Introduction to Real Analysis: Review

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These notes were compiled and summarized in the context of the real numbers ($\mathbb{R}$). Therefore, certain theorems, e.g. Cauchy criterion for convergence, should be interpreted for $\mathbb{R}^n$. Additionally, these notes are not complete and cover only a subset of what is taught in a sequence of real analysis courses; for further reading, please see Dr. Jiří Lebl's website on real analysis.

Definitions

Element

If A is a set and x is an object that belongs to A, we say that x is an element of A and we write that $x \in A$.

If x does not belong to A, we write $x \notin A$

Subsets

If A and B are sets such that each element of A is also an element of B (i.e. x $\in A \Rightarrow x \in B$), we say that A is a subset of B (or A is contained in B), written as $A \subset B$ (also $B \supset A$).

Clearly, $A \subset A$ and $\varnothing \subset A$

Equality of Sets

We use $A = B$ to mean "$A \subset B$ and $B \subset A$"

To show equality of sets, one must show both sides imply the other, i.e., $A \iff B$

Conditionals

Implication: $p \to q$
Converse: $q \to p$
Inverse: $\neg p \to \neg q$
Contrapositive $\neg q \to \neg p$

Complement

For a set A, we use A^C to denote its complement, i.e., $A^C = \{x : x \notin A\}$
The complement of B relative to A is defined as

$A \setminus B := \{x : x \in A \text{ and } x \notin B\}$

Thus, $A \setminus B = A \cap B^C$

Union

If A and B are sets, the union of A and B, written $A \cup B$, is the set of all objects which belong to at least one of the two sets A and B, i.e.

$x \in A \cup B \iff x \in A \text{ or } x \in B$

Intersection

If A and B are sets, the intersection of A and B, written $A \cap B$, is the set of all objects which belong to both A and B, i.e.

$x \in A \cap B \iff x \in A \text{ and } x \in B$

Infinite Unions

Let $A_1, A_2, A_3, \ldots$ be sets. We define their union by

$$\bigcup_{n=1}^{\infty} A_n = \{x : x \in A_n \text{ holds for at least one } n \in \mathbb{N}\}$$

Infinite Intersections

Let $A_1, A_2, A_3, \ldots$ be sets. We define their intersection by

$$\bigcap_{n=1}^{\infty} A_n = \{x : x \in A_n \text{ holds for all } n \in \mathbb{N}\}$$

Cartesian Product

The Cartesian Product of sets A and B, written $A \times B$, is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$, i.e.,

$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

In general, $A \times B \neq B \times A$

Functions

Let A and B be two sets. A function f from A to B (written as $f : A \to B$) is a rule that assigns to each a in A exactly one element b in B (this b will be denoted by $f(a)$).

Graph

Denote the graph of f as $G(f) = \{(a, f(a)) : a \in A\}$, which is a subset of $A \times B$. The two properties of $G(f)$ are

(i) $\forall\ a \in A$, $\exists\ b \in B$ such that $(a, b) \in G(f)$

(ii) If $(a, b) \in G(f)$ and $(a, \tilde{b}) \in G(f)$, then $b = \tilde{b}$

Thus, $G(f)$ describes all of the function's (input, output) pairs

Composition

Let $f : A \to B$ and $g : B \to C$ be two functions. We define $g \circ f : A \to C$ by $g \circ f(a) = g(f(a))$

Domain, Image, and Range

Let $f : A \to B$ be a function. The set A is called the domain (of definition) of f.

If $E \subset A$, we define the (direct) image of E under f by

$f(E) = \{f(x) : x \in E\}$

By definition, we also see that

$f(E) \subset f(A) \subset B$

We call $f(A)$ the range of f (written as $R(f)$)

Inverse

Let $f : A \to B$ be a function and $H \subset B$. We define the inverse image of H under f by

$f^{-1}(H) = \{x : x \in A \text{ and } f(x) \in H\}$

In general, $f^{-1}(f(E)) \neq E$

This is only true if f is a bijection

Injectivity and Surjectivity

Let $f : A \to B$ be a function. Then f is Injective if for any x_1, $x_2 \in A$,

$f(x_1) = f(x_2) \Rightarrow (x_1 = x_2)$

If f is Injective, then $|A| \leq |B|$

If $f(A) = B$ holds, we say that f is Surjective

Bijection

If f is both Injective and Surjective, we call f a Bijection from A to B.

In general, if $f : A \to B$ is a Bijection from A to B, then $\forall y \in B$, $\exists$ a unique $x \in A$ such that $f(x) = y$

$\forall$ Bijection $f : A \to B$, $x \in A$, $y \in B$,

$f(x) = y \iff x = f^{-1}(y)$

Cardinality

If there exists a Bijection from A to B, we say that A and B have the same Cardinality, written as $|A| = |B|$
Generally, for any two finite sets A and B,
$\exists$ a Bijection from A to $B \iff A$ and B have the same number of elements

Principle of Mathematical Induction

For each $n \in \mathbb{N}$, let $P(n)$ be a statement about n. Suppose that
(i) $P(1)$ is true;
(ii) For every $k \in \mathbb{N}$, $P(k)$ implies $P(k+1)$. Then $P(n)$ is true for every $n \in \mathbb{N}$
If $n_0 \in \mathbb{N}$, and
(i) $P(n_0)$ is true;
(ii) For every $k \in \mathbb{N}$ satisfying $k \geq n_0$, $P(k)$ implies $P(k+1)$.
Then $P(n)$ is true for every $n \in \mathbb{N}$ satisfying $n \geq n_0$

Divides

If a and $b \in \mathbb{Z}$ and $a \neq 0$, we say that a divides b if there exists a $c \in \mathbb{Z}$ such that $b = ac$. We write $a \mid b$ to say that a divides b.
$a \mid b \iff c \in \mathbb{Z} : (b = ac)$

