Numer Algor (2016) 72:749–767
DOI 10.1007/s11075-015-0065-8

ORIGINAL PAPER

Finite element method for space-time fractional diffusion equation

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Received: 15 October 2014 / Accepted: 29 September 2015 / Published online: 21 October 2015

Abstract In this paper, we consider two types of space-time fractional diffusion equations(STFDE) on a finite domain. The equation can be obtained from the standard diffusion equation by replacing the second order space derivative by a Riemann-Liouville fractional derivative of order β $(1 < \beta \leq 2)$, and the first order time derivative by a Caputo fractional derivative of order γ $(0 < \gamma \leq 1)$. For the $0 < \gamma < 1$ case, we present two schemes to approximate the time derivative and finite element methods for the space derivative, the optimal convergence rate can be reached $O(\tau^{2-\gamma} + h^2)$ and $O(\tau^2 + h^2)$, respectively, in which τ is the time step size and h is the space step size. And for the case $\gamma = 1$, we use the Crank-Nicolson

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scheme to approximate the time derivative and obtain the optimal convergence rate $O(\tau^2 + h^2)$ as well. Some numerical examples are given and the numerical results are in good agreement with the theoretical analysis.

Keywords Finite element method · Space-time fractional diffusion equation · Riesz derivative · Caputo derivative · Riemann-Liouville derivative

1 Introduction

Fractional differential equations have attracted considerable interest because of their ability to model many phenomena in fractal media, mathematical biology, chemistry, statistical mechanics, engineering and so on. Fractional derivatives play a key role in modeling particle transport in anomalous diffusion including the space fractional diffusion equation (FDE)/space fractional advection-dispersion equation (FADE) describing Lévy flights (see [2]), the time FDE/FADE depicting traps, and the time-space FDE/FADE characterizing the competition between Lévy flights and traps (see [7]). A class of FDE/FADE have been successfully used to describe non-local dependence on either time or space, to explain the development of anomalous dispersion. Some different numerical methods for solving the space or time fractional partial differential equations have been proposed. Liu et al. [2] considered the numerical solution of the space fractional Fokker-Planck equation. They transformed the space fractional Fokker-Planck equation into a system of ordinary differential equations (method of lines) that was then solved using backward differentiation formulas. Meerschaert et al. [13] presented a finite difference method to solve the one dimensional fractional advection-dispersion equations with a Riemann-Liouville fractional derivative on a finite domain. Meerschaert et al. [14] also proposed shifted Gruwald formula to solve the two-sided space-fractional partial differential equations. Liu et al. [4] considered both numerical and analytical techniques for the modified anomalous subdiffusion equation with a nonlinear source term. Fix and Roop [6] developed a least squares finite element solution of a fractional order two-point boundary value problem. Zhang et al. [9] considered the Galerkin finite element approximations of space fractional PDE. Zhuang et al. [15] proposed a new solution and implicit numerical methods for the anomalous subdiffusion equation, which involves one fractional temporal derivative in the diffusion term. Zheng et al. [23] gave a note on the finite element method for the space-fractional advection diffusion equation with non-homogeneous initial boundary condition. Roop [11] investigated the computational aspects for the Galerkin approximation using continuous piecewise polynomial basis functions on a regular triangulation of a bounded domain in R^2. Zhang et al. [19] investigated the space-fractional advection-dispersion equation (SFADE) with space dependent coefficients. Zeng et al. [20] proposed fractional linear multistep methods for time-fractional subdiffusion equation with second-order accuracy. Zheng et al. [21] proposed a novel high order space-time spectral method for the time-fractional Fokker-Planck equation. Zheng et al. [22] discussed the discontinuous Galerkin method for nonlinear fractional Fokker-Planck equation.

However, numerical methods and analysis of the fractional order partial differential equations are limited to date, methods and analysis for space-time fractional equations are less. Liu et al. [3, 5] considered the space-time fractional diffusion and advection-diffusion equation with Caputo time fractional derivative and Riemann-Liouville space fractional derivatives. Shen et al. [16] presented explicit and implicit difference approximations for the space-time Riesz-Caputo fractional advection-diffusion equation. Li et al. [1] studied the Galerkin finite element method of the time-space fractional order nonlinear subdiffusion and superdiffusion equations. Hejazi et al. [8] proposed a finite volume method to solve the time-space two sided fractional advection-dispersion equation on a one-dimensional domain. Chen et al. [12] discuss the finite difference scheme for two-dimensional space-time Caputo-Riesz fractional diffusion equation. Deng [17] developed a finite element method for the numerical resolution of the space and time fractional Fokker-Plank equation.

So far, it seems no effective finite element method for the fractional equation with time derivative. In this paper, we develop the finite element method for the STFDE by utilizing the nodal base functions and the general time discretization scheme, of which the convergence rate is $O(\tau^{2-\gamma}+h^2)$. However, the optimal convergence rate of the general time discretization scheme is at most $2-\gamma$ order. Novelly, combing the finite element method, we propose a second order scheme and apply it into the time derivative, of which the convergence rate can be $O(\tau^2+h^2)$. Both of the two schemes are verified by the numerical examples.

In this paper, we consider the following two types of STFDE:

$$\frac{\partial^\gamma u(x,t)}{\partial t^\gamma} = c\frac{\partial^\beta u(x,t)}{\partial x^\beta} + f(x,t) \tag{1}$$

with initial condition and boundary condition

$$u(x,0) = \psi(x), \qquad a \le x \le b, \tag{2}$$

$$u(a,t) = 0, \quad u(b,t) = \phi_b(t), \quad 0 \le t \le T \tag{3}$$

and

$$\frac{\partial^\gamma u(x,t)}{\partial t^\gamma} = c\frac{\partial^\beta u(x,t)}{\partial |x|^\beta} + f(x,t) \tag{4}$$

with initial condition and boundary condition

$$u(x,0) = \psi(x), \qquad a \le x \le b, \tag{5}$$

$$u(a,t) = 0, \quad u(b,t) = 0, \qquad 0 \le t \le T. \tag{6}$$

Here, the time fractional derivative $\frac{\partial^\gamma u(x,t)}{\partial t^\gamma}$ is the Caputo fractional derivative of order γ $(0 < \gamma \le 1)$ defined by

$$\frac{\partial^\gamma u(x,t)}{\partial t^\gamma} = \begin{cases} \frac{1}{\Gamma(1-\gamma)}\int_0^t \frac{\partial u(x,\eta)}{\partial \eta}\frac{d\eta}{(t-\eta)^\gamma}, & 0 < \gamma < 1, \\ \frac{\partial u(x,t)}{\partial t}, & \gamma = 1 \end{cases} \tag{7}$$

while the derivative $\frac{\partial^\beta u(x,t)}{\partial |x|^\beta}$ is Riesz space-fractional derivative of order β $(1 < \beta \le 2)$, defined by

$$\frac{\partial^\beta u(x,t)}{\partial |x|^\beta} = -\frac{1}{2\cos\frac{\pi\beta}{2}}\left[\frac{\partial^\beta u(x,t)}{\partial x^\beta} + \frac{\partial^\beta u(x,t)}{\partial (-x)^\beta}\right], \tag{8}$$

and

$$\frac{\partial^\beta u(x,t)}{\partial x^\beta} = \frac{1}{\Gamma(2-\beta)}\frac{\partial^2}{\partial x^2}\int_a^x (x-\xi)^{1-\beta}u(\xi,t)d\xi,$$

$$\frac{\partial^\beta u(x,t)}{\partial (-x)^\beta} = \frac{1}{\Gamma(2-\beta)}\frac{\partial^2}{\partial x^2}\int_x^b (\xi-x)^{1-\beta}u(\xi,t)d\xi.$$

The outline of the paper is as follows: several lemmas of fractional derivative are introduced in Section 2. Finite element method with two different time discretization schemes for solving the STFDE is proposed in Section 3. In Section 4, some numerical experiments are carried out and the results are compared with the exact solution. Finally, the conclusions are drawn.

