

# Assignment 2

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Download all python codes from

<https://github.com/yashwanthguguloth24/EE3025-DSP-lab/tree/main/Assignment2/codes>

and latex-tikz codes from

<https://github.com/yashwanthguguloth24/EE3025-DSP-lab/tree/main/Assignment2>

## 1 INTRODUCTION

We are supposed to design the equivalent FIR and IIR filter realizations for filter number 114. This is a bandpass filter whose specifications are available below.

## 2 FILTER SPECIFICATIONS

The sampling rate for the filter has been specified as  $F_s = 48$  kHz. Let the un-normalized discrete-time (natural) frequency is  $F$ , the corresponding normalized digital filter (angular) frequency is given by  $\omega = 2\pi \left(\frac{F}{F_s}\right)$ .

### 2.1 The Digital Filter

- 1) **Tolerances:** The passband ( $\delta_1$ ) and stopband ( $\delta_2$ ) tolerances are given to be equal, so let  $\delta_1 = \delta_2 = \delta = 0.15$ .
- 2) **Passband:** The passband of filter number  $j$ ,  $j$  going from 109 to 135 is from  $\{3 + 0.6(j-109)\}$  kHz to  $\{3 + 0.6(j-107)\}$  kHz. Since our filter number is 114, Substituting  $j = 114$  gives the passband range as 6 kHz - 7.2 kHz. Hence, the un-normalized discrete time filter passband frequencies are

$$F_{p1} = 7.2 \text{ kHz} \quad (1)$$

$$F_{p2} = 6 \text{ kHz} \quad (2)$$

and corresponding normalized digital filter passband frequencies are

$$\omega_{p1} = 2\pi \frac{F_{p1}}{F_s} = 0.3\pi \quad (3)$$

$$\omega_{p2} = 2\pi \frac{F_{p2}}{F_s} = 0.25\pi \quad (4)$$

Center Frequency is given by,

$$\omega_c = \frac{\omega_{p1} + \omega_{p2}}{2} = 0.275\pi \quad (5)$$

- 3) **Stopband:** The *transition band* for bandpass filters is  $\Delta F = 0.3$  kHz on either side of the passband. Hence, the un-normalized *stopband* frequencies are

$$F_{s1} = 7.2 + 0.3 = 7.5 \quad (6)$$

$$F_{s2} = 6.0 - 0.3 = 5.7 \quad (7)$$

and their corresponding Normalized frequencies are,

$$\omega_{s1} = 0.3125\pi \quad (8)$$

$$\omega_{s2} = 0.2375\pi \quad (9)$$

### 2.2 The Analog filter

In the bilinear transform, the analog filter is related to the corresponding digital filter as.,

$$s = \frac{2}{T} \left[ \frac{z-1}{z+1} \right] \quad (10)$$

Substitute.,

$$z = e^{j\omega} \quad (11)$$

$$s = j\Omega \quad (12)$$

where,

$\Omega$  is analog filter frequency

$\omega$  is digital filter frequency

The equation.?? becomes,

$$\Omega = \frac{2}{T} \tan \frac{\omega}{2} \quad (13)$$

Using the above relation, we obtain the analog passband and stopband frequencies as

$$\Omega_{p1} = 0.5095 \quad (14)$$

$$\Omega_{p2} = 0.4142 \quad (15)$$

$$\Omega_{s1} = 0.5345 \quad (16)$$

$$\Omega_{s2} = 0.3914 \quad (17)$$

### 3 IIR FILTER DESIGN

**Filter Type:** We are supposed to design filters whose stopband is monotonic and passband equiripple. Hence, we use the *Chebyshev approximation* to design our bandpass IIR filter.

