TOPIC 2 LINEAR PROGRAMMING

TEXAS
The University of Texas at Austin

Example

- A glass manufacturer makes 6 oz and 10 oz juice glasses
- How many cases of each type of glass should be produced per week in order to maximize profits?
- 100 cases of 6 Oz juice glasses require 6 production hours & 100 cases of 10 Oz juice glasses require 5 hours
- 60 hours of production capacity available per week
- A case of 6 Oz glasses requires 10 cubic feet of storage space and a case of 10 Oz glasses requires 20 cubic feet
- Storage capacity of 15,000 cubic feet is available per week
- 6 Oz glasses make $5.00 per case and 10 Oz glasses make $4.50 per case
- Customers will not accept more than 800 cases per week of 6 Oz glasses
- There is no limit on the amount that can be sold of 10 Oz glasses

y_1 = # cases small y_2 = # large cases

$$\frac{10y_1 + 20y_2}{100} \le \frac{15000}{100}$$

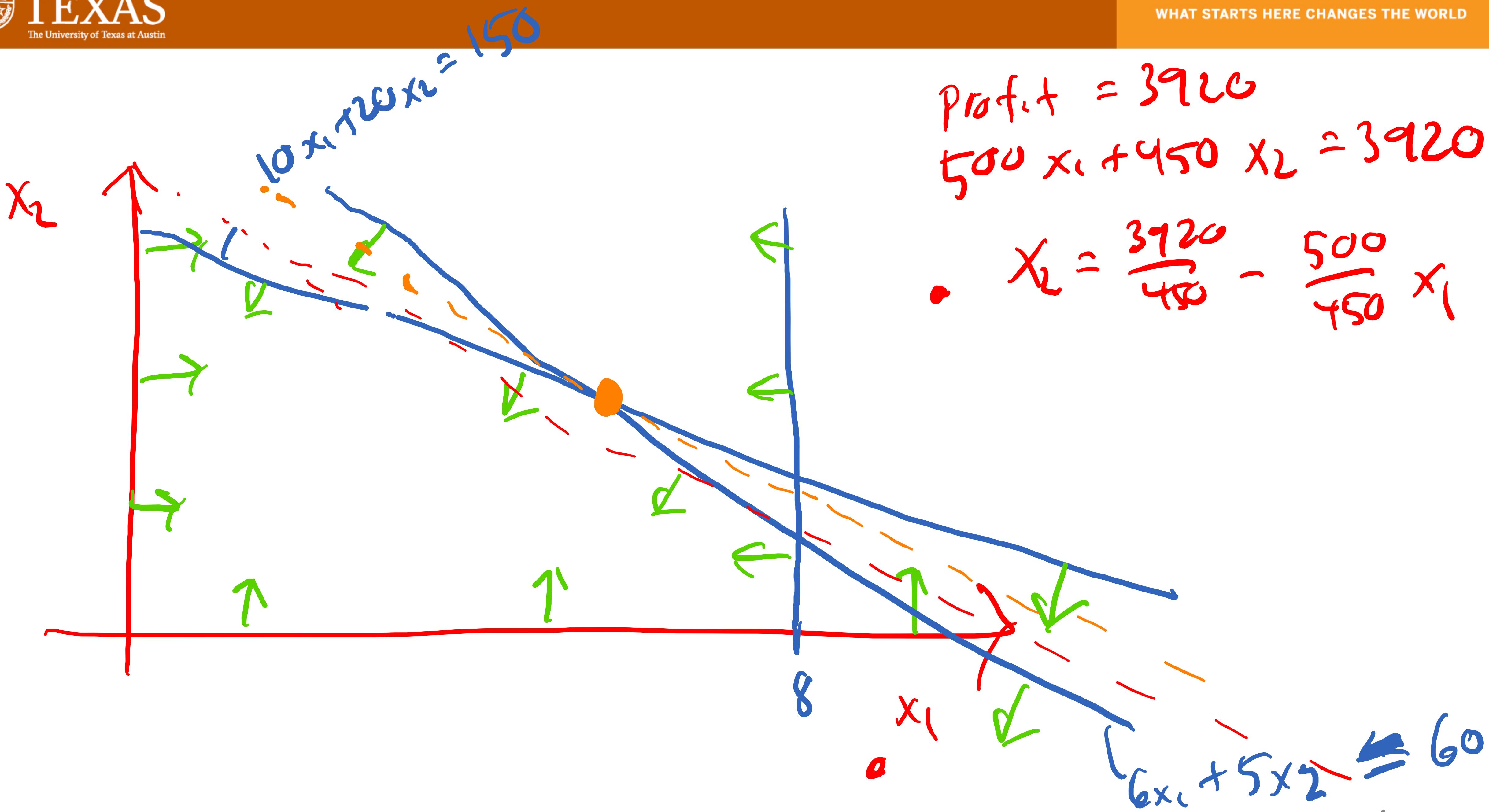
Example

- Choose
  - x_1, x_2 (# cases of 6oz and 10oz, in 100s)
- To maximize
  - $500x_1 + 450x_2$ (total profit)
- Subject to
  - $6x_1 + 5x_2 \le 60$ (production hours)
  - $10x_1 + 20x_2 \le 150$ (storage)
  - $x_1 \le 8$ (6oz demand)
  - $x_1 ,\ x_2 \ge 0$ (non-negativity)

Profit = 3920

$500x_1 + 450x_2 = 3920$

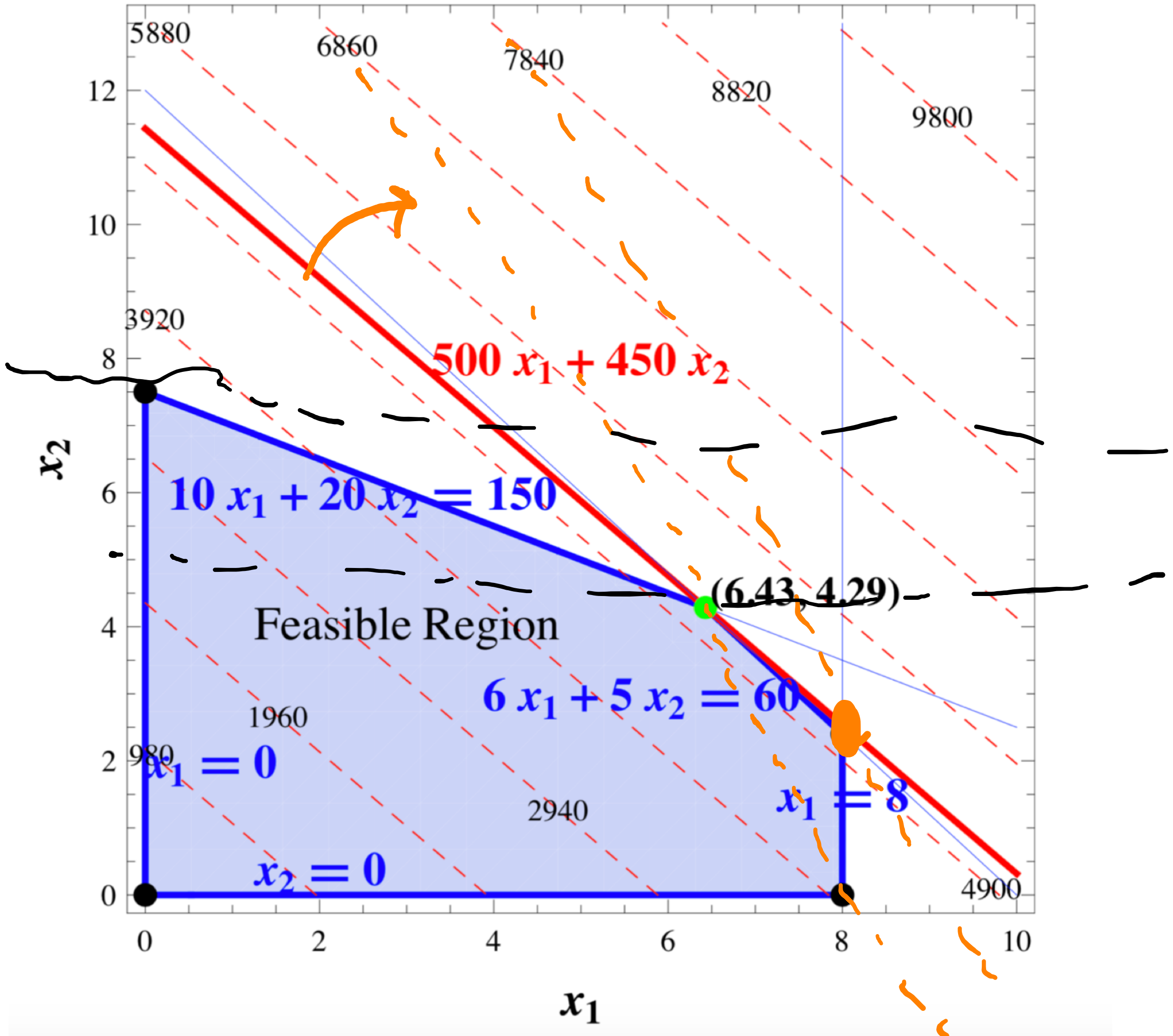
- $x_2 = \frac{3920}{450} - \frac{500}{450}x_1$



