Numerical Methods

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Subject Overview

1. Solution of Nonlinear Equations
2. Interpolation and Approximation
3. Numerical Differentiation and integration
4. Solution of Linear Algebraic Equations
5. Solution of Ordinary Differential Equations
6. Solution of Partial Differential Equations

Chapter 1 Coverage

- An introduction to numerical methods
- Errors in Numerical Computing
- Solution of Nonlinear Equations
  - Bisection method
  - Secant Method
  - Newton Raphson method
  - Fixed point iteration

Numerical methods/analysis

- A numerical method is a complete and definite set of procedures for the solution of a problem, together with computable error estimates. The study and implementation of such methods is the field of numerical analysis/ numerical methods.
- Numerical methods find solutions ***close*** to the answer without ever knowing what that answer is. As such, an important part of every numerical method is a proof that it works.

Applications of Numerical methods

- **Root finding:** Numerical methods like the bisection method, Newton-Raphson method, and secant method are used to find the roots (solutions) of nonlinear equations. Methods like Gaussian elimination and matrix factorization are used to solve systems of linear equations, which are common in engineering and physics.
- **Simulation and modeling:** Numerical methods are used to simulate and model physical, biological, and social systems. These simulations can be used to test hypotheses, predict outcomes, and optimize performance.
- **Optimization:** Numerical methods are used to optimize systems by minimizing or maximizing a given objective function subject to constraints. Examples include linear programming, nonlinear programming, and integer programming.

Applications of Numerical methods

- Numerical differentiation methods, such as finite differences, are employed to **estimate derivatives** when analytical derivatives are unavailable or too complicated.
- **Computer graphics:** Numerical methods are used to create realistic images and animations in computer graphics. Examples include ray tracing, rendering, and animation.
- **Signal Processing:** Numerical methods are used in signal processing to filter, transform, and analyze signals and data, such as in image processing, audio processing, and communications.

Characteristics of Numerical Methods

- **Accuracy:** Accuracy is defined as the closeness or nearness of the measurements to the true or actual value of the quantity being measured.
- **Rate of convergence:** Convergence refers to the tendency of a numerical method to approach the correct solution as the number of iterations increases. A convergent method is one that produces results that get closer to the true solution with each iteration.
- **Numerical Stability:** Stability is a crucial characteristic, especially in iterative methods. A stable method ensures that small errors in one iteration do not lead to significant errors in subsequent iterations.
- **Applicability:** Numerical methods are versatile and applicable to a wide range of mathematical problems, including differential equations, integrals, optimization, and linear algebra problems.

Accuracy of Numbers

- **Exact Numbers**

  Exact number gives it's real value correctly without any rounding or without truncating any digits. For example 23, 200, -56, 0, 0.73, 1/3, 3.143 , $\sqrt{2}$ are all exact numbers.

- **Approximate numbers**

  An approximate number is a number which is an approximation of a true value. The numbers $\frac{1}{3} = 0.33333 \ldots and\ \pi = 3.141592 \ldots$ do not have finite expansion hence they are approximated to some finite digits called **significant digits** for the purpose of calculation

Errors in Numerical Computation

- In numerical analysis we approximate the exact solution of the problem by using numerical method and consequently an error is committed.

- The **numerical error** is the difference between the exact solution and the approximate solution.

- Let **x** be the exact solution of the underlying problem and $\boldsymbol{x}^*$ its approximate solution, then the error (denoted by e) in solving this problem is $\boldsymbol{e} = \boldsymbol{x} - \boldsymbol{x}^*$