Power Set

Let S be a set. The power set, $P(S)$ of S, is given by
$P(S) = \{A : A \subset S\}$
Generally, if S is finite and $|S| = n$, then $|P(S)| = 2^n$

Less Than or Equal To ($\leq$)

Let A and B be two sets. If there exists an Injective function from A to B, we write $|A| \leq |B|$
For elements, we use $x \leq y$ to mean that "$x < y$" or "$x = y$"
"$x < y$" may also be written as "$y > x$"
"$x \leq y$" may also be written as "$y \geq x$"

Finite and Infinite

Suppose that a set A has the same Cardinality as $\{1, 2, 3, ..., n\}$ for some $n \in \mathbb{N}$. We then write $|A| := n$ and that A is Finite. While the Cardinality of $\varnothing$ is 0, if $A = \varnothing$, we also say that A is Finite.
If A is not Finite, then we say A is Infinite or "of Infinite Cardinality"

Countably Infinite

If $|A| = |\mathbb{N}|$, then A is Countably Infinite
Examples) $\mathbb{N}, 2\mathbb{N}, \mathbb{Z}, \mathbb{Q}$

Countable and Uncountable

If $|A|$ is Finite or Countably Infinite, we say that A is Countable. If A is not Countable, then A is said to be Uncountable.

Binary Relations

Let S be a set. If W is a subset of $S \times S$, we call W a Binary Relation on the set S

Ordered Sets

A set S is called an Ordered Set if there exists a Relation $<$ on S such that
(i - Trichotomy) $\forall\ x,\ y \in S$. exactly one of the following holds:
$x < y,\ x = y,\ y < x$
(ii - Transitivity) $\forall\ x,\ y,\ z \in S$, if $x < y$ and $y < z$,
then $x < z$
(iii - Def. of $\leq$) if $x \leq y$ and $y \leq x$, then $x = y$
where $x \leq y \iff x < y$ or $x = y$

Equivalence Relations

Let $\sim$ be a binary relation on A

(i) We say that $\sim$ is *reflective* if $a \sim a$ for every $a \in A$

(ii) We say that $\sim$ is *symmetric* if for every a, $b \in A$ with $a \sim b$ we also have $b \sim a$

(iii) We say that $\sim$ is *transitive* if for every a, b, $c \in A$ satisfying $a \sim b$ and $b \sim c$ we also have $a \sim c$

A binary relation on the set A which is reflexive, symmetric, and transitive is an *equivalence* relation

Equivalence Class

Let $\sim$ be an equivalence relation on A. For any $a \in A$, the equivalence class of a is $[a] = \{b \in A : a \sim b\}$

In other words, the equivalence class of a is all elements $b \in A$ such that $a \sim b$

Note: Let $\sim$ be an equivalence relation on A. Then every member of A is in one and only one equivalence class.

Partition

A *partition* of a set A is a collection P of subsets of A such that every element in A is in exactly one of these subsets

Let $\sim$ be an equivalence relation on a set A. The equivalence classes of $\sim$ form a partition of A. In fact, if P is a partition of A, then we can use it to define an equivalence relation on A with equivalence classes given by P.

Upper and Lower Bounds

Let S be an Ordered Set and $E \subset S$.

If for a certain $u \in S$, $x \leq u$ holds $\forall$ $x \in E$, we call u an Upper Bound of E (and we say that E is bounded above and write $E \leq u$).

If for a certain $v \in S$, $x \geq v$ holds $\forall$ $x \in E$, we call v a Lower Bound of E (and we say that E is bounded below and write $E \geq v$).

If for a subset S is bounded above and below, we say that S is Bounded

Least Upper Bound (Supremum)

Let S be an Ordered set and $E \subset S$. An element b is called the Least Upper Bound of E (written as $b = sup(E)$ or $b = supE$ if (i) and (ii) are true:

(i) $E \leq b$ (i.e., b is an Upper Bound of E)

(ii) if $\tilde{b} \in S$ and $E \leq \tilde{b}$, then $b \leq \tilde{b}$ (i.e., b is the Least Upper Bound)

(ii') if $\tilde{b} \in S$ and $\tilde{b} < b$, then $\tilde{b}$ is not an Upper Bound of E

(ii") if $\tilde{b} \in S$ and $\tilde{b} < b$, then $\exists$ $t \in E$ such that $t > \tilde{b}$

Thus, (ii) $\iff$ (ii') $\iff$ (ii")

Greatest Lower Bound (Infimum)

Let S be an Ordered Set and $E \subset S$. An element d is called the Greatest Lower Bound of E (written as $d = inf(E)$ or $d = infE$ if the following are true:

(i) $E \geq d$ (i.e., d is a Lower Bound of E)

(ii) if $\tilde{d} \in S$ and $E \geq \tilde{d}$, then $d \geq \tilde{d}$ (i.e., d is the Greatest Lower Bound)

Binary Operations

A Binary Operation on a set S is a function from $S \times S$ to S.

Thus, it takes a pair of elements in S and maps it to an element also in S

Let a, b, $c \in S$ and let $*$ be a Binary Operator. Then $(a = b) \rightarrow (a * c = b * c)$

Symmetric Difference

Given sets X, Y, the Symmetric Difference is $(X \setminus Y) \cup (Y \setminus X)$

Least Upper Bound Property

If every non-empty subset $E \in A$ is bounded above and has a least upper bound ($sup(E) \in A$), then A has the least upper bound property

Fields

A set F is called a field if it has two operations defined on it, addition and multiplication, and if $\forall\ x, y, z \in F$, the following axioms are satisfied:

A1, A2, A3, A4, A5, M1, M2, M3, M4, M5, D

Ordered Field

A set F is called an ordered field if it has two operations defined on it, addition and multiplication, and if $\forall\ x, y, z \in F$, the following axioms are satisfied:

(i) For $x, y, z \in F$, $x < y$ implies $x + z < y + z$
(ii) For $x, y \in F$, $x > 0$ and $y > 0$ implies $xy > 0$
(iii) $x \leq x$

