TOPIC 3 INTEGER PROGRAMMING

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Integer Programming

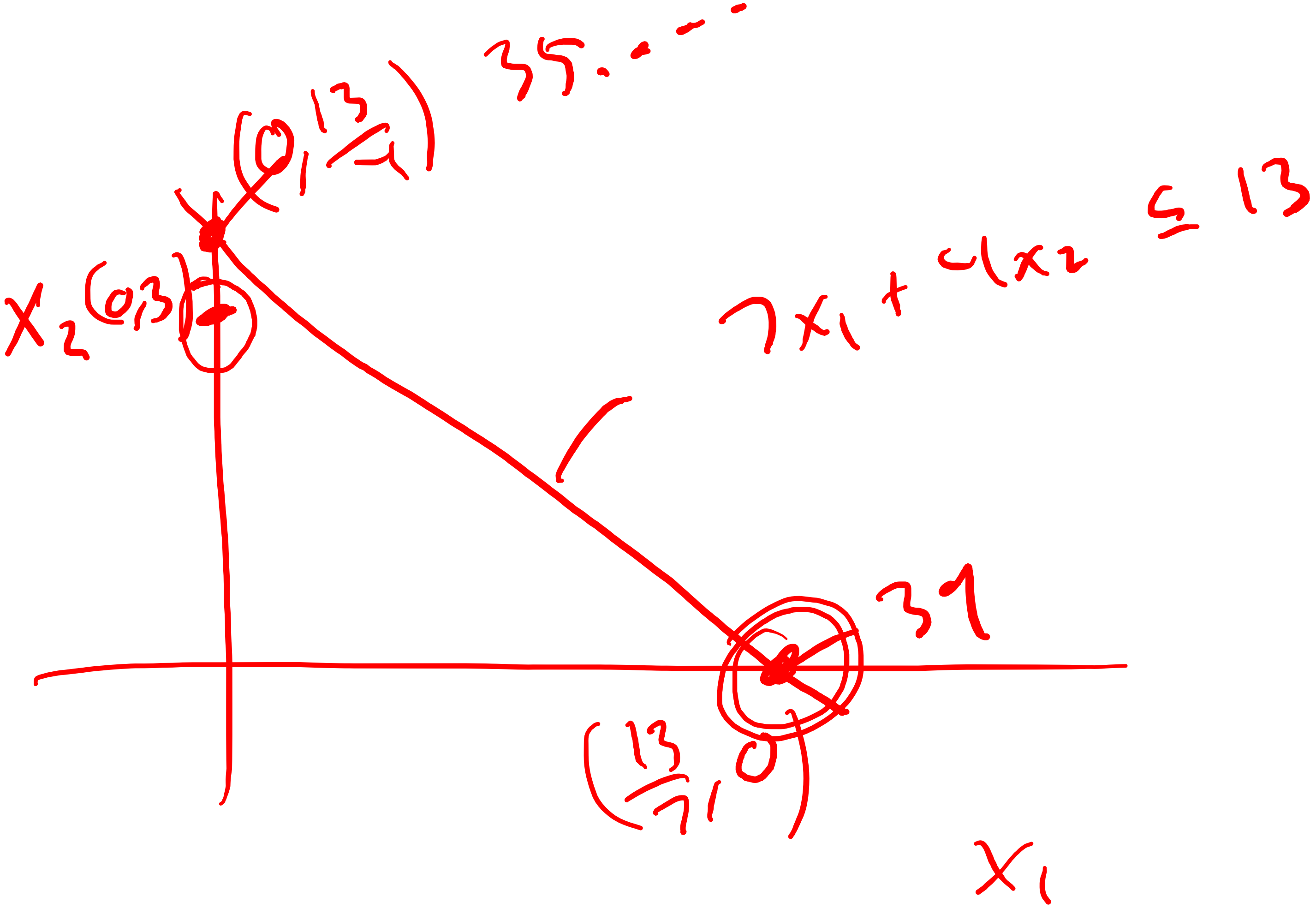
- Integer Programming (IP) : Optimization models in which some or all of the variables must be integers
  - Generally, when we say ‘IP’, we mean integer restricted linear programs
  - If only some variables are restricted to be integers, we call it a mixed integer program (MIP)
- Main difference between LP and IP models
  - LP models allow fractional values
  - IP models allow only integer values for integer-constrained variables

Integer Programming

- Let's start with a simple problem
- $\max 21x_1 + 11x_2$
- Such that
  - $7x_1 + 4x_2 \leq 13$
  - $x_1, x_2 \geq 0$
  - x_1, x_2 are integers

Class Participation

- Solve that problem!