Errors in Numerical Computation

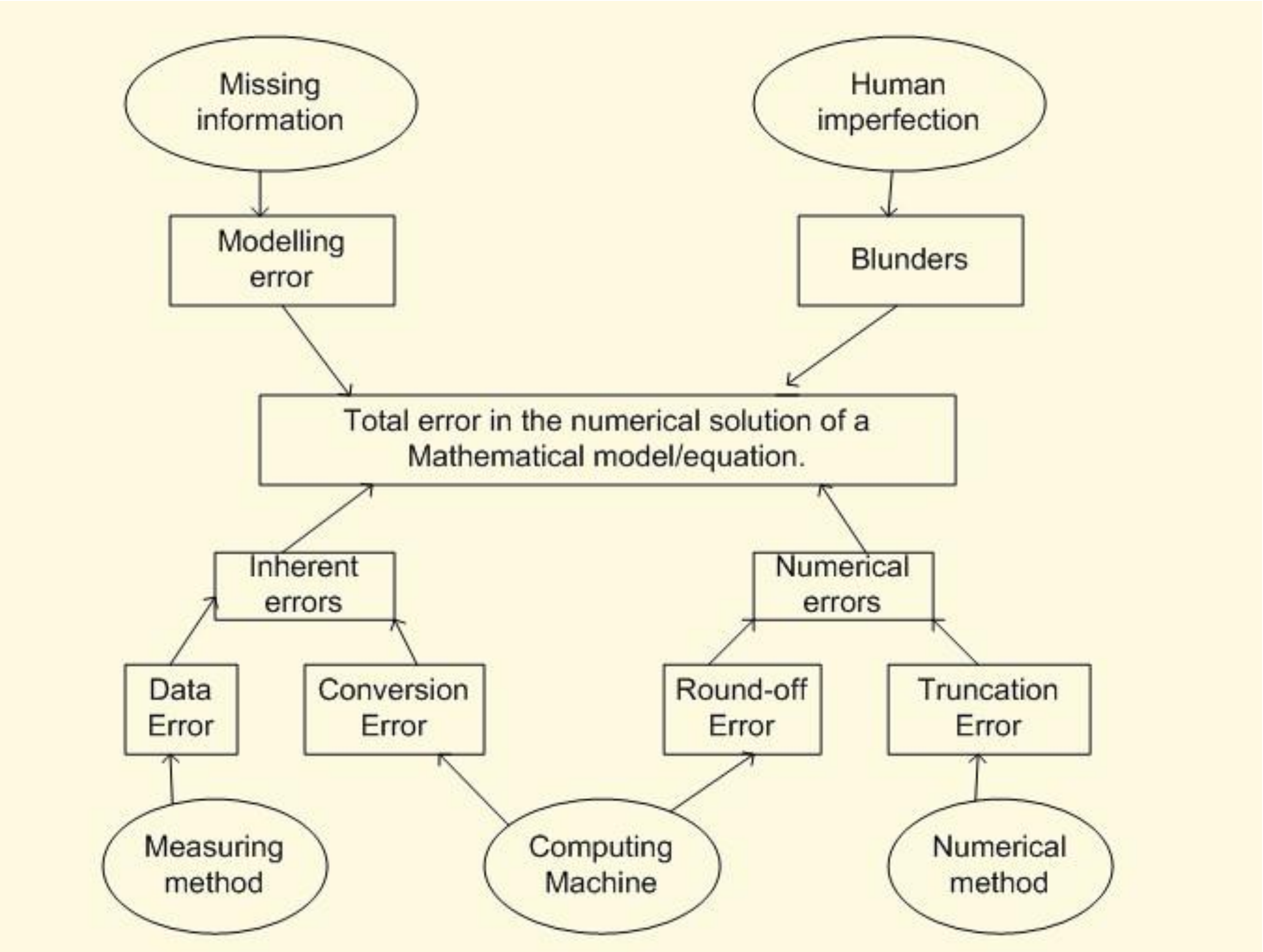


Fig: Anatomy of Errors

Errors in Numerical Computation

1. **Blunders (Gross Errors):** These errors also called **humans errors**, and are caused by humans mistakes and oversight and can be minimized by taking care during scientific investigations. These errors will add to the total error of the underlying problem and can significantly affect the accuracy of solution.
2. **Modelling Errors:** These errors arise during the modelling process when scientists ignore effecting factors in the model to simplify the problem. Also, these errors known as formulation errors.
3. **Inherent Error:** Error that are present in the data supplied to the model.
   a) **Data Errors:** Data errors arise when data to be input into a model are acquired using experimental methods. These are also called **empirical errors.** Such errors occur mostly due to the limitations or errors in the instrumentation.

Errors in Numerical Computation

b. **Conversion Error:** Arises due to limitation of the computer to store the data exactly.

4. Numerical Error: This error introduced during the process of implementation of numerical method.

a) **Round of Error**: Arises when a fixed no. of digit are used to represent exact number. It occurs because of the computing device's inability to deal with certain numbers.

$$1/3 = 0.3333333\ldots$$

Such numbers need to be rounded off to some near approximation which is dependent on the word size used to represent numbers of the device.

Errors in Numerical Computation

b. Truncation Error: Truncation errors arise from using an approximation in place of an exact mathematical procedure. Typically, it is the error resulting from the truncation of numerical process.

For example, consider the following infinite series:

$$\boldsymbol{sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots}$$

When we calculate the sine of an angle using this series. We usually terminate the process after a certain term is calculated. The term “truncated” introduces an error which is called truncation error.

Absolute and Relative Errors

- **True Error:** True error is denoted by E_t, and is defined as the difference between the true value and approximate value

  True error = True value - Approximate value

- **Relative Error:** Relative error is denoted by E_r, and is defined as the ratio between the true error and the true value

  $$\textbf{\textit{Relative error}} = \frac{\textbf{\textit{True Error}}}{\textbf{\textit{True Value}}}$$

Absolute and Relative Errors

- **Approximate Error:** Approximate error is denoted by E_a, and is defined as the difference between the present approximation and previous approximation

$$\textit{Approximate error} = \textit{Present approximation} - \textit{previous approximation}$$

- **Relative Approximate Error:** Relative approximate error is denoted by E_{ra}, and is defined as the ratio between the approximate error and the present approximation.

$$\textit{Relative approximate error} = \frac{\textit{Approximate Error}}{\textit{Present Approximation}}$$

Question:

The derivative of the function $f(x)$ at a particular value of x can be approximately calculated by

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

For $f(x) = x^2 \;\; and \; h = 0.3$

a. Find the approximate value of $f'(2)$, true value of $f'(2)$ and the true error.

b. Find the relative true error at x=2

c. Find approximate error of $f'(2)$ using $h = 0.3$ and $h = 0.15$

d. Find relative approximate Error.