A Graphical Solution

- The constraints together dictate the feasible region. That is, the set of x_1, x_2 that satisfy all the constraints
- To see which feasible point maximizes the objective, it is useful to draw a sequence of lines where, for each line, the objective is a constant.
- A typical line is of the form $500\ x_1 + 450\ x_2 = c$, where c is a constant.
- We want to find the biggest constant, c, with at least one point on the line satisfying all the constraints

Graphical Solution



- The blue region is the feasible region
- Each solid blue line represents a constraint in the problem
- The dashed red lines are the iso-lines of the objective function
- The solid red line is the iso curve with the largest constant that has 1 point in the feasible region
- The greed dot represents the optimal x value

Class Participation

- If we increased the profits of 6oz glasses a bit, what do you think would happen to the optimal policy?
- If we increased the profits of 10oz glasses a bit, what do you think would happen to the optimal policy?

Linear Programming

- All we have to do is check the corners!!!

Solving LPs

- Graphical solutions are intuitive but can only handle toy problems with two decision variables
- Almost all optimization software, including gurobi, use a standard form to express linear programs
- The standard form for linear programs is:
  - maximize/minimize $c^T * x$
  - subject to:
    - $A*x$ $(\leq, =, \geq)$ b, $x>=0$
- What are c, A and b in the OJ example?

$$\underbrace{\max}\quad \underbrace{(500,\ 450)}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

s.t.

$$\underbrace{\begin{bmatrix} 6 & 5 \\ 10 & 20 \\ 1 & 0 \end{bmatrix}}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \underbrace{\begin{pmatrix} \le \\ \le \\ \le \end{pmatrix}} \underbrace{\begin{pmatrix} 60 \\ 150 \\ 8 \end{pmatrix}}$$

$$x_1, x_2 \ge 0$$

Solving with Gurobi

- Once we have c, A, b we can plug these into a solver
- Then the software will solve the problem for us
- Let’s double check the previous solution

Linear Programming

- We just introduced optimization, one of the most powerful and flexible methods of quantitative analysis.
- The specific type of optimization we discussed is linear programming (LP).
- LP is used in many organizations, often daily, to solve a variety of problems:
  - Risk management
  - Labor scheduling
  - Inventory management
  - Selection of advertising media
  - Bond trading, etc.

Optimization

- There are two basic steps in solving an optimization problem:
  - Model development step
  - Optimization step
- Model development is where we will spend almost all our time
- To optimize means that you systematically choose the values of the decision variables that make the objective as large (for maximization) or small (for minimization) as possible and cause all the constraints to be satisfied.
- Most of the published research is about the optimization step
  - One algorithm for searching through the feasible region is called the simplex method. George Dantzig

Optimization

- All optimization models have several common elements:
  - Decision variables, the variable whose values the decision maker is allowed to choose. The values of these variables determine outputs such as total cost, revenue, and profit.
  - An objective function (objective, for short) to be optimized – minimized or maximized.
  - Constraints that must be satisfied. They are usually physical, logical, or economic restrictions, depending on the nature of the problem.
  - Non-negativity constraints are very common. They state that decision variables must have nonnegative (zero or positive) values. Non-negativity constraints are usually included for physical reasons. For example, it is impossible to produce a negative number of automobiles.