#### 3.1 The Analog Filter

- 1) **Low Pass Analog Filter Specifications:** If  $H_{a,BP}(j\Omega)$  be the desired analog band pass filter, with the specifications provided in Section 2.2, and  $H_{a,LP}(j\Omega_L)$  be the equivalent low pass filter, then we map the frequencies as following.,

$$\Omega_L \leftarrow A(\Omega^2 - \Omega_0^2) \quad (18)$$

Whenever the  $\Omega$  on the right (which is the BPF) is equal to either  $\Omega_0$  or  $-\Omega_0$ , the  $\Omega_L$  on the left is zero. Now, suppose we set  $A=Q/\Omega_0\Omega$  which is still non-zero and non-infinite for  $\Omega = \Omega_0$ .

$$\Omega_L = Q \left[ \frac{\Omega}{\Omega_0} - \frac{\Omega_0}{\Omega} \right] \quad (19)$$

Where,

$$Q = \frac{\Omega_0}{\Omega_{p1} - \Omega_{p2}} \quad (20)$$

$$\Omega_0 = \sqrt{\Omega_{p1}\Omega_{p2}} \quad (21)$$

The equation.?? can be rewritten as the following.,

$$\Omega_L = \frac{\Omega^2 - \Omega_0^2}{B\Omega} \quad (22)$$

where.,  $B = \Omega_{p1} - \Omega_{p2}$ . The above equation maps bandpass frequencies to low pass frequencies. Substituting the values, we get.,

$$\Omega_0 = \sqrt{\Omega_{p1}\Omega_{p2}} = 0.4594 \quad (23)$$

$$B = \Omega_{p1} - \Omega_{p2} = 0.0953 \quad (24)$$

The low pass filter has the passband edge at  $\Omega_{LP} = 1$  and stopband edges at  $\Omega_{Ls1} = 1.4653$  and  $\Omega_{Ls2} = -1.5511$ . We choose the stopband edge of the analog low pass filter as  $\Omega_{Ls} = \min(|\Omega_{Ls1}|, |\Omega_{Ls2}|)$ .

$$\Omega_{LP} = 1 \quad (25)$$

$$\Omega_{Ls} = 1.4653 \quad (26)$$

- 2) **The Low Pass Chebyshev Filter Paramters:** The magnitude squared of the Chebyshev low pass filter is given by

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + \varepsilon^2 c_N^2(\Omega_L/\Omega_{LP})} \quad (27)$$

where  $c_N(x) = \cosh(N \cosh^{-1} x)$  and the integer  $N$ , which is the order of the filter, and  $\varepsilon$  are design paramters. Since  $\Omega_{LP} = 1$ , (??) may be rewritten as

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + \varepsilon^2 c_N^2(\Omega_L)} \quad (28)$$

Also, the design paramters have the following constraints

$$\frac{\sqrt{D_2}}{c_N(\Omega_{Ls})} \leq \varepsilon \leq \sqrt{D_1},$$

$$N \geq \left\lceil \frac{\cosh^{-1} \sqrt{D_2/D_1}}{\cosh^{-1} \Omega_{Ls}} \right\rceil, \quad (29)$$

where,

$$D_1 = \frac{1}{(1 - \delta)^2} - 1 \quad (30)$$

$$D_2 = \frac{1}{\delta^2} - 1 \quad (31)$$

After appropriate substitutions, we obtain,

$$N \geq 4 \quad (32)$$

$$0.3184 \leq \varepsilon \leq 0.6197 \quad (33)$$

iir/paraplot.py

In Figure.??, we plot  $|H(j\Omega)|$  for a range of values of  $\varepsilon$ , for  $N = 4$ . We find that for larger values of  $\varepsilon$ ,  $|H(j\Omega)|$  decreases in the transition band. We choose  $\varepsilon = 0.4$  for our IIR filter design. The following code generates the values of all parameters.

iir/para.py

- 3) **The Low Pass Chebyshev Filter:** Thus, we obtain

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + 0.16c_4^2(\Omega_L)} \quad (34)$$

where

$$c_4(x) = 8x^4 + 8x^2 + 1. \quad (35)$$

The poles of the frequency response in (??)

lying in the left half plane are in general obtained as  $r_1 \cos \phi_k + jr_2 \sin \phi_k$ , where

$$\phi_k = \frac{\pi}{2} + \frac{(2k+1)\pi}{2N}, k = 0, 1, \dots, N-1$$

$$r_1 = \frac{\beta^2 - 1}{2\beta} \quad (36)$$

$$r_2 = \frac{\beta^2 + 1}{2\beta} \quad (37)$$

$$\beta = \left[ \frac{\sqrt{1+\varepsilon^2} + 1}{\varepsilon} \right]^{\frac{1}{N}} \quad (38)$$

