# Chapter 2
# Interpolation and Approximation

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# Overview:

- Lagrange’s Interpolation
- Newton’s Divide and Difference Interpolation
- Cubic Spline Interpolation
- Curve Fitting

# Interpolation

Let ***f(x)*** be continuous function may be used to represent the ***n + 1*** data values with passing ***n + 1*** points. The process of computing the value of ***f(x)*** or ***y*** for given ***x*** inside the given range is called Interpolation.

| $x$ | $x_0$ | $x_1$ | .... | $x_n$ |
|---|---|---|---|---|
| $f(x)$ | $f(x_0)$ | $f(x_1)$ | .... | $f(x_n)$ |

Then the process of finding the value of ***f(x)*** corresponding to any value of ***x*** is called interpolation

# Interpolation with unequal intervals

- Lagrange’s interpolation
- Newton’s divide and difference interpolation

# Lagrange's Interpolation

Suppose $f(x)$ be a function with $f(x_0)$, $f(x_1)$, $f(x_2)$,..........,$f(\,x_n)$ corresponding to the values $x_0$, $x_1$, $x_2$.........., $x_n$ then the Lagrange's interpolation formula is given by:

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times f(x_1) + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times f(x_3) + \cdots\cdots + \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})} \times f(x_n)$$

# Solved Example 1:

*Q. Use the Lagrange's interpolation formula to find the value of* $f\ (x)$ *at* $x = 10$ $ie$ $f(10)$ *from the following data.*

| x | 5 | 6 | 9 | 11 |
|---|---|---|---|---|
| f(x) | 12 | 13 | 14 | 16 |

$Solution$: $x_0 = 5, x_1 = 6, x_2 = 9, x_3 = 11$

$f(x_0) = 12, f(x_1) = 13, f(x_2) = 14, f(x_3) = 16$

By Lagrange's interpolation we have

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times f(x_1) + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times f(x_3)$$

$$\therefore f(10) = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)} \times 12 + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)} \times 13 + \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)} \times 14 + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)} \times 16$$

=2-4.3333+11.6666+5.3333

=14.6666

## Exercise:

1. Use Lagrange interpolation to estimate the square of 3.25 if

| **X** | **1** | **2** | **3** | **4** | **5** |
|---|---|---|---|---|---|
| $f(x) = x^2$ | 1 | 4 | 9 | 16 | 25 |

2. Find the missing value from the following table using Lagrange interpolation.

| **X** | **1** | **3** | **6.5** | **10** |
|---|---|---|---|---|
| $f(x)$ | 23 | 61 | ? | 203 |

| **X** | **-2** | **-1** | **0** | **1** | **6** |
|---|---|---|---|---|---|
| $f(x) = x^2$ | -20 | ? | 2 | ? | 70 |

# Lagrange's Interpolation Algorithm

```
1.    Read the number of data 'n'
2.     Read the value at which value is needed(say x)
3.     Read the available data points x and f(x)
4.     Initialize sum=0
5.     for i=0 to n
         initialize L[i]=1.
        for j=0 to n
             if(i!=j)
               L[i]=L[i]*(x-x[j])/(x[i]-x[j])
             end if
        end for
5.      sum=sum+L[i]*f[i]
        end for
7.    Print interpolation value "sum" at x

8.  END
```

# Newton's Divide and Difference Interpolation

Let ***f(x)*** be a function with $\boldsymbol{f(x_0), f(x_1), f(x_2), \ldots, f(x_n)}$ corresponding to values $\boldsymbol{x_0}$, $\boldsymbol{x_1}$ $\boldsymbol{x_2}$,…,$\boldsymbol{x_n}$ then Newton's divide difference interpolation is given by;

$$f(x) = f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] \\ + \cdots \ldots\ldots\ldots (x - x_0)(x - x_1) \ldots\ldots (x - x_n)f[x_0, x_1, \ldots\ldots x_n]$$

or, It can be written as;

$$f(x) = f_0 + (x - x_0)\Delta f_0 + (x - x_0)(x - x_1)\Delta^2 f_0 + (x - x_0)(x - x_1)(x - x_2)\Delta^3 f_0 + \cdots \ldots$$

