

TOPIC 4 NONLINEAR PROGRAMMING

Optimization

- A general optimization problem is of the form
 - $\min_x f(x)$
 - s.t. $x \in S$
- So far, we have addressed a very specific class of optimization problems
 - $f(x) = c_1x_1 + \cdots + c_nx_n$ (linear objective)
 - S is defined by $Ax \leq b$ (linear constraints)
 - S could also have integer constraints

Optimization

- What if $f(x)$ is some function that isn't linear
 - $f(x) = x_1^2 - x_2 + e^{x_3 x_2} - \sin(x_1 x_2^2)$
- What if S isn't defined by linear constraints
 - $x_1^2 + x_2^2 + x_3^2 \leq 1$
- If either of these occur, then we call the optimization problem a **nonlinear program (NLP)**

Nonlinear Programming

- Why can models become nonlinear?
 - Non-constant returns to scale (diminishing returns)
 - In supply/demand models when profit is maximized
 - In measuring goodness of fit with the sum of the squared differences
 - Models where risk is measured as the variance
 - Neural Networks
- Nonlinear models are often more realistic than linear models, but they are also more difficult to solve.

Nonlinear Programming

- With NLP we can't rely on corner hopping anymore!
 - The feasible space might not have corners!
 - The optimal value might not be on the edge of the feasible space
- For example
 - $\max \cos(|x_1| + |x_2|)$
 - s.t. $x_1^2 + x_2^2 \leq \pi$

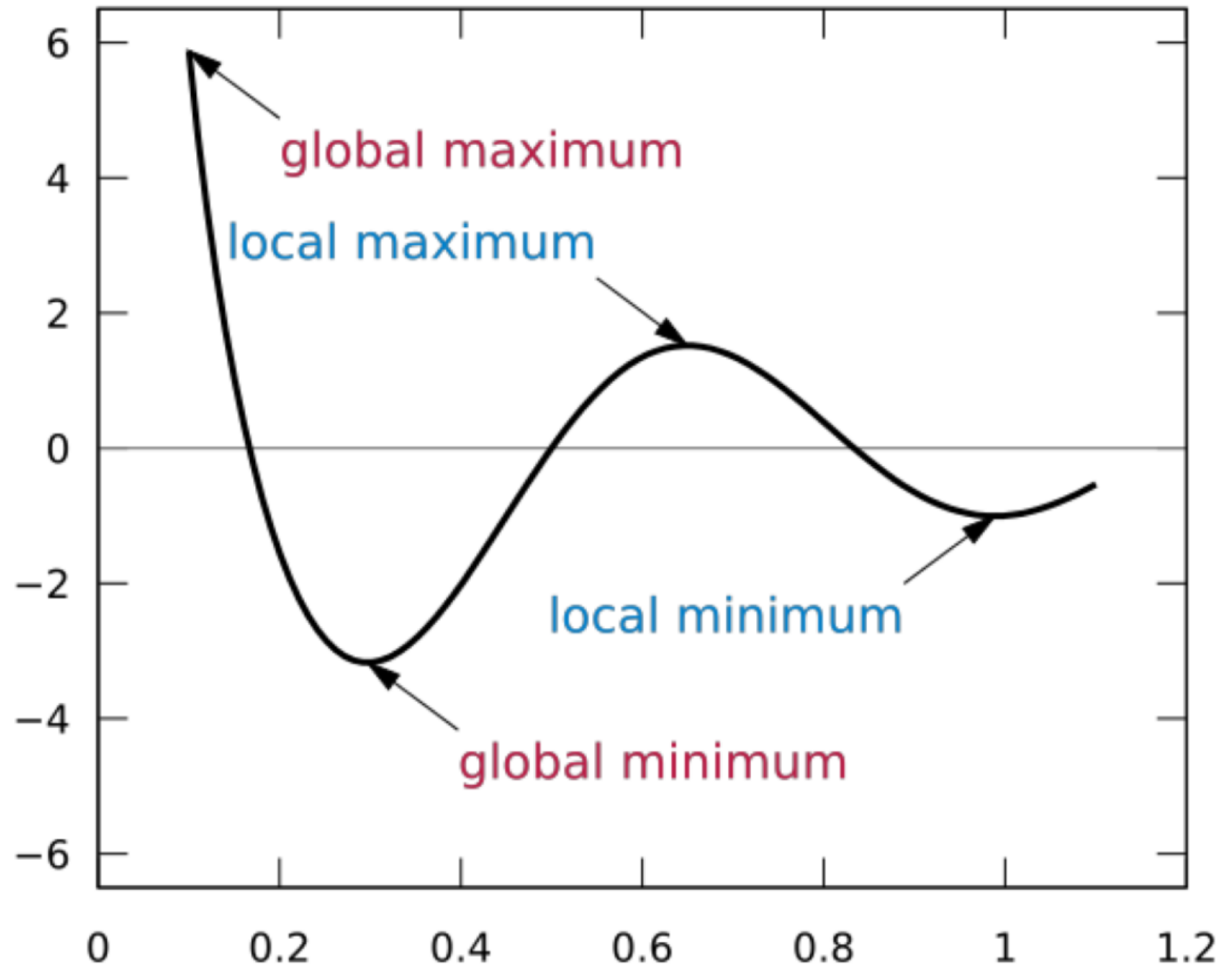
Calculus

- This brings us back to your first semester of calculus!
 - Find the slope of $f(x)$
 - Set the slope equal to zero
- $\min x^2 - 2x$

Limitations of Calculus

- What if the function is non-differentiable?
- What if the function is unknown?
- How do we handle constraints
 - Lagrange multipliers???
- Multiple places where slope is zero
- Min? Max? Saddle?
 - Second derivative test?