2 The properties of the nodal base functions and their fractional derivatives

In this section, we mainly state several important notations and lemmas used in the subsequent sections of this paper.

Let $\Omega = [a,b]$ be a finite domain and $(\cdot,\cdot)$ denotes the inner product on the space $L_2(\Omega)$ with the L_2 norm $||\cdot||_2$. Setting S_h be a uniform partition of Ω, which is given by

$$a = x_0 < x_1 < \ldots < x_{m-1} < x_m = b,$$

where m is a positive integer. Let $h = (b-a)/m = x_i - x_{i-1}$ and $\Omega_i = [x_{i-1}, x_i]$ for $i = 1, 2, \ldots, m$.

Define the space V_h as the set of piecewise-linear polynomials on the mesh S_h, which can be expressed by

$$V_h = \{v : v|_{\Omega_i} \in P_1(\Omega_i),\ v \in C(\Omega)\}$$

where $P_1(\Omega_i)$ is the space of linear polynomials defined on Ω_i.

Let $V_{h0} = V_h \cap H_0^1(\Omega)$. The nodal based functions $\phi_0, \phi_1, \ldots, \phi_m$ of V_h can be expressed in the form

$$\phi_i(x) = \begin{cases} \frac{x-x_{i-1}}{h}, & x \in [x_{i-1}, x_i], \\ \frac{x_{i+1}-x}{h}, & x \in [x_i, x_{i+1}], \\ 0, & \text{elsewhere}. \end{cases} \tag{9}$$

where $i = 1, 2, \ldots, m-1$, and

$$\phi_0(x) = \begin{cases} \frac{x_1-x}{h}, & x \in [x_0, x_1], \\ 0, & \text{elsewhere}, \end{cases} \tag{10}$$

$$\phi_m(x) = \begin{cases} \frac{x-x_{m-1}}{h}, & x \in [x_{m-1}, x_m], \\ 0, & \text{elsewhere}. \end{cases} \tag{11}$$

Lemma 1 *For $i = 1, 2, \ldots, m-1$, we have*

$$(\phi_i(x), \phi_j(x)) = \frac{h}{6}\begin{cases} 1, & |j-i| = 1, \\ 4, & j = i, \qquad j = 0, 1, 2, \ldots, m; \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

We define $\mu = \frac{1}{h\cdot\Gamma(2-\alpha)}$, $\lambda = \frac{h^{1-\alpha}}{\Gamma(3-\alpha)}$, and $b_i = -(i-2)^{2-\alpha} + 3(i-1)^{2-\alpha} - 3i^{2-\alpha} + (i+1)^{2-\alpha}$, then the following lemmas hold, which can be derived by directly calculation.

Lemma 2 *For $i = 1, 2, \ldots, m-1$, we have*

$$\frac{\partial^\alpha \phi_i(x)}{\partial x^\alpha} = \mu\begin{cases} 0, & a \le x \le x_{i-1}, \\ (x-x_{i-1})^{1-\alpha}, & x_{i-1} \le x \le x_i, \\ (x-x_{i-1})^{1-\alpha} - 2(x-x_i)^{1-\alpha}, & x_i \le x \le x_{i+1}, \\ (x-x_{i-1})^{1-\alpha} - 2(x-x_i)^{1-\alpha} + (x-x_{i+1})^{1-\alpha}, & x_{i+1} \le x \le b. \end{cases} \tag{13}$$

$$\frac{\partial^\alpha \phi_i(x)}{\partial (-x)^\alpha} = \mu\begin{cases} (x_{i+1}-x)^{1-\alpha} - 2(x_i-x)^{1-\alpha} + (x_{i-1}-x)^{1-\alpha}, & a \le x \le x_{i-1}, \\ (x_{i+1}-x)^{1-\alpha} - 2(x_i-x)^{1-\alpha}, & x_{i-1} \le x \le x_i, \\ (x_{i+1}-x)^{1-\alpha}, & x_i \le x \le x_{i+1} \\ 0, & x_{i+1} \le x \le b. \end{cases} \tag{14}$$

Lemma 3 *For $i = 1, 2, \ldots, m-1$, we have*

$$\int_{x_{i-1}}^{x_i} \frac{\partial^\alpha \phi_j(x)}{\partial x^\alpha} dx = \lambda \cdot g_{i,j}^{(1)}, \tag{15}$$

$$\int_{x_i}^{x_{i+1}} \frac{\partial^\alpha \phi_j(x)}{\partial x^\alpha} dx = \lambda \cdot g_{i,j}^{(2)}, \tag{16}$$

with

$$g_{i,j}^{(1)} = \begin{cases} b_{i-j}, & j \le i-2 \\ 2^{2-\alpha} - 3, & j = i-1 \\ 1, & j = i \\ 0, & j \ge i+1 \end{cases}, \qquad g_{i,j}^{(2)} = \begin{cases} b_{i-j+1}, & j \le i-1 \\ 2^{2-\alpha} - 3, & j = i \\ 1, & j = i+1 \\ 0, & j \ge i+2 \end{cases}.$$

Lemma 4 *For $i = 1, 2, \ldots, m-1$, we have*

$$\int_{x_{i-1}}^{x_i} \left(\frac{\partial^\alpha \phi_j(x)}{\partial x^\alpha} - \frac{\partial^\alpha \phi_j(x)}{\partial (-x)^\alpha}\right) dx = \lambda \cdot g_{i,j}^{(3)}, \tag{17}$$

$$\int_{x_i}^{x_{i+1}} \left(\frac{\partial^\alpha \phi_j(x)}{\partial x^\alpha} - \frac{\partial^\alpha \phi_j(x)}{\partial (-x)^\alpha}\right) dx = \lambda \cdot g_{i,j}^{(4)}, \tag{18}$$

with

$$g_{i,j}^{(3)} = \begin{cases} b_{i-j}, & j \le i-2 \\ 2^{2-\alpha}-4, & j=i-1 \\ 4-2^{2-\alpha}, & j=i \\ -b_{j-i+1}, & j \ge i+1 \end{cases}, \qquad g_{i,j}^{(4)} = \begin{cases} b_{i-j+1}, & j \le i-1 \\ 2^{2-\alpha}-4, & j=i \\ 4-2^{2-\alpha}, & j=i+1 \\ -b_{j-i}, & j \ge i+2 \end{cases}.$$

3 Finite element method for the STFDE

3.1 The STFDE with Riemann-Liouville space fractional derivative

We first consider (1) and rewrite it as

$$\frac{\partial^\gamma u(x,t)}{\partial t^\gamma} = c\frac{\partial^{1+\alpha} u(x,t)}{\partial x^{1+\alpha}} + f(x,t) \tag{19}$$

with initial condition

$$u(x,0) = \psi(x), \quad a \le x \le b, \tag{20}$$

and boundary condition

$$u(a,t) = 0, \quad u(b,t) = \phi_b(t), \quad 0 \le t \le T \tag{21}$$

where $0 < \alpha \le 1, 0 < \gamma \le 1$.

We define $t_k = k\tau,\ k = 0, 1, \ldots, n$; $x_i = a + ih$ for $i = 0, 1, \ldots, m$, where $\tau = T/n$ and $h = (b-a)/m$ are the time and space steps, respectively. Let $P(a, b)$ denote the space of continuous and piecewise-linear functions with respect to the spatial partition, which vanish at the boundary $x = a$ and $x = b$. The nodal basis functions $\phi_i(x)$ for $i = 0, 1, \ldots, m$ are defined the same as (9)–(11).

