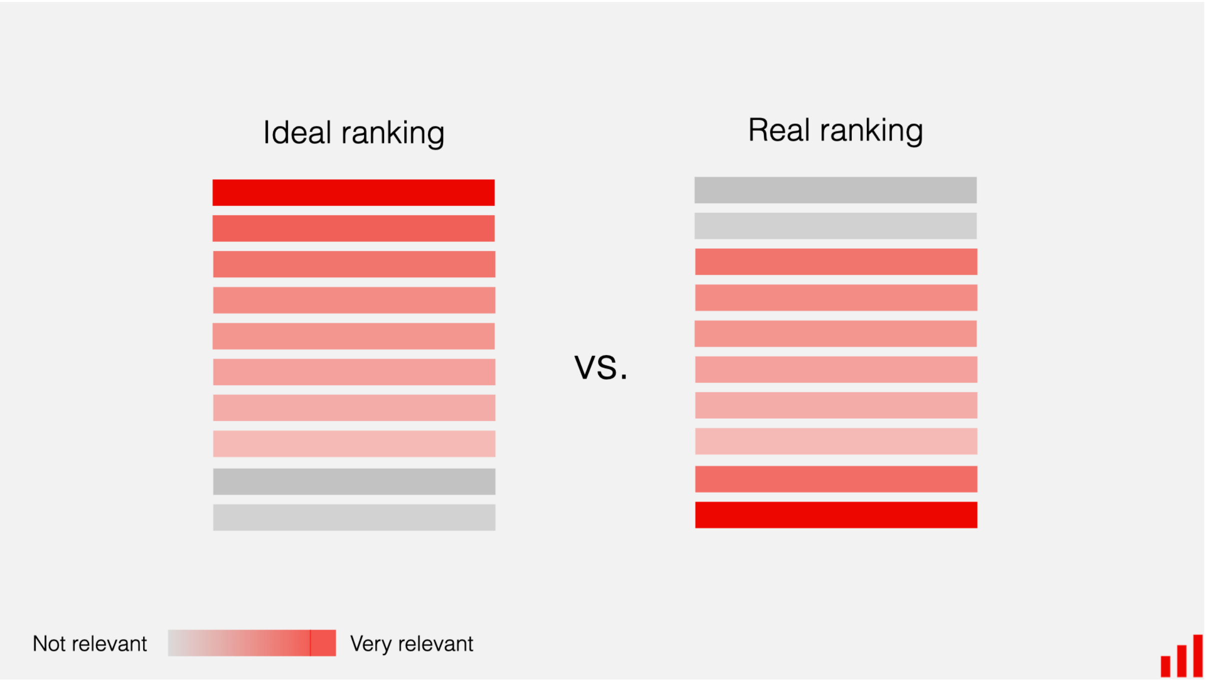
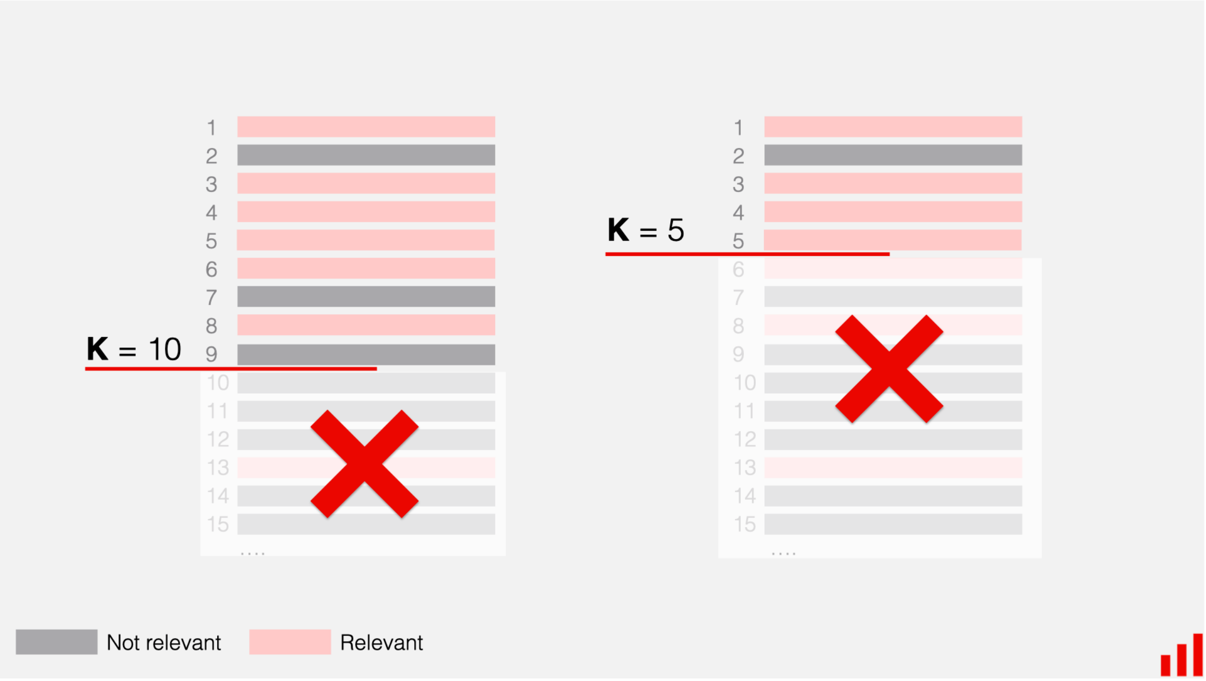
Normalized Discounted Cumulative Gain