Optimization

- Any set of values of the decision variables that satisfies all the constraints is called a feasible solution.
- The set of all feasible solutions is called the feasible region.
- An infeasible solution is a solution that violates at least one constraint.
- The desired feasible solution is the one that provides the best value – minimum for a minimization problem, maximum for a maximization problem – for the objective. This solution is called the optimal solution.

Example

- A trading company is looking for a way to maximize profit per transportation of their goods. The company has a train available with 3 wagons.
- When stocking the wagons they can choose between 4 types of cargo, each with its own specifications.
- How much of each cargo type should be loaded on which wagon in order to maximize profit?
- The following constraints must be taken in consideration;
  - Weight capacity per train wagon
  - Volume capacity per train wagon
  - Limited availability per cargo type

TRAIN WAGON	WEIGHT CAPACITY (TONNE)	SPACE CAPACITY (M^3)
w1	10	5000
w2	8	4000
w3	12	8000

CARGO TYPE	AVAILABLE (TONNE)	VOLUME (M^3 /Ton)	PROFIT (PER TONNE)
c1	18	400	2000
c2	10	300	2500
c3	5	200	5000
c4	20	500	3500

Let x_i = tons of good i in wagon 1, y_i = tons of good i in wagon 2, z_i = tons of good i in wagon 3

max

$$\left.\begin{array}{l} 2000x_1 + 2500x_2 + 5000x_3 + 3500x_4 + \\ 2000y_1 + 2500y_2 + 5000y_3 + 3500x_4 + \\ 2000z_1 + 2500z_2 + 5000z_3 + 3500z_4 \end{array}\right\} \text{obj}$$

$$\left.\begin{array}{l} x_1 + x_2 + x_3 + x_4 \le 10 \\ y_1 + y_2 + y_3 + y_4 \le 8 \\ z_1 + z_2 + z_3 + z_4 \le 12 \end{array}\right\} \text{weight capacity constraints}$$

$$400x_1 + 300x_2 + 200x_3 + 500x_4 \le 5000$$
$$400y_1 + 300y_2 + 200y_3 + 500y_4 \le 4000$$
$$400z_1 + 300z_2 + 200z_3 + 500z_4 \le 8000$$

volume constraints

$$x_1 + y_1 + z_1 \le 18$$
$$x_2 + y_2 + z_2 \le 10$$
$$x_3 + y_3 + z_3 \le 5$$
$$x_4 + y_4 + z_4 \le 20$$