In view of the time derivative γ, we first consider the case $0 < \gamma < 1$, and employ two schemes with finite element method to approximate the STFDE. Then we discuss the trivial situation, i.e., the situation $\gamma = 1$.

3.1.1 The general time discretization scheme (GTDS)

The time fractional derivative $\frac{\partial^\gamma u(x,t)}{\partial t^\gamma}$ at t_n can be approximated by the general time discretization scheme (see [18])

$$\frac{\partial^\gamma u(x,t_n)}{\partial t^\gamma} = \frac{\tau^{1-\gamma}}{\Gamma(2-\gamma)}\sum_{k=0}^{n-1} b_k \frac{u(x,t_{n-k}) - u(x,t_{n-k-1})}{\tau} \tag{22}$$

with $b_k = (k+1)^{1-\gamma} - k^{1-\gamma}$. The weak form of (19)–(21) is given by

$$\left(\frac{\partial^\gamma}{\partial t^\gamma}u, v\right) = -c\left(\frac{\partial^\alpha}{\partial x^\alpha}u, \frac{\partial v}{\partial x}\right) + (f, v),\ \forall\, v \in H_0^1(\Omega) \tag{23}$$

Denoting $f_n(x) = f(x, t_n)$,u^n be the approximation solution of $u(x, t_n)$,

$$\partial u(x, t_k) = \frac{u(x,t_k) - u(x,t_{k-1})}{\tau}.$$

Consider the discretization of the problem (23) as follows: find $u^n \in P(a,b)$, such that for all $v \in P(a,b)$,

$$\frac{\tau^{1-\gamma}}{\Gamma(2-\gamma)}\sum_{k=0}^{n-1} b_k(\partial u^{n-k}, v) + c\left(\frac{\partial^\alpha u^n}{\partial x^\alpha}, \frac{\partial v}{\partial x}\right) = (f_n, v), \tag{24}$$

setting $p = \Gamma(2-\gamma)\cdot\tau^\gamma$, $\omega_k = b_{k-1} - b_k$,then

$$\begin{aligned}&(u^n, v) + cp\left(\tfrac{\partial^\alpha u^n}{\partial x^\alpha}, \tfrac{\partial v}{\partial x}\right)\\ &= \textstyle\sum_{k=1}^{n-1}\omega_k(u^{n-k}, v) + b_{n-1}(u^0, v) + p(f_n, v).\end{aligned} \tag{25}$$

Let $u^n = \sum\limits_{j=0}^{m} u_j^n\phi_j(x) \in P(a,b)$, and choosing each test function v to be $\phi_i(x)$, $i = 1, 2, \ldots, m-1$, we obtain

$$\begin{aligned}&\left(\textstyle\sum_{j=0}^m u_j^n\phi_j(x), \phi_i(x)\right) + cp\left(\textstyle\sum_{j=0}^m u_j^n\frac{\partial^\alpha\phi_j(x)}{\partial x^\alpha}, \frac{\partial\phi_i(x)}{\partial x}\right)\\ &= \textstyle\sum_{k=1}^{n-1}\omega_k\left(\sum_{j=0}^m u_j^{n-k}\phi_j(x), \phi_i(x)\right)\\ &+ b_{n-1}(\psi(x), \phi_i(x)) + p(f(x, t_n), \phi_i(x)).\end{aligned} \tag{26}$$

Applying Lemma 1, we obtain

$$\begin{aligned}&\tfrac{h}{6}(u_{i-1}^n + 4u_i^n + u_{i+1}^n) + \tfrac{cp}{h}[\textstyle\sum_{j=0}^m u_j^n\int_{x_{i-1}}^{x_i}\frac{\partial^\alpha\phi_j(x)}{\partial x^\alpha}dx\\ &- \textstyle\sum_{j=0}^m u_j^n\int_{x_i}^{x_{i+1}}\frac{\partial^\alpha\phi_j(x)}{\partial x^\alpha}dx] = \sum_{k=1}^{n-1}\omega_k[\frac{h}{6}(u_{i-1}^{n-k} + 4u_i^{n-k} + u_{i+1}^{n-k})]\\ &+ b_{n-1}(\psi(x), \phi_i(x)) + p(f(x, t_n), \phi_i(x)).\end{aligned} \tag{27}$$

Applying Lemma 3, yields

$$\begin{aligned}&\tfrac{h}{6}(u_{i-1}^n + 4u_i^n + u_{i+1}^n) + r_1\left[\textstyle\sum_{j=0}^m(g_{i,j}^{(1)} - g_{i,j}^{(2)})u_j^n\right]\\ &= \tfrac{h}{6}\textstyle\sum_{k=1}^{n-1}\omega_k(u_{i-1}^{n-k} + 4u_i^{n-k} + u_{i+1}^{n-k})\\ &+ b_{n-1}(\psi(x), \phi_i(x)) + p(f(x, t_n), \phi_i(x)),\end{aligned} \tag{28}$$

where $r_1 = \lambda\cdot\frac{cp}{h}$.
The initial and boundary conditions are

$$u_i^0 = \psi(ih), \quad u_0^n = 0, \quad u_m^n = \phi_b(n\tau). \tag{29}$$

The convergence rate of above numerical scheme is $O(\tau^{2-\gamma} + h^2)$. In the below we give a second order scheme, which can be reached second order in both time and space direction, i.e., $O(\tau^2 + h^2)$.

3.1.2 New time discretization scheme (NTDS)

In order to show the second order scheme, we need the following result.

Lemma 5 *(see* [10]*) If $\alpha > 0$ then the following equality holds for any $f \in C([a,b])$.*

$$D_{a^+}^\alpha I_{a^+}^\alpha f = f(x), \tag{30}$$

where D^{α}_{a+} and I^{α}_{a+} are the Riemann-Liouville fractional derivative and integral, respectively, which are given by

$$D^{\alpha}_{a+}f(x)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x\frac{f(t)}{(x-t)^{\alpha}}\,dt,\quad I^{\alpha}_{a+}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\frac{f(t)}{(x-t)^{1-\alpha}}\,dt.$$

Then (19) can be transformed into the following equivalent form (see [15])

$$\frac{\partial u(x,t)}{\partial t}=\frac{\partial^{1-\gamma}}{\partial t^{1-\gamma}}\left[c\frac{\partial^{1+\alpha}u(x,t)}{\partial x^{1+\alpha}}+f(x,t)\right], \tag{31}$$

with initial condition

$$u(x,0)=\psi(x),\quad a\le x\le b, \tag{32}$$

and boundary condition :

$$u(a,t)=0,\quad u(b,t)=\phi_b(t),\quad 0\le t\le T, \tag{33}$$

where $0<\alpha\le 1$ and $\frac{\partial^{1-\gamma}u}{\partial t^{1-\gamma}}$ denotes the Riemann-Liouville fractional derivative of order $1-\gamma$ defined by

$$\frac{\partial^{1-\gamma}}{\partial t^{1-\gamma}}u(x,t)=\frac{1}{\Gamma(\gamma)}\frac{\partial}{\partial t}\int_0^t\frac{u(x,\eta)}{(t-\eta)^{1-\gamma}}d\eta,$$

with $0<\gamma\le 1$. By integrating both sides of (31) for t, we obtain

$$u(x_i,t_{n+1})=u(x_i,0)+\frac{1}{\Gamma(\gamma)}\sum_{q=0}^{n}\int_{t_q}^{t_{q+1}}\frac{c\frac{\partial^{1+\alpha}u(x_i,\eta)}{\partial x^{1+\alpha}}+f(x_i,\eta)}{(t_{n+1}-\eta)^{1-\gamma}}d\eta. \tag{34}$$

Now we use the following numerical scheme and lemma.