Now, we can construct the divide and difference table as follows

| $x$ | $f$ | $\Delta f$ | $\Delta^2 f$ | $\Delta^3 f$ |
|---|---|---|---|---|
| $x_0$ | $f_0$ | | | |
| | | $\frac{f_1 - f_0}{x_1 - x_0} = \Delta f_0$ | | |
| $x_1$ | $f_1$ | | $\frac{\Delta f_1 - \Delta f_0}{x_2 - x_0} = \Delta^2 f_0$ | |
| | | $\frac{f_2 - f_1}{x_2 - x_1} = \Delta f_1$ | | $\frac{\Delta^2 f_1 - \Delta^2 f_0}{x_3 - x_0} = \Delta^3 f_0$ |
| $x_2$ | $f_2$ | | $\frac{\Delta f_2 - \Delta f_1}{x_3 - x_1} = \Delta^2 f_1$ | |
| | | $\frac{f_3 - f_2}{x_3 - x_2} = \Delta f_2$ | | |
| $x_3$ | $f_3$ | | | |

## Solved Example 1:

*Q. Using Newton's divide difference interpolation estimate the value of f(x) at x=4 for the function defined below.*

| $x$ | 0 | 2 | 3 | 6 |
|---|---|---|---|---|
| $f$ | 648 | 704 | 729 | 792 |

Solution: The divide difference table for the given data;

| $x$ | $f$ | $\Delta f$ | $\Delta^2 f$ | $\Delta^3 f$ |
|---|---|---|---|---|
| 0 | 648 | | | |
| | | $\frac{704-648}{2-0}=28$ | | |
| 2 | 704 | | $\frac{25-28}{3-0}=-1$ | |
| | | $\frac{729-704}{3-2}=25$ | | $\frac{-1+1}{6-0}=0$ |
| 3 | 729 | | $\frac{21-25}{6-2}=-1$ | |
| | | $\frac{792-729}{6-3}=21$ | | |
| 6 | 792 | | | |

We have,

$$f(x) = f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)f[x_0, x_1, x_2, x_3]$$

$$= 648 + (x - 0)28 + (x - 0)(x - 2)(-1) + 0$$

$$\therefore f(4) = 648 + (4 - 0)28 + (4 - 0)(4 - 2)(-1)$$

$$= 752$$

# Exercise:

1. *Using Newton's divide difference interpolation estimate the value of f(x) at x=1.75 for the function defined below.*

| x | 1.1 | 2.0 | 3.5 | 5 | 7.1 |
|---|---|---|---|---|---|
| f | 0.6981 | 1.4715 | 2.1287 | 2.0521 | 1.4480 |

# Interpolation with equal intervals

- Newton’s forward interpolation
- Newton’s backward interpolation

# Newton's forward interpolation

Newton's forward interpolation formula

$$y_s = y_0 + s\Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 y_0 + \frac{s(s-1)(s-2)(s-3)}{4!}\Delta^4 y_0 + \cdots \ldots$$

Where s = $\frac{x_s - x_0}{h}$, $x_s$= value at which interpolation is to be found

$x_0$=initial value, h= interval of 'x'

| x | y | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ |
|---|---|---|---|---|
| $x_0$ | $y_0$ | | | |
| | | $\Delta y_0 = y_1 - y_0$ | | |
| $x_1$ | $y_1$ | | $\Delta^2 y_0 = \Delta y_1 - \Delta y_0$ | |
| | | $\Delta y_1 = y_2 - y_1$ | | $\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$ |
| $x_2$ | $y_2$ | | $\Delta^2 y_1 = \Delta y_2 - \Delta y_1$ | |
| | | $\Delta y_2 = y_3 - y_2$ | | |
| $x_3$ | $y_3$ | | | |

# Newton's backward interpolation

Newton's backward interpolation formula

$$y_s = y_n + s\nabla y_n + \frac{s(s+1)}{2!}\nabla^2 y_n + \frac{s(s+1)(s+2)}{3!}\nabla^3 y_n + \frac{s(s+1)(s+2)(s+3)}{4!}\nabla^4 y_n + \cdots \ldots$$

Where s = $\frac{x_s - x_n}{h}$, $x_s$= value at which interpolation is to be found