Normalized Discounted Cumulative Gain (NDCG) is a metric that evaluates the quality of recommendation and information retrieval systems. It compares rankings to an ideal order where all relevant items are at the top of the list.



$$\text{NDCG@K} = \frac{DCG@K}{IDCG@K}$$

K parameter



Cumulative gain

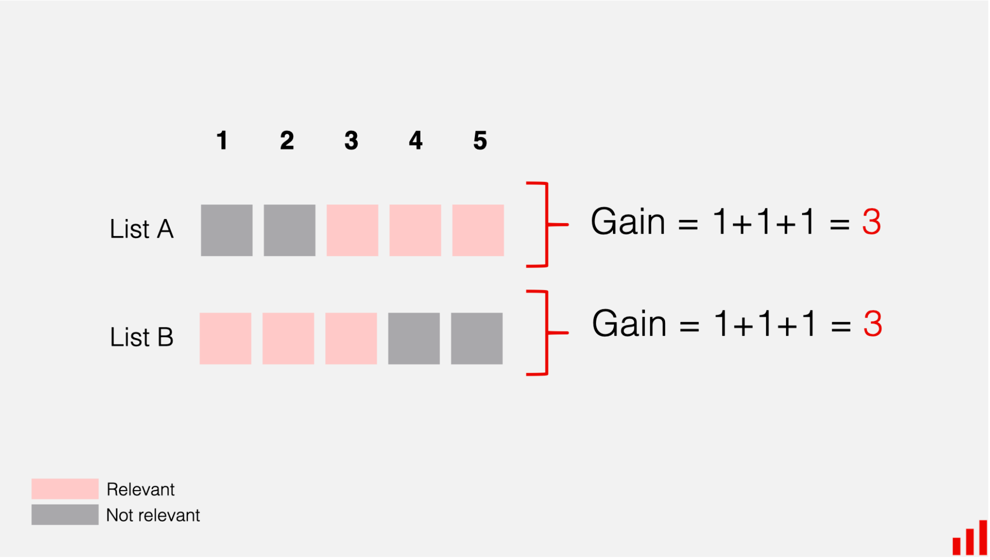
A **gain** of a single recommended item is its relevance score, be it binary or graded.

The **cumulative gain** is a sum of scores of all relevant items among the top-K results in the list.

$$\text{CG@K} = \sum_{k=1}^{K} G_i$$

For example, if you have a binary notion of relevance, and out of 5 recommended items, 3 got a click, the cumulative gain of a given list will be 1+1+1=3. If you have graded scores from 1 to 5, and all your top 5 recommendations got a 5-star, the total gain will be 25.

This does not take into account the ordering. If you change the position of the relevant items, the outcome will be the same.



Intuitively, the second result seems like a much better outcome. The ranking often matters: having the most relevant results at the top might be just what we expect. For example, the best search result should be the first, not the last, on the page.

Therefore, we need to give credit to the system when it can place more relevant items higher and penalize it for placing relevant results lower on the list.

Discounted gain (DCG)

Discounted Cumulative Gain (DCG) is the metric of measuring ranking quality.

To compute the DCG - discounted cumulative gain – we introduce the discount that gives less weight to relevant items that appear lower. This way, instead of simply summing up the item's relevance, we adjust the contribution of each item based on its position.

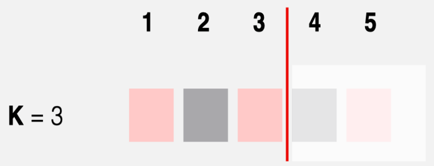
A common way to introduce this discount is a logarithmic penalty. As you move down the list, you divide each item’s gain by a growing number, computed as an inverse logarithm of the position number.

$$\text{DCG@K} = \sum_{k=1}^{K} \frac{rel_i}{\log_2(i+1)}$$

Where **rel (i)** is the relevance score of the item at position i.