Solution of Nonlinear Equations

The general form of nonlinear equation is,

$$ax^2 + by^2 = c$$

Where, x and y are variables and a, b, and c are the constant values

Methods of solution

I. Direct analytical methods
II. Graphical method
III. Trial and Error methods
IV. Iterative method

Iterative method

- Bracketing methods
- Open end methods

Backtracking method and open End method

Bracketing Method

- Bracketing methods start with a bounded interval which is guaranteed to bracket a root. The size of initial bracket is reduced step by step until it encloses the root in a desired tolerance.
- Slower than open end method.
- Example:

i. Bisection method

ii. False position method

Open End Method

- Open methods begin with an initial guess of the root and then improves the guess iteratively.
- Faster than bracketing method
- Example

i. Newton Raphson method

ii. Fixed point method

iii. Secant method

The Bisection Method

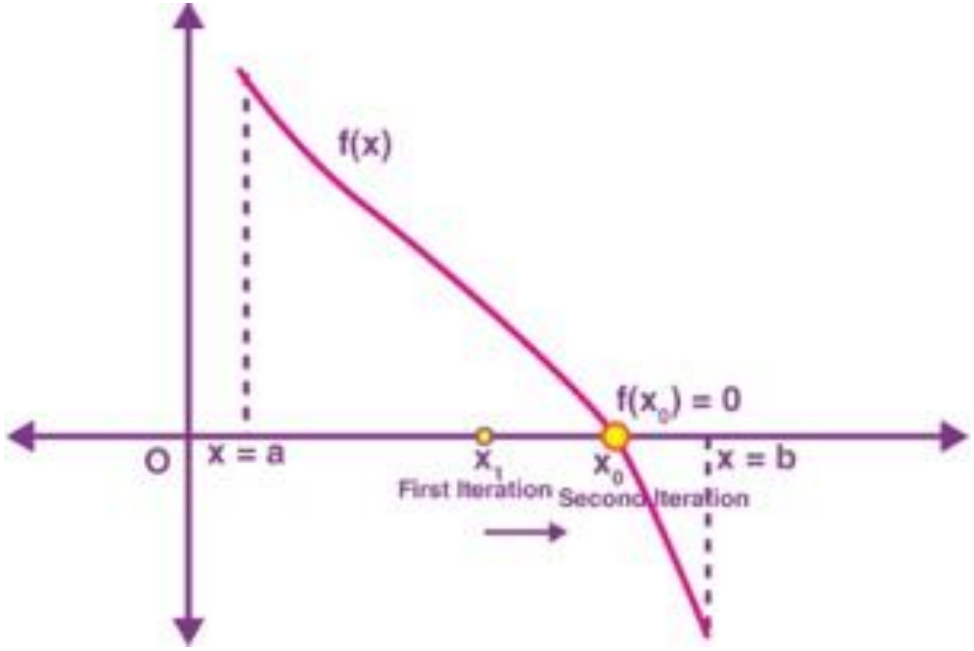
One of the simples and most reliable of iterative methods for the solution of non linear equations

The method is also known as binary chopping or half-interval method.

If $f(x)$ is real and continuous in the interval $[a, b]$ and $f(a)$ and $f(b)$ are of opposite signs, that is

$$f(a) * f(b) < 0$$

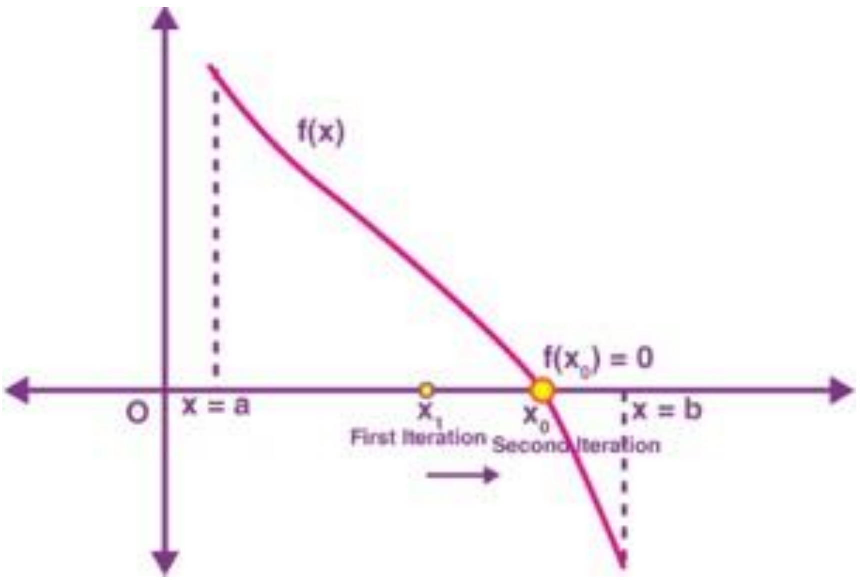
Then there is at least one real root in the interval between a and b.



The Bisection Method

Let $x_1 = \frac{a+b}{2}$ be the mid point of a and b

- If $f(x_1) = 0$, then x_1 is the root of $f(x) = 0$
- If $f(a)f(x_1) < 0$ root of $f(x)$ lies in $[a, x_1]$
- If $f(x_1)f(b) < 0$ root of $f(x)$ lies in $[x_1, b]$



Then we bisect the interval as before and continue the process until the desired accuracy achieved.