$x_n$=final value, h= interval of 'x'

| x | y | $\nabla y$ | $\nabla^2 y$ | $\nabla^3 y$ |
|---|---|---|---|---|
| $x_0$ | $y_0$ | | | |
| | | $\nabla y_0 = y_1 - y_0$ | | |
| $x_1$ | $y_1$ | | $\nabla^2 y_0 = \nabla y_1 - \nabla y_0$ | |
| | | $\nabla y_1 = y_2 - y_1$ | | $\nabla^3 y_0 = \nabla^2 y_1 - \nabla^2 y_0$ |
| $x_2$ | $y_2$ | | $\nabla^2 y_1 = \nabla y_2 - \nabla y_1$ | |
| | | $\nabla y_2 = y_3 - y_2$ | | |
| $x_3$ | $y_3$ | | | |

# When to use forward/Backward interpolation?

- **Forward Interpolation:** When data is given in ascending order or the required point is close to the starting point of the table.
- **Backward interpolation:** when the data is given in descending order or the required point is close to the end point of the table.

## Solved Example 1:

*Q. Using Newton's forward interpolation formula for the given table to evaluate f(5).*

| $x$ | 4 | 6 | 8 | 10 |
|---|---|---|---|---|
| $y$ | 1 | 3 | 8 | 16 |

Solution: The difference table for the given data;

| x | y | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ |
|---|---|---|---|---|
| 4 | 1 | | | |
| | | 2 | | |
| 6 | 3 | | 3 | |
| | | 5 | | 0 |
| 8 | 8 | | 3 | |
| | | 8 | | |
| 10 | 16 | | | |

Here, $x_s$=5, $x_0$=4, and h= 2

$$s = \frac{x_s - x_0}{h} = \frac{5-4}{2} = 0.5$$

Newton's forward interpolation formula

$$y_s = y_0 + s\Delta y_0 + \frac{s(s-1)}{2!}\Delta^2 y_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 y_0$$

$$y_5 = 1 + (0.5)(2) + \frac{0.5(0.5-1)}{2!}(3) + \frac{0.5(0.5-1)(0.5-2)}{3!}(0)$$

$$= 1 + 1 - 0.375 + 0$$

$$= 1.625$$

$$\therefore f(5) = 1.625$$

# Solved Example 2:

*Q. Using Newton's backward interpolation formula for the given table to evaluate f(17).*

| | | | | | |
|---|---|---|---|---|---|
| ***x*** | ***0*** | ***5*** | ***10*** | ***15*** | ***20*** |
| ***f(x)*** | ***1.0*** | ***1.6*** | ***3.8*** | ***8.2*** | ***15.4*** |

Solution: The difference table for the given data;

| $x$ | $y=f(x)$ | $\nabla y$ | $\nabla^2 y$ | $\nabla^3 y$ | $\nabla^4 y$ |
|---|---|---|---|---|---|
| 0 | 1.0 | | | | |
| | | 0.6 | | | |
| 5 | 1.6 | | 1.6 | | |
| | | 2.2 | | 0.6 | |
| 10 | 3.8 | | 2.2 | | 0 |
| | | 4.4 | | 0.6 | |
| 15 | 8.2 | | 2.8 | | |
| | | 7.2 | | | |
| 20 | 15.4 | | | | |

Here, $x_s$=17, $x_n$=20, and h= 5

$$s = \frac{x_s - x_n}{h} = \frac{17-20}{5} = -0.6$$

# Newton's backward interpolation formula

$$y_s = y_n + s\nabla y_n + \frac{s(s+1)}{2!}\nabla^2 y_n + \frac{s(s+1)(s+2)}{3!}\nabla^3 y_n + \frac{s(s+1)(s+2)(s+3)}{4!}\nabla^4 y_n$$

$$\therefore y_{17} = 15.4 + (-0.6)(7.2) + \frac{(-0.6)(-0.6+1)}{2!}(2.8) + \frac{(-0.6)(-0.6+1)(-0.6+2)}{3!}(0.6) + 0$$

$$= 15.4 - 4.32 - 0.336 - 0.0336 + 0$$

$$= 10.7104$$

$$\therefore y_{17} = 10.7104$$

## Exercise:

1. *Using Newton's forward difference formula estimate the value of f(x) at x=3.6 for the function defined below.*

| x | 2 | 2.5 | 3 | 3.5 | 4 | 4.5 |
|---|---|---|---|---|---|---|
| f(x) | 1.43 | 1.03 | 0.76 | 0.6 | 0.48 | 0.39 |