(0, 13/4)
35....
7x1 + 4x2 ≤ 13
x2 (0,3)
37
(13/7, 0)
x1

The Solution

- The optimal LP solution is at (13/7,0) and yields an objective value of 39.
- Should we just round this?
  - up or down?
  - Or should we look simply for the closest integer point to (13/7,0)?
- The optimal integer value is at (0,3) yielding an objective value of 33.
- Clearly the corner hopping method that worked for LPs will not work. IPs are combinatorial problems and enumeration mechanisms will generally be needed.

Integer Programming

- Solving IPs requires enumeration
- It is an NP-hard problem!
- Enumerating integer solutions is close to the order 2^N
  - N is number of variables…
- However, we can be smart about this, leverage on our ability to solve LPs very easily and get a way-better-than brute force (usually) mechanisms to solve IPs
- We still won't have guaranteed converge rates, but practically we can solve many IPs quickly
  - For the project, there will be one VERY slow IP

Branch and Bound

- Branch and Bound is an algorithm for solving IPs
- How does the algorithm work
  - Solve the problem as an LP
  - If the optimal corner is non-integer
    - Add some constraints that cut off that corner and replace it with corners that have integers
  - Repeat until optimality
- I will post a video of me doing branch and bound

IP Example

- Let’s use gurobi to solve the following problem
- $\max 5x_1 + 4x_2$
- Such that
  - $x_1 + x_2 \leq 5$
  - $10\ x_1 + 6\ x_2 \leq 45$
  - $x_1, x_2 \geq 0$
  - x_1, x_2 are integers

obj $= (5, 4)$

$A = \begin{pmatrix} 1 & 1 \\ 10 & 6 \end{pmatrix}$ direction $= \begin{pmatrix} \leq \\ \leq \end{pmatrix}$

rhs $= \begin{pmatrix} 5 \\ 45 \end{pmatrix}$

Binary Integer Programming

- To think of IPs as LPs with integer restrictions, while quite right, does not adequately describe the power of IPs to model various kinds on constraints that are not possible in LP formulations
- A very powerful class of IPs are called Binary-IPs. This is when variables are restricted to be binary (that is, 0 or 1)
- Binary variables are extremely powerful in modeling real life situations. The binary variable usually corresponds to an activity that either is or is not undertaken
- When we use binary variables, we can create a whole new type of constraint called logical constraints

Binary Program

- Say you are considering seven investments.
- Cash required for each investment and the net present value (NPV) each investment add is given.
- The cash available is $15,000.
- You seek an investment policy to maximize NPV.
- You must “go all the way” or not at all on all investments.

Investment cost	5000	2500	3500	6000	7000	4500	3000
NPV	16000	8000	10000	19500	22000	12000	7500

Class Participation

- Formulate and solve this as an IP

Logical Constraints

- To pose this as an integer programming problem
  - Just put an upper bound of 1 on all variables $x_1 + x_2 + \cdots + x_7 \leq 2$
- But what if we can pick a maximum of 2 projects?
- What if project 2 is picked, 3 has to be picked?
- What if project 2 is picked, 5 cannot be picked?
- What if project 2 can only be picked if project 4 is?
- These types of constraints are called logical constraints

Logical Constraints

- What if we can pick a maximum of 3 projects?

$x_1 + x_2 + x_3 + \cdots + x_7 \le 3$

- What if project 2 is picked, 5 cannot be picked?

$x_2 \le 1 - x_5$

0 or 1

- What if project 2 is picked, 3 has to be picked?

$x_2 \le x_3$

1 or 0

- What if project 2 can only be picked if project 4 is?

$x_2 \le x_4$

- Project 3 can only be picked only if 1 and 2 are

1 or 0

$x_3 \le \frac{x_1 + x_2}{2} \Rightarrow 2x_3 \le x_1 + x_2$

Another Example

- There are five opportunities to build factories labeled 1, 2, 3, 4, and 5. The factories build different items and have the costs and payoffs listed in the following table.
- The problem is to maximize profit, which is payoff minus cost.
- The payoff is the sum of the units manufactured times the payoff/unit.
- You have $125 available to build and produce items.
- The cost per item manufactured is the buy-in cost plus the cost/unit times the number of units if you manufacture at least one unit; otherwise, it is 0.
- The cost is the sum of the costs per item.

