnections under stress σ.

functions of "the first kind" to RC-networks and "the second kind" to RL-networks.

Going beyond analytic functions, Zemanian investigated passive networks from an operator theoretic perspective: instead of viewing R, L and C as positive numbers, he studied them as positive bounded linear operators acting on a Hilbert space $\mathcal{H}$ [39]. He later adapted this framework to Meixner's RL and RC relaxation systems, characterizing them by their numerical ranges. Zemanian also showed that an analytic function G agreed with his definitions of relaxation only if an inner product of the form $\langle s^{-1}G(s^2)h, h\rangle$ and $\langle sG(s^2)h, h\rangle$ for all $h \in \mathcal{H}$ satisfied a positivity property. Somewhat confusingly, he numerically subscripts the RL-networks with "1" and the RC-networks with "2," opposite that of Meixner.

When Jan Willems introduced the dissipativity framework, Meixner's relaxation systems reappeared at the end of the second article [8]. He only addressed the RC-type relaxation systems and defined them by having a completely monotonic (symmetric) impulse response. It was shown that relaxation systems are highly structured: they admit state-space realizations externally and internally symmetric. External symmetry corresponds to the reciprocity property of electrical networks whereas internal symmetry represents how each energy storage element is of the same type [40]. A unique property of the relaxation systems, discovered by Willems, is that their storage is determined via a Hankel operator mapping of the past input; therefore, the storage is input-dependent, rather than state-dependent.

Recently, Pates et al. utilized the highly-structured state-space realizations of relaxation systems to solve H_∞-type optimal control problems [10]. It was shown that the resultant controller is sparse and of the relaxation type, thus applicable to large-scale problems like inductive-networks. Grussler and Sepulchre studied the variation-diminishing properties of discrete-time relaxation systems and found that they are Hankel totally positive [41].

Closest to the work presented in this thesis is from Chaffey et. al. [35]. Therein, the authors demonstrate an equivalence between a stable state-space system being of the relaxation type and cyclic monotonicity of the corresponding Hankel operator. This equivalence permits the authors to construct external storage functionals for the system from past input trajectories as detailed in Definition 2.5.1. Despite the numerous similarities between [35] and this work, a key difference is that we develop a theory of external storage functionals in the Laplace-domain, rather than through a state-space description; therefore, our approach is purely input-output. This is advantageous because state-space methods do not readily extend themselves to the nonlinear setting, whereas input-output approaches, predicated on properties like cyclic monotonicity which are well-defined for nonlinear systems, do.

Relative to the prior work aforementioned, this thesis expands the theory of linear relaxation by seeking a characterization in terms of cone invariance properties (as will be described in later sections and Chapter 4). Throughout this work, we have designated relaxation systems of Type-1 for RL networks and Type-2 for RC networks with respect to the impedance function $Z(s) = v(s)/i(s)$. Indeed, due to the duality of network theory, the network response and its classification as Type-1 or Type-2 depends upon the driving-point input, and whether one investigates impedance or admittance. In discussion, we will present Type-2 relaxation systems first because more results exist for this class of functions in the literature.

Before beginning our discussion, we denote various sets of functions.

$$\begin{aligned}
\overline{\mathcal{P}} &\coloneqq \{\text{set of positive-real functions}\} \\
\mathcal{P} &\coloneqq \{\text{set of rational positive-real functions}\} \\
\overline{\mathcal{R}_i} &\coloneqq \{\text{set of relaxation functions of Type-}i\} \\
\mathcal{R}_i &\coloneqq \{\text{set of rational relaxation functions of Type-}i\} \\
\overline{\mathcal{R}_{is}} &\coloneqq \{\text{set of strictly positive-real relaxation functions of Type-}i\} \\
\mathcal{R}_{is} &\coloneqq \{\text{set of strictly positive-real rational relaxation functions of Type-}i\}
\end{aligned}$$

It is remarked that a rational positive real function is necessarily real-rational. Restriction to strictly positive-real functions in sets $\overline{\mathcal{R}_{is}}$ and $\mathcal{R}_{is}$ eliminate functions with poles on the imaginary axis, e.g., the differentiator s and integrator $1/s$.

3.2 Stieltjes-Type Integrals of Relaxation Systems

3.2.1 Type-2 Relaxation Systems

Type-2 relaxation systems arise as the driving-point impedance of RC networks and as the driving-point admittance of RL networks. We have already encountered the form these systems take on in Equation 2.4. However, we will approach relaxation systems of Type-2 from the perspective of completely monotonic functions. Meixner connected these two concepts via his spectral function, whereas Willems via their impulse response behavior [8, 38].

Definition 3.2.1 (Completely Monotonic Function)**.** A function $g : [0, \infty) \to \mathbb{R}$ is completely monotonic if for all $n \in \mathbb{N}$ and all $t \in (0, \infty)$,

$$(-1)^n \frac{d^n}{dt^n} g(t) \geq 0 \tag{3.1}$$

Differentiability holds for all $t \in (0, \infty)$. Continuity holds for all $t \in [0, \infty)$.

For example, an elementary completely monotonic function is e^{-pt} for $p, t \geq 0$. From a control theoretic point of view, the signal corresponds to a (marginally) stable first-order lag. The class of completely monotonic functions were originally studied in the context of the Stieltjes moment problem. It asks: what property should a sequence of elements $\{g_i\}_{i=0}^{\infty}$ have such that

$$g_i = \int_0^{\infty} t^i dM(t) \tag{3.2}$$

where $M(t)$ is a non-decreasing function of bounded variation; in measure-theoretic terms, $M(t)$ is a positive Borel measure [42, 43]. We are interested in a slightly different problem, namely characterizing functions $g(t)$ rather than sequences $\{g_i\}$. It is clear the Laplace transform will play a role; indeed, Equation 3.2 is the discrete analog of the Laplace transform. A change of variables from $i \to s$ and $t \to e^{-p}$ in Equation 3.2 yields

$$g_s = \int_0^{\infty} e^{-sp} d\left[-M(e^{-p})\right] \tag{3.3}$$

Complete monotonicity is a well-studied property of functions and Bernstein gave necessary and sufficient conditions for a function to exhibit complete monotonicity in terms of a Stieltjes integral.

Theorem 3.2.1 (Bernstein-Widder[43, 44])**.** A function $g : [0, \infty) \to \mathbb{R}$ is completely monotonic if and only if

$$g(t) = \int_0^{\infty} e^{-tp} dM(p) \tag{3.4}$$

where $M(p)$ is bounded and non-decreasing. The integral converges for $0 \leq t < \infty$

For a proof of Bernstein's theorem, see Widder [43] page 160. Let $g(t)$ be completely monotonic and denote $G(s) = (\Lambda g)(t)$ by its Laplace transform. To stay consistent with the literature, we replace the generating function $M(p)$ with $\alpha(p)$. Then from Bernstein's theorem, we have the following results within the region of convergence of the Laplace transform:

$$g(t) = \int_0^\infty e^{-tp} d\alpha(p) \quad t \geq 0$$
$$G(s) = \int_{0^+}^\infty e^{-st} g(t) dt \quad Re(s) > 0$$

Since $\alpha(p)$ is non-decreasing, by Theorem $4d$ (Widder pg. 335), we may write $G(s)$ as the Stieltjes integral

$$G(s) = \int_0^\infty \frac{d\alpha(t)}{s+t} \tag{3.5}$$

for any s not belonging to the negative real axis. One should note the resemblance of Equation 3.5 with Cauer's representation of positive-real functions ($G \in \overline{\mathcal{P}}$) from Equation 2.7. We include it here again for reference:

$$P(s) = Cs + \int_0^\infty \frac{s}{s^2+t} d\alpha(t) \tag{3.6}$$

where $C \geq 0$ and $\alpha(t)$ is a non-decreasing real function.

If $G(s)$ is in the form of Equation 3.5, one can show that $lim_{s\to\infty} G(s) = 0$ and $sG(s^2)$ is positive real [45]. Likewise, $G(s)$ is itself positive-real. This corroborates with the definition provided by Zemanian [39] and Willems characterization of $sG(s^2)$ being lossless, which also is positive-real [8]. It is clear the condition of $lim_{s\to\infty} G(s) = 0$ is representative of $G(s)$ being strictly proper. Conversely, if $lim_{s\to\infty} G(s) \neq 0$, then we may write

$$sG(s^2) \text{ is positive real} \iff G(s) = C + \int_0^\infty \frac{d\alpha(t)}{s+t} \quad C \geq 0 \tag{3.7}$$

Definition 3.2.2 (Type-2 Relaxation Functions)**.** The set of Type-2 relaxation functions ($\overline{\mathcal{R}}_2$) consists of the set of analytic functions G such that $sG(s^2)$ is positive-real, i.e., $G(s)$ can be written as

$$G(s) = C + \int_0^\infty \frac{d\alpha(t)}{s+t} \tag{3.8}$$

for $C \geq 0$ and $\alpha(t)$ is a non-decreasing real function.

3.2.1.1 Rational Type-2 Relaxation Systems

Recall that approximating Cauer's analytic expression by finite sums results in a real-rational function with an immediate network interpretation [16]. We follow along these lines by presenting the real-rational Type-2 relaxation systems $\mathcal{R}_2$. Seshu et al. presents the rational representation which we now state [45].

Theorem 3.2.2 (Rational Relaxation, [45]). If $G(s)$ is rational and is representable as

$$G(s) = C + \int_0^\infty \frac{d\alpha(t)}{s+t} \tag{3.9}$$

with $C \geq 0$ and $\alpha(t)$ non-decreasing, then it may be written in the form

$$G(s) = G_0 + \sum_{i=1}^{n} \frac{G_i}{s+p_i} \quad G_0, G_i \geq 0 \; p_i \geq 0 \tag{3.10}$$

This equation agrees with Equation 2.5, and expands to obtain the general driving-point impedance of a RC network from Equation 2.4.

3.2.2 Type-1 Relaxation Systems

Type-1 relaxation systems arise as the driving-point impedance of RL networks and as the driving-point admittance of RC networks. We have already encountered the form these systems take on in Equation 2.2. Type-1 relaxation systems are less studied in the literature, presumably because, in the rational case, they may yield improper transfer functions. For example, the impedance function of a pure inductor is $Z(s) = sL$, which is unbounded as s goes to infinity. Yet, because such functions are positive-real, we may characterize them from Cauer's representation.

Theorem 3.2.3 (Representation of Type-1 Relaxation Systems). A function $G(s)$ is representable by a Stieltjes-like integral of the form

$$G(s) = Cs + \int_0^\infty \frac{s}{s+t} d\alpha(t) \tag{3.11}$$

for all s not on the negative real axis, $C \geq 0$ and $\alpha(t)$ non-decreasing if and only if $s^{-1}G(s^2)$ is positive-real.

Proof. Suppose $s^{-1}G(s^2)$ is positive-real. Then by Cauer's Theorem, we may write

$$s^{-1}G(s^2) = Cs + \int_0^\infty \frac{s}{s^2+t} d\alpha(t) \quad Re(s) > 0 \tag{3.12}$$

$$G(s^2) = Cs^2 + \int_0^\infty \frac{s^2}{s^2+t} d\alpha(t) \quad Re(s) > 0 \tag{3.13}$$

$$\implies G(s) = Cs + \int_0^\infty \frac{s}{s+t} d\alpha(t) \quad Re(s^2) > 0 \tag{3.14}$$

and all s not on the negative-real axis, $C \geq 0$ and $\alpha(t)$ non-decreasing. Conversely, suppose the integral in Equation 3.11 converges. Then one may follow the above steps in reverse to arrive at

$$s^{-1}G(s^2) = Cs + \int_0^\infty \frac{s}{s^2+t} d\alpha(t) \tag{3.15}$$

thereby demonstrating that $G(s)$ can be written as in Equation 3.11. □

The above result corroborates with Meixner's definition of relaxation for RL networks, but without use of a "spectral function" [38].

Corollary 3.2.1. If $s^{-1}G(s^2)$ is positive-real, then $G(s)$ is positive-real.

This follows immediately from Equation 3.14. We now may define Type-1 relaxation systems.

Definition 3.2.3. The set of Type-1 relaxation functions $\overline{\mathcal{R}}_1$ consists of the set of functions G such that $s^{-1}G(s^2)$ is positive-real, i.e., $G(s)$ can be written as

$$G(s) = Cs + \int_0^\infty \frac{s}{s+t} d\alpha(t) \tag{3.16}$$

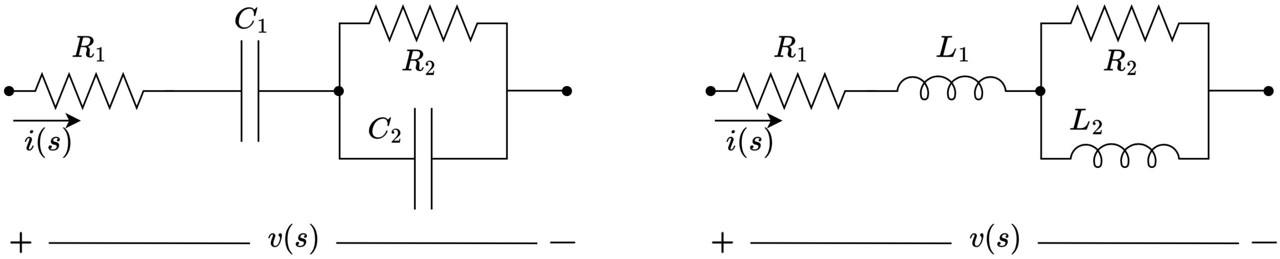
for $C \geq 0$ and $\alpha(t)$ is a non-decreasing real function.