availability constraints

$$x_1, x_2, \ldots, z_4 \ge 0$$

$\max \quad c^T x$

$\text{st} \quad Ax \le b$ any 3 directions

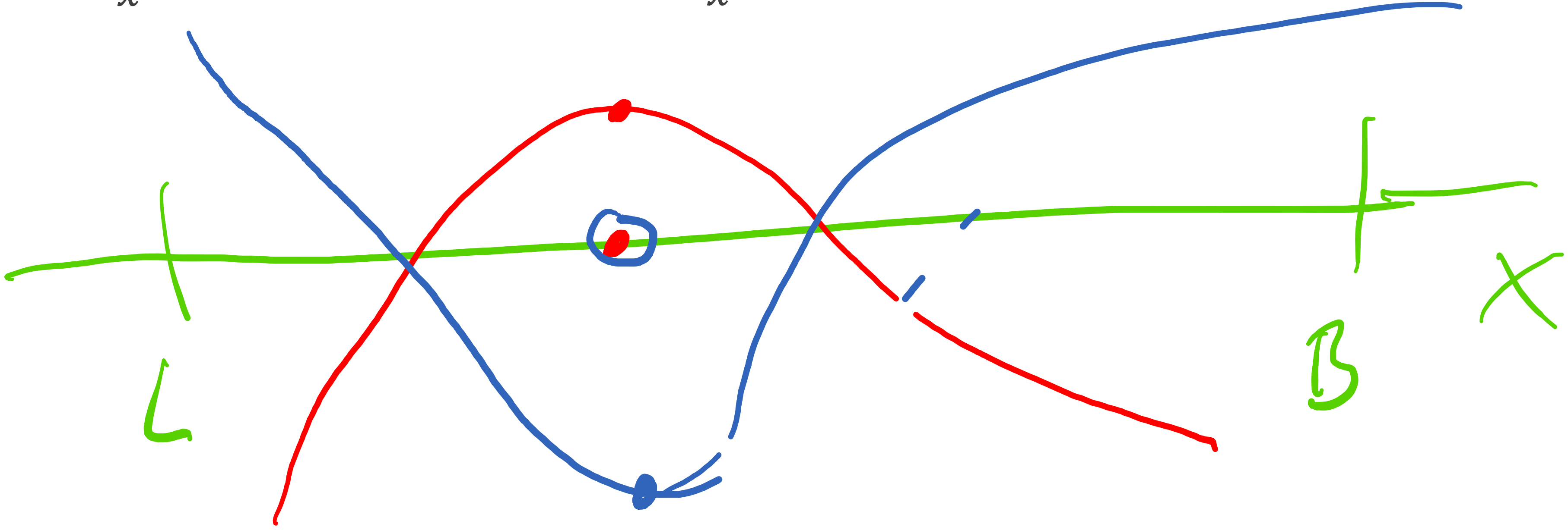
$x \ge 0$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \qquad c = \begin{pmatrix} 2000 \\ 2500 \\ 3000 \\ 3500 \\ 2000 \\ 2500 \\ 5000 \\ 3500 \\ 2000 \\ 2500 \\ 5000 \\ 3500 \end{pmatrix}$$

x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4	z_1	z_2	z_3	z_4	dir	rhs
1	1	1	1	0	0	0	0	0	0	0	0	$\le$	10
0	0	0	0	1	1	1	1	0	0	0	0	$\le$	8
0	0	0	0	0	0	0	0	1	1	1	1	$\le$	12
400	300	200	500	0	0	0	0	0	0	0	0	$\le$	5000
0	0	0	0	400	300	200	500	0	0	0	0	$\le$	4000
0	0	0	0	0	0	0	0	400	300	200	500	$\le$	8000
1	0	0	0	1	0	0	0	1	0	0	0	$\le$	18
0	1	0	0	0	1	0	0	0	1	0	0	$\le$	10
0	0	1	0	0	0	1	0	0	0	1	0	$\le$	5
0	0	0	1	0	0	0	1	0	0	0	1	$\le$	20

Optimization

- Suppose you have a maximization problem
- Could you repose it as a minimization problem?
  - Yes, just take the negative of the objective!
  - $\max_x f(x)$ is the same as $\min_x -f(x)$