Investment	Buy-In Cost	Cost/Unit	Payoff/Unit	Max # Units
1	$25	$5	$15	5
2	$35	$7	$25	4
3	$28	$6	$17	5
4	$20	$4	$13	7
5	$40	$8	$18	3

$x_1, x_2, \dots x_5$ – # manufactured → continuous

$y_1\ y_2 \dots y_5$ – on/off factory → Binary

$$\max_{x,y}\ 16x_1 + 18x_2 + 11x_3 + 9x_4 + 10x_5 - 25y_1 - 35y_2 - 28y_3 - 20y_4 - 40y_5$$

s.t. $5x_1 + 7x_2 + 6x_3 + 4x_4 + 8x_5 + 25y_1 + 35y_2 + 28y_3 + 20y_4 + 40y_5 \le 125$

$UB = (5, 4, 5, 7, 3, 1, 1, 1, 1, 1)$

$$x_i - My_i \le 0 \quad i = 1, 2, 3, 4, 5$$

y_1

0

$x_1 = 0$

$0 \le x_1 \le 0$

$0 \le x_1 \le M y_1$

1

$0 \le x_1 \le \cancel{\infty}\ M$

$0 \le x_1 \le M y_1$

$x_1 - M y_1 \le 0$

big-M constraint

Additional Constraints

- Class participation

- You may not invest in both 2 and 5.

$y_2 + y_5 \leq 1$

- You may invest in 1 only if you invest in at least one of 2 and 3.

$y_1 \leq y_2 + y_3$

- You must invest in at least two of 3, 4, and 5.

$y_3 + y_4 + y_5 \geq 2$

Multi-Period Example – Class Participation

- The Pigskin Company produces footballs. Must decide how many footballs to produce each month.
- Forecasted monthly demands and prod costs are given below. There is also a fixed cost of $50,000 in any month there is production.
- Pigskin wants to meet these demands on time.
- For simplicity, assume production occurs during the month and demand occurs at the end of the month.
- Currently has 5,000 footballs in inventory. Enough storage capacity to store up to 20,000 footballs at the end of the month, after demand has occurred. The holding cost per football held in inventory at the end of any month is figured at 5% of the production cost (per football) for that month.

Month	1	2	3	4	5	6
Demand	10000	15000	30000	35000	25000	10000
Production cost/unit	$12.50	$12.55	$12.70	$12.80	$12.85	$12.95

$X_{1,\dots 6}$ – # manufacture
↳ Contours

$Y_{1,2,\dots 6}$ – on/off factory

$I_{0\,1\,2\,3\,4\,5\,6}$ – Inventory
slack

$$\min \quad 12.50x_1 + 12.55x_2 + 12.7x_3 + 12.8x_4 + 12.85x_5 + 12.95x_6$$
$$+ 50k\,Y_1 + 50k\,Y_2 + 50k\,Y_3 + 50k\,Y_4 + 50k\,Y_5 + 50k\,Y_6$$
$$+ \frac{12.50}{20} I_1 + \frac{12.55}{20} I_2 + \frac{12.7}{20} I_3 + \frac{12.8}{20} I_4 + \frac{12.85}{20} I_5 + \frac{12.95}{20} I_6$$