2. *Using Newton's backward difference formula estimate the value of* ***ln(3.5)*** *for the function defined below.*

| x | 1.0 | 2.0 | 3.0 | 4.0 |
|---|---|---|---|---|
| ln(x) | 0.0 | 0.6931 | 1.0986 | 1.3863 |

# Cubic Spline Interpolation

Cubic interpolation works by constructing the (cubic) polynomial in pieces. Given n points will construct n-1 different (cubic) polynomials. These polynomial have consistent derivatives at the end points.

**Formula**

Formula 1

$$h_i a_{i-1} + 2a_i(h_i + h_{i+1}) + h_{i+1} a_{i+1} = 6\left[\frac{f_{i+1} - f_i}{h_{i+1}} - \frac{f_i - f_{i-1}}{h_i}\right] \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Where $a_0 = a_n = 0$

Formula 2

$$s_i(x) = \frac{a_{i-1}}{6h_i}(h_i^2 U_i - U_i^3) + \frac{a_i}{6h_i}(U_{i-1}^3 - h_i^2 U_{i-1}) + \frac{1}{h_i}(f_i U_{i-1} - f_{i-1} U_i) \ldots\ldots\ldots\ldots (2)$$

$h_i = x_i - x_{i-1}$ and $U_i = x - x_i$

Evaluate equation 1 ***for i= 1to n-1***
Evaluate equation 2 at ***i=a*** value given by position interval

## Solved Example 1:

*Q. Estimate the functional value of f at x=7 using cubic splines from given table.*

| | | | |
|---|---|---|---|
| ***x*** | ***4*** | ***9*** | ***16*** |
| ***f*** | ***2*** | ***3*** | ***4*** |

Solution:

$$h_1 = x_1 - x_0 = 9 - 4 = 5$$

$$h_2 = x_2 - x_1 = 16 - 9 = 7$$

$$f_0 = 2,\ f_1 = 3,\ f_2 = 4$$

$$a_0 = a_2 = 0$$

We have:

$$h_i a_{i-1} + 2a_i(h_i + h_{i+1}) + h_{i+1}a_{i+1} = 6\left[\frac{f_{i+1}-f_i}{h_{i+1}} - \frac{f_i - f_{i-1}}{h_i}\right]$$

For $i = 1$

$$h_1 a_0 + 2a_1(h_1 + h_2) + h_2 a_2 = 6\left[\frac{f_2 - f_1}{h_2} - \frac{f_1 - f_0}{h_1}\right]$$

$$0 + 2a_1(5 + 7) + 0 = 6\left[\frac{4-3}{7} - \frac{3-2}{5}\right]$$

$$24a_1 = 6\left[\frac{1}{7} - \frac{1}{5}\right]$$

$\boldsymbol{a_1}$**= -0.0143**

Again from second formula,

$$s_i(x) = \frac{a_{i-1}}{6h_i}(h_i^2 U_i - U_i^3) + \frac{a_i}{6h_i}(U_{i-1}^3 - h_i^2 U_{i-1}) + \frac{1}{h_i}(f_i U_{i-1} - f_{i-1} U_i)$$

$$s_1(x) = \frac{a_0}{6h_1}(h_1^2 U_1 - U_1^3) + \frac{a_1}{6h_1}(U_0^3 - h_1^2 U_0) + \frac{1}{h_1}(f_1 U_0 - f_0 U_1)$$

$$U_0 = x - x_0 = x - 4$$

$$U_1 = x - x_1 = x - 9$$

$$\therefore s_1(7) = 0 + \frac{-0.0143}{6X5}[(7-4)^3 - 5^2(7-4)] + \frac{1}{5}[3(7-4) - 2(7-9)]$$

$$=2.6229$$

# Curve Fitting

Curve fitting is the process of introducing mathematical relationship between dependent and independent variables in the form of an equation for a given set of data.