Criteria for terminating the computations

- In practical problem the root may not be exact so the criteria
  $f(x_0) = 0$ may not be satisfied.
- Absolute error

$$|b - a| \leq tolerable\ error$$

- Relative error $\left|\frac{b-a}{c}\right| \leq tolerable\ error$
- Absolute error in Functional $|\mathrm{f}(\mathrm{c})| \leq tolerable\ error$

Bisection Method Algorithm

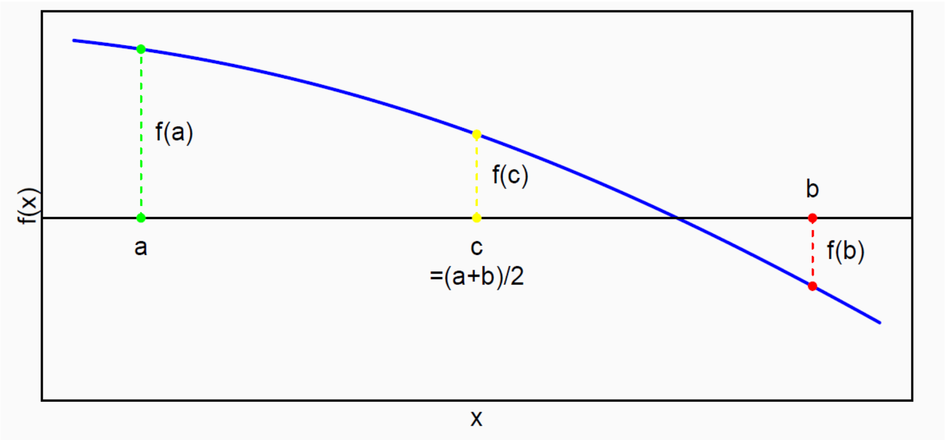
1. *Define function f(x) and error (e)*
2. *Guess initial values **a** and **b***
3. *Calculate* $f(a)$ *and* $f(b)$
4. *If* $\boldsymbol{f(a) * f(b) > 0}$
   goto step 8 otherwise,

5. *Calculate* $\boldsymbol{c = \frac{a+b}{2}}$

6. *Check if* $|f(c)| > error$
   if $f(c) * f(b) > 0$
      b=c
   else
      a=c
   go to step 5

7. *Print root = c*

8. *END*



Pros and Cons of Bisection Method

Advantages

- Convergence is guaranteed.
- The error can be controlled
- Simple and easy to program.

Disadvantages

- A slow, linear rate of convergence.
- Cannot find the root of some equations
- If one of the guesses is closer to the root, it will still take a larger number of iterations

Total number of iteration in Bisection method

Error Tolerance = ε

Initial Interval = (a , b)

$$\frac{|b-a|}{2^n} \leq \varepsilon$$

$$\text{or, } 2^n \geq \frac{|b-a|}{\varepsilon}$$

$$\text{or, } \log(2^n) \geq \log\left(\frac{|b-a|}{\varepsilon}\right)$$

$$\text{or, } nlog\ 2 \geq \log|b-a| - \log(\varepsilon)$$

$$\therefore \boldsymbol{n} \geq \frac{\boldsymbol{log|b-a|-log(\varepsilon)}}{\mathbf{log\ 2}}$$

Solved example 1:

Find the positive roots of the equation $x^3 - 2x - 5 = 0$. Using the bisection method correct to three decimal places.

Given that

$f(x) = x^3 - 2x - 5,$

Let the initial guess be 2 and 3

$f(2) = -1 < 0 \; and \; f(3) = 16 > 0, So \; the \; root \; lise \; in \; between \; 2 \; and \; 3$

Now let us calculate the root by tabulation method

S. N	a	b	c= (a+b) / 2	f(c)	Interval Length (b-a)
1	2	3	2.5	5.625	1
2	2	2.5	2.25	1.890625	0.5
3	2	2.25	2.125	0.34570313	0.25
4	2	2.125	2.0625	-0.3513184	0.125
5	2.0625	2.125	2.09375	-0.0089417	0.0625
6	2.09375	2.125	2.109375	0.16683578	0.03125
7	2.09375	2.109375	2.1015625	0.07856226	0.015625
8	2.09375	2.1015625	2.09765625	0.03471428	0.0078125
9	2.09375	2.09765625	2.09570313	0.01286233	0.00390625
10	2.09375	2.09570313	2.09472656	0.00195435	0.00195313
11	2.09375	2.09472656	2.09423828	-0.0034951	0.00097656
12	2.09423828	2.09472656	2.09448242	-0.0007708	0.00048828

Since x_0, x_1, and x_2 are same up to two decimal place.

∴ The root of the given equation is 2.0948

Solved example 2:

Find the positive roots of the equation $x^2 sinx + e^{-x} = 3$. Using the bisection method correct to three decimal places.