P_1

$\max c^T x$
s.t. $constraints_1$

$$\xrightarrow{\text{constraining}}$$

$$\xleftarrow{\text{relaxation}}$$

P_2

$\max c^T x$
s.t. $constraints_1$
$constraints_2$

$objective = z_1$
$solution = x_1$

$objective = z_2$
$solution = x_2$

$$z_1 \geq z_2$$

if x_1 is feasible to con_2 $\Rightarrow$ x_1 is an optimal solution to P_2

Binding and Non-Binding Constraints

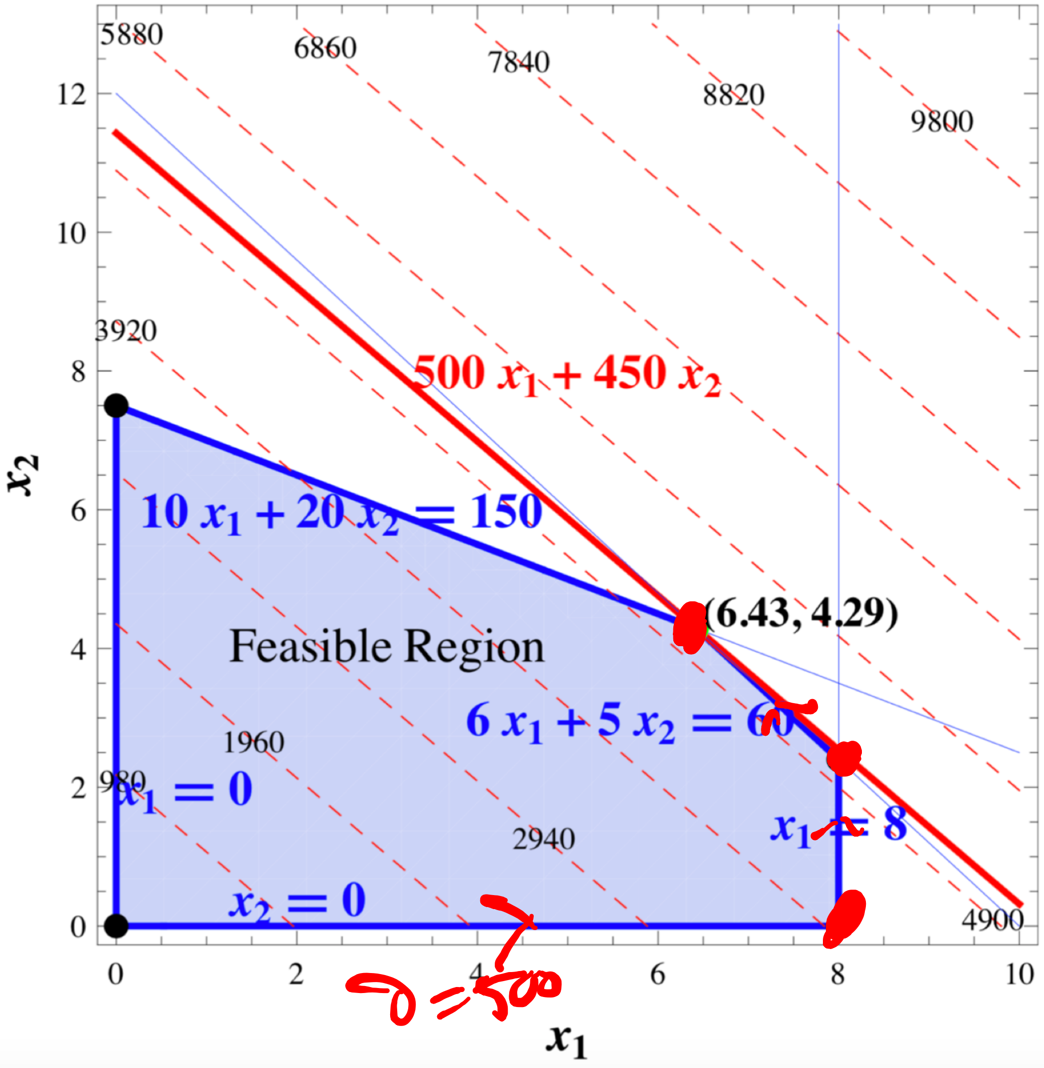
- Of all the inequality constraints, some are satisfied exactly, and others are not
- An inequality constraint is binding if the solution makes it an equality. Otherwise, it is non-binding
- The positive difference between the two sides of the constraint is called the slack

Class Participation

- In the OJ problem, which constraints are binding and which are non-binding?
- For the non-binding constraints, what is the slack?

- Why is this important?

How does the Simplex Method work?