- Linear Curve Fitting
- Non - Linear Curve Fitting

# Linear Curve Fitting

$y = a + bx \; ---(i)$ where ***a*** and ***b*** are constants and are unknown

The normal equation of *(i)* is

$$\sum y_i = na + b\sum x_i \quad ---(ii)$$

$$\sum x_i y_i = a\sum x_i + b\sum x_i^2 \quad ---(iii)$$

Solving equation (ii & iii) we can determine the constants a and b.

$$\boxed{\begin{aligned} b &= \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - \left(\sum x_i\right)^2} \\ a &= \frac{\sum y_i}{n} - b\frac{\sum x_i}{n} = \bar{y} - b\bar{x} \end{aligned}}$$

**Solved Example 1:**

*Q. Fit a straight line to the following set of data.*

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $y$ | 3 | 4 | 5 | 6 | 8 |

Solution:

| $x$ | $y$ | $x^2$ | $xy$ |
|---|---|---|---|
| 1 | 3 | 1 | 3 |
| 2 | 4 | 4 | 8 |
| 3 | 5 | 9 | 15 |
| 4 | 6 | 16 | 24 |
| 5 | 8 | 25 | 40 |
| $\sum x = 15$ | $\sum y = 26$ | $\sum x^2 = 55$ | $\sum xy = 90$ |

We have:

$$b = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - \left(\sum x_i\right)^2} = \frac{5 \times 90 - 15 \times 26}{5 \times 55 - 15^2} = 1.2$$

$$a = \frac{\sum y_i}{n} - b\frac{\sum x_i}{n} = \frac{26}{5} - 1.2 \times \frac{15}{5} = 1.6$$

The linear equation is:

$$y = a + bx$$

$$\therefore \boldsymbol{y = 1.6 + 1.2\,x}$$

# Non-Linear Curve Fitting

a. Exponential Function

Let the curve be $\mathbf{y = a}e^{bx}$**---(i)**

Taking log on both sides

$$\log y = \log a + (b*\log e)*x$$

$$Y = A + Bx \text{ ---(ii)} \quad [\because \log e = 1]$$

Where,

Y = log y

B=loge*b

A = log a

## Solved Example 1:

*Q. Fit the following set of data to the curve of the form* $\mathbf{y = ae^{bx}}$

| *x* | *2* | *4* | *6* | *8* | *10* | *12* |
|---|---|---|---|---|---|---|
| *y* | *16* | *17.1* | *8.7* | *6.4* | *4.7* | *2.6* |

Solution:

Let the curve be $\mathbf{y = ae^{bx}}$**---(i)**

Taking log on both sides

$$\log y = \log a + x\log b$$

$$Y = A + Bx \text{ ---(ii)} \quad [\because \log e = 1]$$

Where,

Y = log y

B= log b

A = log a

We have,

| X | y | Y=logy | $X^2$ | XY |
|---|---|---|---|---|
| 2 | 16 | 1.20412 | 4 | 2.40824 |
| 4 | 11.1 | 1.04532 | 16 | 4.18128 |
| 6 | 8.7 | 0.93952 | 36 | 5.63712 |
| 8 | 6.4 | 0.80618 | 64 | 6.44944 |
| 10 | 4.7 | 0.67209 | 100 | 6.7209 |
| 12 | 2.6 | 0.41497 | 144 | 4.97964 |
| $\sum X = 42$ | | $\sum Y = 5.0822$ | $\sum X^2 = 364$ | $\sum XY = 30.37662$ |

$$B = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - \left(\sum x_i\right)^2} = \frac{6 \times 30.37662 - 42 \times 5.0822}{6 \times 364 - 42^2} = -0.07427$$

$$A = \frac{\sum y_i}{n} - B\frac{\sum x_i}{n} = \frac{5.0822}{6} - (-0.07427) \times \frac{42}{6} = 1.36692$$

Now,

a = antilog(1.36692)= 23.27662

b= (-0.07427)/log e =-0.1710

So the equation (i) becomes,

$$\therefore \mathbf{y} = \mathbf{23.27662}\boldsymbol{e}^{-0.1710\boldsymbol{x}}$$

Thank you!