Lemma 6 *(see [15]) If the function $v(x,t)$ is sufficiently smooth, then*

$$v(x_i,\eta)=\frac{(t_{j+1}-\eta)v(x_i,t_j)+(\eta-t_j)v(x_i,t_{j+1})}{\tau}+O(\tau^2). \tag{35}$$

By applying Lemma 6, we have

$$\begin{aligned}
&u(x_i,t_{n+1})=u(x_i,0)\\
&+\tfrac{c}{\tau\Gamma(\gamma)}\textstyle\sum_{q=0}^{n}\int_{t_q}^{t_{q+1}}\frac{(t_{q+1}-\eta)\frac{\partial^{1+\alpha}u(x_i,t_q)}{\partial x^{1+\alpha}}+(\eta-t_q)\frac{\partial^{1+\alpha}u(x_i,t_{q+1})}{\partial x^{1+\alpha}}}{(t_{n+1}-\eta)^{1-\gamma}}d\eta\\
&+\tfrac{1}{\tau\Gamma(\gamma)}\textstyle\sum_{q=0}^{n}\int_{t_q}^{t_{q+1}}\frac{(t_{q+1}-\eta)f(x_i,t_q)+(\eta-t_q)f(x_i,t_{q+1})}{(t_{n+1}-\eta)^{1-\gamma}}d\eta+O(\tau^2),
\end{aligned}$$

which leads to

$$\begin{aligned}
&u(x_i,t_{n+1})=u(x_i,0)\\
&+r_3\textstyle\sum_{q=0}^{n}\left[C_q^{(1)}\frac{\partial^{1+\alpha}}{\partial x^{1+\alpha}}u(x_i,t_{n-q})+C_q^{(2)}\frac{\partial^{1+\alpha}}{\partial x^{1+\alpha}}u(x_i,t_{n-q+1})\right]\\
&+r_4\textstyle\sum_{q=0}^{n}\left[C_q^{(1)}f(x_i,t_{n-q})+C_q^{(2)}f(x_i,t_{n-q+1})\right]+O(\tau^2),
\end{aligned} \tag{36}$$

where $r_3 = \frac{c\tau^\gamma}{\Gamma(\gamma+1)}$, $r_4 = \frac{\tau^\gamma}{\Gamma(\gamma+1)}$ and

$$C_q^{(1)} = (q+1)^\gamma - \frac{1}{\gamma+1}\Big[(q+1)^{\gamma+1} - q^{\gamma+1}\Big],$$

$$C_q^{(2)} = \frac{1}{\gamma+1}\Big[(q+1)^{\gamma+1} - q^{\gamma+1}\Big] - q^\gamma.$$

The variational (weak) formulation of the above equation subject to the boundary condition (33) reads: find $u^n \in P(a,b)$, such that

$$\begin{aligned}&(u^{n+1}, v) = (u^0, v) - r_3 \textstyle\sum_{q=0}^n \left[C_q^{(1)}(\frac{\partial^\alpha u^{n-q}}{\partial x^\alpha}, \frac{\partial v}{\partial x}) + C_q^{(2)}(\frac{\partial^\alpha u^{n-q+1}}{\partial x^\alpha}, \frac{\partial v}{\partial x})\right]\\ &+r_4 \textstyle\sum_{q=0}^n \left[C_q^{(1)}(f^{n-q}, v) + C_q^{(2)}(f^{n-q+1}, v)\right],\ \forall\, v \in P(a,b).\end{aligned} \tag{37}$$

Let $u^n = \sum_{j=0}^{m} u_j^n \phi_j(x) \in P(a,b)$, and choosing each test function v to be $\phi_i(x)$, $i = 1, 2, \ldots, m-1$, we obtain

$$\begin{aligned}&\big(\textstyle\sum_{j=0}^m u_j^{n+1}\phi_j(x), \phi_i(x)\big) = (\psi(x), \phi_i(x))\\ &-r_3 \textstyle\sum_{q=0}^n \Big[C_q^{(1)}\big(\sum_{j=0}^m u_j^{n-q}\frac{\partial^\alpha \phi_j(x)}{\partial x^\alpha}, \frac{\partial \phi_i(x)}{\partial x}\big)\\ &+C_q^{(2)}\big(\textstyle\sum_{j=0}^m u_j^{n-q+1}\frac{\partial^\alpha \phi_j(x)}{\partial x^\alpha}, \frac{\partial \phi_i(x)}{\partial x}\big)\Big]\\ &+r_4 \textstyle\sum_{q=0}^n \Big[C_q^{(1)}(f^{n-q}, \phi_i(x)) + C_q^{(2)}(f^{n-q+1}, \phi_i(x))\Big].\end{aligned} \tag{38}$$

Applying Lemma 1 and Lemma 3, we obtain

$$\begin{aligned}&\tfrac{h}{6}(u_{i-1}^{n+1} + 4u_i^{n+1} + u_{i+1}^{n+1}) = (\psi(x), \phi_i(x))\\ &-r_5 \textstyle\sum_{q=0}^n \Big\{C_q^{(1)}\big[\sum_{j=0}^m (g_{i,j}^{(1)} - g_{i,j}^{(2)})u_j^{n-q}\big]\\ &+C_q^{(2)}\big[\textstyle\sum_{j=0}^m (g_{i,j}^{(1)} - g_{i,j}^{(2)})u_j^{n-q+1}\big]\Big\}\\ &+r_4 \textstyle\sum_{q=0}^n \Big[C_q^{(1)}\big(f^{n-q}, \phi_i(x)\big) + C_q^{(2)}\big(f^{n-q+1}, \phi_i(x)\big)\Big],\end{aligned} \tag{39}$$

where $r_5 = r_3 \cdot \frac{\lambda}{h}$.

Now we consider the trivial situation. When $\gamma = 1$, (19) can be simplified as

$$\frac{\partial u(x,t)}{\partial t} = c\frac{\partial^{1+\alpha} u(x,t)}{\partial x^{1+\alpha}} + f(x,t). \tag{40}$$

By using the Crank-Nicolson scheme to approximate (40), we can obtain

$$\begin{aligned}&u(x,t_n) = u(x,t_{n-1}) + \tfrac{c\tau}{2}\left[\tfrac{\partial}{\partial x}\left(\tfrac{\partial^\alpha u(x,t_n)}{\partial x^\alpha}\right)\right.\\ &\left.+\tfrac{\partial}{\partial x}\left(\tfrac{\partial^\alpha u(x,t_{n-1})}{\partial x^\alpha}\right)\right] + \tfrac{\tau}{2}\big[f(x,t_n) + f(x,t_{n-1})\big].\end{aligned} \tag{41}$$

The variational (weak) formulation of the above equation subject to the boundary condition (21) reads: find $u^n \in P(a,b)$, such that

$$\begin{aligned}&(u^n, v) = (u^{n-1}, v) - \tfrac{c\tau}{2}\left(\tfrac{\partial^\alpha u^n}{\partial x^\alpha}, \tfrac{\partial v}{\partial x}\right) - \tfrac{c\tau}{2}\left(\tfrac{\partial^\alpha u^{n-1}}{\partial x^\alpha}, \tfrac{\partial v}{\partial x}\right)\\ &+\tfrac{\tau}{2}(f(x,t_n), v) + \tfrac{\tau}{2}(f(x,t_{n-1}), v),\ \forall\, v \in P(a,b),\end{aligned} \tag{42}$$