3.2.2.1 Rational Type-1 Relaxation Systems

Again, let us restrict ourselves to the rational case, i.e., $G(s) \in \mathcal{R}_1$. By approximating Equation 3.16, we may write an arbitrary real-rational Type-1 relaxation system as

$$G(s) = G_0 s + \sum_{i=1}^{n} \frac{sG_i}{s+p_i} \quad G_0, G_i \geq 0 \; p_i \geq 0 \tag{3.17}$$

This equation agrees with Equation 2.3, and expands to obtain the general driving-point impedance of a RC network from Equation 2.2. See Figure 3.2 for a comparison of the rational impedance functions $Z(s)$ between prototypical relaxation systems of Type-1 and Type-2.



$$Z(s) = \frac{v(s)}{i(s)} = R_1 + \frac{1}{sC_1} + \frac{R_2}{sR_2C_2+1} \qquad Z(s) = \frac{v(s)}{i(s)} = R_1 + sL_1 + \frac{sR_2L_2}{sL_2+R_2}$$

Figure 3.2: Prototypical impedances $Z(s)$ for relaxation systems of both types.

It is of interest to compare the impulse responses of two prototypical relaxation systems. Setting $G_1(s) = s/(s+1) \in \mathcal{R}_1$ and $G_2(s) = 1/(s+1) \in \mathcal{R}_2$, their impulse responses are shown in Figure 3.3. Note the *positive* complete monotonic behavior of $g_2(t)$ and the *negative* complete monotonic behavior of $g_1(t)$: these are telltale signs of relaxation systems because their long-term behavior "relaxes" toward a steady-state, hence the namesake. We recall Corollary

2.1.1, which states that so long as the zero-function is excluded, inversion of Type-1 will give a Type-2 relaxation function and vice-versa.

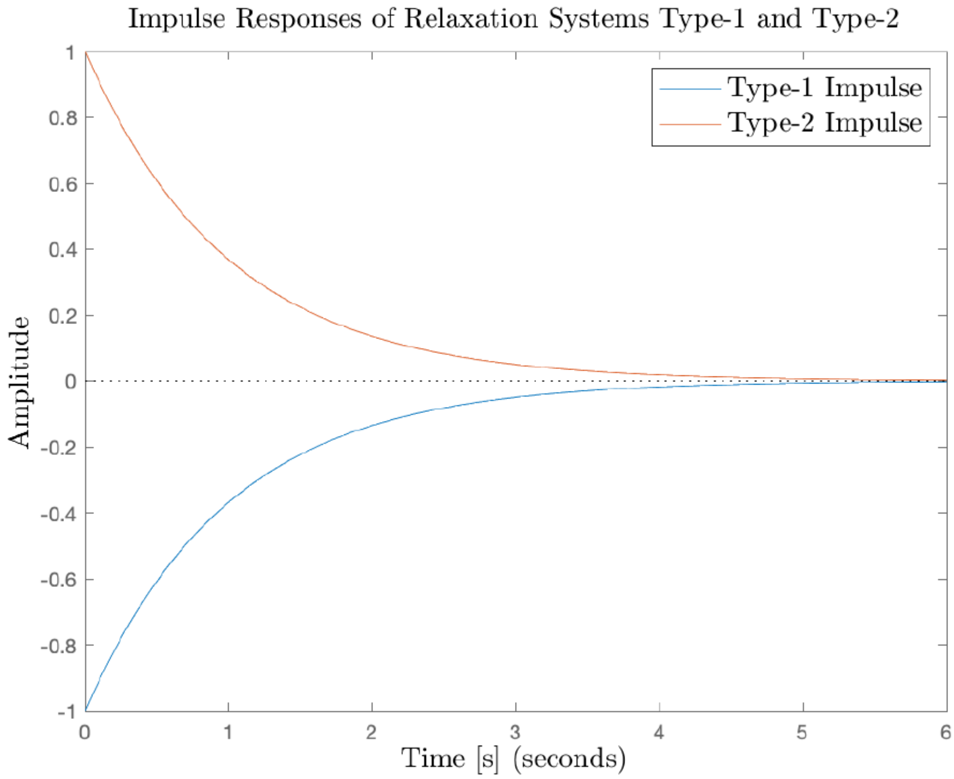


Figure 3.3: Comparison of prototypical relaxation systems impulse responses of Type-1, $G_1(s) = s/(s+1)$ and Type-2, $G_2(s) = 1/(s+1)$. Type-1 systems always maintain a Dirac-delta $\delta(t)$ function.

3.3 Convex Structure and Cone-Invariance of Relaxation Systems

Brune's work on network synthesis provided a near complete characterization of the class of positive-real functions. Notably, Brune demonstrated that the class of positive-real functions $\mathcal{P}$ is closed under addition and inversion [5]. Specifically, a linear combination of positive-real functions with non-negative coefficients will yield another positive-real function. Mathematically, these requirements give rise to a convex invertible cone [46], which we now describe.

3.3.1 Convex (Invertible) Cones of Positive-Real Functions

Definition 3.3.1 (Cone)**.** Let C denote a set. Then C is a *cone* if for all $x \in C$ and $\theta \in \overline{\mathbb{R}^+}$,

$$\theta x \in C \tag{3.18}$$

Definition 3.3.2 (Convex Cone)**.** Let C denote a set. Then C is a *convex cone* is for all $x_1, x_2 \in C$ and $\theta_1, \theta_2 \in \overline{\mathbb{R}^+}$,

$$\theta_1 x_1 + \theta_2 x_2 \in C \tag{3.19}$$

Definition 3.3.3 (Convex Invertible Cone)**.** Let C denote a set. Then C is a *convex invertible cone* if for all $x_1, x_2 \in C$ and $\theta_1, \theta_2 \in \overline{\mathbb{R}^+}$,

$$\begin{aligned}
&\theta_1 x_1 + \theta_2 x_2 \in C \\
&(\theta_1 x_1 + \theta_2 x_2)^{-1} \in C
\end{aligned}$$

when the inverse is well-defined.

It is often of interest to characterize functions and operators which map cones into their interior: such mappings are called positive.

Definition 3.3.4 (Positive and Strictly Positive Operations [47, 48])**.** Let $\mathcal{C}$ be a cone and G a bounded linear operator. Suppose G is non-trivial (not the zero-operator).
We call G *positive* if for all $x \in \mathcal{C}$, $Gx \in C$, i.e., G leaves the cone invariant.
We call G *strictly positive* if for all $x \in \mathcal{C}$, $Gx \in \text{int}(C) \subseteq \mathcal{C}$, i.e., G maps to the interior.

We shall use this notion of positivity below. Now, we state the result alluded to in the beginning of this section.

Proposition 3.3.1 ([46])**.** The sets of (rational) positive-real functions $\mathcal{P}$ and $\overline{\mathcal{P}}$ are convex invertible cones. Moreover, $\mathcal{P} \subset \overline{\mathcal{P}}$.

The convex invertibility property of the set of positive-real functions has a physical interpretation: positive scalings, inversion, addition and closure correspond to transformer ratios, impedance/admittance duality, series/parallel interconnections of impedances/admittances and the passivity theorem (passive interconnections are passive).

The sets of relaxation systems $\mathcal{R}_i$, $\overline{\mathcal{R}_i}$ are subsets of $\mathcal{P}$ and $\overline{\mathcal{P}}$ respectively. As expected, they also enjoy convex structural properties.

Proposition 3.3.2. The sets $\mathcal{R}_i$, $\overline{\mathcal{R}_i}$ for $i \in \{1, 2\}$ are convex cones, but are not invertible.

Proof. We prove the result for Type-2 relaxation systems, as a near identical proof follows for Type-1. Let $G_1(s), G_2(s) \in \overline{\mathcal{R}_2}$. Let $\theta_1, \theta_2 \geq 0$. Then using the analytic expression from

Equation 3.8, we have

$$\begin{aligned}\theta_1 G_1(s) + \theta_2 G_2(s) &= \theta_1 \left(C_1 + \int_0^\infty \frac{d\alpha_1(t)}{s+t} \right) + \theta_2 \left(C_2 + \int_0^\infty \frac{d\alpha_2(t)}{s+t} \right) \\ &= (\theta_1 C_1 + \theta_2 C_2) + \left(\int_0^\infty \frac{d\left(\theta_1 \alpha_1(t)\right) + d\left(\theta_2 \alpha_2(t)\right)}{s+t} \right) \\ &= C + \int_0^\infty \frac{d\alpha(t)}{s+t} \\ &= G(s)\end{aligned}$$

where the second linear follows from linearity of the Stieltjes integral. Clearly, a positive scaling by θ_i will not change the slope of the monotonically increasing $\alpha_i(t)$. Likewise, $\alpha(t) = \alpha_1(t) + \alpha_2(t)$ is nowhere decreasing because monotonicity is preserved under addition. Since $C \geq 0$ and the second term converges (by linearity of Stieltjes integral), we have that $G(s)$ is a Type-2 relaxation function. Since $G_1(s), G_2(s)$ and θ_1, θ_2 were arbitrary, $\overline{\mathcal{R}_2}$ is a convex cone.
To show the set is not a convex *invertible* cone, we proceed by counter example. Let $H(s) = 1/(s) \in \mathcal{R}_2 \subset \overline{\mathcal{R}_2}$. It is evident that $H^{-1}(s) = s \notin \mathcal{R}_2$, hence is not an element of $\overline{\mathcal{R}_2}$. □

The counter example in the above proposition again highlights a signature qualitative property of relaxation systems: only one type of energy storage element is allowed. Alternatively, mixing Type-1 and Type-2 may generate a function whose impulse response is not monotonic.

3.3.2 Cone-Invariance Properties in the Right Half-Plane

The domain and range of positive-real functions is the closed right half-plane, i.e., if G is a positive-real function, then $G : \overline{\mathbb{C}^+} \to \overline{\mathbb{C}^+}$. Recall that all positive-real functions have the following property:

$$|\arg G(s)| \leq |\arg(s)| \text{ for all } |\arg(s)| \leq \frac{\pi}{2} \tag{3.20}$$

This equation specifies a family of convex cones in the right half-plane, parameterized by s, consisting of elements within the phase bounds of s and its complex conjugate (see Figure 3.4). We define a cone in this family by, Θ_s, as follows.

$$\Theta_s = \left\{ z \in \mathbb{C}^+ : |\arg z| \leq |\arg s| \right\} \tag{3.21}$$

For an element $z \in \Theta_s$ in this cone, $G(z)$ is guaranteed to exist on the boundary *or* interior when G is positive real; hence, if $G(s)$ is positive-real, then G is positive on Θ_s in the sense of Definition 3.3.4. Because relaxation systems are positive-real, they also inherit this phase diminishing, positivity property. An additional, unique feature of relaxation systems is one can specify where in Θ_s a complex number $z \in \Theta_s$ will map to (see Figures 3.5 and 3.6. We present

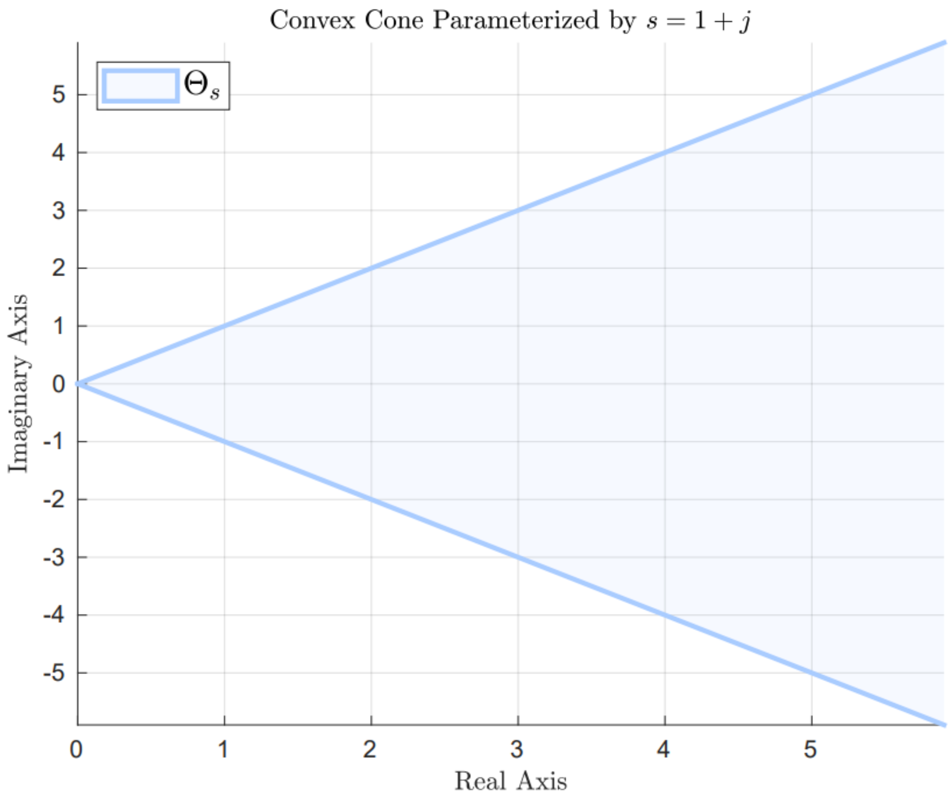


Figure 3.4: Illustration of convex cone Θ_s for $s = 1 + j$.

this result in the next theorem for the rational case.