$$X_i - MY_i \le 0 \quad i = 1,2,3,4,5,6$$

$$X_i + I_{i-1} = D_i + I_i \quad i = 1,2,3,4,5,6$$

UB: $I_i \le 20K$

Traveling Salesperson Problem

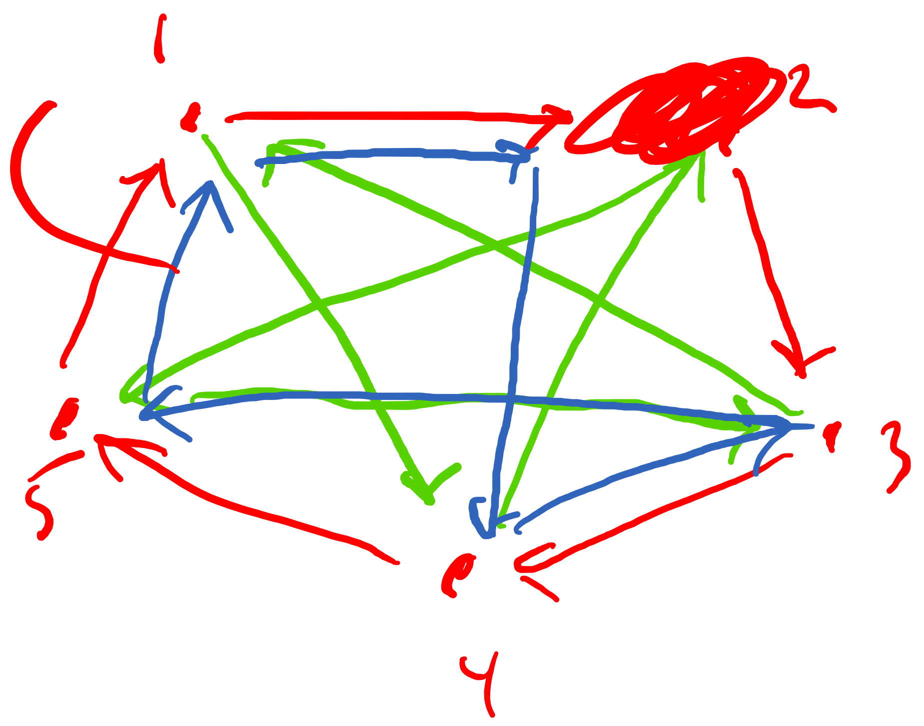
- Jessica lives in Gary, Indiana. She owns insurance agencies in Gary, Fort Wayne, Evansville, Terre Haute and South Bend. Each December she visits each of her insurance agencies. The distance between each agency, in miles, is shown in the table below. What order of visiting her agencies will minimize the total distance travelled?

Day	Gary	Fort Wayne	Evansville	Terre Haute	South Bend
City 1 Gary	1000	132	217	164	58
City 2 Fort Wayne	132	1000	290	201	79
City 3 Evansville	217	290	1000	113	303
City 4 Terre Haute	164	201	113	1000	196
City 5 South Bend	58	79	303	196	1000

decision variables: x_{ij} — travel from city i to j or not

↳ binary

$$\min \sum_{i=1}^{5} \sum_{j=1}^{5} d_{ij} x_{ij}$$



s.t. one path out of each node

one path into each node

$$\sum_{j=1}^{5} x_{ij} = 1 \quad \text{for } i = 1,2,3,4,5$$

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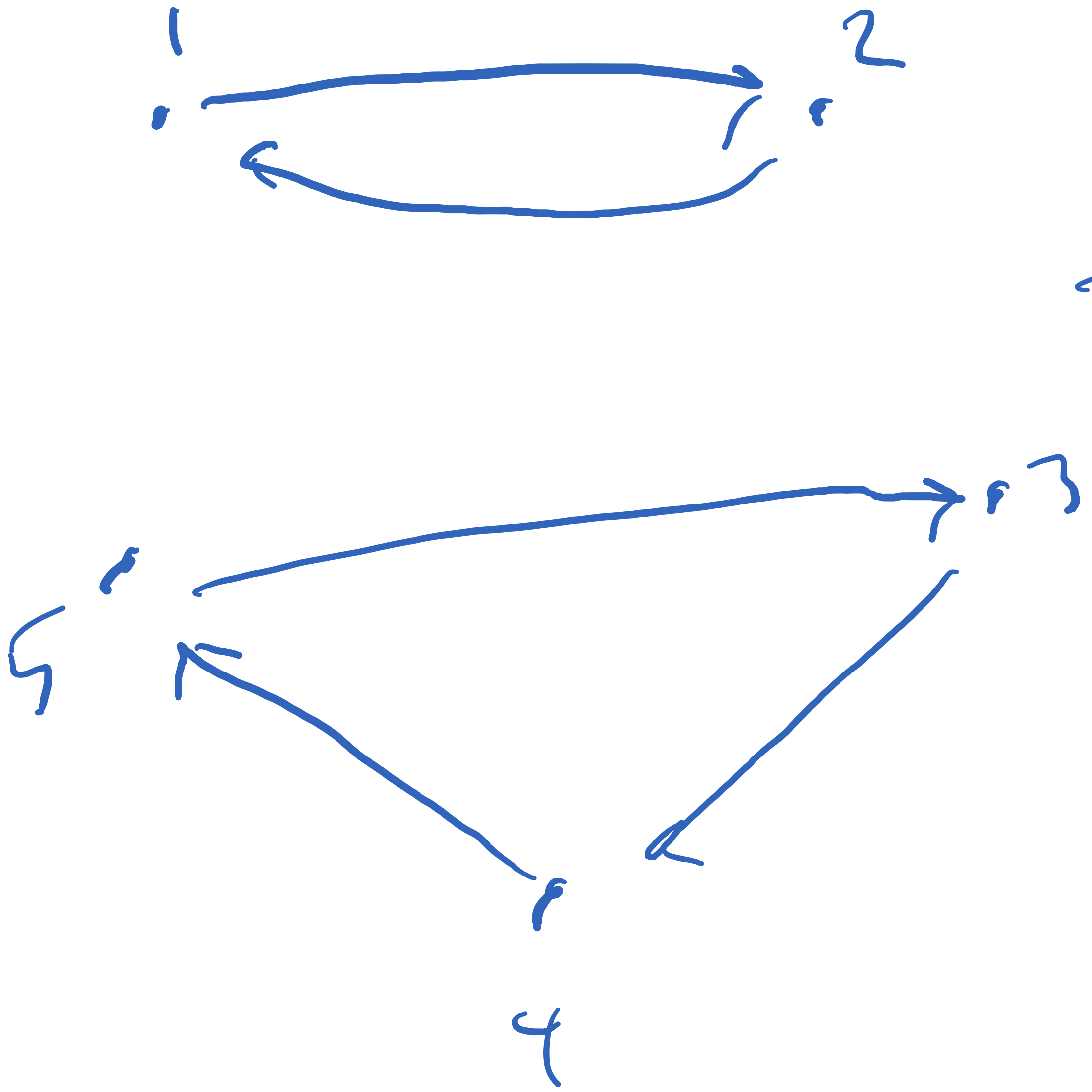