Let $u^n = \sum_{j=0}^{m} u_j^n \phi_j(x) \in P(a,b)$, and choosing each test function v to be $\phi_i(x)$, $i = 1, 2, \ldots, m-1$, we obtain

$$\begin{aligned}&\left(\sum_{j=0}^{m} u_j^n \phi_j(x), \phi_i(x)\right) = \left(\sum_{j=0}^{m} u_j^{n-1} \phi_j(x), \phi_i(x)\right)\\ &-\tfrac{c\tau}{2}\left(\sum_{j=0}^{m} u_j^n \tfrac{\partial^\alpha \phi_j(x)}{\partial x^\alpha}, \tfrac{\partial \phi_i(x)}{\partial x}\right) - \tfrac{c\tau}{2}\left(\sum_{j=0}^{m} u_j^{n-1} \tfrac{\partial^\alpha \phi_j(x)}{\partial x^\alpha}, \tfrac{\partial \phi_i(x)}{\partial x}\right)\\ &+\tfrac{\tau}{2}\left(f(x,t_n), \phi_i(x)\right) + \tfrac{\tau}{2}\left(f(x,t_{n-1}), \phi_i(x)\right).\end{aligned} \tag{43}$$

Applying Lemma 1, we have

$$\begin{aligned}&\tfrac{h}{6}(u_{i-1}^n + 4u_i^n + u_{i+1}^n) = \tfrac{h}{6}(u_{i-1}^{n-1} + 4u_i^{n-1} + u_{i+1}^{n-1})\\ &-\tfrac{c\tau}{2h}\left[\sum_{j=0}^{m} u_j^n \int_{x_{i-1}}^{x_i} \tfrac{\partial^\alpha \phi_j(x)}{\partial x^\alpha} dx - \sum_{j=0}^{m} u_j^n \int_{x_i}^{x_{i+1}} \tfrac{\partial^\alpha \phi_j(x)}{\partial x^\alpha} dx\right]\\ &-\tfrac{c\tau}{2h}\left[\sum_{j=0}^{m} u_j^{n-1} \int_{x_{i-1}}^{x_i} \tfrac{\partial^\alpha \phi_j(x)}{\partial x^\alpha} dx - \sum_{j=0}^{m} u_j^{n-1} \int_{x_i}^{x_{i+1}} \tfrac{\partial^\alpha \phi_j(x)}{\partial x^\alpha} dx\right]\\ &+\tfrac{\tau}{2}\left(f(x,t_n), \phi_i(x)\right) + \tfrac{\tau}{2}\left(f(x,t_{n-1}), \phi_i(x)\right).\end{aligned} \tag{44}$$

Applying Lemma 3, we find

$$\begin{aligned}&\tfrac{h}{6}(u_{i-1}^n + 4u_i^n + u_{i+1}^n) = \tfrac{h}{6}(u_{i-1}^{n-1} + 4u_i^{n-1} + u_{i+1}^{n-1})\\ &-\tfrac{r_2}{2}\left[\sum_{j=0}^{m}\left(g_{i,j}^{(1)} - g_{i,j}^{(2)}\right)u_j^n\right] - \tfrac{r_2}{2}\left[\sum_{j=0}^{m}\left(g_{i,j}^{(1)} - g_{i,j}^{(2)}\right)u_j^{n-1}\right]\\ &+\tfrac{\tau}{2}(f(x,t_n), \phi_i(x)) + \tfrac{\tau}{2}(f(x,t_{n-1}), \phi_i(x)),\end{aligned} \tag{45}$$

where $r_2 = \lambda \cdot \frac{c\tau}{h}$.

3.2 The STFDE with Riesz space fractional derivative

Now, we consider (4)

$$\frac{\partial^\gamma u(x,t)}{\partial t^\gamma} = c\frac{\partial^\beta u(x,t)}{\partial |x|^\beta} + f(x,t), \tag{46}$$

with initial condition

$$u(x,0) = \psi(x), \qquad a \le x \le b, \tag{47}$$

and boundary condition

$$u(a,t) = 0 \qquad u(b,t) = 0, \qquad 0 \le t \le T, \tag{48}$$

where $\beta = 1 + \alpha$, $0 < \alpha \le 1$. Then it can be written as

$$\begin{aligned}\frac{\partial^\gamma u(x,t)}{\partial t^\gamma} &= -\frac{c}{2\cos\frac{\pi(1+\alpha)}{2}}\left[\frac{\partial^{1+\alpha} u(x,t)}{\partial x^{1+\alpha}} + \frac{\partial^{1+\alpha} u(x,t)}{\partial(-x)^{1+\alpha}}\right] + f(x,t)\\ &= -\rho\frac{\partial}{\partial x}\mathcal{H}(x,t) + f(x,t)\end{aligned} \tag{49}$$

with

$$\rho = \frac{c}{2\cos\frac{\pi(1+\alpha)}{2}}, \qquad \mathcal{H}(x,t) = \frac{\partial^\alpha u(x,t)}{\partial x^\alpha} - \frac{\partial^\alpha u(x,t)}{\partial(-x)^\alpha}.$$

We proceed to approximate the Riesz space fractional STFDE in completely analogy to the Riemann-Liouville space fractional STFDE given in the previous part. We first discuss the $0 < \gamma < 1$ case.

3.2.1 The general time discretization scheme (GTDS)

We use (22) to approximate the time fractional derivative $\frac{\partial^\gamma u(x,t)}{\partial t^\gamma}$ at t_n similarly. The weak form of (47)–(49) is given by

$$\left(\frac{\partial^\gamma}{\partial t^\gamma}u, v\right) = \rho\left(\mathcal{H}, \frac{\partial v}{\partial x}\right) + (f, v),\ \forall\, v \in H_0^1(\Omega), \tag{50}$$

Consider the discretization of the problem (50) as follows: find $u^n \in P(a, b)$, such that for all $v \in P(a, b)$,

$$\frac{\tau^{1-\gamma}}{\Gamma(2-\gamma)}\sum_{k=0}^{n-1} b_k(\partial u^{n-k}, v) = \rho\left(\mathcal{H}^n, \frac{\partial v}{\partial x}\right) + (f_n, v), \tag{51}$$

p and ω_k are defined the same as in the previous part, (51) can be written as

$$\begin{aligned}&(u^n, v) - \rho p\left(\mathcal{H}^n, \tfrac{\partial v}{\partial x}\right)\\ &= \textstyle\sum_{k=1}^{n-1}\omega_k(u^{n-k}, v) + b_{n-1}(u^0, v) + p(f_n, v).\end{aligned} \tag{52}$$

Let $\mathcal{R}(x) = \frac{\partial^\alpha \phi_j(x)}{\partial x^\alpha} - \frac{\partial^\alpha \phi_j(x)}{\partial(-x)^\alpha}$, $u^n = \sum\limits_{j=0}^{m} u_j^n\phi_j(x) \in P(a, b)$, and choosing each test function v to be $\phi_i(x)$, $i = 1, 2, \ldots, m-1$, we obtain

$$\begin{aligned}&\left(\textstyle\sum_{j=0}^m u_j^n\phi_j(x), \phi_i(x)\right) - \rho p\left(\sum_{j=0}^m u_j^n\mathcal{R}(x), \frac{\partial\phi_i(x)}{\partial x}\right)\\ &= \textstyle\sum_{k=1}^{n-1}\omega_k\left(\sum_{j=0}^m u_j^{n-k}\phi_j(x), \phi_i(x)\right)\\ &+b_{n-1}(\psi(x), \phi_i(x)) + p(f(x, t_n), \phi_i(x)).\end{aligned} \tag{53}$$