Theorem 3.3.1. Fix a ray $s = \sigma + j\omega \in \overline{\mathbb{C}^+}$ and let G be a rational relaxation system. Define the following (convex) cone parameterized by s in the right half-plane:

$$\Theta_s^+ = \begin{cases} z \in \mathbb{C}^+ : 0 \leq \arg z \leq \arg s & \text{if } Im(s) \geq 0 \\ z \in \mathbb{C}^+ : \arg s \leq \arg z \leq 0 & \text{if } Im(s) \leq 0 \end{cases} \tag{3.22}$$

If G is a rational Type-1 relaxation system ($G \in \mathcal{R}_1$), then G maps Θ_s^+ to Θ_s^+. If G is a rational Type-2 relaxation system ($G \in \mathcal{R}_2$), then G maps Θ_s^+ to $\Theta_{\overline{s}}^+$, where $\overline{s} = \sigma - j\omega$ is the complex conjugate of s.

Remark 3.3.1. Clearly, $\Theta_s = \Theta_s^+ \cup \Theta_{\overline{s}}^+$.

Proof. We present the proof for Type-1 relaxation systems since the proof for Type-2 follows analogously.
Suppose $G \in \mathcal{R}_1 \subset \mathcal{P}$. By definition of a positive-real function,

$$|\arg G(s)| \leq |\arg s| \;\text{ for }\; |\arg s| \leq \pi/2 \tag{3.23}$$

$$\iff -|\arg s| \leq \arg G(s) \leq |\arg s| \;\text{ for }\; -\pi/2 \leq \arg s \leq \pi/2 \tag{3.24}$$

Fix $\overline{\mathbb{C}^+} \ni s = \sigma + j\omega$ with $\omega \geq 0$. Hence, $\arg(s) \geq 0$ because $Im(s) \geq 0$. From Equation 3.24,

it suffices to show that $\arg G(s) \geq 0$. For real-rational Type-1 impedances, there are two cases to consider: $G(s)$ is improper and $G(s)$ proper.
Improper: the general form of a Type-1 relaxation impedance is

$$G(s) = k\frac{(s+p_0)(s+p_2)\cdots(s+p_{2n})}{(s+p_1)(s+p_3)\cdots(s+p_{2n-1})} \tag{3.25}$$

with $k \geq 0$, $0 \leq p_0 < p_1 < \cdots$ and $n \geq 0$. Without loss of generality, we may assume $k = 1$. Fix $n \geq 0$. Since $s \in \mathbb{C}^+$, the argument of $G(s)$ is

$$\begin{aligned}
\arg G(s) &= \sum_{i=0}^{n} \arg(s+p_{2i}) - \sum_{i=1}^{n} \arg(s+p_{2i-1}) && (3.26)\\
&= \sum_{i=0}^{n} \tan^{-1}\left(\frac{\omega}{\sigma+p_{2i}}\right) - \sum_{i=1}^{n} \tan^{-1}\left(\frac{\omega}{\sigma+p_{2i-1}}\right) && (3.27)
\end{aligned}$$

To show $\arg G(s) \geq 0$, we proceed by contradiction. Suppose $\arg G(s) < 0$. Then

$$\begin{aligned}
&\sum_{i=1}^{n} \tan^{-1}\left(\frac{\omega}{\sigma+p_{2i-1}}\right) > \sum_{i=0}^{n} \tan^{-1}\left(\frac{\omega}{\sigma+p_{2i}}\right) > 0 && (3.28)\\
\Longrightarrow &\sum_{i=0}^{n-1} \tan^{-1}\left(\frac{\omega(p_{2i}-p_{2i+1})}{(\sigma+p_{2i+1})(\sigma+p_{2i})+\omega^2}\right) > \tan^{-1}\left(\frac{\omega}{\sigma+p_{2n}}\right) > 0 && (3.29)
\end{aligned}$$

where the second inequality follows from the arctangent difference formula: $\tan^{-1}(a) - \tan^{-1}(b) = \tan^{-1}((a-b)/(1+ab))$. Since $p_{2i} < p_{2i+1}$, the left hand-side of Equation 3.29 is a summation of negative terms, hence negative. This contradicts the right-hand side being positive.
Proper: If $G(s)$ is proper, the proof is the same except that the right hand-side of Equation 3.29 is now 0, still yielding a contradiction. $\square$

We remark that Theorem 3.3.1 is a necessary but not sufficient characterization of relaxation systems. For example, the positive-real function $Z(s) = (s+2)/(s+1)^2 = 1/(s+1) + 1/(s+1)^2$ maps Θ_s to $\Theta_{\overline{s}}$ but is not a Type-2 relaxation system due to the repeated-pole (see Figure 3.7). Alternatively, the impulse response of this positive-real function is monotonic, but not completely monotonic. Restricting to the strictly positive-real rational relaxation systems, one can show that a function $G(s) \in \mathcal{R}_{is}$ is strictly positive on Θ_s.

Theorem 3.3.2. Fix a ray $s = \sigma + j\omega \in \overline{\mathbb{C}^+}$. Suppose $G(s)$ is a strictly positive-real rational relaxation system, i.e., $G \in \mathcal{R}_{is}$ with $i = \{1, 2\}$. Then $G(s)$ is strictly positive on Θ_s.

Proof. Without loss of generality, fix $\overline{\mathbb{C}^+} \ni s = \sigma + j\omega$ with $\omega > 0$. Note that by enforcing $\omega > 0$, we avoid the trivial cone in which $\Theta_s = \mathbb{R}^+$; in this scenario, the interior $\mathrm{int}(\Theta_s)$ coincides with Θ_s. We present the proof for both types of relaxation systems.

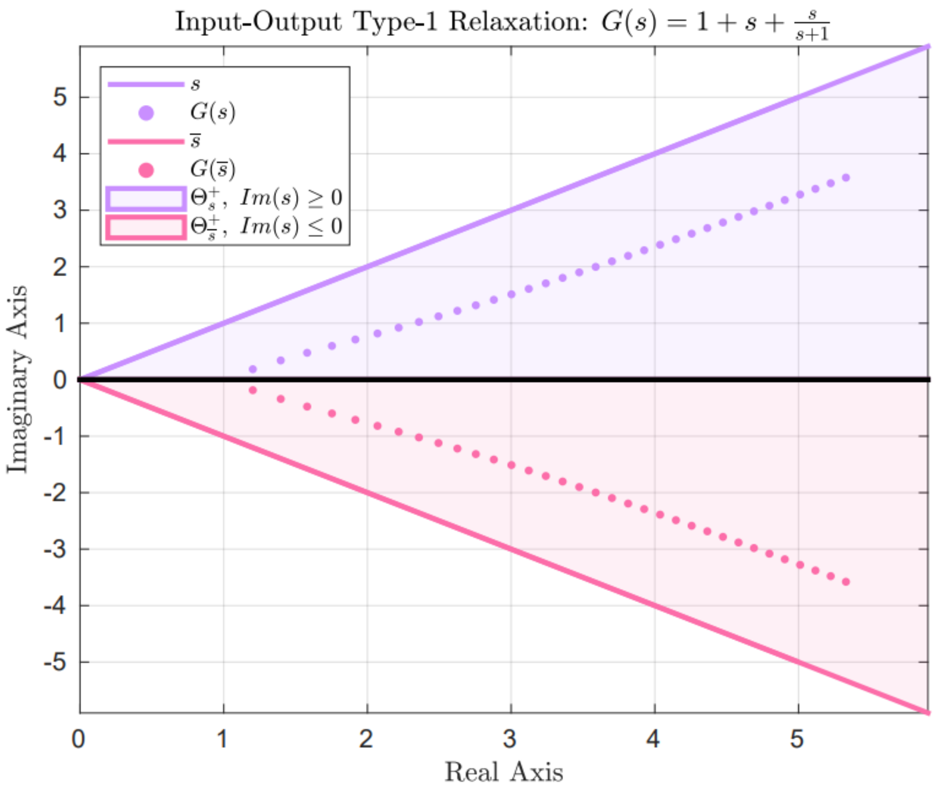


Figure 3.5: Illustration of Θ_s and $\Theta_{\overline{s}}$ in a prototypical Type-1 relaxation system for $G(s)$ evaluated along the ray defined by $= 1 + 1j$.

Type-1: From Theorem 3.3.1, G maps Θ_s^+ to Θ_s^+ and $0 \leq \arg G(s)$. Hence, to show that G maps to the interior of Θ_s, it suffices to show that $\arg G(s) < \arg s$. We proceed by contradiction. Suppose $\arg G(s) \geq \arg s$. Then there are two cases.
Case 1, $\arg G(s) > \arg s$: if this were true, it contradicts the assumption that $G(s)$ is positive-real.
Case 2, $\arg G(s) = \arg s$: If this were true, then $G(s) = \lambda s$ for $\lambda > 0$, contradicting the fact that G is strictly positive-real.

Type-2: From Theorem 3.3.1, G maps Θ_s^+ to $\Theta_{\overline{s}}^+$ and $\arg G(s) \leq 0$. Hence, to show that G maps to the interior of Θ_s, it suffices to show that $\arg G(s) < \arg \overline{s}$. We similarly proceed by contradiction. Suppose $\arg G(s) \geq \arg \overline{s}$. Then there are two cases.
Case 1, $\arg G(s) > \arg \overline{s}$: if this were true, then $|\arg G(s)| > |\arg s|$, contradicting the assumption that $G(s)$ is positive-real.
Case 2, $\arg G(s) = \arg \overline{s}$: write s in polar form as $s = |r_s|e^{j\phi}$ where $|r_s|, \phi > 0$. Then $\arg s = \phi$. Since $1/s = (1/|r_s|)e^{-j\phi}$, one has that $\arg 1/s = \arg \overline{s}$. Therefore

$$\begin{aligned} \arg G(s) = \arg \overline{s} &\iff \arg G(s) = \arg 1/s \\ &\iff G(s) = \lambda/s \quad \lambda > 0 \end{aligned}$$

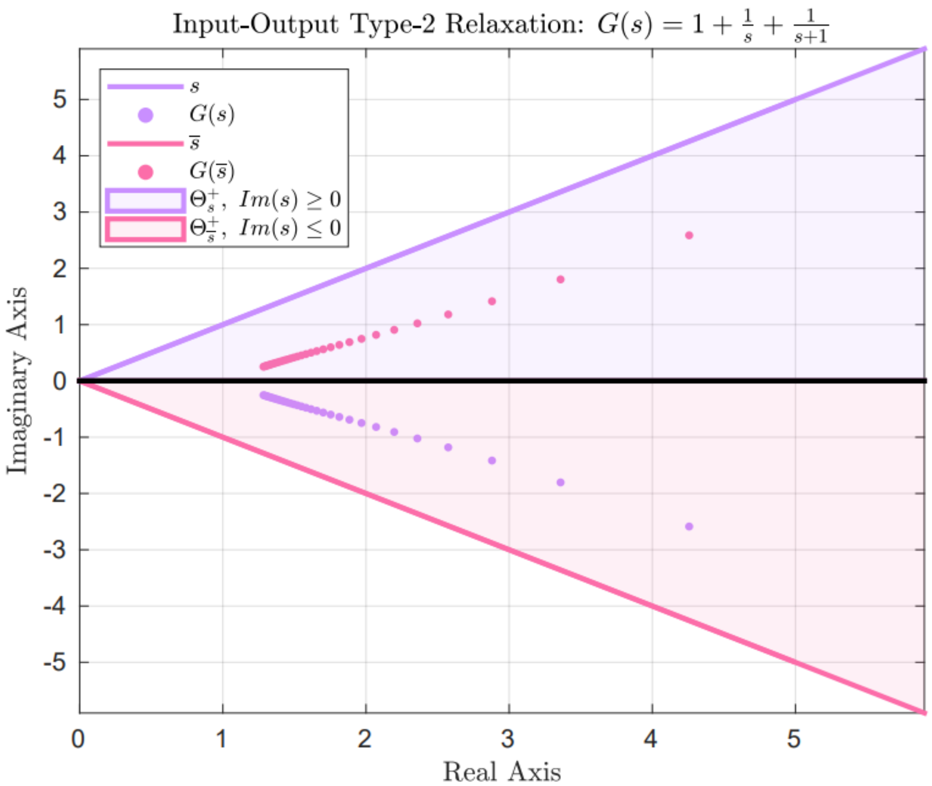


Figure 3.6: Illustration of Θ_s and $\Theta_{\overline{s}}$ in a prototypical Type-2 relaxation system for for $G(s)$ evaluated along the ray defined by $= 1 + 1j$.

contradicting the fact that G is strictly positive-real. $\square$

Corollary 3.3.1. Let $s \in \Theta_s^+$ and suppose G is strictly positive-real. If G is a rational Type-1 relaxation system ($G \in \mathcal{R}_{1s}$), then G maps from Θ_s^+ to the interior of Θ_s^+. If G is a rational Type-2 relaxation system ($G \in \mathcal{R}_{2s}$), then G maps from Θ_s^+ to the interior of $\Theta_{\overline{s}}^+$.

Corollary 3.3.2. Let $s = s_1 \in \Theta_s^+$ and suppose $G \in \mathcal{R}_{is}$ for $i = \{1, 2\}$. Define s_{k+1} as

$$s_{k+1} = G(s_k) \tag{3.30}$$

for $k \in \mathbb{N}$. Then as $k \to \infty$, $s_k \to \mathbb{R}^+$.