Given that,

$$f(x) = x^2 sinx + e^{-x} - 3 = 0$$

Let the initial guess be 1 and 2

$f(1) = -1.79067 < 0$

$f(2) = 0.7725 > 0, So\ the\ root\ lise\ in\ between\ 1\ and\ 2$

Now let us calculate the root by tabulation method

S. N.	a	b	c=(a+b)/2	f(c)	Error |b-a|
1	1	2	1.5	-0.5325	1
2	1.5	2	1.75	0.18723	0.5
3	1.5	1.75	1.625	-0.1663	0.25
4	1.625	1.75	1.6875	0.01327	0.125
5	1.625	1.6875	1.65625	-0.076	0.0625
6	1.65625	1.6875	1.67188	-0.0312	0.03125
7	1.67188	1.6875	1.67969	-0.0089	0.01563
8	1.67969	1.6875	1.68359	0.00218	0.00781
9	1.67969	1.68359	1.68164	-0.0034	0.00391
10	1.68164	1.68359	1.68262	-0.0006	0.00195
11	1.68262	1.68359	1.68311	0.00079	0.00098

Since x_0, x_1, and x_2 are same up to two decimal place.

$\therefore$ The root of the given equation is 1.683

Exercise:

1. Find the positive roots of the equation $e^x - 3x = 0$. Using the bisection method correct to three decimal place.
2. Find the root of equation $x^2-4x-10=0$ using bisection method where root lies between 5 and 6. [T.U. 2019]
3. Find the approximation of $\sqrt{3}$ correct to within 10^{-4} by using bisection method.
4. Find at least one root of the equation $x^2 + tanx + e^x = 0$ correct up to 3 decimal place using bisection method.
5. Find the root of the equation $logx - cosx = 0$ correct up to 4 decimal place using bisection method.
6. Calculate the root of the equation f(x) $=$ $x^2 - 10logx$ correct up to 3 decimal place using bisection method.

The Method of False position (Regula-falsi)

- If $f(x)$ is real and continuous in the interval $[a, b]$ and $f(a)$ and $f(b)$ are of opposite signs, that is

$$\mathbf{f(a) * f(b) < 0}$$

Hence a root must lies in between them

- Join the points (a, f(a)) and (b, f(b)) by a secant line.
- Determine the point (x), which is the point of intersection between the secant line and the x-axis

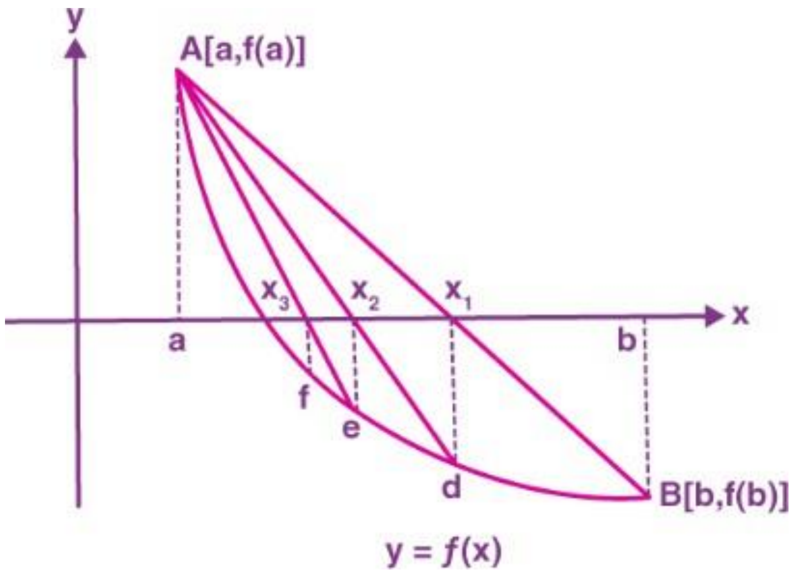
The equation of the secant

$$y - f(a) = \frac{f(b)-f(a)}{b-a}(x-a)$$

substituting $y = 0$ we get

$$x = \frac{a.f(b)-b.f(a)}{f(b)-f(a)}$$

- Similarly we can estimate x_1, x_2, x_3 and so on



The Method of False position (Regula-falsi)

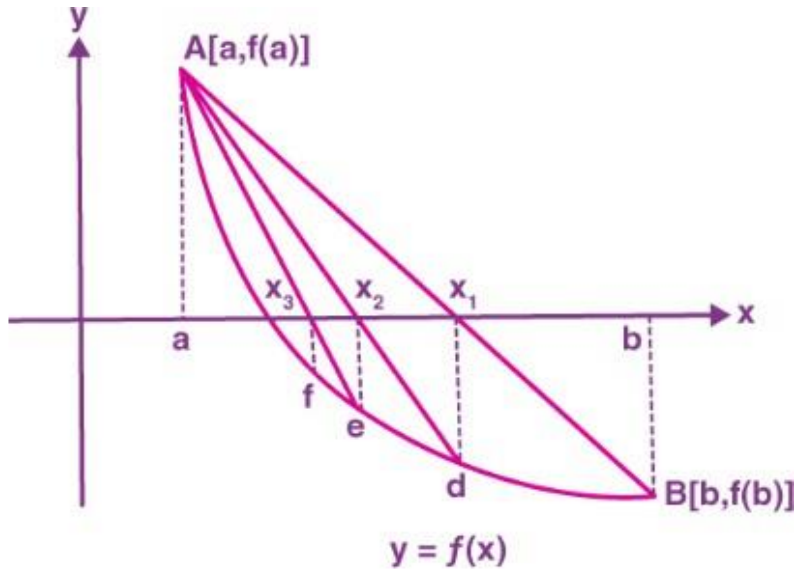
➢ If the sign of f(x) is same as f(b):

move b to x_1

Otherwise:

move a to x_1

➢ Repeat until f(x_n) ≈ 0



Solved examples:

Find the real root of the following equation using false position method correct up to 4 decimal place. $\boldsymbol{f(x) = e^{cosx} - sinx - 1 = 0}$

Given that

$$\boldsymbol{f(x) = e^{cosx} - sinx - 1}$$

Now,

$$f(3) = -0.7695 < 0 \; and \; f(4) = 0.2769 > 0$$

So the root lies in between 3 and 4.