Thus, for  $N$  even, the low-pass stable Chebyshev filter, with a gain  $G$  has the form

$$H_{LP}(s_L) = \frac{G_{LP}}{\prod_k (s_L^2 - 2r_1 C(\phi_k) s_L + r_1^2 C^2(\phi_k) + r_2^2 S^2(\phi_k))} \quad (39)$$

where, (40)

$$C(\phi_k) = \cos(\phi_k) \quad (41)$$

$$S(\phi_k) = \sin(\phi_k) \quad (42)$$

Substituting  $N = 4$ ,  $\varepsilon = 0.5$  and

$H_{a,LP}(j) = \frac{1}{\sqrt{1+\varepsilon^2}}$ , we obtain

$$H_{a,LP}(s_L) = \frac{0.3125}{s_L^4 + 1.12s_L^3 + 1.61s_L^2 + 0.91s_L + 0.34} \quad (43)$$

iir/lpanalog.py

In Figure ?? we plot  $|H(j\Omega)|$  using (??) and (??), thereby verifying that our low-pass

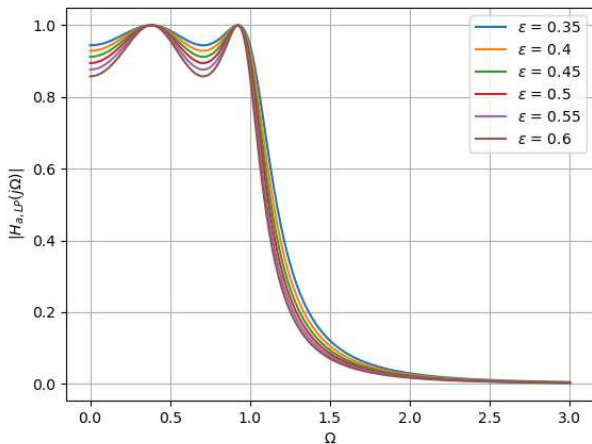


Fig. 2: Analog low pass response for varying epsilon

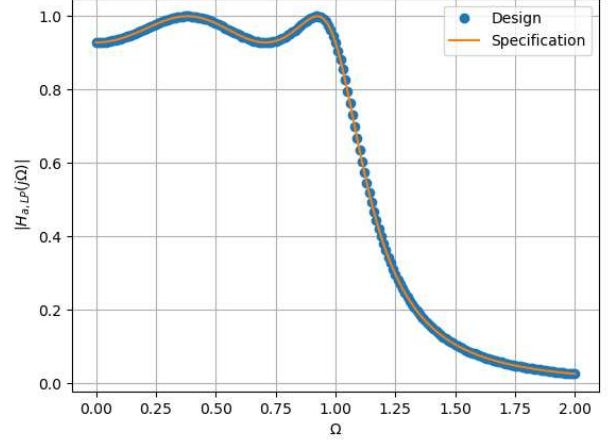


Fig. 3: LP specifications in ??, ??

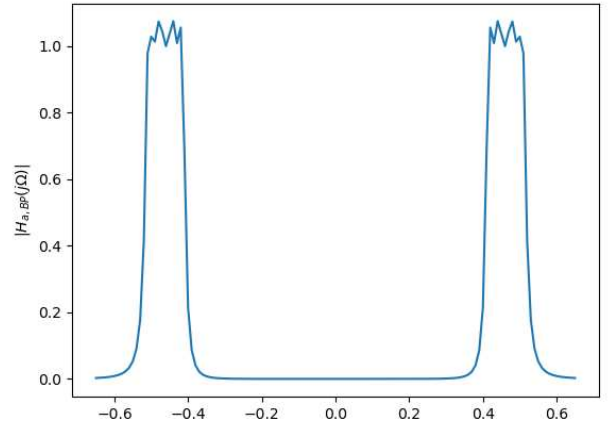


Fig. 4: Analog bandpass from eq.??

Chebyshev filter design meets the specifications.