Proof. This is a simple application of the Banach fixed point theorem. $\square$

3.4 Discussion and Summary

This section was concerned with giving analytical and rational descriptions of the relaxation systems, which are subsets of the class of positive-real functions. Type-1 and Type-2 relaxation systems correspond to the driving-point impedances of RL and RC networks, respectively, and the duality of circuit theory allows one to convert between the two types via inversion. Assuming properness, both Type-1 and Type-2 systems derive from monotonic impulse responses, with

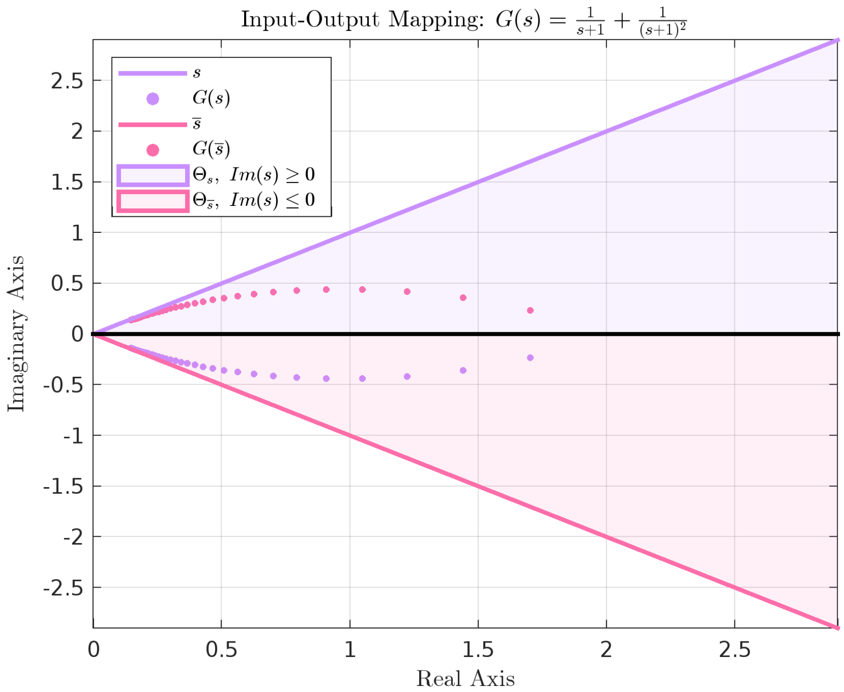


Figure 3.7: Illustration of a positive-real function $G(s)$ which maps Θ_s to $\Theta_{\overline{s}}$ for some points yet is not a relaxation Type-2 function.

Type-1 having a negative-valued impulse response and Type-2 having a positive-valued impulse response. Cohen et al. demonstrated that the positive-real functions form a convex invertible cone [46]. This work showed that relaxation systems are only convex cones.

Fixing a complex number $s \in \mathbb{C}^+$ implicitly defines a convex cone Θ_s in the right half-plane. It was found that both rational Type-1 and Type-2 relaxation systems are positive on Θ_s, leaving the cone invariant. The strictly positive-real relaxation systems are strictly positive on this cone, mapping to the interior.

Depending on the type of relaxation system G, one can specify where in the cone Θ_s a complex number will map to upon action of G. Specifically, given $z \in \Theta_s^+$, then $G(z) \in \Theta_s^+$ if G Type-1 and $G(z) \in \Theta_{\overline{s}}^+$ if G Type-2. If strictly positive-real, then the mapping is to the interior of convex cones of Θ_s^+ or $\Theta_{\overline{s}}^+$, respectively. Repeated application of G on z will eventually yield a real-valued number, consequence of the Banach fixed-point theorem. The relevance of a fixed-point is not yet understood by the author and is under current investigation.

Thus far, we have characterized relaxation systems via the properties exhibited by the functions themselves in the right half plane. Remarkably, the cone-invariance properties and iterative convergence properties to a real-value are mirrored by the Hankel operators with relaxation system symbols. Furthermore, the Hankel operator of a relaxation system is instrumental in construction an external storage functional of the system, as will be shown in the next chapter.

CHAPTER 4

Storage Functionals in LTI Relaxation Systems

This chapter introduces the Hankel operator with rational symbols, mapping signal spaces to signal spaces. Next, the basics of monotone operator theory is discussed, culminating in a discussion of cyclic monotonicity and Rockafellar's theorem [49]. It is shown that a stable Type-2 relaxation system's Hankel operator must derive from a convex functional. Finally, it is discussed how this convex functional is remarkably the system's external storage function up to a constant factor.

4.1 Operators in Dynamical Systems

4.1.1 Hankel Operators

One of the most well-studied operators on spaces of analytic functions are the Hankel operators. They have found numerous applications in problems pertaining to mathematical analysis, such as orthogonal polynomials and moment problems. Hankel operators even appear in proofs for establishing complete monotonicity [43]. In control theory, Hankel operators are utilized in robust stabilization and model reduction [30, 50].

There are numerous definitions for Hankel operators (matrices) depending upon the nature of the continuous (discrete) problem at hand [30, 50, 51]. For instance, it may be advantageous to work in the time-domain or frequency-domain, which affects one's choice of signal space. Of particular importance are the bounded Hankel operators over the l_2 or $H_2(\mathbb{D})$ and $L_2[0,\infty)$ or $H_2(\mathbb{C}^+)$ spaces. Herein, we follow the approach of [51] which mirrors that of [30] via a time-flip operation. As is customary in mathematical analysis, we begin our discussion of the Hankel operator with the discrete, matrix-based approach.

The Hankel operator is nominally introduced as a matrix-valued function as follows: given a sequence $g = \{g_i\}_{i\geq 0}$ with $g_i \in \mathbb{C}$, the (infinite) Hankel matrix $\Gamma = \{g_{j+k}\}_{j,k\geq 0}$ is given by

$$\Gamma = \begin{bmatrix} g_0 & g_1 & g_2 & g_3 & \cdots \\ g_1 & g_2 & g_3 & g_4 & \cdots \\ g_2 & g_3 & g_4 & g_5 & \cdots \\ g_3 & g_4 & g_5 & g_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{4.1}$$

As demonstrated, Hankel matrices have constant ascending skew-diagonal entries. Therefore, if $u = \{u_j\}_{j\geq 0}$, then the Hankel operator yields a sequence y such that

$$y_k = \sum_{j\geq 0} g_{j+k} u_j \quad k \geq 0 \tag{4.2}$$

It is relevant to characterize what sequence $g = \{g_i\}_{i\geq 0}$ yields a bounded Hankel matrix over l_2, meaning $\Gamma : l_2 \to l_2$. As it turns out, the theory of Hankel operators is closely connected to the theory of functions on the unit circle. Nehari's theorem provides this connection and gives necessary and sufficient conditions for boundedness over the space of square summable sequences.

Theorem 4.1.1 (Nehari [50])**.** Let the Hankel operator Γ be given by Equation 4.1, parameterized by the sequence $\{g_i\}_{i\geq 0}$. Then Γ is bounded on l_2 if and only if there exists a function $\psi \in L_\infty$ on the unit circle $\mathbb{T}$ such that

$$g_m = \hat{\psi}(m), \quad m \geq 0 \tag{4.3}$$

where $\hat{\psi}(m)$ denotes the mth Fourier coefficient of ψ. In this case,

$$\|\Gamma\| = \inf\{\|\psi\|_\infty : \hat{\psi}(m) = g_m, \quad m \geq 0\} \tag{4.4}$$

Nehari's theorem reduces the problem of whether a sequence $\{g_i\}_{i\geq 0}$ determines a bounded operator on l_2 to the existence of an extension of $\{g_i\}_{i\geq 0}$ to the sequence of Fourier coefficients of a bounded function ψ, coined the *symbol* of the Hankel operator. This presentation of Hankel operators could be described as "discrete" since we representing operators as matrices and domains/ranges as sequences. The extension to the continuous setting, i.e., operator mappings from $L_2 \to L_2$, presents no difficulty.

Definition 4.1.1 (Hankel Operator, Time-Domain)**.** Let $g \in L_\infty(-\infty, \infty)$ and $u \in L_2[0, \infty)$. Then the Hankel operator with symbol g is defined as a mapping $\Gamma_g : L_2[0, \infty) \to L_2[0, \infty)$ by

$$(\Gamma_g u)(t) := \int_0^\infty g(t + \tau) u(\tau) d\tau \tag{4.5}$$

Via the Laplace transform, one arrives at an equivalent frequency-domain representation.

Definition 4.1.2 (Hankel Operator, Frequency-Domain)**.** Let $G(s) \in L_\infty(j\mathbb{R})$ and $u \in H_2(\mathbb{C}^+)$. Then the Hankel operator with symbol G is defined as a mapping $\Gamma_G : H_2(\mathbb{C}^+) \to H_2(\mathbb{C}^+)$ by

$$(\Gamma_G u)(s) \coloneqq (P_+ M_G)u = P_+ \left(G(s)u(-s)\right) \tag{4.6}$$

Many authors define the Hankel operator as $\Gamma_g : L_2(-\infty, 0] \to L_2[0, \infty)$ and $\Gamma_G : H_2(\mathbb{C}^-) \to H_2(\mathbb{C}^+)$. The definitions provided above are equivalent by applying a time-flip $u(t) \to u(-t)$. By the isometric isomorphism for L_2 spaces in the time- and frequency-domains, clearly $\|\Gamma_g\| = \|\Gamma_G\|$.

The benefit of defining the Hankel operator as mapping between two orthogonal functions spaces is it provides a neat physical interpretation: with respect to LTI dynamical systems, the Hankel operator models the future (free) response from an initial condition which parameterizes the past input [41]. Therefore, one can view the Hankel operator as mapping past input to future output, thereby yielding an "input-output" analog of the system's state or memory.

Remark 4.1.1. Suppose $G(s)$ is rational and denote the rational L_∞ functions as RL_∞. If $G(s) \in RL_\infty$, then G may decomposed as follows:

$$G(s) = G_c(s) + G(\infty) + G_a(s) \tag{4.7}$$

where $G_c(s) \in RH_2(\mathbb{C}^+)$ is the causal component and $G_a(s) \in RH_2(\mathbb{C}^-)$ is the anti-causal component. The constant $G(\infty)$ is reflective of if $G(s)$ is proper or strictly proper in which $G(\infty) < \infty$ or $G(\infty) = 0$, respectively. Therefore, without loss of generality, we may assume $G(s)$ is strictly proper since

$$\Gamma_G u = P_+ \left(G(s)u(-s)\right) = P_+ \left(G_c(s)u(-s)\right) \tag{4.8}$$

by definition and linearity of the projection operator [30].

4.1.2 Relaxation, Hankel Operators, and a Connection to Dissipativity Theory

Recall that in dissipativity theory, the connection between the internal dynamics of a system and its inputs/outputs is traditionally given by the state-dependent storage function. Moreover, the set of storage functionals is not unique - rather, the set of valid storages is a convex set. However, an interesting passage from the author of [8] writes

> Relaxation systems obey the Onsager reciprocal relations and have the very interesting property that for such systems one may always deduce the storage function

from input/output experiments, i.e., the storage function is uniquely determined by the constitutive equations and by the qualitative assumptions that the system is externally and internally of the relaxation type.

Correcting for a slight error in [8], it was found that the (external) storage functional of a relaxation system is given by

$$S(u) = \frac{1}{2}\langle u, y\rangle_{L_2} = \frac{1}{2}\langle u, \Gamma_g u\rangle_{L_2} \tag{4.9}$$

$$= \frac{1}{2}\langle u, \Gamma_G u\rangle_{H_2} \tag{4.10}$$

where $u \in L_2(-\infty, 0]$, $y \in L_2[0, \infty)$ and $g \in L_\infty$ is the completely monotonic impulse response of an LTI (Type-2) relaxation system; the last step follows from the Parseval-Plancherel theorem. Note that Willems characterization of the Hankel operator corresponds to the alternative definition discussed above. It is interesting to note that this comment by Willems in [8] is made almost en passant; no proof is provided and little attention is paid to it moving forward. Indeed, it was not until the recent work of Chaffey et al. which proved the external storage functional in Equation 4.10 is a valid storage with respect to the passive supply rate. Moreover, the storage derives from cyclic monotonicity of the Hankel operator [35].

We argue that this special property of the strictly positive-real (stable) relaxation systems deserves more attention because it provides a constructive approach to writing down external storage functions of a system. The proof connecting cyclic monotonicity, which is made explicit through a famous theorem of Rockafellar [49, 52], is provided below. While similar to Theorem 5 of [35], we proceed in the input-output, Laplace domain, whereas Chaffey et al. developed their proof via a state-space representation. As will be discussed in last chapter, the input-output approach is advantageous because state-space descriptions are not readily amenable to nonlinear nor complex systems.

4.2 Monotonicity and Rockafellar's Theorem

In single-variable calculus, a function is monotonic if "its slope never changes sign," meaning the ratio of the difference between two points is always non-negative. This notion is generalized to inner product spaces by ensuring "the inner product never changes sign," where the inner product of the difference of pairs of elements is always non-negative. The study of operators which preserve this monotonic relation is called monotone operator theory, and it has deep ties to convex analysis [53, 54]. To build our way up to this subject, we first require a few definitions from linear operator theory.

4.2.1 Properties of Bounded Linear Operators

The following theorems and definitions can be found in Axler's text on linear algebra [55]. First, we recall the definition of a linear operator on a Hilbert space being self-adjoint.

Definition 4.2.1 (Adjoint and Self-Adjoint)**.** Suppose $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces and $G : \mathcal{U} \to \mathcal{Y}$ a linear operator. The linear operator $G^* : \mathcal{Y} \to \mathcal{U}$ is the *adjoint* of G if

$$\langle y, Gu \rangle_{\mathcal{Y}} = \langle G^* y, u \rangle_{\mathcal{U}} \tag{4.11}$$

for all $u \in \mathcal{U}$ and $y \in \mathcal{Y}$. If $G^* = G$, then G is called *self-adjoint* and

$$\langle y, Gu \rangle_{\mathcal{Y}} = \langle G y, u \rangle_{\mathcal{U}} \tag{4.12}$$

is satisfied for all $u \in \mathcal{U}$ and $y \in \mathcal{Y}$. This is true if and only if $\mathcal{U} = \mathcal{Y}$. Therefore, self-adjointness of G implies

$$\langle y, Gu \rangle_{\mathcal{U}} = \langle G y, u \rangle_{\mathcal{U}} \tag{4.13}$$

for all $u, y \in \mathcal{U}$.