We have,

$$x_n = \frac{\mathrm{a.f(b) - b.f(a)}}{\mathrm{f(b) - f(a)}}$$

Now let us calculate the root by tabulation method

n	a-	b+	F(a)	F(b)	x_n	f(x_n)
1	3	4	-0.76954	0.27695	3.73535	-0.00396
2	3.73535	4	-0.00396	0.27695	3.73908	0.000043
3	3.73735	3.73908	-0.00396	0.00004	3.73908	-0.00025
4	3.73904	3.73908	-0.00005	0.00004	3.73908	-0.00003

Since the value of two consecutive x_n are same up to four decimal place.
∴ The root of the given equation is 3.73908

Exercise:

1. Find the positive roots of the equation $\boldsymbol{x^3 + 4x + 1 = 0}$. Using the false position method correct to three decimal place.
2. Find the root of the equation $\mathbf{Sinx} - \boldsymbol{x + 2 = 0}$ correct up to 4 decimal place using bisection method.
3. Calculate the root of the equation $\mathbf{f}(\boldsymbol{x}) = \boldsymbol{x^2 - 10logx}$ correct up to 3 decimal place using false position method

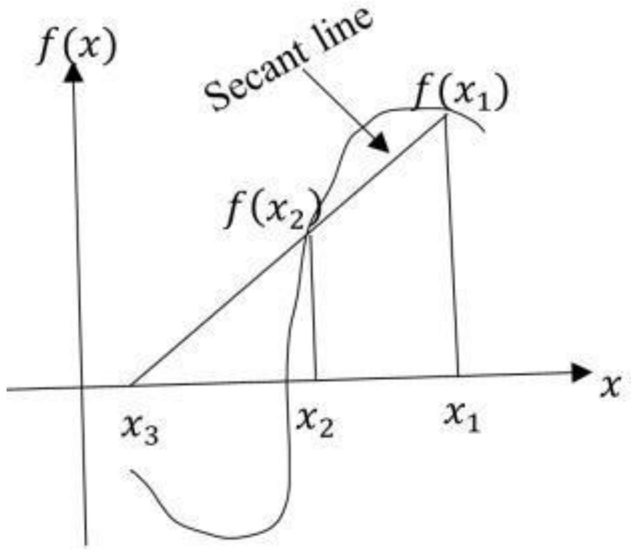
Problem with false position method

- Despite of deciding the new sub-interval using the Intermediate Value Theorem, it can be observed that one of the brackets might remain fixed mostly, resulting in slow convergence.
- Convergence rate can be improved by taking the latest two x-values to compute the next approximation, completely avoiding the intermediate value theorem – the **Secant method.**
- This improves the convergence rate, however, at the cost of reliability.

Secant Method

This method uses two initial estimates but doesn't require that they must bracket the root. Let us consider the following figure:

The point $x_1$and x_2 are starting point, they do not bracket the root. Slope of line (secant line) passing through x_1 and x_2 is given by



$$\frac{f(x_1)}{x_1 - x_3} = \frac{f(x_2)}{x_2 - x_3}$$

$f(x_1)(x_2 - x_3)= f(x_2)(x_1 - x_3)$

$x_3[f(x_2) - f(x_1)] = f(x_2).x_1 - f(x_1).x_2$

$$x_3 = \frac{f(x_2).x_1 - f(x_1).x_2}{[f(x_2) - f(x_1)]}$$

Now by adding and subtracting $f(x_2).x_2$in the numerator and rearranging the terms, we get,

$$x_3 = x_2 - \frac{f(x_2)(x_2 - x_1)}{[f(x_2) - f(x_1)]}$$

This equation is called secant formula.
Here, x_3 represents the approximate root of $f(x)$. The approximate value of root can be refined by repeating this procedure by replacing x_1 by x_2 and x_2 by x_3 above equation
The next approximate value is given by

$$x_4 = x_3 - \frac{f(x_3)(x_3 - x_2)}{[f(x_3) - f(x_2)]}$$

This procedure is continued till the desired level of accuracy is obtained.

The secant formula in general form is

$$\boldsymbol{x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{[f(x_i) - f(x_{i-1})]}}$$

Secant Method Algorithm

1. *Define function f(x) and error (e)*
2. *Guess initial values* x_1 *and* x_2
3. *Calculate* $f(x_1)$ *and* $f(x_2)$
4. *Compute*

$$x_3 = x_2 - \frac{f(x_2)(x_2 - x_1)}{[f(x_2) - f(x_1)]}$$

5. *Set* $x_1 = x_2$ *and* $x_2 = x_3$
6. *Check if* $|f(x_3)| > error$
   goto step 4 otherwise,
7. *Print root =* x_3
8. *END*

Solved example 1:

Q. Find the root of the following equation using secant method correct up to 3 decimal place. $\boldsymbol{f(x) = sinx - 2x + 1}$

Given that

$$f(x) = sinx - 2x + 1$$

Let the initial guess be 0 and 1

Now let us calculate the root by tabulation method

n	x_1	x_2	$f(x_1)$	$f(x_2)$	$x_3 = x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)}$
1	1	0	-0.1585	1	0.863185
2	0	0.863185	1	0.03355	0.89315
3	0.863185	0.89315	0.03355	-0.00725	0.88782
4	0.89315	0.88782	-0.00725	0.000577	0.88786

Since the value of two consecutive x_3 in 3rd and 4th iterations are same up to three decimal place.