Real Numbers

The set of Real Numbers, $\mathbb{R}$, is a nonempty set equipped with the three following properties:

$\mathbb{R}$ is an *ordered set.*
$\mathbb{R}$ is has the *least upper bound property.*
$\mathbb{R}$ is a *ordered field.*

Absolute Value

$\forall\ x \in \mathbb{R}$, we define $|x|$ by

$|x| = x$ if $x > 0$
$|x| = 0$ if $x = 0$
$|x| = -x$ if $x < 0$

By definition,

$|x| > 0$ if $x > 0$ or $x < 0$
$|x| = 0$ if $x = 0$

Be careful when proving statements involving $||$ that you test for negative and non-negative values (need cases)

Intervals

Let a, $b \in \mathbb{R}$ and $a \leq b$

$(a, b) = \{x \in \mathbb{R} : a < x < b\}$
$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$
$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$
$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$
$(a, \infty) = \{x \in \mathbb{R} : x > a\}$
$[a, \infty) = \{x \in \mathbb{R} : x \geq a\}$
$(-\infty, b) = \{x \in \mathbb{R} : x < b\}$
$(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$
$(-\infty, \infty) = \mathbb{R}$

We call $[a, b]$ a Closed and Bounded Interval

Infinite Sequences (of Real Numbers)

An Infinite Sequence of Real Numbers is a Function from $\mathbb{N}$ to $\mathbb{R}$, i.e., $\alpha : \mathbb{N} \to \mathbb{R}$

Notation: $\{\alpha_n\}_{n=1}^{\infty} : \alpha_1, \alpha_2, \alpha_3, \ldots$

Infinite Sequences and Bounds

If $\exists\ u \in \mathbb{R}$ such that $\alpha_n \leq u\ \forall\ n \in \mathbb{N}$, we say that the sequence
$\{\alpha_n\}_{n=1}^{\infty}$ is Bounded Above and we call u and Upper Bound of the sequence $\{\alpha_n\}_{n=1}^{\infty}$

If $\exists\ v \in \mathbb{R}$ such that $\alpha_n \geq v\ \forall\ n \in \mathbb{N}$, we say that the sequence
$\{\alpha_n\}_{n=1}^{\infty}$ is Bounded Below and we call v a Lower Bound of the sequence $\{\alpha_n\}_{n=1}^{\infty}$

A Sequence is Bounded if it is Bounded Above and Bounded Below

Limit

Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence and $L \in \mathbb{R}$. We say that $\{\alpha_n\}_{n=1}^{\infty}$ converges to L if the following is true
$\forall\ \epsilon > 0$, $\exists\ K = K(\epsilon) \in \mathbb{N}$ such that
$|\alpha_n - L| < \epsilon\ \forall\ n \geq K$
In this case, we call L the limit of the sequence $\{\alpha_n\}_{n=1}^{\infty}$ and write $\lim\limits_{x\to\infty} \alpha_n = L$

Convergence and Divergence

If a sequence has a Limit, it is said to be Convergent. Otherwise, it is said to be Divergent

Subsequences

If $\{n_k\}_{k=1}^{\infty}$ is an infinite sequence of natural numbers satisfying
$n_1 < n_2 < n_3 < \cdots < n_k < n_{k+1} < \cdots$
we call $\{x_{n_k}\}_{k=1}^{\infty}$ a subsequence of $\{x_n\}_{n=1}^{\infty}$

Increasing Sequence

If $x_n \leq x_{n+1}\ \forall\ n \in \mathbb{N}$, i.e.,
$x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n \leq x_{n+1} \leq \cdots$,
then we call $\{x_n\}_{n=1}^{\infty}$ an increasing sequence

Decreasing Sequence

If $x_n \geq x_{n+1}\ \forall\ n \in \mathbb{N}$, i.e.,
$x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_n \geq x_{n+1} \geq \cdots$,
then we call $\{x_n\}_{n=1}^{\infty}$ a decreasing sequence

Monotone Sequence

If $\{x_n\}_{n=1}^{\infty}$ is an increasing or decreasing sequence, we call it a Monotone Sequence

Cauchy Sequences

$\{x_n\}_{n=1}^{\infty}$ is called a Cauchy Sequence if it satisfies the following:
$\forall\ \epsilon > 0$, $\exists\ K = K(\epsilon) \in \mathbb{N}$ such that
$|x_n - x_m| < \epsilon\ \forall\ n, m \geq K$

Contractive Sequences

A sequence $\{x_n\}_{n=1}^{\infty}$ is said to be contractive if $\exists\ \lambda$ such that
(i) $0 < \lambda < 1$
(ii) $|x_n - x_{n-1}| \leq \lambda |x_{n-1} - x_{n-2}|\ \forall\ n \geq 3$
Note that in order to verify contractiveness, one needs a λ in $(0, 1)$, which is INDEPENDENT of n

Liminf and Limsup

For every bounded sequence $\{x_n\}_{n=1}^{\infty}$, we define $\liminf\limits_{n\to\infty} x_n$ and $\limsup\limits_{n\to\infty} x_n$ by
$\liminf\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} [inf(T_n)]$
$\limsup\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} [sup(T_n)]$
where T_n is a subset of $\{x_n\}_{n=1}^{\infty}\ \forall\ n \in \mathbb{N}$

Infinite Series

Starting with an infinite sequence, $\{x_n\}_{n=1}^{\infty}$,
$x_1, x_2, x_3, \cdots, x_n, x_{n+1}, x_{n+2} \cdots$, we simply change , to +
$x_1 + x_2 + x_3 + \cdots + x_n + x_{n+1} + x_{n+2} + \cdots$, which is just a formal sum at the moment
Thus, this formal sum is called an Infinite Series and can be expressed as $\sum\limits_{n=1}^{\infty} x_n$