We will drop the subscript on the inner product from now on if the underlying space is clear. If the domain/range is not a Hilbert space, the definition of self-adjointness becomes somewhat more complex because one must now consider the domain of the operator on the underlying Banach or vector space. Since this work is focused on the the L_2 and $H_2(\mathbb{C}^+)$ spaces, we obviate this potential issue.

When working with linear operators often needs to make a distinction between the underlying field, typically either $\mathbb{R}$ and $\mathbb{C}$. Here is one such example.

Theorem 4.2.1. Suppose $\mathcal{U}$ is a complex inner product space and $G : \mathcal{U} \to \mathcal{U}$. Then G is self-adjoint if and only if $\langle u, Gu \rangle \in \mathbb{R}$ for every $u \in \mathcal{U}$.

There are many notions of operator positivity. We present one below which is of use to us.

Definition 4.2.2 (Positive Semi-Definite)**.** A linear operator $G : \mathcal{U} \to \mathcal{U}$ is *positive semi-definite* if G is self-adjoint and

$$\langle u, Gu \rangle \geq 0 \tag{4.14}$$

for all $u \in \mathcal{U}$.

If $\mathcal{U}$ is a complex vector space, then we do not require G to be self-adjoint by Theorem 4.2.1. Many authors use the word "positive" for "positive semi-definite" or even "non-negative definite."

4.2.2 Cyclic Monotonicity

Equipped with the basic results of self-adjoint operators, we now may discuss aspects of monotone operator theory relevant to the rest of this work. Rockafellar illuminated many of the connections between monotone operators and convex analysis, and was the first to develop cyclic monotonicity, a stricter form of monotonicity over inner product spaces [49, 52]. First, let us recall the definition of the dual space.

Definition 4.2.3 (Dual Space [56]). Let $\mathcal{U}$ be a normed vector space and $\mathbb{F}$ a field (typically, $\mathbb{F}$ will be either $\mathbb{R}$ or $\mathbb{C}$). The space of all bounded linear functionals $u^* : \mathcal{U} \to \mathbb{F}$ is termed the *dual space* of $\mathcal{U}$ and is denoted by $\mathcal{U}^*$.

Remark 4.2.1. The value of a linear functional u^* at the element $u \in \mathcal{U}$ is denoted by the symmetric notation as $u^*(u) = \langle u, u^* \rangle$, called the *dual pairing*. This evaluation is not an inner product because $\mathcal{U}$ may fail to be an inner product space. However, in the case that $\mathcal{U}$ is a Hilbert space, all bounded linear functions u^* are generated by elements of $\mathcal{U}$, a consequence of the Riesz representation theorem. In this scenario, $u^*(u) = \langle u, u^* \rangle_{\mathcal{U}}$ is an inner product evaluation, demonstrating that Hilbert spaces are *self-dual*.

Now, denote $\mathcal{U}$ as an arbitrary Hilbert space and $G : \mathcal{U} \to \mathcal{U}$ an operator mapping the Hilbert space to itself. Note that G may possibly be a nonlinear operator. The following definitions are borrowed from [54].

Definition 4.2.4 (Graph of an Operator). The *graph* of G is a subset of $\mathcal{U} \times \mathcal{U}$ and is defined by

$$\operatorname{gra}(G) = \{(u, y) : u \in \mathcal{U}, y = G(u)\} \tag{4.15}$$

Definition 4.2.5 (Monotone Operator). G is termed *monotone* if, for all $u_1, u_2 \in \mathcal{U}$, $y_1 = G(u_1)$, $y_2 = G(u_2)$,

$$\langle y_1 - y_2, u_1 - u_2 \rangle \geq 0 \tag{4.16}$$

Definition 4.2.6 (Maximal Monotone). If $\operatorname{gra}(G)$ is not contained within the graph of any other monotone operator, then G is termed *maximally monotone*.

Monotonicity is defined with respect to pairs of elements within $\mathcal{U}$. The natural generalization of this concept is to consider more than two elements, i.e., n elements - this is called n-cyclic monotonicity.

Definition 4.2.7 (Cyclic Monotonicity). The operator $G : \mathcal{U} \to \mathcal{U}$ is said to be *n-cyclic monotone* if, for all sets of input-output pairs $\{(u_i, y_i) : u_i \in \mathcal{U}, \quad y_i = G(u_i), \quad i = 0, 1, \cdots, n\}$, one has that

$$\langle y_0, u_0 - u_1 \rangle + \langle y_1, u_1 - u_2 \rangle + \cdots + \langle y_n, u_n - u_0 \rangle \geq 0 \tag{4.17}$$

If G is n-cyclic monotone for all $n \geq 1$, then G is *cyclically monotone*. If gra(G) is not contained within the graph of any other monotone operator, then G is *maximal cyclic monotone*.

Remark 4.2.2. Maximality is guaranteed for continuous operators [54], Cor. 20.25. Since this thesis is solely concerned with continuous operators, all operations are automatically maximal. Therefore, when referring to monotonicity or cyclic monotonicity of an operator, we will not mention maximality explicitly.

Cyclic monotonicity was introduced into convex analysis by Rockafellar's landmark work [52] and fully characterized, after correcting for a small mistake, in [57]. Since then, cyclic monotonicity has found numerous applications in dynamical systems, optimal transport and convex analysis [49, 58, 59].

Before stating Rockafellar's characterization of cyclically monotone operators, we first require the notion of a subgradient. Subgradients, as the name suggests, generalize the traditional definition of the gradient of a function. We first give an exposition of subgradients with respect to $\mathbb{R}^n$ for pedagogical purposes, then extend the notion arbitrary real-Banach spaces [57].

For a convex and differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, it is the case that

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) \quad \text{for all } x, y \in \mathrm{dom}(f) = \mathbb{R}^n \tag{4.18}$$

When f is not differentiable, then the subgradient enters the picture.

Definition 4.2.8 (Subgradient and Subdifferential)**.** A *subgradient* of a convex function $f : \mathbb{R}^n \to \mathbb{R}$ at x is any $g \in \mathbb{R}^n$ such that

$$f(y) \geq f(x) + g^T (y - x) \quad \text{for all } y \in \mathrm{dom}(f) = \mathbb{R}^n \tag{4.19}$$

If f is differentiable, then the subgradient is uniquely $g = \nabla f(x)$. If f is not convex, the definition still holds albeit the subgradient may not exist [53, 54]. In general, letting $\mathcal{U}$ be a real Banach space with dual $\mathcal{U}^*$, the *subdifferential* of f is the (possibly multi-valued) mapping $\partial f : \mathcal{U} \to \mathcal{U}^*$ defined by the set of all subgradients:

$$\partial f(x) = \{x^* \in \mathcal{U}^* : f(y) \geq f(x) + \langle y - x, x^* \rangle \quad \forall y \in \mathcal{U}\} \tag{4.20}$$

where $\langle \cdot, \cdot \rangle$ is the dual pairing between $\mathcal{U}$ and $\mathcal{U}^*$.

It is well known in convex analysis that the subgradient of any convex function is a monotone operator. The logical follow-up question is: "when is an operator the subgradient of a convex function?" It turns out that the operator must be cyclically monotone, a result owed to Rockafellar which we now state [52].

Theorem 4.2.2 (Rockafellar's Theorem, [52] (Theorem 3), [57] (Theorem B), [54] (Theorem 22.18))**.** Let $G : \mathcal{U} \to \mathcal{U}$ be a continuous operator on a real Hilbert space $\mathcal{U}$. Then G is a (maximal) cyclically monotone operator if and only if it is the subgradient of a closed, proper and convex function $S : \mathcal{U} \to (-\infty, \infty]$. The solution S is unique up to an arbitrary additive constant.

For completeness, we include definitions of the necessary and sufficient conditions for cyclic monotonicity from Rockafellar's theorem [54]. Denote f as a real-valued function defined over a nonempty set X.

Definition 4.2.9 (Epigraph)**.** The *epigraph* of a function $f : X \to \mathbb{R}$ is the set

$$\mathrm{epi}(f) \coloneqq \{(x, y) \in X \times \mathbb{R} : f(x) \leq y\} \tag{4.21}$$

Definition 4.2.10 (Closed)**.** A function f is *closed* if its epigraph, $\mathrm{epi}(f)$, is a closed set in $X \times \mathbb{R}$. This means that for all sequences $\{(x_i, y_i)\}_{i=1}^{\infty} \subset \mathrm{epi}(f)$ converging to (x, y), one has that $(x, y) \in \mathrm{epi}(f)$.

Definition 4.2.11 (Proper)**.** A function f is *proper* if $-\infty \notin f(X)$ and $\mathrm{dom}(f) \neq \varnothing$, where the *domain* of f is defined as

$$\mathrm{dom} f \coloneqq \{x \in X : f(x) < \infty\} \tag{4.22}$$

Definition 4.2.12 (Convex)**.** A function f is *convex* if for all $x_1, x_2 \in X$ and $\theta \in [0, 1]$,

$$f(\theta x_1 + (1 - \theta)x_2) \leq \theta f(x_1) + (1 - \theta) f(x_2) \tag{4.23}$$

Remark 4.2.3. When stating the necessary and sufficient conditions of Rockafellar's theorem, many authors require S to be proper, lower semicontinuous and convex. Since the epigraph of a real-valued function is closed if and only if it is lower semicontinuous, the conditions here and those presented in Theorem 4.2.2 are equivalent.

While Rockafellar's theorem gives necessary and sufficient conditions for cyclic monotoncity, it is, in general, difficult to construct the closed, proper and convex function S whose subgradient is the operator under study. As a step toward constructing the convex functional S, we borrow a development from Asplund [60]. Therein, the author characterized cyclic monotonicity of linear operators over a real Banach space in terms of the operator's numerical range. To state this theorem, we first define the complexification of an operator.

Definition 4.2.13 (Complexification)**.** Given an operator G acting on a real Hilbert space $\mathcal{U}$, $G : \mathcal{U} \to \mathcal{U}$, define the *complexification* of G, denoted as G_c, as

$$G_c(u) = G_c(\sigma + j\omega) \coloneqq G(\sigma) + jG(\omega) \tag{4.24}$$

for $u = \sigma + j\omega$. The complexification of the Hilbert space, $\mathcal{U}_c$, consequently is endowed with the inner product

$$\langle u_1, u_2 \rangle_{\mathcal{U}_c} := \langle \sigma_1 + j\omega_1, \sigma_2 + j\omega_2 \rangle_{\mathcal{U}_c} = \langle \sigma_1, \sigma_2 \rangle_{\mathcal{U}} + \langle \omega_1, \omega_2 \rangle_{\mathcal{U}} + j\left(\langle \sigma_1, \omega_2 \rangle_{\mathcal{U}} - \langle \sigma_2, \omega_1 \rangle_{\mathcal{U}}\right) \quad (4.25)$$

which follows directly from the definition of the inner product presented in Chapter 2.

The *numerical range*, is the image of the unit sphere over Hilbert space $\mathcal{U}$ under the quadratic form of $u \to \langle u, Gu \rangle$ for $u \in \mathcal{U}$ [61]. For the complexification, the numerical range of G_c on $\mathcal{U}_c$ is

$$W\left(G_c\right) := \left\{ \frac{\langle u, G_c(u) \rangle_{\mathcal{U}_c}}{\|u\|} : u \in \operatorname{dom}\left(G_c\right), \|u\| \neq 0 \right\} \quad (4.26)$$

We now are in a position to state Asplund's characterization of n-cyclically monotone linear operators on real spaces, albeit with a bit of rewording from the original document:

Theorem 4.2.3 (Asplund [60], Theorem 3)**.** Let G be a linear operator on a real Hilbert space $\mathcal{U}$. Then G is n-cyclic monotone if and only if $|\arg(v)| \leq \pi/n$ for all $v \in W(G_c)$.

Remark 4.2.4. Asplund proved this result by only assuming $\mathcal{U}$ to be a real Banach space, equipped with the dual pairing $u^*(u) = \langle u, u^* \rangle$ for $u \in \mathcal{U}$ and $u^* \in \mathcal{U}^*$. Since we are working in Hilbert spaces, the dual pairing naturally takes the form of the inner product by the Riesz representation theorem.

As cyclic monotonicity is the limit of n-cyclic monotonicity, the following corollary is immediate.

Corollary 4.2.1. Let G be a linear operator on a real Hilbert space $\mathcal{U}$. Then G is cyclically monotone if and only if $\arg(v) = 0$ for all $v \in W(G_c)$.

Using the complexification of $\mathcal{U}$ and definition of the numerical range of G_c, a more useful characterization of cyclic monotonicity arises.

Theorem 4.2.4 ([35], Corollary 1)**.** A linear operator G on real Hilbert space $\mathcal{H}$ is cyclically monotone if and only if G is self-adjoint, and for all $u \in \operatorname{dom}(G)$, G is positive semi-definite.