∴ The root of the given equation is 0.088786

Solved example 2:

Q. Using secant method estimate the root of the equation $x^2 - 4x - 10 = 0$ *initial estimates of* x_1 *=4 and* x_2 *=2. Does these points bracket a root?*

Given that

$$f(x) = x^2 - 4x - 10$$

Let the initial guess be x_1 =4 and x_2= 2

Now let us calculate the root by tabulation method

n	x_1	x_2	$f(x_1)$	$f(x_2)$	$x_3 = x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)}$
1	4	2	-10	-14	9
2	2	9	-14	35	4
3	9	4	35	-10	5.11
4	4	5.11	-10	-4.36	5.97
5	5.11	5.97	-4.36	1.761	5.72
6	5.97	5.72	1.761	-0.1616	5.74
7	5.72	5.74	-0.1616	-0.0124	5.74

Since the value of two consecutive x_3 in 6th and 7th iterations are same up to two decimal place.

$\therefore$ The root of the given equation is 5.74

Here the root of the equation does not lie in between x_1 =4 and x_2= 2

Exercise:

Estimate the root of the following equation using secant method correct to three decimal place.

$\boldsymbol{a}.\ \mathbf{3}\boldsymbol{x} + \boldsymbol{sinx} - \boldsymbol{e^x} = \mathbf{0}$

b. $\boldsymbol{xe^x} - \boldsymbol{cosx} = \mathbf{0}$

Newton Raphson Method

Approximate the curve by a straight line tangent drawn on the curve at a point very close to the root x_0

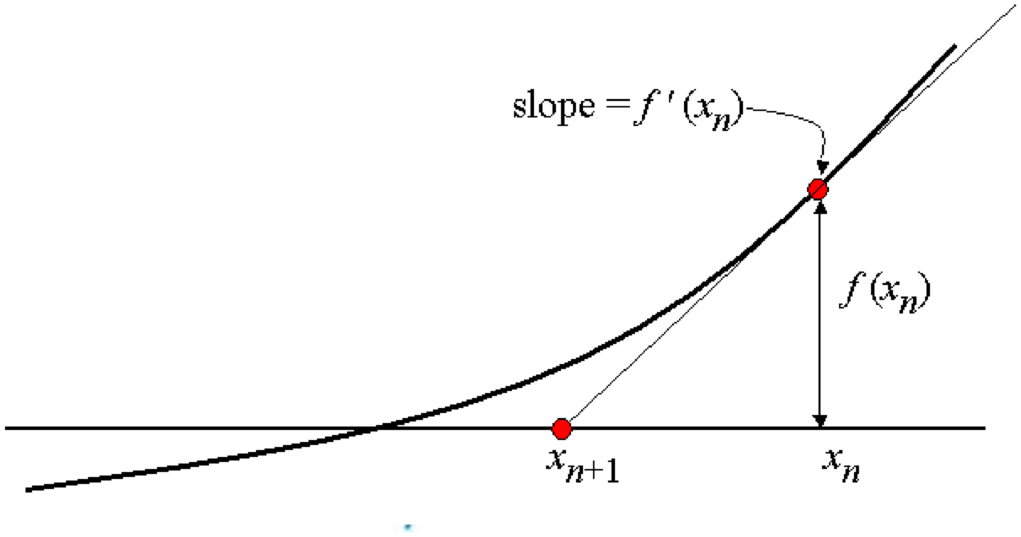
Suppose you need to find the root of a continuous, differentiable function *f*(*x*), and you know the root you are looking for is near the point $x = x_0$ Then Newton's method tells us that a better approximation for the root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

This process may be repeated as many times as necessary to get the desired accuracy.

In general, for any *x*-value x_n, the next value is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$



Derivation via Taylor series

Assume that x_0 be the approximate root of $f(x) = 0$ and x_1=$x_0 + h$ be correct root so that $f(x_1) = f(x_0 + h) = 0$

Where, h is a small interval i.e. $h = x_1 - x_0$

We can express $f(x_1)$ using Taylor series expansion as follows:

$$f(x_1) = f(x_0) + f'(x_0)h + f''(x_0)\frac{h^2}{2!} + \cdots \ldots \ldots \ldots.$$

If we neglect the terms containing the second order and higher derivatives, we get

$$f(x_1) = f(x_0) + f'(x_0)h = 0$$

$$h = \frac{-f(x_0)}{f'(x_0)}$$

$$x_1 - x_0 = \frac{-f(x_0)}{f'(x_0)}$$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Similarly,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Thus,

$$\boldsymbol{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Which is the required formula for Newton Raphson method.