Applying Lemma 1, we obtain

$$\begin{aligned}&\tfrac{h}{6}(u_{i-1}^n + 4u_i^n + u_{i+1}^n) - \tfrac{\rho p}{h}[\textstyle\sum_{j=0}^m u_j^n\int_{x_{i-1}}^{x_i}\mathcal{R}(x)dx\\ &-\textstyle\sum_{j=0}^m u_j^n\int_{x_i}^{x_{i+1}}\mathcal{R}(x)dx] = \sum_{k=1}^{n-1}\omega_k[\tfrac{h}{6}(u_{i-1}^{n-k} + 4u_i^{n-k} + u_{i+1}^{n-k})]\\ &+b_{n-1}(\psi(x), \phi_i(x)) + p(f(x, t_n), \phi_i(x)).\end{aligned} \tag{54}$$

Applying Lemma 4, we obtain

$$\begin{aligned}&\tfrac{h}{6}(u_{i-1}^n + 4u_i^n + u_{i+1}^n) - r_6\left[\textstyle\sum_{j=0}^m(g_{i,j}^{(3)} - g_{i,j}^{(4)})u_j^n\right]\\ &= \tfrac{h}{6}\textstyle\sum_{k=1}^{n-1}\omega_k(u_{i-1}^{n-k} + 4u_i^{n-k} + u_{i+1}^{n-k})\\ &+b_{n-1}(\psi(x), \phi_i(x)) + p(f(x, t_n), \phi_i(x)),\end{aligned} \tag{55}$$

where $r_6 = \lambda \cdot \frac{\rho p}{h}$. The initial and boundary conditions are

$$u_i^0 = \psi(ih), \quad u_0^k = u_m^k = 0. \tag{56}$$

3.2.2 New time discretization scheme (NTDS)

Similarly, (46) can be turned into

$$\frac{\partial u(x, t)}{\partial t} = \frac{\partial^{1-\gamma}}{\partial t^{1-\gamma}}\left[c\frac{\partial^\beta u(x, t)}{\partial |x|^\beta} + f(x, t)\right], \tag{57}$$

then it can be written as

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^{1-\gamma}}{\partial t^{1-\gamma}}\Big[-\rho\frac{\partial}{\partial x}\mathcal{H}(x,t) + f(x,t)\Big], \tag{58}$$

with initial condition

$$u(x,0) = \psi(x), \quad a \le x \le b, \tag{59}$$

and boundary condition :

$$u(a,t) = 0, \qquad u(b,t) = 0, \quad 0 \le t \le T. \tag{60}$$

By integrating both sides of (58) for t, we obtain

$$u(x_i,t_{n+1}) = u(x_i,0) - \frac{1}{\Gamma(\gamma)}\sum_{q=0}^{n}\int_{t_q}^{t_{q+1}}\frac{\rho\frac{\partial}{\partial x}\mathcal{H}(x_i,\eta) + f(x_i,\eta)}{(t_{n+1}-\eta)^{1-\gamma}}d\eta. \tag{61}$$

Then we use the same numerical scheme and applying Lemma 6, we have

$$\begin{aligned}
&u(x_i,t_{n+1}) = u(x_i,0) \\
&-r_3^*\sum_{q=0}^{n}\Big[C_q^{(1)}\tfrac{\partial}{\partial x}\mathcal{H}(x_i,t_{n-q}) + C_q^{(2)}\tfrac{\partial}{\partial x}\mathcal{H}(x_i,t_{n-q+1})\Big] \\
&+r_4\sum_{q=0}^{n}\Big[C_q^{(1)}f(x_i,t_{n-q}) + C_q^{(2)}f(x_i,t_{n-q+1})\Big],
\end{aligned} \tag{62}$$

where $r_3^* = \frac{\rho\tau^\gamma}{\Gamma(\gamma+1)}$ and r_4, $C_q^{(1)}$, $C_q^{(2)}$ are the same as in the previous parts.

The variational (weak) formulation of the above equation subject to the boundary condition (60) reads: find $u^n \in P(a,b)$, such that

$$\begin{aligned}
&(u^{n+1},v) = (u^0,v) + r_3^*\sum_{q=0}^{n}\Big[C_q^{(1)}(\mathcal{H}^{n-q},\tfrac{\partial v}{\partial x}) + C_q^{(2)}(\mathcal{H}^{n-q+1},\tfrac{\partial v}{\partial x})\Big] \\
&+r_4\sum_{q=0}^{n}\Big[C_q^{(1)}(f^{n-q},v) + C_q^{(2)}(f^{n-q+1},v)\Big], \ \forall v \in P(a,b).
\end{aligned} \tag{63}$$

Let $u^n = \sum\limits_{j=0}^{m} u_j^n\phi_j(x) \in P(a,b)$, and choosing each test function v to be $\phi_i(x)$, $i = 1,2,\ldots,m-1$, we obtain

$$\begin{aligned}
&\big(\textstyle\sum_{j=0}^{m} u_j^{n+1}\phi_j(x),\phi_i(x)\big) = (\psi(x),\phi_i(x)) \\
&+r_3^*\sum_{q=0}^{n}\Big[C_q^{(1)}\big(\textstyle\sum_{j=0}^{m} u_j^{n-q}\mathcal{R}(x),\frac{\partial\phi_i(x)}{\partial x}\big) \\
&+C_q^{(2)}\big(\textstyle\sum_{j=0}^{m} u_j^{n-q+1}\mathcal{R}(x),\frac{\partial\phi_i(x)}{\partial x}\big)\Big] \\
&+r_4\sum_{q=0}^{n}\Big[C_q^{(1)}(f^{n-q},\phi_i(x)) + C_q^{(2)}(f^{n-q+1},\phi_i(x))\Big].
\end{aligned} \tag{64}$$

Applying Lemma 1 and Lemma 4, we obtain

$$\begin{aligned}&\tfrac{h}{6}(u_{i-1}^{n+1}+4u_i^{n+1}+u_{i+1}^{n+1})=(\psi(x),\phi_i(x))\\&+r_8\sum_{q=0}^{n}\Big\{C_q^{(1)}\big[\sum_{j=0}^{m}\big(g_{i,j}^{(3)}-g_{i,j}^{(4)}\big)u_j^{n-q}\big]\\&+C_q^{(2)}\big[\sum_{j=0}^{m}\big(g_{i,j}^{(3)}-g_{i,j}^{(4)}\big)u_j^{n-q+1}\big]\Big\}\\&+r_4\sum_{q=0}^{n}\big[C_q^{(1)}\big(f^{n-q},\phi_i(x)\big)+C_q^{(2)}\big(f^{n-q+1},\phi_i(x)\big)\big],\end{aligned}\tag{65}$$

where $r_8=r_3^*\cdot\frac{\lambda}{h}$.