Proof. $\Longleftarrow$: Suppose G is cyclically monotone. Then from Corollary 4.2.1,

$$\arg(v) = \arg\left(\frac{\langle u, G_c(u) \rangle_{\mathcal{U}_c}}{\|u\|}\right) = 0 \quad (4.27)$$

for all $v \in W(G_c)$. Therefore, $v \geq 0$ by definition of argument. Hence,

$$\begin{aligned} \frac{\langle u, G_c(u) \rangle_{\mathcal{U}_c}}{\|u\|} \geq 0 &\iff \langle u, G_c(u) \rangle_{\mathcal{U}_c} \geq 0 \\ &\iff \langle \sigma, G(\sigma) \rangle_{\mathcal{U}} + \langle \omega, G(\omega) \rangle_{\mathcal{U}} + j\langle \sigma, G(\omega) \rangle_{\mathcal{U}} - j\langle \omega, G(\sigma) \rangle_{\mathcal{U}} \geq 0 \end{aligned}$$

Collecting the real and imaginary components yields

$$Re: \ \langle \sigma, G(\sigma) \rangle_{\mathcal{U}} + \langle \omega, G(\omega) \rangle_{\mathcal{U}} \geq 0 \tag{4.28}$$

$$Im: j\langle \sigma, G(\omega) \rangle_{\mathcal{U}} - j\langle \omega, G(\sigma) \rangle_{\mathcal{U}} = 0 \tag{4.29}$$

Since the inner product is over a real Hilbert space, it is symmetric. For the imaginary components, this yields

$$\langle \sigma, G(\omega) \rangle_{\mathcal{U}} = \langle \omega, G(\sigma) \rangle_{\mathcal{U}} = \langle G^*(\omega), \sigma \rangle_{\mathcal{U}} = \langle \sigma, G^*(\omega) \rangle_{\mathcal{U}}$$

thereby showing G is self-adjoint. For the real components, one has

$$\begin{aligned} &\langle \sigma, G(\sigma) \rangle_{\mathcal{U}} \geq 0 \geq -\langle \omega, G(\omega) \rangle_{\mathcal{U}} \text{ for all } \sigma, \omega \in \mathcal{U} \\ \iff &\langle \sigma, G(\sigma) \rangle_{\mathcal{U}} \geq 0 \text{ for all } \sigma \in \mathcal{U} \end{aligned}$$

thereby showing that G is positive semi-definite.

$\Longrightarrow$: Now suppose that G is self adjoint and positive semi-definite. Running the steps above in reverse will yield that $\arg(v) = 0$ for all $v \in W(G_c)$, thereby implying cyclic monotonicity of G by Corollary 4.2.1. □

Equipped with the technical tools from monotone operator theory, we are now ready to explore cyclic monotonicity in the context of Hankel operators.

4.3 Cyclic Monotonicity of Hankel Operators with Stable Relaxation Symbols

In this section, we will be concerned with the sets of rational relaxation systems, $\mathcal{R}_i$. Recall from Nehari's Theorem that L_∞ exhausts the set of symbols for which the Hankel operator is bounded over $H_2(\mathbb{C}^+)$. Therefore, we utilize the sets strictly positive real relaxation systems, $\mathcal{R}_{is}$ from Chapter 3. Doing so ensures the impedance functions defined by these classes are essentially bounded.

We now are interested in computing the external storage functionals of arbitrary strictly positive-real relaxation systems. Equipped with Rockafellar's theorem and Asplund's characterization of cyclic monotonicity, this section culminates in Theorem 4.3.1, demonstrating that external storages of stable Type-2 relaxation systems are unique up to an additive constant, a result attributable to the remarkable fact that Hankel operators with stable Type-2 relaxation symbols are cyclic monotone. In the Laplace-domain, such a result gives an algebraic method to calculate the external storage: under a qualitative assumption that a system behaves like a Type-2 relaxation system, calculation of the storage functional is simply a matter of summing the squares of the input trajectory, evaluated at the poles of the relaxation system.

We begin this section a few technical lemmas characterizing the output from application of the Hankel operator.

4.3.1 Hankel Operator Mappings

Lemma 4.3.1. Suppose $u \in H_2^{\mathbb{R}}(\mathbb{C}^+)$ and $G \in \mathcal{R}_{2s}$. Let

$$G(s) = G_0 + \sum_{i=1}^{n} \frac{G_i}{s+p_i} \tag{4.30}$$

as in Equation 3.10 with $G_0, G_i \geq 0$, $p_i > 0$ and $p_i \neq p_j$ for $i \neq j$. Then the Hankel operator $\Gamma_G : H_2^{\mathbb{R}}(\mathbb{C}^+) \to H_2^{\mathbb{R}}(\mathbb{C}^+)$ is such that

$$\left(\Gamma_G u\right)(s) = \sum_{i=1}^{n} \frac{G_i u(p_i)}{s+p_i} \tag{4.31}$$

Proof. Define $u(s)$ by the Szegő kernel as

$$u(s) = \sum_j \frac{1}{2\pi} \frac{\alpha_j}{s+\overline{\lambda_j}} = \sum_j \frac{\beta_j}{s+\overline{\lambda_j}} \tag{4.32}$$

where $\beta_j = \alpha_j/(2\pi)$ and $\beta_j \in \mathbb{C}$. From Equation 4.8, we must only consider the strictly proper component of G. Hence,

$$\begin{aligned}
(\Gamma_G u)(s) &= P_+\left(G(s)u(-s)\right) \\
&= P_+\left(\left(\sum_{i=1}^{n} \frac{G_i}{s+p_i}\right)\left(\sum_j \frac{-\beta_j}{s-\overline{\lambda_j}}\right)\right) \\
&= P_+\left(\sum_{i=1}^{n}\sum_j \frac{-G_i\beta_j}{(s+p_i)(s-\overline{\lambda_j})}\right)
\end{aligned}$$

Fixing i and j, one can perform partial fractions as

$$\frac{-G_i\beta_j}{(s+p_i)(s-\overline{\lambda_j})} = \frac{A}{s+p_i} + \frac{B}{s-\overline{\lambda_j}}$$

Solving yields

$$A = \frac{G_i\beta_j}{\overline{\lambda_j}+p_i} \quad B = -\frac{G_i\beta_j}{\overline{\lambda_j}+p_i}$$

Thus,

$$(\Gamma_G u)(s) = \sum_{i=1}^{n} \sum_{j} \frac{\frac{G_i \beta_j}{\overline{\lambda_j} + p_i}}{s + p_i} = \sum_{i=1}^{n} \frac{\gamma_i}{s + p_i}$$

for $\gamma_i \in \mathbb{C}$. Fix $i = k$. We can calculate γ_k as

$$\gamma_k = G_k \sum_j \frac{\beta_j}{\overline{\lambda_j} + p_k} = G_k u(p_k) \tag{4.33}$$

Note that because $u \in H_2^{\mathbb{R}}(\mathbb{C}^+)$, $u(p_k) \in \mathbb{R}$. Therefore, $\gamma_k \in \mathbb{R}$. Therefore,

$$(\Gamma_G u)(s) = \sum_{i=1}^{n} \frac{G_i u(p_i)}{s + p_i}$$

thereby concluding the proof. $\square$

We now present an analogous lemma with proof for Type-1 systems.

Lemma 4.3.2. Suppose $u \in H_2^{\mathbb{R}}(\mathbb{C}^+)$ and $G \in \mathcal{R}_{1s}$. Let

$$G(s) = G_0 + \sum_{i=1}^{n} \frac{sG_i}{s + p_i} \tag{4.34}$$

as in Equation 3.17 with $G_0, G_i \geq 0$, $p_i > 0$ and $p_i \neq p_j$ for $i \neq j$. Then the Hankel operator $\Gamma_G : H_2^{\mathbb{R}}(\mathbb{C}^+) \to H_2^{\mathbb{R}}(\mathbb{C}^+)$ is such that

$$(\Gamma_G u)(s) = \sum_{i=1}^{n} \frac{-p_i G_i u(p_i)}{s + p_i} \tag{4.35}$$

Proof. Write u as in Equation 4.32. Application of the Hankel operator yields

$$\begin{aligned}
(\Gamma_G u)(s) &= P_+ \left(G(s) u(-s) \right) \\
&= P_+ \left(\left(\sum_{i=1}^{n} \frac{sG_i}{s + p_i} \right) \left(\sum_j \frac{-\beta_j}{s - \overline{\lambda_j}} \right) \right) \\
&= P_+ \left(\sum_{i=1}^{n} \sum_j \frac{-sG_i \beta_j}{(s + p_i)(s - \overline{\lambda_j})} \right)
\end{aligned}$$

Fixing i, j and performing partial fraction yields

$$\frac{-sG_i \beta_j}{(s + p_i)(s - \overline{\lambda_j})} = \frac{A}{s + p_i} + \frac{B}{s - \overline{\lambda_j}}$$

where A and B are

$$A = \frac{-p_i G_i \beta_j}{p_i + \overline{\lambda_j}} \quad B = \frac{A\overline{\lambda_j}}{p_i} \tag{4.36}$$

Thus,

$$(\Gamma_G u)(s) = \sum_{i=1}^{n} \sum_{j} \frac{\frac{-p_i G_i \beta_j}{p_i + \overline{\lambda_j}}}{s + p_i} = \sum_{i=1}^{n} \frac{\gamma_i}{s + p_i}$$

for $\gamma_i \in \mathbb{C}$. Fix $i = k$. We can calculate γ_k as

$$\gamma_k = -p_k G_k \sum_{j} \frac{\beta_j}{p_k + \overline{\lambda_j}} = -p_k G_k u(p_k) \tag{4.37}$$

Therefore,

$$(\Gamma_G u)(s) = \sum_{i=1}^{n} \frac{-p_i G_i u(p_i)}{s + p_i}$$

thereby concluding the proof. $\square$

The careful reader will note the sign of γ_i depends on the evaluation of $u(p_i)$, as well as whether or not G is a Type-1 or Type-2 relaxation symbol. If we restrict ourselves to only consider real exponentials in the time-domain, a clear description of the signs of γ_i become apparent.

Definition 4.3.1 (Subsets of $H_2^{\mathbb{R}}(\mathbb{C}^+)$)**.** Define the following subsets of $H_2^{\mathbb{R}}(\mathbb{C}^+)$, where $\mathbb{R}^{++}$ ($\mathbb{R}^+$) denotes the positive (non-negative) real axis and $\mathbb{R}^-$ denotes the non-positive real axis. Right-hand side superscripts refer the sign of p_j, whereas right-hand side subscripts refer to sign of α_j. The presentation below follows the Szegő kernel structure of the Hardy space.

$$H_2^{\mathbb{R}}(\mathbb{C}^+)_{\mathbb{R}}^+ := \left\{ u \in H_2^{\mathbb{R}}(\mathbb{C}^+) : u(s) = \sum_j \frac{\alpha_j}{s + p_j} \text{ where } \alpha_j \in \mathbb{R}, p_j \in \mathbb{R}^{++}, p_j \neq p_i \right\} \tag{4.38}$$

$$H_2^{\mathbb{R}}(\mathbb{C}^+)_{\mathbb{R}^+}^+ := \left\{ u \in H_2^{\mathbb{R}}(\mathbb{C}^+) : u(s) = \sum_j \frac{\alpha_j}{s + p_j} \text{ where } \alpha_j \in \mathbb{R}^+, p_j \in \mathbb{R}^{++}, p_j \neq p_i \right\} \tag{4.39}$$

$$H_2^{\mathbb{R}}(\mathbb{C}^+)_{\mathbb{R}^-}^+ := \left\{ u \in H_2^{\mathbb{R}}(\mathbb{C}^+) : u(s) = \sum_j \frac{\alpha_j}{s + p_j} \text{ where } \alpha_j \in \mathbb{R}^-, p_j \in \mathbb{R}^{++}, p_j \neq p_i \right\} \tag{4.40}$$

Note that $H_2^{\mathbb{R}}(\mathbb{C}^+)_{\mathbb{R}}^+$ is a convex cone in the Hardy space; indeed, $u(s) = \theta_1 u_1(s) + \theta_2 u_2(s) \in H_2^{\mathbb{R}}(\mathbb{C}^+)_{\mathbb{R}}^+$ for $\theta_1, \theta_2 \geq 0$ and $u_1(s), u_2(s) \in H_2^{\mathbb{R}}(\mathbb{C}^+)_{\mathbb{R}}^+$ by the vector space structure. We may partition this space as

$$H_2^{\mathbb{R}}(\mathbb{C}^+)^+_{\mathbb{R}} = H_2^{\mathbb{R}}(\mathbb{C}^+)^+_{\mathbb{R}^+} \cup H_2^{\mathbb{R}}(\mathbb{C}^+)^+_{\mathbb{R}^-} \tag{4.41}$$

where both spaces on the right-hand side are also convex cones by their sign restrictions. This construction is *analogous* to the construction of $\Theta_s = \Theta_s^+ \cup \Theta_{\bar{s}}^+$ from Chapter 3. These signal spaces give an immediate corollary.