Partial Sums

$\forall\ n \in \mathbb{N}$, let

$$s_n = x_1 + x_2 + x_3 + \cdots + x_n = \sum_{k=1}^{n} x_k$$

Then $s_n \in \mathbb{R}$ $(A1)$ and we call s_n the n-th partial sum of the series $\sum\limits_{n=1}^{\infty} x_n$

Convergent Series

If the partial sum sequence $\{s_n\}_{n=1}^{\infty}$ is a convergent sequence, we call $\sum\limits_{n=1}^{\infty} x_n$ a convergent series, and we also call $\lim\limits_{n\to\infty} s_n$ the sum of the series $\sum\limits_{n=1}^{\infty} x_n$. Thus,

$$\sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} s_n$$

Divergent Series

If the partial sum sequence $\{s_n\}_{n=1}^{\infty}$ is a divergent sequence, we call $\sum\limits_{n=1}^{\infty} x_n$ a divergent

Geometric Series

The general form of a geometric series is

$$\sum_{n=1}^{\infty} ar^n \text{ or}$$

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1}$$

where a and r are constants

Telescoping Series

A telescoping series is a series whose partial sums consists of only two terms after cancellation

Absolute Convergence

If $\sum\limits_{n=1}^{\infty} |x_n|$ is a convergent series, the series $\sum\limits_{n=1}^{\infty} x_n$ is called absolutely convergent

Conditional Convergence

If $\sum\limits_{n=1}^{\infty} |x_n|$ is a divergent series, but the series $\sum\limits_{n=1}^{\infty} x_n$ is convergent, the series $\sum\limits_{n=1}^{\infty} x_n$ is called conditionally convergent

Theorems

Theorem 1 (Set Complements)

Let A and B be two sets. Then
(i) $A = (A^C)^C$
(ii) $A \subset B \iff B^C \subset A^C$

Theorem 2 (Set Identities)

Let A and B and C and D be sets. Then
(i) $A \cup A = A$, $A \cap A = A$, $A \cup \varnothing = A$, $A \cap \varnothing = \varnothing$, $A \setminus \varnothing = A$,
(ii) $A \cup B = B \cup A$, $A \cap B = B \cap A$
(iii) $A \cap B \subset A \subset A \cup B$, $A \cap B \subset B \subset A \cup B$
(iv) Suppose $A \subset B$ and $C \subset D$
$A \cap C \subset B \cap D$
$A \cup C \subset B \cup D$

Theorem 3 (Generalized DeMorgan's Law)

Let A and B and C be sets. Then
(i) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$
(ii) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

Theorem 4 (Bijective Compositions)

Let $f : A \to B$ and $g : B \to C$ be two functions. Then
(i) If f and g are both Injective, then $g \circ f$ is also Injective
(ii) If f and g are both Surjective, then $g \circ f$ is also Surjective

Theorem 5 (Cantor's Theorem)

Let S be an arbitrary set and $f : S \to P(S)$ be a function. Then f is not Surjective.
$f(S) \neq P(S)$
It then follows that there exists no Bijections from S to $P(S)$, i.e., $|S| \neq |P(S)|$

Theorem 6 (Cantor-Bernstein-Schroeder Theorem)

Let A and B be two sets. If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$

Theorem 7 (Additive and Multiplicative Identities)

(i) $\forall\ x, y \in \mathbb{R}$, if $x + y = x$, then $y = 0$
(ii) $\forall\ x, y \in \mathbb{R}$, if $x + y = 0$, then $y = -x$
(iii) $\forall\ u \in \mathbb{R}$, $u0 = 0 = 0u$
(iv) $\forall\ a, b \in \mathbb{R}$, $a(-b) = -(ab) = (-a)b$

Theorem 8 (Inequality Identities)

Let $x, y, z \in \mathbb{R}$
(i) If $x < 0$ then $-x > 0$
(ii) If $x > 0$ and $y < z$, then $xy < xz$
(ii*) If $x \geq 0$ and $y \leq z$, then $xy \leq xz$
(iii) If $x < 0$ and $y < z$, then $xy > xz$
(iii*) If $x \leq 0$ and $y \leq z$, then $xy \geq xz$
(iv) If $x \neq 0$, then $xx = x^2 > 0$
It follows that $n \geq 1\ \forall\ n \in \mathbb{N}$ by O2
(v) If $0 < x$, and $x < y$ (written as $0 < x < y$), then $\frac{1}{x}, \frac{1}{y}$ both exist in $\mathbb{R}$, and satisfy
$\frac{1}{x} > 0$, $\frac{1}{y} > 0$, and $\frac{1}{x} > \frac{1}{y}$
i.e., $\frac{1}{x} > \frac{1}{y} > 0$,

Theorem 9 (Multiplication of Same Signs are Positive)

If x and $y \in \mathbb{R}$ and $xy > 0$, then either x and y are both positive or both negative

Theorem 10 (No Least Positive Number)

Suppose that $a_o \in \mathbb{R}$ and $a_o \geq 0$. If $a_o < \epsilon$ holds $\forall\ \epsilon \in \mathbb{R}^+$, then $a_o = 0$

Theorem 11a (Bounded Sup Subsets)

Let A and B be two sets such that $A \subset B \subset \mathbb{R}$ and $A \neq \varnothing$. Suppose that B is bounded above.
Then both $sup(A)$ and $sup(B)$ exist in $\mathbb{R}$ and satisfy $sup(A) \leq sup(B)$.

Theorem 11b (Bounded Inf Subsets)

Let A and B be two sets such that $A \subset B \subset \mathbb{R}$ and $A \neq \varnothing$. Suppose that B is bounded below.
Then both $inf(A)$ and $inf(B)$ exist in $\mathbb{R}$ and satisfy $inf(A) \geq inf(B)$.