DCG computation: Suppose we have a ranked list with binary scores [1, 0, 1, 1, 0], and we want to calculate DCG@3.

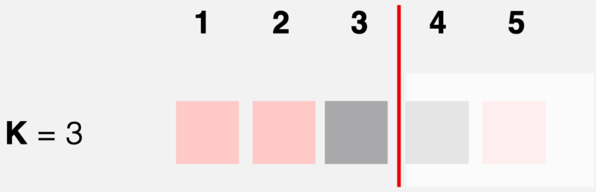
We can sum the relevance scores of all items up to position K = 3, weighted by the corresponding logarithmic discount.



$$\text{DCG@3} = \frac{1}{\log_2(1+1)} + \frac{0}{\log_2(1+2)} + \frac{1}{\log_2(1+3)}$$

$$\text{DCG@3} = 1 + 0 + 0.5 = 1.5$$

if we change the order and put the 3rd relevant item in the second position? Our ranked list looks like a ranked list: [1, 1, 0, 1, 0].



$$\text{DCG@3} = \frac{1}{\log_2(1+1)} + \frac{1}{\log_2(1+2)} + \frac{0}{\log_2(1+3)}$$

$$\text{DCG@3} = 1 + \frac{1}{1.585} + 0 \approx 1.63$$

The Discounted Cumulative Gain (DCG) helps address the ranking order.

Limitation:

Longer lists have higher potential DCG values since more items can contribute to the score. DCG at 10 can be higher than DCG at 3 simply due to the length of the list rather than the inherently better ranking.

To address this, there is need to normalise DCG

To get the normalized DCG (NDCG), you must divide the computed DCG by the ideal DCG (IDCG) for the given list. IDCG represents the maximum achievable DCG with the same set of relevance scores but in the perfect ranking order.

What exactly is ideal?

In the case of binary relevance scores, all relevant items should be at the start of the list.

In the case of graded relevance, you should place all items in a descending order of relevance.

Let's consider that a search engine that outputs 5 documents named (*D_1, D_2, D_3, D_4, D_5*) are output in that order. We need to define the relevance scale (0-3) where:

- **0 :** not relevant
- **1-2 :** somewhat relevant
- **3 :** completely relevant

Suppose these documents have relevance scores:

- D_1 : 3
- D_2 : 2
- D_3 : 0
- D_4 : 0
- D_5 : 1

The ***Cumulative Gain*** is the sum of these relevance scores and can be calculated as:

$$CG = \sum_{i=1}^{5} (rel)_i = 3 + 2 + 0 + 0 + 1 = 6$$

The discounted cumulative gain can be calculated by the formula:

$$DCG = \sum_{i=1}^{5} \frac{rel_i}{\log_2(i+1)}$$

Therefore the discounted cumulative gain of above example is:

$$DCG_5 = \frac{3}{log_2(2)} + \frac{2}{log_2(3)} + \frac{0}{log_2(4)} + \frac{0}{log_2(5)} + \frac{1}{log_2(6)}$$

$$DCG_5 = \frac{3}{log_2(2)} + \frac{2}{log_2(3)} + \frac{0}{log_2(4)} + \frac{0}{log_2(5)} + \frac{1}{log_2(6)}$$

$$DCG_5 = 3 + \frac{2}{1.585} + 0 + 0 + \frac{1}{2.585}$$

$$DCG_5 = 3 + 1.26 + 0.3868$$

$$DCG_5 \simeq 4.67$$

Now we need to arrange these articles in descending order by rankings and calculate DCG to get the Ideal Discounted Cumulative Gain (IDCG) ranking.

- D_1 : 3
- D_2 : 2
- D_5 : 1
- D_3 : 0
- D_4 : 0

$$IDCG_5 = \frac{3}{log_2(2)} + \frac{2}{log_2(3)} + \frac{1}{log_2(4)} + \frac{0}{log_2(5)} + \frac{0}{log_2(6)}$$

$$IDCG_5 = 3 + \frac{2}{1.585} + \frac{1}{2} + 0 + 0$$

$$IDCG_5 = 3 + 1.26 + 0.5$$

$$IDCG_5 = 4.76$$

Now, we calculate our **Normalized DCG** using the following formula

$$nDCG = \frac{DCG_5}{IDCG_5}$$

$$nDCG = \frac{4.67}{4.76}$$

$$nDCG \simeq 0.98$$