Corollary 4.3.1. From lemma, 4.3.1 and 4.3.2, one has the following:

I Suppose $G(s) \in \mathcal{R}_{1s}$. Then

  i If $u(s) \in H_2^{\mathbb{R}}(\mathbb{C}^+)^+_{\mathbb{R}}$, then $(\Gamma_G u)(s) \in H_2^{\mathbb{R}}(\mathbb{C}^+)^+_{\mathbb{R}}$

  ii If $u(s) \in H_2^{\mathbb{R}}(\mathbb{C}^+)^+_{\mathbb{R}^+}$, then $(\Gamma_G u)(s) \in H_2^{\mathbb{R}}(\mathbb{C}^+)^+_{\mathbb{R}^-}$

  iii If $u(s) \in H_2^{\mathbb{R}}(\mathbb{C}^+)^+_{\mathbb{R}^-}$, then $(\Gamma_G u)(s) \in H_2^{\mathbb{R}}(\mathbb{C}^+)^+_{\mathbb{R}^+}$

II Suppose $G(s) \in \mathcal{R}_{2s}$. Then

  i If $u(s) \in H_2^{\mathbb{R}}(\mathbb{C}^+)^+_{\mathbb{R}}$, then $(\Gamma_G u)(s) \in H_2^{\mathbb{R}}(\mathbb{C}^+)^+_{\mathbb{R}}$

  ii If $u(s) \in H_2^{\mathbb{R}}(\mathbb{C}^+)^+_{\mathbb{R}^+}$, then $(\Gamma_G u)(s) \in H_2^{\mathbb{R}}(\mathbb{C}^+)^+_{\mathbb{R}^+}$

  iii If $u(s) \in H_2^{\mathbb{R}}(\mathbb{C}^+)^+_{\mathbb{R}^-}$, then $(\Gamma_G u)(s) \in H_2^{\mathbb{R}}(\mathbb{C}^+)^+_{\mathbb{R}^-}$

In other words, the Hankel operators of strictly positive-real relaxation systems preserve the cone of $H_2^{\mathbb{R}}(\mathbb{C}^+)^+_{\mathbb{R}}$, mapping a decaying exponential in the past to a decaying exponential in the future. Hankel operators of Type-1 systems reverse the sign, whereas Hankel operators of Type-2 systems preserve the sign. This result is *analogous* to the behavior of the *relaxation functions* on Θ_s observed in Theorem 3.3.1, where the "reversal" is now due to Type-1, instead of Type-2.

4.3.2 Cyclic Monotonicity of the Relaxation Hankel Operators

Equipped with the lemmas of the previous subsection, we now proceed to show cyclic monotonicity of the Type-2 relaxation systems. We again state this result was demonstrated by Chaffey et al. in [35] via state-space methods. The Laplace-domain approach is demonstrated below.

Theorem 4.3.1 (Cyclic Monotonicity of Type-2 Hankel Operators)**.** If $G \in \mathcal{R}_{2s}$, then its Hankel operator $\Gamma_G : H_2^{\mathbb{R}}(\mathbb{C}^+) \to H_2^{\mathbb{R}}(\mathbb{C}^+)$ is cyclically monotone.

Proof. It suffices to show that Γ_G is self-adjoint and positive semi-definite. By the residue theorem,

$$
\begin{aligned}
\langle u, \Gamma_G v\rangle_{H_2} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} u^*(j\omega)\left(\Gamma_G v\right)(j\omega)d\omega \\
&= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} u(-s)\left(\Gamma_G v\right)(s)ds \\
&= \frac{1}{2\pi j}\oint_{\mathbb{C}^-} u(-s)\left(\Gamma_G v\right)(s)ds \\
&= \frac{1}{2\pi j}\left(2\pi j\sum_{\text{poles}\in\mathbb{C}^-}\operatorname{Res}\left(u(-s)\left(\Gamma_G v\right)(s)\right)\right) \\
&= \sum_{\text{poles}\in\mathbb{C}^-}\operatorname{Res}\Big(u(-s)\left(\Gamma_G u\right)(s)\Big)
\end{aligned}
$$

where the third equality's integral is a contour integral *up* the imaginary axis and around an infinite semicircle (counterclockwise) in the left half-plane, followed by a return to the origin *up* the imaginary axis. The infinite semicircle's contribution to the integral is zero because $u(-s)\left(\Gamma_G v\right)(s)$ is strictly proper [62]. Therefore

$$
\langle u, \Gamma_G v\rangle = \sum_{\text{poles}\in\mathbb{C}^-}\operatorname{Res}\Big(u(-s)\left(\Gamma_G v\right)(s)\Big) \tag{4.42}
$$

Alternatively, we could have considered the contour integral *up* the imaginary axis and around an infinite semicircle (clockwise) in the right half-plane, which would correspond to

$$
\frac{1}{2\pi j}\oint_{\mathbb{C}^+} v(s)\left(\Gamma_G u\right)(-s)ds = \sum_{\text{poles}\in\mathbb{C}^+}\operatorname{Res}\Big(\left(\Gamma_G u\right)(-s)v(s)\Big) \tag{4.43}
$$

Note the minus signs cancel from this operation due to the counter-clockwise winding number convention. Since Equation 4.42 and 4.43 are equal, it follows that

$$
\langle u, \Gamma_G v\rangle = \langle \Gamma_G u, v\rangle \tag{4.44}
$$

and hence Γ_G is self-adjoint.

Now we seek to prove positivity. Expanding the term inside the residue, and using the representation of $(\Gamma_G u)(s)$ from lemma 4.3.1 we have from partial fractions that

$$\begin{aligned} u(-s)\left(\Gamma_G u\right)(s) &= \left(\sum_j \frac{\beta_j}{-s+\overline{\lambda_j}}\right)\left(\sum_{i=1}^{n} \frac{\gamma_i}{s+p_i}\right) \\ &= \left(\sum_j \frac{-\beta_j}{s-\overline{\lambda_j}}\right)\left(\sum_{i=1}^{n} \frac{\gamma_i}{s+p_i}\right) \\ &= \sum_j \frac{\theta_j}{s-\overline{\lambda_j}} + \sum_{i=1}^{n} \frac{\delta_i}{s+p_i} \end{aligned}$$

for some coefficients $\theta_j, \delta_i \in \mathbb{C}$. Hence

$$\sum_{\text{poles}\in\mathbb{C}^-} \operatorname{Res}\Big(u(-s)\left(\Gamma_G u\right)(s)\Big) = \sum_{i=1}^{n} \delta_i \tag{4.45}$$

Consider $k \in [1, n]$. One can calculate δ_k easily since each pole is simple (by relaxation).

$$\delta_k = \lim_{s\to -p_k} (s+p_k)\left(u(-s)\sum_{i=1}^{n} \frac{\gamma_i}{s+p_i}\right) \tag{4.46}$$

$$= \lim_{s\to -p_k} \left(u(-s)\gamma_k + u(-s)\sum_{i=1}^{n-1} \frac{\gamma_i(s+p_k)}{s+p_i}\right) \tag{4.47}$$

$$= u(p_k)\gamma_k \tag{4.48}$$

The expression for γ_k is given by lemma 4.3.1 in Equation 4.33. Combining Equations 4.48 and 4.33, we have that

$$\delta_k = G_k \left(u(p_k)\right)^2 \geq 0 \tag{4.49}$$

which implies that

$$\langle u, \Gamma_G u\rangle_{H_2} = \sum_{i=1}^{n} \delta_i = \sum_{i=1}^{n} G_i \left(u(p_i)\right)^2 \geq 0 \tag{4.50}$$

Hence, Γ_G is positive semi-definite. □

Theorem 4.3.1 demonstrates that the Hankel operator over the Type-2 relaxation system symbols is cyclically monotone. By Rockafellar's theorem, the Hankel operator Γ_G for $G \in \mathcal{R}_{2s}$ is the (sub)gradient of a closed, convex and proper functional mapping S from $H_2^{\mathbb{R}}(\mathbb{C}^+) \to \mathbb{R}^+$. Chaffey et al. demonstrated that the external storage functional of a relaxation in Equation 4.10 is a valid storage [35] with respect to the passive supply rate in the sense of Hughes' definition of passivity [12]. For pedagogical purposes, we follow Chaffey et al. to show that the subgradient of the external storage S is indeed Γ_G.

Theorem 4.3.2 (Subgradient of External Storage Functional [35])**.** If $G \in \mathcal{R}_{2s}$, then $\Gamma_G = \partial S$ where S is a non-negative storage function $S : H_2^{\mathbb{R}}(\mathbb{C}^+) \to \mathbb{R}^+$.

Proof. Proving this theorem necessitates a notion of a derivative on $H_2(\mathbb{C}^+)$, which is afforded by the functional derivative from the calculus of variations. See [63] for an in-depth treatment of the subject.

Define the functional derivative of $S(u)$ by

$$\left\langle \frac{\partial S}{\partial u}, \phi \right\rangle := \left[\frac{d}{d\epsilon}(S(u+\epsilon\phi)) \right]_{\epsilon=0} \tag{4.51}$$

where ϕ is an arbitrary function and $\epsilon\phi$ is the first variation of u. Computing this quantity yields

$$\begin{aligned}
\left\langle \frac{\partial S}{\partial u}, \phi \right\rangle &= \frac{1}{2}\left[\frac{d}{d\epsilon}\langle u+\epsilon\phi, \Gamma_G(u+\epsilon\phi)\rangle \right]_{\epsilon=0} \\
&= \frac{1}{2}\left[\frac{d}{d\epsilon}\langle u+\epsilon\phi, \Gamma_G(u)+\epsilon\Gamma_G(\phi)\rangle \right]_{\epsilon=0} \\
&= \frac{1}{2}\left[\langle \frac{d}{d\epsilon}(u+\epsilon\phi), \Gamma_G(u)+\epsilon\Gamma_G(\phi)\rangle + \langle u+\epsilon\phi, \frac{d}{d\epsilon}\left(\Gamma_G(u)+\epsilon\Gamma_G(\phi)\right)\rangle \right]_{\epsilon=0} \\
&= \frac{1}{2}\left[\langle \phi, \Gamma_G(u)+\epsilon\Gamma_G(\phi)\rangle + \langle u+\epsilon\phi, \Gamma_G(\phi)\rangle\right]_{\epsilon=0} \\
&= \frac{1}{2}\left[\langle \phi, \Gamma_G(u)\rangle + \langle u, \Gamma_G(\phi)\rangle\right] \\
&= \langle \Gamma_G u, \phi\rangle
\end{aligned}$$

The second line used linearity of the Hankel operator. The third line used the differentiability rule of inner products: $\frac{d}{dt}\langle f(t), g(t)\rangle = \langle f^{'}(t), g(t)\rangle + \langle f(t), g^{'}(t)\rangle$. Finally, the last line used self-adjointness of the Hankel operator with a Type-2 relaxation symbol over real Hilbert space $H_2^{\mathbb{R}}(\mathbb{C}^+)$. Since this result holds for all u, it follows that $\partial S = \Gamma_G$. □

As an example, let us calculate the storage of a simple Type-2 relaxation system in both the time-domain and frequency-domain.

Example 4.1. Let the variables from Figure 4.1 assume the following values: $R_0 = 3[\Omega], R_1 = 1[\Omega], R_2 = 2[\Omega], C_1 = 2[F]$ and $C_2 = 0.25[F]$. By duality, $L_1 = C_1 = 2[H]$ and $L_2 = C_2 = 0.25[H]$. Let $u(t) = e^{-t}$. Then $u(s) = 1/(s+1)$. Given the circuit parameter values, the impulse response $z(t)$ and impedance $Z(s)$ are given by

$$\begin{aligned}
z(t) &= 3\delta(t) + \frac{1}{2}e^{-(1/2)t} + 4e^{-2t} \\
Z(s) &= 3 + \frac{1/2}{s+(1/2)} + \frac{4}{s+2}
\end{aligned}$$

Let us proceed in calculating the storage in the time-domain using the Hankel convolution from

Equation 4.5. All inner products are with respect to L_2.

$$\begin{aligned}
S(u(t)) &= \frac{1}{2}\langle u(t), (\Gamma_g u)(t)\rangle \\
&= \frac{1}{2}\int_0^\infty e^{-t}\left(\int_0^\infty \left(3\delta(t+\tau) + \frac{1}{2}e^{(-1/2)(t+\tau)} + 4e^{-2(t+\tau)}\right) e^{-\tau} d\tau\right) dt \\
&= \frac{1}{2}\int_0^\infty e^{-t}\left(3e^t\theta(-t) + \frac{1}{3}e^{(-1/2)t} + \frac{4}{3}e^{-2t}\right) dt \\
&= \frac{1}{2}\int_0^\infty 3\theta(-t) + \frac{1}{3}e^{(-3/2)t} + \frac{4}{3}e^{-3t} dt \\
&= \frac{1}{2}\left(3(t(1-\theta(t))) + \left(\frac{-2}{9}\right)^{(-3/2)t} + \left(\frac{-4}{9}\right)e^{-3t}\right)\Bigg|_0^\infty \\
&= \frac{1}{2}\left(0 + \frac{2}{9} + \frac{4}{9}\right) \\
&= \frac{1}{3}
\end{aligned}$$

where

$$\theta(t) := \begin{cases} 0 & t < 0 \\ t & t \geq 0 \end{cases}$$

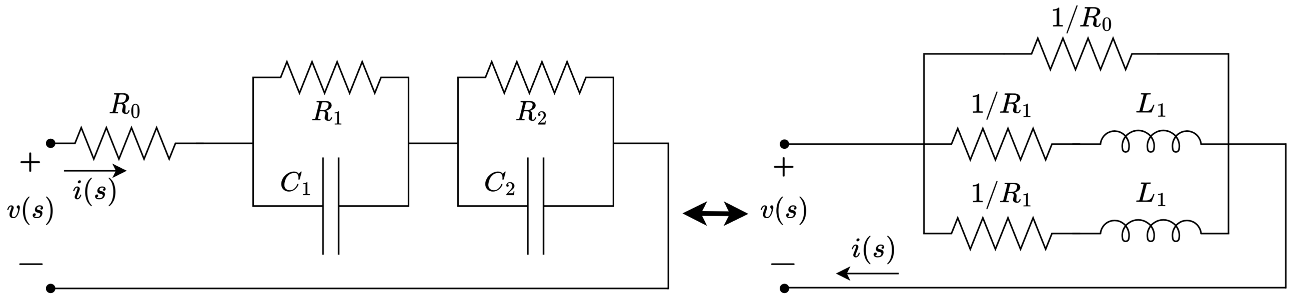
is the Heaviside step-function and $\frac{d}{dt}\theta(t) = \delta(t)$ is the Dirac delta function. Now, let us calculate the storage in the Laplace-domain by Theorem 4.3.1.

$$\begin{aligned}
S(u(s)) &= \frac{1}{2}\langle u(s), (\Gamma_G u)(s)\rangle \\
&= \frac{1}{2}\sum_{i=1}^{n} G_i\,(u(p_i))^2 \\
&= \frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{(1/2)+1}\right)^2 + 4\left(\frac{1}{(2)+1}\right)^2\right) \\
&= \frac{1}{2}\left(\frac{2}{9} + \frac{4}{9}\right) \\
&= \frac{1}{3}
\end{aligned}$$

As expected, we get identical results.