sub loop

TSP Solution

$$\min z = \sum_i \sum_j c_{ij} x_{ij}$$

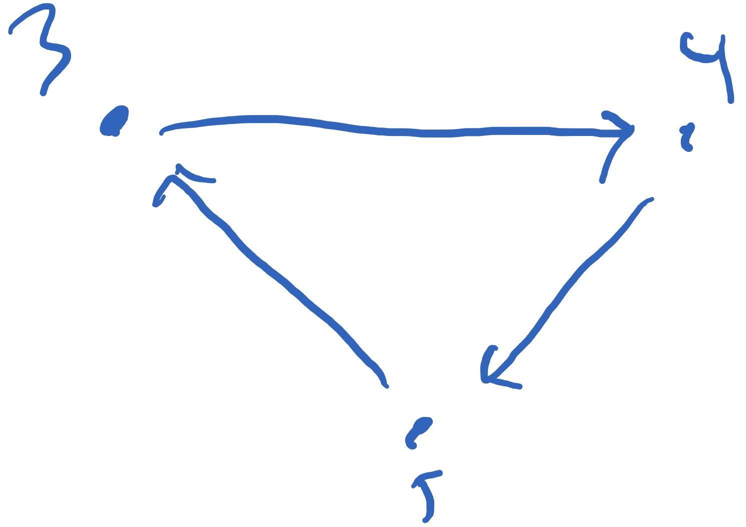
$$\text{s.t.} \quad \sum_{i=1}^{i=N} x_{ij} = 1 \quad (\text{for } j = 1, 2, \ldots, N)$$

$$\sum_{j=1}^{j=N} x_{ij} = 1 \quad (\text{for } i = 1, 2, \ldots, N)$$

$$u_i - u_j + N x_{ij} \leq N - 1 \quad (\text{for } i \neq j;\ i = 2, 3, \ldots, N;\ j = 2, 3, \ldots, N)$$

$$\text{All } x_{ij} = 0 \text{ or } 1, \text{ All } u_j \geq 0$$

$(N-1)^2 - (N-1)$



$$x \le y$$
$$w \le z$$
$$x + w \le y + z$$

$$\cancel{u_3} - \cancel{u_4} + 5x_{34} \le 4$$
$$\cancel{u_4} - \cancel{u_5} + 5x_{45} \le 4$$
$$\cancel{u_5} - \cancel{u_3} + 5x_{53} \le 4$$

$$5(x_{34} + x_{45} + x_{53}) \le 3 * 4$$