- 4) **The Band Pass Chebyshev Filter:** The analog bandpass filter is obtained from (??) by substituting  $s_L = \frac{s^2 + \Omega_0^2}{Bs}$ . Hence

$$H_{a,BP}(s) = G_{BP} H_{a,LP}(s_L) \Big|_{s_L = \frac{s^2 + \Omega_0^2}{Bs}} \quad (44)$$

where  $G_{BP}$  is the gain of the bandpass filter. After appropriate substitutions, and evaluating the gain such that  $H_{a,BP}(j\Omega_{p1}) = 1$ , we obtain

$$H_{a,BP}(s) = \frac{2.78 \times 10^{-5} s^4}{s^8 + 0.11s^7 + 0.8s^6 + 0.07s^5 + 0.3s^4 + 0.01s^3 + 0.04s^2 + 0.001s + 0.002} \quad (45)$$

iir/iirfinal.py

In Figure ??, we plot  $|H_{a,BP}(j\Omega)|$  as a function of  $\Omega$  for both positive as well as negative frequencies. We find that the passband and stopband frequencies in the figure match well with those obtained analytically through BT.

### 3.2 The Digital Filter

From the bilinear transformation, we obtain the digital bandpass filter from the corresponding analog filter as

$$H_{d,BP}(z) = GH_{a,BP}(s) \Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} \quad (46)$$

where  $G$  is the gain of the digital filter. From (??) and (??), we obtain

$$H_{d,BP}(z) = G \frac{N(z)}{D(z)} \quad (47)$$

where  $G = 2.7776 \times 10^{-5}$ ,

$$N(z) = 1 - 4z^{-2} + 6z^{-4} - 4z^{-6} + z^{-8} \quad (48)$$

and

$$D(z) = 2.36 - 12z^{-1} + 31.88z^{-2} - 53.75z^{-3} + 62.81z^{-4} - 51.47z^{-5} + 29.23z^{-6} - 10.53z^{-7} + 1.98z^{-8} \quad (49)$$

The plot of  $|H_{d,BP}(z)|$  with respect to the normalized angular frequency (normalizing factor  $\pi$ ) is available in Figure.???. Again we find that the passband and stopband frequencies meet the specifications well enough.

## 4 THE FIR FILTER

We design the FIR filter by first obtaining the (non-causal) lowpass equivalent using the Kaiser window and then converting it to a causal bandpass filter.

### 4.1 The Equivalent Lowpass Filter

The lowpass filter has a passband frequency  $\omega_l$  and transition band  $\Delta\omega = 2\pi\frac{\Delta F}{F_s} = 0.0125\pi$ . The stopband tolerance is  $\delta$ .

- 1) The *passband frequency*  $\omega_l$  is defined as  $\omega_l = \frac{\omega_{p1} + \omega_{p2}}{2}$ . Substituting the values of  $\omega_{p1}$  and  $\omega_{p2}$  from section 2.1, we obtain  $\omega_l = 0.025\pi$ .

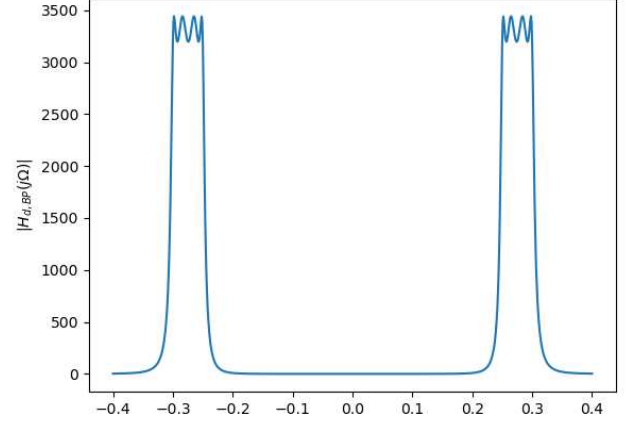


Fig. 4: The magnitude response of the bandpass digital filter designed to meet the given specifications

- 2) The *impulse response*  $h_{lp}(n)$  of the desired lowpass filter with cutoff frequency  $\omega_l$  is given by

$$h_l(n) = \frac{\sin(n\omega_l)}{n\pi} w(n), \quad (50)$$

where  $w(n)$  is the Kaiser window obtained from the design specifications.

### 4.2 The Kaiser Window

The Kaiser window is defined as

$$w(n) = \frac{I_0 \left[ \beta N \sqrt{1 - \left( \frac{n}{N} \right)^2} \right]}{I_0(\beta N)}, \quad -N \leq n \leq N, \beta > 0$$

$$= 0 \quad \text{else,} \quad (51)$$

where  $I_0(x)$  is the modified Bessel function of the first kind of order zero in  $x$  and  $\beta$  and  $N$  are the window shaping factors. In the following, we find  $\beta$  and  $N$  using the design parameters in section 2.1.