We now proceed to prove an analogous theorem for Hankel operators with stable Type-1 relaxation symbols. It was found that such operators admit mirror behavior to their Type-2 counterparts, subject to an additional minus sign.

Theorem 4.3.3 (Hankel Relaxation Type-1)**.** If $G \in \mathcal{R}_{1s}$, then its Hankel operator $\Gamma_G : H_2^{\mathbb{R}}(\mathbb{C}^+) \to H_2^{\mathbb{R}}(\mathbb{C}^+)$ is *negative* cyclically monotone in the following sense: Γ_G is self adjoint



$$Z(s) = \frac{v(s)}{i(s)} = R_0 + \frac{R_1}{sR_1C_1+1} + \frac{R_2}{sR_2C_2+1} \qquad Y(s) = \frac{i(s)}{v(s)} = R_0 + \frac{R_1}{sR_1L_1+1} + \frac{R_2}{sR_2L_2+1}$$

Figure 4.1: Type-2 relaxation system and its dual representation from Example 4.1.

and negative semi-definite.

Proof. As before, since Γ_G is a linear operator, it suffices to show self-adjointness and negative semi-definiteness of Γ_G. Proving self-adjointness follows an identical procedure to Theorem 4.3.1 and is taken as a given. Negative semi-definiteness follows similarly where the main difference is in determining γ_k, where there is an extra s term to account for in the iterated sum. Due to proof similarity, we provide a sketch of the result below.
From Theorem 4.3.1 Equation 4.45, the inner product is

$$\langle u, \Gamma_G u\rangle_{H_2} = \sum_{i=1}^{n} \delta_i$$

where δ_i is given by Equation 4.48 as

$$\delta_i = u(p_i)\gamma_i$$

Lemma 4.3.2 Equation 4.37 gives γ_i as

$$\gamma_i = -p_k G_k \sum_j \frac{\beta_j}{p_i + \overline{\lambda_j}} = -p_i G_i u(p_i)$$

Therefore,

$$\langle u, \Gamma_G u\rangle_{H_2} = \sum_{i=1}^{n} \delta_i = \sum_{i=1}^{n} -p_i G_i \left(u(p_i)\right)^2 \leq 0$$

Hence, Γ_G is negative semi-definite. □

The development of an external storage functional for the Type-1 relaxation systems is tenuous because the negative semi-definiteness of the Hankel operator would suggest a non-positive storage functional. At the time of writing, the author is investigating the physical meaning of such a result, which is reserved for future work.

4.4 Summary

This chapter introduced the Hankel operator with strictly positive-real (stable) relaxation symbols. Rockafellar's theorem was discussed, which associated cyclic monotonicity of a linear operator to a corresponding convex functional, of which it is the subgradient. It was proved that Type-2 relaxation systems have cyclically monotone Hankel operators, and the associated convex functional is an external storage for the system as predicted by dissipativity theory [8, 35]. This result transforms the traditional state-dependent storage function to an input-dependent storage function; furthermore, the Laplace-domain characterization of the external storage gives an algebraic expression to easily calculate the storage.

4.5 Discussion

A number of parallels may be drawn between Chapter 3 and Chapter 4. As mentioned after Corollary 4.3.1 the Hankel operators of strictly positive-real relaxation systems preserve the cone of $H_2^{\mathbb{R}}(\mathbb{C}^+)_{\mathbb{R}}^+$, giving the time-domain interpretation of mapping a decaying exponential in the past to a decaying exponential in the future. The signs of the residues of the exponential in the Laplace-domain, upon application of the Hankel operator, depend upon the relaxation system type: Type-1 systems reverse the sign, whereas Type-2 systems preserve the sign. This result mimics the behavior of the relaxation functions on Θ_s observed in Theorem 3.3.1, where the roles of "reversing sign" are now flipped.

One interpretation is that the positivity properties of relaxation systems in the right half-plane apply almost mutatis mutandis to their Hankel operators. Indeed, one may (cursorily) achieve similar statements by replacing "cone-invariance of complex numbers" with "cone-invariance of decaying exponentials" and "stable relaxation systems" with "Hankel operators of stable relaxation systems.

Chapter 3 demonstrated that iteration in the right-half plane for stable relaxation systems will eventually converge onto the real axis, a result owing to strict positivity and positive-realness. While not proven in this work (and remains to be shown), preliminary results demonstrate that a similar result holds for the stable relaxation Hankel operators. For example, recent work has shown that repeated application of the Hankel operator on a sum of decaying exponentials will reduce the residues (coefficients) of the output to zero, thus eventually converging to a value on the real axis. The benefit of such an observation remains to be seen.

Finally, we remark on cyclic monotonicity and external storage functionals. For Type-2 systems, the equivalence between the convex functional predicted by cyclic monotonicity and the storage function from dissipativity theory has been established [35] and approached from a different perspective in this work. For the Type-1 systems, introduction of a negative sign implies that the storage is non-positive. At the time of writing, the physical interpretation of such a storage is still underway.

As an operator theoretic concept, cyclic monotonicity is well-defined for nonlinear operators. Hence, it is plausible that investigating nonlinear relaxation systems via cyclic monotonicity is a forward step in the tractable analysis of nonlinear circuits. We discuss this issue further in the final chapter, concluding with thoughts on future work.

CHAPTER 5

Conclusions and Outlook

5.1 Summary

This thesis sought to provide a clear qualitative and quantitative description of the linear relaxation systems for use in network and systems analysis. Herein, it was found that two types of relaxation exist: Type-1 relaxation systems correspond to the impedance $Z(s)$ or admittance $Y(s)$ of RL or RC networks by circuit duality, respectively, and vice-versa for Type-2 systems. Conceptually, relaxation systems only maintain one type of energy storage element and differ by how the energy is stored: Type-1 in the magnetic field of the inductor and Type-2 in the electric field of the capacitor, if defined with respect to the impedance function $Z(s) = v(s)/i(s)$.

As subclasses of positive-real functions, the classes of relaxation systems inherit the structural form of a convex cone. This result is a manifestation of the constraints on the realizability of a passive circuit, which has a physical interpretation via positive sums of series (parallel) interconnections of impedances (admittances) through the passivity theorem. Necessary conditions for relaxation were provided in terms of positive operations on specific convex cones in the right half-plane.

Since energy flow and storage is of fundamental concern in RL and RC networks, dissipativity theory provides a natural framework to quantify the energy stored internally by a system in response to external stimuli. Of the passive circuits, the relaxation systems are interesting in that their storage functionals are input-output-dependent, rather than state-dependent. This property was elucidated by modeling a relaxation system as a Hankel operator mapping finite energy signals to finite energy signals ($L_2[0, \infty)$ or $H_2(\mathbb{C}^+)$). Herein, it was revealed that Hankel operators with stable relaxation symbols are positive on the cone of real-valued decaying exponentials. Furthermore, many of the positivity results for relaxation systems on the right half-plane carry over to their correspondent Hankel operators.

For the stable Type-2 systems, these operators are cyclically monotone. A result due to Rockafellar requires that such operators are subgradients of convex functionals. Remarkably, the functional predicted by Rockafellar is identical to the (external) storage observed by Willems for Type-2 relaxation systems in [8] and clarified by Chaffey et al. in [35]. From the perspective of modeling in the Laplace-domain, one acquires an algebraic expression of the external storage as a summation of squares of the past input trajectory, evaluated at the poles of system, under the qualitative assumption the system behaves like a Type-2 system. A similar result holds for the Type-1 systems and a complete characterization of their external storage functionals is underway.

5.2 Discussion and Future Work

The impetus for this work is grounded in the aspirations of neuromorphic engineering, which promises to deliver biologically-inspired computational machines. Yet, many challenges persist in making neuromorphic engineering a reality. In particular, the nonlinearities introduced by biological systems when processing voltage to current impede a tractable analysis. An example of such difficulties was described in Chapter 1, where the nonlinear Hodgkin-Huxley (HH) model of an excitable cell illuminated discrepancies between the governing non-monotonic state-space model and the empirically monotonic input-output data. This mismatch points to the larger issue of nonlinear synthesis: there is no theory detailing what systems are realizable from nonlinear circuit elements.

The linear relaxation systems derive from monotonic impulse responses, thereby serving as a proxy model of biophysical systems which relax toward solutions over a sufficiently long time-horizon (the potassium current of the Hodgkin-Huxley model is one such example). Various notions of monotonicity are fundamental in describing the relaxation systems. Recent work has argued that monotonicity provides a framework to define mixed feedback in networked systems, and that mixed feedback is an essential characteristic of the Hodgkin-Huxley circuit [64, 65]. Hence, its plausible that developing a theory of nonlinear relaxation predicated on notions of monotonicity and positivity would permit a framework for constructing and analyzing nonlinear biological circuits.

5.2.1 Criteria for Nonlinear Relaxation Systems

As a step toward developing a theory of nonlinear relaxation for applications in neuromorphic engineering and conductance-based modeling, we argue that many results presented in this thesis for the linear case readily extend to nonlinear systems. At the time of writing, no agreed upon definition of nonlinear relaxation exists; therefore, the following discussion is based solely on properties the author feels ought to be required of a nonlinear relaxation system.

Firstly, the cone-invariance (positivity) of relaxation systems in the right half-plane, as discussed in Chapter 3, and of relaxation Hankel operators on the spaces of real-valued decaying exponentials, should be enforced for any nonlinear relaxation model. An interpretation of this result is energy dissipation, which accurately describes any physical system that relaxes toward a solution given enough time. Since cone-invariance is an input-output property exhibited without reference to any underlying linearity assumptions, requiring it as a necessary condition for nonlinear relaxation is permissible.

Secondly, cyclic monotonicity ought to be a property exhibited by the Hankel operators of nonlinear relaxation systems. Again, because cyclic monotonicity of an operator is defined without any underlying linearity assumptions, it naturally extends to nonlinear systems. In this work, cyclic monotonicity of the Hankel operator characterized the external storage of a stable relaxation system, thereby describing its internal energy as a function of past input. While the results herein crucially relied on linearity of the Hankel operator, one may extend the definition of the Hankel operators to nonlinear systems in the following way.

Recall that one interpretation of the Hankel operator is a mapping from past input to future output. From this perspective, any underlying linearity is an a priori assumption. Defining a *nonlinear Hankel operator* as any nonlinear mapping of past input to future output, one readily generalizes the Hankel operator to nonlinear systems. As we require cone-invariance in the right half-plane and space of decaying exponential signals, there appear to be two ways of tractably defining nonlinear relaxation.

The first approach would require a single-input, nonlinear Hankel operator which demonstrates cone-invariance for any input. Mapping a past signal to a future signal, this formulation is fundamentally non-incremental. The second approach would require a multi-input, nonlinear Hankel operator which demonstrates cone-invariance for any increment of inputs. Mapping past increments to future increments, this formulation is fundamentally incremental.

The author feels the second approach, based upon incremental mappings, is more amenable to analysis. Many technical results herein crucially relied on reproducing kernel Hilbert space (RKHS) theory, where the structure of the Szegő kernel function on $H_2(\mathbb{C}^+)$ simplified many results. This work demonstrated relaxation systems are intimately related to energy dissipation; indeed it is no surprise that many results in Chapter 4 relied on the Szegő kernel, which has the time-domain representation of a decaying complex exponential and physical interpretation of energy dissipation. How one's choice of kernel affects the signal properties exhibited by elements within the RKHS is a well understood concept in the machine learning community [29, 66].

As the kernel function K is of the form $K(u_1, u_2)$ for inputs u_1 and u_2, it seems plausible to develop a RKHS-based framework for defining nonlinear relaxation systems as incremental nonlinear Hankel operators on inputs u_1, u_2 obeying positivity properties, e.g., cone-invariance and cyclic monotonicity. This is in line with the work pioneered by George Zames in the 1960s, which viewed dynamical systems as operator mappings between signal spaces [23, 24], and of recent work in kernel-based identification of dynamical systems [4].

As RKHS theory is equipped with numerous properties for approximation, interpolation and tractability, such as the Representer Theorem [29], this approach shows promise. Future work will investigate this framework; at the time of writing, utilization of Legendre transform and Fenchel's duality theorem, in combination with RKHS theory, in learning nonlinear Hankel operators from data for nonlinear relaxation systems seems especially fruitful.

In reference to nonlinear relaxation and the nonlinear synthesis question, demonstrating cyclic monotonicity of the Hankel operator for a nonlinear capacitor or parallel interconnections of a nonlinear capacitor in series with an LTI system, reminiscent of the structure of deep neural networks, is reserved for future work. Demonstrating such results will hopefully reveal what systems are realizable from nonlinear elements. Finally, defining nonlinear Hankel operators for the Hodgkin-Huxley model and analyzing its cyclic monotonicity would be a test of the credence of nonlinear relaxation as a route forward in conductance-based modeling and neuromorphic engineering.

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