Theorem 12a (Exact Sup)

Suppose that $\varnothing \neq S \subset \mathbb{R}$, $a_o \in S$ and $S \leq a_o$. Then $sup(S)$ exists in $\mathbb{R}$ and $sup(S) = a_o$

Theorem 12b (Exact Inf)

Suppose that $\varnothing \neq S \subset \mathbb{R}$, $a_o \in S$ and $S \geq a_o$. Then $inf(S)$ exists in $\mathbb{R}$ and $inf(S) = a_o$

Theorem 13 (Archimedean Property)

"Given an element in $\mathbb{R}$, there exists an element in $\mathbb{N}$ that is greater."
(i) If x, $y \in \mathbb{R}$ and $x > 0$, then $\exists\ n = n_{x,y} \in \mathbb{N}$ such that
$nx > y$
(ii) $\forall\ t \in \mathbb{R}$, $\exists\ m_t \in \mathbb{N}$ such that
$m_t > t$
(ii) says that $\forall\ t \in \mathbb{R}$, t cannot be an upper bound of $\mathbb{N}$, which means that $\mathbb{N}$ has no upper bounds in $\mathbb{R}$ (i.e. $\mathbb{N}$ is not bounded above).
We note that $\varnothing \neq \mathbb{N} \subset \mathbb{R}$
(Corollary) If $\delta \in \mathbb{R}$ and $\delta > 0$, then $\exists\ n \in \mathbb{N}$ such that
$0 < \frac{1}{n} < \delta$

Theorem 14 (Existence of Irrational Numbers)

(i) $\forall\ r \in \mathbb{Q}$, $r^2 \neq 2$
(ii) $\exists$ a unique $u \in \mathbb{R}$ such that $u > 0$ and $u^2 = 2$
The set of Irrational Numbers is denoted as $\mathbb{R} \setminus \mathbb{Q}$

Theorem 15 (Unique Bases and Exponents)

Let $b \in \mathbb{R}^+$ and $m \in \mathbb{N}$. Then $\exists$ a unique $d \in \mathbb{R}$ such that $d > 0$ and $d^m = b$
This d will be denoted as $b^{\frac{1}{m}}$ or $\sqrt[m]{b}$

Theorem 16 (Density of Rational Numbers)

$\forall\ x$, $y \in \mathbb{R}$, if $x < y$, then $\exists\ r \in \mathbb{Q}$ such that
$x < r < y$

Theorem 17 (Properties of Rational and Irrational Numbers)

Let r_1, r_2, $r \in \mathbb{Q}$ and $z \in \mathbb{R} \setminus \mathbb{Q}$. Then
(i) $r_1 + r_2$, $r_1 - r_2$, $(r_1)(r_2) \in \mathbb{Q}$
(ii) If $r_2 \neq 0$, then $\frac{r_1}{r_2} \in \mathbb{Q}$
(iii) $r + z$, $r - z$, $z - r \in \mathbb{R} \setminus \mathbb{Q}$
(iv) If $r \neq 0$, then $(r)(z)$, $\frac{r}{z}$, $\frac{z}{r} \in \mathbb{R} \setminus \mathbb{Q}$
Note that $(z)(z)$ is not shown since anything can happen in this case

Theorem 18 (Density of Irrational Numbers)

$\forall\ x,\ y \in \mathbb{R}$, if $x < y$, then $\exists\ z \in \mathbb{R} \setminus \mathbb{Q}$ such that
$x < z < y$

Theorem 19 (Properties of Absolute Values)

Let $x,\ y,\ z \in \mathbb{R}$. Then
(i) $|x| \geq 0$, with "$=$" occurring if and only if $x = 0$
(ii) $|-x| = |x|$
(iii) $|xy| = |x||y|$
(iv) $x^2 = |x^2| = |x|^2$
(v) If $z \geq 0$, then $|x| \leq z \iff -z \leq x \leq z$
(v*) If $z > 0$, then $|x| < z \iff -z < x < z$
(vi) $-|x| \leq x \leq |x|$

Theorem 20 (Triangle Inequality)

$\forall\ x,\ y \in \mathbb{R}$,
(i) $|x + y| \leq |x| + |y|$
(ii) $|x - y| \geq \big||x| - |y|\big|$

Part (i) is commonly referred to as the Triangle Inequality. For the "$\leq$" in (i), "$=$" occurs if $xy > 0$, while "$<$" occurs if $xy < 0$

Theorem 21 (Nested Interval Theorem)

For each $n \in \mathbb{N}$, let $a_n,\ b_n \in \mathbb{R}$ such that $a_n \leq b_n$, and $I_n = [a_n, b_n]$ (Closed and Bounded).
Suppose that $I_1 \supset I_2 \supset I_3 \supset \ldots \supset I_n \supset I_{n+1} \supset \ldots$. Then
$\bigcap\limits_{n=1}^{\infty} I_n \neq \varnothing$
Note: The Nested Interval Theorem only applies to a Closed and Bounded Interval
One can show that $[sup(A), inf(B)] = \bigcap\limits_{n=1}^{\infty} I_n$
Note: It is possible for the Intersection of Nested Open Intervals to be Empty

Theorem 22 ($\mathbb{R}$ is Uncountable)

The interval $[0, 1]$ is an Uncountable set. Thus, since $[0, 1] \subset \mathbb{R}$, $\mathbb{R}$ is also Uncountable

Infinite Sequence Remark (Symmetric Bounds)

If $v \leq \alpha_n \leq u$, one can show that $|\alpha_n| \leq C$
i.e., $-C \leq \alpha_n \leq C$, where $C = |u| + |v|$. Thus,
$\Big(\{\alpha_n\}_{n=1}^{\infty} \text{ is Bounded}\Big) \iff \exists\ C \in \mathbb{R}$ such that $|\alpha_n| \leq C\ \forall\ n \in \mathbb{N}$

Theorem 23 (Repeating Sequences Are Convergent)

Let $C_o \in \mathbb{R}$. Define $\{a_n\}_{n=1}^{\infty}$ by
$a_n = C_o\ \forall\ n \in \mathbb{N}$
Then $\{a_n\}_{n=1}^{\infty}$ converges to C_o, i.e., $\lim\limits_{x\to\infty} a_n = C_o$
Since $a_n = C_o\ \forall\ n \in \mathbb{N}$, $\lim\limits_{x\to\infty} a_n = C_o$ can be written as $\lim\limits_{x\to\infty} C_o = C_o$