- 1)  $N$  is chosen according to

$$N \geq \frac{A - 8}{4.57\Delta\omega}, \quad (52)$$

where  $A = -20\log_{10} \delta$ . Substituting the appropriate values from the design specifications, we obtain  $A = 16.4782$  and  $N \geq 48$ .

- 2)  $\beta$  is chosen according to

$$\beta N = \begin{cases} 0.1(A - 8.7) & A > 50 \\ 0.6(A - 21)^{0.4} + 0.1(A - 21) & 21 \leq A \leq 50 \\ 0 & A < 21 \end{cases} \quad (53)$$

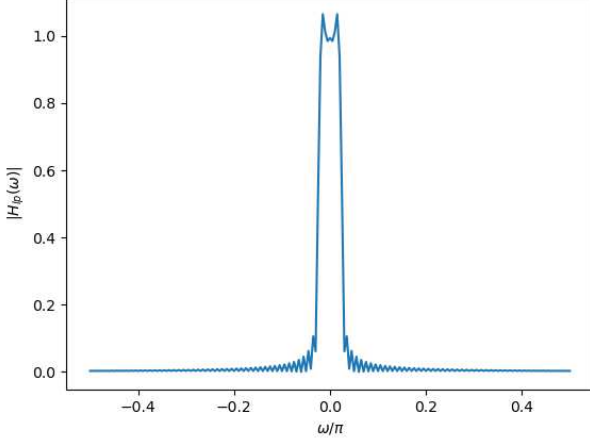


Fig. 3: The magnitude response of the FIR lowpass digital filter designed to meet the given specifications

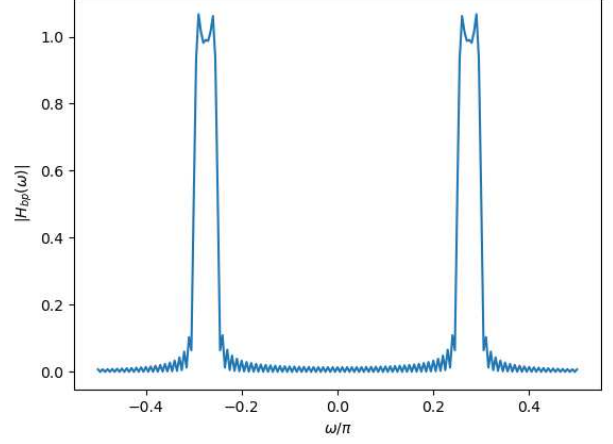


Fig. 3: The magnitude response of the FIR bandpass digital filter designed to meet the given specifications

In our design, we have  $A = 16.4782 < 21$ . Hence, from (??) we obtain  $\beta = 0$ .

- 3) We choose  $N = 100$ , to ensure the desired low pass filter response. Substituting in (??) gives us the rectangular window

$$\begin{aligned} w(n) &= 1, -100 \leq n \leq 100 \\ &= 0 \quad \text{otherwise} \end{aligned} \quad (54)$$

From (??) and (??), we obtain the desired lowpass filter impulse response

$$\begin{aligned} h_{lp}(n) &= \frac{\sin(\frac{n\pi}{40})}{n\pi} - 100 \leq n \leq 100 \\ &= 0, \quad \text{otherwise} \end{aligned} \quad (55)$$

The magnitude response of the filter in (??) is shown in Figure.??.

#### 4.3 The FIR Bandpass Filter

The centre of the passband of the desired bandpass filter was found to be  $\omega_c = 0.275\pi$  in Section 2.1. The impulse response of the desired bandpass filter is obtained from the impulse response of the corresponding lowpass filter as

$$h_{bp}(n) = 2h_{lp}(n)\cos(n\omega_c) \quad (56)$$

Thus, from (??), we obtain

$$\begin{aligned} h_{bp}(n) &= \frac{2\sin(\frac{n\pi}{40})\cos(\frac{11n\pi}{40})}{n\pi} - 100 \leq n \leq 100 \\ &= 0, \quad \text{else} \end{aligned} \quad (57)$$

The magnitude response of the FIR bandpass filter designed to meet the given specifications is plotted in Figure.??.

fir/test\_fir.py