Now we consider the trivial situation. When $\gamma=1$, (49) can be written as:

$$\frac{\partial u(x,t)}{\partial t}=-\rho\frac{\partial}{\partial x}\mathcal{H}(x,t)+f(x,t)\tag{66}$$

By using the Crank-Nicolson scheme to approximate (66), we can obtain

$$\begin{aligned}&u(x,t_n)=u(x,t_{n-1})-\tfrac{\rho\tau}{2}\left[\tfrac{\partial}{\partial x}\mathcal{H}(x,t_n)+\tfrac{\partial}{\partial x}\mathcal{H}(x,t_{n-1})\right]\\&+\tfrac{\tau}{2}\big[f(x,t_n)+f(x,t_{n-1})\big].\end{aligned}\tag{67}$$

The variational (weak) formulation of the above equation subject to the boundary condition (48) reads: find $u^n\in P(a,b)$, such that

$$\begin{aligned}&(u^n,v)=(u^{n-1},v)+\tfrac{\rho\tau}{2}\left(\mathcal{H}^n,\tfrac{\partial v}{\partial x}\right)+\tfrac{\rho\tau}{2}\left(\mathcal{H}^{n-1},\tfrac{\partial v}{\partial x}\right)\\&+\tfrac{\tau}{2}(f(x,t_n),v)+\tfrac{\tau}{2}(f(x,t_{n-1}),v),\ \forall\, v\in P(a,b).\end{aligned}\tag{68}$$

Let $u^n=\sum\limits_{j=0}^{m}u_j^n\phi_j(x)\in P(a,b)$, and choosing each test function v to be $\phi_i(x)$, $i=1,2,\ldots,m-1$, we obtain

$$\begin{aligned}&\Big(\sum_{j=0}^{m}u_j^n\phi_j(x),\phi_i(x)\Big)=\Big(\sum_{j=0}^{m}u_j^{n-1}\phi_j(x),\phi_i(x)\Big)\\&+\tfrac{\rho\tau}{2}\left(\sum_{j=0}^{m}u_j^n\mathcal{R}(x),\tfrac{\partial\phi_i(x)}{\partial x}\right)+\tfrac{\rho\tau}{2}\left(\sum_{j=0}^{m}u_j^{n-1}\mathcal{R}(x),\tfrac{\partial\phi_i(x)}{\partial x}\right)\\&+\tfrac{\tau}{2}\left(f(x,t_n),\phi_i(x)\right)+\tfrac{\tau}{2}\left(f(x,t_{n-1}),\phi_i(x)\right).\end{aligned}\tag{69}$$

Applying Lemma 1 and Lemma 4, we obtain

$$\begin{aligned}&\tfrac{h}{6}(u_{i-1}^n+4u_i^n+u_{i+1}^n)=\tfrac{h}{6}(u_{i-1}^{n-1}+4u_i^{n-1}+u_{i+1}^{n-1})\\&+\tfrac{r_7}{2}\left[\sum_{j=0}^{m}\left(g_{i,j}^{(3)}-g_{i,j}^{(4)}\right)u_j^n\right]+\tfrac{r_7}{2}\left[\sum_{j=0}^{m}\left(g_{i,j}^{(3)}-g_{i,j}^{(4)}\right)u_j^{n-1}\right]\\&+\tfrac{\tau}{2}(f(x,t_n),\phi_i(x))+\tfrac{\tau}{2}(f(x,t_{n-1}),\phi_i(x)),\end{aligned}\tag{70}$$

where $r_7=\lambda\cdot\frac{\rho\tau}{h}$.

4 Numerical examples

In order to demonstrate the effectiveness of our numerical methods, two examples are presented. The main purpose is to check the convergence behavior of the numerical solution with respect to the time step τ and space step h.

Table 1 The errors and convergence order of GTDS about h

$h(\tau = h^2)$	$\alpha = 0.3$		$\alpha = 0.6$		$\alpha = 0.9$	
	$E_2(h, \tau)$	Order	$E_2(h, \tau)$	Order	$E_2(h, \tau)$	Order
1/10	9.1884E-04		7.9884E-04		4.3041E-04	
1/20	2.3119E-04	1.99	2.1026E-04	1.93	1.1889E-04	1.86
1/40	5.8086E-05	1.99	5.4547E-05	1.95	3.2169E-05	1.89
1/80	1.4581E-05	1.99	1.4013E-05	1.96	8.5882E-06	1.91
1/160	3.6570E-06	2.00	3.5749E-06	1.97	2.2714E-06	1.92

Example 1 First we consider the following STFDE with Riemann-Liouville space fractional derivative

$$\frac{\partial^\gamma u(x,t)}{\partial t^\gamma} = \frac{\partial^{1+\alpha} u(x,t)}{\partial x^{1+\alpha}} + f(x,t),$$
$$u(x,0) = 0, \quad 0 \le x \le 1,$$
$$u(0,t) = 0, \qquad u(1,t) = t^2, \quad 0 \le t \le 1.$$

where $f(x,t) = \frac{2 \cdot t^{2-\gamma}}{\Gamma(3-\gamma)} x^2 - 2t^2 \frac{x^{1-\alpha}}{\Gamma(2-\alpha)}$. The exact solution is $u(x,t) = t^2 \cdot x^2$.

We compute the errors in L_2 discrete norm. And all the numerical results in the tables and figures below are evaluated at $T = 1$.

For the $0 < \gamma < 1$ case, the theoretical convergence order of the general time discretization scheme (GTDS) is $O(\tau^{2-\gamma} + h^2)$. Table 1 shows the error between the exact solution and numerical solution with $\tau = h^2$ and the convergence order about h when γ=0.6 for different α values. Table 2 shows the L_2 error and the convergence order of τ when α=0.6 and h=1/2000 for different τ values. The theoretical convergence order of new time discretization scheme (NTDS) is $O(\tau^2 + h^2)$. Table 3 shows the error and the convergence order of NTDS about h with $\tau = h$, γ=0.6 for different α values. Table 4 shows the error and the convergence order about τ when α=0.6 and h=1/2000 for different τ values.

When $\gamma = 1$, the theoretical convergence order of Crank-Nicolson (CN) scheme is $O(\tau^2 + h^2)$. Table 5 shows the error and the convergence order with $\tau = h$ for

Table 2 The errors and convergence order of NTDS about h

$\tau = h$	$\alpha = 0.3$		$\alpha = 0.6$		$\alpha = 0.9$	
	$E_2(h, \tau)$	Order	$E_2(h, \tau)$	Order	$E_2(h, \tau)$	Order
1/10	9.1056E-04		7.9845E-04		4.3166E-04	
1/20	2.2025E-04	2.05	2.0551E-04	1.96	1.1655E-04	1.89
1/40	5.4033E-05	2.03	5.2692E-05	1.96	3.1215E-05	1.90
1/80	1.3372E-05	2.01	1.3455E-05	1.97	8.2990E-06	1.91
1/160	3.3260E-06	2.01	3.4220E-06	1.98	2.1922E-06	1.92

Table 3 The errors and convergence order of GTDS about t

τ	$\gamma = 0.3$		$\gamma = 0.6$		$\gamma = 0.9$	
	$E_2(h,\tau)$	Order	$E_2(h,\tau)$	Order	$E_2(h,\tau)$	Order
1/4	1.2600E-03		4.4059E-03		1.1763E-02	
1/8	4.1550E-04	1.60	1.7190E-03	1.36	5.5813E-03	1.08
1/16	1.3473E-04	1.62	6.6290E-04	1.37	2.6212E-03	1.09
1/32	4.3163E-05	1.64	2.5396E-04	1.38	1.2263E-03	1.10

Table 4 The errors and convergence order of NTDS about t

τ	$\gamma = 0.3$		$\gamma = 0.6$		$\gamma = 0.9$	
	$E_2(h,\tau)$	Order	$E_2(h,\tau)$	Order	$E_2(h,\tau)$	Order
1/4	3.8592E-04		3.2388E-04		9.7790E-05	
1/8	1.0013E-04	1.95	8.0678E-05	2.01	2.3590E-05	2.05
1/16	2.5810E-05	1.96	2.0151E-05	2.00	5.9472E-06	1.99
1/32	6.6030E-06	1.97	5.0245E-06	2.00	1.4870E-06	2.00