Theorem 24 (Convergent Sequences)

If a sequence $\{a_n\}_{n=1}^{\infty}$ is Convergent, then it must be Bounded
Note: If a sequence is Bounded, that does NOT mean it is Convergent
Convergent $\Rightarrow$ Bounded   Not Bounded $\Rightarrow$ Divergent

Theorem 25 (Convergent Sequence Bounded by Convergent Sequence)

Let $n_o \in \mathbb{N}$, $c_o \in \mathbb{R}$, and $L \in \mathbb{R}$. Suppose that $\{x_n\}_{n=1}^{\infty}$ and $\{a_n\}_{n=1}^{\infty}$ satisfy
(i) $|x_n - L| \leq c_o a_n \; \forall \; n \geq n_o$
(ii) $\lim_{n\to\infty} a_n = 0$
Then $\{x_n\}_{n=1}^{\infty}$ converges to L, i.e., $\lim_{n\to\infty} x_n = L$

Theorem 26 (Convergent Subsequences)

A sequence $\{x_n\}_{n=1}^{\infty}$ is convergent if and only if all subsequences of $\{x_n\}_{n=1}^{\infty}$ converge to the same limit.
(i) It follows then that, if $\lim_{n\to\infty} x_n$ exists, then $\lim_{k\to\infty} x_{n_k} = \lim_{n\to\infty} x_n$ holds for every subsequence $\{x_{n_k}\}_{n=1}^{\infty}$
(ii) A sequence $\{x_n\}_{n=1}^{\infty}$ is convergent if and only if if a tail of $\{x_n\}_{n=1}^{\infty}$ is convergent. Moreover, $\forall \; p \in \mathbb{N}$, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+p}$ if the limit exists. Note: ONLY NEED TO SHOW A SINGLE TAIL IS CONVERGENT.
(Remark 1): If $\{x_n\}_{n=1}^{\infty}$ has a divergent subsequence, then $\{x_n\}_{n=1}^{\infty}$ is divergent itself
(Remark 2): If $\{x_n\}_{n=1}^{\infty}$ has two subsequences which converge to different limits, then $\{x_n\}_{n=1}^{\infty}$ must be a divergent sequence

Theorem 27 (Properties of Limits)

Note: You must show $\{x_n\}$ and $\{y_n\}$ to already be convergent.
Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be two convergent sequences. Let $c \in \mathbb{R}$. Then
(i) $\{x_n + y_n\}_{n=1}^{\infty}$ and $\{x_n - y_n\}_{n=1}^{\infty}$ are both convergent and $\lim_{n\to\infty}(x_n \pm y_n) = \lim_{n\to\infty}(x_n) \pm \lim_{n\to\infty}(y_n)$
(ii) $\{x_n y_n\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty}(x_n y_n) = (\lim_{n\to\infty} x_n)(\lim_{n\to\infty} y_n)$
(iii) $\{c y_n\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty}(c y_n) = c(\lim_{n\to\infty} y_n)$
(iv) If $\lim_{n\to\infty} y_n \neq 0$ and $y_n \neq 0$, then $\{\frac{x_n}{y_n}\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty}(\frac{x_n}{y_n}) = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$
(v) $\{|x_n|\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} |x_n| = |\lim_{n\to\infty} x_n|$
(vi) If $x_n \geq 0 \; \forall \; n \in \mathbb{N}$, then $\{\sqrt{x_n}\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{\lim_{n\to\infty} x_n}$

Theorem 28 (If Sequence Less Than, Then Limit Less Than)

If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are both convergent and $a_n \leq b_n \; \forall \; n \in \mathbb{N}$, then
$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$
Note: This does NOT hold for strict inequality

Theorem 29 (Squeeze Theorem)

Let $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, and $\{z_n\}_{n=1}^{\infty}$ be three sequences satisfying $x_n \leq y_n \leq z_n \; \forall \; n \in \mathbb{N}$. Suppose that $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ are both convergent and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = L$. Then
$\{y_n\}_{n=1}^{\infty}$ is also convergent and $\lim_{n\to\infty} y_n = L$
Note: "$\leq$" is preserved under the limit.
Note: "$<$" is NOT preserved under the limit.
Note: STRICT INEQUALITY IS NOT PRESERVED UNDER THE LIMIT

Theorem 30 (Geometric Sequence Theorem)

(i) if $r \in \mathbb{R}$ and $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$
(ii) if $\beta \in \mathbb{R}$ and $\beta > 0$, then $\lim_{n\to\infty} \beta^{\frac{1}{n}} = 1$

Bernoulli Inequality

Let $\alpha \in \mathbb{R}$ and $\alpha > -1$. Then
$(1+\alpha)^n \geq 1 + n\alpha$

Theorem 31 (Ratio Test for Sequences)

Suppose that $x_n \neq 0 \; \forall \; n \in \mathbb{N}$ and $\lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right|$ exists
(i) If $\lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| < 1$, then $\lim_{n\to\infty} x_n = 0$
(ii) If $\lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| > 1$, then $\{x_n\}_{n=1}^{\infty}$ is a divergent sequence
(ii) If $\lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = 1$, then no conclusion can be made

Theorem 32 (Monotone Convergence Theorem)

(i) If $\{x_n\}_{n=1}^{\infty}$ is increasing and bounded above, then $\{x_n\}_{n=1}^{\infty}$ is convergent
Note: x_1 is a lower bound by default
(ii) If $\{x_n\}_{n=1}^{\infty}$ is decreasing and bounded below, then $\{x_n\}_{n=1}^{\infty}$ is convergent
Note: x_1 is an upper bound by default
Thus, every bounded monotone sequence must be convergent

Theorem 33 (Bolzano-Weierstrass)