$\partial = 450$

$\partial = 500$

Example

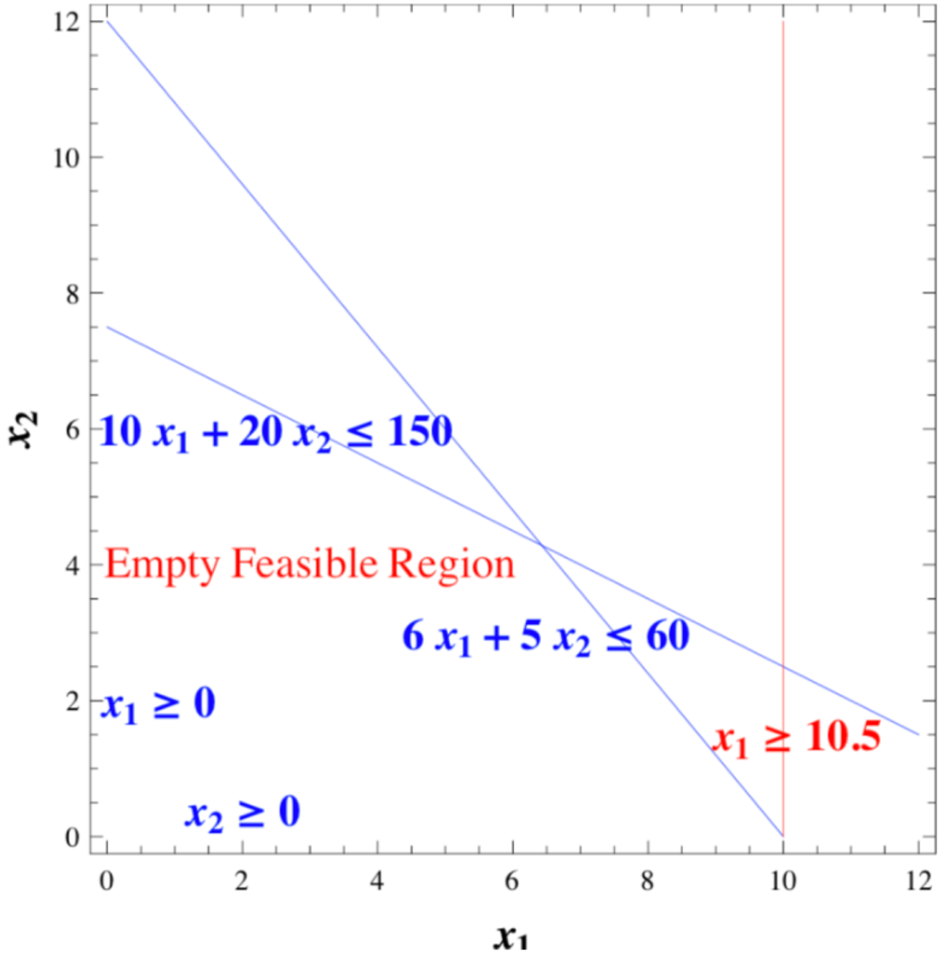
- A Crude oil refining firm distills crude petroleum from two sources, Saudi Arabia and Venezuela, into 3 main products: gasoline, jet fuel and lubricants.
- Each barrel of Saudi crude yields 0.3, 0.4 and 0.2 barrels of gasoline, jet fuel and lubricants, respectively. Similarly, each barrel of Venezuelan crude yields 0.4, 0.2 and 0.3 respectively. Remaining 10% in each case is lost to refining.
- The firm can purchase up to 9000 barrels of Saudi crude per day at $20 per barrel and up to 6000 barrels of Venezuelan crude per day at $15 per barrel.
- The firm's existing contracts require the delivery of 2000, 1500 and 500 barrels of gasoline, jet fuel and lubricants per day.
- How can these requirements be fulfilled most efficiently?
- Do this problem on your own, the R solution is on canvas

Common Problems

- In some cases, the LP returns an error message.
  - This mostly is due to a logical error in formulation.
- Infeasibility and unboundedness are the most common reasons for not finding the optimal solution.
- Because mistakes are common in LP models, you should be aware of the error messages you might encounter.