Table 5 The errors and convergence order of CN scheme with γ=1

$\tau = h$	$\alpha = 0.3$		$\alpha = 0.6$		$\alpha = 0.9$	
	$E_2(h,\tau)$	Order	$E_2(h,\tau)$	Order	$E_2(h,\tau)$	Order
1/10	1.1379E-03		9.1816E-04		5.1314E-04	
1/20	2.7631E-04	2.04	2.3467E-04	1.97	1.3625E-04	1.91
1/40	6.8010E-05	2.02	5.9854E-05	1.97	3.5996E-05	1.92
1/80	1.6870E-05	2.01	1.5221E-05	1.98	9.4621E-06	1.93
1/160	4.2020E-06	2.01	3.8591E-06	1.98	2.4757E-06	1.93

Table 6 The errors and convergence order of GTDS about h

$h(\tau = h^2)$	$\alpha = 0.3$		$\alpha = 0.6$		$\alpha = 0.9$	
	$E_2(h,\tau)$	Order	$E_2(h,\tau)$	Order	$E_2(h,\tau)$	Order
1/10	5.7946E-04		5.1052E-04		2.4580E-04	
1/20	1.5395E-04	1.91	1.4163E-04	1.85	7.3426E-05	1.74
1/40	3.9858E-05	1.95	3.7643E-05	1.91	2.0936E-05	1.81
1/80	1.0233E-05	1.96	9.7981E-06	1.94	5.8097E-06	1.85
1/160	2.6166E-06	1.97	2.5207E-06	1.96	1.5828E-06	1.88

Table 7 The errors and convergence order of NTDS method about h

$\tau = h$	$\alpha = 0.3$		$\alpha = 0.6$		$\alpha = 0.9$	
	$E_2(h,\tau)$	Order	$E_2(h,\tau)$	Order	$E_2(h,\tau)$	Order
1/10	5.7974E-04		5.1046E-04		2.4560E-04	
1/20	1.5468E-04	1.91	1.4216E-04	1.84	7.3765E-05	1.74
1/40	4.0121E-05	1.95	3.7848E-05	1.91	2.1076E-05	1.81
1/80	1.0307E-05	1.96	9.8583E-06	1.94	5.8527E-06	1.85
1/160	2.6355E-06	1.97	2.5367E-06	1.96	1.5946E-06	1.88

Table 8 The errors and convergence order of GTDS about t

τ	$\gamma = 0.3$		$\gamma = 0.6$		$\gamma = 0.9$	
	$E_2(h,\tau)$	Order	$E_2(h,\tau)$	Order	$E_2(h,\tau)$	Order
1/4	1.4374E-04		5.0221E-04		1.3390E-03	
1/8	4.7409E-05	1.60	1.9596E-04	1.36	6.3539E-04	1.08
1/16	1.5382E-05	1.62	7.5575E-05	1.37	2.9847E-04	1.09
1/32	4.9373E-06	1.64	2.8962E-05	1.38	1.3966E-04	1.10

Table 9 The errors and convergence order of NTDS about t

τ	$\gamma = 0.3$		$\gamma = 0.6$		$\gamma = 0.9$	
	$E_2(h,\tau)$	Order	$E_2(h,\tau)$	Order	$E_2(h,\tau)$	Order
1/4	4.4024E-05		3.6704E-05		1.0533E-05	
1/8	1.1433E-05	1.95	9.2001E-06	2.00	2.7011E-06	1.96
1/16	2.9573E-06	1.95	2.3083E-06	1.99	6.9275E-07	1.96
1/32	7.6700E-07	1.95	5.8605E-07	1.98	1.8398E-07	1.91

Table 10 The errors and convergence order of CN scheme with γ=1

$\tau = h$	$\alpha = 0.3$		$\alpha = 0.6$		$\alpha = 0.9$	
	$E_2(h,\tau)$	Order	$E_2(h,\tau)$	Order	$E_2(h,\tau)$	Order
1/10	5.6170E-04		4.9616E-04		2.4708E-04	
1/20	1.5031E-04	1.90	1.3793E-04	1.85	7.3595E-05	1.75
1/40	3.9151E-05	1.94	3.6736E-05	1.91	2.0912E-05	1.82
1/80	1.0100E-05	1.95	9.5843E-06	1.94	5.7851E-06	1.85
1/160	2.5918E-06	1.96	2.4716E-06	1.96	1.5719E-06	1.88

different α values. From these tables we can see that the numerical results are in excellent agreement with the exact solutions.

Example 2 Now we consider the following STFDE with Riesz space fractional derivative

$$\frac{\partial^{\gamma} u(x,t)}{\partial t^{\gamma}} = \frac{\partial^{1+\alpha} u(x,t)}{\partial |x|^{1+\alpha}} + f(x,t),$$

$$u(x,0) = 0, \quad 0 \le x \le 1,$$

$$u(0,t) = 0, \qquad u(1,t) = 0, \quad 0 \le t \le 1.$$

where

$$f(x,t) = \frac{2 \cdot t^{2-\gamma}}{\Gamma(3-\gamma)} x^2 (1-x)^2 + \frac{t^2}{\cos\frac{\pi(1+\alpha)}{2}\Gamma(4-\alpha)} \cdot \Big\{ 12[x^{3-\alpha} + (1-x)^{3-\alpha}] \\ -6(3-\alpha)[x^{2-\alpha} + (1-x)^{2-\alpha}] + (2-\alpha)(3-\alpha)[x^{1-\alpha} + (1-x)^{1-\alpha}] \Big\}.$$

The exact solution is $u(x,t) = t^2 x^2 (1-x)^2$.

First we consider the $0 < \gamma < 1$ case, the theoretical convergence order of GTDS is $O(\tau^{2-\gamma} + h^2)$. Table 6 shows the error and the convergence order about h with $\tau = h^2$, γ=0.6 for different α values. Table 7 shows the error and the convergence order about τ when α=0.6 and h=1/2000 for different τ values. The theoretical convergence order of NTDS is $O(\tau^2 + h^2)$. Table 8 shows the error and the convergence order about h with $\tau = h$, γ=0.6 for different α values. Table 9 shows the error and the convergence order about τ when α=0.6 and h=1/2000 for different τ values.

When $\gamma = 1$, the theoretical convergence order of Crank-Nicolson scheme is $O(\tau^2 + h^2)$. Table 10 shows the error and the convergence order with $\tau = h$ for different α values. From these tables we can see that the numerical results are in good agreement with the exact solutions.

5 Conclusions

In this paper, we have developed and demonstrated two finite element methods for solving two types of space-time fractional diffusion equations(STFDE). First, for the STFDE with Riemann-Liouville space fractional derivative, we discretized the time fractional derivative by using the general time discretization scheme which is of $2-\gamma$ order accuracy. We then derived the variational formation of the semidiscrete scheme. Furthermore, we used the finite element method to approximate the space fractional derivative and obtain the full discretization scheme with convergence order of $O(\tau^{2-\gamma} + h^2)$. To introduce the second order numerical scheme, based on the properties of the Riemann-Liouville and Caputo derivative, we transformed the formulation of the original equation. What follows, we integrated the equation on both sides and employed a technique in the new equation. At last we got the full discretization scheme with convergence order of $O(\tau^2 + h^2)$. For the STFDE with Riesz space fractional derivative, we handled it in completely analogy to the way to the STFDE with Riemann-Liouville space fractional derivative. By numerical examples, the effectiveness of the prospered methods was verified. These methods can be extended to other kinds of space-time fractional diffusion equations.

Acknowledgments The authors wish to thank the referees for many thoughtful and constructive suggestions to improve the paper. This research is partially supported by NSF of China under grant 11471274 and the Natural Science Foundation of Fujian (Grant No. 2013J01021).

Compliance with ethical standards This research does not involve human participants, animals, genetically modified organisms or biosafety. As such, no ethics clearance is required. The authors declare that they have no conflict of interest.

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