Global vs Local Optimum



Global vs Local Optimum

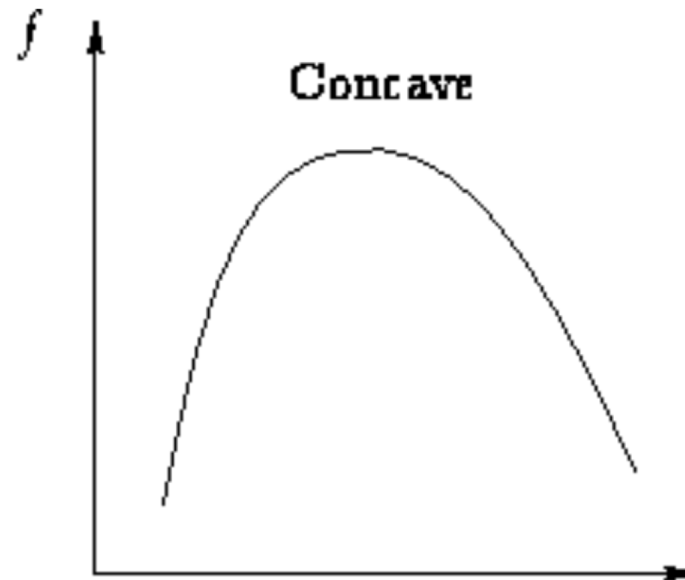
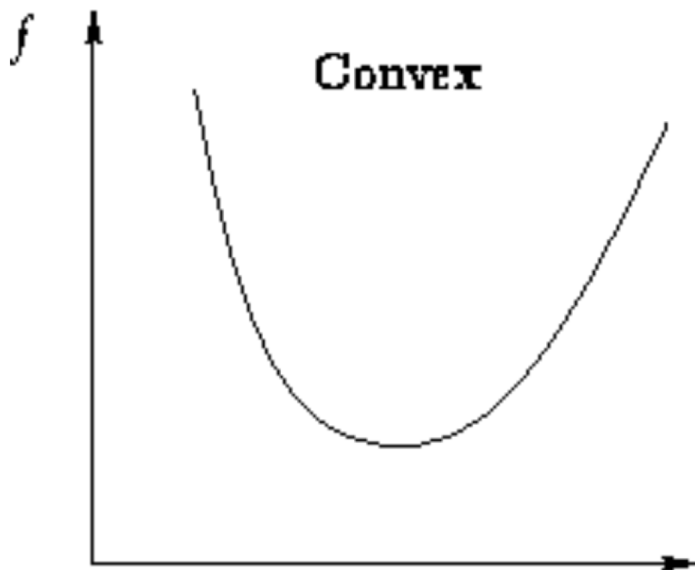
- Finding a local maximum or minimum is relatively much easier than finding the global, since it would be sufficient to check only the points around the local maximum or minimum.
- A local optimum is better than all nearby points.
- A global optimum is the best point in the entire feasible region.
- For NLP problems, some solving methods can get stuck at a local optimum and never find the global optimum.

Global vs Local Optimum

- When is a local optimum guaranteed to be global?
- Before we answer this question, we need to understand 2 concepts
 - Convex/Concave functions
 - Convex sets

Convex/Concave Functions

- A function, f , is convex if $\frac{f(x)+f(y)}{2} \geq f\left(\frac{x+y}{2}\right) \quad \forall x, y$
- A function is concave if $\frac{f(x)+f(y)}{2} \leq f\left(\frac{x+y}{2}\right) \quad \forall x, y$



Convex Set

- Convex sets are a little trickier
- A set, S , is convex if for any 2 points, x and y , that are in S , then the midpoint between them, $\frac{x+y}{2}$, is also in S

Global vs Local Optimum

- If
 - The objective function is convex (concave)
 - The feasible space is convex
- Then a local minimum (maximum) is guaranteed to be global!
- In general, we call both of these cases **convex optimization** (min or max...)

Class Participation

- Is the set defined by all points x , such that $a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b$ convex?

Global vs Local Optimum

- Why do we need the function to be convex?
- Why do we need the feasible space to be convex?

Convex Optimization

- When the optimization problem is convex it is WAY easier to find the optimal solution
 - Regression
- When the problem is non-convex it can be EXTREMELY difficult to solve!!!
 - Neural networks

Solution Algorithms

- There are LOTS of algorithms to solve NLPs
- Most of them are guaranteed to find the optimal solution if the problem is convex
- Very few algorithms exist that are guaranteed to find the solution to non-convex problems
 - Most are VERY slow
- Mostly people just modify convex algorithms to find local optimum of non-convex problems
 - Hopefully, the local optimum you find is good enough?