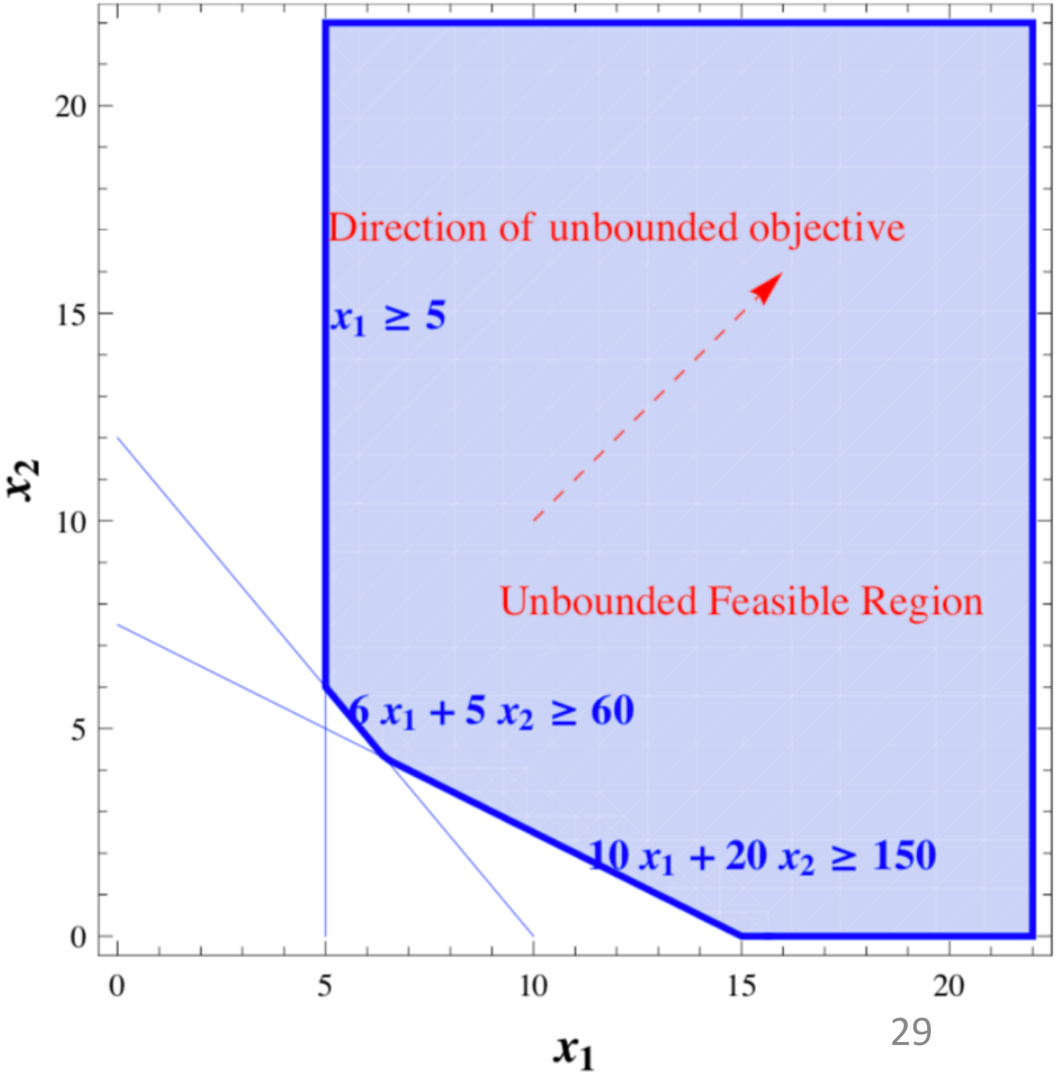
Infeasibility

- Infeasibility occurs when there are no feasible solutions to the model. There are generally two reasons:
- There is a mistake in the model (an input was entered incorrectly, such as a < symbol instead of a >), or
- The problem has been so constrained that there are no solutions left.
- A careful check of the model and finding an error or changing/eliminating some of the constraints might fix the problem.



Unboundedness

- In case of unboundedness, the model has been formulated in such a way that the objective is unbounded
  - It can be made as large (or as small, for minimization problem) as you like.
- If this occurs, you have probably entered a wrong input or forgotten some constraints.



Infeasibility and Unboundedness

- Infeasibility and unboundedness are quite different in a practical sense
  - It is quite possible for a reasonable model to have no feasible solutions. Together, they might constrain the problem so much that there are no feasible solutions left. The only way out is to change or eliminate some of the constraints.
- An unbounded problem is quite different
  - There is no way a realistic model can have an unbounded solution. If you get an unbounded error message, then you must have made a mistake: You entered an input incorrectly, you omitted one or more constraints, or there is a logical error in your model.