Every bounded sequence must have at least one convergent subsequence

Theorem 34 (Cauchy Criterion for Convergent Sequences)

A sequence is convergent if and only if it is a Cauchy Sequence
(Convergent) $\iff$ (Cauchy)

Theorem 35 (Contractive Sequences Converge)

If $\{x_n\}_{n=1}^{\infty}$ is a contractive sequence, then it is a convergent sequence

Theorem 36 (Bounded Sequences and LimSup, LimInf)

A bounded sequence $\{x_n\}_{n=1}^{\infty}$ is convergent if and only if
$\liminf\limits_{n\to\infty} x_n = \limsup\limits_{n\to\infty} x_n$

Theorem 37 (Geometric Series Test)

Let a, $r \in \mathbb{R}$ and $a \neq 0$

(i) If $|r| < 1$, then the series $\sum\limits_{n=1}^{\infty} ar^n$ is a convergent series whose sum is $\frac{ar}{1-r}$

(i*) If $|r| < 1$, then the series $\sum\limits_{n=0}^{\infty} ar^n$ is a convergent series whose sum is $\frac{a}{1-r}$

(i**) If $|r| < 1$, then the series $\sum\limits_{n=}^{\infty} ar^n$ is a convergent series whose sum is $\frac{leadingterm}{1-r}$

(ii) If $|r| \geq 1$, then the series $\sum\limits_{n=1}^{\infty} ar^n$ is a divergent series

Theorem 38 (Divergent Series Test or N-th Term Test)

(i) If $\sum\limits_{n=1}^{\infty} x_n$ is a convergent series, then $\lim\limits_{n\to\infty} x_n = 0$

(ii) If $\lim\limits_{n\to\infty} x_n$ does not exist or it exists but is nonzero, then $\sum\limits_{n=1}^{\infty} x_n$ is a divergent series

Note: $\lim\limits_{n\to\infty} x_n = 0$ DOES NOT IMPLY $\sum\limits_{n=1}^{\infty} x_n$ is a convergent series or a divergent series; thus, we cannot conclude anything in this case

Theorem 39 (Positive Series Test)

Suppose that $x_n \geq 0 \; \forall \; n \in \mathbb{N}$. Then $\sum\limits_{n=1}^{\infty} x_n$ is convergent if and only if its partial sum sequence $\{s_n\}_{n=1}^{\infty}$ is bounded above

Theorem 40 (Comparison Test)

Suppose that $\forall \; n \in \mathbb{N}$, $0 \leq a_n \leq b_n$

(i) If $\sum\limits_{n=1}^{\infty} b_n$ is a convergent series, then $\sum\limits_{n=1}^{\infty} a_n$ is also a convergent series

(ii) If $\sum\limits_{n=1}^{\infty} a_n$ is a divergent series, then $\sum\limits_{n=1}^{\infty} b_n$ is also a divergent series

YOU MUST ENSURE THAT $0 \leq a_n \leq b_n \; \forall \; n \in \mathbb{N}$

Theorem 41 (Linearity of Series)

Let $\sum\limits_{n=1}^{\infty} x_n$ and $\sum\limits_{n=1}^{\infty} y_n$ be two convergent series and $c \in \mathbb{R}$. Then

(i) $\sum\limits_{n=1}^{\infty} (cx_n)$ is a convergent series and

$\sum\limits_{n=1}^{\infty} (cx_n) = c(\sum\limits_{n=1}^{\infty} x_n)$

(ii) $\sum\limits_{n=1}^{\infty} (x_n + y_n)$ and $\sum\limits_{n=1}^{\infty} (x_n - y_n)$ are both convergent series and

$\sum\limits_{n=1}^{\infty} (x_n \pm y_n) = (\sum\limits_{n=1}^{\infty} x_n) \pm (\sum\limits_{n=1}^{\infty} y_n)$

Theorem 42 (P-Series Test)

Let $P \in \mathbb{R}$

(i) If $P > 1$, then $\sum\limits_{n=1}^{\infty} \frac{1}{n^P}$ is a convergent series

(ii) If $P \leq 1$, then $\sum\limits_{n=1}^{\infty} \frac{1}{n^P}$ is a divergent series

Theorem 43 (Limit Comparison Test)

Suppose that $a_n > 0$ and $b_n > 0 \ \forall \ n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \frac{a_n}{b_n}$ exists

(i) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} \neq 0$, then either the two series $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ are both convergent or both divergent (exhibit the same behavior)

(ii) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$, and $\sum\limits_{n=1}^{\infty} b_n$ is convergent, then $\sum\limits_{n=1}^{\infty} a_n$ is also convergent

(iii) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$, and $\sum\limits_{n=1}^{\infty} a_n$ is divergent, then $\sum\limits_{n=1}^{\infty} b_n$ is also divergent

Theorem 44 (Absolute Convergence)

If $\sum\limits_{n=1}^{\infty} |x_n|$ is a convergent series, then $\sum\limits_{n=1}^{\infty} x_n$ is also a convergent series

Theorem 45 (Ratio Test for Series)

Suppose that $x_n \neq 0 \ \forall \ n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right|$ exists

(i) If $\lim\limits_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| < 1$, then $\sum\limits_{n=1}^{\infty} x_n$ is an absolutely convergent series

(ii) If $\lim\limits_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| > 1$, then $\sum\limits_{n=1}^{\infty} x_n$ is a divergent series

(iii) If $\lim\limits_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = 1$, then the test is inconclusive

Theorem 46 (Cauchy Criterion for Series)

A series $\sum\limits_{n=1}^{\infty} x_n$ is convergent if and only the following is true: $\forall \ \epsilon > 0$, $\exists \ K = K(\epsilon) \in \mathbb{N}$ such that

$\left|\sum\limits_{j=q+1}^{q+l} x_j\right| < \epsilon \ \forall \ q \geq K$ and $l \in \mathbb{N}$

"If I start at $(q+1)$ and stop anywhere, then that block of consecutive terms is fine."