Newton Raphson Method Algorithm

1. *Guess initial root =* x_1 *and stopping criteria*
2. *Calculate* $f(x_1)$ *and* $f'(x_1)$
3. *Compute new root* $$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
4. *Set* x_1*=*x_2
5. *Check if* $|f(x_2)| > error$
   go to step 3 otherwise,
7. *Print root =*x_2
8. *END*

Solved example 1:

Q. Find the root of the following equation using Newton Raphson Method method correct up to 3 decimal place. $e^x - 3x = 0$

Given that

$f(x) = e^x - 3x$ and $f'(x) = e^x - 3$

Let the initial guess be 0.5

Now let us calculate the root by tabulation method

n	x_1	$f(x_1)$	$f'(x_1)$	$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$
1	0.5	0.14872	-1.351278	0.61005
2	0.61005	0.010373	-1.15947	0.61899
3	0.61899	0.0081472	-1.14294	0.61273
4	0.61273	0.0035755	-1.14287	0.61903
5	0.61903	0.00003575	-1.14287	0.61906

Since the value of two consecutive x_2 in 4^{th} and 5^{th} iterations are same up to three decimal place.

$\therefore$ The root of the given equation is 0.61906

Solved example 2:

Q. Find the root of the following equation using Newton Raphson Method method correct up to 3 decimal place. $x^2 + 2x - 2 = 0$

Given that
$f(x) = x^2 + 2x - 2$ and $f'(x) = 2x + 2$
Let the initial guess be 0

Now let us calculate the root by tabulation method

n	x_1	$f(x_1)$	$f'(x_1)$	$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$
1	0	-2	2	1
2	1	1	4	0.75
3	0.75	0.0625	3.5	0.73214
4	0.73214	0.000309	3.464	0.73205
5	0.73205	$-2.7975x10^{-6}$	3.464	0.73205

Since the value of two consecutive x_2 in 4th and 5th iterations are same up to three decimal place.
∴ The root of the given equation is 0.73205

Exercise:

1. Estimate the root of the following equation using Newton Raphson method correct to three decimal place.

a. $-4x + \cos x + 2 = 0$ *assuming* $x_0=0.5$

b. $x^2 - 4x - 7 = 0$ *assuming* $x_0 = 5$

c. $f(x)=x^3+3x+1$ *assuming* $x_0 = 3$

2. Estimate the cube root of 12 using Newton Rapson method correct to three decimal place.

Limitations of Newton Raphson Method

- Division by zero may occur if $f'(x)$ is zero or very close to zero.
- If the initial guess is too far away from the required root, the process may converge to some other root.
- Calculating the required derivative for every iteration may be costly tasks for some functions.

Fixed Point Iteration Method

Any function in the form

$$\boldsymbol{f(x) = 0} \text{ ------}\mathbf{1}$$

can be manipulated such that x is on the left hand side of the equation as shown below

$$\mathbf{x = g(}\boldsymbol{x}\mathbf{) = 0} \text{ ------ }\mathbf{2}$$

Equations (1) & (2) are equivalent and therefore a root of equation (2) is a root of equation (1).

If x is the initial guess to a root, the next approximation is given by:

$$\boldsymbol{x_1 = g(x_0)}$$

Further approximation is given by:

$$\boldsymbol{x_2 = g(x_1)}$$

Thus iteration process can be expressed in general form as:

Fixed Point Iteration Method

$$\boldsymbol{x_{i+1} = g(x_i)} \quad \textbf{i = 0,1,2...}$$

Which is fixed point iteration formula.

The iteration process would be terminated when two successive approximation agree within some specified error.

Solved example 1:

Q. Find the root of . $\boldsymbol{sinx - 5x + 2 = 0}$ ***using Fixed point Iteration Method.***

Given that,

$f(x) = sinx - 5x + 2 = 0$ ……………*(1)*

For fixed point iteration method, arranging equation (1) in terms of *g(x)*.

$sinx - 5x + 2 = 0$
or, $sinx = 5x - 2$
or, $x = \frac{1}{5}(sinx + 2)$
$\therefore g(x) = \frac{1}{5}(sinx + 2)$

Now let us calculate the root by tabulation method

n	x_i	$x_{i+1} = g(x_i)$
1	0.5	0.49589
2	0.49589	0.49516
3	0.49516	0.49503
4	0.49503	0.49501

Since the value of two consecutive x_{i+1} in 3rd and 4th iterations are same.

$\therefore$ The root of the given equation is 0.49510

Fixed Point Iteration Method Algorithm

1. *Define function f(x) and error*
2. *Convert the function f(x)=0 in the form of x=g(x)*
3. *Guess the initial value x_0*
4. *Calculate $x_{i+1} = g(x_i)$*

5. Check if $\left|\frac{x_{i+1}}{x_{i+1}-x_i}\right| \leq error$

go to step 7 otherwise,

6. Assign $x_i = x_{i+1}$

go to step 4

7. Print root =x_i

8.END

Exercise:

1. Estimate the root of the following equation using Fixed point Iteration method correct to 4 decimal place.

   a. $\cos x = 3x-1$

   b. $x^3 + x^2 - 1 = 0$

2. Find the square root of 5 using fixed point iteration method. Note that the iteration formula should consist of no other mathematical operations than the basic math operations (addition, subtraction, multiplication, division).

Thank you!