TEXAS
The University of Texas at Austin

TOPIC 4 NONLINEAR PROGRAMMING

Quadratic Form

- What is $\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} c_1 & c_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$?

Quadratic Programming

- A special type of NLP is called quadratic programming
  - Allows quadratic terms in the objective
- The general form of QP is
  - $\min_{x} x^T Q\, x + c^T x$
  - s.t. $A\, x \leq b,\ \ x \geq 0$
- If x has n decision variables, then Q is an n x n matrix, and c is n x 1 (just like for LP)
- In gurobi, the only extra step is to tell Q to the objective

Portfolio Optimization

- Some of you may have seen Markowitz portfolio optimization before
  - Minimize risk (volatility) subject to mean return constraint
  - It is the basis for the finance project
- We will pose it as a QP
- Ordinary Least Squares Regression is also a QP
  - This will be the basis for the non-finance project

Portfolio Optimization

- Our objective is to pick portfolio weights to get the portfolio’s variance/standard deviation to be as small as possible
- Our constraints are
  - Portfolio weights must sum to 1
  - No short selling stock (no negative weights)
  - The mean return of the portfolio must be bigger than some threshold, R

Portfolio Variance

- We can write the portfolio variance using the quadratic form we saw earlier
  - portfolio return = $\sum_{i=1}^{n} w_i r_i$
  - $Var(\sum_{i=1}^{n} w_i r_i) = w^T \Sigma w$
  - Σ is the covariance matrix: $\Sigma_{ij} = Cov(r_i, r_j)$

- Why: Var(aX+bY) = a^2 var(X) + b^2 var(Y) + 2 ab cov(X,Y)

Portfolio Mean

- The portfolio's mean return is
  - $E(\sum_{i=1}^{n} w_i r_i) = \sum_{i=1}^{n} w_i E[r_i] = \bar{r}^T w$
  - $\bar{r}$ is the vector of each stock's mean return

Portfolio Optimization

- We can write the optimization problem as
- $\min_{w} w^T \Sigma w$
- s.t.
  - $\bar{r}^T w \geq R$
  - $1^T w = 1$
  - $w \geq 0$

Example

- Assume there are 3 stocks with mean returns:
  - (10.73%, 7.37%, 6.27%)
- The covariance matrix between the 3 stocks is
  - $\Sigma = \begin{pmatrix} 0.0278 & 0.00387 & 0.000207 \\ 0.00387 & 0.0111 & -0.000195 \\ 0.000207 & -0.000195 & 0.00116 \end{pmatrix}$
- Find the portfolio with the smallest possible variance that achieves a mean return of at least 9%