Note: Absolute Value is not applied to the terms individually

Theorem 47 (Alternating Series Test)

Suppose that $\{a_n\}_{n=1}^{\infty}$ is a decreasing sequence and $\lim\limits_{n\to\infty} a_n = 0$. Then

$\sum\limits_{n=1}^{\infty} (-1)^{n-1} a_n$ is a convergent series

Theorem 48 (Riemann)

(i) If $\sum_{n=1}^{\infty} x_n$ is an absolutely convergent series, then every rearrangement of $\sum_{n=1}^{\infty} x_{\tau(n)}$ is also absolutely convergent. Moreover,

$$\sum_{n=1}^{\infty} x_{\tau(n)} = \sum_{n=1}^{\infty} x_n$$

(ii) If $\sum_{n=1}^{\infty} x_n$ is a conditionally convergent series, then $\forall\ \psi \in \mathbb{R}$, $\exists$ a bijection $\tau : \mathbb{N} \to \mathbb{N}$ such that $\sum_{n=1}^{\infty} x_{\tau(n)}$ is convergent and

$$\sum_{n=1}^{\infty} x_{\tau(n)} = \psi$$

Note: There is also a rearrangement of the series where $\sum_{n=1}^{\infty} x_n$ is divergent

Theorem 49 (Factorials)

(i) $\sum_{n=1}^{\infty} \frac{1}{n!}$ is a convergent series

(ii) Let e = $\sum_{n=1}^{\infty} \frac{1}{n!}$. Then

$2.5 < e < 2.723$

(iii) The sequence $\{(1+\frac{1}{n})^n\}_{n=1}^{\infty} = e$

Theorem 50 (Irrationality of e)

$e \in \mathbb{R} \setminus \mathbb{Q}$

Useful Facts

Stringed Inequalities

If $a \leq b$ and $c \leq d$ then
$a + c \leq b + d$

Subsets of $\mathbb{R}$

$\mathbb{N} \subset \mathbb{R}$
$\mathbb{Z} \subset \mathbb{R}$
$\mathbb{Q} \subset \mathbb{R}$
$\mathbb{R} \setminus \mathbb{Q} \subset \mathbb{R}$

Remarks from the Triangle Inequality

(i) Equivalent Statements: $|x - y| = |-(y - x)| = |y - x|$
(ii) Triangle Inequality on Subtraction: $|x - y| \leq |x| + |y|$
(iii) Triangle Inequality on Three Elements: $|x + y + z| \leq |x| + |y| + |z|$
It follows by PMI that $\forall\ n \in \mathbb{N}$
$|a_1 + a_2 + \ldots + a_n| \leq |a_1| + |a_2| + \ldots + |a_n|$

Remarks of Uncountability (Theorem 22)

(i) The Union of two Countable sets is still Countable
(ii) If A is Uncountable and B is Countable, then $A \setminus B$ is Uncountable
(iii) $\mathbb{Q}$ is Countable
(iv) $\mathbb{R} \setminus \mathbb{Q}$ is Uncountable

Subsequences

$\{x_n\}_{n=1}^{\infty}$ is a subsequence of itself
Tails: the ending subseqences of a sequence
Since the subscripts of n_k of any subsequence must satisfy
$1 \leq n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$, one can use PMI to show that
$n_k \geq k\ \forall\ k \in \mathbb{N}$

Axioms of $\mathbb{R}$

$\forall\ a,\ b,\ c \in \mathbb{R}$
Note: We may write $a \cdot b$ as ab, and x^{-1} as $\frac{1}{x}$

A1 (Closure Under Addition)

$a + b \in \mathbb{R}$

A2 (Commutativity of Addition)

$a + b = b + a$

A3 (Associativity of Addition)

$(a + b) + c = a + (b + c)$

A4 (Existence of an Additive Identity)

$\exists\ 0 \in \mathbb{R}$ such that $\forall\ x \in \mathbb{R}$
$x + 0 = x = 0 + x$

A5 (Existence of Additive Inverses)

$\forall\ x \in \mathbb{R},\ \exists\ -x \in \mathbb{R}$ such that
$x + (-x) = 0 = (-x) + x$

M1 (Closure Under Multiplication)

$a \cdot b \in \mathbb{R}$

M2 (Commutativity of Multiplication)

$a \cdot b = b \cdot a$

M3 (Associativity of Multiplication)

$(a \cdot b) \cdot c = a \cdot (b \cdot c)$

M4 (Existence of a Multiplicative Identity)

$\exists\ 1 \in \mathbb{R} \setminus \{0\}$ such that $\forall\ x \in \mathbb{R}$
$x \cdot 1 = x = 1 \cdot x$

M5 (Existence of a Multiplicative Inverses)

$\forall\ x \in \mathbb{R} \setminus \{0\},\ \exists\ x^{-1} \in \mathbb{R}$ such that
$x \cdot (x^{-1}) = 1 = (x^{-1}) \cdot x$

D (Distributive Law)

$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
$(b + c) \cdot a = (b \cdot a) + (c \cdot a)$

O1 (Trichotomy)

$\mathbb{R}$ is an ordered set under a relation $<$, i.e.,
(i) $\forall\ a,\ b \in \mathbb{R}$. exactly one of the following holds:
$a < b,\ a = b,\ b < a$
(ii) $\forall\ a,\ b,\ c \in S$, if $a < b$ and $b < c$,
then $a < c$

O2 (Less Than)

If a, b, $c \in \mathbb{R}$ and $a < b$, then
$a + c < b + c$

O3 (Mult)

If a, b, $c \in \mathbb{R}$, $a > 0$ and $b > 0$, then
$a \cdot b > 0$

C (Completeness)

If a nonempty subset of E of $\mathbb{R}$ is bounded above, then $sup(E)$ exists in $\mathbb{R}$
If a nonempty subset of E of $\mathbb{R}$ is bounded below, then $inf(E)$ exists in $\mathbb{R}$